QUADRIVIAL PURSUITS:
Case Studies in the Conceptual Foundation of the Mathematical Arts in the Late Middle Ages

by

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Abstract

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The quadrivium, the four mathematical disciplines of the Middle Ages, described the structure of the medieval cosmos, both macrocosm and microcosm. Arithmetic and music were the mathematics of Platonic counting numbers. Geometry and astronomy were the mathematics of continuous magnitude. All four disciplines worked in concert to describe a cohesive and harmonious universe, which in the late Middle Ages incorporated everything from Aristotelian elemental theory to astrology. This dissertation describes the early philosophical formulation of these disciplines from Pythagorean and Platonic roots and the foundation of the quadrivium itself in the mathematical writings of Boethius in the early sixth century. This dissertation then examines the mathematical philosophy of three late medieval authors who were proficient in the quadrivial arts: Nicole Oresme (ca. 1320-1382), Prosdocimo de' Beldomandi (ca. 1375-1428), and Leon Battista Alberti (1404-1472). All three demonstrate that the Boethian quadrivial philosophy continued to be relevant in the late 14th and early 15th centuries, but all three studies point to a significant fault line in the metaphysical structure of the quadrivium itself – the distinction between discrete and continuous, the quadrivial distinction between arithmetic and geometry.

To Dad,

Eliot Manning Newsome
(1934-2010)

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The philosophical content and clarity of the following dissertation benefited immeasurably from the tireless editorial work of my sister, Jennifer Kenyon. I cannot thank her enough, not only for reading it with a critical eye, but also for enthusiastically participating in its development. My wife, Jennifer Udell, and my mother, Shirley Newsome, were also pivotal throughout every stage of this dissertation by providing material assistance and wise counsel. I would also like to thank Joseph Dauben, Ruth DeFord, Jan Herlinger, Allison Kavey, Margaret King, James McClellan, Frederick Purnell, Peter Skiff, and Marvin Trachtenberg for their help and inspiration.

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Finally, I want to thank my father, Eliot M. Newsome. He was a master of
the first two quadrivial disciplines, and instilled in me an interest
in the theory, the structure, and the beauty of the world.
I think he would have enjoyed this dissertation.

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*Unless otherwise indicated, figures and animations are by the author.

Chapter 1:
Introduction and
Pre-Quadrivial Background

General Introduction

For centuries arithmetic, geometry, music, and astronomy have been identified collectively as the quadrivium, literally meaning the four ways or crossroads. Along with the arts of the trivium (grammar, rhetoric, and logic) they form the seven liberal arts that were the curricular backbone of medieval education. The trivium represented the literary arts and the quadrivium, the mathematical. From a modern perspective, the four mathematical arts of the quadrivium seem odd bedfellows, largely because of the inclusion of music. But the modern perspective is misleading. Anyone who looks more closely at the medieval quadrivium quickly learns that the modern disciplines that go by the same names are significantly different from their premodern counterparts. Quadrivial arithmetic is more like modern number theory and numerology, but limited exclusively to the counting numbers. Quadrivial geometry, though generally based on Euclid's *Elements*, could also include such topics as perspective, optics, and surveying. Quadrivial music might better be called music theory, acoustics, and the physics of sound and often included a psychological and astrological component. And quadrivial astronomy included astrology and had no recognizable relationship to what we now call physics. It was purely kinematic, no Newtonian forces or masses, just abstract motion, and it was geocentric in nature. It also often included the study of calendrics, time reckoning, geography, and navigation.

For the casual modern scholar in the humanities, investigations into the premodern quadrivial disciplines is generally not very illuminating. For example, most descriptions of medieval astronomy in non-specialist literature will describe the broad strokes of Aristotelian or Ptolemaic theory, but will not describe the mathematics that made such systems rigorous. Without these details medieval astronomy loses much of what a modern reader might regard as its mathematical character. Without the mathematics, the quadrivium does not hold together and the term simply becomes shorthand for an archaic collection of disciplines.

The objective of this dissertation is to explore what the quadrivium meant to a selection of late medieval scholars chosen for their knowledge of all four quadrivial disciplines. I focus on the period from ca. 1350-1450 because this was a time of great change in the mathematical arts. There appeared in late medieval Latin Europe new number systems, new musical styles, new physical theories, as well as new philosophical and technical texts. Some of these materials may have worked with the quadrivial philosophy, while others may have challenged it. I want to know how those who were proficient in all four quadrivial disciplines conceived of the relationships between and amongst those disciplines. How did late medieval quadrivial scholars respond to the philosphical structure underlying and informing the quadrivium? How did they, or did they not, ground their explorations in quadrivial metaphysics? Did they imagine the mathematical arts as a unified system that structured the universe or as disparate disciplines? The quadrivium lived on as a term, but died out as an organizational principle. Today, math majors no longer study music theory, and astrology has disappeared entirely from the modern curriculum. In the 14^{th} and 15^{th} centuries the quadrivium was a significant part of mainstream university education. To use Thomas Kuhn's term, it was the "normal science" of its day. What happened? Where did it go? Unlike Kuhn, I am not looking at the so-called "birth" of a new movement, I am looking at what seems to be the death of an old one.

In order to understand better the synergy of the quadrivium, the disciplines must be understood on their own terms, mathematically and philosophically. Such treatments exist in the highly specialized literatures on the history of music theory or the history of astronomical tables, but for most scholars in the humanities, these studies are not accessible. They are too specialized and, as a result, they do not address the larger quadrivial picture that I want to describe. The quadrivial forest is difficult to see through all the disciplinary trees. In order to correct this problem, this dissertation includes numerous accessible mathematical examples. These examples are to illustrate quadrivial issues, not to produce medieval mathematicians. I do not intend to analyze every nuance of a particular theory, as this has been done in the specialized literature. I will direct the reader to these studies as they arise. Some of the more complicated mathematical material that is contained in advanced astronomical and musical texts, though quite interesting in its own right, is usually not necessary for my purposes. The mathematics most illustrative of the quadrivial philosophy is often the most basic. I have made every attempt to make the mathematical examples in this dissertation as accessible and as interesting as possible. I want the modern reader to come away with more than simply a fuller understanding of the quadrivium. In this worldview, mathematics was integral, and I want the reader to explore how a concept as simple as counting could be the basis of more than half of a liberal arts education.

The structural philosophy of the quadrivium in the late Middle Ages cannot be understood without first describing the prehistory and history of the quadrivium itself. The quadrivium is the mathematical arts and as such is best understood in mathematical terms. Fortunately, the mathematics needed to understand the quadrivium is relatively simple, but for most people, historians and mathematicians alike, this type of mathematics is unfamiliar. This is the mathematics before the invention of the equals sign, "=".[1]

We call the quadrivium the "mathematical arts," but like the individual disciplines that make up the quadrivium, the word mathematics itself is quite different from our modern understanding

[1] The symbol for equality (=) did not appear until the middle of the 16^{th} century. David M. Burton, *The History of Mathematics: An Introduction*, 6th ed. (New York: McGraw-Hill, 2006), 347.

of the term. To help ease the reader into this world of medieval mathematics, in the first two chapters of this dissertation I will present a short prehistory of the quadrivium and then describe the quadrivial philosophy itself, in broad terms, as it was defined by its early authors and later practitioners. These introductory chapters are meant to prepare the reader for the more complex case studies in Chapters 3-5.

The following overview of the quadrivium is a synthesis fashioned from ideas and materials ranging from the 6th century B.C. to the 17th century A.D., with an emphasis on the period from about 1300 to 1500. Although there are several fine studies on the separate quadrivial disciplines, little has been written on the quadrivium as a coherent and unified philosophy. Most studies break up the quadrivium into its four parts and leave them that way, in pieces.[2] However, the philosophy of the quadrivium cannot be understood in this manner. The quadrivium is more than the sum of its parts. It is also the relationships between those parts. In its totality, it describes the structure of the macrocosm and the microcosm and the links between the two. In the Boethian quadrivial tradition, the tradition that was dominant in the late Middle Ages, the quadrivial disciplines were arranged sequentially: arithmetic, music, geometry, and astronomy,[3] but their interrelationships were not limited by this order. Without a general understanding of these relationships the quadrivium loses most of its meaning. My aim in these introductory chapters is to take a broad synthetic approach so that readers from both the arts and the sciences can meet on some common ground back in a premodern intellectual climate where arts were sciences and vice versa.

These introductory chapters are intended as a primer for the general reader without

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[2] A an overview of these studies is given in Chapter 2.
[3] This is not a hard and fast rule. Martianus Capella starts with geometry. See Martianus Capella, *Martianus Capella and the Seven Liberal Arts: The Marriage of Philology and Mercury [De nuptiis Philologiae et Mercurii]*, trans. William H. Stahl and Richard Johnson, 2 vols., vol. 2 (New York: Columbia University Press, 1977).

background in the specialized study of medieval mathematics and philosophy. It is presented without interference from alternate mathematical or natural philosophical theories. It provides a first approximation of the quadrivium. No synopsis that is sufficiently detailed could describe the beliefs of every premodern scholar. However, the mathematical philosophy presented in this overview would have been recognized by any of them, though they might not embrace each and every element. To provide too much philosophical depth and rigor would be counterproductive in light of this more general purpose. In order to keep this introduction short and broadly accessible, I have taken some liberties in the simplification of this material. These will be noted as they occur.

The Latin quadrivium got its start with the encyclopediasts in the early Middle Ages. Technically speaking, it was the brainchild of Boethius, the early 6^{th}-century philosopher and noted translator of Greek philosophical texts. He first called the four mathematical disciplines the "*quadruvium*"[4] in his treatise *De institutione arithmetica*. In the early to mid-5^{th} century, Martianus Capella grouped the same four mathematical disciplines together along with grammar, dialectic, and rhetoric in his work, *De nuptiis Philologiae et Mercurii,* an allegorical narrative describing the seven liberal arts.

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The Philosophical Pre-History of the Quadrivium

Although the term quadrivium was coined in the early 6^{th} century, these four disciplines had long been grouped together in the ancient Greek mathematical tradition. The philosopher Pythagoras (ca.570-ca.490 B.C.), from the island of Samos, one of the larger and more verdant

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[4] Anicius Manlius Severinus Boethius, *De institutione arithmetica libri duo & De institutione musica libri quinque*, ed. Godofredus Friedlein (Leipzig: B. G. Teubner, 1867), 9.

Greek islands located within view of the Ionian coast where the pre-Socratic Greek philosophers Thales (ca.630-ca.550 B.C.), Anaximander (ca.610-ca.547 B.C.), and Anaximenes (ca.550-ca.475 B.C.) lived, is roughly a century older than Socrates and like him, did not leave behind any written materials. He is only known through the writings of later Pythagoreans, a cult-like brotherhood that formed around the philosophy of their namesake in southern Italy. In general, the Pythagorean mathematical philosophy emphasizes the reality of number as a structural basis for the world as a whole. Its influence has been immense. The theory of Pythagorean mathematical tuning, based on perfect fifths, fourths, and octaves, was an explicit description of the connection between the ideal world of numbers and their structural manifestations earth. This harmonic theory directly influenced music theory and practice well into the 18th century and has continued to be a source of fascination for modern composers such as Olivier Messiaen and Iannis Xenakis. Pythagorean cosmography, as exemplified by Philolaus of Tarentum (d. ca. 390 B.C.), drew upon arithmetic, geometry, and music theory to describe a universe structured on mathematical principles.[5] Although these principles have been amended over time, the idea of a mathematically ordered universe is so basic to modern science that it is difficult to imagine another paradigm. There are, no doubt, antecedents to the Pythagorean philosophy, but it is in these early Pythagorean texts that evidence of the quadrivial philosophy begins to coalesce.

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[5] The universe described by Philolaus of Tarentum/Croton (ca. 470-ca.390 B.C.) consisted of 10 bodies including a spinning earth, which were distributed radially from a central fire at harmonic intervals. George Bosworth Burch, "The Counter-Earth," *Osiris* 11, no. (1954). See also Thomas Little Heath, *A Manual of Greek Mathematics* (Oxford University Press, 1931; reprint, Dover, 1963), 109-110. The Pythagorean Archytas of Tarentum (a contemporary of Plato) similarly promoted the study of the mathematical arts as a way to gain knowledge of truth. For information on Philolaus and Archytas, see Kathleen Freeman and Hermann Diels, eds., *Ancilla to the Pre-Socratic Philosophers* (Cambridge, MA: Harvard University Press, 1996), 78-81; Kenneth Sylvan Guthrie and David R. Fideler, eds., *The Pythagorean Sourcebook and Library: An Anthology of Ancient Writings Which Relate to Pythagoras and Pythagorean Philosophy*, trans. K. S. Guthrie (Grand Rapids, MI: Phanes Press, 1987), 167-202; Robert Navon, *The Pythagorean Writings: Hellenistic Texts from the lst Century B.C.-3rd Century A.D. on Life, Morality, and the World: Comprising a Selection of the Neo-Pythagorean Fragments, Texts, and Testimonia of the Hellenistic Period, Including Those of Philolaus and Archytas* (Kew Gardens, NY: Selene Books, 1986), 131-151.

For the modern scholar, probably the most familiar description of the quadrivial grouping is found in the section from Plato's *Republic* (mid-4th century B.C.) that immediately follows the famous "Allegory of the Cave." In this section Socrates describes the studies that will lead men from the world of change to the world of light. Here he promotes the study of the mathematical arts in a philosophical form that would be called, almost a thousand years later, the quadrivium.[6] His treatment begins with arithmetic,[7] the art of counting, the study of pure number divorced from any material object. His second description is plane geometry,[8] again emphasizing the higher analysis of forms over any terrestrial applications. He then describes solid geometry,[9] which is essentially the three dimensional extension of plane geometry. The fourth discipline is astronomy, which he defines as "the motion of solid bodies" or "solid bodies in circular motion."[10] Plato sees these astronomical bodies as part of the sensible world, the world of change, and thus direct analysis of their motions will only verify their imperfection and irregularity. For Plato, exact knowledge cannot be gained through observation, for our view of the heavens is mediated by our imperfect senses. Yet the general structure of true astronomical knowledge is hinted at by the visible heavens. The arithmetical relationships between the observable motions might be known if studied using the intellect. In very much the same way

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[6] The *Republic*, though well known today, was not read in the Latin Middle Ages. References to and descriptions of Plato's *Republic* can be found in other widely accessible texts such as Macrobius' *Commentary on the Dream of Scipio*. See James Hankins, "Plato in the Middle Ages," in *Dictionary of the Middle Ages,* ed. J. Strayer, (New York: Scribner, 1987); Ernst H. Kantorowicz, "Plato in the Middle Ages," *The Philosophical Review* 51, no. 3 (1942); Raymond Klibansky, *The Continuity of the Platonic Tradition During the Middle Ages: with a New Preface and Four Supplementary Chapters: Together with, Plato's Parmenides in the Middle Ages and the Renaissance* (Millwood, NY: Kraus International Publications, 1982).
[7] Plato, *The Republic of Plato*, trans. F. M. Cornford (New York: Oxford University Press, 1945), VII.524d-526c, pp. 240-243.
[8] Ibid., VII.526c-527c, pp. 243-244.
[9] Ibid., VII.527d-528e, pp. 244-246.
[10] Ibid., VII.528e-530c, pp. 246-249.

that the study of astronomy must transcend our imperfect vision of the heavens, harmonics,[11] the fifth and last mathematical discipline that Plato describes, must transcend the ear. The intellect supersedes the audible. Musical concord is a mathematical idea that is only imperfectly approximated by sounds perceptible to the ear. Each of Plato's descriptions emphasizes the abstract purity of the mathematical arts as a path to a higher understanding of what he refers to as the "essential Form of Goodness." For example, in the case of geometry he writes, "Geometry is the knowledge of the eternally existent .... it will tend to draw the soul towards truth and to direct upwards the philosophical intelligence which is now wrongly turned earthwards."[12] For Plato, "earthward" contemplations, such as land surveying or even celestial observation, are inferior pursuits.

Also of significant consequence to the development of a quadrivial philosophy was the "Myth of Er,"[13] found in the last section of Plato's *Republic*. Here Plato describes the motions of the cosmos in terms of concentric crystalline spheres, or whorls, pierced by a spindle of *adamant*.[14] The outer sphere carries the fixed stars and the inner ones carry the seven planets: Saturn, Jupiter, Mars, Venus, Mercury, the sun, and the moon. At the center is the earth. See Figure 1.1.

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[11] Ibid., VII.530c-531c, pp. 249-250.
[12] Ibid., VII.526c-527c, p. 244.
[13] Ibid., X.613-end, pp. 348-359.
[14] A *sphandulos* is the Greek word usually translated as ‘whorl’ or ‘sphere.’ It is the spherical or semispherical part of a drop spindle used in hand spinning yarn or thread. The analogy of spinning thread is used throughout the "Myth of Er." See Plato, *The Republic*, trans. P. Shorey, 2 vols., Loeb Classical Library, vol. 2 (Cambridge, MA: Harvard University Press, 1942), X616c, p. 500.

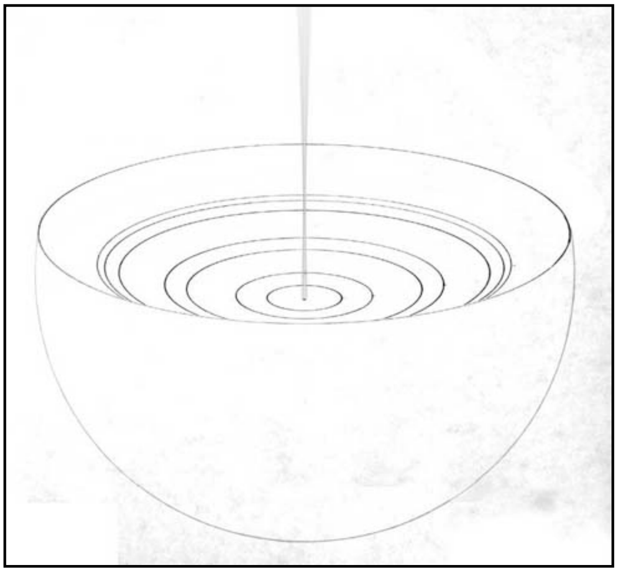

Figure 1.1: Hypothetical Rendering of Plato's Cosmos from the "Myth of Er"

The crystalline spheres are cut away to show their concentric arrangement and the shaft of adamant running through the center.

Each of the eight spheres carries a Siren uttering a single tone, together making "the concord of a single harmony."[15] The Three Fates, all of whom are associated with spinning and weaving, Clotho, Atropos, and Lachesis, regulate the motions of these spheres. Clotho, the fate who traditionally spins the thread of life, spins the crystalline spheres, as a spinner of yarn spins a whorl on a drop spindle. She gives the cosmos its overall diurnal rotation. Atropos reaches in and regulates the specific motions of the inner spheres (sun, moon, and perhaps Venus and Mars) probably accounting for seasonal changes and lunar phases, while Lachesis is responsible for the more irregular motions of the inner and outer spheres. This probably refers to retrograde motions and similar irregular astronomical behaviors.

[15] Ibid., X.617b, pp. 504-505. I follow Shorey's translation here because Cornford's musical terminology is potentially misleading. Cornford has, "the concords of a single scale," but he does not define how he is using the word "scale." See Plato, *The Republic of Plato*, X.617b, p. 354. The descriptions of the nested spheres all merging into a single sphere and the eight tones sounding together as one concordant harmony is also interesting in terms of the general Platonic theory of numbers, which will be described shortly. See Plato, *The Republic of Plato*, VII.524d-526c, p. 240-243.

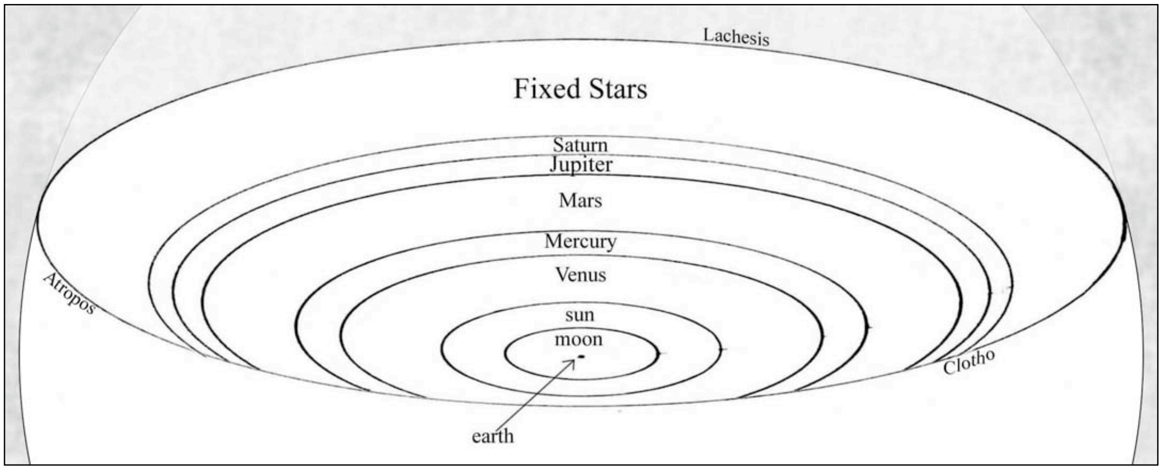

Figure 1.2: Planet-Sirens and Fates from the "Myth of Er"

In Plato's *Timaeus*[16] a similar cosmic structure is described in more metaphysical terms and using more specific mathematical concepts. These concepts joined arithmetic, harmony, geometry, and astronomy together.[17] This quantitative metaphysical structural description would permeate mathematical literature for two millennia.

Aristotle's influence on the pre-history of the quadrivium is significant though more complex. Many of his philosophical and natural philosophical ideas, found in such works as *Physics*, *Prior* and *Posterior Analytics*, *Metaphysics*, and *Meteorology*, reinforce the importance of mathematical studies and the liberal arts in general, yet his epistemological ideas, largely found in *De anima* and *De caelo*, retreated from the overt metaphysical status of number that was insisted upon by the Pythagoreans and Platonists. Some aspects of Aristotelianism even directly contradicted this status. For example, Aristotle's student Aristoxenus of Tarentum (4th century

[16] The pertinent parts of *Timaeus* were available in the Middle Ages via a translation by Chalcidius from the 4th century A.D. See William Harris Stahl, *Roman Science: Origins, Development, and Influence to the Later Middle Ages* (Madison: University of Wisconsin Press, 1962), 142-150. The study of *Timaeus* by Renaissance humanists such as Petrarch and Ficino is discussed in James Hankins, "The Study of the *Timaeus* in Early Renaissance Italy," in *Natural Particulars: Nature and the Disciplines in Renaissance Europe,* ed. Anthony Grafton and Nancy G. Siraisi (Cambridge, MA: MIT Press, 1999): 77-119.
[17] The analogy utilized in *Timaeus* evokes imagery of the forge and blacksmithing instead of spinning and weaving. The mathematical creation of the cosmos starts from Oneness. See Plato, *Plato's Cosmology: The Timaeus of Plato*, trans. F. M. Cornford (New York: Liberal Arts Press, 1937; reprint, 1957), 29d-39e, pp. 33-117.

B.C.), known as "the musician" to Roman scholars, in direct opposition to Platonic philosophy, insisted that the ear be the ultimate arbiter of concord, not mathematics. Aristoxenus finds the Platonic mathematical ideal absurd. In his book on harmonics he writes about some students who went to hear Plato lecture on "the good."

> They all came,..., supposing that they were going to acquire one of the things which people commonly consider good, such as wealth, health, strength – in general, some astounding happiness or other. But when the discourse turned out to be about the mathematical sciences, about numbers and geometry and astronomy, and its conclusion to be that the good is one, it seemed to them, I imagine, altogether contrary to their expectations.[18]

The Aristotelian argues that all we can really have are our perceptions,[19] and we should not fool ourselves into believing that abstract numbers beckon us back to some extra-dimensional life we once had in a world of Forms.[20] However, even though the Aristotelian program

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[18] Aristoxenus, "*Elementa harmonica*," in Andrew Barker, ed., *Greek Musical Writings*, vol. 2, trans. A. Barker (New York: Cambridge University Press, 1984), 148.

[19] From these perceptions we can come to truth in our intellect, not *a priori* from mathematics. Richard Sorabji has a very good essay on Aristotelian perception as it compares to Plato's. See Richard Sorabji, "Intentionality and Physiological Processes: Aristotle's Theory of Sense-Perception," in *Essays on Aristotle's De anima,* ed. Martha Craven Nussbaum and Amélie Rorty (Oxford: Clarendon Press, 1992).

[20] Ironically, Aristoxenus invented and systematized Greek music theory. Even those who did not accept his largely Aristotelian philosophical stance and preferred the Pythagorean number-based theory, adopted his terms and categories without question. See Aristoxenus, "*Elementa harmonica*," in Barker, ed., *Greek Musical Writings*, vol. 2, trans. Andrew Barker, 119-189. Aristoxenus is perhaps the most important counter-argument against quadrivial philosophy from a musical standpoint, yet his impact on the quadrivium proper was not felt until the Renaissance. He was largely representing practical music, not the mathematical/philosophical music of the more numerologically inclined. Norman Cazdan quite reasonably argues that Aristoxenus and the Pythagoreans were arguing apples and oranges. Practical music-making was distinct from a numerological theory of tuning. (The argument between theory and practice comes up in the case study on Prosdocimo de' Beldomandi in Chapter 4.) See Norman Cazden, "Pythagoras and Aristoxenos Reconciled," *Journal of the American Musicological Society* 11, no. 2/3 (1958): 97-105. See also Malcolm Litchfield, "Aristoxenus and Empiricism: A Reevaluation Based on His Theories," *Journal of Music Theory* 32, no. 1 (1988): 51-73. See also Henry Chadwick, *Boethius, the Consolations of Music, Logic, Theology, and Philosophy* (Oxford: Clarendon Press, 1981), 88-91. It is interesting to note that our current understanding of the perception of pitch is in some sense quantized. Our ear acts like a Fourier series machine as well as a wind instrument particular to each individual. The cilia, which sympathetically resonate in response to incoming sound, break the sound into pieces not unlike a Fourier series. Furthermore, the shape of our ear canal affects the timbre of the incoming sound just as the shape of a brass instrument affects the timbre of its outgoing sound. The subjectivity of Aristoxenian theory might in some sense be mathematically objective when viewed through our modern theory of sound production and perception. Aristoxenus' mathematical ideas were not widely available in

occasionally contradicted elements of Plato's mathematical philosophy, many aspects of Aristotle's works, such as his cosmography and elemental theory, were readily incorporated by later quadrivial scholars. And ideas from Aristotle's *Meteorology* and *De anima* were very influential on the development of astrology. Aristotelian influences on the quadrivial disciplines were less pronounced in the early Middle Ages, but by the late Middle Ages a large body of Aristotelian texts were available in Latin and their influence is obvious.

The texts of the Pythagoreans, Plato, and Aristotle combined in this pre-history of the quadrivium to form the basic philosophical tenets that would bind the quadrivial disciplines together. However, none wrote distinct treatises that could serve as quadrivial textbooks. There is no arithmetic by Plato or geometry by Aristotle. Their astronomical works inform quadrivial studies, but tend to be more philosophical than mathematical.

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A Selection of Pre-Quadrivial Texts and Their Philosophical Content

The following selection of texts is by no means complete, but it briefly describes a few of the texts which were either used in quadrivial education, were direct influences on later quadrivial texts, or are uncannily similar to treatments found in later quadrivial texts. The brief descriptions here focus mostly on mathematical philosophy rather than mathematical particulars.

Euclid's *Elements* (ca. 300 B.C.) was and still is the definitive textbook on geometry and was readily available in the late Middle Ages in Campanus of Novara's 13th-century Latin translation. Along with some foundational religious texts, it is one of the most enduring books in the history of literature. We still read the *Elements* as a mathematical textbook. The influence of Euclid's

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the Middle Ages, but his influence on musical/mathematical theory was indirectly propagated via Vitruvius' *De architectura libri decem* (written in the 1st century B.C.), which resurfaced in the Latin West in the 15th century.

Elements has been immense. The *Elements* also contains sophisticated material on number theory (which was frequently incorporated into arithmetical texts), and an analysis of ratios and proportions (which is relevant to speculative music theory, astronomy, and physics). It also has sections on spherics, which are used extensively in astronomy, and it is all bound together with logic, part of the trivium. The material fits perfectly with Plato's instruction in the *Republic* that geometry, and all mathematical instruction, should be abstract and not tied to the world of change.[21]

Euclid is also the supposed author of the *Sectio canonis* (*The Division of the Monochord*), a text which influenced later Greek and Latin music theorists.[22] This text begins by succinctly describing what we might today call frequency. He writes, "it follows that some notes must be higher, since they are composed of closer packed and more numerous movements, and others lower, since they are composed of movements more widely spaced and less numerous."[23] This

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[21] Plato writes, "...it forces the soul to turn towards the region of that beatific reality, which it must by all means behold. ... the true purpose of the whole subject is knowledge– knowledge, moreover, of what eternally exists, not of anything that *comes to be* this or that at some time and ceases to be." Italics mine. Plato, *The Republic of Plato*, VII.527a-c, p. 244.

[22] There is some debate over authorship, but the general consensus is that if it is not by Euclid, it was derived from Euclid.

[23] Euclid (school of), "*Sectio Canonis*," in Barker, ed., *Greek Musical Writings,* vol. 2, trans. Andrew Barker, 192. This part of the *Sectio Canonis* not only suggests a theory of pitch based on a concept of motions per period of time, but also a theory of interval based on these motions per period of time. There is some suggestion that concordant intervals are constructed upon these pseudo-frequencies only if they are commensurate. In his commentary on the *Sectio Canonis*, Andrew Barker suggests that the raising and lowering of pitch described by Euclid happens in discrete steps, not continuously, by noting that the variable in Euclid's pseudo-frequency description limits to an integral number the number of times that a vibrating string can strike the air for a given period of time. However, I see no evidence in the text that Euclid assumed a constant period of time. A variable period of time combined with an integral variable for the beats allows for an infinite regression of frequencies. If the time period is also limited to an integral number, either by some sort of atomic concept or by arbitrary limitation, by my reading, Euclid's theory of pitch would simply eliminate irrational frequencies. Either way, this does not necessarily imply that pitches must raise or lower discretely. At most it evokes the perennial problem between rational and irrational, which will play a large role in the following case studies. It does not necessarily obviate continuity, however neither does it prove it. See also Thomas J. Mathiesen and Euclid, "An Annotated Translation of Euclid's *Division of a Monochord*," *Journal of Music Theory* 19, no. 2 (1975). A more developed theory of vibratory pitch is found in *De audibilibu*s (possibly by Strato of Lampsacus, fl. ca.

pseudo-quantum description of pitch suggests to a modern reader that Euclid's definition of concord or consonance might be described by ratios of these variously spaced "movements." However, the author does not expand upon this physical treatment of pitch, but rather takes a purely mathematical approach, describing consonant intervals in arithmetical terms using only whole number ratios. Owing to its overtly mathematical treatment it is quite clearly anti-Aristoxenian and pro-Pythagorean. The majority of this short text (fewer than 20 pages in the English translation) discusses mathematical harmonic theory in totally abstract terms, with no reference to physical sound beyond the aforementioned description of pseudo-frequency in the beginning and a section at the end where the author suggests that these concordances can be constructed on a stringed instrument to form what, in modern terms, might be called a two-octave diatonic scale.[24] Similar presentations, many inspired by this work, can be found in innumerable music texts well into the 17th century. Euclid also wrote treatises on geometrical optics, conics, and spherical astronomy, all of which are related to quadrivial studies in the Middle Ages, but their direct influence on these topics was generally upstaged by later authors such as Ptolemy and Apollonius of Perga.

The arithmetical and musical texts of Nicomachus of Gerasa (fl. ca. 100 A.D.) were pivotal in the forming of the quadrivium, particularly his *Introduction to Arithmetic* and his *Manual of Harmonics* and to a lesser extent his *Theology of Arithmetic.* His *Introduction to Arithmetic* and

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287 B.C.), extracts of which survive in Porphyry's commentary on Ptolemy's *Harmonica*. This commentary by Porphyry was the main source for Ptolemy's *Harmonica* in the Middle Ages as well as Euclid's *Sectio Canonis*. For *De audibilibus* and information on Porphyry, see Andrew Barker, ed., *Greek Musical Writings,* vol. 2, 98-109 and 229-244.
See also H. B. Gottschalk, "The *De Audibilibus* and Peripatetic Acoustics," *Hermes* 96, no. 3 (1968).

[24] Euclid (school of), "*Sectio Canonis*," 205. He refers to the string of a monochord exactly once in this treatise. Technically speaking, the author constructs the Greater Perfect System, which is a diatonic genus covering two octaves. The diatonic genus is the collection of intervals that we are familiar with in our major and minor scales. However, the intervals described in the *Sectio Canonis* are Pythagorean rather than equal-tempered, a subtle but significant distinction for mathematical metaphysicians.

Manual of Harmonics were profound influences on Boethius and to a great extent form the basis for the latter's 6th-century formulation of the quadrivium. In fact, Boethius' arithmetical and musical texts include large portions of these two books translated straight into Latin. Nicomachus' treatment of this material is that of a Pythagorean, but unlike Euclid, it is not rigorously bound to logical arguments.

The *Introduction to Arithmetic* is a book of number theory, describing such things as odd and even, perfect, and geometrical numbers, means, ratios, and proportions. Boethius' treatise on arithmetic is, for all practical purposes, a translation of Nicomachus' book into Latin, with only a few original additions. A more detailed description of this material will be presented in the next chapter when Boethius is discussed.

In the *Manual of Harmonics*, Nicomachus incorporates many of the structures and terms of Aristoxenian theory, but he presents them from within a Pythagorean/Platonic metaphysics, in effect, Platonizing[25] Aristoxenian music theory. He explains how terrestrial music is derived from the heavens, describing a one-octave planetary "scale" based on Pythagorean tuning and planetary orbital velocities. The harmonic material is very similar to that found in Plato's account of celestial music from *Timaeus* (35b-36b),[26] although Nicomachus gives priority to Pythagoras. Also in this text is the story of Pythagoras and the blacksmith, a story that would be

[25] It might be more accurate to say that Nicomachus Pythagoreanized this material since he tended to downplay Plato, but this distinction is of no apparent importance for Boethius, the primary transmitter of Nicomachus' mathematical works. It is a priority dispute that Plato, justly or unjustly, won. See Flora R. Levin, "Synesis in Aristoxenian Theory," *Transactions and Proceedings of the American Philological Association* 103, no. (1972): 211-234.
[26] See A. Barker's introduction to the chapter on Nicomachus in Barker, ed., *Greek Musical Writings,* vol. 2, 247.

repeated in music theory textbooks for over a thousand years.[27] This story will be discussed in Chapter 2.

In addition to these more technical mathematical manuals, Nicomachus also wrote *The Theology of Arithmetic*. Although the original has been lost, much of it is contained in a book by pseudo-Iamblichus with the same title.[28] This book presents a detailed account of mystical numerological Pythagoreanism, with a particular emphasis on arithmology. The arithmological tradition was often placed under the rubric of arithmetic and was influential in texts throughout the Middle Ages and Renaissance. It begins with the mystical monad, the source of all number, and gives detailed accounts of the numerological and mystical properties that exist in and in-between the numbers of the decad. Of particular relevence to our topic is the section on the number four.

> If number is the form of things, and the terms up to the tetrad are the roots and elements, as it were, of number, then these terms would contain the aforementioned properties and the manifestations of the four mathematical sciences – the monad of arithmetic, the dyad of music, the triad of geometry and the tetrad of astronomy, just as in the text entitled *On the Gods* Pythagoras distinguishes them as follows: "Four are the foundations of wisdom – arithmetic, music, geometry, astronomy – ordered 1, 2, 3, 4." And Cleinias of Tarentum says: "These things when at rest gave rise to arithmetic and geometry, and when moving to harmony and astronomy.[29]

This enumeration of the mathematical arts is the order and content of the Boethian quadrivium as well as the Platonic philosophy of number, which is its basis.

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[27] This story will be discussed below. See Flora R. Levin, *The Harmonics of Nicomachus and the Pythagorean Tradition* (University Park, PA: American Philological Association, 1975); Nicomachus, "The *Enchiridion*," in Barker, ed., *Greek Musical Writings*, vol. 2, trans. Andrew Barker, 245-269.

[28] See Iamblichus (Attributed to), *The Theology of Arithmetic: On the Mystical, Mathematical and Cosmological Symbolism of the First Ten Numbers [Theologoumena Arithmeticae],* trans. R. Waterfield (Grand Rapids, MI: Phanes Press, 1988). According to Robin Waterfield, the translator, the work we have is largely from Nicomachus, but also includes bits from Anatolius (Iamblichus' teacher). It appears to be lecture notes, perhaps from lectures delivered by Iamblichus, but this is not clear. It is dated to the middle of the 4th century A.D.

[29] Ibid., 56.

The author of the *Theology of Arithmetic* goes on to say that arithmetic is like the monad in that it generates the other mathematical sciences as the monad generates number. Music is like the dyad in that it joins together differences, which are only apparent when two elements are compared. Geometry is like the triad because it deals with three dimensions and also because a minimum of three points are required to define a plane. And astronomy is like the tetrad because it deals with the sphere, the most perfect and all-encompassing solid, consisting of center, diameter, circumference, and surface.[30]

The tetrad is also of importance because it is associated with the element of fire, which is associated with the pyramid, a tetrahedral form, with 4 faces and 4 angles. The author points out that the word for fire in Greek is πῠρ (*pur*) and the word for pyramid is πῡρᾰμίς (*puramis*). He further mentions four sources for the universe, four elements, four powers, four directions, four prominent zodiacal positions for the sun, four seasons, four scales of time measurement, four planetary movements, fours senses (disregarding touch which has no identifiable organ), four types of plants, four virtues, four ages of life, four parts to the human body, four sources of rationality, and many more. He also introduces the *tetraktys*, the arrangements of the decad into four rows consisting of 1, 2, 3, and 4 (like bowling pins). This figure will be discussed in Chapter 2. Links are established between the macrocosm and the microcosm (and analogously the soul and the body) via musical relationships found within the first four numbers. Some of the numerical observations in the *Theology of Arithmetic* are perfectly mathematical in a modern sense while others are literary or based on analogies.[31]

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[30] Ibid., 57-58.

[31] Ibid., 54-63. This arithmological tradition is also highly developed by Macrobius. See Ambrosius Aurelius Theodosius Macrobius, *Commentary on the Dream of Scipio*, trans. William Harris Stahl (New York: Columbia University Press, 1990), 1.V.15-VII.9, pp. 98-120. Aristides Quintilianus in his text *De musica* also has an interesting section with arithmological observations. See Aristides Quintilianus, "*De Musica*," in Barker, ed., *Greek Musical Writings,* vol. 2, trans. Andrew Barker, III.6, pp. 502-504.

There are many more pre-quadrivial authors from the Greek and Latin traditions that could be mentioned here such as Eratosthenes (3rd century B.C.), Archimedes (3rd century B.C.), Strabo (2nd-1st centuries B.C.), Vitruvius (1st century B.C.), Varro (1st century B.C.), Cicero (1st century B.C.), Pliny the Elder (1st century A.D.), Theon of Smyrna (2nd century A.D.), Porphyry (3rd century A.D.), Aristides Quintilianus (3rd century A.D.), Plotinus (3rd century A.D.), Augustine (4th-5th centuries A.D.), et al. All of these authors are either directly or indirectly part of the story of the pre-quadrivium and many of these will be discussed as they arise in later chapters. But one more author must be mentioned in this brief survey of pre-quadrivial texts. He is an author who, better than any other pre-quadrivial philosopher, epitomizes the most sophisticated analytical side of the quadrivium, Claudius Ptolemy (fl. mid-2nd century A.D.). Ptolemy wrote hugely influential treatises on astronomy, astrology, geography, optics, and music theory,[32] and his influence in these disciplines remained significant well into the 17th century. Although arithmetic and geometry are not represented as stand-alone texts in his corpus, he addressed these disciplines throughout his works.

Ptolemy embraced both Pythagorean/Platonic numerological philosophy and what might be called Aristotelian empiricism. Unlike many of his mathematical predecessors, he was critical of metaphysical philosophy when it conflicted with observation, but rather than abandon a philosophy of number, he tempered it to allow for new information extracted from experiment and observation. He found a middle road between mathematical and perceptual realism.

He writes,

> [I]t is the proper task of the theoretical scientist to show that the works of nature are crafted with reason and with an orderly cause, and that nothing is produced by nature at random or just anyhow, especially in its most beautiful constructions, the kinds that belong to the more rational of the senses, sight and hearing. To this aim some people

[32] The *Almagest*, *Planetary Hypotheses*, *Tetrabiblos*, *Handy Tables*, *Geography*, *Harmonics*, *Optics,* and the pseudo-Ptolemaic *Centiloquium*. See bibliography for modern editions of these texts.

> seem to have given no thought at all, devoting themselves to nothing but the use of manual techniques and the unadorned and irrational exercise of perception, while others have approached the objective too theoretically. These are, in particular, the Pythagoreans and the Aristoxenians, and both are wrong.[33]

The idealism of Pythagorean metaphysics, which is identifiable in Plato's metaphysics, is intellectually attractive, but difficult to put into practice in the physical world. Platonic and Pythagorean writings are beautiful. The concepts are elegant and poetic, but they lead to mystical transcendence, a return to the world of forms, not earthly knowledge of the shadows. But Ptolemy was interested in what worked here on earth. For Ptolemy, the vague harmonic cosmological descriptions of the cosmos in the *Republic* and *Timaeus* were guidelines, not architectural plans. They were rough sketches of an ideal theory, not finished plans of the ideal world. Similarly the physics, astronomy, and perceptual theories of Aristotle were starting points, not ends in themselves. Ptolemy cannot be fully identified with either school of thought.

Plato claimed that it was a waste of time to listen intently to the sounds of musical instruments or study the motions of the heavens. In the *Republic* he writes,

> They [the harmonicists] lay their ears to the instrument as if they were trying to overhear the conversation from next door. One says he can still detect a note in between, giving the smallest possible interval, which ought to be taken as the unit of measurement, while another insists that there is now no difference between the two notes. Both prefer their ears to their intelligence. ... They are just like the astronomers– intent upon the numerical properties embodied in these audible consonances: they do not rise to the level of formulating problems and inquiring which numbers are inherently consonant and which are not, and for what reasons.[34]

Plato claims that our perceptions of these phenomena will always be imperfect. We can never know these mathematical truths from observation, only through reason and thought.[35] Ptolemy,

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[33] Ptolemy, "*Harmonics*," in Barker, ed., *Greek Musical Writings,* vol. 2, trans. Andrew Barker, 279.
[34] Plato, *The Republic of Plato*, 530c-531c, p. 250.
[35] Ibid., 529d-e, pp. 247-248. Plato does not entirely discount the beauty of our perceptions of the sky and the plucked string. They are imperfect evidence of a higher mathematical truth. But mathematical truths cannot be found by refining data collected from this world of change using our flawed senses. On this point Ptolemy and Plato are not that far apart. Ptolemy also arrives at mathematical truths via reason and

on the other hand, insisted that appearances are trustworthy, but he qualified the manner in which they could be trusted. Perceptions could be useful in evaluating mathematical concepts, but unlike the Aristotelian Aristoxenus, for example, who might claim that perception drives theory and that mathematical theories have little place in music, Ptolemy is more likely to say that perception verifies mathematical theory. In terms of music, he states that the ear can judge differences in pitch and can judge which of two tones is closer to truth, for example, a concordant interval.[36] But the ear is not good at constructing a concordant interval *per se*. The ear can judge. It can hear the true concord located by mathematical construction and differentiate it from a slight deviation in pitch, but the ear cannot dependably construct a concordant interval from scratch. It is unreliable in this task.[37] Similarly, the eye may judge that a circle drawn free-hand is accurate, but will see that it is deficient when compared with a circle constructed by rationality and geometric definition.[38]

For Ptolemy, if the perception did not correspond with theory, perhaps the wrong theory was being used. However, Ptolemy never doubted that a mathematical theory was at work. In this sense his writings on mathematical topics seem quite modern from a methodological point of view.[39] But Ptolemy was also a man of his times. He lived in a world lacking algebraic analysis,

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thought, but he allows perception to verify these truths. Plato only allows perception to enjoy a distorted sensation of some shadow-truth.

[36] Ptolemy, 277.

[37] Ibid.

[38] Ibid., 276-278.

[39] The scientific methods expounded by Ptolemy have been explored by Andrew Barker. See Andrew Barker, *Scientific Method in Ptolemy's Harmonics* (Cambridge: Cambridge University Press, 2000). See also Louis O. Kattsoff, "Ptolemy and Scientific Method: A Note on the History of an Idea," *Isis* 38, no. 1/2 (1947): 18-22.

gravitational theory, and a modern concept of inertia.[40] Even though his tactics are familiar, his theoretical vocabulary is very different from ours.

Generally speaking, Ptolemy worked with an Aristotelian theory of matter[41] in an Aristotelian cosmography and he gave mathematics active properties that today might be given to physics. In this passage from *Harmonics* he describes the active principles of the mathematics of music.

> Since it is natural for a person who reflects on these matters [of harmonics] to be immediately filled with wonder – if he wonders also at other things of beauty – at the extreme rationality of the power of *harmonia*, … , and since it is also natural for him to desire, … , to behold, as it were, the class to which it belongs, and to know with what other things it [*harmonia*] is linked among those included in this world-order, we shall try, …, to display the greatness of this kind of power.
>
> Since all things, then, have as their first principles matter and movement and form, matter corresponding to what underlies a thing and what it comes from, movement to the cause and agency, and form to the end and purpose, we should not accept that *harmonia* is that which underlies (for it is something active, not something passive), nor that it is the end, since on the contrary it is what produces some end, such as good melody, good rhythm, good order and beauty, but that it is the cause, which imposes the appropriate form on the underlying matter.[42]

Citing Aristotle in *Almagest*, Ptolemy divided theoretical philosophy into three parts: physical, mathematical, and theological. The ever-changing qualities (hot, wet, soft, blue, etc.)

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[40] The mathematics and number systems available to Ptolemy were not as inclusive as the mathematics and number systems available for analysis today. Ptolemy utilized a sexagesimal number system (base 60, hence 360º in a circle) of Babylonian origin combined with a Greek system based on 10 for doing the mathematics in *Almagest*. He even utilized a 'place holder' that functioned like a modern zero. This system excelled in addition and subtraction but simple operations like multiplying, dividing, or taking a square root were difficult and time consuming. These limitations dictated much of what was possible to explore and therefore limited the extent to which harmonic theory could be applied to observed phenomena. See O. Neugebauer, *A History of Ancient Mathematical Astronomy*, 3 vols. (New York: Springer-Verlag, 1975); Olaf Pedersen, *A Survey of the Almagest* (Odense: Odense Universitetsforlag, 1974).

[41] Ptolemy's somewhat mechanical theory of perception was different from Aristotle's. See Smith's commentary in Ptolemy, "Ptolemy's Theory of Visual Perception: An English Translation of the *Optics:* English Translation, Introduction, and Commentary by A. Mark Smith," *Transactions of the American Philosophical Society* 86, no. 2 (1996): 21-28; A. Mark Smith, "The Psychology of Visual Perception in Ptolemy's Optics," *Isis* 79, no. 2 (1988): 205-207.

[42] Ptolemy, "*Harmonics*," 371-372.

exhibited by material things located, "for the most part,"[43] in the sublunar world are the domain of physics. Because physics is predominantly a philosophy of terrestrial materials it is subject to the ever-changing nature of the physical world and we can only know about it using our imperfect senses. At the other extreme of theoretical philosophy is theology. This concerns the "first cause of the first motion of the universe."[44] This category can only be imagined, with no manifestation that is knowable in our realm of perception. Because physics is only knowable from perception and theology only knowable in the mind, Ptolemy considers them to be guesswork rather than knowledge. Neither can be verified. He writes, "theology because of its completely invisible and ungraspable nature, physics because of the unstable and unclear nature of matter; hence there is no hope that philosophers will ever be agreed about them."[45]

Mathematics, however, bridges the gap between the philosophy of the physical ever-changing world and the fully abstract philosophy of theology. Mathematics, Ptolemy states in *Almagest*,

> can be conceived of both with and without the aid of the senses, and, secondly, it is an attribute of all existing things without exception, both mortal and immortal: for those things which are perpetually changing in their inseparable form, it changes with them, while for eternal things which have an aethereal nature, it keeps their unchanging form unchanged.[46]

Mathematics is the underlying structure of the terrestrial world and the extraterrestrial heavens, both the world of change and the world of eternal unchanging truth. It is the structure of the macrocosm and the microcosm. It is the formal as well as the motive structure. He writes,

> For in general, each of the things put in order by nature is characterized by some ratio both in its movements and in its underlying materials… [Those things] that are most

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[43] Ptolemy, *Ptolemy's Almagest*, trans. by G. J. Toomer (New York: Springer-Verlag, 1984), 36. This qualification by Ptolemy is part of what makes him difficult to categorize.
[44] Ibid., 35.
[45] Ibid., 36.
[46] Ibid.

> perfect and rational in their natures … are the movements of the heavenly bodies, [divine things] and … those of human souls [mortal things].[47]

Mathematics (as distinct from physics) deals with the form and motion of things that exist. We perceive the mathematical structures and motions of the universe in material substance, but we also understand the mathematics divorced from the physical world, in our intellect.[48] Mathematics is special because it can be observed in the real world. Sheep can be counted, concords heard, physical objects measured, and the heavenly motions predicted. These can be abstracted into intellectual truths or universals. Counting does not require sheep. Harmony can be reduced to the comparisons of numbers and does not require sound. Shapes can be reduced to points, lines, and surfaces, and do not require physical representations. Heavenly cycles can be reduced to counting the motions of an animated geometry and do not require a sun or moon.

Mathematical theory, for Ptolemy, has a foot in both the perceptual world and in the world of forms. Arithmetic, music, geometry, and astronomy can be understood in the intellect, devoid of matter. "For its kind of proof proceeds by undisputable methods, namely arithmetic and geometry."[49] Ptolemy's major astronomical and astrological works were generally available in Latin in the Middle Ages starting in the 12th century; his work on geography was translated and generally available in the early 15th century; but his text on music theory, the most quadrivially robust of his works, was not fully translated into Latin until the late 15th century. However,

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47 Ptolemy, "*Harmonics*," 374.

48 Whether mathematics models reality or drives reality is never unequivocally clarified. In an Aristotelian system, the differences between potential and actual makes such an unequivocal statement difficult, if not impossible to make. A mathematical structure to the universe cannot be actual without stuff in it, therefore such a structure devoid of matter can only be imagined. If that is the case, what would it mean for mathematics to be active? We moderns like to think of forces that exist in space, like gravity or magnetism, but even these are difficult philosophical concepts when the space is empty. Does gravity exist if there is no matter to respond to it? Einstein put the onus on space, and gave it a more active role in physics. In some sense he filled it up with potential activity. Aristotle puts the onus on actual matter. He gave elemental matter an intrinsic motion. Newton and his followers came up with forces that affect matter extrinsically.

49 Ptolemy, *Ptolemy's Almagest*, 36.

Boethius' text on music theory drew upon it heavily and much of Ptolemy's harmonic theory was disseminated in this indirect manner.[50]

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[50] For information on Latin translations of Ptolemy's astronomical texts, see Pedersen, *A Survey of the Almagest,* 16-19; S. J. Tester, *A History of Western Astrology* (Wolfeboro, NH: Boydell Press, 1987), 153. For information on Latin translations of Ptolemy's *Harmonics*, see J. Soloman, introduction to Ptolemy, *Ptolemy Harmonics: Translation and Commentary*, trans. J. Solomon (Boston: Brill, 2000), xxiii-xxv and xxx. For more information on Ptolemy's *Geography*, see Berggren and Jones, introduction in Ptolemy, *Ptolemy's Geography: an Annotated Translation*, trans. J. L. Berggren and A. Jones (Princeton, NJ: Princeton University Press, 2000), 45-52.

Chapter 2:
Introduction and
Quadrivial Background

The following chapter describes the genesis of the quadrivium as a clearly defined program of study in the Middle Ages. Each discipline is treated mathematically and in enough detail such that the larger quadrivial philosophy can be effectively demonstrated. This overview is meant to provide a basis for the case studies in Chapters 3–5. The final part of this chapter outlines the modern scholarship of the quadrivium focusing on the late Middle Ages and Renaissance.

Martianus Capella's Mathematical Arts

The intellectual groundwork for the quadrivium was largely constructed from the enormously important mathematical works of Ptolemy from the 2nd century A.D. Plotinus, Porphyry, Iamblichus, et al. kept the Pythagorean/Platonic philosophy alive and philosophically vibrant in the 3rd and 4th centuries. In the late 4th and early 5th centuries, Augustine recast many fundamental Platonic ideas into the nascent Christian theology.

In the early to mid-5th century, the four mathematical arts come into their own in Martianus Capella's immensely influential *De nuptiis Philologiae et Mercurii [The Marriage of Philology and Mercury]* (between 410 and 439 A.D.). [51] It is in this work that the seven liberal arts are explicitly defined and the mathematical arts that Boethius will soon call the quadrivium are distinguished from the literary arts of the trivium. *De nuptiis* is written in a fanciful allegorical

[51] Just when *De nuptiis Philologiae et Mercurii* was written is the subject of some debate. William Harris Stahl, the foremost expert on Martianus, believes that *De nuptiis* was written some time between 410 and 439. William Harris Stahl and Richard Johnson, *Martianus Capella and the Seven Liberal Arts: The Quadrivium of Martianus Capella*, 2 vols., vol. 1 (New York: Columbia University Press, 1971), 15. Stahl believes that much of *De nuptiis* is derived from a similar encyclopedic work by Varro, now lost, from the early 2nd century B.C. See Stahl and Johnson, *Martianus Capella and the Seven Liberal Arts*, vol. 1, 41-54.

style and takes place in a Neoplatonic celestial arena, where gods (identified as celestial bodies), demigods, and philosophers are assembled to witness the marriage of Mercury to Philology, "an astonishingly erudite young lady."[52] At this wedding, seven learned sisters or handmaidens offered as a dowry[53] (the seven liberal arts), make presentations of their disciplines to the assembly.[54] Each presentation is preceded by a lengthy description of iconography, which has been influential in visual and literary representations of the liberal arts ever since.[55]

His arithmetic (Book VII) is largely derived, either directly or indirectly, from Nicomachus' *Introduction to Arithmetic*, the arithmetical portions of Euclid's *Elements*, and Marcus Terentius Varro's *Disciplinarum libri IX* (2nd century B.C., now lost),[56] and it contains a very heavy dose of arithmology.[57] His geometry (Book VI) is quite literally, geo-metry – earth-measure. It is largely about surveying and geography, with only a short section on Euclidean geometry. His main sources appear to have been Pliny and Varro.[58] His music theory (Book IX) is largely derived from Aristides Quintilianus' *De musica* from the 3rd century and Varro. In this section he discusses the musical motions of the cosmos, the harmonic relationships found in the human body, the musical ethos of various peoples,[59] the arithmetical structures of Pythagorean music

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[52] Stahl and Johnson, *Martianus Capella and the Seven Liberal Arts*, vol. 1, 24.
[53] Ibid.
[54] Medicine and Architeture were not allowed to speak at this wedding for they were considered too mundane. Martianus, writing as Apollo, states, "But since these ladies are concerned with mortal subjects and their skill lies in mundane matters, and they have nothing in common with the celestial deities, it will not be inappropriate to disdain and reject them." Capella, *The Marriage of Philology and Mercury*, vol. 2, 346.
[55] Stahl and Johnson, *Martianus Capella and the Seven Liberal Arts*, vol. 1, 56. For example, see L. D. Ettlinger, "Pollaiuolo's Tomb of Pope Sixtus IV," *Journal of the Warburg and Courtauld Institutes* 16, no. 3/4 (1953): 251.
[56] Stahl and Johnson believe that Varro was a major source for much of Martianus' text. Stahl and Johnson, *Martianus Capella and the Seven Liberal Arts*, vol. 1, 4.
[57] Ibid., 48-49.
[58] Ibid., 44-48.
[59] Capella, *The Marriage of Philology and Mercury*, vol. 2, 356-358.

theory,[60] and rhythm.[61] In comparison to these first three mathematical disciplines, his astronomy (Book VIII) is considerably more sophisticated and had a greater direct influence on later quadrivial scholars. Martianus' section on astronomy is credited, along with Macrobius and Chalcidius, for keeping the theory of a spherical earth alive in the Middle Ages. It also perpetuates both the geo-heliocentric theory in which Venus and Mercury orbit the sun, which then orbits the earth, and it perpetuates the description of Eratosthenes' measurement of the circumference of the earth as 252,000 stadia. Varro is again the most likely source for much of his astronomical theory.[62]

From the 8th through the 12th century, *De nuptiis* was very popular,[63] but by the late Middle Ages only the astronomical part of the disciplinary sections continued to be copied in significant numbers.[64] The elementary presentations of the other mathematical arts had been supplanted by more substantial texts by authors like Euclid, Boethius, and Jordanus de Nemore.[65] In general, the influence of *De nuptiis* was not so much on the individual disciplines, but on the structure of the liberal arts. *De nuptiis* defined the liberal arts in explicit Neopythagorean, Neoplatonic, and Stoic terms.[66] The mathematical part of the liberal arts held the secrets of both the physical and the metaphysical worlds. The stars of astronomy were regarded as human souls, number and music were the structural basis that held the cosmos together, binding the macrocosm to the microcosm. Mathematical pursuits led to the purification of the mind. Studying the

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[60] Ibid., 359-372.
[61] Ibid., 372-381.
[62] Stahl and Johnson, *Martianus Capella and the Seven Liberal Arts*, vol. 1, 50-53 and 69.
[63] Ibid., 56-71.
[64] Ibid., 70-71.
[65] Jordanus de Nemore (fl. mid-13th century) wrote a widely used text on arithmetic and algorism.
[66] Stahl and Johnson, *Martianus Capella and the Seven Liberal Arts*, vol. 1, 85-90.

mathematical arts, as they were described in *De nuptiis*, facilitated the contemplation of truth in a Platonic or Neoplatonic sense and/or an understanding of God in a more Christian setting.[67]

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The Quadrivium of Boethius

The closest thing we have to a quadrivial canon is by the man who coined the word *quadruvium* [sic.] itself, Boethius. In the early 6th century A.D., Boethius set out to write a series of texts for its study,[68] although we now have only two of the projected four: *De institutione arithmetica* and *De institutione musica.* A disputed third book on geometry is, at best, merely a derivation from a lost Boethian original and there is little evidence that the final book on astronomy ever got beyond the planning stages.[69] The two books we do have were cited by nearly every mathematical scholar until the 17th century. The material they contain is well organized, reasonably detailed, reliably accurate, and sets out a coherent structure and program for study.

It is abundantly clear that the Boethian quadrivium was not simply an arbitrary collection of mathematical disciplines. They were not placed in a single category or collection merely as a pedagogical convenience, though the term quadrivium is sometimes used in this way. The quadrivium, as it was defined by Boethius in the first book of *De institutione arithmetica*, refers to a specific philosophical worldview based on mathematics. In this sense, it can be identified as and distinguished from other philosophical systems.

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[67] Ibid., 92-94.

[68] In his text on arithmetic, Boethius vaguely refers to this four-text program in the dedication to his benefactor and father-in-law, Symmachus. See Boethius, *Boethian Number Theory: a Translation of the De institutione arithmetica*, ed. and trans. by Michael Masi (Amsterdam: Rodopi, 1983), 67.

[69] See David Pingree, "Boethius' Geometry and Astronomy," in *Boethius, His Life, Thought, and Influence,* ed. Margaret T. Gibson (Oxford: Blackwell, 1981).

For Boethius and most subsequent quadrivial scholars, the first of the quadrivial disciplines is arithmetic, or the art of counting and number theory.[70] Absolutely fundamental to arithmetic and the quadrivium as a whole is the Platonic concept of Number. Number should not be confused with untrustworthy sensual characteristics like softness or even size. Number is not conserved like Newtonian mass. Number is above and beyond Locke's primary and secondary qualities. If one were to consider Number as a characteristic at all, perhaps it might be considered the *distinctly perfect characteristic of countability*. To count things, is to assign them distinct abstract identities, apart from whatever physical attributes they may have. The Empire State building, an ant, and Australia are three things, regardless of their physical characteristics. We know intuitively that these things are distinct, self-contained Unities, and as such they are countable.

• • •

Quadrivial Arithmetic - Unity to Multitude

A particularly difficult concept that must be acknowledged when discussing most premodern mathematics is that numbers are not points on a number line as we now tend to think. Numbers are quantities, not locations. However, they have no intrinsic quantity by themselves. Their quantity is not fixed. They only have quantity in relation to each other. In the most basic sense, they have quantity in relation to Unity.[71] For Boethius and many subsequent quadrivial writers Unity was not a number. It was "the mother of all numbers."[72] Unity was not one, it was

[70] Boethius, *De institutione arithmetica*, 1.1, pp. 74-75. In the following discussion I will capitalize certain words to remind the reader that these words are very carefully defined in this system.
[71] The question of whether Unity was a number or not was a subject for debate.
[72] Boethius, *De institutione arithmetica*, 1.14, p. 89.

Oneness. This distinction is very important and part of the foundation of the quadrivial philosophy.

Any division of a Unity simply redefines Unity. If you divide Unity in half, the result is not two half-Unities, the result is two Unities. What appears to be division is actually multiplication. Unity is elusive. It cannot technically be divided in two, for if it were, it would no longer be Unity. Division of Unity does not create partial things, it creates more things, a multitude of Unities, and as such, Unity cannot be divided. Plato's concept of Unity is not a fixed measurable, like one meter. In Platonic and Boethian terms, a meter is a magnitude, not a multitude, and as such is infinitely divisible into smaller fractional parts.[73] Unity, on the other hand, is an abstract idea that transcends the world of shadows and change. It is only crudely understandable using our senses here on earth. However, it is intelligible in the world of light or forms. Platonic Numbers are only fully realized as thought, in the abstract. All Numbers imply Unity. As a consequence, all ratios of Numbers must be rational, meaning they are the ratio of two whole numbers. In this system, we do not say "one third of a Unity," what we mean to say is, "one of three Unities." This brings us to Boethius' succinct definition of Number, "Number is a collection of Unities."[74]

Arithmetic is the discipline that formed around this Platonic concept of Number. In addition to defining Unity and Number, arithmetical texts generally discuss a large variety of topics that today might be considered number theory or even numerology. These topics typically categorize numbers and usually include such things as even and odd, perfect, prime, triangular, squared,

[73] Anicius Manlius Severinus Boethius, *Fundamentals of Music [De institutione musica]*, trans. Calvin M. Bower (New Haven, CT: Yale University Press, 1989), 2.3, p. 53. We will return to this issue later in the chapter.
[74] Boethius, *De institutione arithmetica*, 1.3, p. 76. My capitalization for emphasis.

solid or cubed, superparticular, superpartient, a variety of means and averages, and sometimes metaphorical designations like gender or quality.

Boethian arithmetic bears little resemblance to arithmetic today. Addition, subtraction, multiplication, and division were more in the domain of what was called *logistic*. Basic knowledge of these fundamental operations seems to have been assumed by authors of early arithmetical texts. However, a significant shift took place in the late Middle Ages with the introduction of Hindu-Arabic numerals. Basic Platonic numerological theory continued as a quadrivial science, but the operational and algebraic side that had been relegated to practical *logistic* began to develop into a sophisticated discipline in and of itself. Many quadrivial masters by the 14th and 15th centuries had embraced these new methods and incorporated them into the larger Platonic/Boethian arithmetical discipline. These new techniques carried with them their own new forms of analytical abstract theory in addition to offering significantly more computational power that was well suited for astronomical applications. See Figure 2.1.

Figure 2.1: *The Type of Arithmetic* from Gregor Reisch's *Philosophia margarita (1504)*

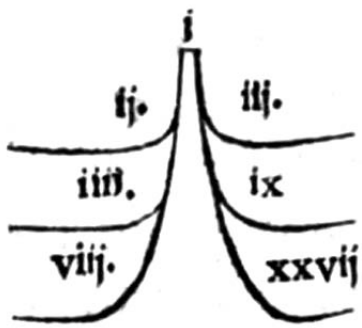
The algoristic method of calculation is shown on the left and the older method of the counting board (similar to an abacus) is shown on the right. Notice how the Platonic Lambda is put on the legs of the arithmetical muse, with the Unity placed at the origin, so to speak. Arithmetic is personified as progenitor.[75]

The metaphysical nature of Platonic Number also inspired a more spiritual realm of numerology, which has lived alongside and even within arithmetic from its earliest origins.[76] Unity only truly exists in the world of ideas. It is the basis for all numbers. In a more metaphorical sense, Unity often took on the role of a mystical progenitor. Plato and many later quadrivial masters observed that a thing can be both one whole thing as well as a collection of many things, simultaneously.[77] For example, the number 8 is both the singular concept of eight-

[75] This woodcut is from Gregor Reisch, *Margarita philosophica* (Joannis Schotti Argentinen, 1504), frontispiece to Book IV. See also Gustave Courbet's painting "The Origin of the World" (1866).
[76] Arithmology is one such mystical form.
[77] Plato, *The Republic of Plato*, 524d-526c, pp. 240-243.

ness and it is a plural conglomerate made up of 8 distinct Unities. Combined with Christianity, this numerological concept could be used to justify Trinitarianism.[78]

In *Timaeus*, Plato metaphorically describes the creation of the cosmos from Unity, as if it were a piece of iron at the forge. He describes a single, unitary, strip of metal being hammered out and then folded in two and three parts, then divided, and then folded again and again, creating a power series based on the first even and odd numbers, 2 and 3.[79] The standard illustration of this configuration, which is virtually ubiquitous in quadrivial literature (though conspicuously absent from Boethius),[80] is called the Platonic Lambda.



1
2 3
4 9
8 27

Figure 2.2: The Platonic Lambda
The woodcut on the left is from the Paris, 1520 ed. of Chalcidius' *Commentary on Timaeus.*[81]

[78] Gillian R. Evans, "The Influence of Quadrivium Studies in the Eleventh- and Twelfth-Century Schools," *Journal of Medieval History* 1, no. (1975): 163.

[79] Plato, *Plato's Cosmology: The Timaeus of Plato*, 35b-36d, pp. 66-77. Recall that one, or Unity, is usually not thought of as a number, but rather as the source of number. Thus, the first odd number is often thought to be 3.

[80] Boethius explicitly refers to Plato and the section in *Timaeus* (35b) where the Platonic Lambda is derived, but Boethius does not describe the mathematics of the Lambda itself. See Boethius, *De institutione musica*, 1.1, p. 2. In *Timaeus*, the description of Unity, in a sense, dividing to generate multiplicity (number) conflicts with a later point that Boethius makes when distinguishing arithmetic from geometry. This may explain why Boethius does not discuss this section.

[81] Plato and Chalcidius, *Chalcidii Viri Clarissimi Luculenta Timaei Platonis traduction, & eiusdem argutissima explanatio* (Paris: 1520), f. XVIIIv. Chalcidius' early 4th-century translation of and commentary on the first two thirds of *Timaeus* was widely available in the late Middle Ages. Stahl, *Roman Science: Origins, Development, and Influence to the Later Middle Ages*, 142.

The Platonic Lambda is both unity and multiplicity. As it folds, recombines, collects and gathers, relationships emerge that do not exist without multiplicity. The Neoplatonists were particularly interested in these relationships and saw them as evidence for "the One." It was thought that these numerical relationships could be followed back, sort of unfolding the metal, (unstirring the cream from the coffee), to the original state, to that first One-ness. In astronomical terms these relationships were reflections of the intelligences or movers of the celestial spheres and Unity was associated with the Prime Mover or *Primum mobile*. Echoes of this creation-from-the-one are evident throughout many western religious traditions including Christianity, Islam, and Judaism, not to mention the obvious singularity concepts in Big Bang theories.[82]

Technically speaking, arithmetic in the quadrivial sense is the study of number in and of itself. With the introduction of ideas about the relationships between numbers, which the Platonic Lambda begins to suggest, we leave the realm of arithmetic and enter the next discipline of the quadrivium – music.

• • •

Quadrivial Music – Multitudes Compared

Boethius' treatise, *De institutione musica* (early 6th century), was undoubtedly the best known music theory text in the Middle Ages and it continued to influence theory well into the Renaissance. Much of its content is based on treatises by Nicomachus and Ptolemy, both of whom were heavily influenced by Pythagorean and Platonic theory. Boethius borrowed from

[82] I am not suggesting that the Big Bang was predicted by Plato, but I do find it interesting that human beings frequently formulate ideas that are so similar, at least in the abstract. And abstract thought is where the real truth lies if you are a Platonist.

this older Greek material, and synthesized a new and very readable text, adding into this mix a few of his own ideas. Boethius' text on music theory was frequently cited by subsequent authors, including Johannes de Muris (ca. 1290-1350), Nicole Oresme (ca. 1320-1382), Prosdocimo de' Beldomandi (ca. 1375-1428), Franchino Gaffurio (1451-1522) and Gioseffo Zarlino (1517-1590). He was both the father of music theory for those after him, and the inheritor of the Pythagorean theory which preceded him, and his organization of this theoretical material was a dominant rubric well into the 17th century.

Boethius divided music into three parts: *musica instrumentalis, musica humana,* and *musica mundana*. *Musica instrumentalis* is essentially instrumental and vocal music. It is covers largely what we now call practical music and music theory. *Musica humana*, human music, is the music of the microcosm and *musica mundana* is the music of the macrocosm. However, music in these last two divisions is a misleading term. Harmony is perhaps a better term. *Armonia* in the ancient Greek sense not only had musical connotations, but also structural meanings such as "a joint" or "a clamp" and was associated with building. Harmony is when things fit together properly. Harmony is structural, it is about multiple things coming together and acting as one, much in the same way that multiple Unities come together as a single number.

Boethius describes *musica humana* as the harmony which "unites the incorporeal nature of reason with the body … [it] unites the parts of the soul, which, according to Aristotle, is composed of the rational and the irrational."[83] This human music holds the disparate and conflicting natures of the body together as one. It allows an immaterial soul to join with a material body. *Musica humana* should be thought of as the well balanced construction, the

[83] Boethius, *De institutione musica*, 1.2, p. 10. Calvin Bower points out in a note to his translation of this passage that although Boethius is associating harmony with Aristotle's concepts of rational and irrational thought (or soul), Aristotle himself does not make this association. See *Nicomachean Ethics* 1.3.1102-3 and *De anima* 432a-b. This issue will come up again and be discussed in more detail in the case study on Prosdocimo de' Beldomandi.

structural oneness, of the human being. Furthermore, in the late Middle Ages, the dominant theory of health depended upon a balance of the four humors (choler, black bile, blood, and melancholy), which were made up of the four elements (air, earth, fire, water). For a healthy person, the distribution of these humors was in harmonic balance. An unhealthy person might have an imbalance of too much blood, for example, causing headaches. A remedy might be leeches or some other method for ex-sanguination, to restore the proper, harmonic, structural balance of the humors.

M*usica mundana*, refers to the structural harmony of the larger cosmos, the macrocosm. In part, it refers to the various theories of the harmony of the spheres made popular by such ancient authorities as Plato, Cicero, and Ptolemy and by later theorists such as Gaffurio, Zarlino, Ficino, Kepler, Fludd, and Kircher, to name but a few. According to Boethius, *musica mundana* "is discernable especially in those things which are observed in heaven itself or in the combination of elements or the diversity of seasons."[84] As with *musica humana, musica mundana* harmonizes the conflicting natures of the four elements, allowing them to coexist in the same body. This harmony did not have to manifest itself in audible sound, although most quadrivial scholars found audible sensation to be a persuasive demonstration of those structurally significant, special numerical relationships that were mutually reflected in the macrocosm and the microcosm. *Musica humana* and *mundana* are two forms of the same harmonic metaphysics that connected the macrocosm to the microcosm and vice versa. A basic mathematical structural principal was shared by both. This connection, using arithmetic and music, later provided astrology with an authoritative metaphysical foundation rooted in the strongest and most basic disciplinary relationship in the quadrivium. Boethius' descriptions of *musica humana* and

[84] Ibid., 1.2, p. 9.

mundana are very brief, but he promises to return to these topics. Unfortunately, he never does.[85]

• • •

The Music of the Monochord

All three of these Boethian musics are at their foundations based on ratios of numbers from arithmetic. The standard instrument for demonstrating these ratios was the monochord. A monochord is simply a string stretched over a sounding box and an assortment of movable bridges. See Figure 2.3.

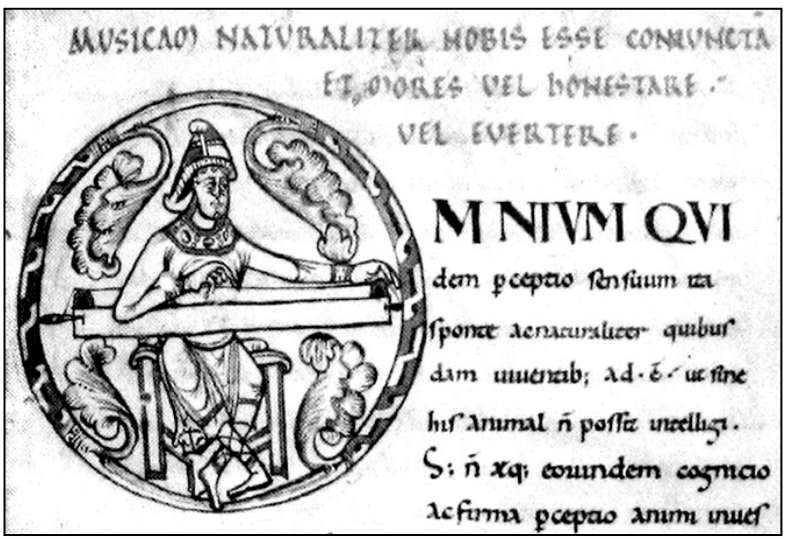


Figure 2.3: Boethius at His Monochord (Paris, Biblioteque Nationale, lat. 7203, f. 8)[86]

Tension on the string is fixed and should not vary. The only variable is the length of a string segment as partitioned by the movable bridges. The monochord is not really an instrument for

[85] Either these sections have been lost or they were never written. Considering how reliant Boethius was on Ptolemy's book on harmonics, it is reasonable to speculate that he would have presented a theory similar to that of Ptolemy. Unfortunately, Ptolemy's astronomical harmonics have not survived in full, although enough has survived to have a good idea of his general approach. However, specifics of Ptolemy's astro-harmonic theories were largely unknown in the late Middle Ages.

[86] Image is from Alison White, "Boethius in the Medieval Quadrivium," in *Boethius, His Life, Thought, and Influence,* ed. Margaret T. Gibson (Oxford: Blackwell, 1981), 173.

performance, it is a philosophical instrument, for testing and demonstrating mathematical theory. Descriptions of harmonic mathematical theory based on this instrument are common.[87] The monochord concretely demonstrated the power of numbers, manifested through the sensation of sound here on earth, and not only in some abstract land of light and thought. It was in audible music that the world of light leaked into the world of shadows. In fact, this is one of the purposes of the quadrivium, as Boethius envisioned it. It was the study of the connections between the world of thought, numbers, and light, and our world of sensation, change, and shadows.[88]

There are several ways to explain this harmonic numerical philosophy, but I think the most interesting and illuminating derivation is from the first four counting numbers, 1, 2, 3, and 4, known as the *tetractys*. See Table 2.1.

4:3:2:1 1 + 2 + 3 + 4 = 10, the decad base-10 number system $10^2 = 1^3 + 2^3 + 3^3 + 4^3$ 10 fingers and 10 toes	i i i i i i + i i i i = X

Table 2.1: Tetractys Quadrivia

Of special importance for music are the superparticular ratios extracted from the *tetractys*, which are in the form (n+1):n, e.g., 4:3.

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87 Christian Meyer lists over 140 texts in the late Middle Ages which specifically deal with monochord theory. See Christian Meyer, *Mensura Monochordi: la division du monocorde (IXe-XVe siècles)* (Paris: Klincksieck, 1996).

88 "This ... is the quadrivium by which we bring a superior mind from knowledge offered by the senses to the more certain things of the intellect. There are various steps and certain dimensions of progressing by which the mind is able to ascend so that by means of the eye of the mind, which (as Plato says) is composed of many corporeal eyes and is of higher dignity than they, truth can be investigated and beheld. This eye, I say, submerged and surrounded by the corporal senses, is in turn illuminated by the disciplines of the quadrivium." Boethius, *De institutione arithmetica*, 1.1, p. 73. According to Aristides Quintilianus, the monochord was recommended by Pythagoras on his death bed, "explaining that the pinnacle of musical excellence is to be achieved intelletually, through numbers, rather than perceptually, through the hearing." See Aristides Quintilianus, "*De musica,*" in Barker, ed., *Greek Musical Writings,* vol. 2, trans. Andrew Barker, 3.2, p. 497.

Referring to Figure 2.4, let us begin by defining a segment of string between B and G and call it 1-unit in length.[89]

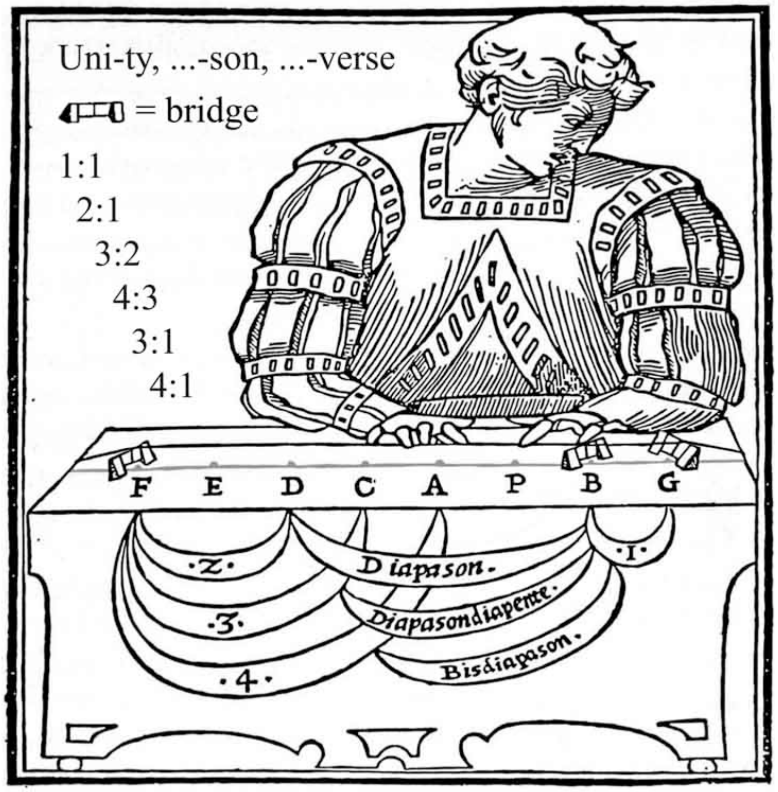

Figure 2.4: Monochord Diagram Derived from a Woodcut in Lodovico Fogliano's *Musica theorica, sectio secunda* (1529), f. 12v.[90]

The banners with the numbers indicate larger divisions as they relate to BG, and the banners with words name the corresponding multiple ratios, 2:1, 3:1, and 4:1.

The initial choice of this unit-length is completely arbitrary. The length between B and G was chosen because it makes it easier and clearer to describe the subsequent intervals all on a single image. This quadrivial system has to do with comparison only; there is no fixed pitch. In fact,

[89] I suggest preparing for this section by running the major scale through in your mind or aloud, *do-re-mi-fa-sol-la-ti-do*, using Rodgers and Hammerstein's "Do-Re-Mi" from *The Sound of Music* if necessary. For authenticity's sake, do not play these intervals on a modern fretted, keyed, or keyboard instrument. The modern equal tempered intervals found on most instruments are slightly different from most of the Pythagorean intervals I will be describing.

[90] The original image is from Lodovico Fogliano, *Musica Theorica* (Venice: J. Antonius, 1529), f. 12v.; facs. Bologna, Forni, 1970. I reworked this image to clarify some points I will be making in the subsequent description. I redistributed the bridges, lightened the measurement line so it would not be confused with a string, added the point labeled "P," added a few observations, and several other minor alterations. Do not let the strange fingers bother you. Not all authors are lucky enough to get woodcuts by Dürer, Holbein, or Calcar.

that is one of the main points of quadrivial music. Everything is scalable up to the macrocosm and/or down to the microcosm.

The string divisions marked out between FE, ED, DC, CA, AP, and PB are the same as the length BG. Imagine placing an additional bridge at P and then plucking the string segment between P and B and comparing that pitch to the one produced by plucking the segment between B and G. Each segment is 1-unit in length. The ratio between them is 1:1. The sonic relationship between these two sounds is unison, uni-son, one-sound. Sing to yourself, *do–do or mi–mi* or any repeated pitch.

Now imagine adding a bridge at D. This will divide off a segment of string that is 2-units long between F and D. The numerical ratio between FD and our unit segment, BG, is 2:1. Pluck the segment FD and compare that sound to BG. The shorter segment, BG, is higher. The difference is an octave or *diapason*.[91] Sing the first and last *do* in the major scale. These two pitches are different and yet somehow the same. This strange feature of the octave was not lost on Boethius, who comments on how much the octave is like unison.[92] The similarity is so uncanny that we often do not bother to give new names to pitches that span an octave. The octave and integral multiples of the octave are the only pure Pythagorean intervals on the modern equal-tempered piano. The octave can also very easily be found by ear, invariably resulting in a 2:1 ratio in string lengths.[93] Whether by ear or by mathematics, there is something very special about this interval.

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[91] Other names for the 2:1 interval are octave, P8, Double, and *Duplex.* It is 1200 cents in modern logarithmic intervalic measurement.

[92] "The consonance of the diapason produces a conjunction of pitch such that the string seems to be one and the same. Even the Pythagoreans agreed on this point." Boethius, *De institutione musica*, 5.9, p. 169. See also Ptolemy, "*Harmonics*," 289.

[93] String harmonics (made by lightly dampening the string) are another way to easily locate the octave, but appear not to have been discussed in premodern literature. I have no doubt that anyone even barely proficient in any stringed instrument would have been aware of dampened-string harmonics, and yet I

Now rearrange the bridges so that they are located at F, C, A, and B. FC is 3-units long, and AB is 2. Pluck FC and then AB. The shorter segment, AB is exactly a fifth above FC. [Sing, *do-sol. Do-re-mi-fa-sol... do-sol.*] This perfect fifth[94] is produced from the measurement ratio 3:2 and like the octave, is a superparticular ratio. Also like the octave, it is possible to find this interval precisely by listening to the two pitches and adjusting the bridges to make the most sonorous interval. This perfect fifth, 3:2, simply resounds more than say 3.1:2. It mysteriously resounds much like the octave, but not to the same degree. The octave is blatantly special. The perfect fifth is subtly special, but noticeably so.[95] This interval is not found on the modern piano which utilizes equal tempered tuning.[96] Be aware that we could have also constructed this ratio with the segments FB and CG, or any other 3:2 configurations. The interval will always be *do-sol* even if we redefine the pitch assigned to *do*. This way of thinking about sound stresses what is in between pitches, the relationships, not the pitches themselves. Like arithmetical Unity, the quadrivial pitches are unfixed and impossible to pin down, but the relationships are absolute.

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have not found this topic seriously discussed in any of the major treatises on music theory before Mersenne discusses the overtone series in the 17th century. See Marin Mersenne, *Harmonie universelle: the Books on Instruments*, trans. Roger E. Chapman (The Hague: M. Nijhoff, 1957), 208-250.

[94] Other names for the perfect fifth are P5, *Sesquialtera*, and *Diapente.* It is 702 cents.

[95] The strings on a violin are spaced in fifths and are often tuned to perfect fifths by listening for this special ring.

[96] In terms of string lengths, the fifth on the modern piano is approximately 2.997:2. Today this interval is measured logarithmically. The modern fifth is 700 cents, compared to the perfect fifth, 3:2, which is 702 cents. Equal tempered tuning has 100-cent half steps. This system allows intervals to be added and subtracted rather than multiplied and divided as the Pythagorean system demands. The introduction of logarithms in the early 17th century significantly changed how people used and thought about mathematics and the mathematics of tuning. The mathematical manipulations of monochord intervals are discussed below.

The last consonant superparticular ratio found amongst the *tetractys* is the 4:3 ratio. Reposition the bridges to G, A, and F. Pluck FA and AG. AG will be a perfect fourth[97] above FA. Sing, *do-fa. Do-re-mi-fa... do-fa.* ("Here comes the bride...")

These three intervals, 4:3, 3:2, and 2:1 were considered consonant intervals. With very little practice, all three can be located exactly by ear. Not only is it miraculous that the three most resounding intervals happen to fall directly out of the relationships of the first four counting numbers, the *tetractys* (4:3:2:1), but these intervals also relate internally and reflexively. If you start at *do*, go up a fifth, 3:2, to *sol*, and then from there go up another fourth, 4:3, you arrive at *do*, an octave, 2:1, above the starting pitch. Mathematically this is done with multiplication. 3:2 times 4:3 is 12:6, which simplifies to 2:1. Note also that division is the way differences between intervals are found. For example, 2:1 divided by 4:3 (which is the same as multiplication by the reciprocal, 3:4) yields 3:2. Not only are these intervals created from the *tetractys*, but the octave contains the other two intervals. The remaining non-superparticular intervals from the *tetractys* are thus easily understood. The 3:1 ratio, is simply an octave and a fifth, 2:1 times 3:2, and the 4:1 ratio is simply the octave and another octave, 2:1 times 2:1. These are generally considered to be consonant intervals.[98]

To construct a diatonic scale, e.g., the major scale I have been using for reference, we need to do a little math. The smallest interval we have looked at is the fourth, 4:3, *do-fa*. To construct a diatonic scale covering one octave, with 7 intervals (and 8 pitches), we need to find smaller intervals than 4:3. We have defined a fourth, *do-fa*, and a fifth, *do-sol*. The interval between *fa* and *sol* is a smaller interval. All we need to do is find its mathematical ratio. We know that

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[97] Other names for the perfect fourth are P4, *Sesquitertia*, and *Diatessaron.* The fourth on the piano is approximately 4.005:3. The Pythagorean perfect fourth is 498 cents, compared to 500 for the modern equal tempered fourth.

[98] Some theorists include other intervals in addition to this set, but this collection is standard across all Pythagorean-based quadrivial literature.

division is how we find the differences between intervals. Thus, 3:2 divided by 4:3 (or multiplied by the reciprocal, 3:4) yields the interval, 9:8. This is the definition of the tone or *tonus*.[99] Miraculously, this happens to be a superparticular interval with both numbers under 10, although it is generally not considered consonant. The diatonic scale is now almost finished. If we start at *do*, go up a *tonus*, 9:8, to *re*, then go up 9:8 again to *mi*, we arrive at the interval of 81:64, *do-mi*. On our next jump we want to land on the fourth, *fa,* 4:3, but if we go up another 9:8 we arrive at 729:512, which is too far. In order to land on 4:3, we must do some more math. The interval between 4:3 (*do-fa*) and 81:64 (*do-mi*) is 256:243, *mi-fa*. This is the Pythagorean semitone.[100] With this we have now mathematically derived all the intervals necessary for a diatonic scale.[101] See Table 2.2.

	do	*tonus*	*re*	*tonus*	*mi*	*semi-tonus*	*fa*	*tonus*	*do*	*tonus*	*re*	*tonus*	*mi*	*semi-tonus*	*fa*
4:3	=	9:8	X	9:8	X	256:243	=								
3:2	=	9:8	X	9:8	X	256:243	X	9:8	=						
2:1	=	9:8	X	9:8	X	256:243	X	9:8	X	9:8	X	9:8	X	256:243	=

Table 2.2: The Pythagorean Diatonic Scale[102]
Indicated in the rows are groupings of 4:3, 3:2, and 2:1.

The cosmological systems of *musica mundana* were generally based on variations of this diatonic arrangement. The general idea was to relate either the radial distances or the orbital angular velocities of the heavenly bodies to some form of the Pythagorean scale similar to the one derived above. There are numerous classical examples, the most complex proposed by

[99] The *tonus* is roughly the equivalent of the modern whole step. The *tonus* is sharper (204 cents) than the equal tempered whole step (200 cents). Measured on a string the equal tempered whole tone is approximately the ratio, 8.980:8.

[100] Technically speaking, it is the minor semitone. This interval (90 cents) is very roughly equivalent to the equal tempered half step (100 cents). Measured on a string it is approximately the ratio, 257.450:243.

[101] This is more specifically called Ptolemy's *Diatonic Ditonic.* The major scale I have been referring to is the ancient Lydian mode, or Ionian using Glarean's 16th-century designations. See James Murray Barbour, *Tuning and Temperament: A Historical Survey*, 2nd ed. (East Lansing: Michigan State College Press, 1953).

[102] The two disjunct diatonic tetrachords (separated by a *tonus*) are shaded in gray.

Ptolemy in his book on music theory, *Harmonics*,[103] and a version based on Plato's "Myth of Er" and *Timaeus* proposed by Cicero in his *De re publica*, known in the Middle Ages via Macrobius' commentary.[104] Later versions such as the one depicted by the frontispiece of Gaffurio's *Practica musicae* (1496) is much simpler than Ptolemy's, but characteristic of the general idea. See Figure 2.5. Gaffurio's harmonic cosmos shows the earth and its four elements down at the bottom and a string/serpent-like creature extending up into the heavens where Apollo, the god of music among other things, sits on a throne holding a stringed instrument.

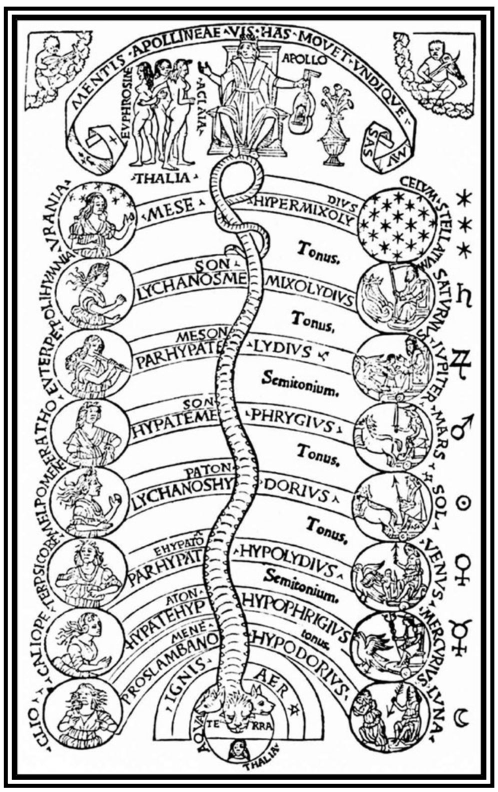

Figure 2.5: Gaffurio's Harmonic Cosmos from *Practica musicae* (1496)[105]

The banner above reads, "The force of Apollo's mind moves the muses in all respects." To the right of the serpent/string the vertical divisions are separated by tones and semitones. Also

[103] Ptolemy, "*Harmonics*," III.4-6, pp. 374-391.
[104] Marcus Tullius Cicero, *De re publica and De legibus*, trans. C. L. Keyes, The Loeb Classical Library (Cambridge, MA: Harvard University Press, 1928; reprint, 1977), VI.xvii-xix, pp. 268-273; Macrobius, 2.I-IV, pp. 185-200.
[105] This is the frontispiece of *Practicae musicae* (1496). Franchinus Gaffurius, *Practica Musicae,* trans. Clement A. Miller (American Institute of Musicology, 1968). See also James Haar, "The Frontispiece of Gafori's *Practica Musicae* (1496)," *Renaissance Quarterly* 27, no. 1 (1974).

indicated in a vertical arrangement are the modes or octave species, e.g., Dorian, Phrygian, Lydian, etc. Farther to the right are the sun, moon, planets and the starry sphere. On the left side are shown the muses and the Greek names of the notes in this diatonic system.

In the early 17th century a very sophisticated version of *musica mundana* was proposed by Johannes Kepler, father of modern planetary kinematics, in his book *Harmonice mundi* (1618). His musical astronomy echoes the structure and the complexity of Ptolemy's, but is modeled on a heliocentric cosmos and uses the algebraic mathematics of his day.[106] In the late 17th and early 18th centuries Isaac Newton explored the idea that the colors of the rainbow were distributed as if along the string of a monochord. See Figure 2.6. The diatonic scale Newton describes is slightly different from the one constructed above,[107] but the consonant intervals are all the same.

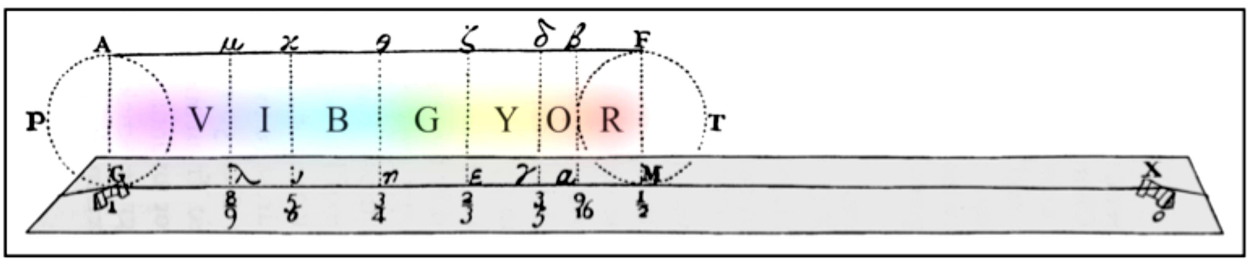

Figure 2.6: Newton's Spectral Monochord
The rainbow's colors, ROYGBIV, are shown in relation to a diatonic scale on a monochord. This is a composite image based on an illustration from Newton's "Opticks" (1704).[108]

There are several other ways to derive the same set of Pythagorean consonant intervals and this variety of paths to the very same consonant ratios only added to their mystique. One of the

[106] See Johannes Kepler, *The Harmony of the World [Harmonice Mundi]*, trans. E. J. Aiton, A. M. Duncan, and J. V. Field (Philadelphia: The American Philosophical Society, 1997).

[107] Newton seems to be using a form of just intonation which was extensively researched from the 15th to the 18th centuries, notably by Fogliano, who coincidentally provided the monochordist woodcut for Figure 2.4. There were many different tuning systems in this period. J. Murray Barbour lists almost 200 different tunings in his book *Tuning and Temperament, a Historical Survey*. In contrast to Newton, Boethius discusses the colors of the rainbow in a section discussing the continuous versus the discrete. See Boethius, *De institutione musica*, 5.5, p. 167.

[108] Original image from Isaac Newton, *Opticks: or, A Treatise of the Reflections, Refractions, Inflections & Colours of Light. Based on the 4th ed., London, 1730* (New York: Dover Publications, 1952), 127. I added bridges, a monochord body, and "ROYGBIV" to Newton's original, which depicts a monochord, but it is difficult to recognize without my alterations.

most popular descriptions of consonance serves as a creation myth of sorts. It is referred to in most music theory texts of the Middle Ages.[109] It is the story about Pythagoras himself walking by a blacksmith. The mystical philosopher is said to have noticed perfect harmonic intervals in the sounds of the clanging hammers. He weighed the hammers and found the sequence: 6, 8, 9, 12. From this sequence the consonant intervals are constructed without any effort. For example, the octave is 12:6, the fifth is 9:6, and the fourth is 12:8. Another source for these intervals is the Platonic Lambda which we saw before in Figure 2.2. Yet another source comes from the arithmetic, geometric, and the harmonic means, which are described in a wide variety of quadrivial texts.

We moderns tend to consider that the variety of paths to these same numbers and ratios are products of coincidence or tautologies, and dismiss them as meaningless trivia. For quadrivial numerologists, numbers and the relationships between them were active. The power of numbers in our world of shadows and change was evidence of their source in the world of light, or the heavens, and thus confirmed the connection between these two worlds.

We can perceive the power of numbers in audible music. We can actually identify with our ears the perfect Pythagorean consonances of the fifth, fourth, and the octave. We can also perceive the power of numbers in the effects that certain modes, arrangements of these intervals, have on our moods. Boethius writes,

> One night, when a whore was closeted in the house of a rival, this frenzied youth wanted to set fire to the house. Pythagoras, being a night owl, was contemplating the courses of the heavens (as was his custom) when he learned that this youth, incited by the sound of the Phrygian mode, would not desist from his action in response to the many warnings of his friends; he ordered that the mode be changed, thereby tempering the disposition of the frenzied youth to a state of absolute calm.[110]

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[109] For example, Boethius, *De institutione musica*, 1.10-11, pp. 17-19; Macrobius, 2.I.9-14, pp. 186-188.
[110] Boethius, *De institutione musica*, 1.1, p. 5.

In more modern terminology we might say that the minor scale is sad, and the major scale is happy, or that the blues are blue… etc.[111]

The consonant intervals are found both abstractly and from observation. Certain intervals just resound more than other intervals and those that resound the most are at the string length ratios of 2:1, 3:2, and 4:3. It is hard to debate whether or not an octave is an interesting interval. It just sounds bigger, but what does this mean? Do numbers have a relationship to some sort of power, or is meaning simply assigned to a coincidence? Is this supposed hierarchy that we perceive in the musical numbers innate or learned, nature or nurture? How much of this is culturally constructed, how much is phenomenological, how much exists if there is nobody around to think about it? These are big questions and they continue to be debated. There were opponents of this Pythagorean/Platonic quadrivial numerology who did raise objections. These often including practical musicians who found much of this material interesting, but not usable in their occupations due to excessive limitations it placed on polyphony and modulation and too much harping over subtleties that were simply too fine to worry about or could not be heard.[112]

The theory of tuning derived from both experimentation and numerology is one of the essential links that holds the quadrivium together. But music theory is much more than just tuning. There are numerous aspects to quadrivial music that I have not discussed at all and I should, at the very least, name them. Musical texts generally contain discussions of the following: tetra- or hexachord structures, classical and/or medieval Church modes, compositional theories, musical ethos, alternative tuning systems, the use of *ficta* or accidentals, rhythmic

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[111] Technically speaking these are modern "modes," but similar observations were made in much of the literature about the emotional characteristics, the ethos, of the classical or medieval modes. See Ibid., 1.1, pp. 2-8.

[112] See the works of Aristoxenus for the most prominent ancient opponent and Vincenzo Galilei, Galileo Galilei's father, for a later example.

theories, and a variety of mathematical shortcuts and tricks. These texts can run into the hundreds of pages.

• • •

Quadrivial Geometry – Static Magnitude

To this point we have looked only at whole numbers, the positive integers, and ratios constructed from them. Boethius refers to the quantities in the first two parts of the quadrivium (arithmetic and music) as multitudes. The second half of the quadrivium, geometry and astronomy, attends to continuous rather than integral measure, what Boethius and Euclid refer to as magnitude. Today we might be tempted to refer to magnitude as the positive real numbers, the rational plus the irrational numbers. However, for Boethius and later quadrivial scholars it was not so simple. Multitudes were one thing, and magnitudes were another. Boethius writes, "That which is continuous is called 'magnitude,' whereas that which is discrete is called 'multitude.' The properties of these are different and even opposite."[113]

Boethius explains that multitude starting from a finite quantity, i.e., unity, can increase indefinitely. There is a limitation on the smallest unit, but there is no limitation on counting. Whereas geometrical multitude can start with a finite quantity, but instead of being able to increase infinitely, it is infinitely divisible. It can be halved, and halved again forever. He writes,

> Magnitude is thus limited insofar as the larger measure is concerned, but it is infinite when it begins to divide. Number (that is, multitude), to the contrary, is limited with regard to the smallest measure but begins to be infinite when it multiplies.[114]

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[113] Boethius, *De institutione musica*, 2.3, p. 53.
[114] Ibid.

Multitude has a smallest quantity, unity, but no limitation on a largest, whereas magnitude has a limitation on its largest quantity, but no limitation on its smallest. As he stated above, "The properties of these are different and even opposite." Given the name of the quadrivial discipline, *geo*-metry, and the limitations Boethius places on the size of magnitude, it would seem that he meant for magnitude to be associated with the measurement of the perceptible world, and that it was fundamentally distinct from arithmetical multitude (number), which was an abstract concept divorced from the world of change. The most likely reason for why multitude would be limited "insofar as the larger measure is concerned," is that the Aristotelian cosmos was determined to be finite in size – finite in magnitude. The standard argument being, if the spherical cosmos were infinitely large, and it were spinning, the result would be an infinite linear velocity, which was nonsensical. Thus, the cosmos must be finite in magnitude.

Because magnitude was not limited to a smallest unit, i.e., unity, it could be divided indefinitely. Aristotelian space was continuous, not discrete. But multitude, unity and the numbers generated from it, were discrete. They existed in the intellect and were the very definition of rationality and ratiocination. The world of change measured by magnitude, however, was not necessarily rational.

Given this apparent mixture of Pythagoreanism, Platonism, and Aristotelianism as a background, Boethius explicitly distinguishes the arts of multitude from those of magnitude.

> Geometry speculates about fixed magnitude, while astronomy pursues knowledge of movable magnitude; arithmetic is the authority concerning quantity that is discrete in itself, whereas music is clearly expert concerning quantities related to other quantities.[115]

Magnitudes are the quantities of geometry and astronomy, while multitudes (numbers) were arithmetic and music.

[115] Ibid., 2.3, p. 54.

In the first approximation, geometry is to magnitude as arithmetic is to multitude. If one wants to measure anything in the world we live in, the world of change, it cannot be done with counting numbers alone, no matter how finely the base unit (unity) is defined. Counting numbers are entities in the world of *ratio*, in the Latin sense of the word, meaning not only a *quantitative relationship*, but also meaning *reason,* as in rational thought. Rational numbers are from the world of light and thought. Quadrivial numbers cannot represent a magnitude like the length of the diagonal of a 1 x 1 square (√2). This magnitude is irrational – ir-*ratio*-nal. It is not a quantitative relationship between two counting numbers and it is not a number in the world of light and thought. It is not a number at all. It is a magnitude. Quadrivial geometry, literally "earth measure," is the mathematical discipline which allows you to analyze, categorize and operate with static magnitudes, magnitudes that are encountered on earth, in the world of shadows and change.[116]

Of all the quadrivial disciplines, geometry is probably the most familiar to the modern student. The undisputed king of geometrical texts was Euclid's *Elements*. Starting with axioms, for ideas like point, line, and plane, Euclid systematically built the discipline of geometry using proofs and constructions. Most students today only study a small portion of the two-dimensional material, plane geometry, but later books in the *Elements* cover solid geometry (e.g., spheres and regular solids), which is vital for astronomy, as well as proportion and number theory similar to treatments that are found in arithmetical and musical texts. Also in the *Elements* is a lengthy discussion of means, such as the arithmetical, geometric, and harmonic means, which are directly

[116] Some authors proposed that geometry, like music, provided a link between the worlds of light and shadow. Where quadrivial music was theoretical and/or aural, quadrivial geometry was theoretical and/or visual. Geometry also extends the ideas of *musica humana* and *mundana* by involving all measurements, not just the integral ones. See Michael Masi, "Boethius and the Iconography of the Liberal Arts," *Latomus* (1974): 66.

related to music theory, as well as the "extreme and mean ratio," also known as the golden ratio.[117]

Geometry is the third in the progression of the quadrivial disciplines, not so much because it is based in music and arithmetic, but because it is necessary for the study of the last quadrivial discipline, astronomy.

• • •

Quadrivial Astronomy – Magnitude in Motion

Imagine lying on your back in a clearing on a cloudless night. Your skyward field of vision could be drawn as a circle, the boundary of which might be a tree-lined horizon all the way around. Assuming you are in a middle latitude in the northern hemisphere with your feet pointing northward, the North Star will be down closer to your feet. See Figure 2.7.

[117] See Euclid's *Elements*, Book 2, Proposition 11, Book 4, Propositions 10-11, and Book 13, Propositions 1-6, 8-11, 16-18.

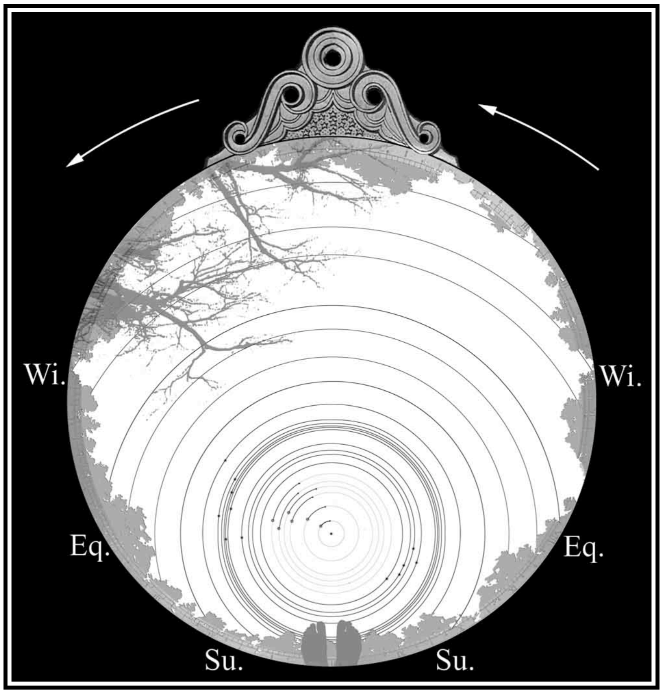

Figure 2.7: The Cosmos from a Terrestrial Point of View[118]

The Little and Big Dippers and Cassiopeia are shown near the rotational center, the North Celestial Pole (NCP). Star-trail segments emphasized for the Little Dipper are ca. 3 hours.

Before the telescope, all you saw up in the sky were specks of light. These specks of light moved. How did they move? They moved in circles, counterclockwise in the northern hemisphere. Modern knowledge of a galaxies, exploding super novae, and a nearly vacuous outer space was unheard of. Mountains on the moon were even an odd idea. To the premodern scholar, these were abstract points of light in the heavens that traced out perfect circles. Obviously, the best way to describe these points that moved in circular paths on the hemispherical dome of the sky was somehow going to involve geometry, the mathematical discipline that explicitly deals with points, circles, and spheres. This is the Boethian definition of astronomy, the discipline that deals with geometry in motion. Static magnitude was the domain of geometry, so astronomy was essentially animated geometry,[119] which lives in the heavens.

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[118] I have oriented the eastern horizon on the right to follow modern conventions. For brevity, I am not going to discuss stereographic projection, which would require a specialized essay of its own. Circles Su. and Wi. are the approximate paths of the summer and winter suns and Eq. is roughly the path of the sun on the vernal and autumnal equinoxes.
[119] Boethius, *De institutione musica*, 2.3, p. 54.

This perfect, shining, moving geometrical diagram in the sky was the key to answering the terrestrial questions *where* and *when*. The motions of the sky seemed to be cyclic and therefore potentially predictable. The center of these concentric circles for the northern hemisphere is the north celestial pole (NCP), currently the star Polaris. See Figure 2.7. How far above the horizon this pole star is, tells you your latitude, how far north of the equator you are. If you are on the equator the north star is on the horizon, and if you are at the north pole the north star is directly overhead. Measure this angle (the declination) and know your latitude. In Figure 2.7, the latitude is roughly 45º. This type of geo-metric, earth-measuring, empirical knowledge combined with abstract geometrical reasoning of the Euclidian style is what allowed Eratosthenes to measure the circumference of the earth in the 3rd century B.C. and is a fundamental concept of cartography. Knowledge of the heavens is knowledge of the earth.

Furthermore, like the hour-hand on a 24-hour clock (that runs counterclockwise in the northern hemisphere), the rotation of stars around this celestial pole tells you how much time has elapsed. In Figure 2.7, the stars of the Little Dipper (Ursa Minor) are shown with a 3-hour time-lapse trail emphasized. This longitudinal motion measured in angles is traditionally apportioned in hours, minutes, and seconds. The sky is a giant, universal clock, and a giant terrestrial GPS.

Generally speaking, the astronomy of the quadrivium is geocentric and geostatic. The earth, as we are naturally accustomed to think, is the center of the cosmos and it does not move in any way, no spinning or orbiting. Quadrivial astronomy is primarily based on the Aristotelian model of the cosmos, see Figure 2.8, in which the stationary, spherical earth, comprised of the four elements (earth,[120] water, air, and fire)[121] is surrounded by a set of spinning concentric spheres,

[120] In preparing this image I flipped and rotated the earth to our usual north-up orientation. The original woodcut had a mirror image of what it should have been, which is not an unusual error due to the mirror-image process of making woodcut prints. Technically speaking, it might be more accurate to show the earth from above (from the pole of the zodiac), rather than the traditional side view as shown here.

each of which carries a celestial body (moon, Mercury, Venus, sun, Mars, Jupiter, and Saturn),[122] with the outer-most sphere carrying the fixed stars, which we previously observed from the point of view of the earth in Figure 2.7.

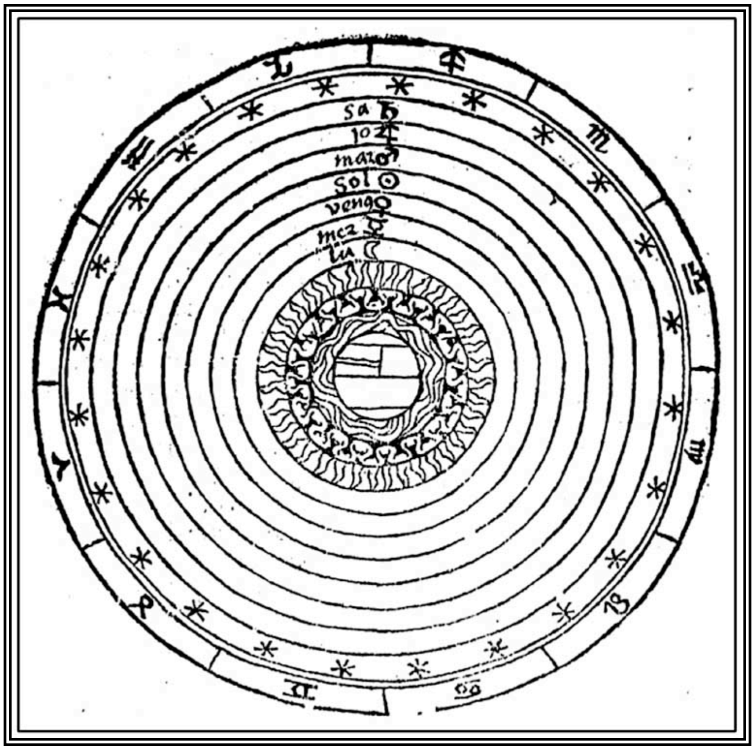

Figure 2.8: Woodcut from a Printed Commentary on Sacrobosco's *Sphere* (1508 ed.)

This shows the geocentric arrangement of spheres that carry the sun, moon, planets, and the stars.[123]

The main point of reference in the sky is this sphere of fixed stars. For all practical purposes, they never change in relation to each other. They are always the same. They just spin silently overhead, one rotation per day. In the first approximation, everything spins once per day, including the sun, moon, and planets. However the sun, moon, and planets are not truly stationary in relation to the fixed stars; they move amongst the stars.

The sun has the simplest of these motions which repeats itself roughly every 365 days. In relation to the fixed stars, the sun travels on its own circular path, a circle called the ecliptic. It is tilted about 23º off the celestial equator, which is essentially the terrestrial equator projected out

[121] Water, Air, and Fire are depicted in concentric rings, using typical medieval iconography.
[122] Uranus, Neptune, and Pluto were not observed until the development of high-powered telescopes.
[123] Woodcut from Johannes de Sacrobosco, *De sphaera* (Venice: 1508), 2v.

to the stars. Describing all of the intricacies of Ptolemaic astronomy would introduce unnecessary complexity into this overview, the primary focus of which is on the relationships between the quadrivial disciplines. With this goal in mind, it will be sufficient to highlight the essential nature of spherical geometry as the mathematical basis of astronomy.

In the central diagram of Figure 2.9 we see the outer sphere of the fixed stars. The entire sphere spins on the axis labeled "A" every 24 hours. This axis pierces the north star (see NCP on Figure 2.7). Circle B, drawn on this starry sphere, is the ecliptic, the path that the sun follows over the course of a year within the fixed stars. The sun has two major motions, the daily motion that it receives from the rotation of the starry sphere, and the annual motion it has in relation to the starry sphere. The woodcut from Sacrobosco's *Sphere* in the upper right hand corner[124] of Figure 2.9 shows these two motions superimposed, combined, and simplified over the course of a year.[125] When the sun is at point C of the ecliptic, at the top of its travels, it is summer on earth in the northern hemisphere, and when it is at the bottom of the ecliptic it is winter. Summer, Spring/Fall, and Winter solar trajectories are indicated with the labels, Su., Eq., and Wi., respectively on Figures 2.7 and 2.9.

The shaded disk in Figure 2.9 is the horizon, positioned to approximate a horizon for southern Europe. This is the dividing plane for star-rise and star-set. Its periphery corresponds to the circular 360º view shown in Figure 2.7. It is stationary, along with the earth, while the starry sphere with its accompanying celestial objects spin. Daylight ends for this southern European location when the sun goes behind this horizontal disk. Note that the summer sun (Su.) has a much longer arc above the horizon than the winter sun (Wi.). For the corresponding arcs from a terrestrial point of view, see Figure 2.7.

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[124] Ibid., 41r.

[125] This woodcut from Sacrobosco's *Sphere* shows only 5 of 365 days. Showing all 365 would have been difficult to carve in such a small woodcut.

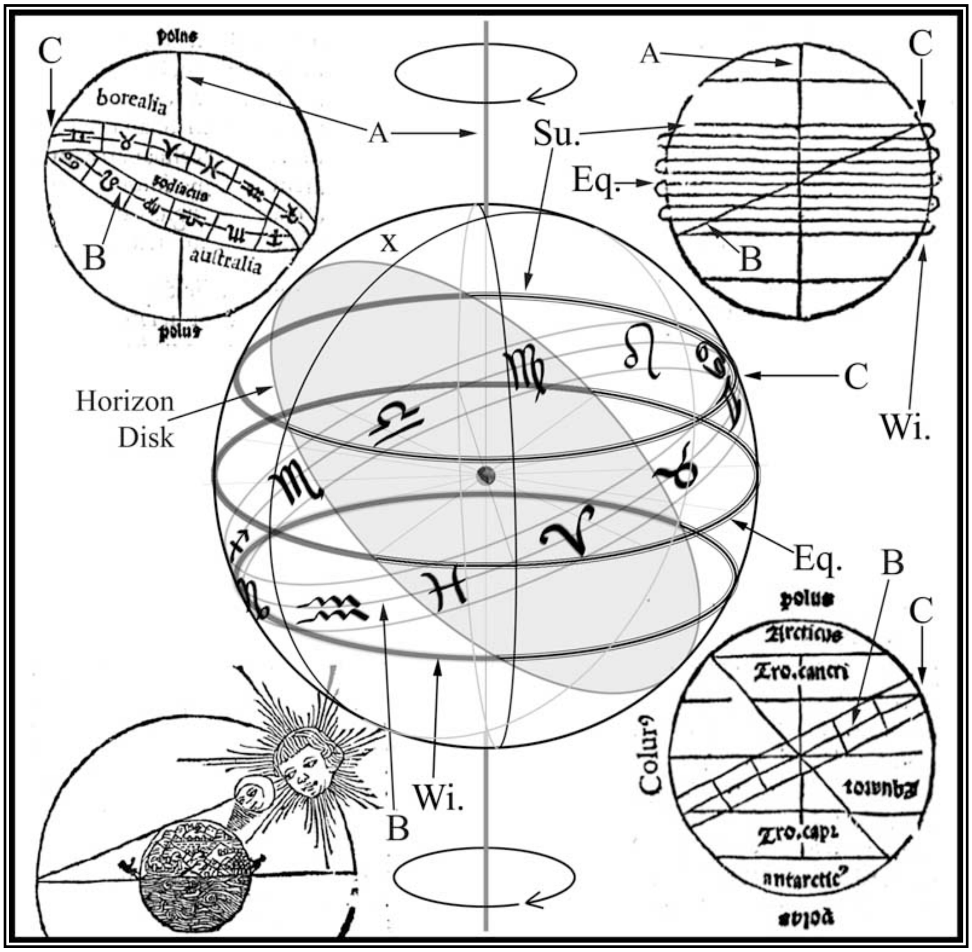

Figure 2.9: Views of the Cosmos

The four woodcuts in the corners are from a printed edition of Sacrobosco's "Sphere" (Venice, 1508). In the central diagram the earth and its horizon are stationary and the outer sphere of the fixed stars along with the sun spin on the vertical axis once per day. The moon and planets also spin with the stars, but are not indicated in this diagram.

The appearance of the horizon changes depending upon the perspective of the viewer, just as the tangent plane does to a sphere. Consequently, a collection of astronomical data for Padua would require translation to be relevant in Paris. All such changes are determined mathematically, using the geometry of spheres and circles.

The parallel circles above and below the ecliptic, B in Figure 2.9, show where the zodiac lives. The zodiac is made up of 12 signs, each taking up a 30º slice of the ecliptical pie. The zodiac is simply a coordinate system. For example, 12º Aries, locates a position within Aries that is 12º from the start of Aries and 18º from the start of Taurus. To make matters more complicated, sometimes a location is not recorded in this coordinate system, which is based on the ecliptic, but in a similar one on the circle of the celestial equator, which is the same circle as the sun at an equinox. More complicated still, locations can also be related to the horizon. These three systems are not equivalent and it takes some relatively complex and at times lengthy spherical geometrical calculations to get from one system to another.

The sun's two major annual motions are not complicated, but the motions of the planets are. All of their motions within the starry sphere occur in the region identified as the zodiac, but their motions are not simple. They speed up and slow down, and they sometimes even reverse direction and do what appears to be a loop-the-loop up in relation to the fixed stars. The standard late medieval explanation for this erratic behavior was the concept of epicycles. For this description it is easiest to view the heavenly situation from the "northern" pole of the ecliptic, looking down at the zodiac where all of the planets, the sun, and the moon live. See Figure 2.10, which singles out the planet Mars. In terms of Figure 2.9, it is as if we were looking down on the zodiacal band from the point marked with an "X." From this vantage point, the motions of the sun, moon, and planets, are counterclockwise in relation to the stars. Keep in mind that the motions described in Figure 2.10 do not include the daily rotation of the sphere of the fixed stars. In Figure 2.10, we are looking down upon the zodiac as if it were stationary. We are in a sense riding on the sphere of the fixed stars as if it were a merry-go-round.

In the first approximation, Mars travels along the black double-lined circle shown in Figure 2.10, the deferent, which is centered on the earth.[126] However, this only roughly describes Mars' approximately two-year circuit.

[126] Using another Ptolemaic mathematical device, the eccentric model, the center of the Martian orbit would not be on the earth.

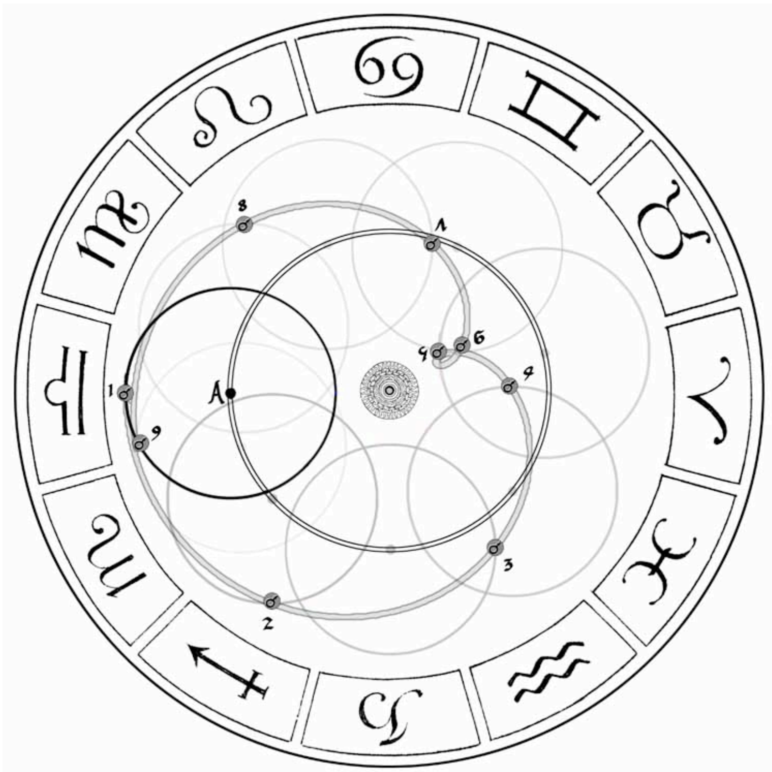

Figure 2.10: Martian Epicycle Seen from the "Northern" Ecliptic Pole

Earth is in the center and Mars is initially located at 9:00 on its epicycle and 9:00 on its deferent. All rotations are counterclockwise. The motion of this epicycle is shown in three-month snapshots all lightly superimposed on this diagram. The retrograde motion is evident in the trajectory at about 2:00 on the deferent, between positions 4 and 6. Numbered Martian locations are three months apart.

A more accurate mathematical description of the irregular motion of Mars utilizes the epicycle, the smaller circle shown centered on A in Figure 2.10. As its name suggests, the deferent ferries the epicycle, both of which rotate at constant angular velocities in a counterclockwise direction. This mathematical device is not unlike a Spirograph. The resultant trajectory, shown with a gray double-line, has a wide variation in angular velocity as seen in the numbered sequential positions of Mars placed at three-month intervals. Between positions 4 and 6 it even traces out the observed loop-the-loop.[127] To more accurately model Mars' behavior, epicycles could in theory carry more epicycles. The more epicycles employed, the more nuanced the motion.[128] But for most astronomical applications, one epicycle on a deferent was enough. Similar treatments were given to each of the planets.

[127] Keep in mind that from the earth we view this from the side, not from the top as seen in Figure 2.10, and the loop-the-loop appears more like a forward-backward-forward motion.
[128] The epicycles-on-epicycles approach is analogous to a Taylor series approximation – the more terms, the more accurate.

Needless to say, even with algebra or any of the mathematics we take for granted today, this model, though quite accurate, is mathematically very cumbersome. The reality was that most students of astronomy did not study this sort of theoretical material in much detail. Figuring out the layers upon layers of geometry was time-consuming and not terribly illuminating. Most students and professionals simply consulted astronomical tables such as the Toledan, the Alfonsine or the Rudolphine, and used methods of interpolation to fine tune the results. However, using these tables required lengthy technical instructions called canons in addition much tedious calculation. Also, parts of these tables were specific to a particular latitude and longitude and had to be recalibrated for any geographic changes, which required that the location be well known in astronomical and geographical terms. Though somewhat byzantine, these tables made it so that one did not have to reinvent the wheel, so to speak, every time one wanted to know where a heavenly body would be on a given date.

Quadrivial astronomy is not just celestial mechanics as described above. It also has an applied side. One major application was chronological– to make more accurate calendars, so that Easter or Passover did not occur in the middle of summer. Another was to make better maps of the globe, and for navigation. But perhaps the main goal for many quadrivial scholars was astrology. Astrological knowledge was in some sense the pinnacle of the quadrivium.

• • •

Quadrivial Astronomy/Astrology

Astrology was very widely practiced in the late Middle Ages and Renaissance. In fact, in the early days of printing, books containing the horoscopes and nativities of famous people were veritable pulp fiction.[129] There were, of course, those adamantly opposed to judicial astrology, the astrology which interfered with the concept of free will upon which Christianity was based, but this was just part of a much larger astrological industry. There used to be many different types of astrology. There was the astrology that dealt with predictions involving nations or kingdoms. This could be used historically like figuring out the chart for the day that Jesus rose from the dead or for predictions into the future, like when the anti-Christ might appear. There was also astro-meteorology, which was basically weather prediction. There was the astrology of nativities (birth charts). Another type of astrology dealt with specific questions, like whether to marry so-and-so's daughter or whether to go into an alliance with King Henry. Another form of astrology dealt with finding propitious times for a particular undertaking.[130] One of the largest domains was medical astrology. Physicians were quite often the most sophisticated astrologers. They used astrological charts to figure out things like good times for bloodletting or how the balance of melancholy in a patient might be affected by the location of Saturn. This required knowledge about planetary locations, their influences, and other esoteric information, which is beyond the scope of this dissertation.

Be that as it may, in order to do astrology, you had to know astronomy. In fact, the words astronomy and astrology were more or less interchangeable in premodern literature and astrologers were frequently called "*mathematici*." In the realm of astrology we will finally see the quadrivium come full circle.

Like its twin sister astronomy, astrology is far too complicated and idiosyncratic to explain here in much detail, but the following description of a simple birth chart gives a hint at its

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[129] See Anthony Grafton, *Cardano's Cosmos: The Worlds and Works of a Renaissance Astrologer* (Cambridge, MA: Harvard University Press, 1999).

[130] See Charles Burnett, "Astrology," in *Medieval Latin: an Introduction and Bibliographical Guide,* ed. Frank Anthony Carl Mantello and A. G. Rigg (Washington, D.C.: Catholic University of America Press, 1996), 375-376.

general mathematical character. Figure 2.11 is a typical format with 12 triangles, one for each of the so-called “houses.” Each house for a birth chart is assigned a property, numbered in Figure 2.11, typically starting with "Life" at the 9:00 position and going around counterclockwise. These houses are the coordinate system that represents the person. They are the microcosm.

Now we place onto the houses, the zodiac, the coordinate system for the macrocosm, which is also conveniently divided into 12 parts. The zodiac is essentially superimposed onto this chart and rotated in relation to the houses depending on several astronomical and geographical factors. Roughly speaking, the first house lines up with where the ecliptic cuts the horizon at the time in question. Simply put, for a nativity, the astrological/zodiacal sign that is on the eastern horizon at the time of birth is aligned with the first house. This fixes the birth chart to a particular time and geographic location. Figuring out this information for a historical figure in a distant land could be quite complicated.[131]

In the nativity chart of Figure 2.11, the zodiac is rotated to correspond to sunrise on June 21st, the summer solstice, of some arbitrary year.[132] The sun in is Cancer, in the first house.

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[131] Astronomical tables often had sections that could help with these sorts of historical horoscopes. To further complicate matters, the actual constellations of the zodiac do not correspond with the zodiacal signs used in astrology. They did in the 2nd century B.C., but another motion not discussed in this chapter called the precession of the equinoxes accounts for this discrepancy. Yet another set of geometric conversions must be made to deal with this.

[132] For simplicity I have chosen a time when the zodiac corresponds neatly with the houses, but in a typical chart it is likely that the signs will straddle the houses.

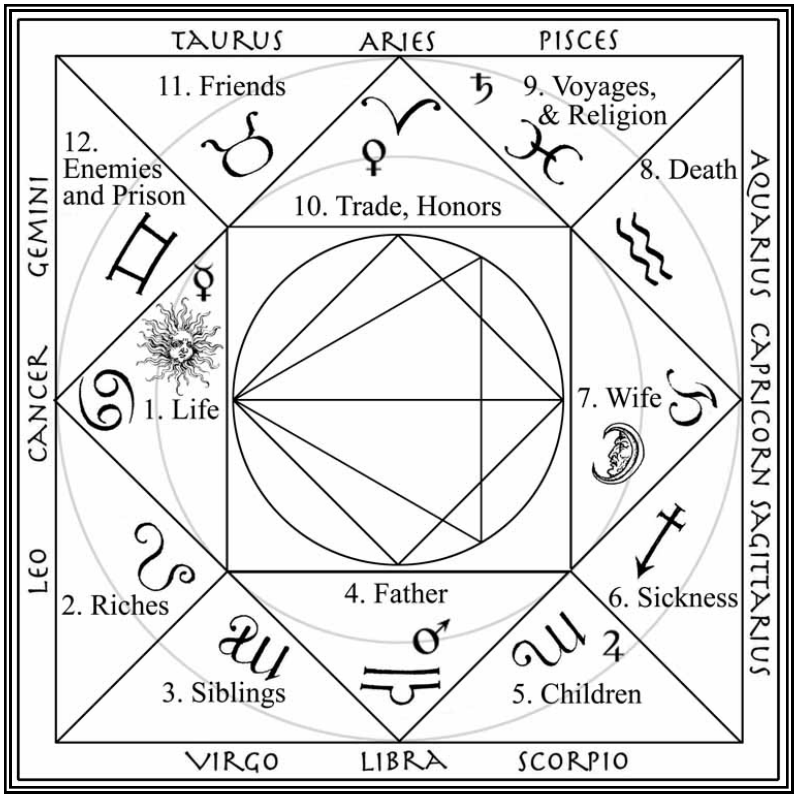

Figure 2.11: A Horoscope

The numbered triangles, starting at 9:00 are the so-called "houses." The part of the sky called the zodiac is rotated in relation to the houses for sunrise on June 21st, the summer solstice. The planetary arrangements in this diagram are arbitrary and thus no specific year is given.[133]

Next the astrologer plots the positions of all the planets, the sun and moon, and some other solar and lunar data of interest. This is where the music and geometry come in. Two heavenly bodies located in the same house or sign are said to be in conjunction. This can be seen as a 1:1, unison, relationship. Two planets separated by six houses or signs, for example the sun and moon in Figure 2.11, are in opposition and correspond to the octave, 2:1. Planets separated by four houses or signs form parts of an equilateral triangle, which corresponds harmonically to the 3:2, perfect fifth or the 3:1, octave and a fifth. Mercury, Jupiter, and Saturn all relate to each other in this way in Figure 2.11. And planets separated by three houses or signs relate to the perfect fourth, 4:3, or the 4:1, double-octave. The same ratios that made consonant intervals

[133] Burnett, "Astrology," 374.

were also the ones that corresponded to significant astrological relationships.[134] Arithmetic, music, geometry, and astronomy all come together in astrological theory.

Like the harmonic consonances, certain astrological relationships are undeniably special. In Figure 2.11, for example, the opposition of the sun and the moon, an octave of sorts, could result in a lunar eclipse. Ptolemy, in his treatise on music theory, analyzed a variety of astronomical cycles such as lunar phases, epicycle stations, and solar ecliptic locations and related them to a variety of theoretical musical concepts. The above description, though lacking detail, shows some of the ways the previous quadrivial disciplines are part of astronomy/astrology. How the astrologer then interprets the mathematics is usually proprietary information and therefore not easy to generalize.

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The Quadrivial Order – The Jump from Multitude to Magnitude

In the introduction to *De institutione arithmetica*, Boethius establishes the quadrivial order of priority: arithmetic to music to geometry to astronomy. Boethius explains that arithmetic is first because "God the creator of the massive structure of the world considered this first discipline as the exemplar of his own thought and established all things in accord with it."[135] He then explains how arithmetic is prior to geometry,

> If you take away numbers, in what will consist the triangle, the quadrangle, or whatever else is treated in geometry? All of those things are the domain of number. If you were to

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[134] The circle with the line, triangle, and square in the center of Figure 2.11 shows how the smallest division resulting from the arrangement of these geometrical constructions (e.g., 12:00 to 1:00), partitions the circle into 30º segments, which divides the circle into 12 parts, the 12 signs of the zodiac or 12 houses.

[135] Boethius, *De institutione arithmetica*, 1.1, p. 74. Theon of Smyrna also makes a very similar statement. See Theon of Smyrna, *Mathematics Useful for Understanding Plato*, trans. Robert and Deborah Lawlor (San Diego, CA: Wizards Bookshelf, 1979), I.2, p. 11.

> remove the triangle and the quadrangle and all of geometry, still *three* and *four* and the terminology of the other numbers would not perish.[136]

Geometry implies numbers, but numbers do not imply geometry. Thus, he concludes, numbers are prior to geometry. This argument is not explained any further by Boethius in his introduction. However, he explains this somewhat cryptic passage later, in Book II, stating, "[that] unity has the potential of a point, the beginning of an interval and longitude."[137] Boethius makes the connection between number and geometry, in part, with dimensional arguments. Unity resembles a point. It has no dimension and is purely abstract. A minimum of 2 points makes a line, a minimum of 3 makes a surface, etc.[138] Boethius also describes a how geometrical shapes like a square can be divided diagonally to produce 4 triangles and similar divisions for other polygons.[139] He describes what he calls triangular numbers, square numbers, pentagonal numbers, etc.[140] These are geometrical arrangements of unities and the series that are produced from them. For example, the triangular numbers are shown in Table 2.3.

1	1 1 1	1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1
1	3	6	10

Table 2.3: Triangular Numbers

[136] Boethius, *De institutione arithmetica*, 1.1, p. 74. A very similar statement is made by Macrobius. See Macrobius, 1.V.5-18, pp. 96-99.
[137] Boethius, *De institutione arithmetica*, 2.4, p. 129.
[138] The analogous section of Nicomachus is slightly more detailed. Nicomachus of Gerasa, "Introduction to Arithmetic," in *Great Books of the Western World: Euclid, Archimedes, Apollonius of Perga, and Nicomachus,* ed. Mortimer Jerome Adler and Robert Maynard Hutchins, trans. M. L. D'Ooge (Chicago: Encyclopaedia Britannica, 1984), 2.7.1-2, pp. 832-833.
[139] Boethius, *De institutione arithmetica*, 2.6, pp. 131-133.
[140] The figured numbers. Ibid., 2.7-19, pp. 133-142.

These triangular numbers produce the series: 1, 3, 6, 10, 15, 21, 28, etc. Similarly, square numbers produce the familiar series: 1, 4, 9, 16, 25, etc. In his discussion on squared numbers he points out that the series of odd numbers always adds up to a square number.[141] For example,

$$1 + 3 + 5 + 7 = 16.$$

These are all interesting manipulations of the numbers of arithmetic, but are they geometric?

Boethius describes these correspondences between arithmetic and geometry, but they are not obvious in the same way that the numbers of arithmetic lead into the ratios of numbers in music. Arithmetic flows quite easily into music. Arithmetic is number *per se*, and music is the comparison of numbers to themselves. But geometry seems to be related to arithmetic in a very different way.

Returning to Boethius' introduction, when he wraps up his description of the quadrivial hierarchy, he makes a final statement that is curious in light of his pseudo-geometric arithmetic.

> In astronomy, the very movement of the stars is celebrated in harmonic intervals. From this it follows that the power of music logically precedes the courses of the stars; and there is no doubt that arithmetic precedes astronomy since it is prior to music, which comes before astronomy.[142]

Everything in this section on the order of the quadrivial disciplines leads up to this one statement. He has linked each of the quadrivial arts to one another in pairs, and he is poised to make one last grand unifying proclamation. The reader expects something like– *Arithmetic is required for music is required for geometry is required for astronomy.* But what Boethius delivers leaves out geometry. What happened to geometry? This, it turns out, is something that other people wondered as well. There is a fundamental division built into the quadrivium that divides multitudes (arithmetic and music) from magnitudes (geometry and astronomy). It appears that this division is like Plato's divided line. Geometry and astronomy are the sciences

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[141] Ibid., 2.12, p. 136.
[142] Ibid., 1.2, p. 75.

of magnitude, ironically the sciences based on human terrestrial perception, while arithmetic and music theory are the sciences of truth in the world of light and forms. And yet astronomy is often portrayed as the pinnacle of the quadrivial arts. This division is complicated and convoluted. As we will see in the upcoming case studies, it is a stumbling block.

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Secondary Literature

This preceding overview of the quadrivial disciplines is by no means comprehensive. It merely sketches out the basic mathematical ideas necessary for understanding the quadrivium as a philosophical system of structure. Without an introduction to this mathematical structure the categories I have used to organize the secondary literature pertaining to the quadrivium and the quadrivial disciplines makes little sense. For this reason I have postponed a discussion of this material until now.

There have been two predominant approaches to the study of the quadrivium found in the secondary literature: the study of early university curricula and the study of the quadrivial disciplines in piece-meal fashion. A third and less pursued approach explores the quadrivium, or aspects of the quadrivium, as a philosophical system. This last approach more closely resembles the objectives of this dissertation, but all three classes of secondary literature have been invaluable for the present study. The following subsections outline the recent literature dealing with the quadrivium that has been most useful in the writing of this dissertation.

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Studies of University Curricula

Works in this category include broad studies of the medieval university in general, studies of particular universities, and studies on various aspects of the university such as specific faculties or legal structures. The classic study on the medieval university in general is Hastings Rashdall's *The Universities of Europe in the Middle Ages* in three volumes first published in 1895 (revised 1936).[143] For this dissertation Rashdall's lengthy sections on the universities of Bologna and Paris and somewhat smaller section on Padua developed a first approximation on the structure and curriculum of the arts faculties, which traditionally presented the quadrivial disciplines. Nancy Siraisi's *The Arts and Sciences at Padua: The Studium of Padua before 1350* (1973)[144] specifically addressed many of the issues surrounding the teaching of the quadrivial disciplines in Padua and provided excellent background material for the case studies of both Prosdocimo de' Beldomandi, who taught at the University of Padua, and Leon Battista Alberti, who studied in its shadow. Other books of note in this category include Paul Grendler's *The Universities of the Italian Renaissance* (2002), Olaf Pedersen's *The First Universities: Studium Generale and the Origins of University Education in Europe* (1997), and Lynn Thorndike's colorful collection of primary documents in *University Records and Life in the Middle Ages* (1944).[145]

In addition to these monographs, there are three collections of essays on the theme of the medieval education that have been particularly useful in the present study. *Arts libéraux et philosophie au Moyen Age: Actes de Quatrième Congrès International de Philosophie Médiévale*

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[143] Hastings Rashdall, *The Universities of Europe in the Middle Ages*, eds. F. M. Powicke and A. B. Emden, 3 vols., vol. 1 (Oxford: The Clarendon Press, 1936).
[144] Nancy G. Siraisi, *Arts and Sciences at Padua: The Studium of Padua Before 1350* (Toronto, Canada: Pontifical Institute of Mediaeval Studies, 1973).
[145] Paul F. Grendler, *The Universities of the Italian Renaissance* (Baltimore: Johns Hopkins University Press, 2002); Olaf Pedersen, *The First Universities: Studium Generale and the Origins of University Education in Europe*, trans. R. North (New York: Cambridge University Press, 1997); Lynn Thorndike, *University Records and Life in the Middle Ages* (New York: W. W. Norton & Company, 1944; reprint, Columbia University Press, 1975).

(1969)[146] contains a wide variety of papers presented at a conference held in Montreal on all aspects of medieval liberal arts education. Included in this volume are Philippe Delhaye's paper "La place des arts libéraux dans les programmes scolaires du XIIIe siècle," James Weisheipl's paper, "The Place of the Liberal Arts in the University Curriculum During the XIVth and XVth Centuries,"[147] and W. H. Stahl's paper, "The *Quadrivium* of Martianus Capella: Its Place in the Intellectual History of Western Europe." Among the many fine essays in this compilation, one in particular should be singled out for its importance to subsequent scholarship. Pearl Kibre's essay "The Quadrivium in the Thirteenth Century Universities (with Special Reference to Paris)"[148] is especially valuable not only for its information on the state of the quadrivium in Paris, but also for its methodology. Kibre notes that while official university statutes may give the appearance that Aristotelian natural philosophy had supplanted the quadrivial disciplines at Paris and Oxford in the 13th century, other sources, namely collections of disputations, the writings of scholars associated with the universities, and examination preparation guides, clearly show that the quadrivial arts flourished along side (and I would maintain, eventually integrated with) the newly introduced Aristotelian sciences. Furthermore, Kibre points out that luminaries associated with the universities of Paris and Oxford such as Robert Grosseteste and Roger Bacon "both maintained that the mathematical arts provide[d] the gateway and key to all other sciences."[149]

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[146] *Arts libéraux et philosophie au Moyen Age: Actes de Quatrième Congrès International de Philosophie Médiévale* (Montréal: Institut d'études Médiévales, 1969). Individual essays found in collections mentioned in this section are included in the bibliography under the author's name.

[147] See also James A. Weisheipl, "The Structure of the Arts Faculty in the Medieval University," *British Journal of Educational Studies* 19, no. 3 (1971): 263-271.

[148] This essay was also published in a collection of Kibre's essays. See Pearl Kibre, *Studies in Medieval Science: Alchemy, Astrology, Mathematics, and Medicine* (London: Hambledon Press, 1984), I.
The published "Questions et discussions" at the end of the section containing Kibre's essay [pages 198-203] records an interesting conversation between Pearl Kibre, Raymond Kiblansky, and James Weisheipl that is not included in the Kibre collection from 1984.

[149] Kibre, "The Quadrivium in the Thirteenth Century Universities," in *Arts Libéraux et Philosophie au Moyen Age*. 177.

Bacon goes even further to suggest that "those who are ignorant of mathematics are unable to perceive their own ignorance and are therefore unable to seek a remedy."[150] Kibre's integration of a variety of primary sources above and beyond official university documents has painted a much more nuanced picture of the medieval university. Subsequent scholars such as Nancy Siraisi and Joseph Dyer[151] have continued with this integrated approach with great success.

A second collection of essays, written in honor of Pearl Kibre, largely pertaining to medieval education and the arts, is found in parts I and II of *Manuscripta* XX *[Science, Medicine and the University: 1200-1550]* (1976).[152] Of particular importance for this dissertation, especially the chapters on Nicole Oresme and Prosdocimo de' Beldomandi, were Richard Lemay's essay, "The Teaching of Astronomy in Medieval Universities, Principally at Paris in the Fourteenth Century" and John Murdoch's essay, "Music and Natural Philosophy: Hitherto Unnoticed *Questiones* by Blasius of Parma (?)" [*sic*].

Lastly, I should mention the valuable collection of essays edited by Hilde de Ridder-Symoens entitled *A History of the University in Europe: Universities in the Middle Ages* (1992).[153] Included in this collection is John D. North's chapter, "The Quadrivium," which provides a general overview of the quadrivial disciplines and a brief tour of the texts used in university quadrivial training as well as Nancy Siraisi's chapter, "The Faculty of Medicine,"

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150 Ibid.

151 Joseph Dyer, "The Place of *Musica* in Medieval Classifications of Knowledge," *The Journal of Musicology* 24, no. 1 (2007): 3-71; Joseph Dyer, "Speculative 'Musica' and the Medieval University of Paris," *Music & Letters* 90, no. 2 (2009): 177-204; Nancy G. Siraisi, "The Music of Pulse in the Writings of Italian Academic Physicians (Fourteenth and Fifteenth Centuries)," *Speculum: A Journal of Mediaeval Studies* 50, no. 4 (1975): 689-710; Nancy G. Siraisi, *Medicine in the Italian Universities, 1275-1600* (Boston: Brill, 2001).

152 Nancy G. Siraisi and Luke Demaitre, eds., *Manuscripta XX*, [Science, Medicine and the University: 1200-1550; Essays in Honor of Pearl Kibre] Parts I and II, vols. 2 and 3 (1976).

153 Hilde de Ridder-Symoens, ed., *A History of the University in Europe: Universities in the Middle Ages*, vol. 1 (New York: Cambridge University Press, 1992).

which describes medical training in the university setting and the relationship between medicine and the liberal arts.

In general these studies of medieval university education have provided information on the structure and curriculum of arts and medical faculties and invaluable information on how and what texts were used in quadrivial instruction. By and large these works do not discuss the philosophical underpinnings of the quadrivium in much detail and as such they have been utilized in this dissertation primarily for the context rather than content of the quadrivium.

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Studies of the Individual Quadrivial Disciplines *per se*.

Scholarly works in this category discuss what the disciplines are and/or how they work. They tend towards what historians of science might call an "internalist" approach. Some in this category are stand-alone essays or books, others are introductions and/or commentaries to premodern works, and some are sections of larger works dealing with mathematics in general. What the members of this category all share in common are disciplinary boundaries. For example, these sources tend to be about arithmetic or astronomy, not the relationship between the two. On occasion a few of these sources discuss other disciplines outside their primary focus, but, by and large, members of this category do not treat the interrelationships between the quadrivial disciplines. A thorough bibliography of all of the available literature in this category is well beyond the scope of this dissertation so I will limit myself to mentioning only those sources which were most useful for the present study.

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Disciplinary Studies Found in Edited Collections on the Liberal Arts

A particularly useful introduction to the four disciplines of the quadrivium is contained in *The Seven Liberal Arts in the Middle Ages* (1983), edited by David Wagner.[154] Essays on arithmetic, music, geometry, and astronomy are written by experts in their respective disciplines: Michael Masi, Theodore Karp, Lon Shelby, and Claudia Kren. What makes these four quadrivial essays stand out are their succinct descriptions of the quadrivial disciplines in terms of mathematical content. These four essays describe what the disciplines were and how they functioned in relatively accessible language. They are excellent introductions to the actual mathematics of the quadrivium. A more specialized collection of essays entitled *Boethius and the Liberal Arts* (1981),[155] edited by Michael Masi, is an excellent introduction to the Boethian quadrivial disciplines of arithmetic and music. Chapters on arithmetic by Pearl Kibre and Michael Masi, and on music by Ubaldo Pizzani and Calvin Bower describe not just some of the general mathematical outlines of Boethian arithmetic and music, but also several of the issues that preoccupied subsequent quadrivial scholars such as the definition of number and the distinction between practical and speculative music. The sort of analysis found in Masi's and Wagner's collections, though done within the confines of separate quadrivial disciplines, partially exemplifies the methodology employed in this dissertation, a methodology which takes into account both the mathematics and the philosophy of the quadrivium.

Also worthy of note is a third collection of essays, edited by Margaret Gibson, *Boethius: His Life, Thought and Influence* (1981).[156] This book includes two chapters on quadrivial topics. The chapter by John Caldwell, "*De institutione arithmetica* and *De institutione musica*,"

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[154] David L. Wagner, ed., *The Seven Liberal Arts in the Middle Ages* (Bloomington: Indiana University Press, 1983).
[155] Michael Masi, ed., *Boethius and the Liberal Arts: A Collection of Essays* (Berne: P. Lang, 1981).
[156] Margaret T. Gibson, ed., *Boethius, His Life, Thought, and Influence* (Oxford: Blackwell, 1981).

discusses earlier sources for these two Boethian texts and places them in a the larger context of Pythagorean mathematics. David Pingree, in his chapter "Boethius' Geometry and Astronomy," explores the fragmentary evidence of the two quadrivial texts that Boethius promised to write, but either no longer exist, or were never completed. Although these two chapters do not discuss the internal workings of the quadrivial disciplines *per se*, they provide useful information on the tradition and historical sources of these Boethian disciplines. Gibson's book also includes an epilogue by Anthony Grafton, "Boethius in the Renaissance," which surveys the ways various Renaissance humanists responded to the legacy of Boethius.

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Disciplinary Studies Arranged by Subject: Arithmetic

Michael Masi is the preeminent expert on Boethian number theory. Besides the sources mentioned above, his numerous essays, and his translation of and commentary on Boethius' *De institutione arithmetica* (1983)[157] have been and continue to be endless sources of insight into the arithmetical philosophy of Boethius, a topic that is fundamental to this dissertation.

Several sources have been indispensible on topic of algorism, the use and operations of Hindu-Arabic numerals. Charles Burnett's article, "The Semantics of Indian Numerals in Arabic, Greek and Latin" (2006)[158] chronicles the entrance of the Hindu-Arabic numeral symbols themselves and briefly describes the Greek and Latin symbols used before their widespread adoption in the Latin West beginning in the 12th century. Also useful were a group of transcriptions/translations of and essays written on Hindu-Arabic numerals and medieval

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157 Boethius, *Boethian Number Theory: a Translation of the De institutione arithmetica*, ed. and trans. by Michael Masi (Amsterdam: Rodopi, 1983); Michael Masi, "A Newberry Diagram of the Liberal Arts," *Gesta* 11, no. 2 (1972): 52-56; Michael Masi, "Boethius and the Iconography of the Liberal Arts," *Latomus* (1974): 56-75; Michael Masi, "The Liberal Arts and Gerardus Ruffus' Commentary on the Boethian *De arithmetica*," *Sixteenth Century Journal* 10, no. 2 (1979): 23-41.

158 Charles Burnett, "The Semantics of Indian Numerals in Arabic, Greek and Latin," *Journal of Indian Philosophy* 34, no. 1-2 (2006): 15-30.

algorisms from the first half of the 20th century by David Eugene Smith, Louis Karpinski, and E. G. R. Waters.[159] Although these writings are nearly a century-old their relevance has not diminished. Frank Swetz's *Capitalism and Arithmetic: the New Math of the 15th Century* (1987)[160] contains excellent explanations and analyses of the earliest forms of Hindu-Arabic numeration and calculation found in the Latin West, as well as a translation (done by David Eugene Smith) of the *Treviso arithmetic* of 1478. In particular, the above works proved invaluable for reading the manuscript and early printed editions of Prosdocimo's quadrivial texts.

Arithmology, the study of the mystical properties of numbers, is perhaps the least examined of the arithmetical subdivisions, even though arithmological observations are quite common in medieval and Renaissance mathematical literature. Essays from the 1920s by Frank Egleston Robbins on classical arithmology provide the most useful survey of the ancient literature.[161] Also very useful are Keith Critchlow's and Robin Waterfield's introductory essays in Waterfield's English translation of the *Theologumena arithmeticae* [*The Theology of Arithmetic*] (1988) attributed to Iamblichus.[162]

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[159] Louis C. Karpinski and Charles N. Staubach, "An Anglo-Norman Algorism of the Fourteenth Century," *Isis* 23, no. 1 (1935): 121-152; Louis C. Karpinski and E. G. R. Waters, "A Thirteenth Century Algorism in French Verse," *Isis* 11, no. 1 (1928): 45-84; David Eugene Smith, *Rara Arithmetica: A Catalogue of the Arithmetics Written Before the Year MDCI* (Boston: Ginn and Company, 1908); David Eugene Smith and Louis Charles Karpinski, *The Hindu-Arabic Numerals* (Boston: Ginn and Company, 1911; reprint, Dover, 2004); E. G. R. Waters, "A Fifteenth Century French Algorism from Liege," *Isis* 12, no. 2 (1929): 194-236.

[160] Frank Swetz and David Eugene Smith, *Capitalism and Arithmetic: the New Math of the 15th Century (including the full text of the 'Treviso arithmetic' of 1478, translated by David Eugene Smith)* (La Salle, IL: Open Court, 1987).

[161] Frank Egleston Robbins, "Posidonius and the Sources of Pythagorean Arithmology," *Classical Philology* 15, no. 4 (1920): 309-322; Frank Egleston Robbins, "The Tradition of Greek Arithmology," *Classical Philology* 16, no. 2 (1921): 97-123.

[162] Iamblichus (Attributed to), *The Theology of Arithmetic: On the Mystical, Mathematical and Cosmological Symbolism of the First Ten Numbers [Theologoumena Arithmeticae]*, trans. and intro. R. Waterfield and K. Critchlow (Grand Rapids, MI: Phanes Press, 1988).

Disciplinary Studies Arranged by Subject: Music

Calvin Bower's commentary and notes for his translation of Boethius' *De institutione musica*[163] have been most useful for the writing of this dissertation as has been an excellent essay on premodern speculative music theory that he wrote for *The Cambridge History of Western Music Theory*, edited by Thomas Christensen (2002).[164] Also included in this volume and of particular relevance for this dissertation have been the chapters on ancient Greek and medieval music theory by Thomas Mathiesen and Jan Herlinger.

Music in the Culture of the Renaissance and Other Essays (1989),[165] a collection of papers by Edward Lowinsky, contains a selection of essays in the section titled, "Music and the History of Ideas," that discuss the place of speculative music theory in the broader context of intellectual history. In these essays Lowinsky finds analogies between such things as tonal space, cosmological theories, and geographical knowledge. Although his primary topic is always music, he freely explores its historical relationship to other forms of intellectual pursuits.

In a similar vein, Claude Palisca has also written a great deal on music in the larger intellectual context of the Renaissance. In his book *Humanism in Italian Renaissance Musical Thought* (1985),[166] Palisca traces various aspects of music theory through the ages, culminating in figures such as Franchino Gaffurio, Vincenzo Galilei, and Gioseffo Zarlino. Although the focus of this book is predominantly on music theorists who flourished later than the chronological limitation of this dissertation, Palisca carefully describes the context out of which

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[163] Anicius Manlius Severinus Boethius, *Fundamentals of Music [De institutione musica]*, trans. Calvin M. Bower (New Haven, CT: Yale University Press, 1989).
[164] Thomas Street Christensen, ed., *The Cambridge History of Western Music Theory*, The Cambridge History of Music (New York: Cambridge University Press, 2001).
[165] Edward E. Lowinsky, *Music in the Culture of the Renaissance and Other Essays*, ed. Bonnie J. Blackburn, 2 vols. (Chicago: University of Chicago Press, 1989).
[166] Claude V. Palisca, *Humanism in Italian Renaissance Musical Thought* (New Haven, CT: Yale University Press, 1985).

these more modern theorists grew and devotes a great deal of this text to detailed descriptions of ancient and medieval systems.

Although the topic of *musica ficta* is only cursorily addressed in the second case study of this dissertation, two books on this topic illuminated a variety of subtleties pertaining to premodern music that had not been well defined in my other musical research. One book is Karol Berger's *Musica Ficta: Theories of Accidental Inflections in Vocal Polyphony from Marchetto da Padova to Gioseffo Zarlino* (1987), and the other is Margaret Bent's *Counterpoint, Composition, and Musica Ficta* (2002).[167] Although Berger and Bent differ on some fundamental points concerning the late medieval and Renaissance theorists' conceptions of fixed pitch, both authors make compelling arguments for their points of view and vividly describe premodern intellectual pitch-spaces that are radically different from the modern perspective. I found both of their descriptions helped me to identify many of my own preconceived notions about music – notions that prevented me from appreciating premodern theoretical issues on their own terms.

My research of premodern music theory has also greatly benefitted from the exceptional organization found in the fields of music history and musicology. Resources such as the multi-volume *Répertoire international des sources musicales* (RISM)[168] and the online *Thesaurus Musicarum Latinarum*[169] are comprehensive and easy to use. If only all historical fields could be so well organized.

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[167] Margaret Bent, *Counterpoint, Composition, and Musica Ficta* (New York: Routledge, 2002); Karol Berger, *Musica Ficta: Theories of Accidental Inflections in Vocal Polyphony from Marchetto da Padova to Gioseffo Zarlino* (New York: Cambridge University Press, 1987). See also Margaret Bent, "Diatonic Ficta," *Early Music History 4*, no. (1984): 1-48; Karol Berger, "The Expanding Universe of Musica Ficta in Theory from 1300 to 1550," *The Journal of Musicology* 4, no. 4 (1985/6): 410-430.

[168] Especially volumes from RISM B/I-III.

[169] The *Thesaurus Musicarum Latinarum* is an online database hosted by the Center for the History of Music Theory and Literature, Jacobs School of Music, Indiana University (Bloomington). It is available online at www.chmtl.indiana.edu/tml/start.html.

Disciplinary Studies Arranged by Subject: Geometry

Euclid's *Elements* has long been considered the most comprehensive text on the topic of theoretical geometry. The commentary written by Thomas Little Heath that accompanies his English translation of the *Elements,* first published in 1908,[170] dominates all geometrical historical literature and was used extensively in the writing of this dissertation. In addition to his work on Euclid, Heath wrote several other books and essays on premodern mathematical topics that are classics in the field and were consulted routinely in the writing of this dissertation.[171] However, as was mentioned above, geometry in the Middle Ages was not only Euclid. Geometry was also the science of terrestrial measurement. For information on this aspect of medieval geometry Evgeny Zaitsev's article, "The Meaning of Early Medieval Geometry: From Euclid and Surveyors' Manuals to Christian Philosophy" (1999) is most useful.[172]

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Disciplinary Studies Arranged by Subject: Astronomy and Astrology

Pre-Copernican astronomical theory has been the subject of numerous studies. Many of these studies, including those by such luminaries as Edward Grant, Alexandre Koyré, Thomas Kuhn, O. Neugebauer, and John David North[173] have been very important for my understanding

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[170] Euclid, *The Thirteen Books of Euclid's Elements*, trans. and notes T. L. Heath, 2nd ed., 3 vols. (Cambridge: Cambridge University Press, 1926; reprint, Dover, 1956).

[171] Thomas Little Heath, *A Manual of Greek Mathematics* (Oxford University Press, 1931; reprint, Dover, 1963); Thomas Little Heath, *Greek Astronomy* (London: J. M. Dent & Sons, 1932; reprint, Dover, 1991); Thomas Little Heath, *Mathematics in Aristotle* (Oxford: Clarendon Press, 1949).

[172] Evgeny A. Zaitsev, "The Meaning of Early Medieval Geometry: From Euclid and Surveyors' Manuals to Christian Philosophy," *Isis* 90, no. 3 (1999): 522-553. See also Berthold Louis Ullman, "Geometry in the Mediaeval Quadrivium," in *Studi di bibliografia e di storia in onore di Tammaro De Marinis*, (Verona: 1964), 263-285. For information on Boethian geometry, see Menso Folkerts, *'Boethius' Geometrie II: ein mathematisches Lehrbuch des Mittelalters* (Wiesbaden: Franz Steiner Verlag, 1970).

[173] Edward Grant, *Planets, Stars, and Orbs: the Medieval Cosmos, 1200-1687* (New York: Cambridge University Press, 1994); Alexandre Koyré, *From the Closed World to the Infinite Universe* (New York: Harper & Brothers, 1958); Thomas S. Kuhn, *The Copernican Revolution* (Cambridge, MA: Harvard University Press, 1985); O. Neugebauer, *A History of Ancient Mathematical Astronomy*, 3 vols. (New

of quadrivial astronomy. In addition to the astronomical historical works by these authors, several additional studies were regularly consulted in the writing of this dissertation. For a general overview of medieval theory and context, Stephen McCluskey's *Astronomies and Cultures in Early Medieval Europe* (1997)[174] provides succinct discussions on such topics as Ptolemaic theory, calendrics, and astrolabes. In a similar vein, Olaf Pedersen's essay on astronomy in the collection *Science in the Middle Ages* (1978), edited by David Lindberg, outlines the nature and applications of medieval astronomy. The early chapters of Pedersen's book *A Survey of the Almagest* (1974) provided a great deal of useful information and insightful observations on Ptolemy's *Almagest* and its influence on medieval astronomical theories.[175]

The astronomical curricula of most late medieval universities focused on both a theoretical understanding of astronomy and a more practical side that involved the use and/or construction of tables, almanacs, and ephemerides. For information on the two primary theoretical astronomical texts used in university instruction I consulted Lynn Thorndike's extensive commentary and notes in his *The Sphere of Sacrobosco and Its Commentators* (1949)[176] and Francis Benjamin's and G. J. Toomer's commentary and notes to the *Theorica planetarum* in their book, *Campanus of Novara and Medieval Planetary Theory* (1971)[177] as well as Olaf

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York: Springer-Verlag, 1975); John David North, *The Norton History of Astronomy and Cosmology*, ed. Roy Porter (New York: Norton, 1994).

[174] Stephen C. McCluskey, *Astronomies and Cultures in Early Medieval Europe* (Cambridge: Cambridge University Press, 1997).

[175] Olaf Pedersen, "Astronomy," in David C. Lindberg, ed., *Science in the Middle Ages* (Chicago: University of Chicago Press, 1978), 303-337; Olaf Pedersen, *A Survey of the Almagest* (Odense: Odense Universitetsforlag, 1974).

[176] Lynn Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, ed. and trans. Lynn Thorndike (Chicago: University of Chicago Press, 1949).

[177] Campanus of Novara, *Campanus of Novara and Medieval Planetary Theory [Theorica planetarum]*, ed., trans, and commentary F. S. Benjamin and G. J. Toomer (Madison, WI: University of Wisconsin Press, 1971).

Pedersen's essay, "The Origins of the *Theorica Planetarum*" (1981).[178] The works of José Chabás and Bernard Goldstein, particulary their book, *The Alfonsine Tables of Toledo* (2003),[179] provide depth and dimension to my descriptions of the *Alfonsine Tables*, the astronomical tables taught in late medieval universities. Also useful in this area of study was Owen Gingrich's short essay, "The Alfonsine Tables in the Age of Printing" (1987),[180] which provides a very basic introduction on how these tables functioned. Outside of the very specialized literature pertaining to astronomical tables and ephemerides, which is largely impenetrable to the non-expert, descriptions of how these tables worked (modern canons) are sorely lacking.

The scholarly study of astrology, once an academic pariah, is currently enjoying a resurgence of interest. Before the 1970s only a few authors addressed this topic in much detail. Lynn Thorndike's classic study, *A History of Magic and Experimental Science* (1923-1958),[181] touches upon astrological topics throughout its eight-volumes. Theodore Otto Wedel's *The Medieval Attitude Toward Astrology* (1920)[182] is a solid stand-alone text devoted to the history of astrology primarily in England. Despite these impressive works, the scholarly study of astrology has frequently been considered a topic for cranks and mystical initiates. The pivotal change of this attitude was formally declared by Otto Neugebauer in his one-page essay, "The Study of

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[178] O. Pedersen, "The Origins of the *Theorica Planetarum*," *Journal for the History of Astronomy* 12, no. (1981): 113-123.
[179] José Chabás and Bernard R. Goldstein, *The Alfonsine Tables of Toledo* (Boston: Kluwer Academic Publishers, 2003).
[180] Owen Gingerich, "The Alfonsine Tables in the Age of Printing," in Mercè Comes, Roser Puig Aguilar, and Julio Samsó, eds., *De astronomia Alphonsi Regis: actas del Simposio sobre Astronomía Alfonsí celebrado en Berkeley (agosto 1985) y otros trabajos sobre el mismo tema = proceedings of the Symposium on Alfonsine Astronomy held at Berkeley (August 1985) together with other papers on the same subject* (Barcelona: Universidad de Barcelona, Instituto "Millás Vallicrosa" de Historia de la Ciencia Arabe, 1987), 89-95.
[181] Lynn Thorndike, *A History of Magic and Experimental Science*, 8 vols. (New York: Columbia University Press, 1923-58).
[182] Theodore Otto Wedel, *The Mediaeval Attitude Toward Astrology: Particularly in England* (New Haven, CT: Yale University Press, 1920).

Wretched Subjects," published in 1951.[183] In this essay Neugebauer chastises George Sarton, the "recognized dean of the History of Science," for his categorical dismissal of astrology as "superstitious flotsam of the Near East." Neugebauer writes, "perhaps [Sarton] does not fully realize how much he is contributing to the destruction of the very foundations of our studies: the recovery and study of the texts as they are, regardless of our own tastes and prejudices."[184] Since Neugebauer's defense of this "wretched" subject, the topic of astrology and associated mystical arts have had something of a scholarly renaissance.

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Studies on Astrology, Astrologers, and Society

In terms of astrology and its function in intellectual and cultural society, two collections of essays have been particularly useful for this dissertation. Patrick Curry's *Astrology, Science, and Society: Historical Essays* (1987)[185] contains a wide variety of essays derived from a conference held at the Warburg Institute in 1984. Included in this volume and of particular importance for this dissertation are essays by Graziella Vescovini on Peter of Abano's astrology, Richard Lemay's essay on astrology and its relationship to the history of science, J. D. North's essay on medieval concepts of astrological forces, and Stefano Caroti's essay on Oresme's polemic against astrology. Another collection, *Horoscopes and Public Spheres: Essays on the History of Astrology* (2005)[186] edited by Günther Oestmann, H. Darrel Rutkin, and Kocku von Stuckrad, contains a very useful and straightforward essay by Monica Azzolini on medical astrology in

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[183] O. Neugebauer, "The Study of Wretched Subjects," *Isis* 42, no. 2 (1951): 111.
[184] Ibid.
[185] Patrick Curry, ed., *Astrology, Science, and Society: Historical Essays* (Wolfeboro, NH: Boydell Press, 1987).
[186] Günther Oestmann, H. Darrel Rutkin, and Kocku von Stuckrad, eds., *Horoscopes and Public Spheres: Essays on the History of Astrology* (New York: Walter de Gruyter, 2005).

Renaissance Milan and a particularly testy assessment of the state of the historiography of astrology by Patrick Curry.

Several monographs devoted to astrology and astrological topics have also proven to be most useful in understanding astrology in the late Middle Ages and Renaissance. Jim Tester's *A History of Western Astrology* (1987)[187] describes the theory and practice of astrology from ancient times to the 17th century. Although this book has been partially superseded by more recent and more topically specific studies, it still holds up as a good all-around first approximation. The first two chapters of Steven Vanden Broecke's *The Limits of Influence: Pico, Louvain, and the Crisis of Renaissance Astrology* (2003)[188] provide excellent overviews of the context and the content of late medieval astrology and contain perhaps the most succinct working definition of astrology I have come across. Vanden Broecke writes, "Astrology aims at predicting and/or studying the power of celestial bodies on earth, and measures their positions by means of astronomy."[189] Studies of individual astrologers such as Laura Ackermann Smoller's *History, Prophecy, and the Stars: The Christian Astrology of Pierre d'Ailly, 1350-1420* (1994),[190] Anthon Grafton's *Cardano's Cosmos: The Worlds and Works of a Renaissance Astrologer* (1999),[191] and Nancy Siraisi's *The Clock and the Mirror: Girolamo Cardano and Renaissance Medicine* (1997)[192] are wonderful books that not only describe in detail the astrological thoughts of d'Ailly and Cardano, but also paint a vivid picture of a world that

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187 S. J. Tester, *A History of Western Astrology* (Wolfeboro, NH: Boydell Press, 1987).
188 Steven Vanden Broecke, *The Limits of Influence: Pico, Louvain, and the Crisis of Renaissance Astrology* (Leiden: Brill, 2003).
189 Ibid., 17.
190 Laura Ackermann Smoller, *History, Prophecy, and the Stars: The Christian Astrology of Pierre d'Ailly, 1350-1420* (Princeton, NJ: Princeton University Press, 1994).
191 Anthony Grafton, *Cardano's Cosmos: The Worlds and Works of a Renaissance Astrologer* (Cambridge, MA: Harvard University Press, 1999).
192 Nancy G. Siraisi, *The Clock and the Mirror: Girolamo Cardano and Renaissance Medicine* (Princeton, NJ: Princeton University Press, 1997).

included astrology as a natural and rational science. Also eminently useful has been Richard Lemay's *Abu Ma'shar and Latin Aristotelianism in the Twelfth Century: The Recovery of Aristotle's Natural Philosophy through Arabic Astrology* (1962),[193] which not only generally describes the influence and incorporation of Aristotelian natural philosophy in both Arabic and Latin astrology in the Middle Ages, but also provides a great deal of information on Abu Ma'shar's 9th-century text, *Introductorium in astronomiam*, as it was known in the Latin world via Hermann of Carinthia's 12th-century translation. Abu Ma'shar's text had a profound influence on the astronomy, astrology, and the natural philosophy of the 14th and 15th centuries.

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Technical Studies on the Practice of Astrology

The technical side of astrology is very complicated and throughout its history has not been practiced consistently. The most valuable introduction I have yet found is J. C. Eade's *The Forgotten Sky: A Guide to Astrology in English Literature* (1984).[194] The first one hundred or so pages are devoted entirely to doing late medieval and early Renaissance astrology and the making of horoscopes. John David North's *Horoscopes and History* (1986)[195] covers much of the same ground as Eade's book, but with much greater mathematical precision and as such it is nearly impenetrable without first reading Eade or by gaining some astrological proficiency from some other source. Lastly, I would mention Charles Burnett's chapter on Latin astrological vocabulary in *Medieval Latin: an Introduction and Bibliographical Guide*, edited by Mantello

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[193] Richard Joseph Lemay, *Abu Ma'shar and Latin Aristotelianism in the Twelfth Century: The Recovery of Aristotle's Natural Philosophy through Arabic Astrology* (Beirut: American University of Beirut, 1962).

[194] J. C. Eade, *The Forgotten Sky: A Guide to Astrology in English Literature* (New York: Clarendon Press, 1984).

[195] John David North, *Horoscopes and History* (London: Warburg Institute, University of London, 1986).

and Rigg (1996).[196] This thirteen-page essay proved to be most useful when confronting the manuscripts and early printed editions of the astronomical texts of Prosdocimo.

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Studies that Address the Mathematical Philosophy Behind the Quadrivium

Few studies have been written on the quadrivium as a whole from a mathematical or philosophical perspective. As was previously mentioned, studies that describe the actual mathematics of the quadrivium, usually do so from inside a specific discipline such as music history or the history of astronomy. I suspect that this is largely due to our modern concept of disciplinary divisions and a tendency towards specialization. The medieval disciplines making up the quadrivium are incommensurable with our modern disciplinary categories. Modern arithmetic is closer to medieval logistic and the mathematics of the abacus than quadrivial arithmetic. The mathematics of strings and pitch are more familiar to modern physicists than to modern musicians. A student of the history of mathematics can complete a Ph.D. without ever stepping foot in the music department, where music historians are generally housed. A reflection of this separation of divisions is easily seen in the secondary literature and the journals in which much of this literature is published. There is, of course, nothing wrong with this. Nearly all of the literature I have surveyed has been solid scholarship. But an uncritical acceptance of our socially constructed modern disciplines can lead to a disjointed view of the medieval quadrivium. In order to understand the quadrivium, one must make an interdisciplinary study of something that was not interdisciplinary in its original form.

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[196] Charles Burnett, "Astrology," in Mantello and Rigg, eds., *Medieval Latin: An Introduction and Bibliographical Guide*, 369-382.

One such study is Guy Beaujouan's "The Transformation of the Quadrivium" found in the collection *Renaissance and Renewal in the Twelfth Century*, edited by Robert Louis Benson, Giles Constable and Carol Dana Lanham (1982).[197] Beaujouan's description of the quadrivium is largely focused on the absorption of new practical and theoretical ideas into the quadrivial disciplines in the 12th century. These include new musical practices incorporated into music theory, the introduction of Hindu-Arabic numerals into arithmetic, and the absorption of Aristotelian natural philosophy from Arabic texts on astrology into quadrivial theory, particularly astronomy. Beaujouan does not describe the actual mathematics of the quadrivium, but he treats the quadrivial disciplines both individually and collectively as a cohesive system.

Although John Murdoch's essay, "The Medieval Language of Proportions: Elements of the Interaction with Greek Foundations and the Development of New Mathematical Techniques,"[198] is not explicitly about the quadrivium, it addresses one of the fundamental issues that defines the quadrivium as a cohesive philosophy– the distinction between multitude and magnitude, arithmetic and geometry, rational and irrational. Murdoch's exploration of this issue in the Middle Ages is based on historical readings of mathematical texts and a comparative analysis of these texts to the mathematical philosophy presented in Euclid. The type of historical mathematical scholarship exemplified in this essay by Murdoch, reading specific examples of medieval mathematics for philosophical content, was one of the primary models for this dissertation.

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[197] Guy Beaujouan, "The Transformation of the Quadrivium," in Robert Louis Benson, Giles Constable, and Carol Dana Lanham, eds., *Renaissance and Renewal in the Twelfth Century* (Cambridge, MA: Harvard University Press, 1982), 463-487.

[198] John Murdoch, "The Medieval Language of Proportions: Elements of the Interaction with Greek Foundations and the Development of New Mathematical Techniques," in A. C. Crombie, ed., *Scientific Change: Historical Studies in the Intellectual, Social, and Technical Conditions for Scientific Discovery and Technical Invention, from Antiquity to the Present* (New York: Basic Books, 1963), 237-271.

In a style similar to Murdoch's essay, but on a much grander scale, is Jacob Klein's *Greek Mathematical Thought and the Origin of Algebra* (first published in German in 1934).[199] The first two parts of this book (approximately 150 pages) are dedicated to a systematic analysis of Greek mathematics and how it differs from medieval and modern thought. Klein not only describes much of the mathematics in detail and without overtly anachronistic mathematical symbols, but he also demonstrates how it is exemplified in Platonic and Aristotelian philosophy, both ontologically and epistemologically. Like Murdoch, Klein specifically addresses the fundamental distinction of multitude and magnitude and how this distinction affected concepts of proportionality and eventually how the elimination of this distinction paved the way for modern algebra. The philosophical information contained in this book was extraordinarily useful for this dissertation, not so much for what it revealed to me about premodern mathematical philosophy, but more because it verified and validated my interpretations of much of what I found in the primary sources associated with my particular case studies. The scholarly method employed by Klein was a model for this dissertation in as much as I have attempted to show how quadrivial mathematics reflects a coherent philosophy, but the degree to which I philosophize in general is limited to my case studies.

Ann E. Moyer's *The Philosophers' Game: Rithmomachia in Medieval and Renaissance Europe* (2001)[200] is the history of a medieval board game designed to teach the proportional mathematical manipulations used in Boethian arithmetic and music theory. In recounting this

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[199] Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. E. Brann (Cambridge, MA: MIT Press, 1968).

[200] Ann E. Moyer, *The Philosophers' Game: Rithmomachia in Medieval and Renaissance Europe, with an edition of Ralph Lever and William Fulke, The Most Noble, Auncient, and Learned Playe (1563)* (Ann Arbor, MI: University of Michigan Press, 2001). Moyer has also written on a similar game used in astronomical instruction and on music as a science in general. See Ann E. Moyer, *Musica Scientia: Musical Scholarship in the Italian Renaissance* (Ithaca, NY: Cornell University Press, 1992); Ann E. Moyer, "The Astronomer's Game: Astrology and University Culture in the Fifteenth and Sixteenth Centuries," *Early Science and Medicine* 4, no. (1999): 228-250.

history Moyer illuminates the philosophical and ethical dimensions of the quadrivium as a whole as envisioned by Boethius and expanded upon by subsequent quadrivial scholars. And by structuring her story around a game, the quadrivium is given life in contexts outside the standard milieus of cathedral schools and universities.

Gabriela Ilnitchi's essay, "*Musica mundana*, Aristotelian Natural Philosophy and Ptolemaic Astronomy" (2002),[201] is structured around a fascinating anonymous late 13th-century manuscript now housed in the Vatican Library. The three Boethian categories of music (*instrumentalis, humana,* and *mundana*) are the subject of this anonymous manuscript and its author is surprisingly forthright in his descriptions of the attitudes of his contemporaries on these topics. The author's discussion of *musica mundana*, the topic of Ilnitchi's study, describes a combination of influences: Aristotelian physics, Neoplatonic and Ptolemaic cosmology, Arabic astrology, and Christian theology.[202] Like Murdoch's essay, described above, this essay is not about the quadrivium *per se*, but it demonstrates the assumption that music and astronomy are intrinsically linked and that the celestial influences of astrology are harmonically structured.

The last work I would like to mention is included largely because of the enormous influence it has had on my initial interests in this topic and my approach to it. The first chapter of D. P. Walker's *Spiritual and Demonic Magic: from Ficino to Campanella* (1st ed., 1958),[203] entitled "Ficino and Music" (originally published as articles in 1953 and 1954), describes Marsilio Ficino's theory of celestial influences based on a combination of Boethian music theory, Aristotelian matter and sensory theory, Platonic and Stoic pneumatic theories of a multi-part soul

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[201] Gabriela Ilnitchi, "*Musica mundana*, Aristotelian Natural Philosophy and Ptolemaic Astronomy," *Early Music History* 21, no. (2002): 37-74.

[202] Although Ilnitchi dismisses the influence of Ptolemy's *Harmonics* on this 13th-century manuscript because it was generally thought to have been unavailable at this time and place, the similarities are striking.

[203] D. P. Walker, *Spiritual and Demonic Magic: from Ficino to Campanella* (University Park, PA: Warburg Institute, University of London, 1958; reprint, Pennsylvania State University Press, 2000).

and perception, Galenic/Hippocratic humoral theory, Neoplatonic cosmology, and astrology. It was my first exposure to an astro-musical system that started to make sense because it involved more than just astronomy and music theory. It described a worldview based on an Aristotelian natural philosophical armature that brought together many disparate ideas that when addressed separately seemed weird and abstract, but they came together as a coherent system when acting in concert. I discovered that the individual parts of the system came into view when Walker addressed the entire system. Walker's analysis of Ficino's spiritual harmonic cosmos eventually led me to my approach to the quadrivium. The parts informed the larger system and vice versa. Although Ficino was not a good candidate for my case studies because he lacked the full complement of the quadrivial arts, his example, as presented by Walker, showed me that even incredibly complicated interconnections of disparate disciplines could be explained and understood if the materials and examples were properly organized and edited. The quadrivium is, in and of itself, a system that does just that. It organizes the disparate topics of arithmetic, music, geometry, and astronomy in such a way that they all work in concert. The system has been flexible enough to incorporate many new ideas such as Hindu-Arabic numerals, a variety of tuning theories, new ways to measure the terrestrial world, and new astronomical models. But it also assumes a world view that is similar to Ficino's in many ways, a world of Galenic medical spirits and humors, Platonic forms, Aristotelian universals, Neoplatonic emanations, and Christian theology. Walker taught me to keep looking deeper. When something does not make sense, there is probably something else at work. "Figure it out," I keep hearing a voice in my head say. This dissertation is a response to that voice in my head, which shouted at me every time I read the word quadrivium.

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The Case Studies

The following case studies were selected from a long list of potential candidates.[204] The first prerequisite was a demonstrated knowledge of all four quadrivial disciplines and access to their relevant texts. Secondly, they had to be active from the period spanning approximately 1350-1450, a time just before the era of printing and the modern authors, such as Nicolaus Copernicus (1473-1543), Franchino Gaffurio (1450-1522), and Ramis de Pareja (1440- ca. 1491). This time period makes it possible to discuss how quadrivial scholars thought about the quadrivial philosophy, without the distractions of revolutions or secondary literature that focuses on progress and novelty. Thirdly, I wanted to examine a selection of scholars whose utilization of quadrivial mathematics was as diverse as possible given the other requirements.

I settled on three case studies: Nicole Oresme (ca. 1320-1382), Prosdocimo de' Beldomandi (ca. 1375-1428), and Leon Battista Alberti (1404-1472). Each one was educated in the quadrivial arts and demonstrates a proficiency in all four disciplines. Each flourished within the time span of 1350-1450. And each has a very distinctive approach to the quadrivium. Oresme pushed against it, Prosdocimo strove to maintain it, and Alberti applied it when it suited his purposes. As such, different approaches were employed to distill the quadrivial philosophies from these each of these authors. Oresme nearly wrote his own chapter on his quadrivial philosophy. Prosdocimo was philosophically guarded and a great deal of his quadrivial philosophy had to be inferred, except for a section where he was provoked by a theory that

[204] This list included Hermann of Carinthia (fl. 1138–1143), Robert Grosseteste (ca. 1168-1253), Johannes de Sacrobosco (ca. 1195- ca. 1256), Georgius Pachymères and Manuel Bryennius (fl. ca. 1300), Peter of Abano (ca. 1250- ca. 1316), Johannes de Muris (ca. 1290- ca. 1350), Johannes de Lignères (fl. early to mid-14th century), Jacques de Liège (fl. mid-14th century), Nicole Oresme (ca. 1320-1382), Blasius of Parma (ca. 1345-1416), Pierre D'Ailly (1350-1420), Prosdocimo de' Beldomandi (ca. 1375-1428), Giorgio Anselmi (d. ca. 1440), Leon Battista Alberti (1404-1472), Jacques Lefèvre D'Etaples (Latinized to Jacobus Faber Stapulensis – ca. 1455-1536), Gregor Reisch (ca. 1467-1525), and Girolamo Cardano (1501-1576).

greatly displeased him. And Alberti required an analysis of his myth before his actual quadrivial philosophy could be reconstructed in a meaningful way.

This dissertation is not about the university curricula of the late Middle Ages, or the influx of Aristotelianism, or the reemergence of Platonism, or the new science of physics, or the beginnings of Humanism. These topics are discussed as they arise, but they are not my focus. This dissertation is about how quadrivial scholars thought about the quadrivium, the four mathematical arts of the Middle Ages, as a unified metaphysical structure of the cosmos.

Chapter 3:
Quadrivial Case Study #1:
Nicole Oresme (ca. 1320-1382)

The animations for this chapter are available in three locations:

1) Quadrivial Pursuits Media Outlet.htm.
[www.mifami.org/quadrivium/pursuits/QuadrivialPursuitMediaOutlet.htm]

2) on the CD that is included with the hard copy of this dissertation

3) as separate files provided by the digital archival service contracted by the CUNY Graduate Center Library.

The following case study on the quadrivial philosophy of Nicole Oresme encounters one of the main themes that both defines the quadrivium and may have subsequently led to its loss of status – the definition of number. As we saw have seen, the quadrivial definition of number, or multitude, was distinct from geometrical magnitude. Boethius went so far as to claim that the properties of multitude and magnitude were "different and even opposite."[205] From a modern point of view it is difficult to understand why such a distinction would be made and why the elimination of this distinction would encounter resistance. Oresme not only eliminates this distinction, but he also sheds light on why such a distinction was so deeply entrenched in mathematical philosophy.

Introduction and Short Biography

Nicole Oresme was born ca. 1320 in Normandy, probably just south of Caen in the town of Allemagne.[206] Little is known of his early education, but it is assumed that he studied at the University of Paris[207] in the 1330s before receiving his master's degree in the arts in 1341 or

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[205] Boethius, *De institutione musica*, 2.3, p. 53.
[206] The modern name for Allemagne is Foeury-sur-Orne. It is about two miles south of the center of Caen. See William J. Courtenay, "The Early Career of Nicole Oresme," *Isis* 91, no. 3 (2000): 542.
[207] Thirteenth-century statutes from the University of Paris give the impression that they had abandoned the traditional seven liberal arts in favor of Aristotle, but other sources such as collections of *Quodlibeta*

1342. After that he presumably began his studies in theology and supported himself by teaching. During this early period of his academic life he may or may not have studied under John Buridan (ca. 1300 – ca. 1358), the famous natural philosopher associated with a theory of impetus, but it is quite clear that, at the very least, he knew Buridan and was most certainly familiar with his work.[208] This general chronology of Oresme's early education puts him at the University of Paris[209] at a time when the radical epistemological and natural philosophical ideas of the English philosophers William of Ockham (d. ca. 1348), Thomas Bradwardine (d. 1349), and William

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and *Questiones disputatae*, manuals of instruction, and innumerable manuscripts clearly demonstrate that the liberal arts in general and the quadrivium in particular remained in the program. Scholars associated with the University of Paris, such as Robert Grosseteste (ca.1168-1253) and Roger Bacon (ca.1215-1292), were staunch supporters of the quadrivium, seeing it as a prerequisite to the other sciences. As was previously noted, Bacon went so far as to write that "men ignorant of [mathematics] do not perceive their own ignorance, and therefore do not seek a remedy." See Roger Bacon, *The Opus majus of Roger Bacon*, 2 vols., vol. 1 (Philadelphia: University of Pennsylvania Press, 1928; reprint, Whitefish, MT: Kessinger Publishing, 2002), Pt. 4, Dist. 1, Ch. 1, p. 116. It may be in response to Bacon that Oresme wrote, "I indeed know nothing except that I know that I know nothing." See Marshall Clagett, "Nicole Oresme," in *The Dictionary of Scientific Biography,* ed. Charles Coulston Gillispie, (New York: Charles Scribner's Sons, 1970-1980), 225. See also Nicole Oresme, *Nicole Oresme and The Marvels of Nature: A Study of his 'De causis mirabilium' with Critical Edition, Translation, and Commentary*, ed. and trans. Bert Hansen (Toronto: Pontifical Institute of Mediaeval Studies, 1985), 98, n7. Like Bacon, Thomas Aquinas (1225-1274) also mentions that the trivium and quadrivium were still part of fundamental education in his time. Generally speaking, in the 13th century the quadrivium was considered the gateway to physics and metaphysics. See Kibre, "The Quadrivium in the Thirteenth Century Universities (with Special Reference to Paris)," 175-191 and 198-203. The extent and quality of quadrivial instruction in Paris is the subject of some debate. Cf. Joseph Dyer, "Speculative 'Musica' and the Medieval University of Paris," *Music & Letters* 90, no. 2 (2009): 177-204. Even if the state of quadrivial instruction in 14th-century Paris was deficient, Oresme demonstrates extensive quadrivial knowledge, with citations to the majority of the standard sources in his writings.

[208] Courtenay, "The Early Career of Nicole Oresme," 548, n20.

[209] For sources on the University Paris in the Middle Ages with special attention paid to the quadrivium, see the following: Pearl Kibre, "The Boethian '*De institutione arithmetica*' and the Quadrivium in the Thirteenth-Century University Milieu at Paris," in *Boethius and the Liberal Arts: A Collection of Essays,* ed. Michael Masi, 67-80; Pearl Kibre, *Studies in Medieval Science: Alchemy, Astrology, Mathematics, and Medicine* (London: Hambledon Press, 1984), XII; Richard Lemay, "The Teaching of Astronomy in Medieval Universities, Principally at Paris in the Fourteenth Century," *Manuscripta* XX, 197-217; John D. North, "The Quadrivium," in *Universities in the Middle Ages,* ed. Hilde de Ridder-Symoens, 337-359; Hastings Rashdall, *The Universities of Europe in the Middle Ages*, 269-584.

Heytesbury (ca. 1313 – ca. 1372) were at their height,[210] and their influence on Oresme, along with that of Buridan, are evident in much of his subsequent thought.

At some point before 1356, Oresme attracted the attention of King John II of France who asked Oresme for advice on monetary issues. This resulted in the treatise *De origine, natura, jure et mutationibus monetarum,* which describes an economic philosophy based on a solid currency and moral behavior. Topics discussed in this text include currency exchange, the causes of inflation, and the evils of usury.[211] In 1356, towards the end of the first phase of the Hundred Years War, John II was captured by English forces.

Also in 1356, after approximately fourteen years of study, Oresme received the modern equivalent of a doctorate in theology and was almost immediately made Grand Master of the College of Navarre, the college in the University of Paris where Oresme had been studying.[212] At about this time he befriended the future king of France, Charles V (r. 1364 – 1380), who tenuously ruled while his father was held hostage in England. John II, still a pseudo-hostage of England, ultimately died in 1364. Charles V succeeded to the throne upon his father's death. As king, Charles implemented many of the policies described in *De origine, natura, jure et*

[210] Stefano Caroti, "Nicole Oresme," in *The Complete Dictionary of Scientific Biography* [electronic resource], ed. Noretta Koertge (Detroit: Charles Scribner's Sons, 2008. *Gale Virtual Reference Library*. Accessed 8 Feb. 2011.), 350-351.
[211] The earliest version of this text dates from ca. 1355. See Kevin B. Bales, "Nicole Oresme and Medieval Social Science: The 14th Century Debunker of Astrology Wrote an Early Monetary Treatise," *American Journal of Economics and Sociology* 42, no. 1 (1983): 105-107. See also Edward Grant, introduction to Nicole Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, ed. and trans. by Edward Grant (Madison: University of Wisconsin Press, 1966), 5.
[212] It was technically called a master's *license*. See Albert Menut, introduction to Nicole Oresme, *Le livre du ciel et du monde*, eds. A. D. Menut and A. J. Denomy, trans. A. D. Menut (Madison: University of Wisconsin Press, 1968), 8-9. See also Courtenay, "The Early Career of Nicole Oresme," 544.

mutationibus monetarum. Charles V remained Oresme's friend and protector for the rest of his life.[213]

The majority of Oresme's Latin works, which include most of his mathematical treatises, probably date from 1348 to 1362, which corresponds to his time at the College of Navarre.[214] In 1362 Oresme was appointed canon of Rouen Cathedral, about 80 miles northwest of Paris, and in 1364 he became its dean.[215] In the 1370s he was commissioned by the king to translate into French a variety of works. These included Latin to French translations and commentaries on Aristotle's *Ethics*, *Politics*, *Economics*, and *On the Heavens*.[216] During this decade Oresme appears to have resided in Paris even though he was still the dean of the Rouen Cathedral. In 1377, with royal assistance, he was named bishop of Lisieux, about 120 miles west of Paris and about 30 miles from his home town, but he did not reside there until September of 1380, which also happens to be the month in which Charles V died. Oresme died two years later in 1382.[217]

Oresme is probably best known for his physical theories which anticipate the concepts of inertia, acceleration, and gravity, as well has his consideration of a spinning, as opposed to, stationary earth. Despite these modern innovations, Oresme is quite clearly entrenched in the Aristotelian world view. He considered many ideas that opposed standard Aristotelian natural philosophy, but when push came to shove, he usually decided in favor of an Aristotelian interpretation. For Oresme, celestial mechanics were fundamentally different from terrestrial

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[213] Bales, "Nicole Oresme and Medieval Social Science," 107. See also Susan M. Babbitt, *Oresme's "Livre de politiques" and the France of Charles V* (Philadelphia: American Philosophical Society, 1985), 3-5.

[214] Edward Grant reviews the literature on this issue in great detail. See Grant, introduction to Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, 5-6.

[215] Ibid., 7.

[216] A description of his translations can be found in Claire Richter Sherman, *Imaging Aristotle: Verbal and Visual Representation in Fourteenth-Century France* (Berkeley: University of California Press, 1995), 3-34.

[217] Grant, introduction to Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, 9-10.

physics; the earth did not really spin, and projectile motion was an unnatural motion requiring a continuous divine mover.

Oresme is also well known today for his dogged opposition to judicial astrology.[218] Although it should be pointed out that Oresme did not necessarily reject many of the claims of traditional astrology, which posited that celestial events influenced terrestrial events, but he did not believe that human beings could know with any degree of certainty when, where, or even how most of these influences would occur. In part his opposition to astrology stemmed from religious concerns, but it also appears to have been exacerbated by Charles V's reliance on astrological advisors.[219] Many of the arguments he formulated against astrology in the 14th century in such works as *Questio contra divinatores horoscopios* and the *Quodlibeta* (also known as *De causis mirabilium*[220]) were later espoused and expanded upon by Pico della Mirandola in the 15th century. A number of his most important mathematical works – including *Ad pauca respicientes, De proportionibus proportionum,* and *Tractatus de commensurabilitate vel incommensurabilitate motuum celi* – were originally written for the purpose of undermining what Oresme saw as the foundations of judicial astrology. Oresme believed that the prognostications of astrology were predicated upon "precise determinations of successively repeating conjunctions, oppositions, and other aspects,"[221] and that astrological tradition held that these determinations were based on mathematical principles – principles based on Platonic

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[218] Oresme writes about these limitations in *Livre de Divinacions*. See Nicole Oresme, *Nicole Oresme and the Astrologers: a Study of His "Livre de Divinacions,"* trans. George William Coopland (Cambridge, MA: Harvard University Press, 1952), ch. 2, pp. 54-57.
[219] Oresme, *Le livre du ciel et du monde*, 5-6; Oresme, *Nicole Oresme and The Marvels of Nature*, 21-24. Oresme writes in *Livre de Divinacions* that astrology is "most dangerous to those of high estate, such as princes." See Oresme, *Nicole Oresme and the Astrologers,"* 50-51.
[220] Oresme, *Nicole Oresme and The Marvels of Nature*, 26.
[221] Clagett, 224.

number, the domain of quadrivial arithmetic and music.[222] So instead of looking at the sky to discover the mathematical principles that presumably structured its motions, a task that Oresme readily admitted was futile to any degree of accuracy, he instead analyzed the types of motions that astrologers claimed were fundamental to their profession. He modeled a hypothetical universe as it was described by the traditions of astrology and found that it bore little resemblance to the motions of the actual heavens.

It is in one of Oresme's more technical works devoted to demolishing judicial astrology that the clearest exposition of his quadrivial philosophy is found, for it pits arithmetic directly against geometry. This work, the *Tractatus de commensurabilitate vel incommensurabilitate motuum celi (De commensurabilitate* for short), as Edward Grant puts it, "is very likely an expanded and revised version of his earlier *Ad pauca respicientes."*[223] It also includes or refers to much of the material from *De proportionibus proportionum*. Grant has translated all three of these works into English and written extensively on each. The following analysis of Oresme's treatment of the quadrivium is structured around *De commensurabilitate*, for this work is an analysis of the quadrivium itself.

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Treatise on the Commensurability or Incommensurability of the Motions of the Heavens

Oresme's *Tractatus de commensurabilitate vel incommensurabilitate motuum celi, [Treatise on the Commensurability or Incommensurability of the Motions of the Heavens]*, written in the

[222] Edward Grant explains Oresme's position on the kinematic assumptions of judicial astrology in more detail in his introduction to Chapter 3 to Nicole Oresme, *Nicole Oresme and the Kinematics of Circular Motion: "Tractatus de commensurabilitate vel incommensurabilitate motuum celi,"* ed. and trans. by Edward Grant (Madison: University of Wisconsin Press, 1971), 142.
[223] Grant, preface to Ibid., xi-xii.

mid-14th century,[224] is a mathematical exploration of the relationships between concentric circular motions[225] and astronomy. Oresme explores under what conditions two or more moving bodies might be in conjunction, i.e., when and/or where two mobiles might be located on the same radial line drawn from the center of the circle). See Figure 3.1.

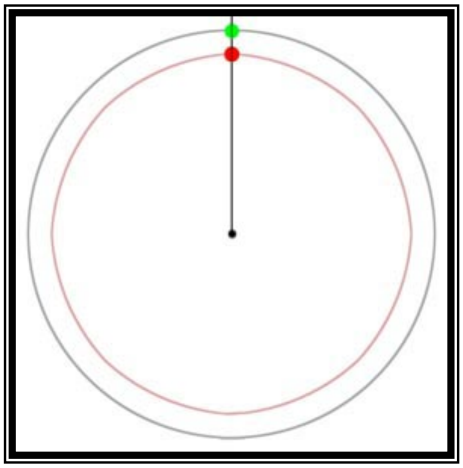

Figure 3.1: Two Mobiles in Conjunction

Red and green (inner and outer) mobiles are in conjunction at 12:00.

The questions Oresme considers are as follows:

–If two bodies conjunct or meet, will they meet at this place only once, or will they meet a finite number of times at this location, or will they meet an infinite number of times at this location?

–Is it possible that they might conjunct at a particular location only once, yet meet an infinite number of times at unique locations?

–What are the implications for the motions of the heavens?

–In short, are celestial motions commensurable or incommensurable?

This text is divided into three parts. Parts I and II are highly abstract and analytical. Only occasionally do they leave the realm of pure mathematics. Part I deals with commensurable motions, those described by ratios of counting numbers, and Part II deals with incommensurable motions, those described by irrational ratios of magnitudes. Part III contains almost no mathematics and is written as a

[224] Grant favors a range between 1351 and 1362. Ibid., 4-5.
[225] Throughout the text it is generally assumed that all circular motions are uniform. This assumes uniform angular velocities, a single center of rotation, and constant radii.

dream wherein Arithmetic and Geometry, representing multitude and magnitude, present their arguments for the commensurability or incommensurability of the motions of the heavenly spheres.

Part I of the *De commensurabilitate* contains one of the most striking examples of harmonic analysis applied to celestial motions in the context of quadrivial philosophy. In Part I Oresme creates an argument, using straightforward quadrivial techniques, to disprove a wide-spread belief that the celestial spheres moved harmonically in relation to one another. He effectively refutes a longstanding and popular belief that a mathematically-structured cosmos must move in accordance with consonant harmonic principles. This particular Pythagorean interpretation of the "music of the spheres" was best known in this period from Macrobius' *Commentary on the Dream of Scipio,* to which Oresme refers in Part III of *De commensurabilitate*. Oresme recounts the story:

> Scipio says : "What is this great and pleasing sound that fills my ears?" And his grandfather, whom Scipio had seen in his dream, answered: "That," he replied, "is a concord of tones separated by unequal but nevertheless carefully proportioned intervals, caused by the rapid motion of the [celestial] spheres themselves. The high and the low tones blended together produce different harmonies."[226]

Macrobius, in his commentary on Cicero, not only describes this Ciceronian theory of harmonic celestial motion, but he also interprets Plato's cosmic description from the "Myth of Er" from the *Republic* and the cosmic creation from *Timaeus* in terms of harmonic celestial motions.

What is most remarkable in Oresme's argument opposing this description of a cosmic harmony is that he uses the disciplinary tools of the quadrivium itself to disprove it. Nevertheless, he does not completely discount the idea that there may be consonant harmonic relationships between the planetary bodies. He leaves open the possibility for a harmonically-structured cosmos, but dismisses the motion-based Pythagorean relationships that he claims had long captured the imaginations of scholars.

[226] Oresme, *De commensurabilitate*, III.161-165, pp. 296-299.

In order to see how Oresme came to his conclusions it will be necessary to outline his mathematical argument, which is simple, elegant, and easily understood. For clarity, I will not present rigorous proofs, but rather plausibility arguments along with illustrations, animations, and simple examples derived from Oresme's text. I have resorted to using some algebraic notation for simplicity and to better reach a modern reader, but it should be emphasized that Oresme's arguments are written entirely in prose.

• • •

Oresme's Argument in Part I of *De commensurabilitate*

The prologue to Part I defines the terms that will be used for the remainder of the book. These are the definitions upon which the subsequent arguments are based and are generally consistent with similar definitions given by Euclid and Boethius.[227] Several basic definitions are not explicitly stated, but are clearly understood from the context. Examples of these include:

-unity – not presented as a number *per se*, although it is sometimes used as one.

-number – implied to be a collection of unities. We call them counting numbers.

-ratio or proportion – implied as a comparison of numbers or ratios.[228]

Of particular interest and significance to the work as a whole, and this study in particular, are Oresme's qualifications for commensurable quantities. He states, "Quantities are said to be commensurable which have some common measure, or which have a ratio of a number to a

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[227] In the commentary to *De Commensurabilitate,* Edward Grant provides many citations to Campanus of Navara's 13th-century edition of Euclid's *Elements*, which was widely available in Oresme's time. See Edward Grant, ed., *A Source Book in Medieval Science*, Source Books in the History of the Sciences (Cambridge, Mass.: Harvard University Press, 1974), 134-150. See also Boethius, *De institutione arithmetica*, passim.

[228] Throughout this text, Edward Grant translates *proportio* as either proportion or ratio. In the context of *De commensurabilitate* there is no need for any distinction. However, it should be noted that Euclid distinguishes ratio from proportion. For Euclid a ratio is a comparison between two quantities of the same kind and a proportion is a comparison of ratios. See Euclid's *Elements*, Book 5, Definitions 3, 6, and 8.

number."[229] The strict definition that a commensurable ratio must be "of a number to a number" is standard, and assumes that it is expressible using only counting numbers. For example, 12:5 or 256:243. The standard Pythagorean consonant harmonic ratios (2:1, 4:3, 3:2, etc.) are all commensurable. His other qualification is of more interest for it seems to say that commensurable quantities of physical measurements are those with a "common measure." The example he gives is "two feet" compared to "three feet."[230] This restriction forbids mixing dimensions, which at times is a tendency of the more numerologically inclined and remains an issue to this day for undergraduate science and math students who forget to keep their dimensions straight. He is saying that one cannot consider a comparison of three feet to six pounds. Their measures are not common and thus they are not commensurable.

This added qualification on commensurability may seem quite obvious and hardly worth stating today, but considering the way in which mathematics was expressed in the 14th century and the popularity of numerology, it seems a reasonable caution. It should be kept in mind that there are no equations in this text as we might now think of them. There are no equals signs or algebraic variables. Everything is written out in prose and it is quite normal for a measurement to be presented without any "common measure," no dimension explicitly stated. However, throughout the text Oresme is quite careful not to compare quantities of different dimensions and his statement to this effect reminds the reader of this restriction.

It should also be kept in mind that historically this is the transition period between kinematics and physical dynamics. Systems of measurement were fluid and self-consistent, but not universally adopted. Proportionality was the main currency. Today we tend to refer to fixed standards of measure: meters, seconds, kilograms, etc. But in the 14th century such standards did

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[229] Oresme, *De commensurabilitate*, I.18-19, pp. 176-177.
[230] "...ut si una est bipedalis et altera trium pedum." Ibid., I.20, pp. 176-177.

not exist. Natural philosophers would instead refer like quantities to each other. They would, in effect, change their coordinate systems to break down relationships to their simplest forms,[231] which explains all the trouble they go through constantly reducing fractions to their lowest "prime" factors. A common example of this proportional way of thinking is Pythagorean tuning. Measurements of the lengths of strings on a monochord were not expressed in inches or centimeters or any other similar standard. Measurements were compared to each other, without any dimension stated. In fact, the lack of dimensions is a consequence of the use of ratios, e.g., 10m:17m equals 10:7. The "m" (i.e. meters) divides out, leaving a dimensionless ratio, which in a way, is more *real* in a Platonic sense. The corruptible-worldly matter is stripped off and what is left is pure form. Before the introduction of algebraic analysis mathematicians tended to frame many arguments in ratios, whereas today, mathematicians tend to think more in terms of equalities. This is a subtle, but illuminating difference – a difference that is almost totally eliminated when premodern mathematics is translated into algebraic notation. An octave was a length of string compared to another of twice its length. The first length could be 8 centimeters or 5 kilometers so long as the second length was 16 centimeters or 10 kilometers respectively. Both are octaves. The measurement system used was calibrated to fit the problem at hand in a neat, integral manner. Furthermore, many measurements were simply not known and thus they could not be compared to some external standard. For example medieval astronomers could determine the angular separation of Venus and Mars, but they did not know how far away these planets were from the Earth. Thus actual linear velocities were unknowable, only angular velocities, which were un-calibrated to anything external.

[231] It was a form of generalized coordinates not dissimilar to the underlying methods of Lagrangian mechanics.

In terms of circular motions, commensurable motions are those that describe commensurable angles around the center in equal times or those that "complete their circulations [full circles] in commensurable times."[232] That is to say, if mobile A moves circularly about a center at x degrees per unit time and mobile B moves at y degrees per unit time, the two motions are commensurable if the ratio x:y is commensurable. For example, using modern notation, if A moves at 90º/hour and B moves at 45º/hour, the ratio of their motions is 90:45, which is clearly commensurable. It is a ratio of two counting numbers. Similarly, if the orbital periods of the two bodies are commensurable, their motions are commensurable. For example, if mobile A has a period of 4 hours per *circulation*[233] and mobile B has a period of 8 hours per circulation, the ratio of the periods is 4:8, which is again commensurable. In fact, these two examples are equivalent, as will be seen shortly.

Incommensurable motions are those whose *circulations* occur in incommensurable times or those which in equal times sweep out incommensurable angles. For example, √2: 3 is incommensurable since √2 is not a number.[234] In Boethian terms √2 is a "magnitude," not a number. Recall that what we now call "irrational numbers" were not considered numbers in a quadrivial sense. "Incommensurability can be found," Oresme states, "in every kind of continuous thing... an angle to an angle, a motion to a motion, a speed to a speed, a time to a time, ..., a pitch to a pitch, and so on for any similar things."[235]

This definition of incommensurability is further, but redundantly qualified in Oresme's earlier work, *Ad pauca respecientes*. In this work Oresme lists four ways in which

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[232] Ibid., I.28-29, pp. 178-179.
[233] An orbital period means the time it takes for the mobile to complete one full "*circulation*" of 360º.
[234] Boethius, *De institutione arithmetica*, I.1-4, pp. 72-77.
[235] Oresme, *De commensurabilitate*, I.38-41, pp. 178-179. "Est namque magnitudo incommensurabilis magnitudini, angulus angulo, motus motui, velocitas velocitati, tempus tempori, proportio proportioni, gradus gradui, et vox voci, et ita de similibus." I altered Grant's translation of "vox voci" from "voice to a voice," to "pitch to a pitch."

incommensurability can occur in a system of mobiles moving on concentric paths. Edward Grant points out that this enumeration of incommensurable situations is technically superfluous to the more complete definition found in *De commensurabilitate*, yet I feel that the situations described can help the modern reader see ramifications that might otherwise be overlooked. The four incommensurable situations are (paraphrased from *Ad pauca respicientes*):[236]

> 1) when the circumferences of the concentric circles are incommensurable and the curvilinear velocities are equal;
>
> 2) when the circumferences are equal or commensurable but the curvilinear velocities are incommensurable;
>
> 3) when the circumferences are equal but the distances traversed are incommensurable;
>
> 4) when the circles are incommensurable and the curvilinear velocities are incommensurable.

Following this brief set of definitions found in the Prologue and the first few pages of Part I, the arguments concerning commensurable motions begin in earnest. Propositions 1–3[237] set the stage for these arguments by explaining a few introductory matters involving prime factorization, geometric progressions, and the relationship between straight-line continua and closed circular continua. These three propositions are of general mathematical interest, but are not necessary for my informal presentation of his argument that refutes consonant harmonic relationships between celestial motions. The critical propositions for this argument are numbers 4-11. The following summary of these propositions is written in hopes of reflecting as much of Oresme's stylistic presentation as possible. However, I have also added algebraic notation for each step to assure the mathematically inclined reader that all of Oresme's steps make sense and that all quantities "have some common measure," which is not always readily apparent in the strictly prose version.

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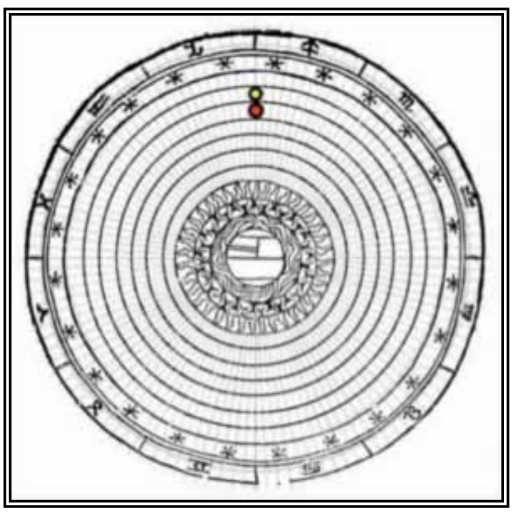
[236] See Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, 394-395 and 432-433.
[237] Oresme, *De commensurabilitate*, I.68-218, pp. 180-193.

Proposition 4 [I.218-235]: Assuming commensurable uniform circular motions Oresme argues that, "*If two mobiles are now in conjunction, it is necessary that they conjunct in the same point at other times.*"[238] Citing Euclid 10.5 as a justification for his definition of commensurability, Oresme argues that two mobiles starting in conjunction and moving circularly with commensurate but unequal motions will repeatedly meet at the same location in the future, and that they have repeatedly met at this location in the past.[239]

For example, given commensurate motions of two mobiles, A and B, in the ratio 5:3, assuming a conjunction of A and B as the initial condition, Oresme states, "When A will have made 5 circulations, B will have made 3, and they will be where they are now."[240] See Animation 3.1.



Animation 3.1: Conjunction and Revolution – 5:3 Ratio of Rotational Speeds

Arbitrarily starting at the 12:00-conjunction, every time mobile A (red) makes 5 circulations, mobile B (yellow) makes 3, and they meet again at 12:00. One could also start this animation from the 6:00-conjunction, with the same result. One could also run this in reverse with the same result.

Thus, any orbiting system consisting of two mobiles, A and B, with commensurate motions in the ratio, *x:y*, starting at a conjunction, will return to the initial configuration after mobile A

[238] Ibid., I.218-219, pp. 192-193.
[239] See also Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, 65-71, pp. 388-389.
[240] Oresme, *De commensurabilitate*, I.231-234, pp. 192-193.

completes *x circulations*.[241] Oresme uses the term *circulation* to mean, "The return of one mobile along a circular path from any point to that same point." In modern terms it means one 360º orbit. Oresme distinguishes *circulation* from the term *revolution*. A *revolution*, Oresme explains, is "[t]he return of several mobiles from any state to a wholly similar state, or aspect." A *revolution* is the return of a system of two or more mobiles to its initial arrangement. One *revolution* of the system shown in the Animation 3.1 occurs from the initial arrangement of both bodies at 12:00 to the proximate arrangement of both bodies at 12:00. This occurs when the red mobile, A, has made 5 *circulations*. Equivalently, this occurs when the yellow mobile, B, has made 3 *circulations*. See Animation 3.1.

$$1 \textit{ revolution} = 5c_A = 3c_B, \qquad \text{Equation 1}$$

where c_A is *circulation[s] of mobile A*,
and c_B is *circulation[s] of mobile B*.

Proposition 5 [I.235-272]: This proposition is by far the most difficult to explain in simple terms and Oresme's argument, lacking algebraic notation, is quite convoluted for the modern reader. Oresme summarizes Proposition 5, "[How] to find the time when the two mobiles will first conjunct in the point in which they are now."[242] In other words, assuming commensurate velocities, how much time elapses between conjunctions at the same location, i.e., what is the period of one *revolution* of the system? He is not referring to just any conjunction, for there may be more than one depending on the ratio of speeds. He is referring to the conjunction at a specific place on the circle, e.g., at 12:00.

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[241] This is the same as saying that it will return to the same configuration after mobile B completes *y* circulations.
[242] Oresme, *De commensurabilitate*, I.236-236, pp. 194-195.

This proposition deals with the relationship between what he calls *velocitates* and *tempora* in a circular system. For clarity, in the following summary, I will call these variables "rotational speeds" and "periods," and they will be referred to as ω and *T* respectively, as these are the more familiar terms and symbols of modern physics. This substitution does not in any way change Oresme's meaning, but it avoids any confusion with the modern meaning of "velocity" and the vagueness of the term "time."

The crux of Oresme's argument in Proposition 5 is the mathematical relationship between the rotational speed and the period, ω and *T*. Rotational speed, ω, is the reciprocal of the period, *T*. Where rotational speed, ω, can be measured in *circulations* per time, the period, *T*, is measured as time per *circulations*. This can be written in modern notation as

$$\omega = c/t \quad \text{and } T = t/c$$

or

$$\omega = 1/T \quad \text{and} \quad T = 1/\omega, \qquad \text{Equation 2}$$

where ω is rotational speed, *T* is the period, and *t* is time.

Oresme continues with the example he used from Proposition 4. If the ratio of rotational speeds for mobile A and mobile B is 5:3, then the ratio of periods is 3:5. We can derive this using modern notation.

If

$$\frac{\omega_A}{\omega_B} = \frac{5}{3} \quad \text{and} \quad \omega_A = 1/T_A \quad \text{and} \quad \omega_B = 1/T_{A,}$$

Then

$$\frac{\omega_A}{\omega_B} = \frac{T_B}{T_A} = \frac{5}{3}.$$

This equation tells us that the relationship between the periods of mobile A and mobile B is 3:5 respectively. We know the periods, T_A and T_B, are related 3:5, but we do not know what measurement of time this might be. It could be 3 minutes to 5 minutes, or 6 minutes to 10 minutes or 3 weeks to 5 weeks, or even 117 years to 195 years.[243] So Oresme supplies us with an arbitrary measurement of time. He chooses days. The ratio, $T_A : T_B$, Oresme proposes for the sake of his example, is 3 days to 5 days.

$$3d : 5d,$$

where d is days.

This means that the period for mobile A, T_A, is 3 days per *circulation* and the period for mobile B, T_B, is 5 days per *circulation*.

$$T_A = 3d/c_A \qquad \text{Equation 3}$$

and

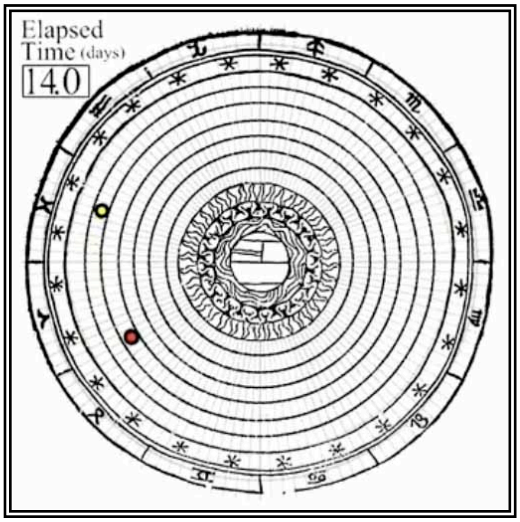
$$T_B = 5d/c_B. \qquad \text{Equation 4}$$

But Oresme is not simply after the periods of the individual motions of the mobiles. He wants to find the *period of the revolution* of the system of both mobiles. From Equation 1 we know that one *revolution* requires 5 *circulations* of mobile A. From Equation 3 we know that each *circulation* of mobile A requires 3 days. Thus, we can now easily determine the *period of revolution*, T_{rev}, of the system made up of mobiles A and B with rotational speeds related 5:3. The *period of revolution* is 5 *circulations* of mobile A multiplied by the period of a *circulation* for mobile A. The *period of revolution* is thus, 15 days.

$$T_{rev} = (5c_A)(T_A) \quad = \quad 5c_A \text{ x } 3d \quad = \quad 15d.$$

See Animation 3.2.

[243] 117:195. Both terms are divisible by 39, yielding 3:5.



Animation 3.2: 15-Day Revolution

The red mobile (A) will cover 5 circulations at 3 days per circulation. The yellow mobile (B) will cover 3 circulations at 5 days per circulation.

Oresme chooses his unit of time, 1 day, out of thin air. In order to clarify this point let me provide a short astronomical example. Let us say that we observe two satellites. We observe that satellite X circulates the heavenly sphere in 195 days and satellite Y circulates the heavenly sphere in 117 days. Their ratio of periods, $T_X:T_Y$ is 195:117, which is commensurate. With a little arithmetic we discover that if we divide both numbers by 39 our ratio, $T_X:T_Y$, simplifies to 5:3.[244] So here we have a hypothetical physical situation that mimics Oresme's example. For this system of satellites Oresme could say that the unit of time describing this system is the collection of 39 days, rather than 1 day, as in his example above. Let us call this collection of 39 days one "blorg." So we can now say that satellite X circulates the heavens in 5 blorgs and satellite Y in 3 blorgs. We also know, due to the reciprocal nature of periods, T, and rotational speeds, ω, that the ratio of rotational speeds, $\omega_X:\omega_Y$, is 3:5. From this we know that a *revolution* of the system requires that satellite X go 3 *circulations* while satellite Y covers 5 *circulations*.

[244] Prosdocimo de' Beldomandi, our next case study, provides instructions for how to do this simplification in his text, *Musica speculativa*. See Prosdocimo de' Beldomandi, *Prosdocimo de' Beldomandi's "Plana musica" and "Musica speculativa,"* trans. J. Herlinger (Urbana, IL: University of Illinois Press, 2008), 2.16, pp. 202-205.

Thus the *period of revolution* will be

$$T_{rev} = \frac{5blorgs}{circulation} \times \frac{3circulations}{revolution}$$

$$T_{rev} = \frac{15blorgs}{revolution}$$

or if we convert back to days,

$$T_{rev} = \frac{585days}{revolution}.$$

In general, given two mobiles, A and B, whose rotational speeds are commensurate and reduced to their prime factors, a:b, the *period of revolution* for this system can be found by multiplying a and b together. This works due the reciprocal nature of rotational velocity and period. For a can be seen as the number of *circulations* that mobile A must complete for a *revolution* and b as the period for a complete *circulation* of mobile A.[245]

$$T_{rev} = ab. \qquad \text{Equation 5}$$

Proposition 6 [I.273-309]: Given a commensurate system of two bodies moving around a circle at uniform speeds, Oresme states that this proposition shows, "[How] to find the time of the first conjunction that will follow a [present] conjunction of two mobiles whose velocities have been given."[246] In other words, starting in conjunction, how long does it take for a system of two mobiles moving at commensurate speeds to be again in conjunction. The proximate conjunction need not be at the same place as the initial conjunction. It need not be the period of a full *revolution* of the system. Here, he is referring to the proximate conjunction, wherever it may be.

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245 Equivalently, b could be the number of circulations that mobile B must complete for a *revolution* and a as the period for one complete circulation of mobile B.
246 Oresme, *De commensurabilitate*, I.273-274, pp. 196-197.

The kernel of Proposition 6 is the concept of relative rotational speed. Oresme determines the rotational speed of one mobile in relation to the other. He does this by subtracting the speed of the slower mobile from the speed of the faster. In a linear situation an example might be a car going 65 miles per hour compared with a car going 50 miles per hour. One car is going 15 miles per hour faster than the other. The relative speed is 15 miles per hour. In a circular scenario Oresme uses this idea of a relative rotational speed to determine how long it will take for the faster mobile to conjunct with the slower one.

Using the example developed in Propositions 4 and 5, we know that the rotational speed of mobile A is 1/3 of a *circulation* per day[247] and that the rotational speed of mobile B is 1/5 of a *circulation* per day.[248]

$$\omega_A = (1/3)c/d \quad \text{and} \quad \omega_B = (1/5)c/d.$$

As Oresme puts it (I.307-309),[249] the relative speeds are determined by, "The motion of one [mobile] ... subtracted from the motion of another..."

$$\omega_{con} = \omega_A - \omega_B,$$

where ω_{con} is what we will call the *relative rotational speed of conjunction*.

$$\omega_{con} = (1/3)c/d - (1/5)c/d = (5/15)c/d - (3/15)c/d =$$

"...and the remainder has a numerator and denominator."[250]

$$\omega_{con} = (2/15)con/d,$$

where *con/d* is conjunctions per day.[251]

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247 We know this because mobile A makes a full circulation in 3 days.

248 From Equations 2 and 3, recall that the period of mobile A is 3 days per *circulation*. Therefore, its rotational speed is 1 *circulation* per 3 days, or 1/3 of a *circulation* per day and similarly for mobile B.

249 "Subtrahatur motus unius a motu alterius..." Oresme, *De commensurabilitate*, I.307-308, pp. 198-199.

250 "...et residuum habet numeratorem et denominatorem." Ibid., I.308, pp. 198-199.

251 Because ω_{con} does not describe the motion of an actual body, but rather the relative motion between two bodies, I have changed the *c* for *circulation* that we had been using, to *con,* for *conjunction,* to eliminate any confusion.

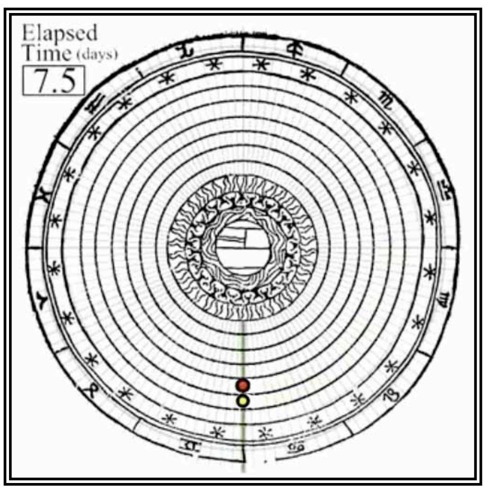
What we want to find using this *relative rotational speed of conjunction*, is how much time is needed, T_{con}, for this relative speed to complete one full circulation. This will be the period of time between two successive conjunctions. Recall Equation 2, that rotational speed, ω, is the reciprocal of the period.

$$T_{con} = 1/(\omega_{con}) = (15/2)\,d/con$$

or

$$= 7.5\ d/con.$$

Thus the period is 7.5 days per conjunction. See Animation 3.3.



Animation 3.3: Conjunction Every 7 and 1/2 Days

Notice that at 7.5 days, the mobiles conjunct at 6:00.

Oresme puts it this way, "The time sought is produced by dividing the denominator by the numerator," [252] which is exactly what we did to the *relative rotational speed of conjunction*, ω_{con}. We flipped it over.

The period between successive conjunctions of a system made up of two mobiles moving circularly at commensurate rotational speeds is

$$T_{con} = 1/(\omega_{con}). \qquad \text{Equation 6}$$

Equation 6 gives us the general equation for the period between all proximate conjunctions.

[252] "Dividatur itaque denominator per numeratorem et exibit tempus quesitum." Oresme, *De commensurabilitate*, I.309, pp. 198-199.

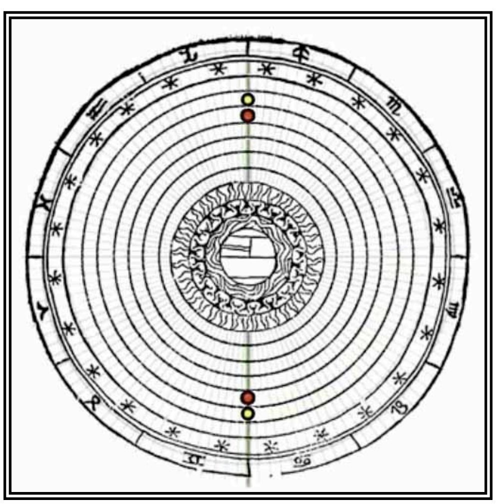
Proposition 7 [I.310-325]: Again, given a commensurate system of two bodies moving around a circle at uniform speeds, Proposition 7 tell us, "[How] to find the number of conjunctions in a complete revolution."[253] Because we have assumed that the rotational speeds are uniform, the time between successive conjunctions will always be equal. Thus to find the total number of conjunctions, N_R, in a given *period of revolution*, we simply divide the *period of revolution*, T_R, by the *period of relative conjunction*, T_{con}. Using the example that Oresme has been developing, we would divide 15 days per revolution by 7.5 days per conjunction, yielding 2 conjunctions per *revolution*.

$$\frac{T_R}{T_{con}} = N_R, \qquad \text{Equation 7}$$

$$\frac{15d/rev}{7.5d/con} = 2con/rev = N_R,$$

where N_R is the number of conjunctions per *revolution (con/rev)*.

See Animation 3.4.



Animation 3.4: Two Conjunctions for 5:3 Motions

Notice that in one complete revolution of the system, there are 2 conjunctions: one at 12:00 and one at 6:00.

[253] Ibid., I.310-311, pp. 198-199.

Proposition 8 [I.326-339]: Expanding on Proposition 6, Oresme now describes, "[How] to determine the place of the first conjunction following the present conjunction of these two mobiles." Using Proposition 6 we can determine the period of time between successive conjunctions. Proposition 8 tells us where on the circle these conjunctions will occur. Simply multiply the rotational speed of a particular mobile by the period of conjunction, T_{con}. Continuing with the same example, we know the rotational speeds of both mobiles:

$$\omega_A = (1/3)c/d \quad \text{and} \quad \omega_B = (1/5)c/d.$$

And we know that the period between conjunctions is

$$T_{con} = (15/2)d/con = 7.5d/con.$$

Thus, we need only multiply these two quantities together,

$$\omega_A(T_{con}) = (1/3)c/d(15/2)d/con = (15/6)c/con = (5/2)c/con$$

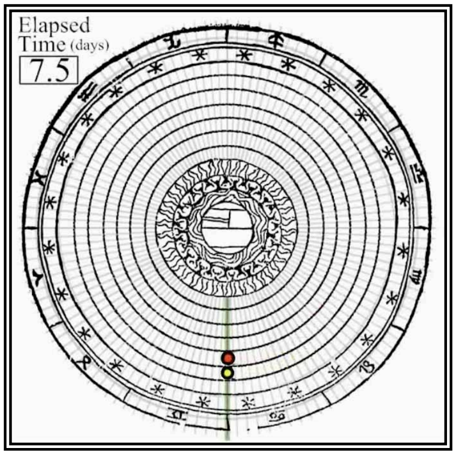
$$= 2.5c/con \text{ (\textit{circulations} per conjunction) for mobile A}$$

and

$$\omega_B(T_{con}) = (1/5)c/d(15/2)d/con = (15/10)c/con$$

$$= 1.5c/con \text{ (\textit{circulations} per conjunction) for mobile B.}$$

In 7.5 days mobile A will travel two full circulations plus a half of a circulation and mobile B will travel one full circulation plus a half of a circulation. If their initial conjunction was at 12:00, their proximate conjunction will be at 6:00, 7.5 days later. Their first relative conjunction after the initial conjunction at 12:00 will occur at a point in direct opposition to the initial point dividing the circle exactly in half. See Animation 3.5.



Animation 3.5: Proximate Conjunction

(Locations of conjunctions for 5:3 ratio of motions.)

Proposition 9 [I.340-363]: Here Oresme shows how to determine the time and place of a conjunction if the initial conditions are not in conjunction. Because this proposition is extraneous to the general argument, it will be omitted.

Proposition 10 [I.364-395]: In this proposition Oresme describes, "[How] to find the number and series of points in which two such mobiles will always conjunct."[254] Here Oresme assembles his larger argument from all of the pieces. Proposition 10 shows how to find all conjunctions and their order of occurrence in a full *revolution* of a system of two mobiles. All subsequent *revolutions* will simply repeat this pattern. Because the periods between relative conjunctions are equal and the rotational speeds of the two bodies are uniform and commensurate, all points of conjunction must be equidistant, resulting in a division of the circle into N_R (Equation 7) equal parts. Oresme uses a 12:5 ratio of rotational speeds for this example because it has a larger number of conjunctions[255] that demonstrate more clearly this proposition.[256]

[254] Ibid., I.364-365, pp. 204-205.
[255] 5:3 has only has 2 conjunctions per revolution.
[256] Oresme's derivation of Proposition 10 is somewhat chaotic and draws upon Proposition 11 for its implementation. For clarity, I will present Proposition 10 without any reference to Proposition 11.

Proposition 5 [Equation 5] is used to determine the *period of revolution* of the system, T_{rev}, and like his previous example, he assumes that days are the unit of measurement.

$$T_{rev} = 12(5) = 60 \text{ days}$$

Proposition 6 [Equation 6] is used to determine the period of relative conjunctions, T_{con}.

$$\begin{aligned} T_{con} &= 1/\omega_{con} \\ &= \frac{1}{\left(\frac{1}{5}\frac{c}{d} - \frac{1}{12}\frac{c}{d}\right)} \\ &= \left(\frac{60}{7}\right)\frac{d}{con}. \end{aligned}$$

Proposition 7 [Equation 7] is used to determine the number of relative conjunctions in one *revolution*:

$$N_R = \frac{T_R}{T_{con}} =$$

$$= \frac{60d/rev}{(60/7)d/con} = 7\ con/rev.$$

We then use Proposition 8 to determine where the conjunctions will be. Their rotational speeds are

$$\omega_A = (1/5)c/d \quad \text{and} \quad \omega_B = (1/12)c/d.$$

Thus, if they start in conjunction, their next conjunction will take place as follows:

Mobile A

$$\omega_A(T_{con}) = (1/5)c/d(60/7)d/con = (60/35)c/con$$

$$= (12/7)c/con$$

and

Mobile B

$$\omega_B(T_{con}) = (1/12)c/d(60/7)d/con = (60/84)c/con =$$

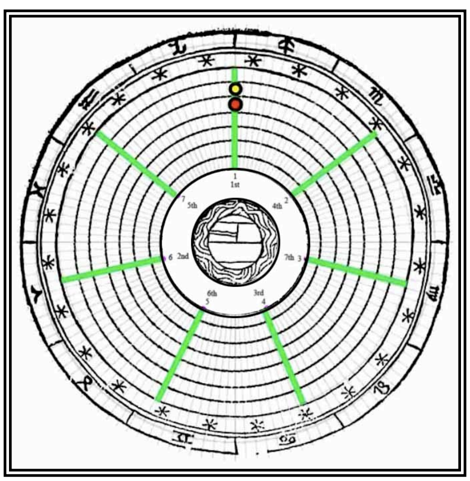
$$= (5/7)c/con$$

Mobile A will travel 1 and 5/7 *circulations* from initial conditions to its proximate conjunction and mobile B will travel 5/7 of a *circulation* in the same amount time. If the initial conjunction was at 12:00, the proximate conjunction would be 5/7 of the way around at approximately 8:43 on the "clock face."

Because we know that there will be 7 equally spaced conjunctions per *revolution,* and we know how they are spaced, it is simply a matter of running through the sequence. See Table 3.1 and Animation 3.6.

	1st conj.	2nd conj.	3rd conj.	4th conj.	5th conj.	6th conj.	7th conj.	1st conj.
Progression of Mobile A	0/7	12/7	24/4	36/7	48/7	60/7	72/7	84/7
Progression of Mobile B	0/7	5/7	10/7	15/7	20/7	25/7	30/7	35/7
Location of Conjunction	0/7	5/7	3/7	1/7	6/7	4/7	2/7	7/7 or 0/7

Table 3.1: Location and Sequence of Conjunctions for the ratio 12:5.



Animation 3.6: Spatial and Temporal Distribution of Conjunctions for a 12:5 Ratio of Motions

Proposition 11 [I.396-422]: This proposition determines the number of relative conjunctions in the *revolution* of a system, not using Proposition 7, but by the difference in velocities as expressed in a ratio of their lowest integral terms. Oresme puts it this way, "The number of conjunctions of any two mobiles in one revolution, and the number of points in which they can always conjunct, is equal to the difference between the least numbers representing the ratio of their velocities."[257] For example, if the ratio of rotational speeds, written in their lowest terms is

$$\omega_A:\omega_B = 12:5,$$

then the number of conjunctions in one revolution can be determined simply by subtracting the lower speed from the higher speed. We are again coming up with a *relative rotational speed*, but instead of measuring the rotational speed in *circulations* per day, as we did before, we measure this speed in *circulations* per period of *revolution*. The rotational speeds are the same, they are just measured against a different standard of time, T_{rev}.

$$T_{rev} = \omega_A - \omega_B, \qquad \text{Equation 11}$$

$$12c/T_{rev} - 5c/\,T_{rev} = 7c/\,T_{rev}$$

There will be 7 distinct points of conjunction in this system. This result was shown above in Propositions 7 and 10.

Thus, given a commensurate ratio of rotational speeds for two mobiles, $\omega_A:\omega_B$ of 12:5, we know from Proposition 5 that mobile A will circulate 12 times and mobile B will circulate 5 times in the completion of one *revolution* of the system. During the time it takes for this *revolution* of the system, mobile A will make 7 more *circulations* than mobile B, thus overtaking mobile B 7 times. Each time it overtakes mobile B, there will be a conjunction. Ergo, there will be 7 conjunctions per *revolution* of the system.

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Oresme's "Principal Harmonic Ratios" and the Motions of the Celestial Bodies

Up until this point in his argument, Oresme was thinking in terms of imaginary mobiles moving in circles at constant rates of commensurate rotational speeds. At this point in his argument Oresme briefly switches to astronomy. The center of the circle becomes the earth, the

[257] Oresme, *De commensurabilitate*, I.396-398, pp. 206-207.

circles become heavenly spheres, and the mobiles become the planets embedded in the spheres.[258]

At the end of Proposition 11 Oresme directly addresses the theory made popular by Macrobius in his *Commentary on the Dream of Scipio*. This theory, quoted above, proposes that the motions of the celestial spheres are related to one another as musical consonances. Oresme writes,

> It follows from this eleventh proposition that if the ratio of velocities of any two celestial mobiles were in any of the principal harmonic ratios in music, namely, the diapason, diapente, diatessaron, and the tone, which make a concord or harmony, the mobiles will never conjunct except in one place only, since the least numbers of such a ratio differ only by a unit.[259]

According to Oresme, the "principal harmonic ratios" are 2:1, 3:2, 4:3, and 9:8, "the diapason, diapente, diatessaron, and the tone" respectively.[260] All of these are superparticular ratios, meaning that each ratio is in the form $n+1:n$, where n is any counting number. If the motions of any two celestial bodies are related to one another in any of Oresme's "principal harmonic ratios," then they must be related by superparticular ratios. According to Proposition 11, any two celestial bodies with orbital speeds in the form of a superparticular ratio will conjunct at one and only one place, repeatedly, into the future, and will have conjuncted at that place and only that place repeatedly in the past. Oresme then concludes, "Since no configuration consisting of two motions is found to occur in only one point in the sky, [it follows] as a consequence that no two celestial motions have velocities related in a principal harmonic ratio." For example, if the motion of Mars were related to the motion of the sun as 2:1, the octave, it

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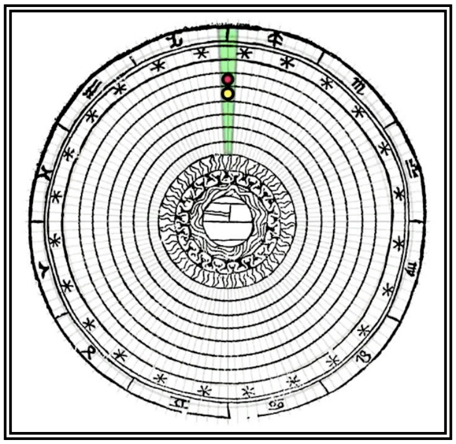
[258] Equivalently, the circular trajectories could be seen as Ptolemaic deferents carrying planets.

[259] Oresme, *De commensurabilitate*, I. 463-468, pp. 212-13.

[260] Oresme's list of "principal harmonic ratios" might better be categorized as the list of traditional superparticular intervals of Pythagorean tuning. Macrobius, clearly a major source for Oresme, claims that the consonant ratios are 2:1, 3:2, 4:3, 3:1, and 4:1, and is somewhat ambiguous concerning the status of 9:8. See Macrobius, 2.I.15-24, pp. 188-189. Oresme explains the source of these four consonant ratios in terms of the "Platonic Lambda" (from *Timaeus*) in much greater detail in Oresme, *Le livre du ciel et du monde*, II.18.125b-126a, pp. 478-481.

would only conjunct with the sun in one place, over and over again, for all of eternity. See Animation 3.7.



Animation 3.7: Conjunctions for a 2:1 Ratio of Orbital Speeds

From Proposition 11, any superparticular ratio will result in a single conjunction location.

However, this pattern does not occur. Centuries of astronomical observations clearly show that Mars does not conjunct with the sun in one and only one place.[261] Therefore, the motions of Mars and the sun are not related in a "principal harmonic ratio." Even if we include the other traditionally Pythagorean consonant ratios like 3:1 and 4:1, or combinations of consonant ratios, like 9:4 or 8:3, the resulting locations of conjunctions would still be fixed to 2, 3, or 5 locations on the zodiac. Such fixed locations of conjunctions were not, and are not, observed. Even if other dissonant, but musical, ratios are allowed, the observed conjunctions do not comply. For example, the dissonant semitone, 256:243, would demand that conjunctions only occur at 13 places on the circle[262] and nowhere else. No celestial motions behave in this manner.

Oresme concludes, "Therefore, if celestial bodies in motion produce a harmony, it is not necessary [to assume] that such a harmony arises from the velocities of their mean motions, but

[261] The *Toledan Tables*, revised by Johannes de Lignères and Johannes de Muris and with canons written by John of Saxony [the Three Johns] were in circulation in Paris in the late 1320s. See José Chabás and Bernard R. Goldstein, *The Alfonsine Tables of Toledo* (Boston: Kluwer Academic Publishers, 2003), 277-289.

[262] Using Proposition 11, 256 – 243 = 13. Oresme, *De commensurabilitate*, I.470-474, pp. 212-213.

perhaps it stems from some other source for other reasons..."[263] He does not give any details in this section about what this "other source" may be, but he returns to it in Part III, which will be discussed shortly.

Following his short digression at the end of Proposition 11 on the impossibility of consonant celestial harmonic motion, Oresme resumes his purely abstract treatment of commensurate circular motion. In Propositions 12 through 19, he discusses systems of three or more mobiles with commensurate motions and runs through many of the same types of arguments he used for the two-body systems.[264] In Propositions 20 through 25 (the end of Part I), Oresme discusses a variety of issues involving commensurable motions including eccentric orbits, the recurrence of various prominent aspects,[265] and the motion of a single mobile with two or more simultaneous motions. The conclusions drawn from these issues are in keeping with those from the earlier

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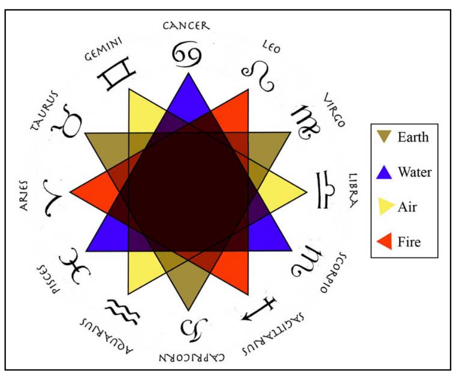
[263] Ibid., I.478-480, pp. 212-215.

[264] Proposition 19 is three-body analogue to the two-body system described in Proposition 10. It finds the successive locations between all conjunctions in a given commensurate system. Oresme specifically discusses the application of this proposition to the astrological concept of "triplicity." Triplicity is yet another counting number organization of the zodiac in relation to the four terrestrial elements.



[265] Astronomical aspects are angular relationships between heavenly locations. For example, the trinal aspect is the circle divided into 3 equal parts, with internal angles of 120º, each celestial body separated by 4 four zodiacal signs. The other prominent aspects are opposition (180º), the quartile (90º), and the sextile (60º). These aspects are of particular importance to astrology. You will note that the regular plane geometric figures described by these prominent aspects are the line (opposition), the equilateral triangle (trinal), the square (quartile) and the regular hexagon (sextile). Notice that each of these aspects divides the circle into parts that evenly divide into 12, the number of signs in the zodiac. Also notice that these aspects divide the circle of the zodiac much like a monochordist divides a string. The conjunction (0º) is also technically an aspect, but in this part Oresme is treating all of the prominent aspects other than the conjunction.

propositions that were outlined above; commensurate motions produce repetitive and predictable systems.

In the case of a mobile with two or more simultaneous motions, Oresme uses the sun as an astronomical example. If the diurnal motion of the sun (from the 9^{th} sphere), the motion of precession from the 8th sphere,[266] and the annual motion of the sun on the ecliptic are commensurate, the full system will return to its initial conditions after the period of a Great Year.[267] This period is analogous to the *period of revolution* for a system of two or more mobiles described above. Oresme uses the term "Great Year" to apply to any system of multiple commensurate planetary motions, be they separate motions of separate mobiles, or multiple simultaneous motions in a single mobile, or a combination of both. Thus, in order to use the term "Great Year" one must define the motions under consideration. Like the belief in commensurate harmonic planetary motions, the belief in a Great Year (either in relation to the three or more motions of the sun described above or in relation to all of the heavenly motions) was widely held in the Middle Ages and its leading authorities were Macrobius, Plato, and Cicero.[268] Oresme points out that the belief in a Great Year in relation to the sun's three motions requires that the sun's simultaneous motions be commensurate. Otherwise, as he later proves in Part II, the system of three motions can and will never repeat any particular configuration, which would make the Great Year infinitely long, and thus, meaningless.

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[266] Oresme does not call the motion of the 8^{th} sphere "precession;" this is the modern term used for this very long cycle.

[267] Oresme, *De commensurabilitate*, I.748-824, pp. 236-243.

[268] See Macrobius, 2.XI, pp. 219-222. See also Marcus Tullius Cicero, *The Nature of the Gods [De natura deorum]*, trans. Horace C. P. McGregor (New York: Penguin, 1972), II.51-52, pp. 143-144; Plato, *Plato's Cosmology: The Timaeus of Plato*, 39D, pp. 116-117. There are various definitions of a Great Year. The 36,000-year Great Year to which Oresme is referring is what we now might call the full revolution through the zodiac of the precession of the equinoxes.

Oresme does not pass judgment on the existence of a Great Year of any variety in Part I. Oresme simply refers to a common estimation of the duration for a Great Year that was popular in his time. He writes, "some say that 36,000 solar years constitute a Great Year of the sun and the eighth sphere," and he goes on to say that a Great Year for all the planets, sun, and moon would be even longer.[269] This second digression into astronomy (the first being his analysis of celestial harmonic motions) constitutes Oresme's second major point in this text and he will return to these two issues in Part III of *De commensurabilitate*.

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Part II of *De commensurabilitate*

In Part II of *De commensurabilitate* Oresme discusses what follows if some of the motions in a circular system are incommensurable. Part II, consisting of twelve propositions, is roughly half the length of Part I with its twenty-five propositions. It follows the general structure of Part I by addressing two-mobile problems in the first five propositions and then expanding to problems with three or more mobiles for Propositions 6 through 10, and then addressing aspect relationships and multiple simultaneous motions in one body in Propositions 11 and 12.

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[269] Oresme, *De commensurabilitate*, I.819-821, pp. 242-3. The figure 36,000 probably refers to Ptolemy's description of Hipparchus' estimate that the 8th sphere rotates "approximately" 1º per 100 years. See Ptolemy, *Ptolemy's Almagest*, VII.2.H15-H16, p. 328. This rate of motion, 1º per 100 years, is given without the modifier "approximately" by Sacrobosco. See Lynn Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, ed. and trans. Lynn Thorndike (Chicago: University of Chicago Press, 1949), 120. A rate of 1º per 100 years will result in a full circulation in 36,000 years. If we accept that just one element of a larger system of multiple motions has a period of circulation of 36,000 years, then the period of *revolution* of the larger system would have to be a multiple of 36,000 years. For a good overview of this particular number, see Thomas Little Heath and Aristarchus, *Aristarchus of Samos, the Ancient Copernicus: A History of Greek Astronomy to Aristarchus, Together with Aristarchus's Treatise on the Sizes and Distances of the Sun and Moon*, trans. by Thomas Little Heath (Oxford: Clarendon Press, 1913), 171-173.

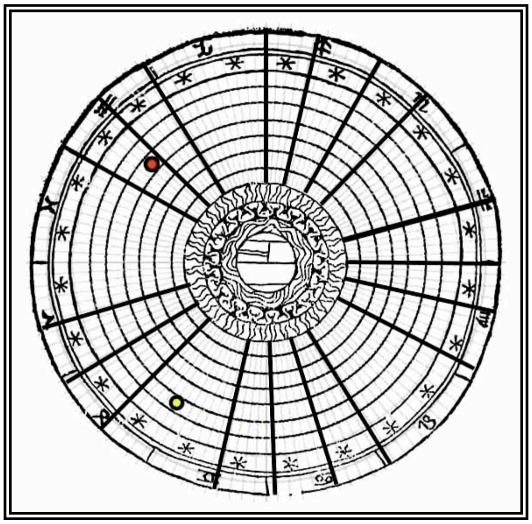
Propositions 1-5 [II.1-143][270] establish that the motions of two mobiles with incommensurable orbital speeds will never conjunct in the same place twice. Furthermore, any two conjunctions will never divide the circle in a commensurable fashion, and, any finite segment on the circle, no matter how small, will eventually contain in the future (or has contained in the past) a conjunction. Oresme also adds that no matter how close together two conjunctions may be, there will always be an infinite number of locations between them available for future conjunctions.

Two mobiles moving at incommensurate orbital speeds conjunct at regular intervals, both spatially and temporally, yet these regular conjunctions never land in the same place twice. They repeat in arc and time intervals, but never repeat any specific location. Oresme was very impressed by what he saw as a paradox in this situation: incommensurate motions, producing regularly spaced conjunctions, in infinitely many locations. Out of disorder comes order *and* limitless variety. While commenting on this apparent miracle, Oresme coins four different somewhat paradoxical terms: "rational irrationality," "regular non-uniformity," "uniform disparity," and "concordant discord."[271]

From this apparent paradox, Oresme further reflects on the implications of a system moving through time with conjunctions in an infinity of discrete locations. Even if, he speculates, the system has been spinning for an eternity in the past, new locations for conjunctions will never be exhausted. The system will always have room for new unique conjunctions into an infinite future.[272] That such unlimited consequences result from a system of lowly incommensurate

[270] Oresme, *De commensurabilitate*, II.1-143, pp. 248-261.
[271] "... ut ita dicam rationalis irrationalitas, regularis difformitas, uniformis disparitas, concors discordia." Ibid., II.112-114, pp. 256-257.
[272] Ibid., II.116-123, pp. 256-259. Oresme discusses the relationship between the infinity in the past and the infinity into the future and notes that they are not the same. Oresme appears to be searching for a way to treat infinity mathematically, but not as a magnitude.

motions is literally mind-boggling for Oresme. The Boethian geometrical magnitude, the non-number of earthly measurement, that which is uncountable, non-musical, and banished from consideration in the structure of a Platonic cosmos, has the ability to do something that no multitude of Unities could ever do. It can create regularity and infinite variety into eternity. See Animation 3.8.



Animation 3.8: The Infinite Variety Demonstrated by Two Bodies Moving with Incommensurable Motions

This animation shows only the first 19 conjunctions, but subsequent conjunctions will always land at new locations. Notice the regularity in both the periods and the arc-distances between conjunctions. The radial lines indicating conjunctions will fill up the diagram, tighter and tighter, forever.

Propositions 6 and 7 [II.144-210][273] establish that under certain conditions it is possible that three-mobile systems with mutually incommensurable motions can conjunct repeatedly in different locations. Under different conditions it is possible that they may conjunct once, but never again at any location. In Proposition 8 [II.211-226][274] Oresme shows that a system starting in conjunction consisting of two mobiles with commensurable motions and a third with a motion incommensurable to the others will never conjunct again.

[273] Ibid., II.144-210, pp. 260-265.
[274] Ibid., II.211-226, pp. 264-267.

In Proposition 9 [II.227-258][275] Oresme draws upon previous propositions from Parts I and II and claims that any system of three mobiles, commensurate or incommensurate, will either never conjunct, conjunct only once, or will conjunct repeatedly forever.

Proposition 10 [II.259-289][276] proposes that a three-mobile system with at least one incommensurable motion will never conjunct in exactly the same place twice. However, in the future all three mobiles will be located in a space approximating conjunction, and that even further into the future the three mobiles will at some time even more closely approximate conjunction at that place, and so on into an infinite future. The three mobiles will get closer and closer to a conjunction in that place, but never ever reach true conjunction.

Proposition 11 [II.290-323],[277] like Proposition 21 from Part I, discusses aspect relationships (opposition, trinal, etc.) and the infinite variety of configurations that result from incommensurable motions. He writes,

> On the assumption of the incommensurability and eternity of motions, it is truly beautiful to contemplate how such a configuration as an exact conjunction occurs only once [at a particular place] through all of infinite time, and how it was necessary through an eternal future that it occur in this [very] instant with no conjunction like it preceding or following. One cannot find reason as to why it happens at that time [rather] than at another time, unless it be because the velocities of motions and the unalterable inclinations of moving bodies are [simply that way].[278]

Oresme then considers how such motions would apply to the motions of the heavens and how these in turn might influence terrestrial matters.

> And if [celestial] configurations are the continuous causes of effects in the lower regions, then, whenever extraordinary aspects concern a whole species, there would occur such a disposition that never again will there be one like it in this world. Speaking naturally, it

[275] Ibid., II.227-258, pp. 266-269.
[276] Ibid., II.259-289, pp. 268-271.
[277] Oresme, *De commensurabilitate*, II.290-323, pp. 270-275.
[278] Ibid., II.307-312, pp. 272-273.

> does not seem inconceivable that a great conjunction of the planets, different from anything that happened before, could produce some individual unlike any other which would begin as a new and previously unseen species in either substance or accident, just as Pliny, in the twenty-sixth book [of his *Natural History*[279]], says that with regard to sickness "the face of a man has been afflicted with new and unknown diseases in every age." And, if the world were eternal, perhaps it is even possible that once such a species has come into being, it would never cease to exist; or it might at some time cease to exist because of the power of another configuration. Similar things may be said about similar corollaries that are deducible from what has been said.[280]

Here Oresme revels in the idea that an infinity of unique astrological configurations would result in an infinity of unique terrestrial effects. The world, macrocosm and microcosm, would forever be marching into new and uncharted territory rather than cycle endlessly through a finite number of configurations. This argument sets the stage for a defense of free will and an attack on astrological determinism, which he will return to in more detail in Part III, but it is hardly a categorical rejection of astrology.[281]

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[279] Grant cites Pliny, *Natural History: Libri XXIV-XXVII*, trans. W.H.S. Jones, 10 vols., Loeb Classical Library, vol. VII (Cambridge, MA: Harvard University Press, 1961), XXVI.i.1.

[280] Oresme, *De commensurabilitate*, II.313-323, pp. 272-275.

[281] This passage seems to endorse the idea that astrological arrangements have influences over disease and health. In *Liver de Divinacions*, Oresme limits the powers of astrology over medical matters. He writes, "So far as medicine is concerned, we can know a certain amount as regards the effects which ensue from the course of the sun and the moon but beyond this little or nothing." Oresme, *Nicole Oresme and the Astrologers,"* 41.r.1, p. 57. See also Stefano Caroti, "Nicole Oresme's Polemic Against Astrology in His *Quodlibeta*," in *Astrology, Science, and Society: Historical Essays,* ed. Patrick Curry (Wolfeboro, NH: Boydell Press, 1987), 79.

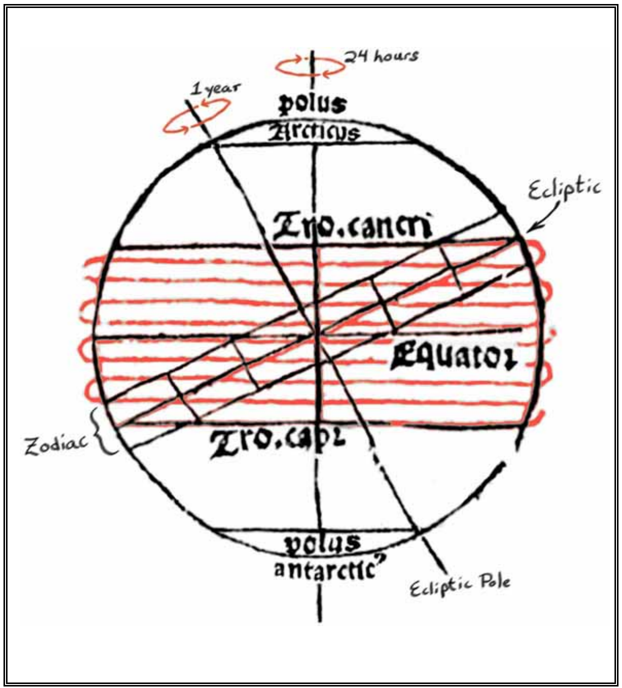

Figure 3.2: Composite Image from Sacrobosco's *Sphere*, 1508.
This diagram shows the two combined motions of the sun: 1) The diurnal rotation around the equatorial pole (the vertical pole) and 2) The annual rotation around the ecliptic pole (off the vertical by ca. 23º). On the longest day of the year (in the northern hemisphere) the sun orbits the earth at the Tropic of Cancer. The annual trip around the ecliptic will take it through the equatorial region, the equinox, on to its southernmost point at the Tropic of Capricorn, and then back up. The ladder-like grid, parallel to the ecliptic on either side, is the zodiacal band. Six of the 12 signs are indicated by the rungs of the zodiacal "ladder." The helical path of the sun, shown in red, is extremely simplified. It shows only 5 days of rotation. To be accurate it should show ca. 185 days of rotation (half of a year) between the two tropics.

In Proposition 12 [II.324-427],[282] the incommensurable analogue to Proposition 22 from Part I, Oresme considers the effects of two or more incommensurable motions acting simultaneously in one and the same mobile. His primary example is once again the sun. If the diurnal and annual motions of the sun are incommensurate, the "spiral"[283] trajectory it would follow over the course of the year would never be the same from one year to the next. The spiral would be infinitely long. The sun would spiral down from the Tropic of Cancer to the Tropic of Capricorn for half the year and then spiral back up for the second half to a new apex on the Tropic of Cancer. See Figure 3.2. This will result in the sun occupying the same locations twice anywhere the downward spiral intersects the upward return spiral. See Figure 3.3.

[282] Oresme, *De commensurabilitate*, II.324-427, pp. 274-283.
[283] The modern term would be helical.

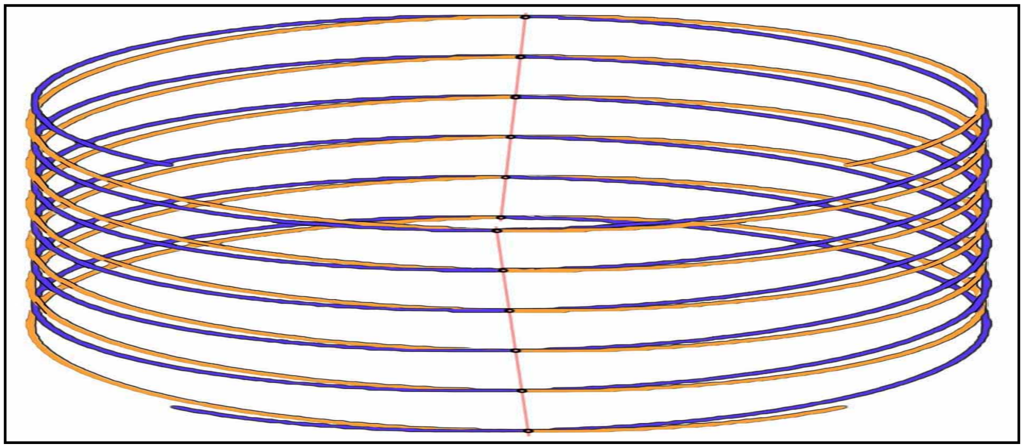

Figure 3.3: The Helical Trajectory of the Sun

The sun is shown making its southward journey in the spring (blue) and its northward journey in the fall (orange). This is only a six-day segment of the helix taken from the equinoctal region showing 6 days going down (blue) and the 6 days going back up (orange). Image is not to scale. The y-axis should be severely squashed in order to approximate scale. However, doing so would make the image very difficult to read. The red line running almost vertically up the near and far sides indicate the points of intersection. This image only shows two motions, the diurnal and the annual. It does not show the motion of the 8th sphere, which would be much to subtle too distinguish at this time scale.[284]

Since the subsequent downward spiral, however, will be a new and unique spiral originating from a new and unique location on the Tropic of Cancer, these points of intersection will never again be traversed by the sun. Thus, the sun can occupy the same location in space twice, but never more than twice. In the closing paragraphs of Part II, Oresme points out several consequences that would result if these motions were incommensurate. Solar and lunar eclipses would not occur in the same places in repeating cycles. Astronomical tables, almanacs, or any form of true calendar could not be expressed in exact numbers. And computations of astronomical conjunctions, oppositions and all other aspects would be, at best, approximations.

The completion of Proposition 12 in Part II marks the end of the technical analyses of *De commensurabilitate*. Oresme has described the consequences of commensurable motions, and

[284] The probably coincidental similarity of this diagram with Ramis de Pareja's cosmic music diagram is striking. See Bartolomeo Ramis de Pareja, *Musica practica* (Bologna: Forni, 1969; reprint, facsimile 1482), Pars III, cap. iii.

the consequences of incommensurable motions. In Part III Oresme is finally ready to settle the debate over "whether [the celestial motions] are commensurable or not."[285]

Before we examine this debate, however, it should be noted that the answer to the question that drives *De commensurabilitate* is unknowable. It is impossible to know if the motions of the heavens are incommensurable, and this was even more true in the time of Oresme. For example, it is possible to imagine that the motions of two planets are commensurable, but that the lowest common ratio of their motions might require enormous numbers. For example, the ratio of two motions might be 1,343,254 : 671,591. This is a commensurable ratio; it is a counting number related to another counting number expressed in its lowest form. Using Proposition 11 from Part I, we know that this ratio will yield 671,663 equally spaced locations for conjunctions around a circle.[286] But observational astronomy in the 14th century was not nearly accurate enough to distinguish this many distinct locations. Each conjunction in this situation would be separated by approximately 0.00053598 degrees. The level of accuracy needed for such a measurement far exceeded the astronomical measurement tools of the day. Oresme was fully aware that proving the incommensurability of the motions of the heavens was beyond his means, if not impossible.[287] This inherent impossibility is probably the reason Part III of *De commensurabilitate* is not written in the proofs and mathematics of Parts I and II, but more in the style of Martianus Capella's *De nuptiis Philologiae et Mercurii.* Oresme presents this final part as a dream.

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285 Oresme, *De commensurabilitate*, I.66-67, pp. 180-181.
286 1,343,254 – 671,591 = 671,663.
287 See Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, A.p.r.-II.239-247, pp. 426-427.

Part III of *De commensurabilitate* – Oresme's Dream

In this dream Oresme encounters "Apollo accompanied by the Muses and the Sciences."[288] Apollo immediately rebukes Oresme for desiring to know that which is not knowable in the world of perception and change. "You should understand that exactness transcends the human mind," he tells Oresme. Then Apollo, making subtle reference to Platonic epistemology, tells Oresme that the senses are unreliable and imprecise. "For if an imperceptible excess – even a part smaller than a thousandth – could destroy an equality and alter a ratio from rational to irrational, how will you be able to know a punctual [or exact] ratio of motions or celestial magnitudes?"[289] Apollo further emphasizes his point by citing Ptolemy and Pliny who each comment on the futility of applying the exact mathematical sciences to sensible things.[290]

Oresme concedes to Apollo that exact knowledge of such things is beyond human ability and that the perfection of mathematical demonstration cannot be verified against the ambiguities of observation. But the nature of being human, Oresme laments, is the desire to know the truth. "I beseech you," Oresme pleads to Apollo, "to disclose to me, ... , your teachings concerning this one doubt."[291]

Apollo responds to Oresme's request by ordering the Muses and Sciences standing in attendance to teach Oresme what he desires to know. Arithmetic responds, "All the celestial motions are commensurable," and Geometry counters by stating, "On the contrary, some celestial motions are incommensurable."[292] Apollo, seeing that there is conflict, orders Arithmetic and

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[288] Oresme, *De commensurabilitate*, III.9-10, pp. 284-285.
[289] Ibid., III.14-17, pp. 284-285.
[290] Apollo actually cites Ptolemy via the Latin translation of al-Battani's *Zīj,* known as *De scientia stellarum*. Grant believes that the original Ptolemaic quote is from *Tetrabiblos* (*Quadripartitum*), I.2. Ibid., III.18-25, pp. 284-287, 336-337, and 108-109; Ptolemy, *Tetrabiblos*, trans. F. E. Robbins, The Loeb Classical Library (Cambridge, MA: Harvard University Press, 1980), I.2, pp. 15-17.
[291] Oresme, *De commensurabilitate*, III.47-49, pp. 288-289.
[292] Ibid., III.50-52, pp. 288-289.

Geometry to present their cases "as if they were litigants in a law suit."[293] Oresme, in his dream, sits and listens to the two presentations.

Arithmetic, in Oresme's dream, is the representative of Number and rational proportion. She is the representative of Boethian multitude, the discrete collections of Unities that create Number. She is the direct source of the ratios which create music: *musica instrumentis*, *musica mundana*, and *musica humana*. If the motions of the cosmos are commensurable or musical or harmonic in any way, they will be represented by Arithmetic. Geometry, on the other hand, is the representative of magnitude. She represents the continuous quantities of geometrical constructions and she is the source of the continuity of magnitudes in motion. If the motions of the heavens are incommensurable, they will be represented by Geometry.

Arithmetic and Geometry represent two sides of the quadrivium: one discrete, the other continuous. From Pythagorean and Platonic roots Arithmetic has dominated the structural ideas of the macrocosm and the microcosm, while Geometry has often been relegated to the world of shadows and change. In the context of Oresme's other writings and even his stated purposes in the prologue to *De commensurabilitate*, this debate between Arithmetic and Geometry is quite clearly an attack on judicial astrology.[294] But this debate can also be seen as a debate over the quadrivium itself for many of the same reasons – Arithmetic with her rational numbers and proportions versus Geometry with her irrational magnitudes and incommensurable proportions. To Oresme an arithmetical/musical structure, or at the very least an arithmetical/commensurate

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293 Ibid., III.52-53, pp. 288-289.

294 Grant states that Oresme wrote *De commensurabilitate* "so that its study would prevent despair and distrust, and enable men to guard themselves against those who would seduce them into believing that they know, or can know, things that are *de facto* unknowable." See Grant's commentary in Oresme, *De commensurabilitate*, 6-7. See also Oresme, *De commensurabilitate*, Proemium 39-42, pp. 174-175.

structure, of the cosmos is a basic tenet of astrology.[295] Without such a commensurate structure the cyclic nature of astronomy is ruined and no configuration would ever recur. Thus astrologers could not amass knowledge of the effects of the stars, because the arrangement of the stars at any given moment would never happen again.

The dichotomies between repetitive and non-repetitive, commensurable and incommensurable, multitude and magnitude, arithmetic and geometry are all related to the quadrivium. The quadrivium is organized around these divisions. There is also a hierarchy. The numbers of arithmetic are always prized over irrational magnitude. Geometry is closest to the divine when it most closely resembles arithmetic. For example, it is frequently pointed out that a cube has 12 edges, 8 corners, 6 faces, 4 sides per face, and 4 corners per face. The relationships between these numbers did not escape the notice of many quadrivial scholars.[296] These countable numbers have meaning, whereas the irrational diagonals across the square faces are of a lower status, their identity is associated with the world of shadows and change. They are earth measures, geo-metry. The irrational magnitudes are not intelligible, and thus not of the higher order. They are not part of the Platonic cosmic structure as described in *Timaeus* or in the *Republic* or in Macrobius/Cicero's "Dream of Scipio." The cosmos was supposedly born from Unity, the progenitor of Number, not from irrationality. Addition and multiplication, the only operations used in the creation story of *Timaeus*, cannot lead to irrationality. Oresme's presentation of Arithmetic and Geometry in his dream come out of this background.[297]

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[295] Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, *A.p.r.*-I.1-13, pp. 382-383, and II.228-250, pp. 424-427. See also Thorndike's commentary in ibid., 122.
[296] Boethius points out that the cube is made up of 12 lines, 8 angles, and 6 faces, and comments that this is a Pythagorean harmonic arrangement. See Boethius, *De institutione arithmetica*, II.49, p. 179. See also Nicomachus of Gerasa, II.xxvi.2, p. 845.
[297] It is possible that Part III was written as a dialogue in a dream because of disputes that arose in the 1340s at the University of Paris in which some masters were censured for misrepresenting the meaning of texts by interpreting them too literally. Particularly egregious were scholars who took some passages in

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The Arguments of Arithmetic and Geometry:[298]

i. On Beauty [III.57-132, pp. 288-295 and III.331-359, pp. 310-313]

Arithmetic begins her oration by attacking Geometry. That the structure of the universe could be governed by irrational magnitudes and incommensurable ratios is, for her, an absurd notion. "It seems unworthy and unreasonable," she claims, "that the divine mind should connect the celestial motions, which organize and regulate the other corporeal motions, in such a haphazard relationship [*indiscreta habitudine*], when, indeed, it ought to arrange them rationally and according to a rule."[299] Arithmetic argues upon the authorities of Aristotle, Pythagoras, and Virgil that "certain figures and certain numbers"[300] are better than others. Similarly, she claims, certain *ratios* are superior to others. For example, she cites Averroes' comment[301] that the double ratio is of particular importance and quotes an unnamed source stating that the *dupla* "unites the lowest things with the highest."[302] She concludes that if the most perfect figure, the sphere, is the shape of the cosmos, it stands to reason that the most perfect ratios would describe its motions.

She again cites Aristotle when she claims that all irrational ratios offend the senses and that the most beautiful colors, sounds, tastes, and odors are produced from certain rational ratios.[303]

Plato, Aristotle, and the Bible too literally. See Ernest A. Moody, *Studies in Medieval Philosophy, Science, and Logic: Collected Papers, 1933-1969* (Berkeley: University of California Press, 1975), 137-160.

[298] I am loosely following Edward Grant's organization by allowing Geometry to respond to each of Arithmetic's major points. In the original presentation, Oresme has Arithmetic give her entire argument followed by Geometry's entire argument. They are not juxtaposed. See Grant, introduction to Oresme, *De commensurabilitate*, 68.

[299] Oresme, *De commensurabilitate*, III.67-69, pp. 288-291.

[300] Ibid., III.72, pp. 290-291.

[301] Identified by Grant as coming from Averroës' commentary on *De caelo*, text 14 of book 3, which comments on 299b32-300a13 of Aristotle.

[302] Oresme, *De commensurabilitate*, III.83-84, pp. 290-291.

[303] Grant cites Aristotle's *De sensu* 3.439b25-440a2 and 442a12-17. Grant points out that the passage from Aristotle, to which this section is probably referring, promotes the idea that certain rational ratios are associated with the most agreeable sensations, but Aristotle does not state that *all* irrational ratios are offensive as Arithmetic claims.

To further emphasize the commensurable nature of beauty, she cites Euclid's *Elements*, where he describes the infinite regression that occurs when one incommensurable magnitude is subtracted from another.[304] "In considering this [infinity], the mind is rendered dull, the reason enfeebled, until that incomprehensible chaos of infinity produces loathsome and afflicted minds," states Arithmetic. "Thus, an irrational ratio is neither suitable or relatable to the understanding, for which reason the ancients said that the mind conforms to a certain numerical and harmonic plan."[305]

Here Arithmetic is making the point that an irrational ratio is inconceivable to the mind. We cannot form such a thought. There is no Platonic *form* for such a thought. We cannot think in terms of irrational numbers, much less irrational ratios. We can only form thoughts in numbers and rational ratios. We can only think ratio-nally. The pun in English is equally valid in Latin. We can only rationally think about rational ratios. Irrational thought is unthinkable. It is irrational. This clearly leads to an idea that the world of forms is dominated by arithmetic and music. This issue is the epistemological basis of the distinction between arithmetical number and irrational geometrical magnitude. The divisions of the quadrivium are defined around this distinction and Arithmetic is defending this distinction.

Arithmetic then bombards the dreaming Oresme with a litany of authoritative references to a commensurate and rational universe. So why, asks Arithmetic, would the "motor intelligences" [*intelligentias motrices*] move the celestial spheres with "such unpleasant and shameful

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[304] "If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable." See Euclid, *The Thirteen Books of Euclid's Elements*, trans. Thomas Little Heath, 2nd ed., 3 vols. (Cambridge: Cambridge University Press, 1908), X.2, p. 17.
[305] Oresme, *De commensurabilitate*, III.108-113, pp. 292-293.

irregularity"?[306] Why would the universal architect [God], who "made all things in number, weight, and measure"[307] design a world that could not be measured with numbers? Boethius, she tells Oresme, asserts that "everything that proceeded from the very origin of things was formed with reference to numbers"[308] and that the elements are bound by numbers.[309] She tells him that Plato states that "the maker of the world joined four corporeal elements in *continuous proportionality* embracing two cube numbers and their two mean proportionals."[310] And she cites Aristotle, who suggests that any occasional incommensurability found amongst the elements on earth are the result of their motions (in "continuous proportionality") and their "remoteness from a proper divinity."[311]

Arithmetic's argument in this section can be generally interpreted as a reference to sections 31b-36a in Plato's *Timaeus* that describes the creation of the world from Unity and the resulting mathematical harmony that the derived multiplicity engenders. The Platonic tradition often expressed this concept in the form of the so-called Platonic Lambda, with powers of 2 and 3, the first even and odd numbers, emanating from Unity, the One.

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306 This may be a reference to the Sirens and/or Fates from Plato's "Myth of Er." Oresme, *De commensurabilitate*, III.114-117, pp. 292-295.

307 Ibid., III.120, pp. 294-295. "Sed omnia in mensura, & numero, & pondere deposuisti." See *Liber Sapientiae*, 11:21, in *Biblia sacra vulgatae editionis* (Antwerp: Aertsens, 1635), 475. This is one of the few obvious references to the Bible, though it is not identified as a Biblical quotation in the text. Grant believes this is because it would have been anachronistic to have had any explicit quotations from the Bible in a classical pagan dialogue. See Grant, introduction to Oresme, *De commensurabilitate*, 76-77.

308 Oresme, *De commensurabilitate*, III.222-223, pp. 295-295. Oresme's quotation of Boethius closely follows Boethius, *De institutione arithmetica*, 1.2, pp. 76-77. This, in all likelihood, is in reference to the Platonic Lambda from *Timaeus*. See Plato, *Plato's Cosmology: The Timaeus of Plato*, 35B-36B, pp. 66-74.

309 "Tu numeris elementa ligas." Grant cites Boethius, "The Consolation of Philosophy," in *The Theological Tractates and The Consolation of Philosophy,* ed. H. F. Stewart, trans. and ed. H. F. Stewart, Loeb Classical Library (New York: G. P. Putnam's Sons, 1918), III.ix(b).10, p. 264.

310 Oresme, *De commensurabilitate*, III.224-226, pp. 294-295.

311 Ibid., III.130-131, pp. 294-295. Grant argues that this passage is from the pseudo-Aristotelian text *De mundo,* 6.397b29-35. See Grant's note in Oresme, *De commensurabilitate*, n23, pp. 341-342.

1
2 6 3
8 12 18 27

The mention of the elemental relationships bounded by cubes and containing two means[312] can be seen in the row 8, 12, 18, 27. The four perfect elements are each separated by a 3:2 ratio. The Aristotelian quotation also fits in well with this model and with the quotation from Plato concerning "elements in continuous proportionality." Although the pure elements themselves are numerically related to one another, the continuous commixture of elements that occurs in the world of change accounts for any incommensurability.[313]

Other parts in this section of Arithmetic's oration appear to refer to the intellegences or Sirens or Fates from Plato's "Myth of Er," as it was transmitted via Cicero via Macrobius' commentary and the frequently quoted passage from the Wisdom of Solomon [*Liber Sapientiae*] that describes how God "ordered all things in measure, and number, and weight."[314]

It is quite clear that Arithmetic is basing her arguments on Platonic and Pythagorean numerological concepts. Even her Aristotelian references are carefully chosen to uphold a fundamentally Platonic interpretation of cosmic structure. The basic premise is aesthetic. An

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[312] The reference to the "two cube numbers and their two mean proportionals" fits perfectly with the so-called Platonic Lambda where 2^3 and 3^3 can be filled in with the means 12 and 18 like so,

1
2 6 3
8 **12** **18** 27

12 is the geometric mean between 8 and 18. 18 is the geometric mean between 12 and 27. And the progression 8, 12, 18, 27 is based on the 3:2 interval, the perfect fifth. A fully developed harmonic cosmos utilizing these elemental numbers and ratios was developed by Zarlino in the 16th century. Many of the classical sources Zarlino cites are the same as those cited by Oresme's Arithmetic. See Gioseffo Zarlino, *Institutioni harmoniche* (Ridgewood, NJ: Gregg Press, 1966; reprint, facsimile of the 1573 Venice ed.), 18-19.

[313] The elements were generally not thought to exist on the earth in pure form. These impure elements were sometimes referred to as "*elementata*." See Richard C. Dales, "The Twelfth-Century Concept of the Natural Order," *Viator* 9, no. (1978): 188. See also Theodore Silverstein, "Elementatum: Its Appearance among the Twelfth-Century Cosmogonists," *Mediaeval Studies* 16, no. (1954).

[314] "Sed omnia mensura et numero et pondere disposuisti." See *Liber Sapientiae*, 11:21, in *Biblia sacra vulgatae editionis*, 475.

incommensurable or irrational structure to the universe, she claims, "would destroy the beauty of the universe and detract from the goodness of the gods."[315] Her *a priori* arguments are from authority and tradition, not observation or even applied mathematics.

Geometry acknowledges that "there is a certain beauty and perfection in [Arithmetic's] rational ratios,"[316] but is not impressed with the idea that rational ratios are "more noble"[317] than irrational ratios and proposes that "an harmonious union"[318] of rational and irrational is "more excellent"[319] than rational alone. For examples she suggests that "a mixture of elements is better than the best element;" that the "sky is more wonderful" in its irregularity than it would be if the stars were equally "distributed everywhere;" that "a song ... is sweeter than if it were constituted continually from the best consonance [that was unvaried], namely, a diapason; that "a picture decorated with different colors is more beautiful than one in which the most beautiful color is spread uniformly over the entire surface."[320] Geometry describes a very dull world if beauty only consisted of rational ratios and Platonic numbers. Her caricature of Arithmetic's ideal describes a world made up of one element, and a sky with a grid of equally spaced stars, a song consisting of a continuously sounding octave, and a monochrome painting.

Geometry then attacks Arithmetic's use of the Biblical quotation from the *Liber Sapientiae* which states that the architect [God] "made all things in number, weight, and measure"[321]

> Even the structure of the sky ... is constituted out of such variety that the bodies are determined by number, and each body by weight–that is, magnitude–, and the motions by measure. If 'measure' were numerical, there would be no good reason to express celestial motions by both number *and* measure. Therefore, this 'measure' is relevant to that

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[315] Oresme, *De commensurabilitate*, III,133-134, pp. 294-295.
[316] Ibid., III.331-332, pp. 310-311.
[317] Ibid., III.340, pp. 310-311.
[318] Ibid., III.341, pp. 312-313.
[319] Ibid.
[320] Ibid., III.342-347, pp. 312-313.
[321] *Biblia sacra vulgatae editionis*, 475. Again, this quotation is not credited to the Bible, but this was a very well known passage amongst mathematicians and would not have needed a citation.

continuity which numbers cannot measure. When measure is indeterminable, we call it irrational and incommensurable.[322]

Geometry's reasoning and rhetoric are clever. Either the Bible is poorly written and redundant when it describes God making "all things in number, weight, and measure," or "measure" means something other than number. If you take numbers out of all measurements of quantity, all that remains is irrational geometric magnitude. God, Geometry explains, uses the irrational ratio in "its proper place"[323] as part of His plan. It is part of the cosmic order. It "makes the celestial revolutions more beautiful."[324]

Geometry's criticisms of Arithmetic's aesthetic arguments do not rely on authority so much as upon common sense and reason. The humorous examples of a world ruled solely by Arithmetic are compelling in their absurdity. Although they are straw man arguments in their ridiculous oversimplifications, they make a point. Beauty tends to be varied. Rationality in the extreme would not be beautiful, it would be boring. The one appeal Geometry makes to authority is to the Bible and her analysis of the quotation is both succinct and sly. Oresme's bias is clear. He favors Geometry. In reading her oration one can almost hear Oresme's voice presenting a *disputatio* at the University of Paris.

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The Arguments of Arithmetic and Geometry:
ii. On Priority [III.133-138, pp. 294-295 and III.360-367, pp. 312-313]

"Among the mathematical disciplines," Arithmetic declares, "I am the firstborn and hold first rank." To back up this claim, Arithmetic cites Macrobius' *Commentary on the Dream of Scipio* where it is written that "numbers precede surfaces and lines."[325] William Harris Stahl explains in

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[322] Oresme, *De commensurabilitate*, III.347-353, pp. 312-313.
[323] Ibid., III.358, pp. 312-313.
[324] Ibid., III.358-359, pp. 312-313.
[325] Ibid., III.137, pp. 294-295. See also Macrobius, 1.V.5-18, pp. 96-99.

a note to this passage from Macrobius' commentary that the classical view was that numbers consist of collections of Unities, which can be represented by dimensionless points. Two or more points can be arranged as a line. Three or more as a plane. Four or more as a solid.[326] In Macrobius' view, numbers precede geometrical figures. Geometry thus depends on numbers for its very existence.[327]

That arithmetic, the science of numbers, comes before geometry, the science of magnitude, was a widely held belief. Boethius is unequivocal on this point in *De institutione arithmetica*. In 1.1 he writes,

> If you take away numbers, in what will consist the triangle, quadrangle, or whatever else is treated in geometry? All of those things are in the domain of number. If you were to remove the triangle and the quadrangle and all of geometry, still *three*, and *four* and the terminology of the other numbers would not perish. Again, when I name some geometrical form, in that term the numbers are implicit. But when I say numbers, I have not implied any geometrical form.[328]

Not surprisingly, Geometry will have none of this argument. Geometry argues that "there is no measure or ratio that is not included within [her] magnitudes."[329] Because the magnitudes of geometry are continuous, they include not only all of the numbers of arithmetic, but infinitely many more ratios and measures not found amongst the numbers. Geometry contains all numbers, all ratios, and all quantities, rational and irrational. From Geometry's point of view, the numbers and commensurate ratios of arithmetic are a subset of geometry. Because of this, Geometry claims to be the "firstborn" of the mathematical sciences.[330]

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326 Stahl notes that Macrobius regarded 4 as a plane number and considered 8 (2^3) as the first solid number. This idiosyncrasy is not important for this argument.

327 See Stahl's note in Macrobius, 97, n10. See also Grant's commentary in Oresme, *De commensurabilitate*, 342-343, n24.

328 Boethius, *De institutione arithmetica*, 1.1, p. 74. As is typical of Boethius' *De institutione arithmetica*, an almost identical passage can be found in Nicomachus' *Introduction to Arithmetic*. See Nicomachus of Gerasa, I.iv.4, p. 813. See also Theon of Smyrna, I.2, p. 11.

329 Oresme, *De commensurabilitate*, III.361-363, pp. 312-313.

330 Chalcidius, in his commentary on Plato's *Timaeus,* 35b (the part that describes the Platonic Lambda), makes an interesting claim that geometry is the foundation of the World Soul. He writes, "from these

Like the previous aesthetic argument, Arithmetic has based her priority argument upon the definitions of arithmetic and geometry from late classical sources, namely Macrobius. Her argument relies on the equivalence between numbers and mathematical points that can be spatially arranged. This equivalence allows the numbers of arithmetic to generate geometrical figures like lines, planes, and solids. In a sense, she is breaking the definition of commensurability that Oresme presented at the very beginning of the treatise where he wrote that commensurable quantities must have "some common measure."[331] Can numbers be compared to geometric points? Do they have a "common measure?"

Oresme's Geometry does not address this issue, but by completely ignoring Arithmetic's number-*qua*-geometrical-point argument, she is essentially dismissing it out of hand, perhaps for the reasons I have suggested. Instead of authority, Geometry appeals to reason. Magnitude is a continuous quantity. Therefore, it must include all of the quantities of arithmetic which are discrete, not continuous. If the magnitudes and ratios of geometry are also included in the ratios of the heavens, "Arithmetic suffers no loss,"[332] for her numbers and ratios will be included.

Oresme is moving towards the position that numbers are just special magnitudes. The distinction between multitude (numbers) and magnitude echoed throughout the quadrivial literature is one of the defining issues of quadrivial philosophy. If numbers are just special cases of magnitudes, does that make arithmetic and music just special cases of geometry? What about

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three particular disciplines having been brought forward, Geometry, Arithmetic, Harmony: out of which Geometry holds the place of the foundation[s], the others [hold the place] of substructure." The Latin reads, "...tribus hoc afferentibus precipuis disciplinis, Geometria, Arithmetic, harmonic: ex quibus Geometria vicem obtinet fundamentorum: ceterae vero substructionis." Plato and Chalcidius, *Chalcidii Viri Clarissimi Luculenta Timaei Platonis traduction, & eiusdem argutissima explanatio*, XXIIIv. Although Geometry is the foundation, arithmetic and harmony appear to be its substructure. The relationship between foundation and substructure is unclear. I speculate that he is referring to the geometric series that generates the legs of the lambda (1, 2, 4, 8 and 1, 3, 9, 27) and that arithmetic and harmonic means can be found in and amongst these numbers.

331 Oresme, *De commensurabilitate*, I.18-19, pp. 176-177.

332 Ibid., III.366-367, pp. 312-313.

the reciprocal? Could magnitudes be special numbers? Given the specific definition of number in all of the quadrivial literature, I would have to say no, but by making number a subset of magnitude, the constant need to carefully distinguish which is which is only occasionally necessary. In fact, all quantities can safely be called magnitudes. The inclusion of quadrivial numbers in geometrical magnitude, it appears to me, has led to where we are today with the concept of real numbers. Real numbers are the geometrical magnitudes of Oresme.[333]

There is another interesting ramification to this revised definition of geometrical magnitude. Numbers had traditionally been a path to Platonic metaphysics. Reality, the world of light, is revealed through mathematics. In Oresme's time this Platonic quadrivial metaphysics was grafted onto Aristotelian epistemology and sensory theory. A theory of intelligibles becomes tricky if irrational ratios are allowed to enter into thought. Arithmetic herself commented that the contemplation of the infinite regressions born from irrational ratios makes "the mind dull [and] the reason enfeebled." She explicitly states that "an irrational ratio is neither suitable or relatable to the understanding, for which reason the ancients said that the mind conforms to a certain numerical and harmonic plan."[334] Allowing irrational ratios to be "relatable to the understanding" would affect the Aristotelian theory of mind significantly. How would that work? If the mind cannot understand it, how can it become actualized? From a Platonic standpoint, if the universe is derived from Unity, how is an irrational numerical form related to Unity? How could the universe be created as it was described in *Timaeus*? Removing the

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[333] Oresme's inclusive treatment of magnitude predates Simon Stevin's similar treatment by about 200 years. See Klein, *Greek Mathematical Thought and the Origin of Algebra*, 186-197. This holds for positive real numbers. At this point in history, negative numbers, real or otherwise, are not part of common practice. See Georges Ifrah, *The Universal History of Numbers from Prehistory to the Invention of the Computer*, trans. David Bellow, E. F. Harding, Sophie Wood, and Ian Monk (New York: Wiley, 2000), 597-598; David Eugene Smith, *History of Mathematics. Volume II: Special Topics of Elementary Mathematics* (Boston: Ginn and Company, 1925), 258-259.
[334] Oresme, *De commensurabilitate*, III.108-113, pp. 292-293.

distinction between multitude and magnitude creates havoc with the classical authorities who rely in this distinction. Starting down this path shakes the very foundation of the quadrivium.

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The Arguments of Arithmetic and Geometry:
iii. On the Three Musics: *mundana-humana-instrumentis*.[335] [III.139-262, pp. 294-305 and III.368-401, pp. 314-317]

a. Musica mundana

Arithmetic begins her arguments for a musical cosmos with an appeal for pity. "Who would deem me worthy of respect," she asks, "if my numbers are incapable of application to celestial motions?"[336] She reasons that if Music can make sounds from numbers, why could Astronomy not join numbers to motions? "Why," she asks Geometry, "are the motions of the stars not measurable by my numbers? What do you want from me?" she pleads, "Do you wish to oust me from my starry throne, to force me from my hereditary home?"[337]

This manipulative entreaty is long on rhetorical flourish and short on rational argument, but in terms of the quadrivium it is quite descriptive of its hierarchical structure. In the Boethian tradition arithmetic is the basis of the subsequent disciplines.[338] It would be odd if arithmetic were not incorporated into all subsequent disciplines. The quadrivial tradition demands it and Arithmetic herself demands it. She even goes on to emphasize her supremacy with a thinly veiled reference to the numerological basis of the Christian Trinity, "one and three everywhere,"

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[335] Arithmetic does not explicitly organize her oration in this part on the three musics of Boethius, but it is evident upon analysis that this is the organizational principle underlying this section. By doing so she very much represents the quadrivial structure described by Boethius. She is clearly aware of the three Boethian divisions. Ibid., III.207, pp. 300-301. Geometry does not respond to Arithmetic's organization in like fashion.
[336] Ibid., III.139-140, pp. 294-297.
[337] Ibid., III.140-144, pp. 296-297.
[338] Boethius, *De institutione arithmetica*, 1.1, p. 74.

and links this to a harmonically structured "triformed universe."[339] Irrational ratios, she claims, are "more appropriate to the wild lamentations of miserable hell than to celestial motions that unite, with marvelous control, the musical melodies soothing a great world."[340]

The majority of Arithmetic's supporting arguments for her dominant influence in the celestial motions are based on the first four chapters of Book II of Macrobius' *Commentary on the Dream of Scipio*.[341] Her first argument consists of a parade of authorities, all found in Macrobius, describing a universe structured on the ratios of musical harmony. Pythagoras is enlisted as the first witness to the celestial harmonies which apparently pervade the earthly realm from above. Then Cicero's Scipio is quoted asking, "What is this great and pleasing sound that fills my ears?"[342] And the answer to his question,

> That... is a concord of tones separated by unequal but nevertheless carefully proportioned intervals, caused by the rapid motion of the spheres themselves. The high and low tones [high and low pitched tones] blended together produce different harmonies .... and the outermost sphere, the star-bearer, with its swifter motion gives forth a high-pitched tone, whereas the lunar sphere, the lowest, has the deepest tone [lowest pitch].[343]

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339 "For the greatest prince of all, himself one and three everywhere–even beyond the stars–who rules all the 'kingdoms of the triformed universe,' extends powerfully from one end to the other and arranges all things pleasantly [and agreeably], that is, harmonically." Oresme, *De commensurabilitate*, III.144-146, pp. 296-297. Grant identifies the phrase "kingdoms of the triformed universe" as being from Ovid's *Metamorphoses*, bk. 15. 859, however in this context it appears to serve as reinforcement for a relationship between the Trinitarian nature of the Christian God and cosmos as a whole. If this is actually Arithmetic's intention, it is a rather weak argument, but it could also be meant as a parody of the types of numerological correspondences that were commonly asserted in Oresme's time.

340 Ibid., III.151-153, pp. 296-297. Zoubov points out Arithmetic's use of poetic alliteration in this passage, "... quam celi **m**otibus **m**iscentibus **m**elodias **m**usicas **m**ulcentes **m**agnum viro **m**oderamine **m**undum." [III.152-153, pp. 296-297] See V. Zoubov, "Nicole Oresme et la Musique," *Mediaeval and Renaissance Studies* V, no. (1961): 98, n1.

341 Macrobius, 2.I-IV, pp. 185-200.

342 Oresme, *De commensurabilitate*, III.161-162, pp. 296-297.

343 Ibid., III.162-167, pp. 296-299. For the corresponding section from Macrobius' commentary, see Macrobius, 2.I.3, p. 185. Edward Grant persuasively argues that Oresme's source for the quotations from Cicero's *De re publica* (which contains the "Dream of Scipio") are from the various quotations from Cicero provided from Macrobius' *Commentary on the Dream of Scipio* and not from an independent copy of *De re publica*. Considering that Oresme follows the order of Macrobius' presentation in this section quite closely, I am inclined to agree with him. Zoubov found a relevant passage in *De configurationibus qualitatum et motuum* where Oresme clearly defines what he calls the "two kinds of intensity" of sound. He uses the terms *gravitas*/*acutus* (heavy/sharp or low/high) for what we now call pitch and the terms *debilitas*/*fortitudo* (weakness/strength) for what we now call volume or loudness. This distinction is

Arithmetic then recounts Macrobius' description of Plato's Sirens from the "Myth of Er," who each sit on a spinning celestial sphere. Macrobius here conflates Cicero's velocity-dependent description with Plato's rather ambiguous description of the Sirens and suggests that "by the motions of the spheres, divinities [Sirens] were provided with song."[344] And then she mentions Hesiod (again via Macrobius) who reinforces the idea of a harmony amongst the nine spheres and their correspondences to the nine Muses.[345]

A bit later in her oration she cites Hermes [Trismegistus], who is quoted, "To know the science of music is nothing else than this – to know how all things are ordered, ..., for the ordered system in which each and all by the supreme Artist's skill are wrought together into a single whole yields a divinely musical harmony."[346]

Each authority cited by Arithmetic in this section [III.139-175] assumes that the cosmos is structured musically. This is the *musica mundana* of Boethius.[347] None of these authorities present a rational or logical argument for this premise. In fact, the references to Plato and Cicero are to works that are overtly allegorical: the "Myth of Er" and "Dream of Scipio."

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frequently confusing in much of the literature. See Nicole Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions; a Treatise on the Uniformity and Difformity of Intensities Known as "Tractatus de configurationibus qualitatum et motuum,"* ed. and trans. by Marshall Clagett (Madison: University of Wisconsin Press, 1968), II.xv, p. 307; Zoubov: 101.

344 Macrobius, 2.III.1, pp. 193-194; Oresme, *De commensurabilitate*, III.168-171, pp. 298-299. In Plato's *Republic*, which was unavailable to Oresme (except via Macrobius via Cicero), the Sirens are described as singing themselves. There is no suggestion that the motions of the spheres create any sounds. See Plato, *The Republic of Plato*, X.617, pp. 354-355.

345 This particular section on Hesiod is so truncated by Arithmetic that it makes little sense. The full description in Macrobius clarifies the meaning significantly. Macrobius, 2.III.1-3, pp. 193-194; Oresme, *De commensurabilitate*, III.171-175, pp. 298-299.

346 This is, of course, not found in Macrobius. Oresme, *De commensurabilitate*, III.238-242, pp. 302-303. Grant cites *Asclepius* 1.9, 13-14a found in Walter Scott, ed., *Hermetica: The Ancient Greek and Latin Writings Which Contain Religious or Philosophic Teachings Ascribed to Hermes Trismegistus*, ed. and trans. Walter Scott, 4 vols., vol. 1 (Boulder, CO: Hermes House, 1985), 310-311.

347 Boethius, *De institutione musica*, 1.2, p. 9. Oresme is aware of these Boethian musical divisions, for Arithmetic later mentions them.

b. Musica humana

In the next section [III.176-232] Arithmetic takes up the Boethian division of *musica humana*. Her general strategy is to claim that the music that structures the larger macrocosm also structures, or at the very least influences, the microcosm. Again, she enlists Macrobius who proceeds to describe how priests use music in a variety of situations to mimic the harmonic structure of the larger divine cosmos. He also describes how music affects all types of people, learned and unlearned. It can "[inflame] them with courage" or sooth them with pleasure. He claims that music even affects animals such as birds.[348] "For the soul," Macrobius explains, "carries with it into the body a memory of the music which it knew in the sky."[349] Macrobius, in Platonic fashion, contends that the soul's origin is in the heavens and that the harmonic structure of the heavens is therefore manifest in the structure of the soul. As a result, the soul sympathetically responds to music.

Arithmetic then cites a variety of authorities who further discuss the musicality of the soul. John of Salisbury in his book, *Policraticus* (1159), describes how the soul resonates harmonically with the "mysteries of nature," via a "kindred element."[350] Macrobius is again used

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[348] Grant thinks that this reference to birds is from Macrobius, 1.III.10, p. 195. If it is from Macrobius, it has been very poorly rendered by Arithmetic.

[349] Oresme, *De commensurabilitate*, III.186-189, pp. 298-299. Lines 176-190 are almost all taken directly from Macrobius. See Macrobius, 2.III.4-7, pp. 194-195.

[350] Oresme, *De commensurabilitate*, III.192-194. John of Salisbury [Joannes Saresberiensis], *Policraticus sive de nugis Curialium, & vestigiis Philosophorum, libri octo* (Leiden: Franciscum Raphelengium, 1595), I.vi, p. 22. Another quotation from *Policraticus* is used by Arithmetic which extends this idea of music as a medium which connects the macrocosm with the microcosm. "Because of the great power exercised by it [music], its many forms, and the harmonies that serve it, it embraces the universe; that is to say, it reconciles the clashing and dissonant relations of all that exists and of all that is thought and expressed in words by a sort of ever-varying but still harmonious law [derived] from its own symmetry. By it the phenomena of the heavens are ruled and men are governed.... Hence, by a kind of course through concealed passages, it pervades the whole universe with its own vital force." John of Salisbury [Joannes Saresberiensis], I.vi.22; Oresme, *De commensurabilitate*, III.214-221, pp. 300-303.

The thoughts expressed by John of Salisbury (ca. 1120-1180) concerning music and the soul are quite similar to those of Marsilio Ficino (1433-1499). For an excellent essay on the metaphysical mechanics of this connection see Walker, *Spiritual and Demonic Magic: from Ficino to Campanella*, 3-11. It is not to describe how the source that gives life to humans and animals is the "World-Soul," whose origins "sprang from music."[351] Similarly, the words of Boethius are used by Arithmetic to endorse Plato's idea that the "World-Soul" is "united in harmony with music," and that the "relationship between our souls and bodies appears, in some measure, to be composed of the same proportions as those which link melodies harmonically."[352]

Arithmetic then describes Boethius' three types of music (*mundana, humana, and instrumentis*) and the way that music in each of these divisions allows contrary powers to unite. For example, the disparate powers and natures of the elements (earth, water, air, and fire) are able to join together into one structure because of certain harmonies.[353]

The power that music has to hold disparate things together is the theme of the preceding authoritative arguments. Just like different numbers with contrary natures, such as even and odd, are held together structurally by music, so too the contrary qualities of elements, the materiality and immateriality of the body and the soul, and the unchangeable heavens with the sub-lunar world of change are joined as one with the concords of harmony.

At this point in the oration, Arithmetic takes up one last particularly thorny issue involving both *musica mundana* and *musica humana*– whether or not these structural harmonies are audible or merely "intelligible to the soul."[354] Cicero's Scipio has already described for

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surprising that Oresme would be quoting from *Policraticus*. John of Salisbury's feelings on astrology were quite similar to Oresme's and his satirical presentation may have been something of an inspiration for Part III of *De commensurabilitate*, which at times seems to be quite sarcastic. See also Theodore Otto Wedel, *The Mediaeval Attitude Toward Astrology: Particularly in England*, 36-41.

Grant's suggestion that Arithmetic and Music never explicitly quote passages from the Bible because it would be anachronistic is problematic in relation to these quotations from *Policraticus*, which was written in the 12th century. Perhaps the fact that both John of Salisbury and Johannes de Hauvilla (whom we will encounter shortly) wrote their works in a classical pagan voice was why they were acceptable to Arithmetic.

351 Macrobius, 2.III.11-12, pp. 195-196.

352 Boethius, *De institutione musica*, 1.1, p. 2; Oresme, *De commensurabilitate*, III.202-206, pp. 300-301.

353 Boethius, *De institutione musica*, 1.2, p. 9; Oresme, *De commensurabilitate*, III.207-213, pp. 300-301.

354 Ibid., III.235-236, 302-303.

Arithmetic "a great and pleasing sound that fills [his] ears"[355] when observing the spinning spheres of the heavens. Now Arithmetic cites Plato, who "proposed that the celestial motions are made rational *without* sound,"[356] as well as Aristotle, who was "not pleased with the idea that motion of the orbs could make noise or audible sound."[357] Arithmetic then quotes Cassiodorus, who says, "The harmony of the heavens is not explicable in human terms, for reason furnishes [the understanding of it] to the mind only, but nature does not make it known to the ears."[358] Arithmetic, completely disregarding Cicero,[359] summarizes these positions by saying, "The philosophers judge that worldly music [*musica mundana*] would not be sensible to the ears, but rather that it would be intelligible to the soul and comprehensible to the mind."[360]

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355 Ibid., III.161-162, pp. 296-297.

356 My italics. Ibid., III.232, pp. 302-303. In a section from Macrobius which refers to Plato frequently, the inaudible nature of the motions of the spheres is discussed, however Plato is not explicitly connected to this idea. Macrobius, 2.II.21, p. 193 and 2.IV.14, p. 199. Grant suggests that this might also refer to Chalcidius' 4th-century translation and commentary on Plato's *Timaeus*, 37B. See Grant's commentary in Oresme, *De commensurabilitate*, 347, n42.

357 Oresme, *De commensurabilitate*, III.233-234, pp. 302-303. See Aristotle, *On the Heavens*, tran. W. K. C. Guthrie, Loeb Classical Library (Cambridge, MA: Harvard University Press, 1939), II.ix.290b12-291a29.

358 Oresme, *De commensurabilitate*, III.244-246, pp. 304-305.

359 Arithmetic also completely disregards Boethius on this point. She has quoted nearly every idea in Book I, chapter 2 of Boethius' *De institutione musica.* Boethius writes, "For how can it happen that so swift a heavenly machine moves on a mute and silent course? Although that sound does not penetrate our ears– which necessarily happens for many reasons –it is nevertheless impossible that such extremely fast motion of such large bodies should produce absolutely no sound, especially since the courses of the stars are joined by such harmonious union that nothing so perfectly united, nothing so perfectly fitted together, can be realized." The "many reasons" that the sound does not "penetrate our ears" are probably, in part, referring to those given in Cicero's *De re publica*: the sound is so ever-present that we have become accustomed to it, and/or the human sense of hearing has been overpowered and selectively deafened by it. Boethius, *De institutione musica*, 1.2, p. 9; Cicero, *De re publica*, VI.xviii.19, pp. 272-273. See also Macrobius, 2.IV.14, pp. 199-200. It is entirely possible that Arithmetic disregards these sources because Oresme is making a straw man for Geometry. Were these arguments included, Geometry would have a much more difficult time pointing out how the theories of classical authors conflict with one another.

360 Oresme, *De commensurabilitate*, III.234-236, pp. 302-303.

c. Musica instrumentis

Arithmetic has assumed throughout her oration that *musica instrumentis* is structured on numbers and ratios of numbers. For her, and for Geometry, this is not in dispute. What is in dispute is the mathematical principle which unifies the three types of Boethian musics. She has argued for the musical structure of the larger cosmos and how that, in turn, structures the sublunar realm including the soul. Now she briefly explains how *musica instrumentis* is evident in the larger macrocosmic structure.

Cassiodorus is again utilized to argue that music exists up in the heavenly realm,[361] explaining that various constellations in the shapes of musical instruments "produce the music of the heavens."[362] Along these same lines Arithmetic cites "a certain poet" who claims that "the cither is the work of the sun, the lyre of Mercury."[363] Furthermore, she continues, the ancients "imagined that a double tetrachord pertains to the seven planets, and a simple monochord to the outermost star-bearing sphere; and with wonderful silence, there is produced from these an harmonic dance."[364]

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[361] Music exists in the heavenly realm even though we cannot hear it.
[362] Oresme, *De commensurabilitate*, III.247-251, pp. 304-305.
[363] "Solis opus cithara studium lyra mercuriale." Ibid., III.252, pp. 304-305. This "certain poet" is Johannes de Hauvilla and the quotation comes directly from his satiric poem *Architrenius* (ca. 1184). Johannes de Hauvilla, *Johannes de Hauvilla: Architrenius*, trans. W. Wetherbee (New York: Cambridge University Press, 2005), IX.20.413, pp. 248-249. Johannes de Hauvilla was a magister at the cathedral school of Rouen. Nearly 200 years later Oresme was made dean of the Cathedral of Rouen. Zoubov incorrectly identifies this "certain poet" as Sidonius Appollinaris, *Carmen* 1, lines 7-8. There are only superficial similarities between this passage in Appollinaris and that of Arithmetic's "certain poet." Grant, rightly so, is not convinced by Zoubov's identification, but suggests no alternative.
[364] Oresme, *De commensurabilitate*, III.256-258, pp. 304-305. A tetrachord covers the interval of a fourth (4:3), and a double tetrachord (in conjunction) will cover a heptachord, one tone shy of an octave. This planetary tone arrangement, in both content and description, is quite close to Nicomachus' account of an intervallic planetary tone system from *The Enchiridion*. Nicomachus describes the overall organization as being "at the interval of a fourth from both extremes, just as the sun also is a fourth from each end among the seven planets, and lies in the middle." However Nicomachus does not give a role for the "star-bearing sphere." See Nicomachus, ch. 3, pp. 250-253. As would be expected, Boethius describes an almost identical system to that of Nicomachus, however Boethius does not describe his system in terms of tetrachords (although they are implied) and his planetary order begins earth-moon-Mercury-Venus,

Arithmetic, using her literary sources, suggests that *musica instrumentis* is structurally embedded, quite literally, in the universe. Constellations can be musical instruments which actually play the cosmic harmony and planets are described as having close sympathetic associations with particular instruments. Even the universe as a whole is described as a gigantic cosmic monochord divided harmonically by the planets.

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The Arguments of Arithmetic and Geometry:
iv. Geometry's Reply to Arithmetic's Three Musics [III.139-262, pp. 294-305 and III.368-401, pp. 314-317]

In response to Arithmetic's lofty references and her impressive parade of classical authors Geometry has a cogent reply,

> In virtue of such mutually conflicting witnesses, she [Arithmetic] ought not to be believed. For one witness insists that the harmony occurs with audible sound, while another denies it; one asserts that the outermost sphere sends forth the sharpest [highest pitch] sound, while another denies it claiming this for the lowest sphere.[365]

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whereas Nicomachus has earth-moon-Venus-Mercury. See Boethius, *De institutione musica*, 1.27, pp. 46-47. Boethius's *De institutione musica* is much more likely to be Oresme's source for he has quoted from it previously and the availability of Nicomachus' *Enchiridion* was limited. Another author who may have been known to Oresme was Aristides Quintilianus (late 3rd century A.D.), who describes a similar planet tone system in Book III of *De musica*, but with his system the pitches are reversed like Cicero's and he gives no indication that the intervals are based on ratios of distances or speeds, rather the intervals are described in terms of astrological and Aristotelian qualities like hotness, impetuousness, or gender. This reliance on qualities other than speed and distance are interesting in terms of Oresme's interest in relating cosmic harmony to weight. Aristides Quintilianus, "*De Musica*," III.21-22, pp. 521-524 and p. 399. Yet another possible source is Theon of Smyrna, who describes a similar system in similar terms and also describes some planet/instrument correspondences as well several references to Hermes. The overall description given by Theon, though differing in several details, most closely resembles Arithmetic's description of planet/tone theory. Theon of Smyrna, III.xv, pp. 91-94. If I had to speculate, I would say that Oresme grafted the specifics of Boethius onto Theon's literary style and content. In *Natural History* Book II, a section cited several times in *De commensurabilitate*, Pliny describes a harmonic Pythagorean planetary system that covers a "*diapason,"* an octave, from the earth to the zodiac. Pliny, *Natural History: Praefatio, Libri I, II*, trans. Harris Rackman, 10 vols., Loeb Classical Library, vol. 1 (Cambridge, MA: Harvard University Press, 1938), II.xx.84, pp. 226-229.

[365] Oresme, *De commensurabilitate*, III.368-371, pp. 314-315.

If all of these authorities are so knowledgeable, why do they not agree with one another? Cicero's Scipio claims that the celestial music is audible, while Plato and Aristotle suggest that it is only understood in the mind. Cassiodorus says that the constellations make instruments, while "a certain poet" associates the planets with instruments. Macrobius/Cicero says the highest sphere makes the highest pitch, others[366] say that the highest sphere makes the lowest pitch.

Geometry requires only one source to undermine Arithmetic's authoritative arguments. She cites Pliny, who complains about Pythagoreans, who claim that "Saturn and Mercury move in the Dorian mode, Jupiter in the Phrygian, and similarly with the other planets." Pliny calls this Pythagorean planetary modal system "a refinement more entertaining than convincing."[367] This incisive quotation uses Arithmetic's own appeal to authority to sweep aside the majority of her arguments.

However, there is one issue concerning *musica mundana* that Geometry wants to address in more detail. She is willing to entertain the idea that there is some sort of "silent harmony"[368] of the cosmos, but she is convinced that "a ratio of tones *does not* vary as a ratio of velocities."[369] And rather than cite authority after authority, she appeals to observation.

> ...for whether a string or a drum is struck strongly or weakly, slowly or quickly, nothing is changed by this. A bass [or deep tone] is consequently never altered; nor is a high [or sharp] tone. [Thus] the tone of a pipe is not made continuously sharper by the continuous

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[366] Both Nicomachus and Boethius describe a celestial music in which higher pitches are closer to earth and lower pitches further away. In their systems, Saturn is the lowest and the moon the highest. See note 364. When Boethius describes his system, he also describes Cicero's system and comments on its reversed order of pitches. Boethius, *De institutione musica*, 1.27, pp. 46-47.

[367] "...iucunda magis quam necessaria subtilitate." Oresme, *De commensurabilitate*, III.372-374, pp. 314-315. Pliny, *Natural History: Praefatio, Libri I, II*, II.xx.84, pp. 226-229. There are minor, but not meaningful, differences between this edition of Pliny and the quotation in Oresme.

[368] Oresme, *De commensurabilitate*, III.375, pp. 314-315. Oresme explains his reasoning for a silent harmony in much greater detail in *Le livre du ciel et du monde.* See Oresme, *Le livre du ciel et du monde*, II.17-18, pp. 468-487.

[369] Oresme, *De commensurabilitate*, III.375-376, pp. 314-315. Italics are mine.

> velocity of the whistling, although sometimes it doubles the sharpness almost instantaneously and this applies to other musical instruments.[370]

Geometry has noticed that in audible terrestrial music pitch is not continuously affected by changes in velocity in any proportional manner. She has, interestingly enough, pointed out the second partial (first overtone) in the modern harmonic series when she describes how a pipe can shift an octave, but this is simply a curiosity for her, an interesting anomaly, not evidence of a pitch/velocity relationship.[371] If objects do not exhibit this relationship in nature, why would it be assumed that the spheres or planets exhibited this behavior in the heavens? There is no physical precedent, only classical (and not-so-classical) authors telling stories that are "more entertaining than convincing" (as Pliny puts it).

Even more damning for Arithmetic's authoritative arguments is Geometry's analysis of the traditional story of Pythagoras discovering the harmonic ratios. "Pythagoras did not measure the motion of the hammering or the force of the blows," states Geometry, "but [instead] sought the ratio of the hammers, a quantity which he knew by their weights."[372] This goes to the kernel of Arithmetic's arguments. It not only highlights a fundamentally flawed assumption about a relationship between velocity and pitch, but it also uses what is arguably the most famous story about the discovery of music in all of classical and medieval history. The story of Pythagoras

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[370] Ibid., III.376-378, pp. 314-315. Vincenzo Galilei (ca. 1525-ca. 1587) experimented with a variety of sound producing instruments and drew many of the same conclusions that Oresme did 300 years previously. His son, Galileo Galilei, observed similar harmonic shifts when scraping brass plates and started down the path towards a quantitative theory. See Vincenzo Galilei, *Dialogue on Ancient and Modern Music*, trans. Claude V. Palisca (New Haven: Yale University Press, 2003), 327-333; D. P. Walker, *Studies in Musical Science in the Late Renaissance* (London: Warburg Institute, 1978), 27-33.

[371] The harmonic series was not theoretically developed until the 17th century by Marin Mersenne.

[372] Oresme, *De commensurabilitate*, III.381-383, pp. 314-315.

and the hammers is told over and over again by countless quadrivial scholars.[373] It is the foundation myth of western music theory.[374]

Geometry is interested in a harmonic cosmos, a *musica mundana*, but she cannot accept a harmonic cosmos based on a direct correspondence between motion and pitch. Her analysis of the Pythagorean story suggests a different variable for pitch correspondence: weight or volume. Geometry writes, "should the celestial spheres produce some concord while moving, this ought not to be measured in terms of the velocities of the motion, but rather by the volumes of the spheres, or the quantities of the orbs."[375] Oresme (as Geometry) does not describe this weight-pitch system any further but it is discussed in *Le livre du ciel et du monde* in greater detail.[376]

Furthermore, if pitch were related to velocity, as Arithmetic claims, these relationships should be evident in the motions of the heavens. Geometry then tells of a belief that the relationship of motions between the sun and Venus is a *diesis*, 256:243, the semi-tone. Not only is this interval not consonant, but Geometry claims that no one really believes this to be true.[377] This claim, that the interval between the sun and Venus is a *diesis,* probably comes from Boethius.[378] In Boethius' theory the sun is identified with the *mese* and Venus with the *trite synemmenon* (paramese), which are separated by a semitone. The reason Oresme might claim

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[373] Palisca, *Humanism in Italian Renaissance Musical Thought*, 226 and 276.
[374] Oresme begins the process of critically analyzing this Pythagorean story by reading it closely. Vincenzo Galilei critically analyzed the physics found in this story and finds that it is severely flawed. In order to produce the same pitch intervals, length-based monochord ratios must be squared in order to determine the ratio for hanging weights off of strings (tension). Vincenzo writes in his *Discorso intorno all'opera di...Zarlino* (Florence, 1589), pp. 103-104, "If someone wished to hear from two strings of equal length, thickness, and quality, the sound of the diapason [2:1], it would be necessary for him to suspend weights, not in the duple but in the quadruple [$4:1=(2:1)^2$] proportion." This quotation from Vincenzo is translated by Palisca and found in his book *Humanism in Italian Renaissance Musical Thought*. Ibid., 275.
[375] Oresme, *De commensurabilitate*, III.394-397, pp. 316-317.
[376] Oresme, *Le livre du ciel et du monde*, II.18.126a, pp. 480-481. In this section, written later than *De commensurabilitate*, Oresme suggests that the faster rotating spheres would produce a louder sound, not a higher pitch, if they made audible sounds.
[377] Oresme, *De commensurabilitate*, III.390-394, pp. 314-317.
[378] Boethius, *De institutione musica*, 1.27, p. 46.

that nobody really believes this may be due to the fact that from simple observation, Venus (and Mercury) were always tied to the mean motion of the sun and were never observed outside of the sun's neighborhood. In other words, the mean motions of the sun, Venus, and Mercury were equal. Thus a ratio of motions, 256:243, is inconceivable. In addition to this, although Geometry does not mention it, this sort of relationship was already addressed in Part I, where Oresme discusses how this ratio of motions would produce only 13 locations of conjunction, a limitation that is simply not observed in nature.[379]

In closing her rebuttal to Arithmetic's arguments in favor of celestial harmony, Geometry reiterates her denial of commensurable celestial motions and reaffirms her belief in a commensurable relationship between the celestial bodies themselves, perhaps weight or volume. However, Geometry feigns no knowledge of the audibility of this harmonic relationship but is pleased that "the most delightful jester should dance with the Muses and, while strumming a spherical cither, appear in full view of Apollo."[380] Is Music this "most delightful jester?"[381]

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[379] Oresme, *De commensurabilitate*, I.470-474, pp. 212-213. It is also possible that this claim is reflecting an opinion voiced by Pliny on the Pythagoreans from Book II of his *Natural History* cited previously. According to Pythagoras, Pliny writes, the distance between Venus and the sun is supposed to be one and a half tones [*sescuplum*]. This description of Pythagorean beliefs ends with a very critical comment from Pliny, quoted above. Pliny calls the Pythagorean harmonic cosmology, "a refinement more entertaining than convincing." This statement of incredulity by Pliny is echoed by the contemptous presentation by Geometry/Oresme in this section. The difference in distances (a semitone compared with a tone and a half) is not important to his argument. However I should point out that Pliny's use of the term "half-tone" [*dimidium tonum*] in this section of *Natural History*, is not even remotely appropriate when discussing Pythagorean intervals. A half tone, $\sqrt{(9/8)}$, is not technically rational. A semitone, 256/243, is. This issue will be discussed in the next case study. Pliny, *Natural History: Praefatio, Libri I, II*, II.xx.84, pp. 226-229.

[380] Oresme, *De commensurabilitate*, III.399-401, pp. 316-317.

[381] In *De nuptiis Philologiae et Mercurii* Martianus Capella describes, Harmony holding a "shield, circular over-all, with many inner circles, the whole interwoven with remarkable configurations. The encompassing circles of this shield were attuned to each other, and from the circular chords there poured forth a concord of all the modes .... the strains coming from that strange rounded form [the shield] surpassed those of all musical instruments." The similarities between this stray comment by Geometry and the scene from *De nuptiis* are certainly intriguing. Capella, *The Marriage of Philology and Mercury,* vol. 2, IX.909-910, pp. 252-253.

The Arguments of Arithmetic and Geometry:
v. On the Conjunctions of Astrology [III.263-317, pp. 304-309 and III.402-466, pp. 316-321]

Arithmetic's final arguments for commensurability have to do with astrology. If the motions of the heavens were not "proportioned by numbers," she argues, " .... no one could ever foresee aspects, or predict conjunction, or learn the effects beforehand .... astronomy would lie hidden [from us] in every age, unknown and even unknowable ... [and astronomy] would no longer be counted among the mathematical disciplines."[382] Without commensurable motions nothing in the heavens could be predictable. It would become a chaos; astronomy could not be a valid member of the quadrivium.

She then wonders why the "maker of the world" (*mundi opifex*) would "give to man an uplifted face ... and turn his eyes to heaven? Of what avail that man derived his intelligence from above?"[383] Here she again refers to the source of the human intellect from the heavens. Again, Arithmetic is confounded by irrationality. Incommensurable motions in the heavens cannot be represented in the mind as she sees it. If the mind/soul is generated from the heavens, and the mind can only conceive of the rational, why would the motion of the heavens be irrational? Arithmetic is making a case for her own narcissism.

Furthermore, Arithmetic adds, what would this say about classical authorities? Did they simply "[invent] things about what is unknown"?[384] As an example she cites Aristotle, who writes that "time is the number [or measure] of celestial motion,"[385] and then comments that this

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[382] Oresme, *De commensurabilitate*, III.263-266, pp. 304-305.
[383] Ibid., III.266-269, pp. 304-305.
[384] Ibid., III.271-272, pp. 306-307.
[385] "tempus est numerus motus celi" Ibid., III.273-274, pp. 306-307. Grant identifies Aristotle's *Physics* 4.12.221a1, 221b7, 221b25 and *De caelo* 1.9.297a15 and 2.4.287a24 as possible sources for this quotation, however Grant traces other Aristotelian possibilities in his lengthy analysis. At issue are the words *numerus* and *celi*. Oresme, *De commensurabilitate*, 349, n51.

"would be impossible if the celestial motions were not measurable by number."[386] Arithmetic concludes that "all astronomical tables would be equally false"[387] if the celestial motions were incommensurable.[388]

The remainder of Arithmetic's arguments all have to do with the existence and astrological significance of the Great Year. She argues that if the celestial motions were incommensurable, "the celestial bodies would never return to the same state, namely, to their present disposition, after the completion of a Great [or Perfect] Year – its existence has been affirmed by many philosophers – embracing the circulations of all [celestial bodies]."[389] As is Arithmetic's general tactic, she draws upon a series of classical authorities to speak in favor of the Great Year. She cites Plato's *Timaeus* for a basic definition– "when the paths traversed by the eight revolutions return to the origin and commencement of another revolution."[390] She cites Apuleius (2nd century A.D.) – "a great year, whose time [or period] is easily known and will be completed when the band of wandering stars [the planets] reaches the same destination and makes a fresh beginning."[391] She quotes Macrobius, who gives a standard definition and speculates on its enormity and mystery – "how many generations are contained in a great year I scarcely dare say."[392] And finally she quotes "divine pronouncements" [the Bible – Ecclesiastes I:5-6] with a

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386 Oresme, *De commensurabilitate*, III.274, pp. 306-307. Arithmetic's conclusion depends upon the word *numerus* in the quotation from Aristotle, and as Grant points out [see previous footnote], this word is the subject of some debate.
387 Ibid., III.275, pp. 306-307.
388 The approximations of astronomical tables are also discussed in Part I, Proposition 25. Ibid., I.852-882, pp. 244-247.
389 Ibid., III.275-278, pp. 306-307.
390 Ibid., III.281-284, pp. 306-307. Grant suggests that this very likely comes from Chalcidius' 4th-century commentary and translation of Plato's *Timaeus*, 39D.
391 Ibid., III.290-292. pp. 306-307.
392 Grant points out that the quotation is actually from Cicero, as found in Macrobius' commentary. Macrobius, 2.XI.2, pp. 219-220; Oresme, *De commensurabilitate*, III.392-393, pp. 308-309.

short, though cryptic, quotation that bears only a vague resemblance to a description of a Great Year.[393]

Again quoting from "divine pronouncements" [Ecclesiastes I:9-10] Arithmetic describes how the Great Year is related to the cyclic history of the cosmos, "What is it that hath been? The same thing that shall be. What is it that hath been done? The same that shall be done. / Nothing under the sun is new, neither is any man able to say: Behold this is new: for it hath already gone before in the ages that were before us."[394] Arithmetic is not only offering evidence for astrological influences, but evidence that the astrological influences are commensurable and therefore cyclic. If the celestial motions were not commensurable, she argues, "constellations and effects would occur that were not of an enduring or perpetual kind."[395] By her reasoning, if the celestial arrangements did not repeat, they would not be enduring. There would be no reference point, no initial conditions, nothing against which to calibrate. She then cites "the Platonists," Claudian, Virgil, and Pythagoras as described by Ovid,[396] who all describe in various poetic ways how "the [very] same men would return again after the completion of this great revolution."[397]

Edward Grant makes a compelling argument that "the Platonists" to whom Arithmetic refers were probably Stoics, who had been confused with Platonists due to their use of the concept of the Platonic Great Year.[398] The very clear presentation by Arithmetic of a doctrine of individual return is quite clearly a Stoic idea. Taken to the extreme, some Stoic philosophers believed that

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[393] "The sun riseth, and goeth down, and returneth to his place: and there rising again, / Maketh his round by the south, and turneth again to the north: the spirit goeth forward surveying all places round about, and retruneth to his circuits." Oresme, *De commensurabilitate*, III.294-297, pp. 308-309.
[394] Ibid., III.305-308, pp. 308-309.
[395] Ibid., III.310-311, pp. 308-309.
[396] Grant gives the following citations for these authors: Claudian, *De raptu Proserpinae*, bk. 1.61-62; Virgil, *Aeneid* 6.748, 751; Ovid, *Fasti*, bk. 3.163-164.
[397] Oresme, *De commensurabilitate*, III.311-312, pp. 308-309.
[398] See Grant's extensive notes on this passage. Ibid., n55, pp. 350-354.

the Great Year was an exact cyclic repetition not only of the motions in the heavens, but everything from the macrocosm to the microcosm.[399] Arithmetic's reasoning in this section is that the arrangement of the heavens causes everything in the world to happen, so when the heavens complete a Great Year, everything in the world will simply happen again. For Arithmetic the stars keep perfect commensurate time, and because the sublunar world is directly caused by the heavens, it will inevitably be ruled by them.

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vi. The Arguments of Arithmetic and Geometry:
Geometry's Reply to Arithmetic's Conjunctions of Astrology [III.263-317, pp. 304-309 and III.402-466, pp. 316-321]

Geometry begins her rebuttal with an aesthetic argument by analogy asking, "What song would please that is frequently or oft repeated? Would not such uniformity [and repetition] produce disgust? It surely would, for novelty is more delightful."[400] Geometry is saying that too much repetition is boring. Then Geometry suggests that if the celestial motions were commensurable, they would repeat over and over again, forever. Like an overly repetitive song, this too would be boring. Furthermore, in the eyes of God, Geometry claims the Great Year "has no existence."[401] A more appropriate motion to the deity would be incommensurable, for with this relationship

> the same event should not be repeated so often, but that [on the contrary] new and dissimilar configurations should emerge from previous ones and always produce different effects .... the far-stretching sequence of ages, which Pythagoras knew as the golden

[399] Alexander Jones, "The Stoics and the Astronomical Sciences," in *The Cambridge Companion to the Stoics,* ed. Brad Inwood, Cambridge Companions to Philosophy (New York: Cambridge University Press, 2003), 337; Michael White, J., "Stoic Natural Philosophy (Physics and Cosmology)," in *The Cambridge Companion to the Stoics,* ed. Brad Inwood, Cambridge Companions to Philosophy (New York: Cambridge University Press, 2003), 141-144.
[400] Oresme, *De commensurabilitate*, III.402-404, pp. 316-317.
[401] Ibid., III.409, pp. 316-317.

> chain, would not return in a circle, but would always proceed endlessly in a straight line.[402]

Geometry is arguing against a Stoic cyclic determinism based on the Great Year proposed by Arithmetic.[403] Not only is it painfully repetitive, but goes against a concept of a human being's free will.[404] Geometry further attacks the authorities who describe a Great Year cited by Arithmetic, calling them "poetic philosophers, who, ..., fail to agree with one another about the length of the Great Year. Nor are their statements compatible with the phenomena observed thus far by astronomers."[405]

At this point Geometry brings in ideas that were developed in Parts I and II. She states, "For it was demonstrated previously that if all celestial motions were commensurable, it would be impossible for the sun and moon to be in conjunction or opposition through all eternity, except in a few points in the sky."[406] This observation refers primarily to Proposition 10 in Part I which shows how to map out all conjunctions of a commensurate system of two bodies. Any system, such as the sun and moon, if commensurate, will have a finite number of places of conjunction, equally spaced, around the zodiacal circle. As a result there will also be places on the zodiac which will never have a conjunction of two bodies with commensurate motions. Geometry asks, "Why should some parts of the ecliptic be deprived of a conjunction between the sun and moon[?] .... Rather, one should be able to say that there is no part of the ecliptic so small that the

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[402] Ibid., III.411-414, pp. 316-317.

[403] It is interesting that she acknowledges the "effects" of the configurations of the stars. See Caroti, "Nicole Oresme's Polemic Against Astrology in His *Quodlibeta*," 79-85.

[404] Oresme's thoughts on free will and astral-determinism are not developed in *De commensurabilitate;* rather, his positions on these matters are discussed in his *Questio contra divinatores horoscopios* and his *Quodlibeta* and are summarized succinctly by Caroti. Ibid., 77. Similar ideas are presented in his *Livre de Divinacions*, where Oresme writes that the complexion of a person may be known astrologically "but cannot be known when it comes to fortune and things which can be hindered by the human will." See Oresme, *Nicole Oresme and the Astrologers,"* 56-57. A similar sentiment is also expressed by Oresme in *Ad pauca respicientes*. See Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, II.xix, pp. 424-427.

[405] Ibid., III.416-419, pp. 316-317.

[406] Oresme, *De commensurabilitate*, III.421-423, pp. 318-319.

sun and moon would not conjunct there sometime, or have not already conjuncted there."[407] If the motions were incommensurable, as was demonstrated in Book II, Propositions 4 and 5, all locations on the ecliptic would be open for conjunction at some point in eternity.

Geometry now addresses Arithmetic's claim that incommensurable motions would make all astronomical predictions impossible. Citing Pliny and Ptolemy,[408] she points out that predictions of future conjunctions are possible, just not exact. And if exact knowledge were known to astronomers, there would be no need to make any more celestial observations, for everything would be known into the future. This, in Geometry's opinion, would dissuade people from transcending their terrestrial occupations and reaching for "higher-minded endeavour[s]."[409]

Furthermore, if the motions were known with exact precision and the motions were commensurable, the Great Year would be real and people would know "all things to come, the whole order of future events .... they would become like immortal gods."[410] This concept strikes Geometry as terribly arrogant. However, if the motions were incommensurable, she suggests, none of these difficulties would result.

As her last argument for incommensurable motion, Geometry makes reference to a mathematical argument "demonstrated elsewhere."

> Indeed, incommensurability is shown in yet another way, for, as demonstrated elsewhere, when any two unknown magnitudes have been designated, it is more probable that they are incommensurable than commensurable, just as it is more probable that any unknown [number] proposed from a multitude of numbers would be non-perfect rather than perfect. Consequently, with regard to any two motions whose ratio is unknown to us, it is more probable that that ratio is irrational than rational.[411]

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[407] Ibid., III.430-433, pp. 318-319.
[408] Grant cites Pliny, *Natural History* II.xxi.87. Grant believes that the reference to Ptolemy comes from the Latin translation of al-Battani's *Zīj*. The original source is from *Tetrabiblos*. Ptolemy, *Tetrabiblos*, 1.2, pp. 5-19.
[409] Oresme, *De commensurabilitate*, III.448, pp. 320-321.
[410] Ibid., III.449-452, pp. 320-321.
[411] Ibid., III.458-464, pp. 320-321.

This argument, as Edward Grant points out,[412] is from Oresme's *De proportionibus proportionum.* Seeing as this argument is pertinent to the argument for incommensurability, is it somewhat surprising that it was not included in Part II. Because this argument also has significant bearing on the structure of the quadrivium, I will briefly describe the relevant proposition from *De proportionibus proportionum.*

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Incommensurability Is More Probable[413]

The title *De proportionibus proportionum* translates to *On the Ratios of Ratios.* As Edward Grant explains, a *proportio proportionum*, a ratio of ratios, refers to the exponent, *g*, in the following expression:[414]

$$8/1 = (2/1)^g. \qquad \text{Eq. 1b}$$

In this example it is clear that g = 3/1 or just 3. This would be called a rational ratio-of-ratios because *g* is rational. There are two other types of rational ratios-of-ratios. In the next example 4/3 is the rational ratio of these two ratios.

$$(4/1)^{1/3} = [(2/1)^{1/2}]^{4/3} = [[(2/1)^{1/2}]^4]^{1/3} = (4/1)^{1/3}. \qquad \text{Eq. 2b}$$

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412 Ibid., 73-76 and 357, n66.

413 The following summary was derived from Edward Grant's commentary on Chapter III of *De proportionibus proportionum*, and focuses on the part leading into and including Proposition X. See Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, 38-42 and III.333-439, pp. 226-255. Grant points out that Oresme's treatment of exponential fractions was initiated by Thomas Bradwardine in his *Tractatus proportionum seu de proportionibus velocitatum in motibus* from 1328. See Grant, introduction to Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, 14-24. Oresme formulated ways to notate and manipulate these exponential fractions in his *Algorismus proportionum*. See Edward Grant and Nicole Oresme, "Part I of Nicole Oresme's *Algorismus proportionum*," *Isis* 56, no. 3 (1965).

414 Grant, introduction to Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, 49-50. Please let me apologize in advance for the egregious amount of alliteration that is necessary for this description. The terminology in this section unfortunately gets in the way of clarity. In order to alleviate as much confusion as I can without losing Oresme's line of reasoning, I will hyphenate the term "ratio-of-ratios" from here on out.

This ratio-of-ratios is rational even though the ratios individually are both irrational: the cube root of 4 and the square root of 2. What makes this ratio-of-ratios rational is the fact that they are related by a rational ratio, the exponent, 4/3.

The last example of a rational ratio-of-ratios is

$$\left(\frac{8}{1}\right)^{\sqrt{2}} = \left[\left(\frac{2}{1}\right)^{\sqrt{2}}\right]^{3}. \qquad \text{Eq. 3b}$$

Oresme himself was unable notate this coherently.[415] Both $(8/1)^{\sqrt{2}}$ and $(2/1)^{\sqrt{2}}$ are irrational ratios, but they are exponentially related rationally by 3/1 or just 3.

Then there are three corresponding irrational ratios-of-ratios. An example of each follows:

$$9/1 \neq (2/1)^{p/q}. \qquad \text{Eq. 4b}$$

Rational ratios related by an irrational exponent,
here shown somewhat misleadingly as p/q,
where p and q are counting numbers.[416]

$$(3/1)^{1/4} \neq [(2/1)^{1/2}]^{p/q} \qquad \text{Eq. 5b}$$

Rational ratios with individual rational exponents
related by an irrational exponent, p/q.

$$\left(\frac{8}{1}\right)^{\sqrt{2}} \neq \left[\left(\frac{5}{1}\right)^{\sqrt{2}}\right]^{p/q} \qquad \text{Eq. 6b}$$

Rational ratios with individual irrational exponents
related by an irrational exponent, p/q.

These six classes of ratios-of-ratios (including mixtures) cover every possibility presented by Oresme. Three are rational ratios-of-ratios and three are irrational.

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[415] Grant, using modern notation and concepts, deciphers what Oresme seemed to be trying to describe. See Grant's introduction. Ibid., 33-36.

[416] I am following Grant's presentation in this section. By using "p/q" as the exponent, these equations become "not equal," for no exponent written as a ratio of two counting numbers could result in an equality for these examples. Another alternative for this presentation would be to express the exponent with a variable that could represent an irrational magnitude and then allow both sides of the equation to be equal. There are advantages and disadvantages to either way.

After describing all of these ratios-of-ratios, at the very end of Proposition IX, leading into Proposition X in *De proportionibus proportionum*, Oresme writes,

> If anyone carefully considered the aforesaid things, and understood geometry and astronomy adequately, he could as a consequence discover many things about ratios, but I wish to delay no longer on these matters.
>
> But, finally, I set forth one other proposition which seems to follow from what has preceded, the fruits of which, by the grace of God, will hardly appear trifling in what follows. Indeed, you will admire it even more as you reflect more deeply upon it and the things which follow from it. The proposition is as follows:
>
> *Proposition X. It is probable that two proposed unknown ratios are incommensurable because if many unknown ratios are proposed it is most probable that any [one] would be incommensurable to any [other].*[417]

Oresme makes his most detailed argument from the classes of ratios-of-ratios formed only from rational ratios, the classes exemplified by Equations 1b and 4b. He starts by creating a set of 100 ratios created from the series, 2/1, 3/1, 4/1, 5/1... 101/1. He then determines that there are 4,950 possible ratios-of-ratios in the form a/b where a>b.[418] From these 4,950 ratios-of-ratios, Oresme determines that only 25 are rational ratios-of-ratios. I will not go any further in explaining his method, but will pick out a few examples to show what this means.

E.g., $(21/1) \neq (13/1)^{p/q}$, because p/q is irrational

E.g., $64 = 16^{3/2}$, p/q is rational

E.g., $(97/1) \neq (3/1)^{p/q}$, because p/q is irrational

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[417] Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, III.226-235, pp. 246-247.

[418] He explains how he comes up with this number in Proposition XI [III.440-498]. Here is a simplified example. Start with the numbers 1-4. There are 4 x 4 unique paired combinations that can be extracted from this set. (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4). Oresme only wants unique pairs in the form (a,b) where a>b. What we are left with are (2,1), (3,1), (3,2), (4,1), (4,2), (4,3). That is 6 pairs. The equation that tells you how many pairs conform to these parameters is (4 x 3)/2. Generalized it is [c x (c-1)]/2, where c is the number of unique members in the set. If you have 100 ratios in your set, the number of ratios-of-ratios that conform to Oresme's parameters are (100 x 99)/2 = 4,950.

Oresme goes on to say that the ratio of rational ratios-of-ratios to irrational ratios-of-ratios constructed from this set of 4,950 ratios-of-ratios is 197:1.[419] Marshall Clagett points out that this result does not prove that irrational ratios-of-ratios outnumber rational ones,[420] but it is compelling anecdotal evidence all the same and Oresme is satisfied that it suits his main purpose– to undercut the foundation of astrological predictions. And this brings us back to the end of Geometry's arguments:

> ...for, as demonstrated elsewhere, when any two unknown magnitudes have been designated, it is more probable that they are incommensurable than commensurable, ... Consequently, with regard to any two motions whose ratio is unknown to us, it is more probable that that ratio is irrational than rational.[421]

What Geometry is clearly implying, is that any two randomly selected motions in the heavens are statistically very likely to be incommensurable. The example given in *De proportionibus proportionum* results in a probability of 197:1 for the sample set of 100 ratios that he examines, and Oresme explains that the higher the number of members in a sample set, the higher the probability.[422] Thus, astrological predictions are based on an assumption that is statistically very unlikely.

Of particular significance for the quadrivium is that Oresme is working with irrational ratios as if they were rational. He is allowing irrational magnitudes and incommensurable ratios, the domain of quadrivial geometry and astronomy, to be compared as if they were rational numbers

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[419] (4,950-25)/25 = 197.

[420] Clagett, 224. Grant points out that if the sample set of ratios was constructed from a geometric series (e.g. 3/1, 9/1, 27/1, 81/1...) all of the ratios-of-ratios would be rational. Grant, introduction to Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, 41-42, n54.

[421] Oresme, *De commensurabilitate*, III.458-464, pp. 320-321. Oresme sums up the corresponding section in *De proportionibus proportionum*, "And so it is clear that with two proposed unknown ratios–whether they are rational or not–it is probable that they are incommensurable." Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, III.426-428, pp. 252-253.

[422] Oresme, *De proportionibus proportionum, and Ad pauca respicientes*, III.355-358, p. 249.

and commensurable ratios, the domain of quadrivial arithmetic and music.[423] He is rationally relating irrational ratios. Not only is this very sophisticated mathematics, but it is also a radical departure from the definitions of number and ratio. Oresme's distinction between multitude and magnitude becomes largely a matter of convenience, rather than an inherent difference between species.

Upon the completion of Geometry's probability argument, Apollo orders silence. The dreaming Oresme wonders to himself why Arithmetic and Geometry presented such "rhetorical persuasions or sophistical proofs" rather than mathematical demonstrations.[424] Apollo reads Oresme's thoughts and tells him, "Do not seriously believe that there is a genuine disagreement between these most illustrious mothers of evident truth. For they amuse themselves and mock the stylistic mode of an inferior science."[425] Apollo then prepares to deliver his judgment on whether the motions of the heavens are commensurable or incommensurable. But just at that moment, the moment of Apollo's final judgment on the matter, Oresme wakes up and the *Treatise on the commensurability or incommensurability of the motions of the heavens* comes to an abrupt and unresolved end.

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[423] John Murdoch sees the incorporation of arithmetical number into geometrical magnitude as a medieval tendency. See John E. Murdoch, "The Medieval Language of Proportions: Elements of the Interaction with Greek Foundations and the Development of New Mathematical Techinques," in *Scientific Change: Historical Studies in the Intellectual, Social, and Technical Conditions for Scientific Discovery and Technical Invention, from Antiquity to the Present,* ed. A. C. Crombie (New York: Basic Books, 1963), 270-271.
[424] Oresme, *De commensurabilitate*, III.471, pp. 322-323.
[425] Ibid., III.474-476, pp. 322-323.

Conclusions

The debate between Arithmetic and Geometry in *De commensurabilitate* is clear evidence of the tensions that could arise in the quadrivial structure. Oresme took standard quadrivial mathematical concepts to their limits and found that the distinction between rational and irrational was not as clear cut as it appeared to be in the simple examples generally given by music theorists and astrologers. He concluded that even irrational quantities could be rationally related to one another. He found that a music of the spheres based on motion would require that conjunctions between heavenly bodies occur in only very limited locations on the ecliptic. Superparticular relationships could only occur in one place. No observed conjunctions behaved in this manner. The motions of the heavens were clearly not arranged musically, at least not consonantly. The motions may have been commensurate in some wildly complicated set of big-number relationships, but there was no evidence for this and Oresme argued compellingly that it was more probable that they were incommensurate. If they were perchance commensurate, the periods of repetitions of various configurations would have had to have been extremely long, much longer than could be extrapolated from any available data. Either way, the subtlety required for an ultimate determination was beyond the scope of human perception. Astrologers simply could not know enough to determine truly analogous configurations. Therefore, as Oresme saw it, their predictions were no better than random guesses.

It appears, from Oresme's presentation, that the quadrivial philosophy was intimately and ultimately linked to astrology. Arithmetic's arguments for the supremacy of number and rational ratios were also arguments for the predictive science of the stars. The quadrivium's champion was also astrology's champion. Judging from Oresme's presentation, their destinies were one and the same.

To date there is little evidence that anyone built upon Oresme's arguments for the incommensurability of celestial motions. Several scholars briefly refer to Oresme's discussion of incommensurability, such as Henry of Hesse (d. 1397), Marsilius of Inghen (d. 1396), and Jean Gerson (1363-1429). Pierre d'Ailly (1350-1420), who was a student at the College of Navarre in the University of Paris six years after Oresme left as Grand Master, goes so far as to plagiarize much of Part III of *De commensurabilitate* in his *De legibus et sectis* (1410).[426] John de Fundis (fl. 15th c.), a proponent of astrology, wrote a critical commentary[427] on Oresme's *Ad pauca respicientes* (which covers roughly the same material as Parts I and II of *De commensurabilitate*). But Edward Grant finds John de Fundis' lack of mathematical fluency so bad as to be almost laughable.[428] Girolamo Cardano's *Opus novum de proportionibus* (1570) contains several of the propositions from Parts I and II of *De commensurabilitate*, but Cardano significantly altered the presentation by incorporating much more algebraic terminology and transliterated many of the Greek mathematical terms.[429] He also never applies these arguments to astronomy, which is odd since Cardano was a famous astrologer. Furthermore, Cardano cites Campanus of Novara (ca. 1220-1296) and an unnamed book given to him by his father Fazio as the sources for these sections on circular motions with no mention of Oresme. Nonetheless, Edward Grant believes that due to the content and structure of Cardano's presentation of these propositions, they must be derived from *De commensurabilitate*.[430]

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[426] Grant refers to this text as Tractatus contra astronomos, but Laura A. Smoller, who studies d'Ailly, refers to this text as *De legibus et sectis*. Smoller, *History, Prophecy, and the Stars: The Christian Astrology of Pierre d'Ailly, 1350-1420*, 38-39.
[427] According to Grant, only one copy of this short treatise exists. See Grant's note in Oresme, *De commensurabilitate*, 137, n131.
[428] Grant's introduction, Ibid., 138.
[429] Grant's introduction, Ibid., 143.
[430] Grant's introduction, Ibid., 159.

It appears that Oresme's work on the commensurability and incommensurability of the celestial motions was largely forgotten. The commensurate musical cosmos danced merrily along for another three hundred years and even longer if you include the likes of Robert Fludd (1547-1637) and Athanasius Kircher (ca. 1601-1680).

Johannes Kepler (1571-1630) made what is generally considered to be the last credible harmonic astronomical theory in his book *Harmonices mundi* from 1619. His three laws of planetary motion are purely kinematic and even if he had known of Oresme's suggestion to find cosmic harmony in weights and volumes like Pythagoras' hammers, he could not have have used this idea, for planetary weights and volumes were as yet unknown. At about the same time, Galileo was developing the mathematics of mass and gravitational acceleration in terrestrial physics, but this concept had yet to be incorporated into astronomy until Newton, in the late 17th century, took Kepler's musical cosmos and introduced the occult force of gravity. Gravity, combined with Kepler's harmonic equations, actually did make a musical cosmos of sorts, but by this time the simple concepts of Pythagorean tuning had been swept away by exponential approaches, not dissimilar to Oresme's ratios of ratios. Also in the details, what was meant by "harmonic" in the 17th century was quite different from "harmonic" in the 14th century. But recall that at its root, *armonia*, in the Greek sense, is a structural concept. It means joining– things coming together and acting in harmony. The gravitational equations of Newton can quite easily be seen as an extension of harmonic theory; they are equations that join the macrocosm to the microcosm. They bind the universe together, and do so mathematically. In this sense, the quadrivium was not demolished by Oresme, it was given new life. Oresme started the ideas that ultimately led to the real number system, whereby the multitudes of quadrivial arithmetic were absorbed into the magnitudes of geometry.

Now we will look at another quadrivial scholar who wrestled with the concept of quadrivial harmonic theory. Unlike Oresme he was not an innovator. He did not push at the boundaries of the traditional Boethian quadrivium, but his staunch reactionary conservatism clearly demonstrates the tensions that were developing in the realm of quadrivial music.

Chapter 4:
Quadrivial Case Study #2:
Prosdocimo de' Beldomandi (ca. 1375-1428)

The quadrivial philosophy of Prosdocimo de' Beldomandi is very different from that of Oresme. Prosdocimo does not expand the definition of quadrivial number; he defends it. The distinction between multitude and magnitude is consistently maintained throughout Prosdocimo's mature quadrivial writings. He is careful never to allow the two to mix. His defense of quadrivial arithmetic and music from a perceived threat by a mathematical novice clearly defines his conservative position. And yet, Prosdocimo was not averse to new ideas. He embraced many of the new mathematical and natural philosophical ideas that were developing all around him, but always in terms of Boethian quadrivial orthodoxy.

Introduction

Prosdocimo de' Beldomandi was a professor of arts and medicine at the University of Padua from ca. 1422 (*terminus ante quem*) until his death in 1428.[431] He is particularly well suited as a

[431] For a more detailed biography of Prosdocimo, see Antonio Favaro, *Intorno alla vita ed alle opere di Prosdocimo de' Beldomandi, matematico padovano del secolo XV*, Bullettino di bibliografia e di storia delle scienze matematiche e fisiche 12 (Rome: Tipografia delle Scienze Matematiche e Fisiche, 1879); Antonio Favaro, *Appendice agli studi Intorno alla vita ed alle opere di Prosdocimo de' Beldomandi, matematico padovano del secolo XV*, Bullettino di bibliografia e di storia delle scienze matematiche e fisiche 18 (Rome: Tipographia della Scienze Matematiche e Fisiche, 1885). See also Alberto F. Gallo, *La tradizione dei trattati musicali de Prosdocimo de Beldemandis* (Bologna: Arti Grafiche Tamari, 1964); Jan Herlinger, introduction to Prosdocimo de' Beldomandi, *Contrapunctus*, ed. and trans. J. Herlinger (Lincoln: University of Nebraska Press, 1984), 1-7. Paul O. Kristeller speculates that Prosdocimo taught Nicolaus Cusanus (1401-1464) while he was at the University of Padua. Paul Oskar Kristeller, *Studies in Renaissance Thought and Letters*, vol. 3 (Rome: Edizioni di Storia e Letteratura, 1993), 29. The mathematical historian J. E. Hofmann definitively states that Cusanus and Toscanelli attended astrology lectures given by Prosdocimo at Padua. See J. E. Hofmann, "Cusa, Nicholas," in *The Complete Dictionary of Scientific Biography* [electronic resource], vol. 3 (Detroit: Charles Scribner's Sons, 2008. *Gale Virtual Reference Library*. Accessed 28 July 2011.), 512-513. It is tempting to think that Prosdocimo knew not only Cusanus, but also Paolo Toscanelli (1397-1482) and Leon Battista Alberti (1404-72).

quadrivial case study for he wrote on all four disciplines of the quadrivium. Prosdocimo was the quintessential 15th-century quadrivial scholar. In his own time, Prosdocimo was identified as a "a famous mathematician, musician, philosopher, and astrologer."[432] His writings exhibit a high level of understanding of the traditional quadrivial disciplines and demonstrate an interest in contemporary trends. His influence on proximate generations is well established.

Perhaps the greatest testament to Prosdocimo's fame is found in Luca Pacioli's *Summa de arithmetica, geometria, proportioni et proportionalita* from 1494, published approximately 65 years after Prosdocimo's death. In the preliminary summary to Book I, Pacioli acknowledges his debt to "the very perspicacious philosopher of Megara, Euclid,[433] and Severinus Boethius, and from our modern mathematicians, *Leonardo pisano, Giordano, Biagio da parma, Giovan sacrobusco, e Prodocimo padoano,* from whom I take the major part of this volume."[434] These modern mathematicians, with whom Prosdocimo is listed, are:

–*Leonardo pisano* [Fibonacci] (d. ca. 1250). He is generally credited with introducing Hindu-Arabic numbers to Europe in his book *Liber abaci*, which also discusses elementary algebra.

–*Giordano* [Jordanus de Nemore] (fl. mid-13th century). He is the author of a widely used book on quadrivial arithmetic, an algorism, texts on triangular and spherical geometry of the astrolabe, and wrote the first advanced algebra in western Europe.

–*Biagio da parma* [Blasius of Parma or Biagio Pelacani da Parma] (ca. 1354-1416).

432 Favaro, *Appendice ... Prosdocimo de' Beldomandi*, 19-20 [421-422].

433 Euclid of Megara (ca. 435–ca.365 B.C.) was actually a disciple of Socrates. Euclid of Megara is how Campanus of Novara referred to Euclid of Alexandria in his 13th-century Latin translation of the *Elements* (first printed in Venice, 1482). This was the ubiquitous Latin translation of Euclid widely available in the late Middle Ages. This confusion between Euclid of Megara and Euclid of Alexandria goes back to Boethius and perhaps further.

434 "Maxime del perspicacissimo phylosopho megarense. Euclide E del severin Boetio. e de nostri moderni Leonardo. pisano. Giordano. Biagio da parma. Giovan sacrobusco. e Prodocimo padoano. da iquali in magior parte cauo el presente volume." Luca Pacioli, "Summa de arithmetica, geometria, proportioni et proportionalita" (Venice: 1494). The Archimedes Project: http://archimedes.mpiwg-berlin.mpg.de/cgi-bin/toc/toc.cgi?step=thumb&dir=pacio_summa_504_it_1494 (accessed 2010). Translation by Nick MacKinnon, "The Portrait of Fra Luca Pacioli," *The Mathematical Gazette* 77, no. 479 (1993): 182.

He was a professor of the quadrivial disciplines (including astrology) and natural philosophy in several Italian universities. He also studied in Paris and brought to Italy many of the geometrical-physical ideas of Nicole Oresme, Thomas Bradwardine, and John Buridan. Late in his life, while at the University of Padua, he was one of Prosdocimo's sponsors for his examination in the arts.[435]

– *Giovan sacrobusco* [John of Holywood] (d. ca. 1256). He was the author of the *Sphere*, the ubiquitous text used to teach introductory Ptolemaic astronomy, and also wrote an influential treatise on algorism. Both of these texts were hugely significant to Prosdocimo, who wrote an extensive commentary on the *Sphere* and also wrote his own algorism largely based on Sacrobosco's version.

Prosdocimo, the most modern among them, finds himself in very good company.[436]

About 65 years after Pacioli's praise, Bernardino Scardeone in his book on the history of Padua from 1560, writes a short (less than 100 words) biographical entry on Prosdocimo, which describes him as being from a noble Paduan family and an "exceptional mathematical music theorist, an excellent philosopher, and a famous astrologer."[437] The only texts mentioned in this

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[435] See Jan Herlinger, introduction to Prosdocimo de' Beldomandi, *Musica speculativa*, 6. See also Meiczyslaw Markowski, "Die kosmologischen Anschauungen des Prosdocimo de' Beldomandi," in *Studi sul XIV Secolo in Memoria de Anneliese Maier,* ed. A. Maierù and A. Paravicini Bagliani (Rome: Edizioni di Storia e Letteratura, 1981), 263. Blasius got in trouble with the church for his acceptance of astrological determinism and he also had rather radical notions about the intellective soul. See Edward Grant, "Blasius of Parma," in *The Complete Dictionary of Scientific Biography* [electronic resource], ed. Charles Coulston Gillispie (Detroit: Charles Scribner's Sons, 2008. *Gale Virtual Reference Library*. Accessed 8 Feb. 2011.), 192-195.

[436] The section of Pacioli's *Summa* for which Prosdocimo is given credit, is on arithmetic, proportion, algorism, and various geometrical measurement issues. The texts by Prosdocimo most relevant to these topics would be his *Algorismus de integris* and/or his *Brevis summula proportionum quantum ad musicam pertinet. Algorismus* had been printed in 1483, and may have been readily available. See Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 65-68.

[437] "...egregius Musicus, & eximius philosophus , & clarus astrologus." Bernardino Scardeone, *Historiae de urbis Patavii* (Leiden: Petrus Vander Aa, 1722; reprint, Bologna: Arnoldo Forni, 1979), II.12.297c. The first edition of Scardeone's *Historiae de urbis Patavii* is generally listed as Basel, 1560. Lynn Thorndike refers to a Venice, 1558 edition, but I have not been able to locate it. See Lynn Thorndike, *A History of Magic and Experimental Science: During the First Thirteen Centuries of Our Era*, 8 vols., vol. 2 (New York: Columbia University Press, 1923), 915, n3. In Souter's *Glossary of Later Latin, "musicus"* is defined specifically as a "scientist who expounds the relations of numbers." Niemeyer's *Mediae Latinitatis Lexicon* defines it as an "expert of musical theory." *Astrologus* could mean either astrologer or

biography are his *Commentary on Sacrobosco's Sphere* and his criticism of Marchetto's music theory (*Musica speculativa*). Although this short biographical sketch is of only minor interest in and of itself, its location in Scardeone's book is more interesting. Given his scholarly output, one would expect to find Prosdocimo in the section "On Famous Paduan Doctors and Philosophers," but he is not mentioned there at all. There is, however, a long entry on the "medical doctor and most famous astrologer," Jacopo de Dondi,[438] the maker of a famous clock in Padua and whose astronomical tables Prosdocimo completed. Instead, Scardeone appears to have thought of Prosdocimo primarily as a music theorist, and most notably the theorist who attacked Marchetto. He situates him in a section on famous Paduan musicians, directly below the entry on Marchetto of Padua (fl. early 14th century), whose biography is nearly twice the length of Prosdocimo's.[439] This placement in Scardeone's text is odd, because at the time only Prosdocimo's *Commentary on Sacrobosco's Sphere* (Venice, 1531) and his *Algorismus de integris* (Padua, 1483 and Venice, 1540) had been printed. One might have expected Prosdocimo's fame, at this point in history, to have been greater in astronomy and arithmetic.

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Short Biography

Prosdocimo's date of birth unknown. Assuming his educational trajectory was typical and that he entered the university as a teenager, it could be assumed that he was born between the mid 1370s to the early 1380s,[440] which would place his birth right about the time of Oresme's

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astronomer or both.

438 Ibid., II.9.232c-233c.

439 Marchetto's biography is twice that of Prosdocimo, who had been dead for half as long. Prosdocimo, as we shall see, was distressed over the popularity of Marchetto's speculative music theory in the early 15th century.

440 Edward Grant writes that students typically entered a university at the age of 14 or 15. Edward Grant, *The Foundations of Modern Science in the Middle Ages: Their Religious, Institutional, and Intellectual*

death.

The bulk of Prosdocimo's university education, starting ca. 1400, took place in Padua, with an additional stint at the University of Bologna.[441] He received his doctorate in the arts at Padua in 1409 and received his medical license there in 1411.[442] The arts and medical faculties were combined at both Padua and Bologna in this period, and both universities were well known for their excellent medical programs.[443]

Scholarly medicine in the early 15th century was inextricably linked to astrology, particularly in the Italian schools.[444] Health was thought to be dependent on an individual's balance of humors, typically four in number: blood, phlegm, choler, and melancholy.[445] A person's general

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Contexts (New York: Cambridge University Press, 1996), 38. Favaro conservatively estimates that Prosdocimo was born sometime between 1370-1380. Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 18.

[441] The reference to his study in Bologna comes from Prosdocimo's own hand in Florence, Biblioteca Medicea Laurenziana, Ashburnham 206, 19r, transcribed in Favaro, *Appendice ... Prosdocimo de' Beldomandi*, 5/407. "Expliciunt canones magistri Johannis de saxsonia super tabulas Alphonsi scripti per me Prosdocimum de beldemandis de padua in artibus Bononie studentem. Amen." ["Here end the Canons of Master John of Saxony on the tables of King Alfonso written by me, Prosdocimo, student in the arts at Bologna.] See also Gallo, 6.

[442] Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 24-25; Favaro, *Appendice ... Prosdocimo de' Beldomandi*, 18-19 [420-421].

[443] Guy Beaujouan, "Motives and Opportunities for Science in the Medieval Universities," in *Scientific Change: Historical Studies in the Intellectual, Social, and Technical Conditions for Scientific Discovery and Technical Invention, from Antiquity to the Present,* ed. A. C. Crombie (New York: Basic Books, 1963), 232-233; Nancy G. Siraisi, *Arts and Sciences at Padua: The Studium of Padua before 1350* (Toronto, Canada: Pontifical Institute of Mediaeval Studies, 1973), 24-26 and 143-166.

[444] Beaujouan, 233. See Siraisi, *Arts and Sciences at Padua,* 67-68 and 149-152; Nancy G. Siraisi, "The Music of Pulse in the Writings of Italian Academic Physicians (Fourteenth and Fifteenth Centuries)," *Speculum: A Journal of Mediaeval Studies* 50, no. 4 (1975): 710. Nancy Siraisi also suggests that music was important for the study of astrology in Padua. The astrology-medicine connection was also a tendency north of the Alps. See Pearl Kibre, "Logic and Medicine in Fourteenth-Century Paris," in *Studi sul XIV Secolo in Memoria de Anneliese Maier,* ed. A. Maierù and A. Paravicini Bagliani (Rome: Edizioni di Storia e Letteratura, 1981), 416. The astrologer Cecco d'Ascoli, a professor at the University of Bologna in the early 14th century, discusses the connection between medicine and astrology in his commentary on Sacrobosco's *Sphere*. See Thorndike, ed., 344-346. See also Lynn White Jr., "Medical Astrologers and Late Medieval Technology," in *Viator 6: Medieval and Renaissance Studies,* ed. Lynn White Jr. (Berkeley, CA: University of California Press, 1975), 295-308. See also Paul F. Grendler, *The Universities of the Italian Renaissance* (Baltimore: Johns Hopkins University Press, 2002), 408-412 and n5; Siraisi, *Arts and Sciences at Padua*, 67-68, 111, 134-136, 149, and 152.

[445] There exist some variations in the names of the humors. There were also humoral theories with more than four basic humors. See Vivian Nutton, "Humoralism," in *Companion Encyclopedia of the History of*

humoral balance, or *complexio*, was established at birth, and was thought by medical astrologers to be dependent on astronomical arrangements. For example, a person born in Aquarius with Jupiter in Capricorn might have a natural and healthy abundance of phlegm. Treatments for physical, and/or what we might now call psychological, problems would typically aim to restore the proper idiosyncratic humoral balance of the patient. For example, treatments might involve phlebotomy, to reduce sanguine, or the eating of veal to lesson melancholy. Astrological considerations often played a role in the timing of treatments. For example, phlebotomy was generally administered at particular times to particular parts of the body, depending on the locations of heavenly bodies. See Figure 4.1.

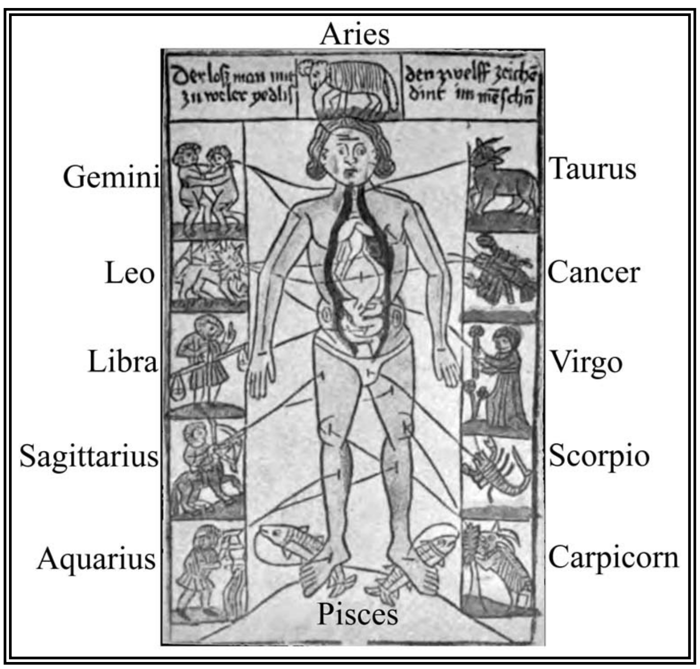

Figure 4.1: Blood-letting Chart (*Aderlassmann*), from the "Calendar of Regiomontanus" (1475)[446]

As a result of the widespread belief in astrological medicine, it is not surprising that the universities of Padua and Bologna each combined their arts and medical faculties into one. In order to practice medicine, a doctor needed to know the motions of the heavens, astronomy, the

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Medicine, ed. W. F. Bynum and R. Porter (New York: Routledge, 1997), 281-291.
[446] Image from Fielding H. Garrison, *An Introduction to the History of Medicine*, 2nd ed. (Philadelphia: W.B. Saunders Company, 1917), 184.

highest and perhaps most difficult quadrivial discipline. And since the prerequisites for astronomy were arithmetic, music theory, and geometry, all four disciplines of the quadrivium were required.

Prosdocimo's own education in the arts and medicine reflect this connection between the quadrivium (including astrology) and medicine. A nearly 150-folio manuscript, written by Prosdocimo himself in the early 15th century when he was still a student at the Universities of Padua and Bologna, provides an amazing inventory of his studies.[447] It is essentially a student notebook with transcriptions, class notes, stray observations, calculations, mathematical and astronomical tables and diagrams, medical and alchemical recipes, poetry, and commentaries. Many of Prosdocimo's later works are suggested by the contents of this manuscript. For example, the algorisms of Sacrobosco and Johannes de Lignères are both transcribed and are later used as primary sources for Prosdocimo's own algorism. The canons to the *Alfonsine Tables* written by John of Saxony are transcribed and are later reworked in his own canons to the *Alfonsine Tables*. Transcriptions of texts by Pseudo-Hippocrates and another by John of Saxony specifically pertain to medical astrology.[448] This manuscript contains many short unidentified astrological/astronomical treatises and medical recipes. It also contains several of Prosdocimo's earliest original texts, including the *Treatise on the Practice of Mensural Music* (which was, at some point, crossed out) and *A Short Summary of Ratios Insofar as They Pertain to Music*. Except for geometry, all of the quadrivial disciplines are addressed in detail. However, it can safely be assumed that a certain amount of geometrical knowledge is contained in the

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[447] Florence, Biblioteca Medicea Laurenziana, Ashburnham 206. 145 folios (early 15th century).

[448] Much of the information on MS Ashburnham 206 comes from Jan Herlinger's description found in Prosdocimo de' Beldomandi, *Brevis summula proportionum quantum ad musicam pertinet and Parvus tractatulus de modo monacordum dividendi = A Short Summary of Ratios Insofar as They Pertain to Music and A Little Treatise on the Method of Dividing the Monochord*, ed. and trans. J. Herlinger (Lincoln: University of Nebraska Press, 1987), 30-39.

astronomical texts, but there is no stand-alone text devoted to it in this particular manuscript. See Appendix 4 for an outline of this manuscript.

In the 19th century, both the manuscript and printed editions of Prosdocimo's works were meticulously catalogued by Antonio Favaro, and were later amended by Alberto Gallo[449] in the mid-20th century, and more recently revised by Jan Herlinger.[450] All of his major writings are quadrivial in nature and all of them, to some degree, show Prosdocimo as a conservative quadrivial scholar. However, unlike Oresme, he does not tend to comment at length on the structure of the quadrivium itself. For the most part he takes the quadrivium for granted and only occasionally refers to its structure. Therefore, it makes sense to provide a brief survey of those instances where Prosdocimo makes specific reference to this structure of the quadrivium from a selection of his major texts. This survey will highlight those instances in terms of quadrivial philosophy. Of his original[451] works, seven are of particular importance for this case study: one work in arithmetic, three works in music, one minor work in geometry, and two works in astronomy.

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[449] Gallo, 5-32.

[450] See Herlinger, introduction to Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 13-43. Giancarlo Truffa also maintains a bibliography of Prosdocimo's astronomical, astrological, and mathematical works along with secondary sources on the world wide web. See Giancarlo Truffa, "Beldemandis (Beldomandi, Beldemando, Beldimando), Prosdocimo de Padua c.1370-1428: Bibliography of Astronomical, Astrological, and Mathematical Works" http://www.asu.cas.cz/~had/prosdoc.html#math1 (accessed 2010).

[451] In the context of medieval authors, "original" is a tricky term. Boethius' major works on arithmetic and music theory are largely paraphrases of Nicomachus and Ptolemy, but are regarded, with some qualifications, as original works by most scholars. I am considering any of Prosdocimo's works that are not simply transcriptions to be original works. These include commentaries and works that are derived from other sources.

Prosdocimo's Arithmetical Multitudes:[452]
***Algorismus de integris* [*Algorism of Whole Numbers*] (1410)**

David Eugene Smith has identified four types of arithmetical texts in the 15th century:[453]

1. Theoretical arithmetical texts: Largely based on Boethius' *De institutione arithmetica* and the sections on number theory in Euclid's *Elements*.
2. Algorisms: These are practical computational texts that teach how to read and work with Hindu-Arabic numerals. They are similar in content to modern elementary arithmetic textbooks.
3. Abacus texts: These are also computational, but use only Roman numerals and an abacus or similar counting board. Such books are relatively uncommon in this period in Italy, having been made obsolete by the algorisms.
4. Computi texts: These gave directions for working with the calendar, e.g., locating Easter.

Prosdocimo wrote on all of these topics except for the abacus. The type of text where one would expect to find the most quadrivial observations would be in the first class, the theoretical arithmetical texts. Unfortunately, the only text on theoretical arithmetic in Prosdocimo's oeuvre is *De arte numerandi* [*On the Art of Counting*], which Favaro describes as simply a transcription of Boethius' *De institutione arithmetica*.[454] Aside from proving that Prosdocimo was intimately familiar with Boethius' arithmetical text, this transcripton would not be expected to reveal Prosdocimo's own quadrivial philosophy. The text with the most promise for this case study is his *Algorismus de integris* [*Algorism on Whole Numbers*].[455] This treatise is largely based on

[452] For Favaro's description of Prosdocimo's arithmetical texts and related topics, see Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 43-128.
[453] David Eugene Smith, *Rara Arithmetica: A Catalogue of the Arithmetics Written Before the Year MDCI* (Boston: Ginn and Company, 1908), 4-7.
[454] Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 116-121 [ca. 155-160]. Only one manuscript copy of *De arte numerandi* is known to exist.
[455] This text appears in numerous manuscripts and was twice printed: once in Padua, 1493, and again in Venice, 1540. For a detailed description of manuscripts and publications, see Ibid., 42-101.

Sacrobosco's *Tractatus de arte numerandi.*[456] Although most of Prosdocimo's text is about adding, multiplying, and finding roots using the relatively new Hindu-Arabic numerals and the decimal system, there are a few stray comments that shed some light on his quadrivial philosophy.

The Venice edition from 1540 of the *Algorismus de integris* covers 56 octavo pages (unnumbered).[457] It was written in 1410,[458] while Prosdocimo was still a student in Padua, but before he received his medical license. Although algorisms often tended to be written for commercial applications, this one was written for more academic purposes.[459] Prosdocimo states in his introduction that the algorisms he had come upon were "wearisome and laborious"[460] and that he intended to correct this by writing a short description of how to employ and calculate with *digiti*, *articuli* and *numeri compositi,*[461] accurately and without error. He is specifically concerned that the methods used for astrological calculations be not only accurate, but recorded in an organized fashion for later examination rather than erased, as was apparently the case with some calculators.[462]

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[456] Johannes de Sacrobosco, "*De arte numerandi*," in *Rara Mathematica, or A Collection of Treatises on the Mathematics and Subjects Connected with Them, from Ancient Inedited Manuscripts,* ed. James Orchard Halliwell (London: Samuel Maynard, 1841), 1-26. A partial English translation of Sacrobosco's text can be found in Grant, ed., *A Source Book in Medieval Science*, 94-101. For a survey of and comparison between early algorisms, including references to Prosdocimo, see Suzan Rose Benedict, "A Comparative Study of the Early Treatises Introducing into Europe the Hindu Art of Reckoning" (Ph.D. Dissertation, University of Michigan, 1914), passim.
[457] Because this edition has no page numbers, I will refer to page numbers starting with "1" where the text by Prosdocimo begins, "Inveni in q[uam] pluribus libris algorismi..." Prosdocimo de' Beldomandi, *Algorismus de integris Magistri Prosdocimi de Belamandis Pataui simul cum Algorismo de minutijs seu fractionibus magistri Ioannis de Liuerij siculi. Reintegratus ab erroribus commissis a scriptoribus, a me Federico Delphio artium, & medicine doctore, mathematicarum disciplinarum...* (Venice: 1540).
[458] Ibid., 56.
[459] Smith, *Rara Arithmetica: A Catalogue of the Arithmetics Written Before the Year MDCI*, 13.
[460] "Et quia haec omnia satis fastidiosa atque laboriosa mihi misa sunt." Prosdocimo de' Beldomandi, *Algorismus de intergris*, 2.
[461] Ibid., 1-2.
[462] D. E. Smith speculates that calculators in the 15th century using slates for calculation might have erased instead of canceling certain figures in the process of division, making the problem unreviewable. Smith,

However, being the quadrivial scholar that he was, he does not get into these matters until he defines his terms. His algorism is about number itself, so he begins with a definition of number. He writes, "Following Euclid in the seventh [book] of his geometry, and following Boethius in the first [book] of his arithmetic, number is defined. Number is a multitude or a discrete quantity issued from unities."[463] He then defines unity as each and everything called one, keeping the concept of unity distinct from multitudes in general.[464] Although Sacrobosco's algorism covers the same material on number in his introduction, Prosdocimo is more specific in his definitions. His reference to astrological calculation suggests his intended use for this method of calculation. Prosdocimo makes it clear that a quadrivial discipline such as astronomy/astrology should be firmly rooted in quadrivial arithmetic, even if new methods for calculation are introduced.

The remainder of the text describes the mechanical processes of manipulating these new numbers. Before introducing the new Hindu-Arabic symbols, he describes the three members [*membra*] into which these new figures are divided: *digiti* (fingers), *articuli* (connecting joints), and *numeri compositi* (composite numbers). The *digiti* are the numbers less than ten, the *articuli* are those numbers which are divisible by ten, and the composite numbers are combinations of the *digiti* and the *articuli*. Numeration is the representation of a number using the appropriate figures as members.[465] He then introduces the Hindu-Arabic figures, "9, 8, 7, 6, 5, 4, 3, 2, 1, and along with something that by itself signifies nothing [*nihil*], although given to mean something,

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Rara Arithmetica: A Catalogue of the Arithmetics Written Before the Year MDCI, 13-14.

[463] "Numerus ergo secundum Euclidem septimo sue geometrie, & secundum Boetium primo sue arismetrice sic diffinitur. Numerus est multitudo sive quantitas discreta ex unitatibus profusa sive ex unitatibus agregata." Prosdocimo de' Beldomandi, *Algorismus de intergris*, 2. This definition of number is also given in much the same way in Sacrobosco's text. "...numerus est multitudo ex unitatibus agregata." Sacrobosco, "*De arte numerandi*," 2. Boethius, *De institutione arithmetica*, 1.3, pp. 76-77; Euclid, VII, Definitions 1 and 2, pp. 277 and 279-280.

[464] After this definition of unity, he states that it should not be considered a number, even though it is presented as the first number. "Per hanc ergo diffinitionem numeri habere potes quomodo unitas non est numerus licet sit principium numeri dato." Prosdocimo de' Beldomandi, *Algorismus de intergris*, 2.

[465] Ibid., 4-5.

which is thus fashioned '0,' and such a figure is called *nihil* or *cifra*."[466] Using this system, he next constructs numbers with the *cifra* as a placeholder. He also explains that the reason numbers are written left to right, large to small, instead of small to large, is because this is "the manner of the Arabs, who are the inventors of this art,"[467] the implication being that if the Arabs wrote left to right, we too would construct numbers small to large. For example, we write 1975, but had we adopted the direction in which the Arabs actually read their numbers, we would write it 5791.

Using these new figures, the remainder of the book describes many of the operations that we still learn in primary school along with methods for finding square and cube roots. There are very few mathematical examples using Hindu-Arabic numerals in the text, most of it being lengthy instructions in prose. See Figure 4.2. He ends his text acknowledging his indebtedness to Sacrobosco.

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[466] "...sunt iste. 9.8.7.6.5.4.3.2.1. & una nihil per se significans licet det aliis significare que[m?] sic figuratur .0. & figura talis vocatur figura nihil vel cifra." Ibid., 5.

[467] "...ad modum Arabum qui huous artis fuerunt inventores." For more information on the *cifra* and order of Hindu-Arabic numerals in Latin, see Charles Burnett, "The Semantics of Indian Numerals in Arabic, Greek and Latin," *Journal of Indian Philosophy* 34, no. 1-2 (2006): 15-30.

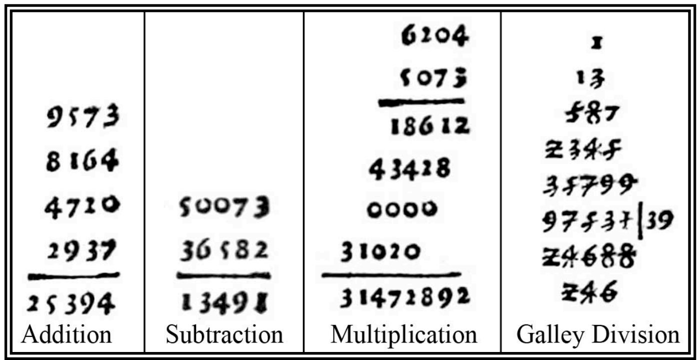

Figure 4.2: The Basic Operations.[468]

Many algorisms were produced in the late Middle Ages. The majority of them appear to be much less rigorous than the algorisms of Prosdocimo and Sacrobosco. Many were written in the vernacular, and some were even written in verse.[469] The merchants and tradespeople who adopted the Hindu-Arabic numerals for doing business would not have been interested in the quadrivial theory of unity and number that structured the cosmos. They would have wanted results. Prosdocimo was not writing for them. He was writing in Latin for an educated Latin-

[468] The selection of images for this figure are from Prosdocimo de' Beldomandi, *Algorismus de intergris*, 9, 13, 29, and 35. Galley division is a form of long division. The diagram on the far right in Figure 4.2 shows 97531÷2468, which equals 39 with a remainder of 1279. This method was also explained by Sacrobosco in his algorism. A history and partial explanation of this method is found in Smith, *History of Mathematics. Volume II: Special Topics of Elementary Mathematics*, 136-140. This method is fully explained in the *Trevisio arithmetic* from 1478. See Frank Swetz and David Eugene Smith, *Capitalism and Arithmetic: the New Math of the 15th Century, including the full text of the Treviso arithmetic of 1478, translated by David Eugene Smith*, 90-100 and 213-221.

[469] The most famous of these texts is Leonardo Pisano's *Liber abaci* from 1202. Fibonacci (Leonardo Pisano), *Fibonacci's Liber abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*, trans. L. E. Sigler (New York: Springer, 2002). James Halliwell's *Rara Mathematica* contains the 13th-century "Carmen de Algorismo" by Alexander de Villa Dei, of which numerous manuscripts survive. James Orchard Halliwell, ed., *Rara Mathematica, or A Collection of Treatises on the Mathematics and Subjects Connected with Them, from Ancient Inedited Manuscripts*, 2nd ed. (London: Samuel Maynard, 1841), 73-83. An anonymous 13th-century French algorism in verse has been published and translated by Karpinski and Staubach and another by Karpinski and Waters. See Louis C. Karpinski and Charles N. Staubach, "An Anglo-Norman Algorism of the Fourteenth Century," *Isis* 23, no. 1 (1935): 121-152; Louis C. Karpinski and E. G. R. Waters, "A Thirteenth Century Algorism in French Verse," *Isis* 11, no. 1 (1928): 45-84. Waters published yet another 15th-century algorism in verse. E. G. R. Waters, "A Fifteenth Century French Algorism from Liege," *Isis* 12, no. 2 (1929): 194-236. See also Martin Levey, "Abraham Savasorda and His Algorism: A Study in Early European Logistic," *Osiris* 11, no. (1954): 50-64. For general information, see Gillian R. Evans, "From Abacus to Algorism: Theory and Practice in Medieval Arithmetic," *The British Journal for the History of Science* 10, no. 2 (1977): 114-131.

speaking audience. He was putting this popular method of calculating in quadrivial terms.

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Prosdocimo's Quadrivial Music[470]

Of Prosdocimo's eight treatises on music, three are of particular importance for their quadrivial content: a) *Brevis summula proportionum quantum ad musicam pertinet* (1409), b) *Parvus tractatulus de modo monacordum dividendi* (1413), and c) *Tractatus musice speculative* (1425). The first two texts clearly define Prosdocimo's theoretical stance on music. The third text, *Tractatus musice speculative,* is the most illuminating, for it shows Prosdocimo reacting to what he perceives as an attack on quadrivial musical orthodoxy.

a) *Brevis summula proportionum quantum ad musicam pertinet* (1409)
[*A Short Summary of Ratios insofar as they Pertain to Music*][471]

Although this text pertains to music, there is never any mention of sound or of a musical instrument. It is purely theoretical. The majority of the text discusses species of musical ratios and defines the vocabulary used to describe them. Terms such as multiple (e.g., 2:1 or 9:3), superparticular (e.g., 3:2 or 9:8), and superpartent (e.g., 5:3 or 13:9) are explained and systematized.

Like the *Algorismus de integris*, the interesting part in terms of the quadrivial philosophy comes at the beginning when he defines his basic terms. In the algorism, a text on arithmetic, his basic term was number. In this text the basic term is ratio. In general he presents a failry standard Boethian/Euclidean quadrivial definition, but with his own empahsis. He writes,

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[470] For Favaro's description of Prosdocimo's musical texts and related topics, see Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 184-211. See also Gallo, passim.
[471] Like Boethius, Prosdocimo does not make a rigid Euclidean distinction between the terms ratio and proportion. The context generally makes it clear. See Masi's commentary to Boethius, *De institutione arithmetica*, 26 and 164, n40.

> [A] ratio as properly so called is the mutual relationship of several quantities of the same proximate genus; "of the same proximate genus" is stated because a continuous quantity cannot properly be taken in a ratio to a discrete quantity since, ... , they are not of the same proximate genus but of ones quite remote.[472]

Prosdocimo is explicitly stating that the continuous magnitudes of geometry cannot be compared with the multitudes of arithmetic. They are different genera.[473] There is no qualification, such as Oresme might make, that the continuous magnitudes of geometry contain all of the numbers of arithmetic. In Prosdocimo's world the magnitudes of geometry and the multitudes of arithmetic are completely separate.

Prosdocimo then describes two sorts of ratios: rational and irrational.

> [A] ratio that is rational is the mutual relationship of several commensurable quantities; [a] ratio that is irrational is the mutual relationship of several incommensurable quantities. For an explanation of these two descriptions, it must be known that those quantities are said to be commensurable among which is found a common measure capable of measuring any of those quantities; those quantities are said to be incommensurable among which no such common measure is found.[474]

An example of a rational ratio would be 3:2, because both quantities share the common measure of 1. An example of an irrational ratio would be $\sqrt{2}:1$, because no common measure can divide both of these quantities evenly. Prosdocimo's definitions for rational and irrational ratios are essentially the same as those give by Oresme in Part I of *De commensurabilitate*[475] and later used for the ratio-of-ratio comparisons in Part III. In terms of basic definitions, both Oresme and

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[472] Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 48-49.

[473] In his book on arithmetic, Boethius states that musical ratios are composed of multitudes. Boethius, *De institutione arithmetica*, 1.1, p. 72-73. See also Boethius, *De institutione musica*, 2.3, pp. 53-54. Jan Herlinger points out that "discrete quantity" and "number" are used interchangeably in Prosdocimo's treatise on counterpoint. See Herlinger, notes to Prosdocimo de' Beldomandi, *Contrapunctus*, 3.6, pp. 50-51.

[474] Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 51.

[475] Oresme states, "Quantities are said to be commensurable which have some common measure, or which have a ratio of a number to a number .... Those [quantities] are incommensurable which have no common measure and do not constitute a ratio of numbers... [they are] found only in continuous quantities and never in numbers." Oresme, *De commensurabilitate*, I.20, pp. 176-177.

Prosdocimo are in agreement with Boethius.[476]

b) *Parvus tractatulus de modo monacordum dividendi* (1413)
[*A Little Treatise on the Method of Dividing the Monochord*]

The standard Pythagorean division of the monochord[477] is designed around perfect octaves (2:1), perfect fifths (3:2), and perfect fourths (4:3). These intervals are all located in their respective harmonic "sweet spots." They resound, because they are harmonically pure. These perfect intervals were then filled in with sub-intervals, the *tonus* and the semitone, to create the familiar diatonic scale: *ut (do), re, mi, fa,* etc.

C D E F G A B C - notes
T T s T T T s - intervals (*Tonus* or semitone)

Prosdocimo begins the treatise by mathematically deriving this tuning on a monochord. In addition to these natural diatonic intervals, in the Middle Ages a Bb was typically inserted between A and B to accomidate a variety of polyphonic issues. The Pythagorean diatonic scale with the addition of Bb makes up what is called *musica vera* or *musica recta*.

T T s T T T s - intervals (*Tonus* or semitone)
C D E F G A b B C - notes (b = Bb)
T T s T T s a s - intervals (T*onus*, semitone, or apotome[478])

Anachronisticly, in this system an octave would have all the white keys of a piano, but only one black key, the Bb. All of these intervals are derived, directly or indirectly, from the tetradic intervals (4:3:2:1). All are ratios of a number to a number.

Medieval parallel organum[479] and a significant portion of the repertoires of the 13th and 14th

[476] Boethius discusses commensurability and common measures in sections on prime and incomposite numbers. See Boethius, *De institutione arithmetica*, 1.17-18, pp. 92-96.
[477] There were other Pythagorean-style divisions of the monochord, but the one presented here is the one that was dominant in the Middle Ages. Jan Herlinger, "Medieval canonics," in Thomas Street Christensen, ed., *The Cambridge History of Western Music Theory* (New York: Cambridge University Press, 2001), 172-173. Other Greek tunings are discussed in Barbour, *Tuning and Temperament: A Historical Survey,* 15-24.
[478] The apotome is the difference between a *tonus* and a semitone: $9/8 \div 256/243 = 2187/2048$.

centuries were perfectly suited to these intervals.[480] The fourth, fifth, and octave were easily emphasized. However, in the 14th century, new compositional styles developed that required more semitone intervals (*mi-fa*) in between the *tonus* intervals besides the standard Bb. In keyboard terms, these styles required more black keys than just Bb. This augmented system was called *musica ficta.* The addition of a these *ficta* intervals (what we might now call accidentals) allowed the *mi-fa* interval (a diatonic semitone) to be inserted into a composition in more places, giving more flexibility to a composer or performer.[481] It was in response to these intervalic demands that Prosdocimo wrote his *Little Treatise on the Method of Dividing the Monochord.*

He explains his reason for devising this system at the very end of the treatise. He writes,

> By this method you will be able to have throughout the whole monochord two semitones between any two adjacent letters on the musical hand that sound a tone [*tonus*]. Granted that these two musica ficta notes ... rarely occur in any melody, it is nonetheless good to place them on the monochord ... in order to be able to play on this monochord any melody in which these two musica ficta notes– or either of them– might be found, if such a piece should happen to be discovered.[482]

These in-between notes are not as simple to insert as might be expected. For one thing, the *tonus*, 9:8, cannot be divided evenly in half by a ratio of two numbers, i.e., $\sqrt{9/8}$ is not rational, and thus such a division is not permitted.[483] As a result, what we now call sharps and flats did

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479 Parallel organum has two parts singing parallel to one another, a fourth, fifth, or octave apart.
480 Herlinger, "Medieval canonics," in Christensen, ed., *The Cambridge History of Western Music Theory*, 176-178.
481 Technically speaking, these additional "accidentals" allowed hexachords (T T S T T – *ut re me fa sol la*) to be moved around, allowing the *mi-fa* (semitone) interval to occur in more places. [I use the term "accidental" for convenience, but the term is not truly applicable to the medieval music.] See David E. Cohen, "Notes, Scales, and Modes in the Earlier Middle Ages," in *The Cambridge History of Western Music Theory,* ed. Thomas Street Christensen (New York: Cambridge University Press, 2001), 340-357. For more information on *musica ficta* and *musica vera* see Margaret Bent, *Counterpoint, Composition, and Musica Ficta*, 61-93; Karol Berger, "The Expanding Universe of *Musica Ficta* in Theory from 1300 to 1550," 410-413; Karol Berger, *Musica Ficta: Theories of Accidental Inflections in Vocal Polyphony from Marchetto da Padova to Gioseffo Zarlino*, 2-55.
482 Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 9.5, pp. 114-115.
483 Prosdocimo cites Boethius and Johannes de Muris when he states that the *tonus* cannot be divided into two equal parts. Ibid., 4.1, pp. 82-83. The division of the *tonus* will be addressed in more detail in the

not converge on the same pitch. G# and Ab were not the same as they are now, and composers were sensitive to this issue.[484] The interval of a *tonus*, was conceived of as containing two *musica ficta* notes, not one, as we have today.[485] The reason for these two *ficta* notes largely stems from the distinction between multitude and magnitude.

Prosdocimo's division inserts two *musica ficta* notes, a sharp and a flat, in-between each *tonus* in the *musica vera* scale. All of the sharps are built from perfect fifths as are all the flats. Essentially the flats are a major semitone lower and the sharps a major semitone higher.[486] For example, in order of ascending pitch, Prosdocimo's notes would proceed G to Ab to G# to A, with the interval of a comma[487] between the Ab and the G#.

This method of dividing the monochord does not deviate from Pythagorean harmonic techniques. All of the *ficta* intervals are still derived from perfect fifths and fourths and the intervals of *musica vera* remain intact. In terms of the quadrivium, it is important to note that like the new Hindu-Arabic numerals, Prosdocimo was not opposed to the new musical styles that required more options for *musica ficta*. His goal was to accommodate modern compositions with a quadrivially divided monochord. At one point he even says that the differences between some of his intervals "are perhaps so slight that they cannot be perceived by the sense of hearing. But it suffices that reason reveals them."[488] His primary concern was to keep music grounded in quadrivial mathematical philosophy. Human perception might not be able to discern the

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next section.

484 Jan Herlinger specifically mentions Francesco Landini (1325-1397), whose compositions distinguish between G# and Ab. See Herlinger's introduction. Ibid., 8.

485 This conception is part of why Marchetto divided the *tonus* into 5 parts. This will be discussed in the next section.

486 A major semitone is the difference between 9:8 and 256:243. 9/8 ⊖ 256/243 = 2187/2048 = major semitone. A minor semitone is 256:243.

487 The comma is 531,441:524,228. It is the difference between a major and a minor semitone.

488 Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 8.2, pp. 110-111. Herlinger points out that differences to which Prosdocimo is referring are actually discernible.

difference between two subtle shades of pitch, but that was not the point. Human perception is flawed. Prosdocimo divided the monochord so that all of the intervals were constructed from ratios of counting numbers. He responded to changing compositional trends by elaborating on Pythagorean intervals to secure a quadrivially sound foundation for modern music, in much the same spirit as he did with his algorism.

c) *Tractatus musica speculative quem Prosdocimus de Beldemando Paduanus contra Marchetum de Padua Complavit* (1425)

Marchetto of Padua was the choirmaster and a singer at the cathedral in Padua in the early 1300s.[489] He was a practicing musician, not a university magister, and probably did not have much formal scholastic education.[490] In 1317 or 1318 he wrote perhaps the most influential text on music theory of the 14th century, *Lucidarium*, *Giver of Light*.[491] His treatment of modes, his proclivity for chromaticism, and his division of the *tonus*[492] (9:8) into five parts pushed the theory of music, long dominated by quadrivial dogma, towards the actual practice of music. However, Marchetto did not present himself as an opponent to the standard quadrivial authorities of Pythagoras, Plato, Macrobius, Martianus, and Boethius. In fact, early in the text he goes to great lengths to promote himself as working from within the standards of the quadrivial music theory. Like a good Platonist, he admits that the sense of hearing can be deceived just like the sense of sight, "[as] when a straight stick is placed in water; it is perceived by the eyes as crooked."[493] From this analogy he observes that the ear can be charmed by "sound devoid of any

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[489] See Jan Herlinger's commentary in Marchetto, *The Lucidarium of Marchetto of Padua: a Critical Edition*, ed. and trans. J. W. Herlinger (Chicago: University of Chicago Press, 1985), 3.
[490] Ibid. In the opening Epistola to *Lucidarium*, Marchetto thanks Brother Syhans [Siffante] of Ferrara for helping with the organization and the "philosophical arguments." Marchetto, Epistola.6, pp. 70-71.
[491] The first printed edition of *Lucidarium* was Martin Gerbert's edition from 1784.
[492] As I have been doing throughout, for clarity, I will use the Latin term "*tonus*" in place of the English translation, "tone" or "whole tone."
[493] Marchetto, 1.4.2, pp. 82-83.

ratio or proportion."[494] But then, citing Boethius, he states that it does "not suffice to enjoy songs unless likewise the various proportions among their notes are investigated."[495] And citing Remigius of Auxerre from his commentary on Martianus Capella he writes, "Truth in music lies in the numbers of proportions."[496] And he continues, "Music is an art both admirable and delightful: it resounds in heaven and on earth. Moreover, music is the science which consists in numbers, proportions, consonances, intervals, measures, and quantities."[497]

Early in the treatise, Marchetto is clearly aligning himself with the quadrivial tradition of music. He describes the flaws of perception and then explains that the truth can be found in numbers and ratios of numbers. But when Marchetto sets up his radical idea to divide the *tonus* into five parts, he makes an interesting statement. He paraphrases from what Jan Herlinger identifies as being Thomas Aquinas' commentary on Aristotle's *Physics*. Marchetto writes,

> ...according to all philosophers and doctors in these matters, number has as its cause the division of a continuum.[498] Into however many parts the continuum can be divided, so

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[494] Ibid., 1.4.3, pp. 82-83.

[495] Ibid., 1.4.4, pp. 84-85. See also Boethius, *De institutione musica*, 1.1, p. 8.

[496] Marchetto, 1.4.5, pp. 84-85. Remigius of Auxerre (ca. 841-908) wrote a commentary on Martianus Capella's *De Nuptiis Philologiae et Mercurii et de septem Artibus liberalibus libri novem.* Martianus writes, "It was I [Harmony] who designated the numerical ratios of perceptible motions and the impulses of perfect will, introducing restraint and harmony into all things." Capella, *The Marriage of Philology and Mercury,* vol. 2, IX.922, p. 357.

[497] Marchetto, 1.5.1-3, pp. 84-85.

[498] This is conceptually similar to the description given in Plato's *Timaeus*, but what Aristotle and Thomas Aquinas call continuum or magnitude, Plato calls unity. Plato says nothing about magnitude in this section of *Timaeus*. Aristotle divides a continuum and Plato multiplies unity. It is an interesting distinction. The section from Thomas Aquinas' commentary is as follows. "For it is clear that division causes multitude. Hence the more a magnitude is divided, the greater is the multitude which arises. And thus the infinite addition of numbers follows upon the infinite division of magnitudes.... *But this number which is thus multiplied to infinity is not a number separated from the division of magnitudes*. ...the division of continuous quantity causes number, ... And this number can be multiplied to infinity just as magnitude is divisible to infinity. ...this number which is multiplied to infinity is not separated from the division of the continuous." Thomas Aquinas, *Commentary on Aristotle's "Physics,"* ed. and trans. R. J. Blackwell, R. J. Spath, and W. E. Thirlkel (Notre Dame, IN: Dumb Ox Books, 1999), 3.393-394, p. 197. Italics mine. The italicized part clearly shows that Thomas views multitude as a subset of magnitude, an idea we saw in the previous chapter on Oresme. The section of Aristotle's *Physics* (207b1-10) upon which Thomas is commenting, is much less explicit about the subservient nature of numbers. Boethius also makes a similar statement in *De institutione musica*, but in this section he does not go so far as to state that multitude is in any way prior to magnitude. Boethius, *De institutione musica*, 1.6, pp. 14-15.

many numbers can there be; and they can be augmented in the same way it can be divided. On account of this they say that because a continuum is infinitely divisible, so is number infinitely augmentable.[499]

Marchetto is describing how numbers (multitude) are generated by dividing a continuum (magnitude). Here again, we see this distinction between multitude and magnitude. Like Oresme, Marchetto appears to be claiming that arithmetical numbers are related to, if not a subset of, geometrical magnitude. Marchetto points the reader towards a very interesting idea, an idea that will ultimately be part of the solution to the problem of the restrictive Pythagorean semitones. But Marchetto does not build upon this idea in a mathematically rigorous fashion. He does not present the reader with a discussion of irrational ratios or discuss how numbers could be considered special magnitudes. His references to mathematical authority appear to be included largely for show. He does not elaborate on the mathematical philosophy to which they allude.

A brief summary of Marchetto's argument for dividing the *tonus* is as follows.[500] Marchetto begins with what appears to be tour down the legs of the Platonic Lambda, but instead of starting with what Plato would identify as unity, Marchetto identifies his source as a continuum. He defines the number 3 to be what he calls "the primary and greater division."[501] From this basis he describes the geometric series 3, 9, 27, etc. Because each set of three, when divided into three parts yields nine parts, he concludes that nine is the basic division of a continuum and that this

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However, in his *De institutione arithmetica* it is quite clear that arithmetic is prior to geometry. Boethius, *De institutione arithmetica*, 1.1, pp. 74-75.

499 Marchetto, 2.4.8-11, pp. 112-115. Cf. Boethius, *De institutione musica*, 2.3, p. 53.

500 The following summary of Marchetto's division of the *tonus* is largely derived from Jan Herlinger, "Fractional Divisions of the Whole Tone," *Music Theory Spectrum* 3, no. (1981), 74-83; Jan W. Herlinger, "Marchetto's Division of the Whole Tone," *Journal of the American Musicological Society* 34, no. 2 (1981), 193-216.

501 Marchetto, 2.4.15, pp. 116-117.

division relates to a string on a stringed instrument.[502]

In a similar trip down the other leg of the Platonic Lambda, Marchetto spins out the geometric series of the lesser primary division of the continuum based on the number 2 and arrives in similar fashion to the number 4. This he doubles for rather incoherent reasons, to get 8, which allows him to construct the *tonus*, 9:8.[503]

Marchetto's mathematical reasoning throughout this part is very confusing. Superficially it looks impressive, but it is extremely chaotic and at times simplistic in the details. He was, after all, not a quadrivial scholar. He was a singer and choirmaster, a detail that Prosdocimo was all too happy to point out in his criticism. However, it is interesting that Marchetto felt the need to couch his ideas in quadrivial terms and concepts, even if they made little mathematical sense.

Now that Marchetto has derived his 9:8 *tonus,* he is ready to make his mark on music history. He begins by claiming that the *tonus* (9:8) cannot be equally divided into an even number of parts, and, he states, "the reason for this is that [nine] is an odd number."[504] Thus, Marchetto deduces that 9 must be divisible by odd numbers only and he lays out those divisions: "1 is its first part, from 1 to 3 the second, from 3 to 5 the third, from 5 to 7 the fourth, and from 7 to 9 the fifth; and this fifth part is the fifth odd number of the whole 9."[505] The reasoning appears to be this; there are five odd numbers leading up to and including 9: 1, 3, 5, 7, and 9. Thus, he concludes, there are five parts to a *tonus*, 9:8. How these numbers and ranges (e.g., "3 to 5") are supposed to be used is not explained. As locations on a monochord, this seems to produce only 4 parts and it is clear that Marchetto wants there to be 5 semitones that "add up" to a whole *tonus*. This discrepancy may be because Marchetto tends to think of pitches, not intervals. He often

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[502] Ibid., 2.4.15-29 , pp. 116-25. His mathematical arguments are notoriously confusing and appear to be more style than substance.
[503] Ibid., 2.4.30-42 , pp. 124-131.
[504] Ibid., 2.5.12, pp. 134-135.
[505] Ibid., 2.5.14, pp. 134-135.

identifies the *tonus* as a place on the monochord, not an interval between two quantities measured on the monochord. This in indicative of his relationship to music. He is a practicing musician, making musical sounds with real, physical instruments. When he puts his finger on the fret board of a lute to make the *tonus*, he puts his finger at a place that is 1/9th of the whole string and plays a *tonus*. When a quadrivial musician puts his finger on a lute to make a *tonus*, his finger does not make the sound of a *tonus*. The quadrivial musician's finger divides a 9-unit long string at the 8th unit. The *tonus* is the interval between the whole string and 8/9 of the string. For a quadrivial musician a *tonus* is a comparison between two measurements. For Marchetto, the *tonus* is frequently referred to as place, in and of itself. He has, in a sense, forgotten that it must be compared to something else to give it any meaning.[506]

With the meaning of these five parts still unclear, Marchetto then states that "any one of these fifth parts is called a *'diesis'* – the last reduction or division, as it were."[507] Two *dieses* he calls an enharmonic semitone, three *dieses* he calls a diatonic semitone, and four *dieses* he calls a chromatic semitone. Marchetto wanted his enharmonic and diatonic semitones to be functionally equivalent to the minor and major semitones of the so-called Pythagorean system.[508] See Table 4.1.

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[506] Marchetto's confusion on this issue is not uncommon in the modern world where we have come to assume that pitches are fixed. We can play an A on a piano and leave it at that. This is not to say that we no longer think in intervals, but we have become accustomed to thinking of places, not differences or relationships. See Bent, *Counterpoint, Composition, and Musica Ficta*, 145. A similar issue exists in regards to our modern conception of numbers. We tend to think of the number 3 without regard to its relationship to another quantity, three times unity. Roman numerals make this explicit, e.g., III. Hindu-Arabic numerals tend to obscure the inherent quadrivial relationships. A number line (analogous to a monochord) calibrated to zero is generally assumed, but rarely acknowledged.
[507] Marchetto, 2.5.23, pp. 138-139.
[508] Ibid., 2.7.2, pp. 144-145.

Marchetto's interval		Pythagorean equivalent	cents[509]
enharmonic semitone = "2/5" of a *tonus*	≈	minor semitone = 256:243 = $(4:3)\div(9:8)^2$	≅ 90
diatonic semitone = "3/5" of a *tonus*	≈	major semitone = 2187:2048 = (9:8)÷(256:243)	≅114

Table 4.1: Marchetto's Semitones Compared to Pythagorean Semitones.[510]

Using these new definitions for semitones, Marchetto goes on to describe how these can be used in practice. Only after this discussion does he return to put some numbers onto his semitones, where, again, his methods are mathematically unorthodox. Realizing (for the wrong quadrivial reasons) that the 9:8 interval cannot be halved, he doubles both parts of the ratio to arrive at 18:16. He then states that the number 17 is between 18 and 16. Thus he arrives at 17:16 for his diatonic (major) semitone made up of 3 *dieses*. In a similarly unorthodox move, he claims 18:17[511] as his enharmonic (minor) semitone, made up of 2 *dieses*.[512] Both of these

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[509] Modern equal temperament tuning is typically measured using a logarithmic system called cents. A modern equal-tempered half-step is 100 cents. Thus there are 1200 cents in an octave. Table 4.1 shows that the Pythagorean minor semitone is significantly smaller than the chromatic half-step on the modern piano. This measuring system does what one naturally wants to do with intervals. You can add two half-steps to get a whole step. E.g., A fifth plus a fourth is 700 cents + 500 cents = 1200 cents, an octave. This cannot be done with monochord ratios; they are multiplied not added. E.g., A fifth plus a fourth is 3:2 x 4:3 = 2:1. The "cent" approach would lend itself well to Marchetto's system, but this logarithmic system had not yet been developed. In the cent system, if one wants to divide an interval into fifths, all you have to do is divide it by 5. E.g., 200 cents ÷ 5 = 40 cents. However, this is terribly anachronistic. In slightly less anachronistic terms, what Marchetto wanted to do was to solve this equation: $x = \sqrt[5]{9/8}$.

[510] For a much more detailed discussion on tuning systems, see Barbour's *Tuning and Temperament*.

[511] Probably by coincidence, Vincenzo Galilei would later promote the 18:17 interval as a very good approximation to equal temperament for tuning a lute. Vincenzo Galilei, *Dialogo della musica antica et della moderna* (Florence: Giorgio Marescotti, 1581), 49.

[512] Marchetto, 2.9.5-12, pp. 158-161. He does not explicitly point out that 18:17 x 17:16 = 9:8, which would have been a coup for his analysis. I suspect he did not point this out because he did not think in these terms. Marchetto does not multiply ratios together when he wants to go up or go down. In his later descriptions of the consonant intervals he describes the traditional Pythagorean ratios in numerical terms (2:1, 3:2, 4:3, etc) but does not build ratios from combinations of ratios. E.g., 3:2 x 4:3 = 2:1. All of these intervals he builds from whole tones (*toni*) and minor semitones. He builds them additively, not multiplicatively. He does not think in terms of ratios and comparative terms. He seems to think in terms of locations on a string, rather than ratios of quantities of string-lengths. For a practicing musician, not a quadrivial music theorist, it may have seemed counterintuitive to multiply intervals to sing from *mi* to *fa* rather than simply add something to *mi*. See Marchetto, 3.1.1-11, pp. 168-181. See also Herlinger, "Marchetto's Division of the Whole Tone," 208.

intervals happen to be superparticular ratios, which gives them a semblance of Pythagorean credibility, but they only approximately relate to the *tonus* in the way that Marchetto claims.[513]

Overall, only a small part of *Lucidarium* is devoted to these pseudo-mathematical discussions. The greater part is about the practice of music: notation, chromatic progressions, the use of his semitones in *musica ficta*, basic counterpoint, modes, accidentals, etc.[514] In these more practical matters, Marchetto is completely competent, if not masterful. Jan Herlinger believes that Marchetto's division of the *tonus* into five parts, clothed though it may be in Pythagorean garb, is a qualitative, not quantitative, description. He is ultimately speaking in approximate terms and the elaborate pseudo-mathematical derivations were added to give his ideas some authoritative legitimacy. After all, in practice, what is the difference between Marchetto's qualitative enharmonic semitone (18:17 or "2/5" of a *tonus*) and the Pythagorean minor semitone = 256:243? The anachronistic answer: about 9 cents. See Table 4.2.

Parts of *tonus*	Modern method $x = \sqrt[5]{9/8}$	Marchetto's Rational Ratios[515]	Pythagorean Ratios
0/5	1.0 = 0 cents		open string
1/5	1.0238... ≈ 40.720 cents	no ratios given	n.a.
2/5	1.0482... ≈ 81.497 cents	18/17= 1.0588... ≈ 98.9546 cents	256/243= 1.0534... ≈90.225 cents
3/5	1.0732... ≈122.303 cents	17/16 = 1.0625 ≈ 104.9554 cents	2187/2048= 1.0678... ≈113.685 cents
4/5	1.0988... ≈163.115 cents	no ratios given	n.a.
5/5	1.125 ≈203.910 cents	9/8	9/8

Table 4.2: Modern Method of Dividing the *Tonus* into 5 Equal Parts
This is compared to Marchetto's and Pythagorean ratios. (Even for a trained ear it is difficult to distinguish the difference between two tones separated by 10 or fewer cents played sequentially.)

What Marchetto did was divide qualitatively the *tonus* into 5 equal parts. He seemed to be grasping at a mathematical solution to this quadrivial paradox when he paraphrased Thomas

513 The minor semitone is too sharp and the major semitone is too flat. See Marchetto, 3.1.1-11, pp. 168-181.
514 Herlinger, "Marchetto's Division of the Whole Tone," 212-213, n46.
515 Marchetto only provides two subdivisions in mathematical ratios.

Aquinas, stating that "number has as its cause the division of a continuum,"[516] but his mathematical methods were not up to the task. Just because he could not make a mathematical argument, however, did not mean that his qualitative idea was wrong. Although his implied mathematics would clearly lead to ratios involving irrational magnitudes, Marchetto's general idea of dividing the *tonus* equally was popular amongst subsequent musicians including Tinctoris, Vicentino, and to some extent, Gaffurius.[517] It was not popular with Prosdocimo, a quadrivial musician.

A century after *Lucidarium* was written, Prosdocimo answered what he saw as its "evils, lies, and mistakes concerning music"[518] in his work, *Tractatus musice speculative* (or *Musica speculativa*).[519] This treatise was written in 1425, a few years before Prosdocimo's death. Although it is an explicit attack on Marchetto's "theoretical or speculative aspects of theory,"[520] in the preface Prosdocimo praises Marchetto's discussion of plainchant, writing, "I have thus far seen nothing more orthodox in what I have read on this subject."[521] In fact, much of Prosdocimo's theory of mode in his *Tractatus plane musice* of 1412 is borrowed from the *Lucidarium*.[522] It was Marchetto's theory of tuning that so angered Prosdocimo. He writes, "In the science of music, you see, this man [Marchetto] was a simple performer, totally lacking in [music's] theoretical or speculative side, which, mistakenly, he thought he understood most perfectly."[523] This rather contemptuous attitude towards the performer has a long history in

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516 Marchetto, 2.4.8-11, pp. 112-115.
517 Herlinger, "Marchetto's Division of the Whole Tone," 215, n49.
518 Prosdocimo de' Beldomandi, *Musica speculativa*, Preface, pp. 158-159.
519 The full title is, *Tractatus musice speculative quem Prosdocimus de Beldemando paduanus contra Marchetum de Padua complavit* [*Treatise on Speculative Music, which the Paduan Prosdocimus de Beldemandis Compiled Against Marchettus of Padua*].
520 Prosdocimo de' Beldomandi, *Musica speculativa*, Preface, pp. 156-159.
521 Ibid., Preface, pp. 158-159.
522 See Herlinger's introduction. Ibid., 7.
523 Ibid., Preface, pp. 158-159.

music.

Jan Herlinger points out that Guido of Arezzo, the great music theorist from the 11th century, wrote, "There is a great difference between musicians and singers. These [merely] perform; those know what music is. And he who sings what he does not understand is considered an animal."[524] Boethius saw a similar distinction between performer and theorist. He writes,

> A musician is one who has gained knowledge of making music by weighing with the reason, not through the servitude of work, but through the sovereignty of speculation. ... [Those] such as kitharists and those who prove their skill on the organ and other musical instruments– are excluded from comprehension of musical knowledge, since, ... , they act as slaves. None of them makes use of reason; rather, they are totally lacking in thought.[525]

This speculative or theoretical music is the basis of quadrivial music. This is the music based on numbers. These are the numbers that exist in Plato's world of forms and light, or as rational thoughts in the mind. Speculative music reflects the structure of reality. A song is just a shadow. From the scholarly quadrivial point of view, performed music ought to be derived from mathematical music, but in and of itself it is not on the same level as theoretical quadrivial music. Describing the medieval attitude, Edward Lowinsky wrote, "only a theorist is a true musician."[526] It is from within this tradition that Prosdocimo describes Marchetto as a "simple performer."

Prosdocimo ends his preface to *Musica speculativa* by stating his main purpose. He does not intend to write on all aspects of practical and speculative music, for he tells the reader that he and others have already written extensively on these topics. He writes, "I shall touch only on some things that seem to me necessary for the explanation of the errors of the aforesaid Marchetto,

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[524] Translation by Edward Lowinsky. Edward E. Lowinsky, "Renaissance Writings on Music Theory (1964)," *Renaissance News* 18, no. 4 (1965): 363.
[525] Boethius, *De institutione musica*, 1.34, p. 51.
[526] Lowinsky makes this statement when describing medieval attitudes. He is not endorsing this idea. Lowinsky: 363.

adding to these the methods by which anyone learned in ratios and the practice of arithmetic might be able to discover and recognize the ratio of any interval."[527] The treatise that follows is in every respect a defense of the first two quadrivial disciplines: arithmetic and music. It is about counting numbers and ratios made from counting numbers.

Prosdocimo sets about to defend what he describes as "pitch related to pitch," and, he states, this cannot be understood "without knowledge of ratios and numbers."[528] He lists sixteen basic intervals that are generally used in the practice of music and found on the musical hand.[529] These are the intervals of *musica vera*. See Figure 4.3 and Table 4.3.

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[527] Prosdocimo de' Beldomandi, *Musica speculativa*, Preface, 158-159.
[528] Ibid., 1.1, pp. 160-161.
[529] He later explains that there are more intervals (*musica ficta*) between the unison and the octave, but that they do not exist on the musical hand. However, they can be mathematically described. Examples of these include the two-semitone "third," the three-semitone "fourth," and the four-semitone "fifth." See also Karol Berger, "The Hand and the Art of Memory," *Musica Disciplina* 35, no. (1981), 87-120. This article is mistakenly credited to and catalogued under the name "Carol Berger."

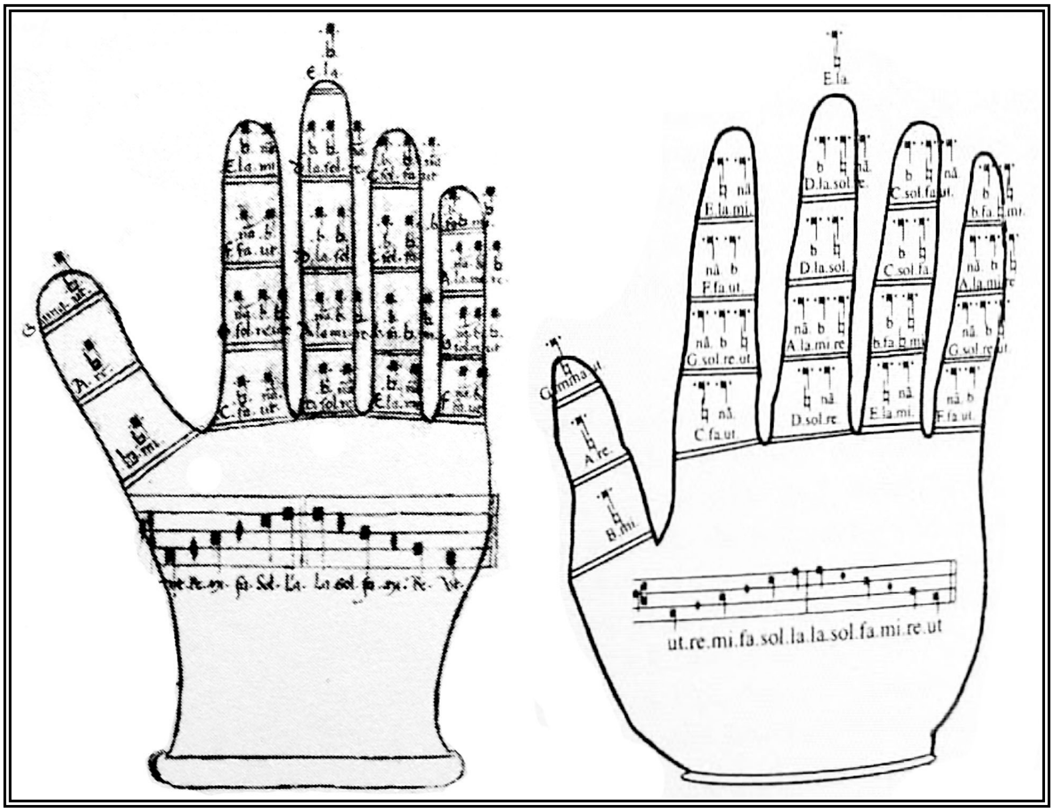

Figure 4.3: The Musical Hand
The hand on the left is from a manuscript of "Musica plana" from 1437
[MS A.56 (Martini, 4) Civico Museo Bibliografico Musicale, Bologna, p. 115].
The hand on the right is a drawing based on the MS from Jan Herlinger's translation.[530]

530 Prosdocimo de' Beldomandi, *Musica speculativa*, 71 and 254.

Interval, Synonym, Example	Intervallic Composition	Intervallic Ratios
1. unison, "one pitch"		1:1
2. *tonus*, "major second," e.g., G to A		**9:8**
3. semitone, "minor second," e.g., B to C		256:243
4. ditone, "major third," e.g., G to B	2 *toni*	81:64
5. semiditone, "minor third," e.g., A to C	*tonus* and semitone	32:27
6. tritone, "major fourth," e.g., F to B	3 *toni*	729:512
7. semitritone, "minor fourth," e.g., G to C. Also called the diatessaron	**2 *toni* and semitone**	**4:3**
8. diatessaron with tone, "major fifth," e.g., G to D. Also called diapente.	**3 *toni* and semitone**	**3:2**
9. diatessaron with semitone, "minor fifth," e.g., B to F	2 *toni* and 2 semitones	1024:729
10. diapente with tone, "major sixth," e.g., G to E	4 *toni* and semitone	27:16
11. diapente with semitone, "minor sixth," e.g., B to G	3 *toni* and 2 semitones	128:81
12. diapente with ditone, "major seventh," e.g., C to B	5 *toni* and semitone	243:128
13. diapente with semiditone, "minor seventh," e.g., B to A	4 *toni* and 2 semitones	16:9
14. diapente with tritone, "major octave," e.g., Bb to B (B-fa to B-mi)	(octave and a semitone)	2187:1024
15. diapente with diatessaron, "medial octave," e.g., G to G. Also called diapason.	**diapente and diatessaron (octave)**	**2:1**
16. double diatessaron with semitone, "minor octave," e.g., B to Bb (B to B-fa)	2 diatessaron and semitone	4096:2187

Table 4.3: Prosdocimo's Sixteen Principal Intervals[531]
*Intervals in **bold** are the "fundamental" ratios of Pythagoras.*

Prosdocimo then briefly explains that Pythagoras first discovered the primary ratios of music by observation, not by demonstration or argument. This point he makes twice, perhaps to emphasize that these ratios are natural phenomena, not arbitrary numerological axioms. For the traditional story of Pythagoras and the hammers, he cites Boethius, Macrobius, Johannes de Muris,[532] and suggests that there are many other sources in addition to these three. [See introduction.] Prosdocimo points out that the four principle ratios discovered by Pythagoras are the *tonus* (*sesquioctave*), the diatessaron (*sesquitertial*), the diapente (*sesquialter*), and the

[531] This table was constructed from Books 1 and 2 of *Musica speculativa*. Ibid., 1.2-2.13, pp. 160-193.
[532] Boethius, *De institutione musica*, 1.10-11, pp. 17-19; Johannes de Muris, *Die Musica speculativa des Johannes de Muris: Kommentar zur Überlieferung und kritische Edition*, ed. Christoph Falkenroth (Stuttgart: Franz Steiner Verlag, 1992), 92-108; Macrobius, 2.I.9-14, pp. 186-188.

diapason (*duple*), 9:8, 4:3, 3:2, and 2:1 respectively. He then describes how to "add" and "subtract" one ratio from another so that all of the intervals in Table 4.3 can be described mathematically. For "adding" intervals he writes,

> So that you grasp the method of adding ratios to each other, note this rule: if you intend to add some ratio to another ratio, take both those ratios in their least numbers and multiply the greater number of one of them by the greater of the other and the lesser by the lesser, and the ratio of the products of these two multiplications will be the ratio of the aggregate of the two aforesaid ratios.[533]

His example for this procedure is to "add"[534] two *toni* together to produce the ditone.

$$9{:}8 \oplus 9{:}8 =$$

$$9/8 \times 9/8 = 81/64$$

To "subtract" intervals he writes,

> If you intend to subtract some ratio from another ratio from which it can be subtracted, take those ratios in their least numbers, multiply the greater number of each of the ratios by the lesser of the other, and the ratio of the products of these two multiplications will be the ratio remaining when the subtraction has been made.[535]

One of his examples "subtracts" the ditone from the diatessaron to get to the semitone.

$$4{:}3 \ominus 81{:}64 =$$

$$4/3 \div 81/64 =$$

$$4/3 \times 64/81 = 256/243$$

Prosdocimo goes on to find all of the ratios for the sixteen intervals he described in the beginning. See Table 4.3. Although Prosdocimo does not mention Marchetto in this long and tedious section, it is clear that he is giving the long dead Marchetto a lesson in basic quadrivial music theory. Marchetto was clearly confused about what it meant to "add" or "subtract"

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[533] Prosdocimo de' Beldomandi, *Musica speculativa*, 2.3, pp. 182-183.
[534] The two special operational symbols, $\oplus$ and $\ominus$, are meant to suggest that they are not really addition and subtraction, even though that is how we tend to talk about these operations. They are actually multiplication and division.
[535] Prosdocimo de' Beldomandi, *Musica speculativa*, 2.3, pp. 180-181.

intervals from one another, and Prosdocimo is giving him a thorough exposition on the topic. It is interesting to note that Prosdocimo uses the terms "add" (*addere*) and "subtract" (*subtrahere*) even though the mathematical operations he describes are multiplication (*multiplicare*) and division (*multiplicare* by reciprocal). Given that Marchetto was not well educated, it makes perfect sense that he botched the mathematics. The terminology is confusing. Marchetto appears to think of pitches as fixed places in some sort of absolute pitch-space. For him, a pitch can exist on its own. A singer can sing a single note. It clearly exists. It can be heard. Prosdocimo, on the other hand, does not think in terms of pitch. His quadrivial training has taught him to divide the sensory world of change out of his rational theory of music leaving only numbers. He thinks in terms of interval, the relationship *between* pitches. For Prosdocimo and quadrivial ontology, pitch has no musical existence except in realtion to another pitch. A diapente (3:2) is always a diapente no matter what pitch initiates the interval.[536] If the relationships are both 3:2, the intervals are equal. The pitches themselves are just shadows in the world of change. They deceive. Prosdocimo adamantly emphasizes this point, writing, "One interval is not said to be equal or unequal to another except according to whether their ratios are found to be equal or unequal to one another."[537] Prosdocimo and Marchetto are not talking about the same thing. It may be that what makes Marchetto's ideas so attractive is that they are based on experience and common sense. His mistake, from Prosdocimo's point of view, was to try to make his ideas look mathematical. Prosdocimo is talking about metaphysics. He never once refers to an instrument or a musical sound in his long and involved mathematical derivations of musical intervals. Prosdocimo is discussing speculative music. He is talking about mathematics. Marchetto is discussing what he hears.

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[536] Ibid., 3.7, pp. 226-227.
[537] Ibid., 2.1, pp. 178-179.

Following Prosdocimo's lengthy tutorial on "adding" and "subtracting" intervals, he finally gets to his main argument. He writes,

> The tone [*tonus*] ... is not divisible into equal parts in any way: inasmuch as it is not divisible into two halves nor into three thirds, four fourths, five fifths, six sixths, and so forth; for no superparticular ratio is divisible into equal parts. Wherefore, neither is the sesquioctave ratio [9:8], nor, consequently, the tone [*tonus*], which consists in this sesquioctave ratio.[538]

Marchetto claimed to have divided the *tonus*, a superparticular ratio, into five parts. Prosdocimo systematically attacks the idea that 9:8 could be divided into equal parts. The Euclidian proof is rather lengthy,[539] and Prosdocimo's presentation is not rigorous. An algebraic example can illustrate this point.[540]

Assume that there is a mean, *x*, that divides the 9:8 ratio into two equal parts. If this were the case, then

$$\frac{9}{x} = \frac{x}{8}.$$

Thus

$$x^2 = 9 \times 8$$

$$x = \left(\sqrt{9}\right)\left(\sqrt{8}\right) = 6(\sqrt{2}).$$

The mean, *x*, for any superparticular ratio has to be irrational, because no two consecutive counting numbers can both have rational square roots. A similar argument extends this irrational result to any number of equal divisions of a superparticular ratio. Prosdocimo points out that this division is the geometric mean,[541] the only mean that divides ratios into equal parts. The issue is not so much that a superparticular ratio cannot be divided into equal parts, but that it cannot be

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[538] Ibid., 2.15, pp. 194-195.

[539] Boethius attributes this proof to the Pythagorean Archytas of Tarentum (4th century B.C.). Archytas is credited by some as the founder of what would later be called the quadrivium. This proof is in the *Euclidean Sectio Canonis*. See Euclid (school of), *Sectio Canonis,* Prop. 3, p. 195. See also Boethius, *De institutione musica*, 3.11, pp. 103-105 and 4.2.iii, p. 118; Euclid, *Euclid's Elements,* VIII.8, pp. 357-358.

[540] This algebraic example is derived from Andrew Barker's footnote to Proposition 3 of Pseudo-Euclid's *Sectio Canonis.* Euclid (school of), *Sectio Canonis,* 195, n12.

[541] Prosdocimo de' Beldomandi, *Musica speculativa*, 2.15, pp. 196-197.

divided into parts that can be represented as a ratio of numbers. The geometric mean produces irrational magnitudes from superparticular ratios, not whole numbers, hence the name "geometric" mean.[542] The issue is, once again, the distinction between multitude and magnitude, arithmetic and geometry. Speculative music is quadrivial music, and quadrivial music is born out of arithmetic, not geometry.

Prosdocimo, again not identifying Marchetto, then attacks the idea that the 9:8 ratio can be divided into two equal parts by rewriting it as 18:16 and interposing an arithmetical mean between the two, i.e., 18:17:16. "It will appear clearly," he writes, "that the two ratios produced by those three numbers are not equal to each other because they differ in name and consequently in nature."[543] Obviously 18:17 is not equivalent to 17:16, and thus this division does not yield two equal parts. Similarly, arithmetically dividing 27:24 into 27:26:25:24 will not yield three equal parts and by extension 45:40 cannot be divided using this method to produce five equal parts.

It is possible that Marchetto is not identified in this passage, because Marchetto did not claim that his 18:17:16 proportion equally divided the *tonus*. Prosdocimo may simply be explaining more clearly what Marchetto did when he divided the *tonus* using this method.

Although *Musica speculativa* is driven by Prosdocimo's need to correct the "evils, lies, and mistakes"[544] propagated by Marchetto's division of the *tonus*, it also contains a wealth of interesting mathematical techniques. In a brief discussion on why the traditional Pythagorean minor semitone does not divide the *tonus* evenly, he shows how to evaluate the relative sizes of two ratios. He writes, "Let two such semitones [minor semitones] or their ratios be added

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542 The geometric mean can function in musical intervals for some non-superparticular ratios, such as those found on the legs of the Platonic Lambda, which are generated from a geometric progression.
543 Prosdocimo de' Beldomandi, *Musica speculativa*, 2.15, pp. 198-199.
544 Ibid., Preface, pp. 158-159.

[*addantur*] together, and the ratio will be produced that is,"[545] 65536:59049. But how does this compare with 9:8, the *tonus*? Is it larger or smaller? He answers this question,

> If there should be two ratios of distinct species and you should wish to know which of them is greater, take them in their least terms; multiply the greater term of one by the lesser term of the other, as you do in the subtraction of one ratio from another; and that one of the two ratios from which the greater number is produced by multiplication is the greater term by the lesser term of the other is said to be the greater ratio.[546]

For example, is 65536:59049 greater than, lesser than, or equal to 9:8? Following Prosdocimo's instructions,

$$65536 \times 8 = 524{,}288$$

and

$$9 \times 59049 = 531{,}441.$$

The product created by the numerator of 65536/59049 is smaller than the product produced by numerator of 9/8. Thus 65536:59049 < 9:8 and two minor semitones are a smaller interval than a *tonus*.[547]

Another useful technique explained by Prosdocimo is how to determine whether or not a given ratio is written in its lowest terms. This is frequently a condition for any number of other mathematical techniques, but instructions on how to know if a ratio is in its lowest terms is generally lacking. The general idea is this. Divide the smaller number of a ratio into the larger number. Then divide whatever remains into the divisor (the smaller number). Take what remains and divide that into the divisor (the first remainder). Do this until the remainder is either 0 or 1. If it is 0, then the ratio is not in its lowest terms. If the remainder is 1, it is in its lowest

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[545] Ibid., 2.16, pp. 200-203. Again, adding intervals is multiplying ratios. Is it any wonder Marchetto was confused?

[546] Ibid., 2.16, pp. 202-203.

[547] Ibid., 2.16, pp. 204-205. This method is essentially the same as making ratios of the ratios. E.g. $\frac{3/2}{4/3} = \frac{9}{8}$. Thus 3:2 is larger than 4:3.

terms.[548]

For example, is 212:79 in its lowest terms?

$$79\overline{)212}^{\,2} \rightarrow \text{remainder of 54.}$$

$$54\overline{)79}^{\,1} \rightarrow \text{remainder of 25}$$

$$25\overline{)54}^{\,2} \rightarrow \text{remainder of 4}$$

$$4\overline{)25}^{\,6} \rightarrow \text{remainder of 1.}$$

Therefore, 212:79 was written in its lowest terms.

An example of a ratio that is not in its lowest terms– 51:39.

$$39\overline{)51}^{\,1} \rightarrow \text{remainder of 12}$$

$$12\overline{)39}^{\,3} \rightarrow \text{remainder of 3}$$

$$3\overline{)12}^{\,4} \rightarrow \text{remainder of 0.}$$

Thus, 51:39 is not in its lowest form. Prosdocimo then explains that the penultimate remainder is the number by which the numbers in the initial ratio should be divided to put it in its lowest form.[549]

For example, 51/3 : 39/3 = 17:13.

Prosdocimo then goes on to use these techniques to derive and analyze the *tonus*, the Pythagorean major and minor semitones, and the Pythagorean comma (the difference between the major and minor semitones). This is all done using quadrivial arithmetic and music– no magnitudes of geometry anywhere. Every musical interval listed in Table 4.3 is constructed

548 Ibid., 2.16, pp. 202-203.
549 Ibid., 2.16, pp. 204-205.

from the primary Pythagorean ratios: 2:1, 3:2, 4:3, 9:8. In fact, only 3:2 and 4:3 are needed to deduce the rest of the system, from the comma to multiple octaves.[550] These Pythagorean intervals are equivalent to the definitions of Euclidean geometry, except as Prosdocimo emphasizes, the primary intervals were discovered through observation. He implies that these ratios are part of the structure of nature.

The rest of *Musica speculativa* continues to criticize various aspects of Marchetto's theory. He complains about the use of various *musica ficta* symbols (sharps, flats, etc.).[551] He complains about how Marchetto misuses such terms as diesis, enharmonic, chromatic, and diatonic.[552] He corrects Marchetto in his claim that Pythagoras "was the first discoverer of music."[553] He criticizes Marchetto's incoherent division of the continuum into powers of 2 and 3 and the way Marchetto speaks of intervals as numbers rather than ratios.[554] In analyzing Marchetto's pseudo-derivation of the numbers that form the *tonus*, 9:8, Prosdocimo is at his wits' end, writing, "But this reasoning is no more valid than this would be: a man is an ass and a goat is a lion; therefore, God exists."[555] Prosdocimo cannot understand why Marchetto chose five divisions for the *tonus* and not some other number. It all seems very arbitrary to him.[556] And finally Prosdocimo criticizes Marchetto for a variety errors in his use of technical terminology.[557] One can almost hear Prosdocimo slam his codex shut after he writes the obligatory, "Here ends the treatise of speculative music that the Paduan Prosdocimus de Beldemando compiled against Marchetus of

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[550] The system could also be deduced from 2:1 and 3:2 or 2:1 and 4:3. No number higher than 4 is required.
[551] Prosdocimo de' Beldomandi, *Musica speculativa*, 2.22, pp. 210-213 and 3.12, pp. 240-241.
[552] Ibid., 2.22, pp. 214-215 and 3.3, pp. 220-225.
[553] Ibid., 3.2, pp. 220-221.
[554] Ibid., 3.8-9, pp. 226-237.
[555] Ibid., 3.9, pp. 234-235.
[556] Ibid., 3.10, pp. 236-239.
[557] Ibid., 3.13-17, pp. 244-253.

Padua. Thanks be to God. Amen."[558]

In the two quadrivial disciplines that deal exclusively with multitude, arithmetic and music, Prosdocimo is very consistent. He never deviates from the strict quadrivial conception of number. His general attitude is best summed up by his statement from *Brevis summula proportionum*, "a continuous quantity cannot properly be taken in a ratio to a discrete quantity since, ... , they are not of the same proximate genus but of ones quite remote."[559] For Prosdocimo, multitude is not comparable to magnitude. They are simply not the same thing– apples and oranges. Throughout the arithmetical and musical texts there is hardly any mention of the existence of the other quadrivial disciplines– no quadrivial progression from arithmetic to astrology. He only mentions the magnitudes of geometry when he sees them as a threat to arithmetic or music.

Prosdocimo's main sources are the quadrivial classics– the two mathematical texts of Boethius. Some of the other standard authors, like Aristotle, Euclid, Cicero, and Macrobius, are only occasionally referred to or implied, and Plato is altogether absent. In fact, in the musical and arithmetical works there is very little hint of Platonism or Neoplatonism. Even Boethius' *musica humana* and *mundana* are absent. In these arithmetical and musical texts he never mentions a musical cosmos or the harmonic structure of the world soul or the healing powers of music or even the Platonic Lambda, which is nearly ubiquitous in any text discussing numbers from this period. It is as if he stripped the more fanciful influences of Plato out of Boethius and just left the Pythagorean number theory.[560] But this is not to say that Prosdocimo did not write on the quadrivial disciplines that deal with magnitude.

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[558] Ibid., 3.19, pp. 252-253.
[559] Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 48-49.
[560] He refers to Pythagoras and the hammers on numerous occasions.

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Prosdocimo's Geometry[561]
[*De parallelogramo, ca. 1410*]

Prosdocimo's lone work in pure geometry fits on one side of a sheet of paper.[562] The problem, Prosdocimo states, "is to describe a parallelogram equal to a given triangle."[563] Using a variety of propositions from Book I of Euclid's *Elements*[564] he proves

> how every latitude uniformly difform beginning from zero degree and continually increasing uniformly until terminated at a certain degree corresponds to its middle degree. And this [is evident] at least in the orthogonal triangular figure when we have bisected the line denoting to us the most intense [degree of] quality and have also bisected similarly the line denoting to us uniform intensity over the subject.[565]

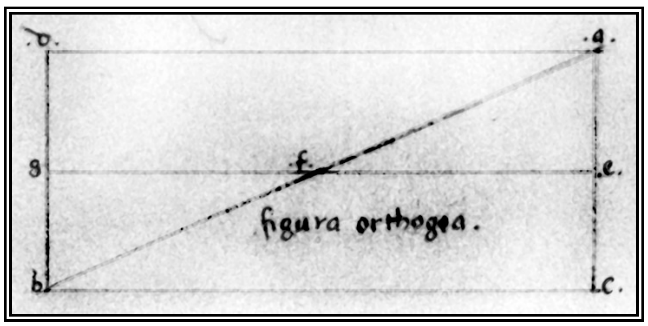

Figure 4.4: Uniformly Difform Latitude - Orthogonal Triangle Example
Detail from Codex A.56, f73r, Civico Museo Bibliografico Musicale of Bologna

In modern terms he is saying that the area of triangle *abc* in Figure 4.4, is equal to the area of rectangle *bceg*, whose altitude is exactly half of line *ca*. The proof shows that triangle *aef* is equal to triangle *bgf*.[566] In and of itself this is not an interesting piece of geometry, but the

[561] For Favaro's description and complete transcription of Prosdocimo's geometrical text, see Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 129-132.
[562] The only example of this work is found in Bologna, Civico Museo Bibliografico Musicale, A.56 (Martini, 4), 73r.
[563] Marshall Clagett has analyzed this short work and transcribed it fully. The translations presented here are from his essay. Marshall Clagett, "Prosdocimus de Beldomandis and Nicole Oresme's Proof of the Merton Rule of Uniformly Difform," *Isis* 60, no. 2 (1969): 223.
[564] Prosdocimo cites propositions 31, 2, 33, 10, 39, 34, 4, and 29, in that order.
[565] Clagett, "Prosdocimus de Beldomandis and Nicole Oresme's Proof of the Merton Rule of Uniformly Difform," 224. I made minor changes to this translation to better reflect the Latin.
[566] This is further generalized to non-orthogonal triangles.

terminology Prosdocimo uses to describe these triangles and quadrilaterals makes it clear that he is referring to what is now called the *Merton Rule of Uniformly Difform Qualities* that was famously described by Oresme in the 14th century. This particular proof was Oresme's approach to uniformly difform qualities involving the use of coordinate graphing, a significant development in mathematical physics.[567] Difform qualities are qualities that change. Uniformly difform qualities are qualities that change uniformly. The standard example is that of a uniform acceleration like gravity.[568] In this context the uniformly difform quality is speed, specifically, a constant acceleration, and the theory is given the name the "mean-speed theorem."

Oresme graphically geometricized this idea by plotting the quantity of the quality of speed vertically and the quantity of elapsed time horizontally. For example, a uniform speed would be graphed as a rectangle. The height of the rectangle is the speed and the length of the rectangle is the elapsed time. Using the example from Figure 4.4, the rectangle *bceg* could be seen to show a steady speed always at the altitude of *bg*, over the duration of time *bc*. The beauty of this system is that the area of the rectangle is the distance traveled. Speed multiplied by time equals distance. This was a significant improvement over the approach taken by Oresme's English predecessors, who were thinking in terms of speed vs. distance. The product of these variables did not result in anything measurable.[569]

A uniform acceleration starting with a speed of zero at *b*, continuing for some specified duration, *bc*, with a final speed of *a*, would be graphed as the triangle *abc* in Figure 4.4. The area of this triangle would be the distance traveled. The mean-speed theorem proves that a point with

[567] A collection of Oresme's writings on this topic along with commentary by Marshall Clagett are included in Grant, ed., *A Source Book in Medieval Science*, 243-253. A description of the precursors to Oresme is found in John E. Murdoch and Edith D. Sylla, "The Science of Motion," in *Science in the Middle Ages,* ed. David C. Lindberg (Chicago: University of Chicago Press, 1978), 206-264.
[568] Uniform gravity assumes small enough changes in altitude such that gravity remains effectively constant. This anachronistic qualification would have made little sense in the Middle Ages.
[569] Murdoch and Sylla, "The Science of Motion," 238-239.

the mean speed *g*, would cover the same distance as a point constantly accelerating as the line *ba*. The rectangle formed by this constant speed of *g* would have the same area as the triangle with the uniformly accelerated speed.[570]

Prosdocimo is clearly referring to uniformly difform qualities, but he does not specifically address the example of speeds and times. What this says about Prosdocimo was that he was aware of the exciting developments going on in Paris and Oxford. It is highly likely that these developments were brought to Padua by Blasius of Parma, who was teaching at the University of Padua and was one of Prosdocimo's examiners the year before this short work was composed.[571] Blasius' *Questiones de latitudinibus formarum* contains this very proof and it seems likely that this text, or Blasius himself, was Prosdocimo's source.[572]

The geometrical skills demonstrated by Prosdocimo in this short proof are very basic. All references to Euclid are from Book I. However this particular proof does not demand any complicated geometrical skill. That is part of its beauty. What makes this proof significant is its explicit applicability to real world problems, the way the calculated areas are measurable quantities, and use of novel graphical techniques. Prosdocimo chose a particularly significant piece of geometry to leave as his one lone example in that discipline.

In terms of the quadrivium, this short piece of geometry shows that Prosdocimo was interested in cutting-edge mathematical natural philosophy. The mean-speed theorem is seen by many as the birth of modern mathematical kinematics. It connects Euclid's geometry to motion. Two centuries later, Galileo would use Euclidean geometry to describe all sorts of motions,

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[570] The significance of this equivalence was not emphasized by Oresme, but it became significant for subsequent natural philosophers, namely Galileo. See Clagett's commentary in n24 in Grant, ed., *A Source Book in Medieval Science*, 252.
[571] Lynn Thorndike, *A History of Magic and Experimental Science: Fourteenth and Fifteenth Centuries*, 8 vols., vol. 4 (New York: Columbia University Press, 1934), 69.
[572] Grant, "Blasius of Parma."

including the motions of pendula and gravitational freefall. It is not clear from this short work how Prosdocimo imagined uniformly difform qualities would be applied, but the very vocabulary used in describing them implies terrestrial application, not just abstract geometry. But, all Prosdocimo left behind was his proof of the abstract geometry.[573]

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Prosdocimo's Astronomy[574]

The two longest, original works on astronomy written by Prosdocimo are his *Commentum sphaerae* (1418)[575] [*Commentary on Sacrobosco's Sphere*] and his C*anones de motibus corporum supercoelestium* (1424) [*Canons on the Motions of the Heavens*] and its associated tables. Judging from his student notebooks, referred to above, he was well read in astrology, but his mature works are largely technical in nature, more what we would now tend to call mathematical astronomy. The astronomy of Prosdocimo was geocentric and geostatic, as was typical in his day. It was based on Aristotle's general conception of concentric spheres and mathematically described using the Ptolemaic theory. The basic medieval treatise explaining this theory was the early 13th-century text, Sacrobosco's *Sphere*.[576] This was the ubiquitous undergraduate-level astronomy textbook[577] that described all of the Aristotelian-Ptolemaic

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573 I cannot help but think that this proof was either copied from a lecture or was some sort of scholastic assignment.
574 For Favaro's description of Prosdocimo's astronomical texts and related material, see Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 133-183.
575 There is some confusion over the date of its composition. Bologna, Civico Museo Bibliografico Musicale, A.56 (Martini, 4), 229b and Milan, Biblioteca Ambrosiana, I. 90 sup, 156v both have 1418. But, Venice, Biblioteca del Museo Correr, VIII, 27, 114r has 1422. See Favaro, *Intorno ... Prosdocimo de' Beldomandi*, 133-149.
576 Thorndike favors a date shortly before 1220. See Thorndike's introduction to Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 14.
577 Thorndike, introduction to Ibid., 42-43. Thorndike's *The Sphere of Sacrobosco and Its Commentators* is the definitive source for Sacrobosco's *Sphere*. For more information on the astronomical curriculum in early universities, see O. Pedersen, "The Origins of the *Theorica Planetarum*," *Journal for the History of*

system, but without getting into too much mathematical detail. It is called the *Sphere* because that was the shape of the medieval cosmos. Recall that quadrivial astronomy was defined by Boethius as geometry in motion. In particular, it was spherical geometry in motion.

Sacrobosco's *Sphere* is a short, efficient, well organized exposition. Chapter 1 defines a sphere and describes several properties associated with the terrestrial and heavenly spheres in Aristotelian terms. It also describes some of the most basic observed motions of the planets. Chapter 2 describes a variety of circles useful in astronomy and geography, such as the zodiacal and arctic circles, and defines where such circles are located on earth and within the greater sphere of the fixed stars. Chapter 3 discusses the rising and setting of stars and signs, the different systems for measuring them, the seasonal motions of the sun, and the perceived solar motions from various latitudes. And finally, Chapter 4 discusses such things as eccentrics, epicycles, deferents and specific situations such as retrograde planetary motion and eclipses. Sprinkled throughout the entire text are a variety of astrological terms and situations, such as quartile, trinal, and opposition, but with no reference to astrological effects. It is a straightforward, somewhat dense, but readable text that stays on its topic– astronomy.[578]

Numerous commentaries were written on the *Sphere* in the late Middle Ages[579] and Renaissance and in 1531 several of these were brought together into one volume.[580] Included in

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Astronomy 12, (1981), 113-123.

[578] The topic is astronomy in the modern sense of the word, but the astronomy of the 15th century is quite alien to the modern reader. The *Sphere*, and any other astronomical text from the Middle Ages, is filled with many unfamiliar terms and frequently refers to astrological arrangements that are difficult to decipher. Most of the modern secondary literature on astronomy of this period is as hard to read as the primary sources. For anyone interested in learning the basics, I suggest J. C. Eade, *The Forgotten Sky: A Guide to Astrology in English Literature* (New York: Clarendon Press, 1984), 1-102.

[579] Thorndike lists more than ten 13th-century commentaries.

[580] Johannes de Sacrobosco, *Spherae tractatus Ioannis de Sacro Busto Anglici uiri clarissimi, Gerardi Cremonensis Theoricae planetarum veteres, Georgii Purbachi Theoricae planetarum novae...* (Venice: L. A. Giunta, 1531).

this volume were the *Sphere* itself, and commentaries on it by Michael Scot[581] (1175-ca. 1232), Robertus Anglicus (fl. 1270s), Cecco d' Ascoli (1257-1327), Pierre d'Ailly (1350-1420), Prosdocimo, Capuanus de Manfredonia (fl. 1475), Jacob Faber Stapulensis (Jacques Lefèvre d'Étaples, ca. 1455-1536), and Bartholomeus Vespucci (fl. early 15th century).[582]

Prosdocimo's commentary is given the pride of place in this volume. It is the first one and it contains within it all of Sacrobosco's *Sphere*, which is given in blocks of text set off from the commentary. Interspersed throughout Prosdocimo's commentary and the *Sphere* itself, is another commentary by Bartholomeo Vespucci.[583] In modern measurements, the *Sphere* is about 25 double-spaced pages of text. Prosdocimo's commentary is well over 200.

Prosdocimo's organization follows Sacrobosco's exactly. Every detail of the *Sphere* is spelled out, line by line, and numerous ramifications are explored. As a work of literature, it is in need of a ruthless editor, but as a script for a series of lectures to undergraduates, it makes more sense. Its length is about 16.5 double-spaced pages per class over 14 classes.[584] The redundancy and frequent references back to Sacrobosco's text are more indicative of a lecture

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[581] It is attributed to Michael Scot. See Lynn Thorndike's introduction to Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 21-23.
[582] Also included in this volume are Gerard of Cremona's *Theoricae planetarum veteres* (ca. 1175), Alpetragius' (al-Bitruji's) *Theorica planetarum* (ca. 1200, *De motibus celorum*?), Georg Purbach's *Theoricae planetarum novae* (1472), and several short essays on astronomical and astrological topics.
[583] The Vespucci commentary may have been folded into Prosdocimo's commentary in order to give a more astrological interpretation. Prosdocimo's commentary is predominantly technical. Bartholomeo Vespucci, the nephew of Amerigo, was a professor of astronomy/astrology (and a medical doctor) at the University of Padua. In 1506 he wrote *Oratio de laudibus Astrologiae* praising astrology and the study of the quadrivium as preparation for it. He also corresponded with Machiavelli on astrology and issues concerning free will. Alison Brown, "Philosophy and Religion in Machiavelli," in *The Cambridge Companion to Machiavelli,* ed. John M. Najemy (New York: Cambridge University Press, 2010), 159. See also Lynn Thorndike, *A History of Magic and Experimental Science: The Sixteenth Century*, 8 vols., vol. 5 (New York: Columbia University Press, 1941), 164-166.
[584] Lynn Thorndike describes lecture frequency, style, and length for medieval universities including Bologna and Paris. See Thorndike's introduction to Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 42-46.

script than a polished piece of literature. If it were used as such, it is possible that it would have been read aloud very slowly, so that the students could take notes or even copy it.[585]

Lynn Thorndike says very little about Prosdocimo's commentary except that it goes "into great or petty detail concerning every matter which can be even most remotely related to [Sacrobosco's] text."[586] This assessment is accurate. Prosdocimo's commentary rarely diverges from the subject matter of Sacrobosco's text, yet it manages to be eight times its length. He discusses every point made by Sacrobosco in depth. Most of his commentary concerns technical issues that are pertinent to the specific quadrivial discipline of astronomy. A few of his discussions, however, do address the structure of the quadrivium itself. In the following section I will point out a few of the interesting passages from Prosdocimo's commentary and elaborate more fully on those parts that address the structure of the quadrivium.

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Prosdocimo's *Commentary on Sacrobosco's Sphere* (1418)

Part I of IV. [1r-19v of 1531 ed.]

In the introduction, Prosdocimo addresses his reader directly and describes his commentary as a "handbook for all... not just for the lovers of astronomy and natural philosophy."[587] His commentary is meant as an introduction for the general undergraduate arts student. The *Sphere*, packaged with Prosdocimo's commentary, even today would make an extremely solid freshmen-

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[585] The practice of university lecturers "speaking slowly until their listeners can catch up with them with the pen" was banned at the University of Paris in 1355. See Lynn Thorndike, *University Records and Life in the Middle Ages* (New York: W. W. Norton & Company, 1944; reprint, Columbia University Press, 1975), 237-238; Thorndike's introduction to Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 44.

[586] Ibid., 40.

[587] "Deberet enim esse omnibus Enchiridion... non solum Astrologiae amatoribus atque philosophis naturalibus." Prosdocimus de Beldemandis, "Super tractatu sphaerico commentaria," in *Spherae tractatus Ioannis de Sacro Busto Anglici uiri clarissimi...* (Venice: L. A. Giunta, 1531), 1r. Unless otherwise indicated, all translations from Prosdocimo's commentary are by the author using the printed edition from 1531 as the primary source.

sophomore class on basic Ptolemaic astronomy.

Following Sacrobosco's lead, and also citing Euclid's *Elements* XI, Prosdocimo begins his commentary by defining the sphere. As is typical throughout the commentary he dissects each and every detail. Here he discusses how Euclid's definition is ambiguous[588] and explores at length issues such as the solidity or concavity of a sphere and the mathematics of three dimensional quantities. As in his *Algorismus de integris* and his *Brevis summula proportionum quantum ad musicam pertinet*, Prosdocimo emphasizes first principles, here being the spherical basis of astronomy.

He then describes the general structure of the cosmos with nine concentric nested spheres (starting from the outermost): the *Primum mobile*, then the fixed stars, then the seven planets, and at the center, the sphere of the earth. See Figure 4.5.

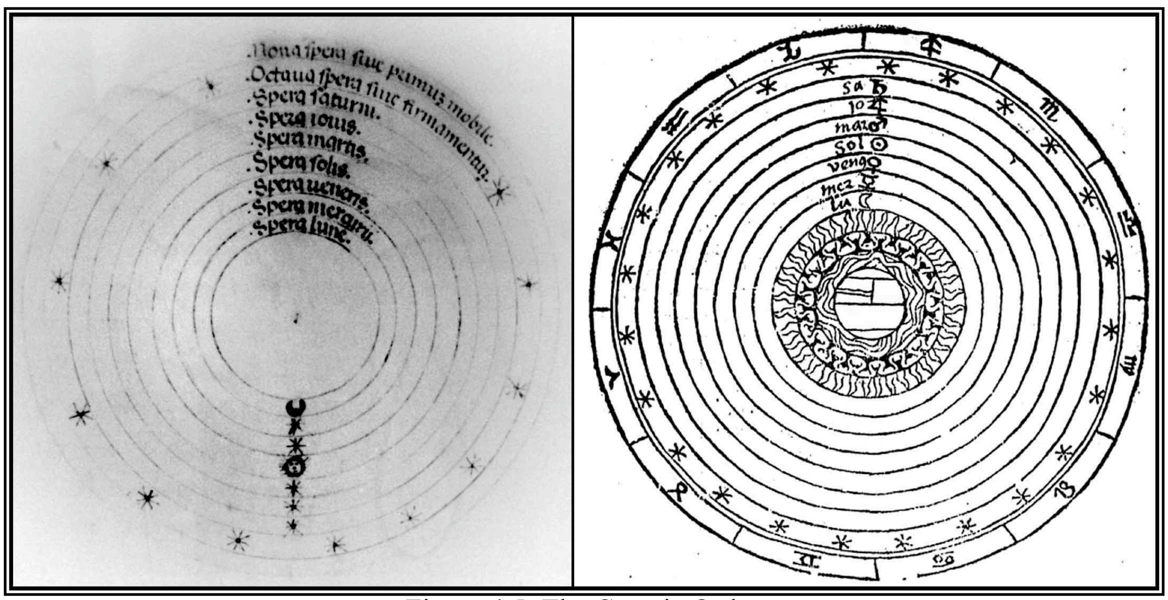

Figure 4.5: The Cosmic Order
The image on the left is from Bologna, Civico Museo Bibliografico Musicale, A.56 (Martini, 4). The image on the right is from Prosdocimo's "Commentary," Venice, 1531, 3v.[589] *(modified slightly for clarity)*

[588] "Quoniam Euclidi non fuit cura an tale corpus sphæricum haberet concavitatem an ne ut clarissime patere potest ex eius diffinitione." "Because it was not a concern for Euclid whether or not such a spherical body had concavity, as is most clearly evident from his definition." Ibid., 2r.
[589] Ibid., 3v.

In Aristotelian fashion he describes the four elements.

> Earth is placed in the middle as the center of the world and immediately around the earth itself is placed water and immediately around water is placed air and immediately around air is placed pure and not turbid fire and (reaching all the way to the sphere-orb) that is to the sphere of the moon, the sphere of the moon which [is] the first of the heavenly super-celestial spheres.[590]

This is followed by a detailed exploration on why the water is unevenly distributed over the surface of the surface of the earth[591] and whether or not the elements are ever encountered as pure substances or only as impure mixtures [*elementata*].[592] One of the few sections of this commentary that has garnered significant attention from modern scholars immediately follows his basic discussion of the four elements, where Prosdocimo considers various arguments for and against the motion of the earth. Mieczyslaw Markowski's essay, "Die kosmologischen Anschauungen des Prosdocimo de' Beldomandi," places Prosdocimo's examination of geocentric and geostatic theories in the context of his contemporaries and subsequent natural philosophers like Copernicus.

His comments on why the sun is "placed in the middle of all the spheres,"[593] found in the middle of Part I, certainly grabs one's attention when first encountered. However, he does not mean for the sun to be at the center of the spheres, as in heliocentric, but rather he means in the center of the order of the spheres, midway between the earth and the *primum mobile*– between Venus and Mars. See Figure 4.5. His four reasons for this order are taken directly from Abu

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[590] "... quod terra ponitur in medio tamquam mundi centrum & circa ipsam terram immediate ponitur aqua & circa aquam immediate ponitur aer: & circa aerem immediate ponitur ignis purus & non turbidus (attingens usque ad orbem) id est sphæram lunæ quæ sphæra lunæ prima sphærarum supercoelestium." Ibid., 6v.42-45.

[591] Prosdocimo's discussion of water is briefly considered in Giuseppe Boffito, *Intorno alla "Quaestio de aqua et terra" Attribuita a Dante*, 2 vols. (Turin: Carlo Clausen, 1902/3), 54-58 and passim.

[592] Prosdocimus de Beldemandis, 7r.56-7v14. Richard Lemay traces this idea to Hermann of Carinthia (fl. 12th century). See Richard Joseph Lemay, *Abu Ma'shar and Latin Aristotelianism in the Twelfth Century: The Recovery of Aristotle's Natural Philosophy through Arabic Astrology* (Beirut: American University of Beirut, 1962), 219-220.

[593] Prosdocimus de Beldemandis, 9r.20. "...sphaera solis in medio omnium sphaerarum poneretur."

Ma'shar [*Albumasar*], whom he cites.[594]

1- Sun must be central to illuminate other celestial bodies equitably
2- By the order of rotational speeds, the sun is central. (It is slower than Venus but faster than Mars.)
3- The sun's heat is necessary for generation, but in moderation.[595]
4- Sun is the ruler of the all the stars, like "the heart in the body of an animal." From a central location it can more easily send "vital spirit" throughout all the body.[596]

Also in Part I, Prosdocimo discusses the general mean motions of the heavens, the spherical shape of the cosmos and the earth, and describes the geometric method used to measure the earth's circumference done by Eratosthenes; Prosdocimo, however, following Sacrobosco, uses an astrolabe.

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Prosdocimo's *Commentary on Sacrobosco's Sphere* (1418)
Part II of IV [19v-33v of 1531 ed.]

The beginning of the second part of Prosdocimo's commentary contains the most interesting section in terms of the quadrivium. In this part he is discussing a section from the *Sphere* where Sacrobosco describes the motions of the *primum mobile* as compared with the motions of the planets. Sacrobosco writes,

> Be it understood that the "first movement" [*primus motus*] means the movement of the *primum mobile*, that is, of the ninth sphere or last heaven, which movement is from east

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594 Ibid., 9r.18-38. These four reasons are derived from Abu Mas'har's *Introductorium in astronomiam*, III.3. See Lemay, *Abu Ma'shar and Latin Aristotelianism in the Twelfth Century*, 254. Macrobius' description of the sun is very similar in many of these details, except he offers additional celestial sources of light, and does not place the sun in the center of the other orbits. He defends the Platonic order: (earth), moon, sun, Mercury, Venus, Mars, Jupiter, Saturn, starry sphere. Macrobius, 1.XX.1-10 and 1.XXI.27, pp. 168-170 and 180. Similarly, Theon of Smyrna describes the Pythagorean arrangement with the sun in the center of the order (forming an octave): (earth), moon, Mercury, Venus, sun, Mars, Jupiter, Saturn, starry sphere. Theon of Smyrna, III.15, pp. 91-94.
595 This is where Prosdocimo cites Abu Ma'shar, though the previous two reasons are also from his text.
596 The vital spirit was part of the general Aristotelian/Galenic conception of life. Copernicus makes very similar observations about the sun, but for heliocentric purposes in *De revolutionibus orbium coelestium* (1543), I.10.

> through west back to east again, which also is called "rational motion" [*motus rationalis*] from resemblance to the rational motion in the microcosm, that is, in man, when thought [*consideratio*] goes from the Creator through creatures to the Creator and there rests.
>
> The second movement is of the firmament and planets contrary to this, from west through east back to west again, which movement is called "irrational" or "sensual" from resemblance to the movement of the microcosm from things corruptible to the Creator and back again to things corruptible.[597]

Physically, Sacrobosco is simply describing the daily motion of the ninth sphere, the *primum mobile*, which carries with it all of the lower spheres. This motion is responsible for the daily rising in the east and setting in the west of all of the stars and other celestial bodies. The second motion, or class of motions, he describes is contrary to this. It is the motion responsible for the mean motions of the planets (including the sun and moon). These motions, in relation to the *primum mobile*, go west to east, but these motions are much slower in their contrary motion. For example, the sun requires a year to travel one circulation, Mars approximately two years, etc.[598] This would be a simple enough concept on its own, except he further identifies the motion of the *primum mobile* as a "rational motion," and the other motion contrary to this as an "irrational" or "sensual" motion.

A rational motion, Sacrobosco seems to claim, is reflected in the microcosm, man. It is the situation by which a thought [*consideratio*] from God, goes through man, and then returns to God. He describes this motion as resembling the east to west to east motion of the *primum mobile*. An irrational motion, on the other hand, originates in man, goes through God, and then

[597] "Unde sciendum quod primus motus dicitur motus primi mobilis, hoc est, spere none sive celi ultimi, qui est ab oriente per occidentem rediens iterum in orientem, qui etiam dicitur motus rationalis ad similitudinem motus rationis qui est in microcosmo, id est, in homine, scilicet quando sit consideratio a creatore per creaturas in creatorem ibi sistendo. Secundus motus est firmamenti et planetarum contrarius huic ab occidente per orientem iterum rediens in occidentem, qui motus dicitur irrationalis sive sensualitatis ad similitudinem motus microcosmi, qui est a corruptibilibus ad creatorem iterum rediens ad corruptibilia." The English translation of this passage is by Thorndike. Johannes de Sacrobosco, "The *Sphere* of Sacrobosco [*Tractatus de spera*]," in *The Sphere of Sacrobosco and Its Commentators,* ed. and trans. Lynn Thorndike (Chicago: University of Chicago Press, 1949), 86 and 123.

[598] This had been discussed in Part I. Sacrobosco (within Prosdocimo's commentary) writes, "Saturn in 30 years, Jupiter 12 years, Mars in 2, the sun in 365 days and 6 hours, Venus and Mercury about the same, the moon in 27 days 8 hours." Prosdocimus de Beldemandis, 8v-9r.

returns to man. It resembles the west to east to west motion of the firmament and planets. This cryptic passage caught the attention of Prosdocimo.

There are some issues with translation that directly affect the philosophical interpretation of this passage. The word "*motus*" readily translates to "motion," however it can also mean "activity of the mind" and/or "reason."[599] This alternative meaning refers to the theories of mind discussed by Plato and Aristotle and numerous commentaries written on them.[600] Also, the word "*consideratio*" can mean "mental examination," "contemplation," and even "astrological conjunction" in some contexts.[601] Lynn Thorndike translates "*consideratio*" as "thought," which fits this passage well.

Prosdocimo's analysis of this passage begins with the obvious description of the astronomy summarized above, i.e., the ninth sphere goes east to west and the rest go west to east. He then goes on to explain why the motion of the *primum mobile* is considered a rational motion (or rational mental activity). Paraphrasing Sacrobosco, he writes, "His [The Creator's] thought begins from the more noble, for example from the highest Creator, and extends to the more ignoble, for example to the creatures, returning again to the more noble, as to the highest Creator,

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599 See *The Oxford Latin Dictionary* (1968) and R. E. Latham, ed., *Revised Medieval Latin Word-List* (1965), s.v. "motus." For example, Thomas Aquinus frequently uses the word in this way.

600 A concise discussion of the relationship between rational thought and the circular motions of the heavens from Aristotle's *Metaphysics*, XII.7-10 is found in Vasilis Politis, *Aristotle and the Metaphysics* (New York: Routledge, 2004), 282-293. See also Aristotle, *De anima*, trans. R. D. Hicks (Cambridge: Cambridge University press, 1907), I.2-3 and III.4, pp. 10-29 and 130-135. Aristotle is largely elaborating on and responding to Plato's description of the division of the World Soul, whose motion is the source of rational thought. See Plato, *Plato's Cosmology: The Timaeus of Plato*, 36B-37C, pp. 72-97. Responding to Aristotle, Macrobius defends Plato's theory of soul/mind as exemplified in motions of the cosmos. Macrobius, 2.XIV-XVI, pp. 227-243. Another excellent source that discusses the connections between numbers, rational ratios, and the intellect as opposed to the irrational magnitudes of sense perception is Jacob Klein's *Greek Mathematical Thought and the Origin of Algebra*. See Klein, 8-9, 89-99, and passim. General background on medieval celestial perfection is found in Edward Grant, *Planets, Stars, and Orbs: the Medieval Cosmos, 1200-1687* (New York: Cambridge University Press, 1994), 220-243.

601 See *The Oxford Latin Dictionary* (1968) and R. E. Latham, ed., *Revised Medieval Latin Word-List* (1965), s.v. "consideratio."

and there rests."[602]

Rational thought in man, Prosdocimo explains, is the effect [*effectus*] from a more noble cause [*causa*]. This rational thought in man is a reflection of the Creator. He writes, "since the cause is said to be more noble than the effect... this thought is deservedly called rational, because it is to proceed from the essence of rationality itself, from cause over its effect, and not the reverse."[603] Irrational or sensual thought, on the other hand, is initiated from the more ignoble, from man, and proceeds to the more noble. As Prosdocimo puts it, "[this thought] proceeds from effect over its cause again returning to the effect."[604]

Rational thought originates from the mind of God.[605] The rational thoughts of man are caused by God. These rational thoughts originate from the Creator and are, in a sense, reflected by the mind of man. A rational thought goes from the Creator, to man, back to the Creator. The irrational or sensual thought is initiated by man. It is man thinking from effects, not causes. It is man drawing conclusions about more noble things from imperfect sensible information from the corruptible [*corruptibilis*] world. An irrational or sensual thought starts from man thinking about the effects of the Creator, but in terms of sensible information. Irrational thought cannot get at the final causes of things, only the effects perceived from our terrestrial vantage point.[606]

Prosdocimo then defines the terms "macrocosm" and "microcosm" in the traditional

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602 "Tunc enim consideratio sua incipit a re nobiliori, ut gratia exempli a summo Creatore & tendit ad ignobiliorem ut gratia exempli ad creaturas rediens iterum ad nobiliorem ut ad summum Creatorem & ibi sistendo." Prosdocimus de Beldemandis, 2.20v.47-49.

603 "[C]um causa suo effectu nobilior esse dicatur, saltem via productionis & noticiae [notitiae] perfectae & haec consideratio merito dicitur rationalis, quonaim de essentia ipsius rationis est procedere a causa supra suum effectum & non econtra." Ibid., 2.20v.57-59. A very similar passage is found in Michael Scot's commentary in Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 302-304.

604 Prosdocimus de Beldemandis, 21r.1-2.

605 This concept is in full agreement with Thomas Aquinas, though Prosdocimo uses very different terminology. See Brian Davies, *The Thought of Thomas Aquinas* (New York: Clarendon Press, 1993), 124-138.

606 Prosdocimo calls the two types of thought *duplex consideratio*. His description, using these terms, is also found in Robert Anglicus' commentary found in Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 163 and 215.

fashion.[607] He writes, "man is called the microcosm or the minor world, because he more than others has communication with all the things of the world.... There is not any being in the world that communicates with everything of the world in the same way as man."[608] Prosdocimo then enumerates the ways in which man connects to all things.

> [B]ecause of intellect [he] comes together with God and the intelligences, because of corporeality he comes together with the spheres, stars and all other corporeal things of the world, on account of sense [animal spirit?] he comes together with all animals both perfect and imperfect, and on account of growth [*vegetationem*] he comes together with all plant-life, and in short, because of his mortality he comes together with all corruptible things, thus, deservedly, man himself is called the microcosm or the lesser world.[609]

In this very interesting passage Prosdocimo lists five ways that man connects to the universe: intellective, corporeal, sensory, vegetative, and mortal. Prosdocimo does not make it clear where these come from or if they are his own invention. For comparison, Aristotle lists five faculties of the human soul, which bear a resemblance: intellective, nutritive/vegetative, sensory, appetitive, and motive.[610] But these do not completely correspond to Prosdocimo's. Thomas Aquinas lists three: rational/intellective, sensory, and vegetative.[611] These are included in Prosdocimo's list, but Thomas' list lacks mortal and corporeal.

Similar to Thomas' theory of the intellect, Prosdocimo refers to intelligences when he

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607 "Notandum tertio quod reperiuntur isti termini macrocosmus & microcosmus, Unde macrocosmus dicitur a macros quod est maior & Cosmos mundus, unde macrocosmus quasi maior mundus, Microcosmus vero dicitur a micros quod est minor & cosmos mundus, unde microcosmus quasi minor mundus." Prosdocimus de Beldemandis, 21r.3-6.

608 "Notandum quarto & ultimo quod pro tanto homo appellatur microcosmus sive minor mundus, quoniam cum omnibus rebus mundi plus aliis communicationem habet, unde si bene consideramus nullum est ens in mundo quod ista cum omni re mundi communicet sicut homo." Ibid., 21r.6-8.

609 "...propter enim intellectum convenit cum deo & intelligentiis, propter vero corporeitatem convenit cum sphaeris, stellis & omnibus aliis rebus mundi corporeis propter autem sensum convenit cum omnibus animalibus tam perfectis quam imperfectis, & propter vegetationem convenit cum omnibus vegetabilibus, & breviter propter eius mortalitatem convenit cum omnibus corruptibilibus, merito ergo ipse homo microcosmus sive minor mundus appellatus est." Ibid., 21r.8-12.

610 Aristotle, *De anima*, II.3.414a29-32, pp. 58-59. Averroës appears to follow Aristotle on this detail, at least in his commentary on *De anima*. See Averroës, *Long Commentary on the "De anima" of Aristotle*, trans. R. C. Taylor (New Haven, CT: Yale University Press, 2009), 134-150.

611 Anthony Kenny, *Aquinas on Mind* (New York: Routledge, 1993), 31-32.

describes how the intellect helps man unite "with God and the intelligences." However, he never uses the standard Thomasian/Aristotelian distinctions such as active, agent, or potential [*actu*, *agens*, or *possibilis*][612] in this or in later references. Like these other lists, Prosdocimo's seems to reflect the standard three spirits of Aristotelian/Galenic medical theory: vegetative (nutritive), mortal (vital), and sensory (animal), along with the Aristotelian-Averroistic-Thomasian intellective "soul." But what Prosdocimo's list also includes is corporeality, which unites man "with the spheres, stars and all other corporeal things of the world." This particular astrological connection between the macrocosm and the microcosm suggests a Neoplatonic influence which, as a general rule is conspicuously absent from Prosdocimo's texts, as was noted in his writings on music. However, in the pages immediately preceding this section, Prosdocimo cites the 9th-century Islamic astrologer Abu Ma'shar,[613] whose reading of Aristotle and Plato combined to make a rational system of astrology. This may help explain Prosdocimo's generally Aristotelian interpretation with additional Platonic or Neoplatonic intelligences.[614]

Prosdocimo then speaks directly to the issue of irrational motion and thought in man.

> In man, ..., since he is moved by an irrational motion [sense-based activity of the mind], ... , then, ... , his thought begins from a more ignoble thing, e.g., the corruptible, and extends to the more noble, e.g., the incorruptible, returning again to the more ignoble, e.g., the corruptible, just as in man a thought goes from the created things to the highest of the creators, returning again to the created things, and there it will make fast.[615]

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612 See Ibid., 41-57.

613 Abu Ma'shar's *Introductorium in astronomiam* is cited regularly by Prosdocimo.

614 Lemay, *Abu Ma'shar and Latin Aristotelianism in the Twelfth Century*, 281-284. A thorough study of Abu Ma'shar's influence on Prosdocimo has not been done.

615 "[I]n homine qui minor mundus appellatus est, quando movetur motu irrationabili, sive sensualitatis inconsiderando, tunc enim eius consideratio incipit a re ignobiliori ut gratia exemplia corruptibilibus & tendit in nobiliores, ut gratia exempli ad incorruptibilia iterum rediens ad ignobiliora, ut gratia exempli ad corruptibilia, sicut est consideratio in homine qui est a creaturis in summum creatorem iterum rediens ad creaturas & ibi se firmando." Prosdocimus de Beldemandis, 21r.24-28. The complete translation, filled with every clause is, "In man, who is called the lesser world, since he is moved by an irrational motion, or sensually thinking, then, that is to say, his thought begins from a more ignoble thing, as for example at the corruptible, and extends to the more noble, as for example to the incorruptible, returning again to the more ignoble, as for example to the corruptible, just as in man a thought goes from the created things to the highest of the creators, returning again to the created things, and there it will make fast." My

Prosdocimo ends this section having made his comparison of rational and irrational motion to rational and irrational thought. Rational thought is superior, from God, and irrational thought is inferior, from man. Prosdocimo's use of the term rational to describe thought whose source is God and irrational as thought whose source is from the corrupted earth, suggests a quadrivial division. Rational thought is of a higher ontological status. In the context of Prosdocimo's musical thought, rational described the numbers of arithmetic and music, while irrational described geometry and astronomy. This extension of the use of the terms rational and irrational is in line with his divisions between multitude and magnitude seen in his criticism of Marchetto; however, in his commentary on the *Sphere* he does not digress directly into other quadrivial topics. Furthermore, his list of ways that man interfaces with the world (intellective, corporeal, sensory, vegetative, and mortal) is suggestive of astrology and medicine, but again he does not elaborate. What this section reveals is that Prosdocimo had an epistemological and ontological philosophical stance that strictly delineated rational thought from irrational thought. It is a philosophy that is a perfectly consistent extension from basic quadrivial philosophy.

The remainder of Part II, describes in technical detail the various poles and circles important to geocentric astronomy and geography, such as the equinoctial, ecliptic, arctic, zodiacal, tropical, the colures, the meridians, etc.

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translation relied heavily on the example of Thorndike's translation of a similarly constructed passage by Sacrobosco. Prosdocimo's excessive use of appositional phrases make this passage very confusing, so the edited translation is presented in the body of this text.

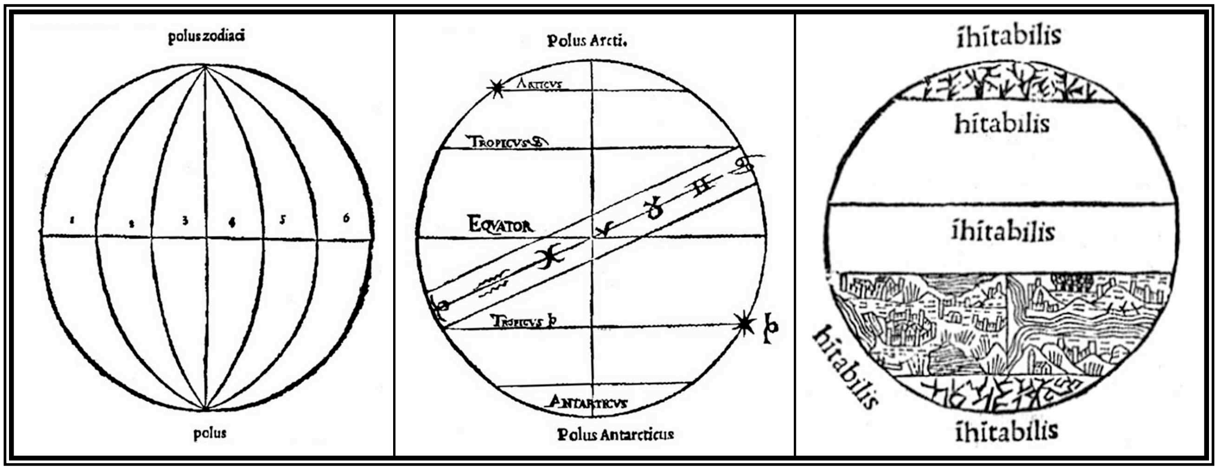

Figure 4.6: Diagrams of the Various Circles of Astronomy
From 23v, 31r, and 32r respectively, from the Venice, 1531 ed.[616]
North is at the bottom of the figure on the right.

In a section on the horizon he describes a simple astrolabe-like instrument used to measure the altitude of the sun and other celestial bodies. See Figure 4.7. This diagram, from Bologna, Civico Museo Bibliografico Musicale, A.56 (Martini, 4), is mentioned here for its similarly to the diagram of what Leon Battista Alberti calls an *orizonte,* which will be discussed in the next case study.

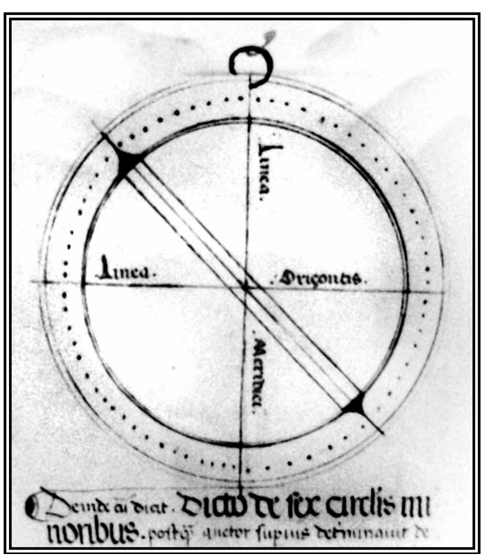

Figure 4.7: Instrument for Astronomical Measurement[617]

An almost identical woodcut image is found on f. 30v of the Venice, 1531 ed.

[616] Ibid., 23v, 31r, and 32r.
[617] Bologna, Civico Museo Bibliografico Musicale, A.56 (Martini, 4), p. 152.

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Prosdocimo's *Commentary on Sacrobosco's Sphere* (1418)
Part III of IV [34r-50v of 1531 ed.]

This part is very technical in nature and describes some of the more complicated aspects of medieval astronomy. Much of it concerns various coordinate systems for locating astronomical events. These include a variety of ways to describe risings and settings of zodiacal signs and a description of the right and oblique ascensions.

In the section discussing the spiral trajectory of the sun (see Figure 4.8), a detail that Oresme used in his argument for incommensurability of the celestial motions, Prosdocimo has little to say. He writes, "they are not circles, but are spirals or circuits [*giri*]... Nevertheless it is not very important if they are called circles or spirals or circuits, because this does not hinder the intent."[618] With this quick dismissal, he moves onto other matters, such as seasonal and latitudinal variations of the lengths of days, and the inhabitability of various zones of the earth which include various digressions into geographical matters. See Figure 4.8.

[618] "...non sint circuli, sed sint spirae sive giri quod idem est, non est tamen multum curandum si circuli appellentur, vel spirae sive giri, quoniam hoc non impedit intentum." Prosdocimus de Beldemandis, 40r. 5-7.

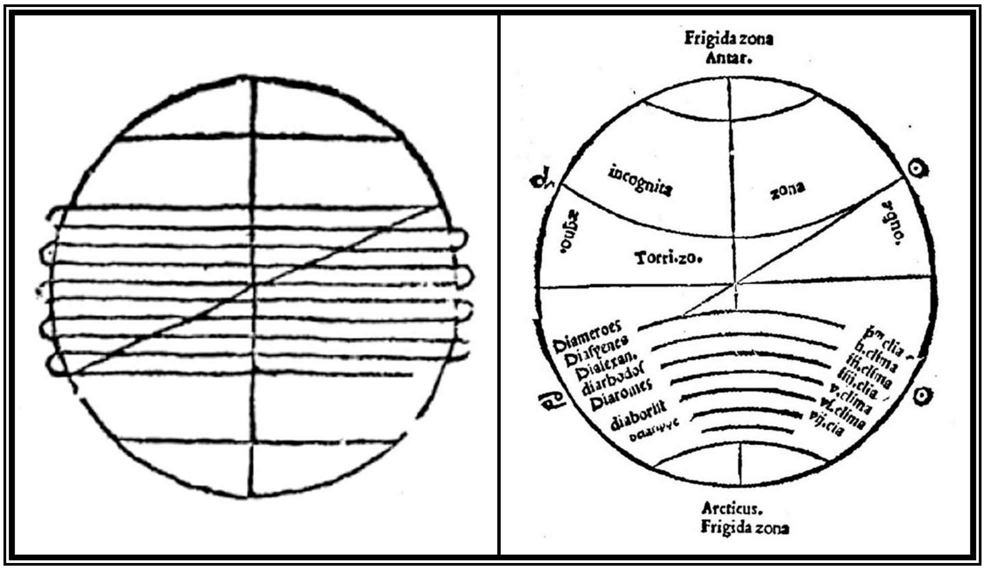

Figure 4.8: Left: The Spiral Trajectory of the Sun Right: The 7 habitable regions or "climes."[619] *North is down.*[620]

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Prosdocimo's *Commentary on Sacrobosco's Sphere* (1418)
Part IV of IV [50v-56r of 1531 ed.]

In this last part, the more complicated Ptolemaic aspects of the celestial motions are described. The annual motion of the sun along the ecliptic is described in more detail as is the motion of the 8th sphere, what we now refer to as the precession of the equinoxes.[621] The three planetary motions (equant, deferent, and epicycle) are described as well as the observed retrograde motions. And finally he describes how lunar and solar eclipses occur and goes into great detail in his analysis of shadows and light. See Figure 4.9.

619 A very similar woodcut is contained in the 1506 edition of Abu Ma'shar's *Introductorium in astronomiam*. The data concerning the climes is identical. See Abu Ma'shar, *Introductorium in astronomiam Albumasaris Abalachi*, trans. (into Latin) Hermann of Carinthia (Venice: 1506), e7r.
620 Prosdocimus de Beldemandis, 38r and 48r.
621 Prosdocimo refers to the Alfonsine Tables in this section.

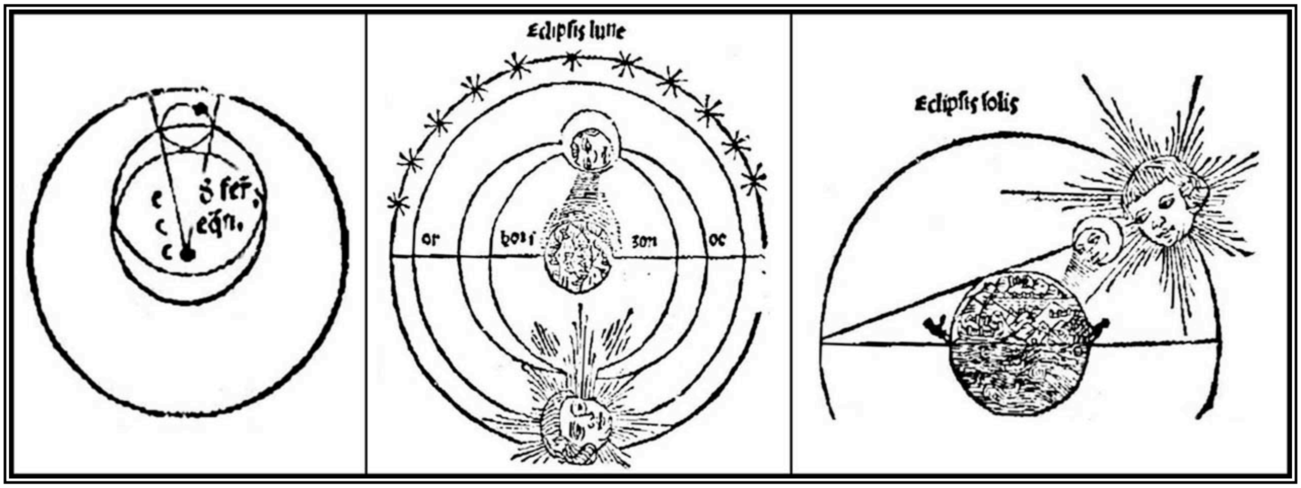

Figure 4.9: Left: Diagram Showing Eccentric, Equant, and Epicycle.
Middle: Lunar Eclipse Diagram. Right: Solar Eclipse Diagram.[622]

In the midst of this discussion he describes two ways that a solar eclipse can be safely observed. The first is to look at the sun through a tiny hole in a card. The second method is much more interesting. He writes,

> Another method is this, take a card as the first, in the middle of which you make a round hole, and when it will be the hour of the solar eclipse, put this card up to the primary solar rays at a distance from some shade such that the rays falling through that hole before the hour of the eclipse show the figure of the hole .... On the shaded object where there was exposed roundness, at the hour of the eclipse [will be] a non-perfect roundness, but deficient by the roundness according to the quantity of the eclipsed sun.[623]

What he is describing, of course, is a camera obscura. This method of observing solar eclipses was not new with Prosdocimo,[624] but his inclusion of a method to safely observe a solar

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[622] Prosdocimus de Beldemandis, 51r, 54v, and 54v.

[623] "Alius modus est iste habeas unam cartam ut prius in cuius medio facias unum foramen rotundum, & quando erit hora eclipsis solaris ponas hanc cartam ad radios solares primarios in distantia ab aliquo opaco in qua radii incidentes per illud foramen ante horam eclipsis ostendebant figuram foraminis & videbis figuram illam quae ante eclipsim in illa certa distantia & in obiecto opaco sibi exposito rotunda erat hora eclipsis non perfecte rotundari sed a rotunditate deficere secundum quantitatem solis eclipsatam." Ibid., 55r.44-48.

[624] The pseudo-Aristotlelain text *Problemata*, XV.5, describes the phenomenon as do Roger Bacon and others in the 13th century. John Peckham, famous for his *Perspectiva communis*, wrote a treatise on the sphere in the mid to late 13th century, in which he also describes the use of a camera obscura. See Thorndike's introduction to Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 24-25.

eclipse is indicative of the type of commentary Prosdocimo was writing– a comprehensive introduction to all things astronomical for the undergraduate.

Perhaps what is most striking in a quadrivial analysis of this commentary is the nearly total absence of music, astrological content, and reference to Plato. Prosdocimo, one of the great writers on music theory, makes no mention of a music of the spheres. There can be no doubt that he was fully cognizant of the idea of *musica mundana*[625] given his use of Boethian mathematical texts. Beyond the standard terms used for various astronomical configurations, he discusses little that can be construed as astrology. This lacuna is unexpected given the preponderance of astrological notes from his student notebooks,[626] and the fact that one of his main sources was Abu Ma'shar, "the principal authority in astrology among the Arabs."[627] The only times he brings up Plato are in response to specific references made by Sacrobosco. These absences are particularly glaring since, as Thorndike puts it, "[Prosdocimo's commentary goes] into great or petty detail concerning every matter which can be even most remotely related to [Sacrobosco's] text."[628]

What little mathematical computation there is in Prosdocimo's commentary refers to geometrical situations, but there are no formal proofs and little more involved than his explanation of Eratosthenes' measurement of the circumference of the earth. However, even this

[625] A synopsis of medieval *musica mundana* is found in Gabriela Ilnitchi's essay from 2002. Of particular interest is Ilnitchi's brief discussion of the cosmic harmonies of Abu Ma'shar and Hermann of Carinthia. See Gabriela Ilnitchi, "'Musica Mundana', Aristotelian Natural Philosophy and Ptolemaic Astronomy," *Early Music History* 21, no. (2002), 37-74.
[626] Biblioteca Medicea Laurenziana, Ashburnham 206.
[627] Lemay, *Abu Ma'shar and Latin Aristotelianism in the Twelfth Century*, xxix.
[628] See Thorndike's introduction to Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 40. Sacrobosco does not mention anything regarding *musica mundana* either, nor do any of the commentaries that Prosdocimo appears to have read, nor does Abu Ma'shar. The only commentary on Sacrobosco's *Sphere* predating Prosdocimo's that I have found that explicitly describes any quadrivial philosophy is an anonymous commentary from the late 13th century, described and partially transcribed by Thorndike, that paraphrases, without attribution, Boethius' quadrivial structure concerning multitude and magnitude and the necessity of studying the quadrivium. See Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 29-31 and 451-453.

argument is surprisingly abbreviated. There is no discussion of the origin of number or any significant numerological correspondences to planets or the zodiac.

The one significant discussion that pertains to the quadrivium was Prosdocimo's analysis of rational and irrational motion and thought. This discussion showed that he drew a clear distinction between the imperfect sensual knowledge born of the earth and the ideal knowledge of true forms or universals born from the *primum mobile* or the Creator. Geometry was, after all, earth-measure, and was inherently tied to knowledge derived from the senses, whereas arithmetic and music theory were derived from numbers, not sensual experience. Given his insistence on these distinctions in his musical texts and his obsessively specific use of terminology, I think it is not too speculative to draw his discussion of rational and irrational motions/thoughts into the realm of quadrivial distinctions between multitude and magnitude. It is perfectly consistent with his other writings.

Overall, Prosdocimo's commentary is quadrivial in the sense that it is about astronomy. But it is not quadrivial in a philosophical sense. Its natural philosophical position is largely Aristotelian, with a small dose of Neoplatonism that probably came from his Arabic sources.

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The Alfonsine Tables and Accompanying Materials

The last astronomical work I want to address is actually three works, but they are often found together:[629]

1. *Canones de motibus corporum supercoelestium* (1424) – 3r-14r

2. *Tabulae mediorum motuum, equationum, stationum et latitudinum planetarum, elavationis signorum, diversitatis aspectus Lunae, mediarum coniunctionum et*

[629] The following folio numbers correspond to Bologna, Biblioteca dell'Università, 2284 (San Salvatore 192), abbreviated MS-BB-2284 hereafter.

oppostiionum lunarium, feriarum, latitudinum climatum, longitudinum et latitudinum civitatum. – 14v-34r.

3. *Stellae fixae verificate per regem Alfonsium annis Ihesu Christi 1251 completis et mensibus 5* – 35v-47r

Taken together these are the instructions and tables necessary to actually do astronomy. These are the theory described in Prosdocimo's *Commentary*, put into action with actual measurements, calibrated to a place, a time, and a coordinate system with fixed stellar locations. Using these, an astronomer/astrologer can do everything from determine future or past dates for Easter,[630] compute the next conjunction of Saturn and Mars, calculate how much daylight there will be on May 24th in a particular location, or cast the horoscope of Julius Caesar. The tables link time to place, both terrestrial and astronomical.

Prosdocimo's tables are derived from the Alfonsine Tables, which were commissioned by King Alfonso X of Castile (r. 1252-1284) and completed in Toledo by Isaac ben Sid and Judah ben Moses ha-Cohen near the end of the 13th century.[631] Much of their data came from Arabic sources, which were readily available in the Iberian Peninsula. Extant copies of actual Castilian tables no longer exist, but ca. 1320 adaptations of it appeared in Paris and these new Alfonsine Tables, and derivations from them,[632] dominated astronomy for the next two centuries. It was not until the appearance of the Erasmus Reinhold's *Prutenicae tabulae*, first printed in 1551 and based on Copernican theory, that the Alfonsine Tables were finally retired.[633]

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[630] Easter was defined as the first Sunday after the first full moon after the first day of spring.
[631] Chabás and Goldstein, *The Alfonsine Tables of Toledo,* 1.
[632] 15th-century derivations from the Alfonsine Tables include the *Tabulae resolutae* from Eastern Europe, the *Bianchini Tables* from Italy and the *Almanach perpetuum* from the Iberian peninsula. See José Chabás and Richard L. Kremer, "Introduction: The Circulation of Astronomical Practices in Late Medieval Europe," *Journal for the History of Astronomy* 38.3, no. 132 (2007): 267. The legacy of the Alfonsine Tables in Europe is discussed in Chabás and Goldstein, *The Alfonsine Tables of Toledo*, 243-306.
[633] Chabás and Goldstein, *The Alfonsine Tables of Toledo*, 6.

At first glance the tables are rather intimidating– page after page of numbers, no diagrams or explanations, just cryptic headings and abbreviated astronomical terms. See Figure 4.10.

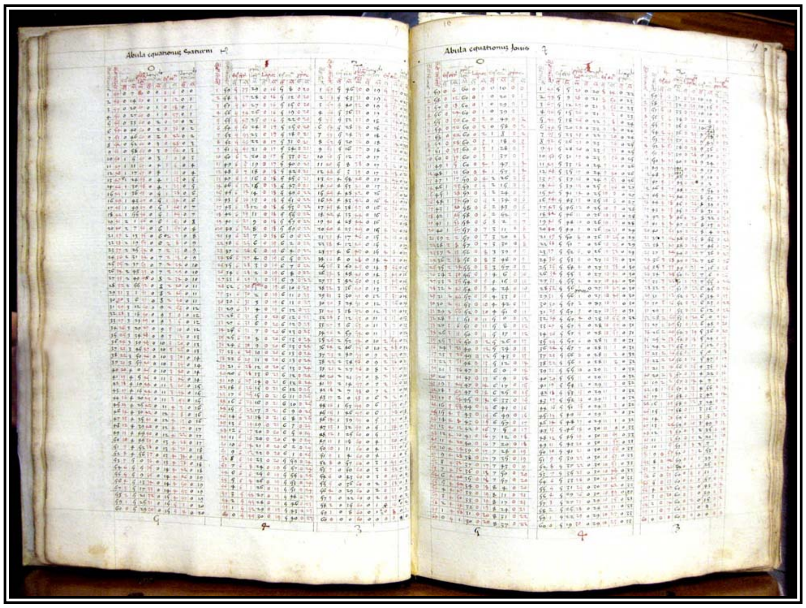

Figure 4.10: Alfonsine Tables[634]

The canons written to accompany them are the instructions for their use. The earliest Parisian canons appear in 1321 by John of Murs (Johannes de Muris) and 1322 by John of Lignères. The most popular of these early canons were by John of Saxony from 1327,[635] a copy of which appears in Prosdocimo's student notebooks.[636]

Even with canons, the Alfonsine Tables were difficult to use and various author-compilers tried to make the computations easier by rearranging the tables and/or converting the data into more convenient numerical systems or alternative coordinate descriptions. By the mid-1400s a new range of astronomical texts appeared, such as ephemerides, calendars, almanacs, and various astrological tracts, all derived from the Alfonsine Tables, but with much of the more difficult

[634] MS-BB-2284, 18v-19r.
[635] Chabás and Goldstein, *The Alfonsine Tables of Toledo*, 247.
[636] Florence, Biblioteca Medicea Laurenziana, Ashburnham 206, 11v-19r. What appear to be a copy of Prosdocimo's copy is found in Modena, Biblioteca Estense Universitaria, Campori 14 (γ.E.6.7), 4r-25r.

and/or tedious mathematics already done, allowing the results to be presented in an easy to read format.[637]

In all likelihood, the Alphonsine Tables and accompanying canons would have been studied in the second year of a university's astronomical curriculum, after having studied the first part of Euclid's *Elements*, Sacrobosco's *Sphere,* and a text on algorism.[638] Without this background the tables and canons would have been virtually unintelligible.

Prosdocimo begins his canons[639] with a short description of his sources and his purposes. He tells the reader that he is expanding a project that was started by Jacopo de Dondi (d. 1359), who had apparently begun to adapt the Alfonsine Tables to the meridian of Padua.[640] Prosdocimo also states that he will modify and reorder his tables and canons "so as to make computation easier to perform."[641] And then, without any further introduction to the subject, Prosdocimo starts into his technical instructions.

The remainder of the text describes how to use each of the tables that follow. His tables describe such things as the mean motions of the planets on their deferents, the "equations" that describe a planets motion on its epicycle, tables for conjunctions and oppositions, and a variety of chronological[642] and geographical information. The following descriptions of Prosdocimo's

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637 José Chabás, "From Toledo to Venice: the Alfonsine Tables of Prosdocimo de'Beldomandi of Padua (1424)," *Journal for the History of Astronomy* 38.3, no. 132 (2007): 267.
638 See Thorndike's introduction to Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 43. Thorndike notes that the statutes of the University of Bologna's program in astrology put the study of the Alfonsine Tables before Sacrobosco's *Algorismus* and *Sphere*, which makes no sense, because knowledge of general astronomical theory and basic mathematical skills are necessary prerequisites for working with the Alfonsine Tables.
639 MS-BB-2284, 3r-14r. None of these works have been printed, but there are several extant copies in manuscript form.
640 "Et quam in mobibus planetarum tabule Jacobi de Dondis paduani ex Alfonsi tabulis extracte leuiores et expeditiores sunt in operando quam Alfonsi tabule." Ibid., 3r.
641 "...ut inde operatio leuior habeatur." Ibid.
642 The Bologna manuscript (MS-BB-2284) does not contain a chronology of eras such as the one found in Florence, Biblioteca Medicea Laurenziana, Ashburnham 206, 19v-20r or in Modena, Biblioteca Estense Universitaria, Campori 14 (γ.E.6.7), 3r. These chronologies typically give the length of time,

tables presented here are extremely abbreviated and meant to convey the general nature of these tables, not the specific technical details that make his tables different from others.[643]

The numbers used on Prosdocimo's tables, as in other Alfonsine Tables, are Hindu-Arabic, but the system is sexagesimal. This system is useful for astronomical computations, because the coordinate systems in astronomy are generally base-60. However the sexagesimal system is a bit awkward in the actual calculation of numbers, since the Hindu-Arabic system is designed for base-10. To aid with these calculations, at the very beginning of the manuscript, before anything else, there is a multiplication table for sexagesimal numbers. See Table 4.4.

Tabula ... quae est valde utilis et parum laborosa															
0;16	0;15	0;14	**0;13**	0;12	0;11	0;10	0;9	0;8	0;7	0;6	0;5	**0;4**	0;3	0;2	0;1
0;32	0;30	0;28	0;26	0;24	0;22	0;20	0;18	0;16	0;14	0;12	0;10	0;8	0;6	0;4	0;2
0;58	0;45	0;42	0;39	0;36	0;33	0;30	0;27	0;24	0;21	0;18	0;15	0;12	0;9	0;6	0;3
1;4	1;0	0;56	0;52	0;48	0;44	0;40	0;36	0;32	0;28	0;24	0;20	**0;16**	0;12	0;8	**0;4**
1;20	1;15	1;10	**1;5**	1;0	0;55	0;50	0;45	0;40	0;35	0;30	0;25	0;20	0;15	0;10	**0;5**

Table 4.4: Excerpt from Multiplication Table

This excerpt shows just the upper right-hand corner of a sexagesimal multiplication table. (1v-2r in MS-BB-2284)

Notice that the semicolon does not work exactly like the decimal point in our modern system. It is simply a separator between degrees and minutes (or any other base-60 divisions you choose). Think of it as multiplying times on a clock. E.g., 12 times 5 minutes equals one hour.

For example, A: 0;4 x 0;4 = 0;16 (i.e., 16 minutes)

For example, B: 0;5 x 0;13 = 1;5 (i.e., 1 degree, 5 minutes)

Using the tables was a process of multiple calculations. For example, if you wanted to know the position of Mars on a particular date, you would have to determine how much time had

measured in days, between such events as the Great Flood, the rule of Caesar, and the era of King Alfonso X. Such chronologies would not only situate the radices of planetary motions into a broader history, but would presumably have facilitated the making of historical horoscopes such as those made by Girolamo Cardano in the 16th century. For example, see Girolamo Cardano's horoscope of Christ in Wayne Shumaker, *Renaissance Curiosa: John Dee's Conversations with Angels, Girolamo Cardano's Horoscope of Christ, Johannes Trithemius and Cryptography, George Dalgarno's Universal Language* (Binghamton, NY: Center for Medieval & Early Renaissance Studies, 1982), 53-90.

643 For an excellent and detailed technical study of Prosdocimo's tables see Chabás, "From Toledo to Venice: the Alfonsine Tables of Prosdocimo de'Beldomandi of Padua (1424)," 269-282.

elapsed from the Incarnation (of Christ) to the particular time in question.[644] The starting position of Mars is given for the Incarnation in a table of "radices." Then, using a series of tables for the mean motion of Mars[645] that give progressively finer and finer detail, you would calculate where Mars would be in x number of years-months-days-hours. See Table 4.5. The tables of "equations" then further refine the position of Mars in terms of the motion of its epicycle.[646]

	Medius motus Martis *[Median motion of Mars]*			
menses	*S.*	*G.*	*M.*	*S.*
Jan	*0*	*16*	*14*	*46*
Feb	*0*	*30*	*55*	*12*
Mar	*0*	*47*	*9*	*58*
Ap	*1*	*2*	*53*	*17*
Ma	*1*	*19*	*8*	*3*
Jun	*1*	*34*	*51*	*23*
Jul	*1*	*51*	*6*	*9*
Aug	*2*	*7*	*20*	*55*
Sept	*2*	*23*	*4*	*14*
Oct	*2*	*39*	*19*	*0*
Nov	*2*	*55*	*2*	*19*
Dec	*3*	*11*	*17*	*5*

Table 4.5: The Mean Motion of Mars over 12 Months (MS-BB-2284, 16r)
This table describes the motion of Mars' deferent over 12 months. Other tables describe the behavior of Mars' epicycle, further refining its motion.

S. = Sign = 60º
G. = Gradus = degree[s] (1/60 of a Sign)
M. = Minute = minute[s] (1/60 of a degree)
S. = Secundua = second[s] (1/60 of a minute)

Rounding to the nearest degree, notice that in 12 months Mars travels from 16º to 191º [3x60º + 11º]. The difference is 175º, slightly less than half of a circulation. This reflects the approximately 2 year cycle of Mars. The annual period for Mars derived from this table is ca. 2.06 years. The modern value is ca. 1.88 years. This discrepancy is due to the difference between geocentric and heliocentric astronomy. The point of view changes in a heliocentric system.

Similar techniques would be used for various other motions, but the general idea was to calculate an elapsed time, and then run that amount of time through various tables to determine where a planet would end up. Further adjustments would have to be made if you were not in

644 In Prosdocimo's tables, the root positions [*radices*] of the planets were given on the date of the Incarnation, but other tables were calibrated to the coronation of Alfonso X, on January 1, 1252.
645 The mean motion of a planet is basically the motion of the center of its epicycle along the deferent. The mean motion is the first approximation of its location.
646 For a simple description of how tables were used, see Owen Gingerich, "The Alfonsine Tables in the Age of Printing," in *De astronomia Alphonsi Regis: Actas del Simposio sobre Astronomía Alfonsí celebrado en Berkeley (Agosto 1985) y otros trabajos sobre el mismo tema,* ed. Mercè Comes, Roser Puig Aguilar, and Julio Samsó (Barcelona: Universidad de Barcelona, Instituto "Millás Vallicrosa" de Historia de la Ciencia Arabe, 1987), 89-95.

Padua, for Prosdocimo's tables were calibrated for Padua, which he locates 32;30º from the so-called western limit.[647] His tables also provide the coordinates for nearly 80 additional cities.

Prosdocimo's star catalogue, called the *Stellae fixae verificate per regem Alfonsium annis Ihesu Christi 1251 completis et mensibus 5* in the Bologna manuscript,[648] is the largest known in medieval times.[649] It contains coordinates for 1022 stars, the same number as given in Ptolemy's *Almagest* in Gerard of Cremona's Latin translation.[650] José Chabás has reported that all of the longitudes differ from those of the Gerard's *Almagest* by 17;8º. This difference is an adjustment for the motion of the 8th sphere (precession) and accounts for the approximately 1300 years that separate Ptolemy from Prosdocimo.[651]

Almost any operation that is performed when using these three astronomical works (canons, Alfonsine Tables, and star catalogue) involves mathematics that was introduced in Prosdocimo's *Algorismus de intergris*. He even mentioned astrological calculations as a motivation for writing his algorism. Oddly enough, geometry, the quadrivial discipline of astronomy, is hardly mentioned. This is because the Alfonsine Tables convert most of the geometry into arithmetic. The geometry is built into the structure of the tables. This allows astronomers to do astronomy, geometry in motion, without actually having to do geometry. They function something like a primitive computer. You ask them a question, turn a metaphorical crank, and get an answer. The complicated spherical geometry of equants, eccentrics, and epicycles, has been turned into tables of numbers that change over time. For the most part, an astronomer only needs geometry

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[647] MS-BB-2284, 34r. Chabás gives the longitude 32:20º for Padua.
[648] MS-BB-2284, 35v-47r.
[649] Chabás, "From Toledo to Venice: the Alfonsine Tables of Prosdocimo de'Beldomandi of Padua (1424)," 277.
[650] Ibid.
[651] Ibid., 278.

in order to conceptually understand how to move from table to table. The actual calculations that must be performed to use these tables are mostly multiplication and interpolation.[652]

The methods necessary for using the Alfonsine Tables based on Ptolemaic astronomy did not change very much with the *Prutenicae tabulae* based on Copernican theory. Both relied on tables of mean motions, leading to tables based on epicycles and eccentrics and other similar refinements. Although the theory changed radically – some call it a "revolution"– the practicing astronomer/astrologer still used basic arithmetical/algoristic methods to get a result. An advanced understanding of the theory was needed for the designers of the tables, not the users.

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Astronomical Conclusions

Thorndike's comment that Prosdocimo's commentary goes "into great or petty detail concerning every matter which can be even most remotely related to [Sacrobosco's] text"[653] may be accurate if analyzed for its novelty, but his commentary provides a very detailed look at what sort of astronomical theory was being taught in Padua in the 15th century. The fact that it was printed in 1531 suggests that its influence continued to be felt more than a century after it was written. Although Prosdocimo presented mostly traditional Aristotelian/Ptolemaic views, his lengthy digressions into matters that challenged these views, such as the possibility that the earth moves, injected such ideas into the conservative scholarly astronomical curriculum of the 15th and 16th centuries in Italy. He seriously considered alternative theories. It is interesting to speculate on Copernicus' exposure to these ideas as he studied at the University of Bologna in

[652] Euclid's name does not even appear in the index of Chabás and Goldstein, *The Alfonsine Tables of Toledo* (2003).
[653] Thorndike's introduction to Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 40.

the late 15th century.[654] Prosdocimo was, in part, responsible for keeping alternative theories alive the academic realm.

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General Conclusions

The combined works of Prosdocimo cover every element of the quadrivial arts. His musical texts are a complete set of music theory and practice[655] and his astronomical texts are a complete set of astronomical theory and practice. His *Algorismus de integris* was specifically designed to address the practical issues of doing astronomy, and his algorism along with *Brevis summula proportionum quantum ad musicam pertinet* and parts of *Musica speculativa* covered the mathematical techniques necessary for doing quadrivial music. His short work on geometry, though not directly useful for his larger quadrivial program, demonstrates that he was not only competent in Euclidean geometry, but that he was also in touch with the physical applications of geometry being developed in the north.

Prosdocimo was not against new ideas. He clearly embraced the new Hindu-Arabic numerals and new styles of musical composition, but he demanded that these new ideas be consistent with quadrivial philosophy. His introduction to his algorism stressed the importance of number and suggested that this new way of manipulating quantities would be useful for astrologers. His reworking of the divisions of the monochord remained strictly within the boundaries of rational ratios and only extended Pythagorean-style divisions. He did not stray outside of those numerical boundaries. He took umbrage with Marchetto for allowing

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[654] Edward Rosen, *Copernicus and His Successors*, ed. Erna Hilfstein (London: Hambledon Press, 1995), 127-138.

[655] See Herlinger, introduction to Prosdocimo de' Beldomandi, *Musica speculativa*, 2-4.

magnitudes to infiltrate quadrivial music and the quadrivial ignorance demonstrated by Marchetto was systematically attacked from his obstinate quadrivial position.

It is unfortunate that Prosdocimo did not write a treatise defending astronomy from a perceived enemy, like he defended music from Marchetto. This might have led to a more direct definition of astronomy from a quadrivial standpoint. But astronomy in the early 15th century had not yet changed to the extent that music had. In hindsight, the proto-physics of Oresme to which Prosdocimo's lone geometrical work alludes, is evidence of a coming change, but the ideas that extended these terrestrial motions out into the heavens had not yet been formulated. These would not come until the chain of events traditionally labeled the "Scientific Revolution" had run its course: Copernicus to Kepler to Galileo to Newton. Prosdocimo's astronomy was safe. What little he says of the quadrivial philosophy in his astronomical works is restricted to the epistemological views demonstrated in his commentary on the *Sphere* that deal with rational and irrational thought. These clearly show a distinction between higher and lower thought, and this distinction is consistent with the quadrivial distinction between multitude and magnitude, but Prosdocimo does not explicitly connect these parallel currents. His description of rational and irrational thought is an interesting example of Neoplatonism grafted onto Aristotelianism (or vice versa), but he does not develop this line of thought into a metaphysical theory of number or magnitude. In general, Prosdocimo seems to be hesitant to delve too far into metaphysical speculation. To use Thomas Kuhn's term, Prosdocimo was the epitome of the "normal science" of the early 15th century.

Prosdocimo's music is kept completely separate from his astronomy. There is no discussion of *musica mundana* or commensurate motions or Great Years. His astronomy starts from the assumption that it is based on magnitude, which, from a terrestrial point of view, is inherently approximate. To think that celestial motions, measured from the earth using the senses, would be otherwise, is not considered. This is in keeping with the ramifications of quadrivial philosophy, if the distinctions between multitude and magnitude are extended to the distinctions between rational and irrational thought. As he states in *Brevis summula proportionum quantum ad musicam pertinet*, "they are not of the same proximate genus but of ones quite remote."[656]

In the next case study we will look at the mathematical works of a man who applied the quadrivial mathematics to the world around him. He was educated in a quadrivial environment exemplified by Prosdocimo, but was not a university professor. He was not invested in the quadrivial philosophy. He was a prominent player in the humanist movement and as such his uses of and references to the quadrivium are more in line with that intellectual movement than to that of medieval scholasticism or Mertonian physics.

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[656] Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 48-49.

Chapter 5:
Quadrivial Case Study #3:
Leon Battista Alberti (1404-72)

Unlike our previous two case studies, Leon Battista Alberti was not invested in the quadrivium as a metaphysical system. He appears to have been entirely uninterested in the subtle issues concerning the distinctions between multitude and magnitude or the idiosyncrasies of Pythagorean-based tunings. He did not delve into such matters. Instead Alberti put the quadrivial disciplines to use. For Alberti, the quadrivium was a storehouse of fascinating mathematical methods and observations that carried with them the authoritative weight of classical history in much the same way that Cicero was a storehouse of the classical Latin language. Alberti was a humanist who both embraced old classical forms and used them to be modern. The quadrivial disciplines are clearly identifiable in Alberti's works, but the quadrivium itself was not his goal.

Short Biography[657]

Leon Battista Alberti was born in 1404, the illegitimate son of Lorenzo di Benedetto Alberti. The Alberti were a powerful mercantile and banking family based in Florence, whose business networks covered most of Europe by the 14th century. A few years before Leon Battista's birth,

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[657] This short sketch of L. B. Alberti's life is largely drawn from the following sources: Spencer's introduction to Leon Battista Alberti, *On Painting (Della Pittura)*, trans. J. R. Spencer (New Haven: Yale University Press, 1966), 13-17; Rykwert's introduction to Leon Battista Alberti, *On the Art of Building in Ten Books*, trans. J. Rykwert, N. Leach, and R. Tavernor (Cambridge, MA: MIT Press, 1988), xi-xix; Paul Davies and David Hemsoll, "Alberti, Leon Battista," *Grove Art Online. Oxford Art Online*. http://www.oxfordartonline.com/subscriber/article/grove/art/T001530 (accessed November 20, 2010); Bertrand Gille, "Alberti, Leone Battista," in *The Complete Dictionary of Scientific Biography* [electronic resource], (Detroit: Charles Scribner's Sons, 2008. *Gale Virtual Reference Library*. Accessed 20 Mar. 2011.), 96-98; Anthony Grafton, *Leon Battista Alberti: Master Builder of the Italian Renaissance* (New York: Farrar, Straus and Giroux, 2000), 6-9; Girolamo Mancini, *Vita di Leon Battista Alberti* (Florence: G. C. Sansoni, 1882), passim.

ca. 1401, Lorenzo and his family were exiled from Florence by the powerful Albizzi family. As a result, Leon Battista was born in Genoa. In ca. 1416 his father Lorenzo sent young Leon Battista to study in Padua under the tutelage of Gasparino Barzizza (1360-1430), the famous grammarian, rhetorician, and Ciceronian scholar.[658] At that time Barzizza taught rhetoric and "moral authors"[659] at the University of Padua and ran a boarding school, where he taught both university and pre-university students, such as Leon Battista.

Alberti's studies with Barzizza lasted from approximately 1416 to 1418 (perhaps longer)[660] and provided the necessary grammatical and rhetorical background necessary for the study of law at the University of Bologna, where he enrolled in 1421. A degree in law was the standard preparation for a high level clerical career. Traditionally, a degree in the arts was the prerequisite for studying at any upper-level faculty such as law, theology, or medicine, but it was not at all unusual for students such as Alberti to be admitted into the law program at Bologna without such a degree.[661]

Once matriculated, a student was free to attend any course he chose. Alberti could have attended lectures on anything the *studium generale* had to offer, from the quadrivial arts, to medicine, to law.[662] By way of comparison, this academic trajectory is quite similar to that of

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[658] Barzizza also taught Francesco Filelfo, Francesco Barbaro, and George of Trebizond. See R. G. G. Mercer, *The Teaching of Gasparino Barzizza: with Special Reference to his Place in Paduan Humanism* (London: Modern Humanities Research Association, 1979), 2-6, 121-122. See also Grendler, *The Universities of the Italian Renaissance*, 210.
[659] "moralibus [auctoribus]." Grendler, *The Universities of the Italian Renaissance*, 207.
[660] There is little information on Alberti from 1418-1421. Barzizza left Padua in 1421, which would have made it possible for Alberti to have continued his studies with Barzizza until that date.
[661] On the arts degree as a prerequisite for graduate study, see Grant, *The Foundations of Modern Science in the Middle Ages*, 37; Olaf Pedersen, *The First Universities: Studium Generale and the Origins of University Education in Europe*, 271; James A. Weisheipl, "The Structure of the Arts Faculty in the Medieval University," 263. Unlike Paris and Oxford, the Italian universities of the 15th century were, by and large, graduate schools and did not grant bachelor's degrees. This may, in part, explain why Italian universities did not generally require a B.A.
[662] Giorgio Cencetti writes, "once matriculation had occurred, the student was free to attend courses, and the professors could not exclude him." Cencetti was the director of the *Archivio di Stato di Bologna*.

Nicolaus Copernicus later in the 15th century. Copernicus enrolled in the law faculty at Bologna, without a degree in the arts, and proceeded to study astronomy with Domenico Maria Novara and even resided in the astronomy professor's house.[663] Although it is speculated that Copernicus did not attend many legal lectures, he nonetheless passed the examinations for a doctorate in canon law.[664] Unfortunately no matriculation records from the University of Bologna survive prior to the 16th century,[665] so it is difficult to determine what Alberti was studying while in Bologna. It is not even clear if he ever received his degree.[666] A short biography (or possibly an autobiography) from the mid-15th century, frequently referred to as the

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Giorgio Cencetti, *Gli Archivi dello Studio Bolognese* (Bologna, Zanichelli, 1938), 66, quoted by Edward Rosen in Edward Rosen, *Copernicus and His Successors*, ed. Erna Hilfstein (London: Hambledon Press, 1995), 129.

663 Rosen, *Copernicus and His Successors*, 127. See also André Goddu, *Copernicus and the Aristotelian Tradition: Education, Reading, and Philosophy in Copernicus's Path to Heliocentrism* (Leiden: Brill, 2010), 18.

664 Copernicus received his doctorate from the University of Ferrara, where the costs of graduation and examination standards were lower. The University of Ferrara was known to be something of a "degree mill." See Goddu, *Copernicus and the Aristotelian Tradition*, 179-180 and 204. A similar trajectory is followed by Witelo (ca. 1230- ca. 1275), who studied law at the University of Padua, but he was clearly more interested in quadrivial topics and mathematical optics. See Siraisi, *Arts and Sciences at Padua*, 72-3; David Lindberg, "Witelo," in *The Complete Dictionary of Scientific Biography* [electronic resource], vol. 14 (Detroit: Charles Scribner's Sons, 2008. *Gale Virtual Reference Library*. Accessed 25 July 2011.), 457-462. Another similar academic trajectory is followed by the mathematician and natural philosopher Nicolaus Cusa who studied canon law at Padua, receiveing his degree in 1423. While studying law, he supposedly attended lectures with Paolo Toscanelli (also a friend of Alberti's) on astrology given by Prosdocimo. See J. E. Hofmann, "Cusa, Nicholas," in *The Complete Dictionary of Scientific Biography* [electronic resource], vol. 3 (Detroit: Charles Scribner's Sons, 2008. *Gale Virtual Reference Library*. Accessed 28 July 2011.), 512-513.

665 Rosen, *Copernicus and His Successors*, 127. Rosen again cites Cencetti as his source for this information.

666 Most secondary sources claim that Alberti received a law degree but are not only inconsistent on whether it was in canon or *in utroque iure* but provide no citations for verification. For examples, see Grafton, *Leon Battista Alberti,* 35-36; Joan Gadol, *Leon Battista Alberti: Universal Man of the Early Renaissance* (Chicago: University of Chicago Press, 1969), 5; Robert Tavernor, *On Alberti and the Art of Building* (New Haven: Yale University Press, 1998), 4; Gille, "Alberti, Leone Battista," 96. Mancini states that he "took a degree in law" [*prese la laurea in decreti*], but cites a passage from Alberti's story *Philodoxus* (1424), written while he was studying law in Bologna, that is neither relevant nor chronologically appropriate. Mancini, *Vita di Leon Battista Alberti* (1882), 59.

Vita di Leon Battista Alberti,[667] describing the first half of Alberti's life, states that after a series of mental and physical breakdowns in the later half of the 1420s, "On the physicians' orders, then, he [Alberti] did give up his legal studies, which had so greatly taxed his memory, *just as they were about to bear fruit*."[668] This biography further claims that "he could not live without intellectual stimulation" so "in his 24th year" (ca. 1428),[669] he "turned to [natural philosophy or medicine] and mathematics,"[670] suggesting that he may have begun attending lectures in the faculty of Arts and Medicine at the University of Bologna at this point. This breakdown and subsequent focus on the study of mathematics would certainly help explain where and how Alberti's quadrivial knowledge was developed, but unfortunately there is little definitive information on this portion of his life.

Also in 1428 the ban on the Alberti family was lifted in Florence and Leon Battista was able to enter the city of his father's ancestors. There he came in contact with much of the art and architecture that would influence him for the rest of his life. In the mid-1430s he was employed by Pope Eugenius IV and appears to have lived for periods of time in both Florence and Rome and became a canon of the Cathedral of Florence.

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[667] This biography is found in L. B. Alberti, *Opere volgari di Leon Battista Alberti*, ed. Anicio Bonucci, 5 vols., vol. 1 (Florence: Tipografia Galileiana, 1843), XC-CXVIII. An English translation was made by Renée Watkins. See L. B. Alberti (attributed to), "The Life of Leon Battista Alberti [The *Vita*]" in Renée Watkins, "L. B. Alberti in the Mirror: An Interpretation of the *Vita* with a New Translation," *Italian Quarterly* XXX, no. 117 (1989): 7-17. This biography will be discussed in more detail in an upcoming section.

[668] "...prope efflorescens intermisit." Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 8. My italics. See also the *Vita* in Alberti, *Opere volgari di Leon Battista Alberti*, XCIV.

[669] "...annos natus quatuor et viginti..." From the *Vita* in Alberti, *Opere volgari di Leon Battista Alberti*, XCIV. Inexplicably, Watkins does not translate this phrase in his English edition. This phrase is also included in the Fubini and Gallorini edition of the *Vita* found in R. Fubini and A. Menci Gallorini, "L'autobiografia di Leon Battista Alberti: Studio e edizione," *Rinascimento* 12, no. 68 (1972): 70. It is interesting to note that Barzizza was teaching at the University of Bologna from 1426-1428. See Grendler, *The Universities of the Italian Renaissance*, 208.

[670] Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 8. Watkins translates the disciplines to which Alberti turned his attention as "physics and mathematics." My alteration to this translation will be discussed in the following section.

From the mid-1430s to 1452 Alberti was extremely productive, writing the majority of his most famous works. These include *I Libri Della Famiglia* (ca. 1434-41), *Canis* (1441-42), *Mosca* (1441-42) and *Momus* (1443-50), as well as those relevant to the quadrivium: *De pictura* (1435), *Elementi di pittura* (between 1432 and 1435),[671] *De statua* (late-1430s-40s), *De re aedificatoria* (late 1440s-50s), and *Ludi rerum mathematicarum*[672] (ca. 1451). In the 1450s he received his first significant architectural commission from Sigismondo Malatesta. This commission was not for a new building, but was essentially a renovation of a preexisting church in Rimini, which is usually referred to as the Tempio Malatestiano (San Francesco). The new Albertian façade, unfinished to this day, clearly shows his interest in the Roman ruins he saw all around him. In the 1460s, with a commission from Ludovico Gonzaga, construction started on another of his major projects, San Sebastiano in Mantua.

In the mid-1460s Alberti, no longer a papal secretary, began to spend more time in Florence and was part of the intellectual circle that formed around Lorenzo de' Medici. He also spent time with the Duke of Urbino in this period and may have been consulted for some of his building projects. Alberti died in Rome in 1472.

The career of Alberti exemplifies that of a humanist scholar. He wrote in both Latin and the Tuscan dialect. He studied not only the classical literature of the ancient Romans but also the remnants of Roman art and architecture which surrounded him in his daily life. His interests were not limited by the traditional boundaries of medieval scholasticism. His literary treatments of painting and architecture helped raise their status from traditional craft to high Renaissance art and his use of both Latin and the vernacular helped soften rigid class distinctions. He was open to new ideas and freely challenged the prevailing distinctions of the past. In terms of the quadrivium, he

[671] A Latin version titled *Elementa picturae* was written some time between 1435 and 1446. See Stephen R. Wassell's commentary in Leon Battista Alberti, *The Mathematical Works of Leon Battista Alberti*, ed. and trans. K. Williams, L. March, and S. R. Wassell (Basel: Birkhäuser, 2010), 153.
[672] Also known as *Ex ludis rerum mathematicarum*.

was unrestrained by the Platonic and Boethian mathematical technicalities that so annoyed Oresme and were so adamantly defended by Prosdocimo. Alberti was a humanist interested in the quadrivial disciplines, not a quadrivial academic. He wrote several treatises on mathematical topics, but as a humanist, not as a professor of the quadrivial arts. Unlike our previous case studies, his mathematical output is relatively simple and straight forward. He is not interested in irrational ratios-of-ratios, uniformly difform motions, or dividing the *tonus* into equal parts. He is, first and foremost, a humanist writer, not a natural philosopher or mathematician. But this distinction is not always recognized in some of the secondary literature that deals with the mathematics of Alberti. To his detriment, all too often Alberti is presented as a quadrivial scholar *par excellence*. This portrayal has done no favors for Alberti.

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Universal Man

Since the late 15th century, scholars writing on Leon Battista Alberti (1404-1472) have generally promoted the idea that he was highly proficient in the mathematical arts. The perpetuation of these assessments has continued to this day, from the classic studies of Jacob Burckhardt and Rudolf Wittkower to the more recent treatments of Joan Gadol [Kelly] and Timothy Anstey.[673]

It is not my intention in the introduction of this chapter to argue for or against Alberti's actual quadrivial abilities. Judging from his works, which will be discussed shortly, he was perfectly capable. My main goal in this introduction is to establish an accurate and credible picture of Alberti's quadrivial knowledge within the intellectual society of 15th-century Italy and to strip away some of the myths that have infected "*Leon Battista Alberti: Universal Man of the Early*

[673] See the bibliography for the works of these authors.

Renaissance."[674] I want to place him back into a category of exceptional "many-sided"[675] men that I think more faithfully represents Alberti, the man. This introduction should allow for a better understanding of both the man and his context. I do not claim in this introduction to be correcting all of the problems I have found, nor am I addressing the literature written on Alberti as a humanist. I am only placing his demonstrated quadrivial skills into a realistic historical context so that Alberti's quadrivial philosophy can be better understood. The following discussion will focus on what I perceive to be one of the more pervasive problems in Albertian mathematical scholarship, the *Vita di Leon Battista Alberti*, also known as the "Autobiography of Leon Battista Alberti."

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The *Vita*

Written in Latin and in the third-person,[676] this 10-page "autobiography" was included along with an Italian translation in the *Opere volgari di Leon Battista Alberti*, published in the middle of the 19th century, and has been a major source of information on Alberti's life and abilities ever since.[677] As a chronology and outline of Alberti's interests, this work readily fits in with other sources on his early career and his own literary output, but as an assessment of Alberti's talents and skills it should be used with caution for many of its claims are inflated.

[674] This is the title to Joan Gadol's book.
[675] Burckhardt refers to Alberti as a "many-sided" man. See Jacob Burckhardt, *The Civilization of the Renaissance in Italy*, trans. Middlemore (New York: Macmillan and Co., 1904), 136.
[676] Martin McLaughlin notes that Caesar wrote of his military campaigns in the third person and may be Alberti's precedent for his autobiography. See Martin McLaughlin, "Burckhardt and Alberti: Restoring the Original," in *Burckhardt's Renaissance, 150 Years Later: Symposium Held at Jesus College, Oxford 19 April 2010,* edited by Owen Margolis (Oxford: Oxford Centre for Medieval History), 1.
[677] The English translation by Watkins is 10 pages long. For the English translation, see Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 7-17. The Latin with facing Italian translation covers 28 pages. See Alberti, *Opere volgari di Leon Battista Alberti*, XC-CXVIII. The Latin edition of the *Vita* by Fubini and Gallorini is 10 pages long. See Fubini and Gallorini, "L'autobiografia di Leon Battista Alberti," 68-78.

This work goes by many names: *The Autobiography of Leon Battista Alberti, The Life of Leon Battista Alberti, The Anonymous Life of Leon Battista Alberti, La Vita di Leon Battista Alberti*, etc. I will henceforth refer to it as simply the *Vita*.[678] Renee Watkins has convincingly argued that it was written in 1437 or 1438 based on what is and is not mentioned of Alberti's written corpus.[679] This would put Alberti in his mid 30s and at a point in his literary career when he had recently written *De pictura* (1435) and had been surveying Rome,[680] but had not yet started writing the bulk of his quadrivial materials such as *De re aedificatoria*,[681] *De statua*, *Ludi rerum mathematicarum,* or *Descriptio urbis Romae* (all three written some time between 1443 and 1452), or *De componendis cifris* (1460s).[682] I will start by describing and commenting upon the quadrivial references found in the *Vita* and then briefly elaborate on the context for some of these quadrivial claims as well as how such claims have become uncritically incorporated in some of the modern literature.

On the very first page of the *Vita*, Alberti is broadly described as a man "devoted to," among other things, "the liberal arts and all recondite and difficult knowledge."[683] That Alberti was

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[678] This appears to be the standard way to refer to this work as of the late 20th century. It should not be confused with Mancini's biography of Alberti, *Vita di Leon Battista Alberti*, cited above.
[679] Watkins, "L. B. Alberti in the Mirror," 6.
[680] Gadol, *Leon Battista Alberti: Universal Man of the Early Renaissance*, 9-10.
[681] *De re aedificatoria* was started in the 1440s and finished ca. 1552 or perhaps even later. See Grafton, *Leon Battista Alberti*, 277-279; Heather Horton, "Authority and Innovation in Alberti's Theory and Practice" (Ph. D. diss., Institute of Fine Arts, New York University, 2010), 48-67.
[682] Alberti also refers to a work called *Historia numeri et linearum* (often translated *Arithmetic and Geometry*) at the end of the Prologue in *De re aedificatoria*. He states that this work was "appended to" or "inserted into" [*additi*] *De re aedificatoria*. Leon Battista Alberti, *L'architettura (De re aedificatoria) [Latin and Italian]*, trans. G. Orlandi and P. Portoghesi (Milan: Edizioni Il Polifilo, 1966), 17; Alberti, *On the Art of Building*, 6. This work has been lost as an independent entity, but Heather Horton argues that its content was incorporated into the larger architectural treatise. Horton, "Authority and Innovation in Alberti's Theory and Practice," 56-57. Alberti also mentions a *Commentaria rerum mathematicarum* (also lost) in Book III, Ch. 2 of *De re aedificatoria*, but this may simply refer to *Historia numeri et linearum.* Cf. L. Lefaivre, *Leon Battista Alberti's Hypnerotomachia Poliphili: Re-Cognizing the Architectural Body in the Early Italian Renaissance* (Cambridge, MA: MIT Press, 2005), 151. See also Robert Tavernor, *On Alberti and the Art of Building* (New Haven: Yale University Press, 1998), 16.
[683] Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 7.

generally interested in the liberal arts in his formative years is not in question. This interest was briefly alluded to by Antonio Beccadelli, a fellow student of Barzizza,[684] in his ribald poem *Hermaphroditus*, written in 1425 while Alberti was at the University of Bologna supposedly studying law. Beccadeli writes of Alberti,

> You are pleasant company, very handsome, witty, wholly dedicated to the liberal arts, born of the true Albertis, incomparable in the nobility of your manners. You are liked for your rare talents, and I like you for your genuine simplicity. You are a true and honest friend. Tell me how you get on with women."[685]

In terms of Alberti's specific quadrivial abilities the problems of the *Vita* begin a few lines further on, where the merits of Alberti's musical skills and sophistication are described. It states,

> He learned music without instructors, and his compositions were commended by learned musicians…he liked to sing, but only in his own house by himself, or in the country with his brother or immediate family. He enjoyed playing the organ, and was considered one of the foremost performers on that instrument. Not few musicians became more learned through his criticism.[686]

That he was a self-taught musician who enjoyed performing for friends and family is hardly surprising for an educated gentleman of the 15th century, but that he might have been considered "one of the foremost performers" on the organ requires further qualification. Although the organs in 15th-century Italy were not the sophisticated, immensely complex machines that they would become in the 17th century, there were numerous examples of fairly complicated large instruments in the cathedrals of Italy in this period, played by famous and proficient organists. The most notable organist of the period and a man certainly known to Alberti was Antonio Squarcialupi (1416-1480), who was the organist at Santa Maria del Fiori from 1432 until his death.[687] It is probably a bit of an overstatement to suggest that Alberti was one of the foremost

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684 Grafton, *Leon Battista Alberti*, 39.
685 Beccadelli quoted in Tavernor, *On Alberti and the Art of Building*, 4
686 Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 7.
687 Kurt von Fischer and Gianluca D'Agostino, "Squarcialupi, Antonio," *Grove Music Online. Oxford Music Online*. http://www.oxfordmusiconline.com/subscriber/article/grove/music/26473 (accessed March 18, 2011). Recent scholarship has questioned the mythic status of Antonio Squarcialupi along similar

performers on these large organs considering there was a highly developed professional class of organists in his day who would have played these instruments. It is more likely that he was proficient at the smaller *portative* organs which were usually pumped with one hand while the other hand played a simple keyboard.[688]

Figure 5.1: Francesco Landini's *Musica son* (left) from the *Squarcialupi Codex* and a detail that shows him playing a portative organ (right).

In fact, the 15th century was the heyday for these portative organs.[689] Due to the limitations on pipe length imposed by their portability, they were high-pitched instruments and commonly found in the types of wealthy households typical of Alberti's patrons. In Trecento and Quattrocento Florence, secular songs were hugely popular and frequently composed by

lines to the criticisms I am leveling at the mathematical myth of Alberti. See James Haar and John Nádas, "Antonio Squarcialupi: Man and Myth," *Early Music History: Studies in Medieval and Early Modern Music* 25, no. (2006): 105-168. I would like to thank Ruth DeFord for calling this essay to my attention.

688 Early portatives probably had diatonic capabilities, but chromatic portatives spanning two octaves had been developed by the late 14th (or early 15th) century. Geoffrey Bridges, "Medieval Portatives: Some Technical Comments," *The Galpin Society Journal* 44, no. (1991): 104-105.

689 Ibid.

professional organists.[690] A vocalist was often accompanied by a portative organ. Because only one hand played the keyboard on these instruments, it is generally thought that only a single part, probably the high part, could have been played.[691] Coincidentally, the blind Francesco Landini (1325-1397), the great-granduncle of Alberti's friend Cristoforo Landino (1424-1498),[692] was the most famous organist in Florence in the late 14th century and was also known to have been a consummate composer and performer of secular songs on these small portative organs.[693] Alberti most certainly would have been familiar with Francesco's legacy not only through his friendship with the blind organist's great-grandnephew, Cristoforo, but also because Francesco was a prominent figure in Giovanni da Prato's *Paradiso degli Alberti,*[694] which describes the cultural,

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[690] For a collection of these songs, see Antonio Squarcialupi, ed., *Squarcialupi Codex,* ed. F. Alberto Gallo (Florence: Giunti Barbèra, 1992).

[691] Johannes Wolf, "Italian Trecento Music," *Proceedings of the Musical Association* 58, no. (1931): 24. For a more elaborate discussion and investigation of what portatives can actually do, see Bridges, "Medieval Portatives: Some Technical Comments," 115. Bridges states that polyphonic playing is possible but requires more pumping of the bellows.

[692] Cristoforo Landino from his *Apologìa di Dante* (1481), singing the quadrivial praises of his older friend Leon Battista Alberti writes, "But what species of mathematics was unknown to him? As a geometer, as an arithmetician, as an astrologer, as a musician, and in perspective he was more marvelous than men from centuries past." [*Ma quale spetie di matematica gli fu incognita? Lui geometra, lui aritmetico, lui astrologo, lui musico, et nella prospettiva maraviglioso più che huomo di molit socoli.*] Cristoforo Landino from his *Apologìa di Dante*, quoted in Girolamo Mancini, *Vita di Leon Battista Alberti*, 2nd ed. (Rome: Bardi Editori, 1967), 442. Like the *Vita*, this passage by Landino is often quoted when discussing the mathematics of Alberti. For examples, see Timothy A. Anstey, "Fictive Harmonies: Music and the Tempio Malatestiano," *RES* 36, no. (1999): 187; Gadol, *Leon Battista Alberti*, 3.

[693] Wolf, "Italian Trecento Music," 17. Wolf identifies Landini as "the very famous organist of Santa Maria del Fiore," but other sources I have consulted list him as the organist at San Lorenzo from 1365-97. This is also where he was buried. His tombstone shows him playing a portative organ. He was involved in the design of the organ for Santa Maria del Fiore in 1387, but I have found no other source making him the organist there. See also Kurt von Fischer and Gianluca D'Agostino, "Landini, Francesco " *Grove Music Online. Oxford Music Online*. http://www.oxfordmusiconline.com/subscriber/article/grove/music/15942 (accessed July 14, 2006). For more information on Landini's involvement on the organ at Santa Maria del Fiore, see Dorothea Baumann and Barbara Haggh, "Musical Acoustics in the Middle Ages," *Early Music* 18, no. 2 (1990): 203. There are 154 extant secular multi-part songs by Landini. Geoffrey Bridges believes that Landini's portative organ was chromatically keyed and spanned about an octave and a half. See Bridges, "Medieval Portatives: Some Technical Comments," 105.

[694] Hans Baron establishes the date of composition for this work in the winter of 1425/6 (or soon thereafter). Giovanni da Prato had lost his lectureship at the Florentine university at this time and also lost his "position as an architect-counselor on the nascent cathedral dome" due to a "violent" argument

political, and intellectual life in and about the Alberti country estate in the late 14th century.[695] In Francesco's oft-quoted madrigal, *Musica son*,[696] he makes a very curious and prescient statement concerning the dilettantes who claim to have skills in composition. Landini writes,[697]

I am Music, who weeping regret to see
Intelligent people desert my sweet
And perfect effects for popular songs:
Because ignorance and vice abound,
Good is deserted, and the worst is seized.

Everyone wants to arrange musical notes,
Compose madrigals (*madrialle*), catches (*cacce*), ballads (*ballate*),
Each holding his to be perfect;
He who would be praised for a virtue
Must first come to earth.

Formerly my sweetnesses were prized
By knights, barons, and great lords;
Now gentle hearts are corrupted.
But I, Music, do not lament alone,
For I see even the other virtues deserted.

Music (the voice of Francesco Landini) laments the invasion of charlatans who "arrange musical notes" into songs which they claim to be perfect. Michael Long in his article, "Francesco Landini and the Florentine Cultural Elite," goes even further and claims that Music is complaining of "composers who ply their trade without the proper liberal-arts training."[698]

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with Brunelleschi over a technical detail. Hans Baron, *The Crisis of the Early Italian Renaissance: Civic Humanism and Republican Liberty in an Age of Classicism and Tyranny*, 2 vols. (Princeton: Princeton University Press, 1955), 297-299. Baron also argues that Giovanni was an early proponent of the Florentine vernacular. Baron, *The Crisis of the Early Italian Renaissance,* 300-309. I find it quite interesting that Giovanni wrote this book just a few years before the lifting of the ban on Alberti family in Florence and less than a decade before Alberti wrote *Della famiglia* (1433-4).

[695] Giovanni del Prato's description of Francesco Landini includes an interesting part describing how the theoretical music of the Boethian quadrivium is closely associated with astrology, and also briefly describes the *musica mundana* of Cicero's "Dream of Scipio" as well as brief descriptions of *musica humana* and its connections to the stars. Giovanni da Prato, *Il Paradiso degli Alberti: ritrovi e ragionamenti del 1389*, ed. Alessandro Wesselofsky (Bologna: 1867), vol. 1, pt. 1, pp. 101-107.

[696] Sometimes called *Musica sono*.

[697] Leonard Ellingwood, "Francesco Landini and His Music," *Musical Quarterly* 22, no. 2 (1936): 194.

[698] Michael P. Long, "Francesco Landini and the Florentine Cultural Elite," *Early Music History* 3, no. (1983): 93. Franco Sacchetti (ca. 1332-1400) also wrote a poem complaining that the world was rife with

Because of the chronology there is no possibility that Landini was referring to Alberti in this song. He is describing the situation he sees in the late 14th century of musical dilettantes who are not even in the same class as himself, a professional musician. He clearly describes unqualified "composers" as poseurs, outrageously bragging about their "perfect" compositions. In light of this comment by Landini, the claims in the *Vita,* (that Alberti was a leading organist and musician and that his compositions were lauded by "learned musicians"), require further evidence to back them up. After all, Alberti was not a professional organist or a trained musician of any kind. The *Vita* itself says as much in this same section, "He learned music without instructors."[699] But there is no other evidence for his exceptional skills. There are no extant compositions by Alberti or any references by musicians or music historians about Alberti and his talents. We have only the word of the *Vita*.

Several lines after the section, the story is told of Alberti's memory problems in law school and subsequent interest in studying the quadrivial arts that was referred to above. We are told that after his breakdown Alberti "devoted himself" [*se contulit*] "*ad phisicam [*or *philosophiam] se atque mathematicas artes.*" There is some confusion concerning whether it should read *phisicam atque mathematicas artes* or *philosophiam atque mathematicas artes*.[700] In this context both *phisica* or *philosophia* would most likely mean natural philosophy, which would have been

rhymers and amateur musicians. "Pieno è il mondo di chi vuol far rime…" See Wolf, "Italian Trecento Music," 18.

[699] Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 7.

[700] Roberto Cardini claims that the Latin *Vita* found in Fubini and Gallorini's 1972 article inaccurately uses the word *phisicam* in place of *philosophiam*. Cardini claims that *philosophiam* is supported by the manuscript source. See Roberto Cardini, Lucia Bertolini, and Mariangela Regoliosi, eds., *Leon Battista Alberti: la biblioteca di un umanista* (Florence: Mandragora, 2005), 153. See also the Latin transcription of the *Vita* and accompanying notes in Fubini and Gallorini, "L'autobiografia di Leon Battista Alberti," 62 and 70. This substitution is evident in Watkins's translation (who uses Fubini and Gallorini) and translates it as, "physics and mathematics." See Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 8. Anthony Grafton seems to translate this phrase, "the arts of nature and medicine," but he fails to cite where the quotation comes from, so I am making an educated guess that it is from the *Vita*. Grafton, *Leon Battista Alberti*, 20.

included in the faculty of arts and medicine at Bologna.[701] Both versions use the term *artes mathematicas*, the mathematical arts, or, in other words, the quadrivium.

Considering Alberti's various writings on mathematical topics, as we shall discuss in the next section, he most certainly had studied the basic quadrivial arts. Between Beccadeli's poem and the *Vita*, it would appear that some of this training occurred at the University of Bologna; however, any specifics are lacking.

Towards the end of the *Vita* another possible quadrivial subject is described, Alberti's divination skills. It states,

> For he used to combine in predicting the future, the wisdom of a learned man with a genius in the arts of divination. There are letters of his to Paolo the Physician in which he describes the fortunes of his country whole years before the event. He predicted the fortunes of the papacy as they were to unfold over the next twelve years, and he foretold the actions of many other cities and princes, as the memories of his friends and intimates will confirm.[702]

These skills of divination are not terribly specific, but the reference to "Paolo the Physician" very strongly suggests that Alberti was practicing astrology. Paolo the Physician (*Paulus phisicus*)[703] is a reference to Paolo Toscanelli (1397-1482) the famous medical-astrologer from Florence whom Alberti knew from at least 1429-1436[704] and perhaps as early as the 1410s.[705] Unfortunately, no letters between Toscanelli and Alberti remain. Explicit evidence of Alberti as

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701 See Jerome J. Bylebyl, "The Medical Meaning of *Physica*," *Osiris* 6, no. (1990), 16-41. See also Lemay, "The Teaching of Astronomy in Medieval Universities," 198.

702 Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 14-15.

703 Fubini and Gallorini, "L'autobiografia di Leon Battista Alberti," 76.

704 Paul Lawrence Rose, "Humanist Culture and Renaissance Mathematics: The Italian Libraries of the Quattrocento," *Studies in the Renaissance* 20, no. 1 (1973): 62-63. Alberti dedicated *Intercoenales* to Toscanelli in 1429.

705 Toscanelli was studying at the University of Padua at the same time that Alberti was studying in Padua with Barzizza. See Kristeller, *Studies in Renaissance Thought and Letters*, 29-30; Rose, "Humanist Culture and Renaissance Mathematics," 63.

an astrologer has only recently come to light.[706] A manuscript copy of Cicero's *De legibus* was found in 1974 containing marginalia by Alberti. In these marginalia are five horoscopes "*di personaggi illustri*," including one for Piero de' Medici and another for Alberti himself.[707]

The predictions about the "fortunes of the papacy" mentioned in the *Vita* probably refer to the post-schism chaos of the Roman Catholic Church and the reformative attempts made by the Council of Basel. It is difficult to gauge the level of proficiency Alberti had in the astrological arts, let alone how his proficiency would compare to that of the average educated man in 15th-century Italy. But the *Vita* does not claim that Alberti was comparable to Toscanelli, only that he corresponded with him. The references summarized above are all of the quadrivial material pertaining to Alberti found in the *Vita*: his general mathematical skills, his musical skills (compositional and as an organist), and his skills as an astrologer. The only implausible quadrivial claims in the *Vita* are the musical ones. There is absolutely no record of his compositions or his performance skills in the musical literature that I have seen, and Francesco Landini makes it quite clear that poseurs were ubiquitous. He would have had to shine among the likes of Guillaume Dufay (c. 1400-1474) and Antonio Squarcialupi (1416-1480),[708] which

[706] Numerous references to astrology in the writings of Alberti have been studied by Cardini. Most of the references to astrology and astronomy in Alberti's written work are in passing and only serve to prove that Alberti was living in a time when astrology was taken seriously by the general population. See Tester, *A History of Western Astrology*, 204-243.

[707] These horoscopes were found in MS. Biblioteca Nazionale Centrale di Firenze, Conventi Soppressi I93. See Roberto Cardini, "Alberti e l'astrologia," in Cardini et al., eds., *Leon Battista Alberti: la biblioteca di un umanista*, 151-152. Unfortunately, these horoscopes are not reproduced or analyzed in Cardini's essay.

[708] Other musicians of note who were either from or working in Florence from the 14th to the early 15th centuries are: Giovanni da Cascia, Donato de Cascia, Ghirardello da Firenze, Lorenzo da Firenze, Andrea da Firenze, Paolo Tenorista (da Firenze), Jacopo da Bologna, Vincenzo da Arimini, Nicolao da Perugia, Bartolino da Padua, and the papal singer Magister Zacherias. See Wolf, "Italian Trecento Music," 17.

seems unlikely for an untrained and amateur musician. Hyperbole of this sort is not difficult to find in an age of hagiographic writers[709] and courtiers jockeying for patronage.

In non-quadrivial matters, the *Vita* further describes the many sides of Alberti. It states, “He practiced ball playing, the use of the javelin with thong, running, wrestling, and above all, the climbing of steep mountains.” It goes on to say that he excelled in equestrian and military exercises. He was interested in all types of knowledge, art, and craft.[710] He painted, sculpted, designed buildings and cities, and wrote fiction and nonfiction in both Latin and the vernacular. One of the strongest impressions from the *Vita* was Alberti's strong moral virtue and stoic continence. Nearly half of the *Vita* is made up of one or two-line remarks or anecdotes, which demonstrate cleverness, wisdom, his moral character, and his sense of humor. The *Vita* paints a very interesting picture of the ideals of a Renaissance humanist.

The hyperbole already encountered in the *Vita* certainly draws suspicion to some of its claims, but the utility/value of the document as a primary source for assessing Alberti's mathematical accomplishments is further compromised when a few of its non-quadrivial claims are examined more closely. If the *Vita* is used broadly to point out that Alberti was an exemplary man of his time, accomplished in all the areas recognized as the hallmarks of an educated and cultivated person, then the legitimacy of each specific claim need not be challenged. However, the *Vita* has occasionally been used as a primary source to say specific things about Alberti's particular accomplishments. When used in this fashion, it is no longer possible to consider the *Vita* in a general sense. The presence of patently unrealistic claims

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[709] See Grafton, *Leon Battista Alberti*, 64-70. Robert Black writes that humanists have an "exaggerated tendency to self-advertisement." See Robert Black, "Humanism," in *The New Cambridge Medieval History, Vol. 7, c.1415-c.1500,* ed. Christopher Allmand (Cambridge: Cambridge University Press, 1998), 258.

[710] Alberti (attributed to), "The Life of Leon Battista Alberti," in Watkins, "L. B. Alberti in the Mirror," 7-10.

undermines its legitimacy as an uncorroborated source for specific attributes of Alberti, the man. As a somewhat comical example, the *Vita* claims that "with his left foot pressed against the very wall of the highest temple, he would throw an apple straight up far above the highest roof of the building; inside the church, he would throw a little silver coin up with such force that whoever was with him could clearly hear it strike the interior of the loftiest dome."[711]

One has to assume that the building to which the author of the *Vita* is referring is the *Duomo* of Santa Maria del Fiore in Florence, for this was by far the tallest "temple" or "church" with the "loftiest dome" in Florence at this time. The exterior and interior height of the *Duomo* without the lantern is approximately 90 meters. Basic kinematic equations determine that an apple thrown straight up to the height of the *Duomo*, 90m, would require an initial velocity of 42m/s (94 mph). This calculation does not account for air resistance, so Alberti's throw in the real world would have had to have been even harder. Only the very best professional baseball pitchers attain speeds of 42m/s (94 mph), and they are throwing horizontally, not vertically, and in a resisting atmosphere.[712] If Alberti lived today, and could do as the *Vita* claims, he could quite possibly be the hardest throwing pitcher in the world.

The *Vita* is thoroughly entertaining to read and it is most certainly filled with interesting material that could be explored by any number of scholarly disciplines. It is clearly of interest as a piece of humanist literature in a vein similar to Petrarch's "Epistle to Posterity" or Erasmus' *Compendium vitae*,[713] but compared to these other humanist autobiographical works, it is much

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711 Ibid., 7.

712 Note that the typical apple is approximately the same weight as a baseball. Rykwert reports Alberti's physical abilities without comment or even a citation to the *Vita*. See Rykwert's introduction to Alberti, *On the Art of Building*, xii.

713 See Francis Petrarch, *Letters from Petrarch*, ed. and trans. Morris Bishop (Bloomington: Indiana University Press, 1966), 5-12; *Collected Works of Erasmus*, ed. and trans. Mynors and Thomson, vol. 4, *The Correspondence of Erasmus. Volume 4, Letters 446 to 593 (1516-1517)* (Buffalo, NY: University of

more hyperbolic. I point out the exaggerations in the *Vita* not to make fun of it as a historical document, but to cast some doubt on its value as an accurate depiction of Alberti's mathematical abilities. That he was interested in mathematics is not in question, but his level of achievement is, and overstating his abilities puts unrealistic expectations on his mathematical works. Alberti was not in the same mathematical league as Oresme or Prosdocimo, but, as we shall see, his mathematical goals were different.

The only reliable sources for evaluating Alberti's quadrivial skills and his mathematical philosophy are his own writings. These demonstrate his actual knowledge and thought and do not require any faith in claims made hagiographers. As such, the remainder of this essay will deal directly with Alberti's written quadrivial material.

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The Quadrivium in the Works of Alberti

There are essentially seven works in the extant corpus of Alberti containing enough quadrivial material to warrant our attention. They are *De pictura* (1435), *Elementa picturae* (between 1432 and 1435), *De re aedificatoria* (principally written in the late-1440s and 50s), *De statua* (late-1430s-1440s),[714] *Descriptio urbis Romae* (1443-1452),[715] *Ludi matematici (Ludi rerum mathematicarum)* (1443-1452),[716] and *De componendis cifris* (1460s).[717]

The work that demonstrates the most theoretical quadrivial knowledge and the work that will be discussed in the most detail is *De re aedificatoria*. The three works written in the period of

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Toronto Press, 1977), 402-410. I would like to thank Margaret King for directing me to these other humanist autobiographies.

714 Jane Aiken reviews many differing theories on the dates for *De statua*. See Jane Andrews Aiken, "Leon Battista Alberti's System of Human Proportions," *Journal of the Warburg and Courtauld Institutes* 43, no. (1980): 95-96.

715 Joan Gadol lists it as 1440s: Gadol, *Leon Battista Alberti*, 167. Robert Tavernor thinks ca. 1444: Tavernor, *On Alberti and the Art of Building,* 13.

716 Tavernor thinks ca. 1450: Ibid.

717 Tavernor thinks ca. 1465: Ibid.

1443-1452, *De statua*, *Descriptio urbis Romae,* and *Ludi rerum mathematicarum,* are mostly geometric in the broad sense of the term. They show how to measure real terrestrial situations and use basic arithmetic and geometry to solve for unknowns. The works on drawing and painting, *De pictura* and *Elementa picturae* are in some sense headed in the opposite direction from *De re aedificatoria* and the geometrical works mentioned above. Rather than the objective of measuring how things actually are, they deal with how things seem. They are an interesting application of mathematics to perception; they describe the geometry of sight. And in the last work, *De componendis cifris*, Alberti develops a substitution cipher and statistically analyses written language for the purpose of breaking codes.

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Geo-metry – Terrestrial Measurements

Alberti copiously annotated his own copy of Euclid's *Elements*,[718] but in general Euclid's strictly syllogistic approach appears to have held little interest for Alberti in his own written works.[719] The purity of Euclid's logic and the elaborate demonstrations built up from a small group of axioms is impressive and in some sense beautiful in the abstract, but it was the application of Euclid's results to the physical world that most interested Alberti and it is this application of geometry to practical tasks that makes Alberti's geometrical work of interest to modern scholars. In particular, he was interested in geometry at its most literal, as measure of the terrestrial world.[720] His interest in physical measurement also led to representing the world proportionally on paper, in other words, mapping to scale. The dominant projection utilized by

[718] Alberti owned a copy of Campanus of Novara's Latin translation of Euclid. See Wassell's commentary in Alberti, *The Mathematical Works of Leon Battista Alberti*, 78 and 153.
[719] Alberti had certainly been taught Euclidian geometry. See Ida Mastrorosa, "Alberti e il sapere scientifico antico: fra i meandri di una biblioteca interdisciplinare," in Cardini et al., eds., *Leon Battista Alberti: la biblioteca di un umanista,* 142.
[720] This definition of geometry as earth-measuring is consistent with Martianus Capella's treatment of the discipline. See Capella, *The Marriage of Philology and Mercury,* vol. 2, 220, line 588.

Alberti in his discussions of maps is what is referred to as the azimuthal equidistant projection. Distances from a central point are measured radially and angularly (plane polar coordinates). These are then plotted two-dimensionally but scaled down in size. It is perhaps the most basic form of mapping and requires only primitive equipment: rope, a central stake, and a tool for measuring angles. In *Descriptio urbis Romae*[721] Alberti describes the procedure for mapping the city of Rome from a central location using a device he refers to as an *orizonte*, a *horizon*, to measure horizontal angles. The *orizonte* is a simplified astrolabe, a common instrument used by astronomers, astrologers, navigators, and surveyors.[722] See Figure 5.2. The radial distances were measured by counting steps from the origin to the location in question.[723]

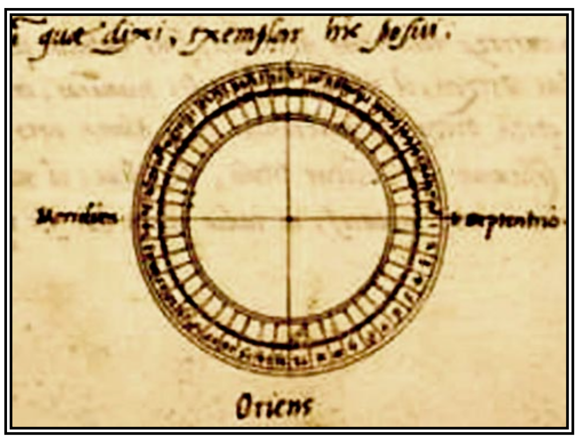


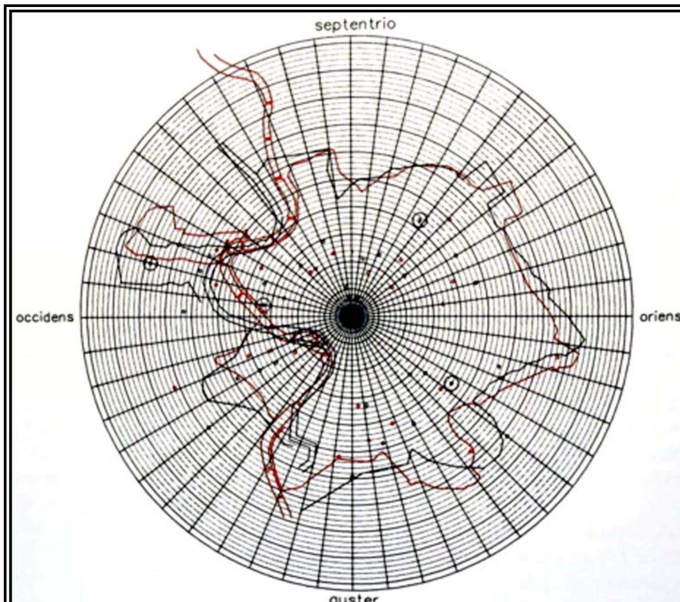
Figure 5.2: Detail from Alberti's *Descriptio urbis Romae* showing an "*orizonte*."[724]

[721] *Descriptio urbis Romae,* in Leonis Baptistae Alberti, *Opera inedita et pauca separatim impressa*, ed. Hieronymo Mancini (Florence: J. C. Sansoni, 1890), 36-46. Seven of these pages are data.
[722] It is the horizontal (the azimuthal) mechanism of a modern theodolite, the measurement instrument used in surveying. Descriptions for constructing paper or cardboard astrolabes are frequently encountered in late medieval astronomical texts. Paper mathematical instruments that could be cut out and put together were commonly part of early printed mathematical texts. See Mario Biagioli, "From Print to Patents: Living on Instruments in Early Modern Europe," *History of Science* 44 (2006): 160-161.
[723] Gadol, *Leon Battista Alberti*, 171-175.
[724] Leon Battista Alberti, *Page from Descriptio urbis Romae*, mid-15th century, Library of Congress Exhibition: Rome Reborn: The Vatican Library & Renaissance Culture, January 8 - April 30, 1993 [http://www.loc.gov/exhibits/vatican/arch.html], Vatican City. East is at the bottom.

The azimuthal equidistant projection had been a standard way of organizing astronomical information for centuries. Such a projection makes perfect sense for the sky which appears to rotate around the north celestial pole (for those observing it in the northern hemisphere). For example, the celestial maps of Conrad of Dyffenbach from 1426 based on Ptolemaic information were constructed using this projection, the same projection used by Alberti for his map of Rome. No period examples of Alberti's map exist, but below is a modern reconstruction using his data for comparison. See Figures 5.3 and 5.4.

Celestial map of northern hemisphere. Rome, Bibl. Vaticana, cod. Palat. 1368, fols. 63v–64r (p. 178)--(1426) Conrad of Dyffenback

Figure 5.3:
Celestial Map of Dyffenbach [725]



After Alberti's *Descriptio urbis Romae* (ca. 1444). Computer pen plot: in red/gray, the city walls of Rome, the Vatican, Tiber and its bridges today; in black, according to Alberti's *Descriptio* (Olivetti/Alberti Group).

Figure 5.4:
Reconstruction of Alberti's Map of Rome[726]

This cartographic projection was not unknown in the 15th century and is implicit in Ptolemy's *Geography* from the mid-2nd century A.D., which had been rediscovered and translated into

[725] Image from Patricia Fortini Brown, "Laetentur Caeli: The Council of Florence and the Astronomical Fresco in the Old Sacristy," *Journal of the Warburg and Courtauld Institutes* 44, no. (1981): p. 21.
[726] Image from Tavernor, *On Alberti and the Art of Building*, 14.

Latin in 1406 by Jacopo Angeli della Scarpetia.[727] The rediscovery of Ptolemy elicited a new era of high-level geographic and cartographic activity in the 15th century which culminated in Mercator's 16th-century projections.

The simple "Albertian projection" (azimuthal equidistant - plane polar coordinates) is not described by Ptolemy directly, but is clearly assumed in all of his discussions of more complicated methods. In terms of cartography and geometry, Ptolemy's *Geography* is significantly more complex than Alberti's *Descriptio urbis Romae*. Ptolemy is attempting to map a sphere (the earth) in two dimensions while having some semblance of regular scale across non-radial measurements. The azimuthal equidistant projection used by Alberti becomes less and less usable as the curvature of the mapped area becomes more pronounced. For example, if Alberti had wanted to make a map of the earth using this technique with the center at the North Pole, local measurements near the pole would be satisfactory, but measurements along the equator would be about 41% too large. South of the equator it would be even worse. The geometrical problems associated with this task are not easily overcome and demonstrate Ptolemy's interest in practical mathematical solutions to real world problems. This interest in applied mathematics by Ptolemy[728] is sometimes overlooked by historians who fall prey to the stereotype that ancient science was mostly metaphysical and that Galileo was the first empirical natural philosopher. The reality is that Ptolemy's *Geography* (*Cosmographia*) is nothing but

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727 Also known as Jacopo d'Angelo. A description of the Greek and Latin editions of Ptolemy's *Geography* in the late Middle Ages and Renaissance is found in Berggren and Jones's introduction to Ptolemy, *Ptolemy's Geography: an Annotated Translation*, 41-53. The Greek title of Ptolemy's work translates to *Guide to Drawing a World*. This title succinctly describes the content of the book. Alberti's *Descriptio urbis Romae* can be seen as a mini-version of Ptolemy's much larger and exhaustive work. More than forty 15th-century manuscript copies of Ptolemy's *Cosmographia/Geography* exist, attesting to its popularity. See Christiane L. Joost-Gaugier, "Ptolemy and Strabo and Their Conversation with Appelles and Protogenes: Cosmography and Painting in Raphael's School of Athens," *Renaissance Quarterly* 51, no. 3 (1998): 769.

728 Similar examples of his empirical scientific method can be found in *Harmonics*, his book on acoustics and music theory. See Barker, *Scientific Method in Ptolemy's Harmonics*.

practical. It is a how-to manual for making a world map and Alberti's *Descriptio* is exactly the same thing, except instead of describing how to make a map of the world, Alberti is describing how to make a map of Rome, which by comparison is approximately flat[729] and thus does not require the complicated projections used by Ptolemy.

In a slightly different context the same plane polar coordinates are employed to measure the physical world in his short essay, *De statua*.[730] This time he measures human beings. But in *De statua* he adds one more dimension, making them cylindrical polar coordinates. In this method every point on the body is located by three variables: angular position, radial distance, and vertical distance. The origin of the *orizonte* is placed on the crown of the head. Alberti describes this modified *orizonte* as being similar to an astrolabe and renames this new instrument a *finitorium*[731] or *diffinitore* in the Italian edition. This instrument sits on the head tangentially, like a halo of sorts. A rod could be attached to the zero-point and extended to whatever radial distance is desired and then a plumb line dropped to any particular altitude. In this way, any point on the body could be fully located in three dimensions.[732] See Figure 5.5. Using this instrument, the human form could be quantified.

Alberti writes,

> I want to establish ... not the particulars of this man or that one, but as far as possible, that exact beauty granted by Nature and given, as if in select portions, to many bodies. ... I have therefore chosen many bodies which are reputed to be the most beautiful by those

[729] I say "approximately flat" because it is a tiny section of the sphere of the earth and effectively flat. Alberti later uses the same approximation for one-point perspective in his texts on painting. Alberti does not comment on this approximation in any of his texts.
[730] *De statua* is found in Leon Battista Alberti, *On Painting and On Sculpture: The Latin Texts of De pictura and De Statua*, ed. and trans. C. Grayson (New York: Phaidon, 1972). It is only eight pages of Latin in Grayson's edition. The Italian version, *Della statua*, along with the table of human dimensions can be found in Alberti, *Opere volgari di Leon Battista Alberti*, 163-186.
[731] *Finitor* is Latin for surveyor.
[732] Using the Pythagorean theorem a comparative analysis of distances between body parts that do not lie parallel to the axes of the coordinate system could be done (e.g., nose to hand), thus allowing additional structures of the human body to be analyzed, but Alberti does not suggest this. He is allowing the choice of coordinate system to dictate his relationships.

> who are knowledgeable, and I have taken the measures and proportions of all of these. Comparing and eliminating the excesses of the extremes, ... I have selected from many bodies and models those mean proportions which seem to me most praiseworthy.[733]

Alberti is not ultimately interested in specific human beings. He is interested in universal ideal proportion. His method for discovering those ideas was to measure a large collection of beautiful examples and then average the results. The culmination of Alberti's investigation of human beauty in three dimensions is the *Tabulai dimensionorum hominis,* which is appended to the end of *De statua.* According to Alberti, the ideal human is six units tall with the center being just above the groin. The base unit is calibrated off the height of the head and neck with the lower boundary being the fork of the throat, which should be also be equivalent to the length of the foot.

Modern scholars are dubious concerning how much Alberti's measurements of real people influenced this ideal man. Jane Aiken in her article on Albertian human proportion suggests that he is more likely following the common practice of contemporary artists, like Donatello, Ghiberti, or Cennini as well as the ideal proportions discussed by Vitruvius.[734]

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[733] Alberti, *Opere volgari di Leon Battista Alberti*, 180-181. English translation by Joan Gadol. See Gadol, *Leon Battista Alberti*, 82.

[734] Aiken, "Leon Battista Alberti's System of Human Proportions," 80-84.

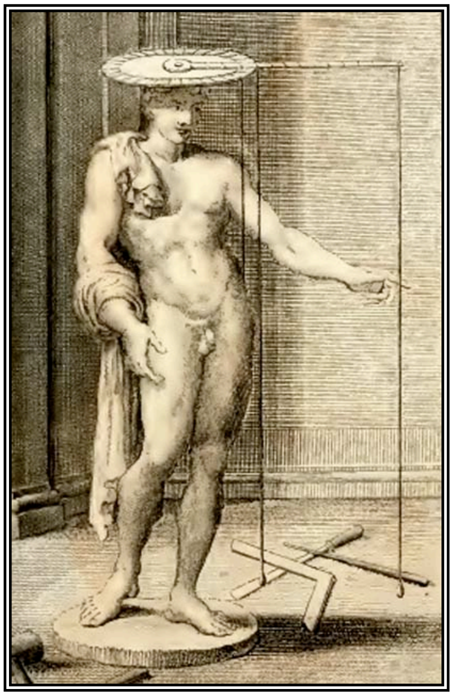


Figure 5.5: The *finitorium*[735]

In *Ludi rerum mathematicarum*[736] mapping is again a topic for exploration and Alberti expands on the simple centrally measured map he describes in his book on Rome. He describes a method of mapping based on nautical charts which significantly reduces the need for linear measurements and relies almost entirely on the angular measurements taken with his *orizonte*.

The method is as follows:[737]

Step 1: Locate yourself at a landmark (a) that is visible from afar and using the "orizonte," find the angular coordinates of as many sites as are desired. Plot the angular coordinates as rays emanating from a point (a) on a piece of paper or similar flat surface. See Figure 5.6.

[735] Illustration from Leonardo da Vinci and Leon Battista Alberti, *Trattato della pittura; Tre libri della pittura; Trattato della statua* (Bologna: Nell' Instituto delle Scienze, 1786), 203.
[736] The Italian version of *Ludi rerum mathematicarum* [*Ludi matematici*] is found in Alberti, *Opere volgari di Leon Battista Alberti*, 405-440. An English translation with commentary can be found in Alberti, *The Mathematical Works of Leon Battista Alberti*, 9-140.
[737] The following is a summary of the method described by Alberti in chapter XVI in *Ludi Matematici.* Alberti, *Opere volgari di Leon Battista Alberti*, 430-434.

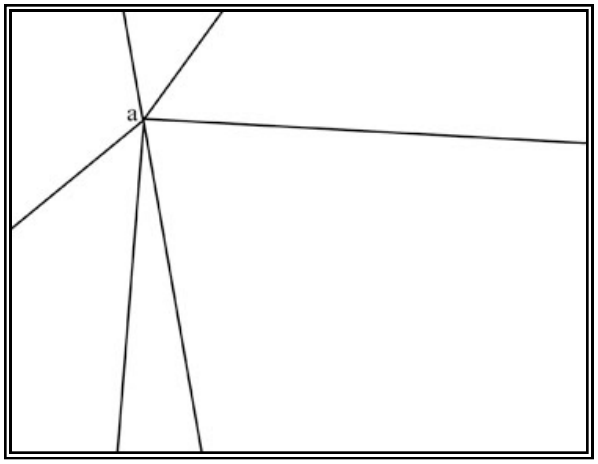


Figure 5.6: Step 1

Step 2: Reposition yourself to a place (b) that was located in step one and measure the actual distance between the two. In this example I will arbitrarily make the distance between points (a) and (b) 1000 feet. Then, using the "orizonte," measure the angular positions of as many of the places that were measured in Step 1 (and other places if they come into view). Plot these angles as rays emanating from point (b). The intersections between the rays from (a) and from (b) will be as accurately located as the angular measurements taken. All measurements between located points will be proportionally related to the distance between points (a) and (b). In this example the ratio between the real world and the map is 1000ft. to 10 units. All other distances can now be measured indirectly from the map. Any distance on the map need only be multiplied by 100 to arrive at the distance in the real world. The map is a scaled representation of the land. See Figure 5.7.

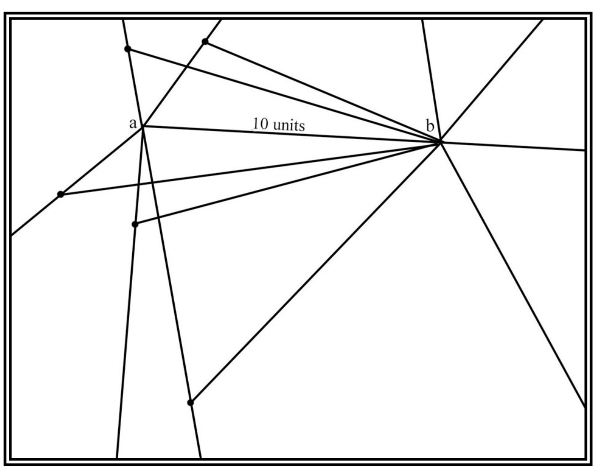


Figure 5.7: Step 2

Step 3: Repeat Step 1 at another point for more accuracy. Shown below is a new point, c. See Figure 5.8. This step may be repeated indefinitely.

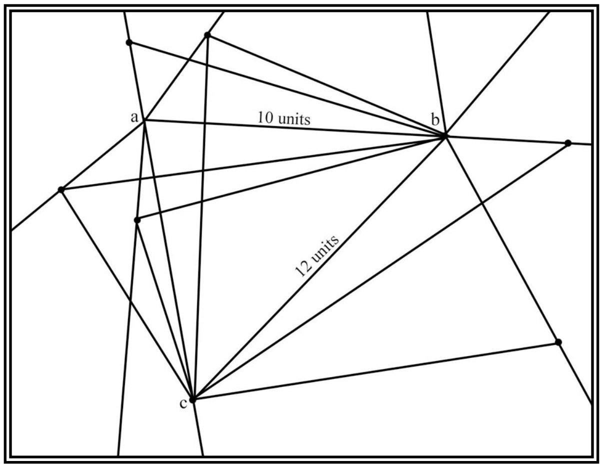
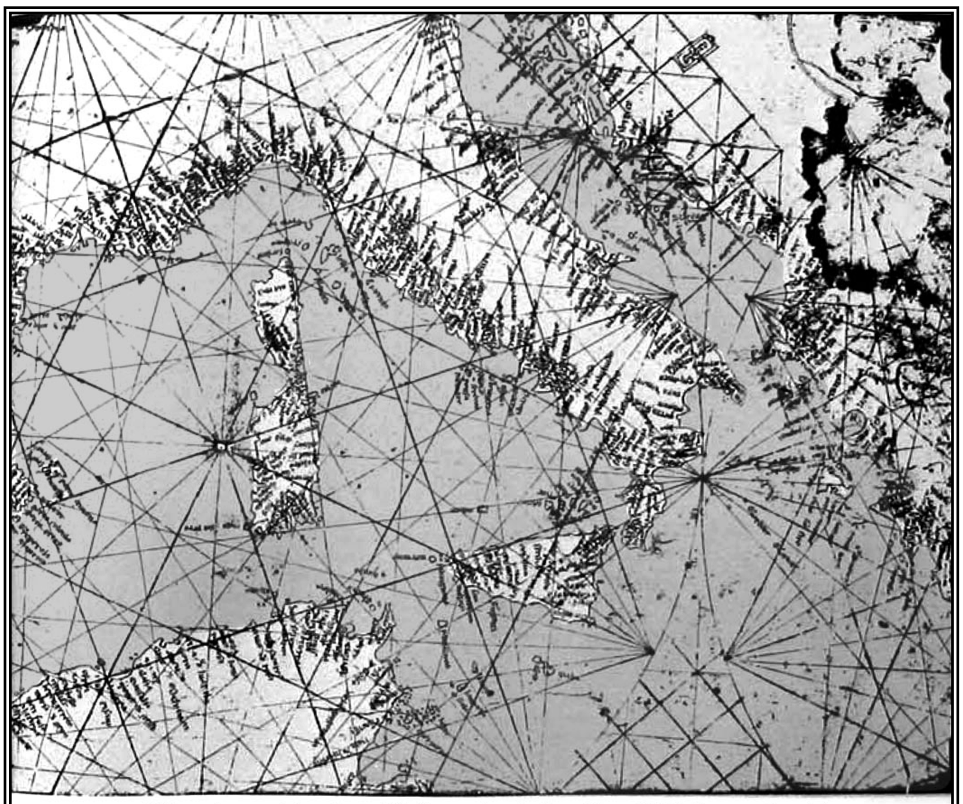


Figure 5.8: Step 3

This method of triangulation has been used since the 12th century by Italian navigators and by military mapmakers from the late 14th century.[738] See Figure 5.9.



13th-century portolan chart of Italian peninsula from Pisa. *(Bibliotèque nationale)*

Figure 5.9:[739] Portolan Chart

[738] Portolan-style land maps of relatively small regions for military purposes were produced in the late 14th century. See Gadol, *Leon Battista Alberti*, 160-162 and 178.

[739] Reproduction from Gadol, *Leon Battista Alberti*, 161. I have shaded it for clarity.

Other exercises in *Ludi rerum mathematicarum* include the determination of the heights of towers, depths of wells and other distant objects by exploiting the proportionality of similar triangles. He describes how to tell time by measuring the altitudes of stars, the construction of a hydro-level, finding ranges for artillery, the use of π (approximated to 22/7), the Archimedean crown problem, and even the design of an odometer that could be attached to a wagon wheel that requires the use of some basic velocity equations. Much, if not all, of the material can be found in older sources such as Euclid, Archimedes, and practical handbooks, all of which were readily available to a scholar in Alberti's day.[740]

It would be wrong to suggest that Alberti was a significant player in the world of cartography or geometry/geography. A survey of the secondary sources written on the history of cartography, geography, or geometry reveals that a few have but a little to say about him, and most do not mention him at all. His simple plane polar coordinate system is certainly practical and perfectly suited to small areas like cities where the consequences of living on a sphere are easily ignored, but he is not describing any mathematical or cartographic ideas that were not already known. He is competent in the basics but does not push into more challenging geometrical territory like Ptolemy, Mercator, and Regiomontanus did. It may be that his most significant addition to the field of geography was literary. He wrote easily understood instructions for how to make scale maps. It is unfortunate that there are no extant maps by his hand. The reconstruction made from the information in *Descriptio* yields a map that is little

[740] For example, an anonymous 14th-century manuscript titled, "A Treatise on the Mensuration of Heights and Distances," covers many of the same techniques described by Alberti. See Halliwell, ed., *Rara Mathematica,* 56-71. The major works of Archimedes had been available in Latin manuscript form since ca. 1300. Archimedes and his famous "crown problem" were discussed by Vitruvius in his book on architecture. Vitruvius, *Ten Books on Architecture (De architectura libri decem)*, trans. I. D. Rowland (Cambridge: Cambridge University Press, 1999), IX.Intro.9-12, 108. See W. R. Laird, "Archimedes among the Humanists," *Isis* 82, no. 4 (1991): 633.

more than a curiosity and has far too little information to be much more than a demonstration. Similarly, his cylindrical polar coordinate system from *De statua* is not new and is rather pedestrian geometry. However, the use to which he put his measurements revealed an interest in ideal proportions. This search for ideal human proportions, a mathematical harmony between and amongst the parts of the body, hints at a quadrivial philosophy that may be guiding Alberti's research, but his statements in these geometrical texts do not explicitly address the issue.

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The Mathematics of *De pictura* [*Della pittura*] and *Elementa picturae* [*Elementi di pittura*][741]

Descriptio urbis Romae, Ludi rerum mathematicarum, De Statua, and *De re aedificatoria* all deal with the mensuration of the physical world. In these works the geometer/mathematician plays the role of a god, measuring things as they actually are in space, not as they seem to be from our individual points of view. The geometer comes to know this world in the abstract, as mathematics. His perception is quantified as abstract numbers (angles, distances, and proportions) and then these data are translated into knowledge of geometric reality. We can see from some point A, that towers B and C are θ^{o} apart using our *orizonte*. We can pace off the distance between A and B, and from B we can then determine the angular difference between A and C. Deductions fill in the rest. When Alberti draws a map of Rome or designs a façade or ground plan he is not drawing it as human being perceives it, but as such geometries are known to be in the *sensorium* of the mind. This sort of geometrical understanding has no perspective, no point of view. It simply is. *De pictura* and *Elementa picturae* assume this type of divine,

[741] Alberti wrote both Latin and Italian versions of these works.

abstract knowledge of the world[742] and then translate this knowledge into a format recognizable to mortals using perspective. The true geometry of the world underlies both presentations, be it a map, or a table of measurements, or a painted picture. The painter of *De pictura* and *Elementa picturae* is not just a man trapped within the prison of his perceptions, copying what is before him with no understanding of what he is seeing, like a prisoner in Plato's cave. Alberti's painter is also a geometer and as such, through his perceptions, can divine true knowledge of the world. I make this statement even though Alberti claims otherwise in the second paragraph of *De pictura*. He writes, "…I beg you to consider me not as a mathematician but as a painter writing of these things. Mathematicians measure with their minds alone the forms of things separated from all matter.[743] Since we wish the object to be seen, we will use a more sensate wisdom."[744] Then Alberti, "as a painter," begins his discussion of geometrical optics. Though the discussion is technical, it is not laden with numbers and equations. Like his geographical/cartographic works, it is more of a how-to guide. The implication is that painters have to be applied/practical mathematicians. But the Albertian painter goes further and, in his fluency of this geometric knowledge, is capable of translating the world of true form into a geometry of individual perception. Alberti writes at the end of Book I in *De pictura*, "It follows that we should teach the painter the way by which what he conceives by his mind, he may copy by his hand."[745]

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[742] While describing the theory and practice of building in *De re aedificatoria*, Alberti writes, "…let lineaments be the precise and correct outline, conceived in the mind, made up of lines and angles, and perfected in the learned intellect and imagination." Alberti, *On the Art of Building*, I.1, p. 7. It is clear that Alberti makes a distinction between perspective and actual geometric knowledge. This issue will be developed in a subsequent section.
[743] Alberti's discussion of ideal architectural spaces in *De re aedificatoria* echoes this description of the mathematician measuring in his mind.
[744] Alberti, *On Painting (Della pittura)*, 43.
[745] My translation. "Sequitur ut pictorem instituamus quemadmodem, quae mente conceperit ea manu imitari queat." Leon Battista Alberti, "*De pictura*," in *M. Vitruvii Pollionis De architectvra libri decem; Elementa architecturae; De pictura; De sculptura* (Antwerp: Joanne de Laet, 1649), 12.

Rather than go too far into the details and minutiae of *De pictura*, as this has been written about in great depth by others,[746] a simple outline of the conceptual mathematical highlights will provide the necessary mathematics for a quadrivial interpretation. Reminiscent of Euclid's *Elements*, Alberti begins *De pictura* by briefly defining point, line, plane, circle, polygon, diameter, acute-obtuse-right angles, curved surfaces, and sphere.[747] Then he briefly describes the geometry of vision. The main point of this part is to establish that the rays that enable vision travel in straight lines to (or from) the eye.[748] These rays form the visual pyramid.

His basic discussion of vision reads much like his description for the use of an *orizonte* for taking angular measurements in mapmaking. He considers the eyes to be instruments that measure angular differences. "The eye," he writes,

> measures these quantities with the visual rays as with a pair of compasses. In every plane there are as many quantities as there are spaces between point and point. Height from top to bottom, width from left to right, breadth from near to far and whatever other dimension or measure which is made by sight makes use of the extreme rays. For this reason it is said that vision makes a triangle. The base of [this triangle] is the quantity seen and the sides are those rays which are extended from the quantity to the eye… Here is the rule: as the angle within the eye becomes more acute, so the quantity seen appears smaller.[749]

The smaller the angle, the smaller the image. All of these angular measurements acting in unison are the visual pyramid. A cross-section of this pyramid is the picture plane.[750]

The actual description of the construction of a picture is less than four pages in the octavo-sized English translation. The manuscript editions by Alberti were not illustrated, which seems odd considering the subject matter, but I will briefly outline the basic procedure with illustrations

[746] See Samuel Y. Edgerton, "Alberti's Perspective: A New Discovery and a New Evaluation," *Art Bulletin* 48, no. 3/4 (1966): 367-378; Samuel Y. Edgerton, *The Renaissance Rediscovery of Linear Perspective* (New York: Basic Books, 1975); Erwin Panofsky, *Renaissance and Renascences in Western Art* (New York: Harper Torchbooks, 1960); Erwin Panofsky, *Perspective as Symbolic Form* (New York: Zone Books, 1991), ch. 3.
[747] Alberti, *On Painting (Della pittura)*, 43-45. All but the sphere are defined by Euclid in Book I of the *Elements*. The sphere is defined in Book XI.
[748] Alberti does not discuss intromissive or extramissive theory. He simply discusses ray optics.
[749] Alberti, *On Painting (Della pittura)*, 46-47.
[750] Ibid., 52.

and some commentary.[751] I will not go into the nuances that Alberti considers necessary for a tasteful picture, but will just provide the basic geometrical layout.[752]

First draw a window through which you want to view the scene to be painted. This window is the cross-section of the visual pyramid mentioned above. Then choose a centric point appropriate to the subject matter. In the following illustrations I have chosen a point at ground level for this centric point. Then draw lines from the centric point to equally spaced divisions at the lower part of the window. I have chosen half-*braccio* divisions, using Alberti's standard for measurement. These lines stemming from the centric point show how equal lateral measurements diminish with distance, like the ties on a railroad track. See Figure 5.10.

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[751] At the end of Book 1 Alberti writes, "I usually explain these things to my friends with certain prolix geometric demonstrations which in this commentary it seemed to me better to omit for the sake of brevity." Ibid., 59.

[752] The following description of the construction of a picture is derived from Spencer's translation. Ibid., 56-59.

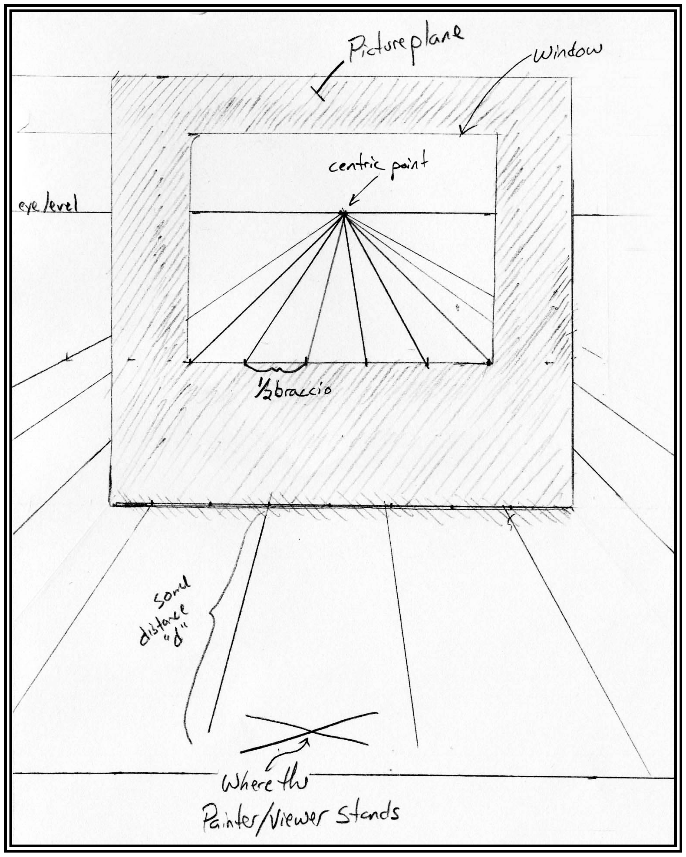

Figure 5.10: Point of View of the Painter – Alberti's Window on the World

After describing this frontal view, Alberti does something very clever in his description. He rotates his point of view 90º on the horizontal plane in order to describe distance relations between the painter, the picture plane, and the scene.[753] See Figures 5.11 and 5.12. Here Alberti plays with the imaginary space he has invented in his prose. He is describing with words, a space which depicts the depiction of a space.

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753 This rotation has caused some confusion in the art historical community, Spencer in particular. Spencer writes, "…Alberti does not explain how he determines the distance from the eye to the plane seen, nor does he give a rule for locating the perpendicular which cuts the height-distance lines." See Spencer's commentary. Ibid., 110-112. Alberti's description of this rotation is not very clear. Edgerton and Lindberg both understand this rotation in point of view, but Edgerton's explanation is hardly any less confusing than Alberti's. Edgerton, "Alberti's Perspective: A New Discovery and a New Evaluation," 367-378; David C. Lindberg, *Theories of Vision from al-Kindi to Kepler* (Chicago: University of Chicago Press, 1976), 150-152. This move by Alberti is fundamental in setting up the picture. Without it, there is no vertical perspective (in the picture plane), only lateral.

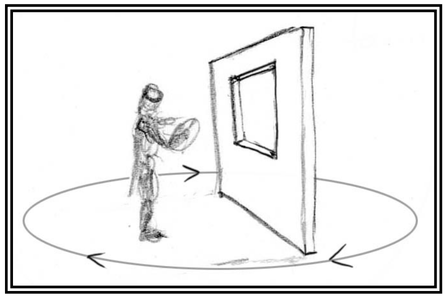

Figure 5.11: Rotation to Side View

Then from this side view Alberti describes how to choose the painter's distance from the window by relating the bottom edge of the window to the ruled "pavement." See Figure 5.12.

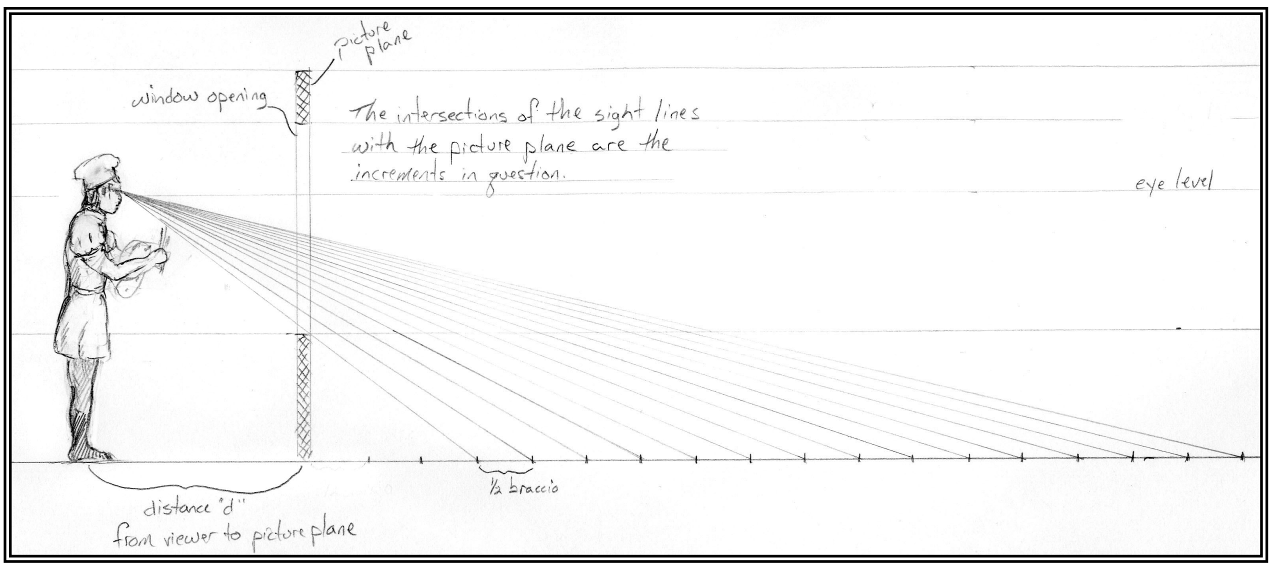

Figure 5.12: Side-View Elevation of the Painter
The picture-plane/window and ground are measured in half-braccio units.

The result is the classic one-point perspective of Renaissance painting, where the grid lines of the pavement look like a tiled floor. See Figure 5.13. There is enough information in this picture to reconstruct the actual geometric space. It is mathematically consistent and no information is lost in the conversion from ideal intellectual space to the constructed pictorial space.

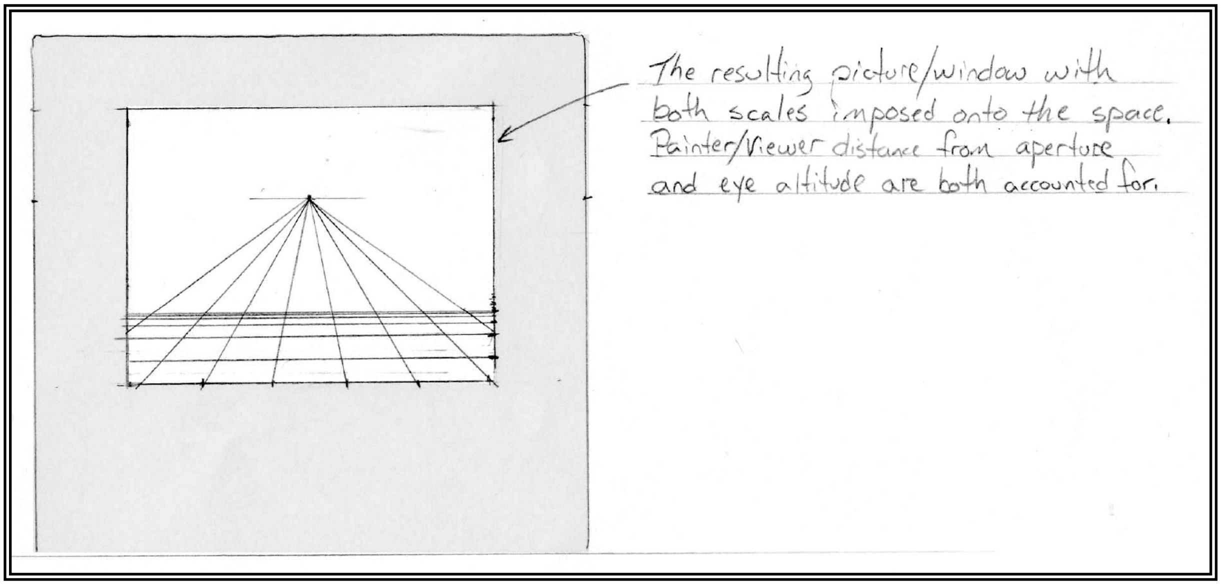

Figure 5.13: The Window with the Grid Measured and Seen in Perspective

The pictures made using this method are very illusionistic, but is not necessarily how or what we really see. The approximation is more accurate around the centric point, but farther from the center, the distortions become apparent. For example, Alberti does not discuss what would happen if the window were very wide, a situation that occurred all around him in the paintings of Giotto, Masaccio, Uccello, and Piero. He does not apply his diminishing size perspective to distances to the left or right of his centric point (or above and below). He only describes a proportional perspective for things directly in front of him. In a way, he is describing the painter as being much like the prisoners in Plato's cave, who "can only see what is in front of them."[754] Alberti is describing a small rectangle taken from a much larger visual sphere. See Figure 5.14.

[754] Plato, *The Republic of Plato*, VII.514, p. 227.

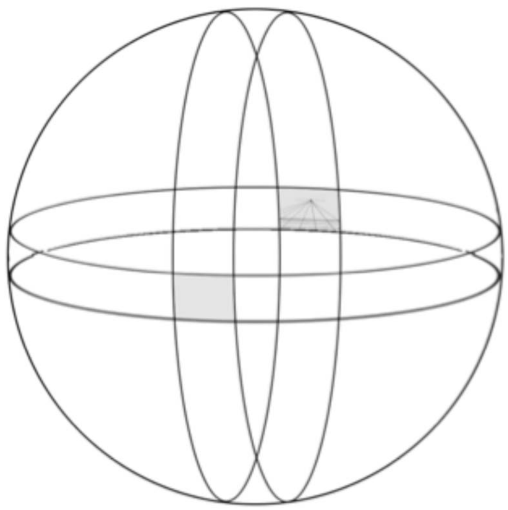

Figure 5.14: Curved Space Treated as Flat

If one were to paint a picture on a very wide panel using this system of perspective, the distortions would become very pronounced the further one looked away from the center point. Imagine the pavement lines that run left to right in a really wide format. They too should converge on a point to the left and right. The only way for this to occur is if those lines are actually curves. This, once again, is the result of Alberti ignoring the spherical geometry of the situation. Like in his map of Rome, he relies on the approximation that these scenes appear flat locally, but globally, they are spherical. And like the map of Rome, the distortions would become more pronounced the farther they are away from the center.

By limiting his approximate perspective to a small window around the centric point, Alberti avoided a much more complicated situation. The geometry of his picture space is flat. This way the equations for the perceived-diminishment-in-size as a function of distance-from-the-viewer are simple linear equations. See Appendix 5A for a mathematical description derived from Alberti's description. The more general equations for perspective in a larger window are quadratics, the mathematics of curves and circles.[755] This is much more difficult math, especially using 15th-century tools.

One of the more striking things about his technical description of how to make a picture is that his instructions contain no analytical mathematics. There are no numbers or equations or charts or tables or variables. His is a system of geometrical construction requiring only a straight edge and a compass. Beyond his very brief mention of basic geometrical concepts like point, line, and circle, he mentions only one Euclidian proposition, Proposition 2 from Book VI of the *Elements*. "Let us add the axiom [proposition] of the mathematicians where it is proved that if a straight line cuts two sides of a triangle, and if this line which forms a triangle is parallel to a side of the first and greater triangle, certainly this lesser triangle will be proportional to the greater. So much say the mathematicians."[756] This is basic, relatively simple Euclid, without the proof.[757] Similar triangles are the stock in trade of all of Alberti's geometrical works justifying all of his scaling. In the picture window of *De pictura,* similar triangles can be found in a few different ways. The most obvious are the similar triangles that are formed by the horizontal lines and the visual triangle (the rail-road tracks with the ties as parallel bases). Similar triangles of more interest are the ones formed not only by the parallel horizontals (the railroad ties) but also parallel sides allowing the topmost angle to be shifted down either the right or the left side. By constructing these similar triangles one can more easily visualize the inverse relationship between distance and apparent size.[758] Given two identical objects, if one object is twice as far as another object, it will appear half as large in Alberti's picture window, in both width and

[755] In Alberti's defense, Albert Einstein's Special Theory of Relativity makes the same approximation, that space is flat locally. His General Relativity equations take into account the curved nature of space (caused by his way of measuring things with light) and things get so messy and the geometry so complicated that General Relativity is usually not taught to undergraduate physics majors.
[756] Alberti, *On Painting (Della pittura)*, 52.
[757] Euclid, vol. 2, VI.2, pp. 194-195.
[758] Euclid's *Elements*, Book I, Proposition 29 can also be used to construct these similar triangles.

height. See Figure 5.15. The highlighted triangle itself shows this trait clearly. The base near the bottom of the window is twice as long as the middle horizontal line serving as a base for a smaller similar triangle upon which the second smaller orange rectangle stands. See Appendix 5A for more details on this characteristic.

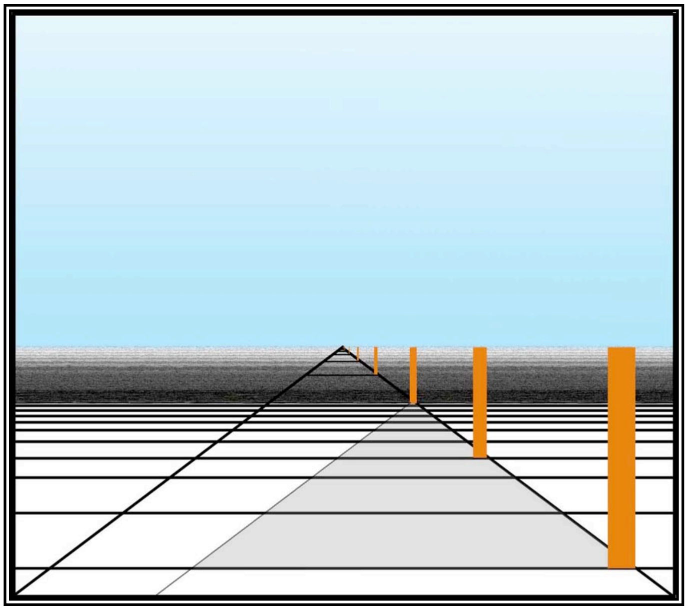

Figure 5.15: Apparent Size to Distance Relationship

Size to distance relationship seen on a triangle constructed using two parallels based on the larger visual triangle. (The left side of the smaller shaded triangle is actually parallel to the left side of the larger triangle. An optical illusion makes them appear skewed.)

Alberti does not point out this characteristic of his own system. However, for a man who compulsively measured everything, it seems unlikely that he did not notice it.

It is generally assumed that Alberti did not invent linear perspective. This honor is usually given to the Florentine architect and mechanical engineer Filippo Brunelleschi (1377-1446), to whom the Italian version, *Della pittura*, is addressed.[759] It has been suggested that Brunelleschi's

[759] Brunelleschi is the first person and most prominently named in the Prologue. Donatello, Della Robbia, Ghiberti and Masaccio are also mentioned. The Milanese architect, Filarete, makes reference to Brunelleschi's invention of a linear perspectival system in his *Treatise on Architecture.* Filarete, *Treatise on Architecture; Being the Treatise by Antonio di Piero Averlino, Known as Filarete*, trans. John Richard Spencer, 2 vols. (New Haven: Yale University Press, 1965), vol. 1, XXIII.178v-179r, p. 305.

perspective was developed in collaboration with Toscanelli (1397-1482), the medical-astrologer and friend of Alberti discussed previously.[760]

Furthermore, artists such as Ghiberti (ca. 1381-1455) and Masaccio (ca. 1401-ca. 1429) appear to have used a system of linear perspective in their works which predate Alberti. In fact, Ghiberti was quite knowledgeable on the topic of medieval optical theory, paraphrasing Alhazen, Bacon, Witelo, and Pecham in a work written in the mid-15th century.[761] David Lindberg makes a compelling argument that Alberti's linear perspective is dependent upon older optical theories. The central ray and the visual pyramid, Lindberg points out, come directly out of medieval theory.[762] What Alberti did to the older geometrical theory, was insert a picture window into the visual pyramid and describe the process of doing so in a relatively simple how-to manual.

Be that as it may, Alberti's system of linear perspective described in *De pictura* is a significant and influential mathematical work and is still read as a basic text for the theory of painting. The literary style of *De pictura* disguises its mathematical subject matter, making it seem less mathematical than it really is. This is part of its charm and its genius. Almost anyone can read and understand *De pictura* and set up a picture with one point linear perspective.

Elementa picturae, on the other hand, is a much shorter (ca. 2000 words), more overtly mathematical treatise. It is composed of ten lists. The first three (A, B, and C) are all definitions for such things as point, line, and body and are clearly inspired by Euclid, but not as rigorous. Alberti defines these geometrical basics in abstract mathematical terms and then in terms that would make sense to a draughtsman. For example, he writes in the third list, "In a painting we call a point that small inscription than which nothing can be smaller."[763]

[760] Rose, "Humanist Culture and Renaissance Mathematics," 60.
[761] Lindberg, *Theories of Vision from al-Kindi to Kepler*, 152-154.
[762] Ibid., 147-154 and 264.
[763] Alberti, *The Mathematical Works of Leon Battista Alberti*, 145.

The fourth list (list D) defines the vocabulary necessary for using Euclid to make perspectival drawings. The critical term in this list is *comminuta.* In the Latin version Alberti defines a comminuted surface as "that which is placed in relation to the gaze in such a way that some of either its lines or its angles, in comparison with each other, appear to be smaller in some part than the thing itself. In the same way an area in a painting which expresses that will be comminuted."[764] As Stephen Wassell has pointed out, this term essentially means foreshortened[765] and it is used numerous times in the subsequent lists. Other terms of note from the fourth list include somewhat idiosyncratic definitions for concentricity, proportional areas, commensurate points, and an assortment of straight and curved lines.

The fifth, sixth, and seventh lists (lists E, F, and G) are exercises for drawing linear, curvilinear, and circular areas. These exercises do not address perspectival (foreshortened) issues, just flat, frontal, geometrical drawings. These frontal drawings he refers to as "concentric," but by concentric he seems to mean that they are viewed and drawn on a plane that is perpendicular to the visual ray that connects the viewer to the centric point described in *De pictura* and discussed above. These exercises include drawing parallel lines, drawing similar geometrical shapes, drawing circles within circles, and other basic geometrical drawing exercises.

The eighth and ninth lists (I and L) culminate in exercises for drawing comminuted (or foreshortened) geometrical shapes in one point perspective. Figure 5.16 succinctly demonstrates the 10th, 11th, and 12th instructions from the eighth list (list L). These instructions pertain to the foreshortened portion of this figure.

> 10th- "Drawing a circular area that is comminuted [foreshortened] and proportionally larger."

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764 Ibid., 145, n17.
765 Wassell's commentary in Ibid., 158.

11th- "Inside this proportionally larger <area> placing the commensurate point, and there drawing a whole circular <area> that is comminuted [foreshortened] and proportionally larger."

12th- "And drawing a circular <area> that is proportionally smaller and comminuted [foreshortened], and in this placing the commensurate point, and there drawing a whole <area> that is proportionally smaller and comminuted [foreshortened] positioned inside it."[766]

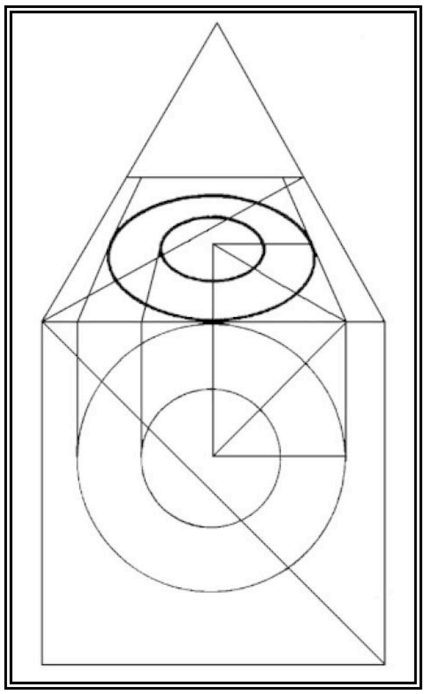

Figure 5.16: Comminuted or Foreshortened Circles (Based on Gambuti's drawing)[767]

The resultant drawing shows both the concentric (or frontal) and the comminuted (or foreshortened) geometrical constructions.

The tenth and final list (list M) is not really a list, it is simply a closing statement to the reader, perhaps included in order to bring the total to ten. In this closing Alberti asks the reader to make corrections and additions if they transcribe it.

[766] Instructions L.10-12 from Alberti, *The Mathematical Works of Leon Battista Alberti*, 151.

[767] Figure 5.16 is from Wassell's commentary in Ibid., 167, and is based on Gambuti's drawing from Alessandro Gambuti, "Nuove ricerche sugli *Elementa picturae*," *Studi e documenti di architettura (Omaggio ad Alberti)* 1, no. (1972): 159, fig. 47.

All of the instructions for the drawing exercises in *Elementa picturae* are extremely abbreviated and nonspecific. These instructions are difficult to follow even knowing before hand what he is attempting to describe and even with the aid of the more detailed descriptions found in *De pictura*. *Elementa picturae* starts with the appearance of Euclidean rigor, but quickly becomes a vague set of instructions for perspectival drawing. It is approximate. It is stated that there are distortions to a foreshortened circle, but no mathematical instructions are given to construct such a circle. These foreshortened shapes must be drawn by eye. In his brief definitions for comminuted areas [D.2] and commensurate points [D.5][768] Alberti seems to suggest that, in theory, mathematical descriptions exist for one-point perspective, but he does not give them. In Alberti's defense, such mathematical descriptions would be quite complicated given the state of analytical geometry in his day and it would seem that his intended audience for this short treatise was an educated gentleman, not a quadrivial expert.

In terms of quadrivial philosophy, *Elementa picturae* is mostly instructive for what Alberti does not say. In particular, his definition of proportional areas in the fourth list [D.3], though vaguely referring to Euclid V, Definition 6 and VII, Definition 20,[769] makes no distinction between rational and irrational, or multitude and magnitude, as Euclid and our previous case studies were careful to do. Alberti's goal in *Elementa picturae* was not to advance quadrivial mathematics, but rather, to measure the world around him and approximately translate these measurements onto a picture plane. Given this practical goal, the quadrivial philosophy was unnecessary. Like Marchetto, Alberti was referring to mathematics largely developed in a quadrivial context and applying it to contemporary tasks. Although Alberti did not invent perspective, he began the task of formulating it in mathematical terms.

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[768] The Latin version of D.5 states, "A commensurate point in a picture will be one with distances in a certain relationship to other points in the picture." Ibid., 146, n20.
[769] See Wassell's commentary in Ibid., 159.

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The Mathematics of *De re aedificatoria*

De re aedificatoria (*On the matter of building*) was begun by Alberti in the 1440s and is thought to have started as a commentary on Vitruvius' *De architectura*, a work from the 1st century B.C. Vitruvius' work had been frequently copied in the Middle Ages but had not been put to much practical use. The discovery of a manuscript copy in 1415 by Giovanni Francesco Poggio Bracciolini[770] (1380-1459) elicited new interest in the classical author, but Vitruvius' references, terminology, and his Latin prose were obscure and difficult to understand for Renaissance readers.[771] Alberti writes,

> I grieved that so many works of such brilliant writers had been destroyed by the hostility of time and of man, and that almost the sole survivor from this vast shipwreck is Vitruvius, an author of unquestioned experience, though one whose writings have been so corrupted by time that there are many omissions and many shortcomings. What he handed down was in any case not refined, and his speech such that the Latins might think that he wanted to appear a Greek, while the Greeks would think that he babbled Latin. However, his very text is evidence that he wrote neither Latin or Greek, so that as far as we are concerned he might just as well not have written at all, rather than write something that we cannot understand.[772]

Despite his complaints, it is clear that Alberti understood Vitruvius' text quite well, since much of *De re aedificatoria* is obviously taken from it.

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[770] Poggio discovered numerous texts, several of which were of interest to Alberti. A partial list of the authors brought to light by Poggio includes: Cicero, Quintilian, Valerius, Priscian, Vitruvius, Lucretius, Manilius, Tacitus, and Plautus. See Poggius Bracciolini and Niccolò Niccoli, *Two Renaissance Book Hunters: The Letters of Poggius Bracciolini to Nicolaus de Niccolis*, trans. P. W. G. Gordan (New York: Columbia University Press, 1991), passim.
[771] Richard Krautheimer, "Alberti and Vitruvius," in *The Renaissance and Mannerism: Studies in Western Art: Acts of the Twentieth International Congress of the History of Art,* ed. Ida E. Rubin (Princeton, N.J.: Princeton University Press, 1963), 48-49.
[772] Alberti, *On the Art of Building*, VI.1, p. 154. If Alberti was the author of *Hypernerotomachia Poliphili*, as some have speculated, this quotation is difficult to explain. Cf. Lefaivre, *Leon Battista Alberti's Hypnerotomachia Poliphili*, passim.

Contrary to a few claims sprinkled throughout *De re aedificatoria*,[773] neither it nor Vitruvius' *De architectura* is directed at the builder himself. Rather these books are written for the educated elite, people who were likely to be patrons of large building projects.[774] *De re aedificatoria* is sort of the dilettante's guide to construction. The intended reader would never actually build anything with his own hands, but would need to know the parameters of good building, good taste, and the theories behind both.

The first three books of *De re aedificatoria* deal largely with the practical rudiments of building, the basic design and function, the materials, and construction, what he refers to generally as the *firmitas*. Alberti discusses such matters as location, weather considerations, proximity to water, drainage, sewage, accessibility, interior climate control, roof pitch, optimal times for harvesting lumber and quarrying stone, brick and mortar production, supply strategies, foundations, basic structural engineering, mechanical fasteners for stones, corbels, joints for splicing beams together, arches and vaults. These topics all have analogues in Vitruvius' *De architectura.*

The fourth and fifth books of *De re aedificatoria* deal with the purpose and utility (*utilitas*) of a construction. They address such topics as city planning and social services, site considerations, fortifications, bridges, harbors, palaces, castles, public buildings, private buildings, religious buildings, country estates, domestic-space considerations, humidity, and light sources. Alberti firmly believed that the design of a building vis-à-vis the environment was not just a matter of

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773 For example, see Ibid., IX.11, p. 318.

774 The word *architectus* is generally translated as "architect," but it can also mean master builder or author. The idea of an *architect* in the modern sense of the word should be entertained with caution. For more information see the following. Horton, "Authority and Innovation in Alberti's Theory and Practice," 345-372; Franklin Toker, "Gothic Architecture by Remote Control: An Illustrated Building Contract of 1340," *The Art Bulletin* 67, no. 1 (1985): 67-95; Carroll William Westfall, "Society, Beauty, and the Humanist Architect in Alberti's *De re aedificatoria*," *Studies in the Renaissance* 16, no. (1969): 61-79.

practical or economic utility but also a physical and mental health issue. Again, similar topics are described by Vitruvius.

Aside from Book X, which is something of a miscellany, Books VI-IX generally deal with beauty, *venustas*. Alberti writes in the beginning of Book VI, “Of the three conditions that apply to every form of construction – that what we construct should be appropriate to its use [*utilitas*], lasting in structure [*firmitas*], and graceful and pleasing in appearance [*venustas*] – the first two have been dealt with, and there remains the third, the noblest and most necessary of all.”[775] It is within this larger conversation on beauty that most of Alberti’s quadrivial observations and descriptions are made. A brief description of the mathematical material of the three books leading into Book IX will set up the necessary context for this discussion, as most of this material appears in these books.[776]

In a digression in Book VI, Alberti very briefly discusses the advantages of a block and tackle for lifting heavy items as well similar observations on the screw and the lever. He also describes the design of cranes used for moving large stones and the like.[777] He writes in this section that his aim is not to describe these mechanisms as a mathematician, but “as a workman.”[778] There is little actual quadrivial material in this part.

In Book VII he deals with the ornaments on sacred buildings. He discusses the regular polygonal structures of ancient temples and other basic layout considerations for doors and windows. Proportional ideas are proposed for the various column orders and overall ground

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775 Alberti, *On the Art of Building*, VI.1, p. 155.
776 Some of this material may be the from *Historia numeri et linearum,* to which he refers at the end of the Prologue.
777 Alberti, *On the Art of Building*, VI.6-8, pp. 164-175. These are all topics discussed by Archimedes whose works were readily available in Latin translation in the Middle Ages. There was a resurgence of interest in Archimedes, especially his machines, in the 15th century by such people as Cusanus, Toscanelli, and Regiomontanus. See Laird, "Archimedes among the Humanists," 628-638; Rose, "Humanist Culture and Renaissance Mathematics," 63-67.
778 Alberti, *On the Art of Building*, VI.7, p. 167.

plans for basilica. Although numbers and some very basic arithmetic are used in several of these discussions, he is not making a philosophical argument that pertains to the structure of the quadrivium.[779]

In much the same way that Book VII deals with sacred buildings, Book VIII deals with public secular buildings such as funerary monuments, towers, theaters, and government buildings. This book contains a bit more proportional information specifically regarding the designs of towers, fora, and triumphal arches. This information has led to much speculation on the measurements of Alberti's own commissions, but again, as in Book VII, the material does not make much of a mathematical argument.

Book IX is the last of the books dealing with beauty, *venustas,* and it is directed specifically to the topic of ornament for private buildings. Judging from the overall geometrical nature of Book IX, ornaments certainly could not be simply decorations or little accents as we would think of them today. In Book VI Alberti defines ornament as "a form of auxiliary light and complement to beauty." He continues, "From this it follows … that beauty is some inherent property, to be found suffused all through the body of that which may be called beautiful."[780] To paraphrase Alberti – beauty is that which is found in beautiful bodies. The early chapters of Book IX deal partially with the history and ethics of ornament, but no details or instructions are given on how to employ them in this part of *De re aedificatoria*. In IX.7 there is a brief passage suggesting that ornaments should be arranged "in their level, alignment, number, shape, and

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[779] I do not deny that there are numerological and aesthetic considerations to take into account in these sections, but many of these same numerological considerations come up in my analysis of Book IX and much more explicitly. So rather than tease out some tidbits of meaning from this mathematically thin material, I will focus on the more philosophically dense sections found in later parts of *De re aedificatoria*.

[780] Alberti, *On the Art of Building*, VI.2, p. 156.

appearance"[781] with vertical and horizontal symmetry. But the majority of Book IX deals with Albertian beauty, specifically the beauty derived from the interrelationships of the parts.

The first three chapters of this book deal with aesthetic moderation and design conveniences to consider, e.g., avoiding stairways because they are bothersome. He also discusses the health benefits of living in the country as weighed against the need to conduct certain types of business in the city. As in all of *De re aedificatoria*, Alberti refers frequently to classical authors such as Cicero and Martial. Chapters 3 and 4 of Book IX describe the designs of porticos and some landscaping ideas. These chapters, like Books VII and VIII, contain some proportional thoughts but are not mathematically significant.

The quadrivially interesting parts start in Chapter 5. It is here that Alberti begins to define beauty rather than just refer to it. After a rather lengthy two-page introduction to the topic, Alberti writes this famous section,

> From this we may conclude, without my pursuing such questions any longer, that the three principal components of that whole theory into which we inquire are **number**, what we might call **outline**, and **position**. But arising from the composition and connection of these three is a further quality in which beauty shines full face: our term for this is *concinnitas*; which we say is nourished with every grace and splendor. It is the task and aim of *concinnitas* to compose parts that are quite separate from each other by their nature, according to some precise rule, so that they correspond to one another in appearance.
>
> That is why when the mind is reached by way of sight or sound, or any other means, *concinnitas* is instantly recognized. It is our nature to desire the best, and to cling to it with pleasure. Neither in the whole body nor in its parts does *concinnitas* flourish as much as it does in Nature herself; thus I might call it the spouse of the soul and of reason. It has a vast range in which to exercise itself and bloom – it runs through man's entire life and government, it molds the whole of Nature. Everything that Nature produces is regulated by the law of *concinnitas*, and her chief concern is that whatever she produces should be absolutely perfect. Without *concinnitas* this could hardly be achieved, for the critical sympathy of the parts would be lost. So much for this.
>
> If this is accepted, let us conclude as follows. Beauty is a form of sympathy and consonance of the parts within a body, according to definite **number**, **outline**, and **position**, as dictated by *concinnitas*, the absolute and fundamental rule in Nature.[782]

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[781] Alberti, *On the Art of Building*, IX.7, p. 310.

[782] Alberti, *On the Art of Building*, IX.5, pp. 302-303. Bold emphasis mine. This aesthetic and structural description of *concinnitas* is reminiscent of Plato's *Timaeus*. Pletho (ca.1360-1452), the father of

In light of this statement, Alberti's studies of nature, his cartography and measurements of the human body and his study of perspective, can be seen as measures either directly or indirectly of *concinnitas*. And, as the following sections will demonstrate, number, outline, and position are the qualities upon which beauty acts as form.

Alberti's definition of *concinnitas* is very much an expression of the overall rationale for the grouping of the quadrivial disciplines, a grouping which reflects the structure of the cosmos as viewed through Platonic theory, which not only reemerged in the 15th century in the newly discovered texts and translations of Plato himself, but also already existed in the texts of Boethius, Ptolemy, Manlius, Macrobius, Cicero, Augustine, Thomas Aquinas, and many others. In the following sections I will discuss the three interdependent qualities affecting beauty: number, outline, and position.

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a. Numbers *(numeri)* from *De re aedificatoria,* IX.5

> Beauty is a form of sympathy and consonance of the parts within a body, according to definite **number**, outline, and position, as dictated by *concinnitas*, the absolute and fundamental rule in Nature.[783]

Quadrivial arithmetic is the science of numbers[784] as they relate only to themselves. In the Greek tradition, arithmetic was composed of two divisions: one being what we moderns more

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Neoplatonism in the Quattrocento, had lectured on Plato in Florence in 1438 and it has been speculated that Alberti was in attendance. Even if he was not in attendance, Alberti certainly knew of Pletho. Pletho's body was brought from the Peloponnesus to Rimini and entombed in the Tempio Malatestiano, which was designed by Alberti. James Hankins, *Plato in the Italian Renaissance*, 3rd ed. (New York: E.J. Brill, 1994), 193-217; Kristeller, *Renaissance Thought and Its Sources*, 50-65. Alberti's teacher, Gasparino Barzizza, owned a copy of *Timaeus* and may have taught from it. See Hankins, "The Study of the *Timaeus* in Early Renaissance Italy," 78-79.

[783] Ibid., IX.5, p. 303. Emphasis mine.

[784] See Ibid., p. 72.

commonly understand to be arithmetic proper, which includes number theory and the abacus,[785] and the other being arithmology. Arithmology is the study of the decad (1-10) and the mystical properties associated with these numbers.[786] In a sense, arithmology is to analytical arithmetic as astrology is to astronomy. Both the mystical and the analytical are aspects of the single quadrivial discipline. The arithmological side makes up most of what Alberti has to say about arithmetic in chapter 5. For the sake of brevity I will describe only his observations on the number 7, the other numbers are outlined in Appendix 5B. For comparison I will also paraphrase the properties of the number 7 found in the 5th-century text, *The Marriage of Philology and Mercury* (*De nuptiis Philologiae et Mercurii),* by Martianus Capella and some observations on the number 7 by Plato from *Timaeus*. See Table 5.1. This is not to suggest that Alberti necessarily took his ideas from Martianus or Plato, for any number of sources contain similar numerological correspondences.[787] But the similarities between them are strong enough to suggest Alberti was not just making this stuff up out of thin air. There was a tradition he was following. Alberti writes, “Architects have used these numbers extensively: yet, especially in the

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785 John Caldwell suggests that simple addition and subtraction were not part of Greek arithmetic at all, but rather were part of what the Greeks called logistics. John Caldwell, "The *De institutione arithmetica* and the *De institutione musica*," in *Boethius, His Life, Thought, and Influence,* ed. Margaret T. Gibson (Oxford: Blackwell, 1981), 136. Whether this is strictly the case or not, it is clear that Boethius assumed basic addition and subtraction in his book on arithmetic.

786 Stahl and Johnson, *Martianus Capella and the Seven Liberal Arts*, vol. 1, 151-152.

787 For example Aristides Quintilian (late 3rd century A.D.) has a section in his text *De musica,* which makes many of the same connections that Martianus makes as well as quite a few different ones. See Aristides Quintilianus, "*De Musica*," 502-504. The *Theologumena Arithmeticae,* attributed to Iamblichus (early 4th century A.D.), may be the source for much of this tradition. Except for the Aristotelian observation, all of Alberti's observations on the number 7 are included in this treatise. See Iamblichus (Attributed to), *The Theology of Arithmetic: On the Mystical, Mathematical and Cosmological Symbolism of the First Ten Numbers [Theologoumena Arithmeticae]*. Other authors with arithmological content include Plato *(Timaeus)*, Hippocrates, Philo of Alexandria, Macrobius, Theon of Smyrna, Nicomachus, and Isidore of Seville. See also Frank Egleston Robbins, "The Tradition of Greek Arithmology," *Classical Philology* 16, no. 2 (1921), 97-123.

temple, they have employed no even number greater than ten, …, nor odd number greater than nine."[788]

<u>Alberti</u> 7 a favorite of God 7 planets 7 stages of man (only lists 4) 7 days to name newborn* *citing Aristotle[789] <u>Plato's *Timaeus*</u> 7 planets (moon, sun, 5 planets) 7 sages 7 movements (6 linear, 1 circular) 7 numbers in structure of soul	<u>Martianus Capella</u> 7 associated with gods: Minerva and Pallas (Athena) 7 is largest prime in decad 7 = sum of male and female (3+4) 7 planets 7 days 7 phases of moon orbit of moon = 1+2+3+4+5+6+7 = 28 (lunar month) 7 climes 7 elemental transmutations 7 months for fully formed offspring 7 apertures in head (2 eyes, 2 ears, 2 nostrils, mouth) 7-months to first teeth 7 stages of man 1 x 7-years to adult teeth 2 x 7 years to puberty 3 x 7 years to beard 4 x 7 years marks end of growth. 5 x 7 years marks full flowering of manhood. 7 vital organs [tongue, heart, lung, spleen, liver, 2 kidneys] 7 parts of the body [head, chest, belly, 2 hands, 2 feet] 7 stars at celestial axis [stars of Ursa Major]

Table 5.1: The Arithmological Number Seven[790]

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b. Three Dimensional Outlines *(finitiones)* – armatures for quantities – from *De re aedificatoria,* IX.5

> "Beauty is a form of sympathy and consonance of the parts within a body, according to definite number, **outline**, and position, as dictated by *concinnitas*, the absolute and fundamental rule in Nature."[791]

The outline (*finitio*), Alberti states, "is a certain correspondence between the lines that define the dimensions."[792] The outline is the geometry of an object in space. Here, Alberti introduces

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[788] Alberti, *On the Art of Building*, IX.5, p. 304. The suggestion is that all proper proportions must be expressible using the numbers 1 through 10.
[789] Aristotle, *History of Animals in Ten Books*, trans. R. Cresswell (London: George Bell and Sons, 1897), VII.xi, p. 193 (588a8-9).
[790] Data for this table is from the following sources: Alberti, *On the Art of Building*, IX.5, p. 304; Capella, *The Marriage of Philology and Mercury,* vol. 2, VII.738-739, pp. 281-283; Plato, *Plato's Cosmology: The Timaeus of Plato*, passim.
[791] Alberti, *On the Art of Building*, IX.5, p. 303. Emphasis mine.

the format for comparing the spatial measurements of objects: length, breadth, and height. Alberti then explains how the three dimensions of outlines relate to music.[793] He identifies sympathies between sound, sight, and the intellect.

> The very same numbers that cause sounds to have that *concinnitas*, pleasing to the ears, can also fill the eyes and mind with wondrous delight. From musicians therefore who have already examined such numbers thoroughly [harmonics - relations between discrete numbers], or from those objects in which Nature has displayed some evident and noble quality [geometrical quantities], the whole method of outlining is derived.[794]

So, from the harmonic concords of music theory as studied by musicians and from the pleasing geometrical relationships found in nature, presumably having been studied by Alberti in his geometrical works discussed above, all outlines partaking in Albertian *concinnitas* can be derived. Alberti is preparing the reader for a discussion of proportion, which in the strict Boethian sense is the comparison of ratios, e.g., 2:1. Alberti starts with the study of arithmetic (the study of counting numbers), and in his section on outline he is moving into music theory (the relationship between arithmetical quantities or ratios). Alberti sets up this proportional argument so that the three dimensions of a solid correspond to three numbers in a proportion, e.g., 4:3:2.

He then launches into a very brief explanation (less than 200 words) of Pythagorean harmonic theory based on string lengths. He extracts from the numbers 1, 2, 3, and 4 his six concordant intervals:[795]

3:2 (*diapente*, *sesquialtera*, the fifth)
4:3 (*diatesseron*, *sesquitertia*, the fourth)
2:1 (*diapason*, double, *duplex*, the octave)

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[792] Ibid., IX.5, p. 305. Boethius describes the three dimensions similarly. See Boethius, *De institutione arithmetica*, II.4, p. 129.
[793] It should be noted that in order to define an interval in Pythagorean music theory, three locations on a string were necessary. Alberti states offhandedly in his numerological section that "Nature is composed in threes all philosophers agree." Alberti, *On the Art of Building*, IX.5, p. 304.
[794] Ibid., IX.5, p. 305.
[795] Ibid.

3:1 (*diapason diapente*, triple, the octave-and-a-fifth)
4:1 (*disdiapason*, quadruple, the double-octave)
9:8 (*tonus*, the tone)

This allows Alberti to construct various harmonic spaces in the following chapter. These intervals are included in nearly every Pythagorean-based music theory textbook, the most prevalent being *De institutione musica* by Boethius, which was the standard text for most all medieval instruction in music theory and no doubt readily available to Alberti.[796] Alberti's choice of Pythagorean music theory for his architecture is a deviation from what had been a largely Vitruvian program in the previous sections of *De re aedificatoria*. Vitruvius' description of music theory is based on Aristoxenian theory, the philosophical antithesis to the number-based Pythagorean system. The philosophical differences between the two schools of musical thought (roughly equivalent to the metaphysical differences between Aristotle versus Plato) are illustrative of the differences in world views that separated Vitruvius from Alberti. Superficially, the music theory presented by Vitruvius looks similar to the theory presented in *De re aedificatoria* and unless Alberti had carefully studied his Pythagorean sources,[797] one wonders if he was fully aware of the philosophical and harmonic gulf separating him from his predecessor. Alberti is silent on this matter.

Alberti's discussion of music theory is quite similar to the harmonic theory found in Plato's *Timaeus* describing the creation and division of the World-Soul.[798] Where Plato is describing the numerological and harmonic structure that binds the World-Soul together, Alberti will use these

[796] Claude V. Palisca, *Studies in the History of Italian Music and Music Theory* (New York: Clarendon Press, 1994), 168-188.
[797] Boethius criticizes Aristoxenian theory on a number of occasions. Aside from a reference by Vitruvius, Aristoxenus was little known in 15th-century Italy, and much of what was known was provided by Boethius. See Vitruvius, *Ten Books on Architecture (De architectura libri decem)*,V.4, pp. 66-67.
[798] Plato, *Plato's Cosmology: The Timaeus of Plato*, 35B-36A, pp. 66-71. This section of *Timaeus* had been available via Chalcidius' Latin translation since the 5th century.

harmonic concords to form a "sympathy and consonance of the parts within a body, ... as dictated by *concinnitas*, the absolute and fundamental rule in Nature."[799]

Like Plato, Alberti is using music as a metaphysical structure. He is using it to make a correspondence between a Platonic world ordered by number and the terrestrial concepts of beauty. These are correspondences that Marsilio Ficino (1433-1499), the Neoplatonic junior contemporary of Alberti, wrote about in great detail.[800] Though there is no evidence that Alberti is taking music theory to the sympathetic extremes that Ficino proposes, he certainly subscribes to a Platonic conception of a harmonically structured physical world that was popular in his day. This system of harmonic proportions may not dictate all structures, but those that partake in concordant *armonia* embody the ideal of *concinnitas* and are to be employed as much as possible by architects.

In the opening to Chapter 6, Alberti further refines where he proposes to find the relationships for proper outlines. "When working in three dimensions," he writes, "we should combine the [ii] universal dimensions, as it were, of the body with [i] numbers naturally harmonic in themselves, or [iii] ones selected from elsewhere by some sure and true method."[801] Once again, Alberti is proposing that three-dimensional proportions should be derived from quantifiable sources. These three sources have been rearranged for clarity:

i. What he calls "numbers naturally harmonic," the Pythagorean ratios derived from the monochord which are summarized above and in Appendix 5C.

ii. What he refers to as "universal dimensions of the body," which are his numbers derived from the geometry of squares and rectangles.

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799 Alberti, *On the Art of Building*, IX.5, p. 303.
800 See Marsilio Ficino, *Three Books on Life: A Critical Edition and Translation*, trans. C. V. Kaske and J. R. Clark (Tempe, AZ: Arizona Center for Medieval and Renaissance Studies, 2002), Book III, pp. 236-393; Walker, *Spiritual and Demonic Magic*, 3-29.
801 Alberti, *On the Art of Building*, IX.6, p. 306. I have added the Roman numerals within the quotation for reference.

iii. What he simply calls "ones selected from elsewhere by some sure and true method," which I take to be the numbers derived from the mathematical means – namely the arithmetic, geometric, and harmonic means. This is the only other method he describes in this chapter, and so by process of elimination it appears that this is what he intends by a "sure and true method."

In the following sub-sections I will explain and comment upon the three sources of three-dimensional measurements presented in *De re aedificatoria,* as all three are quadrivially significant.

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i. Alberti's Three-Dimensional Harmonics or "Numbers naturally harmonic"

These, Alberti writes, "include those [numbers] whose ratios form proportions such as the double, triple, quadruple, and so on."[802] Here he is referring back to the previous chapter, IX.5, which very briefly lays out his Platonic-Pythagorean harmonic theory. In this quotation he is referring to the intervals 3:2 (the fifth) and 4:3 (the fourth), for they can be combined to make the double,[803]

$$2:1 = 3:2 \oplus 4:3,$$

as well as the triple, $3:1 = 2:1 \oplus 3:2$,

and the quadruple, $4:1 = 3:2 \oplus 4:3 \oplus 3:2 \oplus 4:3 = 2:1 \oplus 2:1$,

and, indeed, all of his other concordant intervals. The fourth and the fifth are essentially the prime factors for all of the other Pythagorean consonances. See Appendix 5C.

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[802] The concordant intervals. Ibid.
[803] Recall that I am using the ⊕ symbol to mean the 'addition' of intervals even though mathematically the operation is multiplication.

Alberti then continues to discuss how these natural harmonic intervals interact to produce consonant three-dimensional rectangular spaces.[804] For example, given a 6 x 2 floor area (which harmonically reduces to 3:1) an appropriate height would be 4.[805]

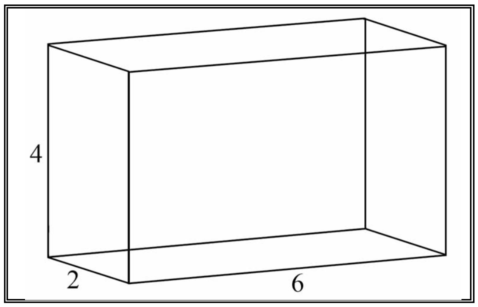

Figure 5.17: The 2x4x6 Space

This is derived from the harmonic interval, 6:2 = 3:1, which has the factors 2:1 (the octave) and 3:2 (the fifth). Recall that 2:1 ⊕ 3:2 = 6:2 = 3:1, or in put another way, an octave plus a fifth is an octave and a fifth. Mathematically this can be seen as follows. Starting with the shorter side of the floor area, 2, double it, raising it by the harmonic factor of an octave to 4;

$$2\left(\frac{2}{1}\right) = 4.$$

Then raise the 4 by a fifth, 3/2, to 6;

$$4\left(\frac{3}{2}\right) = 6.$$

This yields a 2 x 4 x 6 room.[806] The number 4 is the stepping stone on the way to 6.

He describes several other room dimensions in a similar fashion, all derived from floor area dimensions which are Pythagorean harmonic consonances. It should be stressed that the heights

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[804] Macrobius discusses three dimensional spaces and harmonic theory in a rather chaotic section of his *Commentary on the Dream of Scipio*. Like Alberti, his three dimensional spaces are discussed entirely in the abstract, and like Alberti, the harmonic numerical relationships are emphasized, but Macrobius does not develop them in the same manner. See Macrobius, 2.II.1-20, pp. 189-193.

[805] Similar mathematical procedures are described in Boethius, *De institutione arithmetica*, II.2, pp. 126-128 and passim.

[806] One might say that this room 'sings' an octave-and-a-fifth (3:1) as well as a fifth (3:2) over the tonic (using piano keys, A-E-e). Alberti does not suggest that his room literally sings, but that it demonstrates *concinnitas*, a concord or harmony of measurements.

are always derived from the floor-plan dimensions and that the heights are always the median number. He does not explicitly make this point until later in IX.6 where he writes that the sets of three numbers “should be employed with the shortest line serving as the width of the area, the longest as the length, and the intermediate one as the height. But sometimes these may be *modified* to suit the building."[807] This last sentence is quite interesting considering how specific Alberti has been in his preceding descriptions. In the Rykwert translation above, it seems that he is saying, ‘Do it this way, unless you find a more suitable way to do it in a particular context.’ Such a statement introduces subjectivity into a topic which had been strictly in keeping with Platonic numerological idealism. However, in the Leoni edition of *De re aedificatoria,* Alberti’s meaning is more consistent with his previous prescriptions. This sentence is translated as, “These several Rules which we have here set down for the determining of Proportions, are the natural and proper Relations of Numbers and Quantities, and the general Method for the Practice of them all is, that the shortest Line be taken for the Breadth of the Area, the longest for the Length, and the middle Line for the Height, tho’ sometimes for the Convenience of the Structure, they are interchanged.”[808] “Interchanged" has a different meaning from “modified.” Alberti uses the word “*commutabuntur*" in the Latin,[809] which can be translated either way, but its literal meaning tends towards the idea of "change with" rather than simply "change." Rykwert’s translation pushes Alberti into a more Vitruvian-Aristoxenian interpretation, whereas the Leoni translation is more consistent with Pythagorean-Platonic philosophy of Alberti. Therefore, I tend to favor "exchange"

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[807] Alberti, *On the Art of Building*, IX.6, p. 308. Italics mine. In the Leoni translation, this sentence is in a paragraph separated from the discussion of squares and cubes, which I think makes more sense given that it is a general statement that is consistent with all of his examples and not just the ones related to squares and cubes. The original manuscripts of *De re aedificatoria* were totally unformatted, not even into books and chapters. Books and chapters were first imposed on *De re aedificatoria* by a typesetter in 1512. See Horton, 49-50.
[808] Leon Battista Alberti, *The Ten Books of Architecture: The 1755 Leoni Edition (De re aedificatoria)*, ed. and trans. G. Leoni (New York: Dover, 1986), IX.6, p. 199.
[809] Alberti, *L'architettura* (*De re aedificatoria*), Vol. 2, IX.6, p. 831.

or "interchange" for this translation as Leoni has done.[810] See Appendix 5D, Table D1 for the harmonically derived room dimensions described in this section.

There remain several other area configurations which he previously listed in a discussion in IX.6, which I have not here described, but are summarized in Appendix 5D, Table D2. He has not specifically described how to construct several configurations that his method implies. For example, he does not determine the height of a 1:1 square room, or rooms measuring 3:2, 4:3, 8:3 9:4 or 16:9, all of which are significant relationships demonstrating *concinnitas*. Two of these, the 1:1 and the 9:4, he will deal with using methods described in the following sections, the others he will not.

It is assumed that he does not derive the 3:2 or 4:3 configurations using a harmonic factorization method because this method will not work on these intervals. These two intervals are, in effect, prime harmonics. They cannot be factored into more basic rational intervals in the same way that prime numbers cannot be factored into smaller integers. In fact, any ratio that includes a prime number will be a prime harmonic, though not necessarily a consonant interval embodying *concinnitas*.[811] These two intervals also cannot be treated by the methods described in the following sections. The lack of heights for the 8:3 and the 16:9 room dimensions is a mystery to me. Both are capable of being integrally factorized harmonically. The 16:9 floor plan could have a height of 12[812] and the 8:3 area could follow the method Alberti uses for the areas shown in the last two entries in Appendix 5D, Table D1 in which one of two medians is chosen.

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[810] Alberti writes, "Sed interdum pro aedificiorum commoditate commutabuntur." I literally translate this as, "But occasionally these [numbers] will be interchanged for the advantage of buildings." Ibid.

[811] For example, 17:11 or 199:197: these are not consonant intervals but are prime harmonics. It is interesting that at times Alberti deals with dimensions strictly as proportions, meaning that 2:1 and 8:4 are the same, and at other times he is careful to keep them unreduced for no apparent reason.

[812] $9 \times 4/3 = 12$ and $12 \times 4/3 = 16$.

Using this method an 8:3 room could have either a height of 4 or a height of 6.[813] It is curious that Alberti chose to omit these two area-height derivations. They would have provided him with examples for insightful commentary, and yet he included so many others which hardly need any explanation at all, some being trivial or even redundant. The idea of harmonic rectilinear volumes is well explained, but the execution is sloppy.

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ii. Alberti on the Geometry of Quadrilaterals: "The Universal Dimensions of the Body"

In the previous subsection we briefly looked at the naturally harmonic arithmetical numbers and how they can be manipulated to yield harmonic three-dimensional musical proportions demonstrating *concinnitas*. Now let us briefly focus on what Alberti identifies as another source for three-dimensional proportions, what he calls "universal dimensions of the body." Alberti writes, "In establishing dimensions, there are certain natural relationships that cannot be defined as numbers, but that may be obtained through roots and powers."[814] Judging from the material which follows this statement, these "natural relationships" are found in right triangles extracted from a cube. These are the magnitudes of geometry, not the numbers of arithmetic as described above. Alberti is clearly making this distinction.

Alberti begins this discussion with the unit cube, the cube with sides that are one unit in length, faces that have one square unit of area, and a volume of one cubic unit. Alberti marginalizes this particular cube by elevating it out of his analysis, calling it "the primary cube whose root is one, [and] is consecrated to the Godhead, because the cube of one remains one."[815] As is typical of much quadrivial arithmetic, one, or Unity, is often treated as a special case.

[813] 3 x 2/1 = 6 and 6 x 4/3 = 8 or 3 x 4/3 = 4 and 4 x 2/1 = 8.
[814] Alberti, *On the Art of Building*, IX.6, p. 307.
[815] Ibid. A similar dismissal of the unit cube is found in Boethius, *De institutione arithmetica*, II.25, p. 147.

Furthermore, he is probably uninterested in this particular cube because he cannot very easily generate different numbers with it. Ones tend to beget ones.

It is interesting to note that Otto Georg Von Simson, in his book *The Gothic Cathedral: Origins of Gothic Architecture and the Medieval Concept of Order,* suggests that the cube is also numerologically significant in harmonic architectural terms and credits this concept of space-harmony to Boethius in *De institutione arithmetica*.[816] Boethius points out that the cube is made up of 12 lines, 8 angles, and 6 faces, and comments that this is a Pythagorean harmonic arrangement[817] born out of the cube of geometry.[818] It is interesting that Alberti does not mention this in *De re aedificatoria.* This sort of numerological game-playing has been promoted by Alberti in *De re aedificatoria* previously, and this observation about the harmony found in cubes is perfectly suited to enhance his concept of *concinnitas*.

Following his dismissal of the unit cube, Alberti proposes the cube with a side length of 2 as the basic generative cube for finding more three-dimensional measurements. Like Boethius, Alberti points out that the volumetric, planar, and linear measurements of a 2 x 2 x 2 cube yield the numbers 8, 4, and 2, but again he does not suggest that these numbers are meaningful harmonically or geometrically ($x : x^2 : x^3$). Alberti then diverges from his more or less Boethian presentation and puts the cube to work for his applications in architecture. He writes, "From this cube is derived the rule for outlines."[819] It is not necessary to describe his cubic extractions in

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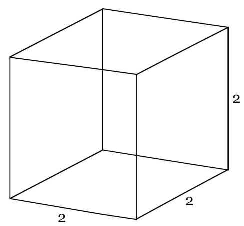
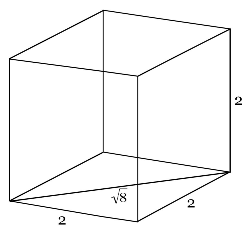
[816] Otto Georg Von Simson, *The Gothic Cathedral: Origins of Gothic Architecture and the Medieval Concept of Order*, 2nd ed. (New York: Harper & Row, 1964), 33, n32.
[817] 12:6 is an octave, 12:8 a fifth, 8:6 a fourth, etc.
[818] Boethius, *De institutione arithmetica*, II.49, p. 179.
[819] Alberti, *On the Art of Building*, IX.6, p. 307. Alberti writes, "Ex cubo istiusmodi finitio num constitutiones habebuntur," which I translate more literally as, "Out of the cube with such an outline [2 x 2] will not rules be had?" Alberti, *L'architettura (De re aedificatoria)*, IX.6, p. 831. Rykwert's "the rule" is too definitive, and the Latin is clearly in the plural. I would also like to point out that Boethius in *De institutione arithmetica* strictly stays within the boundaries of arithmetic in his discussion of cubes in Book II and he does not stray into geometrical measurements requiring square roots. Boethius only uses counting

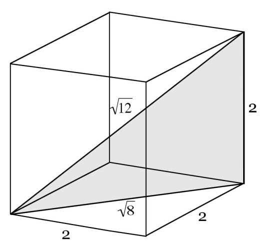
detail. The table below gives the basic features and measurements he extracts from them. See Table 5.2 below and Table D3 in Appendix 5D.



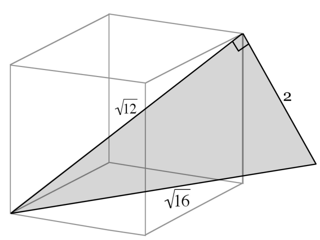
Here is the 2 x 2 x 2 cube that Alberti proposes as the generator of three- dimensional triplets.



The first right triangle he describes is the one created by a diagonal drawn across any of the faces.[820] It measures 2 : 2 : √8.



The second right triangle he describes in the text is this one, which runs corner to corner in the cube and measures 2 : √8 : √12.



The last right triangle mentioned by Alberti is not found within the 2 x 2 x 2 cube. The hypotenuse and the 2-unit-length leg of the previous triangle are used as the legs for this new right triangle. It measures 2 : √12 : 4.

Table 5.2: Derivations from a Cube

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numbers. Alberti, on the other hand, moves from an arithmetical discussion of cubes into a geometrical discussion of irrational quantities.

[820] I am using notation unavailable to Alberti, but he explicitly states √8, even though it was "unknowable." "Haec enim quanta sit ad numerum, ignoratur; sed esse hanc constat *radicem areae octonariae*." My empahsis. Alberti, *L'architettura (De re aedificatoria)*, IX.6.168v, p. 831. It appears that Georges Pachymeres (ca.1242- ca.1310) described similar derivations from the cube in his treatise on geometry. See George Pachymeres, *Quadrivium de Georges Pachymère*, ed. P. Tannery and R. Stéphanou (Vatican City: Biblioteca Apostolica Vaticana, 1940), 201-328. Manuscript copies of this work were available in 15th-century Italy. See Tannery's inventory of manuscripts in Pachymeres, XXXVIII-XCVIII. See also Thomas J. Mathiesen, "Aristides Quintilianus and the *Harmonics* of Manuel Bryennius: A Study in Byzantine Music Theory," *Journal of Music Theory* 27, no. 1 (1983): 41-43.

At this point in the text Alberti is teetering on the edge of a derivation of the golden mean. All of the right triangles used by Alberti in this section were constructed by making the hypotenuse and the shorter leg of the preceding triangle the legs for the subsequent one. Were he to employ this method once more, and use the hypotenuse and the shorter leg of this last triangle (2 : √12 : 4) as the legs for a new right triangle, the resulting series would be 2 : 4 : 2√5. See Figure 5.18.

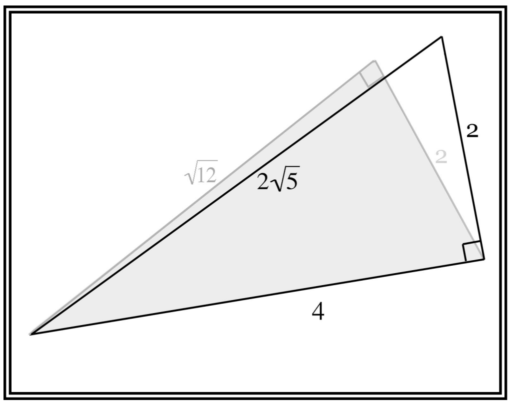

Figure 5.18: Alberti's Missed Golden Step

This triangle is the critical part of a standard derivation of the golden section, often referred to as ϕ. The hypotenuse plus the shorter leg of this triangle is related to the longer leg in the ratio of the golden mean.

$$\frac{2\sqrt{5}+2}{4} = 1.618... = \phi$$

This may seem somewhat cryptic, but this triangle is constructed by Euclid in his derivation of the golden mean, called the "extreme and mean ratio" in XIII.1-5 of the *Elements*.[821] See Figure 5.19.

[821] Euclid, II.11, pp. 402-403; VI.30, pp. 267-268; XIII.1-5, pp. 440-449.

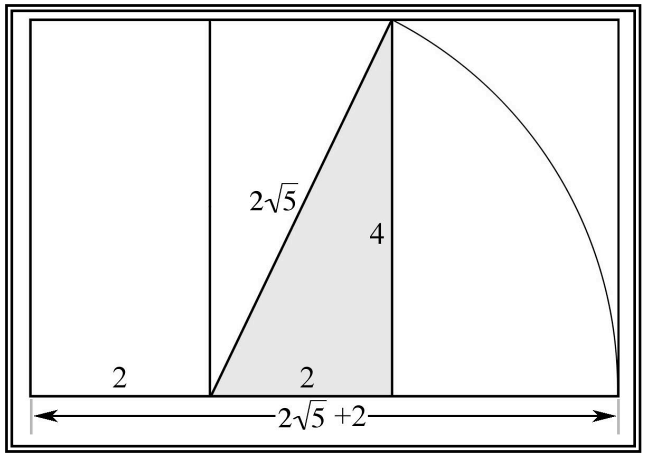

Figure 5.19: A Standard Construction for the Golden Ratio

I will not dwell on this point except to express that Alberti did not take this argument literally one step further to find this famous irrational ratio.[822] Perhaps he does not continue with his method and make this golden triangle (2√5:4:2) because it might compete with the harmonic sequence 6:4:2. Or perhaps he is intentionally avoiding this relationship because the golden ratio was something mysterious and distasteful. Or perhaps he simply overlooked an interesting avenue that he could have explored had he noticed it. All of his geometric dimensions derived from a cube are summarized in Appendix 5D, Table D3.

The main point in this subsection on geometric proportions is not his missed golden step, but rather his inclusion of a variety of irrational ratios from geometry that can be used for three dimensional constructions, what Alberti likes to call "outlines." Alberti presents these irrational ratios of geometry just like he presents his rational ratios of harmony. They appear to be of equal status. That Alberti included these irrational ratios amongst his proportions for three dimensional spaces has been largely ignored by architectural historians. This omission has further obscured the real Alberti. I will comment upon this issue below.

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[822] Constructions of the golden ratio go back to the Pythagoreans. See Heath's commentary in the *Elements*. Ibid., vol. 2, 99.

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iii. Alberti on Means: "A Sure and True Method"

Returning to the initial quote from this section, "When working in three dimensions, we should combine the universal dimensions… of the body with numbers naturally harmonic in themselves…" We looked at both of these above: the arithmetical harmonic dimensions and the geometric square-based dimensions. Now let us look at the "ones selected from elsewhere by some sure and true method."[823] This brings us to Alberti's last source for three-dimensional measurements – the "means." His presentations of these mathematical concepts are not in any way unique as they are standard Pythagorean ideas which date back to Plato and Archytas of Tarentum (4th century B.C.) and no doubt even further. They have also been described innumerable times by subsequent philosophers including Euclid, Ptolemy, Boethius, et al., in their books on arithmetic and music theory. Therefore, rather than describe his method, which is just a rehashing of previous authors, it will be sufficient to summarize the results using modern algebraic notation. See Table 5.3

[823] Alberti, *On the Art of Building*, IX.6, p. 306.

Arithmetic Mean	$c = \frac{a+b}{2}$ *c* is linearly equidistant from both *a* and *b*.	The arithmetic mean is "equidistant between both extremes."[824] Alberti's example: a : c : b = 4 : 6 : 8 The common average.
Geometric Mean	$c^2 = ab$ or $c = \sqrt{ab}$ *a* x *b* is a rectangle, *c* is the square with the same area.	"…the dimension of a side [c] that generates a square of equal [area]" to that of a rectangle with sides measuring a and b.[825] Alberti's example: a : c : b = 4 : 6 : 9
Harmonic Mean	$\frac{\max}{\min} = \frac{b}{a} = \frac{\max - \text{intermediate}}{\text{intermediate} - \min} = \frac{b-c}{c-a}$ $b(c-a) = a(b-c)$ $bc - ab = ab - ac$ $bc + ac = 2ab$ $c = \frac{2ab}{a+b}$	"The proportion between the shortest and the longest dimensions is the same [proportion] as that [quantity] between the shortest and the middle, and … that [quantity] between the middle one and the longest."[826] The ratio between the outer numbers equals the ratio between the differences between the numbers. Alberti's example: a : c : b = 30 : 40 : 60

Table 5.3: The Means [827]

At the end of this section he writes, "By using means like these, whether in the whole building or within its parts, architects have achieved many notable results, too lengthy to mention. And they have employed them principally in establishing the vertical dimension."[828] Here again he suggests that the middle number, the mean, should be favored for the height. The only unique height that he determines in his examples using these means is the geometric mean triplet, 4:6:9. The other two are duplicated in his section using harmonic factorization. He also fails to point out that the geometric mean can be used to determine the height of a 16 x 9 area mentioned above yielding 16:12:9.

[824] Ibid.
[825] Ibid., IX.6, p. 308.
[826] Ibid.
[827] Ibid., IX.6, pp. 308-309.
[828] Ibid., IX.6, p. 309.

What is most striking about this subsection is that Alberti does not mention that these means can duplicate the results found by using the "numbers naturally harmonic," described above. Most medieval music theory texts make this point explicitly and use a variety of paths to get to the same Pythagorean intervals as evidence of the reality and significance of the tuning. For example, the harmonic mean divides the octave 6:3 at 4, yielding an Albertian room of 3:4:6, with 6:4 a musical fifth and 4:3 a musical fourth.[829] By placing the means in a separate category and not pointing out the overlap with harmonics, Alberti gives the impression that he has a tenuous grip on basic Pythagorean music theory. His failure to observe these correspondences, an observation that would have bolstered his argument for the interrelations in his concept of *concinnitas*, make it difficult to believe that Alberti was at all harmonically sophisticated, an assumption frequently made by early modern and modern scholars, no doubt stemming from the *Vita*.

Significantly adding to the mathematical mystique of Alberti harmonic architecture is the highly influential book by Rudolf Wittkower, *Architectural Principles in the Age of Humanism*, first published in 1949. This book is the starting point for any student of the Renaissance with an interest in Platonic harmonic philosophy, although a few misconceptions are contained in Wittkower's presentation of Alberti, and unfortunately these have entrenched themselves in much of the subsequent literature. In part these errors may be explained by his failure to notice that Alberti had three classes of proportions suitable for architectural application rather than two. Wittkower only acknowledges the Pythagorean harmonic proportions and the means that can be used to derive them. But he completely ignores the irrational proportions derived from the triangles found in the 2 x 2 x 2 cube. Let me draw the reader's attention once again to this quote

[829] Boethius uses the series 6, 8, 9, and 12 to demonstrate arithmetic and harmonic means and their relationship to all of the consonant intervals discussed by Alberti. Boethius, *De institutione arithmetica*, II.50-54, pp. 180-188.

from *De re aedificatoria*, "When working in three dimensions, we should combine the universal dimensions, as it were, of the body with numbers naturally harmonic in themselves, or ones selected from elsewhere by some sure and true method."[830] As this sentence, suggests and as the text of *De re aedificatoria* proves, Alberti had three sources for the proportional quantities of three-dimensional spaces: Pythagorean harmonics, various geometrical cube dimensions, and means. Because Wittkower failed to include the irrational cube-triangular proportions he can further claim that the $\sqrt{2}$ and similar irrational numbers were not introduced as a possible architectural proportion until Andrea Palladio (1508-1580) in the mid-16th century and further states that via Palladio the $\sqrt{2}$ was the "only irrational number widely propagated in the Renaissance theory of architectural proportion."[831] Clearly this is not the case. In Alberti's

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[830] Alberti, *On the Art of Building*, IX.6, p. 306. I have relied on this quotation from the Rykwert translation because it has served my purpose very neatly, but I should mention in the interest of full disclosure that every translation of this passage is quite distinct and there appears to be no consensus on its meaning. The Latin reads, "Ternatim autem universos corporis diametros, ut sic loquar, coadiugabimus numeris his, qui aut cum ipsis armoniis innati sunt aut sumpti aliunde certa et recta ratione sunt." *Alberti, L'architettura (De re aedificatoria), IX.6, p. 827.* My own translation of this passage reads, "However, we will conjoin the three universal dimensions of the body, so to speak, by these numbers, any of which are either conceived using the ones harmonic in and of themselves, or are the ones selected elsewhere by definite and proper rules." My translation, admittedly, loses some of the three-source interpretation that I rely on for my analysis, but Alberti goes on to describe three and only three distinct methods for deriving quantities for three dimensional proportions: musical/harmonic, geometric (from the cube), and from means. Therefore, even if the sentence that I have used to emphasize his three methods has been translated improperly, the content of IX.6 supports my interpretation.

[831] Rudolf Wittkower, *Architectural Principles in the Age of Humanism* (New York: W. W. Norton and Company, 1971), 108. Significant insight into Wittkower's conceptions of the Middle Ages, the Renaissance, and aesthetics can be found in an article from 1960 in *Daedalus*. See Rudolf Wittkower, "The Changing Concept of Proportion," *Daedalus* 89, no. 1 (1960): 199-215. In this article Wittkower expands on his theory that in the Middle Ages the geometrical proportions (often relying on the $\sqrt{2}$) (*ad quadratum* or ad *triangulum*) were the measure of the world whereas in the Renaissance the arithmetical harmonics found in whole number ratios were the proportions to be admired. He writes, "Irrational proportions would have presented a dilemma to Renaissance artists, for the Renaissance attitude to proportion was determined by a new organic approach to nature, which aimed at demonstrating that everything was related to everything by integral number." Wittkower, "The Changing Concept of Proportion," 202. This is a vastly oversimplified view of the Renaissance and the Middle Ages and completely contradicts the part on triangles and cubes in *De re aedificatoria*. Also in this article Wittkower reveals his hand as a card-carrying Pythago-Platonist of the first order showing contempt for all things chaotically "organic." He hated Abstract Expressionism. On the last page of this article he writes, "[Bilateral symmetry] is the symmetry of the human body, and for that reason [is] of towering

discussion, every triangle but one contains the $\sqrt{2}$.[832] This insistence by Wittkower that Alberti was exclusively a Pythagorean harmonicist and not a geometer has led to the widespread belief that Alberti only used Pythagorean numerical intervals in his spatial designs. This limitation on the acceptable spaces in the Albertian system has resulted in a cottage industry of harmonic analysis of his supposed architectural projects invariably citing Wittkower as a major source.[833] But if one includes all of the available proportions suggested by Alberti, such analyses become much more difficult as almost every measured proportion can fit his criteria for outlines, and discrepancies in measurement standards allow for very large margins of error. The inclusion of Alberti's irrational proportions makes it possible for any building to fit into the Albertian proportional program of *De re aedificatoria*. See Appendix 5E.

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c. Alberti's Position *(Collocatio)*: The Last Consideration for *Concinnitas*

> "Beauty is a form of sympathy and consonance of the parts within a body, according to definite number, outline, and **position**, as dictated by *concinnitas*, the absolute and fundamental rule in Nature."[834]

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importance to mankind. Again the concordance of the two halves of the body can be expressed in terms of ratios and proportions, and it is these in fact which we perceive without fail. Every disturbance of the balance of parts (e.g., a short leg, a crippled hand) evokes reactions such as pity, irritation, or repulsion." He continues, "When all is said an done, it must be agreed that the quest for symmetry, balance, and proportional relationships lies deep in human nature. It can confidently be predicted that today's 'organic chaos' [Abstract Expressionism] is a passing phase, and that the search for systems of proportion in the arts will continue as long as art remains an endeavor of man." Wittkower, "The Changing Concept of Proportion," 213. I think it would be beneficial if this quotation were added in a preface to new editions of *Architectural Principles in the Age of Humanism*. It might clear up many misconceptions.

[832] "Estque subinde et diameter cubi, quam certo scimus esse radicem numeri duodenarii." Alberti, *L'architettura (De re aedificatoria)*, IX.6, p. 831.

[833] Only a few of the buildings attributed to Alberti are broadly accepted to be his. The debates over which buildings he actually designed are unresolved. Many scholars have invested a great deal of time and effort in some of the more questionable attributions and are thus heavily invested in their positions. From my understanding only Rimini and Mantua have been securely attributed the Alberti and these do not fit his ideal proportions without many significant qualifications.

[834] Alberti, *On the Art of Building*, IX.5, p. 303. Emphases mine.

Collacatio we are told by Alberti, "concerns the site and position of the parts…and relies to a large extent on the judgment Nature instilled in the minds of men, and also has much in common with the rules for outlines."[835] In this brief section on position, Alberti mentions the arrangements of statues and reliefs and other decorative elements but makes no definitive or practical pronouncements. It seems that *concinnitas* has been sufficiently described and that it is up to the reader to figure out how to apply his mathematical schemes to physical arrangements.

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d. Final Thoughts on the Quadrivial Arts in *De re aedificatoria*

Of the three controllable elements affecting *concinnitas* (number, outline, and position) outline is given much more space by Alberti. The other two are almost trivial by comparison. The recommendations Alberti makes for three-dimensional outlines are quite varied and, if his comment about the interchangeability of the dimensions is taken at face value, the proportions available to an Albertian architect are significantly expanded. The actual physical realization of these outlines is not discussed. How is one to measure a room? Should one emphasize interior or exterior measurements? How should one account for the thickness of a wall? Should measurements start and stop in the middle of walls or columns? Nor does he have a coherent section on the relationship between structural integrity and *concinnitas*, although this is implied.[836]

Modern scholars have often tended to draw upon these ratios, at least the harmonic ones, freely and without much qualification and will often apply them to two-dimensional rather than

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[835] Ibid., IX.7, pp. 309-310.

[836] There seems to be some evidence that medieval builders thought that some proportions were structurally more sound than others. See James S. Ackerman, "*Ars Sine Scientia Nihil Est*: Gothic Theory of Architecture at the Cathedral of Milan," *Art Bulletin* 31, no. 2 (1949): 105-106. Whether or not this was due to harmonic powers is not clear.

three dimensional plans. Technically, all of the above outlines are contained in a section devoted to the ornament of private buildings, but it is unclear how to apply the proportions or what to apply them to.

Alberti's own buildings are certainly not good examples of his theory of outlines. Numerous scholars have tried to find some sort of correspondence between the theories in *De re aedificatoria* and his buildings.[837] A prime example of this is the 1999 article "Fictive Harmonies: Music and the Tempio Malatestiano," by Timothy A. Anstey. Combining anachronistic scholarship with unrealistically high expectations for Alberti, Anstey presents him as not merely a high level music theorist, but as a genius knowledgeable in practical compositional and performance matters. It combines the Albertian myth of the *Vita* with the Platonic selective memory of Wittkower. The idiosyncrasies of the Tempio Malatestiano in Rimini are explained as evidence of Alberti's cutting edge compositional and music theoretical knowledge. Much of Anstey's theory is based on an overly simplistic reading of an excellent book by Karol Berger describing the increased use of *musica ficta* (accidentals) in 15th-century music composition[838] and an over-reliance on the *Vita* and *Vita*-derived hagiography. Anstey's

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[837] I contend that if architectural historians were to use the measurement analysis on modern buildings that they use on Albertian buildings they would be forced to think that the moderns are obsessed with the octave, 2:1, since most sheet goods (plywood, sheetrock, subfloor, roofing, etc.) come in 4'x8' sheets. My own experience in furniture design leads me to believe that many designs are influenced by the dimensions of the available material. The more a design deviates from the available material measurements, the more waste there tends to be, raising costs. Furthermore, the math used in design is easier if measurements are rational ratios using whole numbers. I would contend that harmonic proportions tend to be found in a design more out of convenience than from any sort of numerological philosophy.

[838] Berger, *Musica Ficta: Theories of Accidental Inflections in Vocal Polyphony from Marchetto da Padova to Gioseffo Zarlino*. See also Margaret Bent's *Counterpoint, Composition, and Musica Ficta*. Architectural metaphors in music theory were made by late medieval and Renaissance music theorists in discussions on counterpoint, and this is most certainly an appropriate avenue for investigation with proper constraints. See Bonnie J. Blackburn, "The Dispute about Harmony c. 1500 and the Creation of a New Style," in *Théorie et analyse musicales: 1450-1650, Actes du collogue international Louvain-la-Neuve, 23-25 Septembre 1999*, ed. Anne-Emmanuelle Ceulemans and Bonnie J. Blackburn (Louvain-la-Neuve: Départment d'histoire de l'art et d'archéologie, Collège Érasme, 2001), 3-4.

Alberti thinks and designs more like an early 20th-century music historian than a 15th-century polymath. And herein lies a significant problem with modern Albertian scholarship; it is almost invariably written from deep inside a single discipline. Jacob Burckhardt's assessment that Alberti was a "*uomo universale*," "an all-sided man," [839] is accurate. And the study of Alberti requires an all-sided approach. This requirement precludes both cherry picking for supportive materials and excluding exceptions that are unsupportive.

In my opinion, *De re aedificatoria* is far too vague to be used to support attributions of buildings.[840] It seems clear that Alberti did not use his own book for designing buildings. It is doubtful that it could be used at all in the ways that Alberti seems to suggest in *De re aedificatoria*. His prescriptions in Book IX appear to be specific, but deeper examination reveals them to be only superficially so. In fact, careful analysis renders them unintelligible. Derived from noble sources, Alberti leaves his numbers and quantities in the abstract. They float in the imagination and, at best, create dream spaces that, upon waking, make no sense. Alberti himself expresses a similar sentiment in Book IX, "I can say this of myself: I have often conceived of projects in the mind that seemed quite commendable at the time; but when I translated them into drawings, I found several errors in the very parts that delighted me most, and quite serious ones..."[841]

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[839] Jacob Burckhardt, *The Civilization of the Renaissance in Italy*, trans. Middlemore (New York: Macmillan and Co., 1904), 134-138. In Burckhardt's original German edition, he gives both the Italian and the German for this term– "l'uomo universale" he translates to "allseitige Mensch." Middlemore leaves the Italian in Italian, following Burckhardt, and translates the German to "all-sided man." Others English translators (notably Gadol) have translated the Italian term to "universal man," which I find a bit too comic book superhero-esque and misses Burckhardt's distinction between *all-sided* and *many-sided* men in his discussion of the Renaissance individual. See Jacob Burckhardt, *Die Kultur der Renaissance in Italien*, vol. 1 (Leipzig: Seemann, 1885), 149.

[840] In a separate study I argued facetiously for the attribution to Alberti of a church on 33rd St. in New York City. The harmonic correspondences I found in this church correlated very well with Albertian theory as found in *De re aedificatoria*. I could argue that Alberti designed anything if the only criteria is the quadrivial material in *De re aedificatoria*.

[841] Alberti, *On the Art of Building*, IX.10, p. 317.

Furthermore, the presentation of all of the quadrivial material in Book IX is incomplete. The arithmology, harmonics, means, and geometries are only partially developed and superficially treated. His organization of the material is haphazard and as a result he repeats some outline dimensions and ignores others. Interconnections between the derivations are not explained at all.

Alberti writes, "Beauty is a form of sympathy and consonance of the parts within a body, according to definite number, outline, and position, as dictated by concinnitas, the absolute and fundamental rule in Nature."[842] Nature in *De re aedificatoria* is regulated by *concinnitas,* a cosmological organizational principle. How far he takes this cosmology is difficult to ascertain. By comparison, Ficino is quite explicit on these matters in his treatment of similar harmonic material. Though probably lacking the same mystical inclinations as Ficino,[843] Alberti was still a creature of his time. Book IX of *De re aedificatoria* is indicative of his larger worldview. It seems unlikely that Alberti's worldview resembled ours more than that of his own day. In the 15th century it was commonly assumed that astrological influences affected human affairs beyond simple seasonal weather changes. That there are only stray references to astrology in *De re aedificatoria,* and nothing to suggest that Alberti was anything more than a casual observer of the art,[844] does not mean that he was necessarily unsympathetic to its significance. Additionally, he drops no hints that he presents this harmonic and geometric material simply to honor tradition or because he is paying lip service to some unspoken humanist creed. Alberti believed that there was meaning in these ratios and that they were important structural forms.

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842 Ibid., IX.5, pp. 302-303.
843 Alberti ridicules the excess of superstition. Ibid., II.13, p. 59.
844 See Ibid., I.6, p. 18, II.13, p. 59, VIII.5, p. 257, IX.10, p 317. See also Cardini, "Alberti e l'astrologia," 152-156; Grafton, *Leon Battista Alberti*, 257.

The Alberti Code

The last work by Alberti that I want to mention for its quadrivial content is *De componendis cifris*[845] (*On Composing Codes*) or simply *De cifris* written about 1467. In this short essay Alberti describes a machine he calls a "formula" that can be used to compose a polyalphabetic substitution cipher.[846] See Figure 5.20.

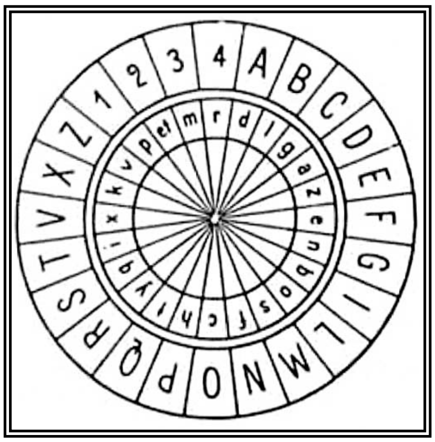

Figure 5.20. Reconstruction of Alberti's 'Formula' from *De cifris*[847]

The inner disc can be rotated to align with the outer alphabet as desired. In the simplest case, two parties, Alice and Bob,[848] would have the same machine similar to the one shown above. Alice would encode a message using the "formula" set at some particular setting. She would then give the coded message to Bob along with the setting she used. Bob could then decode Alice's message using his identical decoder ring. Such substitution systems were not new in the 1460s. The historian of cryptology, Charles Mendelsohn, states that simple letter for letter substitution was used by Julius Caesar and Augustus. The earliest documented instance of a more sophisticated substitution system in which one letter might have several coded equivalents, thus

[845] The proem to *De componendis cifris* can be found in Alberti, *Opera inedita et pauca separatim impressa*, 309-311. An English translation of *De componendis cifris* can be found in Alberti, *The Mathematical Works of Leon Battista Alberti*, 169-187.
[846] Gadol, *Leon Battista Alberti*, 208.
[847] Cipher disc reconstructed by A. Meister and reproduced in Mendelsohn. See Charles J. Mendelsohn, "Blaise de Vigenere and the *Chiffre Carre*," *Proceedings of the American Philosophical Society* 82, no. 2 (1940): 117.
[848] Alice and Bob are the standard actors used in cryptology.

better hiding certain spelling regularities that give away the code, is from 1401 in Mantua, a town well known to Alberti.[849] This system uses multiple symbols for the vowels. There are numerous examples of similar systems expanding to include multiple assignments for consonants starting in 1414. Alberti's improvement upon these systems was his machine, the "formula." Not only did this allow for relatively quick coding and decoding but it also allowed for the easy introduction of a much more complicated cipher, what is commonly identified as the polyalphabetic substitution cipher. Alberti suggested that the discs be realigned periodically, for example, on every occurrence of a capital letter or every three or four words.[850] This realignment, in effect, introduced a whole new alphabet over and over again and made the coded message significantly more complex. Giovanni Battista della Porta (1535-1615) is credited with fully systematizing the polyalphabetic system using the Albertian discs and is considered by Mendelsohn to be "the outstanding cryptographer of the Renaissance."[851]

In *De cifris,* Alberti also wrote about decoding cipher and delved into the probabilities of certain letter combinations. He constructed a table of frequencies for certain combinations and relative locations of letters to aid in the decryption of coded messages.[852] It is not clear if Alberti invented this frequency analysis of language himself or was developing a pre-existing technique.[853] The history of probability in general is closely linked to gambling, and as such many of the innovations were made anonymously. Also, due to the nature of cryptology, primary sources are often secretive and thus the history is somewhat shrouded.

De cifris is included not because it demonstrates some proto-algebraic use of symbols, but because in some sense, Alberti is again measuring something. In this case he has measured and

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849 Mendelsohn, "Blaise de Vigenere and the *Chiffre Carre*," 114-115.
850 Ibid., 113.
851 Ibid.
852 Gadol, *Leon Battista Alberti*, 210.
853 Ibid.

analyzed the frequency of letters and letter combinations in the Latin (or Italian) language. Alberti measures language and attempts to make predictions or decipherings based on what he learns from an analysis of his measurements. An astronomical analogy might be the measurements taken of celestial objects being then used to predict future positions. Furthermore, the cipher itself uses a tool, the "formula," which is clearly a modified astrolabe or similar calculation and measurement tool.

The importance of *De cifris* is difficult to determine. Alberti did not invent substitution cipher, but he did make improvements to it. His use of frequency analysis is also very interesting, but it is difficult to evaluate it in the 15th-century context of cryptology. His impact on subsequent cryptographers is also hard to pin down. Gadol states that his fame as a cryptographer is demonstrated by the fact that the famous cryptologist Blaise de Vigenère (1523-1596) referred to him in his book *Traicté des chiffres* (1586).[854] But this reference, not cited by Gadol, is hardly flattering. Vigenère is actually dismissive of Alberti's skills as a cryptographer and disparages his self-promoting boasts about his cipher being worthy "of an Emperor or of a king."[855] Gadol further builds up Alberti stating, "All of Alberti's rules of ciphering were designed to elude the very 'science' of deciphering which he had been the first to grasp."[856] But it is in no way clear that he was the "first to grasp" the science of deciphering. He may be the first that Gadol has come across, but this is hardly enough to give the entire field to Alberti. The experts on the history of cryptology mention Alberti, and Mendelsohn even made an English

[854] Ibid., 12.
[855] Mendelsohn, "Blaise de Vigenere and the *Chiffre Carre*," 117-118.
[856] Gadol, *Leon Battista Alberti*, 212.

translation of *De Cifris,* but they do not consider him to be the father of any particularly new style of cryptology.[857]

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Conclusions

It should not be a surprise that all of the quadrivial material in Alberti is substantially rooted in the medieval quadrivium. After all, the quadrivium is a medieval system based on classical concepts and Alberti was educated and surrounded by medieval institutions and read large quantities of classical literature. It would be a more of a surprise if he were not influenced by Plato, Boethius, Cicero, and all of the other authors who were important to the quadrivial philosophy.

Researching Alberti's quadrivial background and potential metaphysics is substantially hindered by much of the secondary scholarship. Modern scholars have often been overly impressed with the proficiency of Alberti's mathematical prowess. Although much of the quadrivial material described by Alberti is unfamiliar to us now, it was hardly unfamiliar in Alberti's time. It was the paradigm. Educated people would have been exposed to much, if not all, of the mathematical theory used by Alberti. His presentation of this theoretical material is generally shallow and abbreviated, which is not to say that his knowledge was inferior, but only

[857] David Kahn in his article, "On the Origin of Polyalphabetic Substitution," is impressed with the Albertian system, but ultimately thinks its source is Ramon Lull (ca. 1232-1315) who used a similar 3-disc configuration for more cabalistic/mystical purposes. See David Kahn, "On the Origin of Polyalphabetic Substitution," *Isis* 71, no. 1 (1980): 122-127. Giuliano di Bacco describes a similar set of paper wheels [*volvolles*] for astronomical computation in a 14th-century manuscript. Giuliano Di Bacco, "*Non agunt de musica*: Further footnotes to the Italian Circulation of 14th-century Music Theory," in *Medieval and Renaissance Music Conference Proceedings* (Cambridge: 2006, PDF), 5. For more on Lull's combinatory wheels, see Frances A. Yates, "Ramon Lull and John Scotus Erigena," *Journal of the Warburg and Courtauld Institutes* 23, no. 1/2 (1960): 1-44; Frances Amelia Yates, *The Art of Memory* (Chicago: University of Chicago Press, 1974), 181-183. There are also examples of cosmological "maps" which have two rings which can be adjusted to account for the precession of the vernal equinox, a 26,000 year cycle. This adjustment defines how astrological signs differ from the constellations after which they were named.

that he did not write about his quadrivial ideas using the most sophisticated mathematics at hand. He was not writing for university scholars; he was writing for patrons or other humanists.[858] But even so, modern scholars appear to have become overly enamored with Alberti's depth of quadrivial knowledge. It sometimes seems that for many scholars Alberti's description of Pythagorean harmonics or his directions for determining the height of a tower using simple trigonometry was their first exposure to this type of mathematical material and they hastily jumped to the conclusion that *their* first exposure must be *the* first example in history. Add to this the lingering effects of the *Vita,* and these Albertian scholars have become licensed to think of Alberti as the first and the greatest in nearly everything he did. They forget to look elsewhere for other, more substantial sources, like the most obvious ones: Euclid, Archimedes, Ptolemy, and Boethius. The result is a fanciful image of Alberti as some sort of mathematical genius, when the reality from *De re aedificatoria* or *De pictura* is significantly more humble.

The larger result is that Alberti, the so-called "Universal Man," has been endowed with more quadrivial skill than he really had, to the detriment of his more significant contributions to history. What makes the real Alberti special in the history of the quadrivium is that he, more often than not, used the quadrivial arts as tools for the exploration of the physical world, not the world of light, forms, and numbers described by Plato. In the hands of the real Alberti the quadrivial arts become quadrivial crafts. If the real Alberti is compared to Euclid, Archimedes, Ptolemy, and Boethius, in medieval quadrivial terms, he comes up quite short. His arithmetical, musical, geometrical, and astronomical abilities appear be rather unsophisticated. But then (leaving aside the bravado of the *Vita*) Alberti himself states in *De re aedificatoria* that it is not

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[858] Heather Horton has written a fine analysis on the readership of *De re aedificatoria*. See Horton, 67-74.

expected that the architect be "a Nicomachus in arithmetic, or an Archimedes in geometry,"[859] and that he need not have "an exact understanding of the stars."[860] Alberti recommends only that an architect have, at minimum, a "sufficient knowledge of mathematics for the practical and considered application of angles, numbers, and lines, such as that discussed under the topics of weights and the measurements of surfaces and bodies."[861] Alberti does not insist on absolute proficiency in the mathematical arts, he just requires that an architect be adequate. This adequacy may look like dilettantism if one expects to find a mathematical genius rather than a well rounded architect. A great deal of the secondary literature written on Alberti has undermined his true legacy. It is a shame, for the real Alberti, stripped of his Universal Manliness, is a pleasure to read. He is neither an Aristotelian nor a Platonist nor a Neoplatonist nor a Stoic. He is an all-sided man.

Alberti is quadrivially interesting because he takes his mathematical skills and finds number in the world, not the world in number. Alberti had one foot in the quadrivial past and the other in the empirical movements that would sweep through the scientific circles of Europe; movements that were not simply Platonic or Aristotelian, but were also Epicurean, Stoic, and practical. He was also involved in returning Euclid and Archimedes to the real world of observation, but his work in this realm was more as a popularizer rather than an avant-garde innovator. This emphasis on application is Alberti's mathematical legacy.

Alberti's most explicit and developed quadrivial ideas are associated with his concept of *concinnitas* from Book IX in *De re aedificatoria*. Like Oresme's analysis of celestial motions, Alberti allows both arithmetical/harmonic multitude and geometrical magnitude into his theory of beauty and structure. He does not insist on an absolute division between rational and

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[859] Alberti, *On the Art of Building*, IX.10, p. 317.
[860] Ibid.
[861] Ibid.

irrational. In fact, this distinction is not even mentioned. *Concinnitas* is born out of quadrivial numbers and the concordant relationships between them, *and* it is born from the geometry of cubes. Why is it born from cubes and not any of the other Platonic solids? I can only speculate that he limits himself to cubes because all of his architectural spaces are rectilinear and thus easily divisible into theoretical unit cubes.

In *De statua,* Alberti states that his measurements of the human body serve two purposes: to resemble the universal characteristics of a human being and then to resemble a particular human being.[862] However, the majority of the treatise is given to universal human measurements, not particular ones. He assumes that universal human proportions are based on simple rational ratios, but he discovers these ratios by measuring multiple particular examples and averaging the results. Because his table of measurements for this universal man only lists vertical, horizontal, and angular measurements and not lengths derived from the geometry of triangles, all of his measurements and ratios are rational. Although the approach appears to be Aristotelian and his expectations are at times harmonic, I should emphasize that this treatise is very short and not particularly philosophical. It is an instructional manual, and does not support an in-depth philosophical treatment. Similarly, his other treatises on measurement, *Ludi rerum mathematicarum* and *Descriptio urbis Romae*, are both very basic instruction manuals and not philosophical texts. Alberti's principle interest in these works is in the measure of the primary dimensional qualities of the world and he describes how this is done. Alberti wants to know the dimensions of things in nature, which differs from *De re aedificatoria*, where he explains what those dimensions should be.

The mathematics in *De pictura* and *Elementa picturae*, though again not presented by Alberti in philosophical terms, is quite interesting when placed in the context of his other mathematical

[862] Alberti, *On Painting and On Sculpture*, 122-3, section 5.

works. The works on measurement discussed above are all about how things really are. *De pictura* and *Elementa picturae* are about how they seem. The method for making pictures that he describes is usually called one-point perspective, but it might be better to call it two-point perspective, for there are two fixed points in his system, not just one. There is, of course, the centric point that you see in the picture. This is the point from which all perspective lines emanate, but there is also the static point of view that is not seen in the picture. This is the point where the painter's eye is located. All of Alberti's other descriptions of measurement (bodies, maps, or buildings) are about absolute dimensions that are not dependent on a point of view. They have no need for a centric point nor do they have a favored point for viewing. They are in no way relative to a particular place for observation– if they were, that point would either be infinitely far away or that point would be paradoxically everywhere. And yet, the perspective of *De pictura* and *Elementa picturae* is constructed such that the absolute dimensions of the physical world are derivable from the information in the picture. This is also what some of the exercises in *Ludi rerum mathematicarum* do. They take perspectival information and derive from that, the true dimensions of an object. In a similar manner, pictures constructed using Alberti's perspectival system can, in theory, be analyzed in reverse to determine the absolute dimensions of whatever is depicted. The results will be information like the absolute, perspective-less measurements in *De statua* or those promoted in *De re aedificatoria*. Alberti connects visual perception to true knowledge of the world.[863]

Alberti describes a way to make Platonic perception in the world of shadows into an exact science. Unlike Oresme, Alberti does not struggle with the conflicting natures of magnitude and multitude. They coexist in *De re aedificatoria* without comment. For Alberti, what I'm calling

[863] A Neoplatonic interpretation could be made in terms of the intelligible world "as contained in the mind of man." See Plotinus, *The Essential Plotinus: Representative Treatises from the Enneads,* trans. Elmer O'Brien, 2nd ed. (Indianapolis: Hackett Publishing, 1984), V.9.11, pp. 54-55.

the quadrivium was simply the scholarly mathematics of his day. Hints of quadrivial philosophy found in his various mathematical works come as no surprise. But unlike professors of the mathematical arts, like Oresme or Prosdocimo, Alberti was unencumbered by the philosophy of the quadrivium. He was not constructing a metaphysical system. Nor was he quarreling with judicial astrologers or music theorists. He was not invested in the philosophical coherence of the mathematics he was using. He was measuring the world around him, using the quantitative tools that were available– a mathematics that had thrived in the intellectual world for two millennia. However, quadrivial mathematics was not well suited to the real world. It was designed for an ideal world. But this subtle philosophical conundrum was not of any importance to Alberti. He could go about the dirty and inherently inexact business of measuring the city of Rome one day and dream of perfect Pythagorean spaces the next. The numbers added up either way. In Alberti's hands, the quadrivium was a means to an end, not an end in and of itself.

Chapter 6:
Quadrivial Conclusions

These three case studies describe distinct mathematical approaches, but they all demonstrate an awareness of the quadrivial philosophy as outlined in Chapters 1 and 2 of this dissertation. Each case study engages this philosophy for different purposes. Oresme is the proto-physicist, who wants to undercut astrology, and uses the mathematical techniques of the quadrivium itself to do so. Prosdocimo is the quadrivial purist who wants to protect the metaphysical foundation of music theory from modern practices that prioritize the ear over the soul. And Alberti is the eclectic author who applies aspects of the metaphysical structure of the quadrivium to the world around him.

Case Study Summary and Analysis: Oresme

The primary text used in this dissertation to analyze Oresme's quadrivial philosophy was his *Tractatus de commensurabilitate vel incommensurabilitate motuum celi* (mid-14^{th} century). This text is an explicit exploration of one of the foundational premises of the quadrivium. *De commensurabilitate* is a debate between arithmetic and geometry, multitude and magnitude, discrete and continuous, rational and irrational, commensurable and incommensurable. This debate is the dividing line that Boethius struggled so hard to justify in his quadrivial texts.

Oresme's primary motivation for writing this text was to attack judicial astrology. As Oresme saw it, the astrologers of his day demanded that the celestial motions be commensurate. Their science demanded that arrangements repeat, for if they did not, there would be no basis for comparison, and astrology could not be a science. There would be no evidence from past events to use for the prediction of future events. If the motions of the celestial spheres were incommensurable, any past configuration would never recur. An astrologer could not say, "Based on past conjunctions, when Saturn conjuncts with the sun at exactly 15º Virgo there will be born a new heir to the throne." This would not be possible, because if the motions are incommensurable, Saturn will never conjunct with the sun at exactly 15º Virgo ever again. Oresme's argument to undercut the determinism of astrology required that he articulate a summary of the quadrivial philosophy of his day (as he saw it). In so doing, he developed a variety of techniques, derived from the basic quadrivial disciplines, which led him to question the fundamental metaphysical structure of the quadrivium itself.

In Part I of *De commensurabilitate* Oresme developed the mathematics necessary to analyze celestial motions, assuming they were commensurate. He concluded that the actual motions observed in the sky did not exhibit any of the commensurate characteristics that were often claimed, by both ancient and modern authorities. There was no evidence that the celestial spheres moved in any of the traditional Pythagorean relationships promoted by Cicero and Plato, such as 2:1, 3:2, 4:3 (octaves, fifths, and fourths respectively) or even combinations of these intervals such as 3:1, 8:3, 9:4 or even 256:243. Conjunctions between the heavenly bodies did not appear to occur in a limited number of locations as the results of his analysis of circular kinematics demanded. "Therefore," Oresme stated, "if celestial bodies in motion produce a harmony, it is not necessary [to assume] that such a harmony arises from the velocities of their mean motions, but perhaps it stems from some other source for other reasons..."[864] He later suggests that these "other reasons" might yet be discovered, not in the motions, but in the weights and/or volumes of the celestial orbs themselves.

In Part II of *De commensurabilitate* Oresme developed the mathematics necessary to analyze celestial motions, assuming they were incommensurate. The ramifications of incommensurate

[864] Oresme, *De commensurabilitate*, I.478-480, pp. 212-215.

celestial motions led to an infinitely diverse universe, where configurations were never repeated. The cosmos would always be in a new state– any arrangement would be equally possible. Oresme wrote,

> On the assumption of the incommensurability and eternity of motions, it is truly beautiful to contemplate how such a configuration as an exact conjunction occurs only once [at a particular place] through all of infinite time, and how it was necessary through an eternal future that it occur in this [very] instant with no conjunction like it preceding or following. One cannot find reason as to why it happens at that time [rather] than at another time, unless it be because the velocities of motions and the unalterable inclinations of moving bodies are [simply that way].[865]

Oresme proposed that incommensurable motions should not be shunned, but ought to be embraced. For they would produce both regular repeating cycles, based on irrational orbital periods, and would also produce an infinite variety of unique arrangements. Oresme proposed several terms to describe this seemingly paradoxical wonder: "rational irrationality," "regular non-uniformity," "uniform disparity," and "concordant discord."[866] For Oresme, these incommensurable motions were vastly superior to the dull and predictable cosmos moving commensurably. His incommensurable motions would also make astrology with its Great Years and powerful harmonic celestial arrangements irreconcilable with the heavenly spheres. How could astrologers pretend to predict something based on one, and only one, piece of causal evidence, with no chance that it could ever happen again? The logic would be: if x then y. But if x never ever happens again, the argument is not predictive. It is less than trivial.

In Part III of *De commensurabilitate,* the dream, Oresme directly pits quadrivial arithmetic (and music) against his somewhat modified version of geometry (and astronomy/astrology). All of the arguments presented for arithmetic are orthodox quadrivial philosophy as derived directly from Plato, Cicero/Macrobius, and Boethius. The arguments made by Arithmetic are an

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865 Ibid., II.307-312, pp. 272-273.

866 ″... ut ita dicam rationalis irrationalitas, regularis difformitas, uniformis disparitas, concors discordia." Ibid., II.112-114, pp. 256-257.

excellent view into the mind of a late medieval quadrivial purist, as characterized (or perhaps caricatured) by Oresme. The arguments made by Geometry are Oresme himself. Arithmetic argues that irrational ratios are literally unthinkable, stating, "an irrational ratio is neither suitable or relatable to the understanding, for which reason the ancients said that the mind conforms to a certain numerical and harmonic plan."[867] Counting numbers and the ratios formed with them are intelligible. They can inform the mind. They connect the human *anima* with the upper realms, the Intelligences, the divine realm. Irrational numbers and the ratios made from them are mundane (in the modern sense of the word), they are of this earth, the world of shadow and change. They are not intelligible. They do not inform the human *anima*. They are not part of the structure laid out by God, the universal architect. Oresme (as Geometry) counters this argument by suggesting that the God of Arithmetic is limited to counting numbers and that the true God must be greater than that. A truly great God would produce an infinitely varied world, not a world of endlessly repeating fully predictable cycles and uniformly distributed matter.

Arithmetic and Oresme/Geometry then argue over priority. Arithmetic's argument relies on the analogy of unity and geometric point which allow geometric constructions such as lines, triangles, and cubes to be analyzed in numerical terms. This correspondence allows Arithmetic to claim priority, for without numbers, geometric figures could not exist. It is the same argument Boethius uses to justify the segue from multitude (arithmetic/music) to magnitude (geometry/astronomy) in his description of the philosophy that binds the quadrivium together.[868] As was discussed in Chapter 2, this argument is not presented clearly or forcefully by Boethius and even less so by Oresme's Arithmetic. Oresme/Geometry's counter argument is both clear and forceful. He states, "there is no measure or ratio that is not included within our

[867] Ibid., III.108-113, pp. 292-293.
[868] It is also the same argument given by Macrobius. Macrobius, *Commentary on the Dream of Scipio,* 1.V.5-18, pp. 96-99.

magnitudes."[869] Using the standard quadrivial terminology he clearly states that numbers are a subset of magnitude. This argument feels totally modern and is very easily stated using modern terminology; *the real numbers include all counting numbers, not vice versa.*

Arithmetic then argues for the harmonically structured cosmos, a *musica mundana*, based on celestial motions. All of her arguments are derived from authorities such as Plato, Hermes Trismegistus, Cicero/Macrobius, Cassiodorus, Boethius, and even the satirists Johannes de Hauvilla and John of Salisbury. Oresme simply brushes this argument aside by noting that the various *musica mundana* theories proposed by this parade of authors conflict with one another. Oresme had already explained in Parts I and II that motion-based harmonic theories were incompatible with celestial observation. And then in Part III he explains how motion-based harmonic theories were incompatible with experiential knowledge gained by listening to sounds produced by moving objects on earth. The motion/pitch relationship simply did not behave the way Macrobius et al. claimed. When considered in terms of the quadrivium, Oresme insinuates that the physics here on earth is similar to the physics above the moon. This conflation of terrestrial and celestial physics is counter to the Aristotelian physics and cosmology upon which quadrivial astronomical theory was based. Comparing terrestrial physics to celestial physics was comparing apples to oranges. In the general quadrivial worldview, physics did not transcend the world of shadows and change, only the intellect did. But Oresme seems to suggest that he is comparing apples to apples and that his observations of the relationship between motion and pitch here on earth were evidence against the theory of a harmonic cosmos based on motion. He instead suggests that a harmonic cosmos might yet be discovered, not in the motions, but in the weights and/or volumes of the celestial orbs themselves. Oresme was not troubled with the idea that the celestial spheres might be related harmonically, just not in terms of motion. He still

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[869] Oresme, *De commensurabilitate*, III.361-363, pp. 312-313.

proposed a form of *musica mundana*, just not a form that the standard quadrivial authorities cited by Arithmetic had suggested.

Oresme's final argument against Arithmetic and for incommensurability was based on probability. He claimed that given any random selection of ratios-of-ratios, it was more likely that they would be incommensurable. And thus, it was statistically unlikely that the celestial motions were commensurable. Not only is this reasoning interesting for its sheer mathematical sophistication, but in this argument Oresme rationally compares irrational ratios. In much the same way that he allowed the terrestrial physics to be comparable to celestial physics, in this argument he is acting as if irrational quantities and rational quantities are of the same *genus*, to use Prosdocimo's word. This equivalence is fundamentally contrary to the distinction between geometrical magnitude and arithmetical multitude upon which the quadrivial philosophy was built.

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Case Study Summary and Analysis: Prosdocimo

Unlike Oresme, whose *De commensurabilitate* directly engaged and debated the quadrivial philosophy, much of Prosdocimo's quadrivial philosophy had to be discovered in or inferred from his comprehensive corpus of extant mathematical works. In general, Prosdocimo was a staunch proponent of the traditional quadrivium. His arithmetical and musical texts do not deviate from the Boethian definition of number as multitude. However, this quadrivial conservatism did not stop him from being a modernist. He fully embraced Hindu-Arabic numerals and appreciated the benefits that they afforded calculation, particularly for astronomical applications. He also fully embraced the new musical styles of his day and actively developed tuning systems that would facilitate their production. However, in each of these disciplines he was careful not to disrupt the basic division between multitude and magnitude. In his *Algorismus de integris* (1410) he starts with the Boethian definition of number, writing, "Number is a multitude or a discrete quantity issued from unities."[870] And in his text on basic music theory, *Brevis summula proportionum quantum ad musicam pertinet* (1409), he writes,

> [A] ratio ... is the mutual relationship of several quantities of the same proximate genus; "of the same proximate genus" is stated because a continuous quantity cannot properly be taken in a ratio to a discrete quantity since, ... , they are not of the same proximate genus but of ones quite remote.[871]

In both of these texts dealing with the arithmetical/musical disciplines, Prosdocimo explicitly lays out his quadrivial terms. Multitude and magnitude are not comparable to one another. They are not the same genus. This, of course, differs completely from the mathematics Oresme uses in his comparisons of ratios-of-ratios, where he mixes continuous magnitudes with discrete multitudes. Prosdocimo is adamantly opposed to such a mixture. His most fervent opinions on this matter are expressed in *Musica speculativa* (1425). In this treatise Prosdocimo defends the quadrivial orthodoxy from Marchetto of Padua, who a century earlier had proposed a tuning system that divided the *tonus*, 9:8, into five equal parts. Marchetto's ideas were popular enough in the early 15th century that Prosdocimo dedicated an entire treatise to correcting its "evils, lies, and mistakes concerning music."[872]

At issue was the quadrivial fact, proven by Euclid, that a superparticular ratio like 9:8 could never be divided into any number of equal of parts. This would result in irrational quantities which were for Prosdocimo absolutely out of bounds for music. Why they were out of bounds is

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[870] "Numerus est multitudo sive quantitas discreta ex unitatibus profusa sive ex unitatibus agregata." Prosdocimo de' Beldomandi, *Algorismus de intergris*, 2.
[871] Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 48-49.
[872] Prosdocimo de' Beldomandi, *Musica speculativa*, Preface, pp. 158-159.

not explained by Prosdocimo in his musical texts, but in his astronomical *Commentary on Sacrobosco's Sphere* (1418), the outline of an answer is suggested.

All of the conclusions drawn in Prosdocimo's commentary are, in every way, consistent with the Aristotelian cosmos described in Sacrobosco's *Sphere*, the standard quadrivial astronomical text of the late Middle Ages. But again, as in his music theory texts, he was not averse to novel and interesting ideas, so long as he could incorporate them into the larger quadrivial worldview. He accommodated non-spherical and unevenly distributed water around the terrestrial globe. He readily and in detail discussed the idea of a spinning earth, before dismissing the idea with standard Aristotelian physical objections. And he discussed why the four basic elements of Aristotelian terrestrial physics were not encountered in their pure forms. But his discussion of rational and irrational "motions" (or "activities of the mind") may be the clue as to why Prosdocimo so adamantly maintained the distinction between multitude and magnitude. As Prosdocimo explained, rational thought originated from God, whereas irrational thought originated in man. It is reasonable to conclude that this distinction is also applicable to the multitudes of arithmetic and the magnitudes of geometry. That would mean that the numbers of arithmetic and the rational ratios of music originate in God, and the magnitudes of geometry and astronomy originate from man. This distinction may explain why Prosdocimo was so vehement in his defense of music from the invading magnitudes of Marchetto.

For Prosdocimo, a most devout practitioner of the quadrivial philosophy, the distinction between arithmetical number and geometrical magnitude was much more than simply a distinction between quantities. This division appears to have been fundamental to his metaphysics and his epistemology. In Platonic terms it was the divided line, or the difference between the world of light and the world of shadows in the cave. In Neoplatonic and/or

Aristotelian/Thomistic terms it was the difference between divine thought and human thought. For Prosdocimo there is some suggestion that music was more than just the right Pythagorean-derived ratios on a monochord. There is some suggestion that these quadrivial ratios were the thoughts of God, reflected in man. That it is not possible to distinguish with the ear the difference between Marchetto's minor semitone and Prosdocimo's minor semitone is not the point. Only Prosdocimo's interval is *of God*. Only Prosdocimo's is a reflection of the structure of the divine cosmos, emanated from the mind of God, to use Neoplatonic language. Marchetto's is a cheap imitation, created from man.

Prosdocimo kept his multitudes very separate from his magnitudes. There is no discussion of *musica mundana* (music of the spheres) in his music theory and there is no discussion of music theory in his astronomy. Also conspicuously absent from Prosdocimo's writings is Plato. He is referred to from time to time, but his influence is peripheral. The primary quadrivial philosophical source for Prosdocimo's works on multitude appears to have been Boethius. The sources for his works on magnitude appear to have been largely Abu Ma'shar and Aristotle (and, of course, Sacrobosco). The astrological and medical material that fill much of his student notebooks is almost completely absent from the mature quadrivial works that I have examined. And again, one would have expected these topics to have been discussed in his lengthy and comprehensive *Commentary on Sacrobosco's Sphere.* Other popular and earlier commentaries on the *Sphere*, such as those by Robert Anglicus (fl. 1270s) or Cecco d'Ascoli (d. 1327) digress into astrology, astrological medicine, and other more esoteric topics, and their commentaries are only a fraction of the length of Prosdocimo's commentary.[873]

[873] Thorndike's introduction in Thorndike, ed., *The Sphere of Sacrobosco and Its Commentators*, 48-55. It is possible that Prosdocimo was wary to put his astrological thoughts on record. Cecco d'Ascoli, once a professor at the University of Bologna, was burned at the stake in Florence (along with his commentary

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Case Study Summary and Analysis: Alberti

Unlike both Oresme and Prosdocimo, Alberti was not a professor of the mathematical arts at a university. He was educated within the university system and may even have known Prosdocimo. He demonstrated a proficiency in the quadrivial disciplines, but he was not a professional quadrivial scholar. He might more accurately be called a quadrivial practitioner. Alberti's quadrivial philosophy is of interest because of how and why he uses the quadrivial disciplines.

Most of his mathematical works deal with measurement. *De statua, Descriptio urbis Romae,* and *Ludi rerum mathematicarum* all measure the actual dimensions of the physical world. *De statua* and *Descriptio urbis Romae* directly measure their respective subjects and *Ludi rerum mathematicarum*, with its many different subjects, tends more towards deducing actual measurements using mathematics. *De pictura* and *Elementa picturae* similarly use mathematics to measure the physical world, but in these texts the subject is not the actual measurements of the world, but rather the perspectival appearance of the physical world translated mathematically onto a plane that is perpendicular to a line drawn from the viewer's eye. Like *De componendis cifris*, Alberti's text on codes and code breaking, *De pictura* and *Elementa picturae* translate one form of information into another, using mathematics. This allows pictorial information to be translated back into real physical dimensions. This is what the eye does when it sees depth in a picture designed using these principles. None of the works on measurement mentioned above are very philosophical regarding mathematics. Except for his measurement of particulars to find

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on the *Sphere* according to Thorndike) in 1327, presumably for failing to stop teaching astrological determinism. See also Tester, *A History of Western Astrology,* 193-194.

the universal dimensions of a human body in *De statua*, Alberti does not discuss the structure of the cosmos, physical or metaphysical, in mathematical terms.

The most significant work for his quadrivial philosophy is *De re aedificatoria*. Most of the quadrivial material is from Book IX in which Alberti discusses the best ways to achieve beauty, "the noblest and most necessary"[874] of the conditions that should be considered in building any structure. After some preliminary material which includes discussions on decorating, gardening, and arithmological observations, he describes his concept of *concinnitas*, writing, "Beauty is a form of sympathy and consonance of the parts within a body, according to definite number, outline, and position, as dictated by *concinnitas*, the absolute and fundamental rule in Nature."[875] Later in that same section he writes,

> The very same numbers that cause sounds to have that *concinnitas*, pleasing to the ears, can also fill the eyes and mind with wondrous delight. From musicians therefore who have already examined such numbers thoroughly, or from those objects in which Nature has displayed some evident and noble quality, the whole method of outlining is derived.[876]

As was shown in Chapter 5 of this dissertation, the sources for the numbers that inform *concinnitas* were from traditional Pythagorean monochord tuning, the mathematical means, and from the geometry of cubes. The most surprising of these sources were his geometrical derivations from the cube resulting in a variety of irrational numbers and ratios. Like Oresme, Alberti allows both arithmetical/harmonic multitude and geometrical magnitude into his theory of beauty and structure. Unlike Prosdocimo, Alberti never insists on an absolute division between rational and irrational. *Concinnitas* is born out of quadrivial numbers and the concordant relationships between them, *and* it is born from the geometry of cubes which is inherently irrational.

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[874] Alberti, *On the Art of Building*, VI.1, p. 155.
[875] Ibid., IX.5, p. 303.
[876] Ibid., IX.5, p. 305.

Alberti shares more in common with Prosdocimo's nemesis, Marchetto of Padua. Although Alberti appears to have been better educated in the mathematical arts than Marchetto, both mix multitude with magnitude. Alberti apparently does this not because he thinks that numbers are a subset of magnitude like Oresme, but rather because he is applying quadrivial methods to real world situations. Alberti mixed multitude and magnitude without any philosophical warning whatsoever. He was not invested in the quadrivial worldview like Prosdocimo. He was simply using the quadrivial mathematics that he learned as a student to define his own standard of architectural beauty. For Oresme and Prosdocimo the quadrivial arts were very clearly defined and their discussions of multitude and magnitude are philosophically specific. One can assume, given their professions and their written works, that each had read or lectured on Boethius' *De institutione arithmetica*. For Alberti, the quadrivium was simply the mathematics of his day; he was not conscious of the quadriviality of the mathematics he used.

He defined *concinnitas* in purely quadrivial terms. In many ways it appears that Alberti thought of *concinnitas* as the quadrivial philosophy itself. All of the mathematics used in his description of it are found in Boethius and Euclid. However, Alberti was not writing about the quadrivium or even about a particular quadrivial discipline. He was writing about building. His primary motivation was not to find number in some Aristotelian or Platonic mind space. He was trying to use mathematical quantities to describe proportional beauty here on earth. Like Marchetto, he may have used the intellectual cachet of the quadrivial philosophy to give his architectural theory classical authority, but unlike Marchetto his mathematical discussions were comprehensible. He used the quadrivium and made it work for him.

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Closing Thoughts

In this dissertation I set out to explore how the quadrivial philosophy informed the work of three late medieval scholars. I investigated how they conceived of the relationships between and amongst the four quadrivial disciplines and how they used quadrivial mathematics as a metaphysical system to structured the cosmos– macrocosm and microcosm. I paid particular attention to the way these scholars reconciled new demands placed on quadrivial mathematics, demands that sometimes disrupted the larger cosmological structure that the quadrivium described.

In the process of researching and writing this dissertation I discovered a much deeper and more subtle metaphysics underlying the late medieval quadrivium. What on the surface appeared to be a fundamentally Platonic conception of a single cosmos emanating from Unity, also incorporated healthy doses of Aristotelianism, Stoicism, and Judeo-Christian-Islamic theology. The quadrivium, as a worldview, had evolved from antiquity to support a wide spectrum of philosophical and natural philosophical ideas while maintaining the fundamental premise that the cosmos was structured mathematically.

However, these case studies point to a fault line that was present in the late medieval quadrivial structure– the division between multitude and magnitude. This division was not simply a distinction between the counting numbers and the real numbers. That is the way we may see it now in our modern definition of mathematics, but this division was born in the Pythagorean prehistory of the quadrivium and it took on much more ontological and epistemological significance.[877] Irrational magnitudes were inconceivable. They defied reason.

[877] See Klein, *Greek Mathematical Thought and the Origin of Algebra*, 79-99.

The Pythagorean who supposedly discovered irrational magnitudes, the story goes, was killed in a shipwreck as a punishment for this discovery.[878]

From our modern mathematical perspective, this punishment seems a rather excessive response for simply discovering the √2. But this anecdote provides an excellent example for why modern notation can be deceptive when discussing the quadrivium. The symbol, √2, is just a symbol. It is a picture, and this picture has the number 2 in it. It looks like a number, and the square root symbol even resembles our modern symbol for division (√ and /). But this is not how a late medieval quadrivial scholar would have "thought" about an irrational number. A late medieval mathematician would have thought of Euclid's *Elements*, Book X, which Thomas Heath summarizes writing, "If the diagonal of a square is commensurable with its side, it will follow that one and the same number is both odd and even."[879] A number cannot be both odd *and* even. This breaks one of the foundations of Aristotelian logic. In *Metaphysics*, Aristotle writes, "the same attribute cannot both belong and not belong to the same thing at the same time and in the same respect."[880] This way of thinking of an irrational magnitude is much more irrational than simply writing "√2." An irrational magnitude was just that, irrational. It was unthinkable. As Oresme's Arithmetic argued, "Thus, an irrational ratio is neither suitable or relatable to the understanding, for which reason the ancients said that the mind conforms to a

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[878] This story (or similarly punitive versions of it) is told by many authors including Iamblichus and Proclus. See Carl B. Boyer, *The History of the Calculus and Its Conceptual Development* (New York: Hafner, 1949; reprint, New York: Dover, 1959), 20.
[879] Heath, *A Manual of Greek Mathematics*, 55.
[880] This quotation is from Aristotle's *Metaphysics*, 1005b.11-20, found in Thomas Little Heath, *Mathematics in Aristotle* (Oxford: Clarendon Press, 1949), 203. Several lines later Aristotle says that such a man, who would question this fundamental statement in logic, "is no better than a vegetable." Avicenna goes even further, writing, "Those who deny a first principle [of non-contradiction] should be beaten or exposed to fire until they concede that to burn and not to burn, or to be beaten and not to be beaten, are not identical." This quotation is from Wolter's introduction to *Duns Scotus on the Will and Morality*. See John Duns Scotus, *Duns Scotus on the Will and Morality*, trans. and intro. A. B. Wolter (Catholic University of America Press, 1997), 9.

certain numerical and harmonic plan."[881] The idea of an irrational magnitude could not be formed in the mind and thus had no relationship to the divine creator, who gave human beings rationality, the ability to understand the mind of God.

The elimination of the distinction between rational and irrational may be the first significant crack in the quadrivial philosophy as a coherent mathematical philosophical structure of the macrocosm and the microcosm. Without this distinction, the larger quadrivial structure is severely damaged. For Oresme, the Platonic Numbers simply became a subset of geometric magnitudes. When he noticed that motion and pitch were not related in a way that would produce cosmic harmony, he was suggesting that lessons learned from terrestrial physics, mathematicized using geometry, might be applicable to the physics of the heavens, which had been more closely associated with the mathematics of arithmetic and music. The world of light and truth became part of the world of shadows and change. One usually thinks of this merging of the terrestrial with the celestial in association with Copernicus and Galileo, but a close reading of Oresme's mathematical treatise against astrology, *De commensurabilitate*, clearly indicates that he was already philosophically and mathematically well on his way to eliminating this distinction.

In my study of these quadrivial scholars and the myriad of peripheral material that was required to make sense of them, I have come to realize that in much the same way that the quadrivium was not simply a set of four disparate mathematical disciplines, the quadrivium itself was not separate from other philosophical systems. It worked in concert with Platonic metaphysics, Aristotelian physics and cosmology, and Christian Neoplatonic theology. It even provided a metaphysical foundation for astrology and medicine. It was woven into a unified metaphysical theory of everything.

[881] Oresme, *De commensurabilitate*, III.108-113, pp. 292-293.

Remnants of the quadrivial philosophy are clearly evident in the 16th century and extend well into the 17th and 18th centuries. Johannes Kepler (1571-1630), echoing Ptolemy's elaborate 2nd-century musical cosmos from his text on music theory, described the harmonics of the Copernican solar system using the modern mathematics of his day. This mathematics bears little resemblance to that of Boethius for his mathematical descriptions are filled with irrational "numbers" and ratios. They are the direct offspring of Oresme and his school of physical geometry. Francesco Giorgi (1466-1540), who published a book titled *De harmonia mundi totius cantica tria* (Venice, 1525) designed a church in Venice, San Francesco della Vigna, with a floor plan based on Pythagorean consonance.[882] Musical tuning systems prioritizing some of the Pythagorean consonant ratios continued to be developed by the likes of Franchino Gaffurio (1450-1522), Lodovico Fogliano (1468-1548), and Gioseffo Zarlino (1517-1590), but Vincenzo Galilei (1525- ca. 1587) is generally credited with the first attempt at distributing equal chromatic intervals (roughly equal temperament tuning) throughout the octave, thereby tempering the traditional Pythagorean intervals.[883] Echoes of the quadrivial structure are even evident in the writings of Gottfried Leibniz (1646-1716), who wrote, "Music is a hidden arithmetical exercise for the soul in which the soul counts without being aware of it."[884] And, as

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[882] Francesco Giorgi (not to be confused with Francesco di Giorgio) wrote a memorandum in 1535 concerning the "harmonious proportions" for the design of this church. A translation of this memorandum was made by Wittkower. See Wittkower, *Architectural Principles in the Age of Humanism*, 155-157.

[883] Vincenzo, a lutenist, chose 18:17 as his "halftone" interval. It is equal to ca. 99 cents. Twelve of Vincenzo's chromatic intervals are just shy of a Pythagorean 2:1 octave (≅1.986:1). Incidentally, Kepler fully analyzed Vincenzo's system in *Harmonice mundi*. Barbour, *Tuning and Temperament: A Historical Survey,* 56-58; Kepler, *The Harmony of the World [Harmonice Mundi]*, III.viii, pp. 196-199. Vincenzo's conceptual approach was similar to that of Prosdocimo, but instead of prioritizing the tonus (9:8) he prioritized the "halftone" (18:17). This system did not require multiple *ficta* notes between "whole tones." For example, Ab and G# would be the same. It very closely resembles our modern system, as found on a piano.

[884] This quotation, from a letter Leibniz wrote to Christoph Goldbach, is found in Geza Révész, *Introduction to the Psychology of Music*, tran. G. I. C. de Courcy (Norman, OK: University of Oklahoma Press, 1954; reprint, Dover, 2001), 80. See also R. Katz and C. Dahlhaus, eds., *Contemplating Music:*

shown in Chapter 2, Isaac Newton (1643-1727) divided the rainbow by the divisions of a monochord tuned to a diatonic scale with perfect Pythagorean consonances at 2:1, 3:2, and 4:3. Vestiges of the medieval quadrivial philosophy are abundant.

At its heart, the quadrivium was the description of mathematical connections between the macrocosm and the microcosm, the connections between the world of shadows and the world of light. In this sense the quadrivial mission has not left us. It is the basis of modern science. We continue to assume that there are laws that regulate the structure of everything, and that these laws are mathematical and universally applicable.

Source Readings in the Aesthetics of Music: Essence, ed. and intro. R. Katz and C. Dahlhaus, vol. III (New York: Pendragon Press, 1992), 425-457.

Appendix 4:
Description of Prosdocimo's Student Notebooks

Florence, Biblioteca Medicea Laurenziana, Ashburnham 206, 145 folios. (Early 15th century)
Student Notebooks of Prosdocimo de' Beldomandi

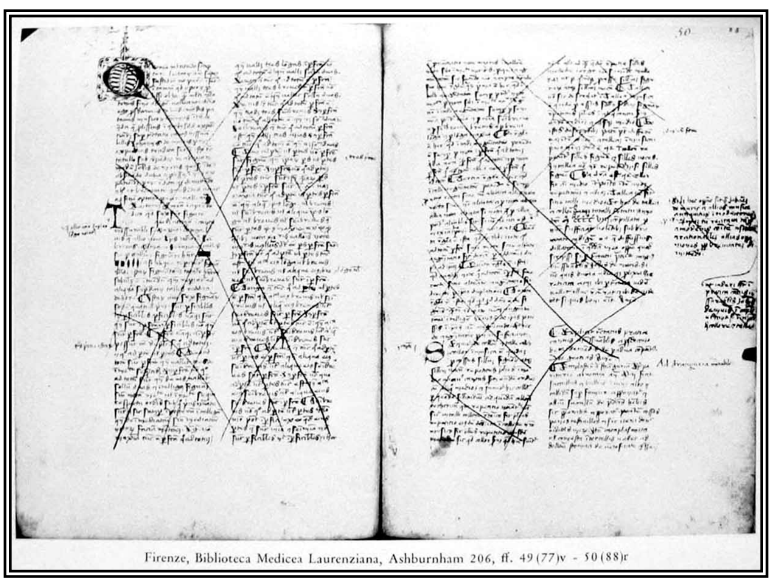
Except for a few folio pages, which will not be described in the following table, all of this manuscript was written in Prosdocimo's hand. It is outlined here because it offers a peak into the world of a university student studying the quadrivium and medicine in the early 15th century. Most of the information in this table is drawn from studies by Favaro, Herlinger, Gallo, and the indices of the Biblioteca Medicea Laurenziana.[885] I have added some commentary as it relates to this dissertation. The vast majority of this manuscript is unedited.

Codex folio #s	Brief Description
1r	Astronomical computations.
1v	Rules for determining the dates of Easter and Passover.
2ra-5vb	Transcription of Sacrobosco's *Algorismus de integris.* It follows Sacrobosco's text with a few minor variations. Prosdocimo added this interesting line to the beginning, "All things that in the beginning are created are formed by the rationality of numbers..." [*...Omninia [sic] que a primera rerum origine creata sunt ratione numerorum formata sunt...*]
6ra-10ra	Transcription of Johannes de Lineriis' *Algorismus de minutiis.* This text was later printed along with Prosdocimo's *Algorismus de integris* in the Padua edition from 1483 and the Venice edition from 1540.
10rb-va	Treatise on fractions or geometry pertinent to the *Alfonsine tables.*
10va	Marginal note on astrology referring to Michael Scot.
10vb-11ra	Poem on astrology, with diagrams.
11ra-va	Treatise on astrology.
11va	Recipe for *aqua imperialis.* Later author has added, "for leprosy."
11vb-19rb	Johannes de Saxony: *Canones super tabulas Alphonsi.* There were the most famous canons of the day. At the end, Prosdocimo identifies himself as a student of the arts in Bologna.
19v-22v	Chronological tables for using the *Alfonsine Tables.*
22v-23r	Medical prescriptions. Inc. appears to be similar to a recipe for a "panacea" pill from Arnoldus de Villa Nova or Joannis Jacobi Manlii de Bosco. Ingredients include: saffron, aloe, cardamom, etc.

[885] Favaro, *Intorno ... Prosdocimo de' Beldomandi*; Favaro, *Appendice ... Prosdocimo de' Beldomandi*. See also Herlinger's introduction to Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 30-39. Incipits and explicits are available in these sources.

23ra-40vb	Medical treatise. Appears to be Avicenna's *Canon*, IV. The beginning describes fever as a heat from the heart, which spreads through the body via the spirit in the arteries and blood in the veins... similar to anger or exertion. [Inc: *Febris est calor extraneus [ac(c)e(ns)sus in corde et procedens ab eo mediantibus spiritus et sanguine per arterias et venas in totum corpus [:] et inflamat [inflammatur] in eo inflamatione quae nocet operationibus naturalibus, non sicut caliditas irae & laboris*.]
42r-44v	Part of a set of astronomical tables for the latitude of Bologna. Perhaps by John of Genoa?
45ra-b	Medical prescriptions. Refers to a "Magistri Johanis sanguinatij de padua." In margin, "Against stomach flow." [*contra fluxum Ventris*] Also a remedy for gall stones and other afflictions.
45va-49rb	Marsilius de Sancta Sophia, *Tractatus de medicinalibus.* In later hand, "Tractatus in medicinalibus editus a magistro marsilio de sancta sophia de padua."
49rb	Medical prescriptions. On expelling phlegm and curing fistulas.
50ra-b	Prosdocimo's *Tractatus practice cantus mensurabilis.* Incomplete and crossed out. See illustration below.
50rb-52vb	Medical prescriptions. Various topics include: on painful urination, on menstrual flow, on combustion, on rabid dogs and bites, on viper bites, on kidney pains, on preservation and pain of brain, stomach, & womb, on drying humors of the brain, on excessive phlegm, on making sound, on epilepsy, on itches, on restoring appetite, on bloating, on the formation of hair, on joint pain, on difficult urination, on pustules on the face, on the best and approved remedy for gout, etc.
53ra-54rb	Prosdocimo's *Brevis summula proportionum quantum ad musicam pertinet*. This text was discussed in Chapter 4.
54va-56r	Prosdocimo's *Canon in quo docetur modus componendi et operandi tabulam quandam*. Instructions for how to calculate roots and use a mathematical table.
56v-58v	Rules for computing chronological issues with tables.
59ra-61ra	Messahala's *De septem planetis*. Referring to the famous astrologer Masha'allah Ibn Athari (fl. ca. 800).
61rb-63ra	Hermes Trismegistus' *Liber aphorismorum*.
63rb-64ra	A poem on the aspects of the moon and planets.
64va-67va	Pseudo-Hippocrates, *De prognosticatione mortis et vite secundum motum lune*, translated [into Latin] by Guillaume de Moerbeke.
67va	Note on the liberation of the fourth age(?)

67vb-69ra	Al-Kindi's *On Rain* [*De pluviis*]. Astrological text.
69ra	Notes on Ptolemy's description of the lunar month.
69rb	Astronomical table associating cities to signs.
72va	Lunar tables with instructions. Dated 1409.
73ra	Statutes of the college of arts and medicine, Padua 1330 (?). Incomplete and crossed out. Nearly illegible. Dated 1409.
73rb-76rb	Johannes de Janua's *Canones eclipsium solis et lune, 1331.* Johannes de Janua was a master of arts and medicine in Paris in the 1330s.
76va-79rb	Treatise on the signs of the zodiac.
79va-84rb	Profatius Judaeus' *Tractatus quadrantis novi*, 1301 revision. Also known as Jacob ben Machir Ibn Tibbon.
84rb	Astronomical tables.
84v	Diagram of a quadrant. [astronomical instrument]
85r	Diagram of astrolabe and quadrant.
85v	Astronomical notes and computations.
86ra-87ra	Profatius Judaeus' *Canones in almanach perpetuum*.
87rb	Table of the motion of the 8th sphere.
87v-88va	Various short treatises on astronomy.
88vb-125va	John of Saxony's *Scriptum super Alkabicium, 1331.* A commentary on Alchabitius' *Introductorium ad judicia astronomiae* [*On Judicial Astrology*] John's text was printed numerous times between 1489 and 1520.

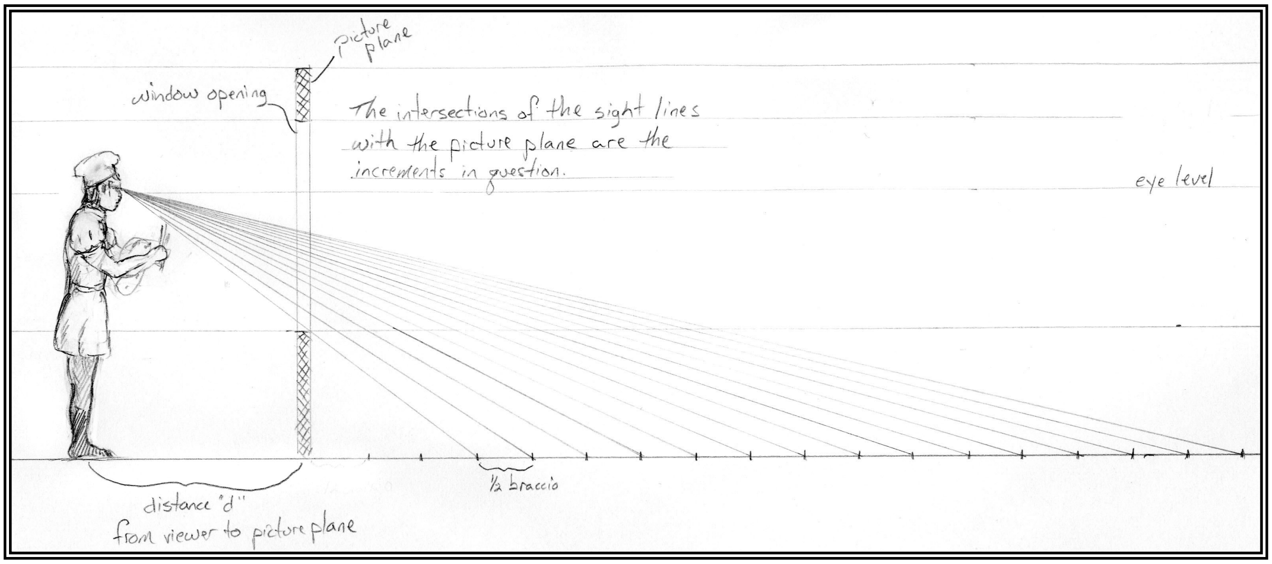


The crossed out *Tractatus practice cantus mensurabilis* by Prosdocimo.[886]

[886] This reproduction is from Gallo, 10b. Gallo's folio references differ from Herlinger's, which was used in the table. See Herlinger's introduction in Prosdocimo de' Beldomandi, *Brevis summula proportionum & De modo monacordum dividendi*, 30.

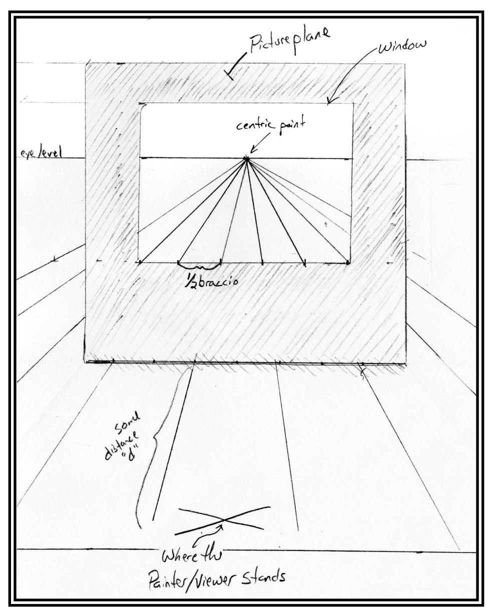
Appendices to Chapter 5

Appendix 5A:
Example of Alberti's Window Analyzed Algebraically



Let a painter standing at (0,0), [where the first number, "a," is the horizontal measure and the second number, "y," is vertical] have his centric point at his eye level, (0,4).
The picture/window plane is located horizontally at a=2, where a is the horizontal.
The measurements of height on this picture plane will be in the form (2, y).
The height of the intersections of the visual rays connecting the painter's eye to the ground he sees through the window on this picture plane, is determined using the general formula,

$$y = C\left(1 - \frac{P}{x_n}\right).$$



where

C=centric point (e.g., y=4)
P=Plane of window (e.g., a=2)
x_n = horizontal distances along the ground

With a centric point of C=4 and a plane for the window of P=2. The equation becomes

$$y = 4\left(1 - \frac{2}{x_n}\right) = 4 - \frac{8}{x_n}.$$

The first several results from this equation…

view of x_n	y coordinate at x=2 (some are approximations)
view of x_n =1	intersects window at y=-4 (probably disregarded)
view of x_n =2	intersects window at y=0
view of x_n =3	intersects window at y=4/3
view of x_n =4	intersects window at y=2
view of x_n =5	intersects window at y=2.4
view of x_n =6	intersects window at y=2.66
view of x_n =7	intersects window at y=2.86
view of x_n =8	intersects window at y=3
view of x_n =9	intersects window at y=3.11
view of x_n =10	intersects window at y=3.2
view of x_n =11	intersects window at y=3.27
view of x_n =12	intersects window at y=3.33
view of x_n =13	intersects window at y=3.38
view of x_n =14	intersects window at y=3.45
view of x_n =15	intersects window at y=3.47
view of x_n =16	intersects window at y=3.5
……………….	………………………..
view of $x_n \rightarrow \infty$	intersects window $y \rightarrow 4$

In the following figure I have plotted some of the these distances in the Albertian window. The numbers to the right are the actual distances from the painter. Orange (or gray) bars were placed at distances 3, 6, 12, 24, etc. These bars are all 4 units tall, the same height as the painter's eye and centric point. You will notice that a bar 3 units away is twice as tall and twice as wide as the bar that is 6 units away. This is why Alberti mentioned the similar triangles theorem.

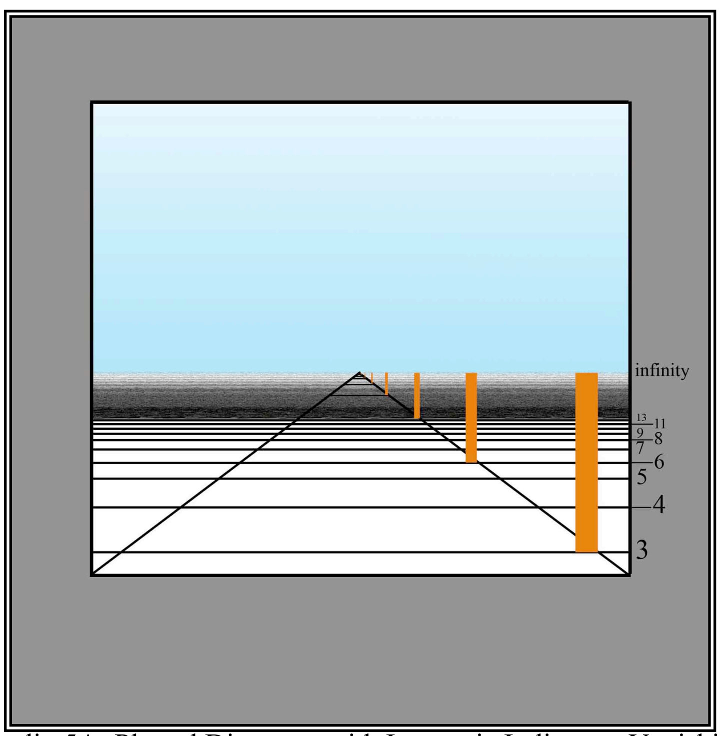


Figure for Appendix 5A: Plotted Distances with Isometric Indicators Vanishing into Infinity

Appendix 5B
Alberti's Arithmology: The Decad from IX.5

"Beauty is a form of sympathy and consonance of the parts within a body, according to definite **number**, outline, and position, as dictated by *concinnitas*, the absolute and fundamental rule in Nature." [*De re aedificatoria* – IX.5, p. 303]

Numbers	Alberti's Comments and Examples	Martianus Capella's Arithmology.*
1	Godhead (referred to in IX.6) Odd number of openings in buildings analogous to openings (doors) found in animal structures, namely the mouth, 1 in the center.	The seminal force of all numbers, the Divine Creator, one God, sun, moon, Jupiter the one primary god, (male)
2	Bones/columns always even in number, 2 legs as an example. (In IX.6 2 is not considered a number.)	Juno, wife, sister, female, source of 4 which is in itself significant, generating matter
3	Nature is composed of 3s, all philosophers agree.	First odd number, first perfect number (1+2), Ideal forms, male
5	Human hand; five fingers, associated with God and the gods of arts, Mercury in particular.	Fifth element (quintessence), marriage (female-2 + male-3), zones of earth, five senses, Pagan and Christian pandering.
7	God likes 7, 7 planets (sun, moon, M,V,M, J, S), 7 stages of man (conception, formation, adolescence, maturity, …etc. He only mentions the first four stages), 7 days until name given to a child according to Aristotle,	Largest prime in decad. Sum of male and female numbers (3 and 4), phases of moon, planets, days, transmutations of elements, stages in life of man, apertures in head,
9	Nature set 9 orbs in the sky, 1/9 of year = 40 days--time for fetal form (Hippocrates), also time to recover from illness, menstruation ceases 40 days after conception if a boy…	Second cube (3x3x3), Mars, 9 muses, zones in universe
4	"Consecrated to divinity"	Tetrahedron as most basic solid body. Decad as sum of 4 numbers (1+2+3+4), the *tetractys*, seasons, directions, elements, ages of man, vices, virtues
6	"Perfect"-- "sum of integral divisors" 1+2+3 = 6. It is also the product of them: 1 x 2 x 3, which is particular to 6 and not all "perfect" numbers. E.g. 28=1+2+4+7+14, these don't multiply to 28.	Perfect (1+2+3), natural properties, Venus (union of sexes (3+2+2?), motions [from *Timaeus*], cube's surfaces, 6 *toni* in octave (5 whole and 2 semi), Mother of Harmony
8	"Those born in 8th month will not survive"—(premature babies). Alberti mentions various humoral problems with 8. (8 has a long history of being bad.)	First cube (2x2x2), Vulcan
10	"Aristotle thought 10 the most perfect of numbers" $10^2=1^3+2^3+3^3+4^3$	Perfect (1+2+3+4), Janus

"Architects have used these numbers extensively: yet, especially in the temple, they have employed no even number greater than 10, nor odd number greater than 9, in the case of openings." (I switched around the clauses in this quotation for clarity.) [*De re aedificatoria* – IX.5, p. 304]

* This has been heavily edited. I extracted the Albertian similarities and a few other observations to give a sense of the text. This list was derived from Book VII, pp. 276-285, in *Martianus Capella and the Seven Liberal Arts: The Marriage of Philology and Mercury (De nuptiis Philologiae et Mercurii)*. Translated by William Harris Stahl and Richard Johnson with E. L. Burge. Vol. 2. 2 vols. New York: Columbia University Press, 1977.

Appendix 5C
The Harmonic Ratios of Alberti from *De re aedificatoria*, IX.6

code	Pytha-gorean Ratios	Harmonic term and synonyms	Decimal values	Inver-sions	Notes (modern logarithmic 'cent' values)ψ Equal tempered values are divisible by 100.
AVcs	3:2	*diapente*, *sesquialtera*, fifth, P5	1.5	$0.\overline{66}$	(and another unit onto that which is divided into halves) (702 cents)
AVcs	4:3	*diatesseron, sesquitertia,* fourth, P4	$1.\overline{33}$	0.75	(and another onto that which is divided into thirds) Large square side: smaller square (498 cents)
AVcs	2:1	*diapason*, octave, P8	2.0	0.5	Boethius describes this as virtually unison. (1200 cents)
AVc	3:1	*diapason-diapente* (or *disdiapente), triple,* 12^{th} (octave and a fifth), P12	3	$\mathbf{0.\overline{33}}$	2/1 x 3/2 = 6/2 = 3/1 (1902 cents)
AVc	4:1	*disdiapson, quadruple*, 15^{th} (two octaves), P15	4	0.25	3/2 x 4/3 x 3/2 x 4/3 = 4 (2400 cents)
As	9:8	*tonus, sesquioctavus*, tone, M2	1.125	$0.\overline{88}$	(and another onto that which is divided into eighths) Alberti isn't clear about its consonance, but he seems to separate it from the previous ones. (204 cents)
A(V) c	1:1	unison	1.0	1.0	Alberti mentions this in the first line of 9.6 in a discussion on areas, but does not include it in his harmonic discussion. Music theorists might consider this ratio to be the most basic consonance, hardly worth mentioning.
(A)V c	8:3	*disdiatesseron,* 11^{th} (octave and a fourth), P11	$2.\overline{66}$	0.375	Alberti mentions this interval as an area though not in his harmonics discussion. Vitruvius mentions this in his harmonic section. (1698 cents)
	256:243	*semitonus, diesis, limma*, m2	≈1.05	≈0.95	(90 cents)
	2187: 2048	*apotome*	≈1.07	≈0.94	This is the difference between a tone and a semitone, for two semitones do not make a tone. $(256/243)^2 \neq 9/8$ This actually comes up in Bb and other "ficta" issues. (114 cents)
	531441: 524288	comma	≈1.01	≈0.99	*apotome – diesis = comma* [hardly ever discussed] (23 cents)
	5:4	M3	1.25	0.8	Just third. Begins to be discussed in mid 15^{th} century. Zarlino promotes this interval in mid 16^{th} century. (386 cents, 408 in Pythagorean tuning)
	6:5	m3	1.2	$0.8\overline{33}$	Just minor third. Also promoted by Zarlino. (316 cents, 294 in Pythagorean tuning.)
	5:3	M6	$1.\overline{66}$	0.6	
	18:17	"chromatic" semi	≈1.06	$0.9\overline{44}$	Vincenzo Galilei's lutenist tuning (mid 16^{th} c.). Very close to modern equal temperament. Octave is 1.986 instead of 2.0.

ψ Modern equal-tempered intonation locates a half step every 100 cents.

Key:
A– mentioned by Alberti
V–mentioned by Vitruvius (but not in numerical terms)
c – consonant in traditional Pythagorean tuning
s – superparticular ratio [i.e. in the form (n+1)/n]

Appendix 5D: Outlines[ϕ]

"The very same numbers that cause sounds to have that *concinnitas*, pleasing to the ears, can also fill the eyes and mind with wondrous delight. From musicians therefore who have already examined such numbers thoroughly [harmonics], or from those objects in which Nature has displayed some evident and noble quality [perhaps he is referring to squares and cubes here], the whole method of outlining is derived." [*De re aedificatoria* – IX.5, p. 305]

Table D1: Alberti's Three-Dimensional Harmonics from *De re aedificatoria*, IX.6.

3-dimensional Harmonic Volumes width:heigth:length	Harmonic derivation given in IX.6	Implied area ratio	Comments
2:3:4	e.g., 2 x 3/2 = 3 3 x 4/3 = 4	2:1-intermediate	He later states that the ratios in 3-D should be width:heigth:length = short:medium:long.* This height is also derivable using the arithmetic mean.**
3:4:6	e.g., 3 x 4/3 = 4 4 x 3/2 = 6	2:1-intermediate	This is also a harmonic mean. In fact it is the same exact 3 x 6 area that he uses in his harmonic mean example.**
2:4:6 (1:2:3)	e.g., 2 x 2/1 = 4 4 x 3/2 = 6	3:1-long	This is also an arithmetic mean.
2:3:6	e.g., 2 x 3/2 = 3 3 x 2/1 = 6	3:1-long	This is also a harmonic mean.
2:4:8 (1:2:4)	e.g., 2 x 2/1 = 4 4 x 2/1 = 8	4:1-long	This is also a geometric mean.
2(:3):4:8	e.g., 2 x 3/2 = 3 3 x 4/3 = 4 4 x 2/1 = 8	4:1-long	Alberti's description of this is convoluted for he steps through the 3 to get to the 4. This is an arithmetic and a geometric mean.
3:6(:9):12 1:2:3:4	e.g., 3 x 2/1 = 6 6 x 3/2 = 9 9 x 4/3 = 12	4:1-long	Again, this is a rather convoluted description in *De re aedificatoria*. He goes to 9 only as a stepping stone for the 12. This exemplifies arithmetic and geometric means.

* He does not consistently use ascending or descending orders when discussing most of these proportions.
** Alberti does not point out the mean equivalencies in this section or in the means section.
[ϕ]Boethius discusses rectilinear shapes, but more generally. See I.31 in *De institutione arithmetica*.

Table D2: Intervals as Applied to Architectural Areas or Floor Plans from *De re aedificatoria*, IX.6

2-dimensional Harmonic Areas	Harmonic derivation given in IX.6	Comments*
Short- 1:1	no derivation given	This is a "simple," prime interval. Axiomatic Height determined *de quadratis.*
Short- 3:2	no derivation given	This is a "simple," prime interval. Axiomatic Height not determined.
Short- 4:3	no derivation given	This is a "simple," prime interval. Axiomatic Height not determined.
Intermediate- 2:1	"best" of the intermediates, no derivation given	Height determined using harmonic factorization method.
Intermediate- 9:4	e.g. 4 x 3/2 x 3/2 = 9	Height determined using geometric mean.
Intermediate- 16:9	e.g. 9 x 4/3 x 4/3 = 16	Alberti also gives this 16 = 2 x (9 x 8/9). For this one the Leoni translation is superior to the Rykwert. Height not determined. Could have been determined using a geometric mean: 16:12:9, but Alberti does not mention this.
Long- 3:1	1 x 2/1 x 3/2 = 3	Height determined using harmonic factorization method.
Long- 4:1	no derivation given	Height determined using harmonic factorization method.
Long- 8:3	3 x 2/1 x 4/3 = 8	Height not determined.

*Many of these harmonic floor plans are later given heights, others are not.

Table D3: Alberti's Proportions from the Cube from *De re aedificatoria*, IX.6

3-D Spaces (length:height :width)	Geometric source	Base 1 factorization	Left interval (high:low) (length:height)	Right interval (high:low) (height:width)	Outer interval; (high:low) (length:width)	Notes
8:4:2	2 x 2 x 2 cube volume:area:line	4:2:1	2:1 = 2.0	2:1 = 2.0	4:1 = 4.0	Alberti is not very specific in the square discussion about how these numbers are to be put to use.*
√8:2:2	right triangle dividing a 2 x 2 square	√2:1:1	√2:1 ≅ 1.41	1:1 = 1.0	√2:1 ≅ 1.41	This is just 2 x 2 square divided diagonally.
√12: √8:2	right triangle inside 2 x 2 cube	√3:√2:1	√3:√2 ≅ 1.22	√2:1 ≅ 1.41	√3:1 ≅ 1.73	This is the diagonally divided rectangle that lives catty-cornered in a 2 x 2 x 2 cube.
√16:√12:√4	right triangle formed from the preceding triangle	2:√3:1	2:√3 ≅ 1.15	√3:1 ≅ 1.73	2:1 = 2.0	This triangle is made from the hypotenuse and one of the legs of the preceding triangle, but unlike the others, it does not live in a 2 x 2 x 2 cube. Alberti baffles me with this one.
2√5 : 4 : 2	right triangle formed from the preceding triangle	√5:2:1	√5:2 ≅ 1.12	2:1 = 2.0	√5 ≅ 2.24	This triangle is **not** described by Alberti, but it follows immediately from his previous argument and yields the golden ratio, $\frac{2\sqrt{5}+2}{4} = 1.62...$

*Comparing volume to area to linear dimension (x^3 to x^2 to x) is an anomaly in Albertian mathematics so far as I can tell. This series, in particular, is dimensionally difficult to justify, though it seems unlikely that Alberti intended these numbers to be used as a set for proportions.

"Beauty is a form of sympathy and consonance of the parts within a body, according to definite number, outline, and position, as dictated by *concinnitas*, the absolute and fundamental rule in Nature."
[*De re aedificatoria* – IX.5, p. 303]

"[Position] … has much in common with rules for outline."
"We must therefore take great care to ensure that even the minutest elements are so arranged in their level, alignment, number, shape, and appearance, that right matches bottom, adjacent matches adjacent, and equal matches equal, and that they are an ornament to that body of which they are to be part."
[*De re aedificatoria* – IX.7, p. 310]

Appendix 5E: How to Find Alberti in Any Structure

Table E1: The Major Ratios from *De re aedificatoria,* Book IX

Class	Ratio	Abbreviation	Decimal♥	Notes on the Ratios♦
All	1	n.a.	**1.0**	Unison
DH	531441/ 524288	DT-ditonic comma	1.01	T⊕T⊕T⊕T⊕T⊕T= P8 ⊕ DT Implicit in Albertian system
DH	256/243	S-*semitonus*	1.05	P4 = T ⊕ T ⊕ S Implicit in Albertian system
DH	2187/2048	L-*lemma*♣	1.07	T = S ⊕ L Implicit in Albertian system
DH	9/8	T-*tonus*	**1.125**	P4 ⊕ T = P5 Explicit in Albertian system
Sq	2/√3	n.a.	1.15	Albertian square derivation
Sq	√3/√2	n.a.	1.22	Albertian square derivation
CH	4/3	P4-fourth	1.33	Albertian/Pythagorean prime consonant harmonic interval
Sq	√2	n.a.	1.41	Albertian Square derivation
CH	3/2	P5-fifth	**1.5**	Albertian/Pythagorean prime consonant harmonic interval
Sq	√3	n.a.	1.73	Albertian square derivation
CHd	16:9	n.a.	1.78	= P4 ⊕ P4
CH	2	P8-octave	**2.0**	P5 ⊕ P4 = P8 Albertian/Pythagorean consonant harmonic interval
CHd	9:4	n.a.	**2.25**	= P5 ⊕ P5
CHd	8:3	n.a.	2.67	= P8 ⊕ P4
CH	3	P12-octave and a fifth	**3**	P8 ⊕ P5 = P12 Albertian/Pythagorean consonant harmonic interval
CH	4	P15-double octave	**4**	P8 ⊕ P8 ⊕ P15 Albertian/Pythagorean consonant harmonic interval
The following are not mentioned by Alberti and are added for comparison.				
	18/17	n.a.	(1.06)*	18:17 Lutenist semitone
	6/5	m3- Just minor 3rd	**(1.2)**	
	5/4	M3- Just major 3rd	**(1.25)**	
	(√5+1)/2	φ	(1.62)	φ (divine/golden ratio)
	(√5-1)/2	φ-1	(0.62)	1/ φ
	5/3	M6- Just major 6th	(1.67)	M6
		e	(2.72)	e (Euler's number)
		π	(3.14)	π

DH-Dissonant Harmonic CH-Consonant Harmonic
CHd-Consonant Harmonic derivative Sq-Square derivative ("from the body")
♥ Numbers not in bold are rounded off *Decimal numbers in parentheses are not mentioned in Book IX.
♦All "⊕"signs should be treated as multiplication for ratio manipulations.
♣ "Lemma" is used here as a generic term and is not always specifically identified with this interval.

Commentary on Table E1

These ratios are in the form x:y and not the proportions that Alberti talks about in the form x:y:z. Modern scholars consistently simplify Alberti's work to conform with this two-number ratio concept rather than using the much more limiting three-number proportions (two intervals expressed using three numbers). In part I think this difference in approach stems from the modern preference on fixed numbers rather than the fluidly scaled proportional systems preferred by classical authors. Today, we tend to think in terms of number locations on a number line rather than the quantity or length in-between the numbers. For example, we tend to think of the number 4 as a location on the number line. The number 4 to Alberti was the length of the number line from 0 to 4. We think of a point and Alberti thinks of a quantity. By thinking in terms of quantity, scaling and comparisons are much more easily understood. The haphazard translation from Alberti's concept of quantity to our modern concept of fixed number is never discussed in the Albertian literature that I have read.

Table E1 is a selection of ratios (in the form x:y) found in Book IX in Alberti's *De re aedificatoria*. It includes both the geometric (derived from cubes in *De re aedificatoria*), and the harmonic (which includes the arithmetic) ratios. It should be kept in mind that any combination made using consonant harmonic intervals is also a consonant interval, so for example P8 ⊕ P5 ⊕ P5 (4.5) would be consonant but I have not listed it in this table. Similarly many ratios can be deconstructed into prime harmonic intervals, for example 8:1 can be factored into $(P4 \oplus P5)^3$. Musically speaking it should also be noted that dissonant intervals are not off limits, they are simply not intervals worthy of emphasis. They are inevitable consequences of the system.

Notice the variety of intervals between 1 and 2. If one wanted to find Albertian ratios in a building, use this table. Depending on one's choice of a margin for error when taking measurements from an actual building and any tolerances allowed for creative interpretations, one could easily fit most measurements into this Albertian system.

I would consider the valid Albertian intervals to be those indicated in the "Class" column in blue (or gray), although, as I stated above, the x:y comparison is ultimately objectionable. I would also include several harmonic consonant interval combinations not shown in the table, e.g., 8:1. As you can see, even with these imposed restrictions, the choices are very well distributed across the gamut. Most any building can be Albertian if you want it to be.

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Prosdocimo de' Beldomandi (?)- Alfonsine Tables: 28r-82r.

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[887] The scribe for this codex was Antonius de Obizis of Lucca and completed this manuscript at Sabbioncello (near Ferrara) in 1437

[888] Full title: Tabulae mediorum motuum, equationum, stationum et latitudinum planetarum, elevationis signorum, diversitatis aspectus lunae, mediarum coniunctionum et oppositionum lunarium, ferarium, latitudinarum, climatum, longitudinum et latitudinum civitatum.

[889] The scribe for all of the works by Prosdocimo in this codex identified himself as Thomas Caroso, October 1456.

[890] The Index to the Campori collection lists Prosdocimo under "Padova," making this codex particularly difficult to find.

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