Pure Yang-Mills, Noncommutative Chern-Simons and Noncommutative Quantum Mechanics: a Hamiltonian Approach

by

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Abstract

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This thesis addresses three topics: calculation of the invariant measure for the pure Yang-Mills configuration space in $(3+1)$ dimensions, Hamiltonian analysis of the pure Chern-Simons theory on the noncommutative plane and noncommutative quantum mechanics in the presence of singular potentials.

In Chapter 1 we consider a gauge-invariant Hamiltonian analysis for Yang-Mills theories in three spatial dimensions. The gauge potentials are parameterized in terms of a matrix variable which facilitates the elimination of the gauge degrees of freedom. We develop an approximate calculation of the volume element on the gauge-invariant configuration space. We also make a rough estimate of the ratio of 0^{++} glueball mass and the square root of string tension by comparison with $(2+1)$-dimensional Yang-Mills theory.

In Chapter 2 the Hamiltonian analysis of the pure Chern-Simons theory on the noncommutative plane is performed. We use the techniques of geometric quantization to show that the classical reduced phase space of the theory has nontrivial topology and that quantization of the symplectic structure on this space is possible only if the Chern-Simons coefficient is quantized. Also we show

that the physical Hilbert space of the theory is one-dimensional and give an explicit expression for the vacuum wavefunction. This vacuum state is found to be related to the noncommutative Wess-Zumino-Witten action.

And finally in Chapter 3 we address the question of two-dimensional quantum mechanics in the presence of delta-function potentials which is known to be plagued by UV divergences resulting from the singular nature of the potentials in question. The two particularly interesting examples of this kind are non-relativistic spin zero particles in delta-function potential and Dirac particles in Aharonov-Bohm magnetic background. We show that by extending the corresponding Schrödinger and Dirac equations onto the flat noncommutative space a well-defined quantum theory can be obtained. Complete analytic solution is found in both cases. In the limit of vanishing noncommutativity we recover the standard expressions corresponding to certain self-adjoint extensions of the Hamiltonians in question.

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Chapter 1

On the invariant measure for the Yang-Mills configuration space in (3+1) dimensions

1.1 Introduction

In a series of recent papers a Hamiltonian analysis of Yang-Mills theories in (2+1) dimensions was developed [1, 2, 3]. This was mainly motivated by the fact that, while it is true that gauge theories of direct physical interest are in (3+1) dimensions, the study of Yang-Mills gauge theories in two spatial dimensions can be useful for two reasons. It can be a guide to the more realistic case of three dimensions, and secondly, gauge theories in two spatial dimensions can be interpreted as an approximation to the high-temperature phase of QCD with the mass gap playing the role of the magnetic mass. (It should be pointed out that, precisely for these reasons, there have been many analyses of (2+1)-dimensional gauge theories starting from the early days [4].) In this chapter, we shall describe a similar Hamiltonian analysis of Yang-Mills theories in (3+1) dimensions [5], carrying over some of the lessons from the lower dimensional analysis.

In the (2+1) dimensional theory, the $A_0 = 0$ gauge was chosen and the complex components of the spatial gauge field, viz., A_z, $A_{\bar{z}}$ were parametrized as $A_z = -\partial_z M\ M^{-1}$, $A_{\bar{z}} = M^{\dagger -1}\partial_{\bar{z}} M^{\dagger}$, where $M, M^{\dagger}$ are $SL(N, \mathbf{C})$-matrices for an $SU(N)$ gauge theory. The basic gauge-invariant variable for the theory is then the hermitian matrix field $H = M^{\dagger}M$. This particular parametrization of the potentials proved to be very useful since the Jacobian for the transformation of variables and the volume element on space of gauge-invariant configurations could be exactly calculated. This invariant volume measure on the physical configuration space, which also determines the inner products for wavefunctions, is given in terms of the Wess-Zumino-Witten (WZW) action for the field H [6, 7, 1]. Considerations of integrable representations of the WZW model then showed that normalizable wavefunctions are functions of the current $J = (N/\pi)\partial_z H\ H^{-1}$. In other words, the wavefunctions have to be more restricted than being just functions of H; they can only depend on H via the specific combination in J. The regularized kinetic energy operator, which is the Laplacian on this infinite-dimensional configuration space, is given in terms of functional derivatives with respect to J; the potential energy can also be written in terms of J [2]. The vacuum wavefunction Ψ_0 of the theory was obtained by solving the (functional) Schrödinger equation in the approximation of keeping all terms in $\log \Psi_0$ which are quadratic in J, with a systematic expansion for the higher order terms. The vacuum wavefunction agrees with perturbation theory for the high momentum modes. The expectation value of the Wilson loop operator and hence the string tension were calculated [3]. The values for the string tension agree within 3% of recent Monte Carlo evaluations [8]. Finally, the propagating particles in the perturbative regime can be shown to have a mass $m = e^2N/2\pi$. This may be taken as a prediction for the magnetic mass of gluons in high temperature QCD [9]. This result compares favorably with resummation calculations of this quantity [10] and with lattice estimates, keeping in mind that this is a difficult lattice calculation as well [11]. Finally, these techniques can also be extended to the Yang-Mills-Chern-Simons theory [12].

While this Hamiltonian analysis still leaves many open questions, it is fair to claim that some progress in understanding the (2+1)-dimensional case has indeed been achieved. It is worth noting that the vacuum wavefunction which was obtained, irrespective of the calculations preceding it, has the desirable features of agreeing with the perturbative vacuum wavefunction in the high momentum limit and giving an area law for the Wilson loop with a string tension which agrees closely with the lattice calculations. Therefore, further study along these lines, in particular exploring a similar strategy in (3+1) dimensions, is warranted.

In section 2, we will introduce the parametrization of the gauge potentials in terms of the matrix variables. The calculation of the volume measure of the configuration space (and hence the inner product for wavefunctions) is taken up in section 3. Section 4 gives a number of remarks on this result and its connection to dimensional transmutation in (3+1)-dimensional Yang-Mills theory. And finally a brief review of [1, 2, 3] is given in Appendix A.

1.2 The parametrization of the gauge potentials

We shall discuss an $SU(N)$-gauge theory and also choose the gauge $A_0 = 0$, as is convenient for a Hamiltonian formulation. The remaining gauge potentials can be written as $A_i = -it^a A_i^a$, $i = 1, 2, 3$, where t^a are hermitian $(N \times N)$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = if^{abc}t^c$, $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$.

We start by recalling that the key ingredients of the (2+1) dimensional analysis were the following:

1. the parametrization of the potentials in terms of the matrix M which allowed the realization of gauge transformations in a homogeneous way, $M^g = gM$.
2. the calculation of the gauge-invariant measure on the configuration space.
3. evaluation of the Hamiltonian in terms of these gauge-invariant variables.
4. solving the functional Schrödinger equation for the vacuum wave function.
5. calculating the string tension and other quantities of interest.

The study of the first two steps in (3+1) dimensions will be taken up in this chapter. Let $\mathcal{A}$ denote the set of all gauge potentials A_i^a. Gauge transformations act on A_i in the standard way, $A_i \to A_i^g$, where

$$A_i^{(g)} = gA_i g^{-1} - \partial_i g g^{-1} \tag{1.1}$$

and $g(\vec{x}) \in SU(N)$. The gauge group $\mathcal{G}_*$ is defined by

$$\mathcal{G}_* = \left\{ \text{set of all } g(\vec{x}) : \mathbf{R}^3 \to SU(N), \quad g \to 1 \text{ as } |\vec{x}| \to \infty \right\} \tag{1.2}$$

The space of gauge-invariant field configurations is $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$. A parametrization of the gauge potentials is equivalent to choosing coordinates on the configurations space. Since the space $\mathcal{C}$ has nontrivial topology, any parametrization is restricted to some open region. We use a parametrization in a region which includes $A = 0$ and calculate (approximately) the volume measure of $\mathcal{C}$ for this region. (Not surprisingly, the geometry and topology of the Yang-Mills configuration space in three spatial dimensions have also been studied by a number of authors, see references [13, 14]. For a recent summary and new results on the metric, see [15].)

Going back to YM_{2+1}, we start by asking why it is possible to parametrize A_z as $-\partial_z M M^{-1}$. Notice that this parametrization may be written as

$$(\partial_z + A_z)M = 0 \tag{1.3}$$

and one can convert it to an integral equation

$$\begin{aligned} M(x) &= 1 - \int_{x'} S(x,x')\, A_z(x')M(x') \\ \partial_z\, S(x,x') &= \delta^{(2)}(x - x') \end{aligned} \tag{1.4}$$

With this equation, we see that, at least iteratively, we can find an M for each given A_z. This establishes a mapping $A_z \to M$. (There are much more elegant and more general ways to justify the parametrization $A_z = -\partial_z M M^{-1}$, but this simple argument is most suitable for what follows [1].) Notice that the key ingredient is the invertibility of ∂_z. The first term involving A in a series expansion for M, namely, $\int (\partial_z)^{-1} A_z$, is a complex matrix which is traceless since A_z has no trace. It is thus an element of the Lie algebra of $SL(N, \mathbf{C})$, showing that M can be taken to be in $SL(N, \mathbf{C})$. Conversely, M contains $\dim[SL(N, \mathbf{C})] = 2 \times \dim[SU(N)]$ independent functions corresponding exactly to the number of independent functions needed for the potential, A_i, $i = 1, 2$, therefore one has the map $M \to A_z$ as well.

Since ∂_z is the chiral Dirac operator in two dimensions, the invertibility of ∂_z is equivalent to the existence of the propagator for the chiral Dirac theory. In three Euclidean dimensions, which is appropriate for the (3+1)-dimensional theory, there is no chirality, but we can use the Dirac operator $\sigma \cdot \partial$ where σ_i, $i = 1, 2, 3$, are the Pauli matrices. We then *define* a matrix M by

$$(\sigma \cdot \partial + \sigma \cdot A)\, M = 0 \tag{1.5}$$

On such a matrix M, gauge transformations act by $M \to M^g = gM$, where g is an element of $SU(N)$. Equation (1.5) has the formal inversion

$$M(x) = 1 - \int_y \left(\frac{1}{\sigma \cdot \partial}\right)_{xy} \sigma \cdot A(y)\ M(y) \tag{1.6}$$

where

$$\begin{aligned}
\left(\frac{1}{\sigma \cdot \partial}\right)_{xy} &= -\int \frac{d^3p}{(2\pi)^3} \frac{i\sigma \cdot p}{p^2}\ e^{ip\cdot(x-y)} \\
&= -\sigma \cdot \partial_x G(x,y) \\
G(x,y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2}\ e^{ip\cdot(x-y)}
\end{aligned} \tag{1.7}$$

To first order in the A's, the solution for M is then

$$\begin{aligned}
M &\approx 1 - it^a\theta^a + \sigma_i t^a \int_y G(x,y)\epsilon_{ijk}\partial_j A^a_k(y) \\
\theta^a &= \int_y G(x,y)\partial \cdot A^a(y)
\end{aligned} \tag{1.8}$$

The term $-it^a\theta^a$ on the right hand side of expression (1.8) for M can be removed by a gauge transformation of the form $M \to \exp(it^a\theta^a)\ M$, consistent with the fact that $\partial \cdot A$ represents the gauge degree of freedom, to linear order in A. The last term shows that the infinitesimal generators, for whatever group M belongs to, must include $\sigma_i t^a$, which are a subset of generators of $SU(2N, \mathbf{C})$. Completion of the algebra under commutation rules shows that we need all of $SU(2N, \mathbf{C})$. Thus generally we must take M to be an element of $SU(2N, \mathbf{C})$. Equation (1.5) thus gives a map $A \to M \in SU(2N, \mathbf{C})$.

An arbitrary element of $SU(2N, \mathbf{C})$ will contain $2 \times (4N^2 - 1)$ parameter functions. Thus arbitrary $SU(2N, \mathbf{C})$-matrix functions M contain too many parameters to give a faithful coordinatization of a region of $\mathcal{A}$, we will need to use constraints on M. We will now work out the required constraints. For most of what follows, it is convenient to use $U(2N)$ rather than $SU(2N)$. We define the set of hermitian matrices $\{t^A\}$, $A = 1, 2, \cdots, 4N^2$ as the set $\{1 \otimes t^a, \sigma_i \otimes t^a\}$,

$a = 1, 2, \cdots, N^2$. t^a are taken as $(N \times N)$ hermitian matrices normalized by $\mathrm{Tr} t^a t^b = \frac{1}{2}\delta^{ab}$. $\{-it^a\}$ form an antihermitian basis for $U(N)$ embedded in $U(2N)$. The set of matrices $\{-it^A, t^A\}$ form a basis for the Lie lagebra of $U(2N, \mathbf{C})$. The normalization condition for the t^A is $\mathrm{Tr} t^A t^B = \delta^{AB}$; they further obey the completeness relation $t^A_{mn} t^A_{pq} = \delta_{np}\delta_{mq}$.

Now let M be an arbitrary $U(2N, \mathbf{C})$ matrix. We can then expand

$$\sigma \cdot \partial M M^{-1} = i\ \phi^a t^a + i\ \sigma \cdot A^a t^a \tag{1.9}$$

where ϕ^a, A^a_i are in general complex functions. If we were to start from real A^a_i and use equation (1.5), then ϕ^a in (1.9) would be zero. Thus, to eliminate unwanted degrees of freedom starting from an arbitrary M, we must impose the conditions $A^a_i - \bar{A}^a_i = 0, \quad \phi^a = 0$. These are equivalent to

$$\mathrm{Tr}(t^a \sigma \cdot \partial M M^{-1}) = 0 \tag{1.10}$$

$$\partial_i \cdot (M^\dagger \sigma_i M) = 0 \tag{1.11}$$

The only remaining degree of freedom in M then corresponds to the real part of A^a_i which is the $U(N)$-gauge potential.

It is instructive to work out these conditions for M close to the identity. Writing $M \approx 1 + it^a\varphi^a + i\sigma_i t^a \Theta^a_i$, we find

$$\sigma \cdot \partial M M^{-1} = it^a \partial_i \Theta^a_i \ + \ i\sigma_k t^a \left(\partial_k \varphi^a + i\epsilon_{ijk}\partial_i \Theta^a_j\right) \tag{1.12}$$

Imposing the constraint (1.11) on this, and separating out the real and imaginary parts of the functions, we get

$$\begin{aligned} \partial_i(\mathrm{Im}\Theta^a_i) &= 0 \\ \partial_i(\mathrm{Im}\varphi^a) + \epsilon_{ijk}\partial_j \mathrm{Re}\Theta^a_k &= 0 \end{aligned} \tag{1.13}$$

The second of these equations gives the Laplace equation for $\mathrm{Im}\varphi^a$, namely, $\partial^2 \mathrm{Im}\varphi^a = 0$, so that with proper boundary conditions, we can take $\mathrm{Im}\varphi^a = 0$.

Further, we find $\text{Re}\Theta_i^a = \partial_i \xi^a$ for some scalar functions ξ^a. Putting this back into (1.12) and comparing with (1.9) we find

$$\begin{aligned} A_i^a &= \partial_i \varphi^a - \epsilon_{ijk} \partial_j \text{Im}\Theta_k^a \\ \phi^a &= \partial^2 \xi^a \end{aligned} \tag{1.14}$$

The constraint (1.10) eliminates ϕ^a (or ξ^a). The functions φ^a (which are now real) represent the gauge degrees of freedom. The gauge invariant degrees of freedom are given by $\text{Im}\Theta_i^a$, which are only two polarizations ($2 \times N^2$ functions) because of the condition $\partial_i(\text{Im}\Theta_i^a) = 0$ in (1.13). It is rather well known that an Abelian gauge potential can be parametrized in the form given in (1.14), $A_i = \partial_i \varphi - \epsilon_{ijk} \partial_j \Theta_k$ with $\partial_i \Theta_i = 0$. Near the zero potential, a similar parametrization will apply to the $U(N)$ potentials as well; equation (1.14) is just this, with the required N^2 replication of the functions.

Since $M = 1$ corresponds to the zero potential, the above analysis shows that an arbitrary $U(2N, \mathbf{C})$-matrix M, subject to the conditions (1.10, 1.11), does give a faithful coordinatization of $\mathcal{A}$ for a region containing the zero potential. For, given any A, we can generate a corresponding M by solving (1.5) and conversely, given any M subject to (1.10, 1.11), we get a general gauge potential with the correct number of degrees of freedom.

How much of $\mathcal{A}$ or $\mathcal{C}$ can be covered by this parametrization is a very valid and interesting question; at this stage there is no clear answer to this. This is also evidently related to the question of Gribov ambiguities and other topological issues for $\mathcal{C}$ [16, 13, 17]. In the (2+1)-dimensional case, a similar question arises for the parametrization $A_z = -\partial_z M M^{-1}$. In that case, there is an ambiguity in M for a given A, namely, M and $MV(\bar{z})$ give the same A; by ensuring invariance under this holomorphic symmetry for all physical quantities, at least some of the difficulties of transitions from one coordinate patch to another could be circumvented [1].

1.3 The volume measure

We now turn to the calculation of the volume measure on the configuration space. In terms of the fields ϕ^a, A_i^a given in (1.9), introduce the Euclidean metric

$$ds^2 = \int d^3x \, \left(\delta \bar{A}_i^a \delta A_i^a + \delta \bar{\phi}^a \delta \phi^a\right) \tag{1.15}$$

For the gauge potential of interest which is the real part of A_i^a, this is the Euclidean metric which is precisely the metric of interest for the gauge theory. The Euclidean volume measure for the real part of A_i^a can be written as

$$[d \ \mathrm{Re} A_i^a] = \int [dA] \ \delta(A_i^a - \bar{A}_i^a) \ \delta(\phi^a) \ \delta(\bar{\phi}^a) \tag{1.16}$$

$[dA]$ involves all components, A_i^a, $\bar{A}_i^a$ ϕ^a and $\bar{\phi}^a$. The functional Dirac delta functions eliminate all except the real part of A_i^a. The volume $[dA]$ corresponds to the metric (1.15). From the definition (1.9), we have

$$\begin{aligned} \delta A_i^a &= -i \mathrm{Tr}[\sigma_i t^a \sigma_j \mathcal{D}_j (\delta M M^{-1})] \\ \delta \phi^a &= -i \mathrm{Tr}[t^a \sigma_j \mathcal{D}_j (\delta M M^{-1})] \end{aligned} \tag{1.17}$$

where $\mathcal{D}_j$ is defined by

$$\mathcal{D}_j \chi = \partial_j \chi + [\mathcal{A}_j, \chi], \qquad \mathcal{A}_j = -\partial_j M M^{-1} \tag{1.18}$$

The equations in (1.17) may be combined as $\delta A^A = -i \, \mathrm{Tr}[t^A \sigma_j \mathcal{D}_j \theta]$, $\theta = \delta M M^{-1}$. Using the completeness of the t^A, the metric (1.15) can then be simplified as

$$\begin{aligned} ds^2 &= \int d^3x \ \mathrm{Tr} \left(\overline{\mathcal{D}_i \theta} \ \sigma_i \sigma_j \ \mathcal{D}_j \theta\right) \\ &= \int d^3x \ \mathrm{Tr} \left(t^A \sigma_i \sigma_j t^B\right) (\overline{\mathcal{D}_i \theta})^A (\mathcal{D}_j \theta)^B \end{aligned} \tag{1.19}$$

where we use the fact that, in terms of components in the Lie algebra, $\mathcal{D}_j\theta = -it^A(\mathcal{D}_j\theta)^A$, $(\mathcal{D}_j\theta)^A = \partial_i\theta^A + f^{ABC}\mathcal{A}_i^B\theta^C$. $(T^A)_{BC} = -if^{ABC}$ are the Lie algebra generators in the adjoint representation of $U(2N, \mathbf{C})$. We now define the $(4N^2 \times 4N^2)$- matrices

$$(\Sigma_i)^{AB} = \mathrm{Tr}(t^A\sigma_i t^B) \tag{1.20}$$

By the completeness relation for the t^A, these are seen to obey the relation

$$\Sigma_i^{AB}\Sigma_j^{BC} = \mathrm{Tr}(t^A\sigma_i\sigma_j t^C) = \delta_{ij}\delta^{AC} + i\epsilon_{ijk}\Sigma_k^{AC} \tag{1.21}$$

The Σ_i are a $(4N^2 \times 4N^2)$ representation of the algebra of σ_i. The metric (1.19) can thus be written as

$$\begin{aligned} ds^2 &= \int d^3x\ (\overline{\mathcal{D}_i\theta})^A\Sigma_i^{AC}\Sigma_j^{CB}(\mathcal{D}_j\theta)^B \\ &= \int d^3x\ (\overline{\Sigma\cdot\mathcal{D}\theta})^A(\Sigma\cdot\mathcal{D}\theta)^A \\ &= \int d^3x\ \bar{\theta}^A\left[(\Sigma\cdot\mathcal{D})^\dagger(\Sigma\cdot\mathcal{D})\right]^{AB}\theta^B \end{aligned} \tag{1.22}$$

A metric of the form

$$ds^2 = \int d^3x\ \bar{\theta}^A\theta^A = \int d^3x\ \mathrm{Tr}(M^{\dagger-1}\delta M^\dagger\ \delta M M^{-1}) \tag{1.23}$$

is the Cartan-Killing metric for $U(2N, \mathbf{C})$ (for each spatial point) and leads to the Haar measure $d\mu(M, M^\dagger)$ for $U(2N, \mathbf{C})$. By comparison with this we see that the volume measure for the metric (1.22) can be written as

$$\begin{aligned} [dA] &= \det\left[(\Sigma\cdot\mathcal{D})^\dagger(\Sigma\cdot\mathcal{D})\right]\ d\mu(M, M^\dagger) \\ &= d\mu(M, M^\dagger)\ \exp\left(\Gamma + \bar{\Gamma}\right) \end{aligned} \tag{1.24}$$

$$\exp(\Gamma) = [\det(\Sigma\cdot\mathcal{D})]_{reg} \tag{1.25}$$

In equation (1.25), we have explicitly indicated that the determinant is to be evaluated with proper regularization. The regularization should be such that

$\Gamma + \bar{\Gamma}$ is gauge-invariant. The volume element for the real part of A_i^a is then given as

$$[d\mathrm{Re}A_i^a] = \int e^{\Gamma+\bar{\Gamma}}\; d\mu(M, M^\dagger)\; \delta[\sigma \cdot \partial M M^{-1} + \mathrm{h.c.}]\; \delta[\mathrm{Tr}(t^a \sigma \cdot \partial M M^{-1}) - \mathrm{h.c.}] \tag{1.26}$$

The calculation of the volume thus involves several distinct steps. The first is the calculation of the determinant $\exp(\Gamma + \bar{\Gamma})$; the second is the reduction of the Haar measure $d\mu(M, M^\dagger)$ by the elimination of the set of gauge transformations and finally we have to address the question of the constraints given by the δ-functions.

The full determinant can be calculated by computing the determinants of the Dirac-like operators $\Sigma \cdot \mathcal{D}$ and its adjoint and putting the results together in a gauge-invariant way. The regulated form of the determinant of $\Sigma \cdot \mathcal{D}$ can be written as

$$\Gamma_{reg} = \mathrm{Tr}\log \Sigma \cdot \mathcal{D} - \frac{M_2}{M_2 - M_1}\mathrm{Tr}\log(\Sigma \cdot \mathcal{D} + M_1) + \frac{M_1}{M_2 - M_1}\mathrm{Tr}\log(\Sigma \cdot \mathcal{D} + M_2) \tag{1.27}$$

where M_1 and M_2 are regulator masses. We will need to use two regulators of the Pauli-Villars type, with coefficients as given, to eliminate all the divergences.

We can calculate the determinant by a series expansion in powers of the gauge potential. The only unusual point is that the simplification of the traces are more involved because the Σ-matrices do not commute with the Lie algebra of the $\mathcal{A}$'s. Indeed if this were not so, the determinat would be trivial, apart from possible anomalies, since $\mathcal{A}$ has the form $-\partial M M^{-1}$.

The term quadratic in the potentials is given by

$$\Gamma^{(2)} = \frac{1}{2}\int_{x,y} \mathrm{Tr}(t^A \partial_i M M^{-1})(x)\, \mathrm{Tr}(t^B \partial_j M M^{-1})(y) \int \frac{d^3k}{(2\pi)^3}\, e^{ik\cdot(x-y)}\, \Pi_{ij}^{AB}(k)$$

$$\Pi_{ij}^{AB}(k) = -\frac{i}{16\pi}\left[\frac{M_2}{M_2 - M_1}(\mathrm{sgn} M_1) - \frac{M_1}{M_2 - M_1}(\mathrm{sgn} M_2)\right] k_r \mathrm{Tr}([\Sigma_r, \Sigma_i T^A]\Sigma_j T^B)$$
$$-\frac{k}{64}\left(\delta_{rs} + \frac{k_r k_s}{k^2}\right) \mathrm{Tr}(\Sigma_r \Sigma_i T^A \Sigma_s \Sigma_j T^B) \qquad (1.28)$$

where $\mathrm{sgn} M = \frac{M}{|M|}$. The first term in Π_{ij}^{AB} corresponds to a Chern-Simons term. The second term will be seen to be similar to the one-loop vacuum polarization result in three dimensions; the factor $\delta_{rs} + k_r k_s / k^2$ is correct with the connecting plus sign, the usual projection operator will emerge once the traces are evaluated. The following two observations help to simplify these expressions. First, notice that $\mathcal{A}_i$ obeys the identity

$$\partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_j] = 0 \qquad (1.29)$$

so that, to the quadratic order in the potentials we have $\partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i \approx 0$. Secondly, the traces are in the adjoint representation of $U(2N)$, but these can be converted to the fundamental representation. For example, using the completeness of the t^A's, we can write

$$\begin{aligned} [T^A, \Sigma_r]^{BC} &= -if^{ABD}\Sigma_r^{DC} + \Sigma^{BD} i f^{ADC} \\ &= -\mathrm{Tr}([t^A, t^B]\sigma_r t^C) - \mathrm{Tr}(t^B \sigma_r [t^A, t^C]) \\ &= \mathrm{Tr}(t^B [t^A, \sigma_r] t^C) \end{aligned} \qquad (1.30)$$

Using the algebra of the Σ's, we then get

$$\begin{aligned} \mathrm{Tr}[\Sigma_r, \Sigma_i T^A]\Sigma_j T^B &= 2i\epsilon_{rik}\mathrm{Tr}(\Sigma_k T^A \Sigma_j T^B) - \\ &\quad -\mathrm{Tr}\left(\sigma_i [t^A, \sigma_r]\sigma_j \tfrac{1}{2} i f^{CMN} t^C\right)(-if^{BMN}) \\ &= 2i\epsilon_{rik}\mathrm{Tr}(\Sigma_k T^A \Sigma_j T^B) - \frac{C_2}{2}\mathrm{Tr}\left(\sigma_i [t^A, \sigma_r]\sigma_j t^B\right) \end{aligned} \qquad (1.31)$$

where C_2 is the quadratic Casimir for the adjoint representation of $U(2N)$, $f^{AMN}f^{BMN} = C_2\delta^{AB}$. With $\epsilon_{rik}k_r\mathcal{A}_i \approx 0$, the first term gives zero for the Chern-Simons contribution, reducing the trace to the trace in the fundamental. Similar simplification can be done for all the other terms and the final result is

$$\begin{aligned}\Gamma^{(2)} &= \tfrac{1}{2}C_2\Bigg[-\frac{i}{16\pi}\left(\frac{M_2}{M_2-M_1}(\mathrm{sgn}M_1) - \frac{M_1}{M_2-M_1}(\mathrm{sgn}M_2)\right)\int \epsilon^{ijk}\partial_i A_j^a\, A_k^a \\ &\qquad -\frac{1}{128}\int F_{ij}^a \frac{1}{\sqrt{-\nabla^2}} F_{ij}^a + \frac{1}{32}\int \phi^a \sqrt{-\nabla^2}\,\phi^a\Bigg] \end{aligned} \tag{1.32}$$

Here $F_{ij}^a \approx \partial_i A_j^a - \partial_j A_i^a$ to the order we have calculated. The terms involving ϕ's are eventually set to zero by the constraints (1.10, 1.11). The form of the ϕ-terms could also change depending on the regulators, but the final answer is unambiguous since we can set them to zero anyway. The Chern-Simons term will cancel out when we take $\Gamma + \bar{\Gamma}$, as it should, since there is no parity violation in pure (3+1)-dimensional gauge theory. Using these simplifications, we get for the volume measure

$$\begin{aligned}[d\mathrm{Re}A] &= \int d\mu(M,M^\dagger)\ \exp\left(\Gamma+\bar{\Gamma}\right)\ \delta[\sigma\cdot\partial M M^{-1} + \mathrm{h.c.}] \\ &\qquad \times\ \delta[\mathrm{Tr}(t^a\sigma\cdot\partial M M^{-1}) - \mathrm{h.c.}] \\ \Gamma+\bar{\Gamma} &= -\frac{C_2}{128}\int F_{ij}^a \frac{1}{\sqrt{-\nabla^2}} F_{ij}^a\ +\ \mathcal{O}(A^3)\end{aligned} \tag{1.33}$$

Based on gauge invariance, we can say that part of the higher order terms will render the first term fully invariant, so that the result is of the form

$$\Gamma+\bar{\Gamma} = -\frac{C_2}{128}\int F_{ij}^a \left[\frac{1}{\sqrt{-(\partial+A)^2}}\right]^{ab} F_{ij}^b\ +\ \mathcal{O}(A^3) \tag{1.34}$$

with $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f^{abc}A_i^b A_j^c$.

We now turn to the Haar measure $d\mu(M,M^\dagger)$. We are interested in factoring out the gauge transformations which act as $M^g = gM$, $g \in U(N)$. Out of M we can construct the gauge-invariant quantities $H = M^\dagger M$ and $W_i = M^\dagger \sigma_i M$. We write a generic M as

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.35}$$

where a, b, c, d are $(N \times N)$-matrices. We can take a and d to be invertible in general [18]. Elements of the combinations H and W_i give $a^\dagger a$, $d^\dagger d$, $a^\dagger d$, $c^\dagger d$, $b^\dagger d$, etc. They can thus be regarded as functions of H, W_i. The square roots of $a^\dagger a$ and $d^\dagger d$ can be defined by diagonalizing them. We then see that we can write

$$\begin{aligned} a &= U\,\sqrt{a^\dagger a}\,, & b &= U\,\beta \\ c &= U\,\gamma, & d &= U\,V\sqrt{d^\dagger d} \end{aligned} \tag{1.36}$$

where U and V are unitary matrices; V is determined from $a^\dagger d$ as a function of H, W_i. Likewise β and γ are given by $c^\dagger d$ and $b^\dagger d$. Thus the matrix M can generally be parametrized as

$$M = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} N \tag{1.37}$$

where N is a function of H, W_i. The Haar measure is given by the top rank differential form $dMM^{-1} \wedge dMM^{-1} \cdots M^{\dagger -1} dM^\dagger \wedge M^{\dagger -1} dM^\dagger \cdots$ where we substitute (1.37) for M. This brings out a factor $d\mu(U) = dUU^{-1} \wedge dUU^{-1} \cdots$ which is the volume of the gauge part, $U(N)$. The remainder is given entirely in terms of the gauge-invariant combinations H, W_i. In other words, we have $d\mu(M, M^\dagger) = d\mu(U) d\mu(H, W)$; $d\mu(H, W)$ is the volume on the coset space $U(2N, \mathbf{C})/U(N)$. By taking the product of this formally over all spatial points, we have

$$d\mu(M, M^\dagger) = \prod_x d\mu(U) \prod_x d\mu(H, W) \tag{1.38}$$

The elimination of the gauge part of the measure is now trivial, we just get $\prod_x d\mu(H, W)$.

Finally, it is easy to see that the U-dependence of the constraints drops out from the δ-functions in (1.26) or (1.33); they can be written in terms of N or H, W_i. Combining results (1.33) and (1.38) and the arguments given above, we can write the gauge-invariant measure as

$$\begin{aligned} d\mu(\mathcal{C}) &= \prod_x d\mu(H,W)\ \delta[\sigma\cdot\partial N N^{-1} + \text{h.c.}]\ \delta[\text{Tr}(t^a\sigma\cdot\partial N N^{-1}) - \text{h.c.}] \\ &\qquad\qquad \times\ \exp\left(\Gamma + \bar{\Gamma}\right) \\ &\equiv\ d\mu\ \exp\left(\Gamma + \bar{\Gamma}\right) \end{aligned} \tag{1.39}$$

where $\Gamma + \bar{\Gamma}$ has the expansion (1.34).

We can now write the inner product for states $|1\rangle$ and $|2\rangle$, with the gauge-invariant wavefunctions Ψ_1 and Ψ_2, as

$$\langle 1|2\rangle = \int d\mu \exp\left(\Gamma + \bar{\Gamma}\right)\ \Psi_1^*\Psi_2 \tag{1.40}$$

The key result of this paper is this formula for the inner product, along with (1.34, 1.39), which summarize our results on the gauge-invariant volume element for the configuration space $\mathcal{C}$. Notice that, as in the (2+1)-dimensional case, the term $\Gamma + \bar{\Gamma}$ is proportional to the quadratic Casimir C_2, which vanishes for the Abelian theory, once again indicating a significant difference between the Abelian and nonabelian cases.

1.4 Discussion

Equation (1.40) for the inner product shows that the matrix elements of the (3+1)-dimensional theory can be reduced to the correlators of a three-dimensional Euclidean gauge theory with the action $\Gamma + \bar{\Gamma}$ and functional measure $d\mu$. We have obtained the quadratic terms in this action, but not yet calculated the terms which will involve gauge-invariant combinations which are cubic and higher order in the fields, although some of these higher terms can be inferred from gauge invariance. Nevertheless, it is still interesting to look ahead and see what implications our results may have for the physics of the gauge theory.

We can establish some properties of the $(3+1)$-dimensional theory by comparison with the $(2+1)$-dimensional theory. The vacuum wave function for the $(2+1)$-dimensional theory was of the form $\exp(-\pi \int B^2/2e^4N)$ for long wavelength modes. With such a wave function, for the Wilson loop $W_F(C)$ in the fundamental representation of $SU(N)$ we find

$$\begin{aligned} \langle W_F(C)\rangle &= \text{constant} \quad \exp\left[-\sigma \mathcal{A}_C\right] \\ \sqrt{\sigma} &= e^2\sqrt{\frac{N^2-1}{8\pi}} \end{aligned} \tag{1.41}$$

This result was obtained in the Hamiltonian description; nevertheless, based on the full Euclidean invariance of the Wick rotated theory, this may be

expressed as

$$\int d\mu(C) \ \exp\left(-\int \frac{F^2}{4e^2}\right) \ W_F(C) = \langle W_F(C)\rangle = \text{constant} \quad \exp\left[-\sigma \mathcal{A}_C\right] \tag{1.42}$$

This version may in turn be interpreted as the equal time correlator in the $(3+1)$-dimensional theory with a vacuum wave function of the form $\sim \exp(-\int F^2/8e^2)$. Thus if the $(3+1)$-dimensional theory has a vacuum wave function

$$\Psi_0 \sim \exp\left(-\int \frac{F^2}{8\Lambda}\right) \tag{1.43}$$

then we get confinement and a string tension

$$\sqrt{\sigma} = \Lambda\sqrt{\frac{N^2-1}{8\pi}} \tag{1.44}$$

We therefore assume that the wave function has the form (1.43) and ask what other implications it may have. The mass of a 0^{++} glueball in the lower dimensional theory is given by

$$\langle B^2(x)B^2(0)\rangle = \int d\mu(C) \ \exp\left(-\int \frac{F^2}{4e^2}\right) \sim \exp(-M_{0^{++}}|x|) \tag{1.45}$$

for large separations $|x|$. The mass $M_{0^{++}} = \alpha e^2 N$, where α is, in principle, calculable in the Hamiltonian formulation of the $(2+1)$-dimensional theory. An explicit calculation is difficult; lattice data show that $\alpha \approx 0.808$ [8]. We can also think of the result (1.45) as an equal time correlator in the $(3+1)$-dimensional theory, for the wave function (1.43) (with $e^2 \to \Lambda$), in which case the glueball mass is given by $M_{0^{++}} = \alpha \Lambda N$. This means that, if the wave function (1.43) is a good decription in the $(3+1)$-dimensional case, the ratio $M_{0^{++}}/\sqrt{\sigma}$ is the same in the $(3+1)$- and $(2+1)$-dimensional theories. Collecting results

$$
\begin{aligned}
\frac{M_{0^{++}}}{\sqrt{\sigma}} &= \alpha\sqrt{8\pi}\frac{N}{\sqrt{N^2-1}} \\
&\approx 4.05\,\frac{N}{\sqrt{N^2-1}}
\end{aligned}
\tag{1.46}
$$

where we have used the lattice value for the $(2+1)$-dimensional theory. This is then a prediction, based on the premise of (1.43), for the $(3+1)$-dimensional theory. The lattice estimate of this quantity for the $(3+1)$-dimensional theory is approximately 3.37 as $N \to \infty$ [19]; the discrepancy is about 20%. Thus equation (1.43) may be considered to be a reasonable ansatz for the wave function.

The approximate dimension-independence of the glueball masses has been noted before in the context of lattice values. In the context of using wave functions, an argument which has some similarity to ours has been given in [20]. In the context of a parton mass for gluons, a similar observation has been made by Philipsen [21].

There is another lesson from the (2+1)-dimensional case that we can use. In three Euclidean dimensions an action of the form (1.42) can generate a mass gap. This is not yet the gap for the (3+1)-dimensional theory, but a cutoff on modes of low momenta when integrations are actually carried out using (1.42). In turn this can generate a mass gap for the (3+1)-dimensional theory in much the same way as the cut-off on low momentum modes due to the measure factors in the

(2+1)-dimensional analysis can lead to a gap [1].

Chapter 2

Hamiltonian analysis of the noncommutative Chern-Simons theory

2.1 Introduction

Chern-Simons (CS) theories have been extensively investigated in various contexts since their appearance in physics literature as topological mass terms for odd dimensional gauge theories [22]. On the other hand, recent progress in understanding connection between string theory and noncommutative geometry [23] has brought much attention to studies of the field theories over the noncommutative spaces. In particular, noncommutative version of the Chern-Simons theory have been proposed in both star-product [24] and operator formalism [25]. And although this theory has been discussed by many authors by now [26], much of the previous analysis was done using conventional Lagrangian formalism and path-integral quantization techniques. It is, however, well-known that at least in the commutative case, hamiltonian approach has been much more useful in illuminating various aspects of the CS theory. Therefore it appears to be interesting to extend the canonical formalism to the noncommutative Chern-Simons theory (NCCS) as well.

Also as it was found recently in [27], the noncommutative $U(1)$ Chern-Simons theory can be quite useful in describing fractional quantum Hall effect. The argument of that paper crucially depends on whether Chern-Simons coefficient (also known as level number) is quantized or not. After some initial controversy [28] it was finally shown in [29] that noncommutative CS theory shares the same property as its commutative counterpart, *i.e.* the quantization of the level number. In fact the result is even stronger in the noncommutative case. CS coefficient is quantized even for the $U(1)$ theory indicating that as the noncommutativity parameter θ approaches zero, the noncommutative $U(1)$ Chern-Simons theory does not go over smoothly to the commutative one.

In the Lagrangian formalism the reason for level quantization is standard. Like in the commutative case, one can show [29] that NCCS action is not invariant under the gauge transformations belonging to nontrivial homotopy classes of the gauge group $\mathcal{G}$. For transformation with winding number n the action changes by $8\pi^2\lambda n$ and the requirement of single-valuedness of the path-integral measure leads to the following quantization condition on the Chern-Simons coefficient λ

$$\lambda = \frac{n}{4\pi}, \qquad n = \pm 1, \pm 2 \ldots. \tag{2.1}$$

In this chapter we would like to consider canonical quantization of the pure $U(N)$ Chern-Simons theory on the noncommutative plane and to show how this quantization condition appears in the Hamiltonian formalism [30]. In Section 2 we give the classical analysis of the phase space of the theory in the framework of geometric quantization. In particular, we show that the reduced phase space is topologically nontrivial and therefore consistent quantization of the symplectic structure on this space is possible only if the level number is quantized. Section 3 describes canonical quantization of the theory in the functional Schrödinger representation. We find that consistent realization of the Gauss law constraint on the wave functionals leads to nontrivial transformation law of physical states under the gauge transformations. And like in the commutative case, this transformation law combined with single-valuedness of wave functionals gives us once again the Chern-Simons coefficient quantization condition (2.1). In Section 4 we explain how to construct the most general functionals obeying the gauge transformation law mentioned above. It turns out that the physical Hilbert space for our choice of the flat space geometry is one-dimensional. We find an explicit expression for the only physical state of the theory and show that it is naturally connected to the noncommutative Wess-Zumino-Witten action. At the end of this section we describe how the results of the ordinary commutative CS theory can be recovered from our expressions in the limit of vanishing noncommutativity parameter θ.

Throughout this chapter we use the following conventions. The noncommutative plane is defined in the usual way [31] by introducing the coordinate operators (x_1, x_2) which satisfy the commutation relation

$$[x_i, x_j] = i\theta\delta_{ij} \qquad i, j = 1, 2, \tag{2.2}$$

where θ is the c-number parameter characterizing the noncommutativity of space. The actual space can be thought of as a representation of the operator algebra generated by (x_1, x_2), and the standard realization of the noncommutative plane is given by the Fock oscillator basis $|n\rangle, n = 0, 1, 2, \ldots$

$$\begin{aligned} \bar{z}|n\rangle &= \sqrt{2\theta}\,\sqrt{n}\,|n+1\rangle \\ z|n\rangle &= \sqrt{2\theta}\,\sqrt{n-1}\,|n-1\rangle \\ z|0\rangle &= 0 \end{aligned} \tag{2.3}$$

$$z = x_1 + ix_2 \qquad \bar{z} = x_1 - ix_2. \tag{2.4}$$

Functions on this space are the elements of the enveloping algebra of (x_1, x_2), while derivatives are given by the inner automorphisms of that algebra

$$\partial_i(\ldots) = [\hat{\partial}_i, \ldots]. \tag{2.5}$$

These automorphisms are generated by derivative operators

$$\hat{\partial}_i = \frac{i}{\theta}\,\epsilon_{ij}\hat{x}_j \tag{2.6}$$

or

$$\hat{\partial} = -\frac{\bar{z}}{2\theta} \qquad \hat{\bar{\partial}} = \frac{z}{2\theta} \tag{2.7}$$

if we use complex coordinates (2.4). Since we are going to analyze noncommutative Chern-Simons theory in operator formulation, it is convenient to think of the functions as infinite matrices with raw and column indices labeling the oscillator states. In the case of the $U(N)$ theory (as opposed to $U(1)$) we should take the direct sum of N copies of the Fock space (which is certainly isomorphic to a single space) and consider all the relevant quantities (gauge connections, covariant derivative operators, *etc*) as operator-valued $N \times N$ matrices or as infinite matrices with double index labeling both oscillator states and internal degrees of freedom. The space integral on the commutative plane becomes a trace in the noncommutative case

$$\int d^2x \longrightarrow 2\pi\theta\ \mathrm{Tr} \tag{2.8}$$

and for $U(N)$ theory Tr stands for the integration over the noncommutative plane as well as for the $U(N)$ group trace.

2.2 Canonical formalism

The starting point of our analysis is the matrix form of the noncommutative Chern-Simons action proposed in [25]

$$S_{NCCS} = 2\pi i\theta\lambda \int dt \mathrm{Tr}(\frac{2}{3} D_\mu D_\nu D_\kappa \epsilon^{\mu\nu\kappa}) + 4\pi\lambda \int dt \mathrm{Tr} D_0. \tag{2.9}$$

Here $D_\mu, \mu = 0, 1, 2$ - are hermitian matrix-valued covariant derivative operators, transforming adjointly under the $U(N)$ gauge transformations

$$D_\mu \longrightarrow D_\mu^U = U D_\mu U^{-1}. \tag{2.10}$$

These operators are related to ordinary noncommutative gauge connections $A_\mu, \mu = 0, 1, 2$ *via*

$$\begin{aligned} D_i &= -i\hat{\partial} + A_i \qquad i = 1, 2 \\ D_0 &= -i\partial_t + A_0. \end{aligned} \tag{2.11}$$

For canonical quantization purposes it is most convenient to choose the time-axial gauge $A_0 = 0$. In this gauge, the action (2.9) is quadratic in D_1, D_2

$$S_{NCCS} = 2\pi\theta\lambda \int dt \mathrm{Tr}(\partial_t D_1 D_2 - \partial_t D_2 D_1) \tag{2.12}$$

and since it is first order in time derivatives, we can immediately write the symplectic 2-form Ω

$$\Omega = 8\pi\theta\lambda i \ \mathrm{Tr}(\delta Z \delta \bar{Z}) \tag{2.13}$$

as well as Poisson brackets

$$\{Z_{ij}, \bar{Z}_{kl}\} = \frac{i}{8\pi\theta\lambda} \delta_{il} \delta_{jk} \tag{2.14}$$

on the space of all covariant derivatives $\mathcal{Z}$. Here we introduced the complex coordinates $Z = \frac{1}{2}(D_1 - iD_2)$, $\bar{Z} = \frac{1}{2}(D_1 + iD_2)$ on $\mathcal{Z}$, and δ in (2.13) is to be interpreted as denoting exterior derivative on this space (we do not write the wedge sign for exterior products on $\mathcal{Z}$ since it is clear from the context). With this choice of coordinates $\mathcal{Z}$ can be considered as a Kähler manifold with Ω being the Kähler form and

$$K = 8\pi\theta\lambda i \mathrm{Tr}(Z\bar{Z}) \tag{2.15}$$

being the Kähler potential.

Moreover, the Hamiltonian is obviously zero meaning that there is no time evolution in this theory. Equivalently, we can say that equations of motion for $Z, \bar{Z}$

$$\partial_t Z = 0 \qquad \partial_t \bar{Z} = 0 \tag{2.16}$$

are satisfied trivially with time-independent matrices. However, these matrices have to satisfy an extra constraint (the Gauss law constraint)

$$\theta[Z, \bar{Z}] + \frac{1}{2} = 0 \tag{2.17}$$

which appears from (2.9) as an equation of motion for A_0. Equation (2.17) is a matrix identity, although matrix indices are suppressed for simplicity. It is also easy to see that this equation can not be satisfied with finite matrices, meaning that we consider an essentially infinite-dimensional matrix model.

At this point we have the two alternatives. One may impose the Gauss law constraint at the classical level. This yields the reduced phase space, the set of matrices $Z, \bar{Z}$ satisfying (2.17) up to gauge transformations, endowed with a symplectic structure inherited from (2.13). This reduced phase space may be quantized using the holomorphic polarization induced by complex structure on the noncommutative plane. In this discussion, however, we will follow an alternative procedure of quantizing the Poisson brackets (2.14) first. In this case $\mathcal{Z}$ may be considered as the phase space of the theory before reduction by the action of the gauge symmetries. Reduction is done by requiring that Gauss law constraint acts on the Hilbert space thus selecting the subspace of physical states. As will be shown shortly, the quantization of the level number appears in this case as a consistency condition for performing such reduction.

But before we can proceed a few words about gauge transformations are in order. The $A_0 = 0$ condition does not fix the gauge completely: one can still make time-independent gauge transformations. However, we have to be careful about what the allowed gauge transformations are. We want to show now that the group $\mathcal{G}$ of the valid gauge transformations is given by those unitary matrices only, which act as identity on the oscillator basis $|n\rangle$ as $n \to \infty$. This property is the noncommutative version of the requirement that gauge transformations go to identity at spatial infinity.

For infinitesimal gauge transformation $U = 1 + \phi + \ldots$ (ϕ - antihermitian matrix) we obtain from (2.10)

$$\begin{aligned} \delta Z &= [\phi, Z] \\ \delta \bar{Z} &= [\phi, \bar{Z}]. \end{aligned} \tag{2.18}$$

The vector field on $\mathcal{Z}$ generating such transformation is

$$\xi = [\phi, Z]_{ij}\frac{\delta}{\delta Z_{ij}} + [\phi, \bar{Z}]_{ij}\frac{\delta}{\delta \bar{Z}_{ij}}. \tag{2.19}$$

By contracting this with Ω we get

$$\begin{aligned} i_\xi \Omega &= 8\pi\theta\lambda i \mathrm{Tr}([\phi, Z]\delta\bar{Z} - [\phi, \bar{Z}]\delta Z) \\ &= 8\pi\theta\lambda i \mathrm{Tr}(\phi[Z, \delta\bar{Z}] + \phi[\delta Z, \bar{Z}]) \\ &= 8\pi\theta\lambda i \mathrm{Tr}(\phi\, \delta[Z, \bar{Z}]). \end{aligned} \tag{2.20}$$

This identifies the generator G of infinitesimal gauge transformation (2.18) up to an arbitrary constant as

$$G_{ij} = 8\pi\theta\lambda i[Z, \bar{Z}]_{ij} + \text{const}\,\delta_{ij}. \tag{2.21}$$

We can fix this constant by requiring that $G(\phi) = 0$ condition is equivalent to the Gauss law constraint (2.17). Therefore, the fixed expression for $G(\phi)$ is

$$G(\phi) = 8\pi\lambda i \text{Tr}\,\phi(\theta[Z, \bar{Z}] + \frac{1}{2}). \tag{2.22}$$

As a consistency check we can evaluate, using the canonical Poisson brackets (2.14), the commutator of two such transformations with infinitesimal parameters ϕ and ρ

$$[G(\phi), G(\rho)] = -iG([\phi, \rho]) - 4\pi\lambda \text{Tr}[\phi, \rho]. \tag{2.23}$$

From this expression we see that the algebra of these generators gives the representation of the Lie algebra of the gauge group $\mathcal{G}$ provided

$$\text{Tr}[\phi, \rho] = 0. \tag{2.24}$$

This last condition is satisfied only by functions ϕ, ρ which act as zero on the oscillator basis states $|n\rangle$ for large n. In this case, ϕ and ρ are essentially finite matrices and we can use the cyclicity of trace to prove (2.24). This also validates the statement we have made above that closure of the algebra of gauge transformations restricts $\mathcal{G}$ only to those unitary matrices which go to identity at spatial infinity.

Once we have identified the group $\mathcal{G}$ we would like to analyze the reduced phase space $\mathcal{Z}/\mathcal{G}$ of covariant derivatives modulo gauge transformations in more detail. We want to show now that this space has nontrivial topology, in particular, that there are closed noncontractible two-surfaces in $\mathcal{Z}/\mathcal{G}$.

Let $(Z_0, \bar{Z}_0)$ denote a specific set of matrices corresponding to a point in $\mathcal{Z}$. Consider now the 2-surface in this space parameterized by σ and τ

$$Z = (1-\sigma)Z_0 + \sigma U Z_0 U^{-1} \quad (2.25)$$
$$\bar{Z} = (1-\sigma)\bar{Z}_0 + \sigma U \bar{Z}_0 U^{-1}$$

where $0 \leq \sigma \leq 1$ and $U(\tau)$ is a one-parameter family of gauge transformations with $U(0) = U(1) \equiv \mathbf{1}$, $0 \leq \tau \leq 1$. Easy to see that in the reduced phase space $\mathcal{Z}/\mathcal{G}$, the boundary of this surface corresponds to the single point $(Z_0, \bar{Z}_0)$ and so we have a closed 2-surface in $\mathcal{Z}/\mathcal{G}$. This closed surface is not contractible if $U(\tau)$ traces a noncontractible path in $\mathcal{G}$. We can now integrate the symplectic 2-form Ω over this surface:

$$\delta Z = \delta\sigma(U Z_0 U^{-1} - Z_0) - \sigma U[Z_0, U^{-1}\delta U]U^{-1} \quad (2.26)$$
$$\delta \bar{Z} = \delta\sigma(U \bar{Z}_0 U^{-1} - \bar{Z}_0) - \sigma U[\bar{Z}_0, U^{-1}\delta U]U^{-1}$$

so

$$\begin{aligned} \int \Omega \;&=\; 8\pi\theta\lambda i \int \delta\sigma\,\sigma \mathrm{Tr}\left[(\bar{Z}_0 - U^{-1}\bar{Z}_0 U)[Z_0, U^{-1}\delta U]\right] - \\ &\quad -8\pi\theta\lambda i \int \delta\sigma\,\sigma \mathrm{Tr}\left[(Z_0 - U^{-1}Z_0 U)[\bar{Z}_0, U^{-1}\delta U]\right] \\ &= -16\pi\theta\lambda i \int \delta\sigma\,\sigma \mathrm{Tr}[Z_0, \bar{Z}_0]U^{-1}\delta U + \\ &\quad +8\pi\theta\lambda i \int \delta\sigma\,\sigma \delta \mathrm{Tr}\left(\bar{Z}_0 U Z_0 U^{-1} - Z_0 U \bar{Z}_0 U^{-1}\right) \end{aligned} \quad (2.27)$$

The last term integrates to $\mathrm{Tr}\left(\bar{Z}_0 U Z_0 U^{-1} - Z_0 U \bar{Z}_0 U^{-1}\right)$ at $\tau = 0$ and $\tau = 1$. Since $U \equiv \mathbf{1}$ at these points, this term should give zero. Therefore,

$$\begin{aligned} \int \Omega \;&=\; -16\pi\theta\lambda i \int \delta\sigma\,\sigma \mathrm{Tr}[Z_0, \bar{Z}_0]U^{-1}\delta U \\ &= -8\pi\theta\lambda i \int \mathrm{Tr}\left[([Z_0, \bar{Z}_0] + \frac{1}{2\theta})U^{-1}\delta U\right] + 4\pi\lambda i \int_0^1 d\tau \mathrm{Tr} U^{-1} d_\tau U \end{aligned} \quad (2.28)$$

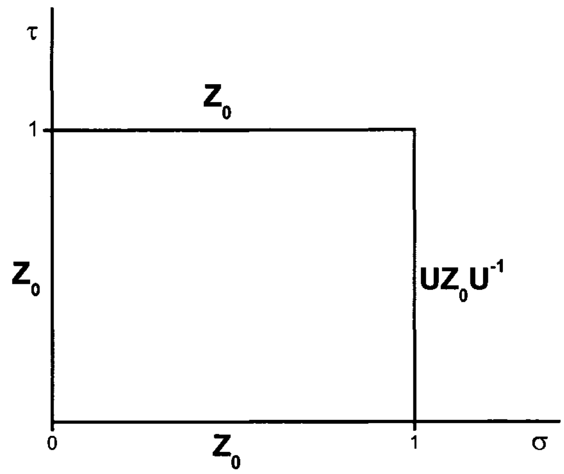

Figure 2.1: Rectangular 2-surface (2.25) in $\mathcal{Z}$ parameterized by σ and τ, $0 \leq \sigma, \tau \leq 1$. In the reduced phase space $\mathcal{Z}/\mathcal{G}$ the boundary of this surface corresponds to a single point Z_0.

and we see that if the Gauss law constraint (2.17) is satisfied, then the first term disappears and we are left with

$$\int \Omega = 4\pi\lambda i \int_0^1 d\tau \mathrm{Tr} U^{-1} d_\tau U. \qquad (2.29)$$

only. As it was shown previously [29], $\Pi_1(\mathcal{G}) = \mathbb{Z}$ and

$$Q[U] = \frac{i}{2\pi} \int_0^1 d\tau \mathrm{Tr} U^{-1} d_\tau U \qquad (2.30)$$

is an integer equal to the winding number of the class in $\Pi_1(\mathcal{G})$ to which $U(\tau)$ belongs. Also from the general principles of geometric quantization we know that the reduced phase space can be quantized only if the integral of $\Omega/2\pi$ over any closed noncontractible surface is an integer. Therefore we can write the following quantization condition

$$4\pi\lambda Q[U] = \text{integer} \qquad (2.31)$$

which can be satisfied for arbitrary $Q[U] \in \mathbb{Z}$ only if the level number is quantized as

$$4\pi\lambda = k \qquad k = 0, \pm 1, \dots \tag{2.32}$$

This is exactly the same quantization condition as was found in [29] using Lagrangian approach to the noncommutative Chern-Simons theory.

2.3 Schrödinger representation

After the preliminary analysis of the classical phase space in the previous section, we would like now to explicitly quantize our theory and to show how does the Chern-Simons coefficient quantization condition (2.32) appear as a requirement of consistency in implementing the Gauss law (2.17) on physical states.

Canonical quantization of the Poisson structure (2.14) leads to the following quantum commutation relations

$$[Z_{ij}, \bar{Z}_{kl}] = -\frac{1}{8\pi\theta\lambda}\delta_{il}\delta_{jk}. \tag{2.33}$$

In order to construct a unitary representation of this canonical algebra, we have to choose polarization on the phase space of the theory. It is convenient to use a holomorphic polarization condition. The wave functionals $\Psi[Z]$ are functionals of Z only; Z is represented trivially as multiplication by Z, while $\bar{Z}$ acts as a functional derivative with respect to Z

$$\bar{Z}_{ij}\Psi[Z] = \frac{1}{8\pi\theta\lambda}\frac{\delta}{\delta Z_{ij}}\Psi[Z]. \tag{2.34}$$

In this representation generator (2.22) of infinitesimal gauge transformations becomes

$$G(\phi) = i[\phi, Z]_{ij}\frac{\delta}{\delta Z_{ij}} + 4\pi\lambda i \operatorname{Tr}\phi. \tag{2.35}$$

It is easy to verify that the algebra of these generators closes

$$[G(\phi), G(\rho)] = -iG([\phi, \rho]) \tag{2.36}$$

provided that we choose ϕ and ρ to satisfy (2.24). Closure of this algebra means that there is no apparent obstruction to demanding that the Gauss law constraint (2.17) be met by requiring that $G(\phi)$ annihilates physical states

$$G(\phi)\Psi[Z] = 0. \tag{2.37}$$

However, as it is known from the ordinary commutative Chern-Simons theory [32, 34] such condition does not necessarily mean gauge-invariance of the physical wave-functionals. In fact, in the commutative case consistent implementation of (2.37) requires that the action of the gauge group on states is realized with a 1-cocycle which leads to multivalued wave-functionals unless the level number λ is quantized. We want to show now that similar arguments apply in the noncommutative case as well.

For an arbitrary gauge transformation $g = e^{\phi}$ its realization on states $\Psi[Z]$ is given by the unitary operator

$$U(g) = e^{-iG(\phi)}. \tag{2.38}$$

As it follows from the definition (2.35), we can split $G(\phi)$ as

$$G(\phi) = G_Z(\phi) + 4\pi\lambda i \mathrm{Tr}\phi, \tag{2.39}$$

where

$$G_Z(\phi) = i[\phi, Z]_{ij}\frac{\delta}{\delta Z_{ij}} \tag{2.40}$$

is the generator of infinitesimal gauge transformations on Z. Therefore,

$$\begin{aligned} \Psi[Z] \longrightarrow U(g)\Psi[Z] &= e^{-iG(\phi)}e^{iG_Z(\phi)}\Psi[Z^g] \\ Z^g &= gZg^{-1}. \end{aligned} \tag{2.41}$$

The prefactor $e^{-iG(\phi)}e^{iG_Z(\phi)}$ can easily be evaluated since $[G_Z(\phi), 4\pi\lambda i \operatorname{Tr}\phi] = 0$ and the result is just $e^{4\pi\lambda \operatorname{Tr}\phi}$. The Gauss law constraint (2.37) requires that physical states $\Psi_{phys}[Z]$ be left unchanged by the action of $U(g)$, since the generator $G(\phi)$ annihilates them (2.37)

$$U(g)\Psi_{phys}[Z] = \Psi_{phys}[Z]. \tag{2.42}$$

Therefore, in the noncommutative Chern-Simons theory, functionals describing physical states are not gauge invariant; rather, according to (2.41), they satisfy

$$\Psi_{phys}[Z^g] = e^{-4\pi\lambda \operatorname{Tr}\phi}\Psi_{phys}[Z]. \tag{2.43}$$

However, the above expression can not be met with single-valued functionals unless λ is quantized. To see this it is useful to rewrite (2.43) in the following equivalent way

$$\Psi_{phys}[Z^g] = (\det g)^{-4\pi\lambda}\ \Psi_{phys}[Z]. \tag{2.44}$$

As was argued in the previous section, the valid gauge transformations are given by essentially finite (although they can be very large) unitary matrices g. For such matrices $\det g$ is well-defined (it is basically a complex number with unit modulus). However, $(\det g)^{-4\pi\lambda}$ is multivalued unless the exponent $-4\pi\lambda$ is an integer. Therefore, for (2.43) to make sense, λ has to be quantized in units of $\frac{1}{4\pi}$

$$4\pi\,\lambda = k, \qquad\qquad k = 0, \pm 1, \ldots \tag{2.45}$$

so again we obtain the same quantization condition as in(2.32).

Finally, to have a well-defined quantum theory we need to define the inner product on the Hilbert space of physical states. The inner product of two wave functionals is given by

$$\langle\Phi|\Psi\rangle = \int [dZ, d\bar{Z}]\, e^{-8\pi\theta\lambda \operatorname{Tr}\bar{Z}Z}\Phi^{\star}[\bar{Z}]\Psi[Z] \tag{2.46}$$

where the exponential prefactor is just the Kähler potential (2.15), as is standard in holomorphic quantization. This prefactor ensures that $\bar{Z}$ is the hermitian conjugate of Z, the quantum version of the classical relation $\bar{Z} = (Z)^{\dagger}$. Also it can be easily verified that this inner product is insensitive to the gauge noninvariance of states meaning that physical expectation values do not depend on the gauge choice as they should.

2.4 Physical states

Given that we know the gauge transformation properties of the physical states, we now want to explicitly construct functionals that obey (2.43). But before we proceed to the details of this construction, we would like to briefly outline our strategy. Equation (2.44) tells us, that under the gauge transformation g the wave functionals are multiplied by some power of $\det g$. Therefore, if h is some noncommutative matrix field parametrizing covariant derivative Z and transforming as

$$h \longrightarrow h^g = gh, \tag{2.47}$$

then $(\det h)^{-4\pi\lambda}$ transforms exactly as (2.44), and the most general functional with the correct transformation properties is given by

$$\Psi_{\text{phys}}[Z] = (\det h)^{-4\pi\lambda}\,\psi[Z]. \tag{2.48}$$

Here $\psi[Z]$ is an arbitrary gauge-invariant functional of Z only. In the case of pure Chern-Simons theory on the noncommutative plane the only such functional is $\psi[Z] \sim 1$ so the physical Hilbert space of the theory is one-dimensional with the only vacuum state given (up to normalization) by

$$\Psi_{VAC}[Z] = (\det h)^{-4\pi\lambda}. \tag{2.49}$$

To make these heuristic arguments precise we need, first of all, parametrization of the covariant derivative Z in terms of the matrix field h obeying (2.47) and then, we need to give an exact meaning to $\det h$ since, unlike the gauge uransformations g, h is an infinite matrix and it is not clear *a priori* what $\det h$ is in this case and whether the usual property of determinants $\det h^g = (\det h)(\det g)$, which is crucial in deriving (2.49), still holds in the noncommutative case.

In the commutative field theory in two space dimensions, the following parametrization of the gauge potential A_z is frequently used

$$A_z = -\partial_z h h^{-1}. \tag{2.50}$$

The reason why such parametrization is possible is that in two dimensions operator ∂_z is invertible and, for any field configuration A_z we can invert (2.50) and find corresponding h at least perturbatively as a series in powers of A_z. It is also easy to verify, that under the gauge transformations h transforms as in (2.47).

With these ideas in mind, we introduce the following parametrization of the noncommutative covariant derivative

$$Z = \frac{i}{2\theta} h\bar{z}h^{-1}. \tag{2.51}$$

One can use the relationship (2.11) between covariant derivative Z and noncommutative gauge potential A_z as well as the definition of the noncommutative derivatives (2.5) to see that (2.51) is the noncommutative analogue of (2.50). It is shown in the Appendix C that (2.51) gives the valid parametrization in the sense that it can be perturbatively solved for h.

Now we have to clarify the meaning of $\det h$ in (2.49) and to do that we start with the well-known expression

$$\det h = e^{\mathrm{Tr}\log h} \tag{2.52}$$

which we use to put $\det h$ into an exponential form. Then $\mathrm{Tr}\log h$ can be written as

$$\mathrm{Tr}\log h = \int_0^1 d\tau\, \mathrm{Tr}\,(\tilde{h}^{-1}\partial_\tau \tilde{h}) \tag{2.53}$$

where $\tilde{h}(\tau)$ is the smooth extension of h onto the line segment $\tau \in [0,1]$ with the following values at the boundary

$$\begin{aligned} \tilde{h}(0) &\equiv \mathbf{1} \\ \tilde{h}(1) &\equiv h. \end{aligned} \tag{2.54}$$

One can use a power series expansion of $\tilde{h}(\tau) = e^{\chi(\tau)}$ to verify (2.53). Under the gauge transformation $U = e^{\phi}$ the value of $\chi(\tau)$ on the $\tau = 1$ boundary transforms as $\chi^U(1) = \chi(1) + \phi + [\phi, \chi(1)] + \ldots$ and if ϕ goes sufficiently fast to zero at infinity (as required by (2.24)) then under the trace all the commutator terms vanish and we get

$$e^{\mathrm{Tr}\log h^U} = e^{\mathrm{Tr}\log h + \mathrm{Tr}\,\phi}. \tag{2.55}$$

This is exactly the transformation we need for the physical states. As a result we see that (2.53) has all the required transformation properties and therefore can be used to write the vacuum state as

$$\Psi_{VAC}[Z] = e^{-4\pi\lambda \int_0^1 d\tau \mathrm{Tr}\,(\tilde{h}^{-1}\partial_\tau \tilde{h})}. \tag{2.56}$$

Although this expression can already be used as the definition of $\Psi_{VAC}[Z]$, we would like to explore it a bit further. In particular, we want to show how it is related to the noncommutative WZW action

$$\begin{aligned} S_{NCWZW} = S_{KIN} + S_{NCWZ} &= 2\pi\theta \mathrm{Tr}\,(h^{-1}\partial_z h)(h^{-1}\partial_{\bar{z}} h) + \\ & \quad 2\pi\theta \int d\tau \mathrm{Tr}\,\left(h^{-1}\partial_\tau h\left[h^{-1}\partial_z h, h^{-1}\partial_{\bar{z}} h\right]\right). \end{aligned} \tag{2.57}$$

Using the definition of derivatives on the noncommutative plane (2.5) and formally expanding the commutators we can transform the Wess-Zumino term as

$$\begin{aligned} S_{NCWZ} &= 2\pi\theta\int d\tau \mathrm{Tr}\left(h^{-1}\partial_\tau h\left[h^{-1}\partial_z h, h^{-1}\partial_{\bar z} h\right]\right) \\ &= \frac{\pi}{2\theta}\int d\tau \mathrm{Tr}\left(h^{-1}\partial_\tau h\left[h^{-1}[z,h], h^{-1}[\bar z, h]\right]\right) \\ &= 2\pi\int d\tau \mathrm{Tr}\left(h^{-1}\partial_\tau h\right) - \frac{\pi}{2\theta}\int d\tau \mathrm{Tr}\, h^{-1}\partial_\tau h\left([h^{-1}zh, \bar z] + [z, h^{-1}\bar z h]\right) \\ &= 2\pi\int d\tau \mathrm{Tr}\left(h^{-1}\partial_\tau h\right) + \frac{\pi}{2\theta}\int d\tau\partial_\tau \mathrm{Tr}\left(h^{-1}zh\bar z - h^{-1}\bar z h z\right) \\ &= 2\pi\int d\tau \mathrm{Tr}\left(h^{-1}\partial_\tau h\right) + \frac{\pi}{2\theta}\mathrm{Tr}(h^{-1}zh\bar z - z\bar z) - \frac{1}{4\theta^2}\mathrm{Tr}(hzh^{-1}\bar z - z\bar z). \end{aligned} \tag{2.58}$$

Similarly the kinetic term becomes

$$\begin{aligned} S_{KIN} &= 2\pi\theta\mathrm{Tr}(h^{-1}\partial_z h)(h^{-1}\partial_{\bar z} h) \\ &= -\frac{\pi}{2\theta}\mathrm{Tr}(h^{-1}zh - z)(h^{-1}\bar z h - \bar z) \\ &= \frac{\pi}{2\theta}\mathrm{Tr}(h^{-1}zh\bar z - z\bar z) + \frac{\pi}{2\theta}\mathrm{Tr}(hzh^{-1}\bar z - z\bar z). \end{aligned} \tag{2.59}$$

Note that we did not use the cyclicity of trace while performing these transformations. Also, although each of the expressions like $\mathrm{Tr}\, z\bar z$ or $\mathrm{Tr}\, h^{-1}zh\bar z$ is divergent, their difference, as it appears in the last lines of (2.58) and (2.59), is, in fact, a well-defined finite quantity. These two simple observations in certain sense validate the formal manipulations that we have done.

Now, if we put the two terms together, we get the following identity

$$2\pi\int d\tau \mathrm{Tr} h^{-1}\partial_\tau h = S_{NCWZW} + 2\pi i \mathrm{Tr}(zA) \tag{2.60}$$

which can be used to rewrite the vacuum state as

$$\Psi_{VAC}[Z] = e^{-2\lambda S_{NCWZW} - 4\pi\lambda i \mathrm{Tr}(zA)}. \tag{2.61}$$

This expression is much more convenient for the analysis of the transformation properties of $\Psi_{VAC}[Z]$ since now we can use the well-known Polyakov-Wiegmann identity [35] (which still holds in the noncommutative theory)

$$S_{NCWZW}(gh) = S_{NCWZW}(g) + S_{NCWZW}(h) + 2\int \mathrm{Tr}\,(g^{-1}\bar{\partial}g\partial h h^{-1}) \qquad (2.62)$$

to prove that (2.61) transforms properly under the unitary gauge transformations. Also it is easy to see that our parametrization of Z in terms of h is somewhat ambiguous. Really, if h is some solution of (2.51), then $hf(\bar{z})$, where $f(\bar{z})$ is some antiholomorphic function, gives an equivalent solution of that equation and can be used to define the vacuum state of the theory as well. $\Psi_{VAC}[Z]$ should certainly be invariant with respect to such ambiguity in the choice of parametrization and again we can use the Polyakov-Wiegmann identity to see that this is indeed the case.

Finally the above expression for the ground state looks very similar to the well-known vacuum wave functional of the commutative Chern-Simons theory [32]

$$\Psi_{VAC_{commut}}[A] = e^{-2\lambda S_{WZW}(h)} \qquad (2.63)$$

except for the last term in the exponential of (2.61). The reason for appearance of such term can be easily tracked down to our choice of the covariant derivative operators $Z, \bar{Z}$ as the fundamental set of variables of the theory. Really, the change in the phase space parametrization from $Z, \bar{Z}$ to $A, \bar{A}$ is essentially the canonical shift of variables according to (2.11). Upon such transformation, the path-integral measure in the inner product (2.46) becomes

$$\int [dZ][d\bar{Z}]\, e^{-8\pi\theta\lambda\,\mathrm{Tr}\,(\bar{Z}Z)} = \int [dA][d\bar{A}]\, e^{-8\pi\theta\lambda\,\mathrm{Tr}\,(\bar{A}A) + 4\pi\lambda i\mathrm{Tr}\,(zA) - 4\pi\lambda i\mathrm{Tr}\,(\bar{z}\bar{A}) - \frac{2\lambda}{\theta}\mathrm{Tr}\,(z\bar{z})} \qquad (2.64)$$

and we see, that $e^{4\pi\lambda i\,\mathrm{Tr}\,(zA)}$ can be absorbed into $\Psi[Z]$ thus cancelling the extra term in the wave function ($e^{-4\pi\lambda i\,\mathrm{Tr}\,(\bar{A}\bar{z})}$ correspondingly is absorbed by $\bar{\Phi}[\bar{Z}]$ in

(2.61)). Therefore, the canonically transformed wave functional of variable A is

$$\Psi_{VAC}[A] = e^{-2\lambda S_{NCWZW}(h)}. \tag{2.65}$$

In the commutative limit $\theta \to 0$ the noncommutative WZW action $S_{NCWZW}(h)$ goes to the commutative one $S_{WZW}(h)$ and we trivially recover the ground state of the commutative Chern-Simons theory (2.63). Similarly, the only remaining term [1] $-8\pi\theta\lambda \operatorname{Tr}(\bar{A}A)$ in the Hilbert space measure (2.46) becomes just $-4\lambda \int d^2z \operatorname{Tr}(\bar{A}A)$ in the limit of vanishing noncommutativity and again we obtain the standard expression for the inner product of states in the commutative case

$$\langle\Phi|\Psi\rangle = \int [dA][d\bar{A}]\, e^{-4\lambda \operatorname{Tr}(\bar{A}A)} \Phi^\star[\bar{A}]\Psi[A]. \tag{2.66}$$

2.5 Summary and conclusions

In this chapter the hamiltonian analysis of the pure $U(N)$ Chern-Simons theory on the noncommutative plane was described. It was found that quantization of the level number in the canonical formalism is a consequence of existence of the closed noncontractible surfaces in the reduced phase space of the theory. The quantization condition (2.32) is exactly the same as was previously obtained in [29] using Lagrangian approach. Also like its commutative counterpart, pure noncommutative CS theory turns out to be exactly solvable. We use the techniques of holomorphic quantization to construct an explicit representation of the quantum commutator algebra (2.33). Furthermore, it is shown that the Gauss law constraint (2.17), which selects the subspace of physical states of our theory, can be solved exactly. The physical Hilbert space for our choice of flat space

[1] The other term $-\frac{2\lambda}{\theta}\operatorname{Tr}(z\bar{z})$, being independent of A, can be absorbed into the wavefunction normalization constant.

geometry is found to be one-dimensional and we give an explicit expression for the only physical state of the theory.

Although pure Chern-Simons theory appears to be trivial, it is well-known that in the commutative case it leads to highly nontrivial results when coupled to external sources. Therefore, it appears to be interesting to include external charges into the noncommutative theory as well. In particular, one might address the question of how does the presence of such charges affect the quantum holonomy of physical states. This is currently under investigation.

Chapter 3

Noncommutative quantum mechanics in the presence of delta-function potentials

3.1 Introduction

Singular interaction potentials were introduced in quantum mechanics more than sixty years ago [36]. Since then they found applications in various areas of solid state [37], particle [38] and nuclear [39] physics. It has also been known for a long time that local nature of these potentials at all scales leads to appearance of ultraviolet divergencies in quantum mechanics similar to those encountered in quantum field theory. However, unlike quantum field theory in which one meets UV divergensies primarilly due to the presence of infinite number of degrees of freedom, in quantum mechanics the infinities occur due to the singular nature of the potentials chosen. Mathematical theory of how to treat such potentials is well known [40], [41] and is based on construction of self-adjoint extensions of the

Hamiltonians in question.

On the other hand there has been a lot of recent interest in noncommutative spaces [42], [43], [44] motivated primarilly by string [45] and field theory [46, 47] applications. In particular, on the field theory side it is believed that in certain cases noncommutative modification of the algebraic structure of space-time can provide a natural regularization of UV divergencies. This is certainly the case for compact manifolds [48], [49] since field theories on the noncommutative generalizations of such manifolds posess only finite number of degrees of freedom thus removing the intrinsic reason for appearance of UV divergencies.

The purpose of the present chapter is to study the quantum mechanics on the flat noncommutative two dimensional space [50] in the presence of singular potentials and to show that nonlocality of interaction induced by fuzziness of space leads to a well-defined quantum theory [51]. In Section 3.2 we consider nonrelativistic spin zero particles in a δ-function potential and after a brief discussion of the main results known from the commutative case [52], [55] we give a complete analytic solution of the corresponding Schrödinger equation over the noncommutative plane. Section 3.3 deals with spin-$\frac{1}{2}$ relativistic particles in the Aharonov-Bohm background magnetic field and we use Fock space formalism to obtain the complete set of eigenfunctions for this problem. Also the relationship between commutative and noncommutative solutions is discussed and we show that in the limit of vanishing noncommutativity parameter θ we recover the same solutions as those given by theory of self-adjoint extensions.

Our notations and conventions as well as a review of star-product and Fock space formalisms in noncommutative geometry are given in Appendix B.

3.2 Bosons in a two-dimensional delta-function potential.

3.2.1 Commutative case.

In the commutative case the Schrödinger equation for spin zero particle moving in a two-dimensional δ-function potential, can be written as ($\hbar = 1$)

$$-\frac{1}{2m}\nabla^2\Psi(\mathbf{r}) + V_0\delta(\mathbf{r})\Psi(\mathbf{r}) = E\Psi(\mathbf{r}) \tag{3.1}$$

or, in terms of new coupling constant $\alpha_0 = 2mV_0$ and energy parameter $\tilde{E} = 2mE$,

$$-\nabla^2\Psi(\mathbf{r}) + \alpha_0\delta(\mathbf{r})\Psi(\mathbf{r}) = \tilde{E}\Psi(\mathbf{r}). \tag{3.2}$$

Both the δ-function potential in two dimensions and the kinetic energy operator scale in polar coordinates (ρ, ϕ) as $1/\rho^2$, therefore the coupling α_0 is dimensionless. As a consequence, the Hamiltonian is scale invariant and we can anticipate the presence of logarithmic ultraviolet divergencies, analogous to those appearing in QED and QCD. The standard way of obtaining an exact solution of (3.2) analytically is to use method of self-adjoint extensions. As explained in [52], the two-dimensional Laplace operator ∇^2 is not self-adjoint on a punctured plane and construction of self-adjoint extension requires relaxing the condition of regularity of wave functions at the origin and allowing $\log\rho$ singularity at $\rho = 0$. However, $\delta(\rho)\log\rho$ is then not well-defined. Thus, we need to define $\delta(\rho)\Psi(\rho, \phi)$ for wavefunctions behaving near the origin as

$$\Psi(\rho, \phi) = \psi_0 \log(\mu\rho) + \psi_1 + O(\rho). \tag{3.3}$$

where μ is an arbitrary dimensional parameter. Formal integration of eq.(3.2) over a small disk of radius ϵ followed by taking the limit $\epsilon \to 0$ gives the correct constraint on coefficients ψ_0 and ψ_1

$$2\pi\psi_0 - \alpha_0\psi_1 = 0 \tag{3.4}$$

which we should take as a boundary condition on a wavefunction. More detailed derivation of (3.4) can be found in [40].

With these ideas in mind we can rewrite (3.2) for an axially symmetric wavefunction as

$$\Psi''(\rho) + \frac{1}{\rho}\Psi'(\rho) + \tilde{E}\Psi(\rho) = 0 \tag{3.5}$$

which for $\tilde{E} < 0$ admits

$$\Psi(\rho) = K_0\left(\sqrt{|\tilde{E}|}\ \rho\right) \tag{3.6}$$

as a solution. Here $K_0(x)$ is the modified Bessel function of the third kind [56]. From the asymptotic behaviour of $\Psi(\rho)$ near the origin, namely,

$$\Psi(\rho) = -\log\left(\frac{\tilde{E}\rho}{2}\right) - \gamma + O(\rho) \tag{3.7}$$

and the boundary condition (3.4) we find an expression for the energy of this bound state

$$\tilde{E} = -4\mu^2 e^{-2\gamma} e^{\frac{4\pi}{\alpha_0}}. \tag{3.8}$$

Therefore this theory provides us with an example of a spontaneous breakedown of scale invariance where bound state energy is set by an arbitrary dimensionful parameter μ.

3.2.2 Noncommutative case

In complete analogy with commutative case we can write the Schrödinger equation for a particle moving in a noncommutative δ-function potential as

$$-\hat{\nabla}^2\hat{\Psi}(\hat{\mathbf{x}}) + \alpha_0\hat{\delta}(\hat{x})\hat{\Psi}(\hat{\mathbf{x}}) = \tilde{E}\hat{\Psi}(\hat{\mathbf{x}}), \tag{3.9}$$

but where $\hat{\Psi}(\hat{\mathbf{x}})$ now is an operator valued wavefunction

$$\hat{\Psi}(\hat{\mathbf{x}}) = \int d^2p \; \Phi(p,\bar{p}) e^{i(pz+\bar{p}\bar{z})}$$

and $\hat{\delta}(\hat{x})$ is defined in (B.11).

In momentum space equation (3.9) becomes

$$(p\bar{p} - \tilde{E})\Phi(p,\bar{p}) = -\frac{\alpha_0}{(2\pi)^2} \int d^2\eta \; \Phi(\eta,\bar{\eta}) \; e^{-\frac{\theta}{4}|p-\eta|^2 + \frac{\theta}{4}(\bar{p}\eta - p\bar{\eta})} \tag{3.10}$$

with parameters $\tilde{E} = 2mE$ and $\alpha_0 = 2mV_0$ defined as in commutative case.

Equation (3.10) is an integral equation for momentum space wave-function which, in general, is difficult to solve. But the solution of this particular equation is greatly simplified if one observes that the differential operator

$$\frac{\partial}{\partial p} + \frac{\theta}{4}\bar{p} \tag{3.11}$$

when applied to the r.h.s of (3.10) gives zero, and, therefore, we are left only with

$$\left[\frac{\partial}{\partial p} + \frac{\theta}{4}\bar{p}\right] (|p|^2 - \tilde{E})\Phi(p,\bar{p}) = 0 \tag{3.12}$$

This last equation is easy to solve and, after we pass to the eigenstates $\Phi_n(p)$ of angular momentum operator

$$\Phi(p,\phi) = \sum_{n=-\infty}^{\infty} \Phi_n(p) e^{in\phi}, \tag{3.13}$$

its solutions are given by

$$\Phi_n(p) = \frac{p^n}{p^2 - \tilde{E}} \, e^{-\frac{\theta}{4}p^2}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{3.14}$$

for $\tilde{E} < 0$, while for positive energies eq.(3.12) admits an extra δ-function term and therefore we write

$$\Phi_n(p) = \delta(p^2 - \tilde{E}) + C_n \frac{p^n}{p^2 - \tilde{E}} \, e^{-\frac{\theta}{4}p^2} \tag{3.15}$$

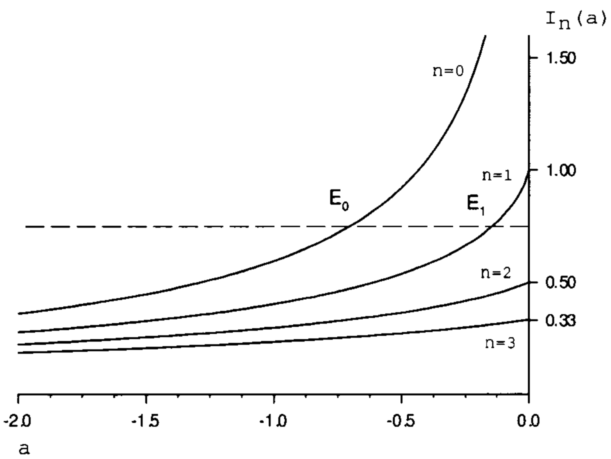

Figure 3.1: Example of graphical solution of eigenvalue equation (3.17). In this case we have two bound states.

By direct substitution, it can be verified that eq.(3.15) gives a complete set of scattering states provided that constants C_n are choosen as the solutions of

$$1 = -\frac{\alpha_0}{4\pi}\left\{C_n I_n\left(\frac{\theta\tilde{E}_n}{2}\right) + \frac{1}{n!}\, e^{-\frac{\theta}{4}\tilde{E}}\left(\frac{\theta}{2}\sqrt{\tilde{E}}\right)^n\right\} \tag{3.16}$$

for $n \geq 0$, and $C_n = 0$ for $n < 0$, with functions I_n defined by

$$I_n(a) = \frac{1}{n!}\int_0^\infty dt \frac{t^n}{t-a} e^{-t} = (-a)^n \Psi(n+1, n+1, -a)$$

and where $\Psi(a, b, c)$ is the confluent hypergeometric function.

Similar analysis shows that Eqs.3.14 also satisfy Schrödinger equation for $n \geq 0$ only, thus giving us bound state solutions the number of which, as well as their energies being given by the following set of eigenvalue equations

$$1 = -\frac{\alpha_0}{4\pi}\, I_n\left(-\frac{\theta|\tilde{E}_n|}{2}\right). \tag{3.17}$$

Example of a graphical solution of these equations is presented in Fig. 3.1, from which it can also be seen, that like in commutative case, the radially symmetric solution ($n = 0$) exists for arbitrary strength of attractive potential but, unlike the commutative case, for sufficiently strong potentials (when dimensionless coupling $|\alpha_0| > 1$) this problem also admits bound states with nonzero angular momentum ($n > 0$), the number of such solutions being a function of α_0 only. It is also instructive to look at limiting cases of very strong and very weak binding potential more closely.

Large coupling ($|\alpha| \gg 1$)

In this case $a = \frac{\theta|\tilde{E}|}{2} \gg 1$ and we can use asymptotic expansion

$$I_n(a) = \frac{1}{a} - \frac{n+1}{a^2} + O\left(\frac{1}{a^3}\right)$$

and write (3.14) as

$$1 = \frac{|\alpha_0|}{4\pi}\left(\frac{1}{a_n} - \frac{n+1}{a_2^2}\right)$$

with solutions

$$a_n = \frac{4\pi}{|\alpha_0|} - (n+1)$$

or

$$\tilde{E}_n = -\frac{2}{\theta}a_n = \frac{8\pi}{\theta\alpha_0} + \frac{2}{\theta}(n+1)$$

and we see that bound state spectrum in this case coincides with that of a simple harmonic ocsillator of frequency $\frac{2}{\theta}$ [53].

Small coupling ($|\alpha| \ll 1$)

In this case only one solution of (3.17) exists corresponding to $n = 0$ and $a = \frac{\theta|\tilde{E}|}{2} \ll 1$. Therefore, we can use the asymptotic form of $I_n(a)$ for small a's which is

$$I_0(a) = -\ln|a| - \gamma + O(|a|\ln|a|) \tag{3.18}$$

$$\gamma = 0.5772\ldots\ , \quad \text{Euler's constant}$$

This gives the following eigenvalue equation

$$1 = \frac{|\alpha_0|}{4\pi}(\ln|a| + \gamma)$$

with binding energy

$$|\tilde{E}| = \frac{2}{\theta}e^{-\gamma}e^{\frac{4\pi}{\alpha_0}}$$

Comparing this last expression with (3.8) we see that this limit corresponds to the commutative case with $\frac{1}{\theta}$ playing the rôle of parameter μ^2 in commutative case.

3.3 Fermions in a magnetic vortex background

3.3.1 Commutative case

In this section we briefly review the solutions of a massive Dirac equation in the Aharonov-Bohm background field of an infinitely thin magnetic vortex carrying magnetic flux Φ. This presentation closely follows Ref.[54]. The electromagnetic potential describing such field configuration can be chosen in polar coordinates (ρ, ϕ) as $\mathbf{A} = -\frac{e\Phi}{2\pi}\frac{1}{\rho}\hat{\phi}$. $\mathbf{A}$ has a well known property of being locally a pure gauge.

The Dirac equation for this problem is

$$(i\,\partial\!\!\!/ + A\!\!\!/ - m)\Psi(t, r, \phi) = 0. \tag{3.19}$$

and allows passing to the eigenstates of angular momentum $n + \frac{1}{2}$. By defining

$$\Psi_{E,n}(t, \rho, \phi) = \begin{pmatrix} \chi^1(\rho)e^{in\phi} \\ \chi^2(\rho)e^{i(n+1)\phi} \end{pmatrix} e^{-iEt} \tag{3.20}$$

the radial eigenvalue problem is

$$\begin{pmatrix} m - E & -i\left(\partial_\rho + \frac{\nu+1}{\rho}\right) \\ -i\left(\partial_\rho - \frac{\nu}{\rho}\right) & -m - E \end{pmatrix} \chi(\rho) = 0 \tag{3.21}$$

with $\nu \equiv n + \Phi$. For $E^2 > m^2$ it has the solutions

$$\chi_\nu(\rho) = \frac{1}{N}\begin{pmatrix} \sqrt{E+m}(\epsilon_n)^n J_{\epsilon_n \nu}(k\rho) \\ i\sqrt{E-m}(\epsilon_n)^{n+1} J_{\epsilon_n(\nu+1)}(k\rho) \end{pmatrix} \tag{3.22}$$

where N is a normalization factor, $k = \sqrt{E^2 - m^2}$, $J_\lambda(x)$ denotes the Bessel functions and ϵ_n is taken to be either$+1$ or -1 to assure regularity of spinor components at the origin. This choice of the sign for ϵ_n can always be done except for the partial wave with

$$-1 < \nu < 0 \tag{3.23}$$

in which case both choices of sign lead to solutions that are square integrable, though singular in one component, at the origin. To avoid a loss of completeness in angular basis, a family of self-adjoint extensions of Dirac Hamiltonian is required. These extensions are parametrized by a single parameter $0 \leq \Theta \leq 2\pi$ (not to be confused with noncommutativity parameter θ) and restrict the behaviour of the wavefunction at $\rho \to 0$ to be

$$\lim_{\rho \to 0} \chi(\rho) \sim \begin{pmatrix} i\rho^\nu \sin\left(\frac{\pi}{4} + \frac{\Theta}{4}\right) \\ \rho^{-\nu-1} \cos\left(\frac{\pi}{4} + \frac{\Theta}{4}\right) \end{pmatrix} \tag{3.24}$$

With the boundary condition established, the energy eigenstates are

$$\chi_\nu(\rho) \sim \begin{pmatrix} \sqrt{E+m}\left[\sin\mu J_\nu(k\rho) + (-1)^n \cos\mu J_{-\nu}(k\rho)\right] \\ i\sqrt{E-m}\left[\sin\mu J_{\nu+1}(k\rho) + (-1)^{n+1} \cos\mu J_{-(\nu+1)}(k\rho)\right] \end{pmatrix} \tag{3.25}$$

with μ related to Θ by the equation

$$\begin{aligned} \tan\left(\frac{\pi}{4} + \frac{\Theta}{2}\right) = \; & (-1)^n \left(\frac{E+m}{E-m}\right)^{1/2} \left(\frac{k}{2m}\right)^{2\nu+1} \\ & \times \frac{\Gamma(-\nu)}{\Gamma(\nu+1)} \tan\mu. \end{aligned} \tag{3.26}$$

In addition, for $\pi/2 < \Theta < 3\pi/2$ there is a bound state

$$B_\nu(\rho) \sim \begin{pmatrix} \sqrt{m+E} K_\nu(\bar{k}\rho) \\ i\sqrt{m-E} K_{\nu+1}(\bar{k}\rho) \end{pmatrix} \tag{3.27}$$

where $\bar{k} = -ik = \sqrt{m^2 - E^2}$ and $K_\nu(x)$ are modified Bessel functions. The bound-state energy is implicitly determined from

$$\frac{(1+E/m)^{\nu+1}}{(1-E/m)^{-\nu}} = -2^{2\nu+1}\frac{\Gamma(\nu+1)}{\Gamma(-\nu)}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right). \tag{3.28}$$

3.3.2 Noncommutative case

On the noncommutative plane it is convenient to use complex notation for the vector potential

$$A_z = \frac{1}{2}(A_1 - iA_2), \qquad A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2), \tag{3.29}$$

so that magnetic field strength can be written as

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z - ie[A_z, A_{\bar{z}}]. \tag{3.30}$$

For a magnetic vortex field

$$B = 2iF_{z\bar{z}} = \frac{\Phi}{2\pi\theta}|0\rangle\langle 0| \tag{3.31}$$

an explicit expression for vector potential can be written if we use the following ansatz

$$A_z = i\frac{\bar{z}}{2\theta}\Delta, \qquad A_{\bar{z}} = -i\Delta\frac{z}{2\theta}, \tag{3.32}$$

with a radially symmetric function

$$\Delta = \frac{1}{e}\sum_{n=0}^{\infty}\left(1 - \sqrt{1 - \frac{e\Phi}{2\pi}\frac{1}{n+1}}\right)|n\rangle\langle n| \tag{3.33}$$

This form is valid for $1 - \frac{e\Phi}{2\pi} > 0$ only.

The Dirac equation for this problem formally coincides with eq.(3.19) in commutative case

$$(i\,\partial\!\!\!/ + A\!\!\!/ - m)\Psi = 0. \tag{3.34}$$

if we require that gauge connection acts from the left on operator-valued spinor wave function Ψ. By defining

$$\Psi_E = \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} e^{-iEt} \tag{3.35}$$

we get the following eigenvalue problem

$$\begin{pmatrix} E-m & 2i\left(\frac{\partial}{\partial_z} - ieA_z\right) \\ -2i\left(\frac{\partial}{\partial_{\bar{z}}} - ieA_{\bar{z}}\right) & -E-m \end{pmatrix} \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} = 0 \tag{3.36}$$

which leads to the second order equations for each of the two components of the Dirac spinor

$$\left(\frac{\partial}{\partial_z} - ieA_z\right)\left(\frac{\partial}{\partial_{\bar{z}}} - ieA_{\bar{z}}\right)\chi^1 = -\frac{k^2}{4}\chi^1, \tag{3.37}$$

$$\left(\frac{\partial}{\partial_{\bar{z}}} - ieA_{\bar{z}}\right)\left(\frac{\partial}{\partial_z} - ieA_z\right)\chi^2 = -\frac{k^2}{4}\chi^2. \tag{3.38}$$

It should be noted here that due to the noncommutativity of covariant derivatives

$$\left[\left(\frac{\partial}{\partial_z} - ieA_z\right), \left(\frac{\partial}{\partial_{\bar{z}}} - ieA_{\bar{z}}\right)\right] = ieF_{z\bar{z}} \tag{3.39}$$

the equation for χ^1 component contains an extra term $ieF_{z\bar{z}}\chi^1$ as compared to the equation satisfied by χ^2. In a commutative limit this term is proportional to $\delta(x)\chi^1(x)$ and is equal to zero as long as χ^1 is regular at the origin.

To solve (3.37, 3.38) we use the following ansatz of angular momentum $n + \frac{1}{2}$

$$\chi = \begin{pmatrix} \sum\limits_{l=0}^{\infty} \chi_l^1 |l-n\rangle\langle l| \\ \sum\limits_{l=0}^{\infty} \chi_l^2 |l-n-1\rangle\langle l| \end{pmatrix} \tag{3.40}$$

for $n = -1, -2, \ldots$ and

$$\chi = \begin{pmatrix} \sum\limits_{l=n}^{\infty} \chi_l^1 |l-n\rangle\langle l| \\ \sum\limits_{l=n+1}^{\infty} \chi_l^2 |l-n-1\rangle\langle l| \end{pmatrix} \tag{3.41}$$

for $n = 0, 1, 2, \ldots$.

For negative n's this gives us recursion relations on coefficients $\chi_l^{1,2}$

$$\begin{aligned} \chi_{l+1}^{1,2}\sqrt{l+1}\sqrt{l-\nu+1} \quad &+ \chi_{l-1}^{1,2}\sqrt{l}\sqrt{l-\nu} \\ &-\chi_l^{1,2}\ (2l-\nu+1-\tfrac{\theta}{2}k^2) = 0 \end{aligned} \tag{3.42}$$

$$\chi_1^{1,2}\sqrt{-\nu+1} - \chi_0^{1,2}(-\nu+1-\frac{\theta}{2}k^2) = 0 \tag{3.43}$$

where again we used $\nu = n + \Phi$. These recursion relations are quite easy to solve with the solutions given by (up to a normalization factor)

$$\chi_l^1 = \frac{i(E+m)}{\sqrt{-\nu}}\sqrt{\frac{\theta}{2}}\sqrt{\frac{l!}{(-\nu+1)_l}}L_l^{-\nu}\left(\frac{k^2\theta}{2}\right) \tag{3.44a}$$

$$\chi_l^2 = \sqrt{\frac{l!}{(-\nu)_l}}L_l^{-\nu-1}\left(\frac{k^2\theta}{2}\right) \tag{3.44b}$$

with $(\alpha)_l = \alpha(\alpha+1)\ldots(\alpha+l-1)$ the Pochhammer symbol and $L_l^\alpha(x)$ the generalized Laguerre polynomial.

For $n = 0, 1, 2, \ldots$ recursion relations are the same as (3.42) but "boundary" conditions (3.43) are different (note that in this case the series expansion in (3.41) begins with χ_n^1 and χ_{n+1}^2 terms for the first and second spinor components respectively)

$$\chi_{n+1}^1\sqrt{n+1}\sqrt{n-\nu+1} - \chi_n^1(n+1-\tfrac{\theta}{2}k^2) = 0 \tag{3.45}$$

$$\begin{aligned} &\chi_{n+2}^2\sqrt{n+2}\sqrt{n-\nu+1} - \\ &\qquad -\chi_{n+1}^2\ (2(n+1)-\nu-\tfrac{\theta}{2}k^2) = 0 \end{aligned} \tag{3.46}$$

These equations are also easy to solve

$$\chi = \frac{1}{N}\begin{pmatrix} \frac{i(E+m)}{\sqrt{-\nu}}\sqrt{\frac{\theta}{2}}\sum\limits_{l=n}^{\infty}\sqrt{\frac{l!}{(-\nu+1)_l}}\,\rho_n^1\,|l-n\rangle\langle l| \\ \sum\limits_{l=n+1}^{\infty}\sqrt{\frac{l!}{(-\nu)_l}}\,\rho_n^2\,|l-n-1\rangle\langle l| \end{pmatrix} \tag{3.47}$$

$$\begin{aligned} \rho_n^1 &= \left\{L_l^{-\nu-1}(\tfrac{k^2\theta}{2}) + \tfrac{-2\nu A^\nu}{k^2\theta}(-\nu+1)_l\Psi(l+1, 1+\nu, -\tfrac{k^2\theta}{2})\right\} \\ \rho_n^2 &= \left\{L_l^{-\nu-1}(\tfrac{k^2\theta}{2}) + A^\nu(-\nu)_l\Psi(l+1, 2+\nu, -\tfrac{k^2\theta}{2})\right\} \end{aligned} \tag{3.48}$$

where coefficient A^ν is a solution of the linear equation

$$(n-\nu)L_n^{-\nu-1}(\frac{k^2\theta}{2}) + A^\nu(-\nu)_{n+1}\Psi(n+1, 2+\nu, -\frac{k^2\theta}{2}) = 0 \tag{3.49}$$

and ensures that conditions (3.45, 3.46) are obeyed. It is an easy task now to check that eqs.(3.44), (3.47) do also satisfy the first order Dirac equations (3.36) and, therefore, give a complete set of angular momentum eigenstates for our problem.

3.4 Conclusions

In this chapter we have studied noncommutative generalizations of quantum mechanics in the presence of δ - function potentials. It was found that noncommutativity of space-time can be used to provide an intrinsic regularization of the theories in question. Using the star product formalism we found analytically all the solutions of that problem . The folowing remarks, however, are in order:

1. The apparent asymmetry between holomorphic and antiholomorphic solutions in, for example (3.15), can be understood if one notes that action of noncommutative δ-function operator on antiholomorphic wavefunctions is trivial

$$\hat{\delta}(\hat{x})\hat{\bar{\Psi}} = 0 \tag{3.50}$$

   and, therefore, in our model these modes are free, i.e. they are described by Schrödinger equation (3.9) with kinetic term only. However, the highly nontrivial action of the same operator on holomorphic wavefunctions gives rise to a finite number of extra bound states with nonzero angular momentum. These states do not have any commutative analogues and disappear from our theory in the limit of vanishing θ as well, while in the limit of strong noncommutativity the spectrum of these states coinsides with the spectrum of a harmonic oscillator with frequency $\frac{2}{\theta}$.

2. For Dirac particles we can use the correspondence between Fock space operators and ordinary functions

$$|n\rangle\langle m| \sim \frac{\bar{z}^n}{\sqrt{n!}}\frac{z^m}{\sqrt{m!}}\ e^{-\frac{1}{\theta}r^2} \tag{3.51}$$

to show that commutative limit of our solutions (3.44),(3.47) for critical value of $-1 < \nu < 0$ is

$$\chi_\nu(r) \sim \begin{pmatrix} \sqrt{E+m}J_{-\nu}(kr) \\ -i\sqrt{E-m}J_{-\nu-1}(kr) \end{pmatrix} \tag{3.52}$$

which after camparison with eq.(3.25) tells us that commutative limit of our model corresponds to $\Theta = 3\pi/2$ which probably explains the absence of bound states in our model, since in commutative limit bound states exist only if $\pi/2 < \Theta < 3\pi/2$.

3. The simple ansatz used to find vector potential is valid only if condition $1 - \frac{e\Phi}{2\pi} > 0$ is satisfied. It is not clear at present if it is possible to extend our approach to $1 - \frac{e\Phi}{2\pi} < 0$ region.

Appendix A

Review of Karabali-Nair theory in $2+1$ dimensions

A.1 The parametrization of the gauge potentials

Consider a gauge theory with group $G = SU(N)$ in the $A_0 = 0$ gauge. The gauge potential can be written as $A_i = -it^a A_i^a$, $i = 1, 2$, where t^a are hermitian $N \times N$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = if^{abc}t^c$, $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. The spatial coordinates x_1, x_2 will be combined into the complex combinations $z = x_1 - ix_2$, $\bar{z} = x_1 + ix_2$ with the corresponding components for the potential $A \equiv A_z = \frac{1}{2}(A_1 + iA_2)$, $\bar{A} \equiv A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2) = -(A_z)^\dagger$. The starting point of Karabali-Nair analysis is a change of variables given by

$$A_z = -\partial_z M M^{-1}, \qquad A_{\bar{z}} = M^{\dagger -1}\partial_{\bar{z}} M^\dagger \tag{A.1}$$

Here M, $M^\dagger$ are complex matrices in general, not unitary. If they are unitary, the potential is a pure gauge. The parametrization (A.1) is standard in many discussions of two-dimensional gauge fields. A particular advantage of this parametrization is the way gauge transformations are realized. A gauge transformation $A_i \to A_i^{(g)} = g^{-1}A_i g + g^{-1}\partial_i g,\ g(x) \in SU(N)$ is obtained by the transformation $M \to M^{(g)} = gM$. The gauge-invariant degrees of freedom are parametrized by the hermitian matrix $H = M^\dagger M$. Physical state wavefunctions are functions of H.

The measure of integration over the fields $A, \bar{A}$ is $d\mu(\mathcal{A})/vol(\mathcal{G}_*)$ where $d\mu(\mathcal{A}) = \prod_{x,a} dA^a(x) d\bar{A}^a(x)$ is the Euclidean volume element on the space of gauge potentials $\mathcal{A}$ and $vol\mathcal{G}_*$ is the volume of gauge transformations, viz., volume of $SU(N)$-valued functions on space. From (A.1) we see that

$$\begin{aligned} \delta A &= -D(\delta M M^{-1}) \\ &= -\left(\partial(\delta M M^{-1}) + [A, \delta M M^{-1}]\,\right) \\ \delta \bar{A} &= \bar{D}(M^{\dagger -1}\delta M^\dagger) \end{aligned} \tag{A.2}$$

which gives

$$d\mu(\mathcal{A}) = (\det D\bar{D})\ d\mu(M, M^\dagger) \tag{A.3}$$

where $d\mu(M, M^\dagger)$ is the volume for the complex matrices $M, M^\dagger$, which is associated with the metric $ds^2_M \ = 8\int \mathrm{Tr}(\delta M M^{-1}\ M^{\dagger -1}\delta M^\dagger)$. This is given by the highest order differential form dV as $d\mu(M, M^\dagger) = \prod_x dV(M, M^\dagger)$ where

$$\begin{aligned} dV(M, M^\dagger) \propto & \epsilon_{a_1 \ldots a_n} (dMM^{-1})_{a_1} \wedge \ldots (dMM^{-1})_{a_n} \\ & \times \epsilon_{b_1 \ldots b_n} (M^{\dagger -1} dM^\dagger)_{b_1} \cdots \end{aligned} \tag{A.4}$$

where $n = \mathrm{dim}G = \mathrm{dim}SU(N) = N^2 - 1$. (There are some constant numerical factors which are irrelevant for our discussion.) The complex matrix M can be written as $M = U\rho$, where U is unitary and ρ is hermitian. This is the matrix analogue of the modulus and phase decomposition for a complex number. Since gauge transformations act as $M \to M^{(g)} = gM$, we see that U represents the gauge degrees of freedom and ρ represents the gauge-invariant degrees of freedom on M. Substituting $M = U\rho$, (A.4) becomes

$$\begin{aligned} dV(M, M^\dagger) &\propto \epsilon_{a_1 \ldots a_n}(d\rho\rho^{-1} + \rho^{-1}d\rho)_{a_1} \wedge \ldots \\ &\times \epsilon_{b_1 \ldots b_n}(U^{-1}dU)_{b_1} \wedge \cdots \\ &\propto d\mu(U)\epsilon_{a_1 \ldots a_n}(H^{-1}dH)_{a_1} \wedge \cdots \end{aligned} \tag{A.5}$$

Here $d\mu(U)$ is the standard group volume measure (the Haar measure) for $SU(N)$. Upon taking the product over all points, $d\mu(U)$ gives the volume of the entire gauge group, namely $vol(\mathcal{G}_*)$, and thus

$$\begin{aligned} d\mu(M, M^\dagger) &= \prod_x dV(M, M^\dagger)\ vol(\mathcal{G}_*) \\ &= d\mu(H)\ vol(\mathcal{G}_*) \end{aligned} \tag{A.6}$$

$$d\mu(H) == \epsilon_{a_1 \ldots a_n}(H^{-1}dH)_{a_1} \ldots (H^{-1}dH)_{a_n} \tag{A.7}$$

The volume element or the integration measure for the gauge-invariant configurations can now be written as

$$\begin{aligned} \frac{d\mu(\mathcal{A})}{vol(\mathcal{G}_*)} &= \frac{[dA_z dA_{\bar{z}}]}{vol(\mathcal{G}_*)} \\ &= (\det D_z D_{\bar{z}}) \frac{d\mu(M, M^\dagger)}{vol(\mathcal{G}_*)} \\ &= (\det D\bar{D}) d\mu(H) \end{aligned} \tag{A.8}$$

where we have used (A.6). The problem is thus reduced to the calculation of the determinant of the two-dimensional operator $D\bar{D}$. This is well known [35]. The simplest way to evaluate this is to define $\Gamma = \log\ \det D\bar{D}$, which gives

$$\frac{\delta\Gamma}{\delta\bar{A}^a} = -i\ \mathrm{Tr}\left[\bar{D}^{-1}(x,y)T^a\right]_{y\to x} \tag{A.9}$$

$(T^a)_{mn} = -if^a_{mn}$ are the generators of the Lie algebra in the adjoint representation. The coincident-point limit of $\bar{D}^{-1}(x,y)$ is singular and needs regularization. With a gauge-invariant regulator,one finds

$$\mathrm{Tr}\left[\bar{D}^{-1}_{reg}(x,y)T^a\right]_{y\to x} = \frac{2c_A}{\pi}\mathrm{Tr}\left[(A - M^{\dagger -1}\partial M^\dagger)t^a\right] \tag{A.10}$$

where $c_A\delta^{ab} = f^{amn}f^{bmn}$; it is equal to N for $SU(N)$. Using this result in (A.9) and integrating we get

$$(\det D\bar{D}) = \left[\frac{\det'\partial\bar{\partial}}{\int d^2x}\right]^{\dim G} \exp\left[2c_A\ \mathcal{S}(H)\right] \tag{A.11}$$

$\mathcal{S}(H)$ is the Wess-Zumino-Witten (WZW) action for the hermitian matrix field H given by [6]

$$\mathcal{S}(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H\bar{\partial}H^{-1}) + \frac{i}{12\pi}\int \epsilon^{\mu\nu\alpha} \times \mathrm{Tr}(H^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\alpha H) \tag{A.12}$$

We can now write the inner product for states $|1\rangle$ and $|2\rangle$, represented by the wavefunctions Ψ_1 and Ψ_2, as [7]

$$\langle 1|2\rangle = \int d\mu(H)e^{2c_A\ I(H)}\ \ \Psi_1^*\Psi_2 \tag{A.13}$$

A.2 The Hamiltonian

The next step is the change of variables in the Hamiltonian. However, there is some further simplification we can do before taking up the Hamiltonian. One would expect the wavefunctions to be functionals of the matrix field H, but actually we can take them to be functionals of the current of the WZW model (A.12) given by $J = (c_A/\pi)\partial_z H\ H^{-1}$. Notice that matrix elements calculated with (A.13) are correlators of the hermitian WZW model of level number $2c_A$. The properties of the hermitian model of level number $k + 2c_A$ can be obtained by comparison with the $SU(N)$-model defined by $e^{kS(U)}$, $U(\vec{x}) \in SU(N)$. The hermitian analogue of the renormalized level $\kappa = (k + c_A)$ of the $SU(N)$-model is $-(k + c_A)$. Since the correlators involve only the renormalized level κ, we see that the correlators of the hermitian model (of level $(k + 2c_A)$) can be obtained from the correlators of the $SU(N)$-model (of level k) by the analytic continuation $\kappa \to -\kappa$. For the $SU(N)_k$-model there are the so-called integrable representations whose highest weights are limited by k (spin $\leq k/2$ for $SU(2)$, for example). Correlators involving the nonintegrable representations vanish. For the hermitian model the corresponding statement is that the correlators involving nonintegrable representations are infinite. In our case, $k = 0$, and we have only one integrable representation corresponding to the identity operator (and its current algebra descendents). Therefore, for states of finite norm, it is sufficient to consider J.

This means that we can transform the Hamiltonian $\mathcal{H} = T + V$ to express it in terms of J and functional derivatives with respect to J. By the chain rule of differentiation

$$
\begin{aligned}
T\Psi &= \frac{e^2}{2}\int E_i^a E_i^a\ \Psi \\
&= -\frac{e^2}{2}\left[\int_{x,u} \frac{\delta J^a(u)}{\delta A_i^c(x)\delta A_i^c(x)}\frac{\delta\Psi}{\delta J^a(u)}\right. \\
&\quad \left. + \int_{x,u,v} \frac{\delta J^a(u)}{\delta A_i^c(x)}\frac{\delta J^b(v)}{\delta A_i^c(x)}\ \frac{\delta}{\delta J^a(u)}\frac{\delta}{\delta J^b(v)}\Psi\right] \\
V &= \frac{1}{2e^2}\int B^a B^a
\end{aligned}
\tag{A.14}
$$

where $B^a = \frac{1}{2}\epsilon_{ij}(\partial_i A_j^a - \partial_j A_i^a + f^{abc}A_i^b A_j^c)$. Regularization is important in calculating the coefficients of the two terms in T. Carrying this out we find

$$
\begin{aligned}
T \quad &= m\left[\int_u J^a(u)\frac{\delta}{\delta J^a(u)}\right. \\
&\quad \left. + \int \Omega_{ab}(u,v)\frac{\delta}{\delta J^a(u)}\frac{\delta}{\delta J^b(v)}\right]
\end{aligned}
\tag{A.15}
$$

$$
V \quad = \frac{\pi}{mc_A}\int \bar{\partial}J_a(\vec{x})\bar{\partial}J_a(\vec{x}) \tag{A.16}
$$

where $m = e^2 c_A/2\pi$ and

$$
\Omega_{ab}(u,v) = \frac{c_A}{\pi^2}\frac{\delta_{ab}}{(u-v)^2} \quad - \quad i\frac{f_{abc}J^c(v)}{\pi(u-v)} \tag{A.17}
$$

The first term in T shows that every power of J in the wavefunction gives a value m to the energy, suggesting the existence of a mass gap. The calculation of this term involves exactly the same quantity as in (A.9) and with the same regulator leads to (A.15), i.e.,

$$
\begin{aligned}
-\frac{e^2}{2}&\int d^2y \frac{\delta^2 J_a(x)}{\delta \bar{A}^b(y)\delta A^b(y)} \\
&= \frac{e^2 c_A}{2\pi} M_{am}^{\dagger}\,\mathrm{Tr}\left[T^m \bar{D}^{-1}(y,x)\right]_{y\to x} \\
&= m\ J_a(x)
\end{aligned}
\tag{A.18}
$$

Finally, (A.15,A.16) (with regularizations taken account of) give a self-adjoint Hamiltonian which is a nice consistency check.

A.3 The Vacuum State

Let us now consider the eigenstates of the theory. The vacuum wavefunction is presumably the simplest to calculate. Ignoring the potential term V for the moment, since T involves derivatives, we see immediately that the ground state wavefunction for T is $\Phi_0 = 1$. This may seem like a trivial statement, but the key point is that it is normalizable with the inner product (A.13); in fact, the normalization integral is just the partition function for the WZW action and is finite. Starting with this, we can solve the Schrödinger equation taking Ψ_0 to be of the form $\exp(P)$, where P is a perturbative series in the potential term V (equivalent to a $1/m$-expansion). We then get

$$\begin{aligned} P = & -\frac{\pi}{m^2 c_A} \mathrm{Tr} \int :\bar{\partial} J \bar{\partial} J: \\ & -\left(\frac{\pi}{m^2 c_A}\right)^2 \mathrm{Tr} \int [:\bar{\partial} J (\mathcal{D}\bar{\partial}) \bar{\partial} J \\ & +\frac{1}{3} \bar{\partial} J [J, \bar{\partial}^2 J] :] \\ & -2\left(\frac{\pi}{m^2 c_A}\right)^3 \mathrm{Tr} \int [:\bar{\partial} J (\mathcal{D}\bar{\partial})^2 \bar{\partial} J \\ & +\frac{2}{9}[\mathcal{D}\bar{\partial} J,\ \bar{\partial} J]\bar{\partial}^2 J + \frac{8}{9}[\mathcal{D}\bar{\partial}^2 J,\ J]\bar{\partial}^2 J \\ & -\frac{1}{6}[J,\ \bar{\partial} J][\bar{\partial} J,\ \bar{\partial}^2 J] - \frac{2}{9}[J, \bar{\partial} J][J, \bar{\partial}^3 J] :] \\ & +\mathcal{O}(\frac{1}{m^8}) \end{aligned} \tag{A.19}$$

where $\mathcal{D}h = (c_A/\pi)\partial h - [J, h]$. The series is naturally grouped as terms with 2 J's, terms with 3 J's, etc. These terms can be summed up; for the $2J$-terms we find

$$
\begin{aligned}
\Psi_0 &= \exp\left[P\right] \\
P &= -\frac{1}{2e^2}\int_{x,y} B_a(x)K(x,y)B_a(y) + \mathcal{O}(3J) \\
K(x,y) &= \left[\frac{1}{\left(m+\sqrt{m^2-\nabla^2}\right)}\right]_{x,y}
\end{aligned}
\tag{A.20}
$$

The first term in (A.20) has the correct (perturbative) high momentum limit, viz.,

$$
\Psi_0 \approx \exp\left[-\frac{1}{2e^2}\int_{x,y} B_a(x)\left[\frac{1}{\sqrt{-\nabla^2}}\right]_{x,y} B_a(y) + \mathcal{O}(3J)\right] \tag{A.21}
$$

Thus although we started with the high m (or low momentum) limit, the result (A.20) does match onto the perturbative limit. The higher terms are also small for the low momentum limit.

We can now use this result to calculate the expectation value of the Wilson loop operator which is given as

$$
\begin{aligned}
W(C) &= \mathrm{Tr}P\ e^{-\oint_C (Adz+\bar{A}d\bar{z})} \\
&= \mathrm{Tr}P\ e^{(\pi/c_A)\oint_C J}
\end{aligned}
\tag{A.22}
$$

For the fundamental representation, its expectation value is given by

$$
\begin{aligned}
\langle W_F(C)\rangle &= \text{constant}\quad \exp\left[-\sigma\mathcal{A}_C\right] \\
\sqrt{\sigma} &= e^2\sqrt{\frac{N^2-1}{8\pi}}
\end{aligned}
\tag{A.23}
$$

where $\mathcal{A}_C$ is the area of the loop C. σ is the string tension. This is a prediction of Karabali-Nair analysis starting from first principles with no adjustable parameters. Notice that the dependence on e^2 and N is in agreement with large-N

expectations, with σ depending only on the combination e^2N as $N \to \infty$. (The first correction to the large-N limit is negative, viz., $-(e^2N)/2N^2\sqrt{8\pi}$ which may be interesting in the context of large-N analyses.) Formula (A.23) gives the values $\sqrt{\sigma}/e^2 = 0.345, 0.564, 0.772, 0.977$ for $N = 2, 3, 4, 5$. There are estimates for σ based on Monte Carlo simulations of lattice gauge theory. The results for the gauge groups $SU(2)$, $SU(3)$, $SU(4)$ and $SU(5)$ are $\sqrt{\sigma}/e^2 = 0.335$, 0.553, 0.758, 0.966 [57]. We see that Karabali-Nair result agrees with the lattice result to within $\sim 3\%$.

Appendix B

Noncommutative plane: notations and conventions

Troughout this thesis we work in $2+1$ dimensional flat noncommutative space with usual commutation relations:

$$[x^1, x^2] = i\theta. \tag{B.1}$$

It is convenient to introduce complex variables z and $\bar{z}$

$$z = x^1 + ix^2, \qquad \bar{z} = x^1 - ix^2 \tag{B.2}$$

so that (B.1) becomes

$$[z, \bar{z}] = 2\theta \tag{B.3}$$

and $z, \bar{z}$ can be thought of as a pair of creation-annihilation operators acting in the space of Fock states $\{|0\rangle, |1\rangle, \ldots, |l\rangle, \ldots\}$ as

$$\bar{z}|n\rangle = \sqrt{2\theta}\sqrt{n+1}|n+1\rangle \tag{B.4a}$$

$$z|n\rangle = \sqrt{2\theta}\sqrt{n}|n-1\rangle \tag{B.4b}$$

$$z|0\rangle = 0. \tag{B.4c}$$

Algebra of functions on the noncommutative plane is then equivalent to the algebra of linear operators in Fock space. Derivatives on noncommutative plane are the inner derivations

$$\partial_z = -\frac{1}{2\theta}[\bar{z}, \ldots], \quad \partial_{\bar{z}} = \frac{1}{2\theta}[z, \ldots] \tag{B.5}$$

while integration is the same as trace of operator

$$\int f(x) \to 2\pi\theta \mathrm{Tr} f \tag{B.6}$$

The elements of the algebra of functions on noncommutative space can also be identified with ordinary functions on $\mathbb{R}^2$ through the Weyl-Moyal correspondence

$$f(x) \mapsto \hat{f}(\hat{x}) = \int d^2p \; f(p) e^{i(p_1\hat{x}_1 + p_2\hat{x}_2)} \tag{B.7}$$

where

$$f(p) = \int \frac{d^2x}{(2\pi)^2} f(x) e^{-i(p_1x_1+p_2x_2)} \tag{B.8}$$

is the usual Fourier transform of $f(x)$. The product of two functions f and g which corresponds to the product of operators $\hat{f}\hat{g}$ is given by the Moyal (or star product) formula

$$f * g(x) = \exp\left[\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x_i^1}\frac{\partial}{\partial x_j^2}\right] f(x^1)g(x^2)|_{x^1=x^2=x} \tag{B.9}$$

We also need a generalization of the concept of a δ - function to noncommutative space. In usual field theory δ - functions are used to describe localized sources. But because in noncommutative case the space is smeared at small distances we cannot construct a truly localized source. Direct application of transform (B.7) to $\delta^2(x)$ gives an operator which is spread out over all of space. Therefore, the most localized source we can construct in the noncommutative case is a Gaussian wave packet [47]

$$\delta_\theta(x) = \frac{1}{\theta\pi} e^{-\frac{1}{\theta}(x_1^2+x_1^2)} \tag{B.10}$$

whose transform is

$$\hat{\delta} = |0\rangle\langle 0| \tag{B.11}$$

meaning that noncommutative δ-function is in fact a projection operator onto the Fock space groundstate $|0\rangle$. We also note that

$$\int d^2x \delta_\theta(x) = 1$$

and in $\theta \to 0$ limit we recover the ussual δ-function.

Appendix C

On matrix parametrization of the noncommutative covariant derivative operators

In this Appendix we would like to show that parametrization (2.51) of the covariant derivative operator Z in terms of an infinite matrix h is well-defined in the sense that for any given matrix Z it is possible, at least perturbatively, to find corresponding matrix h. It is useful to rewrite (2.51) in an equivalent form

$$A = -\frac{-i}{2\theta}\,[\bar{z}, h]\,h^{-1}, \tag{C.1}$$

where A is the noncommutative gauge potential as defined in (2.11). To be able to solve this equation we have to prove that operator $\frac{-i}{2\theta}[\bar{z}, \ldots]$ is invertible, i.e. that there exists a map (we call it $D(\ldots)$)

$$B \xrightarrow{D} D(B) \tag{C.2}$$

which associates to any given noncommutative function B another function $D(B)$ such that

$$-\frac{i}{2\theta}[\bar{z}, D(B)] = B. \tag{C.3}$$

This map is the noncommutative analogue of $\int dw\, G(z,w)\ldots$ with $G(z,w)$ being the Green's function of the ordinary commutative derivative operator ∂_z. In terms of $D(\ldots)$ we can write then the solution of (C.1) as

$$h = 1 + D(Ah) = 1 + D(A) + D(AD(A)) + \ldots. \qquad \text{(C.4)}$$

However, the validity of this expression crucially depends on the existence of map $D(\ldots)$, so we give now the proof that $D(\ldots)$ is indeed a well-defined operation on the noncommutative plane.

Since B and $D(B)$ are both infinite-dimensional matrices, we can represent them in the oscillator basis as

$$\begin{aligned} B &= \sum_{i,j=0}^{\infty} B_{ij} |i\rangle\langle j| \qquad \text{(C.5)} \\ D(B) &= i\sqrt{2\theta} \sum_{i,j=0}^{\infty} C_{ij} |i\rangle\langle j| \end{aligned}$$

With this expansion eq.(C.3) now gives the following set of recursion relations for matrix elements of $D(B)$

$$C_{i-1\,j}\sqrt{i} - C_{i\,j+1}\sqrt{j+1} = B_{ij} \qquad i,j = 0,1,2,\ldots \qquad \text{(C.6)}$$

From these we find

$$\begin{aligned} B_{00} &= -C_{01} \qquad \text{(C.7)} \\ B_{01} &= -C_{02}\sqrt{2} \\ &\ldots\ldots \\ B_{0l} &= -C_{0l+1}\sqrt{l+1}, \end{aligned}$$

which means that we can immediately find all C_{0l} coefficients with $l \geq 1$. Next we consider the following set of equations

$$\begin{aligned} B_{11} &= C_{01} - C_{12}\sqrt{2} \\ B_{12} &= C_{02} - C_{13}\sqrt{3} \\ &\ldots\ldots \\ B_{1l} &= C_{0l} - C_{1l+1}\sqrt{l+1} \end{aligned} \tag{C.8}$$

and obtain C_{1l}, $l \geq 2$. Now we can proceed iteratively and see that given that we have already found $C_{i-1l}, l \geq i$ for some i, we can always find $C_{il}, l \geq i+1$ from

$$\begin{aligned} B_{ii} &= C_{i-1i}\sqrt{i} - C_{ii+1}\sqrt{i+1} \\ &\ldots\ldots \\ B_{il} &= C_{i-1l}\sqrt{i} - C_{il+1}\sqrt{l+1}. \end{aligned} \tag{C.9}$$

Therefore, all the matrix elements C_{ij} of $D(B)$ with $i < j$ can be uniquely determined from the above equations.

For those C_{ij} with $i \geq j$ we may consider the following set of equations

$$\begin{aligned} B_{10} &= C_{00} - C_{11} \\ B_{21} &= C_{11}\sqrt{2} - C_{22}\sqrt{2} \\ &\ldots\ldots \\ B_{i+1i} &= C_{ii}\sqrt{i+1} - C_{i+1i+1}\sqrt{i+1}. \end{aligned}$$

From these equations we can find all diagonal matrix elements C_{ii}, $i \geq 0$ provided that we fix arbitrarily the value of C_{00}. Easy to see that this freedom in choosing C_{00} translates into the following ambiguity of $D(B)$

$$C_{00}\sum_{0}^{\infty}|i\rangle\langle i| = C_{00}\mathbf{1} \tag{C.10}$$

so we can add an arbitrary constant function to $D(B)$. Similarly, from

$$\begin{aligned} B_{20} &= C_{10}\sqrt{2} - C_{21} \\ B_{31} &= C_{21}\sqrt{3} - C_{32}\sqrt{2} \\ &\ldots\ldots \\ B_{i+2i} &= C_{i+1i}\sqrt{i+2} - C_{i+2i+1}\sqrt{i+1} \end{aligned} \tag{C.11}$$

we can find all C_{i+1i}, $i \geq 0$, however, solution is not unique again; we can add

$$C_{10}\sum_{0}^{\infty}\sqrt{i+1}|i+1\rangle\langle i| = C_{10}\bar{z} \tag{C.12}$$

to $D(B)$. In exactly the same way one can show that all the remaining matrix elements C_{ij} with $i \geq j$ can be found from (C.6) and this completes the proof of existence of the map (C.2). This map, however, is not unique; D(B) is defined up to

$$C_{00}\mathbf{1} + C_{10}\bar{z} + C_{20}\bar{z}^2 + C_{30}\bar{z}^3 + \ldots \tag{C.13}$$

with arbitrary coefficients $C_{00}, C_{10}, \ldots$, *i.e.* we can add any noncommutative antiholomorphic function $f(\bar{z})$ to $D(B)$ and still satisfy (C.3).

One can also see that solution (C.4) of equation (C.1) is not unique as well. Really, because of the ambiguity in the definition of $D(\ldots)$ we can write an alternative solution of (C.1) as

$$h = f(\bar{z}) + D(Ah) = f(\bar{z}) + D\left(Af(\bar{z})\right) + D\left(AD\left(Af(\bar{z})\right)\right) + \ldots \tag{C.14}$$

However, this can also be written as

$$h = \left[1 + D(Af(\bar{z}))f^{-1}(\bar{z}) + \ldots\right] f(\bar{z}) = \left[1 + D^f(A) + D^f(AD^f(A)) + \ldots\right] f^{-1}(\bar{z}) \tag{C.15}$$

where $D^f(\ldots) = D(\ldots f(\bar{z}))f^{-1}(\bar{z})$ satisfies (C.3) and expression in brackets

$$h^f = 1 + D^f(A) + D^f(AD^f(A)) + \ldots \tag{C.16}$$

is the solution of (C.1) as well. Therefore, we see that if h is some solution of (C.1) then

$$h^f = hf(\bar{z}) \tag{C.17}$$

is another solution of that equation. This means that our parametrization of the covariant derivative Z in terms of the matrix field h is defined up to right multiplication by an arbitrary antiholomorphic function only. In fact, this can be seen directly from (2.51) since any such function obviously commutes with the antiholomorphic coordinate operator $\bar{z}$.

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