77-20,497

SCHOLNICK, Allen, 1947-
REPRESENTABILITY AND LOCALIZATION THEOREMS.

City University of New York, Ph.D., 1977
Mathematics

Xerox University Microfilms, Ann Arbor, Michigan 48106

REPRESENTABILITY AND LOCALIZATION THEOREMS

by

ALLEN SCHOLNICK

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1977

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

18 May 1977
date

CHAIRMAN OF EXAMINING COMMITTEE

date

Alphonse Thomas Vasquez
EXECUTIVE OFFICER

Eldon Dyer

Alphonse Thomas Vasquez
Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

ABSTRACT

The Representability and Adjoint Functor Theorems of Freyd [10,12] have proved to be useful in the functorial construction of objects in complete (cocomplete) categories that satisfy prescribed universal mapping properties. Brown's Representability Theorem [4] yields necessary and sufficient conditions for the representability of contravariant functors on the category $hoCW$ which is badly behaved with respect to limits. Brown's theorem was the first to study representability in a category lacking limits.

Recently there have appeared several constructions [1,2,3] which yield adjoint functors. But there are no adjoint functor theorems to explain their existence.

In I. generalizations of the theorems of Freyd are proved.

II. is devoted to applications of I. In particular the h_*, HZ and HR-localizations of Bousfield are shown (by adjoint functor methods) to exist. Furthermore the existence of the HZ and HR-localizations are proved

independently of the localizations of spaces. Also an adjoint functor approach to the $+$-construction of Quillen is given here.

In III. the notion of a localization is discussed in detail. With the use of Theorem 1.9. of I a generalization of the results of Deleanu [5,6] are proved.

To

MOM and DAD

ACKNOWLEDGEMENTS

I would like to express my gratitude to my family for the encouragement they gave me during the preparation of this work.

To my wife, Faby, for putting up with my unfavorable moods and my mother for too many things to mention here.

Also I would like to thank all of my past instructors, in particular Professor Joseph Roitberg for introducing me to localization theory.

Foremost, my thanks to Professor Alex Heller, for his concerned interest and without whom this work would have been impossible.

TABLE OF CONTENTS

Table of Contents (continued) Page Number

§0. Preliminaries.

The following is a summary of notations and some basic facts to be used. Whenever possible references will be made to text books.

0.1. Category Theory [10,12].

(i). Representability.

If C is a category denote the set of morphisms from X to Y by C(X,Y). Then C(X,_),C(_,Y) will be the representable functors determined by X and Y. A functor naturally isomorphic to C(X,_) or C(_,Y) is termed representable.

Lemma 0.1. (Yoneda)

Let F:C→Sets (the category of sets) be a functor and X an object of C. Then the map $\xi \mapsto \xi_X(Id_X)$ from Nat(C(_,X),F) (the class of natural transformations from C(_,X) to F) to F(X) is a bijection.

(ii). Adjoint functors.

Definition 0.2

One says that the pair of functors $F:C\longrightarrow D$, $G:D\longrightarrow C$ are an adjoint pair if there exists a natural isomorphism θ: $(F\_,\_)\longrightarrow C(\_,G\_)$. This is written $F \dashv G$ and one says that F is left adjoint to G (or G is right adjoint to F). The isomorphism will be called the adjunction and if we wish to stress θ one writes $F \xrightarrow{\theta} G$.

By Yoneda's lemma F has a right adjoint if and only if $D(F\_,X)$ is representable and similarly for G.

An adjunction determines isomorphisms

$$\theta_{X,FX}: D(F(X),F(X)) \to C(X,GFX)$$
$$\theta_{GX,X}: D(FGX,X) \to C(GX,GX)$$

which induce natural transformations

called resp. the unit and counit of the adjunction. Moreover, both the following composites are the identities (of G, resp. F)

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G \;,\; F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

Theorem 0.3. 10

Each adjunction $F \overset{\theta}{\longrightarrow\!\!\!\!|}\, G$ is completely determined by the items in any of the following lists:

(i) Functors F,G and a natural transformation $\eta : Id_C \longrightarrow GF$ such that each $\eta_X : X \longrightarrow GFX$ is universal from G to X.

(ii) The functor $G : D \longrightarrow C$ and for each X of C an object $F_0 X$ of D and a universal map $\eta_X : X \longrightarrow GF_0 X$. Then the functor F has object function F and is defined on morphisms $h : X \longrightarrow X'$ by $GFh \cdot \eta_X = \eta_{X'} \cdot h$

(iii) Functors F,G and a natural transformation such that $\varepsilon_Y : FGY \longrightarrow Y$ is universal from F to Y.

(iv) The functor $F : C \longrightarrow D$ and for each $Y \in obD$ an object $G_0 Y \in obC$ and a morphism $\varepsilon_Y : FG_0 Y \longrightarrow Y$ universal from F to Y.

(v) Functors F,G and natural transformations $\eta : Id_C \longrightarrow GF$, $\varepsilon : FG \longrightarrow Id_D$ such that the composites (1) are identities.

0.2. Compactly generated housdorff spaces and CW complexes [9,13]

One calls a topological space compactly generated hausdorff if:

(i) The space is hausdorff.

(ii) $C \subseteq X$ is closed if $C \cap K$ is closed for each compact $K \subseteq X$

The category of compactly generated hausdorff spaces will be denoted by $\mathcal{T}$ Note that $\mathcal{T}$ has an internal hom functor $\overline{\mathbb{T}}$: $\mathcal{T}^{op} \times \mathcal{T} \longrightarrow \mathcal{T}$ i.e. $\mathcal{T}$ is cartesian closed.

<u>Lemma 0.4.</u>

$\mathcal{T}$ is complete and cocomplete.

<u>Lemma 0.5.</u>

(i) $\overline{\mathbb{T}}(X, \overline{\mathbb{T}}(Y, \ ))$ is naturally isomorphic to $\overline{\mathbb{T}}(X \times Y, Z)$.

(ii) If $*$ is the space with one point, then $\overline{\mathbb{T}}(*, X)$ is naturally isomorphic to X.

Given $\mathcal{T}$ we can also consider the pointed category formed from $\mathcal{T}$ where we consider spaces equipped with a good (i.e. cofibred) base point and based maps. In either of these categories one has a homotopy congruence. Thus one may consider the categories $\underline{H \bullet \mathcal{T}}$ and $\underline{h o \mathcal{T}}$ which are respectively the unpointed and pointed homotopy categories.

Denote by $\underline{CW}$ the category of cell complexes and

$\underline{iCW}$ the subcategory consisting of those morphisms which are inclusions of subcomplexes (cell complexes come with a fixed cellular structure and all maps are to be cellular). One also has the unpointed and pointed categories of CW complexes (where the basepoint is assumed to be a vertex) and their respective homotopy congruences. Moreover, $HoCW \overset{HoJ}{\hookrightarrow} HoT$ and $hoCW \overset{hoJ}{\hookrightarrow} hoT$ as full subcategories.

If one restricts attention to connected CW complexes, then those compactly generated spaces isomorphic in hoT to $hoJK$ (K connected) form a category $\underline{ho\mathcal{J}}$. $\mathcal{J}$ is <u>enriched</u> over T i.e. one has $\mathcal{J}: \mathcal{J}^{op} \times \mathcal{J} \to T$ (we are considering everything now as being pointed).

0.3. <u>Homology theory [7,11]</u>.

A <u>homology theory</u> will be a sequence of functors defined on pairs of CW complexes which satisfies the Eilenberg-Steenrod Axioms (except perhaps for the dimension axiom). In addition we shall assume that h_* satisfies Milnor's Axiom.

We shall need the theory h_* to be extended to objects other than CW complexes. In this case we consider our theory as being defined by taking the singular extension of one defined on $\mathcal{CW}$. The extended theory will again be denoted by h_*.

A reduced homology theory is defined only for pairs $(X, *)$. There is a well known process for constructing a reduced homology theory from a homology theory and vice versa. Again we will use h_* to denote either of these.

§ I. Representability and Adjoint Functors.

1.1. Definition of the embedding and basic properties.

Let H be a category. We will denote by $[\mathfrak{2},H]$ the category of functors from $\mathfrak{2}$ to H where $\mathfrak{2}$ is the category with two objects and one morphism between them.

Denote by $\hat{H}$ the category obtained by introducing the following equivalence relation. Given two morphisms $f=(f_1,f_2)$ and $g=(g_1,g_2)$

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\alpha"'] & X_2 \arrow[d, "\beta"] \\ Y_1 \arrow[r, "f_2"] & Y_2 \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "g_1"] \arrow[d, "\alpha"'] & X_2 \arrow[d, "\beta"] \\ Y_1 \arrow[r, "g_2"] & Y_2 \end{tikzcd}$$

will say that $f \sim g$ if $\beta f_1 = \beta g_1$ Notice that H is a full subcategory of $\hat{H}$ via the embedding $X \mapsto Id_X$

Lemma 1.1

(i) If H has products, then $\hat{H}$ has products.

(ii) If H has products and weak pullbacks, then $\widehat{H}$ has equalizers.

<u>proof</u>

(i) is trivial.

For (ii) let $f, g : (X_1 \xrightarrow{\alpha} Y_1) \longrightarrow (X_2 \xrightarrow{\beta} Y_2)$. Since H has weak pullbacks and products it has weak equalizers. Consider the weak equalizer,

$$\begin{tikzcd} X_3 \arrow[r, "h_1"] & X_1 \arrow[r, shift left, "\beta f_1"] \arrow[r, shift right, "\beta g_1"'] & Y_2 \end{tikzcd}$$

We will show that $h = (h_1, Id_{Y_1})$

$$\begin{tikzcd} X_3 \arrow[r, "h_1"] \arrow[d, "\alpha h_1"] & X_1 \arrow[d, "\alpha"] \\ Y_1 \arrow[r, "Id_{Y_1}"] & Y_1 \end{tikzcd}$$

is an equalizer for f and g. Since morphisms of the form (k, Id) are monomorphisms we need only show that every equalizer $h' = (h'_1, h'_2)$,

$$\begin{tikzcd} X_4 \arrow[r, "h'_1"] \arrow[d, "\gamma"] & X_1 \arrow[d, "\alpha"] \\ Y_4 \arrow[r, "h'_2"] & Y_1 \end{tikzcd}$$

factors through $h=(h_1, Id_{Y_1})$.

Let j be such that the diagram,

$$\begin{tikzcd} X_3 \arrow[r, "h_1"] & X_1 \arrow[r, shift left, "\beta f_1"] \arrow[r, shift right, "\beta g_1"'] & Y_2 \\ & X_4 \arrow[ul, "j"] \arrow[u, "h_1'"'] & \end{tikzcd}$$

is a factorization of h_1' Since $(\alpha h_1)j = \alpha(h_1 j) = \alpha h_1' = h_2'\gamma$ we see that (j, h_2') yields the required factorization.

Cor. 1.2.

If H has products and weak pullbacks, then $\hat{H}$ is complete.

1.2 Detection of objects of H.

Definition 1.3.

Call $f=(f_1, f_2)$ a *proper epimorphism* if f_1 is an isomorphism.

Definition 1.4.

If $e: X \to X$ and $e^2 = e$, then one calls e an *idempotent*. One says that the *idempotent splits* if there exist Y, f, g such that $f: Y \to X$, $g: X \to Y$ and $fg = e$, gf is an isomorphism.

Lemma 1.5.

If idempotents split in H, then $\alpha \in ob\,\hat{H}$ is an object of H if and only if $\hat{H}(\alpha, \_)$ takes proper epimorphisms into surjections.

proof

Let $\alpha \in ob\,\hat{H}$ be such that $\hat{H}(\alpha, \_)$ takes proper epimorphisms into surjections. Then $\alpha_*: X_1 \to \alpha$

$$\alpha_* = \begin{tikzcd} X_1 \arrow[r, "Id_{X_1}"] \arrow[d, "Id_{X_1}"] & X_1 \arrow[d, "\alpha"] \\ X_1 \arrow[r, "\alpha"] & Y_1 \end{tikzcd}$$

is a proper epimorphism. Then $\hat{H}(\alpha, \alpha_*): \hat{H}(\alpha, X_1) \to \hat{H}(\alpha, \alpha)$ is surjective. Thus there exist $g: \alpha \to X_1$ such that $\alpha_* g = Id_\alpha$. Let $e = g\alpha_*: X_1 \to X_1$, then $e^2 = (g\alpha_*)^2$ $(g\alpha_*)^2 = g(\alpha_* g)\alpha_* = g\alpha_* = e$ Hence e is an idempotent. Therefore there exist Y, h, j such that $j: Y \to X_1$, $h: X_1 \to Y$, $e = jh$ and hj is an isomorphism.

Consider $\alpha_* j : Y \longrightarrow \alpha$ and $hg : \alpha \longrightarrow Y$ Then

$$
\begin{aligned}
(\alpha_* j)(hg) &= \alpha_* e g \\
&= \alpha_* (g \alpha_*) g \\
&= (\alpha_* g)(\alpha_* g) \\
&= Id_\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
(hg)\cdot(\alpha_* j) &= hej \\
&= (hj)(hj) \\
&= (hj)^2
\end{aligned}
$$

which is an isomorphism. Hence $\alpha \approx Y$

Conversly let $X \in ob\,\hat{H}$, $f : \alpha \longrightarrow \beta$ be a proper epimorphism and $g = (g_1, g_2) \in \hat{H}(X,\beta)$. Define $g' : X \to \alpha$ by $g' = (f_1^{-1} g_1, \alpha f_1^{-1} g_1)$ Then

$$
\begin{aligned}
\hat{H}(X,f)\, g' &= (f_1, f_2)(f_1^{-1} g_1, \alpha f_1^{-1} g_1) \\
&= (g_1, f_2 \alpha f_1^{-1} g_1) \\
&= (g_1, \beta g_1) \\
&= (g_1, g_2) \\
&= g
\end{aligned}
$$

and $\hat{H}(X,f)$ is surjective.

1.3. The solution set condition and representability.

Definition 1.6.

One says that a functor $F:H \to$ Sets *satisfies a solution set condition* if there exists a set $\mathcal{A}$ of objects of H with the property that for each $x \in F(X)$ there exist f, with domain in $\mathcal{A}$ and codomain X, such that $x \in \operatorname{Im}(Ff)$.

Given a functor $F:C \to D$ one says that F *satisfies a solution set condition* if for each Y $\mathcal{D}(F(\ ),Y):\mathcal{C} \to$ Sets satisfies one.

We now get the following extension of Freyd's Representability Theorem [10,12].

Theorem (Representability) 1.7.

Let $F:H \to$ Sets be such that

H1- H has products,

H2- idempotents split in H,

H3- H has weak pullbacks,

F1- F preserves products and weak pullbacks,

F2- F satisfies a solution set condition.

Then F is representable.

<u>proof</u>

The idea of the proof is to define an extension $\hat{F}:\hat{H}\to$ Sets in such a way that $\hat{F}$ satisfies the conditions of Freyd's Representability Theorem. Then we show that for the representing object α, $\hat{H}(\alpha,\_)$ takes proper epimorphisms into surjections. Thus α is isomorphic to an object of H.

We define the required extension by $\hat{F}(\alpha)=\mathrm{Im}\,F(\alpha)$ In the category of sets image commutes with product, hence one sees that $\hat{F}$ preserves products. Consider a diagram of equalizers in $\hat{H}$

(1)
$$\begin{tikzcd}
X_3 \arrow[r, "h_1"] \arrow[d, "\alpha h_1"] & X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] \arrow[d, "\alpha"] & X_2 \arrow[d, "\beta"] \\
Y_1 \arrow[r, "\mathrm{Id}\,Y_1"] & Y_1 \arrow[r, shift left, "f_2"] \arrow[r, shift right, "g_2"'] & Y_2
\end{tikzcd}$$

where

$$\begin{tikzcd}
X_3 \arrow[r, "h_1"] & X_1 \arrow[r, shift left, "\beta f_1"] \arrow[r, shift right, "\beta g_1"'] & Y_2
\end{tikzcd}$$

is a weak equalizer in H. Application of F to diagram (1)

assures us that $\hat{F}(h_1\alpha) \longrightarrow \hat{F}(\alpha)$ is an injection.
Thus we need only show that

$$\begin{tikzcd} \hat{F}(\alpha h_1) \arrow[r] & \hat{F}(\alpha) \arrow[r, shift left, "\hat{F}(f)"] \arrow[r, shift right, "\hat{F}(g)"'] & \hat{F}(\beta) \end{tikzcd}$$

is a weak equalizer. Since in the category of sets we need only show this pointwise, let $x \in \hat{F}(\alpha)$ such that $\hat{F}(f)x = \hat{F}(g)x$. Let $x' \in F(X_1)$ such that $F(\alpha)x' = x$, then we see that $F(\beta f_1)x' = \hat{F}(f)x = \hat{F}(g)x = F(\beta g_1)x'$. Thus there exist $x'' \in F(X_3)$ such that $F(h_1)x'' = x'$. However commutivity of the diagram

$$\begin{tikzcd} F(X_3) \arrow[r] \arrow[d] & F(X_1) \arrow[d] \\ \hat{F}(\alpha h_1) \arrow[r] & \hat{F}(\alpha) \end{tikzcd}$$

assures us that $F(\alpha h_1)x'' = x$ lies in $\hat{F}(\alpha h_1)$

The solution set condition for $\hat{F}$ follows immediately from that of F. Therefore $\hat{F}$ is represented by some $\gamma \in ob\,\hat{F}$ However by its definition we see that $\hat{F}$ takes proper epimorphisms into surjections. Thus γ is isomorphic to an object of H.

In complete categories idempotents split, the splitting being given by the equalizer f of e and Id_X

$$
\begin{tikzcd}
Y \arrow[r, "f"] & X \arrow[r, shift left, "e"] \arrow[r, shift right, "Id_X"'] & X \\
 & X \arrow[u, "e"'] \arrow[ul, "g"] &
\end{tikzcd}
$$

where g is the unique map which makes the triangle commute. Therefore we get the following corollary.

<u>Cor 1.8.</u>

Let $F:H \to$ Sets be such that,

(i) H is complete,

(ii) F preserves products,

(iii) F takes pullbacks to weak pullbacks,

(iv) F satisfies a solution set condition.

Then F is representable.

I shall now prove a generalization of the adjoint functor Theorem [10,12].

Theorem 1.9.

Let H be a category such that,

(i) H has products,

(ii) idempotents split in H,

(iii) H has weak pullbacks.

Then $F: H \longrightarrow C$ has a left adjoint G if and only if,

(i) F preserves products,

(ii) F preserves weak pullbacks,

(iii) F satisfies a solution set condition.

proof

The necessity of the conditions follows immediately from the adjunction $H(G(\_),\_) \approx C(\_, F(\_))$

To show sufficiency consider the functor $C(X, F(\_))$ from H to Set defined for any $X \in ob\, C$. The hypotheses then assure us that $C(X, F(\_))$ satisfies the hypotheses of Theorem 1.7. and therefore is represented by an object GX

Yoneda's Lemma assures us that G is a functor.

Cor 1.10.

Let H be a complete category and $F:H \longrightarrow \mathcal{C}$ Then F has a left adjoint if and only if,

(i) F preserves products,

(ii) F takes pullbacks to weak pullbacks,

(iii) F satisfies a solution set condition.

1.4. The dual of 1.3.

Since a map e is an idempotent in H (and splits) if and only if e^{op} is an idempotent in H^{op} (and splits) the results of 1.3. dualize.

Theorem D 1.7.

Let $F:H \to$ Sets be a contravariant functor such that,

DH1- H has coproducts,

DH2- idempotents split

DH3- H has weak pushouts,

DF1- F takes coproducts to products,

DF2- F takes weak pushouts to weak pullbacks,

DF3- F satisfies a solution set condition.

Then F is representable.

Cor D 1.8.

Let $F:H \to Sets$ be contravariant such that,

(i) H is cocomplete,

(ii) F takes coproducts to products,

(iii) F takes pushouts to weak pullbacks,

(iv) F satisfies a solution set condition.

Then F is representable.

Theorem D 1.9.

Let H be a category such that,

(i) H has coproducts,

(ii) idempotents split in H,

(iii) H has weak pushouts,

Then $F:H \to \mathcal{C}$ has a right adjoint if and only if,

(i) F preserves coproducts,

(ii) F preserves weak pushouts,

(iii) F satisfies a solution set condition.

Cor D 1.10.

Let H be a complete category and $F:H \to \mathcal{C}$ Then F has a right adjoint if and only if,

(i) F preserves products,

(ii) F takes pullbacks to weak pullbacks,

(iii) F satisfies a solution set condition.

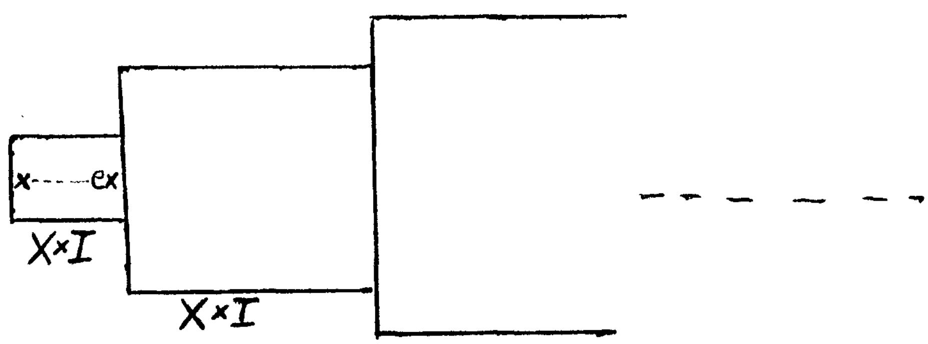
1.5. Primary example.

Consider the category $hoCW$ It has coproducts, products, weak pushouts and weak pullbacks. Products and coproducts are formed in the obvious manner. To form weak pushouts (weak pullbacks) one must first turn one of the maps into a cofibration (fibration) and then form the usual pushout (pullback) in spaces.

Furthermore idempotents split in $hoCW$ For let e be an idempotent and H: $e \sim e^2$ be a homotopy between e and e^2 Consider the sequence of maps

$$X \xrightarrow{e} X \xrightarrow{e^2} X \xrightarrow{e^3} X \rightarrow \cdots\cdots$$

and form the Milnor Telescope $Tel(e)$ on this sequence



where $Tel(e) = \coprod_{n=0}^{\infty} X \times [n,n+1] / \sim$ where $(x,n+1) \in X \times [n,n+1]$ is identified with $(e^{n+1}x, n+1) \in [n+1, n+2]$. We have a map $i: X \to Tel(e)$ induced by the inclusion $X \times 0 \hookrightarrow X \times [0,1]$ into the left side. Furthermore H defines a map $Tel(e) \xrightarrow{f} X$ defined by $f(x,\ell) = e^{n+1}H(x,\ell-n)$ where $(x,\ell) \in X \times [n,n+1]$ Clearly $fi = e: X \to X$ However $if: Tel(e) \to Tele$ is an isomorphism since

$$\pi_k(if) = \operatorname*{colim}_{n \to \infty} \pi_k(e^{n+1})$$

which is clearly an isomorphism because

$$\pi_k(Tel(e)) \approx Im\, \pi_k(e).$$

1.6. <u>A special adjoint functor theorem.</u>

In general the solution set condition is difficult to verify. The following theorem shows that often it need not be verified. We therefore get a sort of "Special Adjoint Functor Theorem" 10 which when applied to specializes to Brown's Representability Theorem 4 .

<u>Definition 1.11.</u>

An object X of a category $\mathcal{C}$ is called <u>left (right) adequate</u> if whenever $\mathcal{C}(X,f)$ is an isomorphism, f is an isomorphism.

example -1.

$$\bigvee_{n=1}^{\infty} S^n \text{ is left adequate in } \mathit{hoCW}$$

example -2.

$\mathbb{Z}$ is left adequate in Groups and Abelian Groups.

Let γ be an ordinal number. Then consider γ as a category with objects γ' such that $\gamma' < \gamma$ and let the ordering induce the morphisms.

Definition 1.12.

An object X of $\mathcal{C}$ will be called $\mathcal{S}$-definite if there exists an ordinal β such that for every functor $F: \beta \longrightarrow \mathcal{C}$ one can find a weak colimit N of F with the property that $\operatorname{colim}_{\beta' < \beta} \mathcal{C}(X, F\beta') \cong \mathcal{C}(X, N)$, Where the isomorphism is induced by $\mathcal{C}(X, i\beta')$ and $i\beta' : F\beta' \longrightarrow N$ is a structure map.

example -3.

All objects of hoCW are $\mathcal{S}$-definite.

Theorem 1.13.

Let C be a category such that,

(i) C has coproducts,

(ii) C has weak pushouts,

(iii) There exists a left adequate, $\mathcal{S}$-definite object A of C.

Suppose that $F:C\longrightarrow Set^{\bullet}$ is a contravariant functor (where $Set^{\bullet}$ is the category of pointed sets) such that,

(i) $F(\amalg X_\alpha)=\prod F(X_\alpha)$,

(ii) F takes weak pushouts to weak pullbacks,

(iii) for each functor $\mathfrak{X}:\beta\longrightarrow C$ (where β is an ordinal) $F(L)\longrightarrow \varprojlim_{\beta'<\beta} F(\mathfrak{X}_{\beta'})$ is surjective where L is a weak colimit of F.

(Note that if a category has coproducts and weak pushouts, then it has weak colimits. If $F(L)\rightarrow \varprojlim_{\beta'<\beta} F(\mathfrak{X}_{\beta'})$ then it is surjective for any weak colimit of F.)

proof

The proof of Theorem 1.13. will follow immediately from Lemmas 1.14. and 1.15.

<u>Lemma 1.14.</u>

If Y is an object of $\mathcal{C}$ and $y \in F(Y)$ then there exists $f: Y \longrightarrow \tilde{Y}$ and $\tilde{y} \in F(\tilde{Y})$ such that,

(i) $\tilde{y}_*: [A, \tilde{Y}] \longrightarrow F(A)$, is an isomorphism, where $\tilde{y}_*(g) = F(g)\tilde{y}$ is the Yoneda transformation induced by $\tilde{y}$,

(ii) $F(f)\tilde{y} = y$.

<u>proof of Lemma 1.14.</u>

Let $Y^0 = Y$, $y^0 = y$ and $Y^1 = Y^0 \amalg (\coprod_{x \in FA} A)$ There exists $y^1 \in F(Y^1)$ such that $F({}_1f_0)y^1 = y^0$ (where ${}_1f_0: Y^0 \longrightarrow Y^1$ is the canonical injection) and y^1_* from $[A, Y^1]$ to $F(A)$ is surjective. Construct Y^2 by requiring that

$$\coprod_{\substack{(h_1, h_2),\\ h_i: A \to Y^1\\ F(h_1)y^1 = F(h_2)y^1}} A \;\underset{\amalg h_2}{\overset{\amalg h_1}{\rightrightarrows}}\; Y^1 \longrightarrow Y^2$$

be a weak coequalizer.

Since F takes coproducts to products and weak pushouts to pullbacks it takes weak colimits to weak limits. Furthermore $F(\coprod h_1)y' = F(\coprod h_2)y'$. Therefore there exists $y^2 \in F(Y^2)$ such that $F({}_2f_1)y^2 = y^1$ Define inductively $Y^{\beta'}$ and $y^{\beta'} \in F(Y^{\beta'})$ such that $F({}_{\beta''}f_{\beta'})y^{\beta''} = y^{\beta'}$ for $\beta' < \beta'' < \beta$ by requiring

$$\coprod_{\substack{(h_1, h_2) \\ h_i : A \to Y^{\beta'} \\ F(h_1)y^{\beta'}}} A \;\underset{\coprod h_2}{\overset{\coprod h_1}{\rightrightarrows}}\; Y^{\beta'} \xrightarrow{\;{}_{\beta'+1}f_{\beta'}\;} Y^{\beta'+1}$$

to be a weak coequalizer for nonlimit ordinals $\beta'+1$. If β' is a limit ordinal let $Y^{\beta'}$ be a weak colimit of $\{{}_{\beta'''}f_{\beta''}\}_{\beta''<\beta'''<\beta'}$ and $y^{\beta'} \in F(Y^{\beta'})$ an element such that $F({}_{\beta'}f_{\beta''})y^{\beta'} = y^{\beta''}$.

Let $Y : \beta \longrightarrow C$ be the functor constructed and $\tilde{Y}$ a weak colimit of Y such that $C(A, \tilde{Y}) \simeq \operatorname*{colim}_{\beta'<\beta} C(A, Y_{\beta'})$

Take $\tilde{y} \in F(\tilde{Y})$ to be such that $F(f^{\beta'})\tilde{y} = y^{\beta'}$ where $f^{\beta'} : Y^{\beta'} \to \tilde{Y}$ is a structure map. Taking $f = f^0$ we see that $F(f)\tilde{y} = y$ Comutivity of the diagram

$$\begin{tikzcd}
{[A, Y']} \arrow[rr, "{[A, f']}"] \arrow[dr] & & {[A, \tilde{Y}]} \arrow[dl] \\
& F(A) &
\end{tikzcd}$$

assures us that $\tilde{y}_*$ is surjective (since y'_* was). To show injectivity let g_i be such that $\tilde{y}_* g_1 = \tilde{y}_* g_2$. Then there exists an ordinal $\beta' < \beta$ such that $f^{\beta'} g_i^{\beta'} = g_i$, $i = 1, 2$. But then ${}_{\beta'+1}f_{\beta'}\, g_1^{\beta'} = {}_{\beta'+1}f_{\beta'}\, g_2^{\beta'}$ and hence $g_1 = f^{\beta'+1}\,{}_{\beta'+1}f_{\beta'}\, g_1 = f^{\beta'+1}\,{}_{\beta'+1}f_{\beta'}\, g_2 = g_2$.

<u>Lemma 1.15.</u>

If $Y \in ob\, C$ and $y \in F(Y)$ are such that $y_* : [A, Y] \to F(A)$ is an isomorphism, then $y_* : [-, Y] \to F(\_)$ is an isomorphism of functors.

<u>proof</u>

First we will prove surjectivity. Let $X \in ob\, C$ and $x \in F(X)$ Consider $Y \to X \amalg Y$. By Lemma 1.13 there exists Y', $y' \in F(Y')$, $f: X \amalg Y \to Y'$ such that $F(f)y' = (x, y)$ and $y_*: [N, Y'] \to F(N)$ is an isomorphism. On considering the commutative diagram

$$\begin{tikzcd}
C(A,Y) \arrow[r] \arrow[dr, "y_*"'] & C(A, X \amalg Y) \arrow[r] & C(A,Y') \arrow[dl, "y'_*"] \\
& F(A) &
\end{tikzcd}$$

one observes that $Y \xrightarrow{j} X \amalg Y \xrightarrow{f} Y'$ is an isomorphism. Let g be an inverse of this isomorphism. Thus $F(g)y = y'$ and $F(fi)y' = x$ where $i: X \to X \amalg Y$. Therefore $F(gfi)y = x$ and hence y_* is surjective.

To show injectivity suppose that $f_0, f_1 \in C(X,Y)$ are such that $y_* f_0 = y_* f_1$, i.e. $F(f_0)y = F(f_1)y$. Consider the weak coequalizer

$$\begin{tikzcd}
X \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & Y \arrow[r] & Z
\end{tikzcd}$$

Then

$$F(X) \underset{F(f_1)}{\overset{F(f_0)}{\longleftleftarrows}} F(Y) \xleftarrow{F(g)} F(Z)$$

is a weak equalizer and since $C(Z,Y) \to F(Z)$ is surjective there exists $h: Z \to Y$ such that

$$F(g)y_* h = y$$
$$F(g)F(h)y = y$$
$$F(gh)\, y = y$$

Now consider the natural transformation,

$$\begin{tikzcd} C(-,Y) \arrow[r, "{C(-,hg)}"] \arrow[rr, bend right, "\psi"'] & C(-,Y) \arrow[r, "y_*"] & F(-) \end{tikzcd}$$

It is determined by an element of $F(Y)$ and this element is $y_* C(Y,hg) Id_Y = F(hg)y = y$ Therefore $y_* C(-,hg) = y_*$ and hence $y_* C(A,hg) = y_*$ But since y_* is an isomorphism $C(A,hg)$ is too and therefore hg is an isomorphism. However $hgf_0 = hgf_1$ and thus $f_0 = f_1$

Cor 1.16.

Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$ such that,

(i) A is an object of $\mathcal{C}'$

(ii) $\mathcal{C}'$ has products and $J : \mathcal{C}' \hookrightarrow \mathcal{C}$ preserves products,

(iii) $\mathcal{C}'$ has weak pushouts and $J : \mathcal{C}' \hookrightarrow \mathcal{C}$ preserves weak pushouts.

Then

§ II. Applications

2.1. Preliminary remarks.

The following lemma of Bousfield 3 will be useful in verifying the solution set condition. For completeness I have included a proof.

Lemma 2.1.

Let D be a cocomplete category, $T : D \longrightarrow \text{Sets}^{\bullet}$ a functor that preserves small filtered colimits. Suppose that there exists a set of objects K such that every object in D is a small filtered colimit of object in K Let us denote by L the Set of objects X such that,

(i) $T(X) = *$

and (ii) $X = \operatorname{colim}_{I} F(i)$ (where I ranges over those filtered sets such that $\operatorname{card} I \leqslant \alpha$ and α is an infinite cardinal such that for every $Y \in K$ $\operatorname{card} T(Y) \leqslant \alpha$)

Then every object Z Such that $T(Z) = *$ is a small filtered colimit of objects of L.

proof

Let α and L be as above. Let X be such that $T(X) = *$ and $X = \operatorname{colim}_{J} F(j)$ where J is a filtered set and $F(j) \in K$. I claim that, for each subset $G \subseteq J$ such that $\operatorname{card} G \leqslant \alpha$ there exists a filtered subset H of J such that,

(i) $G \subseteq H$

(ii) $\operatorname{card} H \leqslant \alpha$

(iii) $\operatorname{colim}_{H} F(h) = *$

The lemma then follows immediately since $X = \operatorname{colim}_{H \subseteq J} \left(\operatorname{colim}_{h \in H} F(h) \right)$ where H runs over the set $\mathcal{H}$ of filtered subsets of J satisfying $\operatorname{card} H \leqslant \alpha$ and $\operatorname{colim}_{h \in H} F(h) = *$ which, by the claim, is a filtered set.

To show the existence of H I will construct a sequence of filtered subsets $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$, such that

(i) $G \subseteq H_1$

(ii) $\operatorname{card} H_i \leqslant \alpha$

(iii) $\operatorname{colim}_{h \in H_i} F(h) \to \operatorname{colim}_{h \in H_{i+1}} F(h)$ is trivial.

Given these one need only take $H = \bigcup_{i=1}^{\infty} H_i$. Let H_1 = filt G be the smallest filtered subset of J which contains G. Since card $H_1 \leq \alpha$ it follows that card $\operatorname{colim}_{h \in H_1} TF(h) \leq \alpha$. For each element ξ of $\operatorname{colim}_{h \in H_1} F(h)$ choose a representative $x \in F(h_\xi)$ and an element $\bar{h}_\xi \in J$ such that, $h_\xi \leq \bar{h}_\xi$ and $TF(h_\xi) \longrightarrow TF(\bar{h}_\xi)$ takes x to $*$ Let X_2 be the filtered subset generated by $H_1 \cup \{\bar{h}_\xi \mid \xi \in \operatorname{colim}_{h \in H_1} TF(h)\}$ and proceed inductively.

2.2. Primary examples.

Example 1.

Let $T : [2, \text{Groups}] \longrightarrow \text{Sets}^\bullet$ be given by

$T(f) = \ker H_1(f,R) \vee \operatorname{coker} H_1(f,R) \vee \operatorname{coker} H_2(f,R)$

(where the groups in question act trivially on the module R). Since $H_*(\ ,R)$ commutes with filtered colimits and finite limits commute with filtered colimits in the category $\text{Sets}^\bullet$ [10] one sees that T commutes with filtered colimits. Thus, taking $K = [2, \text{f.g.Groups}]$ application of Lemma 2.1. shows that the set L consisting of all morphisms f satisfying,

(i) $H_1(f,R)$ is an isomorphism,

(ii) $H_2(f,R)$ is an epimorphism,

(iii) card codomain $f \leqslant \alpha$,

(iv) card domain $f \leqslant \alpha$,

(where α is an infinite cardinal such that card $R \leqslant \alpha$) has the property that every morphism g which satisfies $H_1(g,R)$ is an isomorphism and $H_2(g,R)$ is an epimorphism, is a small filtered colimit of objects of L.

Example 2.

Let $T: [2, \Pi\text{- Module}] \longrightarrow \text{Sets}^{\bullet}$ (where Π is a group) be given by,

$T(f) = \ker H_0(\Pi, f) \vee \operatorname{coker} H_0(\Pi, f) \vee \operatorname{coker} H_1(\Pi, f)$.

Taking $K = [2, f \cdot g \cdot \Pi - \text{Module}]$ one sees as in example 1 that the set L consisting of those morphisms satisfying

(i) $H_0(\Pi, f)$ is an isomorphism,

(ii) $H_1(\Pi, f)$ is an epimorphism,

(iii) card codomain $f \leqslant \alpha$,

(iv) card domain $f \leqslant \alpha$,

(where α is an infinite cardinal such that card $\Pi \leqslant \alpha$) has the property that every morphism g with $H_0(\Pi, g$) an isomorphism and $H_1(\pi, \mathcal{G})$ an epimorphism is a filtered colimit of objects of $\mathcal{L}$.

Example 3.

Let T: $[2, {}^{\bullet}CW] \longrightarrow$ Sets (see §0 Preliminaries.) be given by $T(f) = \left(\bigvee_{i=-\infty}^{\infty} \ker h_i(f)\right) \vee \left(\bigvee_{i=-\infty}^{\infty} \operatorname{coker} h_i(f)\right)$ where h_* is a reduced homology theory which satisfies the limit axiom. Take $K = [2, \text{finite-}{}^{\bullet}CW]$ and let α be an infinite cardinal number such that card $h_*(S^0) \leq \alpha$. Then Lemma 2.1 assures us of the existence of a set of morphisms such that,

(i) $h_*(f)$ is an isomorphism,

(ii) $\#$ domain $f \leq \alpha$

(iii) $\#$ codomain $f \leq \alpha$

(where $\#$ X is the cardinality of the set of cells of X) with the property that every h_*- isomorphism f of $\mathrm{Hom}({}^{\bullet}CW)$ is a filtered colimit of objects of $\mathcal{L}$.

2.3. Definition of localization.

I will now give independent constructions of the HZ, HR and h_* localizations of Bousfield[2] These constructions will all be made in analogous manners, by application of Theorem 1.9. However before doing so I will give a brief discussion due to Adams and Bousfield [1 2] of the meaning of localization.

Let $\mathcal{C}$ be a category and $\mathcal{S} \subseteq \operatorname{Hom}\mathcal{C}$ a class of morphisms.

<u>Definition 2.2.</u>

An object X of $\mathcal{C}$ will be called <u>$\mathcal{S}$-local</u> if for every $f: Y \longrightarrow Z$ where $f \in \mathcal{S}$ we have $\mathcal{C}(f, X) : \mathcal{C}(Z, X) \longrightarrow \mathcal{C}(Y, X)$ is a bijection. We will denote the full subcategory of $\mathcal{S}$-local objects by $L_{\mathcal{S}}\mathcal{C}$

<u>Definition 2.3.</u>

Let $J: L_{\mathcal{S}}\mathcal{C} \hookrightarrow \mathcal{C}$ be the inclusion functor. We will call $\mathcal{S}$ <u>localizing</u> if there exists $E: \mathcal{C} \to L_{\mathcal{S}}\mathcal{C}$ such that $E \dashv J$ and in that case we will call EX the <u>$\mathcal{S}$-localization of X</u>

<u>Lemma 2.4.</u>

If X is $\mathcal{S}$-local and $\mathcal{S}$ is localizing, then $EX \approx X$ and the isomorphism is given by η_X the unit of the adjunction.

proof

$\eta_X : X \longrightarrow EX$ is characterized by the property that for any $g : X \to Y$ where Y is S-local there exists a unique map $\bar{g} : EX \longrightarrow Y$ such that

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "g"'] & EX \arrow[d, "\bar{g}"] \\ & Y \end{tikzcd}$$

commutes. However $Id_X : X \to X$ satisfies this universal property. Hence Id_X and η_X are isomorphic as maps.

Cor 2.5.

If S is localizing, then we may choose E such that $E \cdot J = Id_{L_S C}$ and $\eta_Y = Id_Y$ for $Y \in ob\, L_S C$

2.4. Calculus of fractions.

I would like to recall the notion (Gabriel-Zisman [12]) that a class of morphisms admits a calculus of left fractions. This notion will aid us in the detection of $\mathcal{S}$-local objects. Later we will see that the properties of admitting a calculus of left fractions and of being localizing are closely related.

Definition 2.6.

In a category $\mathcal{C}$ a class of morphisms $\mathcal{S}$ is said to admit a calculus of left fractions if:

(i) $\mathcal{S}$ is closed under compositions and contains the identities of $\mathcal{C}$.

(ii) Given $X_2 \xleftarrow{s} X_1 \longrightarrow X_3$ a diagram in $\mathcal{C}$ with $s \in \mathcal{S}$ there exists a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "s"] & X_2 \arrow[d, "s'"] \\ X_3 \arrow[r] & X_4 \end{tikzcd}$$

with $s' \in \mathcal{S}$

(iii) Given $X_1 \xrightarrow{s} X_2 \overset{f_0}{\underset{f_1}{\rightrightarrows}} X_3$ such that $f_1 s = f_0 s$ and $s \in \mathcal{S}$ then there exists $X_3 \xrightarrow{s'} X_4$ such that $s' f_1 = s' f_0$.

The following lemma of Bousfield[2] is easily proved.

<u>Lemma 2.7.</u>

If S admits a calculus of left fractions then the following are equivalent.

(i) D is S-local.

(ii) Each morphism $s: X \to Y$ in S induces a surjection $\mathrm{Hom}(s,D): \mathrm{Hom}(Y, D) \to \mathrm{Hom}(X,D)$.

(iii) Each morphism $D \to Y$ in S has a left inverse.

2.5. h_*-Localization of $ho\widetilde{\mathcal{T}}$.

Let h_* be a reduced homology theory [7] which satisfies Milnor's Axiom. Then h_* determines a homology theory on the category of pairs of spaces [7]. We will denote this theory also by h_* Let S be the class of morphisms of $ho\widetilde{\mathcal{T}}$ which are h_* isomorphisms. This class determines the full subcategory $L_S ho\widetilde{\mathcal{T}}$ of S-local spaces. We will use Theorem 1.9. to construct the Bousfield localization[2] corresponding to h_* The difficult part in the application of Theorem 1.9. will be the verification of the solution set condition. For this we will need to use Lemma 2.1.

The technique used to verify the solution set condition for the h_*-localization of spaces will be adaptable to getting the solution set condition for Bousfield's [2] HR-localization of groups and HZ-localization of π-Modules. I shall show that application of Theorem 1.9. (or Freyd's adjoint functor theorem) implies the existence of the HR and HZ localizations.

Before verifying that $L_S(ho\mathcal{T}) \hookrightarrow ho\mathcal{T}$ satisfies the hypothesis of Theorem 1.9. I will need the following lemma of Adams [1].

<u>Lemma 2.8.</u>

Let h_* and S be as above. Then S admits a calculus of left fractions (see definition 2.5.).

<u>proof</u>

(i) is obvious. For (ii) note that by using homotopy equivalent CW complexes and taking mapping cylinders we may assume without loss of generality that

f and S are inclusions of subcomplexes. Taking $X_4 = X_2 \cup X_3$ we get $h_*(X_4, X_3) = h_*(X_4/X_3, *) = h_*(X_2/X_1, *) = 0$ and hence $(X_3 \longrightarrow X_4) \in S$ For (iii) once again we may assume that S is an inclusion of a subcomplex. Then $h_*(Cyl, Spool) = 0$ where

$$Spool = 0 \times X_2 \cup I \times X_1 \cup 1 \times X_2 \big/ I \times *$$

$$Cyl = I \times X_2 \big/ I \times *$$

Take S' to be the right hand vertical map in the pushout diagram

$$\begin{tikzcd}
Spool \arrow[r, "b"] \arrow[d] & X_3 \arrow[d] \\
Cyl \arrow[r] & X_4
\end{tikzcd}$$

where $b/0\times X_2 = f_0$, $b/1\times X_1 = f_1$ and $b/I\times X$ is a homotopy between $f_0 S$ and $f_1 S$

Thus Lemma 2.7. assures us of the following simpler criterion for $X\in ho\mathcal{T}$ to be h_*-local.

Lemma 2.9.

X is h_*-local if for every h_*- isomorphism $ho\mathcal{T}(f,X)$ is surjective.

(i) $L_S(ho\mathcal{T})$ has products.

The usual product in $ho\mathcal{T}$ yields a product in $L_S(ho\mathcal{T})$ since $ho\mathcal{T}(f,\Pi X_\alpha) = \Pi(ho\mathcal{T})(f,X_\alpha)$ is a bijection if $ho\mathcal{T}(f,X_\alpha)$ is.

(ii) $L_S(ho\mathcal{T})$ has weak pullbacks.

In order to show that $L_S(ho\mathcal{T})$ has weak pullbacks we will need the following lemmas.

Lemma 2.10.

Let $f: Y_0 \to Y_1$ be an h_*-isomorphism. Then $\mathcal{U}\times f$ is an h_*-isomorphism (Connectedness assumptions are not needed here.).

<u>proof</u>

We may assume that $\mathcal{U}$ is a CW complex. The argument will proceed inductively on the skeletons of $\mathcal{U}$ Clearly $\Delta^m \times f$ is an h_*-isomorphism. Thus $\mathcal{U}^0 \times f$ is an h_*-isomorphism. Suppose that $\mathcal{U}^n \times f$ is an h_*-isomorphism for all $\mathcal{U}$. From the pushout,

$$\begin{tikzcd}
\coprod \dot{\Delta}^{n+1} \arrow[r] \arrow[d] & \mathcal{U}^n \arrow[d] \\
\coprod \Delta^{n+1} \arrow[r] & \mathcal{U}^{n+1}
\end{tikzcd}$$

We get the pushouts

$$\begin{tikzcd}
\coprod \dot{\Delta}^{n+1} \times Y_i \arrow[r] \arrow[d] & \mathcal{U}^n \times Y_i \arrow[d] \\
\coprod \Delta^{n+1} \times Y_i \arrow[r] & \mathcal{U}^{n+1} \times Y_i
\end{tikzcd}$$

Using the Mayer-Vietoris sequence we see that $\mathcal{U}^{n+1} \times f$ is an h_*- isomorphism. Therefore $h_*(\mathcal{U} \times f) = h_*(\operatorname{colim} (\mathcal{U}^n \times f)) = \operatorname{colim} h_*(\mathcal{U}^n \times f)$ is an isomorphism.

Cor 2.11.

Let f be an h_*- isomorphism. Then $\mathcal{U} \wedge f$ is an h_*- isomorphism (No connectedness assumptions are made here).

proof

Since $\mathcal{U} \times f$ is an h_*- isomorphism the result follows by considering the long exact sequence of the pair $(\mathcal{U} \times Y_i, \mathcal{U} \vee Y_i)$

Lemma 2.12.

If f is an h_*- isomorphism and X is h_*- local, then, $\mathcal{F}(f, X)$ is a homotopy equivalence. (Note that $\mathcal{F}(Y_i, X)$ is the function space of pointed maps).

proof

$(ho\mathcal{T})(\mathcal{U}, \mathcal{F}(f, X))$ is adjoint to

$(ho\mathcal{T})(U \wedge f, X)$ which by Cor 2.11. is a bijection. By Yoneda's lemma $\mathcal{T}(f, X)$ is an isomorphism.

<u>proof that $L_S(ho\mathcal{T})$ has weak pullbacks.</u>

Let $X_1 \xrightarrow{f_1} X_0 \xleftarrow{f_2} X_2$ be a diagram in $L_S(ho\mathcal{T})$. We may assume without loss of generality that f_i are fibrations. Let

$$\begin{tikzcd}
X_3 \arrow[r] \arrow[d] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"] & X_0
\end{tikzcd}$$

be a pullback in $\mathcal{T}$. This will yield a weak pullback in $ho\mathcal{T}$. We need only show that X_3 is h_*- local.

Let $f: Y_0 \rightarrow Y_1$ be an h_*- isomorphism. Then the diagrams

$$\begin{tikzcd} \mathcal{J}(Y_i, X_3) \arrow[r] \arrow[d] & \mathcal{J}(Y_i, X_2) \arrow[d, "\mathcal{J}(Y_i, f_2)"] \\ \mathcal{J}(Y_i, X_1) \arrow[r, "\mathcal{J}(Y_i, f_1)"] & \mathcal{J}(Y_i, X_0) \end{tikzcd}$$

of compactly generated housdorff spaces (not necessarily connected) are pullbacks and $\mathcal{J}(Y_i, f_i)$ are fibrations. If we knew that $\mathcal{J}(f, X_3)$ was a homotopy equivalence, then since $\pi_0 \mathcal{J}(Y_i, X_3) = (ho\mathcal{J})(Y_i, X_3)$ we would get the result that X_3 is h_*- local. However $\mathcal{J}(f, X_i)$ $(i = 0, 1, 2)$ are homotopy equivalences. It then follows from [8] that $\mathcal{J}(f, X_3)$ is a homotopy equivalence.

(iii) <u>Idempotents split in $Ls(ho\mathcal{J})$.</u>

<u>Lemma 2.13.</u>

Retracts of h_*- local spaces are h_*- local.

<u>proof</u>

Let $X \xrightarrow{i} Y \xrightarrow{r} X$ define a retraction in $ho\mathcal{J}$ where Y is h_*- local. If f is an h_*- isomorphism

We get the commutative diagram,

$$\begin{tikzcd}
(ho\mathcal{T})(Z_0,X) \arrow[r] \arrow[d] & (ho\mathcal{T})(Z_0,Y) \arrow[r] \arrow[d] & (ho\mathcal{T})(Z_0,X) \arrow[d] \\
(ho\mathcal{T})(Z_1,X) \arrow[r] & (ho\mathcal{T})(Z_1,Y) \arrow[r] & (ho\mathcal{T})(Z_1,Y)
\end{tikzcd}$$

Where the horizontal compositions are identities and the center vertical map is a bijection. An easy diagram chase assures us that $ho\,\mathcal{T}(f,X)$ is a bijection. Therefore X is h_*- local.

The verification of the fact that $J: L_S(ho\mathcal{T}) \hookrightarrow ho\mathcal{T}$ preserves products and weak pullbacks proceeds immediately from their constructions in $L_S(ho\mathcal{T})$

Before I verify the solution set condition I should like to point out the following property of h_*- local spaces.

<u>Lemma 2.14.</u>

Let X be an h_*- local space. Then $\mathcal{T}(\mathcal{U},X)$

is h_* – local (Note that $\mathcal{T}(\mathcal{U},X)$ may be neither connected nor of the homotopy type of a CW complex but the above terminology still makes sense).

proof

Let f be an h_*- isomorphism. Then since $(h_0\mathcal{T})(f,\mathcal{T}(\mathcal{U},X))$ is adjoint to $(h_0\mathcal{T})(f\wedge\mathcal{U},X)$ which is a bijection, the lemma follows.

Cor 2.15.

If $\mathcal{T}(\mathcal{U},X)^0$ denotes the component of the constant map to the base point, then $\mathcal{T}(\mathcal{U},X)^0$ is an h_*- local space.

proof

The corollary follows from the lemma with the observation that for V connected $\mathcal{T}(V,\mathcal{T}(\mathcal{U},X)^0)= \mathcal{T}(V,\mathcal{T}(\mathcal{U},X))$.

(iv) The solution set condition.

The following two lemmas will be needed to verify the solution set condition.

Lemma 2.16.

Let X be a CW complex and A a proper subcomplex of X such that $h_*(X,A) = 0$ Then there exists a subcomplex B of X such that $B \not\subseteq A, \# B \leq \alpha$ and $h_*(B, B \cap A) = 0$ (see Lemma 2.1, ex. 3.).

proof

Consider the category $\mathbb{K}$ of subcomplexes of X and inclusions, and the functor $T: \mathbb{K} \longrightarrow$ Sets defined by $T(B') = \bigvee_n h_n(B', A \cap B')$ Then lemma 2.1. is applicable with $K = \{$ finite subcomplexes of $X \}$ Thus X is a filtered colimit of objects of L (see lemma 2.1.). Therefore we need only take $B = F(i)$ for some index i such that $F(i)$ contains some cell not in A

Lemma 2.17.

With X and A as in Lemma 2.6:

(i) If X is connected we can choose B to be connected.

(ii) If A is connected we can choose B such that $A \cap B$ is connected.

(iii) If both X and A are connected we can choose B such that both B and $A \cap B$ are connected.

<u>proof</u>

For (i) repeat the proof of lemma 2.16. but now take $\mathbb{K}$ to consist of the connected subcomplexes of X and $K = \{$ finite connected subcomplexes of $X\}$

To show (ii), take a vertex v in A. For each component $(B \cap A)_i$ of $B \cap A$ choose a path P_i from v to $(B \cap A)_i$ that is cellular and lies in the one skeleton of A Then $B' = (\bigcup P_i) \cup B$ suffices.

For (iii) choose B connected and then use the process in the proof of (ii) to construct B', which will be connected.

<u>Lemma 2.18.</u>

With the hypothesis as above.

If X has a base point (which we assume to be a vertex) and A contains this point, then we may assume that B contains this point.

proof

Apply the process used to show (ii) Cor 2.17..

Lemma 2.19.

Let X be such that $(h\circ\mathcal{T})(i,X)$ is surjective for all $(i: L \hookrightarrow K) \in i\,C.W.$ with L, K connected, $\# K \leq \alpha$ and $h_*(K,L.) = 0$ Then $(h\circ\mathcal{T})(i',X)$ is surjective for all $i': L' \hookrightarrow K'$ with L', K' connected and $h_*(K',L') = 0$

proof

Let $f: L' \longrightarrow X$, Then I will construct inductively $\{(M_\beta, f_\beta)\}$ where β is an ordinal $f_\beta: M_\beta \to X$, $L' \subseteq M_\beta \subseteq K'$ and $M_{\beta'} \subseteq M_\beta$ for $\beta' \leq \beta$ such that

(i) $h_*(M_\beta, L') = 0$,

(ii) $f_\beta | M_{\beta'} = f_{\beta'}$ for $\beta' \leq \beta$,

(iii) $M_{\beta'} \neq M_\beta$, unless perhaps $M_{\beta'} = K'$.

Now take β such that card $\beta > \# K'$ Then f_β is the required extension of f.

Let $M_0 = L'$ and suppose $M_{\beta'}, f_{\beta'}$ have been defined for all $\beta' < \beta$. If β is a nonlimit ordinal, then since $h_*(M_{\beta-1}, L') = 0$ we have $h_*(K', M_{\beta-1}) = 0$. Corollaries 2.16 and 2.17 then assure us of the existence of A such that $\# A \leq \alpha$, A connected, A contains the base point of K', $M_{\beta-1} \cap A$ is connected, $h_*(A, M_{\beta-1} \cap A) = 0$ and $A \nsubseteq M_{\beta-1}$. Thus one may take $M_\beta = M_{\beta-1} \cup A$ and let f_β be an extension of $f_{\beta-1} \mid M_{\beta-1} \cap A$

In the event that β is a limit ordinal take $M_\beta = \bigcup_{\beta' < \beta} M_{\beta'}$, $f_\beta = \bigcup_{\beta' < \beta} f_{\beta'}$ and note that

$$h_*(M_\beta, L') = h_*\left(\bigcup M_{\beta'}/L', *\right) = \operatorname{colim} h_*\left(M_{\beta'}/L', *\right) = 0.$$

By taking mapping cylinders, we see that to verify the solution set condition we need only prove Proposition 2.20.

<u>Proposition 2.20.</u>

Let Y be a CW complex which is h_*-local and X a subcomplex of Y. Then there exists a subcomplex Y' of Y which contains X, is h_*-local and is such that $\# Y'$ is bounded by a cardinal which depends only on $\# X$ and card $h_*(S^0)$

<u>proof</u>

As in example -3. §2. the set L, consists of the h_*-isomorphisms which are inclusions of subcomplexes $(i: A \hookrightarrow B)$ with $\# B \leq \alpha$ (see Lemma 2.1. and example 3).

Let λ be an ordinal such that card $\lambda > \alpha$
I shall inductively define Y_β for $\beta < \lambda$ in such a way that if $\beta' < \beta < \lambda$ then $Y_{\beta'} \subseteq Y_\beta \subseteq Y$
Let $X = Y_0$ and consider all diagrams,

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "i"] & Y_0 \arrow[d] \\
\mathcal{U} \arrow[r, "h"] & Y
\end{tikzcd}$$

Where $i \in L$ Define $Y_1 = Y_0 \cup h(\mathcal{U})$ where the union is over all such diagrams (note that $h(\mathcal{U})$ is a subcomplex of Y). Similarly we define $Y_{\beta+1}$ from Y_β by considering all diagrams.

$$\begin{tikzcd} V \arrow[r] \arrow[d, "i"] & Y_\beta \arrow[d] \\ \mathcal{U} \arrow[r, "h"] & Y \end{tikzcd}$$

For limit ordinals β define $Y_\beta = \bigcup_{\beta'<\beta} Y_{\beta'}$

Let $Y' = \bigcup_{\beta<\lambda} Y_\beta$.

Since every $V \longrightarrow Y$ factors through some Y_β it follows by construction that $CW(i, Y')$ is surjective for all $(i: V \longrightarrow \mathcal{U}) \in L$ Lemma 2.19. then assures us that Y' is h_*—local.

2.6. <u>HR - localization of groups.</u>

In this section I shall prove the existence of the Bousfield HR - localization of groups 2 by using an adjoint functor theorem approach. To be more precise.

<u>Definition 2.21.</u>

Let $f: G_0 \longrightarrow G_1$ be a group homomorphism and R a trivial G, module. We will say $\underline{f \in HR}$ if,

(i) $H_1(\mathcal{f},R)$ is an isomorphism,

(ii) $H_2(\mathcal{f},R)$ is an epimorphism.

Let $\underline{\mathcal{G}p(HR)}$ be the full subcategory of HR-local groups and $\mathcal{J}: \mathcal{G}p(HR) \hookrightarrow \mathcal{G}p$ be the inclusion functor.

I shall show that the hypothesis of Theorem 1.9. are satisfied (or, equivalently in this case, the adjoint functor theorem of Freyd [10,12]). The solution set condition will be the most troublesome condition to verify. The following lemmas will take care of all of the other hypothesis of Theorem 1.9.

<u>Lemma 2.22.</u>

$\mathcal{G}p$(HR) is complete and $\mathcal{J}$ preserves limits.

<u>proof</u>

Let F be a diagram in $\mathcal{G}p$(HR). Then $\varprojlim \mathcal{J}F$ exists in $\mathcal{G}p$. Therefore we need only show that $\varprojlim \mathcal{J}F$ is HR - local. However this follows immediately since

$$\mathcal{G}p(\mathcal{f}, \varprojlim \mathcal{J}F) = \varprojlim \mathcal{G}p(\mathcal{f}, \mathcal{J}F).$$

Since idempotents split in $\mathcal{G}p$ the splitting of idempotents in $\mathcal{G}p$(HR) follows from Lemma 2.23.

Lemma 2.23.

Retracts of HR-local groups are HR-local.

proof

This argument is analogous to that of Lemma 2.13.

We will now show that the solution set condition holds for the HR-localization of groups. The argument will be similar to that of Proposition 2.20.; in fact Lemma 2.1. will play a parallel role. However since $\mathcal{G}p$ and $\mathcal{G}p$ (HR) are complete the argument will be somewhat less complicated than that of Proposition 2.20.

The solution set condition follow easily from

Proposition 2.24.

For every group G and homomorphism $f: G \to G'$ where G' is HR-local, there exists a subgroup G'' of G' which contains $f(G)$, is HR-local, and is such that card G'' is bounded by a cardinal number dependent only upon card G and card R.

proof

As in example -1 there exists a set L of morphisms such that $H_1(f,R)$ is an isomorphism, $H_2(f,R)$ an epimorphism and every such morphism is a filtered colimit of these. In fact since card $H_*(\pi, R) \leqslant \alpha$ for every finitely generated group, where α is an infinite cardinal such that card $R \leqslant \alpha$ we observe (Lemma 2.1.) that we may take L to consist of all such morphisms f with card (domain f), (codomain f) $\leqslant \alpha$.

Let λ be an ordinal such that card $\lambda > \alpha$ We shall define inductively a sequence of subgroups $\{G_\beta\}_{\beta<\lambda}$ such that $G_\beta \subseteq G_{\beta'}$, $\beta \leqslant \beta' < \lambda$ and then show that $G'' = \bigcup_{\beta<\lambda} G_\beta$ has the required properties.

Consider all diagrams,

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"] & G \arrow[d, "f"] \\ V \arrow[r, "h"] & G' \end{tikzcd}$$

such that $g \in L$ Note that since G' is local h is uniquely determined by $U \longrightarrow G \xrightarrow{f} G'$ Let

Let $G''_1 = gp(f(G), \cup h(V))$ where the union is taken over all such diagrams. In the same manner define $G''_{\beta+1}$ by considering all diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & G''_{\beta} \arrow[d] \\ V \arrow[r] & G' \end{tikzcd}$$

For limit ordinals take $G''_{\beta} = \bigcup_{\beta' < \beta} G''_{\beta'}$. Clearly card G''_{β} is bounded by a cardinal which depends only upon α and card G Therefore card G'' is bounded by some cardinal that depends only on α and card G where $G'' = \bigcup_{\beta < \lambda} G''_{\beta}$

We shall be done as soon as we show that G'' is HR-local. Given $g \in L$ and $f : Dom\, g \to G''$ one observes that f must factor through some G''_{β} and therefore $Gp(g, G'')$ is bijective. The proposition follows immediately from the following lemma.

Lemma 2.25.

If $Gp(g, G)$ is bijective for all $g \in L$ then $Gp(f, G)$ is bijective for all $f \in HR$.

proof

As in example -1, $f = \operatorname{colim}_J f_j$ where J is a filtered set and $f_j \in L$. In fact examination of the proof of Lemma 2.1. assures us that all the maps that occur in this colimit are inclusions. If K: domain $f \longrightarrow G$ then define K_j^1 from codomain f_j to G by requiring the following diagram to commute.

$$\begin{tikzcd}
\text{domain } f_j \arrow[r, "\phi_j"] \arrow[d, "f_j"'] & \text{domain } f \arrow[d, "K"] \\
\text{codomain } f_j \arrow[r, "K_j^1"] & G
\end{tikzcd}$$

Note that since $f_j \in L$, K_j^1 is uniquely determined. Let ${}_j\Phi_{j'}: f_{j'} \longrightarrow f_j$ be a structure map where ${}_j\Phi_{j'} = ({}_j\phi^0_{j'}, {}_j\phi^1_{j'})$,

$$\begin{tikzcd}
\text{domain } f_{j'} \arrow[r, "{}_j\phi^0_{j'}"] \arrow[d, "f_{j'}"] & \text{domain } f_j \arrow[d, "f_j"] \\
\text{codomain } f_{j'} \arrow[r, "{}_j\phi^1_{j'}"] & \text{codomain } f_j
\end{tikzcd}$$

Since the diagram

$$\begin{tikzcd}
\text{domain } f_{j'} \arrow[r, "{}_j\phi^0_{j'}"] \arrow[d] & \text{domain } f_j \arrow[r, "\phi_j"] \arrow[d] & \text{domain } f \arrow[d] \\
\text{codomain } f_{j'} \arrow[r, "{}_j\phi^1_{j'}"] & \text{codomain } f_j \arrow[r, "K^1_j"] & \text{codomain } f
\end{tikzcd}$$

commutes one is assured (by the uniqueness of the $K^1_{j'}$) that $K^1_j \cdot {}_j\phi^1_{j'} = K^1_{j'}$ Therefore $\{K^1_j\}_{j \in J}$ defines a map K^1: codomain $f \longrightarrow G$ such that $K^1 f = K$

Uniqueness of K^1 follows immediately since the K^1_j are uniquely determined by K.

2.7 HZ-localization of π-Modules

I shall now exhibit an adjoint functor approach to the construction of Bousfield's HZ-localization of π-Modules [2].

Definition 2.26.

Let $f : M_1 \longrightarrow M_2$ be a morphism of π-Modules. We will say $f \in HZ$ if,

(i) $H_0(\pi, f)$ is an isomorphism,

(ii) $H_1(\pi, f)$ is an epimorphism.

As before we will denote the full subcategory of HZ-local π-Modules by $\mathrm{Mod}_\pi(HZ)$ and $J : \mathrm{Mod}_\pi(HZ) \to \mathrm{Mod}_\pi$ to be the inclusion functor.

The verification that J satisfies the hypothesis of the adjoint functor theorem is completely analogous to the verification made for the HR-localization of groups. The only changes to be made in the arguments are to replace the categorical notions used in $\mathcal{G}p$ by these analogous ones in π-Modules and to substitute example -2 for example 1. Note also that card $H_*(\pi, M) \leq \alpha$ for finitely generated π-Modules M, where α is any infinite cardinal such that card $\pi \leq \alpha$ Hence we may take L to consist of all $f \in HZ$ such that card domain f, card codomain $f \leqslant \alpha$. The analogous lemmas are.

Lemma 2.27.

Mod_{π}(HZ) is complete and J preserves limits.

Lemma 2.28.

Retracts of HZ-local π-Modules are HZ-local.

Proposition 2.29.

For every π-Module M and morphism $f: M \to M'$ where M' is HZ-local, there exists a submodule M'' of M' which is HZ-local, such that f factors through M'' and card M'' is bounded by a cardinal dependent only upon card M and card π.

<u>Lemma 2.30.</u>

If M is such that $\mathrm{Mod}_{\pi}(g, M)$ is a bijection for all $g \in L$ then $\mathrm{Mod}_{\pi}(f, M)$ is a bijection for all $f \in HZ$.

2.8. <u>The Quillen $+$- construction</u>

<u>Definition 2.31.</u>

One says that a group A is <u>perfect</u> if it is equal to its commutator subgroup.

Recall that Quillen's $+$- construction yields for each $X \in ho\mathcal{T}$ and normal perfect subgroup $A \trianglelefteq \pi_1 X$ a space $X^+ \in ho\mathcal{T}$ and a map $\eta : X \longrightarrow X^+$ such that,

(i) $\pi_1(\eta)A = \{1\}$,

(ii) η is universal with respect to (i),

i.e. given $f : X \longrightarrow Y$ such that $\pi_1(f)A = \{1\}$ there exists a unique $g : X^+ \longrightarrow Y$ such that the diagram,

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & X^{+} \arrow[d, "g"] \\ & Y \end{tikzcd}$$

commutes.

If one considers the category $\mathcal{C}$ whose objects consist of (X, A) with $X \in ho\mathcal{T}$ and A a perfect normal subgroup of $\pi_1(X)$ and whose morphisms are $(f: X \longrightarrow X') \in ho\mathcal{T}$ such that $\pi_1(f)A \subseteq A'$ then (taking $A = \{1\}$ one sees that) $ho\mathcal{T}$ is embedded by $X \longmapsto (X, \{1\})$ as a full subcategory of $\mathcal{C}$ Furthermore it follows immediately that the $+$ - construction yields a functor "$+$": $\mathcal{C} \longrightarrow ho\mathcal{T}$ which is left adjoint to the inclusion $J: ho\mathcal{T} \hookrightarrow \mathcal{C}$

Before using Theorem 1.9. to show the existence of the left adjoint "$+$" to J I will make some preliminary remarks about perfect groups.

Lemma 2.32.

If A and B are perfect normal subgroups of G then $gp(A \cup B)$ is a perfect normal subgroup of G.

proof

Since $gp(A \cup B) = B \cdot A$ and B, A are normal it follows that $gp(A \cup B)$ is normal. It is perfect since $A \cdot B \subseteq [A \cdot A] \cdot [B \cdot B] \leq [A \cdot B, A \cdot B] \leq A \cdot B$

Lemma 2.33.

Given a group G there exists a unique maximal perfect normal subgroup PG of G.

Furthermore PG is a characteristic subgroup.

proof

If one orders the perfect normal subgroups by inclusion it follows immediately from Zorn's Lemma that there exists a maximal perfect normal subgroup and

Lemma 2.32. assures us that it is unique. Furthermore since homomorphic images of perfect subgroups are perfect one sees immediately that $f(PG) \subseteq PG$ for every automorphism,

Cor. 2.34.

If A is a normal subgroup of G then PA is a perfect normal subgroup of G contained in A (in fact PA is the unique maximal perfect normal subgroup of G that is contained in A).

I will now show that $J: ho\tilde{J}C \longrightarrow C$ satisfies the hypothesis of Theorem 1.9. and thus show the existence of "$+$"

Lemma 2.35.

(i) C has products.

(ii) C has weak pullbacks.

(iii) Idempotents split in C

(iv) J preserves products and weak pullbacks.

proof

(i) follows by taking $\prod(X_i, A_i) = (\prod X_i, P(\prod A_i))$ where $\prod X_i$ is the usual product in $ho\mathcal{T}$

To prove (ii) let $(X_1, A_1) \xrightarrow{f_1} (X_0, A_0) \xleftarrow{f_2} (X_2, A_2)$ be a diagram in $\mathcal{C}$ and let X be a weak pullback of $X_1 \xrightarrow{f_1} X_0 \xleftarrow{f_2} X_2$ in $ho\mathcal{T}$ (we may for instance take f_2 to be a fibration in $\mathcal{T}$ and X_3 to be the pullback in $\mathcal{T}$). Then we observe that $(X_3, P(\pi_1(f_1)^{-1}A_1 \cap \pi_1(f_2)^{-1}A_2)$ is a weak pullback of $(X_1, A_1) \xrightarrow{f_1} (X_0, A_0) \xleftarrow{f_2} (X_2, A_2)$

Let $e: (X, A) \longrightarrow (X, A)$ be an idempotent in $\mathcal{C}$ and let Y split e in $ho\mathcal{T}$ Thus there exist $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ such that $gf = e$ and fg is an isomorphism. It then follows that $\pi_1(f)A$ is a perfect normal subgroup of $\pi_1(Y)$ and hence $(Y, \pi_1(f)A)$ splits e.

That J preserves products follows from the construction of products in $ho\mathcal{T}$ and $\mathcal{C}$ Therefore we need only show that $P(\ker \pi_1 f_1' \cap \ker \pi_1 f_2') = 0$ where

$$\begin{tikzcd} X_3 \arrow[r, "f_1'"] \arrow[d, "f_2'"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"] & X_0 \end{tikzcd}$$

is a pullback in $\mathcal{T}$ and f_2 is a fibration. However by considering the Mayer-Vietoris sequence for homotopy groups we observe that

$$\pi_2 X_0 \longrightarrow \operatorname{Ker} \pi_1 f_1' \cap \operatorname{Ker} \pi_2 f_2'$$

is surjective and hence $\operatorname{Ker} \pi_1 f_1' \cap \operatorname{Ker} \pi_2 f_2'$ is Abelian, thus $P(\operatorname{Ker} \pi_1 f_1' \cap \operatorname{Ker} \pi_2 f_2') = 0$

<u>Theorem 2.36.</u>

$J : ho\mathcal{T} \hookrightarrow C$ has a left adjoint.

<u>proof</u>

We need only show that the solution set condition is satisfied for $(X, A) \in ob\,\mathcal{C}$.
Let $\{f_i : S^1 \to X\}$ represent a set of generators for A Then the set $\{Y\}$ provides the solution set condition for (X, A) where Y is defined by the pushout in $\mathcal{T}$

$$\begin{tikzcd}
\bigvee_{i\in I} S^1 \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\bigvee_{i\in I} D^2 \arrow[r] & Y
\end{tikzcd}$$

where D^2 is the disk with boundary S^1.

III Localization

3.1. Categories of fractions

Definition 3.1.

Given a class of morphisms $\mathcal{S}$ of a category $\mathcal{C}$ one says that a functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ <u>inverts $\mathcal{S}$</u> if $F(s)$ is an isomorphism for all s of $\mathcal{S}$

Definition 3.2. 12

With $\mathcal{S}$ as before, suppose that there exists a functor $I: \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}]^{-1}$ which inverts $\mathcal{S}$ such that given any functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ which inverts $\mathcal{S}$ there exists a unique functor G, $G: \mathcal{C}[\mathcal{S}]^{-1} \longrightarrow \mathcal{D}$ such that $GI = F$ Then one call $\mathcal{C}[\mathcal{S}]^{-1}$ <u>a category of fractions</u> of $\mathcal{C}$.

Notice that if $I: \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}]^{-1}$ exists then it is determined modulo a canonical isomorphism.

I would like to recall the following facts concerning categories of fractions [12].

<u>Theorem 3.3.</u>

Let $\mathcal{S}$ be as before and suppose that,

(i) $\mathcal{S}$ admits a calculus of left fractions,

(ii) for each object X of $\mathcal{C}$ there exist a set $\mathcal{S}_X$ of morphisms of $\mathcal{S}$ with domain X which is cofinal in the class of morphisms of $\mathcal{S}$ with domain X That is for each $(s: X \longrightarrow Y) \in \mathcal{S}$ there exist $s' \in \mathcal{S}_X$ and u such that $s' = us.$

Then $\mathcal{C}[\mathcal{S}]^{-1}$ exists. Furthermore $\mathcal{C}[\mathcal{S}]^{-1}$ has the following description:

(i) $ob\,\mathcal{C}[\mathcal{S}]^{-1} = ob\,\mathcal{C}$

(ii) Morphisms of $\mathcal{C}[\mathcal{S}]^{-1}$ can be represented in the form $I(s)^{-1}I(g)$, $s \in \mathcal{S}$ (thus we will denote them by (g, s)).

(iii) If $(f_1, s_1) = (f_2, s_2)$ then there exists g_1 and g_2 such that $g_1 s_1 = g_2 s_2 \in \mathcal{S}$

and $g_1 f_1 = g_2 f_2$.

$$\begin{tikzcd}
 & Z_1 \arrow[dr, "g_1"] & \\
X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & Y \arrow[u, "s_1"'] \arrow[d, "s_2"] & U \\
 & Z_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

(iii) The composition

$(f_2, s_2)\cdot(f_1, s_1) = (f, s)$ may be calculated by (use (iii) definition 2.4.) completing the diagram

$$\begin{tikzcd}
 & & U \\
 & Y' \arrow[ur, dashed, "g"] & Z' \arrow[u, dashed, "s_3"'] \\
X \arrow[ur, "f_1"] & Y \arrow[u, "s_1"'] \arrow[ur, "f_2"] & Z \arrow[u, "s_2"']
\end{tikzcd}$$

and then setting $f = g f_1$ and $s = s_3 s_2$

Definition 3.4.

Suppose that $I : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}]^{-1}$ exists. One calls the class of morphisms which are inverted by I the saturation of $\mathcal{S}$ and denote it by $\operatorname{sat}\mathcal{S}$ If $\operatorname{sat}\mathcal{S} = \mathcal{S}$ then we say $\mathcal{S}$ is saturated.

Notice that if $\mathcal{C}[\mathcal{S}]^{-1}$ exists, then $\mathcal{C}[\operatorname{sat}\mathcal{S}]^{-1}$ exists, in fact $\mathcal{C}[\operatorname{sat}\mathcal{S}]^{-1} = \mathcal{C}[\mathcal{S}]^{-1}$

3.2. Definitions and summary of results.

Definition 3.5.

Suppose that $\mathcal{C}'$ is a reflective full subcategory of $\mathcal{C}$ with reflection $\mathcal{R}$ One calls the class of morphisms f of $\mathcal{C}$ such that $\mathcal{R}f$ is an isomorphism the closure of $\mathcal{C}'$ and denote it by $\mathcal{E}(\mathcal{C}')$

Definition 3.6.

One says that a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is closed under isomorphic objects if whenever A' is isomorphic to A and A' is an object of $\mathcal{C}'$ then A is an object of $\mathcal{C}'$.

In sections 3.3. - 3.6. we shall be concerned with showing the relationship between full reflective subcategories and categories of fraction. It will be proved (Theorem 3.21.) that $\mathcal{E}(\_)$ defines a bijective correspondence between the class of full reflective subcategories of $\mathcal{C}$ which are closed under isomorphic objects and saturated classes of morphisms $\mathcal{S}'$ such that $I : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}']^{-1}$ has a right adjoint. Furthermore the inverse of $\mathcal{E}(\_)$ is given by $L_{(-)}\mathcal{C}$ (see 2.3.).

The proof of Theorem 3.21. will involve us in the verification of several details which will give us much more detailed information concerning the relationship between categories of fractions, localization and reflective subcategories than is stated in Theorem 3.21.

In the final section Theorem D 1.9. will be used to give sufficient conditions for the existence of a right adjoint to $I : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}']^{-1}$ which by

Theorem 3.21. gives sufficient conditions for $\mathcal{S}$ to be localizing. That is (Theorem 3.22.), if $\mathcal{C}$ is a category such that $\mathcal{C}$ has coproducts, weak pushouts, idempotents split and $\mathcal{S}$ is a class of morphisms closed under coproducts and satisfying the cofinality condition of Theorem 3.3., then $I : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ has a right adjoint.

Theorem 3. generalizes two results of Deleanu [5, 6] one of which is a similar theorem for cocomplete categories and the other is concerned with $ho\mathcal{CW}$.

3.3. Reflective subcategories and localization.

I will now show that $L_{\mathcal{E}(\mathcal{C}')}\mathcal{C} = \mathcal{C}'$ where $\mathcal{C}'$ is a reflective full subcategories which are closed under isomorphic objects.

Throughout this section I will assume that $\mathcal{C}'$ is as previously stated. Furthermore since $\mathcal{C}'$ is full we can assume, for convenience, that the reflection $\mathcal{R}$ restricted to $\mathcal{C}'$ is the identity.

Lemma 3.7.

Let $f\colon Y \longrightarrow Z$ be such that $\mathcal{R}f$ is an isomorphism and suppose that Y is $\mathcal{E}(\mathcal{C}')$- local. Then f has a left inverse g.

proof

Let g correspond to Id_Y via the bijection $\mathcal{C}(f, Y)$

Cor. 3.8.

Suppose that $f\colon Y \longrightarrow Z$ is such that $\mathcal{R}f$ is an isomorphism and Y, Z are $\mathcal{E}(\mathcal{C}')$- local.

Then f is an isomorphism.

proof

By 3.7 there exists g such that $gf = Id_Y$ However $\mathcal{R}g\mathcal{R}f = Id_{\mathcal{R}Y}$ therefore g has a left inverse. Thus g is an isomorphism and so f is also.

Lemma 3.9.

Objects of $\mathcal{C}'$ are $\mathcal{E}(\mathcal{C}')$-local.

proof

If X is an object of $\mathcal{C}'$ and f is such that $\mathcal{R}f$ is an isomorphism, then commutivity of the diagram,

$$\begin{tikzcd}
\mathcal{C}(Y,X) & \mathcal{C}(Z,X) \arrow[l, "\mathcal{C}(f,X)"'] \\
\mathcal{C}(\mathcal{R}Y,X) \arrow[u] & \mathcal{C}(\mathcal{R}Z,X) \arrow[l, "\mathcal{C}(\mathcal{R}f,X)"'] \arrow[u]
\end{tikzcd}$$

, where the vertical arrows are the bijections given by the adjunction, implies the lemma.

<u>Lemma 3.10.</u>

If X is $\mathcal{E}(\mathcal{C}')$-local, then X is an object of $\mathcal{C}'$.

<u>proof</u>

Consider $\eta_X : X \longrightarrow \mathcal{R}X$ Since $\mathcal{R}\eta_X$ is an isomorphism η_X is an isomorphism.

Proposition 3.11.

The $\mathcal{E}(\mathcal{C}')$-local objects are precisely the objects of $\mathcal{C}'$.

3.4. $\mathcal{E}(\mathcal{C}')$ admits a calculus of left fractions.

Throughout this section $\mathcal{C}'$ will be as in 3.3. I will show that $\mathcal{E}(\mathcal{C}')$ admits a calculus of left fractions and satisfies the cofinality condition of Theorem 3.3. Therefore $\mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1}$ exists. I will then show that $\mathcal{E}(\mathcal{C}')$ is saturated.

Lemma 3.12.

$\mathcal{E}(\mathcal{C}')$ admits a calculus of left fractions (see definition 2.4.).

proof

(i) is trivial. To show (ii) let $X_2 \xleftarrow{s} X_1 \xrightarrow{f} X_3$ be such that $\mathcal{R}s$ is an isomorphism. Since $\mathcal{R}(\eta_{X_2} \cdot s)$ is an isomorphism there exists $g: \mathcal{R}X_2 \longrightarrow \mathcal{R}X_3$ such that the diagram

$$\begin{tikzcd}
\mathcal{R}X_1 \arrow[r, "\mathcal{R}s"] \arrow[dd, "\mathcal{R}f"] & \mathcal{R}X_2 \arrow[d, "\mathcal{R}\eta_{X_2}"] \\
 & \mathcal{R}^2X_2 = \mathcal{R}X_2 \arrow[d, "\mathcal{R}g"] \\
\mathcal{R}X \arrow[r, "\mathcal{R}\eta_{X_3}"] & \mathcal{R}X_3
\end{tikzcd}$$

commutes. Adjointness then assures us that the diagram

$$\begin{tikzcd}
X_1 \arrow[r, "s"] \arrow[dd, "f"] & X_2 \arrow[d, "\eta_{X_2}"] \\
 & \mathcal{R}X_2 \arrow[d, "g"] \\
X_3 \arrow[r, "\eta_{X_3}"] & \mathcal{R}X_3
\end{tikzcd}$$

commutes.

For (iii) let $X_1 \xrightarrow{s} X_2 \overset{f}{\underset{g}{\rightrightarrows}} X_3$ be such that $\mathcal{R}s$ is an isomorphism and $fs = gs$ Thus $\mathcal{R}f \cdot \mathcal{R}s = \mathcal{R}g\mathcal{R}s$ and since $\mathcal{R}s$ is an isomorphism

Commutivity of the diagrams

$$\begin{tikzcd} X_2 \arrow[r, "f"] \arrow[d, "\eta_{X_2}"] & X_3 \arrow[d, "\eta_{X_3}"] \\ \mathcal{R}X_2 \arrow[r, "\mathcal{R}f"] & \mathcal{R}X_3 \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "g"] \arrow[d, "\eta_{X_2}"] & X_3 \arrow[d, "\eta_{X_3}"] \\ \mathcal{R}X_2 \arrow[r, "\mathcal{R}g"] & \mathcal{R}X_3 \end{tikzcd}$$

assures us that $\eta_{X_3} f = \eta_{X_3} g$

<u>Lemma 3.13.</u>

With $\mathcal{C}'$ and $\mathcal{C}$ as before, the class $\mathcal{E}(\mathcal{C}')$ satisfies the cofinality condition of Theorem 3.3.

<u>proof</u>

Let $S_X = \{\eta_X : X \longrightarrow \mathcal{R}X\}$ If $(s : X \longrightarrow Y) \in \mathcal{E}(\mathcal{C}')$ then $\eta_X = ((\mathcal{R}s)^{-1} \eta_Y)\, s$ We may now construct $I : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1}$.

The following lemma assures us that $\mathcal{E}(\mathcal{C}')$ is saturated.

Lemma 3.14.

Suppose that $\mathcal{R}: \mathcal{C} \to \mathcal{C}'$ is a reflection and $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ which is closed under isomorphic objects. Then $\mathcal{E}(\mathcal{C}')$ is saturated.

proof

Let (f, Id) be an isomorphism. Then there exist (g, s) such that $(g, s)\cdot(f, Id)$ and $(f, Id)\cdot(g, s)$ are identity maps. However $(g, s)(f, Id) = (gf, s)$. Thus there exists h_1, h_2 such that $h_1 s = h_2 \in \mathcal{E}(\mathcal{C}')$ and $h_1 g f = h_2 s$. Since $s, h_2 \in \mathcal{E}(\mathcal{C}')$ both h_1 and gf are elements of $\mathcal{E}(\mathcal{C}')$ Let f' and s' be such that $s' \in \mathcal{E}(\mathcal{C}')$ and $f's = s'f$ Thus $(f, Id)\cdot(g, s) = (f'g, s')$. An analogous argument assures us that $f'g \in \mathcal{E}(\mathcal{C}')$ Therefore $\mathcal{R}f \cdot \mathcal{R}g$ and $\mathcal{R}g \cdot \mathcal{R}f$ are isomorphisms. Hence $\mathcal{R}g$ is an isomorphism.

3.5. <u>$I : \mathcal{C} \longrightarrow [\mathcal{E}(\mathcal{C}')]^{-1}$ has a right adjoint.</u>

Suppose that $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{R}$ are as before. Let the functor $\mathbb{E}'(\mathcal{C}') : \mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1} \longrightarrow \mathcal{C}'$ be defined by requiring that the diagram

$$\begin{tikzcd}
\mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1} \arrow[rr, "\mathbb{E}'(\mathcal{C}')"] & & \mathcal{C}' \\
\mathcal{C} \arrow[u, "I"] \arrow[urr, "\mathcal{R}"'] & &
\end{tikzcd}$$

commute. If $J : \mathcal{C}' \hookrightarrow \mathcal{C}$ is the inclusion functor, then let $\mathbb{E}(\mathcal{C}') = J\,\mathbb{E}'(\mathcal{C}')$

<u>Lemma 3.15.</u>

$$I \dashv \mathbb{E}(\mathcal{C}').$$

<u>proof</u>

Let $\eta : Id_{\mathcal{C}} \longrightarrow J\mathcal{R}$ be the unit of the adjunction

$\mathcal{R} \dashv \mathcal{J}$ Let $\varepsilon : \mathrm{I}\,\mathbb{E}(\mathcal{C}') \longrightarrow \mathrm{Id}_{\mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1}}$ be defined by $\varepsilon_X = (\mathrm{I}\eta_X)^{-1}$ Since $\mathbb{E}(\mathcal{C}')\,\mathrm{I} = \mathcal{J}\mathcal{R}$ we have $\eta : \mathrm{Id}_{\mathcal{C}} \longrightarrow \mathbb{E}(\mathcal{C}') \cdot \mathrm{I}$. It is easily shown that (η, ε) are then unit and counit of an adjunction.

Proposition 3.16.

If $\mathcal{C}'$ is a full reflexive subcategory of $\mathcal{C}$ which is closed under isomorphic objects, then $\mathcal{E}(\mathcal{C}')$ admits a calculus of left fractions, satisfies the cofinality condition of Theorem 3.3. and is saturated. Furthermore $\mathrm{I} : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1}$ has a right adjoint.

I would now like to point out the following relationship between $\mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1}$ and $\mathcal{C}'$

Proposition 3.17.

With $\mathcal{C}'$ as before, the functor $\mathbb{E}'(\mathcal{C}') : \mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1} \longrightarrow \mathcal{C}'$ is an equivalence of categories with inverse $\mathrm{I}\mathcal{J}$.

proof

η defines a natural transformation $\eta' : Id_{\mathcal{C}'} \longrightarrow \mathbb{E}'(\mathcal{C}') \cdot I\,J$ such that $\eta'_X \in \mathcal{E}(\mathcal{C}')$ Since X and $(\mathbb{E}(\mathcal{C}')IJ)X$ are $\mathcal{E}(\mathcal{C}')$ - local η'_X is an isomorphism. Thus we need only show that $I\,J$ is an equivalence of categories. Since $\eta_X \in \mathcal{E}(\mathcal{C}')$ each object of $\mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1}$ is isomorphic to an object in the image of $I\,J$.

Furthermore since the isomorphism

$$\begin{aligned}\mathcal{C}[\mathcal{E}(\mathcal{C}')]^{-1}(IJX, IJY) &\approx \mathcal{C}(JX, \mathbb{E}(\mathcal{C}')IJY) \\ &\approx \mathcal{C}(JX, JY) \\ &\approx \mathcal{C}(X, Y)\end{aligned}$$

is induced by $I\,J$ we see that $I\,J$ is an equivalence of categories.

3.6. Categories of fractions and reflective subcategories.

To complete the proof of Theorem 3.21. we need only show that $L_{\mathcal{S}}(\mathcal{C})$ is a reflexive subcategory for

$\mathcal{S}$ a class of morphisms of $\mathcal{C}$ such that,

(i) $\mathcal{C}[\mathcal{S}]^{-1}$ exists,

(ii) $I: \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}]^{-1}$ has a right adjoint,

(iii) $\mathcal{S}$ is saturated.

Throughout this section we will assume that $\mathcal{S}$ satisfies the above conditions.

Lemma 3.18.

If $\eta : Id_{\mathcal{C}} \longrightarrow \mathcal{S}I$ is the unit of the adjunction $I \longrightarrow \mathcal{S}$ then $I\eta_X$ is an isomorphism and hence an element of $\mathcal{S}$.

proof

Let $\varepsilon : I\,\mathcal{S} \longrightarrow Id_{\mathcal{C}[\mathcal{S}]^{-1}}$ be the counit of the adjunction. Then $\varepsilon_{IX} \cdot I\eta_X = Id_{IX}$ and $\mathcal{S}\varepsilon_Y \cdot \eta_{\mathcal{S}Y} = Id_{\mathcal{S}Y}$. If we let $\zeta = I\eta$ then ζ_X has a left inverse and we need only show that it has a right inverse. Naturality of ζ assures us of the commutative diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
I\mathcal{S}Y \arrow[r, "\varepsilon_Y"] \arrow[d, "\zeta_{\mathcal{S}Y}"] & Y \arrow[d, "\zeta_X"] \\
I\mathcal{S}I\mathcal{S}Y \arrow[r, "I\mathcal{S}\varepsilon_Y"] & I\mathcal{S}Y
\end{tikzcd}, \quad Y = IX$$

Therefore, $\zeta_X \varepsilon_Y = I\mathcal{S}\varepsilon_Y = I\mathcal{S}\varepsilon_Y \cdot I\eta_{\mathcal{S}Y}$

$$= I(\mathcal{S}\varepsilon_Y \cdot \eta_{\mathcal{S}Y})$$

$$= I\,Id_{\mathcal{S}Y}$$

<u>Lemma 3.19.</u>

With $\mathcal{S}$ as before, $\mathcal{S}X$ is $\mathcal{S}$ local.

<u>proof</u>

Let $f \in \mathcal{S}$ Then the commutative diagram,

$$\begin{tikzcd}
\mathcal{C}[S]^{-1}(IZ,X) \arrow[r] \arrow[d, "{\mathcal{C}[S]^{-1}(I\xi,X)}"] & \mathcal{C}(Z,\mathcal{S}X) \arrow[d, "{\mathcal{C}(\xi,\mathcal{S}X)}"] \\
\mathcal{C}[S]^{-1}(IY,X) \arrow[r] & \mathcal{C}(Y,\mathcal{S}X)
\end{tikzcd}$$

assures us that $\mathcal{C}(\xi, \mathcal{S}X)$ is a bijection.

Since $\mathcal{S}X$ is $\mathcal{S}$-local we may define a functor $\mathcal{R} : \mathcal{C} \longrightarrow L_{\mathcal{S}}(\mathcal{C})$ by $\mathcal{R} = \mathcal{S}I$

The next lemma will then complete the proof of Theorem 3.21.

<u>Lemma 3.20.</u>

With the notations as above, $\mathcal{R} \dashv J$ where $J : L_{\mathcal{S}}(\mathcal{C}) \hookrightarrow \mathcal{C}$ is the inclusion functor.

<u>proof</u>

Let η and ε be the unit and counit of the adjunction $I \dashv \mathcal{S}$ Then they define and

which one sees easily are the unit and counit of an adjunction

Therefore theorem 3.21. follows.

Theorem 3.21.

$\mathcal{E}(\_)$ defines a bijective correspondence between the class of full reflective subcategories of $\mathcal{C}$ which are closed under isomorphic objects and the class of saturated classes of morphisms such that $I : \mathcal{C} \longrightarrow \mathcal{C}[S]^{-1}$ has a right adjoint. Furthermore the inverse of $\mathcal{E}(\_)$ is given by $L_{(\_)}\mathcal{C}$

3.7. A sufficient condition for the existence of localization.

As a consequence of Theorem 3.21., Theorem 3.22. will give us a sufficient condition for the existence of a localization. Theorem 3.22. generalizes a result of Deleanu for complete categories and will be proved with the aid of Theorem D 1.9..

<u>Theorem 3.22.</u>

Let $\mathcal{C}$ be a category such that,

$\mathcal{C}$1- $\mathcal{C}$ has coproducts,

$\mathcal{C}$2- $\mathcal{C}$ has weak pushouts,

$\mathcal{C}$3- Idempotents split in $\mathcal{C}$

Let $\mathcal{S}$ be a class of morphisms of such that,

$\mathcal{S}$1- $\mathcal{S}$ is closed under coproducts,

$\mathcal{S}$2- $\mathcal{S}$ admits a calculus of left fractions,

$\mathcal{S}$3- For every X an object of $\mathcal{C}$ there exists a set $\mathcal{S}_X$ such that,

(i) if $f \in \mathcal{S}_X$ then domain $f = X$,

(ii) $\mathcal{S}_X$ is cofinal in the set of morphisms of $\mathcal{S}$ with domain X.

Then $I : \mathcal{C} \longrightarrow [\mathcal{S}]^{-1}$ has a right adjoint

The proof of Theorem 3.22. follows from Theorem D 1.9. Verification that the hypothesis of Theorem D 1.9. are satisfied will be a consequence of the following Lemmas.

Lemma 3.23. (Wedge Axiom)

Let $\{X_i\}$ be objects of $\mathcal{C}$ and $k_i : X_i \longrightarrow \coprod X_i$. Then

$$\mathcal{C}[S]^{-1}(I(\coprod X_i), Y) \approx \prod \mathcal{C}[S]^{-1}(I X_i, Y)$$

where $pr_i = \mathcal{C}[S]^{-1}(I(k_i), Y)$ are the projections.

proof

Let $\phi_i = (f_i, s_i)$ be morphisms from X_i to Y in $\mathcal{C}[S]^{-1}$. Consider the map $\langle \phi_i \rangle : \coprod X_i \longrightarrow Y$ in $\mathcal{C}[S]^{-1}$ defined by $\langle \phi_i \rangle = I\nabla \cdot I(\coprod s_i)^{-1} \cdot I(\coprod f_i)$ where ∇ is the folding map.

Since S admits a calculus of left fractions one can construct the diagram,

$$\begin{tikzcd}
& & \coprod Y \arrow[r, "\nabla"] \arrow[d, "\coprod s_i"] & Y \arrow[d, "s"] \\
X_i \arrow[r, "k_i"] & \coprod X_i \arrow[r, "\coprod f_i"] & \coprod Z_i \arrow[r, "e"] & V
\end{tikzcd}$$

where $s \in \mathcal{S}$. It then follows that

$$\langle \phi_i \rangle I k_i = (e(\coprod f_i)k_i, s)$$

However from the diagram,

$$\begin{tikzcd}
& Z_i \arrow[dr, "ej_i"] & \\
X_i \arrow[ur, "f_i"] \arrow[dr, "e(\coprod f_i)k_i"'] & Y \arrow[u, "s_i"'] \arrow[d, "s"] & V \\
& V \arrow[ur, "Id_V"'] &
\end{tikzcd}$$

$, \; j_i : Z_i \to \coprod Z$

one observes that,

$$\langle \phi_i \rangle I k_i = (e(\amalg f_i) k_i, S)$$
$$= (f_i, S_i)$$

Note that $\langle \phi_i \rangle$ depends only on $\{\phi_i\}$ and not the representation of ϕ_i. For if $(f_i, S_i) = (f_i', S_i')$ then there exists t_i, t_i' such that $t_i S_i = t_i' S_i' \in \mathcal{S}$ and $t_i f_i = t_i' f_i'$.

Therefore $(\amalg t_i)(\amalg f_i) = (\amalg t_i')(\amalg f_i')$

and $(\amalg t_i)(\amalg S_i) = (\amalg t_i')(\amalg S_i') \in \mathcal{S}$.

Hence $(\amalg f_i, \amalg S_i) = (\amalg f_i', \amalg S_i')$.

Let $(\varphi : \amalg X_i \longrightarrow Y) \in \mathcal{C}[\mathcal{S}]^{-1}(I \amalg X_i, Y)$ and $\varphi k_i = \varphi_i$. I will show that $\varphi = \langle \varphi_i \rangle$

Represent φ by (f, S). Then,

$$\varphi_i = \varphi k_i = (f, S)(k_i, 1) = (f k_i, S)$$

From the diagram,

$$\begin{tikzcd}
 & \coprod Y \arrow[r, "\nabla"] \arrow[d, "\coprod s"] & Y \arrow[d, "s"] \\
\coprod X_i \arrow[r, "\coprod fk_i"] & \coprod Z \arrow[r, "\nabla"] & Z
\end{tikzcd}$$

one observes that,

$$\begin{aligned}
\langle \varphi_i \rangle &= (I \nabla) \cdot (\coprod fk_i, \coprod s) \\
&= (\nabla \cdot \coprod fk_i, s) \\
&= (f, s).
\end{aligned}$$

<u>Lemma 3.24.</u>

$\mathcal{C} \longrightarrow \mathcal{C}[S]^{-1}$ preserves weak pushouts

<u>proof</u>

Let the diagram,

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "k"] \\
Z \arrow[r, "h"] & U
\end{tikzcd}$$

be a weak pushout in $\mathcal{C}$. Let $(\ell, s') : Y \to T$ and $(m, s'') : Z \longrightarrow T$ be such that $(\ell g, s') = (mh, s'')$. Hence there exist $t' : T' \longrightarrow T'''$ and $t'' : T'' \longrightarrow T'''$ ($s' : T \to T'$ and $s'' : T \longrightarrow T''$) such that $t's' = t''s'' \in \mathcal{S}$ and $t'\ell = t''m$

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "k"] \arrow[dr, "\ell"] & & \\
Z \arrow[r, "h"] \arrow[dr, "m"'] & U & T' \arrow[ddr, "t'"] & \\
& T'' \arrow[drr, "t''"'] & T \arrow[l, "s''"'] \arrow[u, "s'"'] & \\
& & & T'''
\end{tikzcd}$$

Hence there exist $\eta : U \longrightarrow T'''$ such that $\eta k = t'\ell$ and $\eta h = t''m$. Therefore $(\eta, t's') \in \mathcal{C}[\mathcal{S}^{-1}](U, T)$ is such that $(\eta, t's')Ik = (\ell, s')$ and $(\eta, t's')Ih = (m, s'')$.

To complete the proof of Theorem 3.22. we need only show that $I : \mathcal{C} \longrightarrow \mathcal{C}[S]^{-1}$ satisfies the solution set condition. Let $\alpha \in \mathcal{C}[S]^{-1}(X,Y)$ be represented by (f, s), $s: Y \longrightarrow Z$. Then there exists $u: Z \longrightarrow Z'$ such that $us \in S_Y$ If we let $\beta = (Id_{Z'}, s')$ then from the diagram,

$$\begin{tikzcd}
 & Z \arrow[r, "u"] & Z' \\
X \arrow[ur, "f"] & Y \arrow[u, "s"] \arrow[ur, "s'"'] &
\end{tikzcd}$$

one observes that $\beta(uf, Id_{Z'}) = \alpha$ Those objects Z' which are codomains of morphisms of S_Y form a set. Thus the solution set condition is satisfied.

BIBLIOGRAPHY

[1] J. F. Adams, Mathematical lecture notes, Univ. of Chicago, 1973.

[2] A. K. Bousfield, The localization of spaces with respect to homology, Topology 14, 1975.

[3] A. K. Bousfield, Constructions of factorization systems in categories, preprint.

[4] E. H. Brown, Jr., Cohomology theories, Ann. of Math 75, 1962.

[5] A. Deleanu, Existence of the Adams completion for objects of cocomplete categories, J. Pure and Applied Alg. 6, 1975

[6] A. Deleanu, Existence of the Adams completion for C. W. complexes, J. Pure and Applied Alg., 1976.

[7] E. Dyer, Cohomology Theories, W. A. Benjamin, 1969.

[8] A. Heller, Completions in abstract homotopy theory, Trans of the A.M.S., 1970.

[9] Lundell and Weingram, The Topology of C. W. Complexes, Van Nostrand, 1969.

[10] S. Mac Lane, Categories for the Working Mathematician, Springer-Verlag, 1971.

[11] J. Milnor, _On axiomatic homology theory_, Pacific J. Math., 1962.

[12] H. Schubert, _Categories_, Springer-Verlag, 1972.

[13] N. E. Steenrod, _A convenient category of topological spaces_, Michigan Math. J. 14, 1967.