



On String Topology of Three Manifolds

by
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Abstract

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Hossein Abbaspour

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In this dissertation we establish a connection between some aspects of the string topology of three dimensional manifolds and their topology and geometry using the theory of the prime decomposition and characteristic surfaces.

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Et à la fin, *Isa...*

*À ma famille
et Isabelle*

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1 An Introduction to String Topology and the Statement of the Main Theorem

Throughout this manuscript M denotes a connected, oriented smooth manifold unless otherwise stated. We shall follow the terminology and notations of [5].

We think of the unit circle, \mathbb{S}^1 , as the quotient \mathbb{R}/\mathbb{Z} . The *marked point* of the circle is the image of 0 in this quotient.

A *loop* in M is a continuous map from \mathbb{S}^1 to M . The *free loop space* $\mathbb{L}M$ of M is the space of all loops in M . If $f : \mathbb{S}^1 \rightarrow M$ is a loop in M then the image of the marked point of \mathbb{S}^1 is called the *marked point* of the loop f .

We denote the integral homology of $\mathbb{L}M$ by homology $\mathbb{H}_*(M)$ and is equipped with an *associative product* called *loop product* (see [5]) denoted by \bullet . The loop product has degree $-d$, where d is the dimension M namely

$$\bullet : \mathbb{H}_i(M) \otimes \mathbb{H}_j(M) \rightarrow \mathbb{H}_{i+j-d}(M).$$

Theorem (Chas-Sullivan [5]) For a connected closed oriented smooth manifold M , $(\mathbb{H}_{*-d}(M), \bullet)$ is a graded commutative algebra.

Let $i : M \rightarrow \mathbb{L}M$ be the inclusion map that sends each point m in M to the constant loop at m . This induces a map in homology, also denoted by i ,

$$i : H_*(M) \rightarrow \mathbb{H}_*(M).$$

Also, there is a map p which sends every loop in $\mathbb{L}M$ to its marked points. We denote the induced map on homologies also by p . It is clear that $p \circ i = id_M$ even at the chain level, therefore

$$p \circ i = id_{H_*(M)}$$

proving that i is injective. There is a canonical decomposition

$$\begin{aligned}\mathbb{H}_*(M) &= i(H_*(M)) \oplus A_M, \\ A_M &= \text{Ker } p \text{ and } p_{A_M} = \text{id}_{\mathbb{H}_*(M)} - p\end{aligned}$$

Proposition 1.1 (See [5]) $i : H_*(M) \rightarrow \mathbb{H}_*(M)$ is a map of graded algebras where the multiplication on $H_*(M)$ is the intersection product denoted by \cap . Therefore, one can regard $(\mathbb{H}_*(M), \bullet)$ as an extension of $(H_*(M), \wedge)$.

Definition 1.2 We shall say that an oriented closed manifold M has *nontrivial extended loop products* if the restriction of the loop product to A_M is nontrivial.

Definition 1.3 Let M be a closed 3-manifold. We call M *algebraically hyperbolic* if it is a $K(\pi, 1)$ (aspherical) and its fundamental group has no rank 2 abelian subgroup.

Note that any finite cover of an algebraically hyperbolic manifold is also algebraically hyperbolic.

The following is the main theorem of this manuscript:

Theorem 1.4 *Let M be an oriented closed 3-manifold. If M is not algebraically hyperbolic then M or a double cover of M has nontrivial extended loop products.*

If M is algebraically hyperbolic then M and all its finite covers have trivial extended loop products.

From now on we identify $H_*(M)$ with its image under i . Consequently p projects $\mathbb{H}_*(M)$ onto $i(H_*(M)) \subset \mathbb{H}_*(M)$.

We refer to $\mathbb{H}_*(M)$ by *loop homology*. If $[\alpha] \in \hat{\pi}_1(M)$ is a conjugacy class then $(\mathbb{L}M)_{[\alpha]}$ denotes the corresponding connected component in $\mathbb{L}M$ and $\mathbb{H}_*(M)_\alpha$ its integral homology (See Appendix I).

If α and β are two loops with the same marked points, their *composition*, $\alpha\beta$, is also a loop defined as follows:

$$\alpha\beta = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

We give an informal description of loop product: given two homology classes in $\mathbb{H}_*(M)$, choose a chain representative for each one of them and consider the corresponding set of marked points. They form two chains in M . If these two chains are not in transversal position, replace the representatives with appropriate ones so that the chains of their marked points intersect transversally. At each intersection point, compose the corresponding loops of each chain of loops. This process produces a new chain of loops. It is proven that this operation passes to homology (see [5]).

Remark 1.5 If M is a closed manifold then the algebra $(\mathbb{H}_*(M), \bullet)$ has a *unit*, described as follows. Since each point in M can be regarded as a constant loop, a chain in M gives rise to a chain in $\mathbb{L}M$. This map passes to homology. The image of the fundamental class of M under this map is the unit of $(\mathbb{H}_*(M), \bullet)$.

Remark 1.6 When M is closed then the unit $\mu_M \in \mathbb{H}_*(M)$ is the image of the fundamental class of M under p .

$\mathbb{H}_*(M)$ is also equipped with a self map of degree 1, denoted Δ , which can be described as follows. Consider the natural action of \mathbb{S}^1 on $\mathbb{L}M$: $f \in \mathbb{L}M$

and $r \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$,

$$\begin{aligned} r.f : \mathbb{S}^1 &\rightarrow M \\ (rf)(t) &= f(r+t) \end{aligned}$$

This action induces a map

$$H_*(\mathbb{S}^1 \times \mathbb{L}M) \rightarrow H_*(\mathbb{L}M).$$

Applying the Künneth formula one gets a map $H_1(\mathbb{S}^1) \times \mathbb{H}_*(M) \rightarrow \mathbb{H}_{*+1}(M)$,
or

$$\Delta : \mathbb{H}_*(M) \rightarrow \mathbb{H}_{*+1}(M)$$

Δ has the property that $\Delta^2 = 0$ (see [5]).

Consider deviation of Δ from being a derivation with respect to \bullet , namely

$$\{a, b\} = \Delta a \bullet b + (-1)^{(i-d)} a \bullet \Delta b - \Delta(a \bullet b). \quad (1.1)$$

It turns out (see [5]) that $\{.,.\}$ is a derivation in each variable for \bullet *i.e.*

$$\{a, b \bullet c\} = \{a, b\} \bullet c \pm b \bullet \{a, c\}$$

and

$$\{b \bullet c, a\} = \{b, a\} \bullet c \pm b \bullet \{c, a\}.$$

Moreover $\{.,.\}$ satisfies the Jacobi identity for graded algebras (see [5]).
Therefore $\mathbb{H}_*(M)$ is also equipped with a graded Lie bracket of degree $1-d$,

$$\{, \} : \mathbb{H}_i(M) \otimes \mathbb{H}_j(M) \rightarrow \mathbb{H}_{i+j+1-d}(M).$$

It follows from the definition of $\{, \}$ and the fact that $\Delta^2 = 0$ that Δ is a derivation with respect to $\{, \}$ or

$$\Delta\{a, b\} = \{\Delta a, b\} + (-1)^{(i+1-d)} \{a, \Delta b\} \quad (1.2)$$

A *Batalin-Vilkovsky* algebra is a graded commutative algebra (A, \bullet) with an operator of degree one Δ such that the binary operator $\{, \}$,

$$\{a, b\} = \Delta a \bullet b + (-1)^{(i-d)} a \bullet \Delta b - \Delta(a \bullet b),$$

is a derivation for \bullet in each variable. This plus $\Delta^2 = 0$ implies that $\{, \}$ satisfies Jacobi identity.

Theorem (Chas-Sullivan [5]): For a closed oriented smooth connected manifold M , $(\mathbb{H}_{-d}(M), \bullet, \Delta)$ is a Batalin-Vilkovsky algebra.

Example 1.7 (see [5]) Let M be a closed hyperbolic oriented 3-manifold. As M is aspherical then by lemma 3 in Appendix I, each component of $(\mathbb{L}M)_{[\alpha]}$ is a $K(\pi_\alpha, 1)$ where π_α is the centralizer of α in $\pi_1(M)$. If $\alpha \neq 1 \in \pi_1(M)$ then it can be regarded as a hyperbolic element of a hyperbolic subgroup of $PSL(2, \mathbb{C})$. So its centralizer is the cyclic group generated by α . Hence $(\mathbb{L}M)_{[\alpha]}$ is a $K(\mathbb{Z}, 1)$ and $A_M \simeq H_*(\coprod_{[\alpha] \in \pi_1(M)} K(\mathbb{Z}, 1))$. Therefore A_M has homological dimension 1 which implies that the loop product is identically zero on $A_M \otimes A_M$ as a consequence of the absence of homology class of sufficiently large dimension. So oriented hyperbolic manifolds have trivial extended loop product.

The following Lemma will be useful to us and the proof can found in Appendix I.

Lemma:

- (i) There is a one-to-one correspondence between the connected components of $\mathbb{L}M$ and the conjugacy classes of $\pi_1(M)$.

(ii) Let $\alpha \in \pi_1(M)$ and $f \in \mathbb{L}M$ be a loop representing α . Then there exists a short exact sequence

$$\pi_2(M) \cong \pi_1((\Omega M)_\alpha, f) \xrightarrow{S} \pi_1((\mathbb{L}M)_{[\alpha]}, f) \xrightarrow{R} \pi_\alpha \longrightarrow 0 \quad (1.3)$$

2 3-Manifolds with Finite Fundamental Group

In this section we show that 3-manifolds with finite fundamental group have nontrivial extended loop products.

The following construction is valid for any oriented closed manifold M . Let p be the based point for M . Consider the 0-chain in $\mathbb{L}M$ that consists of the constant loop at p . It represents a homology class $\epsilon \in \mathbb{H}_0(M)$. Given a chain $c \in \mathbb{L}M$ whose set of marked points is transversal to p , then the loops in c whose marked points are p form a chain in the base loop space of ΩM . This induces a map of degree $-d$

$$\cap : \mathbb{H}_*(M) \rightarrow H_{*-d}(\Omega M).$$

On the other $H_*(\Omega M)$ is equipped with Pontryagin product. Indeed,

Lemma 2.1 (see [5]) $\cap : (\mathbb{H}(M), \bullet) \rightarrow (H_{*-d}(\Omega M), \times)$ is an algebra map where \times is the Pontryagin multiplication. In particular if M is a Lie group then \cap is surjective.

Proposition 2.2 S^3 has nontrivial extended loop products.

Proof. Since S^3 is a Lie group, by previous lemma \cap is surjective. $(H_*(\Omega S^3), \times)$ is a polynomial algebra with one generator of degree 2. Let $x_1 \in H_p(\Omega S^3)$ and $x_2 \in H_q(\Omega S^3)$ be non-zero elements such that $p, q > 3$. Let $a_i \in \cap^{-1}(x_i)$ for $i = 1, 2$. Since $H_i(S^3) = 0$ for $i > 3$ we have $a_i \in A_{S^3}$. By Lemma 2.1 $\cap(a_1 \bullet a_2) = \cap(a_1) \times \cap(a_2) = x_1 \times x_2 \neq 0$, therefore $a_1 \bullet a_2 \neq 0$ \square

Generally, we do not know if \cap is a natural map, but under certain assumptions it is. The next lemma deals with this aspect of \cap .

Lemma 2.3 *Let M and N be two closed oriented n -manifolds and $p \in N$. For $f : M \rightarrow N$ which is transversal to p and $f^{-1}(p)$ consists of only one point, the following diagram commutes for:*

$$\begin{array}{ccc} \mathbb{H}_*(M) & \xrightarrow{\cap} & H_*(\Omega M) \\ f_{\mathbb{L}} \downarrow & & f_{\Omega} \downarrow \\ \mathbb{H}_*(N) & \xrightarrow{\cap} & H_*(\Omega N) \end{array} \quad (2.1)$$

f_{Ω} and $f_{\mathbb{L}}$ are the induced maps by f on the homologies of the based loop space and the free loop space.

Proof. We choose p and $f^{-1}(p)$ respectively as the base point of N and M . We show that the above diagram is commutative even at the chain level.

Let $k : \Delta^m \rightarrow \mathbb{L}M$ be a m -simplex in $\mathbb{L}M$. So for each x

$$k(x) : \mathbb{S}^1 \rightarrow M.$$

If $k_0 : \Delta^m \rightarrow M$,

$$k_0(x) = k(x)(0),$$

is transversal to $f^{-1}(p)$ then consider the set $\Lambda \subset \Delta^m$,

$$\Lambda = \{x \in \Delta^m \mid k(x)(0) = f^{-1}(p)\}.$$

$k|_{\Lambda}$ is a chain in $\Omega_{f^{-1}(p)}M$ and $\cap(k) = k|_{\Lambda}$. After composing with f , $f \circ k|_{\Lambda}$ is a chain in $\Omega_p N$ and

$$(f_{\Omega} \circ \cap)(k) = f \circ k|_{\Lambda}.$$

On the other hand $f_{\mathbb{L}}(k) = f \circ k$ is a chain in $\mathbb{L}N$. Since k_0 is transversal to $f^{-1}(p)$ and f is transversal to p thus $f \circ k_0$ is transversal to p . Since f^{-1}

consists of only one point in M then

$$\{x \in \Delta^m \mid (f \circ k)(x) = p\} = \{x \in \Delta^m \mid k(x)(0) = f^{-1}(p)\} = \Lambda,$$

therefore,

$$\cap \circ f_{\mathbf{L}}(k) = f \circ k|_{\Lambda} = f_{\Omega} \circ \cap(k).$$

□

Lemma 2.4 *Let M be a 3-manifold which is a homotopy 3-sphere and $p_1, p_2, \dots, p_r \in M$. Then there is a homotopy equivalence $f : S^3 \rightarrow M$ which is transversal to all p_i 's and $f^{-1}(p_i)$ consists of only one point for each i .*

Proof. Consider a closed ball $D \subset S^3$. Let f be a diffeomorphism from D to a closed ball in $D' \subset M$ where $p_i \in D'$ for all i .

$f|_{\partial D} : \partial D \rightarrow \partial D'$ can be extended to $f : S^3 \setminus \text{Int}D' \rightarrow M \setminus \text{Int}D$ as all the obstructions which are cohomology classes in

$$H^{q+1}(S^3 \setminus \text{Int}D', \partial D', \pi_q(M \setminus \text{Int}D)).$$

vanish since $M \setminus \text{Int}D'$ is contractible¹.

Therefore we have a map that $f : S^3 \rightarrow M$ which sends D' to D diffeomorphically and sends the complement of D' to the complement of D and in particular it is transversal to p_i and $f^{-1}(p_i)$ consists of only one point in S^3 .

We must show that f is a homotopy equivalent. For this, it is enough to observe that it has degree one.

□

¹This can be proved using Mayer-Vietoris exact sequence

Proposition 2.5 *A closed simply connected 3-manifold M , has nontrivial extended loop products.*

Proof. Let $p \in M$ and $f : S^3 \rightarrow M$ be the homotopy equivalence provided by lemma 2.4 for $r = 1$.

f induces a homotopy equivalences $f_\Omega : \Omega S^3 \rightarrow \Omega M$ and $f_L : LS^3 \rightarrow LM$.

We have the following commutative diagram by lemma 2.3:

$$\begin{array}{ccc} \mathbb{H}_*(S^3) & \xrightarrow{\cap_M} & H_*(\Omega_{f^{-1}(p)}S^3) \\ f_* \downarrow & & f_\Omega \downarrow \\ \mathbb{H}_*(M) & \xrightarrow{\cap_{S^3}} & H_*(\Omega_p M) \end{array} \quad (2.2)$$

Since f_Ω and f_L are isomorphisms and \cap_{S^3} is surjective, therefore \cap_M is also surjective.

Now the same argument which was used for S^3 can be applied for M to show that M has nontrivial extended loop product. To be more explicit, let $0 \neq x_1 \in H_m(\Omega M)$ ² and $0 \neq x_2 \in H_n(\Omega M)$ where $n, m > 3$ and $a_i \in \cap^{-1}(x_i)$, $i = 1, 2$.

Then, $a_i \in A_M$ since $H_k(M) = 0$ for all $k > 3$. By commutativity of diagram (2.2).

$$f(a_1 \bullet a_2) = f(a_1) \times f(a_2) = x_1 \times x_2 \neq 0.$$

Hence $a_1 \bullet a_2 \neq 0$.

□

Proposition 2.6 *If M is a closed oriented 3-manifold with finite fundamental group then M has nontrivial extended loop products.*

² $H_*(\Omega M) \cong H_*(\Omega S^3) \cong \mathbb{Z}[x]$ and degree $x=2$

Proof. Let \tilde{M} be the universal cover of M and q be the covering map and $r = \text{degree}(q) = |\pi_1(M)| < \infty$. Choose $p \in M$ and let $\{p_1, p_2, \dots, p_r\} \in q^{-1}(p)$ where p_i 's are distinct.

Since \tilde{M} is homotopy equivalent to S^3 , by lemma 2.4 there is a homotopy equivalence $f : S^3 \rightarrow \tilde{M}$ which is transversal at each p_i and $f^{-1}(p_i)$ consists of only point in S^3 in M . Let $m_i = f^{-1}(p_i)$, $i = 1, 2, \dots, r$. Hence $(q \circ f)^{-1}(p) = \{m_1, m_2, \dots, m_r\}$ and $q \circ f$ is transversal to p .

f composed by the covering map $q : \tilde{M} \rightarrow M$ induces the maps

$$(q \circ f)_{\mathbb{L}} : \mathbb{H}_*(S^3) \rightarrow \mathbb{H}_*(M)$$

and

$$(q \circ f)_{\Omega} : H_*(\Omega_{f^{-1}(p)}S^3) \rightarrow H_*(\Omega_p M)$$

where the latter is an isomorphism.

As we know, generally ΩM and $\mathbb{L}M$ are not connected. So we should emphasize that the images of $(f \circ p)_{\mathbb{L}}$ and $(f \circ p)_{\Omega}$ are homology classes of the connected components corresponding to the trivial loop in M . So in the diagram below, the sub-index 0 means the homology of the component corresponding to trivial loops.

We claim that

$$\begin{array}{ccc} \mathbb{H}_*(S^3) & \xrightarrow{r \cap_{S^3}} & H_*(\Omega S^3) \\ (f \circ q)_{\mathbb{L}} \downarrow & & (f \circ q)_{\Omega} \downarrow \\ (\mathbb{H}_*(M))_0 & \xrightarrow{\cap_M} & H_*(\Omega M)_0 \end{array} \quad (2.3)$$

is commutative. To prove this note that S^3 has a group structure hence

$$\mathbb{L}S^3 \stackrel{\text{homeo.}}{\simeq} S^3 \times \Omega S^3,$$

therefore

$$\mathbb{H}_*(S^3) \simeq H_*(S^3) \otimes H_*(\Omega S^3). \quad (2.4)$$

There are two types of homology classes in $\mathbb{H}_*(S^3)$.

- (1) The classes that under the isomorphism (2.4) correspond to homology classes in $H_0(S^3) \otimes H_*(\Omega S^3)$. These classes can be represented by cycles in $\mathbb{L}S^3$ whose sets of marked points consist of a single point in S^3 .

Type (1) homology classes are mapped to 0 by \cap_{S^3} , since we can choose a chain representative whose set of marked point, which consists of one point, is different from all m_i 's. Therefore they are mapped to 0 by $(f \circ q)_\Omega \circ (r \cap_{S^3})$

Indeed their image under $(f \circ q)_\mathbb{L}$ have sets of marked consist of only one point, p . Thus, they are mapped to zero under $\cap_M \circ (f \circ q)_\mathbb{L}$. Hence the diagram commutes in this case

- (2) The classes that under isomorphism (2.4) correspond to homology classes in $H_3(S^3) \otimes H_*(\Omega S^3)$. They can be represented by a cycle in $\mathbb{L}S^3$ whose set of marked points is all of S^3 .

Let $\theta = a \otimes b \in \mathbb{H}_3(S^3) \simeq H_3(S^3) \otimes H_*(\Omega S^3)$. Then

$$\cap_{S^3}(\theta) = b$$

so

$$(f \circ q)_\Omega \circ (r \cap_{S^3}) = r(f \circ q)(b)$$

and this is exactly $\cap_M \circ (f \circ q)(b)$. Note that r copies of b under $f \circ q$ which are based at each one of m_i 's come together.

Now consider $a_1, a_2 \in \mathbb{H}_*(S^3)$ constructed in proof of proposition 2.2.

Then from commutativity of diagram (2.3) and the fact that $(q \circ f)_\Omega$ is an isomorphism it follows that

$$(q \circ f)_L(a_1) \bullet (q \circ f)_L(a_2) \neq 0$$

□

3 3-Manifolds with Non Separating 2-Sphere or 2-Torus

Proposition 3.1 $S^1 \times S^2$ has nontrivial extended loop products.

Proof. Fix $a \in S^1$ and $b, p \in S^2$ and $p \neq b$. Consider the following maps:

$$\alpha : S^1 \rightarrow S^1 \times S^2, \quad \alpha(x) = (x, b),$$

$$\beta : S^2 \rightarrow S^1 \times S^2, \quad \beta(y) = (a, y)$$

The loop α with marked point (a, b) can be regarded as an element of $\mathbb{H}_0(M)$. The loop $\gamma\alpha\gamma^{-1}$ represents an element $g_1 \in \pi_1(M)$, where γ is a path connecting (a, b) and (a, p) . The free homotopy type of α is $[g_1] \in \hat{\pi}_1(M)$.

Also, β can be regarded as an element of $\pi_1(\Omega_{(a,p)}(S^1 \times S^2)) \simeq \pi_2(S^1 \times S^2)$ or in other words we think of β as a 1-dimensional family of loops. The free homotopy type of the loops of β is the one of the trivial loop. We compose each loop of β with a fixed loop γ' whose marked point is (a, p) and it represents a nontrivial element $g_2 \in \pi_1(S^2 \times S^1)$, $g_2 \neq g_1^{\pm 1}$. We continue to denote the result by β .

The image of β under the Hurewicz map

$$\pi_2(\Omega_{(a,p)}S^2 \times S^1) \rightarrow H_2(\Omega_{(a,p)}S^2 \times S^1)$$

and then followed by the induced map by inclusion

$$H_2(\Omega_{(a,p)}S^2 \times S^1) \rightarrow \mathbb{H}_2(S^2 \times S^1)$$

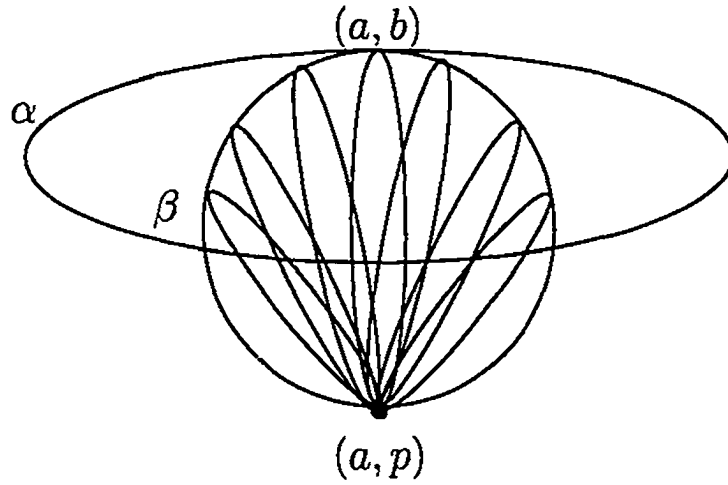


Figure 1: A non separating 2-sphere

gives rise to an element in $\mathbb{H}_2(S^1 \times S^2)$. Again we continue to denote this element by β .

We claim that

$$p_{A_{S^1 \times S^2}}(\Delta\alpha) \bullet p_{A_{S^1 \times S^2}}(\Delta\beta) \neq 0 \in \mathbb{H}_0$$

and this proves that $S^1 \times S^2$ has nontrivial extended loop products.

Note that

$$p_{A_{S^1 \times S^2}}(\Delta\alpha) \bullet p_{A_{S^1 \times S^2}}(\Delta\beta) = (\Delta\alpha - p_{S^1 \times S^2}(\Delta\alpha)) \bullet (\Delta\beta - p_{S^1 \times S^2}(\Delta\beta)).$$

In expanding this expression, we get four terms, each one is the loop product of two homology classes, a 1-homology class and a 2-homology class. The set of the marked points of the chain representatives of the homology classes involved in each one of these four terms intersect at one point, namely (a, b) . Hence $p_{A_{S^1 \times S^2}}(\Delta\alpha) \bullet p_{A_{S^1 \times S^2}}(\Delta\beta)$ is equal to the sum of 4 homology classes in $\mathbb{H}_0(S^1 \times S^2)$, each represented by a loop and a \pm sign.

The free homotopy types of these loops are $[g_1g_2], [g_1], [g_2], 1$ which are different. Therefore the loop product is nontrivial. \square

Corollary 3.2 *An oriented closed 3-manifold with a non separating 2-sphere has nontrivial extended loop products.*

Proof. The proof is similar to the one of proposition 3.1. The only fact we used in the proof of proposition 3.1 was that there was a non-separating 2-sphere (a 2-sphere that meets some closed simple curve only once). Every oriented closed 3-manifold with non separating 2-sphere is indeed a connected of $S^2 \times S^1$ with another 3-manifold. So $\pi_1(M)$ is the free product of $\pi_1(S^2 \times S^1)$ and the fundamental group of another 3-manifold. Therefore the choices $g_1, g_2 \neq 1, g_2 \neq g_1$ of elements of $\pi_1(M)$, as in the proof of proposition 3.1, are also possible in this case. \square

Proposition 3.3 *If a closed oriented 3-manifold M has a non separating two sided³ incompressible torus, then M has nontrivial extended loop products.*

Proof. Let $\varphi : S^1 \times S^1 \rightarrow M$ be a non separating torus. H gives rise to a homology class H in $\mathbb{H}_1(M)$, where the marked points are $\varphi(t, 1)$, $t \in S^1$ and the loop passing through $\varphi(t, 1)$ is

$$\varphi_t : S^1 \rightarrow M$$

$$\varphi_t(s) = h(t, s).$$

³The normal bundle is trivial.

The loop φ_1 represents an element $h \in \pi_1(M)$ which is nontrivial as the torus is incompressible.

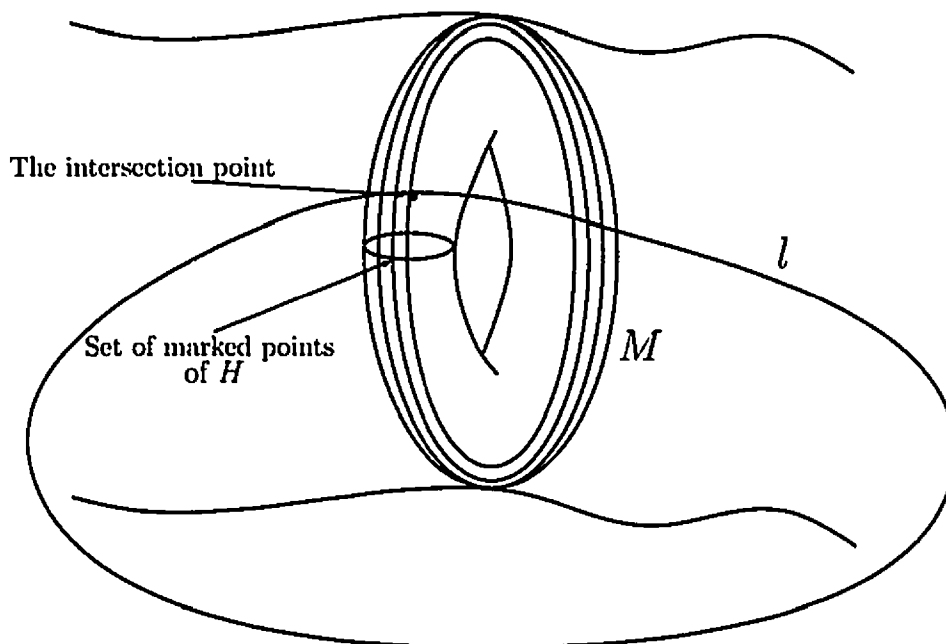


Figure 2: A non separating torus

Let $\lambda : S^1 \rightarrow M$ be a loop in M which intersects the non separating torus transversally at exactly one point m (see Figure 2). We can assume that the intersection point m is the marked point of λ and we consider this point as the base point of M . Without loss of generality we can assume $m = \varphi(1, 1)$.

λ represents a nontrivial element $l \in \pi_1(M)$ since the homology class it represents is nontrivial. To see that this homology class is a nontrivial notice

that it has nontrivial intersection with homology class represented by the torus φ .

Also, λ gives rise to a homology class $L \in \mathbb{H}_0(M)$. The free homotopy type of the loops of L is $[l] \in \hat{\pi}_1(M)$.

We claim that

$$p_{A_M}(\Delta L) \bullet p_{A_M}(\Delta H) \neq 0$$

which proves the assertion.

In the expansion of $p_{A_M}(\Delta L) \bullet p_{A_M}(\Delta H) \in \mathbb{H}_0(M)$ we get four terms, each represented by a single loop and $+$ or $-$ sign. The loops have free homotopy type $[l.h]$, $[l]$, $[h]$ and $[1]$. If we show three out of four terms are different then there cannot be a complete cancellation, thus the product is not trivial.

Assume $[l]$ and $[h]$ are the same. Then the 1-homologies represented by λ and φ are the same.

Note that the intersection of the 1-homology represented by λ intersects the 2-homology represented by the torus $\varphi(\dots)$ nontrivially, while the 1-homology class represented by φ_1 intersects trivially with the same 2-homology trivially⁴. This is a contradiction, hence $[l]$ and $[h]$ are different. Similar argument shows that $[l] \neq [1]$. To see that $[h] \neq [1]$ note that the non separating torus does not bound in M . \square

⁴Since the torus is two sided (it normal bundle is trivial) and we can push h_1 off the torus.

4 Closed Seifert Manifolds

In this section we prove that closed Seifert manifolds have nontrivial extended loop products.

Before we state and prove the result it we recall the definition of Seifert manifolds and some facts about them.

A 3-manifold is called *Seifert manifold* if it admits a foliation by circles S^1 in such a way that each leaf has a foliated tubular neighborhood $D^2 \times S^1$ which is $D^2 \times S^1$ either with its trivial foliation or its quotient by a standard action of a finite cyclic group \mathbb{Z}_p (acting on D^2 factor). The fibers (leaves) which have a trivially foliated $D^2 \times S^1$ neighborhood are called *normal fibers* and the others are called *singular fibers* and p in above is its *multiplicity*. The space of the fibers is a surface and it is called the *base surface*.

Notation: Let M be an oriented Seifert manifold with p exceptional fibers and k boundary components. If S , the base surface of M , is orientable, then it has genus $g \in \mathbb{Z}^+$ otherwise S is non orientable then g is the number of cross caps in S . $\pi_1(M)$ has the presentation (see [8]):

(1) If S is orientable and of genus g ,

$$\begin{aligned} \pi_1(M) = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_p, d_1, d_2, \dots, d_k, h | a_i h a_i^{-1} = h, \quad b_i h b_i^{-1} = h, \\ c_i h c_i^{-1} = h, \quad d_i h d_i^{-1} = h, \\ c_i^{\alpha_i} h^{\beta_i} = 1 \\ 1 = \prod_{i=1}^g [a_i, b_i] \prod_{i=1}^p c_i \prod_{i=1}^k d_i \rangle, \end{aligned} \tag{4.1}$$

(2) If S is nonorientable and has g cross caps.

$$\begin{aligned}
\pi_1(M) = \langle a_1, \dots, a_g, c_1, \dots, c_p, d_1, d_2, \dots, d_k, h \mid & a_i h a_i^{-1} = h^{-1}, \\
& c_i h c_i^{-1} = h^{\delta_i}, \quad d_i h d_i^{-1} = h, \\
& c_i^{\alpha_i} h^{\beta_i} = 1 \\
1 = \prod_{i=1}^g a_i^2 \prod_{i=1}^p c_i \prod_{i=1}^k d_i & \\
\end{aligned} \tag{4.2}$$

where $0 < \beta_i < \alpha_i$ are integers and $\delta_i = \pm 1$. h corresponds to the normal fiber. Each c_i can be represented by a loop on the base surface going around the singular fiber once. The d_i 's correspond to the boundary components. α_i is the multiplicity of the singular fiber corresponding to c_i and we say that it has type (α_i, β_i) .

Note that $\langle h \rangle$ is a normal subgroup of $\pi_1(M)$ and it is central if the S is orientable. $\pi_1(M)$ is finite if and only if $\pi_1(M)/\langle h \rangle$ is finite and M is not $S^2 \times S^1$ or $P^3 \# P^3$; So $\pi_1(M)$ is finite if $M \neq S^2 \times S^1$, and

1) S is orientable, $g = 0$, $k = 0$ and one of the following holds:

(a) $p \leq 2$ and M is a Lens space.

(b) $p = 3$ and

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1,$$

$(\alpha_1, \alpha_2, \alpha_3)$ is one of the triples $(2, 2, \alpha_3)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$.

2) S is not orientable, $g = 1$, $q = 0$ and $p \leq 1$. These Seifert manifolds have already appeared in the list above. Indeed they correspond to the triples $(2, 2, \alpha_3)$

Remark 4.1 An orientable Seifert manifold M may not be oriented as a fibration but it always has a double cover which is Seifert and is an orientable fibration or in other words its base surface is orientable. This double cover \tilde{M} can be described as following:

$$\tilde{M} = \{(m, o) | m \in M \ \& \ o \text{ an orientation for the fiber passing through } m\}$$

The covering map $\tilde{M} \xrightarrow{q} M$ is

$$q(m, o) = m.$$

The Seifert fibration on \tilde{M} is the pull back of the one of M . If M has boundary then each boundary, a torus, gives rise to 2 boundary components each of which is a torus. If M has g cross cap then \tilde{M} has genus $g - 1$.

Proposition 4.2 *If M is a closed oriented Seifert 3-manifold then M or a double cover of M has nontrivial extended loop products.*

Proof. By remark 4.1 we may assume that the base surface of M is orientable. Also we assume $\pi_1(M) \not\cong \mathbb{Z}_2$, since Corollary 2.6 deals with this case. Let h be the generator of $\pi_1(M)$ corresponding to the normal fiber according the presentation (4.1).

There is a natural 3-homology class in $\mathbb{H}_3(M)$ which has a representative whose set of marked points is exactly M and the homotopy type of the loops is $[h]$. The loop associated with a point in M that is on a normal fiber, is the fiber passing through the point. The loop passing through a point on a singular fiber as a map is a multiple of the singular fiber.

We denote this homology class by Θ_M . Loops of Θ_M have the free homotopy type $[h]$. We have $p_{A_M}(\Theta_M) = \Theta_M - p_M(\Theta_M) = \Theta_M - \mu_M$ where μ_M is the unit of $\mathbb{H}_*(M)$.

Consider a loop l representing an element $\alpha \in \pi_1(M)$, $\alpha \neq h, 1$. This is possible since we are assuming $\pi_1(M) \neq \mathbb{Z}_2$. Since h is central, the free homotopy class $[\alpha]$ is different from $[h]$ and $[1]$.

α represents a 0-homology class $\hat{\alpha}$ in $\mathbb{H}_0(M)$. We have

$$p_{A_M}(\hat{\alpha}) = \hat{\alpha} - p(\hat{\alpha})$$

where $p(\hat{\alpha})$ is just the constant loop at the marked point of l .

We claim that

$$p_{A_M}(\Theta_M) \bullet p_{A_M}(\hat{\alpha}) \neq 0.$$

Indeed expanding $p_{A_M}(\Theta_M) \bullet p_{A_M}(\hat{\alpha})$ provides four terms, all in $\mathbb{H}_0(M)$ and each one is represented by a single loop and a \pm sign. These four loops have free homotopy type $[\alpha.h], [h], [\alpha]$ and $[1]$. Since there are at least three different classes, any linear combination of them with coefficient ± 1 is nonzero. Therefore M has nontrivial extended loop products. \square

5 Seifert Manifolds with boundary

By torus decomposition, Seifert manifolds with incompressible⁵ boundary are among the building blocks of 3-manifolds. In this section we provide examples of nontrivial extended loop product for various Seifert manifold with boundary. The examples are organized according to the number of boundary components and singular fibers.

Throughout this section M is a compact oriented Seifert manifold with orientable base surface unless otherwise it is stated (see corollary 5.7).

M has $b \geq 1$ boundary components and p singular fibers and p' denotes the number of the singular fibers of multiplicity greater than 2. We assume⁶ that

$$b + p \geq 3.$$

g denotes the genus of the base surface of M . In the proofs of the statements in this section g play no role. So we assume that

$$g = 0.$$

We present two main ideas in constructing nontrivial extended loop products,

Two curves argument and Chas' figure eight argument.

The following table indicates the cases where we apply these arguments.

⁵ π_1 -injective.

⁶This assumption will be justified later, in section 10, when we recall the torus decomposition.

	argument
$p + b \geq 4$	Two curves argument
$p' + b \geq 3$	Chas' figure eight argument

5.1 $p + b \geq 3$ (Two curves argument)

Proposition 5.1 *Let M be a compact oriented Seifert manifold with $p > 2$ and $b \geq 1$. Then M has nontrivial extended loop products.*

Proof. Let c_1, c_2 and c_3 be generators of $\pi_1(M)$ corresponding to three of the singular fibers. Consider a simple curve γ_1 , on the base surface away from the singular fibers and representing the free homotopy class $[c_1c_2]$. Similarly, consider a simple curve γ_2 on the base surface, away from the singular fibers and representing the free homotopy class $[c_2c_3]$. Moreover, γ_2 can be chosen such that it has exactly 2 intersection points with γ_1 .

Since γ_1 is away from the singular point, the fibration can be trivialized over γ_1 (see Figure 3). Therefore, we obtain a map

$$f : \mathbb{T}^2 \rightarrow M$$

from the torus to M , where $f(0, \cdot) = \gamma_1$.

f gives rise to a homology class $\Gamma_1 \in \mathbb{H}_1(M)$, by declaring $f(t, 0)$, $t \in \mathbb{S}^1$, as the set of marked points. The loop passing through $f(t, 0)$ is $f(t, s)$, $s \in \mathbb{S}^1$ and it has the free homotopy type $[c_1c_2]$. Also, the loop γ_2 gives rise to a homology class $\Gamma_2 \in \mathbb{H}_0(M)$.

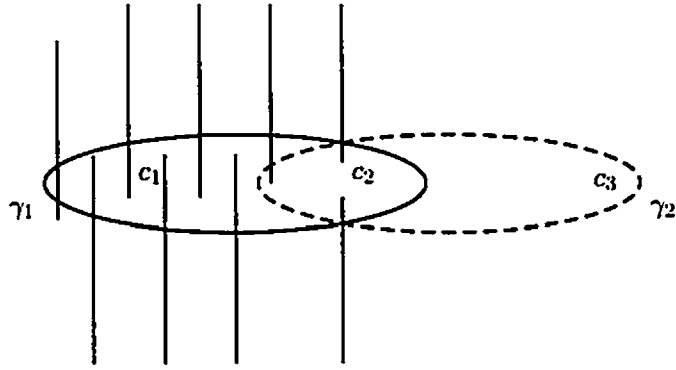


Figure 3: γ_1 and γ_2 with a trivialized fibration over γ_1

We claim that

$$p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in \mathbb{H}_0(M)$$

which proves the proposition.

The sets of marked points of $\Delta\Gamma_1 \in \mathbb{H}_2(M)$ and $\Delta\Gamma_2 \in \mathbb{H}_1(M)$ intersect at two points. Each point contributes four terms in the expansion of $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2)$. All in all 8 terms emerge when we calculate $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2)$. Each is represented by a conjugacy class in $\hat{\pi}_1(M)$ and a \pm sign. These conjugacy classes are

$$[c_1 c_2 c_3 c_2], [c_1 c_2], [c_2 c_3], [1]$$

and

$$[c_1 c_2^2 c_3], [c_1 c_2], [c_2 c_3], [1].$$

Calculating $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \pmod{2}$, only two terms remain, namely (see Figure 4)

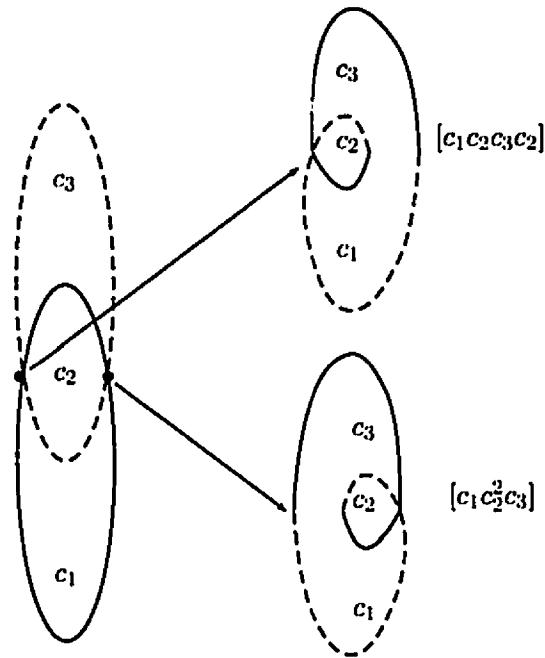


Figure 4: Two curves argument

$$[c_1 c_2^2 c_3] \quad \text{and} \quad [c_1 c_2 c_3 c_2].$$

To prove the claim it is sufficient to show that these two conjugacy classes are different. For this, consider the group

$$H = \langle c_1, c_2, c_3, d \mid c_1 c_2 c_3 d_1 = 1, c_i^{\alpha_i} = 1, 1 \leq i \leq 3 \rangle,$$

where d_1 is a generator corresponding to a boundary component. Indeed H is the free product of the groups $\mathbb{Z}_{\alpha_1} * \mathbb{Z}_{\alpha_2} * \mathbb{Z}_{\alpha_3}$.

Consider the homomorphism $\phi : \pi_1(M) \rightarrow H$ which is identity on c_1, c_2, c_3, d and sends the other generators to the trivial element of H . By the presenta-

tion (4.1) ϕ is well-defined.

Note that image of the conjugacy classes $[c_1c_2^2c_3]$ and $[c_1c_2c_3c_2]$ under the ϕ are different in \hat{H} as $c_1c_2^2c_3$ and $c_1c_2c_3c_2$ are cyclically different. Therefore $[c_1c_2^2c_3]$ and $[c_1c_2c_3c_2]$ are different conjugacy classes in of $\pi_1(M)$.

□

Remark 5.2 Indeed, in the proof of the proposition above one can replace a singular fiber by a boundary component or in other words a boundary component works as well as a singular fiber of multiplicity zero. To be more explicit, the curves γ_1 and γ_2 can go around a boundary component of the base surface instead of the singular fiber. In the free product H , one of finite cyclic group is replaced by a infinite cyclic group. The rest of the proof remains the same.

Corollary 5.3 *Let M be a compact oriented Seifert manifold with p singular fibers and q boundary components. If $p + b > 3$ then M has nontrivial extended loop products.*

Proof. If $p \geq 3$ this is just the statement of proposition 5.1. In the case $p < 3$, by Remark 5.2, in the proof of proposition 5.1 one replaces the missing singular fiber with one of the extra boundary components. □

5.2 $p' + b \geq 3$, (Chas' figure eight argument)

Now we turn to compact oriented Seifert manifolds with the total number of singular fibers and boundary components 3.

The idea of the proof of this case is due to Moira Chas . She found this idea while reformulating a conjecture of Turaev (see [4]) on Lie bi-algebras

of surfaces, characterizing non self intersecting closed curves.

Proposition 5.4 *Let M be an oriented Seifert manifold with $p = 2$ singular fibers and $b = 1$ boundary component. Suppose that none of its singular fibers is of multiplicity 2. Then M has nontrivial extended loop products.*

Proof. Let c_1 and c_2 be generators of $\pi_1(M)$ corresponding to the singular fibers. Let γ_1 be a smooth curve on the base surface, with one self intersection point, away from singular fibers and representing the free homotopy class $[c_1 c_2^{-1}]$. Similarly, consider a curve γ_2 on the base surface, away from the singular fibers, with one intersection point and representing the free homotopy class $[c_1^{-1} c_2]$. Moreover, γ_2 can be chosen such that it has exactly 2 intersection points with γ_1 .

The fibration can be trivialized over the γ_1 since it is away from the singular point. Therefore, we obtain a map

$$f : \mathbb{T}^2 \rightarrow M$$

from torus to M , where $f(\cdot, 0) = \gamma_1$. This map gives rise to a homology class $\Gamma_1 \in \mathbb{H}_1(M)$, by declaring $f(t, 0)$, $t \in \mathbb{S}^1$, as the set of marked points. The loop passing through $f(t, 0)$ is $f(s, t)$, $s \in S^1$. All the loops of this family have the free homotopy type $[c_1 c_2^{-1}]$. The loop γ_2 gives rise to a homology class $\Gamma_2 \in \mathbb{H}_0(M)$. We claim that

$$p_{\Lambda_M}(\Delta\Gamma_1) \bullet p_{\Lambda_M}(\Delta\Gamma_2) \neq 0 \in \mathbb{H}_0(M)$$

which proves the proposition.

The set of marked points of $\Delta\Gamma_1 \in \mathbb{H}_2(M)$ and $\Delta\Gamma_2 \in H_1(M)$ intersect at two points. Each point contributes four terms in the expansion

of $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2)$. All in all 8 terms emerge in the expansion of $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2)$. Each one of them is represented by a conjugacy class in $\hat{\pi}_1(M)$ and a \pm sign. These conjugacy classes are

$$[c_1c_2c_1^{-1}c_2^{-1}], [c_1^{-1}c_2], [c_1c_2^{-1}], [1]$$

and

$$[c_1c_2^{-1}c_1^{-1}c_2], [c_1^{-1}c_2], [c_1c_2^{-1}], [1].$$

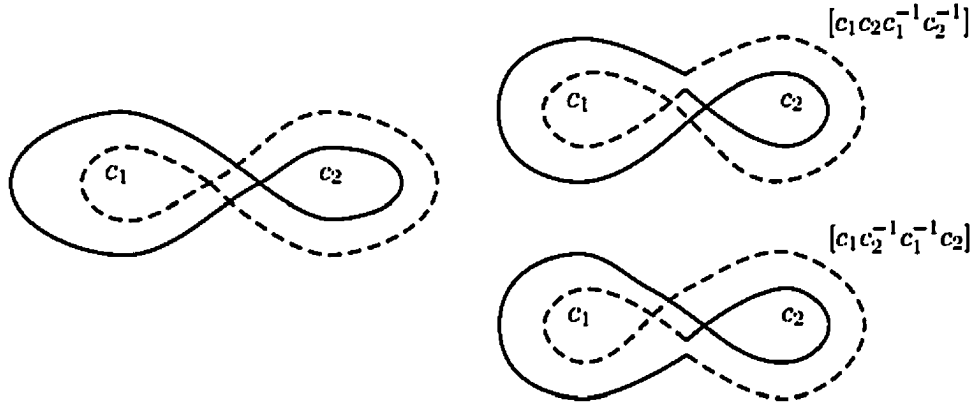


Figure 5: Figure eight argument

Calculating $p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \pmod{2}$, only two terms remain (see Figure 5), namely

$$[c_1c_2c_1^{-1}c_2^{-1}] \quad \text{and} \quad [c_1c_2^{-1}c_1^{-1}c_2].$$

To prove the claim it is sufficient to show that these two homology classes are different. For this, consider the subgroup

$$H = \langle c_1, c_2, d_1 \mid c_1c_2d_1 = 1, c_i^{a_i} = 1, i = 1, 2 \rangle$$

where d_1 corresponds to one of the boundary components in presentation (4.1). Indeed H is the free product of cyclic groups generated by c_1 , c_2 and d_1 .

Consider the homomorphism $\phi : \pi_1(M) \rightarrow H$ which is identity on c_1, c_2, d_1 and sends the other generators of $\pi_1(M)$ in presentation (4.1) to the identity element of H .

Since $c_i, i = 1, 2$, have nonzero multiplicity then $c_i^2 \neq 1 \in H$ or $c_i \neq c_i^{-1} \in H$. Thus the images of $c_1 c_2 c_1^{-1} c_2^{-1}$ and $c_1 c_2^{-1} c_1^{-1} c_2$ under ϕ in the free product H are not conjugate as they are cyclically different. Therefore the conjugacy classes $[c_1 c_2 c_1^{-1} c_2^{-1}]$ and $[c_1 c_2^{-1} c_1^{-1} c_2]$ in $\hat{\pi}_1(M)$ are different as well. \square

Remark 5.5 The idea of the proof above works unless there is a singular fiber of multiplicity 2 and this case need special treatment.

Corollary 5.6 *Let M be a compact oriented Seifert manifold with $p'+b \geq 3$. Then M has nontrivial extended loop products.*

Proof. If $p' \geq 2$ then it is just proposition 5.4. Otherwise, by remark 5.2 a boundary component works as well as a singular fiber with multiplicity zero. The proof is the same as of the one of proposition 5.4 except that one has to replace one of the generators of $\pi_1(M)$ contributed by a singular fiber to a generator corresponding to a boundary component. \square

Corollary 5.7 *Let M be a compact oriented Seifert manifold with non oriented base and $b \geq 2$ boundary components, then M has nontrivial extended loop products.*

Proof. We can modify the proof of proposition 5.4 considering that by remark 5.2 a boundary component serves our purposes as well as a singular

fiber. Note that the fibration is oriented once restricted to the boundary components so it can be trivialized over a simple curve representing the free homotopy type $[d_1 d_2^{-1}]$ where d_1 and d_2 are generators of $\pi_1(M)$ contributed by the two boundary components (see the presentation 4.2). The rest of the proof is similar to the one of proposition 5.4. Calculating $\mod 2$ we get two terms

$$[d_1 d_2 d_1^{-1} d_2^{-1}] \quad \& \quad [d_1 d_2^{-1} d_1^{-1} d_2].$$

If a_1 is a generators of $\pi_1(M)$ by a cross cap (see the presentation 4.2) then consider the group

$$H = \langle d_1, d_2, a_1 \mid d_1 d_2 a_1^2 = 1 \rangle \simeq \langle d_1 \rangle * \langle a_1 \rangle$$

and the homomorphism $\varphi : \pi_1(M) \rightarrow H$ which is identity on d_1, d_2, a_1 and sends the other generators of $\pi_1(M)$. The image of the two conjugacy classes above in the free product H is $[d_1 a_1^2 d_1^{-1} a_1^{-2}] \quad \& \quad [d_1 a_1^{-2} d_1^{-1} a_1^2]$ which are distinct. Therefore the above conjugacy classes are distinct conjugacy classes of $\pi_1(M)$.

□

6 Connected Sum of 3-Manifolds

In this section we consider the connected sum of 3-manifolds. We shall show how one can obtain nontrivial loop products in this kind of manifold. For this part, the author has benefited from conversations with numerous colleagues ([1]).

Proposition 6.1 *If M_1 and M_2 are two 3-manifolds such that $\pi_1(M_i) \neq 1$, $i = 1, 2$, then $M_1 \# M_2$ has nontrivial extended loop products.*

Proof. Let $m_i \in M_i$, $i = 1, 2$ be two points which are identified in the connected sum $M_1 \# M_2$ and considered as the base point m of M_1 , M_2 and $M_1 \# M_2$. Let $g_1 \in \pi_1(M_1)$ and $g_2 \in \pi_1(M_2)$ be two non-trivial elements. Consider the loops γ_i in M_i with marked point m_i and representing g_i , $i = 1, 2$.

By composing these two loops at the base point m in $M_1 \# M_2$, we obtain a loop $\gamma_1 \gamma_2$ representing $g_1 g_2 \in \pi_1(M_1 \# M_2) \simeq \pi_1(M_1) * \pi_1(M_2)$ and it has the free homotopy type $[g_1 g_2] \in \hat{\pi}_1(M)$ (see Figure 6). Notice that $\gamma_1 \gamma_2$ also represents an element of $\mathbb{H}_0(M_1 \# M_2)$. We denote this homology class by $g_1 \# g_2$.

Consider the embedding $\beta : S^2 \hookrightarrow M_1 \# M_2$ where the connected sum happens (see Figure 6). This gives rise to an element in $\pi_2(M_1 \# M_2) \simeq \pi_1(\Omega(M_1 \# M_2))$ and therefore an element in $\pi_1(\mathbb{L}(M_1 \# M_2))$. The image of this element under the Hurewicz map is an element of $\mathbb{H}_1(M_1 \# M_2)$.

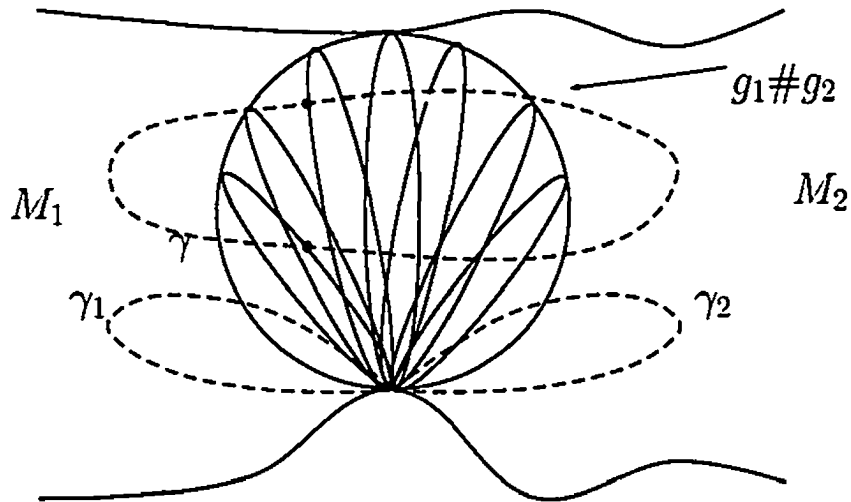


Figure 6: Loop product in connected sum

By composing the loops of this 1-cycle of loops with a fixed loop whose marked point is m and represents a nontrivial element h in $\pi_1(M_1 \# M_2)$, we can obtain a new element in $\mathbb{H}_1(M_1 \# M_2)$. We denote this element by $\beta_{M_1 \# M_2}$ the homotopy type of its loops is $[h] \in \hat{\pi}_1(M_1 \# M_2)$. We claim that

$$p_{A_M}(\Delta(g_1 \# g_1)) \bullet p_{A_M}(\Delta(\beta_{M_1 \# M_2})) \neq 0$$

which proves that M has nontrivial extended loop product. To compute

$$\begin{aligned} p_{A_M}(\Delta(g_1 \# g_2)) \bullet p_{A_M}(\Delta(\beta_{M_1 \# M_2})) &= (\Delta(g_1 \# g_2) - p_M(\Delta(g_1 \# g_2))) \\ &\bullet (\Delta(\beta_{M_1 \# M_2}) - p_M(\Delta(\beta_{M_1 \# M_2}))) \end{aligned}$$

one has to homotop $\gamma_1 \gamma_2$ to a simple curve γ which intersects the imbedded S^2 in $M_1 \# M_2$ at two different points and away from the base point m .

So in calculating the expansion above we get eight terms, four terms for each intersection point of γ and the imbedded 2-sphere. Each one of these eight terms, which are elements of $\mathbb{H}_0(M)$, is represented by a single loop

and a \pm sign. These loops have free homotopy types,

$$[g_1g_2h], [g_1g_2], [h], [1], [g_2g_1h], [g_2g_1], [h], [1].$$

So calculating $p_{A_M}(\Delta(g_1\#g_2)) \bullet p_{A_M}(\Delta(\beta_{M_1\#M_2})) \pmod{\mathbb{Z}_2}$, only two terms remain,

$$[g_1g_2h] \& [g_2g_1h].$$

If we choose $h = g_1g_2$ in $\pi_1(M_1\#M_2)$, the two conjugacy classes

$$[g_1g_2h] = [g_1g_2g_1g_2]$$

and

$$[g_2g_1h] = [g_1g_1^2g_2] = [g_2^2g_1^2]$$

are different as two cyclically reduced sequences (g_1, g_2, g_1, g_2) and (g_2^2, g_1^2) are not cyclic permutation of one another.

This proves that for $h = g_1g_2$, $p_{A_M}(\Delta(g_1\#g_2)) \bullet p_{A_M}(\Delta(\beta_{M_1\#M_2}))$ is non zero $\pmod{2}$, therefore it is non zero in $\mathbb{H}_0(M_1\#M_2)$. \square

7 An Injectivity Lemma

Suppose that M is a closed oriented manifold and \mathcal{T} is a collection of incompressible $\bar{\tau}$ separating tori in M . Let $M \setminus \mathcal{T} = M_1 \cup M_2 \dots \cup M_n$. We associate with (M, \mathcal{T}) a tree of groups (G, T) (See Appendix II for definitions).

The vertices of T are in one-to-one correspondence with M_i 's. Two vertices v_i and v_j are connected by an edge if the corresponding M_i and M_j are glued along a torus. With each vertex v_i we associate the fundamental group of $\pi_1(M_i)$ and with each edge we associate the fundamental group of the corresponding torus. As the tori are incompressible there are two monomorphisms from each edge group to the vertex groups of the corresponding vertices. Namely if e is an edge (a torus in M) and v a vertex of e (some M_i s.t. $T \subset \partial \overline{M_i}$) then there is an homomorphism

$$f_e^v : G_e \rightarrow G_v$$

which is indeed the map induced by inclusion

$$i : \pi_1(T) \rightarrow \pi_1(\overline{M_i}) \cong \pi_1(M_i).$$

By Van-Kampen theorem, $\pi_1(M)$ is the amalgamation of G_v 's along G_e 's (see Appendix II, tree of groups).

Combining all these with lemma 5 in Appendix II, we have,

Lemma 7.1 (*Injectivity Lemma*) *Let M, \mathcal{T} and M_i 's be as above. Suppose that $[a]$ and $[b]$ are two distinct conjugacy classes in $\pi_1(M_i)$ such that a is not conjugate to the image of any element of the edge groups in $\pi_1(M_i)$. Then a and b represent distinct conjugacy classes of $\pi_1(M)$.*

$\bar{\tau}$ π_1 -injective

The Injectivity Lemma will be very useful to show that certain examples of nontrivial loop product in submanifold of M give rise to nontrivial extended loop product in M .

8 Gluing Seifert Manifolds along Tori

In this section we consider the gluing of the Seifert manifolds along tori. We prove that the result of gluing Seifert manifolds along incompressible *separating* tori has nontrivial extended loop products except certain classes for which there are double covers that have nontrivial extended loop products.

The idea of the proof is to apply either the *two curves argument* or the *figure eight argument* to a Seifert piece of the gluing that fulfills the condition of the arguments. Then we show that the examples of nontrivial loop product in the piece gives rise to a nontrivial example of loop product in the manifold, for that we need the injectivity lemma, namely lemma 7.1.

When we cannot apply either the two curves argument or the figure eight argument we consider an appropriate double cover of the manifold which either has a non separating torus or so that one of the two arguments can be applied to one of the pieces in its torus decomposition ⁸.

Before stating the result and demonstrating the proof we introduce the following notations.

Convention and Notation: Let M be a closed 3-manifold and $\mathcal{T} \neq \emptyset$ be a collection of tori in M and

$$M \setminus \mathcal{T} = M_1 \cup M_2 \cdots \cup M_n,$$

where all the \overline{M}_i 's are oriented Seifert manifolds with incompressible boundary⁹.

Let $b_i \geq 1$ and p_i denote respectively the number of boundary components

⁸See Section for the definition of torus decomposition

⁹We say that the torus decomposition of M has only Seifert pieces.

and singular fibers of \overline{M}_i and $A_i \subset \mathbb{Z}^+$ is the set of all the multiplicities of the singular fibers of \overline{M}_i . The base surface of M_i is denoted S_i and the genus of S_i is g_i if S_i is oriented otherwise g_i is the number of cross caps in S_i .

We assume that all tori in T are separating since in section 3 we proved that 3-manifolds with non separating torus have nontrivial extended loop products.

We assume¹⁰ that for all i

$$b_i + p_i \geq 3$$

if S_i is oriented. Also we assume that

$$g_i = 0$$

if S_i is oriented as if $g_i > 0$, then there is a non separating torus in M and this case has already been studied in section 3.

Proposition 8.1 *Let M be a closed 3-manifold described as above. Then M or a double cover of M has nontrivial extended loop products.*

Proof. We divide the proof into four parts. The detailed proof will follow.

- (1) When $b_i \geq 3$ and S_i is orientable for some i , we use the figure eight argument. Note that if $b_i > 3$ we can use the two curves argument.
- (2) When $b_i \geq 2$ and S_i is not orientable for some i , for this case we also use the figure eight argument (see corollary 5.7).
- (3) When $p_i + b_i > 3$ and S_i orientable for some i , we can apply the two curves argument. This part is organized as follows:

¹⁰see section 10 for justification.

- (i) For all k , $1 \leq b_k \leq 2$, but there is a piece with $p_i > 2$ singular fibers.
- (ii) For all k , $1 \leq b_k \leq 2$ and $p_k \leq 2$ but there exists i such that $b_i = p_i = 2$, case the two curves argument can be applied.
- (4) When $b_i + p_i = 3$, $1 \leq b_i \leq 2$ and S_i orientable for all i and $2 \notin A_i$ for some i .
In this case we use figure eight argument without facing any trouble.
- (5) When for all i , $b_i + p_i = 3$, $1 \leq b_i \leq 2$, S_i orientable and $2 \in A_i$ for all i . For this case we consider certain double cover of M .
- (6) When $b_i + p_i = 3$ and $2 \in A_i$ if S_i is orientable otherwise $b_i = 1$, we also consider a double cover.

We verify the statement for each cases:

- (1) $b_i \geq 3$ and S_i orientable for some i :

Suppose that $b_1 \geq 3$. By corollary 5.6 \overline{M}_1 has nontrivial extended loop products. Consider the homology classes Γ_1 and $\Gamma_2 \in \mathbb{H}_1(\overline{M}_1)$ constructed in the proof of corollary 5.6. Γ_1 and Γ_2 can be regarded as the elements of $\mathbb{H}_*(M)$. We claim that

$$p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in \mathbb{H}_0(M)$$

In calculating the above product 4 terms merge all in $\mathbb{H}_0(M)$. To prove the product is nonzero it is sufficient to show it is nonzero mod 2. Having done the calculation mod 2 just like in the proof of proposition 5.4 and corollary 5.6, only two terms survive.

$$[d_1 d_2 d_1^{-1} d_2^{-1}] \text{ and } [d_1 d_2^{-1} d_1^{-1} d_2],$$

where d_1 and d_2 are generators contributed by two boundary components (see presentation (4.1)).

By the proof of corollary 5.6 (indeed proposition 5.4) we know these two are distinct conjugacy classes of $\pi_1(M_1)$. So if we show that $d_1 d_2 d_1^{-1} d_2^{-1}$ is not conjugate to any element of the fundamental group of one of the component of ∂M_1 , then by the injectivity lemma $[d_1 d_2 d_1^{-1} d_2^{-1}]$ and $[d_1 d_2^{-1} d_1^{-1} d_2]$ are different as the conjugacy classes of $\pi_1(M)$.

Suppose that

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [h^r d^s] \quad (8.1)$$

where h is the generator of $\pi_1(M_1)$ that corresponds to a normal fiber of M_1 and d is a generator of $\pi_1(M)$ contributed by a boundary component.

Consider the group H

$$H = \langle d_1, d_2, d_3 \mid d_1 d_2 d_3 = 1 \rangle \cong \langle d_1 \rangle * \langle d_2 \rangle$$

and

$$\phi : \pi_1(M_1) \rightarrow H$$

the homomorphism which is identity on d_1, d_2, d_3 and trivial on all other generators of $\pi_1(M_1)$. It follows from the presentation (4.1) of $\pi_1(M_1)$ that ϕ is well-defined.

After applying ϕ to (8.1) we get

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [1]$$

If $d \neq d_1, d_2, d_3$ otherwise

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [d^s].$$

The first case is impossible as $d_1 d_2 d_1^{-1} d_2^{-1}$ is a cyclically reduced word in the free product $\langle d_1 \rangle * \langle d_2 \rangle$ thus it represents a nontrivial conjugacy class.

In the latter case, If $d = d_1$ (or d_2) then

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [d_1^s],$$

and this is not possible either as $d_1 d_2 d_1^{-1} d_2^{-1}$ is cyclically reduced of length 4 and d_1^s (or d_2^s) has length 1, hence they represent different conjugacy classes.

If $d = d_3 = d_2^{-1} d^{-1}$ then we have

$$[d_1 d_2 d_1^{-1} d_2^{-1}] = [(d_1 d_2)^{-s}].$$

Considering the length of the word we have to have $s = \pm 2$. In either case this inequality does not hold as $d_1 \neq d_1^{-1}$ and $d_2 \neq d_2^{-1}$ (H is a free group). Therefore these two conjugacy classes are always distinct. This proves the claim.

(2) $b_i \geq 2$ and S_i is not orientable.

Similar to the previous case we use the figure eight argument in M_i .

Suppose that $b_1 \geq 2$ and S_1 is not orientable. By corollary 5.7 \overline{M}_1 has nontrivial extended loop products. The homology classes Γ_1 and Γ_2 in $\mathbb{H}_*(\overline{M}_1)$ constructed in the proof of corollary 5.7 can be regarded as homology classes of $\mathbb{H}_*(\overline{M})$. We claim that

$$p_{A_M}(\Delta\Gamma_1) \bullet p_{A_M}(\Delta\Gamma_2) \neq 0 \in \mathbb{H}_0(M).$$

Calculating mod 2 we get two terms represented by the conjugacy classes $[d_1 a_1^2 d_1^{-1} a_1^{-2}]$ and $[d_1 a_1^{-2} d_1^{-1} a_1^2]$ (see the proof of corollary 5.7). We must prove that these two conjugacy are distinct as conjugacy classes of M . For that, by injectivity lemma, it is sufficient to prove that one of them has no boundary representative. We show that $[d_1 a_1^2 d_1^{-1} a_1^{-2}]$ has no boundary representative. Suppose we have

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [h^r d^s]. \quad (8.2)$$

where d is a generator of $\pi_1(M_1)$ contributed by a boundary component and h corresponds to the normal fiber. Consider the group

$$H = \langle d_1, d_2, a_1 \mid d_1 d_2 a_1^2 = 1 \rangle \simeq \langle d_1 \rangle * \langle a_1 \rangle \simeq \langle d_2 \rangle * \langle a_1 \rangle$$

and the homomorphism $\varphi : \pi_1(M_1) \rightarrow H$ which is identity on d_1, d_2, a_1 and trivial on the other generators of $\pi_1(M_1)$. It follows from the presentation (4.2) that φ is well defined.

Applying φ to (8.2) we have

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [1] \quad (8.3)$$

if $d \neq d_1, d_2$, otherwise

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [d^s]$$

in the free product $H \simeq \langle d_1 \rangle * \langle a_1 \rangle$. The first case is impossible as $d_1 a_1^2 d_1^{-1} a_1^{-2}$ is a cyclically reduced word of length 4 so it represents a nontrivial conjugacy class. If $d = d_1$ then

$$[d_1 a_1^2 d_1^{-1} a_1^{-2}] = [d_1^s]$$

which is again impossible in the free product $\langle d_1 \rangle * \langle a_1 \rangle$ as the cyclically reduced words $d_1 a_1^2 d_1^{-1} a_1^{-2}$ and d_1^s are different in the free product $\langle d_1 \rangle * \langle a_1 \rangle$.

(3) $p_i + b_i \geq 4$ and S_i orientable for some i :

Since case 1 deals with the case $b_i \geq 3$ for some i so we may assume that $b_i \leq 2$. So we are dealing with one the following cases:

(i) $p_i > 2$ for some i

(ii) $b_i = p_i = 2$ for some i .

The proof of both cases are similar and used the two curves argument. We just have to remember that a boundary component works as a singular fiber of multiplicity 0. Here we only present the proof of case (i).

Suppose that $p_1 > 2$. Consider the homology classes Γ_1 and Γ_2 in $\mathbb{H}_*(\overline{M}_1)$ introduced in the proof of proposition 5.1. They can be considered as the elements of $\mathbb{H}_*(M)$. We claim that

$$p_{\Lambda_M}(\Delta\Gamma_1) \bullet p_{\Lambda_M}(\Delta\Gamma_2) \neq 0 \in \mathbb{H}_0(M)$$

The calculation is the same and the results is a sum of conjugacy classes

of $\pi_1(M)$. Computing mod 2 only two terms

$$[c_1c_2c_3c_2] \text{ and } [c_1c_2^2c_3].$$

We must prove that they are different conjugacy classes of $\pi_1(M)$. By the proof of Proposition 3 we know that they are distinct as the conjugacy classes of $\pi_1(M_1)$. If we show that at least one of them has no representative in the fundamental group of any of the boundary components of M_1 then by injectivity lemma they are different as conjugacy classes of $\pi_1(M)$.

Suppose that

$$[c_1c_2c_3c_2] = [h^r d^s] \tag{8.4}$$

where h stands for the normal fiber and d is the generator coming from a boundary component.

Consider the group,

$$H = \langle c_1, c_2, c_3 | c_1c_2c_3 = 1, c_i^{\alpha_i} = 1, 1 \leq i \leq 3 \rangle,$$

where α_i is the multiplicity of the singular fiber corresponding to c_i , and the homomorphism $\phi : \pi_1(M_1) \rightarrow H$ which is identity on c_1, c_2, c_3 and sends the other generators of $\pi_1(M_1)$ to the trivial element of H .

Applying ϕ to (8.4) we have

$$[c_1c_2c_3c_2] = [1]$$

or

$$c_1c_2c_3c_2 = 1$$

in H . Since $c_1c_2c_3 = 1$ in H we conclude that $c_2 = 1$ which is a contradiction.

(4) $b_i + p_i = 3, 1 \leq b_i \leq 2$, S_i orientable and $2 \notin A_i$ for some i .

Here we shall use the figure eight argument. Suppose $i = 1$. By assumption, one of the following holds:

$$b_1 = 1 \text{ and } p_1 = 2$$

or

$$b_1 = 2 \text{ and } p_1 = 1.$$

The proof is similar for both cases, considering that a boundary component behaves just like a singular fiber of multiplicity 0. So we only present the proof of the first case.

Suppose $p_1 = 2$ and $b_1 = 1$. Consider the homology classes Γ_1 and $\Gamma_2 \in \mathbb{H}_*(\overline{M}_1)$ as constructed in the proof of the proposition 5.4. They can be regarded as homology classes in $\mathbb{H}_*(M)$. We prove that

$$p_{AM}(\Delta\Gamma_1) \bullet p_{AM}(\Delta\Gamma_2) \neq 0 \in \mathbb{H}_0(M)$$

which proves that M has nontrivial extended products.

Having done the same mod 2 calculation as the proof of proposition 5.4 we get two conjugacy classes

$$[c_1 c_2 c_1^{-1} c_2^{-1}] \quad \text{and} \quad [c_1 c_2^{-1} c_1^{-1} c_2].$$

Just like the previous cases, in order to show that these two conjugacy classes are distinct, we must prove that at least one of them has no representative in the fundamental group of the boundary component.

Suppose that

$$[c_1 c_2 c_1^{-1} c_2^{-1}] = [h^r d^s] \tag{8.5}$$

where h is the generator corresponding to the normal fiber and d is the generator corresponding to a boundary component.

Consider the group

$$H = \langle c_1, c_2, d \mid c_1 c_2 d = 1, c_1^{\alpha_1} = c_2^{\alpha_2} = 1 \rangle$$

and the homomorphism $\phi : \pi_1(M_1) \rightarrow H$ which is identity on c_1, c_2, d and sends other generators of $\pi_1(M)$ to the trivial element of H . It follows from presentation (4.1) that ϕ is well-defined.

By applying ϕ to (8.5) we have

$$[c_1 c_2 c_1^{-1} c_2^{-1}] = [d^s].$$

in H , or

$$[c_1 c_2 c_1^{-1} c_2^{-1}] = [(c_1 c_2)^s].$$

Notice that H is the free product $\mathbb{Z}_{\alpha_1} * \mathbb{Z}_{\alpha_2}$ with c_1 and c_2 as the generators of the corresponding factors.

So if $c_1 c_2 c_1^{-1} c_2^{-1}$ and $(c_1 c_2)^s$ are conjugate then $s = 2$. Even if $s = 2$ then two cyclically reduced words $c_1 c_2 c_1^{-1} c_2^{-1}$ and $c_1 c_2 c_1 c_2$ are cyclically different since c_1 and c_2 are not of multiplicity 2.

- (5) $b_i + p_i = 3$, $1 \leq b_i \leq 2$, S_i orientable and $2 \in A_i$ for all i :

This is case where the figure eight argument does not work. The following lemma provides us a double cover of the manifold which has nontrivial extended products by previous cases or the non separating torus argument. Appendix III is devoted to the proof of this lemma.

Lemma 8.2 *Let M be as above, $|\mathcal{J}| = n$, $b_i + p_i = 3$, $1 \leq b_i \leq 2$ and $2 \in A_i$ and S_i orientable for all i .*

(i) *If $n \geq 3$ then M has a double cover with no non separating torus whose torus decomposition has a Seifert piece with 3 boundary components.*

(ii) *If $n = 2$ and $r \in A_1$, $r \neq 2$, then M has a double cover with no non separating torus whose torus decomposition has a Seifert piece two singular fibers of multiplicity $r \neq 2$.*

(iii) *If $n = 2$ and $A_1 = A_2 = \{2\}$ then M has a double cover with a non separating torus.*

(6) **Suppose that:**

- $b_i = 2$, $p_i = 1$, $A_i = \{2\}$ and S_i is orientable for $i \neq 1, n$
- $b_1 = b_n = 1$;
- S_1 is non orientable;
- If S_n is orientable then $p_n = 1$ and $2 \in A_n$

If \overline{M}_1 has a singular fiber or S_1 has more than one cross cap: then consider M'_1 the double cover of \overline{M}_1 which is Seifert manifold and is oriented as fibration (see remark 4.1). M'_1 has at least 2 singular fibers and 2 boundary components or its base surface has genus greater than zero. We can construct \tilde{M} , a double cover of M , by gluing two copies of $M \setminus M_1$ to M'_1 along the boundary components of M'_1 such that the result double covers M . The covering on the complement of M'_1 is the trivial double cover. Note that the torus decomposition of \tilde{M} has a Seifert piece M'_1 which has either two boundary components

and 2 singular fibers thus by case (3) \tilde{M} has nontrivial extended loop products or its base surface has genus greater than 0 thus it has a non separating torus and therefore nontrivial extended loop products.

If \overline{M}_1 has no singular fiber and S_1 has one cross cap: then S_1 is indeed Möbius strip and \overline{M}_1 has another Seifert fibration model whose base surface is a disk and has 2 singular fibers of multiplicity 2. Now if S_n is orientable this is just case (5). If S_n is not orientable then it is the above case.

□

9 Gluing to Hyperbolic Manifolds along Torus

In this section we analyze various gluing of Seifert and hyperbolic 3-manifolds along torus and for that we need the following lemma. The proof can be found in Appendix II.

Lemma 9.1 *Suppose that G_1, G_2 and H are three groups with monomorphisms $\phi_i : H \rightarrow G_i$, $i = 1, 2$ denotes the amalgamated free product of G_1 and G_2 over H .*

Let $g_1 \in G_1 \setminus H$ and $g_2 \in G_2 \setminus H$ and $h \neq 1$ in H such that:

$$(i) \ g_1^{-1} H g_1 \cap H = 1$$

$$(ii) \ g_2 h \neq h g_2.$$

*Then two conjugacy classes $[h g_1 g_2]$ and $[h g_2 g_1]$ of $G_1 *_H G_2$ are different.*

Proposition 9.2 *Let M be an oriented (not necessarily closed) 3-manifold that contains an incompressible torus T such that:*

(i) $M - T$ is disconnected with two components M_1 and M_2 .

(ii) M_1 has a hyperbolic interior.

(iii) $\overline{M_2}$ either has a hyperbolic interior or is Seifert fibred not homeomorphic to $S^1 \times S^1 \times I$.

Then M has nontrivial extended loop products.

Proof. Let $\varphi : S^1 \times S^1 \rightarrow T$ be the embedded torus. φ gives rise to an element $H \in \mathbb{H}_1(M_3)$, whose marked points are $\varphi(s, 1)$, $s \in S^1$ and the loop passing through $\varphi(s, 1)$ is

$$\begin{aligned} \varphi_s : S^1 &\rightarrow M \\ \varphi_s(t) &= \varphi(s, t) \end{aligned}$$

Take $\varphi(1, 1)$ as the base point for M , M_1 and M_2 . Let $h = [\varphi_1] \in \pi_1(M)$.

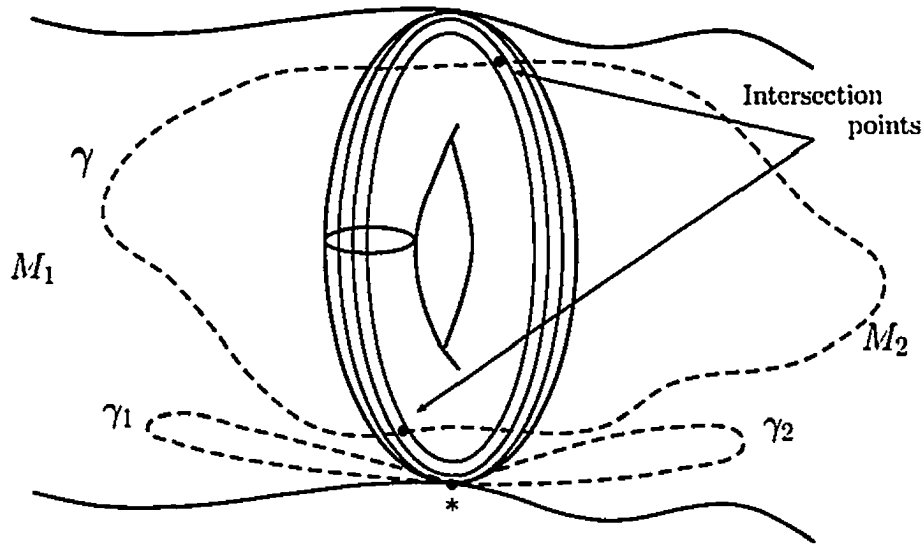


Figure 7: M_1 and M_2 glued along T

We recall that $\pi_1(M)$ is the amalgamated free product of $\pi_1(M_1)$ and $\pi_1(M_2)$ over $\pi_1(T)$. Pick two arbitrary elements $g_1 \in \pi_1(M_1)$ and $g_2 \in \pi_1(M_2)$. Let $g_1 g_2 \in \pi_1(M)$ be their product in the amalgamated free product $\pi_1(M_1) *_{\pi_1(T)} \pi_1(M_2)$. Let γ_1 and γ_2 be two loops in M_1 and M_2 representing g_1 and g_2 . The composition $\gamma_1 \gamma_2$ of these two loops is a representative for $g_1 g_2$. This loop has free homotopy type $[g_1 g_2]$ (see Figure 7).

Replace $\gamma_1 \gamma_2$ by a loop γ with the same free homotopy type which intersects transversally the torus T at two points (see Figure 7). γ can be regarded as an element of $\mathbb{H}_0(M)$. We denote this homology class by Γ .

We claim that there is a choice of $g_i \in \pi_1(M_i) \setminus \pi_1(T)$, $i = 1, 2$, and

$h \in \pi_1(T)$ such that:

$$p_{A_M}(\Delta\Gamma) \bullet p_{A_M}(\Delta H) \neq 0.$$

In the expansion of $p_{A_M}(\Delta\Gamma) \bullet p_{A_M}(\Delta H)$, we get 4 terms for each intersection point, just like in the proof of proposition 3.3, therefore 8 terms overall. Each term is an element of $\mathbb{H}_0(M)$ and represented by a loop. They have the free homotopy types:

$$[hg_1g_2], [h], [g_1g_2], 1, [hg_2g_1], [h], [g_1g_2], 1.$$

Calculating mod 2, we get $[hg_1g_2] - [hg_2g_1]$. By lemma 9.1 $[hg_1g_2]$ and $[hg_2g_1]$ are different conjugacy classes if the fundamental groups of the components of $M - T$ satisfy the conditions of lemma 9.1.

By assumption the interior of M_1 is hyperbolic. $\pi_1(M_1)$, can be regarded as a subgroup of $PSL(2, \mathbb{C})$ where the image of $\pi_1(T)$ in $\pi_1(M_1)$ consists of parabolic elements which have a common fixed point.

For $g \in \pi_1(M_1) \setminus \pi_1(T)$ (we have denoted the image of $\pi_1(T)$ in $\pi_1(M_1)$ simply by $\pi_1(T)$), $g^{-1}\pi_1(T)g \cap \pi_1(T) = \{1\}$, since if the fixed point of the elements of $\pi_1(T)$ is m then the elements of $g^{-1}\pi_1(T)g$ fix $g^{-1}(m)$. So if $g^{-1}\pi_1(T)g \cap \pi_1(T)$ is nontrivial then m must be a fixed point of g also but this implies that $g \in \pi_1(T)$ which is a contradiction. Thus $G_1 = \pi_1(M)$ satisfies condition (i) of lemma 9.1.

Also, the argument above proves that if M_2 has hyperbolic interior then for any $g_2 \in \pi_1(M_2) \setminus \pi_1(T)$ and $h \in \pi_1(T) \setminus \{1\}$ the condition (ii) is satisfied. So the assertion is proven if M_1 and M_2 are both hyperbolic.

Now consider the case when M_2 is Seifert fibred. If h is not an power of the element by represented by the normal fiber of M_2 , then it is not in

the centralizer of $\pi_1(M_2)$ and the condition (ii) holds for an appropriate choice of g_2 . Otherwise exchange the role of longitude and meridian in the construction of the homology class H . Namely, the set of marked points is:

$$\varphi(1, t), 0 \leq t \leq 1$$

and the loop passing through $\varphi(1, t)$ is

$$\varphi_t(s) = \varphi(s, t), 0 \leq s \leq 1.$$

Then h , the element of $\pi_1(M_2)$ represented by loop passing through $\varphi(1, 1)$, is not in the centralizer of $\pi_1(M_2)$ and one can choose $g_2 \in \pi_1(M_2) \setminus \pi_1(T)$ such that $g_2 h \neq h g_2$. This finishes the proof. \square

10 Proof of Main Theorem: Part I

In this section we prove the first part of the main theorem, namely that if a closed oriented manifold M which is not algebraically hyperbolic, then M or a double cover of M has nontrivial extended loop products.

For this we use the theory of *the prime decomposition and the characteristic surface* for 3-manifolds. We recall some of this theory.

A closed oriented 3-manifold M is said to be *prime* if it cannot be obtained as the connected sum of two 3-manifolds that none of them is homeomorphic to 3-sphere. By the Alexander's theorem S^3 is prime, as every imbedded 2-sphere in S^3 bounds a cell.

Prime Decomposition (see [13]): Let M be a compact, connected and oriented 3-manifold. Then there is a decomposition

$$M = P_1 \# P_2 \# \dots \# P_n$$

with each P_i prime and this decomposition is unique up to permutation, insertion or deletion of S^3 's.

A closed 3-manifold is said to be *irreducible* if any embedded 2-sphere S^2 bounds a 3-ball. Again by the Alexander's theorem, S^3 is irreducible. An oriented irreducible 3-manifold is prime. The converse is not true but the only prime oriented 3-manifold which is not irreducible, is $S^1 \times S^2$.

One can deduce a necessary condition for irreducibility from the *sphere theorem*. Here, we state a simplified version of the theorem.

Sphere Theorem (see [6]): Let M be an oriented connected 3-manifold. If $\pi_2(M) \neq 0$ then there is an imbedded S^2 in M representing a nontrivial element in $\pi_2(M)$.

The topology of an irreducible 3-manifold M is coarsely determined by the cardinality of the fundamental group. By the *sphere theorem* (see [7]) $\pi_2(M) = 0$. Let \tilde{M} be the universal cover of M . If $\pi_1(M)$ is finite, then \tilde{M} is closed and simply connected. Therefore \tilde{M} is a homotopy sphere. If $\pi_1(M)$ is infinite then \tilde{M} is not closed and we have $\pi_i(\tilde{M}) = 0$ for all i , by Hurewicz theorem. Therefore \tilde{M} is contractible or in other words M is a $K(\pi, 1)$ (aspherical).

Now by the prime decomposition theorem, every 3-manifold M can be written as

$$M = (K_1 \# K_2 \# \dots \# K_p) \# (L_1 \# L_2 \dots \# L_q) \# (\#_i (S^1 \times S^2)) \quad (10.1)$$

where K_i 's are closed aspherical 3-manifolds and L_i are closed 3-manifolds that are finitely covered by a homotopy 3-sphere.

Let S be an imbedded surface in a 3-manifold M . We shall say that S is *properly imbedded* if $S \cap \partial M = \partial S$. Then S is *two-sided* in M if the normal bundle of S in M is trivial.

A 2 sided surface S in M is said to be *compressible* if either:

(i) S is a 2-sphere and it bounds a 3-cell in M

or

(ii) there is a disk $D \in M$ such that $D \cap S = \partial D$ and ∂D is non contractible in S .

Otherwise S is said to be *incompressible*. An imbedded disk is considered to be incompressible. For an incompressible surface S in M , the induced map by inclusion $\pi_1(S) \rightarrow \pi_1(M)$ is injective.

A 2-sided surface S is incompressible if and only if the map $\pi_1(S) \rightarrow$

$\pi_1(M)$ induced by inclusion is injective.

An oriented 3-manifold M , possibly with boundary is called *atoroidal* if any incompressible torus in M is ∂ -parallel, meaning that it can be isotoped to the boundary of M .

Torus decomposition (see [9] or [10]): For a compact irreducible oriented 3-manifold M there exists a collection $T \subset M$ of disjoint incompressible tori such that the closure of each component of $M \setminus T$ is either atoroidal or a Seifert manifold with T tangent to the fibration. A minimal such a collection T is unique up to isotopy.

As we are interested in rank 2 abelian subgroups of fundamental groups, torus theorems by Scott and Seifert fibred space theorem by Casson-Jungreis [3] and Gabai [6] will be useful to us.

Torus Theorem (see [16]): Let M be a compact, connected oriented, irreducible 3-manifold. Suppose that $\pi_1(M)$ has a free abelian subgroup of rank 2. Then either

- (i) M contains a two sided torus such that the inclusion homomorphism $\pi_1(T) \rightarrow \pi_1(M)$ is injective, or
- (ii) $\pi_1(M)$ contains an infinite cyclic normal subgroup.

Seifert Fibred Space Theorem (see [3] or [6]) Let M be a compact connected, oriented, irreducible 3-manifold with infinite π_1 , then M is a Seifert fibred space if and only if $\pi_1(M)$ has a cyclic normal subgroup.

We recall the definition of algebraically hyperbolic manifolds.

Definition An oriented 3-manifold is said to be *algebraically hyperbolic* if it is aspherical ($K(\pi, 1)$) and $\pi_1(M)$ has no rank 2 abelian subgroup.

If M is not algebraically hyperbolic then at least one of the following

holds:

10.1 If M is not aspherical

By prime decomposition theorem, every 3-manifold M can be written as

$$M = (K_1 \# K_2 \# \dots \# K_p) \# (L_1 \# L_2 \# \dots \# L_q) \# (\#_1^r(S^1 \times S^2))$$

where K_i 's are closed aspherical 3-manifolds and L_i are closed 3-manifolds that are finitely covered by a homotopy 3-sphere. If M is not aspherical then one the following holds:

- (i) If M is a connected sum of manifolds with nontrivial fundamental group then proposition 6.1 says that M has nontrivial extended loop products.
- (ii) If M has a non separating 2-sphere or in other words there is a $S^2 \times S^1$ factor in its prime decomposition, then by corollary 3.2 M has nontrivial extended loop products.
- (iii) M has finite fundamental group; Then by proposition 2.6 M has nontrivial extended loop products.

Thus in either cases M has nontrivial extended loop product.

10.2 If M aspherical and $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$

We turn to aspherical 3-manifolds with rank 2 abelian groups. By Torus theorem and Seifert theorem one of the following holds:

a) M is a closed Seifert manifold

If M is Seifert manifold then by proposition 4.2 it has nontrivial extended loop products.

b) The collection of tori in the torus decomposition of M is nonempty.

This case itself is divided into 3 cases:

1. M contains a non separating torus:

If there is a non separating torus in the collection T then by proposition 3.3 M has nontrivial string topology.

2. Torus decomposition of M has only Seifert pieces :

Let \mathcal{T} be the collection of tori provided by the torus decomposition of M and

$$M \setminus \mathcal{T} = M_1 \cup M_2 \dots \cup M_n.$$

\overline{M}_i is a Seifert manifold. Let p_i and b_i be the number of singular fibers and boundary components of \overline{M}_i . If S_i , the base surface of \overline{M}_i , is orientable then g_i denotes its genus otherwise g_i is number of cross caps in S_i .

If for some i , S_i is orientable and $g_i \geq 1$, then M has a non separating torus and by case 1 we know that M has nontrivial extended loop product. So we assume that $g_i = 0$ if S_i is oriented.

If S_i is orientable then,

$$p_i + b_i \geq 3.$$

To see that suppose that $p_i + b_i \leq 2$. Then one of the following holds:

- (a) $b_i = 1$ and $p_i = 1$;
- (b) $b_i = 2$ and $p_i = 0$

In the first case it follows that $\pi_1(\overline{M}_i) \simeq \mathbb{Z}$ so the boundary components of \overline{M}_i can not be incompressible. In the latter case it follows that $\overline{M}_i \simeq S^1 \times S^1 \times [0, 1]$ this contradicts the minimality of the collection \mathcal{T} as one can extend the Seifert fibration of one of the neighboring \overline{M}_j to \overline{M}_i .

Therefore M satisfies all the conditions of proposition 8.1, hence M or a double cover of M has nontrivial extended loop product.

3. If M has an atoroidal component:

Suppose that \mathcal{T} is the collection of tori provided by the torus decomposition of M and every torus in \mathcal{T} is separating.

By Thurston theorem ¹¹[17],

Theorem *A compact, oriented, irreducible atoroidal 3-manifold whose boundary consists of a finite number of tori admits a complete hyperbolic metric of finite volume;*

atoroidal components of $M \setminus \mathcal{T}$ are hyperbolic.

Let M_1 be a hyperbolic component of $M \setminus \mathcal{T}$ and M_2 is another component that is attached to M_1 along a torus T . By proposition 9.2, $M' = M_1 \cup T \cup M_2$ has nontrivial extended loop products.

Consider the same homology classes Γ and H in $\mathbb{H}_*(M')$ constructed as in the proof of proposition 9.2. They can be regarded as elements of $\mathbb{H}_*(M)$. We claim that the loop product

$$p_{A_M}(\Delta(\Gamma)) \bullet p_{A_M}(\Delta H) \neq 0,$$

so M has nontrivial extended loop product and string topology.

¹¹Geometrization of Haken manifolds.

The calculation of the loop product is the same as of the proof Proposition 9.2. Calculating $\text{mod } 2$ only two terms remain, each one an element of $\mathbb{H}_0(M)$ which is represented by a conjugacy class in $\pi_1(M)$. These conjugacy classes are,

$$[hg_1g_2] \text{ and } [hg_2g_1].$$

We prove that these two conjugacy classes are different. By the proof of proposition 9.2 we know that hg_1g_2 and hg_2g_1 are not conjugate in $\pi_1(M')$. So by the injectivity lemma (lemma 7.1), if we show that hg_1g_2 is not conjugate to an element of the fundamental group of one the boundary components of $\overline{M'}$ it follows that hg_1g_2 and hg_2g_1 are not conjugate in $\pi_1(M)$.

Let $T_1 \cup T_2 \dots \cup T_k = \partial\overline{M'}$ and suppose that hg_1g_2 is conjugate to an element of $\pi_1(T_1)$ in $\pi_1(M')$. Since one of the

$$T_1 \subset \overline{M_1} \text{ or } \overline{M_2}$$

holds let's assume that $T_1 \subset \overline{M_1}$. Then

$$\pi_1(T_1) \subset \pi_1(M_1) = \pi_1(\overline{M_1})$$

as T_1 is incompressible. If hg_1g_2 is conjugate to an element to $\pi_1(T_1)$ then it is conjugate to an element in $\pi_1(M_1)$. But this is a contradiction since $hg_1g_2 = (hg_1)g_2$ as an element of

$$\pi_1(M') = \pi_1(M_1) *_{\pi_1(T)} \pi_1(M_2)$$

is a cyclically reduced word of length 2 and it is cannot be conjugate to an

element of the factor $\pi_1(M_1)$. Now by the injectivity lemma hg_1g_2 and hg_2g_1 are not conjugate in $\pi_1(M)$.

11 Proof of Main Theorem: Part II, Algebraically Hyperbolic Manifolds

In this section we prove the second part of the main theorem, namely that algebraically hyperbolic manifolds have trivial extended loop products. Before that we recall the definition of algebraically hyperbolic.

Definition An oriented 3-manifold is said to be *algebraically hyperbolic* if it is aspherical ($K(\pi, 1)$) and $\pi_1(M)$ has no rank 2 abelian subgroup.

We then need the following lemma. The proof can be found in the Appendix I, Lemma 3.

Lemma 11.1 *If M is a $K(\pi, 1)$ (aspherical) then $\mathbb{L}M$ is homotopy equivalent to*

$$\coprod_{[\alpha] \in \hat{\pi}_1(M)} K(\pi_\alpha, 1)$$

where we choose only one representative for each conjugacy class and π_α is the centralizer of α in $\pi_1(M)$.

Therefore to prove the second part of the theorem we need a good understanding of the centralizers. For that we need following theorem.

Compact Realization Theorem (see [7]): Let M be a connected, oriented 3-manifold, and let K be a subgroup of $\pi_1(M)$ which is finitely generated. Then there is a compact, oriented irreducible 3-manifold M_0 with $\pi_1(M_0) \simeq K$.

Lemma 11.2 *If a closed oriented 3-manifold M is algebraically hyperbolic then the fundamental group of M does not have any nontrivial finite sub-*

groups and the centralizer π_α of an element $\alpha \in \pi_1(M)$ is isomorphic to an additive subgroup of \mathbb{Q} .

Proof. Let \mathbb{Z}_p be a nontrivial finite subgroup of $\pi_1(M)$. Then $K(\mathbb{Z}_p, 1)$ is a covering of M , so it has to be a 3-manifold. But one knows that $K(\mathbb{Z}_p, 1)$ has homology in infinitely many dimensions. So $\pi_1(M)$ has no finite subgroup.

For the second part we first prove that any finitely generated subgroup of $\pi_1(M)$ is isomorphic to \mathbb{Z} . Suppose $\beta_1, \beta_2, \dots, \beta_n$ are some elements of π_α . Consider the subgroup $\langle \beta_1, \beta_2, \dots, \beta_n, \alpha \rangle$ which is finitely generated and it is not a free product since it has a nontrivial center. Therefore by compact realization theorem it is isomorphic to a fundamental group of a compact, oriented irreducible 3-manifold. The cyclic subgroup $\langle \alpha \rangle$ is in its center and therefore it is normal. Thus by the Seifert fibred space theorem, the compact realization should be a Seifert manifold. Since $\pi_1(M)$ has no finite subgroup or rank 2 abelian subgroup this Seifert manifold has to be a solid torus¹². As the fundamental group of a solid torus is \mathbb{Z} , we have $\langle \beta_1, \beta_2, \dots, \beta_n, \alpha \rangle \simeq \mathbb{Z}$. To prove that it is isomorphic to a subgroup of \mathbb{Q} , notice that since $\pi_1(M)$ is countable and every finitely generated subgroup is cyclic, we can write $\pi_1(M) = G_1 \cup G_2 \dots$ where each G_i is an infinite cyclic subgroup and $G_i \subset G_{i+1}$. Let x_i be a generator for G_i ; then we have $x_i = x_{i+1}^{n_i}$ for some n_i . We construct a map $\varphi : \pi_1(M) \rightarrow \mathbb{Q}$ by letting $\varphi(z_1) = 1$ and $\varphi(z_{i+1}) = \frac{1}{n_1 n_2 \dots n_i}$. This is a well defined map because of the way we chose n_i 's. φ is injective, since every $x \in \pi_1(M)$ then $x \in G_i$ for some i . So $x = kz_i$ for some i and k . $\varphi(x) = \frac{k}{n_1 n_2 \dots n_i}$. Therefore $\varphi(x) = 0$ implies $k = 0$ or $x = 0$. \square

¹²It cannot be $S^2 \times S^1$ since it is irreducible.

Lemma 11.3 *If G is an additive subgroup of \mathbb{Q} then $k(G, 1)$ has homological dimension one.*

Proof. One can write $G = \varinjlim G_n$, where G_i 's are cyclic subgroups of G . So $K(G, 1) = \varinjlim K(G_n, 1)$. Since each $K(G_n, 1) \simeq K(\mathbb{Z}, 1)$ has homological dimension 1 therefore the direct limit $K(G, 1)$ has homological dimension one as taking homology commutes with taking direct limit. \square

Proposition 11.4 *If M is algebraically hyperbolic then any finite cover of M has trivial extended loop products.*

Proof. Let \widetilde{M} be a finite cover of M . Notice that \widetilde{M} is also algebraically hyperbolic. Since \widetilde{M} is a $K(\pi', 1)$ then its free loop space is homotopy equivalent to

$$\coprod_{[\alpha] \in \widetilde{\pi}'} K(\pi'_\alpha, 1)$$

where we choose only one representative for each conjugacy and π'_α is the centralizer of α in $\pi_1(\widetilde{M})$. So

$$A_{\widetilde{M}} \simeq H_*\left(\coprod_{[\alpha] \in \widetilde{\pi}', \alpha \neq 1} K(\pi'_\alpha, 1)\right) \simeq \bigoplus_{[\alpha] \in \widetilde{\pi}', \alpha \neq 1} H_*(K(\pi'_\alpha, 1))$$

Now by lemma 11.2, we see that each $K(\pi'_\alpha, 1)$, $\alpha \neq 1$, is of homological dimension 1. This proves that the loop product \bullet on $A_{\widetilde{M}}$ is zero because of the absence of homology classes of sufficiently high degree. \square

Appendix I: Some Algebraic Topology of Free Loop Space

Lemma 1 There is a one-to-one correspondence between the connected components of $\mathbb{L}M$ and the conjugacy classes of $\pi_1(M)$.

Proof. Choose a base point m for M . Any loop in $\mathbb{L}M$ is freely homotopic to a loop whose marked point is m . Indeed, consider a loop f in $\mathbb{L}M$. Let p be a path connecting the base point of m to f . The loop that starts from m , follows p and f and then comes back to m by following p again, is freely homotopic to f . Hence it is in the same connected component as f and its marked point is m . In other words, each connected component of $\mathbb{L}M$ has a representative whose marked point is m .

Let α and β be two loops in $\mathbb{L}M$ with the same marked point m . Suppose they have the same free homotopy type or, equivalently, they are in the same connected component of $\mathbb{L}M$. This means that there is a map $F : [0, 1] \rightarrow \mathbb{L}M$ such that $F_0 = \alpha$ and $F_1 = \beta$. Consider $\gamma : [0, 1] \rightarrow M$ where

$$\gamma(t) = F_t(0).$$

In fact γ is the path of marked points of the path of loops from α and β defined by F .

We have

$$\gamma(0) = \gamma(1) = m$$

and

$$\alpha = \gamma\beta\gamma^{-1}$$

in $\pi_1(M)$. To prove the latter, consider the following family of paths for $0 \leq t \leq 1$,

$$\gamma_t : [0, 1] \rightarrow M$$

$$\gamma_t(s) = \gamma(st).$$

γ_0 is the constant loop at m and $\gamma_1 = \gamma$. Now

$$G_t(\cdot) = \gamma_t F_t(\cdot) \gamma_t^{-1}$$

is a homotopy between α and $\gamma\beta\gamma^{-1}$, as $G_0 = \alpha$ and $G_1 = \gamma\beta\gamma^{-1}$. This finishes the proof of the assertion. \square

Lemma 2 Let $\alpha \in \pi_1(M, m)$ and $f \in \mathbb{L}M$ be a loop representing α . Then there exists a short exact sequence

$$\pi_2(M) \cong \pi_1((\Omega M)_\alpha, f) \xrightarrow{S} \pi_1((\mathbb{L}M)_{[\alpha]}, f) \xrightarrow{R} \pi_\alpha \longrightarrow 0 \quad (11.1)$$

where π_α is the centralizer of α in $\pi_1(M)$. In particular $\pi_1((\mathbb{L}M)_{[\alpha]}, f) \cong \pi_\alpha$ if $\pi_2(M)$ is trivial.

Proof. First we describe the map R . Let $\gamma \in \pi_1((\mathbb{L}M)_{[\alpha]}, f)$ and $F : \mathbb{S}^1 \rightarrow \mathbb{L}M$ be a representative for γ . F can be regarded as a map $[0, 1] \times [0, 1] \rightarrow M$ such that $F(0, \cdot) = F(1, \cdot) = f(\cdot)$ and $F(\cdot, 0) = F(\cdot, 1)$. Consider the loop $g(\cdot) = F(\cdot, 0) = F(\cdot, 1)$. We have $g(0) = g(1) = f(0) = f(1) = m$. We define $R(\gamma) = [g] \in \pi_1(M)$. We shall prove that $[g]$ belongs to the centralizer of α or in other words we shall construct a homotopy between fg and gf . G is such a homotopy (see Figure 8):

$$\begin{aligned}
 G(s, t) &= F(x, y) \\
 x &= \frac{1}{2}((1 - 2s)|1 - 2t| + 2s + 2t - 1) \\
 y &= \frac{1}{2}((2s - 1)|1 - 2t| - 2s + 2t + 1)
 \end{aligned}
 \tag{11.2}$$

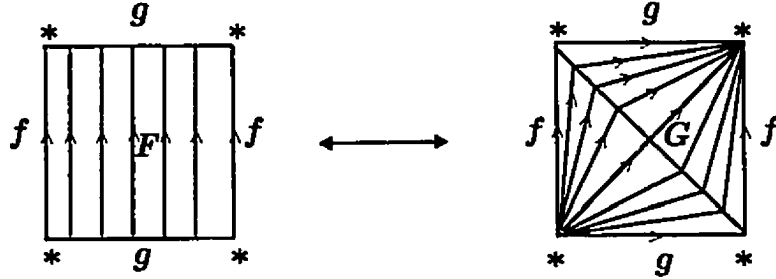


Figure 8: Homotopy G obtained from F

We have $G(0, t) = gf$ and $G(1, t) = fg$.

- R is well-defined: let $F_1, F_2 : \mathbb{S}^1 \rightarrow \mathbb{L}M$ represent the same element in $\pi_1(\mathbb{L}M, f)$. Let G be a homotopy between F_1 and F_2 .

G can be regarded as a map

$$G : [0, 1] \times [0, 1] \times [0, 1] \rightarrow M$$

such that

$$G(0, \cdot, \cdot) = F_1(\cdot, \cdot) \text{ and } G(1, \cdot, \cdot) = F_2(\cdot, \cdot).$$

Then H defined by,

$$H : [0, 1] \times [0, 1] \rightarrow M,$$

$$H(r, s) = G(r, s, 0),$$

is a homotopy between g_1 and g_2 where $g_i(\cdot) = F_i(\cdot, 1) = F_i(\cdot, 0)$, $i = 1, 2$.

- *R is onto*: Suppose $\gamma = [g] \in \pi_\alpha \subset \pi_1(M)$. Let $G : [0, 1] \times [0, 1] \rightarrow M$ be a homotopy between fg and gf . Choose G such that:

$$G(0, t) = f(2t), \quad 0 \leq t \leq 1/2, \quad G(0, t) = g(2t - 1), \quad 1/2 \leq t \leq 1$$

and

$$G(1, t) = g(2t), \quad 0 \leq t \leq 1/2, \quad G(1, t) = f(2t - 1), \quad 1/2 \leq t \leq 1.$$

By solving (11.2) for s and t one can define a self free-homotopy F for f which gives rises to an element of $\pi_1((\mathbb{L}M)_{[\alpha]}, f)$. To be brief, we only provide the result of the calculation:

$$\begin{aligned} F(x, y) &= G(s(x, y), t(x, y)) \\ s &= \frac{1}{2} \left(\frac{y-x-1+|1-x-y|}{|1-x-y|-1} \right) \\ t &= \frac{x+y}{2} \\ s(0, 0) &= 0, s(1, 1) = 1. \end{aligned} \tag{11.3}$$

A simple calculation shows that $F(0, y) = F(1, y) = f(y)$ and $F(x, 0) = F(x, 1) = g(x)$.

- $\text{Ker } R = \text{Im } S$: S is induced by the inclusion of $(\Omega M)_\alpha \hookrightarrow \mathbb{L}M_{[\alpha]}$. It is obvious that $R \circ S = 0$, as R is basically the loop consisting of the marked points of a given loop of loops. For the image of S , this will be a constant loop so $R \circ S = 0$. This shows that $\text{Im } S \subset \text{Ker } R$.

Let $\gamma \in \text{Ker } R$ and $F : \mathbb{S}^1 \rightarrow \mathbb{L}M$ be a representative for γ . As usual, we regard F as a function $F : [0, 1] \times [0, 1] \rightarrow M$ such that $F(0, \cdot) = F(1, \cdot) = f(\cdot)$. We have $R(\gamma) = [g]$ for $g(\cdot) = F(\cdot, 0) = F(\cdot, 1)$. Let $G : S^1 \rightarrow M$ be a homotopy between $F(\cdot, 0)$ and the constant loop,

$G(1, \cdot) = F(\cdot, 0) = F(\cdot, 1)$, $G(0, \cdot) = m$ and $G(\cdot, 0) = G(\cdot, 1) = m$.

Consider $H : [0, 1] \times [0, 1] \rightarrow M$ as follows (see Figure 9):

$$H(s, t) = \begin{cases} F(0, t) = F(1, t) & \text{if } s = 0, 1 \\ G(s, \frac{t}{r_1(s)}) & \text{if } 0 \leq t \leq r_1(s), \quad s \neq 0, 1 \\ F(s, \frac{t-r_1(s)}{r_2(s)-r_1(s)}) & \text{if } r_1(s) \leq t \leq r_2(s), \quad s \neq 0, 1 \\ G(s, \frac{t-r_2(s)}{1-r_2(s)}) & \text{if } r_2(s) \leq t \leq 1, \quad s \neq 0, 1 \end{cases} \quad (11.4)$$

where

$$r_1(s) = \frac{1}{3}(1 - |2s - 1|)$$

$$r_2(s) = 1 - r_1(s)$$

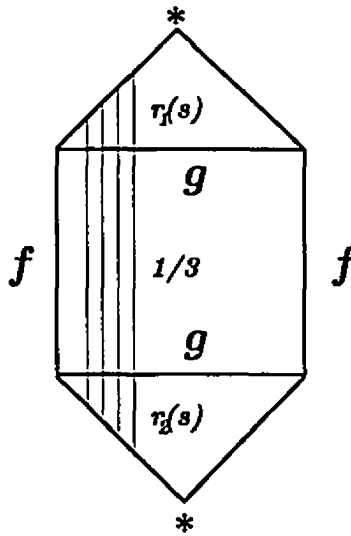


Figure 9: H

We have

$$H(s, 0) = H(s, 1) = G(s, 0) = G(s, 1) = m$$

and

$$H(0, t) = H(1, t),$$

so H can be regarded as a map $H : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow M$ and H is in the image of S .

We claim that $\gamma = [H] \in \pi_1((\mathbb{L}M)_\alpha, f)$. Consider $K : [0, 1] \times [0, 1] \times [0, 1] \rightarrow M$ defined as follows:

$$K(q, s, t) = \begin{cases} F(0, t) = F(1, t) & \text{if } s = 0, 1 \\ G(s, \frac{t}{r_1(q, s)}) & \text{if } 0 \leq t \leq r_1(q, s), \quad s \neq 0, 1 \\ F(s, \frac{t-r_1(q, s)}{r_2(q, s)-r_1(q, s)}) & \text{if } r_1(q, s) \leq t \leq r_2(q, s), \quad s \neq 0, 1 \\ G(s, \frac{t-r_2(q, s)}{1-r_2(q, s)}) & \text{if } r_2(q, s) \leq t \leq 1, \quad s \neq 0, 1 \end{cases}$$

$$r_1(q, s) = \frac{q}{3}(1 - |2s - 1|)$$

$$r_2(q, s) = 1 - r_1(q, s)$$

K is a homotopy between F and H . □

Remark 1 Different components of $\mathbb{L}M$ can have different homotopy types. For instance, let M be a surface with a non abelian fundamental group. Since $\pi_2(M)$ is trivial then the fundamental group of each component of $(\mathbb{L}M)_\alpha$ is the centralizer π_α of α . Hence if $\alpha \notin Z(\pi_1(M))$ then $(\mathbb{L}M)_{[\alpha]}$ is homotopy typically different from $\mathbb{L}M_{[0]}$.

A useful picture is the following. Fix a base point for M and consider ΩM the based loop space of M , with respect to the point. There is a natural map which sends

$$\begin{array}{ccc} (\Omega M)_\alpha & \xrightarrow{i} & (\mathbb{L}M)_{[\alpha]} \\ & & \downarrow p \\ & & M \end{array} \quad (11.5)$$

where p is the projection to the marked point.

Lemma 3 If M is a $K(\pi, 1)$ (aspherical) then $\mathbb{L}M$ is homotopy equivalent to

$$\coprod_{[\alpha] \in \hat{\pi}_1(M)} K(\pi_\alpha, 1)$$

where we choose only one representative for each conjugacy class and π_α is the centralizer of α in $\pi_1(M)$.

Proof. We have the following long exact sequence associated with this fibration (11.5):

$$\pi_i((\Omega M)_\alpha) \longrightarrow \pi_i((\mathbb{L}M)_{[\alpha]}) \xrightarrow{p} \pi_i(M) \xrightarrow{\partial_\alpha} \pi_{i-1}((\Omega M)_\alpha) \quad (11.6)$$

Each component of ΩM is contractible since M is a $K(\pi, 1)$. Now it is obvious from the long exact sequence above that $\mathbb{L}M$ is also aspherical. By lemma 2 we have that $\pi_1(\mathbb{L}M)_{[\alpha]} \simeq \pi_\alpha$. \square

Appendix II: Some Facts in Free Products with Amalgamation

1. Some Basic Facts about Free Products with Amalgamation:

We recall some definitions and theorems from group theory related to the free product with amalgamation. We follow the terminology and notations of [12] and [11].

Let G_1 , G_2 and H be three groups and $\phi_i : H \rightarrow G_i$, $i = 1, 2$, be two injective homomorphisms. We can regard H as a subgroup of each G_i , $i = 1, 2$ and we identify each element of H with its image under ϕ_i while we are in G_i . So if $h \in H$, then h can be considered as an element of both G_1 and G_2 . Following [11], we can form the amalgamated free product $G_1 *_H G_2$ of G_1 and G_2 over H .

There is no canonical way of presenting the elements of $G_1 *_H G_2$ uniquely. Once a choice of left coset representatives for G_1/H and G_2/H is made then every element can be represented uniquely. Let c_1, \dots, c_n be a right coset representative for G_1/H and G_2/H . Here is an informal description of the representation:

Let g be an element in $G_1 *_H G_2$ and $g = g_1 g_2 \dots g_r$ be a reduced presentation of g . Start with g_r , there is an $h_r \in H$ and $c_r \in G_1/H$ or G_2/H (depending in which factor is g_r) such that

$$g_r = h_r c_r.$$

Now by replacing g_r by $h_r c_r$ we have $g = g_1 g_2 \dots g_{r-1} h_r c_r$. Again there is a $h_{r-1} \in H$ and c_{r-1} such that

$$g_{r-1} h_r = h_{r-1} c_{r-1}.$$

By replacing $g_{r-1} h_r$ by $h_{r-1} c_{r-1}$ we have $g = g_1 g_2 \dots g_{r-2} h_{r-1} c_{r-1} c_r$. Continuing this procedure we end up with $g = h_1 c_1 c_2 \dots c_r$. This is the desired presentation. While stating it more precisely, the following theorem also asserts its uniqueness (*i.e.* its independence from the reduced word we used in the construction).

Theorem 1 (See [12] pp.201) Let $G = G_1 *_H G_2$ where H is a mutual subgroup of G_1 and G_2 . Suppose a specific right coset representative systems for G_1/H and G_2/H have been selected. Then with each element of g of G we can associate a unique sequence¹³

$$(h, c_1, c_2, \dots, c_r)$$

such that:

- (i) h is an element, possibly 1, of H ;
- (ii) c_i is a coset representative of G_1/H or G_2/H ;
- (iii) $c_i \neq 1$
- (iv) c_i and c_{i+1} are not both in G_1 or G_2 .
- (v) If h' and c'_i 's are the images of h or c_i 's under the homomorphisms of G_1 or G_2 into G , then $g = h' c'_1 c'_2 \dots c'_r$ in G .

A sequence of elements $g_1, g_2, \dots, g_n \in G_1 *_H G_2$ is called *reduced* if:

- (1) Each g_i is in one of the factors G_1 or G_2 ;

¹³This presentation is called *reduced form* (see [12]).

- (2) g_i, g_{i+1} come from different factors; So if $n > 1$ then $g_i \notin H$ for all i .
 (3) If $n = 1$ then $g_1 \neq 1$.

$g_1 g_2 \cdots g_n \in G_1 *_H G_2$ is called a *reduced word* if g_1, g_2, \dots, g_n is a reduced sequence.

Every element of G can be written as $g = g_1 g_2 \cdots g_n$ where g_1, g_2, \dots, g_n is reduced sequence. Indeed let $g = g_1 g_2 \cdots g_n$ be an element in $G_1 *_H G_2$. If g_i and g_{i+1} are in the same factor G_1 (resp. G_2) then we replace $g_i g_{i+1}$ by $g'_i = g_i g_{i+1} \in G_1$ (resp. G_2). In this way we end up with a reduced sequence g'_1, g'_2, \dots, g'_k and $g'_1 g'_2 \cdots g'_k = g_1 g_2 \cdots g_n = g$

A sequence of elements $g_1, g_2, \dots, g_n \in G_1 *_H G_2$ is called *cyclically reduced* if all cyclic permutations of the sequence g_1, g_2, \dots, g_n are also reduced.

$g_1 g_2 \cdots g_n \in G_1 *_H G_2$ is called a *cyclically reduced word* if g_1, g_2, \dots, g_n is a cyclically reduced sequence.

Also every element of $G_1 *_H G_2$ is conjugate to $g_1 g_2 \cdots g_n$ where the sequence g_1, g_2, \dots, g_n is cyclically. Indeed if $g = g'_1 g'_2 \cdots g'_k$ where g'_1, g'_2, \dots, g'_k is a reduced sequence. If g'_i and g'_k are from different factors, then clearly $g'_1 g'_2 \cdots g'_k$ is cyclically reduced. Otherwise consider $g'_k g g'_k^{-1} = g'_k g'_1 g'_2 \cdots g'_{k-1}$ and replace $g'_k g_1$ by $g' = g'_k g'_1$ which is from the same factor. Now $g', g'_2, \dots, g'_{k-1}$ is cyclically.

Theorem 2(see [11] p.187) If $c_1, c_2, \dots, c_n, n \geq 1$, is a reduced sequence in $G = G_1 *_H G_2$, then the product $c_1 c_2 \cdots c_n \neq 1$ in G .

The following theorem describes the conjugacy classes of $G_1 *_H G_2$. Only part (i) is needed here.

Theorem 3(See [12] pp.212)

Let $G = G_1 *_H G_2$. Then every element of G is conjugate to a cyclically

reduced element of G . Moreover, suppose that g is a cyclically reduced element of G . Then:

- (i) If g is conjugate to a cyclically reduced word $p_1 p_2 \dots p_r$ where $r \geq 2$; then g can be obtained by cyclically permuting p_1, p_2, \dots, p_r and then conjugating by an element of H .
- (ii) If g is conjugate to an element h in H , then g is in some factor and there is a sequence of h, h_1, \dots, h_t, g where h_i is in H and consecutive terms of the sequence are conjugate in some factor.
- (iii) If g is conjugate to an element g' in some factor but not in a conjugate of H then g and g' are in the same factor and are conjugate in that factor.

Lemma 4 Suppose that G_1, G_2 and H are three groups with monomorphisms $\phi_i : H \rightarrow G_i, i = 1, 2$ denotes the amalgamated free product of G_1 and G_2 over H .

Let $g_1 \in G_1 \setminus H$ and $g_2 \in G_2 \setminus H$ and $h \neq 1$ in H such that:

- (i) $g_1^{-1} H g_1 \cap H = 1$
- (ii) $g_2 h \neq h g_2$.

Then two conjugacy classes $[h g_1 g_2]$ and $[h g_2 g_1]$ of $G_1 *_H G_2$ are different.

Proof. Suppose that $[h g_1 g_2] = [h g_2 g_1]$. It is clear that

$$[h g_1 g_2] = [g_2 h g_1] \tag{11.7}$$

$$[h g_2 g_1] = [g_2 g_1 h] \tag{11.8}$$

So $g_2(h g_1)$ is conjugate to $g_2(g_1 h)$. Since $g_2(g_1 h)$ is cyclically reduced (of length 2, $p_1 = g_2$ & $p_2 = g_1 h$) by Theorem 3 part (ii), $g_2(g_1 h)$ is conjugate

to $g_2(g_1h)$ or $(g_1h)g_2$ by an element in H .

In the latter case we have

$$g_2(hg_1) = h_1(g_1h)g_2h_1^{-1}$$

which is equivalent to

$$g_2^{-1}(h_1g_1h)^{-1}g_2(hg_1h_1) = 1.$$

But this contradicts Theorem 2. Since the sequence

$$g_2^{-1}, (h_1g_1h)^{-1}, g_2, (hg_1h_1)$$

is a reduced sequence.

Therefore there should be an element $h_1 \in H$ such that

$$g_2hg_1 = h_1g_2g_1hh_1^{-1}$$

or equivalently

$$g_2hg_1h_1 = h_1g_2g_1h$$

or

$$(g_2h)(g_1h_1) = (h_1g_2)(g_1h) \quad (11.9)$$

Now using the uniqueness of representation in terms of left cosets of H Theorem 1 (also see the construction) the equation (11.9) implies that g_1h_1 and g_1h are in the same left coset in G_1/H or in other words, there exists $h_2 \in H$ such that

$$g_1h_1 = h_2g_1h$$

or equivalently

$$h_1 h^{-1} = g_1^{-1} h_2 g_1$$

So $h_1 h^{-1} = g_1^{-1} h_2 g_1 \in g_1^{-1} H g_1 \cap H = 1$ hence $h_2 = h_1 h^{-1} = 1$ which implies $h_1 = h$. By letting $h_1 = h$ in (11.9) and simplifying, it follows

$$g_2 h = h g_2$$

which is a contradiction. Therefore $[h g_1 g_2] \neq [h g_2 g_1]$

□

2. Tree of Groups

A *tree of groups* (G, T) consists of a finite tree T and G a collection of groups

$$G = \{G_e\}_{e \in \text{edge} T} \amalg \{G_v\}_{v \in \text{vert} T}.$$

a group G_v for every vertex $v \in \text{vert} T$, a group G_e for every edge e in $\text{edge} T$ and a monomorphism

$$f_e^v : G_e \rightarrow G_v$$

if v is a vertex of the edge e and

We call G_v a *vertex group* where v is a vertex of T and G_e a *edge group* if e is an edge of T . If v is a vertex of the edge e then G_e can be as considered as a subgroup of G_v .

Suppose that (G, T) is a tree of groups, v_0 a vertex of T . Let $n = |\text{edge} T|$. Consider a sequence of trees T_i , $1 \leq i \leq n$ such that

- (i) $|\text{edge} T_i| = i$, $0 \leq i \leq n$
- (ii) $\text{vert} T_0 = \{v_0\}$,

(iii) $T_i \subset T_{i+1}$ and

(iv) $T_n = T$.

Let e_1, e_2, \dots, e_n be the sequences of the edges and v_1, v_2, \dots, v_n be the sequence of vertices such that

- $\text{edg}T_1 = \{e_1\}$
- $\text{vert}T_1 = \{v_0, v_1\}$
- $\text{edg}T_{i+1} = \text{edg}T_i \cup \{e_{i+1}\}, \quad 1 \leq i \leq n-1.$
- $\text{vert}T_{i+1} = \text{vert}T_i \cup \{v_{i+1}\} \quad 1 \leq i \leq n-1.$

Let

$$G_{T_0} = G_{v_0},$$

and

$$G_{T_i} = G_{T_{i-1}} *_{G_{e_i}} G_{v_i}$$

$$1 \leq i \leq n$$

$G_T = G_{T_n}$ is called the amalgamation G_v 's along G_e 's and is independent of the choice of the sequence T_i and depends only on (G, T) and there is an inclusion

$$\varphi_k : G_k \rightarrow G_T$$

where k is a vertex or an edge.

The following lemma is a generalization of theorem 3.

Lemma 5 Let (G, T) be a tree of groups and v_0 a vertex of T . Consider the construction above. Suppose that $[a]$ and $[b]$ are two distinct conjugacy

classes of G_{v_0} and a is not conjugate in G_{v_0} to an element of any the edge group $G_e \subseteq G_{v_0}$ where v_0 is a vertex of e . Then $[a]$ and $[b]$ as the conjugacy classes of G_T are distinct.

Proof. T_i 's, v_i 's and e_i 's are as above.

We prove the lemma by induction. We prove that for every i , $1 \leq i \leq n$

- (1) $[a]$ and $[b]$ are distinct as conjugacy classes of G_{T_i} .
- (2) a is not conjugate in G_{T_i} to an element in the vertex group G_w , where $w \neq v_0$ is a vertex of T_i .

By the assumption and part (iii) of theorem 3 it follows that that the statement of induction is true. For the second part, a is conjugate in G_{T_1} to an element in G_{v_1} , then again by the third part of theorem 3, a has to be conjugate to an element of G_{e_1} and then by part (ii) it has to be conjugate in G_{v_0} to an element of the vertex group G_{e_1} which contradicts our assumption.

Suppose that the statement is true for i . By the second statement of the induction for i , a is not conjugate in G_{T_i} to an element of $G_{e_{i+1}} \subseteq G_{T_i}$. Therefore by part (iii), a and b represent different conjugacy classes in $G_{T_{i+1}}$ as

$$G_{T_{i+1}} = G_{T_i} *_{G_{e_{i+1}}} G_{v_{i+1}},$$

proving the first statement of the induction for $i + 1$. To prove the second statement for $i + 1$, suppose that by contrary a is conjugate to an element of G_w , where $w \neq v_0$ and $w \in T_{i+1}$.

Since

$$G_{T_{i+1}} = G_{T_i} *_{G_{e_{i+1}}} G_{v_{i+1}},$$

and a is not conjugate in G_{T_i} to an element of $G_{e_{i+1}}$, then again by part (iii) of theorem 3 we must have $w \in T_i$ and a is conjugate in G_{T_i} to an element of G_w . This contradicts the assumption of the induction for i . Hence the second statement of the induction holds for $i + 1$.

Finally for $i = n$ we get the statement of the lemma. □

Appendix III: Finite Covers of Certain Seifert Manifolds with Incompressible Boundary

This appendix is devoted to explaining some facts that are used in the proof of main theorem, section 9. We follow the convention introduced section 8.

Construction 1: *Suppose M is a compact orientable Seifert manifold with $g = 0$, $p = 1$ and $b = 2$ and where the singular fiber is of type $(2, 1)$. Then M has a double cover M' with $g' = 0$, $p' = 0$ and $b' = 3$ boundary components and two of the boundary components are identified under the covering map :*

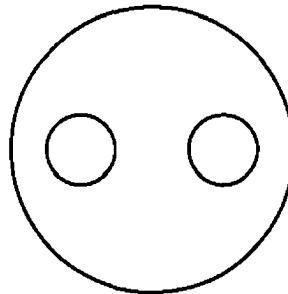


Figure 10: N

Consider $D \subset \mathbb{R}^2$ the unit disk centered at the origin and F a union of two open disks of radius $1/3$ centered at $(1/2, 0)$ and $(-1/2, 0)$ (Figure 10).

Let $M' = N \times \mathbb{S}^1$ where

$$N = D \setminus F.$$

\mathbb{Z}_2 acts on M' freely where the action on both factors is realized by 180° rotation.

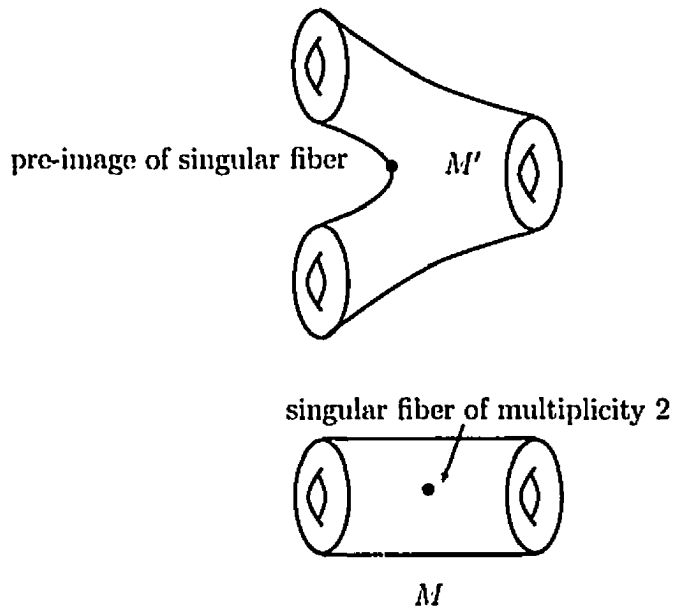


Figure 11: Double cover M' with 3 boundary components

Indeed,

$$M = M'/\mathbb{Z}^2$$

and M' is the desired double covering (Figure 11). M' has 3 boundary components, of which two are identified under the covering map. \square

Construction 2: Let M be a compact orientable Seifert manifold with $g = 0$, $b = 1$ and $p = 2$ singular fibers of type $(2, 1)$ and (r, s) . Then M has

a double cover M' with $g' = 0$, $b' = 1$ and $p' = 2$ and the singular fibers are of type (r, s) :

Let N be the 2-sphere $S^2 \setminus D \subset \mathbb{R}^3$, where D is a set of n open disks located on the equator in xy -plane which is invariant under $2\pi/s$ rotation around z -axis and 180° rotation; and If s is even then D does not meet the x axis otherwise D meets the x -axis exactly at one point.

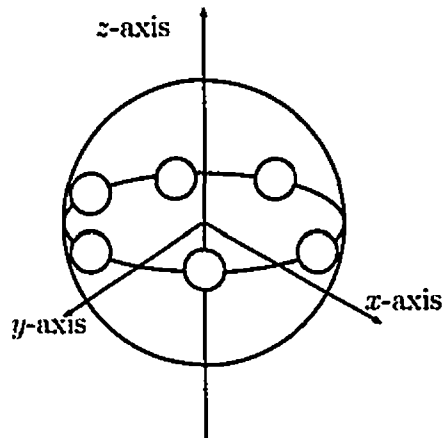


Figure 12: Double cover \widetilde{M}

The $2s$ -Dihedral group,

$$D_{2s} = \langle a, b \mid a^2 = 1, b^s = 1, aba = b^{-1} \rangle$$

acts on $M' = N \times \mathbb{S}^1$ freely (see Figure 12);

- (i) a acts by 180° rotation about x -axis in the first factor and 180° rotation in the second factor.

(ii) b acts by $2\pi/s$ rotation in the first factor and $2r\pi/s$ rotation in the second factor.

Let $M' = N / \langle b \rangle$, then M' is a Seifert manifold with two singular fibers of type (r, s) and one boundary component and base surface a disk. $\langle a \rangle$ acts on M' freely since $\langle b \rangle$ is normal subgroup of D_{2s} . Indeed,

$$M = M' / \langle a \rangle$$

Construction 3: Let M be a compact Seifert manifold with $g = 0$, $b = 1$ and $p = 2$ singular fibers of type $(2, 1)$. Then M has a double cover M' with $g' = 0$, $b' = 2$ and $p' = 0$ and the boundary components of M are identified homeomorphically under the covering map:

Consider the annulus $C = S^1 \times [-1, 1] \subset \mathbb{R}^3$, where S^1 is the unit circle in (x, y) plane and centered at the origin (Figure 13).

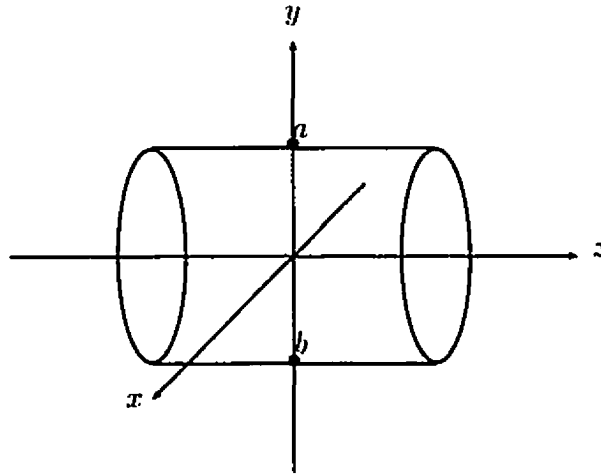


Figure 13: Annulus $C \subset \mathbb{R}^3$

\mathbb{Z}_2 acts on C with a and b as the only fixed points. The action is realized by 180° rotation about y axis.

\mathbb{Z}_2 acts on

$$M' = C \times \mathbb{S}^1$$

freely where the action on the first factor as described above and on the second factor is rotation by 180° . Indeed,

$$M \simeq M'/\mathbb{Z}_2$$

and the boundary components of M' are identified under homeomorphically.

Lemma 1 *Let M be a closed oriented 3-manifold and T be the collection of tori provided by torus decomposition of M . Suppose that $M \setminus T = M_1 \cup M_2 \cup \dots \cup M_n$ with all M_i 's Seifert manifold such that:*

- (1) $n \geq 3$ and $g_i = 0$ for all i 's;
- (2) $b_1 = b_n = 1$ and $b_i = 2$ for $i \neq 1, n$;
- (3) $p_1 = 2$, $A_1 = \{2, r\}$;
- (4) $p_2 = 1$ and $A_2 = \{2\}$;

Then M has a double cover M whose torus decomposition has a Seifert component with 3 boundary components.

Proof. Let M'_1 be the double cover of \overline{M}_1 provided by Construction 2 and M'_2 be the double cover of M_2 by Construction 1. M'_1 has one boundary S component with an induced involution by the covering map. M'_2 has 3

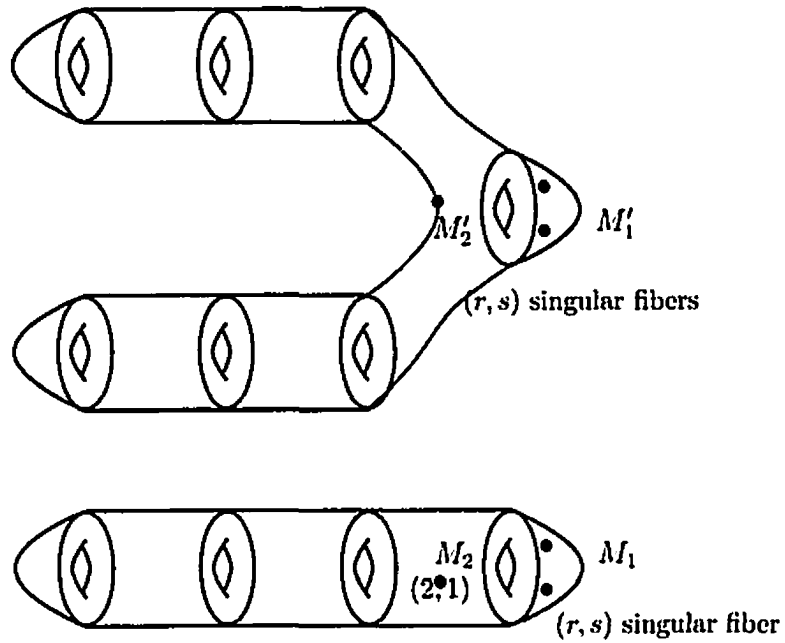


Figure 14: Double cover \widetilde{M}

boundary components T_1 , T_2 and T_3 where T_2 and T_3 are identified under the covering map and T_1 has an induced involution.

Glue M'_1 and M'_2 along along S and T_1 and then two copies of $\overline{M}_3 \cup \overline{M}_4 \cdots \overline{M}_n$ to $M'_1 \cup_{S=T_1} M'_2$ along T_2 and T_3 in such a way that the result is a covering of M . This is the double covering which has a Seifert component with two singular fibers of multiplicity r (Figure 14). \square

Lemma 2 *Let M be a closed oriented 3-manifold and T be the collection of tori provided by torus decomposition of M . Suppose that $M \setminus T = M_1 \cup M_2$ with all M_i 's Seifert manifold such that:*

- (1) $b_1 = 1$, $p_1 = 2$ and $A_1 = \{2, r\}$;

(2) $b_2 = 1$, $p_2 = 2$ and $A_2 = \{2, s\}$.

Then M has a double cover whose torus decomposition 2 Seifert components with 2 singular fibers of multiplicity r and s .

Proof. Let M'_1 and M'_2 be 2-covers of \overline{M}_1 and \overline{M}_2 provided by Construction 2. M'_1 and M'_2 have one boundary component where the covering map induces an involution on the boundary. Glue M'_1 and M'_2 along the boundary in such a way that the result is a double cover of M . This is the desired 2-cover. \square

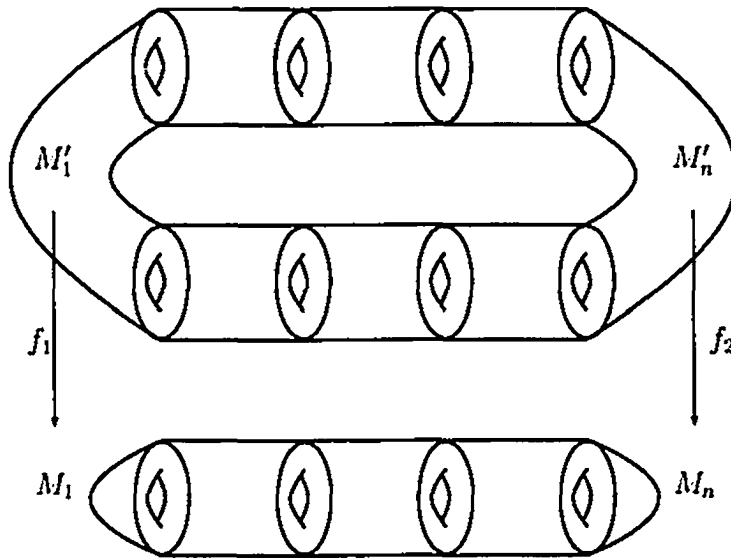


Figure 15: Double cover \widetilde{M} with a non separating torus

Lemma 3 Let M be a closed oriented 3-manifold and T be the collection of tori provided by torus decomposition of M . Suppose that $M \setminus T = M_1 \cup M_2 \cup \dots \cup M_n$ with all M_i 's Seifert manifold; and

(1) $b_1 = b_n = 1; b_i = 2$ for $i \neq 1, n$;

(2) $p_1 = p_n = 2$;

(3) $A_1 = A_n = \{2\}$.

Then M has a double cover with a non separating torus.

Proof. The desired 2-cover of M is constructed as follows: Let $M'_1 \xrightarrow{f_1} \overline{M}_1$ and $M'_n \xrightarrow{f_2} \overline{M}_n$ be the double covers of \overline{M}_1 and \overline{M}_n provided by Construction 3. Consider two copies of $M \setminus (M_1 \cup M_n)$ and glued them to $M'_1 \cup M'_n$ so that the result \widetilde{M} double-covers M (Figure 15) in such a ways that covering map restricted to M'_1 and M'_n is f_1 and f_2 and on the rest of \widetilde{M} it is the trivial covering map. This can be done since the covering map f_1 (f_n) identifies different components of M'_1 (resp. M'_n). \square

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