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Path Integral Formulation of Field Theories with Second-Class Constraints

by

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Abstract

Path Integral Formulation of Field Theories with Second-Class Constraints

by

Pavao Senjanovic

Adviser: Professor Bunji Sakita

Faddeev's Hamiltonian path integral method for singular Lagrangians is generalized to the case when second-class constraints appear in the theory. The general formalism is then applied to a variety of problems: quantization of first-order field theories, quantization of the massive Yang-Mills field theory, light-cone quantization of the self-interacting scalar field theory and quantization of a local field theory of magnetic monopoles.

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References

Introduction

The first systematic study of field theories with constraints was done by Dirac [1,2,3]. He showed that the algebra of Poisson brackets determines a division of constraints into two classes: the so-called first class constraints and second class ones. The first class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are not first class are by definition second class. Dirac also showed how to redefine the Poisson brackets in such a way that all new redefined brackets (so-called Dirac brackets) of second class constraints are zero. As we shall see, this is indeed necessary if transition to quantum theory is to be made.

The quantization of field theories with constraints based on Dirac's formalism and using the method of functional integration was performed by Faddeev[4]. Faddeev restricted his discussion to the case when only first-class constraints are present. We shall see that there are several interesting examples of field theories which contain second-class constraints, so that a generalization of Faddeev's method to these cases is warranted. The development of this generalization and its applications are the main concern of this thesis.

The thesis will be organized as follows: In chapter I, we shall present the general theory. This chapter will consist of three sections. In Section I, we shall review Dirac's method. In Section II , we shall give a short description of Faddeev's method. In Section III , a generalization of Faddeev's method to the case when second class constraints are present will be formulated. Chapter II will contain various applications of the general formalism developed in Chapter I.

In Section II-1, we shall undertake the quantization of two field theories in the first order formulation: the self-interacting scalar theory and the free massless vector field theory. The next two sections (II.2 and II.3) will be devoted to the quantization of the self-interacting Yang-Mills field theory in the first-order formulation and the quantization of the massive Yang-Mills field theory in the standard formulation.

In Section II.4, the light-cone quantization of the self-interacting scalar field theory will be performed using our method. In Section II.5, we shall quantize a local field theory of magnetic monopoles. In both Section II.4 and Section II.5, second-class constraints will appear naturally.

Section II.6 will be devoted to a conclusion.

Chapter I. General Theory

I. Dirac's theory of systems with constraints

a. Singular Lagrangians

Given a mechanical system (of N degrees of freedom) with a Lagrangian L,

$$L = L(q, \dot{q}) \tag{I.1}$$

one defines the conjugate momenta by

$$p_n = \frac{\partial L}{\partial \dot{q}_n} \qquad (n = 1, \ldots\ldots N) \tag{I.2}$$

We shall dwell on the case when the expressions $\frac{\partial L}{\partial \dot{q}_n}$ are not independent functions of $\dot{q}_n$. Eliminating the $\dot{q}$'s one obtains a certain number of independent constraints

$$\phi_m(q, p) = 0 \qquad (m = 1, 2, \ldots k) \quad . \tag{I.3}$$

Thus, some of the p's are not independent. Solving (I.3) one writes

$$p_\alpha = \psi_\alpha(q, p_i) \qquad (\alpha = 1, 2 \ldots k) \quad , \tag{I.4}$$

where the p_i's ($i=1, \ldots r$, $r+k=N$) are independent.

Because we are dealing with a singular Lagrangian, it is impossible to solve (I.2) for <u>all</u> the $\dot{q}$'s. However, if we take some of the $\dot{q}$'s to be independent (and undetermined) [we shall take those to be $\dot{q}_\alpha$ ($\alpha = 1, \ldots k$)], we can use Eq. (I.2) to solve for the remaining $\dot{q}$'s:

$$\dot{q}_i = \zeta_i(q, p_i, \dot{q}_\alpha) \quad (i = 1, 2, \ldots r \quad ; r = N - k) . \tag{I.5}$$

Eqs. (I.4) and (I.5) together have the same content as Eq. (I.2).

b. Equations of motion

Define the Hamiltonian by

$$\bar{H} = p_n \dot{q}_n - L \equiv \tilde{W}(q, p_i, \dot{q}_\alpha) \quad . \tag{I.6}$$

In view of Eqs. (I.4) and (I.5) the Hamiltonian defined by (I.6) can be considered in general as a function of q, p_i, $\dot{q}_\alpha$. However, because of the nature of the Legendre transformation of (I.6) and in view of (I.2), it does not depend on $\dot{q}_\alpha$, which can be checked directly

$$\frac{\partial \tilde{W}}{\partial \dot{q}_\alpha} = p_\alpha + p_i \frac{\partial \mathfrak{J}_i}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \mathfrak{J}_i}{\partial \dot{q}_\alpha} = 0 \quad .$$

Hamilton's principle

$$\delta \int L \, dt = 0 \tag{I.7}$$

can be written as

$$\delta \int (p_n \dot{q}_n - \tilde{W}(q, p_i)) \, dt = 0 \tag{I.8}$$

with the constraints

$$\phi_m (q, p) = 0 \quad . \tag{I.3}$$

Using the Lagrange multiplier method, one is led to the equations of motion

$$\dot{p}_n = - \frac{\partial \tilde{W}}{\partial q_n} - v_m \frac{\partial \phi_m}{\partial q_n}$$

$$\dot{q}_n = \frac{\partial \tilde{W}}{\partial p_n} + v_m \frac{\partial \phi_m}{\partial q_n} \quad , \tag{I.9}$$

where v_m are the multipliers, which are arbitrary at this stage. These equations, together with the constraints (I.3) form a complete set of equations of motion.

Before proceeding with the development of the formalism, we shall introduce several important definitions. First, let us define the phase space Γ as a set whose elements are ordered 2N-tuples $(q_1, \ldots q_N, p_1 \ldots p_N)$. Then, introduce the submanifold $\overline{M}$ in Γ, which by definition is the subset of Γ for which Eqs. (I.3) hold. Note that Eqs. (I.9) hold only in $\overline{M}$, which is clear from the way they were derived.

The Poisson bracket of two functions f and g of the q's and p's is defined by

$$\{f, g\} = \sum_{n=1}^{N} \left(\frac{\partial f}{\partial q_n} \frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n} \frac{\partial g}{\partial q_n} \right) .$$

It is convenient to introduce the "total" Hamiltonian H in the following way:

$$H = \widetilde{W} + v_m \phi_m \qquad \text{(I.10)}$$

It is then easy to see that Eqs. (I.9) can be transcribed into the form

$$\dot{p}_n\Big|_{\overline{M}} = \{p_n, H\}\Big|_{\overline{M}}$$

$$\dot{q}_n\Big|_{\overline{M}} = \{q_n, H\}\Big|_{\overline{M}} , \qquad \text{(I.11)}$$

where our notation is intended to emphasize that those equations hold in $\overline{M}$. For a function g of p_n and q_n we find the equation of motion

$$\dot{g}\Big|_{\overline{M}} = \{g, H\}\Big|_{\overline{M}} . \qquad \text{(I.12)}$$

Hence, H as given by (I.10) is the generator of time translation.

c. First-class and second-class constraints

The constraints ϕ_m must remain zero at all times, which implies:

$$\dot{\phi}_m\Big|_{\bar{M}} = \left[\{ \phi_m , H|_{\bar{M}} \} + v_n \{ \phi_m , \phi_n \} \right]\Big|_{\bar{M}} \quad . \tag{I.13}$$

Excluding the case when those equations are contradictory either among themselves or with Eqn. (I.3) as uninteresting, those equations may be (a) a trivial identity, (b) independent of the v's , (c) may involve some of the v's .

In case they are of type (b), they represent new constraints (called secondary constraints) and may be written in the form

$$\rho_i (q, p) = 0 \quad . \tag{I.14}$$

Obviously, we can continue this process of generating secondary constraints until we arrive at the point when no more independent equations of type (b) are produced. After eliminating as many v's as possible from (c) type equations, we can use the remaining equations to solve for some (or all) of the v's .

Let us denote by M the subspace of phase space in which all constraints hold (i.e. both primary and secondary ones) . We shall assume the irreducibility of all constraints with respect to M , i.e. a function of q's and p's vanishing in M will be expressible as a linear function of the ϕ's and the ρ's with functions of q's and p's as coefficients. We thus have in particular

$$H = H|_M (q, p) + v_\ell (q, p) \psi_\ell (q, p) \quad , \tag{I.15}$$

where we have denoted by a common symbol $\psi_\ell(q,p)$ all the constraints, i.e. $(\psi_\ell) = ((\phi_m), (\rho_i))$.

By definition, a <u>first-class</u> constraint φ^a (secondary or primary) satisfies

$$\{\varphi^a, \psi_\ell\}\Big|_M = 0 \tag{I.16}$$

for all ψ_ℓ, and thus in view of our irreducibility hypothesis

$$\{\varphi^a, \psi_\ell\} = \lambda^{ak}_{\ell}\, \psi_k \quad . \tag{I.17}$$

We call a constraint θ^a <u>second class</u> if it is not first-class. Performing suitable linear transformations on the constraints, i.e. choosing new constraints which are linear functions of the old ones with functions of q's and p's as coefficients, let us bring as many ψ's as possible into the first-class. We then claim that the following theorem holds:

<u>Theorem</u>

$$\left[\det \| \{\theta_a, \theta_b\} \| \right]\Big|_M \neq 0 \quad , \tag{I.18}$$

where we have denoted the remaining constraints (all second-class) by θ_a . To prove (I.18) assume the contrary, i.e.

$$\left[\det \| \{\theta_a, \theta_b\} \| \right]\Big|_M = 0 \quad .$$

Then there exists a set of functions λ_a , not all equal to zero, such that

$$\lambda_a \left[\{\theta_a, \theta_b\}\right]\Big|_M = 0 \qquad \text{for all b} \quad ,$$

and thus

$$\{\lambda_a \theta_a, \theta_b\}\Big|_M = 0 \quad ,$$

so $\lambda_a \theta_a$ is first-class, contrary to the assumption that we have put as many constraints as possible into the first class. This constitutes the proof of the theorem.

Corollary 1

Eq. (I.18) implies that the number of second-class constraints for a mechanical system is even, since $\{ \theta_a, \theta_b \}$ is an antisymmetric matrix.

Corollary 2

All those v's in (I.15) which multiply second-class constraints (let us call them $v_b^{(\theta)}$) are determined in M .

Indeed, we have a set of consistency conditions

$$\dot{\theta}_a \Big|_M = \{ \theta_a, H \}\Big|_M = \{ \theta_a, H|_M \}\Big|_M + v_b^{(\theta)}\Big|_M \{ \theta_a, \theta_b \}\Big|_M = 0 \tag{I.19}$$

and we can solve for $v_b^{(\theta)}\Big|_M$ in view of (I.18).

d. Dirac brackets and quantization

A naive transition to quantum theory would consist in imposing the constraints as conditions on the quantum state vectors and replacing standard Poisson brackets by "- i " times commutators. But then if

$$\psi_1 |a\rangle = 0 \quad , \quad \psi_2 |a\rangle = 0 \quad ,$$

we find

$$[\psi_2, \psi_1] |a\rangle = 0 \quad ,$$

which corresponds to a classical equation

$$\{ \psi_2, \psi_1 \}\Big|_M = 0$$

Thus, for the naive passage to quantum theory to be possible, all constraints must be first-class. In case a mechanical system has second-class constraints, the remedy consists in redefining the Poisson brackets in a suitable manner:

$$\{ \xi, \eta \}^* = \{ \xi, \eta \} - \{ \xi, \theta_a \} c_{ab} \{ \theta_b, \eta \} \quad , \tag{I.20}$$

where

$$C_{ab}\{\theta_b, \theta_c\} = \delta_{ac} \tag{I.21}$$

$\{\xi, \eta\}^*$ is the new bracket, while the brackets on the right-hand side are standard Poisson brackets. It can be shown that the new brackets have all the standard properties of Poisson brackets.

As a consequence of the definitions (I.20) and (I.21), we find

$$\{\xi, \theta_a\}^* = \{\xi, \theta_a\} - \{\xi, \theta_b\} C_{bc} \{\theta_c, \theta_a\} =$$

$$= \{\xi, \theta_a\} - \{\xi, \theta_b\} \delta_{ba} = 0 \quad . \tag{I.22}$$

The passage to quantum theory can now be made by replacing the new brackets by "$-i$" times commutators. Then, in quantum theory, we can take $\theta_a = 0$ to hold as operator equations without any contradiction, since in view of (I.22), $[\theta_a, \xi] = 0$ for any operator ξ. The consistency condition (I.19) implies

$$\{g, H\}^*\Big|_M = \{g, H\}\Big|_M = \dot{g}\Big|_M \quad ,$$

which means that the new bracket may be used to give the Hamiltonian equations of motion.

The generalization of this formalism to field theory, i.e. a mechanical system with a continuously infinite number of degrees of freedom, presents no difficulty.

II. The Feyman path integral for singular Lagrangians with first-class constraints only[4] (Faddeev's method)

a. Introduction

As discussed in Section I, given a certain Lagrangian, it can happen that the equations

$$p_n = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_n} \tag{II.1}$$

cannot be solved for all of the $\dot{q}$'s. As a result (direct or indirect), the q's and the p's are constrained:

$$\varphi^a(q, p) = 0 \qquad a = 1, 2, \cdots m \quad . \tag{II.2}$$

The constraints (II.2) are either primary or secondary; in this section we shall limit ourselves to the case when there are no second-class ones among them. Thus,

$$\{\varphi^a, \varphi^b\} = C^{ab}_c \varphi^c \quad . \tag{II.3}$$

In view of the discussion in Section I, we must have

$$\{H, \varphi^a\} = C^a_b \varphi^b \quad , \tag{II.4}$$

where we have changed the notation somewhat: H now stands for $H|_M$. Note that the hypersurface M in the phase space Γ is of dimension 2N-m.

b. Observables and gauge conditions

Only those functions on M are observable quantities whose equations of motion contain no arbitrariness. The equation of motion of a quantity f is

$$\dot{f}\big|_M = \{f, H\}\big|_M + v_a\big|_M \{f, \varphi^a\}\big|_M \quad . \tag{II.5}$$

This will be unique in M if

$$\{f, \varphi^a\}\big|_M = 0 \tag{II.6}$$

or equivalently

$$\{f, \varphi^a\} = d^a{}_b \varphi^b \quad . \tag{II.7}$$

The function f occuring in (II.5), (II.6) and (II.7) is an arbitrary continuation in Γ of a function defined on M . Since the constraints (II.2) are irreducible by assumption, any two such continuations will differ by a linear combination of the constraints and Eqs. (II.6) and (II.7) are independent of the continuation. Equations (II.6) can be viewed as a set of m first-order differential equations on M , with Eqs. (II.3) serving as integrability conditions. To see that, let us write Eq. (II.6) in terms of a noncanonical system of variables (φ^a, η^b, q^*_i , p^*_i). Using (II.3) we obtain

$$\frac{\partial f(X)}{\partial X_\ell} g^\ell_a(X) = 0 \quad , \tag{II.8}$$

where $X = (\eta, q^*, p^*)$ and

$$g^\ell_a(X) = \{X^\ell, \varphi^a\}\big|_M \quad . \tag{II.9}$$

The term containing $\frac{\partial f}{\partial \varphi_a}$ vanishes on account of (II.3). It is in this sense that (II.3) serve as integrability conditions for (II.6); if (II.3) did not hold, we would obtain a set of rather nonstandard differential equations with the nonmanageable first term:

$$\frac{\partial f}{\partial \varphi^b}\Big|_{\varphi^c=0} \{\varphi^b, \varphi^a\}\Big|_M + \frac{\partial f}{\partial X_\ell} g^\ell_a(X) = 0 \quad . \tag{II.10}$$

Since f satisfies a set of m first-order linear differential equations, it is completely determined by its values in the submanifold of the initial conditions for (II.6) (or (II.8)). This submanifold is of dimension (2N-m)-m = 2(N-m). We can choose this submanifold to be the surface Γ^*($\Gamma^* \subset$ M), defined by the equations

$$\chi_a(q,p) = 0 \qquad a = 1, \ldots\ldots m \quad . \tag{II.11}$$

It is essential for later developments to assume that

$$\{\chi_a, \chi_b\} = 0 \quad . \tag{II.12}$$

In order to achieve a canonical description in Γ^*, it is necessary to require, as will be seen below (see Eq. (II.14))

$$\det \| \{\chi_a, \varphi^b\} \| \neq 0 \quad . \tag{II.13}$$

c. Independent canonical variables

If (II.12) holds, we can perform a canonical transformation in Γ and make a transition to new variables in which

$$\chi_a(q,p) = p_a \quad .$$

In these new variables, (II.13) becomes

$$\det \left\| \frac{\partial \varphi^a}{\partial q_b} \right\| \neq 0 \quad , \tag{II.14}$$

where q^a are the coordinates conjugate to p_a . Thus Eqs. (II.2) can be solved for q^a . Hence, Γ^* is defined in Γ by

$$p_a = 0, \quad q^a = \bar{q}^a(q^*, p^*) \quad ,$$

where

$$\varphi^b(\bar{q}^a, 0, q^*, p^*) = 0$$

and q^* and p^* are the remaining canonical variables which act as independent variables on Γ^*.

One can show that [4]

$$\{f, g\}\Big|_M = \frac{\partial f^*}{\partial q_i^*}\frac{\partial g^*}{\partial p_i^*} - \frac{\partial f^*}{\partial p_i^*}\frac{\partial g^*}{\partial q_i^*} \quad , \tag{II.15}$$

where

$$f^* = f(\bar{q}^a(q^*, p^*), 0, q^*, p^*) \tag{II.16}$$

Thus q^* and p^* are canonical variables in Γ^*.

d. Passage to quantum theory

We now prove the central result of this section: For the mechanical system described hitherto, the expression for the matrix element of the S - matrix is

$$\langle out|S|in\rangle = \int \exp\{i\int_{-\infty}^{+\infty}(p_i \dot{q}^i - H)\,dt\}\prod_t d\mu(q(t), p(t)) \quad , \tag{II.17}$$

where the measure of integration is given by

$$d\mu(q,p) = \Big[\prod_a \delta(\chi_a)\,\delta(\varphi^a)\Big]\det\|\{\chi_a, \varphi^b\}\|\prod_i dp_i\,dq^i \quad . \tag{II.18}$$

The trajectories $q(t)$ coincide as $t \to \pm\infty$ with the solutions $q_{in}(t)$ and $q_{out}(t)$ of the equations describing the asymptotic motion.

The proof of the theorem goes as follows: Let us perform a canonical transformation to achieve a canonical description with the coordinates q^a, p^a, q^*, p^* as discussed above. The factor $\pi\, dp_i\, dq^i$ is invariant under a canonical transformation. In addition, we have

$$\int_{-\infty}^{+\infty}(p_i'\dot{q}^{i\prime} - H')\,dt = \int_{-\infty}^{+\infty}(p_i\dot{q}^i - H)\,dt + \left(p_i\frac{\partial\Phi}{\partial p_i} - \Phi\right)\Big|_{-\infty}^{+\infty} .$$

Φ is the generating function for the canonical transformation, in the sense that

$$\delta q^i = \frac{\partial\Phi}{\partial p_i} , \quad \delta p_i = -\frac{\partial\Phi}{\partial q^i} .$$

In field theory the interesting canonical transformations are linearized asymptotically as $t \to \pm\infty$, and then it can be shown that the change is equivalent to a unitary transformation in the operator formalism. Thus this change produces no change in the matrix element $\langle \text{out} | S | \text{in} \rangle$.

In the new canonical representation the measure becomes:

$$\prod_a \delta(p_a)\,\delta(\varphi^a)\,\det\left\|\frac{\partial\varphi^b}{\partial q^a}\right\| \prod_i dp_i\,dq^i =$$

$$= \prod_a \delta(p_a)\,\delta(q^a - \bar{q}^a(q^*, p^*))\,dp_a\,dq^a \prod dp^*\,dq^* . \tag{II.19}$$

After a trivial integration over p_a and q^a the integral takes the form

$$\int \exp\left\{i\int_{-\infty}^{+\infty}(\Sigma p^*\dot{q}^* - H^*)\,dt\right\}\prod dp^*\,dq^* , \tag{II.20}$$

and this is indeed the functional integral representation for $\langle \text{out} | S | \text{in} \rangle$ in terms of an integration over the independent variables q^* and p^*. This constitutes the proof of our assertion.

e. Independence of the choice of gauge conditions

The integral (II.17) is independent of the choice of χ_a. Indeed, for an infinitesimal change in χ_a, we easily find

$$\delta \chi_a = \{\Phi, \chi_a\} + C_{ab}\varphi^b \quad , \tag{II.21}$$

where $\Phi = h_a\varphi^a$ and the h_a's are the solution of the system of equations

$$\{\chi_a, \varphi^b\} h_b = -\delta\chi_a \quad , \tag{II.22}$$

which by (II.13) has a unique solution.

The second term is of no relevance, because of the first-class nature of the φ^b's and the factors $\delta(\varphi^a)$ in the integral, and the first term represents a canonical transformation. In this canonical transformation:

$$\delta\varphi^a = \{\Phi, \varphi^a\} = \{h_b\varphi^b, \varphi^a\} = A^a{}_b\varphi^b \quad , \tag{II.23}$$

so

$$\chi \to \chi + \delta\chi \, , \quad \varphi \to (1+A)\varphi \, , \quad H \to H$$
$$\prod_a \delta(\varphi^a) \to (1 + \mathrm{tr}\, A)^{-1} \prod_a \delta(\varphi^a)$$
$$\det \| \{\chi_a, \varphi^b\} \| \to (1 + \mathrm{tr}\, A) \det \| \chi_a + \delta\chi_a, \varphi^b\} \| \, . \tag{II.24}$$

We thus see that the integral (II.17) is independent of the choice of χ_a.

f. Example: Quantization of the electromagnetic field

We shall illustrate the general formalism developed in this section by applying it to a particular example: the quantization of the free electromagnetic field. The inclusion of interactions is rather trivial, so we take the free field in order to center our attention on the essentials. The Lagrangian density is thus:

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \quad . \tag{II.25}$$

Therefore

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \qquad , \tag{II.26}$$

so we have the first constraint

$$\varphi_1 = \pi^0 \qquad . \tag{II.27}$$

Following the formalism discussed in Section I, we find the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\pi^i\pi^i - A_0\,\partial_i\pi^i + \frac{1}{2}H_iH_i + u\,\pi^0 \qquad . \tag{II.28}$$

In Eq. (II.28) π^i is the momentum conjugate to A_i , $H_i = \epsilon_{ijk}\,\partial_j A_k$, and u is the multiplier for the constraint (II.27). Imposing the consistency condition

$$\dot{\varphi}_1\Big|_{\bar{M}} = \{\varphi_1, H\}\Big|_{\bar{M}} = 0 \tag{II.29}$$

we find a secondary constraint

$$\varphi_2 = \partial_i\pi^i \tag{II.30}$$

No new constraint can be produced, since $\dot{\varphi}_2\big|_M = 0$ is satisfied trivially in view of

$$\{\varphi_2(\vec{x}), H_i(\vec{y})\} = 0 \qquad , \tag{II.31}$$

as can be checked easily, Since

$$\begin{aligned} \{\varphi_1(\vec{x}), \varphi_2(\vec{y})\} &= 0 \\ \{\varphi_1(\vec{x}), \varphi_1(\vec{y})\} &= 0 \\ \{\varphi_2(\vec{x}), \varphi_2(\vec{y}) &= 0 \qquad , \end{aligned} \tag{II.32}$$

our constraints are first class. With these constraints we can associate the gauge conditions

$$\chi_1 = A^0$$
$$\chi_2 = \partial_i A_i \quad . \tag{II.33}$$

This choice fullfills the conditions (II.12) and (II.13) and moreover, in our case

$$\det \| \{ \chi_a , \varphi^b \} \| = \det \vec{\nabla}^2 \quad . \tag{II.34}$$

The determinant in (II.34) is to be understood as a functional determinant. The S-matrix element (II.17), (admittedly trivial in our case, since we are dealing with a free field, but nevertheless interesting in view of an easy extension to the interacting case), therefore reads:

$$< out | S | in > = \int exp \{ i \int (\pi^0 \dot{A}_0 + \pi^i \dot{A}_i - \mathcal{H}) d^4x \} \times$$
$$\times \left[\prod_{\vec{x},t} \delta(A_0) \, \delta(\partial_i A_i) \, \delta(\partial_i \pi^i) \, \delta(\pi^0) \right] \det \vec{\nabla}^2 \times \mathcal{D} A_0 \, \mathcal{D} \pi^0 \times$$
$$\times \prod_i \mathcal{D} A_i \, \mathcal{D} \pi^i \quad , \tag{II.35}$$

where $\mathcal{H}$ is given by (II.28). One can now first integrate trivially over A_o and π^o and then write

$$\prod_{\vec{x},t} \delta(\partial_i \pi^i) = \int \mathcal{D} A_0 \, exp \{ i \int d^4x \, A_0 \, \partial_i \pi^i \} \quad . \tag{II.36}$$

After a simple Gaussian integration over π^i, one recovers the expression

$$\langle out | S | in \rangle = \int exp\left\{ i \int \mathcal{L} d^4x \right\} \prod_{\vec{x},t} \delta(\partial_i A_i) \prod_{\mu} \mathcal{D} A_\mu \det \vec{\nabla}^2 , \tag{II.37}$$

where $\mathcal{L}$ is given by (II.25).

The generalization of (II.37) to the case of quantum electrodynamics reads

$$\langle out | S | in \rangle = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \prod_{\mu} \mathcal{D} A_\mu \det \vec{\nabla}^2 \prod_{\vec{x},t} \delta(\vec{\nabla}\cdot\vec{A}) exp\left\{ i \int \mathcal{L} d^4x \right\} , \tag{II.38}$$

where $\mathcal{L}$ is the standard Lagrangian in quantum electrodynamics. This is the well known expression for the matrix element in quantum electrodynamics in the Coulomb gauge [5].

The Coulomb gauge condition appearing in (II.38) violates manifest Lorentz invariance, so it is desirable to make a transition to the covariant gauge described by the δ-function $\delta(\partial_\mu A_\mu)$. This can be achieved most easily in the following way: first, one notes that

$$\det \Box \int \mathcal{D}\Omega \prod_{\vec{x},t} \delta(\partial_\mu A_\mu + \Box \Omega) = 1 , \tag{II.39}$$

which is seen by performing a change of variables

$$\Omega \to \Omega' = \Omega - \Box^{-1} \partial_\mu A_\mu \tag{II.40}$$

and performing the integration. Thus one can insert the left hand side of (II.39) as a factor in the integral (II.38). After that one can change variables as follows:

$$A_\mu \to A_\mu - \partial_\mu \Omega$$
$$\psi \to e^{-ie\Omega}\psi$$
$$\bar{\psi} \to e^{ie\Omega}\bar{\psi} \quad . \tag{II.41}$$

Since the Lagrangian in (II.38) is invariant under this change of variables, and since the Jacobian of the transformation is 1, one is immediately led to the expression

$$\langle out|S|in\rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \prod_\mu \mathcal{D}A_\mu \det \Box \prod_{\vec{x},t} \delta(\partial_\mu A_\mu) \times$$
$$\times \int \mathcal{D}\Omega \prod_{\vec{x},t} \delta(\vec{\nabla}\cdot\vec{A} - \vec{\nabla}^2\Omega) \det \vec{\nabla}^2 \exp\{i\int \mathcal{L} d^4x\} \quad . \tag{II.42}$$

The integration over Ω can now be performed trivially, producing the desired final expression:

$$\langle out|S|in\rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \prod_\mu \mathcal{D}A_\mu \det \Box \prod_{\vec{x},t} \delta(\partial_\mu A_\mu) \times$$
$$\times \exp\{i\int \mathcal{L} d^4x\} \quad , \tag{II.43}$$

which completes the transition to the covariant gauge.

III. The Generalization of Faddeev's Method to the Case When Second Class Constraints are Present

In this section we shall generalize Faddeev's results described in Section II to the case when second class constraints are present. Thus, in our case the canonical variables do not vary throughout the phase space Γ, but satisfy the equations

$$\begin{aligned} \varphi^a(q,p) &= 0 \qquad & a &= 1, \ldots m \\ \theta^\ell(q,p) &= 0 \qquad & \ell &= 1,2, \ldots 2n \end{aligned} \qquad \text{(III.1)}$$

We shall assume that the φ's and the θ's are independent and also irreducible in the sense that an arbitrary function h in Γ which vanishes in the subspace M in which Eqs. (III.1) hold is a linear combination of the constraints:

$$h = c_a(q,p)\,\varphi^a(q,p) + d_\ell(q,p)\,\theta^\ell(q,p) \quad . \qquad \text{(III.2)}$$

The φ^a's are first class constraints, i.e.:

$$\begin{aligned} \{\varphi^a, \varphi^b\}\Big|_M &= 0 \\ \{\varphi^a, \theta^\ell\}\Big|_M &= 0 \end{aligned} \quad , \qquad \text{(III.3)}$$

while the θ^ℓ's are second class:

$$\Big[\det \big\|\{\theta^\ell, \theta^k\}\big\|\Big]\Big|_M \neq 0 \quad . \qquad \text{(III.4)}$$

Thus, in view of the irreducibility hypothesis we have

$$\begin{aligned} \{\varphi^a, \varphi^b\} &= c^{ab}_c\,\varphi^c + d^{ab}_\ell\,\theta^\ell \\ \{\varphi^a, \theta^\ell\} &= e^{a\ell}_c\,\varphi^c + f^{a\ell}_k\,\theta^k \end{aligned} \quad . \qquad \text{(III.5)}$$

Self-consistency requires

$$d_{\ell}^{ab}\Big|_M = 0 \qquad , \tag{III.6}$$

as we shall now prove; indeed, from (III.5) we find immediately

$$\{\theta^s, \{\varphi^a, \varphi^b\}\} = d_{\ell}^{ab}\{\theta^s, \theta^\ell\} + \bar{c}_c^{sab}\varphi^c + \bar{d}_{\ell}^{sab}\theta^\ell$$

$$\{\varphi^b, \{\varphi^a, \theta^s\}\} = \bar{e}_c^{bas}\varphi^c + \bar{f}_k^{bas}\theta^k \qquad , \tag{III.7}$$

so that Jacobi's identity implies

$$d_{\ell}^{ab}\Big|_M \{\theta^s, \theta^\ell\}\Big|_M = 0 \qquad ,$$

and thus, in view of (III.4)

$$d_{\ell}^{ab}\Big|_M = 0 \qquad .$$

which was to be proved.

The equation of motion for an arbitrary quantity f is found to be, in a manner similar to that of Section II:

$$\dot{f}\Big|_M = \{f, H\}\Big|_M \qquad , \tag{III.8}$$

where

$$H = H\Big|_M + v_a\varphi^a + u_\ell\theta^\ell \qquad . \tag{III.9}$$

Self-consistency requires

$$\dot{\varphi}^a\Big|_M = \{\varphi^a, H\}\Big|_M = 0 \tag{III.10}$$

and

$$\dot{\theta}^k\Big|_M = \{\theta^k, H\}\Big|_M = 0 \quad . \tag{III.11}$$

One can easily see that (III.5), (III.10) and the irreducibility hypothesis imply

$$\{H\big|_M, \varphi^a\} = C^a_b\,\varphi^b + d^a_\ell\,\theta^\ell \quad . \tag{III.12}$$

Eq. (III.11), in turn, simply determines $u_\ell\big|_M$ in view of (III.4).

Not all quantities are observable (physical), but only those whose variation in time is not affected by the arbitrariness in the choice of the v_a 's . Thus for physical quantities we must impose the requirement that

$$\dot{f}\Big|_M = \{f, H\big|_M\}\Big|_M + v_a\Big|_M\{f,\varphi^a\}\Big|_M + u_\ell\Big|_M\{f,\theta^\ell\}\Big|_M \tag{III.13}$$

is a uniquely determined quantity, which implies

$$\{f, \varphi^a\}\Big|_M = 0 \tag{III.14}$$

or

$$\{f, \varphi^a\} = C^a_b\,\varphi^b + d^a_\ell\,\theta^\ell \quad , \tag{III.15}$$

while there is no such restriction on $\{f, \theta^\ell\}\big|_M$ in view of the corollary preceding Eq. (I.19). (See also the comment following (III.12)). Condition (III.6) plays an important role in the self-consistency of those requirements. Indeed, from

(III.15), (III.5) and the Jacobi identity one can deduce

$$\{ \{ \varphi^a, \varphi^b \}, f \} \big|_M = 0 \ , \qquad \text{(III.16)}$$

which, in view of (III.5), results in

$$d^{ab}_{\ell} \big|_M \{ \theta^{\ell}, f \} \big|_M = 0 \ . \qquad \text{(III.17)}$$

so that, if (III.6) did not hold, we would have to impose $\{ \theta^{\ell}, f \} \big|_M = 0$, which is unacceptable, as we shall see below.

Note that condition (III.15) is independent of the choice of continuation of a function f defined in M, that is $f \big|_M$, into the whole phase space Γ. Indeed, any two such continuations may differ only by a linear combination of constraints and then (III.5) implies that (III.15) holds for any such continuation. Unlike (III.15), the condition $\{ \theta^{\ell}, f \} \big|_M = 0$ does not possess this feature and thus is unacceptable.

Eqs. (III.14) can be thought of as a set of m first-order differential equations on M with (III.5) serving as integrability conditions. The proof of this statement is a straightforward extension of the corresponding proof in Section II. Hence the function f is defined uniquely by its values in the submanifold of the initial conditions of the equations which is of dimension (2N-m-2n) -m = 2(N-n-m) (2N = number of canonical coordinates and momenta in Γ).

We can take this submanifold to be a surface Γ^* in M defined by

$$\chi_a \, (q, p) = 0 \qquad a = 1, \ldots\ldots m \ . \qquad \text{(III.18)}$$

We shall call Eqs. (III.18) gauge conditions. We thus see that gauge conditions are associated with first class constraints only. It is essential for later developments to require (see Eq. (III.20)):

$$\det \| \{ \chi_a , \varphi^b \} \| \neq 0 \ .$$

We now prove the following theorem.

Theorem:

Let there be given a mechanical system with m first-class constraints and 2n second-class constraints. Let the first-class constraints be called φ_a, the second-class θ_a, and the gauge conditions associated with the first-class constraints χ_a [*1]. Let the χ_a's be chosen in such a way that $\{\chi_a , \chi_b\} = 0$.

Then the expression for the S-matrix element is [*2]

$$\langle out | S | in \rangle = \int \exp\left\{ i \int_{-\infty}^{+\infty} (p_i \dot{q}_i - H)\, dt \right\} \prod_t d\mu(q(t), p(t)) \ , \qquad \text{(III.19)}$$

where H is the Hamiltonian of the system and the measure of integration is defined by

$$\begin{aligned} d\mu(q,p) = & \prod_a \delta(\chi_a)\delta(\varphi_a) \, |\det \| \{\chi_a , \varphi_b\} \|| \times \\ & \times \prod_c \delta(\theta_c) \, | \, |\det \| \{\theta_a , \theta_b\} \| |^{1/2} \prod_i dp_i \, dq_i \ . \end{aligned} \qquad \text{(III.20)}$$

In order to prove this theorem, we need the following lemma:

Let M_ϵ be that region of phase space in which $\theta_a(p,q)$ and $\varphi_a(p,q)$ are of

---

*1 In what follows, we shall not distinguish between the indices associated with the second-class constraints and those associated with the first-class constraints. The range of the respective indices will be clear in all the expressions we shall write.

*2 Again, as in Section II, the trajectories q(t) coincide as $t \to \pm\infty$ with the solutions $q_{in}(t)$ and $q_{out}(t)$ of the equations describing the asymptotic motion.

order ϵ (ϵ is an infinitesimal quantity). Note that $M_\epsilon \supset M$. In the region M_ϵ there exists a set of functions Λ_{ab} and μ_{ab} such that if we define

$$\theta_a' = \Lambda_{ab}\,\theta_b + \mu_{ab}\,\varphi_b \quad , \tag{III.21}$$

then

$$\begin{aligned} \{\theta_a', \theta_b'\} &= (\Theta')_{ab} + O(\epsilon^2) \quad \text{in } M_\epsilon \,, \\ \{\theta_a', \chi_b\} &= O(\epsilon^2) \quad \text{in } M_\epsilon \,, \end{aligned} \tag{III.22}$$

where

$$\Theta' = \left[\begin{array}{ccc|ccc} & & & & & 1 \\ & 0 & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & 0 & \\ -1 & & & & & \end{array}\right] . \tag{III.22a}$$

The proof of the lemma is given in the Appendix following Chapter I.

Now note that, due to the presence of the δ- functions $\delta(\varphi_a)$ and $\delta(\theta_c)$, the integral in (III.19) extends only over the region M_ϵ. Secondly, note that by (A.I-8) and (A.I-12)

$$\prod_a \delta(\varphi_a) \prod_c \delta(\theta_c') = \left| \det \left\| \Lambda_{ab} + \frac{\partial \Lambda_{ac}}{\partial \theta_b}\theta_c + \frac{\partial \mu_{ac}}{\partial \theta_b}\varphi_c \right\| \right|^{-1} \prod_d \delta(\theta_d) \prod_a \delta(\varphi_a)$$

$$= \left|\det \| \Lambda_{ab} \| \right|^{-1} \prod_d \delta(\theta_d) \prod_a \delta(\varphi_a) = \left|\det \| \Lambda^0_{ab} \| \right|^{-1} \prod_d \delta(\theta_d) \prod_a \delta(\varphi_a) =$$

$$= \left| \det \| \{\theta_a, \theta_b\} \| \right|^{1/2} \prod_d \delta(\theta_d) \prod_a \delta(\varphi_a) , \tag{III.23}$$

where we have used the relation

$$\left[\det \|\Lambda^0\|\right]^2 \left|\det \|\{\theta_a, \theta_b\}\|\right| = |\det \Theta'| = 1, \tag{III.24}$$

which is a consequence of (A.I-1) and (A.I-2).

In view of Eq. (III.22) we can perform a canonical transformation in M_ε such that the new variables are $P_a = \chi_a$ $1 \leq a \leq m$ $Q_{m+a} = \theta'_a$, $P_{m+a} = \theta'_{2n-a+1}$; $1 \leq a \leq n$ (The discussion of canonical invariance following (II.18) applies in our case as well.) . We thus find, using (III.19), (III.20) and (III.23), and integrating trivially over P_a , P_{m+a} and Q_{m+a}

$$\langle out | S | in \rangle = \int \exp\left\{ i \int (\bar{P}_i \dot{\bar{Q}}_i - \bar{\bar{H}})\, dt \right. \times$$

$$\times \prod_t \left(\prod_a \delta(\varphi_a) \left|\det \left\| \frac{\partial \varphi_b}{\partial Q_a} \right\|\right| \right) \prod_i \partial \bar{P}_i \partial \bar{Q}_i \prod_a \partial Q_a , \tag{III.25}$$

where

$$\bar{\bar{H}} = H(\bar{P}_i, \bar{Q}_i, P_a = 0, Q_a, Q_{m+a} = 0, P_{m+a} = 0) \tag{III.26}$$

and $\bar{P}_i$'s and $\bar{Q}_i$'s are the remaining canonical variables. Finally, noting that

$$\prod_a \delta(\varphi_a) \left|\det \left\| \frac{\partial \varphi_b}{\partial Q_a} \right\|\right| = \prod_a \delta(Q_a - Q_a^*(\bar{P}_i, \bar{Q}_i)), \tag{III.27}$$

where $Q_a^*(\bar{P}_i, \bar{Q}_i)$ is the solution of the equation

$$\varphi_a(\bar{P}_i, \bar{Q}_i, Q_a^*, P_a = 0, Q_{m+a} = 0, P_{m+a} = 0) = 0, \tag{III.28}$$

we can write

$$< out | S | in > = \int \exp\left\{ i \int (\bar{P}_i \dot{\bar{Q}}_i - \bar{H}) dt \right\} \prod_i \mathcal{D}\bar{P}_i \mathcal{D}\bar{Q}_i , \tag{III.29}$$

$$\bar{H} = \bar{H}\,(Q_a = Q_a^*(\bar{P}_i, \bar{Q}_i)) . \tag{III.30}$$

Thus, in full analogy with the case described in Section II, we have obtained an expression for the S-matrix elements as a functional integral over the independent canonical variables only. The weight of this integral is one, as it should be, and therefore (III.29) provides a justification of Eq. (III.19) and thus a proof of our theorem *. Note the crucial role played by the determinants $|\det \| \{ \chi_a , \varphi_b \} \| |$ and $|\det \| \{ \theta_a , \theta_b \} \| |^{\frac{1}{2}}$.

It remains to prove that the matrix element in (III.19) with the measure given by (III.20) is independent of the choice of the gauge conditions χ_a. Again, as in Section II, we find

$$\delta \chi_a = \{ \Phi , \chi_a \} + C_{ab} \varphi_b \tag{III.31}$$

with

$$\Phi = h_a \varphi_a \tag{III.32}$$

---

* From our proof it is clear that (II.17) and (III.19) still hold if one relaxes Faddeev's requirement $\{ \chi_a , \chi_b \} = 0$ to read:

$$\{ \chi_a , \chi_b \} = \alpha_{ab}^{cd} \chi_c \chi_d + \beta_{ab}^{cd} \chi_c \varphi_d + \gamma_{ab}^{cd} \varphi_c \varphi_d$$

and the h's are the solution of the system of equations

$$\{\chi_a, \varphi_b\} h_b = -\delta \chi_a \,. \tag{III.33}$$

The first term in (III.31) represents a canonical transformation. Performing such a canonical transformation results in changing φ_a and θ_a by

$$\delta \varphi_a = \{\Phi, \varphi_a\} = A_{ab} \varphi_b \quad , \tag{III.34}$$

$$\delta \theta_a = \{\Phi, \theta_a\} = \{h_b \varphi_b, \theta_a\} = B_{ab} \varphi_b + D_{ab} \theta_b \,. \tag{III.35}$$

Calling:

$$\delta_0 \chi_a = \{\Phi, \chi_a\} \tag{III.36}$$

we find:

$$\{\chi_a, \varphi_b\}\Big|_M \rightarrow \{\chi_a + \delta_0 \chi_a, \varphi_b + A_{bc} \varphi_c\}\Big|_M =$$

$$= \{\chi_a + \delta \chi_a, \varphi_b + A_{bc} \varphi_c\}\Big|_M , \tag{III.37}$$

where we have used the first-class nature of φ_a. Thus:

$$\prod_a \delta(\varphi_a) \longrightarrow (1 + \mathrm{tr} A)^{-1} \prod_a \delta(\varphi_a) \,, \tag{III.38}$$

$$\left[\det \| \{\chi_a, \varphi_b\} \|\right]\Big|_M \rightarrow (1 + \mathrm{tr} A)\left[\det \| \{\chi_a + \delta \chi_a, \varphi_b\| \right]\Big|_M \,, \tag{III.39}$$

$$\prod_a \delta(\theta_a) \longrightarrow (1 + \mathrm{tr}\, D)^{-1} \prod_a \delta(\theta_a) \tag{III.40}$$

and

$$\left| \det \| \{ \theta_a, \theta_b \} \| \right|_M^{1/2} \rightarrow (1 + \mathrm{tr}\, D) \left| \det \| \{ \theta_a, \theta_b \} \| \right|_M^{1/2} . \tag{III.41}$$

We can thus conclude that the integral (III.19) is independent of χ_a, due to canonical invariance and Eqs. (III.38) - (III.41).

We wish to alert the reader to the fact that the extension of our results to field theory, as in Sections I and II, is rather straight forward.

Appendix I

In this Appendix we wish to prove the lemma we needed in Section III (See Eq. (III.22)). We shall first prove that there exists a matrix Λ^0 such that

$$\Lambda^0 \Theta (\Lambda^0)^T = \Theta', \qquad \text{(A.I-1)}$$

where

$$\Theta_{ab} = \{ \theta_a, \theta_b \} \qquad \text{(A.I-2)}$$

and Θ' is given by (III.22 a). The proof of (A.I-1) proceeds as follows: First, since Θ is an antisymmetric matrix of even order then by a general theorem [6] there exists an orthogonal matrix R such that

$$R \Theta R^T = \tilde{\Theta}$$

and

$$\tilde{\Theta} = \begin{bmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & 0 & \lambda_2 & & & \\ & & -\lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \lambda_n \\ & & & & & -\lambda_n & 0 \end{bmatrix} \qquad \text{(A.I-3)}$$

Note that in our case all the λ_i's are different from zero, since by (III. 4)

$$\det \tilde{\Theta} = \prod_i \lambda_i^2 = \det \Theta \neq 0 .$$

Then define the matrix R_2 :

$$
R_2 = \begin{bmatrix}
1 & 0 & 0 & \cdots & & & \\
0 & 0 & 1 & 0 & \cdots & & \\
0 & 0 & 0 & 0 & 1 & \cdots & \\
\cdots & & & & & & \cdots \\
0 & 0 & 0 & 0 & \cdots & & \frac{1}{\lambda_n} \\
0 & 0 & \cdots & & \frac{1}{\lambda_{n-1}} & 0 & 0 \\
\cdots & & & & & & \cdots \\
0 & 0 & 0 & \frac{1}{\lambda_2} & & & \\
0 & \frac{1}{\lambda_1} & 0 & \cdots & & & \cdots
\end{bmatrix},
\tag{A.I-4}
$$

that is

$$
\begin{aligned}
(R_2)_{k,2k-1} &= 1 \qquad \text{for} \quad 1 \le k \le n \\
(R_2)_{2n-k,2k+2} &= \frac{1}{\lambda_{k+1}} \qquad \text{for} \quad 0 \le k \le n-1 \\
(R_2)_{m,n} &= 0 \qquad \text{otherwise}.
\end{aligned}
\tag{A.I-5}
$$

One can then check that

$$
R_2 \tilde{\Theta} R_2^T = \Theta'. \tag{A.I-6}
$$

This constitutes the proof of (A.I-1), since we can take

$$
\Lambda^0 = R_2 R . \tag{A.I-7}
$$

To prove our lemma given by (III.21), (III.22) and (III.22a), let us look for solutions for Λ_{ab} and μ_{ab} in the form

$$
\begin{aligned}
\Lambda_{ab} &= \Lambda^0_{ab} + \bar{\Lambda}_{ab} , \\
\mu_{ab} &= \mu^0_{ab} + \bar{\mu}_{ab} ,
\end{aligned}
\tag{A.I-8}
$$

where Λ^0 is the matrix introduced in (A.I-1) while μ^o_{ab} is chosen in such a way that

$$\{\chi_c, \theta_b\} \Lambda^0_{ab} + \mu^0_{ab} \{\chi_c, \varphi_b\} = 0 . \tag{A.I-9}$$

Note that μ^o_{ab} can be found by solving (A.I-9), since our choice of χ_a satisfies the condition

$$\det \| \{\chi_c, \varphi_b\} \| \neq 0 \quad \text{in } M_\epsilon .$$

If we denote by Φ_{bc} a set of functions which satisfy

$$\Phi_{bc} \{\chi_c, \varphi_a\} = \delta_{ba} , \tag{A.I-10}$$

we find

$$\mu^0_{ad} = - \{\chi_c, \theta_b\} \Lambda^0_{ab} \Phi_{dc} . \tag{A.I-11}$$

We shall look for solutions of the form

$$\bar{\Lambda}_{ab} = W^e_{ab} \theta_e + u^e_{ab} \varphi_e , \tag{A.I-12}$$

$$\bar{\mu}_{ab} = Z^e_{ab} \theta_e + V^e_{ab} \varphi_e . \tag{A.I-13}$$

Neglecting terms of the order of ϵ^2, comparing the coefficients of θ_a and φ_a and using (III.5), (III.6), (A.I-1) and (A.I-9), we see that a way of satisfying (III.22) is having

$$\{\chi_b, \theta_e\} u^c_{ae} + \{\chi_b, \theta_e\} Z^e_{ac} + \{\chi_b, \varphi_e\} V^e_{ac} +$$

$$+ \{\chi_b, \varphi_e\} V^c_{ae} = - \{\chi_b, \mu^0_{ac}\} , \tag{A.I-14}$$

$$\{X_b, \theta_e\} W^e_{ac} + \{X_b, \theta_e\} W^c_{ae} + \{X_b, \varphi_e\} U^e_{ac} +$$

$$+ \{X_b, \varphi_e\} Z^c_{ae} = - \{X_b, \Lambda^0_{ac}\} , \qquad \text{(A.I-15)}$$

$$W^e_{ab} \Lambda^0_{cd} \{\theta_b, \theta_d\} + W^e_{cd} \Lambda^0_{ab} \{\theta_b, \theta_d\}$$

$$+ \Lambda^0_{ab} \{\theta_b, \theta_f\} W^f_{ce} + \Lambda^0_{cd} W^f_{ae} \{\theta_f, \theta_d\} = \Xi_{ac,e} , \qquad \text{(A.I-16)}$$

and

$$U^e_{ab} \Lambda^0_{cd} \{\theta_b, \theta_d\} + U^e_{cd} \Lambda^0_{ab} \{\theta_b, \theta_d\} +$$

$$+ Z^f_{ae} \{\theta_f, \theta_d\} \Lambda^0_{cd} - Z^f_{ce} \{\theta_f, \theta_b\} \Lambda^0_{ab} = \rho_{ac,e} , \qquad \text{(A.I-17)}$$

where

$$\Xi_{ac,e} = - \Lambda^0_{ab} \{\theta_b, \Lambda^0_{ce}\} - \Lambda^0_{cd} \{\Lambda^0_{ae}, \theta_d\} -$$

$$- \mu^0_{ab} \{\varphi_b, \Lambda^0_{ce}\} + \mu^0_{cb} \{\varphi_b, \Lambda^0_{ae}\} -$$

$$- \mu^0_{ab} \Lambda^0_{cd} f^e_{bd} + \mu^0_{cb} \Lambda^0_{ad} f^e_{bd} ,$$

$$\text{(A.I-18)}$$

and

$$\rho_{ac,e} = -\mu^0_{ab}\mu^0_{cd}\, c^e_{bd} - \{\mu^0_{ae}, \varphi_d\}\mu^0_{cd} -$$

$$- \{\varphi_b, \mu^0_{ce}\}\mu^0_{ab} - \{\mu^0_{ae}, \theta_d\}\Lambda^0_{cd} -$$

$$-\mu^0_{ab}\Lambda^0_{cd}\, e^e_{bd} + \mu^0_{cb}\Lambda^0_{ad}\, e^e_{bd} + \{\mu^0_{ce}, \theta_d\}\Lambda^0_{ad} \,. \qquad \text{(A.I-19)}$$

In (A.I-18) and A.I-19), c^e_{bd}, e^e_{bd}, f^e_{bd} are the functions appearing in (III.5); we have changed the index notation somewhat, e.g.

$$\{\varphi_b, \theta_d\} = e^e_{bd}\varphi_e + f^e_{bd}\theta_e \,. \qquad \text{(A.I-20)}$$

Defining now

$$u^c_{ae} + Z^e_{ac} = y^c_{ae} \,, \qquad \text{(A.I-21)}$$

$$t^e_{cb} = W^e_{cb} + W^b_{ce} \,, \qquad \text{(A.I-22)}$$

(Note that $t^e_{cb} = t^b_{ce}$.) and

$$q^e_{ac} = V^e_{ac} + V^c_{ae} \,, \qquad \text{(A.I-23)}$$

we see that (A.I-14) merely provides a solution for q^e_{ac} once the y^c_{ac}'s are known, while the remaining equations can be written as

$$t^e_{ab}\chi_{bc} - t^e_{cb}\chi_{ba} = \Xi_{ac,e} \,, \qquad \text{(A.I-24)}$$

$$\{\chi_b, \theta_e\}\, t^e_{ac} + \{\chi_b, \varphi_e\}\, y^e_{ac} = -\{\chi_b, \Lambda^0_{ac}\} \,, \qquad \text{(A.I-25)}$$

$$y^e_{ab}\chi_{bc} - y^e_{cb}\chi_{ba} = \rho_{ac,e} \,, \qquad \text{(A.I-26)}$$

$$t^{e}_{ac} = t^{c}_{ae} \quad , \tag{A.I-27}$$

where

$$X_{bc} = \{\theta_b, \theta_d\} \Lambda^{0}_{cd} \; . \tag{A.I-28}$$

From (A.I-25) we can find y^{e}_{ac}, once t^{e}_{ac} is known. Solving for y^{e}_{ac} from (A.I-25), and inserting into (A.I-26), we obtain (A.I-24), to the lowest order in ϵ. We omit the lengthy and straight forward proof of this statement. The basic prerequisite for this proof are Eqs. (III.5) and (A.I-9), the distributive law for Poisson brackets, i.e.

$$\{A, BC\} = \{A, B\} C + \{A, C\} B \quad , \tag{A.I-29}$$

the Jacobi identity for Poisson brackets, i.e.

$$\{\{A, B\}, C\} + \{\{C, A\}, B\} + \{\{B, C\}, A\} = 0 \quad , \tag{A.I-30}$$

Eqs. (A.I-18), (A.I-19) and (A.I-28), and finally the relation

$$\{X_f, (\Theta')_{ac}\} = \{X_f, \Lambda^{0}_{ab} \Lambda^{0}_{cd} \{\theta_b, \theta_d\}\} = 0 \; , \tag{A.I-31}$$

which follows from (III.22a), (A.I-1) and (A.I-2).

Thus it remains to show that there exists a set of functions t^{e}_{ac} such that (A.I-24)and (A.I-27) hold. In turn, Eqs. (A.I-24) and (A.I-27) are equivalent to

$$h_{fac} - h_{fca} = b_{f,ca} \tag{A.I-32}$$

$$h_{afc} = h_{cfa} \quad , \tag{A.I-33}$$

where

$$b_{f,ca} = X_{ef} \, \Xi_{ac,e} \tag{A.I-34}$$

and

$$h_{fac} = X_{ef}\, t^{e}_{ab}\, X_{bc} \quad . \tag{A.I-35}$$

A solution for the h's is found by the following procedure:

a) fix h_{afc} for $a \leq f \leq c$ arbitrarily

b) use (A.I-32) and (A.I-33) to solve for the remaining h_{afc} .

Chapter II. Applications

II.1 Quantization of some first-order field theories

In this section we shall quantize some first-order field theories, as promised in the introduction*. In the first subsection we shall analyze the quantization of the first-order formulation of the scalar field theory with a quartic interaction. Besides studying the path integral formulation of the theory as an application of the formalism developed in Section III, we shall also calculate the Dirac brackets of fundamental variables to have an idea of how quantization would proceed in the operator formalism (by Dirac's method). In the second subsection we shall repeat the same analysis for the case of the first-order version of the field theory of the free electromagnetic field. The example which will be considered in the first subsection will be found to contain only second-class constraints, while the second example will contain both first-class and second-class ones.

II.1.A Quantization of the first-order version of the φ^4 scalar theory

Let us analyze first the first-order theory described by the Lagrangian density

$$\mathcal{L} = A^\mu \partial_\mu \varphi - \frac{1}{2} A^\mu A_\mu - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4 . \qquad \text{(II.1.A.1)}$$

This first-order theory is an equivalent formulation of the φ^4 scalar theory.

For the momenta conjugate to A_μ one finds immediately

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = 0 , \qquad \text{(II.1.A.2)}$$

---

*For an account of first-order field theories, the reader is referred to Deser's lecture notes [7].

while the momentum conjugate to φ is found to be

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = A^0 \ . \qquad \text{(II.1.A.3)}$$

Thus one establishes five primary constraints

$$\theta^\mu = \pi^\mu$$

$$\theta = \pi - A^0 \ . \qquad \text{(II.1.A.4)}$$

The Hamiltonian density of the system can be obtained by the method introduced by Dirac and described in Section I. It has the following form:

$$\mathcal{H} = \frac{1}{2}\pi^2 - \frac{1}{2}\vec{A}^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \vec{A}\cdot\vec{\nabla}\varphi + v_\mu \pi^\mu + v(\pi - A^0) \ . \qquad \text{(II.1.A.5)}$$

As explained in Section I, (see Eq. (I.13)) consistency requires that

$$\dot{\theta}^\mu\Big|_{\overline{M}} = \{\theta^\mu, H\}\Big|_{\overline{M}} = 0 \qquad \text{(II.1.A.6)}$$

and

$$\dot{\theta}\Big|_{\overline{M}} = \{\theta, H\}\Big|_{\overline{M}} = 0 \ . \qquad \text{(II.1.A.7)}$$

Eq. (II.1.A.7) leads to

$$\vec{\nabla}\cdot\vec{A} - m^2\varphi - \frac{\lambda}{3!}\varphi^3 - v_0 = 0 \ . \qquad \text{(II.1.A.8)}$$

This simply determines v_0 and leads to no new constraints.

From $\dot{\theta}^0\Big|_{\overline{M}} = \{\theta^0, H\}\Big|_{\overline{M}} = 0$, one finds $v = 0$ (no new constraints), while the remaining conditions in (II.1.A.6) produce the following secondary constraints:

$$g_i = -\partial_i \varphi + A_i \ . \qquad \text{(II.1.A.9)}$$

Again, consistency requires that

$$\dot{g}_i = \{ g_i , H \} = 0, \tag{II.1.A.10}$$

which results in

$$v_i = \partial_i \pi \tag{II.1.A.11}$$

and thus leads to no additional secondary constraints.

One can easily establish the following values of the Poisson brackets of constraints:

$$\begin{aligned} &\{ \theta_\mu(\vec{x}), \theta_\nu(\vec{y}) \} = 0 \; , \\ &\{ \theta_\mu(\vec{x}), \theta(\vec{y}) \} = \delta_{\mu 0}\, \delta(\vec{x}-\vec{y}) \, , \end{aligned} \tag{II.1.A.12}$$

$$\{ \theta(\vec{x}), g_i(\vec{y}) \} = - \partial_i^{\vec{x}} \delta(\vec{x}-\vec{y}), \tag{II.1.A.13}$$

$$\{ \theta_\mu(\vec{x}), g_i(\vec{y}) \} = \delta_{\mu i}\, \delta(\vec{x}-\vec{y}), \tag{II.1.A.14}$$

all other Poisson brackets being zero.

The matrix $\{ \theta_a(\vec{x}), \theta_b(\vec{y}) \}$ is thus found to be

$$\{ \theta_a(\vec{x}), \theta_b(\vec{y}) \} = \left[\begin{array}{cccc|cccc} & & & & \delta & & & \\ & 0 & & & & \delta & & \\ & & & & & & \delta & \\ & & & & & & & \delta \\ \hline -\delta & & & & 0 & -\partial_1\delta & -\partial_2\delta & -\partial_3\delta \\ & -\delta & & & \partial_1\delta & & & \\ & & -\delta & & \partial_2\delta & & 0 & \\ & & & -\delta & \partial_3\delta & & & \end{array} \right] \tag{II.1.A.15}$$

where $\delta = \delta(\vec{x} - \vec{y})$, $\partial_i \delta = \partial_i^{\vec{x}} \delta(\vec{x} - \vec{y})$ and we have ordered the constraints in the following way: $\theta_0 \ldots\ldots \theta_3$, θ, $g_1 \ldots\ldots g_3$.

The determinant of this matrix is seen to be 1, i.e. different from zero, and thus all constraints are second-class.

Even though the main task of this thesis is to quantize certain field theories by means of phase space path integrals, we deem it instructive nevertheless to show how quantization would proceed in the operator formalism. We are thus led to consider the Dirac brackets of fundamental variables. Using (I.20) and (I.21) one finds for instance, that

$$\{\pi(\vec{x}), \varphi(\vec{y})\}^* = \{\pi(\vec{x}), \varphi(\vec{y})\} - \\ - \int d\vec{z}\, d\vec{u}\, \{\pi(\vec{x}), g_i(\vec{z})\}\, C_i(\vec{z}, \vec{u})\, \{\theta(\vec{u}), \varphi(\vec{y})\} \tag{II.1.A.16}$$

where $c_i(\vec{z}, \vec{u})$ are the relevant elements of the matrix inverse to (II.1.A.15), which are the cofactors of $\{\theta_a(\vec{z}), \theta_5(\vec{u})\}$, $a = 6,7,8$. By inspection of (II.1.A.15) these cofactors are found to be zero, since they have one **row** consisting solely of zeros. Thus

$$\{\pi(\vec{x}), \varphi(\vec{y})\}^* = \{\pi(\vec{x}), \varphi(\vec{y})\} = -\delta(\vec{x} - \vec{y}) \tag{II.1.A.17}$$

To find the value of $\{\pi(\vec{x}), \vec{A}(\vec{y})\}$ we can use the fact that the Dirac bracket of any second-class constraint with anything is zero (see Eq. (I.22)). Thus, using (II.1.A.9), we find

$$\{\pi(\vec{x}), \vec{A}(\vec{y})\}^* = \{\pi(\vec{x}), \vec{\nabla}_{\vec{y}}\, \varphi(\vec{y})\} = \vec{\nabla}_{\vec{x}}\, \delta(\vec{x} - \vec{y}) \tag{II.1.A.18}$$

Quite analogously to (II.1.A.17), one finds

$$\{\pi(\vec{x}), \pi(\vec{y})\}^* = 0 \tag{II.1.A.19}$$

$$\{\pi(\vec{x}), A^0(\vec{y})\}^* = 0 \,. \tag{II.1.A.20}$$

In a similar manner one obtains the remaining Dirac brackets:

$$X_b = \text{(all canonical variables)}$$

$$\{\pi^\mu(\vec{x}), X_b(\vec{y})\}^* = 0$$
$$\{\varphi(\vec{x}), \varphi(\vec{y})\}^* = 0$$
$$\{A_0(\vec{x}), \varphi(\vec{y})\}^* = -\delta(\vec{x}-\vec{y})$$
$$\{\vec{A}(\vec{x}), \varphi(\vec{y})\}^* = 0$$
$$\{A_0(\vec{x}), \vec{A}(\vec{y})\}^* = \vec{\nabla}_{\vec{x}}\,\delta(\vec{x}-\vec{y})$$
$$\{A_i(\vec{x}), A_j(\vec{y})\}^* = 0$$
$$\{A_0(\vec{x}), A_0(\vec{y})\}^* = 0 \,. \tag{II.1.A.21}$$

The vacuum-vacuum transition amplitude can, by use of the results of Section III, be expressed as a functional integral

$$<0|S|0> = \int \prod_\mu (\mathcal{D}A_\mu \mathcal{D}\pi_\mu)\, \mathcal{D}\pi \mathcal{D}\varphi \prod_x \delta(\pi - A^0) \prod_{x,\mu} \delta(\pi^\mu) \times$$
$$\times \prod_{x,i} \delta(A_i - \partial_i \varphi)\, exp\left\{ i \int d^4x \left(\pi^\mu \dot{A}_\mu + \pi\dot{\varphi} - \frac{1}{2}\pi^2 + \frac{1}{2}\vec{A}^2 - \right.\right.$$
$$\left.\left. - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4 - \vec{A}\cdot\vec{\nabla}\varphi \right) \right\},$$

or

$$<0|S|0> = \int \prod_\mu \mathcal{D}A_\mu\, \mathcal{D}\varphi \prod_{x,i} \delta(A_i - \partial_i \varphi) \times$$

$$\times\, exp\left\{ i \int d^4x \left(A_0 \dot{\varphi} - \vec{A}\cdot\vec{\nabla}\varphi - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\vec{A}^2 - \frac{1}{2}(A_0)^2 \right) \right\}. \tag{II.1.A.22}$$

Writing

$$\prod_{x,i} \delta(A_i - \partial_i \varphi) = \int \mathcal{D}\lambda_1 \mathcal{D}\lambda_2 \mathcal{D}\lambda_3 \, \exp\{ i \int \lambda_i (A_i - \partial_i \varphi) d^4x \}, \qquad \text{(II.1.A.23)}$$

making a change of variables $A_i \to A_i - \lambda_i$, and integrating trivially over λ_i , we find

$$\langle 0|S|0\rangle = \int \prod_\mu \mathcal{D}A_\mu \mathcal{D}\varphi \exp\{ i \int d^4x (A^\mu \partial_\mu \varphi - \tfrac{1}{2} A^\mu A_\mu - \tfrac{1}{2} m^2 \varphi^2 - \tfrac{\lambda}{4!} \varphi^4)\}. \qquad \text{(II.1.A.24)}$$

The action appearing in (II.1.A.24) is precisely the one corresponding to the Lagrangian density (II.1.A.1). By integrating further over the A_μ 's , one can convince oneself that the theory considered is equivalent to the self-interacting scalar theory with a quartic coupling.

This completes our discussion of the quantization of the first-order version of the self-interacting scalar field theory with a quartic coupling.

II.1.B. Quantization of the first-order version of the field theory of the free electromagnetic field

The second example of a field theory we shall study is the one described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} . \qquad \text{(II.1.B.1)}$$

In (II.1.B.1) the variables $F_{\mu\nu}$ are taken to be antisymmetric, i.e. $F_{\mu\nu} = - F_{\nu\mu}$. In this case the primary constraints are seen to be

$$\begin{aligned} \theta^{\mu\nu} &= \pi^{\mu\nu} , \\ \theta^{i} &= \pi^{i} + F^{0i} , \\ \theta^{0} &= \pi^{0} . \end{aligned} \qquad \text{(II.1.B.2)}$$

In Eq. (II.1.B.2) the $\pi^{\mu\nu}$'s are antisymmetric. $\pi^{\mu\nu}$ are the momenta conjugate to $F_{\mu\nu}$, π^i are conjugate to A_i, and π^o is conjugate to A_o. Proceeding in the same fashion as in the previous example, we find the Hamiltonian density:

$$\mathcal{H} = \pi^i \partial_i A_0 + \frac{1}{2} \pi^i \pi^i + \frac{1}{2} F_{ij} (\partial_i A_j - \partial_j A_i) -$$

$$- \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} u_{\mu\nu} \pi^{\mu\nu} + u_0 \pi^0 + u_i (\pi^i + F^{0i}) . \tag{II.1.B.3}$$

We take the Poisson brackets of the canonical variables to be:

$$\{ \pi^{\mu\nu}(\vec{x}), F_{\lambda\sigma}(\vec{y}) \} = - (\delta^{\mu}_{\lambda} \delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma} \delta^{\nu}_{\lambda}) \delta(\vec{x}-\vec{y}). \tag{II.1.B.4}$$

Imposing the conditions

$$\{ \theta^{lm}, H \} = 0 , \{ \theta^0, H \} = 0 , \tag{II.1.B.5}$$

results in new constraints

$$g^{lm} \equiv F^{lm} - \partial^l A^m + \partial^m A^l = 0 , \tag{II.1.B.6}$$

$$g^0 \equiv \partial_i F^{0i} = 0 . \tag{II.1.B.7}$$

Furthermore, the following conditions must be fullfilled:

$$\{ \theta^{0j}, H \} = \{ \pi^{0j}, H \} = - u^j = 0 , \tag{II.1.B.8}$$

$$\{ \theta^i, H \} = \{ \pi^i + F^{0i}, H \} = \partial_k F^{ki} + u^{0i} = 0 . \tag{II.1.B.9}$$

We must further require

$$\dot{g}_0 = \{ g_0, H \} = 0 , \tag{II.1.B.10}$$

which results in

$$0 = \partial_i u^{0i} = -\partial_i \partial_k F^{ki}.$$

This is trivially fullfilled in view of the antisymmetry of F^{ki}.

Equating the Poisson bracket of $g^{\ell m}$ with H with zero, and using (II.1.B.8) one finds $u^{\ell m}$ to be:

$$u^{\ell m} = -\partial^\ell \pi^m + \partial^m \pi^\ell . \qquad \text{(II.1.B.11)}$$

Thus the complete set of constraints is

$$\theta^{\ell m} = \pi^{\ell m}, \quad \theta^{0i} = \pi^{0i}, \quad \theta^0 = \pi^0,$$

$$\theta^i = \pi^i + F^{0i}, \quad g_0 = \partial_i F^{0i}, \quad g^{\ell m} = -\partial^\ell A^m + \partial^m A^\ell + F^{\ell m}. \qquad \text{(II.1.B.12)}$$

Out of these fourteen constraints, two of them (actually one is a linear combination of constraints) are found to be first-class

$$\theta^0 = \pi^0,$$

$$f^0 = \partial_i \theta^i - g^0 = \partial_i \pi^i . \qquad \text{(II.1.B.13)}$$

The first-class nature of θ^0 is trivially established. As for f^0, one finds:

$$\{f^0(\vec{x}), g^{\ell m}(\vec{y})\} = \partial^i_{\vec{x}} \{\pi_i(\vec{x}), -\partial^\ell_{\vec{y}} A^m(\vec{y}) + \partial^m_{\vec{y}} A^\ell(\vec{y})\} =$$

$$= \{\partial^i_{\vec{x}} \partial^\ell_{\vec{y}} \delta_i{}^m - \partial^i_{\vec{x}} \partial^m_{\vec{y}} \delta_i{}^\ell\} \delta(\vec{x}-\vec{y}) =$$

$$= \{\partial^m_{\vec{x}} \partial^\ell_{\vec{y}} - \partial^\ell_{\vec{x}} \partial^m_{\vec{y}}\} \delta(\vec{x}-\vec{y}) = 0 . \qquad \text{(II.1.B.14)}$$

The Poisson brackets of f^0 with other constraints are also zero.

The second-class constraints satisfy the following Poisson bracket algebra:

$$\{\theta^{\ell m}(\vec{x}), \theta^{0i}(\vec{y})\} = 0 ,$$

$$\{\theta^{\ell m}(\vec{x}), g_{rs}(\vec{y})\} = -(\delta^{\ell}_{r}\delta^{m}_{s} - \delta^{\ell}_{s}\delta^{m}_{r})\delta(\vec{x}-\vec{y}), \quad \text{(II.1.B.15)}$$

$$\{\theta^{\ell m}(\vec{x}), \theta^{j}(\vec{y})\} = 0 ,$$

$$\{\theta^{0i}(\vec{x}), \theta_{j}(\vec{y})\} = -\delta^{i}_{j}\,\delta(\vec{x}-\vec{y}) , \quad \text{(II.1.B.16)}$$

$$\{\theta^{0i}(\vec{x}), g^{\ell m}(\vec{y})\} = 0 ,$$

$$\{g^{\ell m}(\vec{x}), \theta_{i}(\vec{y})\} = -\{-\partial^{\ell}_{\vec{x}}\delta^{m}_{i} + \partial^{m}_{\vec{x}}\delta^{\ell}_{i}\}\delta(\vec{x}-\vec{y}), \quad \text{(II.1.B.17)}$$

We shall arrange the second-class constraints in the following way:

$$\theta_1 = \pi^{12}, \ \theta_2 = \pi^{23}, \ \theta_3 = \pi^{31}, \ \theta_4 = \pi^{01} \ldots \ \theta_6 = \pi^{03}$$

$$\theta_7 = F^{12} - \partial^1 A^2 + \partial^2 A_1 \ \ldots \ \theta_9 = F^{31} - \partial^3 A^1 + \partial^1 A^3$$

$$\theta_{10} = \pi^1 + F^{01} \ \ldots\ldots \ \theta_{12} = \pi^3 + F^{03} .$$

(II.1.B.18)

Eqs. (II.1.B.15) to (II.1.B.18) determine the matrix $\{\theta_a(\vec{x}), \theta_b(\vec{y})\}$:

$$
\{\theta_a(\vec{x}),\theta_b(\vec{y})\} = \left(\begin{array}{c|c}
0 & \begin{matrix} -\delta &&&&& \\ & -\delta &&&& \\ && -\delta &&& \\ &&& -\delta && \\ &&&& -\delta & \\ &&&&& -\delta \end{matrix} \\
\hline
\begin{matrix} \delta &&&&& \\ & \delta &&&& \\ && \delta &&& \\ &&& \delta && \\ &&&& \delta & \\ &&&&& \delta \end{matrix} &
\begin{array}{c|c} 0 & \begin{matrix} -\partial^2\delta & \partial^1\delta & 0 \\ 0 & -\partial^3\delta & \partial^2\delta \\ \partial^3\delta & 0 & -\partial^1\delta \end{matrix} \\ \hline \begin{matrix} \partial^2\delta & 0 & -\partial^3\delta \\ -\partial^1\delta & \partial^3\delta & 0 \\ 0 & -\partial^2\delta & \partial^1\delta \end{matrix} & 0 \end{array}
\end{array}\right) . \qquad \text{(II.1.B.19)}
$$

The determinant of the matrix is immediately found to be 1.

The next task consists in finding the Dirac brackets of the canonical variables. First, of course,

$$
\{\pi^{\mu\nu}(\vec{x}), X_b(\vec{y})\}^* = 0 , \qquad \text{(II.1.B.20)}
$$

where $X_b(\vec{y})$ are all the canonical variables.

Next, since A_o and π^o have zero Poisson brackets with all second-class constraints:

$$
\{A_0(\vec{x}), Z_b(\vec{y})\}^* = 0 ,
$$

$$
\{\pi_0(\vec{x}), Y_b(\vec{y})\}^* = 0 ,
$$

$$
\{A_0(\vec{x}), \pi_0(\vec{y})\}^* = \delta(\vec{x}-\vec{y}) , \qquad \text{(II.1.B.21)}
$$

where $Z_b(\vec{y})$ are all canonical variables except $\pi^o(\vec{y})$, and $Y_b(\vec{y})$ are all canonical variables except $A_o(\vec{y})$.

Proceeding as in the first example treated in this section, we find the rest of the Dirac brackets to be:

$$\{A_m(\vec{x}), \pi^i(\vec{y})\}^* = \delta_m{}^i\, \delta(\vec{x}-\vec{y}) \,,$$

$$\{A_m(\vec{x}), A_i(\vec{y})\}^* = 0 \quad , \tag{II.1.B.22}$$

$$\{A_m(\vec{x}), F^{oi}(\vec{y})\}^* = -\delta^i{}_m\, \delta(\vec{x}-\vec{y}) \,,$$

$$\{A_m(\vec{x}), F^{\ell k}(\vec{y})\}^* = 0 \,,$$

$$\{F_{om}(\vec{x}), F_{oj}(\vec{y})\}^* = \{\Pi_m(\vec{x}), \Pi_s(\vec{y})\}^* = -\{F_{om}(\vec{x}), \Pi_s(\vec{y})\}^* = 0 \,,$$

$$-\{F^{rs}(\vec{x}), F_{om}(\vec{y})\}^* = \{F^{rs}(\vec{x}), \Pi_m(\vec{y})\}^* =$$

$$= \{\partial^r_{\vec{x}}\, \delta^s_m - \partial^s_{\vec{x}}\, \delta^r_m\}\, \delta(\vec{x}-\vec{y}) \,,$$

$$\{F^{rs}(\vec{x}), F^{\ell m}(\vec{y})\}^* = 0 \,. \tag{II.1.B.23}$$

Standard quantization is then performed by replacing the Dirac brackets by " - i " times commutators.

We also wish to quantize the theory using the generalization of Faddeev's method described in Section I-3. We associate the following gauge conditions with the first-class constraints (II.1.B.13):

$$\chi_1 = A^0 \,, \quad \chi_2 = \partial_j A^j \,. \tag{II.1.B.24}$$

The Faddeev-Popov determinant is thus found to be:

$$\det \| \{\chi_a, \varphi_b\} \| = \det \vec{\nabla}^2 \,. \tag{II.1.B.25}$$

Therefore, in view of (III.19) and (III.20), the vacuum-vacuum transition amplitude for this theory is given by the expression

$$\langle 0|S|0\rangle = \int \det \vec{\nabla}^2 \prod_{\mu>\nu} \mathcal{D}\pi^{\mu\nu}\mathcal{D}F^{\mu\nu} \prod_{\mu} \mathcal{D}\pi^{\mu}\mathcal{D}A^{\mu} \prod_{x,\mu>\nu} \delta(\pi^{\mu\nu}) \cdot$$

$$\times \prod_{x,k} \delta(\pi^k + F^{0k}) \prod_{x} \delta(\pi^0) \prod_{x,l>m} \delta(F^{lm} - \partial^l A^m + \partial^m A^l) \prod_{x} \{\delta(\partial_i \pi^i) \cdot$$

$$\times \delta(A^0)\, \delta(\partial_j A^j)\} \exp\{i \int d^4x\, (\pi^{\mu\nu} \dot{F}_{\mu\nu} + \pi^0 \dot{A}_0 +$$

$$+ \pi^i \dot{A}_i - \pi^i \partial_i A_0 - \frac{1}{2}\pi^i \pi^i - \frac{1}{2} F^{ij}(\partial_i A_j - \partial_j A_i) + \frac{1}{4} F_{ij} F^{ij})\}.$$

(II.1.B.26)

After a trivial integration over A_0, $\pi^{\mu\nu}$ and π^{μ}, the following expression is produced:

$$\langle 0|S|0\rangle = \int \det \vec{\nabla}^2 \prod_{\mu>\nu} \mathcal{D}F^{\mu\nu} \prod_{i} \mathcal{D}A_i \prod_{x,l>m} \delta(F^{lm} -$$

$$- \partial^l A^m + \partial^m A^l) \prod_{x} \delta(\partial_i F^{0i})\, \delta(\partial_j A^j) \cdot$$

$$\times \exp\{i \int d^4x\, (\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} F^{0i} F_{0i} -$$

$$- \frac{1}{2} F^{ij}(\partial_i A_j - \partial_j A_i) - F^{0i}\, \partial_0 A_i)\}.$$

(II.1.B.27)

Writing

$$\prod_{x} \delta(\partial_i F^{0i}) = \int \mathcal{D}A_0 \exp\{i \int d^4x\, F^{0i} \partial_i A_0\} \qquad \text{(II.1.B.28)}$$

and imitating the derivation that led to Eq. (II.1.A.24) we can bring Eq. (II.1.B.27) into the form

$$< 0|S|0 > = \int \prod_{\mu>\nu} \mathcal{D} F_{\mu\nu} \prod_{\mu} \mathcal{D} A_\mu \prod_x \delta(\vec{\nabla}\cdot\vec{A}) \det \vec{\nabla}^2 \times$$

$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{2} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] \right\} \qquad \text{(II.1.B.29)}$$

Integrating further over the $F^{\mu\nu}$'s one establishes the expression (II.37). Thus our first-order theory is seen to be equivalent to the field theory of the free electromagnetic field. The inclusion of interactions is seen to be rather straightforward.

II.2. Quantization of the first-order formulation of the Yang-Mills field theory

In analogy with the case of the free vector field, the first-order Lagrangian for a Yang-Mills field theory is

$$\mathcal{L} = \frac{1}{4} F^{\alpha}_{\mu\nu} F^{\mu\nu}_{\alpha} - \frac{1}{2} F^{\alpha}_{\mu\nu} (\partial^{\mu} A^{\nu}_{\alpha} - \partial^{\nu} A^{\mu}_{\alpha} + g f_{\alpha\beta\gamma} A^{\mu}_{\beta} A^{\nu}_{\gamma}) . \quad \text{(II.2.1)}$$

The transition to the Hamiltonian formulation starts with finding the conjugate momenta:

$$\begin{aligned} \pi^{\mu\nu}_{\alpha} &= \frac{\partial \mathcal{L}}{\partial \dot{F}^{\alpha}_{\mu\nu}} = 0 , \\ \pi^{0}_{\alpha} &= \frac{\partial \mathcal{L}}{\partial \dot{A}^{\alpha}_{0}} = 0 , \\ \pi^{i}_{\alpha} &= \frac{\partial \mathcal{L}}{\partial \dot{A}^{\alpha}_{i}} = - F^{0i}_{\alpha} , \end{aligned} \quad \text{(II.2.2)}$$

which results in the primary constraints

$$\begin{aligned} \theta^{\mu\nu}_{\alpha} &\equiv \pi^{\mu\nu}_{\alpha} = 0 , \\ \theta^{0}_{\alpha} &\equiv \pi^{0}_{\alpha} = 0 , \end{aligned} \quad \text{(II.2.3)}$$

$$\theta^{i}_{\alpha} \equiv \pi^{i}_{\alpha} + F^{0i}_{\alpha} = 0 . \quad \text{(II.2.4)}$$

The Hamiltonian density is determined by the standard procedure described in Section I. It is given by the following expression:

$$\begin{aligned} \mathcal{H} = & - \frac{1}{2} \pi^{i}_{\alpha} \pi^{\alpha}_{i} - \frac{1}{4} F^{\alpha}_{ij} F^{ij}_{\alpha} + \pi^{\alpha}_{i} \partial^{i} A^{0}_{\alpha} + \frac{1}{2} F^{\alpha}_{\ell m} \times \\ & \times (\partial^{\ell} A^{m}_{\alpha} - \partial^{m} A^{\ell}_{\alpha}) + \frac{1}{2} g F^{\alpha}_{ij} f_{\alpha\beta\gamma} A^{i}_{\beta} A^{j}_{\gamma} - g \pi^{\alpha}_{i} f_{\alpha\beta\gamma} A^{0}_{\beta} A^{i}_{\gamma} \\ & + \frac{1}{2} u^{\alpha}_{\mu\nu} \pi^{\mu\nu}_{\alpha} + u^{\alpha}_{0} \pi^{0}_{\alpha} + u^{\alpha}_{i} (\pi^{i}_{\alpha} + F^{0i}_{\alpha}) , \end{aligned} \quad \text{(II.2.5)}$$

where the u's are the arbitrary (at this stage) multipliers.

As before, the next step consists in equating to zero the time derivations of the hitherto established constraints in order to obtain the secondary constraints:

$$\dot{\pi}^{lm}_{\alpha} = \{\pi^{lm}_{\alpha}, H\} = F^{lm}_{\alpha} - \partial^l A^m_{\alpha} + \partial^m A^l_{\alpha} - \\ - g f_{\alpha\beta\gamma} A^l_{\beta} A^m_{\gamma} \equiv g^{lm}_{\alpha} = 0 \quad , \tag{II.2.6}$$

$$\dot{\pi}^{0i}_{\alpha} = \{\pi^{0i}_{\alpha}, H\} = - u^i_{\alpha} = 0 \quad , \tag{II.2.7}$$

$$\dot{g}^{lm}_{\alpha} = \{g^{lm}_{\alpha}, H\} = u^{lm}_{\alpha} + \partial^l \pi^m_{\alpha} - \partial^m \pi^l_{\alpha} - \\ - g f_{\alpha\beta\gamma}(A^l_{\beta}\pi^m_{\gamma} - \pi^l_{\gamma} A^m_{\beta}) - g f_{\alpha\beta\gamma} A^0_{\beta} F^{lm}_{\gamma} = 0 \, . \tag{II.2.8}$$

To prove (II.2.8), one uses (II.2.7) and the Jacobi identity for the structure constants of the compact semisimple Lie algebra corresponding to the global group of invariance of the Yang-Mills field theory:

$$f_{\alpha\beta\gamma} f_{\gamma\varepsilon\eta} + f_{\alpha\gamma\eta} f_{\gamma\varepsilon\beta} + f_{\beta\eta\gamma} f_{\gamma\varepsilon\alpha} = 0 \, . \tag{II.2.9}$$

The calculation leading to (II.2 . 8) is lengthy but straightforward, and we choose to omit it. The same is true of the calculation leading to:

$$\dot{\theta}^i_{\delta} = \{\theta^i_{\delta}, H\} = u^{0i}_{\delta} + \partial_l F^{li}_{\delta} + g \pi^i_{\alpha} f_{\alpha\beta\delta} A^0_{\beta} - \\ - g F^{ij}_{\alpha} f_{\alpha\delta\gamma} A^{\gamma}_j = 0 \tag{II.2.10}$$

Equations (II.2.7), (II.2.8) and (II.2.10) simply determine the hitherto unknown multipliers and produce no new constraints. Next, we find

$$\dot{\pi}^0_\alpha = \{\pi^0_\alpha, H\} = \partial^i \pi^\alpha_i + g\, \pi^\delta_i f_{\delta\alpha\gamma} A^i_\gamma \equiv g^\alpha_0 = 0 \tag{II.2.11}$$

It is then a cumbersome but straightforward exercise to prove, using (II.2.9), (II.2.10) and (II.2.11), that

$$\dot{g}^\alpha_0 = \{g^\alpha_0, H\} = 0 \tag{II.2.12}$$

is satisfied identically in the subspace M in which constraints hold.

Our next objective is to determine the Poisson bracket algebra of all the constraints. We start with

$$\{g^\alpha_0(\vec{x}), -g^{\ell m}_\beta(\vec{y})\} = \{(\partial^i \pi^\alpha_i + g\, \pi^\delta_i f_{\delta\alpha\gamma} A^i_\gamma)_{\vec{x}},$$
$$(\partial^\ell A^m_\beta - \partial^m A^\ell_\beta + g\, f_{\beta\eta\gamma} A^0_\eta A^m_\gamma)_{\vec{y}}\} \tag{II.2.13}$$

After some calculation, using again (II.2.9), one can show that

$$\{g^\alpha_0(\vec{x}), -g^{\ell m}_\beta(\vec{y})\}\Big|_M = -g\, f_{\beta\alpha\gamma} F^{\ell m}_\gamma(\vec{x})\, \delta(\vec{x}-\vec{y}) \tag{II.2.14}$$

This leads us to consider the following linear combination of the constraints:

$$h^\alpha_0 = g^\alpha_0 - \frac{1}{2} g\, f_{\alpha\beta\gamma} \pi^{rs}_\beta F^\gamma_{rs} \tag{II.2.15}$$

since

$$\{h^\alpha_0(\vec{x}), g^{\ell m}_\beta(\vec{y})\}\Big|_M = g\, f_{\alpha\beta\gamma} F^{\ell m}_\gamma + g\, f_{\beta\alpha\gamma} F^{\ell m}_\gamma = 0 \tag{II.2.16}$$

and we wish to put as many constraints as possible in the first-class in accordance with Dirac's prescription. If we define

$$k_0^\alpha = h_0^\alpha + g f_{\alpha\beta\gamma} \pi^{0i}_\beta F^\gamma_{0i} \tag{II.2.17}$$

we obtain, in view of (II.2.3)

$$\{\theta^j_\beta(\vec{x}), k_0^\alpha(\vec{y})\}\Big|_M = g f_{\alpha\beta\gamma} \{\pi^j_\gamma(\vec{x}) + F^{0j}_\gamma(\vec{x})\}\Big|_M \delta(\vec{x}-\vec{y}) = 0. \tag{II.2.18}$$

One can easily check that the remaining Poisson brackets of k_α^0 with other constraints are zero, therefore, $k_\alpha^0(\vec{y})$ are first-class constraints. The same is true of $\theta_\alpha^0 = \pi_\alpha^0$.

The following constraints are second-class:

$$\theta^{\mu\nu}_\alpha = \pi^{\mu\nu}_\alpha ,$$

$$\theta^i_\alpha = \pi^i_\alpha + F^{0i}_\alpha ,$$

$$g^{lm}_\alpha = F^{lm}_\alpha - \partial^l A^m_\alpha + \partial^m A^l_\alpha - g f_{\alpha\beta\gamma} A^l_\beta A^m_\gamma , \tag{II.2.19}$$

in full analogy with the case of the free massless vector field. We shall display the only nontrivial Poisson bracket of constraints:

$$\{\theta^i_\delta(\vec{x}), g^{lm}_\alpha(\vec{y})\} = [\delta_{\alpha\delta}(\partial^l_{\vec{x}} \delta_i^{\ m} - \partial^m_{\vec{x}} \delta_i^{\ l}) -$$
$$- g f_{\alpha\delta\gamma}(\delta_i^{\ l} A^m_\gamma(\vec{x}) - \delta_i^{\ m} A^l_\gamma(\vec{x}))] \delta(\vec{x}-\vec{y}) . \tag{II.2.20}$$

Introducing the shorthand notation

$$(A^{\ell}_{F})_{\alpha\delta} = g f_{\alpha\delta\gamma} A^{\ell}_{\gamma}$$

$$(1)_{\alpha\delta} = \delta_{\alpha\delta}$$

$$\partial^{\ell}_{F} = 1\,\partial^{\ell} + A^{\ell}_{F} \quad , \tag{II.2.21}$$

we find for the matrix $\{ \theta_a(\vec{x}) , \theta_b(\vec{y}) \}$

$$\{\theta_a(\vec{x}),\theta_b(\vec{y})\} = \left[\begin{array}{cccccc|cccccc}
0&0&0&0&0&0 & -1\delta&0&0&0&0&0\\
0&0&0&0&0&0 & 0&-1\delta&0&0&0&0\\
0&0&0&0&0&0 & 0&0&-1\delta&0&0&0\\
0&0&0&0&0&0 & 0&0&0&-1\delta&0&0\\
0&0&0&0&0&0 & 0&0&0&0&-1\delta&0\\
0&0&0&0&0&0 & 0&0&0&0&0&-1\delta\\
\hline
1\delta&0&0&0&0&0 & 0&0&0&-\partial^2_F\delta&\partial^1_F\delta&0\\
0&1\delta&0&0&0&0 & 0&0&0&0&-\partial^3_F\delta&\partial^2_F\delta\\
0&0&1\delta&0&0&0 & 0&0&0&\partial^3_F\delta&0&-\partial^1_F\delta\\
0&0&0&1\delta&0&0 & \partial^2_F\delta&0&-\partial^3_F\delta&0&0&0\\
0&0&0&0&1\delta&0 & -\partial^1_F\delta&\partial^3_F\delta&0&0&0&0\\
0&0&0&0&0&1\delta & 0&-\partial^2_F\delta&\partial^1_F\delta&0&0&0
\end{array}\right] . \tag{II.2.22}$$

As in the previous section, the nonzero Dirac brackets of fundamental variables are:

$$\{ A^{\alpha}_{0}(\vec{x}), \pi^{0}_{\beta}(\vec{y})\}^{*} = \delta^{\alpha}_{\beta}\, \delta(\vec{x}-\vec{y}) \quad ,$$

$$\{A^\alpha_m(\vec{x}), \pi^i_\beta(\vec{y})\}^* = \delta^\alpha_\beta\, \delta_m^{\;i}\, \delta(\vec{x}-\vec{y}) ,$$

$$\{A^\alpha_m(\vec{x}), F^{0i}_\beta(\vec{y})\}^* = -\delta^{\;i}_m\, \delta^\alpha_\beta\, \delta(\vec{x}-\vec{y}) ,$$

$$-\{F^{\gamma s}_\alpha(\vec{x}), F^\beta_{0m}(\vec{y})\}^* = \{F^{\gamma s}_\alpha(\vec{x}), \pi^\beta_m(\vec{y})\}^* =$$

$$= [(\partial^\gamma_F)^{\vec{x}}_{\alpha\beta}\, \delta^s_{\;m} - (\partial^s_F)^{\vec{x}}_{\alpha\beta}\, \delta^\gamma_{\;m}]\, \delta(\vec{x}-\vec{y}) . \tag{II.2.23}$$

Standard quantization is performed by replacing Dirac brackets by " - i " times the corresponding commutator.

To quantize the theory by means of the path integral method, we need to specify the gauge conditions corresponding to the first-class $k_o^{\,a}$ and $\theta_a^{\,o}$ (see (II.23) and (II.2.17)). We choose the gauge conditions

$$\begin{aligned} \chi^0_\alpha &= A^0_\alpha , \\ \chi_\alpha &= \partial_i A^i_\alpha . \end{aligned} \tag{II.2.24}$$

The Faddeev–Popov determinant $|\det \| \{\chi_a , \varphi_b\} \|$ is therefore:

$$\left|\det \| \{\chi_a, \varphi_b\} \|\right| = |\det(\vec{\nabla}^2 \delta_{\alpha\beta} + g f_{\alpha\beta\gamma} A^i_\gamma\, \partial_i)| . \tag{II.2.25}$$

Noting that the determinant $\det \{\theta_a(\vec{x}), \theta_b(\vec{y})\}$ is equal to 1 in this case, as follows directly from (II.2.22), we can write for the vacuum - vacuum transition amplitude:

$$\langle 0 | S | 0 \rangle = \int \prod_{\alpha, \mu > \nu} \mathcal{D}\pi^{\mu\nu}_\alpha\, \mathcal{D}F^{\mu\nu}_\alpha \prod_\alpha \mathcal{D}A^\alpha_0\, \mathcal{D}\pi^\alpha_0 \;\times$$

$$\times \prod_{\alpha,i} \partial A_i^\alpha \partial \pi_i^\alpha \prod_{x,\alpha,\mu>\nu} \delta(\pi^{\mu\nu}_\alpha) \prod_{x,\alpha} \delta(\pi^0_\alpha) \prod_{x,\alpha,i} \delta(\pi^i_\alpha + F^{0i}_\alpha) \times$$

$$\times \prod_{x,\alpha,\ell>m} \delta(F_\alpha^{\ell m} - \partial^\ell A^m_\alpha + \partial^m A^\ell_\alpha - g f_{\alpha\beta\gamma} A^\ell_\beta A^m_\gamma) \prod_{x,\alpha} \{ \delta(\partial^i \pi_i^\alpha +$$

$$+ g \pi_i^\delta f_{\delta\alpha\gamma} A^i_\gamma) \delta(A^0_\alpha) \delta(\partial^i A_i^\alpha) \} \times |det(\vec{\nabla}^2 \delta_{\alpha\beta} +$$

$$+ g f_{\alpha\beta\gamma} A^i_\gamma \partial_i)| \exp \{ i \int d^4x [\pi^i_\alpha \dot{A}_i^\alpha +$$

$$+ \frac{1}{2} \pi^i_\alpha \pi_i^\alpha - \frac{1}{2} F_\alpha^{\ell m} (\partial_\ell A_m^\alpha - \partial_m A_\ell^\alpha + g f_{\alpha\beta\gamma} A_\ell^\beta A_m^\gamma) +$$

$$+ \frac{1}{4} F^\alpha_{\ell m} F^{\ell m}_\alpha] \} . \tag{II.2.26}$$

Upon trivial integration over some of the variables we obtain:

$$\langle 0 | S | 0 \rangle = \int \prod_{x,\alpha,\ell>m} \delta(F_\alpha^{\ell m} - \partial^\ell A^m_\alpha + \partial^m A^\ell_\alpha - g f_{\alpha\beta\gamma} A^\ell_\beta A^m_\gamma) \times$$

$$\times \prod_{x,\alpha} \{ \delta(\partial^i F^\alpha_{0i} + g F^\delta_{0i} f_{\delta\alpha\gamma} A^i_\gamma) \delta(\partial_i A^i_\alpha) \} | det(\vec{\nabla}^2 \delta_{\alpha\beta} +$$

$$+ g f_{\alpha\beta\gamma} A^i_\gamma \partial_i)| \prod_{\alpha,\ell>m} \partial F^{\ell m}_\alpha \prod_{\alpha,i} \partial F^{0i}_\alpha \partial A_i^\alpha \exp\{ i \int d^4x [-F^{0i}_\alpha \dot{A}_i^\alpha$$

$$+ \frac{1}{2} F^{0i}_\alpha F^\alpha_{0i} - \frac{1}{2} F^{\ell m}_\alpha (\partial_\ell A^\alpha_m - \partial_m A^\alpha_\ell + g f_{\alpha\beta\gamma} A^\beta_\ell A^\gamma_m) + \frac{1}{4} F^\alpha_{\ell m} F^{\ell m}_\alpha] \} . \tag{II.2.27}$$

Writing

$$\prod_{x,\alpha} \delta(\partial^i F^\alpha_{0i} + g F^\delta_{0i} f_{\delta\alpha\gamma} A^i_\gamma) = \int \prod_\alpha \partial A^\alpha_0 \exp\{ i \int (F^\alpha_{0i} \partial^i A^0_\alpha - g A^\alpha_0 F^\delta_{0i} f_{\delta\alpha\gamma} A^i_\gamma) d^4x \} \tag{II.2.28}$$

and imitating again the derivation that led to Eq. (II.1.A.24) we can easily obtain

$$
\langle 0|S|0\rangle = \int \prod_{\alpha,\mu>\nu} \mathcal{D}F^{\alpha}_{\mu\nu} \prod_{\alpha,\mu} \mathcal{D}A^{\alpha}_{\mu} \prod_{x,\alpha} \delta(\partial_i A^i_\alpha) \times \left|\det(\vec{\nabla}^2\delta_{\alpha\beta} + g f_{\alpha\beta\gamma} A^i_\gamma \partial_i)\right| \exp\Big\{ i\int\Big[-\tfrac{1}{2}F^{\alpha}_{\mu\nu}(\partial^\mu A^\nu_\alpha - \partial^\nu A^\mu_\alpha + g f_{\alpha\beta\gamma} A^\mu_\beta A^\nu_\gamma) + \tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha\Big] d^4x \Big\}, \tag{II.2.29}
$$

which completes our path integral quantization scheme for the first-order formulation of the Yang-Mills field theory. A further integration over $F^{\alpha}_{\mu\nu}$ establishes the equivalence of this formulation with the standard one [5].

We shall close this section with the analysis of the perturbation expansion for the self-interacting Yang-Mills field in the first-order formulation.

In view of (II.2.29) the generating functional of Green's functions is given by

$$
Z[J^\alpha_\mu, I^\beta_{\lambda\nu}] = \int \prod_{\alpha,\mu>\nu} \mathcal{D}F^\alpha_{\mu\nu} \prod_{\beta,\lambda} \mathcal{D}A^\beta_\lambda \prod_{x,\gamma} \delta(\partial_\mu A^\mu_\gamma) \left|\det[\Box\delta_{\alpha\beta} + g f_{\alpha\beta\gamma} A^\mu_\gamma \partial_\mu]\right| \times \exp\Big\{ i\int\Big[-\tfrac{1}{2}F^\alpha_{\mu\nu}(\partial^\mu A^\nu_\alpha - \partial^\nu A^\mu_\alpha + g f_{\alpha\beta\gamma} A^\mu_\beta A^\nu_\gamma) + \tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha + \tfrac{1}{2}F^\alpha_{\mu\nu} I^{\mu\nu}_\alpha + A^\alpha_\mu J^\mu_\alpha\Big] d^4x\Big\}, \tag{II.2.30}
$$

where we have made a transition to the covariant gauge [5] $\partial_\mu A^\mu_\gamma = 0$.

We now note that

$$
\left|\det(\Box\delta_{\alpha\beta} + g f_{\alpha\beta\gamma} A^\mu_\gamma \partial_\mu)\right| = \int \prod_\alpha \mathcal{D}\Psi_\alpha \mathcal{D}\bar{\Psi}_\alpha \exp\Big\{ i\int \bar{\Psi}_\alpha(-\Box\delta_{\alpha\beta} - g f_{\alpha\beta\gamma} A^\mu_\gamma \partial_\mu)\Psi_\beta \, d^4x\Big\}. \tag{II.2.31}
$$

In (II.2.31) ψ_a and $\overline{\psi}_a$ are anticommuting c-number scalar fields [8]. They represent the famous Faddeev-Popov ghost [5].

The term $-\frac{1}{2} g\, F^a_{\mu\nu}\, f_{\alpha\beta\gamma}\, A^\mu_\beta\, A^\nu_\gamma$ can be treated perturbatively in the standard fashion. The same can be said of the term $-\overline{\psi}_a\, g\, f_{\alpha\beta\gamma}\, A^\mu_\gamma\, \partial_\mu \psi_\beta$ if we add the source terms $\overline{\rho}_a\, \psi_a$ and $\overline{\psi}_a\, \rho_a$ to the action displayed in (II.2.31). Once the interaction terms are represented in terms of appropriate functional derivatives, we can integrate over $\overline{\psi}_a$, ψ_a, $F^a_{\mu\nu}$ and A^μ_a. First, we perform the integration over $\overline{\psi}_a$ and ψ_a with the result

$$Z'(\overline{\rho}_\alpha, \rho_\alpha) = \exp\left\{-i \int \overline{\rho}_\alpha(x)\, D_F(x-y)\, \rho_\alpha(y)\, d^4x\, d^4y \right\}, \qquad \text{(II.2.32)}$$

where $D_F(x-y)$ is the massless scalar propagator

$$D_F(x-y) = \frac{1}{(2\pi)^4} \int \frac{e^{ik\cdot(x-y)}}{k^2+i\epsilon}\, d^4k\,. \qquad \text{(II.2.33)}$$

In the second step we integrate over $F^{\mu\nu}_a$ with the result:

$$Z^0(J^\alpha_\mu, I^\beta_{\lambda\nu}, \overline{\rho}_\alpha, \rho_\alpha) = \int \prod_{\alpha,\lambda} \mathcal{D} A^\alpha_\lambda \prod_{x,\alpha} (\partial_\mu A^\mu_\alpha) \times$$

$$\times \exp\left\{ i \int \left[-\frac{1}{4} (\partial^\mu A^\nu_\alpha - \partial^\nu A^\mu_\alpha - I^{\mu\nu}_\alpha)(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - I^\alpha_{\mu\nu}) + \right.\right.$$

$$\left.\left. + A^\alpha_\mu J^\mu_\alpha \right] d^4x \right\} Z'(\overline{\rho}_\alpha, \rho_\alpha). \qquad \text{(II.2.34)}$$

At this point we are ready to integrate over A^a_λ. We find:

$$Z^0(J^\alpha_\mu, I^\beta_{\lambda\nu}, \overline{\rho}_\alpha, \rho_\alpha) = \exp\left\{-\frac{i}{4} \int d^4x\, I^{\mu\nu}_\alpha(x)\, I^\alpha_{\mu\nu}(x)\right\} \times$$

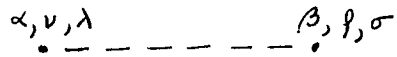
$$\times \exp\left\{-i \int \overline{\rho}_\alpha(x)\, D_F(x-y)\, \rho_\alpha(y)\, d^4x\, d^4y\right\} \times \exp\left\{\frac{i}{2} \int d^4x \,\times\right.$$

$$\left. \times\, d^4y\, (J^\alpha_\lambda - \partial^\nu I^\alpha_{\nu\lambda})_x\, D^{\lambda\sigma}_F(x-y)\, (J^\alpha_\sigma - \partial^\rho I^\alpha_{\rho\sigma})_y \right\}, \qquad \text{(II.2.35)}$$

where

$$D_F^{\lambda\sigma}(x-y) = \left(g^{\lambda\sigma} - \frac{\partial^\lambda \partial^\sigma}{\Box}\right) D_F(x-y) \tag{II.2.36}$$

One can therefore establish the following Feynman rules (we ignore the kinematic factors in external lines):

F-propagator 

$$\frac{i}{(2\pi)^4}\, \delta_{\alpha\beta} \left(G_{\nu\rho}\, G_{\lambda\sigma} - G_{\nu\sigma}\, G_{\lambda\rho} \right) , \tag{II.2.37}$$

where

$$G_{\nu\rho} = g_{\nu\rho} - \frac{k_\nu k_\rho}{k^2 + i\epsilon} \quad ;$$

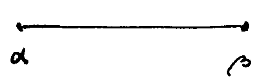
A-propagator

$$\frac{-i}{(2\pi)^4}\, \frac{g_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2}}{k^2 + i\epsilon}\, \delta_{\alpha\beta} \quad ; \tag{II.2.38}$$

mixed A-F propagator 

$$\delta_{\alpha\beta}\, \frac{1}{(2\pi)^4}\, \frac{g_{\lambda\sigma} - \frac{k_\lambda k_\sigma}{k^2}}{k^2 + i\epsilon}\, k_\nu - (\nu \leftrightarrow \lambda) \quad ; \tag{II.2.39}$$

scalar ghost propagator



$$\frac{1}{(2\pi)^4}\, \delta_{\alpha\beta}\, \frac{i}{k^2 + i\epsilon} \quad ; \tag{II.2.40}$$

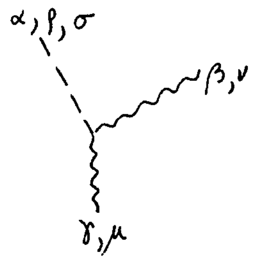
mixed A-F external line 

$$\left[i\, \epsilon_\lambda^{(i)}\, k_\nu - i\, \epsilon_\nu^{(i)}\, k_\lambda \right] \delta_{\alpha\beta} \quad ; \tag{II.2.41}$$

A-external line

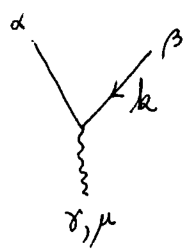
$$\epsilon^{(i)}_{\sigma}\,\delta_{\alpha\beta} \quad ; \tag{II.2.42}$$

F-A-A vertex



$$\frac{1}{2}\, g\, f_{\alpha\beta\gamma}\, g_{\rho\nu}\, g_{\sigma\mu} ; \tag{II.2.43}$$

S-S-A vertex



$$i\, g\, f_{\alpha\beta\gamma}\, k_{\mu} . \tag{II.2.44}$$

II.3. Quantization of the massive Yang-Mills field

We shall devote this section to the study of the quantization of the massive Yang-Mills field as another example of a field theory containing second-class constraints. For an account of a different quantization scheme corresponding to the same problem, the reader is referred to the paper of Finkelstein, Kwitky and Mouton [9].

The formalism is based on the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^{\alpha}_{\mu\nu} F^{\mu\nu}_{\alpha} + \frac{1}{2} M^2 A^{\mu}_{\alpha} A^{\alpha}_{\mu} \quad . \tag{II.3.1}$$

In (II.3.1) $F^{\mu\nu}_{a}$ is given by the formula

$$F^{\mu\nu}_{\alpha} = \partial^{\mu} A^{\nu}_{\alpha} - \partial^{\nu} A^{\mu}_{\alpha} + g f_{\alpha\beta\gamma} A^{\mu}_{\beta} A^{\nu}_{\gamma} \quad . \tag{II.3.2}$$

As usual, one begins by calculating the conjugate momenta:

$$\pi^{0}_{\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{A}^{\alpha}_{0}} \quad , \tag{II.3.3}$$

$$\pi^{i}_{\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{A}^{\alpha}_{i}} = - F^{0i}_{\alpha} \quad . \tag{II.3.4}$$

By inspecting (II.3.2), (II.3.3) and (II.3.4) one concludes that the only primary constraint is given by (II.3.3) . To calculate the secondary constraints one needs the expression for the Hamiltonian density, which is obtained in a straightforward manner.

$$\mathcal{H} = \frac{1}{2} \pi^{\alpha}_{i} \pi^{\alpha}_{i} - \pi^{\alpha}_{i} \partial_{i} A^{0}_{\alpha} - g f^{\alpha\beta\gamma} \pi^{i}_{\alpha} A^{\beta}_{0} A^{\gamma}_{i} -$$
$$- \frac{1}{2} M^2 A^{0}_{\alpha} A^{0}_{\alpha} + \frac{1}{2} M^2 A^{i}_{\alpha} A^{i}_{\alpha} + \frac{1}{4} F^{\alpha}_{\ell m} F^{\ell m}_{\alpha} + u^{0}_{\alpha} \pi^{0}_{\alpha} \quad . \tag{II.3.5}$$

The secondary constraints are then

$$\varphi^{\alpha} = \{\Pi_0^{\alpha}, H\} = \partial_i \Pi_{\alpha}^i - g f^{\alpha\beta\gamma} \pi_{\beta}^i A_i^{\gamma} + M^2 A_{\alpha}^0 . \tag{II.3.6}$$

Imposing the consistency condition $\dot{\varphi}^{a} = \{\varphi^{a}, H\} = 0$ leads to

$$M^2 u_{\alpha}^0 + \{\partial_i \Pi_{\alpha}^i - g f^{\alpha\beta\gamma} \pi_{\beta}^i A_i^{\gamma}, H\} = 0 , \tag{II.3.7}$$

which merely determines u_{a}^{o} and thus does not produce any new constraints.

The algebra of Poisson brackets of all the constraints is found to be:

$$\{\pi_{\alpha}^0(\vec{x}), \pi_{\beta}^0(\vec{y})\} = 0 , \tag{II.3.8}$$

$$\{\pi_{\alpha}^0(\vec{x}), \varphi^{\beta}(\vec{y})\} = -M^2 \delta_{\alpha\beta}\, \delta(\vec{x}-\vec{y}) , \tag{II.3.9}$$

$$\{\varphi^{\alpha}(\vec{x}), \varphi^{\beta}(\vec{y})\} = g f^{\alpha\beta\gamma} \varphi^{\gamma}(\vec{x})\, \delta(\vec{x}-\vec{y}) . \tag{II.3.10}$$

The proof of (II.3.8) and (II.3.9) is trivial. To prove (II.3.10), one uses the Jacobi identity and the antisymmetry of the structure constraints of the compact semisimple Lie algebra corresponding to our Yang-Mills field theory. One can thus convince oneself that all the constraints are second-class. The characteristic weight in the phase space functional integral is

$$\left| \det \| \{\theta_a, \theta_b\} \| \right|^{1/2} = \prod_{\vec{x}, t, \alpha} (M^2) = \det M^2 . \tag{II.3.11}$$

The general discussion of Section III (Eqs.(III.19) and (III.20) in particular) then leads to the expression for the vacuum-vacuum S-matrix element:

$$\begin{aligned} \langle 0|S|0\rangle = \int \prod_{\alpha,i} \mathcal{D}A_i^{\alpha} \mathcal{D}\pi_{\alpha}^i \prod_{\alpha} \mathcal{D}A_{\alpha}^0 \mathcal{D}\Pi_{\alpha}^0 \det M^2 \prod_{x,\alpha} \{\delta(\Pi_{\alpha}^0) \\ \delta(\partial_i \pi_{\alpha}^i - g f^{\alpha\beta\gamma} \pi_{\beta}^i A_i^{\gamma} + M^2 A_{\alpha}^0)\} \exp\{i \int [\pi_0^{\alpha} \dot{A}_0^{\alpha} + \pi_{\alpha}^i \dot{A}_i^{\alpha} - \\ - \tfrac{1}{2} \pi_{\alpha}^i \Pi_{\alpha}^i - \pi_{\alpha}^i (\partial_i A_{\alpha}^0 - g f^{\alpha\beta\gamma} A_0^{\beta} A_i^{\gamma}) + \tfrac{1}{2} M^2 A_{\alpha}^{\mu} A_{\mu}^{\alpha} - \tfrac{1}{4} F_{\ell m}^{\alpha} F_{\alpha}^{\ell m}] d^4x\} . \end{aligned} \tag{II.3.12}$$

This can obviously be written as

$$< 0|S|0 > = \int \prod_{\alpha,i} \mathcal{D}A_i^\alpha \mathcal{D}\pi_\alpha^i \prod_\alpha \mathcal{D}\lambda_\alpha \mathcal{D}A_\alpha^0 \det M^2 \times$$

$$\times \exp\{i \int [\pi_\alpha^i (F_{0i}^\alpha - \partial_i \lambda_\alpha + g f^{\alpha\beta\gamma} \lambda^\beta A_i^\gamma) + M^2 \lambda_\alpha A_\alpha^0 -$$

$$- \frac{1}{2} \pi_\alpha^i \pi_\alpha^i + \frac{1}{2} M^2 A_\alpha^0 A_\alpha^0 - \frac{1}{2} M^2 A_\alpha^i A_\alpha^i - \frac{1}{4} F_{\ell m}^\alpha F_\alpha^{\ell m}] d^4x \} . \quad \text{(II.3.13)}$$

After a change of variables

$$A_\alpha^0 \rightarrow A_\alpha^0 - \lambda_\alpha \quad \text{(II.3.14)}$$

and a Gaussian integration over λ_a and π_a^i, we can establish the following result:

$$< 0|S|0 > = \int \prod_\alpha \mathcal{D}A_\alpha^0 \prod_{\alpha,i} \mathcal{D}A_i^\alpha \det M \exp\{i \int [\frac{1}{2} F_{0i}^\alpha F_{0i}^\alpha -$$

$$- \frac{1}{4} F_{\ell m}^\alpha F_\alpha^{\ell m} + \frac{1}{2} M^2 A_\alpha^0 A_\alpha^0 - \frac{1}{2} M^2 A_\alpha^i A_\alpha^i] d^4x \} . \quad \text{(II.3.15)}$$

The expression in the exponential is just the Lagrangian (II.3.1).

The result (II.3.15) is the basic result of this section. It shows that S-matrix elements (note that (II.3.15) can be generalized to arbitrary matrix elements by inserting appropriate wave functionals corresponding to the initial and final state or else appropriate boundary conditions in the functional integral) are expressible as functional integrals solely over the basic fields of the theory. The characteristic weight of such integrals is $\det M = \prod_{\vec{x},t,a} (M)$. A proper generalization of (II.3.15) to arbitrary S-matrix elements serves as a basis for developing perturbation theory in the path integral formalism and finding out the Feynman rules. We shall not dwell on this, since the Feynman rules have already been found by Finkelstein, Kwitky and Mouton [9]. We merely note that, since the functional measure in (II.3.15) is independent of the field variables, there will be no modification of the simple Feynman rules due to a nontrivial functional measure. We have thus rederived the basic result of Finkelstein, Kwitky and Mouton in a more economic fashion.

The reduction to independent variables is particularly simple in this case. Integrating over A^0_a and π^0_a, one can write (II.3.12) as

$$
\begin{aligned}
\langle 0 | S | 0 \rangle = \int \prod_{\alpha, i} \mathcal{D} A^\alpha_i \mathcal{D} \pi^i_\alpha \; \times \\
\times \exp \Big\{ i \int \Big[\pi^i_\alpha \dot{A}^\alpha_i - \tfrac{1}{2} \pi^i_\alpha \pi^i_\alpha - \tfrac{1}{2M^2} (\partial_i \pi^i_\alpha - g f^{\alpha\beta\gamma} \pi^i_\beta A^\gamma_i) \times \\
\times (\partial_j \pi^j_\alpha - g f^{\alpha\delta\varepsilon} \pi^k_\delta A^\varepsilon_k) - \tfrac{1}{2} M^2 A^i_\alpha A^i_\alpha - \tfrac{1}{4} F^\alpha_{\ell m} F^{\ell m}_\alpha \Big] d^4 x \Big\} .
\end{aligned}
\tag{II.3.16}
$$

The weight of integration is one, as expected from our general discussion in Section III.

It is interesting to determine the Dirac brackets for our canonical variables. One finds first

$$
\| C_{ab} (\vec{x}, \vec{y}) \| = \| \{ \theta_a (\vec{x}), \theta_b (\vec{y}) \} \|^{-1} =
$$

$$
= \left\| \begin{array}{c|c} \frac{1}{M^4} \Phi(\vec{x}) \delta(\vec{x} - \vec{y}) & \frac{1}{M^2} \delta(\vec{x} - \vec{y}) \\ \hline - \frac{1}{M^2} \delta(\vec{x} - \vec{y}) & 0 \end{array} \right\| . \tag{II.3.17}
$$

Here,

$$
\Phi_{\alpha\beta} (\vec{x}) = g f_{\alpha\beta\gamma} \varphi_\gamma (\vec{x}) . \tag{II.3.18}
$$

Let us introduce the notation

$$
a = (\tilde{a}, \bar{a}) , \tag{II.3.19}
$$

so that

$$C_{\bar{a}\bar{b}}(\vec{x},\vec{y}) = 0 \quad ,$$

$$C_{\underset{(\alpha)}{\tilde{a}}\underset{(\beta)}{\tilde{b}}}(\vec{x},\vec{y}) = \frac{g}{M^4} f_{\alpha\beta\gamma}\, \varphi_\gamma(\vec{x})\, \delta(\vec{x}-\vec{y}) ,$$

$$C_{\underset{(\alpha)}{\tilde{a}}\underset{(\beta)}{\bar{b}}}(\vec{x},\vec{y}) = \frac{1}{M^2}\, \delta_{\alpha\beta}\, \delta(\vec{x}-\vec{y}) \quad ,$$

$$C_{\underset{(\alpha)}{\bar{a}}\underset{(\beta)}{\tilde{b}}}(\vec{x},\vec{y}) = -\frac{1}{M^2}\, \delta_{\alpha\beta}\, \delta(\vec{x}-\vec{y}) \quad . \tag{II.3.20}$$

Obviously

$$\{\, \pi_a^0 \,,\ \text{all canonical variables}\, \}^* = 0 \quad , \tag{11.3.21}$$

since $\theta_a = \pi_a^0$ are second-class constraints.

Next, using Eq. (II.1.20), we obtain

$$\{\pi_\alpha^i(\vec{x}), A_\beta^0(\vec{y})\}^* = -\int d\vec{z}\, d\vec{u}\, \{\pi_\alpha^i(\vec{x}), \theta_a(\vec{z})\}\, C_{ab}(\vec{z},\vec{u}) \times$$

$$\times \{\theta_b(\vec{u}), A_\beta^0(\vec{y})\} = -\int d\vec{z}\, d\vec{u}\, \{\pi_\alpha^i(\vec{x}), \varphi_\delta(\vec{z})\}\, C_{\underset{(\delta)}{\bar{a}}\underset{(\gamma)}{\tilde{b}}}(\vec{z},\vec{u}) \times$$

$$\times \{\pi_\gamma^0(\vec{u}), A_\beta^0(\vec{y})\} = -\frac{1}{M^2}\, \{\pi_\alpha^i(\vec{x}), \varphi_\beta(\vec{y})\} . \tag{II.3.22}$$

Using (II.3.6) one finds

$$\{\pi_\alpha^i(\vec{x}), A_\beta^0(\vec{y})\}^* = -\frac{g}{M^2} f_{\alpha\beta\gamma}\, \pi_\gamma^i(\vec{x})\, \delta(\vec{x}-\vec{y}) . \tag{II.3.23}$$

Since $C_{\bar{a}\bar{b}}(\vec{x},\vec{y}) = 0$, one has

$$\{\pi_\alpha^i(\vec{x}), A_j^\beta(\vec{y})\}^* = -\delta_\alpha^{\ \beta}\, \delta_j^{\ i}\, \delta(\vec{x}-\vec{y}) . \tag{II.3.24}$$

Quite similarly, we obtain

$$\{\pi^i_\alpha(\vec{x}), \pi^j_\beta(\vec{y})\}^* = 0 \quad ,$$

$$\{A^\alpha_i(\vec{x}), A^\beta_j(\vec{y})\}^* = 0 \quad . \tag{II.3.25}$$

Next, taking into account (II.3.20), one finds the remaining Poisson brackets

$$\{A^\alpha_0(\vec{x}), A^\beta_i(\vec{y})\}^* = -\frac{1}{M^2}\{\varphi^\alpha(\vec{x}), A^\beta_i(\vec{y})\} =$$

$$= \frac{1}{M^2}\partial^{\vec{x}}_i \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y}) - \frac{g}{M^2} f_{\alpha\beta\gamma} A^\gamma_i(\vec{x})\,\delta(\vec{x}-\vec{y}) \quad , \tag{II.3.26}$$

$$\{A^0_\alpha(\vec{x}), A^0_\beta(\vec{y})\}^* = \frac{g}{M^4} f_{\alpha\beta\gamma}\,\varphi_\gamma(\vec{x})\,\delta(\vec{x}-\vec{y}) \quad . \tag{II.3.27}$$

This completes the quantization of the massive Yang-Mills field theory, since in operator formalism one can obtain the basic commutators from the Dirac brackets, while in path integral formalism one uses (II.3.15) (properly generalized) to obtain S-matrix elements.

II.4. Light-cone quantization of the self-interacting scalar theory*

As we shall see in this section, quantization of field theories on the null plane leads naturally to second-class constraints. Therefore, the method we developed in Section III is applicable in this case. For different methods of quantization on the null plane, the reader is referred to the existing literature [10,11,12]. For completeness and because of the fact that we analyzed it independently (although much later), we shall also present the method of Banyai and Mezincescu [13,14]. Both our method and the method of Banyai and Mezincescu will be illustrated in the example of the self-interacting scalar theory with a quartic coupling.

Let us begin by reminding ourselves of the expression for the vacuum functional (generating functional of Green's functions) in the first-order formulation of this theory which can be obtained by a proper generalization of Eq. (II.1.24)

$$Z(J) = \langle 0|S|0\rangle_J = \int \prod_\mu \mathcal{D}A_\mu \mathcal{D}\varphi \exp\Big\{ i\int d^4x \Big[A^\mu \partial_\mu \varphi - \frac{1}{2} A^\mu A_\mu - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4 + \varphi J \Big] \Big\}. \tag{II.4.1}$$

Introducing the null plane variables

$$X_+ = \frac{X_0 + X_3}{\sqrt{2}}, \quad X_- = \frac{X_0 - X_3}{\sqrt{2}}, \quad (\underset{\sim}{X})_{1,2} = X_{1,2}$$

$$A_+ = \frac{A_0 + A_3}{\sqrt{2}}, \quad A_- = \frac{A_0 - A_3}{\sqrt{2}} \quad (\underset{\sim}{A})_{1,2} = A_{1,2}, \tag{II.4.2}$$

we can write (II.4.1) in the following form:

$$Z(J) = \int \mathcal{D}A_+ \mathcal{D}A_- \mathcal{D}\underset{\sim}{A} \mathcal{D}\varphi \exp\Big\{ i\int dX_- dX_+ d\underset{\sim}{X} \Big[A_+ \frac{\partial\varphi}{\partial X_-} + \underset{\sim}{A}\cdot\frac{\partial\varphi}{\partial \underset{\sim}{X}} + A_- \frac{\partial\varphi}{\partial X_+} - \frac{1}{2}[2A_+A_- - \underset{\sim}{A}^2] - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \varphi J \Big] \Big\}. \tag{II.4.3}$$

---

* This section is a product of research done in collaboration with Antal Jevicki.

Integrating over $\underset{\sim}{A}$ and A_+, we find

$$Z(J) = \int \mathcal{D}A_- \mathcal{D}\varphi \prod_x \delta\left(A_- - \frac{\partial \varphi}{\partial x_-}\right) \exp\left\{ i \int dx_- \, dx_+ \, d\underset{\sim}{x} \cdot \right.$$

$$\left. \cdot \left[A_- \frac{\partial \varphi}{\partial x_+} - \mathcal{H} + \varphi J \right] \right\}, \tag{II.4.4}$$

where

$$\mathcal{H} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial \underset{\sim}{x}}\right)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!}\varphi^4 . \tag{II.4.5}$$

This is a phase space functional integral for a theory with the Hamiltonian density (II.4.5). The role of the time variable is played by the variable x_+, and the conjugate momentum is seen to be A_-. The constraint

$$\theta = A_- - \frac{\partial \varphi}{\partial x_-} = 0 \tag{II.4.6}$$

is properly accounted for by the δ-function in (II.4.4). Henceforth, we shall label A_- simply by π.

To be systematic, let us repeat the analysis we described in the previous sections to this case. Thus, one starts with Lagrangian

$$\mathcal{L} = \frac{\partial \varphi}{\partial x_+}\frac{\partial \varphi}{\partial x_-} - \frac{1}{2}\left(\frac{\partial \varphi}{\partial \underset{\sim}{x}}\right)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4 \tag{II.4.7}$$

and deduces the conjugate momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x_+}\right)} = \frac{\partial \varphi}{\partial x_-} . \tag{II.4.8}$$

Eq. (II.4.8) obviously represents a constraint.

The Hamiltonian is immediately found to be

$$H = \int dx_- \, d\underset{\sim}{x} \left[\frac{1}{2}\left(\frac{\partial \varphi}{\partial \underset{\sim}{x}}\right)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!}\varphi^4 + u\left(\pi - \frac{\partial \varphi}{\partial x_-}\right) \right] , \tag{II.4.9}$$

As usual we require that

$$\dot{\theta} = \{\theta, H\} = 0 \ , \tag{II.4.10}$$

where

$$\theta = \pi - \frac{\partial \varphi}{\partial x_-} \ . \tag{II.4.11}$$

Eq. (II.4.10) results in

$$\underset{\sim}{\nabla}^2 \varphi - m^2 \varphi - \frac{\lambda}{3!}\varphi^3 - 2\,\frac{\partial u}{\partial x_-} = 0 \ , \tag{II.4.12}$$

since

$$\{\pi(\underset{\sim}{y}, y_-) - \partial_-^y \varphi(\underset{\sim}{y}, y_-), \pi(\underset{\sim}{x}, x_-) - \partial_-^x \varphi(\underset{\sim}{x}, x_-)\} = 2\partial_-^x \delta(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) \ . \tag{II.4.13}$$

Eq. (II.4.12) does not represent a new constraint but merely serves to determine u . One possible determination of u is:

$$u(x) = \frac{1}{4}\int \epsilon(x_- - \xi)\left[\underset{\sim}{\nabla}^2 \varphi(\underset{\sim}{x}, \xi, x_+) - m^2 \varphi(\underset{\sim}{x}, \xi, x_+) - \right.$$
$$\left. - \frac{\lambda}{3!}\varphi^3(\underset{\sim}{x}, \xi, x_+)\right] d\xi \ . \tag{II.4.14}$$

Thus (II.4.11), taken at all $(x_-, \underset{\sim}{x})$, represents a complete set of constraints. Due to (II.4.13) these constraints are second-class.

To quantize the theory in operator formalism, we have to find the Dirac brackets (this is precisely the method of Banyai and Mezincescu [13,14]). By (I.20) these are given by

$$\{a(\vec{u}), b(\vec{v})\}^* = \{a(\vec{u}), b(\vec{v})\} - \int d\vec{x}\, d\vec{y}\, \{a(\vec{u}), \theta(\vec{x})\} \cdot$$
$$\cdot\, C(\vec{x}, \vec{y})\, \{\theta(\vec{y}), b(\vec{v})\} \ , \tag{II.4.15}$$

where, e.g. $\vec{x} = (x_-, \underset{\sim}{x})$ and

$$\int c(\vec{x},\vec{z})\, d(\vec{z},\vec{y})\, d\vec{z} = \delta(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) \,, \tag{II.4.16}$$

$$\int d(\vec{x},\vec{z})\, c(\vec{z},\vec{y})\, d\vec{z} = \delta(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) \,, \tag{II.4.17}$$

$$d(\vec{z},\vec{y}) = \{\theta(\vec{z}),\theta(\vec{y})\} = 2\partial_-^{y}\,\delta(z_- - y_-)\,\delta(\underset{\sim}{z} - \underset{\sim}{y}) \,. \tag{II.4.19}$$

Conditions (II.4.16) - (II.4.19) imply a special solution*

$$c(\vec{x},\vec{y}) = -\frac{1}{4}\,\epsilon(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) \,, \tag{II.4.20}$$

which enables us to find the basic Dirac brackets:

$$\{\varphi(\vec{x}),\varphi(\vec{y})\}^* = -\frac{1}{4}\,\epsilon(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) \,, \tag{II.4.21}$$

$$\{\varphi(\vec{x}),\pi(\vec{y})\}^* = \frac{1}{2}\,\delta(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) \,, \tag{II.4.22}$$

$$\{\pi(\vec{x}),\pi(\vec{y})\}^* = \frac{1}{2}\,\partial_-^{x}\,\delta(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) \,. \tag{II.4.23}$$

The transition to quantum theory is then effected by replacing Dirac brackets by " - i " times the corresponding commutators, so one finds for the commutators:

---

*The general solution is

$$C(\vec{x},\vec{y}) = -\frac{1}{4}\,\epsilon(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) + h(\underset{\sim}{x},\underset{\sim}{y})$$

where $h(\underset{\sim}{x},\underset{\sim}{y})$ is an arbitrary function of the transverse components antisymmetric in $(\underset{\sim}{x} \leftrightarrow \underset{\sim}{y})$. We choose to discuss only the simplest solution characterized by $h(\underset{\sim}{x},\underset{\sim}{y}) = 0$.

$$[\varphi(x),\varphi(y)]\Big|_{x_+=y_+} = -\frac{i}{4}\epsilon(x_- - y_-)\,\delta(\underset{\sim}{x}-\underset{\sim}{y}) \quad ,$$

$$[\varphi(x),\pi(y)]\Big|_{x_+=y_+} = \frac{1}{2}\,i\,\delta(x_- - y_-)\,\delta(\underset{\sim}{x}-\underset{\sim}{y}) \quad ,$$

$$[\pi(x),\pi(y)]\Big|_{x_+=y_+} = \frac{1}{2}\,i\,\partial_-^x\,\delta(x_- - y_-)\,\delta(\underset{\sim}{x}-\underset{\sim}{y}) \,. \tag{II.4.24}$$

To quantize the theory by the method of Section III of this thesis, we need to know the value of the determinant $|\det\{\theta(\vec{x}),\theta(\vec{y})\}|$. From (II.4.13) it is seen to be

$$|\det\{\theta(\vec{x}),\theta(\vec{y})\}| = \det(2\,\partial_-)\,. \tag{II.4.25}$$

Thus we have all the ingredients necessary to write the functional integral for the generating functional of Green's functions:

$$Z(J) = \int \mathcal{D}\pi\,\mathcal{D}\varphi\,|\det(2\partial_-)|^{1/2}\prod_x \delta(\pi-\partial_-\varphi)\exp\left\{i\int(\pi\partial_+\varphi-\mathcal{H}+\varphi J)\,d^4x\right\}, \tag{II.4.26}$$

where

$$\mathcal{H} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial\underset{\sim}{x}}\right)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4\,. \tag{II.4.27}$$

To within the ignorable infinite constant $|\det(2\,\partial_-)|^{\frac{1}{2}}$ this is seen to agree with (II.4.4). In turn, (II.4.4) is equivalent to (II.4.1). If we integrate over A_1 , A_2 and A_3 in (II.4.1) and relabel $A^0 \to \bar{\pi}$, we can write (II.4.1) as

$$Z(J) = \int \mathcal{D}\bar{\pi}\,\mathcal{D}\varphi\,\exp\left\{i\int d^4x\left(\bar{\pi}\frac{\partial\varphi}{\partial x^0} - \bar{\mathcal{H}} + \varphi J\right)\right\}, \tag{II.4.28}$$

where

$$\bar{\mathcal{H}} = \frac{1}{2}\bar{\pi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 . \qquad \text{(II.4.29)}$$

In this formal sense, then, the two quantization schemes, namely the equal-time quantization corresponding to (II.4.28) and the null plane quantization corresponding to (II.4.26) are equivalent.

II.5. Local Lagrangian theory of electric and magnetic charges and its quantization*

While the theory of the electromagnetic field interacting with both electric and magnetic charges has a long history [15,16], its local Lagrangian formulation is comparatively new [17]. It is based on the introduction of two potentials A_μ and B_μ. The electromagnetic field tensor $F_{\mu\nu}$ is suitably expressed in terms of these. As we shall see, this necessitates the introduction of a fixed four-vector n_μ into the theory; thus, explicit Lorentz invariance is lost and it remains to be proved that the theory is in fact n - independent.

In the case when magnetic charges are present, Maxwell's equations read

$$\partial_\mu F^{\mu\nu} = j_e^\nu \quad , \tag{II.5.1}$$

$$\partial_\mu F^{d\mu\nu} = j_g^\nu \quad . \tag{II.5.2}$$

In (II.5.1), $F^{d\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\,F^{\kappa\lambda}$ and $\epsilon^{\mu\nu\kappa\lambda}$ is the completely antisymmetric symbol with $\epsilon^{0123} = 1$. j_e^ν is the electric current, while j_g^ν is the magnetic current. Both currents are conserved:

$$\partial_\mu j_e^\mu = \partial_\mu j_g^\mu = 0 \quad . \tag{II.5.3}$$

The general solution to (II.5.2) may be written as

$$F = \partial \wedge A - (n\cdot\partial)^{-1}(n \wedge j_g)^d \quad . \tag{II.5.4}$$

n_μ is an arbitrary fixed four-vector and $(n\cdot\partial)^{-1}$ is an integral operator with the kernel $(n\cdot\partial)^{-1}(x-y)$ satisfying $n\cdot\partial\,(n\cdot\partial)^{-1}(x) = \delta^{(4)}(x)$.

---

* This section was done in collaboration with Antal Jevicki.

For arbitrary two vectors C^{μ} and D^{ν}, $(C \wedge D)^{\mu\nu} = C^{\mu}D^{\nu} - C^{\nu}D^{\mu}$. A^{μ} is a four-potential which depends on the choice of gauge, the choice of n and the determination of $(n \cdot \partial)^{-1}$.

The general solution to (Eq. II.5.1) is

$$F = -(\partial \wedge B)^{d} + (n \cdot \partial)^{-1}(n \wedge j_e) \ . \tag{II.5.5}$$

Since any antisymmetric tensor G satisfies the identity

$$G = \frac{1}{n^2}\left\{ [n \wedge (n \cdot G)] - [n \wedge (n \cdot G^{d})]^{d} \right\}, \tag{II.5.6}$$

we can obtain from (II.5.4) and (II.5.5)

$$F = \frac{1}{n^2}\left(\{n \wedge [n \cdot (\partial \wedge A)]\} - \{n \wedge [n \cdot (\partial \wedge B)]\}^{d} \right) \tag{II.5.7}$$

$$F^{d} = \frac{1}{n^2}\left(\{n \wedge [n \cdot (\partial \wedge A)]\}^{d} + \{n \wedge [n \cdot (\partial \wedge B)]\} \right). \tag{II.5.8}$$

These expressions, when substituted into (II.5.1) and (II.5.2), yield

$$\begin{aligned}\frac{1}{n^2}\big(&n\cdot\partial\, n\cdot\partial\, A^{\mu} - n\cdot\partial\, \partial^{\mu} n\cdot A - n^{\mu} n\cdot\partial\, \partial\cdot A \, + \\ &+ n^{\mu}\partial^{2} n\cdot A - n\cdot\partial\, \epsilon^{\mu}{}_{\nu\kappa\lambda} n^{\nu}\partial^{\kappa}B^{\lambda}\big) = j_e^{\mu} \ ,\end{aligned} \tag{II.5.9}$$

$$\begin{aligned}\frac{1}{n^2}\big(&n\cdot\partial\, n\cdot\partial\, B^{\mu} - n\cdot\partial\, \partial^{\mu} n\cdot B - n^{\mu} n\cdot\partial\, \partial\cdot B \, + \\ &+ n^{\mu}\partial^{2} n\cdot B + n\cdot\partial\, \epsilon^{\mu}{}_{\nu\kappa\lambda} n^{\nu}\partial^{\kappa}A^{\lambda}\big) = j_g^{\mu} \ .\end{aligned} \tag{II.5.10}$$

These equations of motion follow from the Lagrangian density:

$$\mathcal{L} = \mathcal{L}_{\gamma} + \mathcal{L}_{I} \ , \tag{II.5.11}$$

where

$$\begin{aligned}\mathcal{L}_{\gamma} = &-\frac{1}{2n^2}[n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^{d}] + \frac{1}{2n^2}[n\cdot(\partial\wedge B)]\cdot \\ &\cdot [n\cdot(\partial\wedge A)^{d}] - \frac{1}{2n^2}[n\cdot(\partial\wedge A)]^{2} - \frac{1}{2n^2}[n\cdot(\partial\wedge B)]^{2}.\end{aligned} \tag{II.5.12}$$

and

$$\mathcal{L}_I = -j_e\cdot A - j_g\cdot B \tag{II.5.13}$$

Using the identity

$$tr(G\cdot G) = G_{\mu\nu}G^{\nu\mu} = \frac{2}{n^2}\left[-(n\cdot G)^2 + (n\cdot G^d)^2\right] \tag{II.5.14}$$

which follows from (II.5.6), we can obtain a different form for $\mathcal{L}_\gamma$:

$$\mathcal{L}_\gamma = \frac{1}{8}tr[(\partial\wedge A)\cdot(\partial\wedge A)] + \frac{1}{8}tr[(\partial\wedge B)\cdot(\partial\wedge B)] -$$

$$-\frac{1}{4n^2}\{n\cdot[(\partial\wedge A)+(\partial\wedge B)^d]\}^2 - \frac{1}{4n^2}\{n\cdot[(\partial\wedge B)-(\partial\wedge A)^d]\}^2 \tag{II.5.15}$$

We have thus obtained a Lagrangian formulation for the electromagnetic field interacting with both electric and magnetic charges.

We shall now quantize this theory using the method of phase space functional integration developed in Section III of Chapter I.

The quantization procedure starts, as usual, from the Lagrangian. One then goes on to derive the conjugate momenta, the Hamiltonian and the constraints. In our case the photon Lagrangian is given by (II.5.15). After some algebra the total Lagrangian can be recast into a form more suitable for transition into the Hamiltonian formalism:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_1 + \mathcal{L}_\psi \\ \mathcal{L}_1 &= \mathcal{L}_\gamma + \mathcal{L}_I\end{aligned} \tag{II.5.16}$$

$$\mathcal{L}_1 = \frac{1}{4}(\vec{F}_0^{\,2} - \vec{F}^2) + \frac{1}{4}(\vec{G}_0^{\,2} - \vec{G}^2) + \frac{1}{4}(\vec{a}^2 + \vec{b}^2) -$$

$$-\frac{1}{2n^2}[(\vec{n}\cdot\vec{a})^2 - \vec{n}^2\vec{a}^2] - \frac{1}{2n^2}[(\vec{n}\cdot\vec{b})^2 - \vec{n}^2\vec{b}^2] + \frac{n_0}{n^2}\epsilon_{ijk}a_i b_j n_k$$

$$-j_e^0 A_0 + \vec{j}_e\cdot\vec{A} - j_g^0 B_0 + \vec{j}_g\cdot\vec{B} \tag{II.5.17}$$

$\mathcal{L}_\psi$ is the free Lagrangian of the charged and monopole fields (we assume here they are spin $\frac{1}{2}$ fields). The explanation of the notation is as follows:

$$
\begin{aligned}
n^\mu &= (n_0, \vec{n}) , \\
a_i &= \bar{F}_{0i} + G_i \quad , \quad b_i = \bar{G}_{0i} - F_i \\
\bar{F}_{ij} &= \epsilon_{ijk} F_k \quad , \quad \bar{G}_{ij} = \epsilon_{ijk} G_k , \\
(\vec{F}_0)_i &= \bar{F}_{0i} = \partial_0 A_i - \partial_i A_0 , \\
(\vec{G}_0)_i &= \bar{G}_{0i} = \partial_0 B_i - \partial_i B_0 , \\
\bar{F}_{ij} &= \partial_i A_j - \partial_j A_i , \\
\bar{G}_{ij} &= \partial_i B_j - \partial_j B_i .
\end{aligned}
\tag{II.5.18}
$$

Equivalently:

$$
\begin{aligned}
\mathcal{L}_1 = {} & \frac{1}{4}(\bar{F}_{0i}\bar{F}_{0i} - F_i F_i) + \frac{1}{4}(\bar{G}_{0i}\bar{G}_{0i} - G_i G_i) + \\
& + \frac{1}{4}(a_i a_i + b_i b_i) + \frac{1}{2}\alpha_{ij}(a_i a_j + b_i b_j) + \\
& + \beta_{ij} a_i b_j - j_e^0 A_0 + \vec{j}_e \cdot \vec{A} - j_g^0 B_0 + \vec{j}_g \cdot \vec{B} ,
\end{aligned}
\tag{II.5.19}
$$

where

$$
\begin{aligned}
\alpha_{ij} &= \frac{\vec{n}^{\,2}}{n^2}\left(\delta_{ij} - \frac{n_i n_j}{\vec{n}^{\,2}}\right) = p\left(\delta_{ij} - \frac{n_i n_j}{\vec{n}^{\,2}}\right) , \\
\beta_{ij} &= \frac{n_0}{n^2}\epsilon_{ijk} n_k = \frac{q}{n_0}\epsilon_{ijk} n_k .
\end{aligned}
\tag{II.5.20}
$$

(Note that $q - p = 1$)

The derivation of conjugate momenta proceeds now as usual

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \bar{F}_{0i} + \alpha_{ij}\, a_j + \beta_{ij}\, b_j + \frac{1}{2} G_i \ ,$$

$$\sigma^i = \frac{\partial \mathcal{L}}{\partial \dot{B}_i} = \bar{G}_{0i} + \alpha_{ij}\, b_j - \beta_{ij}\, a_j - \frac{1}{2} F_i \ , \qquad \text{(II.5.21)}$$

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \ ,$$

$$\sigma^0 = \frac{\partial \mathcal{L}}{\partial \dot{B}_0} = 0 \ . \qquad \text{(II.5.22)}$$

Defining new quantities $\tilde{\pi}^i$ and $\tilde{\sigma}^i$ by

$$\tilde{\pi}^i = \pi^i - \frac{1}{2} G_i - \alpha_{ij}\, G_j + \beta_{ij}\, F_j \ ,$$

$$\tilde{\sigma}^i = \sigma^i + \frac{1}{2} F_i + \alpha_{ij}\, F_j + \beta_{ij}\, G_j \ , \qquad \text{(II.5.23)}$$

we obtain, making use of (II.5.18), (II.5.20), (II.5.21) and (II.5.23):

$$\vec{\tilde{\pi}} = q\left(\vec{F}_0 - \frac{\vec{n} \times \vec{G}_0}{n_0}\right) - \frac{p\vec{n}}{\vec{n}^2}(\vec{n}\cdot\vec{F}_0) \ ,$$

$$\vec{\tilde{\sigma}} = q\left(\vec{G}_0 + \frac{\vec{n} \times \vec{F}_0}{n_0}\right) - \frac{p\vec{n}}{\vec{n}^2}(\vec{n}\cdot\vec{G}_0) \ . \qquad \text{(II.5.24)}$$

At this stage we have to distinguish between two cases: $n_0 = 0$ and $n_0 \neq 0$. As far as the quantization procedure is concerned, the two cases are fundamentally different. For $n_0 = 0$ we obtain both first and second-class constraints; for $n_0 \neq 0$ only first-class constraints appear. For completeness, we shall include the second case in our discussion.

The case $n_o = 0$ was studied by Balachandran, Rupertsberger and Schecter[18]. We wish to emphasize that, while those authors quantize the theory by Dirac's formalism, we perform a phase space path integral quantization as an application of Section III of this thesis.

We choose to discuss the second case first, since it is in fact simpler than the first case. In the case $n_o \neq 0$ Eqs. (II.5.24) can be inverted, with the result

$$\vec{F}_o = \tilde{\vec{\pi}} + \frac{\vec{n} \times \tilde{\vec{\sigma}}}{n_o} ,$$
$$\vec{G}_o = \tilde{\vec{\sigma}} - \frac{\vec{n} \times \tilde{\vec{\pi}}}{n_o} . \quad \text{(II.5.25)}$$

In view of (II.5.23) this can be written as:

$$\vec{F}_o = \tilde{\vec{\pi}} - \frac{1}{2} \vec{G} + \frac{\vec{n}}{n_o} \times (\vec{\sigma} - \frac{1}{2} \vec{F}) ,$$
$$\vec{G}_o = \vec{\sigma} + \frac{1}{2} \vec{F} - \frac{\vec{n}}{n_o} \times (\vec{\pi} + \frac{1}{2} \vec{G}) . \quad \text{(II.5.26)}$$

Since by (II.5.18) $\vec{F}_o = \dot{\vec{A}} - \vec{\nabla} A_o$ and $\vec{G}_o = \dot{\vec{B}} - \vec{\nabla} B_o$, we have in fact expressed the time derivatives of the spatial components of potentials in terms of the conjugate momenta $\vec{\pi}$ and $\vec{\sigma}$. Eqs. (II.5.22) result in constraints

$$\varphi_1 = \pi^0 ,$$
$$\varphi_2 = \sigma^0 . \quad \text{(II.5.27)}$$

The next step consists in finding the Hamiltonian density:

$$\mathcal{H}_1 = \vec{\pi} \cdot \vec{F}^o + \vec{\sigma} \cdot \vec{G}^o + \vec{\pi} \cdot \vec{\nabla} A_o + \vec{\sigma} \cdot \vec{\nabla} B_o + u \pi^o + v \sigma^o - \mathcal{L}_1 . \quad \text{(II.5.28)}$$

A rather straightforward, though tedious derivation produces the final expression for the Hamiltonian density:

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_\psi \quad , \qquad \text{(II.5.29)}$$

$$\mathcal{H}_1 = \frac{1}{2}(\vec{\pi}'^2 + \vec{\sigma}'^2 + \vec{F}^2 + \vec{G}^2) - \vec{\pi}'\cdot\vec{G} + \vec{\sigma}'\cdot\vec{F} +$$

$$+ \frac{\vec{\pi}'}{m_0}\cdot[\vec{m}\times\vec{\sigma}'] + \vec{\pi}\cdot\vec{\nabla}A_0 + \vec{\sigma}\cdot\vec{\nabla}B_0 + u\pi^0 + v\sigma^0$$

$$+ j_e^0 A_0 - \vec{j}_e\cdot\vec{A} + j_g^0 B_0 - \vec{j}_g\cdot\vec{B} \quad , \qquad \text{(II.5.30)}$$

where

$$\vec{\pi}' = \vec{\pi} + \frac{1}{2}\vec{G} \quad ,$$

$$\vec{\sigma}' = \vec{\sigma} - \frac{1}{2}\vec{F} \qquad \text{(II.5.31)}$$

and $\mathcal{H}_\psi$ is the free Hamiltonian density for the spinor fields. We are now ready to find the secondary constraints by imposing consistency conditions of the familiar type:

$$\varphi_3 \equiv \{\pi^0, \int \mathcal{H} d\vec{x}\} = \vec{\nabla}\cdot\vec{\pi} - j_e^0 = 0 \quad ,$$

$$\varphi_4 \equiv \{\sigma^0, \int \mathcal{H} d\vec{x}\} = \vec{\nabla}\cdot\vec{\sigma} - j_g^0 = 0 \quad . \qquad \text{(II.5.32)}$$

Before continuing, let us prove an auxiliary result

$$\{\vec{\nabla}_{\vec{x}}\cdot\vec{\pi}(\vec{x}), \vec{F}(\vec{y})\} = 0 \quad . \qquad \text{(II.5.33)}$$

Indeed, we have

$$\{ \vec{\nabla}_{\vec{x}} \cdot \vec{\pi}(\vec{x}), F_i(\vec{y}) \} = \partial_\ell^{\vec{x}} \epsilon_{imn} \partial_m^{\vec{y}} \{ \pi_\ell(\vec{x}), A_n(\vec{y}) \} =$$

$$= \partial_\ell^{\vec{x}} \partial_m^{\vec{y}} \epsilon_{im\ell} \, \delta(\vec{x}-\vec{y}) \; .$$

Using (II.5.33) we immediately find

$$\{ \vec{\nabla} \cdot \vec{\pi}, \int \mathcal{H} d\vec{x} \} = \vec{\nabla} \cdot \vec{j}_e \; . \tag{II.5.34}$$

On the other hand,

$$\{ j_e^0, \int \mathcal{H} d\vec{x} \} = \vec{\nabla} \cdot \vec{j}_e \; . \tag{II.5.35}$$

Eq. (II.5.35) follows from the equation of motion

$$\dot{j}_e^0 = \{ j_e^0, \int \mathcal{H} d\vec{x} \} \; , \tag{II.5.36}$$

and from the fact that the current is conserved, which implies the equation of motion for the current

$$\dot{j}_e^0 = \vec{\nabla} \cdot \vec{j}_e \; . \tag{II.5.37}$$

Eq. (II.5.35) can, of course, be checked explicitly. Thus, no further constraints are generated. We have

$$\begin{aligned} \varphi_1 &= \pi^0 \; , & \varphi_3 &= \vec{\nabla} \cdot \vec{\pi} - j_e^0 \; , \\ \varphi_2 &= \sigma^0 \; , & \varphi_4 &= \vec{\nabla} \cdot \vec{\sigma} - j_g^0 \; . \end{aligned} \tag{II.5.38}$$

Now

$$\begin{aligned} j_e^0 &= \sum_n e_n \dot{j}_n^0 \; , \\ j_g^0 &= \sum_n g_n \dot{j}_n^0 \end{aligned} \tag{II.5.39}$$

and

$$\{ j^0_m(\vec{x}), j^0_m(\vec{y}) \} = 0 \quad . \tag{II.5.40}$$

We have therefore established that all constraints (II.5.39) are first-class. In view of the general discussion in Section III, there must be four gauge conditions. There are four interesting choices:

$$\chi_1 = A^0 \qquad \begin{array}{ll} \chi_3 = \vec{\nabla}\cdot\vec{A} & \\ \chi_4 = \vec{\nabla}\cdot\vec{B} \ , & \text{(I)} \\ \chi_3 = \vec{m}\cdot\vec{A} & \\ \chi_4 = \vec{m}\cdot\vec{B} \ , & \text{(II)} \end{array}$$

$$\chi_2 = B^0 \qquad \begin{array}{ll} \chi_3 = \vec{m}\cdot\vec{A} & \\ \chi_4 = \vec{\nabla}\cdot\vec{B} \ , & \text{(III)} \\ \chi_3 = \vec{\nabla}\cdot\vec{A} & \\ \chi_4 = \vec{m}\cdot\vec{B} \ . & \text{(IV)} \end{array}$$

(II.5.40)

For definiteness we shall consider the choice corresponding to I. The Faddeev-Popov determinant is trivially found to be $(\det \vec{\nabla}^2)^2 = \Delta_f$.

The expression for $\langle 0 | S | 0 \rangle$ in terms of a phase space functional integral is then (for notational simplicity we ignore the spinor variables):

$$\langle 0|S|0\rangle = \int \mathcal{D}\vec{\pi}\,\mathcal{D}\pi^0\,\mathcal{D}\vec{A}\,\mathcal{D}A^0\,\mathcal{D}\vec{\sigma}\,\mathcal{D}\sigma^0\,\mathcal{D}\vec{B}\,\mathcal{D}B^0\,\Delta_f \times$$
$$\times \prod_x \{ \delta(A_0)\,\delta(B_0)\,\delta(\pi_0)\,\delta(\sigma_0)\,\delta(\vec{\nabla}\cdot\vec{A})\,\delta(\vec{\nabla}\cdot\vec{B})\,\delta(\vec{\nabla}\cdot\vec{\pi} - j^0_e) \times$$
$$\times\, \delta(\vec{\nabla}\cdot\vec{\sigma} - j^0_g)\} \exp\{ i \int (\vec{\pi}\cdot\dot{\vec{A}} + \pi^0\dot{A}^0 + \vec{\sigma}\cdot\dot{\vec{B}} + \sigma^0\dot{B}_0 - \mathcal{H})\, d^4x \} \tag{II.5.41}$$

with $\mathcal{H}$ given by (II.5.29).

It is the expression of the vacuum functional in terms of a phase space functional integral that is fundamental and corresponds directly to the operator formalism. As a rule, the Lagrangian functional integral has to be recovered from expressions like (II.5.41) by integration over the conjugate momenta. It is to this point that we now devote our attention.

We first integrate trivially over π^0, σ^0, A^0 and B^0 to obtain:

$$\langle 0|S|0\rangle = \int \mathcal{D}\vec{\pi}\,\mathcal{D}\vec{A}\,\mathcal{D}\vec{\sigma}\,\mathcal{D}\vec{B}\,\Delta_f \prod_x \{\delta(\vec{\nabla}\cdot\vec{A})\,\delta(\vec{\nabla}\cdot\vec{B})\,\times$$

$$\times\,\delta(\vec{\nabla}\cdot\vec{\pi}-j_e^0)\,\delta(\vec{\nabla}\cdot\vec{\sigma}-j_g^0)\}\exp\{i\int(\vec{\pi}\cdot\dot{\vec{A}}+\vec{\sigma}\cdot\dot{\vec{B}}-\bar{\mathcal{H}})d^4x\}, \tag{II.5.42}$$

where

$$\bar{\mathcal{H}} = \frac{1}{2}(\vec{\pi}'^2+\vec{\sigma}'^2+\vec{F}^2+\vec{G}^2)-\vec{\pi}'\cdot\vec{G}+\vec{\sigma}'\cdot\vec{F}\,+$$

$$+\,\frac{\vec{\pi}'}{m_0}\cdot[\vec{m}\times\vec{\sigma}'] - \vec{j}_e\cdot\vec{A}-\vec{j}_g\cdot\vec{B}\,. \tag{II.5.43}$$

After writing

$$\prod_x \delta(\vec{\nabla}\cdot\vec{\pi}-j_e^0) = \int\mathcal{D}A^0\,e^{i\int[-\vec{\pi}\cdot\vec{\nabla}A_0-A_0 j_e^0]d^4x},$$

$$\prod_x \delta(\vec{\nabla}\cdot\vec{\sigma}-j_g^0) = \int\mathcal{D}B^0\,e^{i\int[-\vec{\sigma}\cdot\vec{\nabla}B_0-B_0 j_g^0]d^4x}, \tag{II.5.44}$$

we can bring (II.5.42) into the form:

$$\langle 0|S|0\rangle = \int\mathcal{D}\vec{\pi}\,\mathcal{D}\vec{A}\,\mathcal{D}\vec{\sigma}\,\mathcal{D}\vec{B}\,\mathcal{D}A^0\mathcal{D}B^0\,\Delta_f\prod_x\{\delta(\vec{\nabla}\cdot\vec{A})\,\times$$

$$\times\,\delta(\vec{\nabla}\cdot\vec{B})\}\exp\{i\int[\vec{\pi}\cdot(\dot{\vec{A}}-\vec{\nabla}A_0)+\vec{\sigma}\cdot(\dot{\vec{B}}-\vec{\nabla}B_0)-\bar{\bar{\mathcal{H}}}]d^4x\}, \tag{II.5.45}$$

where

$$\bar{\bar{\mathcal{H}}} = \bar{\mathcal{H}} + A_0 j_e^0 + B_0 j_g^0 . \tag{II.5.46}$$

We are thus led to calculate the subintegral

$$I = \int \mathcal{D}\vec{\pi}\mathcal{D}\vec{\sigma}\, exp\{i\int[\vec{\pi}\cdot\vec{F}^0 + \vec{\sigma}\cdot\vec{G}^0 - \frac{1}{2}(\vec{\pi}'^2+\vec{\sigma}'^2) + \vec{\pi}'\cdot\vec{G} - \vec{\sigma}'\cdot\vec{F} - \frac{\vec{\pi}'}{m_0}\cdot(\vec{m}\times\vec{\sigma}')]\},$$

Performing the change of variables:

$$\vec{\pi} = \vec{\pi}' - \frac{1}{2}\vec{G} ,$$

$$\vec{\sigma} = \vec{\sigma}' + \frac{1}{2}\vec{F} , \tag{II.5.47}$$

we find

$$\langle 0|S|0\rangle = \int \mathcal{D}\vec{A}\mathcal{D}\vec{B}\mathcal{D}A^0\mathcal{D}B^0\Delta_f \prod_x\{\delta(\vec{\nabla}\cdot\vec{A})\,\delta(\vec{\nabla}\cdot\vec{B})\} \times$$

$$\times\, exp\,\{i\int d^4x[-\frac{1}{2}\vec{F}^2 - \frac{1}{2}\vec{G}^2 - \frac{1}{2}\vec{G}\cdot\vec{F}^0 + \frac{1}{2}\vec{F}\cdot\vec{G}^0 - j_e^\mu A_\mu - j_g^\mu B_\mu]\} \times$$

$$\times\, J(\vec{F}^0, \vec{G}^0, \vec{F}, \vec{G}) , \tag{II.5.48}$$

where

$$J = \int \mathcal{D}\vec{\pi}'\mathcal{D}\vec{\sigma}' exp\{i\int[\vec{\pi}'\cdot(\vec{F}^0+\vec{G}) + \vec{\sigma}'\cdot(G^0 - \vec{F}) - \frac{1}{2}(\vec{\pi}'^2+\vec{\sigma}'^2) - \tag{II.5.49}$$
$$- \frac{\vec{\pi}'}{m_0}\cdot[\vec{m}\times\vec{\sigma}']]\,d^4x\}.$$

A straightforward Gaussian integration, over $\vec{\pi}'$ gives

$$J = \int \mathcal{D}\vec{\sigma}'\, exp\{i\int[\frac{1}{2}(\vec{a} - \frac{\vec{m}\times\vec{\sigma}'}{m_0})^2 + \vec{\sigma}'\cdot\vec{b} - \frac{1}{2}\vec{\sigma}'^2]\,d^4x\} =$$

$$= exp\{i\int\frac{1}{2}\vec{a}^2 d^4x\}\int\mathcal{D}\vec{\sigma}' exp\{i\int[\vec{\sigma}'\cdot\vec{c} - \frac{1}{2}[1-\frac{\vec{m}^2}{m_0^2}]\vec{\sigma}'^2 - \frac{1}{2m_0^2}(\vec{m}\cdot\vec{\sigma}')^2]d^4x\}, \tag{II.5.50}$$

where

$$\vec{c} = \vec{b} - \frac{\vec{a} \times \vec{M}}{M_0} \tag{II.5.51}$$

and $\vec{a}$ and $\vec{b}$ are given by (II.5.18). Now

$$\int \mathcal{D}\vec{\sigma}' \exp\left\{ i \int \left[\vec{\sigma}' \cdot \vec{c} - \frac{1}{2}\left(1 - \frac{\vec{M}^2}{M_0^2}\right)\vec{\sigma}'^2 - \frac{1}{2M_0^2}(\vec{M}\cdot\vec{\sigma}')^2 \right] d^4x \right\} =$$

$$= \int \mathcal{D}\vec{\sigma}' \exp\left\{ i \int \left[-\frac{1}{2}\sigma'^T O \sigma' + \sigma'^T c \right] d^4x \right\} =$$

$$= \prod_x \left\{ (\det O^{-1})^{1/2} \right\} \exp\left\{ i \int \left(\frac{1}{2} c^T O^{-1} c \right) d^4x \right\}, \tag{II.5.52}$$

where

$$O_{ij} = \frac{1}{g}\left[\delta_{ij} + \frac{M_i M_j}{M^2} \right] \tag{II.5.53}$$

and

$$(O^{-1})_{ij} = g\left[\delta_{ij} - \frac{M_i M_j}{M_0^2} \right]. \tag{II.5.54}$$

One can calculate $\det O^{-1}$ by noting that it is rotationally invariant and equal to

$$g^3 \det \begin{Vmatrix} 1 & & \\ & 1 & \\ & & 1 - \frac{\vec{M}^2}{M_0^2} \end{Vmatrix} = g^2$$

in the reference frame in which $(\vec{n})_i = |\vec{n}|\,\delta_{i3}$. Thus

$$(\det O^{-1})^{1/2} = \left| \frac{M_0^2}{M^2} \right|, \tag{II.5.55}$$

Combining the hitherto derived results, we obtain for the vacuum-vacuum S-matrix element,

$$\langle 0|S|0\rangle = \int \mathcal{D}\vec{A}\,\mathcal{D}\vec{B}\,\mathcal{D}A^0\,\mathcal{D}B^0\,\Delta_f \prod_x \{\delta(\vec{\nabla}\cdot\vec{A})\,\delta(\vec{\nabla}\cdot\vec{B})\} \cdot$$

$$\times \det\left|\frac{n_0^2}{n^2}\right| \exp\Big\{ i\int d^4x \Big[-\frac{1}{2}\vec{F}^2 - \frac{1}{2}\vec{G}^2 + \frac{1}{2}\vec{a}^2 +$$

$$+ \frac{1}{2}\vec{F}\,\vec{G}^0 - \frac{1}{2}\vec{G}\,\vec{F}^0 - j_e^\mu A_\mu - j_g^\mu B_\mu + \frac{g}{2}\Big[\vec{b} -$$

$$- \frac{(\vec{a}\times\vec{n})}{n_0}\Big]^2 - \frac{g}{2n_0^2}(\vec{n}\cdot\vec{b})^2\Big]\Big\} \quad . \qquad \text{(II.5.56)}$$

It is then a matter of some straightforward algebra to show that the Lagrangian density appearing in (II.5.56) is in fact equal to the Lagrangian density in (II.5.17).

If we change the gauge conditions to covariant ones (by the standard Faddeev-Popov [5] procedure), say

$$\chi_1 = \partial_\mu A^\mu \qquad \chi_2 = \partial_\mu B^\mu \quad ,$$

we find

$$\bar{\Delta}_f = [\det \partial^2]^2 \quad . \qquad \text{(II.5.57)}$$

In addition, there is the weight $\frac{n_o^2}{n^2}$ not originating from the gauge conditions, so that the expression for the vacuum-vacuum S-matrix element reads

$$\langle 0|S|0\rangle = \int \prod_\mu \mathcal{D}A_\mu\,\mathcal{D}B_\mu \prod_x \{\delta(\partial_\nu B^\nu)\,\delta(\partial_\nu A^\nu)\}(\det\partial^2)^2 \det\left|\frac{n_0^2}{n^2}\right| e^{i\int \mathcal{L}\, d^4x} \quad . \qquad \text{(II.5.58)}$$

There are other possibilities for the choice of gauge conditions, e.g. $\chi_1 = \partial \cdot A$, $\chi_2 = n \cdot B$, but we shall not dwell on these.

We now turn to the case $n_o = 0$, which is more interesting from our point of view, since it contains second-class constraints. For $n_o \to 0$ we have $\frac{q}{n_o} = \frac{n_o}{n^2} \to 0$ and also $q \to 0$, so $p = -1$. Eq. (II.5.24) becomes in this case:

$$\tilde{\vec{\pi}} = \frac{\vec{n}}{\vec{n}^2}(\vec{n}\cdot\vec{F}_o) \quad ,$$

$$\tilde{\vec{\sigma}} = \frac{\vec{n}}{\vec{n}^2}(\vec{n}\cdot\vec{G}_o) \quad . \tag{II.5.59}$$

Therefore

$$\vec{n}\cdot\vec{F}_o = \vec{n}\cdot\tilde{\vec{\pi}} = \vec{n}\cdot(\vec{\pi} - \tfrac{1}{2}\vec{G}) \quad ,$$

$$\vec{n}\cdot\vec{G}_o = \vec{n}\cdot\tilde{\vec{\sigma}} = \vec{n}\cdot(\vec{\sigma} + \tfrac{1}{2}\vec{F}) \, , \tag{II.5.60}$$

and

$$\vec{\tau}^{(\ell)}\cdot\tilde{\vec{\pi}} = \vec{\tau}^{(\ell)}\cdot\tilde{\vec{\sigma}} = 0 \quad , \tag{II.5.61}$$

or, in other words

$$\vec{\tau}^{(\ell)}\cdot\vec{\pi}' = \vec{\tau}^{(\ell)}\cdot\vec{\sigma}' = 0 \quad , \tag{II.5.62}$$

where we have used (II.5.20), (II.5.23) and (II.2.31). In (II.5.61) and (II.5.62), $\vec{\tau}^{(\ell)}$ are two vectors ($\ell = 1,2$) orthogonal to $\vec{n}$, that is:

$$\vec{\tau}^{(\ell)}\cdot\vec{n} = 0 \quad . \tag{II.5.63}$$

We choose them in such a way that

$$\vec{\tau}^{(\ell)}\cdot\vec{\tau}^{(k)} = \delta_{\ell k} \quad . \tag{II.5.64}$$

As usual, we have to calculate the Hamiltonian at this point. Setting $n_o = 0$ in the Lagrangian yields:

$$\mathcal{L}_1 = -\tfrac{1}{2}\vec{a}\cdot\vec{G} + \tfrac{1}{2}\vec{b}\cdot\vec{F} + \frac{1}{2\vec{n}^2}\left[(\vec{n}\cdot\vec{a})^2 + (\vec{n}\cdot\vec{b})^2\right] - j_e\cdot A - j_g\cdot B \, . \tag{II.5.65}$$

Therefore, calling $\vec{\pi}_1' = \vec{\pi}', \vec{\pi}_2' = \vec{\sigma}'$, one obtains

$$\mathcal{H}_1\Big|_{\bar{M}} = \frac{1}{2}(\vec{\pi}_a' \cdot \vec{\pi}_a') - \vec{\pi}_1' \cdot \vec{G} + \vec{\pi}_2' \cdot \vec{F} + \frac{1}{2}\vec{F}^2 +$$

$$+ \frac{1}{2}\vec{G}^2 + j^e \cdot A + j^g \cdot B + \vec{\pi} \cdot \vec{\nabla} A_0 + \vec{\sigma} \cdot \vec{\nabla} B_0 \ , \tag{II.5.66}$$

where we have used (II.5.18), (II.5.60) and (II.5.62). Therefore,

$$\mathcal{H}_1 = \mathcal{H}_1\Big|_{\bar{M}} + u\pi^0 + v\sigma^0 + u_b^\ell \vec{\tau}^{(\ell)} \cdot \vec{\pi}_b' \ , \tag{II.5.67}$$

where u,v and u_b^ℓ are, for the moment, arbitrary multipliers. As in the case $n_o \neq 0$, taking Poisson brackets of primary constraints with the Hamiltonian and setting these equal to zero, one generates the following secondary first-class constraints:

$$\varphi_3 \equiv \vec{\nabla} \cdot \vec{\sigma} - j_g^0 = 0 \ ,$$

$$\varphi_4 \equiv \vec{\nabla} \cdot \vec{\pi} - j_e^0 = 0 \ . \tag{II.5.68}$$

Imposing the consistency conditions

$$\{ \vec{\tau}^{(\ell)} \cdot \vec{\pi}_b' , \int \mathcal{H} d\vec{x} \} = 0 \tag{II.5.69}$$

merely determines u_b^ℓ , since

$$| \det \| \{ \theta_b^\ell, \theta_c^j \} \| |^{1/2} = \det \{ \frac{(\vec{n}\cdot\vec{\nabla})^2}{\vec{n}^2} \} \neq 0 \tag{II.5.70}$$

as demonstrated in Appendix II. In Eq. (II.5.70):

$$\theta_b^\ell = \vec{\tau}^{(\ell)} \cdot \vec{\pi}_b' \ . \tag{II.5.71}$$

Therefore, calling $\vec{\pi}'_1 = \vec{\pi}', \vec{\pi}'_2 = \vec{\sigma}'$, one obtains

$$\mathcal{H}_1\Big|_{\bar{M}} = \frac{1}{2}(\vec{\pi}'_a \cdot \vec{\pi}'_a) - \vec{\pi}'_1 \cdot \vec{G} + \vec{\pi}'_2 \cdot \vec{F} + \frac{1}{2}\vec{F}^2 + $$

$$+ \frac{1}{2}\vec{G}^2 + j^e \cdot A + j^g \cdot B + \vec{\pi} \cdot \vec{\nabla} A_0 + \vec{\sigma} \cdot \vec{\nabla} B_0 , \tag{II.5.66}$$

where we have used (II.5.18), (II.5.60) and (II.5.62). Therefore,

$$\mathcal{H}_1 = \mathcal{H}_1\Big|_{\bar{M}} + u\pi^0 + v\sigma^0 + u_b^\ell \vec{\tau}^{(\ell)} \cdot \vec{\pi}'_b , \tag{II.5.67}$$

where u,v and u_b^ℓ are, for the moment, arbitrary multipliers. As in the case $n_o \neq 0$, taking Poisson brackets of primary constraints with the Hamiltonian and setting these equal to zero, one generates the following secondary first-class constraints:

$$\begin{aligned} \varphi_3 &\equiv \vec{\nabla} \cdot \vec{\sigma} - j_g^0 = 0 , \\ \varphi_4 &\equiv \vec{\nabla} \cdot \vec{\pi} - j_e^0 = 0 . \end{aligned} \tag{II.5.68}$$

Imposing the consistency conditions

$$\{ \vec{\tau}^{(\ell)} \cdot \vec{\pi}'_b , \int \mathcal{H} d\vec{x} \} = 0 \tag{II.5.69}$$

merely determines u_b^ℓ, since

$$| \det \| \{ \theta_b^\ell , \theta_c^j \} \| |^{1/2} = \det \{ \frac{(\vec{n} \cdot \vec{\nabla})^2}{\vec{n}^2} \} \neq 0 \tag{II.5.70}$$

as demonstrated in Appendix II. In Eq. (II.5.70):

$$\theta_b^\ell = \vec{\tau}^{(\ell)} \cdot \vec{\pi}'_b . \tag{II.5.71}$$

In view of (II.5.70), the constraints θ_b^{ℓ} are second-class.

If we choose the gauge conditions to be

$$\chi_3 = \vec{\nabla}\cdot\vec{B} \qquad \chi_4 = \vec{\nabla}\cdot\vec{A}$$
$$\chi_1 = A_0$$
$$\chi_2 = B_0 \qquad , \tag{II.5.72}$$

the Faddeev-Popov determinant is found to be

$$\Delta_f = (\det \vec{\nabla}^2)^2 \quad , \tag{II.5.73}$$

On the basis of (II.5.70), (II.5.73) and the general results of Section III, we are led to the following expression for the vacuum to vacuum S-matrix element:

$$\begin{aligned}
\langle 0|S|0\rangle = &\int \mathcal{D}\vec{\pi}\,\mathcal{D}\vec{\sigma}\,\mathcal{D}\vec{A}\,\mathcal{D}\vec{B}\,\mathcal{D}\pi^0\mathcal{D}\sigma^0\mathcal{D}A^0\mathcal{D}B^0\,\Delta_f \,\times \\
&\times \det\frac{(\vec{m}\cdot\vec{\nabla})^2}{m^2}\prod_x\{\delta(\vec{\nabla}\cdot\vec{\pi}-j_e^0)\,\delta(\vec{\nabla}\cdot\vec{\sigma}-j_g^0)\,\delta(\vec{\nabla}\cdot\vec{A})\,\delta(\vec{\nabla}\cdot\vec{B})\,\times \\
&\times \delta(\pi^0)\delta(\sigma^0)\delta(A^0)\delta(B^0)\}\prod_{x,\ell,a}\delta(\vec{\tau}^{(\ell)}\cdot\vec{\pi}_a')\,\exp\{i\int[\vec{\pi}\cdot\dot{\vec{A}}+ \\
&+\vec{\sigma}\cdot\dot{\vec{B}}-\tfrac{1}{2}(\vec{\pi}'^2+\vec{\sigma}'^2)+\vec{\pi}'\cdot\vec{G}-\vec{\sigma}'\cdot\vec{F}-\tfrac{1}{2}\vec{F}^2- \\
&-\tfrac{1}{2}\vec{G}^2-j_e\cdot A-j_g\cdot B]\,d^4x\}\quad .
\end{aligned} \tag{II.5.74}$$

In order to avoid cumbersome expressions, we have ignored the spinor variables except in the coupling terms $-j^e\cdot A$ and $-j^m\cdot B$.

Integrating over π^0, σ^0, A^0, B^0 and writing, e.g.

$$\prod_x \delta(\vec{\nabla}\cdot\vec{\pi}-j_e^0) = \int\mathcal{D}A_0\,\exp\{i\int(-\vec{\pi}\cdot\vec{\nabla}A_0 - A_0 j_e^0)\,d^4x\}\,, \tag{II.5.75}$$

we obtain

$$<0|S|0> = \int \mathcal{D}\vec{\pi}\, \mathcal{D}\vec{\sigma} \prod_{\mu} \mathcal{D}A_\mu \mathcal{D}B_\mu \prod_{\ell,a} \mathcal{D}\lambda_a^\ell \; \Delta_f \;\cdot$$

$$\times \det \frac{(\vec{n}\cdot\vec{\nabla})^2}{\vec{n}^2} \prod_x \{\delta(\vec{\nabla}\cdot\vec{A})\,\delta(\vec{\nabla}\cdot\vec{B})\} \exp\{i\int d^4x(\vec{\pi}'\cdot\vec{a} +$$

$$+ \vec{\sigma}'\cdot\vec{b} - \frac{1}{2}\vec{G}\cdot\vec{F}_0 + \frac{1}{2}\vec{F}\vec{G}_0 - \frac{1}{2}\vec{\pi}'^2 - \frac{1}{2}\vec{\sigma}'^2 - \frac{1}{2}\vec{F}^2 -$$

$$- \frac{1}{2}\vec{G}^2 - j^e\cdot A - j^g\cdot B + \lambda_a^\ell\, \vec{\tau}^{(\ell)}\cdot\vec{\pi}_a')\} \,. \tag{II.5.76}$$

One can now integrate first over $\vec{\pi}'$ and $\vec{\sigma}'$ and then over λ_ℓ^a , and use the fact that

$$\frac{1}{2}\vec{a}^2 + \frac{1}{2}\vec{b}^2 - \frac{1}{2}(\vec{\tau}^{(\ell)}\cdot\vec{a})\,(\vec{\tau}^{(\ell)}\cdot\vec{b}) - \frac{1}{2}(\vec{\tau}^{(\ell)}\cdot\vec{b})(\vec{\tau}^{(\ell)}\cdot\vec{b}) =$$

$$= \frac{1}{2\vec{n}^2}\left[(\vec{n}\cdot\vec{a})^2 + (\vec{n}\cdot\vec{b})^2\right] . \tag{II.5.77}$$

The final result of these manipulations is

$$<0|S|0> = \int \prod_\mu \mathcal{D}A_\mu \mathcal{D}B_\mu \,\Delta_f \det \frac{(\vec{n}\cdot\vec{\nabla})^2}{\vec{n}^2} \prod_x \delta(\vec{\nabla}\cdot\vec{A})\,\delta(\vec{\nabla}\cdot\vec{B})\, e^{i\int \mathcal{L}_1 d^4x} , \tag{II.5.78}$$

where $\mathcal{L}_1$ is given by (II.5.65). Eqs. (II.5.58) and (II.5.78) represent the central result of this section. When properly generalized to the case of $\langle 0 | S | 0 \rangle_J$, i.e. the vacuum-to-vacuum S-matrix element in the presence of external sources, they yield the generating functional of Green's functions of the theory:

$$\tilde{Z}(J) = \frac{<0|S|0>_J}{<0|S|0>} \tag{II.5.79}$$

Since all constant factors in the integration measure cancel in (II.5.79), one finds for both $n_o = 0$ and $n_o \neq 0$, after making a transition to the covariant gauge [5] $\partial \cdot A = \partial \cdot B = 0$ in (II.5.78):

$$\tilde{Z}(J) = \frac{\int \prod_{\mu} \mathcal{D}A_{\mu} \mathcal{D}B_{\mu} \prod_{x} \delta(\partial \cdot A)\, \delta(\partial \cdot B) \exp\{i \int (\mathcal{L} + J) d^4x\}}{\int \prod_{\mu} \mathcal{D}A_{\mu} \mathcal{D}B_{\mu} \prod_{x} \delta(\partial \cdot A)\, \delta(\partial \cdot B) \exp\{i \int \mathcal{L} d^4x\}} \tag{II.5.80}$$

where J is the sum of all external source terms. The integration over spinor variables, although not denoted explicitly, is understood. $\mathcal{L}$ is the Zwanziger Lagrangian density. We have thus recovered the standard path integral formulation of the theory in terms of a functional integral over the field variables only.

II.6. Conclusion

The basic original result of this thesis is the generalization of Faddeev's phase space (or Hamiltonian) path integral method [4] to the case when second-class constraints are present in the theory. This is the result we have obtained in Section III. It is a general result, applicable not only to the examples worked out in Chapter II, but any field theory containing second-class constraints [*1]. Being a generalization of Faddeev's phase space method, it has all its merits, in particular: a) it provides a bridge between the operator formalism and the Lagrangian path integral method, b) it is a canonical method in the sense that S-matrix elements are expressed in terms of path integrals over canonical coordinates and momenta and that such path integrals are invariant under canonical transformations of those coordinates and momenta, c) it supplies the functional measure for path integrals over the coordinates after integration over the canonical momenta is performed [*2]. The most interesting applications of the general result obtained in Section III are: a) an alternative (and simpler) derivation of the basic result of Finkelstein, Kwitky and Mouton [9] concerning the quantization of the massive Yang-Mills field b) light-cone quantization of the φ^4 scalar field theory via the phase space (Hamiltonian) path integral method c) quantization of Zwanziger's local Lagrangian formulation of magnetic monopole theory by the same method.

[*1] Currently, the author is working with M. Kaku on the problem of functional measure in another such theory: quantum gravity, when quantized in the light-cone gauge and reduced to independent field variables [19] is a theory with second-class constraints. Part of this work will be an application of the author's thesis.

[*2] In the application described in Footnote 1, preliminary results indicate that this functional measure is a function of the field variables.

Appendix II

Introducing a suitable notation

$$V_\gamma^1 = A_\gamma \qquad V_\gamma^2 = B_\gamma$$

$$\epsilon_{11} = \epsilon_{22} = 0 \qquad \epsilon_{12} = -\epsilon_{21} = 1 \quad , \tag{A.II.1}$$

we can write a chain of identities:

$$\{\theta_b^\ell(\vec{x}), \theta_c^k(\vec{y})\} = \tau_i^{(\ell)} \tau_j^{(k)} \{\pi'^i_b(\vec{x}), \pi'^j_c(\vec{y})\} =$$

$$= \tau_i^{(\ell)} \tau_j^{(k)} \{\pi^i_b(\vec{x}) + \frac{1}{2}\epsilon_{be}\,\epsilon^{is\gamma}\,\partial_s^{\vec{x}} V_\gamma^e(\vec{x}), \pi^j_c(\vec{y}) +$$

$$+ \frac{1}{2}\epsilon_{cd}\,\epsilon^{ju\nu}\,\partial_u^{\vec{y}} V_\nu^d(\vec{y})\} =$$

$$= -\epsilon_{bc}\,\tau_i^{(\ell)}\,\tau_j^{(k)}\,\epsilon_{ijs}\,\partial_s^{\vec{x}}\,\delta(\vec{x}-\vec{y}) =$$

$$= -\epsilon_{bc}\,[\vec{\tau}^{(\ell)} \times \vec{\tau}^{(k)}] \cdot \vec{\nabla}_{\vec{x}}\,\delta(\vec{x}-\vec{y}) \ . \tag{A.II.2}$$

Because of our choice (II.5.63) and (II.5.64) for the vectors $\vec{\tau}^{(\ell)}$, we can write

$$\{\theta_b^\ell(\vec{x}), \theta_c^k(\vec{y})\} = -\epsilon_{bc}\,\epsilon^{\ell k}\,\frac{\vec{n}\cdot\vec{\nabla}_{\vec{x}}}{|\vec{n}|}\,\delta(\vec{x}-\vec{y}) \ , \tag{A.II.3}$$

whereupon Eq. (II.5.70) follows directly.

References

1. P.A.M. Dirac, Can. J. Math. 2 , 129 (1950).
2. P,A.M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science (Yeshiva University, New York, 1964).
3. P.A.M. Dirac, Phys. Rev. 114 , 924 (1959).
4. L.D. Faddeev, Teoreticheskaya i Matematicheskaya Fizika, 1 , 3 (1969) [English translation: Theoretical and Mathematical Physics 1 , 1 (1970), Copyright by Consultants Bureau] .
5. V.N. Popov and L.D. Faddeev, Kiev Report ITP 67-36 (1967) (Unpublished) [English translation: NAL Report THY-57 (1972), edited by D. Gordon and B.W. Lee] .
6. F.A. Berezin, The Method of Second Quantization (Academic Press, New York, 1966).
7. S. Deser, The Gravitational Field, Lectures at Orsay, [Universites de Paris VI, VII et XI (Physique Theorique), 1971-1972] .
8. See reference 6. for the definition and mathematical properties of anticommuting c-numbers.
9. R.J. Finkelstein, J.S. Kvitky and J.O. Mouton, Phys. Rev. D4 , 2220 (1971).
10. R. Jackiw, Canonical light-cone Commutators and their Applications, in Springer Tracts in Modern Physics, vol. 62 (Berlin, Heidelberg, New York, 1972).
11. F.L. Feinberg, Phys. Rev. D6 , 2852 (1972); S.J. Chang, R. Root and T.-M. Yan, Phys. Rev. D7 , 1133 (1973); S.J. Chang and T.-M. Yan, ibid 7 , 1147 (1973); T.-M. Yan, ibid 7 , 1760 (1973).
12. E. Tomboulis, Phys. Rev. D8 , 2736 (1973).
13. L. Banyai and L. Mezincescu, Phys. Rev. D8 , 417 (1973).

References (cont.)

14. L. Banyai and L. Mezincescu, Rev. Roum. Phys. 18, 1035 (1973).

15. P.A.M. Dirac, Proc. Roy. Soc. A133, 60 (1931); P.A.M. Dirac, Phys. Rev. 74, 817 (1948).

16. J. Schwinger, Phys. Rev. 144, 1087 (1966).

17. D. Zwanziger, Phys. Rev. D3, 880 (1971).

18. A.P. Balachandran, H. Rupertsberger and S. Schechter, Monopole Theories with Massless and Massive Gauge Fields, to be published in Phys. Rev. D, (15 April 1975).

19. M. Kaku, Ghost-Free Formulation of Quantum Gravity in the Light-Cone Gauge, CCNY Preprint HEP 74/7, December 1974.

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SOLID/LIQUID PHASE TRANSFORMATIONS

by

James H. Whittam

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1975

This manuscript has been read and accepted for the Graduate Committee in Chemistry in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

Solid/Liquid Phase Transformations

by

James H. Whittam

Adviser: Professor Henri L. Rosano

Evidence of the importance of phase transformations and the solid/liquid interface is observed from studies on three model systems:

A. Aqueous Solutions

In depth experimentation on the dendrite interface structure of aqueous solutions has established that not only are diffusion and concentration factors important to the interface configuration but the rate of heat extraction from the interface (heat transfer) and associated water play important roles.

B. Enzyme Systems

The change in the activity of α amylase is controlled by the rate of freezing, rate of thawing and the concentration of so-called protective agents. This appears to be due to the solute building up at the solid/liquid interface. Very small concentrations of small molecular weight protective agents lower the activity of the enzyme

after a given freeze thaw cycle more than the unprotected enzyme system would do under similar conditions.

C. Saturated Monoacid Triglycerides

Triglycerides were chosen as a third area of study since they exist in almost all fatty materials and their polymorphic states are due to three principle cross sectional arrangements of long chains α, β' and β. This dissertation presents evidence on the physical aging of pure monoacid triglycerides. A technique is also described using a polarizing microscope and a temperature gradient microscope stage which supplies supplementary information to compliment thermal, x-ray I.R. and NMR methods. This apparatus may also be used as a simple device to determine the temperature stability of various fatty systems. More important, it eliminates many three dimensional heat transfer problems and provides a direct method for observing the phenomena of physical aging of triglycerides.

Physical aging or solid/solid phase transformations occurs as a result of the formation of the meta-stable α and β' polymorphs. Prevention of physical aging may be accomplished by solidifying the triglyceride below the critical rate of freezing for α formation or by the addition of low melting triglyceride impurities (such as trilaurin).

Finally, nucleation data is presented for these pure saturated monoacid triglycerides. Estimates of γ, the surface free energy, for these triglycerides is of the order of 14-17 ergs/cm^2.

Scientific

Involvement

Bridges

Yesterday's

Legends to

Learned

Explanations

Preface

Many individuals, pursuing a project of this magnitude alone and accomplishing its goal, feel they achieve ultimate satisfaction. To this author few individuals reach this pinnacle of success and enjoy it without the help of some excellent support.

Likewise, this dissertation at hand is the product of one person striving to attain formal acknowledgment in the field of Chemistry; however, the support which served as the foundation and guidance of this individual effort must not go unrecognized. To all those individuals involved, I express my deepest appreciation.

I am extremely grateful to my family who throughout the years has helped guide, advise, support, and "put up" with my many antics and activities that have brought me to this very promising frontier of my life. To them I always will be indebted.

Beyond no reasonable doubt, the next most influential person in my life has been Professor Henri L. Rosano. Since my undergraduate days his persistance "to solve zee problum" combined with his overall dynamic personality has left an indelible mark in my mind. I have learned much and benefited greatly from his broad knowledge and experience not only in chemistry but in all aspects of life. According to the words of Gibran, "he who is wise does not bid you enter the house of wisdom but rather leads you to the threshold of your own mind." This sums up my education with Henri L. Rosano.

To thank in one line all the other people whom I have had the fortune to meet during the development of this thesis would be a gross injustice since no one would realize the role they played. The following is a brief account of three years at City College.

The genesis of this thesis first "saw light" during the summer of 1971 at C.C.N.Y. At this time I was doing research as an undergraduate with H.L.R. in the Marlies Lab of Old Baskerville Hall. My summer project was suggested by Mr. R. Pfluger of the Maxwell Division, General Foods Corporation and involved a study on the freeze drying of coffee. The following academic year I assisted Mr. M. Freedman (who was working on his Master's thesis) with research involving the freezing of aqueous solutions. Along with the deligent help of Dr. K.A. Jackson and Mr. W. Miller of the Bell Telephone Laboratories, Murray Hill, New Jersey we designed a unique microscope freeze drying stage. Dr. Jackson's expertise in the field of solidification also served as a bank of knowledge where I could often refer for discussion in my current work.

In June 1972 I made an important decision in my educational career. I had just graduated with my bachelor's degree in chemical engineering when I decided (with some convincing from H.L.R.) to accept a teaching fellowship at the City University of New York and progress on towards a Ph'D in Physical Chemistry. At this time it was also decided that I continue my work on aqueous solutions. In addition, the investigation of aqueous enzyme systems (the second phase of this work) was provoked in discussions by Dr. C. J. Cante and Dr. R. Guardia of the General Foods Corporation, Tarrytown, New York and Dr. Rosano. As a result, α amylase was chosen as a model system which could be related to a practical problem namely, freezing of foods.

During the summer of '72, I was fortunate enough to be employed with the Lever Bros Co. in Edgewater, New Jersey. Under the direction of Dr. E. D. Goddard (now with Union Carbide) I learned some of the techniques to study fats and fat emulsion systems. This experience was invaluable since I could relate some of the concepts I learned in college to solve some practical problems.

That September, the start of the new school year I was admitted into the PhD Program. I began to extend my study on freezing and solidification problems in solutions and enzyme systems. During this year, thoughts on the third system discussed in this thesis were developed by my good friend Dr. Cante, Dr. Rosano and myself. The crystal behavior of triglyceride systems was chosen because of the importance of solidification processes on their crystal form and because of the many inherent problems associated with products containing fats. With the help of my undergraduate friends H. Ragin, M. Petko and A. Garuba a great deal of information was collected on the kinetics of phase transformations also referred as physical aging.

The following years at City, I had the real fortune of working with some fine people. Dr. William Gerbacia who received his PhD in February 1974, and is now working for Chevron Oil Field Research Co. in La Habra California, en riched my educational background in this field of micro-emulsions. Today he is using these systems in hope of developing an economically feasible process for tertiary oil recovery. Bill and I also played a fine game of tennis together especially when our opponents didn't show up. When the other team did manage to play us our usual emulsified continuity would rapidly break down. Then there was Shu Hsien Chen (The Oriental Flower) who enlightened me in the field of monolayers to such an extent that we published the "world famous theory" on Surface Drag Viscosity. The original dubious theory was developed by a Bulgarian scientist and smuggled out of that country by Dr. Rosano who was working for the French government which in turn was paid by the C.I.A.

One often hears that the environment for a PhD must have a certain critical mass. In Rosano's lab, I'm sure we always ran above this critical. I therefore wish to extend thanks to all the other members of our lab; E. LaGamma, M. Mughelli, S. Schecter, A. Weiss, T. Forman and L. Kennberg. If there is anything we will all remember, it is that we had the best lunch hours in the entire college. This group also exhibited the greatest enthusiasm for having a party. No matter what the situation was we could always find an excuse to celebrate.

I must also acknowledge Dr. J. Rennert and Dr. A. Santoro for their thoughts on this project especially in the final stages of preparation and oh yes, the vocal renditions of Prof. H. Salzberg as he walked through the halls singing, Figaro Figaro, the Russian National Anthem or Manon.

Finally, I wish to thank the "shop", especially J. Cannella and B. Cope for their help in designing some of the equipment used in this project and the tireless efforts of Mrs. C. Silver in the Chemistry office for translating "our scribbled thoughts.

Respectfully,

James H. Whittam

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