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SYMMETRY PROPERTIES OF THE ZERO SETS OF NIL-THETA FUNCTIONS

by

Sharon Goodman

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Abstract

SYMMETRY PROPERTIES
OF THE ZERO SETS OF NIL-THETA FUNCTIONS

by

Sharon Goodman

Advisor: Professor Louis Auslander

Let N denote the three dimensional Heisenberg group, and let Γ be a discrete co-compact subgroup of N. One can decompose $L^2(N/\Gamma)$ into primary summands $H_m(\Gamma)$, $m \in \mathbb{Z}$, with respect to the right regular representation R of N on $L^2(N/\Gamma)$.

The space $\textcircled{H}$ of <u>nil-theta</u> functions on N is defined as the solution space of the differential operator $\frac{\partial}{\partial \bar{Z}} - \frac{i}{2} Z \frac{\partial}{\partial t}$, and the relationship between the functions of $\textcircled{H}_m(\Gamma) = \textcircled{H} \cap H_m(\Gamma)$ and the classical theta functions is reviewed.

In section 1, we give a characterization of the zero set of a function in $\textcircled{H}_m(\Gamma)$, and use this to refine a classical theorem regarding the expression of an elliptic function as a quotient of theta functions. In section 2, we give a new characterization of the indexing on the irreducible closed R-invariant subspaces of $H_m(\Gamma)$, as defined by

L. Auslander and J. Brezin, using the zero set of the nil-theta function lying in this subspace. In section 3, we show how the index behaves under multiplication of theta functions. Finally, we include an appendix relating the distinguished subspace theory of Auslander and Brezin, and the rationality theory on N.

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TABLE OF CONTENTS

Section

SECTION 0

INTRODUCTION

Let N denote the three dimensional Heisenberg group, that is the connected, simply connected Lie group whose underlying manifold is $\mathbb{R}^3$, and whose group multiplication is given by the rule

$$(x, y, t)(a, b, c) = (x+a,\ y+b,\ t+c+\tfrac{1}{2}(ya - xb)) ,$$

where (x, y, t), $(a, b, c) \in N$. It is easily verified that the center $Z(N)$ of N equals the commutator subgroup $[N, N]$ of N, and consists of all the elements of N which are of the form $(0, 0, t)$ for $t \in \mathbb{R}$.

Let F be a complex valued function on N, and let $g \in N$. We define the left and right translates of F by g to be the functions

$$(L(g)F)(h) = F(g^{-1}h) \quad \text{and} \quad (R(g)F)(h) = F(hg) ,$$

$h \in N$, respectively. A family $\mathfrak{F}$ of functions on N will be called R-invariant if $R(g)\mathfrak{F} = \mathfrak{F}$ for all $g \in N$.

Let Γ be a discrete subgroup of N such that the space N/Γ of right Γ-cosets, is compact. N/Γ has a unique (up to constant multiple) Haar measure ω, and we may consider the space of functions, $L^2(N/\Gamma, \omega)$.

We fix the measure ω and omit it from all further notation. We note that a function F on N/Γ can alternatively be thought of as a function on N for which $L(\gamma)F = F$ for all $\gamma \in \Gamma$, i.e., which is constant on Γ-cosets. It makes sense then to define the right translate of such a function F by an element g of N to be

$$(R(g)F)(\Gamma h) = F(\Gamma hg) \quad , \quad h \in N \quad .$$

This yields, moreover, a unitary representation, R, of N on $L^2(N/\Gamma)$ defined by

$$R: \quad g \mapsto R(g) \quad ,$$

which we call the <u>right regular representation</u>.

It is well known (see [3]), that we can decompose $L^2(N/\Gamma)$ into an orthogonal direct sum of R-invariant subspaces:

$$L^2(N/\Gamma) = \sum_{m \in \mathbb{Z}} \oplus H_m(\Gamma) \quad .$$

where $H_m(\Gamma)$ is the closed subspace of $L^2(N/\Gamma)$ spanned by those continuous functions F satisfying

$$F(x, y, t+c) = \exp\left(\frac{2\pi imc}{\beta(\Gamma)}\right) F(x, y, t) \quad ,$$

where $\beta(\Gamma)$ is the positive real number such that $(0, 0, \beta(\Gamma))$ generates the discrete subgroup $\Gamma \cap Z(N)$ of $Z(N)$.

It follows from the Stone-von-Neumann theorem (see [7] p. 71), that the restriction of the representation R to the space $H_m(\Gamma)$ is a multiple of a single irreducible unitary representation $U(m)$ of N, as long as $m \neq 0$. Moreover, for $m \neq n$, $U(m)$ and $U(n)$ are not unitarily equivalent. It follows then that two irreducible closed

R-invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $L^2(N/\Gamma)$ are unitarily equivalent if and only if both $\mathcal{H}_1$ and $\mathcal{H}_2$ are subspaces of $H_m(\Gamma)$, for some integer m .

The space $H_0(\Gamma)$ can be identified with $L^2(\mathbb{R}^2/\pi(\Gamma))$, where $\pi: N \to N/Z(N) = \mathbb{R}^2$ is the canonical projection: $\pi(x, y, t) = (x, y)$. $H_0(\Gamma)$ is thus essentially $L^2(T^2)$, where T^2 denotes the 2-torus. Therefore, it can be decomposed in one and only one way into an orthogonal direct sum of irreducible (in fact, one dimensional) closed R-invariant subspaces.

For $m \neq 0$, one may show (see [3]), that R restricted to $H_m(\Gamma)$ is precisely the $|m|$-fold multiple of $U(m)$. There are, however, an infinite number of ways in which one can decompose $H_m(\Gamma)$ into a direct sum of irreducible closed R-invariant subspaces. The distinguished subspace theory of L. Auslander and J. Brezin [3], picks out a finite number of these decompositions that are in some ways "nicer" than others. In [3], they define an indexing on the set of irreducible closed R-invariant subspaces of $H_m(\Gamma)$ in such a way that distinguished subspaces have index 1 , and the smaller the index of a subspace is, the "nicer" the space is in various senses. Several other characterizations of the distinguished subspaces are given in [2] and [4].

In section 2 of this paper, we will discuss the indexing in $H_m(\Gamma)$, for m a positive integer, from a different point of view. It has been shown, first in [4] and then in [10] and [2], that a certain class of functions in $L^2(N/\Gamma)$, called <u>nil-theta functions</u>, that arise as solutions of a certain differential operator, are very closely related to the classical theta functions of a certain "type," as discussed in [9] and [6]. Every irreducible closed R-invariant subspace $\mathcal{H}$ of $H_m(\Gamma)$ contains a unique (up to constant multiple) nil-theta function, and conversely, every nil-theta function in $H_m(\Gamma)$ is contained in a unique irreducible closed R-invariant subspace. Since the classical theta functions are completely determined by their zero sets, it is natural to attempt to classify the irreducible subspaces of $H_m(\Gamma)$ according to the properties of the zero set of the nil-theta function lying in this subspace. We do this in section 2. In particular, we show that the zero sets of the nil-theta functions lying in the distinguished subspaces are particularly nice.

Let $\textcircled{H}_m(\Gamma)$ denote the space of nil-theta functions that lie in $H_m(\Gamma)$. It is shown in [2] that this is an m dimensional complex vector space. Let

$$\mathcal{A} = \sum_{m\in\mathbb{Z}} \oplus \textcircled{H}_m(\Gamma) .$$

It is proven in [5] that $\mathcal{A}$ is an algebra without zero divisors. In particular, if $F \in \textcircled{H}_m(\Gamma)$ and $G \in \textcircled{H}_n(\Gamma)$, then $FG \in \textcircled{H}_{m+n}(\Gamma)$. In section 3, we investigate the relationship between this algebra structure and the indexing structure; that is, we consider how the index of the irreducible subspace of $H_{m+n}(\Gamma)$ containing FG relates to the indices of the irreducible subspaces of $H_m(\Gamma)$ and $H_n(\Gamma)$ containing F and G, respectively. Some relationship is indicated, since for example, the index of the irreducible subspace containing the nil-theta function F^s for $F \in \textcircled{H}_m(\Gamma)$, s any positive integer, is easily seen to be sk, where k is the index of the subspace containing F. This suggests the possibility that the indices might add as the functions multiply. Actually, the behavior of the index under multiplication is, in general, more complicated than this, and we discuss this in section 3.

In section 1, we give a complete characterization of the zero sets of the functions in $\textcircled{H}_m(\Gamma)$, and use this to give a refinement of a certain classical theorem regarding the expression of an elliptic function as a quotient of theta functions.

Finally, we include an appendix discussing the relationship between the distinguished subspace theory and the rational structure theory of N.

SECTION 1

NIL-THETA FUNCTIONS AND ELLIPTIC FUNCTIONS

Let Γ_1 be a discrete co-compact subgroup of N. It is shown in [2] that Γ_1 is a non-abelian subgroup of N with either two or three generators. We will call Γ_1 a lifted subgroup of N if it has two generators. This is equivalent to the condition:

$$\Gamma_1 \cap Z(N) = [\Gamma_1, \Gamma_1] = \{(0, 0, m\beta(\Gamma_1)) \mid m \in \mathbb{Z}\} .$$

It is furthermore shown in [2] that for any discrete co-compact subgroup Γ_1 of N, there exists a maximal lifted subgroup Γ of Γ_1 such that

$$\beta(\Gamma) = d\beta(\Gamma_1) \quad \text{and} \quad H_{md}(\Gamma) = H_m(\Gamma_1) ,$$

where d is the order of Γ_1/Γ. It therefore suffices, for our purposes, to consider only the lifted subgroups of N.

A detailed discussion of the automorphism group, $\mathcal{A}$, of N is given in [10]. We will mention, without proof, certain facts that we will need.

1. Every automorphism α of N can be described by a 3×3-matrix of the form

$$\alpha = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{S}} & 0 \\ & & 0 \\ a_1 & a_2 & c \end{pmatrix}$$

where a_1, a_2 and c are real numbers, and S is a real 2×2-matrix such that

$$S^t J S = c \cdot J$$

where S^t denotes the transpose of the matrix S, and the J denotes the linear transformation on $\mathbb{R}^2$ that arises from the complex structure of $\mathbb{C}$, given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

2. We can write the identity component, $\mathfrak{A}^0$, of $\mathfrak{A}$ as a semi-direct product:

$$\mathfrak{A}^0 = (\mathrm{Sp}(2, \mathbb{R}) \ltimes D_2 \rtimes)\mathrm{inn}(N)$$

where

$$\mathrm{Sp}(2, \mathbb{R}) = \left\{ \begin{pmatrix} S & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} : \ S^t J S = J \right\} = \mathrm{SL}(2, \mathbb{R})$$

$$D_2 = \left\{ \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^2 \end{pmatrix} : \ c > 0 \right\}$$

$$\mathrm{inn}(N) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_1 & a_2 & 1 \end{pmatrix} : \ a_1, a_2 \in \mathbb{R} \right\}$$

The group $\mathrm{inn}(N)$ is easily seen to be the group of inner automorphisms of N.

Let $L(N)$ denote the Lie algebra of N. Viewing $L(N)$ as the space of left invariant vector fields on N, we have the following basis for this space:

$$X(x, y, t) = \frac{\partial}{\partial x} + \frac{1}{2}\, y \frac{\partial}{\partial t} \,\Big|\, (x, y, t)$$

$$Y(x, y, t) = \frac{\partial}{\partial y} - \frac{1}{2}\, x \frac{\partial}{\partial t} \,\Big|\, (x, y, t)$$

$$T(x, y, t) = \frac{\partial}{\partial t} \,\Big|\, (x, y, t)$$

for $(x, y, t) \in N$. Let α be an automorphism of N. Then α induces an automorphism on the complexification, $L_{\mathbb{C}}(N)$, of $L(N)$, whose matrix with respect to the basis X, Y, T, is the same as the matrix of α. In particular, let J be the automorphism of N whose matrix is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and let the corresponding automorphism induced on $L_{\mathbb{C}}(N)$ also be denoted by J. Since J^4 is the identity matrix, J determines a direct sum decomposition of $L_{\mathbb{C}}(N)$:

$$L_{\mathbb{C}}(N) = V_i + V_{-i} + Z_{\mathbb{C}}$$

where $V_{\pm i}$ denotes the eigenvalue $\pm i$ subspace of $L_{\mathbb{C}}(N)$ with respect to J, and $Z_{\mathbb{C}}$ denotes the center of $L_{\mathbb{C}}(N)$. V_i can be shown (see [10]), to have the basis vector, which we also denote by V_i,

$$V_i = X + iY = \frac{\partial}{\partial Z} - \frac{i}{2}\, Z \frac{\partial}{\partial t}$$

where $Z = x + iy$, $\frac{\partial}{\partial \bar{Z}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$.

We define the space $\textcircled{H}$ of <u>nil-theta functions</u> on N to be

$$\textcircled{H} = \{F \in C^{\infty}(N) \mid V_i F = 0\} ,$$

where $C^{\infty}(N)$ denotes the C^{∞} functions on N .

Suppose that Γ is a lifted subgroup of N . Because V_i is a left invariant operator, it follows that it is a well-defined operator on $C^{\infty}(N/\Gamma)$. Thus, we can define

$$\textcircled{H}(\Gamma) = C^{\infty}(N/\Gamma) \cap \textcircled{H} .$$

Let P_m denote the orthogonal projection of $L^2(N/\Gamma)$ onto $H_m(\Gamma)$, $m \neq 0$. It is shown in [2] that $P P_m(\textcircled{H}(\Gamma)) \subseteq \textcircled{H}(\Gamma)$. Therefore, we can define

$$\textcircled{H}_m(\Gamma) = \textcircled{H} \cap H_m(\Gamma)$$

and we have the following L^2 decomposition

$$\textcircled{H}(\Gamma) = \sum_{m \in \mathbb{Z}} \oplus \textcircled{H}_m(\Gamma) .$$

We now recall the relationship between the nil-theta functions and the classical theta functions. For proofs not given here, see [2] or [10].

<u>Lemma 1.1</u>: Let $F \in C^{\infty}(N)$ be such that $V_i F = 0$ and such that

$$(*) \qquad F(x, y, t) = \exp(2\pi i a t)\, F(x, y, 0) , \quad a \in \mathbb{R} .$$

Then the function

$$(M(a)F)(Z) = \exp(-2\pi i a t)\, \exp(-\pi i a x y)\, \exp(\pi a y^2)\, F(x, y, t)$$

is an entire function of $Z = x + iy$.

Conversely, if $H(z)$ is an entire function of $Z = x + iy$, then

$$(M(a)^{-1}H)(x, y, t) = \exp(2\pi iat)\exp(\pi iaxy)\exp(-\pi ay^2)\,H(x + iy)$$

is a solution of V_i satisfying (*).

<u>Corollary 1.2</u>: Let Γ be a lifted subgroup of N. Then $\textcircled{H}_m(\Gamma)$ is trivial for $m < 0$, and consists of the constant functions for $m = 0$.

Let Γ be a lifted subgroup of N, with generators γ_1 and γ_2, and let $\pi: N \to \mathbb{R}^2$ be the canonical projection defined above. Then $L = \pi(\Gamma)$ is the lattice in $\mathbb{R}^2$ generated by the linearly independent vectors $\pi(\gamma_1)$ and $\pi(\gamma_2)$. We will henceforth use the notation

$$G = \mathrm{grp}\{g_1, g_2\}$$

to mean that G is a group with generators g_1 and g_2.

<u>Corollary 1.3</u>: Let F and G be elements of $\textcircled{H}_m(\Gamma)$, for $m < 0$. Then $\frac{F}{G}$ is a meromorphic function on the torus $\mathbb{C}/L$, $L = \pi(\Gamma)$.

We will now discuss what the Γ-invariance of a function F in $\textcircled{H}_m(\Gamma)$ implies for the entire function, $M(\frac{m}{\beta(\Gamma)})F$. If Γ is an arbitrary lifted subgroup of N, it is shown in [10] that the function $G = M(\frac{m}{\beta(\Gamma)})F$ is an entire theta function of a certain "type," as defined in [9] and [6], with respect to the lattice $L = \pi(\Gamma)$; i.e., there exists an inhomogeneous linear function $\varphi(Z, \ell)$ of Z such that

$$G(Z + \ell) = \exp(\varphi(Z, \ell))\, F(Z) \quad , \text{ for all } \ell \in L \ .$$

We note that if $L = \mathrm{grp}\{\ell_1, \ell_2\}$ is a fixed lattice in the plane, then there are many lifted subgroups Γ of N such that $\pi(\Gamma) = L$. In fact, it is shown in [2] , that the set of lifted subgroups Γ such that $\pi(\Gamma) = L$, is in one-one correspondence with the group, $\mathrm{inn}(N)$, of inner automorphisms of N . As Γ runs through the set of these lifted subgroups, $M(\frac{m}{\beta(\Gamma)})\,\textcircled{H}_m(\Gamma)$, $m > 0$, runs through the spaces of entire theta functions of different types with respect to the lattice L . Let us mention that if L is generated by $\ell_1 = (a_1, a_2)$ and $\ell_2 = (b_1, b_2)$ then for any $\Gamma = \mathrm{grp}\{\gamma_1, \gamma_2\}$ such that $\pi(\Gamma) = L$, we have that

$$\beta(\Gamma) = \gamma_1^{-1}\ \gamma_2^{-2}\ \gamma_1\ \gamma_2 = a_1 b_2 - a_2 b_1 \ .$$

For our purposes, the following special case of our preceding remarks will suffice. This theorem is proven in [2].

<u>Theorem 1.4</u>: Let $\tau = \alpha + i\beta$ be an element of the complex upper half-plane, and let $a, b \in \mathbb{R}$. We define the lifted subgroup:

$$\Gamma(\tau;\ a,\ b) = \mathrm{grp}\{(1,\ 0,\ b\beta),\ (\alpha,\ \beta,\ -a\beta)\} \ .$$

Let $F \in \textcircled{H}_m(\Gamma(\tau;\ a,\ b))$, $m > 0$, and let

$H(Z) = (M(\frac{m}{\beta})\, F)(x, y, t)$, $Z = x + iy$. Then $H(Z)$ satisfies the following functional equations:

(a) $H(Z + 1) = \exp(2\pi imb)\, H(Z)$

(b) $H(Z + \tau) = \exp(-2\pi ima)\, \exp(-\pi im(2Z + \tau))\, H(Z)$.

Conversely, if $H(Z)$ is analytic and satisfies (a) and (b), then $M(\frac{m}{\beta})^{-1}H \in \textcircled{H}_m(\Gamma(\tau; a, b))$.

We see then that $M(\frac{m}{\beta})$ is an isomorphism between the space $\textcircled{H}_m(\Gamma(\tau; a, b))$ and the m-dimensional complex vector space of classical Jacobi theta functions of period τ and characteristic $\begin{bmatrix} 2mb \\ 2ma \end{bmatrix}$ in the variable $Z = x + iy$, as discussed in [8]. Adopting the notation of [8], we denote the latter space by $\textcircled{H}_m \begin{bmatrix} 2mb \\ 2ma \end{bmatrix} (Z, \tau)$.

Let $\Gamma(\tau) = \Gamma(\tau; 0, 0) = \mathrm{grp}\{(1, 0, 0), (\alpha, \beta, 0)\}$. Then $\Gamma(\tau; a, b) = n^{-1}\Gamma(\tau)n$, for $n = (a+b\alpha, b\beta, \frac{ab\beta}{2})$. Now $M(\frac{m}{\beta})\ \textcircled{H}_m(\Gamma(\tau)) = \textcircled{H}_m \begin{bmatrix} 0 \\ 0 \end{bmatrix} (Z, \tau)$, so that the automorphism, i_n , of N defined by

$$i_n(g) = n^{-1}gn$$

results in a shift in the characteristic of the corresponding theta functions.

We will need the following results in what follows. Anything not proven here can be found in [2] or [10].

<u>Theorem 1.5</u>: Let α be an automorphism of N , and let Γ be a lifted subgroup of N . Then $\alpha^{-1}(\Gamma)$ is a lifted subgroup of N and

$$H_m(\Gamma) \circ \alpha = H_m(\alpha^{-1}(\Gamma)) \quad .$$

Corollary 1.6: Let $\tau = \alpha + i\beta$, $\beta > 0$ and let $a, b \in \mathbb{R}$. Then

$$H_m(\Gamma(\tau;\, a,\, b)) = H_m(\Gamma(\tau)) \circ i_n$$

where $n = (a+b\alpha,\, b\beta,\, \frac{ab\beta}{2})$.

Theorem 1.7: Let $g \in N$. Then $L(g)\textcircled{H} = \textcircled{H}$.

Theorem 1.8: Let σ be an automorphism of N of the form

$$\sigma = \begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & u^2 + v^2 \end{pmatrix}$$

Then $\textcircled{H} \circ \sigma = \textcircled{H}$.

Proof: The proof of this theorem follows by a simple application of the chain rule.

Let us note that since the automorphism J of N which gives rise to the nil-theta functions comes from the complex structure on $\mathbb{C}$, it makes sense that the automorphism σ in Theorem 1.8 which arises from complex multiplication should preserve the space $\textcircled{H}$.

Theorem 1.9: The isomorphisms

$$M(\tfrac{m}{\beta}):\ \textcircled{H}_m(\Gamma(\tau)) \to \textcircled{H}_m \begin{bmatrix} 0 \\ 0 \end{bmatrix} (Z,\, \tau) \ , \quad m \in \mathbb{Z}$$

extend to an algebra isomorphism

$$M:\ \sum_{m \in \mathbb{Z}} \oplus\, \textcircled{H}_m(\Gamma(\tau)) \to \sum_{m \in \mathbb{Z}} \oplus\, \textcircled{H}_m \begin{bmatrix} 0 \\ 0 \end{bmatrix} (Z,\, \tau) \ .$$

Corollary 1.10: The algebra $\mathcal{A} = \sum_{m \in \mathbb{Z}} \oplus \textcircled{H}_m(\Gamma(\tau))$ has no divisors of zero.

Theorem 1.11: Let $\mathfrak{h}$ be an irreducible closed R-invariant subspace of $H_m(\Gamma)$, $m > 0$. Then

$$\dim_{\mathbb{C}}(\mathfrak{h} \cap \textcircled{H}) = 1 \quad .$$

Proof: Let γ_1 and γ_2 generate Γ, and let Γ_1 be the lifted subgroup of N , containing Γ, whose generators are $\frac{1}{m}\gamma_1$ and $\frac{1}{m}\gamma_2$. By the results in [2], $\mathfrak{h}_1 = H_1(\Gamma_1)$ is an irreducible subspace of $H_m(\Gamma)$. Since $H_m(\Gamma)$ is a multiplicity space for the representation R , of multiplicity m , there exists an isomorphism U making the following diagram commute

$$\begin{tikzcd}
H_m(\Gamma) \arrow[r, "U"] \arrow[d, "R(g)"'] & \mathfrak{h}_1 \otimes \mathbb{C}^m \arrow[d, "R(g) \otimes 1"] \\
H_m(\Gamma) \arrow[r, "U"] & \mathfrak{h}_1 \otimes \mathbb{C}^m
\end{tikzcd}$$

for all $g \in N$, where $1(v) = v$, $v \in \mathbb{C}^m$. The isomorphism U is described explicitly in [3] in the following way. We refer the reader to [3] for further details and proofs.

Let $\Gamma[m] = \mathrm{grp}\{\gamma_1, \gamma_2, \frac{\beta(\Gamma)}{m}\}$. The group

$$\Omega = \{\Gamma[m](\tfrac{j}{m}, 0, 0) \mid j \in \mathbb{Z}\}.$$

is a subgroup of $N/\Gamma[m]$ of order m , and we can decompose

$$H_m(\Gamma) = \sum_{\omega\in\Omega} \oplus \; L(\omega)\mathcal{J}_1 \quad .$$

Let $\mathcal{H}$ denote the Hilbert space of all functions from Ω to $\mathbb{C}$ with the usual inner product. For each $\omega \in \Omega$, we set f_ω equal to the function on Ω vanishing everywhere except at ω , where it takes the value 1 . Thus, $\{f_\omega \mid \omega \in \Omega\}$ forms a basis for $\mathcal{H}$.

We now define the unitary operator

$$U: \; H_m(\Gamma) = \sum_{\omega\in\Omega} \oplus \, L(\omega)\,\mathcal{J}_1 \to \mathcal{J}_1 \otimes \mathcal{H}$$

by

$$U: \; \sum_{\omega\in\Omega} L(\omega)\varphi_\omega \mapsto \sum_{\omega\in\Omega} \varphi_\omega \otimes f_\omega \quad , \qquad \varphi_\omega \in \mathcal{J}_1 \; .$$

It is shown in [3] that this isomorphism satisfies the required properties.

It is proven in [2] that the space $\mathcal{J}_1 \cap \textcircled{H}_m(\Gamma) = \textcircled{H}_1(\Gamma_1)$ is a one-dimensional complex vector space. Let θ_1 be a basis vector in this space. By theorem 1.7,

$$L(\omega)\theta_1 \in \textcircled{H}_m(\Gamma) \quad , \quad \omega \in \Omega \, ,$$

so that

$$\textcircled{H}_m(\Gamma) = \sum_{\omega\in\Omega} \oplus \, L(\omega)\theta_1$$

and

$$U(\textcircled{H}_m(\Gamma)) = \theta_1 \otimes \mathcal{H} \quad .$$

Suppose now that $\mathcal{J} \subseteq H_m(\Gamma)$ is an arbitrary irreducible closed R-invariant subspace. Then clearly, $U(\mathcal{J}) = \mathcal{J}_1 \otimes u$, for some fixed $u \in \mathcal{H}$. Therefore, $U^{-1}(\theta_1 \otimes u)$ is the unique (up to constant multiple) element of $\mathcal{J} \cap \textcircled{H}$.

We now discuss the zero sets of the nil-theta functions.

<u>Theorem 1.12</u>: The zeroes of a classical theta function of a fixed type, or equivalently, the zeroes of an element of $M(\frac{m}{\beta(\Gamma)})\textcircled{H}_m(\Gamma)$, $m > 0$, for a lifted subgroup Γ of N , are isolated, and determine the function up to constant multiple.

<u>Proof</u>: Let F and G be two classical theta functions of the same type with precisely the same zeroes. Then $\frac{F}{G}$ is a bounded entire function, and hence constant by Liouiville's Theorem.

<u>Theorem 1.13</u>: Let $\tilde{S}$ be the set of zeroes of the function $F \in \textcircled{H}_m(\Gamma)$, Γ a lifted subgroup of N , and let S be the zero set of the function $M(\frac{m}{\beta(\Gamma)})\,F$. Then

$$\tilde{S} = \{(x, y, t) \mid x + iy \in S \text{ and all } t \in \mathbb{R}\} ;$$

i.e.,

$$\tilde{S} = S \cdot Z(N) .$$

<u>Proof</u>: This follows easily from the fact that the multiplier $M(\frac{m}{\beta(\Gamma)})$ never vanishes.

Theorem 1.14: Let H be a classical theta function with respect to the lattice $L \subseteq \mathbb{C}$. Then the zero set S of H is invariant under translations by elements of L ; i.e., $S + \ell = S$, for all $\ell \in L$.

Proof: For any $Z \in \mathbb{C}$ and $\ell \in L$, we have that

$$H(Z + \ell) = \exp(\varphi(Z, \ell))\, H(Z) \quad ,$$

and so we have our result.

Definition: Let L be a lattice in the plane. The fundamental parallelogram, D , of L is the open parallelogram spanned by the two generators of L , together with two sides of the parallelogram that contain 0 .

The previous two theorems indicate that the zero set of a function $F \in \textcircled{H}_m(\Gamma)$, $m > 0$, is completely determined by the subset of the complex plane consisting of the zeroes of the analytic function $M(\frac{m}{\beta(\Gamma)})F$ which are comtained in the fundamental parallelogram D of the lattice $L = \pi(\Gamma)$.

Theorem 1.15: Let $F \in \textcircled{H}_m(\Gamma)$, $m > 0$. Then $M(\frac{m}{\beta(\Gamma)})F$ has m zeroes in the fundamental parallelogram D , of $L = \pi(\Gamma)$.

Proof: For $\Gamma = \Gamma(\tau;\, a, b)$, $\tau = \alpha + i\beta$, $\beta > 0$, $a, b, \in \mathbb{R}$, the theorem can be proven by a straightforward application of the logarithmic derivative principle, similar to the computation on page 79 of [8]. The proof for arbitrary Γ follows from theorem 1.16, corollary 1.17 and lemma 1.18 below.

<u>Theorem 1.16</u>: Let Γ be an arbitrary lifted subgroup of N. Then, there exists an element $\tau = \alpha + i\beta$ in the complex upper half-plane, real numbers a and b, and an automorphism σ of N of the form

$$\sigma = \begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & u^2 + v^2 \end{pmatrix} \qquad u, v \in \mathbb{R}$$

such that $\sigma(\Gamma) = \Gamma(\tau; a, b)$.

<u>Proof</u>: Suppose that Γ is generated by the elements (a_1, a_2, a_3) and (b_1, b_2, b_3). We take

$$u = \frac{a_1}{a_1^2 + a_2^2}, \quad v = \frac{a_2}{a_1^2 + a_2^2}, \quad \alpha = \frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$$

$$\beta = \frac{b_2 a_1 - a_2 b_1}{a_1^2 + a_2^2} = \frac{\beta(\Gamma)}{a_1^2 + a_2^2}, \quad a = \frac{-b_3}{b_2 a_1 - a_2 b_1}$$

and

$$b = \frac{a_3}{b_2 a_1 - a_2 b_1}.$$

A straightforward computation shows that

$$\sigma(a_1, a_2, a_3) = (1, 0, b\beta)$$

and

$$\sigma(b_1, b_2, b_3) = (\alpha, \beta, -a\beta)$$

so that $\sigma(\Gamma) = \Gamma(\tau; a, b)$.

<u>Corollary 1.17</u>: Let σ be as in theorem 1.16 .

Then

$$\textcircled{H}_m(\Gamma) = \textcircled{H}_m(\Gamma(\tau;\ a,\ b)) \circ \sigma$$

<u>Proof</u>: This follows from theorems 1.5, 1.8, and 1.16.

<u>Lemma 1.18</u>: Let F be a function on N , and σ an automorphism of N . Then g is a zero of F if and only if $\sigma^{-1}(g)$ is a zero of $F \circ \sigma$.

We are now ready to state our first main result.

<u>Theorem 1.19</u>: Let Γ be a lifted subgroup of N with generators (a_1, a_2, a_3) and (b_1, b_2, b_3) , and let D be the fundamental parallelogram of the lattice $L = \pi(\Gamma) \subseteq \mathbb{C}$. Let $S = \{Z_1, Z_2, \ldots, Z_m\}$ be a subset of D , $m > 0$. Then, there exists a function $F \in \textcircled{H}_m(\Gamma)$ such that $M(\frac{m}{\beta(\Gamma)})F$ has S as the set of its zeroes in D , if and only if $\sum_{i=1}^{m} Z_i$ is congruent modulo L to

$$\begin{cases} mv_0 & ,\ m \text{ even} \\ \dfrac{a_1 + b_1}{2} + i(\dfrac{a_2 + b_2}{2}) + mv_0 & ,\ m \text{ odd} \end{cases}$$

where $v_0 = \dfrac{a_3b_1 - a_1b_3}{b_2a_1 - a_2b_1} + i\ (\dfrac{a_3b_2 - a_2b_3}{b_2a_1 - a_2b_1})$,

and where w is congruent to u modulo L means that $w - u \in L$.

Proof: We consider several cases.

Case I: Suppose that $\Gamma = \Gamma(\tau) = grp\{(1, 0, 0,), (\alpha, \beta, 0)\}$ $\tau = \alpha + i\beta$, $\beta > 0$. Let $L(\tau) = \pi(\Gamma(\tau))$ and let $D(\tau)$ be the fundamental parallelogram of $L(\tau)$. If $S = \{Z_1, Z_2, \ldots, Z_m\} \subseteq D(\tau)$, we may write $Z_i = r_i + s_i\tau$ where $0 \leq r_i$, $s_i \leq 1$, $i = 1, 2, \ldots, m$. It is a well known fact (see [8] page 81), that the classical Jacobi theta function, $\theta\begin{bmatrix}\epsilon \\ \epsilon'\end{bmatrix}(Z, \tau)$, of period τ, and characteristic $\begin{bmatrix}\epsilon \\ \epsilon'\end{bmatrix}$, $(\epsilon, \epsilon' \in \mathbb{R})$, defined by

$$\theta\begin{bmatrix}\epsilon \\ \epsilon'\end{bmatrix}(Z, \tau) = \sum_{n \in \mathbb{Z}} \exp \pi i\{\tau(n + \tfrac{\epsilon}{2})^2 + 2(n + \tfrac{\epsilon}{2})(Z + \tfrac{\epsilon'}{2})\}$$

has a unique zero in $D(\tau)$, that is at the point $Z_0 \in D(\tau)$, where Z_0 is congruent modulo $L(\tau)$ to $(\frac{1}{2} + \frac{\tau}{2}) + (\frac{\epsilon'}{2} - \frac{\epsilon}{2}\tau)$. It follows then, that up to constant multiple, the unique classical theta function with zero set S in $D(\tau)$ is the function

$$H(Z) = \prod_{i=1}^{m} \theta\begin{bmatrix}2(\frac{1}{2} - s_i) \\ -2(\frac{1}{2} - r_i)\end{bmatrix}(Z, \tau) \in \textcircled{H}_m \begin{bmatrix}2 \sum_{i=1}^{m}(\frac{1}{2} - s_i) \\ -2 \sum_{i=1}^{m}(\frac{1}{2} - r_i)\end{bmatrix}(Z, \tau) .$$

Let $F = M(\frac{m}{\beta})^{-1} H$. By theorem 1.4, we have that $F \in \textcircled{H}_m(\Gamma(\tau; r, s))$, where $r = -\frac{1}{m}\sum_{i=1}^{m}(\frac{1}{2} - r_i)$, and $s = \frac{1}{m}\sum_{i=1}^{m}(\frac{1}{2} - s_i)$. F will then be the required nil-theta function if and only if

$$F \in \textcircled{H}_m(\Gamma(\tau)) \cap \textcircled{H}_m(\Gamma(\tau; r, s)) .$$

In that case, however, we have that $L(\gamma)F = F$ for all γ in the group generated by $\Gamma(\tau)$ and $\Gamma(\tau; r, s)$; i.e., in $\mathrm{grp}\{(1, 0, 0), (\alpha, \beta, 0), (1, 0, s\beta), (\alpha, \beta, -r\beta)\}$. Hence, for any $g \in N$

$$F(g) = F[(1, 0, s\beta)g] = F[(1, 0, 0)(0, 0, s\beta)g] = F[(0, 0, s\beta)g] = \exp\left(\frac{2\pi i m s\beta}{\beta}\right)F(g) = \exp(2\pi i m s)F(g)$$

and similarly

$$F(g) = F[(\alpha, \beta, -r\beta)g] = \exp(-2\pi i m r)F(g) \quad .$$

Then we must have that both mr and ms are integers, or, equivalently, that

$$\sum_{i=1}^{m} \left(\frac{1}{2} - s_i\right) \quad \text{and} \quad \sum_{i=1}^{m} \left(\frac{1}{2} - r_i\right)$$

are integers. This holds if and only if both $\sum_{i=1}^{m} r_i$ and $\sum_{i=1}^{m} s_i$ are congruent modulo $\mathbb{Z}$ to

$$\begin{cases} 0 \ , & m \text{ even} \\ \frac{1}{2} \ , & m \text{ odd} \end{cases}$$

or, if and only if $\sum_{i=1}^{m} Z_i = \sum_{i=1}^{m} (r_i + s_i\tau)$ is congruent modulo $L(\tau)$ to

$$\begin{cases} 0 \ , & m \text{ even} \\ \frac{1}{2} + \frac{\tau}{2} \ , & m \text{ odd} \end{cases} \quad .$$

This proves the theorem in this special case.

Case II: Suppose that $\Gamma = \Gamma(\tau; a, b)$, $\tau = \alpha + i\beta$, $\beta > 0$, $a, b \in \mathbb{R}$, and let $L(\tau) = \pi(\Gamma)$ and $D(\tau)$ be as above. By corollary 1.6,

$$H_m(\Gamma(\tau; a, b)) = H_m(\Gamma(\tau)) \circ i_n \text{ , where}$$

i_n denotes inner automorphism by $n = (a+b\alpha, b\beta, \frac{ab\beta}{2})$. Thus,

$$H_m(\Gamma(\tau; a, b)) = H_m(\Gamma(\tau)) \circ i_n = L(n)R(n)\textcircled{H}_m(\Gamma(\tau)) =$$
$$= L(n)H_m(\Gamma(\tau)) ,$$

where the last equality follows from the R-invariance of $H_m(\Gamma(\tau))$. It follows then, by theorem 1.7, that

$$\textcircled{H}_m(\Gamma(\tau, a, b)) = L(n)\textcircled{H}_m(\Gamma(\tau)) .$$

We clearly have then that F is a function in $\textcircled{H}_m(\Gamma(\tau, a, b))$ such that the zero set in $D(\tau)$ of $M(\frac{m}{\beta})F$ is $S = \{Z_1, Z_2, \ldots, Z_m\}$, if and only if $L(n^{-1})F$ is a function in $\textcircled{H}_m(\Gamma(\tau))$ such that the zero set in $D(\tau)$ of $M(\frac{m}{\beta})(L(n^{-1})F)$ is the set

$$\{-(a + b\tau) + Z_1, -(a + b\tau) + Z_2, \ldots, -(a + b\tau) + Z_m\}$$

which is true, using our previous case, precisely when $-m(a + b\tau) + \sum_{i=1}^{m} Z_i$ is congruent modulo $L(\tau)$ to

$$\begin{cases} 0 , & m \text{ even} \\ \frac{1}{2} + \frac{\tau}{2} , & m \text{ odd} \end{cases} ,$$

or equivalently, if and only if $\sum_{i=1}^{m} Z_i$ is congruent modulo $L(\tau)$ to

$$\begin{cases} m(a + b\tau) & , \quad m \text{ even} \\ (\frac{1}{2} + \frac{\tau}{2}) + m(a + b\tau) & , \quad m \text{ odd} . \end{cases}$$

This proves the theorem in this case.

<u>Case III</u>: Suppose now that Γ is an arbitrary lifted subgroup of N with generators (a_1, a_2, a_3) and (b_1, b_2, b_3), and let $L = \pi(\Gamma)$ have the fundamental parallelogram D. Let σ be the automorphism of N, and let $\tau = \alpha + i\beta$ and $a, b \in \mathbb{R}$ be as in theorem 1.16. Then $\sigma(\Gamma) = \Gamma(\tau; a, b)$, and by corollary 1.17 $\textcircled{H}_m(\Gamma) = \textcircled{H}_m(\Gamma(\tau; a, b)) \circ \sigma$. Let $S = \{Z_1, Z_2, \ldots, Z_m\} \subseteq D$. Then $F \in \textcircled{H}_m(\Gamma)$ is such that $M(\frac{m}{\beta(\Gamma)})$ has the zero set S in D if and only if $F \circ \sigma^{-1} \in \textcircled{H}_m(\Gamma(\tau; a, b))$ is such that $M(\frac{m}{\beta})(F \circ \sigma^{-1})$ has the zero set $\{\sigma(Z_1), \sigma(Z_2), \ldots, \sigma(Z_m)\}$ in $D(\tau)$. But this happens, by our previous case, if and only if $\sum_{i=1}^{m} \sigma(Z_i) = \sigma(\sum_{i=1}^{m} Z_i)$ is congruent modulo $L(\tau)$ to

$$\begin{cases} m(a + b\tau) & , \quad m \text{ even} \\ (\frac{1}{2} + \frac{\tau}{2}) + m(a + b\tau) & , \quad m \text{ odd} \end{cases}$$

or equivalently, if and only if $\sum_{i=1}^{m} Z_i$ is congruent modulo L to

$$\begin{cases} \sigma^{-1}(m(a + b\tau)) = mv_0 & , \quad m \text{ even} \\ \sigma^{-1}(\frac{1}{2} + \frac{\tau}{2}) + \sigma^{-1}(m(a + b\tau)) = \frac{a_1+b_1}{2} + i(\frac{a_2+b_2}{2}) + mv_0 , & m \text{ odd} \end{cases}$$

and the theorem is proven.

It is a well known fact that if L is a lattice in the complex plane, then the field, $\mathfrak{F}(\mathbb{C}/L)$, of meromorphic functions on $\mathbb{C}/L$ or equivalently, the field of elliptic functions with period lattice L is given by

$$\mathfrak{F}(\mathbb{C}/L) = \sum_{\pi(\Gamma)=L} Q(\textcircled{H}_m(\Gamma)) \quad ,$$

where we sum over all lifted subgroups Γ such that $\pi(\Gamma) = L$, and where Q denotes the quotients of elements of $\textcircled{H}_m(\Gamma)$, which by corollary 1.3, can be regarded as functions on $\mathbb{C}/L$. In more classical language an elliptic function $\varphi \in \mathfrak{F}(\mathbb{C}/L)$ can be written as the sum of quotients of theta functions where various "types" in the sense of [9] are allowed.

This result is refined in [10], where it is proven that we can, in fact, locate all of the theta functions necessary to form $\mathfrak{F}(\mathbb{C}/L)$ on a single nilmanifold or equivalently by using only those of a single type. There it is proven that if Γ is a fixed lifted subgroup of N such that $\pi(\Gamma) = L$, then

$$\mathfrak{F}(\mathbb{C}/L) = \sum_{m>0} Q(\textcircled{H}_m(\Gamma)) \quad ,$$

or equivalently, if $\varphi \in \mathfrak{F}(\mathbb{C}/L)$, then we may write

$$\varphi = \sum_{i\in I} \frac{F_i}{G_i} \quad , \qquad F_i, G_i \in \textcircled{H}_i(\Gamma) \quad ,$$

where I is a finite set of positive integers.

We now use theorem 1.19 to prove a further refinement of this result - that is, we can eliminate the need for the summation in the expression of φ as the quotient of theta functions:

<u>Theorem 1.20</u>: Let L be a lattice in the complex plane (with two linearly independent generators), and let $\varphi \in \mathfrak{F}(\mathbb{C}/L)$. Then if $\Gamma = \mathrm{grp}\{(a_1, a_2, a_3), (b_1, b_2, b_3)\}$ is any fixed lifted subgroup of N such that $\pi(\Gamma) = L$, we may choose an integer m, and functions $F, G \in \textcircled{H}_m(\Gamma)$, such that $\varphi = \frac{F}{G}$. Moreover, if n is the number of zeroes (and hence also the number of poles) of φ in the fundamental parallelogram D of L, then $n \leq m \leq n+1$.

<u>Proof</u>: Let $S_1 = \{Z_1, Z_2, \ldots, Z_n\}$ and $S_2 = \{W_1, W_2, \ldots, W_n\}$ denote the sets of zeroes and poles of φ, respectively, that are contained in D. It is a well known fact that $\sum_{i=1}^{n} Z_i$ is congruent modulo L to $\sum_{i=1}^{n} W_i$. Let Z' denote the unique element in D congruent modulo L to $\sum_{i=1}^{n} Z_i$.

If Z' is congruent modulo L to

$$\begin{cases} n\, v_0 , & n \text{ even} \\ \frac{a_1+b_1}{2} + i\left(\frac{a_2+b_2}{2}\right) + n\, v_0 , & n \text{ odd} \end{cases}$$

where v_0 is as in theorem 1.19, then we let F and G be the elements (unique up to constant multiple) of $\textcircled{H}_m(\Gamma)$, as in theorem 1.19, with zero sets S_1 and S_2, respectively. Clearly, we may choose these constants so that $\varphi = \frac{F}{G}$.

If Z' is not congruent modulo L to the above elements, we let μ be the unique element of D such that $Z' + \mu$ is congruent modulo L to

$$\begin{cases} (n+1)v_0 & , \quad n \text{ odd} \\ \frac{a_1+b_1}{2} + i\left(\frac{a_2+b_2}{2}\right) + (n+1)v_0 & , \quad n \text{ even} \end{cases} .$$

Let $S_1' = \{Z_1, Z_2, \ldots, Z_n, \mu\}$, $S_2' = \{W_1, W_2, \ldots, W_n, \mu\}$. By theorem 1.19, we may choose functions $F, G \in \textcircled{H}_{n+1}(\Gamma)$ with zero sets S_1' and S_2' , respectively, and we can write $\varphi = \frac{F}{G}$ as desired.

SECTION 2

THE INDEX AND THE DISTINGUISHED SUBSPACES

Let $\Gamma = \mathrm{grp}\{\gamma_1, \gamma_2\}$ be a lifted subgroup of N, and let $L = \pi(\Gamma)$ be the corresponding lattice in $\mathbb{R}^2$. We now recall the definition, given in [3], of the indexing on the set, Ω_m, of irreducible closed R-invariant subspaces of $H_m(\Gamma)$, for m a non-negative integer:

Let

$$\Lambda[m] = \mathrm{grp}\{\tfrac{1}{m}\gamma_1, \tfrac{1}{m}\gamma_2\}$$

$$\Gamma[m] = \mathrm{grp}\{\gamma_1, \gamma_2, \tfrac{\beta(\Gamma)}{m}\} \quad , \quad \text{and}$$

$$G_m(\Gamma) = \Lambda[m]/\Gamma[m] \quad , \quad \text{the space of right cosets.}$$

We note that if $Z(G_m(\Gamma))$ denotes the center of $G_m(\Gamma)$, then $G_m(\Gamma)/Z(G_m(\Gamma)) = \pi(\Lambda[m])/\pi(\Gamma[m]) = \frac{1}{m}L/L$ is a discrete group of order $|m|^2$. Since $\Lambda[m]$ normalizes $\Gamma[m]$, the map defined on $\Lambda[m]$ by

$$L: \lambda \to L(\lambda)$$

defines a representation of $\Lambda[m]$ on $H_m(\Gamma)$. Since $\Gamma[m]$ is contained in the kernel of this map, we think of L as a representation of $G_m(\Gamma)$ on $H_m(\Gamma)$.

Let $\mathcal{J} \in \Omega_m$. We define the <u>left stabilizer</u>, $\tilde{\Delta}_\Gamma(\mathcal{J})$, of $\mathcal{J}$ to be

$$\{\lambda \in G_m(\Gamma) \mid L(\lambda)\mathcal{H} = \mathcal{H}\} \quad .$$

The following theorems are proven in [3]:

<u>Lemma 2.1</u>: $\tilde{\Delta}_\Gamma(\mathcal{H})$ is an abelian subgroup of $G_m(\Gamma)$ containing the center $Z(G_m(\Gamma))$ of $G_m(\Gamma)$. Moreover, there exists a character $\tilde{\zeta}$ of $\tilde{\Delta}_\Gamma(\mathcal{H})$ such that $L(\lambda)F = \tilde{\zeta}(\lambda)F$ whenever $F \in \mathcal{H}$ and $\lambda \in \tilde{\Delta}_\Gamma(\mathcal{H})$, and where $\tilde{\zeta}$ restricted to $Z(G_m(\Gamma))$ is the character

$$\eta : \quad \Gamma[m](0,\ 0,\ \frac{a\beta(\Gamma)}{m^2}) \to \exp(\frac{2\pi i a\beta(\Gamma)}{m}) \quad .$$

<u>Lemma 2.2</u>: Let $\tilde{\Delta}$ be an abelian subgroup of $G_m(\Gamma)$. Then the order, $|\tilde{\Delta}|$, of $\tilde{\Delta}$ divides $|m|^2$, and $\tilde{\Delta}$ is maximal abelian if and only if $|\tilde{\Delta}| = |m|^2$.

Since $\tilde{\Delta}_\Gamma(\mathcal{H})$ contains $Z(G_m(\Gamma))$, we may consider

$$\Delta_\Gamma(\mathcal{H}) = \tilde{\Delta}_\Gamma(\mathcal{H})/Z(G_m(\Gamma)) \quad .$$

Let ζ denote the character on $\Delta_\Gamma(\mathcal{H})$ such that $\tilde{\zeta} = \zeta\eta$, i.e., such that

$$\tilde{\zeta}(\Gamma[m](a,\ b,\ c)) = \zeta(\Gamma[m](a,\ b,\ 0)\eta(\Gamma[m](0,\ 0,\ c)) \quad ,$$

$(a,\ b,\ c) \in \Lambda[m]$. Then

$$\Delta_\Gamma(\mathcal{H}) = \{\mu \in \frac{1}{m} L/L \mid L(\mu)F = \zeta(\mu)F \quad \text{for all } F \in \mathcal{H}\} \quad ,$$

and $\Delta_\Gamma(\mathcal{H})$ is a subgroup of $\frac{1}{m} L/L$ of order dividing $|m|$.

We now define the index, <u>$\mathrm{ind}_\Gamma(\mathcal{H})$</u> , of $\mathcal{H}$ as follows:

$$\mathrm{ind}_\Gamma(\mathcal{H}) = \frac{|m|^2}{|\tilde{\Delta}_\Gamma(\mathcal{H})|} = \frac{|m|}{|\Delta_\Gamma(\mathcal{H})|} \quad ,$$

where $|G|$ denotes the order of the group G , and $|m|$

denotes the absolute value of the integer m . We say that $\mathfrak{H}$ is a <u>distinguished</u> subspace of $H_m(\Gamma)$ if $\mathrm{ind}_\Gamma(\mathfrak{H}) = 1$. We see then that $\mathfrak{H}$ is distinguished if and only if $\widetilde{\Delta}_\Gamma(\mathfrak{H})$ is a maximal abelian subgroup of $G_m(\Gamma)$.

In [3] various characterizations of the distinguished subspaces are given, and the differences between these subspaces and those of higher index is discussed. Several other characterizations are discussed in [4] and [2]. The following characterization given in [2] will be useful:

$\mathfrak{H}$ is a distinguished subspace of $H_m(\Gamma)$ if and only if $\mathfrak{H} = H_1(\Gamma_1)$, where Γ_1 is a lifted subgroup of N .

The condition $H_1(\Gamma_1) \subseteq H_m(\Gamma)$ implies (see [10]) moreover, that $\Gamma \subseteq \Gamma_1$ and $\beta(\Gamma) = m\beta(\Gamma_1)$.

In this section, we will discuss the indexing from a different point of view, and in the process, we will see another way in which the distinguished subspaces are nicer than those of higher index. For later reference, let us first quote the following theorem, proven in [3]:

<u>Theorem 2.3</u>: Let $\widetilde{\Delta}$ be an abelian subgroup of $G_m(\Gamma)$ that contains $Z(G_m(\Gamma))$. For each character $\widetilde{\zeta}$ of $\widetilde{\Delta}$ restricting to η on $Z(G_m(\Gamma))$, set

$$H(m, \widetilde{\zeta}) = \{F \in H_m(\Gamma) \mid L(\lambda)F = \widetilde{\zeta}(\lambda)F \text{ for all } \lambda \in \widetilde{\Delta}\} .$$

Then

(1) $H(m, \tilde{\zeta})$ is R-invariant

(2) The multiplicity of $U(m)$ in R restricted to $H(m, \tilde{\zeta})$ is equal to $\frac{|m|^{2}}{|\tilde{\Delta}|}$

(3) $H_m(\Gamma) = \sum_{\tilde{\zeta}} \oplus H(m, \tilde{\zeta})$, the sum being over all characters $\tilde{\zeta}$ of $\tilde{\Delta}$ restricting to η on $Z(G_m(\Gamma))$.

We see from this theorem that $H(m, \tilde{\zeta})$ is irreducible precisely when $\tilde{\Delta}$ is maximal abelian, in which case $H(m, \tilde{\zeta})$, whose stabilizer is $\tilde{\Delta}$, is a distinguished subspace of $H_m(\Gamma)$. If $\tilde{\Delta}$ is not maximal abelian, and if $\mathcal{J} \in \Omega_m$ has $\tilde{\Delta}$ as its stabilizer, then $\mathcal{J}$ is properly included in $H(m, \tilde{\zeta})$, for some character $\tilde{\zeta}$ on $\tilde{\Delta}$, and, in fact, the multiplicity of $U(m)$ in $H(m, \tilde{\zeta})$ equals $\mathrm{ind}_\Gamma(\mathcal{J})$.

Let $\theta \in \textcircled{H}_m(\Gamma)$. As mentioned above, θ lies in a unique $\mathcal{J} \in \Omega_m$ which we denote by $\mathcal{J}(\theta)$. In fact, we can easily see that $\mathcal{J}(\theta)$ is the closure (in the topology arising from the L^2 norm) of the linear span of

$$\{R(g)\theta \mid g \in N\} \quad .$$

We define the <u>stabilizer</u> and <u>index</u> of θ to be $\tilde{\Delta}_\Gamma(\theta) = \tilde{\Delta}_\Gamma(\mathcal{J}(\theta))$ and $\mathrm{ind}_\Gamma(\theta) = \mathrm{ind}_\Gamma(\mathcal{J}(\theta))$, respectively. Similarly, we let $\Delta_\Gamma(\theta) = \Delta_\Gamma(\mathcal{J}(\theta))$. We will now discuss the relationship between the index of θ and the properties of its zero set.

<u>Lemma 2.4</u>: Let Γ and Γ' be lifted subgroups of N and suppose that $\mathcal{J} \subseteq H_m(\Gamma) \subseteq H_n(\Gamma')$ is an irreducible closed R-invariant space, $m, n \in \mathbb{Z}$. Then $\mathrm{ind}_\Gamma(\mathcal{J}) = \mathrm{ind}_{\Gamma'}(\mathcal{J})$, so that we may denote this number simply by $\mathrm{ind}(\mathcal{J})$.

<u>Proof</u>: It is proven in [2] that $H_m(\Gamma) \subseteq H_n(\Gamma')$ precisely when $n = sm$, $\Gamma \supseteq \Gamma'$, and $|\pi(\Gamma)/\pi(\Gamma')| = s$, for some positive integer s . It is not difficult to see that

$$\Delta_{\Gamma'}(\mathcal{J}) = \{\mu + \Delta_\Gamma(\mathcal{J}) \mid \mu \in \pi(\Gamma) \mid \pi(\Gamma')\} ,$$

so that

$$\mathrm{ind}_{\Gamma'}(\mathcal{J}) = \frac{sm}{|\Delta_{\Gamma'}(\mathcal{J})|} = \frac{sm}{s|\Delta_\Gamma(\mathcal{J})|} = \frac{m}{|\Delta_\Gamma(\mathcal{J})|} = \mathrm{ind}_\Gamma(\mathcal{J}) .$$

<u>Theorem 2.5</u>: Let $\theta \in \textcircled{H}_m(\Gamma)$, $m > 0$, and let S denote the zeroes of the function $\theta' = M(\frac{m}{\beta(\Gamma)})\theta$ that lie in the fundamental parallelogram D of $L = \pi(\Gamma)$. Then

$$\Delta_\Gamma(\theta) = \{\mu \in \tfrac{1}{m} L/L \mid -\mu + S = S\}$$

or equivalently

$$\tilde{\Delta}_\Gamma(\theta) = \{\lambda \in G_m(\Gamma) \mid \lambda^{-1} \tilde{S} = \tilde{S} \}$$

where $\tilde{S} = S \cdot Z(N)$ is the zero set of θ lying in $D \cdot Z(N)$.

Before we prove this theorem, let us mention that what it says is that the index of a subspace $\mathcal{J} \in \Omega_m$, $m > 0$, is completely determined by the "symmetry" properties of the zero set of a single function lying in this space - the unique $\theta \in \mathcal{J} \cap \textcircled{H}$.

Proof of Theorem 2.5: Let

$$\tilde{\Delta} = \{\lambda \in G_m(\Gamma) \mid \lambda^{-1}\tilde{S} = \tilde{S}\ ] \quad \text{and}$$

$$\Delta = \{\mu \in \frac{1}{m} L/L \mid -\mu + S = S \quad .$$

a) We show first that $\tilde{\Delta}_\Gamma(\theta) \subseteq \tilde{\Delta}$ and $\Delta_\Gamma(\theta) \subseteq \Delta$.

Let $\lambda \in \tilde{\Delta}_\Gamma(\theta)$. Then, for all $g \in N$,

$$L(\lambda)\theta(g) = \theta(\lambda^{-1}g) = \tilde{\zeta}(\lambda)\theta(g) \quad .$$

Therefore, $g \in \tilde{S}$ if and only if $\lambda^{-1}g \in \tilde{S}$, and thus $\lambda^{-1} \cdot \tilde{S} = \tilde{S}$ and $\lambda \in \tilde{\Delta}$. Moreover,

$$\Delta_\Gamma(\theta) = \tilde{\Delta}_\Gamma(\theta)/Z(G_m(\Gamma)) \subseteq \tilde{\Delta}/Z(G_m(\Gamma)) = \Delta \quad .$$

b) We will now show, conversely, that $\tilde{\Delta} \subseteq \tilde{\Delta}_\Gamma(\theta)$ and that $\Delta \subseteq \Delta_\Gamma(\theta)$.

Let $\lambda \in \tilde{\Delta}$. We must show that

$$L(\lambda)\mathcal{J}(\theta) = \mathcal{J}(\theta) \quad .$$

Let $\theta' = M(\frac{m}{\beta(\Gamma)})\theta$ and $\theta'_\lambda = M(\frac{m}{\beta(\Gamma)})(L(\lambda)\theta)$. Now θ' and θ'_λ are entire theta functions with respect to the lattice $L = \pi(\Gamma)$ of the same type, with the same zero set. Therefore, by theorem 1.12, there exists a complex number C_λ such that $\theta'_\lambda = C_\lambda\theta'$. But then

$$L(\lambda)\theta = M(\frac{m}{\beta(\Gamma)})^{-1}\theta'_\lambda = M(\frac{m}{\beta(\Gamma)})^{-1}(C_\lambda\theta') =$$

$$= C_\lambda M(\frac{m}{\beta(\Gamma)})^{-1}\theta' = C_\lambda\theta \in \mathcal{J}(\theta) \quad .$$

Suppose now that $g_1, g_2, \ldots, g_n \in N$. Then

$$L(\lambda)(\sum_{i=1}^{n} R(g_i)\theta) = \sum_{i=1}^{n} L(\lambda)R(g_i)\theta = \sum_{i=1}^{n} R(g_i)L(\lambda)\theta$$

$$= C_\lambda \sum_{i=1}^{n} R(g_i)\theta \in \mathcal{J}(\theta) \quad .$$

Let $F \in \mathcal{J}(\theta)$. Then $F = \lim_{k} F_k$, where F_k is a function of the form $\sum_{i=1}^{n} R(g_i)\theta$, and where the limit is taken in the topology induced from the L^2 norm. Then $L(\lambda)F_k \in \mathcal{J}(\theta)$, and by the continuity of the operator $L(\lambda)$ in the same topology, we have that

$$L(\lambda)F = L(\lambda)(\lim F_k) = \lim(L(\lambda)F_k) \in \mathcal{J}(\theta) ,$$

since $\mathcal{J}(\theta)$ is closed. Hence $\lambda \in \tilde{\Delta}_\Gamma(\theta)$, and $\tilde{\Delta} \subseteq \tilde{\Delta}_\Gamma(\theta)$. As above, this implies that $\Delta \subseteq \Delta_\Gamma(\mathcal{J})$, and the theorem is proven.

We will now use this theorem to discuss further the relationship between the indexing on Ω_m , when $m > 0$, and the properties of the zero set of the corresponding theta functions. We will see to what extent the index and the stabilizer of $\mathcal{J}$ determine the location of the zeroes of the nil-theta function lying in $\mathcal{J}$.

Lemma 2.6: Let L be a lattice in $\mathbb{R}^2$ with the two linearly independent generators γ_1 and γ_2 , let m be a positive integer and let k be a positive divisor of m . Suppose that Δ is a subgroup of $\frac{1}{m} L/L$ of order $\frac{m}{k}$. Then Δ has generators β_1 and β_2 such that

(a) $\beta_1 = \frac{k_{11}}{m}\gamma_1$, $\beta_2 = \frac{k_{21}}{m}\gamma_1 + \frac{k_{22}}{m}\gamma_2$,

where $k_{ij} \in \{0, 1, 2, \ldots, m-1\}$, $i, j = 1, 2$.

(b) $k_{11}k_{22} = km$

(c) k divides k_{ij} for all i, j .

(d) k_{ii} divides m , $i = 1, 2$.

Proof: The Fundamental Theorem of Abelian Groups guarantees the existence of generators β_1 and β_2 of Δ satisfying (a). Moreover, it is not difficult to see that $\frac{m}{k} = |\Delta| = \frac{1}{d}$, where d is the determinant of the matrix

$$\begin{pmatrix} \frac{k_{11}}{m} & 0 \\ \frac{k_{21}}{m} & \frac{k_{22}}{m} \end{pmatrix} .$$

Thus, $\frac{m}{k} = \frac{m^2}{k_{11}k_{22}}$ or $k_{11}k_{22} = km$, which proves (b).

To prove (c), we note that, since β_1 and β_2 are elements of the group Δ which is of order $\frac{m}{k}$, we must have that $\frac{m}{k} \cdot \frac{k_{ij}}{m} = \frac{k_{ij}}{k} \in \mathbb{Z}$, and k divides k_{ij} , for all i and j .

Finally, by (b) and (c), $\frac{m}{k_{ii}} = \frac{k_{jj}}{k} \in \mathbb{Z}$, for $i, j \in \{1, 2\}$, $i \neq j$, which proves (d) .

Suppose now that $\theta \in \textcircled{H}_m(\Gamma)$, $m > 0$, for some lifted subgroup Γ . Let $L = \pi(\Gamma)$ have generators γ_1 and γ_2 . If $\text{ind}(\theta) = k$, then the stabilizer, $\Delta(\theta)$, of θ is a subgroup of $\frac{1}{m} L/L$ of order $\frac{m}{k}$, and therefore, has generators β_1 and β_2 as in lemma 2.6.

Let L' be the lattice in $\mathbb{R}^2$ spanned by β_1 and β_2 , and let D and D' be the fundamental parallelograms of L and L' , respectively. If $S = \{Z_1, Z_2, \ldots, Z_m\}$ is the set of zeroes of $M(\frac{m}{\beta(\Gamma)})\theta$

in D, then

$$S = \bigcup_{\delta \in L'/L} \{\delta + S'\} ,$$

where $S' = \{Z_1, Z_2, \ldots, Z_k\} \subseteq S$ is the set of zeroes of $M(\frac{m}{\beta(\Gamma)})\theta$ in D'. We will now investigate the zero set S'.

Theorem 2.7: Let Γ be a lifted subgroup of N with generators $\gamma_1 = (a_1, a_2, a_3)$ and $\gamma_2 = (b_1, b_2, b_3)$. Let $\theta \in \textcircled{H}_m(\Gamma)$, $m > 0$, be of index k, and let β_1 and β_2 be generators of $\Delta(\theta)$ satisfying the conditions of lemma 2.6. Let L' be the lattice generated by β_1 and β_2, and let D and D' be the fundamental parallelograms of the lattices $L = \pi(\Gamma)$ and L', respectively, and suppose that $S = \{Z_1, Z_2, \ldots, Z_m\}$ and $S' = \{Z_1, Z_2, \ldots, Z_k\}$ are the zero sets of the function $M(\frac{m}{\beta(\Gamma)})\theta$ in D and D', respectively. Then, $\sum_{i=1}^{k} Z_i$ is congruent modulo L' to

$$\frac{k}{m}\ell + kv_0 + \begin{cases} 0 , & k \text{ even} \\ \dfrac{\beta_1 + \beta_2}{2} , & k \text{ odd, } k_{11} \text{ and } k_{22} \text{ even} \\ \dfrac{\beta_2}{2} , & k \text{ odd, } k_{11} \text{ even, } k_{22} \text{ odd} \\ \dfrac{\beta_1}{2} , & k \text{ odd, } k_{11} \text{ odd, } k_{22} \text{ even} \\ \dfrac{1}{2m}(k_{22}\beta_1 + k_{11}\beta_2) - \dfrac{k_{21}}{2m}\beta_1 , & k, k_{11} \text{ and } k_{22} \text{ odd} \end{cases} ,$$

where mv_0 is the unique element in D congruent modulo L to $m(\frac{a_3b_1 - a_1b_3}{\beta(\Gamma)} + i\,\frac{(a_3b_2 - a_2b_3}{\beta(\Gamma)})$, and ℓ is the fixed element of L , as in theorem 1.19, such that $\sum_{i=1}^{m} Z_i - \ell$ is the unique element in D congruent modulo L to

$$\begin{cases} mv_0 & , \quad m \text{ even} \\ \frac{\gamma_1 + \gamma_2}{2} + mv_0 & , \quad m \text{ odd} \end{cases} .$$

<u>Proof</u>: As we mentioned above

$$S = \bigcup_{\delta \in \Delta(\theta)} \{\delta + S'\} .$$

Now $\Delta(\theta) = \mathrm{grp}\{\beta_1, \beta_2\} = \{a\beta_1 + b\beta_2 \mid a \in \{0, 1, \ldots, \frac{k_{22}}{k} - 1\},$
$b \in \{0, 1, \ldots, \frac{k_{11}}{k} - 1\}\}$.

Thus, summing all the zeroes in S , we have that

$$\sum_{i=1}^{m} Z_i = \frac{m}{k} \sum_{i=1}^{k} Z_i + k \sum_{a=0}^{\frac{k_{22}}{k}-1} \sum_{b=0}^{\frac{k_{11}}{k}-1} (a\beta_1 + b\beta_2)$$

$$= \frac{m}{k} \sum_{i=1}^{k} Z_i + \frac{m}{2}[(\frac{k_{22}}{k} - 1)\beta_1 + (\frac{k_{11}}{k} - 1)\beta_2]$$

$$= \frac{m}{k} \sum_{i=1}^{k} Z_i - \frac{m}{2}(\beta_1 + \beta_2) + \frac{m}{2}[\frac{k_{22}}{k}\beta_1 + \frac{k_{11}}{k}\beta_2] .$$

Therefore,

$$\sum_{i=1}^{k} Z_i = \frac{k}{m} \sum_{i=1}^{m} Z_i + \frac{k}{2}(\beta_1 + \beta_2) - \tfrac{1}{2}[k_{22}\beta_1 + k_{11}\beta_2] .$$

Let ω_0 denote the unique element of D congruent modulo L to

$$\begin{cases} mv_0 & , \quad m \text{ even} \\ \dfrac{\gamma_1 + \gamma_2}{2} + mv_0 & , \quad m \text{ odd} \end{cases} ,$$

and let $\ell = \ell_1\gamma_1 + \ell_2\gamma_2$ be the element of L such that by theorem 1.19 ,

$$\sum_{i=1}^{m} Z_i = \ell + \omega_0 \quad .$$

(We note, for later use, that in fact, $\ell_i \in \{0, 1, 2, \ldots, m-1\}$ $i = 1, 2$, since we may write the elements $Z_i \in D$ in the form $Z_i = x_i\gamma_1 + y_i\gamma_2$, where $x_i, y_i \in [0, 1)$, $i = 1, 2, \ldots, m$, and similarly, $\omega_0 = s\gamma_1 + t\gamma_2$, $s, t \in [0, 1)$. Thus

$$\sum_{i=1}^{m} Z_i = \gamma_1 \sum_{i=1}^{m} x_i + \gamma_2 \sum_{i=1}^{m} y_i < m\gamma_1 + m\gamma_2 \quad ,$$

and $\ell = \sum_{i=1}^{m} Z_i - \omega_0 < m\gamma_1 + m\gamma_2$.)

We have now that

$$\sum_{i=1}^{k} Z_i = \frac{k}{m}\,\omega_0 + \frac{k}{m}\,\ell + \frac{k}{2}(\beta_1 + \beta_2) - \tfrac{1}{2}(k_{22}\beta_1 + k_{11}\beta_2) \quad ,$$

and this is congruent modulo L to

$$\begin{cases} \dfrac{k}{m}\,\ell + kv_0 + \dfrac{k}{2}(\beta_1 + \beta_2) - \tfrac{1}{2}(k_{22}\beta_1 + k_{11}\beta_2) & , \quad m \text{ even} \\ \dfrac{k}{m}\,\ell + kv_0 + \dfrac{k}{m}\left(\dfrac{\gamma_1 + \gamma_2}{2}\right) + \dfrac{k}{2}(\beta_1 + \beta_2) - \tfrac{1}{2}(k_{22}\beta_1 + k_{11}\beta_2) , & \quad m \text{ odd} \end{cases} .$$

Now $\gamma_1 = \dfrac{k_{22}}{k}\,\beta_1$ and $\gamma_2 = \dfrac{-k_{21}}{k}\,\beta_1 + \dfrac{k_{11}}{k}\,\beta_2$,

so the above is equal to

$$kv_0 + \frac{k}{m}\ell + \frac{k}{2}(\beta_1+\beta_2) + \begin{cases} -\frac{1}{2}(k_{22}\beta_1 + k_{11}\beta_2) , & m \text{ even} \\ -\frac{(m-1)}{2m}(k_{22}\beta_1 + k_{11}\beta_2) - \frac{k_{21}}{2m}\beta_1 , & m \text{ odd} \end{cases} .$$

Reducing modulo L', we have that $\sum_{i=1}^{k} Z_i$ is congruent modulo L' to

$$kv_0 + \frac{k}{m}\ell + \begin{cases} 0 , & k \text{ even} \\ \frac{\beta_1+\beta_2}{2} , & k \text{ odd, } k_{11} \text{ and } k_{22} \text{ even} \\ \frac{\beta_2}{2} , & k \text{ odd, } k_{11} \text{ even, } k_{22} \text{ odd} \\ \frac{\beta_1}{2} , & k \text{ odd, } k_{11} \text{ odd, } k_{22} \text{ even} \\ \frac{1}{2m}(k_{22}\beta_1 + k_{11}\beta_2) - \frac{k_{21}}{2m}\beta_1 , & k, k_{11}, k_{22} \text{ all odd} \end{cases}$$

where we note that

(a) if k is even, then m and k_{ij} are all even since k divides m and k_{ij}, for all i, j,

(b) if either k_{11} or k_{22} is even, then m is even, since k_{ii} divides m, $i = 1, 2$,

(c) if k, k_{11}, and k_{22} are all odd, then m is odd.

<u>Corollary 2.8</u>: Let $\theta \in \textcircled{H}_m(\Gamma)$, $m > 0$, be distinguished, with stabilizer $\Delta(\theta) = \mathrm{grp}\{\beta_1, \beta_2\}$ as in lemma 2.6. Let L' be the lattice spanned by β_1 and β_2, and let D and D' be the fundamental parallelograms of the lattices $L = \pi(\Gamma)$ and L', respectively. Then, the zero set S of the function $M(\frac{m}{\beta(\Gamma)})\theta$ inside D consists of

$$S = \bigcup_{\delta \in \Delta(\theta)} \{\delta + Z_0\} ,$$

where Z_0 is the unique zero of this function inside D'. Moreover, Z_0 is congruent modulo L' to

$$\frac{1}{m}\ell + v_0 + \begin{cases} \frac{\beta_1+\beta_2}{2} & , k_{11} \text{ and } k_{22} \text{ even} \\ \frac{\beta_2}{2} & , k_{11} \text{ even and } k_{22} \text{ odd} \\ \frac{\beta_1}{2} & , k_{11} \text{ odd and } k_{21} \text{ even} \\ \frac{1}{2m}(k_{22}\beta_1 + k_{11}\beta_2) - \frac{k_{21}}{2m}\beta_1 & , k_{11} \text{ and } k_{22} \text{ odd} \end{cases}$$

where ℓ and v_0 are as in the previous theorem.

Let us now interpret these results. Suppose that Δ is a subgroup of order $\frac{m}{k}$ of $\frac{1}{m}L/L$. We would like to characterize all of the subspaces $\mathcal{H} \in \Omega_m$ that have Δ as their stabilizer, by describing the zero sets of the nil-theta functions lying in these subspaces. What we have shown is that if $S = \{Z_1, Z_2, \ldots, Z_m\}$ are the zeroes of $M(\frac{m}{\beta(\Gamma)})\theta$ in D, then we may partition the set S as follows:

$$S = S_1 \cup S_2 \cup \dots \quad S_{\frac{m}{k}} \quad , \text{ where}$$

(a) $|S_i| = k, \qquad i = 1, 2, \dots, \frac{m}{k}$

(b) $S_i = \delta_i + S_1 \quad , \quad \delta_i \in \Delta(\theta) \quad , \quad i = 1, 2, \dots \frac{m}{k}$

(c) $S_1 = \{Z_1, Z_2, \dots, Z_k\}$ is the zero set of $M(\frac{m}{\beta(\Gamma)})\theta$ in the fundamental parallelogram D' of the lattice L' spanned by Δ .

Moreover, it is not possible to further partition all of the S_i , $i = 1, 2, \dots, \frac{m}{k}$, into sets with these same properties.

Summing all the zeroes in S_i yields

$$\sum_{Z \in S_i} Z = \sum_{i=1}^{k} Z_i + k\delta_i$$

or equivalently

$$\lambda_i = \frac{1}{k} \sum_{Z \in S_i} Z = \frac{1}{k} \sum_{i=1}^{k} Z_i + \delta_i \, , \quad i = 1, 2, \dots, \frac{m}{k} \quad .$$

We see that λ_i is the center of gravity of the zeroes of $M(\frac{m}{\beta(\Gamma)})\theta$ lying in the translate $\delta_i + D'$ of D' , $i = 1, 2, \dots, \frac{m}{k}$, and that $\{\lambda_1, \lambda_2, \dots, \lambda_{\frac{m}{k}}\}$ forms a group isomorphic to Δ . Moreover, from theorem 2.7, we know exactly what these λ_i's are up to the addition of the element $\frac{k}{m} \ell \in \frac{1}{\frac{m}{k}} L$. We now interpret this uncertainty.

Let $\hat{\Delta} = \frac{1}{\frac{m}{k}} L / L_1$. Then $|\hat{\Delta}| = \frac{m}{k}$. For every $\mu \in \hat{\Delta}$, by theorem 2.7, we have a different possibility for the sum $\sum_{i=1}^{k} Z_i$. Let us denote this possibility by ν_μ . Then the possibilities for λ_1 are $\frac{1}{k}\nu_\mu$, $\mu \in \hat{\Delta}$.

Let us recall the decomposition of $H_m(\Gamma)$ described in theorem 2.3: $H_m(\Gamma) = \sum_{\tilde{\zeta}} \oplus H(m, \tilde{\zeta})$. It is not difficult to show that the group $\hat{\Delta}$ is in fact isomorphic to the character group of Δ , (see [2]). Now $\textcircled{H}_m(\Gamma) = \sum_{\zeta} \oplus \textcircled{H}(m, \tilde{\zeta})$, where $\textcircled{H}(m, \tilde{\zeta}) = \textcircled{H} \cap H(m, \zeta)$ is a complex vector space of dimension k . It follows then, that there exists a lifted subgroup Γ' containing Γ , such that $\textcircled{H}(m, \tilde{\zeta}) = \textcircled{H}_k(\Gamma')$, and $H(m, \tilde{\zeta}) = H_k(\Gamma')$. Thus, we may write

$$H_m(\Gamma) = \sum_{\mu \in \hat{\Delta}} \oplus H_k(\Gamma'_\mu) .$$

Moreover, one can show (see [10]), that $\Gamma'_\mu = \tilde{\mu}^{-1}\Gamma'\tilde{\mu}$, where $\pi(\tilde{\mu}) = \mu$, and $\Gamma' = \Gamma'_{\mu_0}$ when $\mu_0 = 0 \in \tilde{\Delta}$. It is clear, then, that the $\frac{m}{k}$ possibilities for $\nu_\mu = \sum_{i=1}^{k} Z_i$ - the sum of the zeroes of the theta function in D' - correspond to the $\frac{m}{k}$ spaces $\textcircled{H}_k(\Gamma'_\mu)$; i.e., if the sum of the zeroes of $M(\frac{m}{\beta(\Gamma)})\theta$ in D' is ν_μ , then $\theta \in \textcircled{H}_k(\Gamma'_\mu)$.

Let us note now the difference between the distinguished theta functions and those of higher index. When we are given the stabilizer of the distinguished theta, we know precisely where each of its zeroes lies (up to translation by an element of $\hat{\Delta}$). In particular, these zeroes clearly occur at points whose coordinates are rationally related to the coordinates of the lattice L. For thetas of index $k > 1$, however, we cannot locate these zeroes precisely, now can we even insure that they are rational. We can only locate (the $\frac{m}{k}$ possibilities for) the center of gravities, λ_i, of the $\frac{m}{k}$ subsets S_i occurring in the partition of S described above, and these λ_i are rational, although the individual zeroes need not be. Clearly, the larger the index is, the larger the sets S_i are, and the less information we have about the precise location of the zeroes.

In [2] and [4], using the distinguished subspace decompositions of $H_m(\Gamma)$, various natural basis are given for $\textcircled{H}_m(\Gamma)$, $m > 0$, consisting only of theta functions of index 1. Let us close this section by giving a basis for this space, all of whose elements are theta functions of index k, where k is any positive divisor of m.

We let Δ be a subgroup of $\frac{1}{m} L/L$ of order $\frac{m}{k}$, and we decompose

$$H_m(\Gamma) = \sum_{\mu\in\hat{\Delta}} \oplus H_k(\Gamma'_\mu)$$

as above. We indicate how to select a basis of $\textcircled{H}_k(\Gamma'_\mu)$ with the necessary properties, $\mu \in \hat{\Delta}$, and the union of these basis as μ ranges over $\hat{\Delta}$ yields the desired basis of $\textcircled{H}_m(\Gamma)$.

We select k points $\xi_1, \xi_2, \ldots, \xi_k$ in D' such that

(a) $\xi_i \neq \xi_j$ for $i \neq j$

(b) $\sum_{i=1}^{k} \xi_i \neq \nu_\mu$

(c) $\{\xi_1, \xi_2, \ldots, \xi_k\}$ is invariant under addition of elements of a group of order k .

We may easily find such points. Now for $i = 1, 2, \ldots, k$, we let $T_i = \sum_{j \neq i} \xi_j$ and x_i be the element of D' congruent modulo L' to $\nu_{\mu_i} - T_i$, $\mu_i \in \hat{\Delta}$. Let θ'_i be the entire theta function with respect to the lattice L' , with zeroes $\{\xi_1, \xi_2, \ldots, \xi_{i-1}, x_i, \xi_{i+1}, \ldots, \xi\}$ in D' , and let $\theta_i = M(\frac{m}{\beta(\Gamma)})^{-1}\theta'_i$. Then $\theta_i \in \textcircled{H}_k(\Gamma'_{\mu_i})$. Moreover, we note that since $\xi_i + T_i$ is not congruent modulo L' to ν_{μ_i} , but $x_i + T_i$ is congruent to this element, we have that x_i is not congruent modulo L' to ξ_i . Therefore, the zero set of θ'_i cannot be invariant under any non-zero element of D' , and by theorem 2.5, θ_i is of index k .

To see that $\{\theta'_1, \theta'_2, \ldots, \theta'_{\frac{m}{k}}\}$, and hence also $\{\theta_1, \theta_2, \ldots, \theta_{\frac{m}{k}}\}$ are linearly independant, we suppose that

$$a_1\theta'_1 + a_2\theta'_2 + \ldots + a_{\frac{m}{k}}\theta'_{\frac{m}{k}} = 0 \quad .$$

Evaluating both sides of this expression at ξ_j , yields $a_j\theta'_j(\xi_j) = 0$, but $\theta'_j(\xi_j) \neq 0$, so that $a_j = 0$, $j = 1, 2, \ldots, \frac{m}{k}$. Thus, we have our basis.

SECTION 3

MULTIPLICATION THEORY

We would now like to consider the behavior of the index under the multiplication of nil-theta functions - how does $\mathrm{ind}(\theta_1\theta_2)$ relate to $\mathrm{ind}(\theta_1)$ and $\mathrm{ind}(\theta_2)$? For convenience of notation, we will restrict ourselves to the lifted subgroup $\Gamma_0 = \mathrm{grp}\{(1, 0, 0), (0, 1, 0)\}$. We let $M = \pi(\Gamma_0)$, and let D be the fundamental parallelogram of M . For a nil-theta function θ , we let $\Delta(\theta) = \Delta_{\Gamma_0}(\theta)$. The simplest behavior of the index under multiplication is illustrated by corollary 3.2 below.

Lemma 3.1: Let $\theta \in \textcircled{H}_m(\Gamma_0)$ and $\chi \in \textcircled{H}_n(\Gamma_0)$, $m, n \in \mathbb{Z}^+$, have index j and k , respectively. Then for any $t \in \mathbb{Z}^+$,

$$\Delta((\theta\chi)^t) = \Delta(\theta\chi) \quad .$$

Proof: Let S denote the set of zeroes of the function $\theta\chi \in \textcircled{H}_{m+n}(\Gamma_0)$ that are contained in D . Then clearly, the zero set of the function $(\theta\chi)^t \in \textcircled{H}_{t(m+n)}(\Gamma_0)$ in D is S^t , where this notation means that every element in the set S is counted with multiplicity t . By theorem 2.5, we have that

$$\Delta(\theta\chi) = \{\mu \mid -\mu + S = S\} \cap \frac{1}{m+n} M \quad \text{, and}$$

$$\Delta((\theta\chi)^t) = \{\mu \mid -\mu + S^t = S^t\} \cap \frac{1}{t(m+n)} M \quad .$$

Clearly, $\{\mu \mid -\mu + S = S\} = \{\mu \mid -\mu + S^t = S^t\}$. Moreover, $\frac{1}{m+n} M \subseteq \frac{1}{t(m+n)} M$, so that $\Delta(\theta\chi) \subseteq \Delta((\theta\chi)^t)$. It remains to show that $\Delta((\theta\chi)^t) \subseteq \Delta(\theta\chi)$.

We show now that if $-\mu + S = S$, then in fact $\mu \in \frac{1}{m+n} M$, which will imply that for any $\mu \in \Delta((\theta\chi)^t)$, $-\mu + S^t = S^t$ which implies that $-\mu + S = S$ which in turn implies that $\mu \in \frac{1}{m+n} M$. Hence, $\mu \in \Delta(\theta\chi)$, and $\Delta((\theta\chi)^t) \subseteq \Delta(\theta\chi)$.

Suppose then that $-\mu + S = S$, and let $Z \in S$. There exists an integer p such that $Z - \mu$, $Z - 2\mu$, ..., $Z - p\mu$ are all elements of S , and such that $Z - p\mu$ is congruent to Z modulo M . Therefore, $p\mu \in M$, or $\mu \in \frac{1}{p} M$. Since $|S| = m + n$, it is clear that p must divide $m + n$, so that $\frac{1}{p} M \subseteq \frac{1}{m+n} M$ and $\mu \in \frac{1}{m+n} M$ as desired.

<u>Corollary 3.2</u>: Let $\theta \in \textcircled{H}_m(\Gamma_0)$ have index j . Then for any positive integer t ,

$$\operatorname{ind}(\theta^t) = tj \quad .$$

<u>Proof</u>:

$$\operatorname{ind}(\theta^t) = \frac{tm}{|\Delta(\theta^t)|} = \frac{tm}{|\Delta(\theta)|} = \frac{tm}{\frac{m}{j}} = tj \quad .$$

We now consider the general multiplication theory.

Lemma 3.3: Let $\theta \in \textcircled{H}_m(\Gamma_0)$, $\chi \in \textcircled{H}_n(\Gamma_0)$. Then

$$\Delta(\theta) \cap \Delta(\chi) \subseteq \Delta(\theta\chi) \ .$$

Proof: Let S_1 and S_2 denote the zero sets in D of θ and χ respectively. Then the zero set in D of $\theta\chi \in \textcircled{H}_{m+n}(\Gamma_0)$ is $S_1 \cup S_2$. Therefore, by theorem 2.5,

$$\Delta(\theta\chi) = \{\mu \in \tfrac{1}{m+n} M \mid -\mu + (S_1 \cup S_2) = S_1 \cup S_2\} \ .$$

Let $\mu \in \Delta(\theta) \cap \Delta(\chi)$. Then $-\mu + S_i = S_i$, $i = 1, 2$, and $\mu \in \frac{1}{m} M \cap \frac{1}{n} M \subseteq \frac{1}{m+n} M$, so that clearly, $-\mu + (S_1 \cup S_2) = S_1 \cup S_2$ and $\mu \in \Delta(\theta\chi)$.

Lemma 3.4: Let L_1 and L_2 be two lattices in $\mathbb{R}^2$ containing M , with generators as in lemma 2.6:

$$L_1 = \text{grp}\{(\tfrac{j_{11}}{m}, 0), (\tfrac{j_{21}}{m}, \tfrac{j_{22}}{m}) \mid j_{11}j_{22} = jm, \ j \text{ divides } j_{st}\}$$

$$L_2 = \text{grp}\{(\tfrac{k_{11}}{m}, 0), (\tfrac{k_{21}}{m}, \tfrac{k_{22}}{m}) \mid k_{11}k_{22} = km, \ k \text{ divides } k_{st}\} .$$

Then $L_1 \cap L_2$ has generators of the form $(\frac{1}{d_2}, 0)$ and $(x, \frac{A}{d_1})$, where d_i is the greatest common divisor (g.c.d.) of $\frac{j_{ii}}{j}$ and $\frac{k_{ii}}{k}$, $i = 1, 2$, and A is the smallest positive divisor of $d_1 d_2$, less than or equal to d_1 , such that $\frac{d_1 d_2}{A}$ divides $Q = \frac{j_{22}}{j}\frac{k_{21}}{k} - \frac{k_{22}}{k}\frac{j_{21}}{j}$.

Proof: Since the generators of the lattice L_1 and L_2 are rationally related, it is clear that $L_1 \cap L_2$ will be a lattice in $\mathbb{R}^2$, containing M , with two linearly independent generators. Moreover, by the Fundamental Theorem of Abelian Groups, we may choose these generators to be of the form $(v, 0), (x, y)$. To find v , we look for the smallest non-zero integers a and b for which

$$a\,\frac{j_{11}}{m} = b\,\frac{k_{11}}{n} \quad \text{, if and only if,} \quad a\,\frac{j}{j_{22}} = b\,\frac{k}{k_{22}} \quad ,$$

if and only if, $\frac{a}{b} = \frac{\frac{j_{22}}{j}}{\frac{k_{22}}{k}}$. The smallest integers a and b that work are clearly $a = j'_{22} = \frac{j_{22}}{jd_2}$, $b = k'_{22} = \frac{k_{22}}{kd_2}$, where $d_2 = \text{g.c.d.}(\frac{j_{22}}{j}, \frac{k_{22}}{k})$, and thus $v = \frac{aj_{11}}{m} = \frac{bk_{11}}{n} = \frac{1}{d_2}$.

To find (x, y) , we must find integers c, d, e, and f , with d and f non-zero and minimal, such that

$$c(\frac{j_{11}}{m}, 0) + d(\frac{j_{21}}{m}, \frac{j_{22}}{m}) = e(\frac{k_{11}}{n}, 0) + f(\frac{k_{21}}{n}, \frac{k_{22}}{n}) \quad .$$

Considering first the second coordinates, we must have that

$$d\,\frac{j_{22}}{m} = f\,\frac{k_{22}}{n} \quad \text{, if and only if ,}$$

$\frac{d}{f} = \frac{\frac{j_{11}}{j}}{\frac{k_{11}}{k}}$, and we may take $d = A\,j'_{11}$, $f = A\,k'_{11}$, where $j'_{11} = \frac{j_{11}}{jd_1}$, $k'_{11} = \frac{k_{11}}{kd_1}$, $d_1 = \text{g.c.d.}(\frac{j_{11}}{j}, \frac{k_{11}}{k})$, and where A is an integer less than or equal to d_1 , whose value will be determined below. Then $y = d\,\frac{j_{22}}{m} = f\,\frac{k_{22}}{n} = \frac{A}{d_1}$. The order of $(L_1 \cap L_2)/M$ is clearly $\frac{d_1 d_2}{A}$. Therefore, we must have that A is a divisor of $d_1 d_2$.

Considering now the first coordinates, we must have that

$$c\,\frac{j_{11}}{m} + d\,\frac{j_{21}}{m} = e\,\frac{k_{11}}{n} + f\,\frac{k_{21}}{n}$$, equivalently

$$c\,\frac{j_{11}}{m} + Aj'_{11}\,\frac{j_{21}}{m} = e\,\frac{k_{11}}{n} + Ak'_{11}\,\frac{k_{21}}{n}$$, equivalently

$$c\,j_{11}n - ek_{11}m = Ak'_{11}k_{21}m - Aj'_{11}j_{21}n$$, equivalently

$$\frac{cj_{11}k_{11}k_{22}}{k} - \frac{ek_{11}j_{11}j_{22}}{j} = \frac{Ak_{11}k_{21}j_{11}j_{22}}{k\;jd_1} - \frac{Aj_{11}j_{21}k_{11}k_{22}}{kjd_1}$$

equivalently

$$j_{11}k_{11}\left(c\,\frac{k_{22}}{k} - e\,\frac{j_{22}}{j}\right) = \frac{Aj_{11}k_{11}}{d_1}\left(\frac{k_{21}}{k}\,\frac{j_{22}}{j} - \frac{j_{21}}{j}\,\frac{k_{22}}{k}\right) ,$$

equivalently

$$\frac{d_1}{A} = \frac{\frac{k_{21}}{k}\,\frac{j_{22}}{j} - \frac{j_{21}}{j}\,\frac{k_{22}}{k}}{c\,\frac{k_{22}}{k} - e\,\frac{j_{22}}{j}} = \frac{Q}{c\,\frac{k_{22}}{k} - e\,\frac{j_{22}}{j}} ,$$

if and only if

$$\frac{d_1d_2}{A} = \frac{Q}{ck'_{22} - ej'_{22}} .$$

Now we have already remarked that we must select A so that $\frac{d_1d_2}{A}$ is an integer. Let A be the smallest divisor of d_1d_2 , less than or equal to d_1 , such that $\frac{d_1d_2}{A}$ divides Q , and let $r \in \mathbb{Z}$ be such that $Q = \frac{d_1d_2}{A}\,r$. We note that such an A exists, since we may certainly take $A = d_1$. We must be able to choose c and e so that $ck'_{22} - ej'_{22} = r$. But this is certainly possible, since k'_{22} and j'_{22} are relatively prime. Selecting c and e , we then have then have that $x = c\,\frac{j_{11}}{m} + Aj'_{11}\,\frac{j_{21}}{m}$, and the theorem is proven.

<u>Theorem 3.5</u>: Let $\theta \in \textcircled{H}_m(\Gamma_0)$ and $\chi \in \textcircled{H}_n(\Gamma_0)$ have index j and k, respectively. Then

$$\mathrm{ind}(\theta\chi) = \frac{p_1}{w}\, j + \frac{p_2}{w}\, k \ ,$$

where $p_1 = |\Delta(\theta)/(\Delta(\theta) \cap \Delta(\chi))| = \dfrac{mA}{jd_1d_2}$,

$p_2 = |\Delta(\chi)/(\Delta(\theta) \cap \Delta(\chi))| = \dfrac{nA}{kd_1d_2}$,

where d_1, d_2, and A are obtained by applying theorem 3.4 to the lattices L_1 and L_2 which are generated by $\Delta(\theta)$ and $\Delta(\chi)$, respectively, and where w is the order of the set

$$W = \{\mu \in \tfrac{1}{m+n} M/(L_1 \cap L_2) \mid -\mu + (S_1 \cup S_2) = S_1 \cup S_2\},$$

where S_1 and S_2 are the zero sets of θ and χ, respectively, in D.

(i.e., $W = \Delta(\theta\chi)/(\Delta(\theta) \cap \Delta(\chi))$.)

<u>Proof</u>:

$$\mathrm{ind}(\theta\chi) = \frac{m+n}{|\Delta(\theta x)|} = \frac{m+n}{w\,|\Delta(\theta) \cap \Delta(\chi)|}$$

$$= \frac{m+n}{w\,\dfrac{d_1d_2}{A}} = \frac{p_1}{w}\, j + \frac{p_2}{w}\, k \ .$$

<u>Corollary 3.6</u>: We will have that $\mathrm{ind}(\theta\chi) = \mathrm{ind}(\theta) + \mathrm{ind}(\chi)$, if and only if $\dfrac{p_1}{w} = \dfrac{p_2}{w} = 1$.

Let us make the following remark: the numbers p_1, j, p_2, and k depend only on the indices and stabilizers of the functions θ and χ, and can therefore be computed easily using lemma 3.4. The number w, however, measures the interaction between the zero sets of θ and χ - it counts those elements in $\frac{1}{m+n} M$ that send $S_1 \cup S_2$ to itself, without sending S_1 to itself and S_2 to itself. This is obviously a function, not merely of the stabilizers of the separate functions θ and χ, but of the actual position of the zeroes of θ and χ relative to each other.

Although our previous remarks show that we can group the zeroes of θ and χ in such a way that sums of certain subsets of these zeroes are completely determined, this does not determine the actual location of each of the zeroes. In fact, suppose that we know that θ has j zeroes, whose sum in some parallelogram $D' \subseteq D$, equals some fixed number, ν_μ. The actual location of these zeroes can be quite arbitrary; given any $j-1$ points of D', we may choose a jth point such that the sum of all j points equals ν_μ.

It is clear then, that the number w cannot be computed in general, by looking separately at the zeroes of the functions θ and χ. We will now discuss to what extent we can determine w, and what factors are involved in this determination. Our next theorem shows that if θ and χ are both of index 1, then w is quite easily determined.

<u>Theorem 3.7</u>: Suppose that $\theta \in \textcircled{H}_m(\Gamma_0)$ and $\chi \in \textcircled{H}_n(\Gamma_0)$ are both of index 1 . Let L_1 and L_2 denote the lattices in $\mathbb{R}^2$ determined by $\Delta(\theta)$ and $\Delta(\chi)$, respectively and let D_1 and D_2 be the fundamental parallelograms of L_1 and L_2 respectively. Then

$$w = \begin{cases} 2 \ , & \text{if } \Delta(\theta) = \Delta(\chi) \text{ and } Z_0 - w_0 \in \tfrac{1}{2}L_1 \\ 1 \ , & \text{otherwise} \ , \end{cases}$$

where Z_0 is the unique zero of θ in D_1 , and w_0 is the unique zero of χ in D_2 .

<u>Proof</u>: Let S_1 and S_2 denote the zeroes in D of θ and χ , respectively. It follows from corollary 2.8 that

$$S_1 = \{Z_0 + \gamma \mid \gamma \in \Delta(\theta)\} \text{ and } S_2 = \{w_0 + \beta \mid \beta \in \Delta(\chi)\} \ .$$

We show that if $\mu \in \frac{1}{m+n} M$ is such that $-\mu + (S_1 \cup S_2) = S_1 \cup S_2$, and $-\mu + Z_1$ intersects S_1 for any $Z_1 \in S_1$, then $\mu \in \Delta(\theta) \cap \Delta(\chi)$. Suppose then that for $Z_1 = Z_0 + \gamma_1 \in S_1$, we have that $-\mu + Z_1 = Z_0 + \gamma$, $\gamma \in \Delta(\theta)$. Then clearly $\mu = \gamma - \gamma_1 \in \Delta(\theta)$. Since $-\mu + (S_1 \cup S_2) = S_1 \cup S_2$, we have $-\mu + S_2 = S_2$, and also $\mu \in \frac{1}{m+n} M \cap \frac{1}{m} M \subseteq \frac{1}{n} M$. Hence $\mu \in \Delta(\theta) \cap \Delta(\chi)$.

Suppose now that $\mu \in \frac{1}{m+n} M$, and $-\mu + (S_1 \cup S_2) = S_1 \cup S_2$. We have then that either $\mu \in \Delta(\theta) \cap \Delta(\chi)$, or else $-\mu + S_1 = S_2$ and $-\mu + S_2 = S_1$. Therefore,

$$W = \{0\} \cup \{\mu \in \tfrac{1}{m+n} M \mid -\mu + S_1 = S_2 \text{ and } -\mu + S_2 = S_1\} \ .$$

We examine two cases:

<u>Case I</u>: $m \neq n$. Then clearly, there can exist no μ such that $-\mu + S_1 = S_2$, and $w = 1$.

<u>Case II</u>: $m = n$. Suppose that $\mu \in \frac{1}{m+n} M$ is such that $-\mu + S_1 = S_2$. Let $\beta \in \Delta(\chi)$ be such that $-\mu + Z_0 = w_0 + \beta$. Then, for any $\gamma \in \Delta(\theta)$, we have $-\mu + Z_0 + \gamma = w_0 + \beta + \gamma$. But this must be an element of S_2 , which implies that γ must be an element of $\Delta(\chi)$. Hence, we must have that $\Delta(\theta) \subseteq \Delta(\chi)$. Equivalently, $-\mu + S_2 = S_1$ implies that $\Delta(\chi) \subseteq \Delta(\theta)$. Thus, in order for W to contain a non-zero element, it is necessary that $\Delta(\theta) = \Delta(\chi)$.

We assume then that $\Delta(\theta) = \Delta(\chi)$. It follows then, from corollary 2.8, that $w_0 = Z_0 - \mu$ for some element μ of $\widehat{\Delta(\theta)} = \frac{1}{m} L/L_1$, so that $-\mu + S_1 = S_2$. In order for this element μ to belong to W , we need that $-\mu + S_2 = S_1$, and this holds if and only if $2\mu \in \Delta(\theta) \cap \Delta(\chi) = \Delta(\theta)$; i.e., if $\mu \in \frac{1}{2}\Delta(\theta)$ or equivalently, if $\mu \in \frac{1}{2}L_1$. In this case $W = \{0, \mu\}$ and $w = 2$. Otherwise, $w = 1$.

<u>Example 1</u>: Let $\theta \in \textcircled{H}_4(\Gamma_0)$ and $\chi \in \textcircled{H}_4(\Gamma_0)$ have zero sets $S_1 = \{(\frac{1}{8}, \frac{1}{2}), (\frac{3}{8}, \frac{1}{2}), (\frac{5}{8}, \frac{1}{2}), (\frac{7}{8}, \frac{1}{2})\}$ and $S_2 = \{(\frac{1}{8}, 0), (\frac{3}{8}, 0), (\frac{5}{8}, 0), (\frac{7}{8}, 0)\}$, respectively. Then $\Delta(\theta) = \Delta(\chi) = \{(\frac{1}{4}, 0), (\frac{2}{4}, 0), (\frac{3}{4}, 0), (0, 0)\}$, and $\operatorname{ind}(\theta) = \operatorname{ind}(\chi) = 1$. Now $Z_0 = (\frac{1}{8}, \frac{1}{2})$ and $w_0 = (\frac{1}{8}, 0)$, so that $Z_0 - w_0 = (0, \frac{1}{2}) \in \frac{1}{2}L_1$, $W = \{(0, 0), (0, \frac{1}{2})\}$, and $w = 2$. In this case $\operatorname{ind}(\theta\chi) = 1$.

<u>Example 2</u>: Let $\theta \in \textcircled{H}_4(\Gamma_0)$ and $\chi \in \textcircled{H}_4(\Gamma_0)$ have zero sets $S_1 = \{(\frac{1}{8}, \frac{1}{2}), (\frac{3}{8}, \frac{1}{2}), (\frac{5}{8}, \frac{1}{2}), (\frac{7}{8}, \frac{1}{2})\}$ and $S_2 = \{(\frac{1}{8}, \frac{1}{4}), (\frac{3}{8}, \frac{1}{4}), (\frac{5}{8}, \frac{1}{4}), (\frac{7}{8}, \frac{1}{4})\}$ respectively. Again $\Delta(\theta) = \Delta(\chi) = \{(0, 0), (\frac{1}{4}, 0), (\frac{2}{4}, 0), (\frac{3}{4}, 0)$, and $\mathrm{ind}(\theta) = \mathrm{ind}(\chi) = 1$. In this case, however, $Z_0 = (\frac{1}{8}, \frac{1}{2})$ and $w_0 = (\frac{1}{8}, \frac{1}{4})$, so that $Z_0 - w_0 = (0, \frac{1}{4}) \notin \frac{1}{2}L_1$ so that $W = \{(0, 0)\}$, $w = 1$ and $\mathrm{ind}(\theta\chi) = 2$.

We now consider the case in which $\theta \in \textcircled{H}_m(\Gamma_0)$ is of index 1 , and $\chi \in \textcircled{H}_n(\Gamma_0)$ is of index $k > 1$. Let S_1 and S_2 denote the zeroes in D of θ and χ , respectively. We will see below, that in this case, $w = |W|$ may be determined by an inspection of the set S_2 .

Let $W' = \{\lambda \in \frac{1}{m+n} M \mid -\lambda + (S_1 \cup S_2) = S_1 \cup S_2\}$. Then $W = W'/(\Delta(\theta) \cap \Delta(\chi))$. We will view W , not as the space of $\Delta(\theta) \cap \Delta(\chi)$ - cosets, but rather as the set of coset representatives obtained by selecting from each coset the element of $\frac{1}{m+n} M$ lying in the fundamental parallelogram, $\tilde{D}$, of the lattice spanned by the elements of $\Delta(\theta) \cap \Delta(\chi)$.

The Fundamental Theorem of Abelian Groups insures that W' has generators $\mu_1 = (x, 0)$ and $\mu_2 = (y, Z)$, where x and Z are non-zero and minimal. Let γ_1 and γ_2 generate $\Delta(\theta) \cap \Delta(\chi)$. We may clearly choose μ_1 and μ_2 from the set $\tilde{D} \cup \{\gamma_1, \gamma_2\}$. If μ_1 and μ_2

are both elements of $\{\gamma_1, \gamma_2\}$, then $W = (0)$, and $w = 1$. If one of the two generators μ_1 and μ_2 belongs to $\{\gamma_1, \gamma_2\}$, then W is generated by the other one, and therefore, has one linearly independent non-trivial generator. If neither μ_1 nor μ_2 is an element of $\{\gamma_1, \gamma_2\}$, then W has these two linearly independent generators.

We now determine how $w = |W|$ can be computed. Suppose that μ is a non-trivial element of W. It follows, as in the proof of theorem 3.7, that $-\mu + S_1$ is disjoint from S_1. Therefore, $-\mu + S_1 \subseteq S_2$. A similar argument shows that the set $-2\mu + S_1$ is disjoint from $-\mu + S_1$, and that it is either disjoint from, or else equal to the set S_1. Containing in this manner, we obtain a disjoint union:

$$(-\mu + S_1) \cup (-2\mu + S_1) \cup (-3\mu + S_1) \cup \ldots$$

contained in S_2. Since $\mu \in \frac{1}{m+n} M$, it is clear that there exists a smallest positive integer r such that $-(r + 1)\mu + S_1 = S_1$. Since $-(r + 1)\mu + (S_1 \cup S_2) = S_1 \cup S_2$, it follows that $-(r + 1)\mu + S_2 = S_2$, so that $(r + 1)\mu \in \Delta(\theta) \cap \Delta(\chi)$. Thus, for any non-zero element μ of W, there exists a smallest positive integer r such that

(a) $(r + 1)\mu \in \Delta(\theta) \cap \Delta(\chi)$

(b) $U(\mu, r) = (-\mu + S_1) \cup (-2\mu + S_1) \cup \ldots \cup (-r\mu + S_1) \subseteq S_2$.

Similarly, if $\mu \in \Delta(\theta) \cap \Delta(\chi)$ is the zero element of W, then (a) and (b) are trivially satistied by taking $r = 0$, and defining $U(\mu, 0) = \emptyset$.

We can now describe the set S_2 precisely. One simple possibility is that $S_2 = U(\mu, r)$, in which case $W = \{0, \mu, 2\mu, \ldots r\mu\}$, $w = r + 1$, and by theorem 2.5, $\mathrm{ind}(\theta_\chi) = 1$. More generally, we let V denote the complement in S_2 of $U(\mu, r)$. (V may be empty as in the case just mentioned.) Now S_2 may contain zeroes of order greater than 1, so we note that if $Z \in S_2$ has order s, and $Z \in U(\mu, r)$, then Z appears in V with order $s - 1$.

Since $-\mu + (S_1 \cup U(\mu, r)) = S_1 \cup U(\mu, r)$, we must clearly have that $-\mu + V = V$, with the further requirement that $-\mu + v$ occurs in S_2 with the same order as does v, for all $v \in V$. It follows then, that if V intersects either S_1 or $U(\mu, r)$, it cannot do so arbitrarily. Close examination indicates that we can decompose V, and therefore S_2 as well, into a disjoint union of subsets in one of the following two ways:

(1) There exists a non-negative integer s, a set T disjoint from S_1 and $U(\mu, r)$, with $-\mu + T = T$, such that

$$S_2 = S_1^s \cup (U(\mu, r))^{s+1} \cup T ,$$

where we recall that for s a positive integer, S^s means that every element in the set S occurs with order s, and where $S^0 = \emptyset$.

(2) Let $\tilde{S}_1$ be a subset of S_1 such that $\{\lambda \in \frac{1}{m+n} M \mid -\lambda + \tilde{S}_1 = \tilde{S}_1\}$ is a non-trivial subgroup of $\Delta(\theta)$, one of whose generators is $(r + 1)\mu$, let $S_1'' = S_1 - \tilde{S}_1$,

$\tilde{U}(\mu, r) = (-\mu + \tilde{S}_1) \cup (-2\mu + \tilde{S}_1) \cup \ldots \cup (-r\mu + \tilde{S}_1)$

and

$U''(\mu, r) = (-\mu + S_1'') \cup (-2\mu + S_1'') \cup \ldots \cup (-r\mu + S_1'')$.

Then, there can exist non-negative integers s and t , a set T disjoint from S_1 and $U(\mu, r)$, with $-\mu + T = T$, such that we can write

$S_2 = \tilde{S}_1^s \cup (\tilde{U}(\mu, r))^{s+1} \cup (S'')^t \cup (U''(\mu, r))^{t+1} \cup T$.

We note that if $\tilde{S}_1 = S_1$, then (2) degenerates into (1).

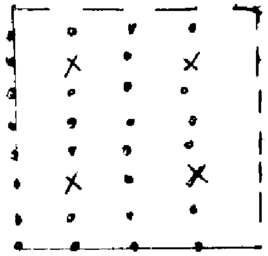
Suppose now that μ_1 , and μ_2 generate W' , and let r_1 and r_2 be non-negative integers such that $(r_i + 1)\mu_i \in \Delta(\theta) \cap \Delta(\chi)$, $i = 1, 2$. If $\mu_1 \in \Delta(\theta) \cap \Delta(\chi)$, so that $r_1 = 0$, then we can trivially decompose S_2 with respect to μ_1 as in (1), taking $s = 0$, and $T = S_2$. If neither μ_1 nor μ_2 belongs to $\Delta(\theta) \cap \Delta(\chi)$, then we can decompose S_2 as in either (1) or (2), with respect to <u>both</u> μ_1 and μ_2 . In all cases, we have that

$$W = \{a\mu_1 + b\mu_2 \mid a \in \{0, 1, \ldots, r_1\}, b \in \{0, 1, \ldots, r_2\}\},$$

so that $w = |W| = (r_1 + 1)(r_2 + 1)$.

We illustrate our remarks by describing three simple examples. In the first one, W will have two linearly independent non-trivial generators, in the second, W will have one non-trivial generator, and in the third, $W = (0)$. In each of these examples, we indicate the zeroes of S_1 by an "x" , the simple zeroes of S_2 by an " . " , those of $\tilde{S}_1$ by an " ⊗ " , and those zeroes of S_2 taken with order 2 by " ⊙ " .

<u>Example 3</u>: $\theta \in \textcircled{H}_4(\Gamma_0)$, $\Delta(\theta) = \mathrm{grp}\{(\frac{1}{2}, 0), (0, \frac{1}{2})\}$, $\mathrm{ind}(\theta) = 1$. $\chi \in \textcircled{H}_{28}(\Gamma_0)$, $\Delta(\chi) = \mathrm{grp}\{(\frac{1}{4}, 0), (0, \frac{1}{8})\}$ $\mathrm{ind}(\chi) = 14$.



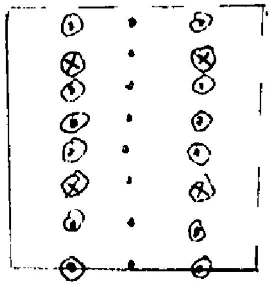
$\mu_1 = (\frac{1}{4}, 0)$, $r_1 = 1$

$\mu_2 = (0, \frac{1}{8})$, $r_2 = 3$

$w = 8$, $\mathrm{ind}(\theta\chi) = 1$

<u>Example 4</u>: $\theta \in H_4(\Gamma_0)$, $\Delta(\theta) = \mathrm{grp}\{(\frac{1}{2}, 0), (0, \frac{1}{2})\}$, $\mathrm{ind}(\theta) = 1$. $\chi \in H_{28}(\Gamma_0)$, $\Delta(\chi) = \mathrm{grp}\{(1, 0), (0, \frac{1}{2})\}$, $\mathrm{ind}(\chi) = 14$.

$\{\lambda \mid -\lambda + \tilde{S}_1 = \tilde{S}_1\} = \mathrm{grp}\{(1, 0), (0, \frac{1}{2})\}$



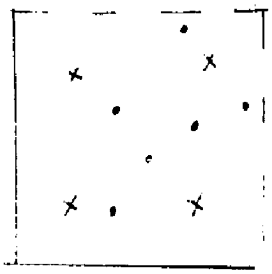
$\mu_1 = (1, 0)$, $r_1 = 0$

$\mu_2 = (0, \frac{1}{8})$, $r_2 = 3$

$w = 4$, $\mathrm{ind}(\theta\chi) = 4$

<u>Example 5</u>: $\theta \in \textcircled{H}_4(\Gamma_0)$, $\Delta(\theta) = \mathrm{grp}\{(\frac{1}{2}, 0), (0, \frac{1}{2})\}$, $\mathrm{ind}(\theta) = 1$.

$\chi \in \textcircled{H}_6(\Gamma_0)$, $\Delta(\chi) = \mathrm{grp}\{(1, 0), (0, 1)\}$, $\mathrm{ind}(\chi) = 6$.



$\mu_1 = (1, 0)$, $r_1 = 0$

$\mu_2 = (0, 1)$, $r_2 = 0$

$w = 1$, $\mathrm{ind}(\theta\chi) = 10$.

In order to state the theorem we have proven, we will need to make the following definitions:

<u>Definition 3.1</u>: Let r_1 and r_2 be non-negative integers, and let S_1 and S_2 be the zero sets in D of $\theta \in \textcircled{H}_m(\Gamma_0)$ and $\chi \in \textcircled{H}_n(\Gamma_0)$, respectively, where $\mathrm{ind}(\theta) = 1$, and $\mathrm{ind}(\chi) = k > 1$. We will say that the pair (S_1, S_2) is of <u>Type (I; r_1, r_2)</u> if there exist elements μ_1 and μ_2 of $\tilde{D} \cup \{\gamma_1, \gamma_2\}$, where γ_1 and γ_2 generate $\Delta(\theta) \cap \Delta(\chi)$, of the form $(x, 0)$ and (y, Z) , respectively, sets T_1 and T_2 of points in D , (orders greater than one allowed), and a non-negative integer s , such that the following conditions hold:

(a) r_1 and r_2 are the smallest non-negative integers such that $(r_i + 1)\mu_i \in \Delta(\theta) \cap \Delta(\chi)$, $i = 1, 2$.

(b) $-\mu_i + T_i = T_i$, with the order of $-\mu_i + v$ in S_2 being the same as the order of v in S_2 , for all $v \in T_i$, $i = 1, 2$.

(c) μ_1 is the element of $\tilde{D} \cup \{\gamma_1, \gamma_2\}$ of the form $(x, 0)$, with the smallest non-zero value of x, and μ_2 is the element of $\tilde{D} \cup \{\gamma_1, \gamma_2\}$ of the form (y, Z) with the smallest non-zero value of Z, so that we can write S_2 as a disjoint union in each of the following two ways:

(1) $S_2 = S_1^s \cup (U(\mu_1, r_1))^{s+1} \cup T_1$

(2) $S_2 = S_1^s \cup (U(\mu_2, r_2))^{s+1} \cup T_2$, where

$$U(\mu_i, r_i) = (-\mu_i + S_1) \cup (-2\mu_i + S_1) \cup \ldots \cup (-r_i\mu_i + S_1) \quad , \quad i = 1, 2 \quad .$$

We note that in particular, if (S_1, S_2) is of Type(I; 0, 0), then condition (a) implies that μ_1 and μ_2 are both elements of $\Delta(\theta) \cap \Delta(\chi)$, and it follows then from (c) and our previous remarks that $W = (0)$. If (S_1, S_2) is of Type(I; r_1, 0), then $\mu_2 \in \Delta(\theta) \cap \Delta(\chi)$ by (a), (2) of (c) holds trivially, and W has one non-trivial generator $-\mu_1$. If (S_1, S_2) is of Type(I; r_1, r_2), where r_1 and r_2 are both non-zero, then μ_1 and μ_2 are linearly-independent non-trivial generators of W.

<u>Definition 3.2</u>: Let r_1, r_2, S_1 and S_2 be as in the previous definition. We will say that the pair (S_1, S_2) is of <u>Type(II; r_1, r_2</u>) if there exist

i) non-negative integers s and t,

ii) elements $\mu_1 = (x, 0)$ and $\mu_2 = (y, Z)$ of $\tilde{D} \cup \{\gamma_1, \gamma_2\}$ satisfying conditions (a) and (b) of definition 3.1 ,

iii) subsets T_1 and T_2 of D with $-\mu_i + T_i = T_i$, and orders being preserved under this addition, $i = 1, 2$,

iv) a subset $\tilde{S}_1$ of S_1 such that

$$\{\lambda \mid -\lambda + \tilde{S}_1 = \tilde{S}_1\} = \mathrm{grp}\{(r_1 + 1)\mu_1, (r_2 + 1)\mu_2\} ,$$

so that μ_1 and μ_2 are the elements of the form $(x, 0)$ and (y, Z) , respectively, with x and Z non-zero and minimal, such that we can write S_2 in each of the following two ways:

$$(1) \quad S_2 = (\tilde{S}_1)^s \cup (\tilde{U}(\mu_1, r_1))^{s+1} \cup (S_1'')^t \cup (U''(\mu_1, r_1)^{t+1} \cup T_1$$

$$(2) \quad S_2 = (\tilde{S}_1)^s \cup (\tilde{U}(\mu_2, r_2))^{s+1} \cup (S_1'')^t \cup (U''(\mu_2, r_2)^{t+1} \cup T_2 ,$$

where $S_1'' = S_1 - \tilde{S}_1$,

$$\tilde{U}_i(\mu_i, r_i) = (-\mu_i + \tilde{S}_1) \cup (-2\mu_i + \tilde{S}_1) \cup \ldots \cup (-r\mu_i + \tilde{S}_1)$$

$$U_i''(\mu_i, r_i) = (-\mu_i + S_1'') \cup (-2\mu_i + S_1'') \cup \ldots \cup (-r_i\mu_i + S_1'') , \quad i = 1, 2 .$$

Our previous remarks indicate that if S_1 is the zero set of $\theta \in \textcircled{H}_m(\Gamma_0)$, where $\mathrm{ind}(\theta) = 1$, and S_2 is the zero set of $\chi \in \textcircled{H}_n(\Gamma_0)$, where $\mathrm{ind}(\chi) = k > 1$, then the pair (S_1, S_2) is of Type$(I; r_1, r_2)$ or of Type$(II; r_1, r_2)$ for some non-negative integers r_1 and

r_2 . We have therefore proven the following theorem:

Theorem 3.8: Let S_1 and S_2 be the zero sets in D of $\theta \in \textcircled{H}_m(\Gamma_0)$ and $\chi \in \textcircled{H}_n(\Gamma_0)$, respectively, where $\mathrm{ind}(\theta) = 1$ and $\mathrm{ind}(\chi) = k > 1$, and let the pair (S_1, S_2) be of Type(I; r_1, r_2) or of Type(II; r_1, r_2). Then

$$w = (r_1 + 1)(r_2 + 1) \quad .$$

Suppose now that $\theta \in \textcircled{H}_m(\Gamma_0)$ is of index $j > 1$, and $\chi \in \textcircled{H}_n(\Gamma_0)$ is of index $k > 1$, with zero sets S_1 and S_2 , respectively, in D . In this case, the fact that $-\mu + (S_1 \cup S_2) = S_1 \cup S_2$, does not imply the dicotomy that $-\mu + S_1$ must be disjoint from, or else equal to S_1 . All kinds of interactions between S_1 and S_2 are possible. When this happens, the only way to determine w is to examine the set $S_1 \cup S_2$ directly, and look for generators of W . We cannot, then, give as explicit a discription of w in this case, as we did in the case covered in theorem 3.8.

Suppose, however, that in our more general case, one of the functions - say χ , is such that all of its zeroes have rational coordinates. We show that, in this case, w can be computed by successive applications of theorem 3.8. We include this result, not so much for its practical importance, as in reality the computation of w in theorem 3.8 by examining the pair (S_1, S_2) to examine its type is not much easier than the examination of $S_1 \cup S_2$ to compute w directly, but because it illustrates the importance of "rationality," an idea which we will take up further in the appendix.

We must first mention that although we have assumed in this section, for convenience of notation, that all of our functions are in $\textcircled{H}(\Gamma_0)$, for the fixed lifted subgroup $\Gamma_0 = \mathrm{grp}\{(1, 0, 0), (0, 1, 0)\}$, a brief examination indicates that our results are equally valid when θ and χ both lie in $\textcircled{H}(\Gamma)$, for any lifted subgroup Γ of N. Moreover, if Γ_1 and Γ_2 are lifted subgroups of N with generators that are rationally related; i.e., $\Gamma_1 = \mathrm{grp}\{(a_1, a_2, a_3), (b_1, b_2, b_3)\}$ and $\Gamma_2 = \mathrm{grp}\{(a_1', a_2', a_3'), (b_1', b_2', b_3')\}$, with $\frac{a_i}{a_i'} \in \mathbb{Q}$ and $\frac{b_i}{b_i'} \in \mathbb{Q}$, $i = 1, 2, 3$. It follows easily from the results in [1] and [2], (see our appendix for further discussion), that there exists a lifted subgroup Γ of N such that $L^2(N/\Gamma_i) \subset L^2(N/\Gamma)$; $i = 1, 2$. Therefore, if $\theta \in \textcircled{H}_m(\Gamma_1)$ and $\chi \in \textcircled{H}_n(\Gamma_2)$, we may think of both of these functions as elements of $\textcircled{H}(\Gamma)$. Moreover, since by lemma 2.4, we have that $\mathrm{ind}_{\Gamma_1}(\theta) = \mathrm{ind}_{\Gamma}(\theta)$ and $\mathrm{ind}_{\Gamma_2}(\chi) = \mathrm{ind}_{\Gamma}(\chi)$, we can consider the multiplication theory even in the case when θ and χ live on two different nilmanifolds - N/Γ_1 and N/Γ_2, respectively, as long as Γ_1 and Γ_2 are rationally related.

Suppose now that $\mathrm{ind}(\theta) = j$ and $\mathrm{ind}(\chi) = k$, where j and k are greater than one. Let

$S_2' = \{v_1, v_2, \ldots, v_k\}$ be the zero set of χ in the fundamental parallelogram D_2 of the lattice, L_2, determined by $\Delta(\chi)$. We assume moreover, that the coordinates of all the v_i are rational. Let χ_i' denote the entire theta function with respect to the lattice L_2, whose unique zero in D_2 is v_i ; $i = 1, 2, \ldots, k$. The "type" of this theta function, or equivalently, the lifted subgroup Γ_i', with $\pi(\Gamma_i) = L_2$, such that $\chi_i = M(\frac{1}{\beta(\Gamma_1')})^{-1} \chi_i'$ is an element of $H_1(\Gamma_i')$ can be determined using theorem 1.19. It is not difficult to see that Γ_i' will have generators that are rationally related to the generators of Γ_0 for all $i \in \{1, 2, \ldots, k\}$. Therefore, we can choose a lifted subgroup Γ of N so that $L^2(N/\Gamma_i') \subseteq L^2(N/\Gamma)$, $i = 1, 2, \ldots, k$, and so that $L^2(N/\Gamma_0) \subseteq L^2(N/\Gamma)$.

Moreover, $\chi = \chi_1 \chi_2 \ldots \chi_k$, and χ_i is of index 1, for all i. Therefore, we can calculate

$$\operatorname{ind}(\theta\chi) = \operatorname{ind}(\ldots(\theta\chi_1)\chi_2) \ldots \chi_k)$$

by successive applications of theorem 3.8.

APPENDIX

DISTINGUISHED SUBSPACES AND THE RATIONAL STRUCTURES OF N

We have seen above that if $\mathcal{D}$ is a distinguished subspace of $H_m(\Gamma)$, where Γ is a lifted subgroup of N , then the zeroes of the function $M(\frac{m}{\beta(\Gamma)})\theta$, where $\theta \in \mathcal{D} \cap \textcircled{H}$, are at points in the plane whose coordinates are rationally related to the coordinates of the generators of Γ , where what we mean by x and y being rationally related is that $\frac{x}{y}$ is rational. This seems to indicate some relationship between the distinguished subspace theory in $L^2(N/\Gamma)$, and the rational structures of N containing Γ . In this section, we explore this idea, and we prove the following:

Two lifted subgroups Γ_1 and Γ_2 of N determine the same rational form on N , if and only if $L^2(N/\Gamma_1)$ and $L^2(N/\Gamma_2)$ share a distinguished subspace.

Let us first define what we mean by a rational form on N . Let (x, y) be a coordinate system on $\mathbb{R}^2$, and let $B((x, y), (x', y'))$ denote the symplectic form $\frac{1}{2}(yx' - xy')$ on $\mathbb{R}^2$. The group operation on $N = \mathbb{R}^2 \times \mathbb{R}$ can then be expressed in this coordinate system, as

$$(x, y, t)(x', y', t') = (x+x', y+y', t+t' + B((x, y),(x',y')).$$

Let us take any coordinate system, now on $\mathbb{R}^2$, with a basis $\{e_1, e_2\}$ such that the matrix representing the bilinear form B with respect to this basis has rational entries. Let

$$V_{\mathbb{Q}} = \{r_1e_1 + r_2e_2 \mid r_i \in \mathbb{Q}, \quad i = 1, 2\} \quad .$$

We define a group structure on $V_{\mathbb{Q}} \times \mathbb{Q}$ by the usual formula

$$(v_1, t_1)(v_2, t_2) = (v_1+v_2, t_1+t_2, + B(v_1, v_2)) \quad ,$$

for $(v_i, t_i) \in V_{\mathbb{Q}} \times \mathbb{Q}$, $i = 1, 2$. The resulting group will be called a rational form of N, and denoted by $N_{\mathbb{Q}}$. Clearly $N_{\mathbb{Q}} \subseteq N$, and N has many rational forms which can be shown to be isomorphic.

We now list several facts from [1] and [2] that we will need. Unless otherwise indicated, N denotes any connected, simply connected, nilpotent Lie group.

1. Let Γ be a discrete co-compact subgroup of N such that N/Γ is compact. There exists a unique rational form $N_{\mathbb{Q}}$ of N such that $\Gamma \subseteq N_{\mathbb{Q}}$.
2. Let $N_{\mathbb{Q}}$ be a rational form of N. Let Γ be a finitely generated subgroup of $N_{\mathbb{Q}}$: Then Γ is a discrete subgroup of N. Further, $N_{\mathbb{Q}}$ contains a finitely-generated subgroup Γ such that N/Γ is compact.
3. Let N denote the Heisenberg group and let Γ be a discrete co-compact subgroup of N. Then,
   a) There exists a minimal lifted subgroup Γ^L containing Γ, such that

$\Gamma \cap Z(N) = \Gamma^{L} \cap Z(N)$.

Note that Γ and Γ^{L} are subsets of the same rational form of N .

b) There exists a maximal lifted subgroup $\Gamma_{L} \subseteq \Gamma$ such that $\pi(\Gamma) = \pi(\Gamma_{L})$, and $[\Gamma, \Gamma] = [\Gamma_{L}, \Gamma_{L}]$. Γ and Γ_{L} also clearly belong to the same rational form of N .

This fact allows us to consider lifted subgroups rather than the more general theory of all discrete co-compact subgroups of N .

4. Let Γ_1 and Γ_2 be lifted subgroups of N such that $\Gamma_1, \Gamma_2 \subseteq N_{\mathbb{Q}}$, for some rational form $N_{\mathbb{Q}}$ of N . Then

   (a) There exists a lifted subgroup $\Gamma \subseteq N_{\mathbb{Q}}$ such that $\Gamma \supseteq \Gamma_i$, $i = 1, 2$.

   (b) There exists a lifted subgroup $\Gamma' \subseteq N_{\mathbb{Q}}$ such that $\Gamma' \subseteq \Gamma_i$, $i = 1, 2$.

Suppose now that Γ is a lifted subgroup of N , and let $N_{\mathbb{Q}}$ be a rational form of N such that $\Gamma \subseteq N_{\mathbb{Q}}$. We recall that a subspace $\mathfrak{h} \subseteq H_m(\Gamma)$ is distinguished if and only if $\mathfrak{h} = H_1(\Gamma_1)$, where Γ_1 is a lifted subgroup of $N_{\mathbb{Q}}$. The following lemma is proven in [2]:

<u>Lemma A.1</u>: Let Γ , Γ_1 be lifted subgroups of N , and $m \in \mathbb{Z}$. Then $H_1(\Gamma_1) \subseteq H_m(\Gamma)$ if and only if

(a) $\Gamma \subseteq \Gamma_1$

(b) $\beta(\Gamma) = m\beta(\Gamma_1)$.

Now, if $\Gamma \subseteq \Gamma_1$ and $|\Gamma_1/\Gamma| < \infty$, we may clearly identify $L^2(N/\Gamma_1)$ with a subspace of $L^2(N/\Gamma)$. Moreover, if R and R_1 denote the right regular representations of N on $L^2(N/\Gamma)$ and $L^2(N/\Gamma_1)$, respectively, and if $\mathcal{H}$ is irreducible, closed, and R-invariant as a subspace of $L^2(N/\Gamma)$, then it is irreducible, closed, and R_1-invariant as a subspace of $L^2(N/\Gamma_1)$. We are now ready to state and prove the main result of this section.

<u>Theorem A.2</u>: Let Γ_1 and Γ_2 be lifted subgroups of N , and let

$$L^2(N/\Gamma_1) \cap L^2(N/\Gamma_2) = \{f : N \to \mathbb{C} \,/\, f(\gamma g) = f(g) ,$$

for all $g \in N$, $\gamma \in \mathrm{grp}\{\Gamma_1, \Gamma_2\}$, and such that $\int |f|^2 d\omega_i < \infty\}$, where ω_i denotes Haar measure on N/Γ_i , $i = 1, 2$. Then

(a) There exists a subspace $\mathcal{H}$ of $L^2(N/\Gamma_1) \cap L^2(N/\Gamma_2)$, and integers m_1 and m_2 , such that $\mathcal{H}$ is a distinguished subspace of both $H_{m_1}(\Gamma_1)$ and $H_{m_2}(\Gamma_2)$, if and only if, there exists a rational form $N_{\mathbb{Q}}$ of N such that $\Gamma_i \subseteq N_{\mathbb{Q}}$, $i = 1, 2$.

(b) Let $\Gamma_1, \Gamma_2 \subseteq N_{\mathbb{Q}}$, be lifted subgroups, and let $\mathcal{H}_i$ be a distinguished subspace of $H_{m_i}(\Gamma_i)$,

$i = 1, 2$. Then, either $\mathfrak{H}_1 \cap \mathfrak{H}_2 = (0)$ or $\mathfrak{H}_1 = \mathfrak{H}_2$.

<u>Proof:</u> (a) Suppose first that $\mathfrak{H} \subseteq H_{m_1}(\Gamma_1) \cap H_{m_2}(\Gamma_2)$ is distinguished. Then, there exists a lifted subgroup Γ of N such that $\mathfrak{H} = H_1(\Gamma)$. Therefore, by lemma A.1, we have that $\Gamma_i \subseteq \Gamma$, $i = 1, 2$. Let $N_{\mathbb{Q}}$ be the rational form of N determined by Γ . Then clearly $\Gamma_i \subset N_{\mathbb{Q}}$, $i = 1, 2$.

Conversely, suppose that $\Gamma_i \subseteq N_{\mathbb{Q}}$, $i = 1, 2$, for some rational form $N_{\mathbb{Q}}$ of N . By fact (4) of this appendix, we can choose a lifted subgroup Γ of $N_{\mathbb{Q}}$ such that $\Gamma \supseteq \Gamma_i$, $i = 1, 2$. By lemma A.1, there exist integers m_i such that $H_1(\Gamma)$ is a distinguished subspace of $H_{m_i}(\Gamma_i)$, $i = 1, 2$.

(b) Let $\Gamma_1, \Gamma_2 \subseteq N_{\mathbb{Q}}$. Choose a lifted subgroup Γ' as in fact (4), so that $\Gamma' \subset \Gamma_i$, $i = 1, 2$. Then $L^2(N/\Gamma_i) \subseteq L^2(N/\Gamma)$, $i = 1, 2$. If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are distinguished subspaces of $L^2(N/\Gamma_1)$ and $L^2(N/\Gamma_2)$, respectively, then they are each irreducible, closed R-invariant subspaces of $L^2(N/\Gamma)$, where R denotes the right regular representation of N on $L^2(N/\Gamma)$, and therefore, either $\mathfrak{H}_1 \cap \mathfrak{H}_2 = (0)$ or $\mathfrak{H}_1 = \mathfrak{H}_2$.

<u>Corollary:</u> Let Γ be a lifted subgroup of N , and let the following be two distinguished subspace decompositions of $H_m(\Gamma)$:

$$H_m(\Gamma) = \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots + \mathfrak{H}_m$$

$$H_m(\Gamma) = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \dots \oplus \mathcal{K}_m \ .$$

Then either $\mathcal{J}_i \cap \mathcal{K}_j = (0)$ for all i, j or else the two decompositions are identical, up to the order of the factors.

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