IA-automorphisms and localization of nilpotent groups

by

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Abstract

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Advisor: Professor Joseph Roitberg

A group is called *p-local*, where p is a prime number, if every element in the group has a unique nth root for each n relatively prime to p. Given a nilpotent group G and a prime p, there is a unique p-local group $G_{(p)}$ which is, in some sense, the "best approximation" to G among all p-local nilpotent groups. $G_{(p)}$ is called the *p-localization* of G.

Let $G_{(p)}$ be the p-localization of a nilpotent group G, and let $IA(G)$ be the subgroup of $AutG$ consisting of those automorphisms of G that induce the identity on G/G', where G' denotes the commutator subgroup of G. $IA(G)$ turns out to be nilpotent, so its p-localization exists. Two groups G and H are said to be in the same *localization genus* if $G_{(p)}$ is isomorphic to $H_{(p)}$ for all primes p. The main result of this thesis is that if two finitely generated, torsion-free, nilpotent, and metabelian groups lie in the same localization genus, their IA-groups also lie in the same localization genus. The method of proof involves basic sequences and commutator calculus.

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Chapter 1

Background

1.1 Nilpotent groups and basic sequences

We begin with a brief account on nilpotent groups and basic commutators. For a detailed presentation of this material, refer to [2] and [16].

Let G be any group. The *lower central series* of G is defined as:

$$G = \gamma_1 G \geq \gamma_2 G \geq \cdots$$

where $\gamma_i G$ is the normal subgroup of G generated by all elements of G of the form $[g_1, \ldots, g_i]$. Here,

$$[g_1, \ldots, g_i] = [[g_1, \ldots, g_{i-1}], g_i], \ \ [g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2.$$

G is called *nilpotent* if $\gamma_{c+1} G = 1$ for some c. In case $\gamma_c G \neq 1$, we say that G is *nilpotent of class* c.

Lemma 1.1.1. *The following commutator identities hold in any group:*

1. $[x, yz] = [x, z]\,[x, y]^z$,

2. $[xy, z] = [x, z]^y [y, z]$,

3. $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1$,

where $a^b = b^{-1}ab$.

Proof. The first two identities are straightforward so we only verify the third. Let

$$u = xzx^{-1}yx,\ v = yxy^{-1}zy,\ \text{and } w = zyz^{-1}xz.$$

(v and w are obtained from u as cyclic permutations of x, y, and z). From

$$\left[x, y^{-1}, z\right] = \left[y^{-1}, x\right] \left[x, y^{-1}\right] \left[x, y^{-1}, z\right] = \left[y^{-1}, x\right] \left[x, y^{-1}\right]^z,$$

it is easy to check that

$$\left[x, y^{-1}, z\right]^y = u^{-1}v.$$

Similarly,

$$\left[y, z^{-1}, x\right]^z = v^{-1}w,$$

and

$$\left[z, x^{-1}, y\right]^x = w^{-1}u.$$

The lemma now follows from this. □

In order to construct a sequence of basic commutators in a nilpotent group we consider a more general setup, an approach taken in [2]. Let

$$X = \{x_1, \ldots, x_q\}$$

be a finite set. We define the *free groupoid G, freely generated by X*, as the set of all "bracketed" finite words on X, under a binary operation called *bracket*. Observe that if g and h lie in G, their bracket $[g, h]$ also lies in G. Observe further that G does not satisfy associativity, commutativity, or any other group-theoretic law, except closure under the bracket.

For example, take $q = 3$. Then $x_1, x_2 \in G$, $(x_1x_2)x_1 \in G$, $((x_1x_2)\,x_1)\,(x_2x_3) \in G$, and so on.

The *length* of $g \in G$, denoted $|g|$, is the number of letters appearing in g.

Definition 1.1.2. *Let G be a free groupoid on X. Let $b_1, b_2, \ldots$ be an infinite sequence of elements in G. This sequence is called basic if the following holds:*

1. *The elements of X appear in the sequence.*

2. *If $|b_i| < |b_j|$ then $i < j$.*

3. *Let $u = vw \in G$ be of length at least 2. Then u belongs to the basic sequence if and only if*

    (a) *$v = b_i$, $w = b_j$, and $j < i$; and*

    (b) *either $|v| = 1$ (in which case $|w| = 1$ by the previous items) or $v = b_k b_l$ where $l \leq j$.*

To construct a basic sequence, we define the so-called "rep" operation. Let $A \subset G$ and $a \in A$. Define

$$A\ rep\ a := \{(bia) : i = 0, 1, \ldots; b \in A - \{a\}\} \subset G$$

where, by definition,

- $b0a = b$, and

- $b\,(i+1)\,a = (bia)\,a$ for $i \geq 0$.

Consider the free groupoid G on $X = \{x_1, \dots, x_q\}$. We construct a sequence of subsets of G as follows:

- Put $X_1 = X$. Choose $b_1 \in X_1$.
- Put $X_2 = X_1 \ rep \ b_1$.

Suppose we have constructed X_n. Choose $b_n \in X_n$ of minimal length and put

$$X_{n+1} = X_n \ rep \ b_n.$$

Then

$$b_1, b_2, \dots$$

is a basic sequence on X.

In general, the first q terms of the sequence can be chosen to be $x_1, \dots, x_q$.

Let Γ be a finitely generated groupoid (not necessarily free) with generating set $\{\mu_1, \dots, \mu_q\}$. We now explain what is meant by a basic sequence on Γ.

Definition 1.1.3. *A sequence $\beta_1, \beta_2, \dots$ of elements of Γ is called a basic sequence on $\{\mu_1, \dots, \mu_q\}$ if and only if there exists a basic sequence $b_1, b_2, \dots$ on X such that the groupoid homomorphism*

$$G \to \Gamma$$

$$x_i \mapsto \mu_i$$

sends b_i to β_i.

Let G be a group, generated by the finite set

$$Y = \{y_1, \dots, y_q\}.$$

Ignoring the fact that G is a group, we can construct a groupoid relative to the binary operation "commutation":

$$[g, h] = g^{-1}h^{-1}gh.$$

Note that Y no longer generates G as a groupoid, but rather, it generates (under commutation) a subgroupoid of G.

Use the "rep" operation (as before) to construct a basic sequence $c_1, c_2, \ldots$ on Y. We call the terms of this sequence *basic commutators* on Y. We define the weight of c_i as $|b_i|$ where b_i is the "canonical" pre-image of c_i in the free groupoid on X.

The first of the following results can be found in [7], and the other two in [15]:

Theorem 1.1.4. *Let G be a group generated by $Y = \{y_1, \ldots, y_q\}$ and let*

$$c_1, c_2, \ldots$$

be a basic sequence on Y.

Then $\gamma_r G/\gamma_{r+1}G$ is generated by the basic commutators of weight r ($r = 1, 2, \ldots$).

Corollary 1.1.5. *Let G be a group generated by $Y = \{y_1, \ldots, y_q\}$. G is nilpotent if and only if all but finitely many terms on any basic sequence on Y are equal to 1.*

Theorem 1.1.6. *Let F be a free nilpotent group, freely generated by $X = \{x_1, \ldots, x_q\}$. Let*

$$b_1, b_2, \ldots$$

be a basic sequence on X. Then $\gamma_r F/\gamma_{r+1}F$ is free abelian on the basic commutators of weight r.

1.2 IA groups of nilpotent groups

For any group G, define $IA(G)$ to be the subgroup of $AutG$ consisting of all automorphisms of G inducing the identity on G/G' where $G' = \gamma_2 G$. We now discuss a series of fundamental results from [7], leading to a major fact about $IA(G)$, when G is nilpotent.

Lemma 1.2.1. *Let X, Y, and Z be subgroups of G. Let $X^* = [Y, Z, X]$, $Y^* = [Z, X, Y]$, and $Z^* = [X, Y, Z]$. If N is normal in G and both X^* and Y^* are subgroups of N, then Z^* is a subgroup of N.*

Proof. A typical generator of Z^* can be written as $[x, y^{-1}, z]$. From lemma 1.1.1, we have:

$$\left[x, y^{-1}, z\right]^y \left[y, z^{-1}, x\right]^z \left[z, x^{-1}, y\right]^x = 1.$$

Since N is normal in G, and both X^* and Y^* are subgroups of N, this means that $[x, y^{-1}, z] \in N$, so that Z^* is a subgroup of N. □

Theorem 1.2.2. *Let H and K be subgroups of a group G. Let*

$$H = H_0 \geq H_1 \geq \ldots$$

be a normal series for H such that

$$[H_i, K] \leq H_{i+1}$$

for each i.

Put $K = K_1$. *Define*

$$K_j = \{x \in K : [H_i, x] \leq H_{i+j}\}$$

for all i. *Then*

1. $[K_j, K_l] \leq K_{j+l}$ *for all* j, l;

2. $[H_i, \gamma_j K] \leq H_{i+j}$ *for all* i, j.

Proof. We show first that $[K_j, K_l] \leq K_{j+l}$. The second part will follow from this. By definition, $[H_i, K_j] \leq H_{i+j}$ and $[H_{i+j}, K_l] \leq H_{i+j+l}$. Hence

$$[H_i, K_j, K_l] \leq [H_{i+j}, K_l] \leq H_{i+j+l}. \tag{1.1}$$

Likewise,

$$[K_l, H_i, K_j] = [H_i, K_l, K_j] \leq H_{i+j+l}. \tag{1.2}$$

By hypothesis, H_{i+j+l} is normal in H. Moreover, for $x \in K$

$$x^{-1} H_{i+j+l} x = H_{i+j+l} \left[H_{i+j+l}, x\right].$$

Since $[H_{i+j+l}, x] \subset H_{i+j+l+1}$, we conclude that

$$x^{-1} H_{i+j+l} x \leq H_{i+j+l}.$$

This shows that H_{i+j+l} is normal in the subgroup of G generated by H and K.

Applying lemma 1.2.1 to this situation we get:

$$[K_j, K_l, H_i] = [H_i, [K_j, K_l]] \leq H_{i+j+l}. \tag{1.3}$$

Therefore, by definition,

$$[K_j, K_l] \leq K_{j+l}. \tag{1.4}$$

In particular, $[K_j, K] \leq K_{j+1}$ so

$$K = K_1 \geq K_2 \geq \cdots$$

is a central series for K. This implies that $\gamma_j K \leq K_j$ and hence

$$[H_i, \gamma_j K] \leq [H_i, K_j] \leq H_{i+j}. \tag{1.5}$$

This completes the proof.

□

Corollary 1.2.3. *For any group G,*

$$[\gamma_i, \gamma_j] \leq \gamma_{i+j}.$$

Proof. This follows from theorem 1.2.2 by taking $H_i = \gamma_{i+1}$ and $K = G$. □

In order to express what follows in the correct language we need to construct the so-called "holomorph" of a group G (see [10]). Let

$$G^* = \{\phi g : \phi \in AutG,\, g \in G\}.$$

G^* can be regarded as the cartesian product

$$AutG \times G.$$

This set becomes a group under the operation

$$(\phi g)(\phi' g') = \phi\phi' g^{\phi'} g',$$

where $g^{\phi'} = \phi'(g) \in G$. ($G^*$ is in fact a semi-direct product of G by $AutG$). The notation

$$G^* = HolG$$

is customary, and we call $HolG$ the *holomorph* of G.

Lemma 1.2.4. *Let*

$$G = G_0 \geq G_1 \ldots \geq G_r = 1$$

be a series of normal subgroups in G.

Let A be the group of automorphisms of G leaving each G_i invariant and transforming

$$G_i/G_{i+1}$$

identically. Then A and $[G, A]$ are nilpotent of class less than r. Here, $[G, A] \leq HolG$.

Proof. Notice that $[G_i, A] \leq G_{i+1}$ for each i: if $x \in G_i$ and $\alpha \in A$, we have

$$[x, \alpha] = x^{-1}\alpha^{-1}x\alpha = x^{-1}x^{\alpha} \in G_{i+1}$$

since α transforms G_i/G_{i+1} identically. By theorem 1.2.2 with $H = G$ and $A = K$,

$$[G, \gamma_r A] = [G_0, \gamma_r A] \leq G_r = 1;$$

so that

$$[G, \gamma_r A] = 1.$$

Now let $\alpha \in \gamma_r A$ and $x \in G$. Then

$$[x, \alpha] = x^{-1}x^{\alpha} = 1,$$

which shows that $\alpha = 1$. As a consequence, $\gamma_r A = 1$ so A is nilpotent of class less than r.

To show that $[G, A]$ is nilpotent of class less that r we proceed as follows: since G_{i-1} is normal in G, then $[G_{i-1}, G] \leq G_{i-1}$. Hence

$$[G_{i-1}, G, A] \leq [G_{i-1}, A] \leq G_i.$$

Also,

$$[A, G_{i-1}, G] \leq [G_i, G] \leq G_i.$$

By assumption, $x^\alpha \in G_i$ for $x \in G_i$ and $\alpha \in A$, so that G_i is normal in the subgroup of $HolG$ generated by A and G. We can then apply lemma 1.2.1 to obtain

$$[G, A, G_{i-1}] = [G_{i-1}, [G, A]] \leq G_i.$$

Thus, by theorem 1.2.2,

$$[G_1, \gamma_{r-1}[G, A]] \leq G_r = 1.$$

But

$$[G, A] = [G_0, A] \leq G_1,$$

so that

$$[[G, A], \gamma_{r-1}[G, A]] = 1.$$

By definition of the lower central series, $\gamma_r[G, A] = 1$, as required. □

Lemma 1.2.5. *Let H and K be subgroups of a group G such that*

$$[H, K] \leq H'.$$

Then

$$[\gamma_i H, \gamma_j K] \leq \gamma_{i+j} H$$

for all i, j.

Proof. We do "double induction." Assume $j = 1$. We prove that

$$[\gamma_i H, K] \leq \gamma_{i+1} H$$

by induction on i. The case $i = 1$ follows from the hypothesis. Assume that

$$[\gamma_{i-1} H, K] \leq \gamma_i H.$$

Notice that:

$$[\gamma_{i-1} H, K, H] \leq [\gamma_i H, H] = \gamma_{i+1} H \quad \text{(by the induction hypothesis), and}$$

$$[K, H, \gamma_{i-1} H] \leq [\gamma_2 H, \gamma_{i-1} H] \leq \gamma_{i+1} H \quad \text{(by hypothesis and corollary 1.2.3).}$$

Using lemma 1.2.1 and the definition of $\gamma_i H$,

$$[\gamma_i H, K] = [H, \gamma_{i-1} H, K] \leq \gamma_{i+1} H.$$

The basis of induction for j now follows.

The induction hypothesis for j is

$$[\gamma_i H, \gamma_{j-1} K] \leq \gamma_{i+j-1} H$$

for all i. Note that

$$[\gamma_i H, \gamma_j K] = [\gamma_i H, [K, \gamma_{j-1} K]] = [K, \gamma_{j-1} K, \gamma_i H].$$

By the induction hypothesis on j:

$$[\gamma_i H, K, \gamma_{j-1} K] \leq [\gamma_{i+1} H, \gamma_{j-1} K] \leq \gamma_{i+j} H \quad \text{and}$$

$$[\gamma_{j-1} K, \gamma_i H, K] \leq [\gamma_{i+j-1} H, K] \leq \gamma_{i+j} H.$$

Using lemma 1.2.1 once again, we conclude that

$$[\gamma_i H, \gamma_j K] \leq \gamma_{i+j} H.$$

This completes the double induction. □

Corollary 1.2.6. *Let H be nilpotent of class c. Then*

- *$\gamma_j IA(H)$ transforms each $\gamma_i H/\gamma_{i+j}H$ identically, and*
- *$IA(H)$ is nilpotent of class $c-1$.*

Proof. To prove the first assertion, notice that if we choose $x \in H$ and $\psi \in IA(H)$, then

$$[x, \psi] = x^{-1}x^{\psi} \in H',$$

in $HolH$. Hence

$$[H, IA(H)] \leq H'$$

and by lemma 1.2.5,

$$[\gamma_i H, \gamma_j IA(H)] \leq \gamma_{i+j} H.$$

This means that $\gamma_j IA(H)$ transforms each $\gamma_i H/\gamma_{i+j}H$ identically.

To see that $IA(H)$ is nilpotent of class less than c, let $\alpha \in \gamma_c IA(H)$ and $x \in H = \gamma_1 H$. By the first assertion, α transforms $\gamma_1 H/\gamma_{c+1}H \cong H$ identically. This proves that $\gamma_c IA(H) = 1$ so that $IA(H)$ is nilpotent of class less than c.

Finally recall that the inner automorphisms of H constitute a subgroup of $IA(H)$ which is isomorphic to H modulo its center. This shows that $IA(H)$ must have class exactly $c-1$. □

We end this section with additional results about $IA(G)$. We state and prove some pertinent facts about nilpotent groups first.

Lemma 1.2.7. *The upper central quotients of a torsion-free nilpotent group are torsion-free.*

Proof. Let

$$\zeta_i = \{g \in G : [g, x] \in \zeta_{i-1} \text{ for all } x \in G\}$$

be the i-th center of G; where $\zeta_1 = \zeta$, the center of G. The proof is by induction on i, the basis of induction being obvious since ζ is also torsion-free. Assume ζ_{i-1}/ζ_{i-2} is torsion-free. We will show that so is ζ_i/ζ_{i-1}. Let $g \in \zeta_i$ and $m > 0$ such that $g^m \in \zeta_{i-1}$. We wish to show that $g \in \zeta_{i-1}$. Let $x \in G$. Then

$$1 = [g^m, x] = \left[gg^{m-1}, x\right] =$$

$$[g, x]\left[g, x, g^{m-1}\right]\left[g^{m-1}, x\right] = [g, x]\left[g^{m-1}, x\right] (\text{modulo } \zeta_{i-2}).$$

Continue reducing in this way to finally obtain

$$1 = [g^m, x] = [g, x]^m (\text{modulo } \zeta_{i-2}).$$

By the induction hypothesis, $[g, x] \in \zeta_{i-2}$ for every x, so that $g \in \zeta_{i-1}$, as required. □

Corollary 1.2.8. *G/ζ is torsion-free.*

Proof. Let $g \in G$ and $g^m \in \zeta$ with $m > 0$. Since G is nilpotent there is an i such that $g \in \zeta_i$. The fact that $\zeta \leq \zeta_{i-1}$ yields $g^m \in \zeta_{i-1}$. By lemma 1.2.7, g itself belongs to ζ_{i-1}. Continue this argument down the upper central series of G to finally conclude that $g \in \zeta$. This gives that G/ζ is torsion-free. □

The fact that G/ζ is isomorphic to the group of inner automorphisms of G, suggests that $IA(G)$ may be torsion-free. This is in fact true. We have:

Lemma 1.2.9. *If G is a torsion-free nilpotent group, $IA(G)$ is torsion-free.*

Proof. Define

$$\overline{\gamma_c} = \{g \in G : \text{there exists } n > 0 \text{ with } g^n \in \gamma_c G\},$$

where G is a torsion-free nilpotent group of class c. It is immediate that $\overline{\gamma_c}$ is a normal and central subgroup of G containing γ_c. $G/\overline{\gamma_c}$ is nilpotent of class less than c since c-fold commutators are trivial in $G/\overline{\gamma_c}$. Moreover, $G/\overline{\gamma_c}$ is torsion-free, for if $g \in G$ with $g^m \in \overline{\gamma_c}$, there is an n such that $g^{nm} \in \gamma_c$; hence $g \in \overline{\gamma_c}$.

The proof that $IA(G)$ is torsion-free is by induction on the class of G. If G has class 2, $IA(G)$ is abelian. Let $\varphi \in IA(G)$ and x be an arbitrary generator of G. Then

$$\varphi(x) = xd$$

where $d \in G'$. Suppose $\varphi^m = 1$ where $m > 0$. Then

$$\varphi^m(x) = xd^m = x,$$

and by torsion-freeness of G', $d = 1$. This completes the basis of induction.

Assume now that the IA-group of a torsion-free nilpotent group of class less that c is always torsion-free. Let G be of class c. Since $G/\overline{\gamma_c}$ is nilpotent, torsion-free, and of class less than c, the induction hypothesis gives that $IA\left(G/\overline{\gamma_c}\right)$ is torsion-free. To prove that $IA(G)$ is torsion-free, let $\varphi \in IA(G)$

and assume that $\varphi^m = 1$ where $m > 0$. Consider the canonical homomorphism

$$IA(G) \to IA\left(G/\overline{\gamma_c}\right),$$

$$\tau \mapsto \hat{\tau}$$

where

$$\hat{\tau}(\langle g \rangle) = \langle \tau(g) \rangle \,.$$

For $x \in G$, $\langle x \rangle$ denotes the equivalence class of x in $G/\overline{\gamma_c}$. Since φ^m is the identity and $IA\left(G/\overline{\gamma_c}\right)$ is torsion-free, $\hat{\varphi}$ is the identity. Let x be an arbitrary generator of G and write

$$\varphi(x) = xd$$

where $d \in G'$. Then

$$\hat{\varphi}\left(\langle x \rangle\right) = \langle \varphi(x) \rangle = \langle xd \rangle = \langle x \rangle \,.$$

Hence

$$d \in \overline{\gamma_c}.$$

We claim that φ acts trivially on $\overline{\gamma_c}$. Let y be any element in $\overline{\gamma_c}$. There exists $y_1 \in G'$ such that

$$\varphi(y) = yy_1.$$

Moreover, there is a positive integer m such that $y^m \in \gamma_c$. Hence

$$y^m = \varphi(y^m) = \varphi(y)^m = (yy_1)^m = y^m y_1^m$$

(recall that $\overline{\gamma_c}$ is central). Since G is torsion-free, this means $y_1 = 1$, so that φ acts trivially on $\overline{\gamma_c}$. This, together with $\varphi(x) = xd$ and $\varphi^m = 1$ gives

$$\varphi^m(x) = xd^m = x,$$

so that $d^m = 1$, and since G is torsion-free, $d = 1$. This completes the proof. $\square$

Lemma 1.2.10. *If G is a finitely generated nilpotent group, $IA(G)$ is finitely generated.*

Proof. The proof is by induction on the class of G. If G has class 2, each element of $IA(G)$ acts trivially on G' and $IA(G)$ is abelian. A typical member of a generating set for $IA(G)$ can be constructed as follows: for each generator x_i of G choose a generator y_j of G'. Construct the IA-automorphism that sends x_i to $x_i y_j$ and each remaining generator of G to itself. Since the generating sets for G and G' can be chosen to be finite, this generating set for the abelian group $IA(G)$ will also be finite.

Assume the induction hypothesis: the IA-group of finitely generated nilpotent groups of class less than c is finitely generated.

Let G be of class c. Consider the following subgroup of $IA(G)$:

$$I_c = \left\{ \alpha \in IA(G) : g^{-1}\alpha(g) \in \gamma_c \text{ for all } g \in G \right\}.$$

If $\alpha \in I_c$, then

$$\alpha(x_i) = x_i h_i$$

where x_i is a typical generator of G and $h_i \in \gamma_c$. It follows from this that the elements of I_c act trivially on G' and I_c is in fact an abelian subgroup of $IA(G)$.

Consider the natural homomorphism

$$\phi : IA(G) \to IA\left(G/\gamma_c\right)$$

$$\varphi \mapsto \hat{\varphi}$$

where

$$\hat{\varphi}(\langle g \rangle) = \langle \varphi(g) \rangle .$$

We prove that

1. $I_c = \ker \phi$, and

2. I_c is finitely generated.

The fact that $I_c = \ker \phi$ follows from the definition of I_c. The fact that I_c is finitely generated can be established by an analogous construction as in the basis of induction: for each generator x_i of G choose a generator y_j of γ_c. Construct the IA-automorphism that sends x_i to $x_i y_j$ and each remaining generator of G to itself. Since these generating sets are finite, the generating set for I_c so obtained is also finite.

By our induction hypothesis, $IA\,(G/\gamma_c)$ is finitely generated so the image of ϕ (being a subgroup of a finitely generated nilpotent group) is also finitely generated. This image is isomorphic to $IA(G)/I_c$.

Finally, a set consisting of one representative for each equivalence class of $IA(G)/I_c$, together with the generating set for I_c gives a generating set for $IA(G)$. This completes the proof. □

Remark. Assuming nilpotency of G is essential here. For example, if G is a two generator metabelian group, C.K. Gupta proved that $IA(G)$ is still metabelian (see reference [7] in [1]). However, $IA(G)$ need not be finitely generated (see theorem C in [1]).

Corollary 1.2.11. *If G is finitely generated, torsion-free nilpotent of class c, $IA(G)$ is finitely generated, torsion-free nilpotent of class $c-1$.*

Proof. This follows from lemmas 1.2.9 and 1.2.10 together with corollary 1.2.6. ☐

It is well known that if G is finitely generated, torsion-free nilpotent, then any Eilenberg-MacLane $K(G,1)$ space has the homotopy type of a finite complex. Our results now imply:

Corollary 1.2.12. *Let G be a finitely generated, torsion-free, nilpotent group, then any $K(IA(G),1)$ space has the homotopy type of a finite complex.*

1.3 Localization and completion of nilpotent groups

Refer to [9] and [16] for fine accounts of the material on localization. We use the following notation:

- P denotes a set of primes.
- P' denotes the set of primes not in P.
- $n \in P'$ means that the natural number n only involves primes from P'.

Definition 1.3.1. *A group G is called P-local if and only if the map*

$$\begin{aligned} G &\to G \\ x &\mapsto x^n \end{aligned}$$

is a bijection for all $n \in P'$.

Definition 1.3.2. *Let $\mathcal{H}$ be a subcategory of the category of groups. A morphism*

$$e : G \to G_P$$

in $\mathcal{H}$ is said to be P-universal or a P-localizing map if

1. *G_P is P-local.*

2. *For any P-local group $K \in \mathcal{H}$, the map:*

$$e^* : Hom\,(G_P, K) \to Hom\,(G, K)$$

$$\varphi \mapsto e^*(\varphi)$$

   where

$$e^*(\varphi)(g) = \varphi e(g).$$

is a bijection.

Assume next that each group in $\mathcal{H}$ admits a P-localizing map. Given a morphism

$$\varphi : G \to K$$

in $\mathcal{H}$, there exists a unique morphism

$$\varphi_P : G_P \to K_P$$

making the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & K \\ \big\downarrow e & & \big\downarrow e \\ G_P & \xrightarrow{\varphi_P} & K_P \end{array}$$

commute. Here, e denotes the localization map.

The above diagram gives a functor L from $\mathcal{H}$ to itself. The pair (L, e) is called a *localization theory* in $\mathcal{H}$.

Definition 1.3.3. *Let G be any group. An element $x \in G$ is said to be P'-torsion if there is an $n \in P'$ such that $x^n = 1$.*

Definition 1.3.4. *A homomorphism*

$$\varphi : G \to K$$

is called P-injective if

$$\ker \varphi = \{x \in G : x \textit{ is } P'\textit{-torsion}\}.$$

Definition 1.3.5. *A homomorphism*

$$\varphi : G \to K$$

is called P-surjective if for all $y \in K$ there exists $n \in P'$ such that y^n lies in the image of φ.

Definition 1.3.6. *A P-isomorphism is a homomorphism which is P-injective and P-surjective.*

Refer to [9] for proofs of the following results.

Lemma 1.3.7. *Let $\varphi : G \to G'$ be a homomorphism of P-local groups.*

1. *If φ is P-injective then φ is one-to-one.*
2. *If φ is P-surjective then φ is onto.*

Fundamental theorem of P-localization of nilpotent groups

For a nilpotent group G, write nil$G$ for its nilpotency class.

1. There exists a localization theory (L, e) in the category of nilpotent groups $\mathcal{N}$.

2. If $c \geq 1$, (L, e) restricts to a localization theory in $\mathcal{N}_c$, the category of nilpotent groups of class at most c.

3. From (2),

$$\text{nil}LG \leq \text{nil}G$$

where $G \in \mathcal{N}$.

4. Let $\varphi : G \to K$ be a morphism in $\mathcal{N}$. φ is a P-localizing map if and only if

   (a) K is P-local, and

   (b) φ is a P-isomorphism.

The last item is a very useful fact.

Theorem 1.3.8. *Let*

$$1 \to G' \to G \to G'' \to 1$$

be a short exact sequence of nilpotent groups. If any two of these groups are P-local, so is the third.

Theorem 1.3.9. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \varphi'} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \varphi''} & & \\ 1 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 1 \end{array}$$

be a map of short exact sequences of nilpotent groups. If any two of φ', φ, φ'' P-localizes, so does the third.

Theorem 1.3.10. *P-localization is an exact functor. That is, applying P-localization to a short exact sequence of nilpotent groups*

$$1 \to G' \to G \to G'' \to 1$$

yields a short exact sequence of P-local nilpotent groups

$$1 \to G'_P \to G_P \to G''_P \to 1.$$

If P consists of a single prime p, we write $G_{(p)}$ for the P-localization of G (soon we will discuss G_p, the "p-completion" of G). For a single prime p, a homomorphism may be p-injective, p-surjective, or a p-isomorphism.

Definition 1.3.11. *Two nilpotent groups G and H are said to be in the same localization genus if they are p-isomorphic for every prime p.*

A concept related to p-localization is that of p-completion. To define the p-completion of a nilpotent group we recall the notion of inverse systems of groups and their limits (see [6]). A *quasi-order* in a set M is a relation on M which is reflexive and transitive. In general, the relation need not be symmetric. A set M furnished with a quasi-order "$<$" is called *directed* if for every α, β in M, there exists γ in M such that $\alpha < \gamma$ and $\beta < \gamma$.

Let M be a directed set. An *inverse system of groups* over M is an assignment $\{G, \Pi\}$ such that for each $\alpha \in M$, G_α is a group and for each $\alpha < \beta$,

$$\pi_\alpha^\beta : G_\beta \to G_\alpha$$

is a group homomorphism satisfying:

1. π_α^α is the identity for all $\alpha \in M$ and

2. $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$ for $\alpha < \beta < \gamma$.

The homomorphisms π_α^β are called *projections* of the system.

The *inverse limit* of the system $\{G, \Pi\}$, denoted G_∞, is the subgroup of the product $\Pi_{\alpha \in M} G_\alpha$ consisting of those functions $g = \{g_\alpha\}$ satisfying $\pi_\alpha^\beta (g_\beta) = g_\alpha$ for each $\alpha < \beta$.

Let G be a nilpotent group, p a prime, and N the set of non-negative integers. Then N is a directed set under the relation "$\leq$." For each $\alpha \in N$, the subgroup

$$G^{p^\alpha} = gp\left(g^{p^\alpha} : g \in G\right)$$

is normal in G. Let

$$G_\alpha = G/G^{p^\alpha}$$

and

$$\pi_\alpha^\beta : G_\beta \to G_\alpha,$$

for $\alpha < \beta$ be the obvious homomorphism. Each projection clearly satisfies the above conditions, so $\{G, \Pi\}$ is an inverse system of p-groups. We call the nilpotent group G_∞ the *p-adic completion* of G (or *p-completion* for short) and use the notation $G_\infty = G_p$.

Remark. If G is finitely generated, the G_α are torsion, finitely generated nilpotent groups, so each G_α is a finite p-group.

For every nilpotent group G, there is a natural homomorphism $G \to G_p$ with kernel

$$G^{p^\omega} = \cap_{\alpha \in N} G^{p^\alpha}.$$

If G is finitely generated and torsion-free, G_p is also torsion-free and the natural homomorphism $G \to G_p$ is injective [12].

Definition 1.3.12. *Two groups are in the same completion genus if their p-completions are isomorphic for every p.*

The following results will help us compare the localization genus with the completion genus.

Lemma 1.3.13. *Let G be a torsion nilpotent group. Then G is p-local if and only if G has no n-torsion for n relatively prime to p.*

Proof. Suppose first that G is p-local. For n relatively prime to p, $x \mapsto x^n$ is a bijection. Hence G has no n-torsion. Conversely, suppose G has no n-torsion for all n relatively prime to p and consider the localization map

$$e : G \to G_{(p)}.$$

We show first that e is 1-1: since e is p-injective, if $g \in \ker e$ there exists n, relatively prime to p, such that $g^n = 1$. As G has no n-torsion it follows that $g = 1$. Hence e is injective.

Next, we show that "$G_{(p)}$ is a torsion group." Let $g \in G_{(p)}$. There exist n relatively prime to p and $x \in G$ such that

$$e(x) = g^n.$$

Since G is a torsion group, but has no n-torsion for n relatively prime to p, there are integers m and α such that $x^{mp^\alpha} = 1$ with $(m, p) = 1$. Hence

$$1 = e\left(x^{mp^\alpha}\right) = e(x)^{mp^\alpha} = g^{mnp^\alpha}$$

in $G_{(p)}$. As $G_{(p)}$ is p-local, $g^{p^\alpha} = 1$. This shows that $G_{(p)}$ is a torsion group.

Let $t(H)$ be the torsion subgroup of a nilpotent group H. We thus have

$$t(G_{(p)}) = G_{(p)}.$$

Since all of the torsion elements of a localization come from the original group (corollary 8.6 in [16]),

$$G_{(p)} = t(G_{(p)}) = e(t(G)) = e(G),$$

so that e is onto. This proves that e is an isomorphism and G is p-local. □

Corollary 1.3.14. *Every p-group is p-local.*

Theorem 1.3.15. *For any nilpotent group G*

$$(G_{(p)})_p \cong G_p.$$

Proof. G_p is the inverse limit of (the inverse system of) its p-group quotients

$$G/G^{p^\alpha}$$

for $\alpha \in N$. Similarly, $(G_{(p)})_p$ (the p-completion of the p-localization of G) is the inverse limit of (the inverse system of) its p-group quotients

$$G_{(p)}/G_{(p)}^{p^\alpha}$$

for $\alpha \in N$. Since p-groups are p-local, we have that for each α,

$$G/G^{p^\alpha} \cong \left(G/G^{p^\alpha}\right)_{(p)}.$$

To show that the inverse systems of G and $G_{(p)}$ agree, and their corresponding limits are therefore isomorphic, it suffices to check that the p-localization of G/G^{p^α} is $G_{(p)}/G_{(p)}^{p^\alpha}$. But p-localization is an exact functor (see theorem 1.3.10), so the sequence

$$1 \to G_{(p)}^{p^\alpha} \to G_{(p)} \to \left(G/G^{p^\alpha}\right)_{(p)} \to 1$$

is exact, and the assertion follows. □

Remark. We have shown that in order to obtain $(G_{(p)})_p$ it suffices to p-localize the inverse system corresponding to G (groups and projections maps) and then find its limit.

Corollary 1.3.16. *The localization genus of any nilpotent group is contained in its completion genus.*

Pickel proved that the completion genus of a finitely generated nilpotent group is finite ([14], [12]). Our corollary now yields a localization version of Pickel's result:

Theorem 1.3.17. *The localization genus of any finitely generated nilpotent group is finite.*

Chapter 2

Localization of the IA-group of finitely generated, torsion-free, nilpotent, and metabelian groups

2.1 Powers of IA-automorphisms of torsion-free, metabelian, p-local, and nilpotent groups of class c

Let G be the p-localization of a finitely generated, torsion-free, metabelian, and nilpotent group of class c.

Let $X = \{x_1, \ldots, x_r\}$ be a finite set that generates G as a p-local group. Then G is also generated, as a p-local group, by the set of basic commutators

$B = \{b_1, \ldots, b_m\}$ on X; where $b_i = x_i$ for $i = 1, \ldots, r$. Since G is metabelian, any basic commutator on B is of the form

$$[\ldots[[[x_{i_1}, x_{i_2}], x_{i_3}], x_{i_4}], \ldots], x_{i_t}].$$

Denote the weight of b_i as $wt(b_i)$.

Let $\varphi \in IA(G)$ and put

$$\varphi(b_i) = b_i A_i$$

where $wt(b_i) \leq c - 1$ and $A_i \in G'$. A_i can be expressed as a product of rational powers of basic commutators of weight at least 2, and at most c. So we can write

$$A_i = \prod_{k>l} [b_k, x_l]^{\frac{i_{kl}}{v(i)}}$$

where $wt(b_k) \leq c - 1$ for each k; and $v(i)$ is relatively prime to p.

For each $k > l$, standard commutator calculus in this metabelian group gives:

$$\varphi([b_k, x_l]) = [b_k, x_l][b_k, A_l][A_k, x_l].$$

In order to find an expression for $\varphi^m([b_k, x_l])$, we do as follows:

$$\varphi^2([b_k, x_l]) = \varphi([b_k, x_l])\varphi([b_k, A_l][A_k, x_l])$$
$$= [b_k, x_l][b_k, A_l][A_k, x_l][b_k A_k, \varphi(A_l)][\varphi(A_k), x_l A_l]$$
$$= [b_k, x_l][b_k, A_l][A_k, x_l][b_k, \varphi(A_l)][\varphi(A_k), x_l].$$

Continuing in this way we obtain:

$$\varphi^m([b_k, x_l]) = [b_k, x_l] \prod_{i=0}^{m-1} [b_k, \varphi^i(A_l)][\varphi^i(A_k), x_l]. \tag{2.1}$$

We now wish to obtain a closed formula for $\varphi^m([b_k, x_l])$, one that does not depend on lower powers of φ. The following computation gives this:

$$\begin{aligned}
\varphi(A_i) &= \prod_{k>l} \varphi\left([b_k, x_l]\right)^{\frac{i_{kl}}{v(i)}} \\
&= \prod_{k>l} \left([b_k, x_l][b_k, A_l][A_k, x_l]\right)^{\frac{i_{kl}}{v(i)}} \\
&= \prod_{k>l} [b_k, x_l]^{\frac{i_{kl}}{v(i)}} \prod_{k>l} ([b_k, A_l][A_k, x_l])^{\frac{i_{kl}}{v(i)}} \\
&= A_i \prod_{k>l} ([b_k, A_l][A_k, x_l])^{\frac{i_{kl}}{v(i)}}.
\end{aligned}$$

Let

$$\delta_{i1} = \prod_{k>l} ([b_k, A_l][A_k, x_l])^{\frac{i_{kl}}{v(i)}}.$$

Then

$$\varphi(A_i) = A_i \delta_{i1}.$$

Next,

$$\begin{aligned}
\varphi(\delta_{i1}) &= \prod_{k>l} ([\varphi(b_k), \varphi(A_l)][\varphi(A_k), \varphi(x_l)])^{\frac{i_{kl}}{v(i)}} \\
&= \prod_{k>l} ([b_k A_k, A_l \delta_{l1}][A_k \delta_{k1}, x_l A_l])^{\frac{i_{kl}}{v(i)}} \\
&= \prod_{k>l} ([b_k, A_l][b_k, \delta_{l1}][A_k, x_l][\delta_{k1}, x_l])^{\frac{i_{kl}}{v(i)}} \\
&= \prod_{k>l} ([b_k, A_l][A_k, x_l])^{\frac{i_{kl}}{v(i)}} \prod_{k>l} ([b_k, \delta_{l1}][\delta_{k1}, x_l])^{\frac{i_{kl}}{v(i)}};
\end{aligned}$$

so that

$$\varphi(\delta_{i1}) = \delta_{i1} \delta_{i2}$$

where

$$\delta_{i2} = \prod_{k>l} ([b_k, \delta_{l1}][\delta_{k1}, x_l])^{\frac{i_{kl}}{v(i)}}.$$

In general, for each i, we may construct a sequence of elements of G':

$$A_i, \delta_{i1}, \delta_{i2}, \ldots \tag{2.2}$$

where

$$\delta_{i1} = \prod_{k>l}([b_k, A_l][A_k, x_l])^{\frac{i_{kl}}{v(i)}},$$

$$\delta_{ij} = \prod_{k>l}([b_k, \delta_{l(j-1)}][\delta_{k(j-1)}, x_l])^{\frac{i_{kl}}{v(i)}} \text{ for } j > 1,$$

$\varphi(A_i) = A_i\delta_{i1}$, and $\varphi(\delta_{ij}) = \delta_{ij}\delta_{i(j+1)}$.

Suppose that $A_i \in \gamma_z G$ for some integer z. A typical basic commutator $[b_k, x_l]$ appearing as a factor in A_i has weight at least z, so that b_k has weight at least $z-1$ and $A_k \in \gamma_z$. This means that $[b_k, A_l][A_k, x_l] \in \gamma_{z+1}$, so $\delta_{i1} \in \gamma_{z+1}$. By making similar observations about higher terms of sequence 2.2, we see that $\delta_{ij} \in \gamma_{z+j}$.

We henceforth refer to sequence 2.2 as the *δ-sequence* associated to A_i.

Consider the following computations:

- $\varphi(A_i) = A_i\delta_{i1}$
- $\varphi^2(A_i) = \varphi(A_i)\varphi(\delta_{i1}) = A_i\delta_{i1}\delta_{i1}\delta_{i2} = A_i\delta_{i1}^2\delta_{i2}$
- $\varphi^3(A_i) = A_i\delta_{i1}(\delta_{i1}\delta_{i2})^2\delta_{i2}\delta_{i3} = A_i\delta_{i1}^3\delta_{i2}^3\delta_{i3}$
- $\varphi^4(A_i) = A_i\delta_{i1}(\delta_{i1}\delta_{i2})^3(\delta_{i2}\delta_{i3})^3\delta_{i3}\delta_{i4} = A_i\delta_{i1}^4\delta_{i2}^6\delta_{i3}^4\delta_{i4}$
- $\varphi^5(A_i) = A_i\delta_{i1}(\delta_{i1}\delta_{i2})^4(\delta_{i2}\delta_{i3})^6(\delta_{i3}\delta_{i4})^4\delta_{i4}\delta_{i5} = A_i\delta_{i1}^5\delta_{i2}^{10}\delta_{i3}^{10}\delta_{i4}^5\delta_{i5}$.

In general, we have

Lemma 2.1.1.

$$\varphi^m(A_i) = A_i \delta_{i1}^{c_1(m)} \delta_{i2}^{c_2(m)} \cdots \delta_{im}^{c_m(m)}$$

where the numbers $c_1(m), \ldots, c_m(m)$ correspond to to the m-th row of Pascal's triangle (ignoring the left-most diagonal of 1's):

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$\vdots$

More precisely: $c_j(m) = c_{j-1}(m-1) + c_j(m-1)$ if $j > 1$ and $c_1(m) = m$. Thus, $c_j(m)$ is simply the binomial coefficient

$$c_j(m) = \binom{m}{j} = \frac{m!}{n!(m-n)!}.$$

To find a formula for $\varphi^m(b_i)$ we proceed as follows:

$$A_i \varphi(A_i) \varphi^2(A_i) \cdots \varphi^{m-1}(A_i) =$$

$$A_i \left(A_i \delta_{i1}^{c_1(1)} \right) \left(A_i \delta_{i1}^{c_1(2)} \delta_{i2}^{c_2(2)} \right) \cdots \left(A_i \delta_{i1}^{c_1(m-1)} \delta_{i2}^{c_2(m-1)} \cdots \delta_{i(m-1)}^{c_{m-1}(m-1)} \right) =$$
$$A_i^m \delta_{i1}^{c_1(1)+\cdots+c_1(m-1)} \delta_{i2}^{c_2(2)+\cdots+c_2(m-1)} \cdots \delta_{i(m-1)}^{c_{m-1}(m-1)}. \tag{2.3}$$

Now put

$$d_j(m) = c_j(j) + \cdots c_j(m-1).$$

The following simple lemma will be extremely useful.

Lemma 2.1.2.

$$d_j(m) = \binom{m}{j+1}. \tag{2.4}$$

Proof. This follows from direct calculation:

$$\binom{m}{j+1} = \binom{m-1}{j} + \binom{m-1}{j+1} =$$
$$\binom{m-1}{j} + \binom{m-2}{j} + \binom{m-2}{j+1} =$$
$$\binom{m-1}{j} + \binom{m-2}{j} + \binom{m-3}{j} + \binom{m-3}{j+1} =$$
$$\binom{m-1}{j} + \binom{m-2}{j} + \binom{m-3}{j} + \cdots + \binom{j}{j} = d_j(m)$$

since $\binom{j}{j+1} = 0$. □

In particular

$$d_1(m) = c_1(1) + \cdots + c_1(m-1) = 1 + 2 + \cdots + m - 1 = \frac{m(m-1)}{2}.$$

Rewriting equation 2.3 we obtain:

$$A_i\varphi(A_i)\varphi^2(A_i)\cdots\varphi^{m-1}(A_i) = A_i\delta_{i1}^{d_1(m)}\delta_{i2}^{d_2(m)}\cdots\delta_{i(m-1)}^{d_{m-1}(m)}.$$

Since the $d_j(m)$ depend only on m, and not on i, we have proved:

Theorem 2.1.3. *If*

$$\varphi(b_i) = b_iA_i,$$

then

$$\varphi^m(b_i) = b_iA_i^m\delta_{i1}^{d_1(m)}\delta_{i2}^{d_2(m)}\cdots\delta_{i(m-1)}^{d_{m-1}(m)},$$

where

$$d_j(m) = \binom{m}{j+1}.$$

2.2 Proof that $IA(G)$ is p-local

Consider the map

$$IA(G) \to IA(G)$$

$$\varphi \mapsto \varphi^n$$

where $(n, p) = 1$, and G is as in section 2.1.

There are two results in this section. We begin with

Theorem 2.2.1.

$$IA(G) \to IA(G)$$

is one-to-one.

Proof. To see this, let

$$\varphi^n = \psi^n.$$

We wish to prove that $\varphi = \psi$. For this purpose put

$$\varphi(b_i) = b_i A_i,$$

$$\psi(b_i) = b_i \hat{A}_i$$

where $1 \leq wt(b_i) \leq c - 1$.

As usual we write

$$A_i = \prod_{k>l} [b_k, x_l]^{\frac{i_{kl}}{v}} \tag{2.5}$$

and

$$\hat{A}_i = \prod_{k>l} [b_k, x_l]^{\frac{\hat{i}_{kl}}{w}}. \tag{2.6}$$

In order to show that $\varphi = \psi$ we proceed by reverse induction on $wt(b_i)$:

Suppose $wt(b_i) = c-1$. Then both A_i and $\hat{A}_i$ belong to γ_c. $\varphi^n(b_i) = \psi^n(b_i)$ implies that $b_i A_i^n = b_i \hat{A}_i^n$, so that $A_i^n = \hat{A}_i^n$. By p-locality this means that $A_i = \hat{A}_i$.

Now suppose that b_i satisfies

$$1 \leq j = wt(b_i) \leq c-2.$$

Assume the induction hypothesis that $\varphi = \psi$ on γ_{j+1}. Our goal is to show that $\varphi(b_i) = \psi(b_i)$. Since $\varphi^n(b_i) = \psi^n(b_i)$ then

$$b_i A_i^n \delta_{i1}^{d_1(n)} \cdots \delta_{i(n-1)}^{d_{n-1}(n)} = b_i \hat{A}_i^n \hat{\delta}_{i1}^{d_1(n)} \cdots \hat{\delta}_{i(n-1)}^{d_{n-1}(n)}.$$

Using the fact that we are in a p-local group, we obtain:

$$A_i \hat{A}_i^{-1} = (\delta_{i1}^{-1} \hat{\delta}_{i1})^{\frac{d_1(n)}{n}} \cdots (\delta_{i(n-1)}^{-1} \hat{\delta}_{i(n-1)})^{\frac{d_{n-1}(n)}{n}}. \tag{2.7}$$

Since $wt(b_i) = j$, A_i and $\hat{A}_i$ each belongs to γ_{j+1}. In fact, by equation 2.7, the product $A_i \hat{A}_i^{-1}$ actually lies in γ_{j+2}. By induction:

$$\varphi(\hat{A}_i) = \psi(\hat{A}_i) = \hat{A}_i \hat{\delta}_{i1}.$$

This means that

$$\varphi(A_i \hat{A}_i^{-1}) = A_i \delta_{i1} (\hat{A}_i \hat{\delta}_{i1})^{-1} = A_i \hat{A}_i^{-1} \delta_{i1} \hat{\delta}_{i1}^{-1}.$$

Simply because φ is and IA-automorphism, it follows that

$$\delta_{i1} \hat{\delta}_{i1}^{-1} \in \gamma_{j+3}.$$

Similarly, if we evaluate φ on any $\delta_{im} \hat{\delta}_{im}^{-1}$ in the right hand side of 2.7 we have (again by induction on j) that

$$\varphi(\delta_{im} \hat{\delta}_{im}^{-1}) = \delta_{im} \hat{\delta}_{im}^{-1} \delta_{i(m+1)} \hat{\delta}_{i(m+1)}^{-1} \tag{2.8}$$

since

$$\varphi(\hat{\delta}_{im}) = \psi(\hat{\delta}_{im}).$$

Again, because φ is and IA-automorphism, equation 2.8 implies that $\delta_{i(m+1)}\hat{\delta}_{i(m+1)}^{-1}$ lies in a higher commutator subgroup than the commutator subgroup to which $\delta_{im}\hat{\delta}_{im}^{-1}$ belongs. Hence the entire right-hand side of equation 2.7 belongs to γ_{j+3}. We have established that

$$A_i\hat{A}_i^{-1} \in \gamma_{j+3}.$$

Apply the same argument repeatedly to finally conclude that

$$A_i\hat{A}_i^{-1} \in \gamma_{c+1} = 1.$$

This completes the proof that $IA(G) \to IA(G)$ is one-to-one. □

The second result of this section is that $IA(G) \to IA(G)$ is onto. To see this we prove

Theorem 2.2.2. *Let $\{b_1, b_2, \ldots\}$ be the basic commutators on $X = \{x_1, \ldots, x_r\}$ of weight at most $c-1$. Let $\varphi(b_i) = b_iA_i \in IA(G)$. Then, there exists an IA-automorphism ψ such that $\psi^n = \varphi$.*

Proof. **Part I.** Let $\delta_{i1}, \delta_{i2}, \ldots, \delta_{i(c-2)}$ be the δ-sequence associated to A_i.[1] We claim that it is possible to find p-local integers $\epsilon_1(n), \ldots, \epsilon_{c-2}(n)$ that depend on n (and c), such that

- $\psi(b_i) = b_iA_i^{\frac{1}{n}}\delta_{i1}^{\epsilon_1(n)}\cdots\delta_{i(c-2)}^{\epsilon_{c-2}(n)} \in IA(G)$, and
- $\psi^n = \varphi$.

[1]Since G has class c, we can henceforth assume that $\delta_{il} = 1$ for $l > c-2$.

Construction of ψ:

Put

$$A_i = \prod_{k>l} [b_k, x_l]^{i_{kl}},$$

where the i_{kl} are p-local integers.

We wish to find p-local integers $\epsilon_1, \ldots, \epsilon_{c-2}$, depending on n and c alone, such that

$$\psi(b_i) = b_i \alpha_i$$

gives an IA-automorphism where $\alpha_i = A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}}$ and $\psi^n = \varphi$. We will show that these ϵ's can in fact be found by "solving" the equation $\psi^n = \varphi$.

Associated with α_i we have the corresponding sequence of deltas: $\hat{\delta}_{i1}, \cdots, \hat{\delta}_{i(c-2)}$. To relate the $\hat{\delta}$'s with the δ's we do as follows:

$$\psi(A_i) = \prod_{k>l} [\psi(b_k), \psi(x_l)]^{i_{kl}} = \prod_{k>l} \left[b_k A_k^{\frac{1}{n}} \delta_{k1}^{\epsilon_1} \cdots \delta_{k(c-2)}^{\epsilon_{c-2}}, x_l A_l^{\frac{1}{n}} \delta_{l1}^{\epsilon_1} \cdots \delta_{l(c-2)}^{\epsilon_{c-2}} \right]^{i_{kl}}.$$

Using commutator calculus in this metabelian group, and recalling the definitions of $A_i, \delta_{i1}, \ldots, \delta_{i(c-2)}$ we see that

$$\psi(A_i) = A_i \delta_{i1}^{\frac{1}{n}} \delta_{i2}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-3}} \delta_{i(c-1)}^{\epsilon_{c-2}},$$

where $\delta_{i(c-1)} = 1$.

Analogous calculations give:

$$\begin{aligned}
\psi(b_i) &= b_i A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots\cdots\cdots\cdots \delta_{i(c-2)}^{\epsilon_{c-2}} \\
\psi(A_i) &= A_i \delta_{i1}^{\frac{1}{n}} \delta_{i2}^{\epsilon_1} \cdots\cdots\cdots \delta_{i(c-2)}^{\epsilon_{c-3}} \\
\psi(\delta_{i1}) &= \delta_{i1} \delta_{i2}^{\frac{1}{n}} \delta_{i3}^{\epsilon_1} \cdots\cdots \delta_{i(c-2)}^{\epsilon_{c-4}} \\
\psi(\delta_{i2}) &= \delta_{i2} \delta_{i3}^{\frac{1}{n}} \delta_{i4}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-5}} \\
&\vdots \\
\psi(\delta_{i(c-2)}) &= \delta_{i(c-2)}.
\end{aligned}$$

Next, we find the $\hat{\delta}_{ij}$:

$$\psi(\alpha_i) = \psi\left(A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}}\right) =$$
$$\left(A_i^{\frac{1}{n}} \delta_{i1}^{\frac{1}{n}\frac{1}{n}} \delta_{i2}^{\frac{1}{n}\epsilon_1} \delta_{i3}^{\frac{1}{n}\epsilon_2} \cdots\cdots\cdots \delta_{i(c-2)}^{\frac{1}{n}\epsilon_{c-3}}\right)$$
$$\left(\delta_{i1}^{\epsilon_1} \delta_{i2}^{\epsilon_1\frac{1}{n}} \delta_{i3}^{\epsilon_1\epsilon_1} \cdots \delta_{i(c-3)}^{\epsilon_1\epsilon_{c-5}} \delta_{i(c-2)}^{\epsilon_1\epsilon_{c-4}}\right)$$
$$\left(\delta_{i2}^{\epsilon_2} \delta_{i3}^{\epsilon_2\frac{1}{n}} \cdots \delta_{i(c-3)}^{\epsilon_2\epsilon_{c-6}} \delta_{i(c-2)}^{\epsilon_2\epsilon_{c-5}}\right)$$
$$\vdots$$
$$\delta_{i(c-2)}^{\epsilon_{c-2}}.$$

By rearranging the above expression:

$$\psi(\alpha_i) = \left(A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \delta_{i2}^{\epsilon_2} \cdots\cdots\cdots \delta_{i(c-2)}^{\epsilon_{c-2}}\right)$$
$$\left(\delta_{i1}^{\frac{1}{n}\frac{1}{n}} \delta_{i2}^{\epsilon_1\frac{1}{n}} \delta_{i3}^{\epsilon_2\frac{1}{n}} \cdots\cdots \delta_{i(c-2)}^{\epsilon_{c-3}\frac{1}{n}}\right)$$
$$\left(\delta_{i2}^{\frac{1}{n}\epsilon_1} \delta_{i3}^{\epsilon_1\epsilon_1} \delta_{i4}^{\epsilon_2\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-4}\epsilon_1}\right)$$
$$\left(\delta_{i3}^{\frac{1}{n}\epsilon_2} \delta_{i4}^{\epsilon_1\epsilon_2} \cdots \delta_{i(c-2)}^{\epsilon_{c-5}\epsilon_2}\right)$$
$$\vdots$$
$$\left(\delta_{i(c-3)}^{\frac{1}{n}\epsilon_{c-4}} \delta_{i(c-2)}^{\epsilon_1\epsilon_{c-4}}\right)$$
$$\delta_{i(c-2)}^{\frac{1}{n}\epsilon_{c-3}}.$$

From this we see that:

$$\hat{\delta}_{i1} = \left(\delta_{i1}^{\frac{1}{n}\frac{1}{n}}\right)\left(\delta_{i2}^{\epsilon_1\frac{1}{n}+\frac{1}{n}\epsilon_1}\right)\left(\delta_{i3}^{\epsilon_2\frac{1}{n}+\epsilon_1\epsilon_1+\frac{1}{n}\epsilon_2}\right)\left(\delta_{i4}^{\epsilon_3\frac{1}{n}+\epsilon_2\epsilon_1+\epsilon_1\epsilon_2+\frac{1}{n}\epsilon_3}\right)\cdots\left(\delta_{i(c-2)}^{\epsilon_{c-3}\frac{1}{n}+\epsilon_{c-4}\epsilon_1+\cdots+\frac{1}{n}\epsilon_{c-3}}\right). \tag{2.9}$$

This expresses $\hat{\delta}_{i1}$ in terms of the δ_{ij}'s.

Rewriting the exponents in equation 2.9 we get

$$\hat{\delta}_{i1} = \delta_{i1}^{\frac{1}{n^2}} \delta_{i2}^{\alpha_1} \delta_{i3}^{\alpha_2} \cdots \delta_{i(c-2)}^{\alpha_{c-3}},$$

where each α_j depends on $n, \epsilon_1, \ldots, \epsilon_j$. Symbolically:

$$\alpha_j = \alpha_j(n, \epsilon_1, \epsilon_2, \ldots, \epsilon_j)$$

for $j = 1, 2, \ldots, c-3$.

In order to find an expression for $\hat{\delta}_{i2}$, evaluate ψ on $\hat{\delta}_{i1}$:

$$\psi\left(\hat{\delta}_{i1}\right) = \psi\left(\delta_{i1}^{\frac{1}{n^2}} \delta_{i2}^{\alpha_1} \delta_{i3}^{\alpha_2} \cdots \delta_{i(c-2)}^{\alpha_{c-3}}\right) =$$

$$\left(\delta_{i1} \delta_{i2}^{\frac{1}{n}} \delta_{i3}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-4}}\right)^{\frac{1}{n}\frac{1}{n}} \left(\delta_{i2} \delta_{i3}^{\frac{1}{n}} \cdots \delta_{i(c-2)}^{\epsilon_{c-5}}\right)^{\alpha_1} \cdots \left(\delta_{i(c-3)} \delta_{i(c-2)}^{\frac{1}{n}}\right)^{\alpha_{c-4}} \left(\delta_{i(c-2)}\right)^{\alpha_{c-3}} =$$

$$\left(\delta_{i1}^{\frac{1}{n}\frac{1}{n}} \delta_{i2}^{\alpha_1} \cdots \delta_{i(c-2)}^{\alpha_{c-3}}\right) \left(\delta_{i2}^{\frac{1}{n^3}} \delta_{i3}^{\frac{1}{n}\alpha_1} \cdots \delta_{i(c-2)}^{\frac{1}{n}\alpha_{c-4}}\right) \left(\delta_{i3}^{\epsilon_1 \frac{1}{n^2}} \delta_{i4}^{\epsilon_1 \alpha_1} \cdots \delta_{i(c-2)}^{\epsilon_1 \alpha_{c-5}}\right) \cdots \left(\delta_{i(c-2)}^{\epsilon_{c-4} \frac{1}{n^2}}\right).$$

Since the first term of the right-hand side is $\hat{\delta}_{i1}$, we readily obtain

$$\hat{\delta}_{i2} = \delta_{i2}^{\frac{1}{n^3}} \delta_{i3}^{\frac{1}{n}\alpha_1 + \epsilon_1 \frac{1}{n^2}} \delta_{i4}^{\frac{1}{n}\alpha_2 + \epsilon_1 \alpha_1 + \epsilon_2 \frac{1}{n^2}} \cdots \delta_{i(c-2)}^{\frac{1}{n}\alpha_{c-4} + \cdots + \epsilon_{c-4}\frac{1}{n^2}}.$$

Again, we rewrite the exponents in this equation to obtain:

$$\hat{\delta}_{i2} = \delta_{i2}^{\frac{1}{n^3}} \delta_{i3}^{\beta_1} \cdots \delta_{i(c-2)}^{\beta_{c-4}},$$

where

$$\beta_j = \beta_j(n, \epsilon_1, \ldots, \epsilon_j; \alpha_1, \ldots, \alpha_j).$$

Since $\alpha_j = \alpha_j(n, \epsilon_1, \ldots, \epsilon_j)$, we can in fact write:

$$\beta_j = \beta_j(n, \epsilon_1, \ldots, \epsilon_j)$$

where $j = 1, \ldots, c-4$.

Continue computing the $\hat{\delta}_{ij}$'s in this way, to finally get

$$\hat{\delta}_{i(c-3)} = \delta_{i(c-3)}^{\left(\frac{1}{n}\right)^{c-2}} \delta_{i(c-2)}^{\tau_1},$$

where $\tau_1 = \tau_1(n, \epsilon_1)$, and

$$\hat{\delta}_{i(c-2)} = \delta_{i(c-2)}^{\left(\frac{1}{n}\right)^{c-1}}.$$

We now show that the ϵ's appearing in the equation

$$\psi^n(b_i) = \varphi(b_i) \tag{2.10}$$

can be chosen so that they depend only on n. For this purpose we use the formula that describes powers of IA-automorphisms (see theorem 2.1.3). Equation 2.10 becomes:

$$A_i \delta_{i1}^{n\epsilon_1} \cdots \delta_{i(c-2)}^{n\epsilon_{c-2}} \hat{\delta}_{i1}^{d_1} \cdots \hat{\delta}_{i(c-2)}^{d_{(c-2)}} = A_i.$$

Canceling A_i and rewriting the $\hat{\delta}$'s in terms of the δ's yields:

$$\delta_{i1}^{n\epsilon_1} \cdots \delta_{i(c-2)}^{n\epsilon_{c-2}} \left(\delta_{i1}^{\frac{1}{n^2}} \delta_{i2}^{\alpha_1} \cdots \delta_{i(c-2)}^{\alpha_{c-3}}\right)^{d_1} \left(\delta_{i2}^{\frac{1}{n^3}} \delta_{i3}^{\beta_1} \cdots \delta_{i(c-2)}^{\beta_{c-4}}\right)^{d_2} \cdots \left(\delta_{i(c-2)}^{\frac{1}{n^{c-1}}}\right)^{d_{c-2}} = 1.$$

Now solve for each ϵ_i in the following way:

- $n\epsilon_1 + \frac{1}{n^2} d_1 = 0 \Rightarrow \epsilon_1 = -\frac{1}{n^3} d_1$, which means that ϵ_1 can be chosen so that it depends on n only.

- $n\epsilon_2 + \alpha_1 d_1 + \frac{1}{n^3} d_2 = 0 \Rightarrow \epsilon_2 = -\frac{1}{n}\alpha_1 d_1 - \frac{1}{n^4} d_2$. Since α_1 depends on n and ϵ_1, we conclude that ϵ_2 can be chosen so that it depends on n only.

- $n\epsilon_3 + \alpha_2 d_1 + \beta_1 d_2 + \frac{1}{n^4} d_3 = 0 \Rightarrow \epsilon_3 = -\frac{1}{n}\alpha_2 d_1 - \frac{1}{n}\beta_1 d_2 - \frac{1}{n^5} d_3$. Again, α_2 depends on n, ϵ_1, and ϵ_2; so ϵ_3 can be chosen as to depend on n only.

Continue this process to choose $\epsilon_1, \ldots, \epsilon_{c-2}$ so that they depend on n alone.

Part II. Consider the following map, defined on the "original" p-generators $\{x_1, \ldots, x_r\}$ of G:

$$\psi(x_i) = x_i A_i^{\frac{1}{n}} \delta_{i1}^{\epsilon_1} \cdots \delta_{i(c-2)}^{\epsilon_{c-2}}.$$

Our first task is to show that ψ can be extended to a well defined homomorphism

$$gp(x_1, \ldots, x_r) \to G$$

(and can therefore be "lifted" to a self-homomorphism of G).

For this purpose, suppose that r is a trivial word in the x's:

$$r = x_{t_1} \cdots x_{t_s} = 1.$$

We need to show that

$$\text{``}\psi(x_{t_1}) \cdots \psi(x_{t_s}) = 1.\text{''}$$

A straightforward computation yields:

$$\psi(x_{t_1})\psi(x_{t_2}) \cdots \psi(x_{t_s}) =$$
$$\left(x_{t_1} A_{t_1}^{\frac{1}{n}} \delta_{t_1 1}^{\epsilon_1} \cdots \delta_{t_1(c-2)}^{\epsilon_{c-2}}\right) \left(x_{t_2} A_{t_2}^{\frac{1}{n}} \delta_{t_2 1}^{\epsilon_1} \cdots \delta_{t_2(c-2)}^{\epsilon_{c-2}}\right) \cdots \left(x_{t_s} A_{t_s}^{\frac{1}{n}} \delta_{t_s 1}^{\epsilon_1} \cdots \delta_{t_s(c-2)}^{\epsilon_{c-2}}\right) =$$
$$x_{t_1} \cdots x_{t_s} (A_{t_1} \cdots A_{t_s})^{\frac{1}{n}} (\delta_{t_1 1} \cdots \delta_{t_s 1})^{\epsilon_1} \cdots \left(\delta_{t_1(c-2)} \cdots \delta_{t_s(c-2)}\right)^{\epsilon_{c-2}} \prod_{k<l} [\omega_{t_k}, x_{t_l}],$$

where

$$\omega_{t_j} = A_{t_j}^{\frac{1}{n}} \delta_{t_j 1}^{\epsilon_1} \cdots \delta_{t_{j(c-2)}}^{\epsilon_{c-2}}.$$

Remembering that $r = 1$, we have:

$$\psi(x_{t_1}) \cdots \psi(x_{t_s}) = (A_{t_1} \cdots A_{t_s})^{\frac{1}{n}} (\delta_{t_1 1} \cdots \delta_{t_s 1})^{\epsilon_1} \cdots \left(\delta_{t_1(c-2)} \cdots \delta_{t_s(c-2)}\right)^{\epsilon_{c-2}} \prod_{k<l} [\omega_{t_k}, x_{t_l}]. \tag{2.11}$$

On the other hand:

$$1 = \varphi(x_{t_1} \cdots x_{ts}) = x_{t_1} A_{t_1} \cdots x_{t_s} A_{t_s} =$$
$$x_{t_1} \cdots x_{t_s} A_{t_1} \cdots A_{t_s} \prod_{k<l} [A_{t_k}, x_{t_l}].$$

Again, since $x_{t_1} \cdots x_{t_s} = 1$ we find that

$$1 = A_{t_1} \cdots A_{t_s} \prod_{k<l} [A_{t_k}, x_{t_l}]\,. \tag{2.12}$$

Taking the nth power on both sides of equation 2.11 yields

$$(\psi(x_{t_1}) \cdots \psi(x_{t_s}))^n =$$

$$(A_{t_1} \cdots A_{t_s})\,(\delta_{t_1 1} \cdots \delta_{t_s 1})^{n\epsilon_1} \cdots \left(\delta_{t_1(c-2)} \cdots \delta_{t_s(c-2)}\right)^{n\epsilon_{c-2}} \prod_{k<l} [\omega_{t_k}, x_{t_l}]^n =$$

$$(A_{t_1} \cdots A_{t_s})\,(\delta_{t_1 1} \cdots \delta_{t_s 1})^{n\epsilon_1} \cdots \left(\delta_{t_1(c-2)} \cdots \delta_{t_s(c-2)}\right)^{n\epsilon_{c-2}} \prod_{k<l} \left[\omega_{t_k}^n, x_{t_l}\right]. \tag{2.13}$$

Now, since

$$\omega_{t_k} = A_{t_k}^{\frac{1}{n}} \delta_{t_k 1}^{\epsilon_1} \cdots \delta_{t_k(c-2)}^{\epsilon_{c-2}}, \text{ then } \omega_{t_k}^n = A_{t_k} \delta_{t_k 1}^{n\epsilon_1} \cdots \delta_{t_k(c-2)}^{n\epsilon_{c-2}};$$

so

$$\prod_{k<l} \left[\omega_{t_k}^n, x_{t_l}\right] = \prod_{k<l} [A_{t_k}, x_{t_l}] \left[\delta_{t_k 1}^{n\epsilon_1}, x_{t_l}\right] \cdots \left[\delta_{t_k(c-2)}^{n\epsilon_{c-2}}, x_{t_l}\right]. \tag{2.14}$$

Substituting 2.14 in 2.13, together with equation 2.12 and commutator calculus gives

$$(\psi(x_{t_1}) \cdots \psi(x_{t_s}))^n =$$

$$\left((\delta_{t_1 1} \cdots \delta_{t_s 1})^{\epsilon_1} \cdots \left(\delta_{t_1(c-2)} \cdots \delta_{t_s(c-2)}\right)^{\epsilon_{c-2}} \prod_{k<l} [\delta_{t_k 1}, x_{t_l}]^{\epsilon_1} \cdots \left[\delta_{t_k(c-2)}, x_{t_l}\right]^{\epsilon_{c-2}} \right)^n. \tag{2.15}$$

Since we are in a p-local group, we may take nth roots on both sides, so that

$$\psi(x_{t_1}) \cdots \psi(x_{t_s}) =$$

$$(\delta_{t_1 1} \cdots \delta_{t_s 1})^{\epsilon_1} \cdots \left(\delta_{t_1(c-2)} \cdots \delta_{t_s(c-2)}\right)^{\epsilon_{c-2}} \prod_{k<l} [\delta_{t_k 1}, x_{t_l}]^{\epsilon_1} \cdots \left[\delta_{t_k(c-2)}, x_{t_l}\right]^{\epsilon_{c-2}}. \tag{2.16}$$

Next, apply φ to equation 2.12 to obtain

$$1 = A_{t_1}\delta_{t_1 1}\cdots A_{t_s}\delta_{t_s 1}\prod_{k<l}[A_{t_k}\delta_{t_k 1}, x_{t_l}A_{t_l}] =$$
$$A_{t_1}\cdots A_{t_s}\prod_{k<l}[A_{t_k}, x_{t_l}]\,\delta_{t_1 1}\cdots\delta_{t_s 1}\prod_{k<l}[\delta_{t_k 1}, x_{t_l}]\,.$$

Using equation 2.12 itself we conclude that

$$\delta_{t_1 1}\cdots\delta_{t_s 1}\prod_{k<l}[\delta_{t_k 1}, x_{t_l}] = 1. \tag{2.17}$$

Apply φ to 2.17 to get

$$\delta_{t_1 2}\cdots\delta_{t_s 2}\prod_{k<l}[\delta_{t_k 2}, x_{t_l}] = 1. \tag{2.18}$$

We do this repeatedly to finally conclude (from equation 2.16) that

$$\psi(x_{t_1})\cdots\psi(x_{t_s}) = 1,$$

as promised.

We have proved the following two major facts:

1. The map ψ defined on the p-generators of G as

$$\psi(x_i) = x_i A_i^{\frac{1}{n}}\delta_{i1}^{\epsilon_1}\cdots\delta_{i(c-2)}^{\epsilon_{c-2}}$$

   extends to a well-defined homomorphism from $gp(x_1,\ldots,x_r)$ to G, and can therefore be lifted to a self-homomorphism of G.

2. ψ satisfies

$$x_i^{-1}\psi(x_i) \in G' = [G,G]$$

   for all i, and

$$\psi^n = \varphi.$$

We now give a very simple argument to establish that ψ is in fact an automorphism, and therefore an IA-automorphism. To show that ψ is one-to-one, choose $g \in \ker \psi$. Then $\psi(g) = 1$, so that $\psi^n(g) = 1$. Since $\psi^n = \varphi$, $\varphi(g) = 1$. We conclude that $g \in \ker \varphi$. Since φ is one-to-one, $g = 1$. Hence, ψ is one-to-one.

Next choose $g \in G$. Since φ is onto, there exists a $g' \in G$ so that $\varphi(g') = g$. Thus, $\psi^n(g') = g$, which implies that $\psi\left(\psi^{n-1}(g')\right) = g$. This means that ψ is onto.

Theorem 2.2.2 is now proved. □

2.3 $IA(G) \to IA(G_{(p)})$ is a p-isomorphism

Let $G = gp(x_1, \ldots, x_m)$ be a finitely generated, torsion-free, metabelian, and nilpotent group of class c. Let $G_{(p)}$ be its localization at the prime p. Consider the localization diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ G_{(p)} & \xrightarrow{f_p} & G_{(p)} \end{array}$$

and observe that if $f \in IA(G)$, then $f_p \in IA(G_{(p)})$.

Lemma 2.3.1. *The homomorphism*

$$IA(G) \to IA(G_{(p)})$$

$$f \mapsto f_p$$

is a monomorphism (and hence p-injective).

Proof. Let $f \in \ker\big(IA(G) \to IA(G_{(p)})\big)$ and put $f(x_i) = x_i A_i$ where A_i lies in $G' = [G, G]$. For $g \in G$, write $e(g) = \bar{g}$. If x_i is a generator of G, $\bar{x}_i$ belongs to $G_{(p)}$ and

$$\bar{x}_i = f_p(\bar{x}_i) = \overline{f(x_i)} = \bar{x}_i \bar{A}_i.$$

Hence, $\bar{A}_i = \bar{1}$ in $G'_{(p)}$, which means that A_i belongs to $\ker\Big(e : G' \to G'_{(p)}\Big)$.

Since G' is torsion-free and $e : G' \to G'_{(p)}$ is a localization map, e is one-to-one so that $A_i = 1$. Hence $f(x_i) = x_i$; and therefore

$$IA(G) \to IA(G_{(p)})$$

$$f \mapsto f_p$$

is indeed a monomorphism. □

Remark. The discussion that led to the δ-sequence (see 2.2) and theorem 2.1.3 is also valid for $IA(G)$, where G is finitely generated, torsion-free, nilpotent, and metabelian (the condition of G being p-local is dropped). This observation will be used in the sequel.

Lemma 2.3.2. *$IA(G) \to IA(G_{(p)})$ is p-surjective.*

Proof. Let $\varphi \in IA(G_{(p)})$. Consider the action of φ on the "p-generators" of $G_{(p)}$: $\varphi(\bar{x}_i) = \bar{x}_i A_i$, where $A_i \in G'_{(p)}$. Since $e : G' \to G'_{(p)}$ is p-surjective, there exists and integer s_i, relatively prime to p, such that $A_i^{s_i}$ belongs to the image of $e : G' \to G'_{(p)}$ for each $i = 1, 2, \ldots, m$. Put

$$\sigma_1 = s_1 s_2 \ldots s_m.$$

$A_i^{\sigma_1}$ clearly lies in the image of $e : G' \to G'_{(p)}$ (for each $i = 1, 2, \ldots, m$) because such image is a subgroup of $G'_{(p)}$. Similarly, choose $\sigma_2, \ldots, \sigma_{c-1}$ (independent if i) so that $\delta_{ik}^{\sigma_{k+1}}$ belongs to the image of $e : \gamma_{k+2}(G) \to \gamma_{k+2}(G_{(p)})$, for $k = 1, \ldots, c-2$. Let

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_{c-1}.$$

We have the following:

1. σ is relatively prime to p,

2. A_i^{σ} belongs to the image of $e : G' \to G'_{(p)}$, and

3. δ_{ik}^{σ} belongs to the image of $e : \gamma_{k+2}(G) \to \gamma_{k+2}(G_{(p)})$, $(k = 1, \ldots, c-2)$.

Using lemma 2.1.2, we see that

- $d_1(\sigma) = \binom{\sigma}{2} = \frac{\sigma(\sigma-1)}{2}$
- $d_2(\sigma) = \binom{\sigma}{3} = \frac{\sigma(\sigma-1)(\sigma-2)}{3!}$
- $d_3(\sigma) = \binom{\sigma}{4} = \frac{\sigma(\sigma-1)(\sigma-2)(\sigma-3)}{4!}$
- $\vdots$
- $d_{c-2}(\sigma) = \binom{\sigma}{c-1} = \frac{\sigma(\sigma-1)(\sigma-2)\cdots(\sigma-c+2)}{(c-1)!}$.

Fix $1 \leq j \leq c-2$ and consider the number

$$d_j(\sigma) = \binom{\sigma}{j+1} = \frac{\sigma(\sigma-1)(\sigma-2)\cdots(\sigma-j)}{(j+1)!}.$$

Write

$$(j+1)! = p^{\alpha_j} \epsilon_j$$

where p and ϵ_j are relatively prime. (If p does not divide $(j+1)!$ take $\alpha_j = 0$.) Next, let

$$\epsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_{c-2}.$$

Notice that p and ϵ are relatively prime. Now put

$$s = \epsilon\sigma.$$

Again, p and s are relatively prime. Invoking lemma 2.1.2 once more, we see that for each j:

$$d_j(s) = \frac{\epsilon_1\epsilon_2\cdots\epsilon_{c-2}\sigma(s-1)(s-2)\cdots(s-j)}{p^{\alpha_j}\epsilon_j} = \frac{\epsilon_1\epsilon_2\cdots\epsilon_{j-1}\epsilon_{j+1}\cdots\epsilon_{c-2}\sigma(s-1)(s-2)\cdots(s-j)}{p^{\alpha_j}}.$$

As $d_j(s)$ is an integer, and p^{α_j} does not divide $\epsilon_1\epsilon_2\cdots\epsilon_{j-1}\epsilon_{j+1}\cdots\epsilon_{c-2}\sigma$, p^{α_j} has to divide $(s-1)(s-2)\cdots(s-j)$. Hence, each integer $s, d_1(s), d_2(s), \ldots, d_{c-2}(s)$ is a multiple of σ. The crucial conclusion is that

1. s is relatively prime to p,
2. A_i^s belongs to the image of $e : G' \to G'_{(p)}$, and
3. $\delta_{ij}^{d_j(s)}$ lies in the image of $e : \gamma_{j+2}(G) \to \gamma_{j+2}(G_{(p)})$, for $j = 1, 2, \ldots, c-2$.

We can therefore choose $\alpha_i \in G'$ such that $\overline{\alpha}_i = A_i^s$, and $D_{ij} \in \gamma_{j+2}(G)$ such that $\overline{D}_{ij} = \delta_{ij}^{d_j(s)}$ for each $j = 1, 2, \ldots, c-2$; where $\overline{g} = e(g)$ for $g \in G$.

Using theorem 2.1.3 we see that

$$\varphi^s(\bar{x}_i) = \bar{x}_i A_i^s \delta_{i1}^{d_1(s)} \cdots \delta_{i(c-2)}^{d_{c-2}(s)}.$$

Let

$$\beta_i = \alpha_i D_{i1} \cdots D_{i(c-2)} \in G'.$$

Define the following map on the generators of G:

$$f(x_i) = x_i\beta_i.$$

If we can show that f can be extended to an element of $IA(G)$, then f_p and φ^s will coincide on the p-generators of $G_{(p)}$ and will therefore be equal as IA-automorphisms.

Using similar techniques as before, we prove first that f extends to a self-homomorphism of G. For this purpose let $r = x_{t_1}x_{t_2}\cdots x_{t_v}$ be a trivial word in the generators of G. Then

$$\begin{aligned} f(x_{t_1})\cdots f(x_{t_v}) = (x_{t_1}\beta_{t1})\cdots(x_{t_v}\beta_{t_v}) = \\ (x_{t_1}\cdots x_{t_v})(\beta_{t_1}\cdots\beta_{t_v})\prod_{l<m}[\beta_{t_l}, x_{t_m}] = \\ (\beta_{t_1}\cdots\beta_{t_v})\prod_{l<m}[\beta_{t_l}, x_{t_m}]\,. \end{aligned}$$

We thus have:

$$f(x_{t_1})\cdots f(x_{t_v}) = (\beta_{t_1}\cdots\beta_{t_v})\prod_{l<m}[\beta_{t_l}, x_{t_m}]\,. \tag{2.19}$$

Next we work with φ. Since $r = x_{t_1}\cdots x_{t_v} = 1$, then

$$\overline{r} = \overline{x_{t_1}x_{t_2}\cdots x_{t_v}} = \overline{1}.$$

Thus

$$\begin{aligned} \overline{1} = \varphi(\overline{r}) = (\overline{x}_{t_1}A_{t_1})\cdots(\overline{x}_{t_v}A_{t_v}) = \\ \overline{x}_{t_1}\cdots\overline{x}_{t_v}A_{t_1}\cdots A_{t_v}\prod_{l<m}[A_{t_l}, \overline{x}_{t_m}] = \\ A_{t_1}\cdots A_{t_v}\prod_{l<m}[A_{t_l}, \overline{x}_{t_m}]\,. \end{aligned}$$

Hence

$$\overline{1} = A_{t_1}\cdots A_{t_v}\prod_{l<m}[A_{t_l}, \overline{x}_{t_m}]\,. \tag{2.20}$$

Apply φ to both sides of equation 2.20 to get

$$\overline{1} = (A_{t_1}\delta_{t_1 1})\cdots(A_{t_v}\delta_{t_v 1})\prod_{l<m}[A_{t_l}\delta_{t_l 1}, \overline{x}_{t_m}A_{t_m}] =$$

$$A_{t_1}\cdots A_{t_v}\prod_{l<m}[A_{t_l}, \overline{x}_{t_m}]\,\delta_{t_1 1}\cdots\delta_{t_v 1}\prod_{l<m}[\delta_{t_l 1}, \overline{x}_{t_m}] = \overline{1}.$$

We conclude that

$$\delta_{t_1 1}\cdots\delta_{t_v 1}\prod_{l<m}[\delta_{t_l 1}, \overline{x}_{t_m}] = \overline{1}.$$

Iterate this process to obtain

$$\delta_{t_1 j}\cdots\delta_{t_v j}\prod_{l<m}[\delta_{t_l j}, \overline{x}_{t_m}] = \overline{1} \tag{2.21}$$

for all $j = 1, 2, \ldots, c-2$.

By equation 2.19 and the definition of $\overline{\beta}_i$:

$$\overline{f(x_{t_1})\cdots f(x_{t_v})} =$$

$$(A_{t_1}\cdots A_{t_v})^s(\delta_{t_1 1}\cdots\delta_{t_v 1})^{d_1}\cdots\left(\delta_{t_1(c-2)}\cdots\delta_{t_v(c-2)}\right)^{d_{c-2}}\prod_{l<m}\left[A_{t_l}^s\delta_{t_l 1}^{d_1}\cdots\delta_{t_l(c-2)}^{d_{c-2}}, \overline{x}_{t_m}\right] =$$

$$(A_{t_1}\cdots A_{t_v})^s(\delta_{t_1 1}\cdots\delta_{t_v 1})^{d_1}\cdots\left(\delta_{t_1(c-2)}\cdots\delta_{t_v(c-2)}\right)^{d_{c-2}}$$

$$\prod_{l<m}\left([A_{t_l}, \overline{x}_{t_m}]^s[\delta_{t_l 1}, \overline{x}_{t_m}]^{d_1}\cdots\left[\delta_{t_l(c-2)}, \overline{x}_{t_m}\right]^{d_{c-2}}\right) =$$

$$\left(A_{t_1}\cdots A_{t_v}\prod_{l<m}[A_{t_l}, \overline{x}_{t_m}]\right)^s\left(\delta_{t_1 1}\cdots\delta_{t_v 1}\prod_{l<m}[\delta_{t_l 1}, \overline{x}_{t_m}]\right)^{d_1}\cdots$$

$$\left(\delta_{t_1(c-2)}\cdots\delta_{t_v(c-2)}\prod_{l<m}\left[\delta_{l(c-2)}, \overline{x}_{t_m}\right]\right)^{d_{c-2}} = \overline{1}$$

(by equations 2.20 and 2.21). We have shown that

$$\overline{f(x_{t_1})\cdots f(x_{t_v})} = \overline{1};$$

in other words

$$f(x_{t_1}) \cdots f(x_{t_v}) \in \ker\left(e : G \to G_{(p)}\right).$$

As G is torsion-free, e is in fact one-to-one; so that

$$f(x_{t_1}) \cdots f(x_{t_v}) = 1.$$

f thus extendes to a self-homomorphism of G.

Besides proving that f extends to a self-homomorphism of G, we have seen (by construction) that

$$g^{-1} f(g) \in G'$$

for all $g \in G$ and

$$f_p = \varphi^s$$

on the p-generators of $G_{(p)}$ (and hence on all of $G_{(p)}$).

We now provide a very simple argument to show that f is an IA-automorphism. Contemplate the localization diagram:

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ G_{(p)} & \xrightarrow{f_p} & G_{(p)}. \end{array}$$

Since f_p is already an IA-automorphism and G is torsion-free, both e and f_p are one-to-one. Choose $g \in \ker(f)$. Then, by commutativity of the diagram, $f_p\left(e(g)\right) = \overline{1}$. It follows readily that $g = 1$ and f is one-to-one.

To prove that f is onto we argue as follows: using commutator calculus one can prove that if $g_j \in \gamma_j$ then $g_j^{-1} f(g_j) \in \gamma_{j+1}$. In particular, f is the identity on γ_c and hence (trivially) onto there. We now do reverse induction on j. Suppose f is onto on γ_{j+1}. We show that "f is onto on γ_j": let $g_j \in \gamma_j$, then, by induction

$$f(g_j) = g_j g_{j+1} = g_j f(h_{j+1})$$

for some $g_{j+1}, h_{j+1} \in \gamma_{j+1}$. Hence

$$f\left(g_j h_{j+1}^{-1}\right) = g_j$$

where clearly $g_j h_{j+1}^{-1} \in \gamma_j$. Thus f is onto on γ_j. This completes the proof of the lemma. □

The main result of this thesis follows, which we state as

Theorem 2.3.3. *Let G be finitely generated, metabelian, and nilpotent of class c; and $G_{(p)}$ its localization at the prime p. Then, the natural map*

$$IA(G) \to IA(G_{(p)})$$

is a p-isomorphism.

If it were the case that $InnG = IA(G)$ for all nilpotent groups G, theorem 2.3.3 would be trivial since we would have

$$IA(G)_{(p)} \cong (G/\zeta)_{(p)} \cong G_{(p)}/\zeta_{(p)} \cong IA(G_{(p)}).$$

To demonstrate that our theorem is nontrivial in general we compute $InnG$ and $IA(G)$ where G is free nilpotent of class 2 and rank 3. It will then be clear that $InnG \neq IA(G)$.

Let

$$G = \langle x, y, z \rangle$$

be free nilpotent of class 2 on the generators $\{x, y, z\}$. Put $c_{12} = [x, y]$, $c_{13} = [x, z]$, and $c_{23} = [y, z]$. Then

$$\{x, y, z, c_{12}, c_{13}, c_{23}\}$$

is a basic sequence of basic commutators on $\{x, y, z\}$. Since G is free nilpotent, every $g \in G$ can be uniquely written as

$$g = x^{e_1} y^{e_2} z^{e_3} c_{12}^{e_{12}} c_{13}^{e_{13}} c_{23}^{e_{23}}.$$

If $g' = x^{e'_1} y^{e'_2} z^{e'_3} c_{12}^{e'_{12}} c_{13}^{e'_{13}} c_{23}^{e'_{23}}$ is another element of G, standard commutator calculus gives

$$gg' = x^{e_1+e'_1} y^{e_2+e'_2} z^{e_3+e'_3} c_{12}^{e_{12}+e'_{12}-e'_1 e_2} c_{13}^{e_{13}+e'_{13}-e'_1 e_3} c_{23}^{e_{23}+e'_{23}-e'_2 e_3}. \tag{2.22}$$

Consider the following nine elements of $IA(G)$:

$$\varphi_1(x) = xc_{12},\ \varphi_1(y) = y,\ \varphi_1(z) = z;\quad \varphi_2(x) = x,\ \varphi_2(y) = yc_{12},\ \varphi_2(z) = z;$$

$$\varphi_3(x) = x,\ \varphi_3(y) = y,\ \varphi_3(z) = zc_{12};\quad \varphi_4(x) = xc_{13},\ \varphi_4(y) = y,\ \varphi_4(z) = z;$$

$$\varphi_5(x) = x,\ \varphi_5(y) = yc_{13},\ \varphi_5(z) = z;\quad \varphi_6(x) = x,\ \varphi_6(y) = y,\ \varphi_6(z) = zc_{13};$$

$$\varphi_7(x) = xc_{23},\ \varphi_7(y) = y,\ \varphi_7(z) = z;\quad \varphi_8(x) = x,\ \varphi_8(y) = yc_{23},\ \varphi_8(z) = z;$$

$$\varphi_9(x) = x,\ \varphi_9(y) = y,\ \varphi_9(z) = zc_{23}.$$

For any $\varphi \in IA(G)$ we can write

$$\varphi(x) = xc_{12}^{a_x} c_{13}^{b_x} c_{23}^{c_x},$$

$$\varphi(y) = yc_{12}^{a_y} c_{13}^{b_y} c_{23}^{c_y},$$

$$\varphi(z) = zc_{12}^{a_z} c_{13}^{b_z} c_{23}^{c_z}.$$

It is straightforward to show that φ is uniquely expressed as

$$\varphi = \varphi_1^{a_x}\varphi_2^{a_y}\varphi_3^{a_z}\varphi_4^{b_x}\varphi_5^{b_y}\varphi_6^{b_z}\varphi_7^{c_x}\varphi_8^{c_y}\varphi_9^{c_z}.$$

Since $IA(G)$ is a torsion-free abelian group, this proves that

$$IA(G) \text{ is free abelian of rank } 9. \tag{2.23}$$

Now choose $\varphi \in InnG$. By definition of $InnG$, there exists $g = x^{e_1}y^{e_2}z^{e_3}c_{12}^{e_{12}}c_{13}^{e_{13}}c_{23}^{e_{23}} \in G$ such that

$$\varphi(x) = g^{-1}xg,\ \varphi(y) = g^{-1}yg,\ \varphi(z) = g^{-1}zg.$$

By the normal form 2.22 we readily find that

$$g^{-1} = x^{-e_1}y^{-e_2}z^{-e_3}c_{12}^{-e_{12}-e_1e_2}c_{13}^{-e_{13}-e_1e_3}c_{23}^{-e_{23}-e_2e_3}, \tag{2.24}$$

and further use of 2.22 ultimately gives

$$\varphi(x) = xc_{12}^{e_2}c_{13}^{e_3},\ \varphi(y) = yc_{12}^{-e_1}c_{23}^{e_3},\ \varphi(z) = zc_{13}^{-e_1}c_{23}^{-e_2}.$$

We obtain three specific elements of $InnG$ by setting, in turn, $e_1 = 1$, $e_2 = e_3 = 0$; $e_2 = 1$, $e_1 = e_3 = 0$; and $e_3 = 1$, $e_1 = e_2 = 0$. These are:

$$\varphi_1(x) = x,\ \varphi_1(y) = yc_{12}^{-1},\ \varphi_1(z) = zc_{13}^{-1};$$

$$\varphi_2(x) = xc_{12},\ \varphi_2(y) = y,\ \varphi_2(z) = zc_{23}^{-1};$$

$$\varphi_3(x) = xc_{13},\ \varphi_3(y) = yc_{23},\ \varphi_3(z) = z.$$

It is straightforward that

$$\varphi = \varphi_1^{e_1}\varphi_2^{e_2}\varphi_3^{e_3};$$

and that this expression is unique. This proves that

$$InnG \text{ is free abelian of rank } 3$$

and hence

$$InnG \neq IA(G)$$

for G free nilpotent of class 2 and rank 3.

2.4 Connections with homotopy theory

We may think of $IA(G)$ as the group of automorphisms of G inducing the identity on H_1G, the first homology group of G. Denote by Aut_*G the group of automorphisms of G that induce the identity on *all* homology groups. It is not hard to show that $InnG \leq Aut_*G \leq IA(G)$. It would be worthwhile to attempt the proof that if G and H are suitable nilpotent groups in the same localization genus, Aut_*G and Aut_*H also belong to the same localization genus.

In fact, the main motivation for theorem 2.3.3 in group theory comes from a related result by Maruyama [11] in homotopy theory involving some version of Aut_*G: let X be a simply connected CW-complex and denote by $\varepsilon_0(X)$ the group of homotopy classes of self-homotopy equivalences of X that induce the identity on all homology groups. E. Dror and A. Zabrodsky proved that $\varepsilon_0(X)$ is nilpotent [5], so its p-localization makes sense. Maruyama's result is that the homomorphism

$$\varepsilon_0(X) \to \varepsilon_0(X_{(p)})$$

obtained by localizing each homotopy class is in fact the localization homomorphism of nilpotent groups

$$\varepsilon_0(X) \to \varepsilon_0(X)_{(p)}.$$

It is important to observe that our result does not follow from Maruyama's. If we attempted to derive our theorem from his, we would have to consider a CW-complex X whose homotopy type is that of a $K(G,1)$, with G finitely generated, torsion-free, nilpotent, and metabelian. Already, X is not simply connected so his result does not apply. In addition, his theorem is about the group of homotopy classes of self-homotopy equivalences of X inducing the identity on homology, a group that corresponds to Aut_*G, which may be smaller than $IA(G)$ in general.

Chapter 3

Examples

3.1 Background

There is a very useful technique developed by Pickel [14] involving certain multilinear forms to study the so-called "one-relator nilpotent groups." These are finitely generated nilpotent groups of class c, each arising as a quotient of a free nilpotent group modulo an infinite cyclic group generated by an element of γ_c which is not a proper power. He uses this technique to study such groups, along with their p-completions. We will use it to discuss the localization genus of two nilpotent groups of class 4. Pickel's results in p-completions carry over to p-localizations mainly because the p-localization of a finitely generated, torsion-free nilpotent group G, generated by X, is a $\mathbb{Z}_{(p)}$-*group*; meaning that $G_{(p)}$ is generated by X over the p-local integers $\mathbb{Z}_{(p)}$.

To explain the details, let F be a free nilpotent group of class c on n generators. Let a be an element of $\gamma_c F$, where a is not a proper power.

Choose a basis $\{\omega_\alpha\}$ for the free abelian group $\gamma_c F$ (for example, the ω_α can be chosen to be the basic commutators of weight c), and write

$$a! = \prod_\alpha \omega_\alpha^{j_\alpha}, \quad (j_\alpha \in \mathbb{Z}).$$

Define a map $\varphi_a : \gamma_c F \to \mathbb{Z}$ on the basis of $\gamma_c F$ as

$$\varphi_a(\omega_\alpha) = j_\alpha.$$

φ_a clearly extends to a homomorphism.

Now consider the function β from the cartesian product of abelian groups $F/F' \times \cdots \times F/F'$ (c copies) to $\gamma_c F$ given by

$$\beta\left(\overline{f_1}, \ldots, \overline{f_c}\right) = [f_1, \ldots, f_c]\,.$$

β is a well-defined, multilinear function (see lemma 7.3 in [14]). Associate to a the multilinear function

$$f_a = \varphi_a \circ \beta : F/F' \times \cdots \times F/F' \text{ (c times)} \to \mathbb{Z}. \tag{3.1}$$

We show that the function f_a corresponds to a unique a in $\gamma_c F$. To see this, observe that the image of $\beta : F/F' \times \cdots \times F/F' \to \gamma_c F$ generates $\gamma_c F$. Next, assume that $f_a = f_{a'}$ where a and a' lie in $\gamma_c F$. That is,

$$\varphi_a \circ \beta = \varphi_{a'} \circ \beta.$$

We will show that φ_a and $\varphi_{a'}$ agree on the left-normed commutators of length c (and therefore agree everywhere on $\gamma_c F$). Indeed, let $x = [x_1, \ldots, x_c]$ be a typical left-normed commutator of length c. Choose $y \in F/F' \times \cdots \times F/F'$ such that $\beta(y) = x$. Thus

$$\varphi_a(x) = \varphi_a \circ \beta(y) = \varphi_{a'} \circ \beta(y) = \varphi_{a'}(x).$$

This proves that $\varphi_a = \varphi_{a'}$. By the definition of the φ's, we conclude that $a = a'$.

Following Pickel, we call f_a the *c-form* associated with $a \in \gamma_c F$.

For elements a and b in $\gamma_c F$ which are not proper powers, we say that the associated c-forms f_a and f_b are *equivalent* if there exists and automorphism φ of F/F' such that

$$f_a = f_b \circ (\varphi \times \cdots \times \varphi) . \tag{3.2}$$

Similarly, each $a \in \gamma_c F_{(p)}$ gives rise to a c-form

$$f_a : F_{(p)}/F^{'}_{(p)} \times \cdots \times F_{(p)}/F^{'}_{(p)} \ (c \text{ times}) \to \mathbb{Z}_{(p)} \ .$$

Given a and b in $\gamma_c F_{(p)}$ we say that f_a is *p-equivalent* to f_b if there is an automorphism φ of $F_{(p)}/F^{'}_{(p)}$ such that

$$f_a = f_b \circ (\varphi \times \cdots \times \varphi) .$$

Now, any automorphism of F/F' can be represented by an invertible matrix φ_{ij} with integral coefficients, relative to the generating set $x_1, \ldots, x_n$ of F. The map given by

$$x_i \mapsto \prod_j x_j^{\varphi_{ij}} \tag{3.3}$$

clearly extends to an endomorphism of F and induces the given automorphism of F/F'. Lemma 7.2 in [14] gives that the map on the x_i is in fact an automorphism of F. The proof of lemma 7.2 depends on the fact that an endomorphism is completely determined by its action on the generators of the group. Since endomorphisms of $F_{(p)}$ and $(F/F')_{(p)}$ are also determined by their actions on the p-generators, we have a similar statement for the localized groups. In the localized case the invertible matrix φ_{ij} has entries in $\mathbb{Z}_{(p)}$ and the map 3.3 is an automorphism of $F_{(p)}$.

Consider again a and b in $\gamma_c F$. Proposition 7.1 in [14], which is also valid for localized groups, gives that $F/gp(a)$ is isomorphic to $F/gp(b)$ (resp. $(F/gp(a))_{(p)}$ is isomorphic to $(F/gp(b))_{(p)}$) if and only if there is automorphism of F (resp. an automorphism of $F_{(p)}$) sending a to b^μ where μ is 1 or -1 (resp. μ is a p-local unit).

Suppose that f_a is equivalent (resp. p-equivalent) to f_b. By definition there is an automorphism φ of F/F' (resp. $(F/F')_{(p)}$) such that equation 3.2 holds. Let $\hat{\varphi}$ be the "lift" of φ given by 3.3, which is an automorphism of F. For each $(\overline{g_1}, \ldots, \overline{g_c})$ in the c-fold cartesian product of copies of F/F', we have

$$f_a(\overline{g_1}, \ldots, \overline{g_c}) = f_b\left(\overline{\hat{\varphi}(g_1)}, \ldots, \overline{\hat{\varphi}(g_c)}\right).$$

By definition (see 3.1) we conclude that

$$\varphi_a\left[g_1, \ldots, g_c\right] = \varphi_b \hat{\varphi}\left[g_1, \ldots, g_c\right].$$

This means that $\hat{\varphi}$ must send a to b in γ_c. Notice that the converse is also true: if φ is an automorphism of F (resp. $F_{(p)}$) such that $\hat{\varphi}$ sends a to b, then f_a and f_b are equivalent (resp. p-equivalent). Finally, it is clear that the form associated to the element a^μ where $\mu = 1, -1$ (resp. μ is a p-local unit) is μf_a. All this gives the following lemma (analogous to proposition 8.1 in [14]).

Lemma 3.1.1. *$F/gp(a)$ is isomorphic (resp. p-isomorphic) to $F/gp(b)$ if and only if f_a is equivalent (resp. p-equivalent) to μf_b where μ is 1 or -1 (resp. a p-local unit).*

We will be working with one-relator nilpotent groups of class 4 that are also metabelian. The advantage of the 4-forms associated to these groups is that they can be reduced to symmetric bilinear forms. The following result (see lemma 9.1 in [14]) is very useful in this direction. We give a detailed proof here.

Lemma 3.1.2. *Let F be a free nilpotent group of class c, which is also metabelian. Then any left-normed commutator of length c satisfies*

$$[a_1, a_2, a_3, \ldots, a_c] = \left[a_1, a_2, a_{\sigma(3)}, \ldots, a_{\sigma(c)}\right]$$

where σ is a permutation of $\{3, 4, \ldots, c\}$. Put another way, left-normed commutators of length c are symmetric in the last $c-2$ entries.

Proof. We begin by defining an action of the group ring $\mathbb{Z}(F/F')$ on F'. For $\overline{g} \in F/F'$ and $x \in F'$ define

$$\overline{g}x = x^g \in F'.$$

Similarly, integers act on F' by exponentiation. It suffices to check that this alleged action restricted to the basis F/F' of the group ring is well defined. To show that this definition does not depend on the chosen representative g suppose that

$$\overline{g} = \overline{g'}$$

in F/F'. Then

$$x^g = x^{g'} \Leftrightarrow g^{-1}xg = {g'}^{-1}xg' \Leftrightarrow g'g^{-1}xg{g'}^{-1} = x \Leftrightarrow \left(g{g'}^{-1}\right)^{-1} xg{g'}^{-1} = x.$$

But this clearly holds since both $g{g'}^{-1}$ and x lie in the abelian group F'. Notice that:

- $[a_1, a_2]^{(-1+a_3)} = [a_1, a_2]^{-1} [a_1, a_2]^{a_3} = [a_1, a_2]^{-1} [a_1, a_2] [a_1, a_2, a_3] = [a_1, a_2, a_3]$.

- $[a_1, a_2, a_3]^{(-1+a_4)} = [a_1, a_2, a_3, a_4]$.

- $[a_1, a_2, a_3, \ldots, a_k]^{(-1+a_{k+1})} = [a_1, a_2, a_3, \ldots, a_{k+1}]$.

Hence, the left-normed commutator

$$[a_1, a_2, \ldots, a_c] = [[\ldots [a_1, a_2], a_3], \ldots, a_c]$$

can be rewritten as

$$\left(\cdots\left(\left([a_1, a_2]^{(-1+a_3)}\right)^{(-1+a_4)}\right)^{(-1+a_5)}\cdots\right)^{(-1+a_c)} = [a_1, a_2]^{(-1+a_3)\cdots(-1+a_c)}.$$

Finally, let σ be a permutation of $\{3, \ldots, c\}$. Since the group ring $\mathbb{Z}(F/F')$ is commutative we conclude that

$$(-1 + a_3) \cdots (-1 + a_c) = \left(-1 + a_{\sigma(3)}\right) \cdots \left(-1 + a_{\sigma(c)}\right),$$

so the lemma follows. □

Let $F = gp(x, y)$ be the two-generator free nilpotent group of class 4 and consider the basic sequence of basic commutators on $\{x, y\}$,

$$\{x, y, c_{21}, c_{212}, c_{211}, c_{2122}, c_{2112}, c_{2111}\}$$

where $c_{21} = [y, x]$, $c_{212} = [y, x, y]$, $c_{211} = [y, x, x]$, $c_{2122} = [y, x, y, y]$, $c_{2112} = [y, x, x, y]$, and $c_{2111} = [y, x, x, x]$. (We will use this notation repeatedly).

We now describe how to associate a symmetric bilinear form to a quotient of F by a cyclic subgroup of $\gamma_4 F$.

The following is a simple (but crucial) observation:

Lemma 3.1.3. *F is metabelian.*

Proof. This follows from the fact that F has a very low rank. Since c_{21} is the only basic commutator of weight 2, all basic commutators commute. Therefore, F' is abelian and F is metabelian. $\square$

A basis for $\gamma_4 F$ is given by

$$\{c_{2122}, c_{2112}, c_{2111}\}.$$

Choose

$$a! = c_{2122}^{a_1} c_{2112}^{a_2} c_{2111}^{a_3}$$

in $\gamma_4 F$. Then the function

$$f_a : F/F' \times F/F' \times F/F' \times F/F' \to \mathbb{Z}$$

is a 4-form. By lemma 3.1.2, this 4-form induces the symmetric bilinear form

$$\hat{f}_a : F/F' \times F/F' \to \mathbb{Z}$$

$$\hat{f}_a(\overline{g}_1, \overline{g}_2) = f_a(\overline{y}, \overline{x}, \overline{g}_1, \overline{g}_2).$$

Let

$$F_a = F/gp(a).$$

$\hat{f}_a$ is the symmetric bilinear form associated to F_a.

Consider two elements a and b of $\gamma_4 F$ and their corresponding symmetric bilinear forms $\hat{f}_a$ and $\hat{f}_b$. Let A_a and A_b be the symmetric integral matrices associated with $\hat{f}_a$ and $\hat{f}_b$ respectively, relative to the basis of F/F' induced by the generators of F. The notion of equivalence (p-equivalence) of symmetric bilinear forms now translates into equivalence (p-equivalence) of their corresponding matrices. We may therefore say that $\hat{f}_a$ is *equivalent* (*p-equivalent*) to $\hat{f}_b$ if there is a $\mathbb{Z}$-invertible ($\mathbb{Z}_{(p)}$-invertible) matrix M such that

$$A_a = M A_b M^t.$$

The matrices A_a and A_b themselves may be regarded as equivalent (p-equivalent).

The following result is analogous to lemma 3.1.1, replacing forms with matrices.

Lemma 3.1.4. *F_a is isomorphic (p-isomorphic) to F_b if and only if μA_a is equivalent (p-equivalent) to A_b where μ is 1 or -1 (a unit in $\mathbb{Z}_{(p)}$).*

According to the definition, to show that two integral symmetric matrices are equivalent over $\mathbb{Z}_{(p)}$ one needs to find an invertible matrix M with entries in $\mathbb{Z}_{(p)}$. However, the following result of Bokor reveals that our search can be restricted to integral matrices. We include the proof given in [3].

Following Bokor, call two integral symmetric matrices *weakly equivalent* (*weakly p-equivalent*) if they are equivalent up to a unit in $\mathbb{Z}$ (in $\mathbb{Z}_{(p)}$).

Lemma 3.1.5. (lemma 1 in [3]) *Two integral symmetric matrices G and H are weakly p-equivalent if and only if there exists an integral matrix A (not necessarily invertible over $\mathbb{Z}$ but with non-zero determinant) and an integer m such that*

- *m and p are relatively prime,*
- $\det A$ *and p are relatively prime,*
- *$mG = AHA^t$.*

Proof. Assume that G and H are weakly p-equivalent. Then there exists a $\mathbb{Z}_{(p)}$-invertible matrix A' and a unit m' of $\mathbb{Z}_{(p)}$ such that

$$m'G = A'H\left(A'\right)^t.$$

We now construct m and A with the desired properties.

Let k be the least common multiple of the denominators of the entries of A' and the denominator of m'. Observe that the prime decomposition of k does not include p. Therefore, k and p are relatively prime and k is a unit in $\mathbb{Z}_{(p)}$. Also k^2 and p are relatively prime. Let

$$m = k^2m'$$

and

$$A = kA'.$$

m is an integer which is relatively prime to p. A is an integral matrix, invertible over $\mathbb{Z}_{(p)}$. Therefore the integer $\det A$ is a *unit* in $\mathbb{Z}_{(p)}$, so that $\det A$ is relatively prime to p. Finally,

$$mG = k^2m'G = kA'Hk(A')^t = AHA^t.$$

Conversely, assume now that

$$mG = AHA^t$$

where $\det A$ and m are each relatively prime to p. These two integers are units in $\mathbb{Z}_{(p)}$ so that G and H are weakly p-equivalent. □

3.2 Remeslennikov's groups

We now describe two non-isomorphic, class 4 nilpotent and metabelian groups which lie in the same localization genus.

In the category of class 4 nilpotent groups let

$$F = \langle x, y \rangle ,$$

$$F_S = \left\langle x, y; c_{2122}^3 c_{2112} c_{2111}^2 \right\rangle ,$$

and

$$F_T = \left\langle x, y; c_{2122}^6 c_{2112} c_{2111} \right\rangle .$$

Remark. Neither F_S nor F_T is isomorphic to F. On the one hand, $\gamma_4 F$ is free abelian of rank 3, freely generated by $\{c_{2122}, c_{2112}, c_{2111}\}$. On the other, both $\gamma_4 F_S$ and $\gamma_4 F_T$ are free abelian of rank 2: it is easy to show that $\gamma_4 F_S$ is freely generated by $\{c_{2122}^3 c_{2112} c_{2111}, c_{2122}\}$ and F_T is freely generated by $\{c_{2122}^{-6} c_{2112}, c_{2122}\}$.

Consider the symmetric bilinear forms

$$\hat{f}_S, \hat{f}_T : F/F' \times F/F' \to \mathbb{Z}$$

associated to F_S and F_T respectively.

Let $M_S = (m_{ij})$ and $M_T = (m'_{ij})$ be the symmetric matrices of these forms with respect to the ordered basis $\{\overline{x}, \overline{y}\}$ of the $\mathbb{Z}$-module F/F'. M_S is found as follows:

- $m_{11} = \hat{f}_S(\overline{x}, \overline{x}) = f_S(\overline{y}, \overline{x}, \overline{x}, \overline{x}) = 2$
- $m_{12} = \hat{f}_S(\overline{x}, \overline{y}) = f_S(\overline{y}, \overline{x}, \overline{x}, \overline{y}) = 1$

- $m_{21} = \hat{f}_S(\overline{y}, \overline{x}) = f_S(\overline{y}, \overline{x}, \overline{y}, \overline{x}) = f_S(\overline{y}, \overline{x}, \overline{x}, \overline{y}) = 1$ (see lemma 3.1.2)

- $m_{22} = \hat{f}_S(\overline{y}, \overline{y}) = f_S(\overline{y}, \overline{x}, \overline{y}, \overline{y}) = 3.$

Similarly, we find M_T:

- $m'_{11} = \hat{f}_T(\overline{x}, \overline{x}) = f_T(\overline{y}, \overline{x}, \overline{x}, \overline{x}) = 1$

- $m'_{12} = \hat{f}_T(\overline{x}, \overline{y}) = f_T(\overline{y}, \overline{x}, \overline{x}, \overline{y}) = 1$

- $m'_{21} = \hat{f}_T(\overline{y}, \overline{x}) = f_T(\overline{y}, \overline{x}, \overline{y}, \overline{x}) = f_T(\overline{y}, \overline{x}, \overline{x}, \overline{y}) = 1$

- $m'_{22} = \hat{f}_T(\overline{y}, \overline{y}) = f_T(\overline{y}, \overline{x}, \overline{y}, \overline{y}) = 6.$

In an attempt to diagonalize each matrix over the integers, we perform row/column elementary operations:

1. $M_S|I = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ 1 & -6 & 0 & 1 \end{array}\right) \rightarrow$
   $\left(\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ -2 & 12 & 0 & -2 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ -2 & 10 & 0 & -2 \end{array}\right) \rightarrow$
   $\left(\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 10 & 1 & -2 \end{array}\right) = D|Q^t$ where $Q^t M_S Q = D$.

   Since $\det Q = -2$, a unit in $Z_{(p)}$ for $p \neq 2$, this diagonalization process is valid for $p \neq 2$. It is not valid over $Z_{(2)}$ or over the integers.

2. $M_T|I = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 6 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right) \rightarrow$
   $\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 5 & -1 & 1 \end{array}\right) = D|Q^t$ where $Q^t M_T Q = D$. In this case $\det Q = 1$, so this diagonalization process is valid over $\mathbb{Z}$, as well as over $\mathbb{Z}_{(p)}$ for every p.

Lemma 3.2.1. *The matrices* $\begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix}$ *and* $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ *are not weakly equivalent over* $\mathbb{Z}$*.*

Proof. By the above diagonalization process it suffices to show that $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ are not weakly equivalent over the integers. Suppose first that there is a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det M = 1$ and such that

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = M^t \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} M.$$

This gives

$$a^2 + 5b^2 = 3,$$

an equation with no integer solutions.

To show that $-\begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ are not equivalent either, we diagonalize the first matrix to get $\begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$ and assume that

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = M^t \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix} M.$$

This yields

$$-b^2 - 5d^2 = 3.$$

Again, an impossible situation. This completes the proof. □

Lemma 3.2.2. *The matrices* $\begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix}$ *and* $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ *are weakly* p*-equivalent for all* p*.*

Proof. Assume first that $p \neq 2$. As we did in the computations above, we find that $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$, and these are valid (strong) p-equivalences for all $p \neq 2$. Since

$$\begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

and 2 is a unit in $\mathbb{Z}_{(p)}$, the lemma follows in this case.

It suffices to prove now that $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ are weakly 2-equivalent. By lemma 3.1.5 the search for an invertible "transition" matrix Q and a unit μ of $\mathbb{Z}_{(2)}$ can be restricted to the integers. Let $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integral matrix with odd determinant and μ and odd integer. Consider the integral matrix equation

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

This gives:

$$a^2 + 5c^2 = 2\mu$$

$$ab + 5cd = \mu$$

$$b^2 + 5d^2 = 3\mu.$$

The first of these equations has no solutions for $\mu = 1$ so we try the next odd number $\mu = 3$ (notice that μ can never be negative). For this value of μ we get

$$a^2 + 5c^2 = 6$$
$$ab + 5cd = 3$$
$$b^2 + 5d^2 = 9.$$

The first equation is satisfied by $c = 1 = a$ and the third by $d = 0$, $b = 3$. These values also satisfy the second equation. Put

$$M = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}.$$

Since $\det M = -3$, M is invertible over $\mathbb{Z}_{(2)}$ and

$$M^t \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} M = 3 \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

This completes the proof. □

Corollary 3.2.3. *Remeslennikov's groups F_S and F_T are not isomorphic but they lie in the same localization genus.*

In fact, if two integral symmetric matrices are non-singular (by this we mean that they have non-zero determinant, although they need not be invertible over $\mathbb{Z}$), they are equivalent over the p-local integers if and only if they are equivalent over the p-adic integers (See [3] and reference [2] there). Hence, we have also shown that the groups F_S and F_T lie in the same completion genus.

3.3 IA-automorphisms of Remeslennikov's groups

Consider again the class 4 nilpotent groups

$$F = \langle x, y \rangle\,,$$

$$F_S = \left\langle x, y; c_{2122}^3 c_{2112} c_{2111}^2 \right\rangle,$$

and

$$F_T = \left\langle x, y; c_{2122}^6 c_{2112} c_{2111} \right\rangle.$$

We record some facts about their groups of IA-automorphisms. We need the following lemma:

Lemma 3.3.1. *F_S and F_T are torsion-free.*

Proof. We first show that that $F_S^{'}$ and $F_S/F_S^{'}$ are torsion-free. Let

$$x = c_{21}^a c_{212}^b c_{211}^c c_{2122}^d c_{2112}^e c_{2111}^f \in F_S^{'}$$

and suppose that there exists a positive integer n with

$$x^n = c_{21}^{na} c_{212}^{nb} c_{211}^{nc} c_{2122}^{nd} c_{2112}^{ne} c_{2111}^{nf} = 1 \in F_S^{'}.$$

Then there exists an integer ω such that

$$c_{21}^{na} c_{212}^{nb} c_{211}^{nc} c_{2122}^{nd} c_{2112}^{ne} c_{2111}^{nf} = c_{2122}^{3\omega} c_{2112}^{\omega} c_{2111}^{2\omega} \in F.$$

From this we see that $a = b = c = 0$, $d = 3e$, and $f = 2e$. Hence

$$x = c_{2122}^{3e} c_{2112}^{e} c_{2111}^{2e} = 1 \in F_S.$$

This proves that $F_S^{'}$ is torsion-free.

Next observe that F_S/F_S' is free abelian of rank 2, so it is torsion-free. Finally let $x \in F_S$ such that $x^n = 1 \in F_S$ for some positive n. Then $\overline{x}^n = 1 \in F_S/F_S'$. As F_S/F_S' is torsion-free, $\overline{x} = 1 \in F_S/F_S'$ so that $x \in F_S'$. Since F_S' is torsion-free, this implies that $x = 1$, completing the argument that F_S is torsion-free.

For F_T it suffices to show that F_T' is torsion-free; the rest of the argument is the same. To see this let

$$x = c_{21}^a c_{212}^b c_{211}^c c_{2122}^d c_{2112}^e c_{2111}^f \in F_T',$$

and assume that $x^n = 1$ with $n \neq 0$. Then, there exists an integer ω such that

$$c_{21}^a c_{212}^b c_{211}^c c_{2122}^d c_{2112}^e c_{2111}^f = c_{2122}^{6\omega} c_{2112}^{\omega} c_{2111}^{\omega} \in F.$$

By equating exponents we get $a = b = c = 0$, $d = 6e$, and $f = e$. Hence

$$x = c_{2122}^{6e} c_{2112}^{e} c_{2111}^{e} = 1 \in F_T'.$$

This completes the proof. □

Let G denote F_S or F_T. By corollary 1.2.11, $IA(G)$ is torsion-free and nilpotent of class 3. Since nilpotent groups of class 3 are always metabelian, so is $IA(G)$. As a corollary of theorem 2.3.3 we have:

Lemma 3.3.2. *$IA(F_S)$ and $IA(F_T)$ are finitely generated, torsion-free, metabelian, and nilpotent of class 3, which lie in the same localization genus.*

Using computational techniques based on our δ-sequences from chapter 2, it is possible (but tedious) to find normal forms for $IA(F_S)$ and $IA(F_T)$ involving, in each case, generators of the form

$$\varphi_i(x) = xc_i, \ \varphi_i(y) = y$$

and

$$\varphi_j(x) = x, \ \varphi_j(y) = yc_j;$$

where c_i and c_j range over the free generators of the free abelian groups $F_S^{'}$ and $F_T^{'}$, respectively. These normal forms lead to presentations for $IA(F_S)$ and $IA(F_T)$. Even though it is still unclear from this whether $IA(F_S)$ and $IA(F_T)$ are isomorphic or not, our results certainly imply that $IA(F_S)$ and $IA(F_T)$ lie in the same localization genus.

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