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THE ANALYSIS OF DATA INVOLVING ORDINAL VARIABLES IN WHICH  
SOME OF THE DATA ARE MISSING

by

SHIELA MAIMAN STRAUSS

A dissertation submitted to the Graduate Faculty in Educational Psychology in partial  
fulfillment of the requirements for the degree of Doctor of Philosophy, The City  
University of New York

1996

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## Abstract

# THE ANALYSIS OF DATA INVOLVING ORDINAL VARIABLES IN WHICH SOME OF THE DATA ARE MISSING

by

Shiela Maiman Strauss

Adviser: Professor David Rindskopf

The current work explores the analysis of categorical data in which some of the values are missing. For a given data set, each missing data point is viewed as only partially categorized, having an unknown value on at least one, but not all of the variables. It is not necessarily assumed that the data are missing at random. Hypothesized associations among the variables in a data set and the hypothesized nature (random or nonrandom) of the missing data mechanism are represented using loglinear models. Expected values of cell frequencies and sums of expected frequencies (marginals) are expressed by using an extension of the generalized linear model with composite links.

For each of three kinds of data set structures, models which hypothesize nonignorable nonresponse are first explored for their identification status. Then, for each of these three kinds of data set structures, a specific example is chosen from the literature for a determination of the adequacy of fit of a variety of loglinear models. Models proposed to represent relationships among the data for each of these specific data sets include both those which postulate ignorable nonresponse and those for which the missingness is hypothesized to be nonrandom. Maximum likelihood estimates for

parameters and expected cell frequencies and marginals are obtained for each of the models using the EM algorithm. Chi square goodness of fit tests are performed, and for each of the three data sets, comparisons of predicted cell frequencies are made for those models which adequately fit the data.

All data sets analyzed in the current work contain at least one ordinal variable. Statistical modeling involving ordinal variables allows for the consideration of a greater number of patterns of association among the variables than if all the variables are nominal, and often requires fewer model parameters. More parameters can then be used to represent the missing data mechanism, and the possibility of nonignorable nonresponse can be investigated. The current work adds to the small body of literature in which a nonrandom missing data mechanism is considered in modeling data with missing values.

## Dedication

This work is dedicated to the memory of my mother, Rose Bleier Maiman. Mom died in December, 1990, just three months after I began the doctoral program.

Though formally educated only through sixth grade, mom was one of the most astute and intelligent people I've ever known. Life did not always treat her kindly; she lost a long awaited child to hyaline membrane disease, her husband to a massive coronary at an early age, and her vision and health to macular degeneration and breast and colon cancers. But I do not remember her indulgence in sorrow, only her optimism. I particularly recall her joy in her family, especially her grandchildren. In fact, not a day has gone by without my thinking of her and rejoicing in the gift of the years we shared.

Mom taught me so many life lessons: the precious bonds of a mother-daughter relationship, the importance of treating others with dignity and compassion, of enjoying and celebrating the moment, of striving for personal excellence. She was everyone's friend, bringing sunshine and cheer to so many lives. She taught me to value what is given from the heart, and to prize love above materialism. My pride in her was matched only by her pride in me; I believe that she would have been pleased that this child of an elementary school graduate has earned a doctorate degree. I am blessed to be the daughter of this extraordinary woman and to be able to always carry her memory with me. I know that I will continue to be uplifted by mom's faith and pride in me all the days of my life.

Let this work stand as my lasting tribute to her and to all that we were to each other. I dedicate to mom the care and excellence I brought to this dissertation, and all subsequent works and recognition that may come as a result of it.

## Acknowledgments

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Fellow students Helen Ishofsky and Maureen Kennedy provided support and encouragement and made the path easier and memorable. I am proud to call them 'friends'.

Dr. Barry Zimmerman helped me to see the relevance of my own personal and professional experiences to the study of many aspects of the learning process. His wisdom and excellence continue to inspire me. Dr. Carol Tittle encouraged me to begin the doctoral program and has been a supportive mentor and friend throughout the years. Her guidance continues to be a constant source of strength. Drs. Alan Gross and Roger Millsap provided a clear and solid structure for the statistics I studied. They each helped me to recognize the usefulness and uniqueness of my mathematics background, and how it could be used to advantage in applied statistics and psychometrics. I was consistently buoyed by their faith in my skills.

Special thanks go to my mentor, Dr. David Rindskopf. Over the years we have worked together, he has never failed to be encouraging and supportive. I always left my meetings with him feeling uplifted and empowered, ready to apply myself to the development of the statistical theory and its applications. He often articulated his pride in me and in my work, helping me to continue past the inevitable 'dead ends'. Rare among mentors, he was, and continues to be, a true friend, as concerned about my professional development as my dissertation progress. He has been an excellent teacher, and taught me to trust my instincts in statistics, even as I learned the techniques and applications he instructed with such clarity. This dissertation is a

reflection of the goals and high standards he helped me to set and to realize. I am proud and privileged to have been his student.

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## CHAPTER 1

### INTRODUCTION AND OVERVIEW: MISSING CATEGORICAL DATA

The problem of missing data is one which is ubiquitous in most data analyses. In the social sciences, for example, data may be missing due to incomplete questionnaire responses, or to the failure of some subjects to complete an experimental program. Data involving human subjects can also be considered to be missing if, for some or all of the subjects, data from several categories are collapsed into fewer groupings. In this case, the richness of the original data is lost, its detail missing for all future analyses.

Data can be missing for both continuous and categorical variables. The current work focuses on the analysis of data which is missing on categorical variables. A brief description of the traditional approach to the analysis of categorical variables is given. This is followed by a presentation of the classification conventions for missing data. In the second chapter, an overview of the literature on the methods, models, and assumptions which have guided the analysis of missing categorical data is presented. In particular, most researchers have made the assumption that missing data are missing in a random fashion. The present area of study involves the creation of models to represent data involving ordered categorical variables in which some of the data are missing. It does not necessarily assume that the data are missing in a random fashion.

### Analysis of Categorical Data - Loglinear Models

Categorical variables are those variables which can assume only a small number of discrete values. These variables may be nominal or ordinal. Nominal variables have no inherent order; they merely group the data into separate categories. Examples include political party affiliation (Democrat, Republican, Liberal or Conservative), sex (male or female), and marital status (married, widowed, divorced, never married). Ordinal variables, on the other hand, establish a sequencing of the data. Examples include time of year (summer, fall, winter, or spring) and socioeconomic class (upper, middle or lower). Since ordinal variables can be viewed as nominal variables that have an ordering of categories, analyses involving nominal variables are applicable to those involving ordinal variables, as well.

### Analyses Involving Nominal Variables

Tables which cross-tabulate frequency counts for categorical variables (contingency tables) are generally created as a first step in analyzing the categorical data. Loglinear models are then formed to describe possible association patterns among the variables represented in the table, with the table's cell counts modeled in terms of these associations. The goodness of fit of each of these models is often measured and tested using either the Pearson statistic or the likelihood ratio statistic. For each of the models, the Pearson statistic is given by

$$X^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$$

Here,  $O_i$  is the observed frequency in cell  $i$  of the contingency table, and  $E_i$  is the expected frequency in cell  $i$  as predicted by the model. The likelihood ratio statistic is given by

$$G^2 = 2 \sum_i O_i \ln \left( \frac{O_i}{E_i} \right)$$

Here,  $O_i$  and  $E_i$  are as defined above. Each of these two statistics has a chi square distribution if the model fits the data. Thus, chi square goodness of fit tests can be used to determine how adequately the model describes the hypothesized relationships among the variables.

In the case of two categorical variables, loglinear models can explore the hypotheses of equiprobability, exactly one main effect, and two main effects but no interaction. One typically seeks the most parsimonious model which describes the data. If none of the models is found to fit the data, one concludes that the two categorical variables are related and that the appropriate model is the saturated one. In the case of three categorical variables, models of equiprobability, one or more main effects, and one or more two way interactions can be tested using chi square goodness of fit tests. In practice, the models investigated are generally hierarchical; i. e., if there is a term in the model that represents an interaction between two of the variables, for example, terms must appear in the model to account for the main effects of each of these variables. Loglinear models can also be created to attempt to explain association patterns among  $k$  categorical variables,  $k > 3$ . In such models, main effects of each of the variables, as well as two- through  $k-1$  way interactions may be included. Again, hierarchical models are generally the ones investigated.

To understand why the models are termed *loglinear*, one may consider, for example, the independence model for two categorical variables (i.e., no interaction term).

Suppose  $N$  represents the sample size. Let  $p_i$  represent the proportion of  $N$  in row  $i$  of the contingency table, and let  $p_j$  represent the proportion of  $N$  in column  $j$ . Then the expected cell count in row  $i$  and column  $j$  is given by

$$E_{ij} = Np_i p_j.$$

Taking the natural logarithm of both sides of this equation gives,

$$\ln(E_{ij}) = \ln(N) + \ln(p_i) + \ln(p_j).$$

The right hand side of this latter equation resembles the model of a two way analysis of variance (ANOVA) without an interaction term. The ANOVA model is an example of a general *linear* model; the *log* in the term *loglinear* model reflects the fact that *logarithms* of expected cell counts, sample size and sample proportions appear in the model. If there are three categorical variables, there are many more possible loglinear models than in the two variable case. As indicated above, the models considered are generally hierarchical. For example, suppose the three categorical variables are  $A$ ,  $B$ , and  $C$ , and suppose that  $F_{ijk}$  is the observed frequency in the cell of the contingency table for which  $A=i$ ,  $B=j$ , and  $C=k$ . One possible loglinear model which could represent the data is given symbolically by

$$\ln(F_{ijk}) = \theta + \lambda_i^A + \lambda_j^B + \lambda_k^C + \lambda_{ij}^{AB} + \lambda_{ik}^{AC}$$

In this model, there are terms  $(\lambda_i^A, \lambda_j^B, \text{ and } \lambda_k^C)$  which account for the main effects of each of the categorical variables,  $A$ ,  $B$ , and  $C$ . There are also terms which account for the two way interactions between  $A$  and  $B$  and between  $A$  and  $C$   $(\lambda_{ij}^{AB} \text{ and } \lambda_{ik}^{AC})$ .

There are no terms in the given model which account for a two way interaction between  $B$  and  $C$ , nor a three way interaction between  $A$ ,  $B$ , and  $C$ . This model may or may not adequately fit the data; a chi square goodness of fit test would help to establish the adequacy of fit.

### Analyses Involving Ordinal Variables

In the case of ordinal variables, additional loglinear models may be considered. Some of these models are described below. For each model, we will assume that there are two ordinal variables and that  $F_{ij}$  represents the observed frequency count corresponding to row  $i$  and column  $j$ , for  $i = 1, 2, \dots, I, j = 1, 2, \dots, J$ . Further,  $\{\mu_i\}$  and  $\{v_j\}$  will be the scores assigned to the row and column variables in the contingency table, respectively, with  $\mu_1 < \mu_2 < \dots < \mu_I$  and  $v_1 < v_2 < \dots < v_J$ . Also,  $\bar{\mu}$  and  $\bar{v}$  will represent the means of the  $\{\mu_i\}$  and  $\{v_j\}$ , respectively.

#### *Association Models*

A first group of such loglinear models can be described as *association models*. These model relationships among the data concerning *local odds ratios* among 2x2 tables formed using cells in adjacent rows and adjacent columns in the contingency table. Here, the local odds ratio is given by

$$\frac{F_{i,j}F_{i+1,j+1}}{F_{i,j+1}F_{i+1,j}} = \theta_{ij}, \text{ for } i = 1, 2, \dots, I-1, j = 1, 2, \dots, J-1.$$

The *uniform association model* posits that

$$\theta_{ij} = \theta, \text{ for some constant } \theta, \text{ and for } i = 1, 2, \dots, I-1, j = 1, 2, \dots, J-1.$$

(Goodman, 1979)

As a loglinear model, this uniform association can be expressed by

$$\ln(F_{ij}) = \theta + \lambda_i^A + \lambda_j^B + \pi(\mu_i - \bar{\mu})(v_j - \bar{v}) \text{ for } i = 1, 2, \dots, I, j = 1, 2, \dots, J \text{ (Agresti, 1984).}$$

The *row effects model* is given by

$$\theta_{ij} = \theta_i, \text{ for } i = 1, 2, \dots, I-1, j = 1, 2, \dots, J-1.$$

(Goodman, 1979)

As a loglinear model, the row effects model is given by

$$\ln(F_{ij}) = \theta + \lambda_i^A + \lambda_j^B + \tau_i(v_j - \bar{v}) \text{ for } i = 1, 2, \dots, I, j = 1, 2, \dots, J \text{ (Agresti, 1984).}$$

The row effects model does not make use of the ordinal nature of the row variable. Thus, the model would have the same form if only the column variable was ordinal and the row variable was nominal.

Similarly, the *column effects model* is given by

$$\theta_{ij} = \theta_{.j}, \text{ for } i = 1, 2, \dots, I-1, j = 1, 2, \dots, J-1. \quad (\text{Goodman, 1979})$$

As a loglinear model, the column effects model is given by

$$\ln(F_{ij}) = \theta + \lambda_i^A + \lambda_j^B + \tau_j(\mu_i - \bar{\mu}) \text{ for } i = 1, 2, \dots, I, j = 1, 2, \dots, J \text{ (Agresti, 1984).}$$

The column effects model does not make use of the ordinal nature of the column variable. Thus, the model would have the same form if only the row variable was ordinal and the column variable was nominal.

Goodman (1979) discusses the *R+C model*, given by

$$\theta_{ij} = \theta_{i.}\theta_{.j}, \text{ for } i = 1, 2, \dots, I-1, j = 1, 2, \dots, J-1$$

In loglinear form, it is given by

$$\ln(F_{ij}) = \theta + \lambda_i^A + \lambda_j^B + \beta(\mu_i - \bar{\mu})(v_j - \bar{v}) + \tau_{1i}(v_j - \bar{v}) + \tau_{2j}(\mu_i - \bar{\mu})$$

for  $i = 1, 2, \dots, I, j = 1, 2, \dots, J$  (Agresti, 1984)

Unlike the row and column effects models, the *R + C* model makes use of the ordinal nature of both the row and column variables.

### *Models for Square Tables*

In addition to the association models, other loglinear models involving ordinal variables may be considered. These models are appropriate when the contingency table is a square table; i.e., there are the same number of rows and columns.

Consider, first, the *symmetry model*, given by

$$F_{ij} = F_{ji}, \text{ for all } i \text{ and } j.$$

As a loglinear model, the symmetry model is given by

$$\ln(F_{ij}) = \theta + \lambda_i + \lambda_j + \lambda_{ij}^{AB}, \quad \lambda_{ij}^{AB} = \lambda_{ji}^{AB} \text{ for all } i \text{ and } j. \quad (\text{Agresti, 1984})$$

We may also consider the *quasi-symmetry model*. This model assumes that symmetry would hold if the marginal distribution of the row and column variables were not unequal. As a loglinear model, the quasi-symmetry model is given by

$$\ln(F_{ij}) = \theta + \lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB}, \quad \lambda_{ij}^{AB} = \lambda_{ji}^{AB} \text{ for all } i \text{ and } j. \quad (\text{Agresti, 1984})$$

It has the property that  $\theta_{ij} = \theta_{ji}$  for  $i = 1, 2, \dots, I-1, j = 1, 2, \dots, J-1$  (Goodman, 1979)

The frequency counts in a square contingency table may also fit the *quasi-independence model*. This model postulates independence if some of the cells in the contingency table (usually those on the main diagonal) are "ignored". The loglinear model for the quasi-independence model, with main diagonal cells omitted, is given by

$$\ln(F_{ij}) = \theta + \lambda_i^A + \lambda_j^B \text{ if } i \neq j. \quad (\text{Agresti, 1984})$$

Rindskopf (1990) demonstrated that the quasi-symmetry and quasi-independence models are equivalent in a 3x3 table.

All of the models described above for ordinal data can be tested for goodness of fit using chi square tests. It is to be emphasized that all of these models are *non-hierarchical* and, thus, *nonstandard* loglinear models. Rindskopf (1990) showed that it is important to consider nonstandard loglinear models in addition to traditional hierarchical ones in an attempt to explore relationships involving data described by nominal and ordinal variables.

### Classification and Statistical Analyses with Missing Data

The seriousness of the problem created by missing values in a data analysis is influenced by such things as the proportion of data that is missing, whether values are missing on one or on more than one of the variables, and on the degree to which the data can be assumed to be missing in a random fashion. In the case of a data set with two variables,  $X$  and  $Y$ , in which data is complete in  $X$ , but not in  $Y$ , the missing data mechanism can be characterized in a number of ways (Little and Rubin, 1987). If the probability that a particular value of  $Y$  is missing is independent of both the  $X$  and  $Y$  values, the missing data are said to be missing completely at random (MCAR). If the probability that a particular value of  $Y$  is missing depends on  $X$ , but not on  $Y$ , the missing data are said to be missing at random (MAR). If this probability depends on  $Y$  and possibly on  $X$ , the missing data mechanism is nonignorable.

When data are missing on more than one variable, the pattern of missingness may be monotonic. For example, if the variables in the analysis are  $X$ ,  $Y$ , and  $Z$ , one monotonic pattern is exhibited if the data is complete in  $X$ , missing some values of  $Y$ , and missing some values of  $Z$  only if also missing corresponding values of  $Y$ . The validity of statistical analyses involving missing data, and the appropriateness of the



method chosen to deal with the issue of missingness, are dependent on the mechanism which causes this missingness and the pattern the missing data exhibits.

Missing data patterns can be explored through the creation and use of missing data indicator variables. For example, one can examine the frequency distribution of each indicator variable to determine the percentage of missing data. The extent of correlation among the indicator variables suggests the degree to which missingness on one variable is related to missingness on the others.

Various approaches have been used to deal with incomplete data involving categorical variables. The data analyst may simply choose to omit those cases in which there are missing values on one or more of the variables. Alternatively, methods such as maximum likelihood estimation may be employed. The next chapter reviews the approaches taken by statisticians to analyze categorical data which are missing in one or more of the variables.

## CHAPTER 2

### REVIEW OF THE METHODS, MODELS, AND ASSUMPTIONS WHICH HAVE BEEN MADE IN THE ANALYSIS OF MISSING CATEGORICAL DATA

#### Approaches to Estimation With Missing Data

Little and Rubin (1987) discuss how various imputation methods and weighting cell estimators have been used in the past to deal with the problem of missing data. The more typical approach, however, has been to employ maximum likelihood estimation of the parameters which model the data. Proposed methods to maximize this likelihood have been based on the structure of the data (including the pattern of missingness in the data), as well as on the computational tools available to the analyst.

If the pattern of missingness in the data is monotonic, maximum likelihood estimates for the parameters in the model can be readily obtained from the likelihood function. In this case, the logarithm of the likelihood function factors into two separate components: one for the complete data and one for the missing data. Maximum likelihood estimates for the parameters of the model can then be found in a straightforward manner by analyzing each of the two components separately.

Nonmonotonic patterns of missingness in the data are more typical, and cannot generally be handled in so direct a fashion. In such cases, iterative procedures have been used to find maximum likelihood estimates for model parameters. For example,

the Newton Raphson Method uses initial estimates of model parameters to generate subsequent estimates. In fact, this method generally results in convergence to the desired maximum likelihood estimates for the model parameters. However, it requires that the logarithm of the likelihood function be both concave down and unimodal, and involves complicated computations with the second derivative of this function. Little and Rubin discuss an alternate algorithm, the expectation - maximization algorithm, which is conceptually and computationally simpler than many other iterative procedures. This latter procedure is based on maximization using the complete data likelihood. It has been used extensively, as reflected in the literature on missing data.

#### Maximum Likelihood Estimation and the EM Algorithm for Data Missing at Random

Dempster, Laird and Rubin (1977) provided a synthesis and gave a structure to some of the research on missing data as presented in the literature to that point in time. Theirs was a general approach to the computation of maximum likelihood estimates of parameters of models when observations could be viewed as incomplete data. Each iteration of their algorithmic method contained an expectation step, followed by a maximization step. They thus termed their iterative approach the 'EM algorithm'. They stressed the intuitiveness and generality of this approach and demonstrated its broad applicability to many types of missing data situations. They showed how successive iterations always increased the likelihood, gave sufficient conditions for the convergence of the likelihood, and described the rate of this convergence.

Almost two decades before the Dempster, Laird and Rubin (1977) paper, Hartley (1958) presented an iterative expectation - maximization method for maximum likelihood estimation when data are incomplete. He discussed this method in the context of a Poisson distribution truncated at the lower tail end, and for grouped data

for a variable modeled by a binomial distribution. Hartley described the process of forming initial estimates for the frequency counts in the incomplete contingency table, and then of estimating the parameters for the entire sample by usual maximum likelihood estimation techniques. These new parameter estimates were then used to improve on the estimates for the frequency counts in the incomplete contingency table. These estimated frequencies, in turn, could be used to generate new approximations of the parameters for the entire sample. Hartley showed that this iterative method, repeated until convergence, yielded maximum likelihood estimates of the parameters in the case of censored or truncated data. He stressed the simplicity and generality of the method, though he emphasized that it might not be the most computationally efficient one in all cases.

Also before the Dempster, Laird and Rubin (1977) paper appeared, Chen and Fienberg (1974) used the EM algorithm. They obtained maximum likelihood estimates for expected cell values in contingency tables having partially cross classified data with both Poisson and multinomial sampling. Their data were presented in two separate contingency tables: one fully categorized and the other as a supplemental margin. Chen and Fienberg considered both the saturated and independence models for the fully categorized data, and explored how these models fit the data in a specific example.

Clearly, the work and synthesis of Dempster et al. (1977) sparked analyses that used the EM algorithm. Fuchs (1982) reanalyzed data from the Protective Services Project for Older Persons (Blenkner, Bloom and Nielsen 1971; Blenkner, Bloom and Weber 1974) using the algorithm. Fuchs showed how different conclusions concerning associations among the categorical variables would be reached depending on whether or not partially classified data were included in the analysis. In

summarizing his findings, Fuchs emphasized the increased reliability of results, the ease of computation and the intuitive appeal of the EM algorithm.

Espeland and Odoroff (1985) showed how models for doubly sampled categorical data could be incorporated within the structure of loglinear models and the EM algorithm. For example, in the case of a single variable, a sample of size  $N$  may be categorized with respect to  $A^*$ , giving an imprecise measurement of a variable  $A$ . In addition, a random subsample of size  $n$  from  $N$  may also be categorized with respect to  $A$ , using a precise measurement of variable  $A$ . The random subsample could then be used to form an  $A$  by  $A^*$  contingency table with total count,  $n$ , while a supplemental margin on  $A^*$  of size  $N - n$  would summarize the rest of the sample. Espeland and Odoroff used the EM algorithm to estimate model parameters for doubly sampled data originally analyzed by Hochberg (1977). They showed, as did Fuchs (1982), that the use of supplemental margins in addition to data from a completely categorized contingency table can provide a better understanding of the relationships of the variables in a study.

#### Data Subject to Nonignorable Nonresponse

Much of the literature on missing data has assumed that the data are missing in a random fashion. Little (1982), was among those who considered the situation where the data might not be missing at random. He emphasized that an analysis of data subject to nonignorable nonresponse should be performed based on a model that reflects the mechanism which caused the lack of response. For Little, this was accomplished by specifying prior odds, and thus effectively specifying the parameters of the model. For each missing variable, and for each pair of categories that the variable could assume, the odds of the response being in one category rather than in

the other were specified. The EM algorithm was then employed. As in the case of ignorable nonresponse, the 'E' step distributed the partially classified units among cells that matched on the observed variables. These were adjusted in Little's method by multiplying the conditional probabilities for the ignorable case by these prior odds. In this manner, an adjustment for nonresponse bias was accomplished.

Nordheim (1984) also explored data subject to nonignorable nonresponse. Like Little (1982), Nordheim assumed that some prior information was available to the analyst concerning nonresponse probabilities. Nordheim examined data that were classified according to two factors, one of which was dichotomous and subject to nonresponse. He assumed that the ratio of probabilities of uncertain classification to one or the other of the two dichotomous categories was known. He then showed that the parameters which defined the model could be determined through maximum likelihood estimation. Nordheim acknowledged that this ratio might not be known explicitly by the researcher beforehand. He suggested, however, that a range of possible values for this ratio might be available as a result of past experience or experimentation. This range of possible ratios, in turn, would give rise to a corresponding range of possible values of the parameters which define the model. Thus, Nordheim's approach was that of a sensitivity analysis. He extended his approach to data involving more than two categories and to the presence of possible misclassifications among the data.

Fay (1986) took a different perspective on the analysis of data with nonignorable nonresponse. Rather than approximate a probability of nonresponse, Fay presented a series of explicit alternative models for the causes of nonresponse. He acknowledged that the causes of such nonresponse may be many and complex, and noted that different causal models of nonresponse would yield different estimates for the

marginal distributions of the variables. In his approach, Fay allowed for the possibilities of response (or nonresponse) for a given variable as a consequence of the values of that variable as well as response (or nonresponse) for a different variable. Fay demonstrated how these causal models could be specified in terms of equations for odds, and how they could be expressed as loglinear models. Fay further showed how the EM algorithm could be used to find maximum likelihood estimates for the parameters in some of the causal models. When many such models are feasible in a data analysis, Fay suggests that the researcher carefully examine the different plausible models. This will help to determine a bound for the range of potential effects of nonresponse in the data analysis.

Baker and Laird (1988) took an approach similar to that of Fay to explore the modeling of incomplete data subject to nonignorable nonresponse. As an example, they considered the Roper polling data which were used to predict the proportion of voters who indicated a preference for Truman in the 1948 presidential election. Though Truman won with 52% of the vote, polling results predicted that Dewey would win. It is to be noted that the polls made no adjustment for nonresponse, though there were prior indications that nonrespondents were heavily pro-Truman. The assumption made by the pollsters was that the nonrespondents would not vote. Baker and Laird investigated how modeling for nonignorable nonresponse could have adjusted for nonresponse bias. They constructed models in which there were covariates,  $X$  (time of survey and economic class of voter in the polling data), a polychotomous outcome variable,  $Y$  (vote for Truman, Dewey, or other, in the polling data), and a response mechanism,  $R$  (expression of preference - yes or no - in the polling data). Baker and Laird then expressed the likelihood of these models in terms of expected cell counts in both the  $XY$  marginal table (for the complete data) and in the full table involving  $X$ ,  $Y$  and  $R$ . The ultimate aim of the analysis was to estimate

parameters in the marginal table so as to make inferences about the regression of Y on X. They demonstrated how the EM algorithm could be used to find estimates for the parameters in both the marginal and full tables. Baker and Laird used their method to explore a number of models for nonresponse, in each case allowing for nonresponse to depend on some combination of X and Y. (In their example, nonignorable nonresponse implies a YR interaction). Using the estimated parameters that result from the EM algorithm as applied to each of their models, Baker and Laird computed, by economic class, estimated percentages of those voting for Truman in the poll conducted closest to the 1948 presidential election. They demonstrated that all models which accounted for nonignorable nonresponse predicted percentages which were close to the actual election results. By contrast, those models which assumed ignorable nonresponse incorrectly underestimated these percentages. It is to be noted that, though both of the covariates, X, were ordinal variables in the polling data analyzed by Baker and Laird, no use was made of the ordering of these variables.

Conaway (1993), however, did make some use of the ordinal nature of the variables in the data set he explored. He reexamined data analyzed by Stasny (1990) from the National Crime Survey. For each of the years 1975, 1976, 1977 and 1978, sampled households were asked to report their crime victimization (crime free, single crime, or multiple crime). Interviews were held twice each year, but not all sampled households were interviewed each time. Thus, some of the data were missing. Conaway made use of the time ordering of the two interviews in his analysis of the data. In particular, he assumed that a household's victimization status at the second interview was related to the victimization status at the first interview, but not on whether the person responded at the first interview. Conaway focused on the 1977 data, and used only the symmetry model (one possible model for ordinal data) to represent relationships among the complete data for that year. He examined models for the entire data set which were



missing completely at random, missing at random, and three which had nonignorable nonresponse. The EM algorithm was used to estimate parameters for the five models he considered. Conaway showed that one of the nonignorable nonresponse models gave an especially good fit for the 1977 data. (This particular year's data was of interest because Stasny's (1990) approach failed to yield a model which fit the data.) As did Baker and Laird (1988), Conaway showed how modeling nonignorable nonresponse can result in models which better fit some data with missing values.

### Missing Data Analysis Through the Use of Generalized Linear Models With Composite Links

The classical linear model can be summarized by expressing  $E(Y) = \mu$ , with  $\mu = X\beta$ . Here,  $Y$  is assumed to be a vector of independent normal variables with means  $\mu$ ,  $X$  is the design (or model) matrix, and  $\beta$  is a vector of parameters whose values are unknown and must be estimated from the data. Nelder and Wedderburn (1972) presented an extension of this model, allowing for the specification of a *link* function. In particular, if we let  $\eta = X\beta$  and  $g(\mu) = \eta$ , where  $g$  is any monotonic differentiable function, then  $g$  is called a *link* function.

Thompson and Baker (1981) developed an extension of the generalized linear model which allowed for the modeling of expected values of composites of elements of  $\mu$ . A specific application of the theory they developed, written in matrix form, would allow the creation of  $\mu^* = C\mu$  where  $C$  is a matrix of 1's and 0's. Thus, the elements of  $\mu^*$  are sums of selected elements of  $\mu$ . If  $Y$  is now considered to be a vector of sums of selected elements of  $\mu$ ,  $E(Y) = \mu^* = C\mu$ . Further, if  $\eta = g(\mu) = X\beta$ , we can define  $\mu = h(\eta)$ , where  $h = g^{-1}$ . In sum,  $E(Y) = \mu^* = C\mu = Ch(\eta) = Ch(X\beta)$ . In the specific case of loglinear models for categorical data,  $g$  is chosen to be the natural

logarithm function, and  $h$ , its inverse, is the exponential function. Thus, for the loglinear model application,  $E(Y) = \mu^* = \text{Cexp}(X\beta)$ . Rindskopf (1992) applied Thompson and Baker's (1981) generalization of the linear model, in the context of models for categorical data described above, to analyses of data sets in which some of the data were missing. Rindskopf showed how this approach could be used to estimate parameters of contingency tables in which there were two supplemental margins. He further explored the fitting of both saturated and unsaturated models, as well as the estimation of parameters in which the data were more coarsely classified in some units than in others. Rindskopf demonstrated how this approach could also be used to find model parameters in which the contingency table had fused cells, and in latent class analysis. In his examples, Rindskopf considered only data that were missing at random.

Rindskopf (1992) suggested, however, that this method of analysis could also be used to explore data with nonignorable nonresponse. He cautioned that hierarchical models subject to nonignorable nonresponse often have inestimable parameters. For example, some parameters in the model may not be estimable since they do not appear in the likelihood function (Little and Rubin, 1987). This situation may occur even though there are fewer parameters than degrees of freedom in the model. When a model has inestimable parameters, it is said to be not identified. By contrast, nonstandard models (Rindskopf, 1990) for some data sets might have estimable parameters, and so be identified, in those cases where standard (hierarchical) models are not. Demonstrating that nonstandard models can be likened to contrast coding in ANOVA, Rindskopf (1990) describes these nonstandard models by defining model matrices to represent them. Viewed in this fashion, the columns of these matrices represent hypotheses about the structure of the data. As noted earlier, a number of such nonstandard loglinear models can be used to describe relationships in contingency tables involving

ordinal variables. The current work explores the analysis of data involving ordinal variables subject to nonignorable nonresponse through the use of such nonstandard loglinear models.

## CHAPTER 3

### METHODS

#### Overview

#### Determination of the Identification Status of Nonignorable Nonresponse Models for Three Kinds of Data Sets

Through the use of nonstandard loglinear models, the current work explores the analysis of data involving ordinal variables in which some of the data are missing. The possibility of nonignorable nonresponse is investigated and emphasized. In particular, for each of three kinds of data sets, the identification status of specific models postulating nonignorable nonresponse is first determined. The first kind involves a data set which contains two ordinal variables, each having three categories, and for which there is one supplemental margin of partially classified data. The second kind involves a data set which also contains two ordinal variables, each having three categories, and for which there are two supplemental margins of partially classified data. The third kind involves a data set which contains both an ordered and an unordered categorical variable, and for which there is one supplemental margin of partially classified data.

All models involving partially classified data require one or more missing data indicators, one for each of the supplemental margins. For each of the data sets considered in the current work, missing values occur for the column variable. For the second data set, there are also missing values for the row variable. In this work, for this second data set, the

column variable is viewed as the dependent variable. Thus, for each of the three kinds of data sets, in order to model the situation of nonignorable nonresponse it is necessary to include interaction terms for the missing data indicator(s) and the column variable. Nonignorable nonresponse, however, does not preclude the additional possible interactions of the missing data indicator(s) and the row variable, and the three way interactions of the missing data indicator(s), the column variable, and the row variable. Thus, nonignorable nonresponse is categorized as follows:

Type I: interactions of the missing data indicator(s) and the column variable, but no interactions of the missing data indicator(s) and the row variable, and no three way interactions of the missing data indicator(s), the row variable and the column variable.

Type II: interactions of the missing data indicator(s) and the column variable and interactions of the missing data indicator(s) and the row variable, but no three way interactions of the missing data indicator(s), the row variable and the column variable.

Type III: interactions of the missing data indicator(s) and the column variable, interactions of the missing data indicator(s) and the row variable, and three way interactions of the missing data indicator(s), the row variable and the column variable.

In addition to the models involving the three types of nonignorable nonresponse described above, when appropriate, modifications of some of these models are also considered for identification. These modified models do not include all of the interaction terms involving the missing data indicator(s) and the column variable.

### Empirical Analyses of Each of the Three Kinds of Data Sets

A specific example of each of the three kinds of data sets considered above is then chosen from the literature. Each of these data sets is examined empirically for the goodness of fit of the independence and nonstandard loglinear models. Parameters and contingency table frequencies are first estimated for models using the fully categorized data, only. These estimates are then also obtained for a variety of models which contain the incomplete data. This latter group of models includes those which hypothesize that the partially classified data are missing completely at random (MCAR), missing at random (MAR), or identified and subject to nonignorable nonresponse.

A computer program employing the EM algorithm was written in SAS to obtain estimates for parameters and expected frequencies for the models considered in this work. The program also computes likelihood ratio fit statistics for each of the models. The expectation step is adapted from the work of Espeland and Odoroff (1985), while the maximization step is adapted from that of Rindskopf (1992). For each model, one supplies the observed frequencies (including the supplemental margin(s)), and the  $C$  and  $X$  matrices (page 17). One also enters initial estimates for the model parameters ( $\beta$  vector). The computer program then iteratively fits the data to the model. The iterative process terminates when successive likelihood ratio fit statistics differ by less than .00001. The convergence criterion is very strict, and often many iterations are required. An example of the computer program is given in the appendix.

In the discussion section (chapter 4), a comparison and contrast of model estimates, both with and without the partially classified data, clarifies the extent to which the partially classified data are important in understanding relationships among the variables. For those models which include the partially classified data and which adequately fit the data,

estimates in models for which the missing data mechanism is ignorable are contrasted with those for which the mechanism is nonignorable.

## DATA SET 1

The first kind of data set explored in this work contains two ordinal variables, each having three categories. In addition to the fully classified data, there is one supplemental margin (for the column variable).

### Determination of the Identification Status of Nonignorable Nonresponse Models

As discussed earlier (page 17), we use the extension of the generalized linear model with composite links as conceptualized by Thompson and Baker (1981) and as applied by Rindskopf (1992) to the loglinear model. We have,  $E(Y) = C \exp(X\beta)$ .

Here,  $Y$  is a column vector of twelve elements. The first nine are the cell frequencies in the 3x3 completely classified contingency table. The last three are the supplemental row marginals for the partially classified data.

$\beta$  is a vector of parameters whose values are unknown and must be estimated from the data.

$C$  is a matrix of 1's and 0's. For all models for Data Set 1,  $C$  is the same 12 by 18 matrix, whose elements indicate how the observed and unobserved cell counts are to be added to make use of the observed information. More specifically, for these data, the matrix  $C$  has the following form:

$$\begin{pmatrix} I_{9 \times 9} & 0_{9 \times 9} \\ 0_{3 \times 9} & A \end{pmatrix}, \text{ where } I \text{ and } 0 \text{ are the identity and zero matrices, respectively, and}$$



$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The rows of the  $C$  matrix correspond to the pattern of the given cell counts and the supplemental margin provided. The first nine rows of  $C$  correspond to the completely classified data. The tenth row of  $C$  indicates that three specific cell counts need to be added in order to produce the first observed supplemental row margin. The 1's in the tenth row appear in columns 10, 11, and 12. These correspond to the (unobserved) frequency counts in the first row of the contingency table in which only the supplementary row margin is observed. Thus, the tenth row of  $C$  indicates that these three (unobserved) cell counts must be added to produce the first observed supplementary row margin. Similarly, the eleventh and twelfth rows of  $C$  indicate the need to add the three (unobserved) frequency counts in the second and third rows, respectively, of the contingency table in which only the supplemental row margins are observed.

Finally,  $X$  is the design (or model) matrix. The  $X$  matrix has 18 rows. The first nine of these rows correspond to each of the nine cells in the fully classified 3x3 contingency table, while the last nine rows correspond to each of the nine cells in the (hypothetical) 3x3 contingency table for the partially classified data. The columns of the  $X$  matrix reflect hypotheses concerning the relationships among the row and column variables and the missing data mechanism in the model.

As the data are categorized by ordinal variables, integer coding will be used as follows:

row variable:  $r = -1, 0, 1$

column variable:  $c = -1, 0, 1$

As is usual for ordinal variables which assume three values and which have integer coding, when the row (or column) variable appears in any of the loglinear models, there will be both a linear and a quadratic term.

One missing data indicator is required. It will assume the value '1' if the data are completely classified and '0' if partially classified. Data cells corresponding to the completely classified data will be designated by  $k = 1$ , and those for which the data is partially classified by  $k = 2$ .

The identification status of models for each of the three types of nonignorable nonresponse will be explored for this first data set. Initially, relationships among the data categorized by the two ordinal variables will be described by each of seven models : the independence, uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry models. To the terms of each of these initial models will be added terms which represent the missing data indicator and nonignorable nonresponse for the partially categorized data. The terms representing nonignorable nonresponse may be of three types (page 21). Thus, for each of the seven models describing relationships among the row and column variables, three nonignorable nonresponse models will be considered for identification. Further, if a model is determined to be not identified, a modified version of this model will be evaluated for identification. This modified model will include only the linear interaction terms involving the missing data indicator. Thus, for each of the seven models describing relationships among the row and column variables, up to six nonignorable nonresponse models may be considered for identification.

As noted above, the columns of the  $X$  matrix reflect hypotheses concerning the relationships among the row and column variables and the missing data mechanism in the model. For each nonignorable nonresponse model considered for identification, the model

matrix (page 17) will first be presented. Cell indices will be displayed to the right of this matrix, with  $i$  = row of cell,  $j$  = column of cell, and  $k$  = classification of contingency table ('1' if completely classified, '2' if partially classified). An analysis to determine the identification status of the model will then follow.

### Empirical Analyses for Data Set 1

Empirical model fitting of a specific data set containing two ordinal variables (each having three categories) and one supplemental margin (for the column variable) will then be performed. This data set is part of the one presented by Stasny (1990) from the National Crime Survey. In particular, reported victimization classifications for households sampled in 1977 are presented. These classifications include crime free, single crime (one victimization) and multiple crime (two or more victimizations). Most of the sampled households were interviewed twice during 1977, though some were only interviewed once. The data set includes frequency counts for those who were either interviewed twice in 1977, or only at the second interview. It is to be noted that Stasny's original presentation of the data included an additional supplemental margin of those who were interviewed only at the first interview. This second supplemental margin is omitted for Data Set 1. Given below is the contingency table for the distribution of responses at each of the interviews:

|                     |                 | First Interview |              |                 | missing | total |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|-------|
|                     |                 | crime free      | single crime | multiple crimes |         |       |
| Second<br>Interview | crime free      | 1742            | 260          | 66              | 994     | 3062  |
|                     | single crime    | 228             | 56           | 31              | 177     | 492   |
|                     | multiple crimes | 63              | 31           | 10              | 76      | 180   |
|                     | total           | 2033            | 347          | 107             | 1247    | 3734  |

Using the computer program written in SAS (page 22), likelihood ratio fit statistics and estimates for parameter and expected frequencies will be determined for models hypothesized to fit this data set. Estimates will be obtained for the independence, uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry models using the both the fully categorized data and the supplemental row margin, as well as only the completely classified data. This former group of models will include those which hypothesize that the partially classified data are missing completely at random (MCAR), missing at random (MAR), and identified and subject to nonignorable nonresponse.

#### Organization of Presentation of Analyses for Data Set 1

For each of the seven types of relationships among the variables considered for Data Set 1 (independence, uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry), the identification status of the nonignorable nonresponse models will first be established. For each type of relationship, this will be followed by the determination of likelihood ratio fit statistics and parameter and frequency estimates for those models involving the complete data only, as well as for those involving the partially classified data. In particular, this latter group will include those nonignorable nonresponse models determined to be identified.

### Independence Models - Data Set 1

Little and Rubin (1987) explored the identification status of hierarchical loglinear models in which nonignorable nonresponse was hypothesized. In particular, the independence model with nonignorable nonresponse was found to have inestimable parameters, and so was determined to be not identified. The approach to the determination of the identification status of loglinear models with nonignorable nonresponse taken in this work is different from that of Little and Rubin (1987). Further, the approach taken here allows for the evaluation of the identification status of many models which Little and Rubin (1987) did not consider. For the independence model with hypothesized nonignorable nonresponse, this work should be viewed as a corroboration of the results of Little and Rubin (1987).

Type I: independence; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | 1   | 1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | -1  | 1   | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 1   | 1   | 1   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the missing data indicator, and the seventh and eighth columns, the interaction of the missing data indicator and the column variable. The eight parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order. The row and column variables are each coded to reflect linear and quadratic terms in the model.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the

contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier (page 17), we use the extension of the generalized linear model with composite links as conceptualized by Thompson and Baker (1981) and as applied by Rindskopf (1992) to the loglinear model. We have,  $E(Y) = C \exp(X\beta)$ . Recall that  $Y$  is a vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a vector of parameters whose values are unknown and must be estimated from the data. The upper left hand  $9 \times 9$  section of matrix  $C$  is an identity matrix (page 24). It can therefore be seen that for the portion of vector  $Y$  corresponding to the completely classified data (and the first nine rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Let us assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 = \ln f1 \quad (1.a.1)$$

$$b1 - b2 + b3 + b6 = \ln f2 \quad (1.a.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b7 + b8 = \ln f3 \quad (1.a.3)$$

$$b1 - b4 + b5 + b6 - b7 + b8 = \ln f4 \quad (1.a.4)$$

$$b1 + b6 = \ln f5 \quad (1.a.5)$$

$$b1 + b4 + b5 + b6 + b7 + b8 = \ln f6 \quad (1.a.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b7 + b8 = \ln f7 \quad (1.a.7)$$

$$b1 + b2 + b3 + b6 = \ln f8 \quad (1.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 = \ln f9 \quad (1.a.9)$$

Here,  $b1, b2, \dots, b8$  are the eight parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's.

$$\text{From (1.a.5), } b1 + b6 = \ln f5 \quad (1.a.10)$$

Substituting (1.a.10) into (1.a.2) and into (1.a.8), we have

$$-b2 + b3 = \ln f2 - \ln f5$$

$$b2 + b3 = \ln f8 - \ln f5$$

$$\text{Thus, } b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (1.a.11)$$

$$\text{and } b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (1.a.12)$$

Subtracting (1.a.2) from (1.a.1), and subtracting (1.a.2) from (1.a.3), we have

$$-b4 + b5 - b7 + b8 = \ln f1 - \ln f2$$

$$b4 + b5 + b7 + b8 = \ln f3 - \ln f2$$

$$\text{Thus, } b5 + b8 = \frac{1}{2} \ln \frac{f1 f3}{(f2)^2} \quad (1.a.13)$$

$$\text{Also, } b4 + b7 = \frac{1}{2} \ln \frac{f3}{f1} \quad (1.a.14)$$



Note that  $b1$  and  $b6$  cannot be determined uniquely from equations (1.a.1) - (1.a.9), since whenever  $b1$  appears in an equation, so does  $b6$ ; that is  $b1 + b6$  appears as a unit in each of the equations. Similarly,  $b5$  and  $b8$  cannot be determined uniquely, nor can  $b4$  and  $b7$ . Only two of the eight model parameters ( $b2$  and  $b3$ ) have been uniquely determined. Equations (1.a.10) - (1.a.14) provide the most complete solution for the  $b$ 's which can be determined solely from equations (1.a.1) - (1.a.9).

The last three elements of vector  $Y$  correspond to the partially classified data (and the last nine rows of the  $X$  matrix). Each of these last three elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f10, \dots, f18$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5) = s1 \quad (1.a.15)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (1.a.16)$$

$$\exp(b1 + b2 + b3 - b4 + b5) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (1.a.17)$$

Here,  $s1 = f10 + f11 + f12$ , the first row marginal for the partially classified data,

$s2 = f13 + f14 + f15$ , the second row marginal for the partially classified data, and

$s3 = f16 + f17 + f18$ , the third row marginal for the partially classified data.

We assume that we can obtain estimates  $s1$ ,  $s2$ , and  $s3$ , for the last three elements of vector  $Y$ .

Equations (1.a.15) - (1.a.17) can be rewritten as

$$\begin{aligned} \exp(-b_2 + b_3)[\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5)] &= s_1 \\ \exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) &= s_2 \\ \exp(b_2 + b_3)[\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5)] &= s_3 \end{aligned}$$

From equations (1.a.11) and (1.a.12),  $\exp(-b_2 + b_3) = f_2 / f_5$ , and  $\exp(b_2 + b_3) = f_8 / f_5$ .

Thus, equations (1.a.15) - (1.a.17) can be further rewritten as

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_1 \frac{f_5}{f_2} \quad (1.a.18)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (1.a.19)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_3 \frac{f_5}{f_8} \quad (1.a.20)$$

The left hand sides of equations (1.a.18) - (1.a.20) are identical, and thus provide only one relationship between  $b_1$ ,  $b_4$  and  $b_5$ . This, together with equations (1.a.10), (1.a.13) and (1.a.14) thus give four relationships among the six remaining unknown parameters. These four relationships are insufficient to uniquely determine the six unknowns. Thus, the independence model for this data set, with nonignorable nonresponse of Type I is not identified.

While the independence model for this data set with nonignorable nonresponse of Type I is not identified, it may be that a modified Type I model will be identified. We form this modified model by eliminating the term involving the interaction of the missing data indicator and the quadratic term for the column variable. This modified nonignorable nonresponse model is considered next.

Type I: independence; 3x3 table with one supplemental margin

- modified (with only the linear interaction term of the missing data indicator and the column variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | 1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | -1  | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 1   | 1   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 0   | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the missing data indicator, and the seventh column, the interaction of the missing data indicator and the linear column variable term. The seven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

Using the same conceptual rationale and procedures as for Type I nonignorable nonresponse with both missing data indicator and column variable interaction terms, the identification status of this model is now explored. The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Let us assume that we can obtain estimates  $f1, f2, \dots, f9$  for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$  for the three row marginals. Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 = \ln f1 \quad (1.b.1)$$

$$b1 - b2 + b3 + b6 = \ln f2 \quad (1.b.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b7 = \ln f3 \quad (1.b.3)$$

$$b1 - b4 + b5 + b6 - b7 = \ln f4 \quad (1.b.4)$$

$$b1 + b6 = \ln f5 \quad (1.b.5)$$

$$b1 + b4 + b5 + b6 + b7 = \ln f6 \quad (1.b.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b7 = \ln f7 \quad (1.b.7)$$

$$b1 + b2 + b3 + b6 = \ln f8 \quad (1.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 = \ln f9 \quad (1.b.9)$$

Here,  $b1, b2, \dots, b7$  are the seven parameters of matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's. (It is to be noted that equations (1.b.1) - (1.b.9) are identical to equations (1.a.1) - (1.a.9) on page 32, but with  $b8 = 0$ .)

In a procedure identical to that used for the independence model, nonignorable nonresponse Type I (page 32), we have

$$b1 + b6 = \ln f5 \quad (1.b.10)$$

$$b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (1.b.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (1.b.12)$$

$$b5 = \frac{1}{2} \ln \frac{f1 f3}{(f2)^2} \quad (1.b.13)$$

$$b4 + b7 = \frac{1}{2} \ln \frac{f3}{f1} \quad (1.b.14)$$

Note that  $b1$  and  $b6$  cannot be determined uniquely from equations (1.b.1) - (1.b.9), nor can  $b4$  and  $b7$ . Only three of the seven model parameters ( $b2$ ,  $b3$  and  $b5$ ) have been uniquely determined. Equations (1.b.10) - (1.b.14) provide the most complete solution for the  $b$ 's which can be determined solely from equations (1.b.1) - (1.b.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Type I, with both interaction terms for the missing data indicator and column variable (page 33). Thus,

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5) = s1 \quad (1.b.15)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (1.b.16)$$

$$\exp(b1 + b2 + b3 - b4 + b5) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (1.b.17)$$

As was noted earlier (page 34), these equations produce only one relationship between  $b1$ ,  $b4$  and  $b5$ . As  $b5$  has been determined uniquely (equation 1.b.13), equations (1.b.15) - (1.b.17) provide only one relationship between  $b1$  and  $b4$ , the remaining unknowns. This, together with equations (1.b.10) and (1.b.14) thus give three relationships among the four remaining unknown parameters. These three relationships are insufficient to uniquely determine the four unknowns. Thus, the independence model for this data set, with

nonignorable nonresponse of Type I but with only a linear interaction term for the missing data indicator and the column variable, is therefore not identified.

We next consider the identification status of the independence model with nonignorable nonresponse Type II.

Type II: independence; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | -1  | 1    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | 1   | 1   | -1  | 1    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | -1  | 1   | 0   | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 1   | 1   | 1   | 0   | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 1   | 1   | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 1   | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the missing data indicator, the seventh and eighth columns, the interaction of the missing data indicator and the column variable, and the ninth and tenth columns, the interaction of the missing data indicator and the row variable. The ten parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 + b10 = \ln f1 \quad (1.c.1)$$

$$b1 - b2 + b3 + b6 - b9 + b10 = \ln f2 \quad (1.c.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b7 + b8 - b9 + b10 = \ln f3 \quad (1.c.3)$$

$$b1 - b4 + b5 + b6 - b7 + b8 = \ln f4 \quad (1.c.4)$$

$$b1 + b6 = \ln f5 \quad (1.c.5)$$

$$b1 + b4 + b5 + b6 + b7 + b8 = \ln f6 \quad (1.c.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b7 + b8 + b9 + b10 = \ln f7 \quad (1.c.7)$$

$$b1 + b2 + b3 + b6 + b9 + b10 = \ln f8 \quad (1.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 = \ln f9 \quad (1.c.9)$$

Here,  $b1, b2, \dots, b10$  are the ten parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as for the independence model with nonignorable nonresponse Type I (page 32), we obtain

$$b1 + b6 = \ln f5 \quad (1.c.10)$$



$$b_2 + b_9 = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (1.c.11)$$

$$b_3 + b_{10} = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (1.c.12)$$

$$b_5 + b_8 = \frac{1}{2} \ln \frac{f_1 f_3}{(f_2)^2} \quad (1.c.13)$$

$$b_4 + b_7 = \frac{1}{2} \ln \frac{f_3}{f_1} \quad (1.c.14)$$

Note that  $b_1$  and  $b_6$  cannot be determined uniquely from equations (1.c.1) - (1.c.9), nor can  $b_2$  and  $b_9$ ,  $b_3$  and  $b_{10}$ ,  $b_4$  and  $b_7$ , and  $b_5$  and  $b_8$ . None of the ten parameters have been uniquely determined. Equations (1.c.10) - (1.c.14) provide the most complete solution for the  $b$ 's which can be determined solely from equations (1.c.1) - (1.c.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Type I (page 33). They are given by

$$\exp(b_1 - b_2 + b_3 - b_4 + b_5) + \exp(b_1 - b_2 + b_3) + \exp(b_1 - b_2 + b_3 + b_4 + b_5) = s_1 \quad (1.c.15)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (1.c.16)$$

$$\exp(b_1 + b_2 + b_3 - b_4 + b_5) + \exp(b_1 + b_2 + b_3) + \exp(b_1 + b_2 + b_3 + b_4 + b_5) = s_3 \quad (1.c.17)$$

As was done on page 34, equations (1.c.15) - (1.c.17) can be rewritten as

$$\exp(-b_2 + b_3) [\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5)] = s_1 \quad (1.c.18)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (1.c.19)$$

$$\exp(b_2 + b_3) [\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5)] = s_3 \quad (1.c.20)$$

Dividing (1.c.18) by (1.c.19) and (1.c.20) by (1.c.19) gives

$$\exp(-b_2 + b_3) = \frac{s_1}{s_2} \quad \text{and} \quad \exp(b_2 + b_3) = \frac{s_3}{s_2}, \text{ or}$$

$$-b_2 + b_3 = \ln \frac{s_1}{s_2} \quad \text{and} \quad b_2 + b_3 = \ln \frac{s_3}{s_2}.$$

$$\text{Therefore, } b_2 = \frac{1}{2} \ln \frac{s_3}{s_1} \quad (1.c.21)$$

$$\text{and } b_3 = \frac{1}{2} \ln \frac{s_1 s_3}{(s_2)^2} \quad (1.c.22)$$

Substituting these values for  $b_2$  and  $b_3$  in equations (1.c.11) and (1.c.12) allows for the unique determination of  $b_9$  and  $b_{10}$ . Thus, four of the ten model parameters can be uniquely determined, while six still remain unknown.

When (1.c.21) and (1.c.22) are substituted into equations (1.c.18) - (1.c.20), only one relationship is provided between three of the remaining unknowns,  $b_1$ ,  $b_4$  and  $b_5$ . This relationship, together with equations (1.c.10), (1.c.13) and (1.c.14) thus give four equations to determine the six remaining unknown model parameters. Since this is insufficient to solve for the six unknowns, this model is not identified.

While the independence model for this data set with nonignorable nonresponse of Type II is not identified, it may be that the elimination of the interaction term of the missing data indicator and the quadratic term for the column variable, and the interaction term of the missing data indicator and the quadratic term for the row variable will result in an identified model. This modified nonignorable nonresponse model is considered next.

Type II: independence; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, and the missing data indicator and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | -1  | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | -1  | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | 1   | -1  | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | -1  | 0   | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 1   | 1   | 0   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 1   | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the missing data indicator, the seventh column, the interaction of the missing data indicator and the linear column variable term, and the eighth column, the interaction of the missing data indicator and the linear row variable term. The eight parameters of the model, comprising the  $\beta$  matrix, reflect the same model parameters and interactions as do the columns of the  $X$  matrix, in the same order.

Using the same conceptual rationale and procedures as for Type II nonignorable nonresponse with both missing data indicator and column variable interaction terms, and both missing data indicator and row variable interaction terms, the identification status of this model is now explored. The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Let us assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 - b8 = \ln f1 \quad (1.d.1)$$

$$b1 - b2 + b3 + b6 - b8 = \ln f2 \quad (1.d.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b7 - b8 = \ln f3 \quad (1.d.3)$$

$$b1 - b4 + b5 + b6 - b7 = \ln f4 \quad (1.d.4)$$

$$b1 + b6 = \ln f5 \quad (1.d.5)$$

$$b1 + b4 + b5 + b6 + b7 = \ln f6 \quad (1.d.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b7 + b8 = \ln f7 \quad (1.d.7)$$

$$b1 + b2 + b3 + b6 + b8 = \ln f8 \quad (1.d.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 = \ln f9 \quad (1.d.9)$$

Here,  $b1, b2, \dots, b8$  are the eight parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. In a procedure identical to that used for the independence model with nonignorable nonresponse Type I (page 32), we have

$$b1 + b6 = \ln f5 \quad (1.d.10)$$

$$b2 + b8 = \frac{1}{2} \ln \frac{f8}{f2} \quad (1.d.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (1.d.12)$$

$$b5 = \frac{1}{2} \ln \frac{f1 f3}{(f2)^2} \quad (1.d.13)$$

$$b4 + b7 = \frac{1}{2} \ln \frac{f3}{f1} \quad (1.d.14)$$

Note that  $b1$  and  $b6$  cannot be determined uniquely from equations (1.d.1) - (1.d.9), nor can  $b2$  and  $b8$  and  $b4$  and  $b7$ . Only two of the eight model parameters ( $b3$  and  $b5$ ) have been uniquely determined. Equations (1.d.10) - (1.d.14) provide the most complete solution for the  $b$ 's which can be determined solely from equations (1.d.1) - (1.d.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type II with both interaction terms for the missing data indicator and the column variable, and the missing data indicator and the row variable. They are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5) = s1 \quad (1.d.15)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (1.d.16)$$

$$\exp(b1 + b2 + b3 - b4 + b5) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (1.d.17)$$

As was done on page 34, equations (1.d.15) - (1.d.17) can be rewritten as

$$\exp(-b_2 + b_3)[\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5)] = s_1 \quad (1.d.18)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (1.d.19)$$

$$\exp(b_2 + b_3)[\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5)] = s_3 \quad (1.d.20)$$

Substituting (1.d.20) into (1.d.19), we have

$$s_2 = \frac{s_3}{\exp(b_2 + b_3)}, \text{ or, alternatively, } b_2 + b_3 = \ln \frac{s_3}{s_2}.$$

$$\text{But from equation (1.d.12), } b_3 = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2}.$$

$$\text{Thus, } b_2 = \frac{1}{2} \ln \frac{(s_3)^2 (f_5)^2}{(s_2)^2 f_2 f_8} \quad (1.d.21)$$

Substituting this value for  $b_2$  in equation (1.d.11) allows for the unique determination of  $b_8$ . Thus, four of the eight model parameters can be uniquely determined, while four still remain unknown.

When (1.d.12), (1.d.13) and (1.d.21) are substituted into equations (1.d.18) - (1.d.20), only one relationship is provided between two of the remaining unknowns,  $b_1$  and  $b_4$ . This relationship, together with equations (1.d.10) and (1.d.14) thus give three equations to determine the four remaining unknown model parameters. Since this is insufficient to solve for the four unknowns, this model is not identified.

**Type III:**

A total of twelve observed data frequencies (nine from the completely classified data and three from the row marginals) are assumed to be estimable. If all the interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable, and the row variable are included in the loglinear model, fourteen parameters are required. Thus, Type III nonignorable nonresponse with independence for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

Type III: independence; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interaction of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | -1  | -1  | 1   | 1   | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 1   | 0   | -1  | 0   | 1   | 2 | 1 | 1 |
| 1      | -1  | 1   | 1   | 1   | 1   | 1   | -1  | -1  | 1   | 3 | 1 | 1 |
| 1      | 0   | 0   | -1  | 1   | 1   | -1  | 0   | 0   | 2   | 1 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 2   | 2 | 1 | 1 |
| 1      | 0   | 0   | 1   | 1   | 1   | 1   | 0   | 0   | 2   | 3 | 1 | 1 |
| 1      | 1   | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 3   | 1 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 1   | 0   | 1   | 0   | 3   | 2 | 1 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 3   | 3 | 1 | 1 |
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 1 | 2 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 2 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 3 | 2 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 2   | 1 | 2 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2   | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 2   | 3 | 2 | 2 |
| 1      | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 3   | 1 | 2 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 3   | 2 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 3   | 3 | 2 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the missing data indicator, the seventh and eighth columns, the interactions of the missing data indicator and the linear column variable term, and the linear row variable term, respectively, and the ninth column, the three way interaction of the missing data indicator and the linear column variable term and the linear row variable term. The nine parameters



of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 - b8 + b9 = \ln f1 \quad (1.e.1)$$

$$b1 - b2 + b3 + b6 - b8 = \ln f2 \quad (1.e.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b7 - b8 - b9 = \ln f3 \quad (1.e.3)$$

$$b1 - b4 + b5 + b6 - b7 = \ln f4 \quad (1.e.4)$$

$$b1 + b6 = \ln f5 \quad (1.e.5)$$

$$b1 + b4 + b5 + b6 + b7 = \ln f6 \quad (1.e.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 = \ln f7 \quad (1.e.7)$$

$$b1 + b2 + b3 + b6 + b8 = \ln f8 \quad (1.e.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 = \ln f9 \quad (1.e.9)$$

Here,  $b1, b2, \dots, b9$  are the nine parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. In a procedure identical to that used for the independence model with nonignorable nonresponse Type I (page 32), we have

$$b1 + b6 = \ln f5 \quad (1.e.10)$$

$$b2 + b8 = \frac{1}{2} \ln \frac{f8}{f2} \quad (1.e.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (1.e.12)$$

$$b5 = \frac{1}{2} \ln \frac{f1 f3}{(f2)^2} \quad (1.e.13)$$

Subtracting (1.e.4) from (1.e.6),

$$b4 + b7 = \frac{1}{2} \ln \frac{f6}{f4} \quad (1.e.14)$$

Subtracting (1.e.3) from (1.e.1) and substituting (1.e.14) gives

$$b9 = \frac{1}{2} \ln \frac{f1 f6}{f3 f4} \quad (1.e.15)$$

Note that  $b1$  and  $b6$  cannot be determined uniquely from equations (1.e.1) - (1.e.9), nor can  $b2$  and  $b8$ , and  $b4$  and  $b7$ . Only three of the nine model parameters ( $b3$ ,  $b5$  and  $b9$ ) have been uniquely determined. Equations (1.e.10) - (1.e.15) provide the most complete solution for the  $b$ 's which can be determined solely from equations (1.e.1) - (1.e.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type II. They are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5) = s1 \quad (1.e.16)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (1.e.17)$$

$$\exp(b1 + b2 + b3 - b4 + b5) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (1.e.18)$$

As was done on page 34, equations (1.e.16) - (1.e.18) can be rewritten as

$$\exp(-b_2 + b_3)[\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5)] = s_1 \quad (1.e.19)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (1.e.20)$$

$$\exp(b_2 + b_3)[\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5)] = s_3 \quad (1.e.21)$$

Substituting (1.e.21) into (1.e.20), we have

$$s_2 = \frac{s_3}{\exp(b_2 + b_3)}, \text{ or, alternatively, } b_2 + b_3 = \ln \frac{s_3}{s_2}.$$

$$\text{Together with (1.e.12), } b_2 = \frac{1}{2} \ln \frac{(s_3)^2 (f_5)^2}{(s_2)^2 f_2 f_8} \quad (1.e.22)$$

Having determined  $b_2$ ,  $b_8$  can now be found from (1.e.11).

Five of the nine model parameters can thus be uniquely determined.

If (1.e.12), (1.e.13) and (1.e.22) are substituted into equations (1.e.19) - (1.e.21), only one relationship between  $b_1$  and  $b_4$  is provided. Together with equations (1.e.10) and (1.e.14), three equations are available to find the remaining four unknown model parameters. These equations are insufficient to uniquely determine the four unknowns. This model, therefore, is not identified.

In summary, none of the five models considered above which postulate independence among the row and column variables and which model nonignorable nonresponse is identified.

### Empirical Analysis for Data Set 1

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the independence models hypothesized to fit Data Set 1 (page 27). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). (Note that none of the nonignorable nonresponse models was found to be identified.)

#### Data Set 1

##### Independence Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1690.49         | 288.54       | 88.97           | 2068    |
|                     | single crime    | 257.50          | 43.95        | 13.55           | 315     |
|                     | multiple crimes | 85.01           | 14.51        | 4.47            | 103.99  |
|                     |                 | 2033            | 347          | 106.99          | 2486.99 |

$$\beta = (3.78, -1.49, .39, -1.47, .30)$$

Likelihood Ratio fit statistic = 59.3404

Degrees of freedom = 4

**Data Set 1**

Independence Model - One Supplemental Margin

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1667.13         | 284.55       | 87.74           | 2039.42 |
|                     | single crime    | 267.87          | 45.72        | 14.10           | 327.69  |
|                     | multiple crimes | 98.00           | 16.73        | 5.16            | 119.89  |
|                     |                 | 2033            | 347          | 107             | 2487    |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 835.91          | 142.68       | 44.00           | 1022.58 |
|                     | single crime    | 134.31          | 22.93        | 7.07            | 164.31  |
|                     | multiple crimes | 49.14           | 8.39         | 2.59            | 60.11   |
|                     |                 | 1019.36         | 174          | 53.66           | 1247.02 |

$$\beta = (3.13, -1.42, .41, -1.47, .30, .69)$$

Likelihood Ratio fit statistic = 68.0761

Degrees of freedom = 6

**Data Set 1**

Independence Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1690.49    | 288.54       | 88.97           | 2068    |
|                     | single crime    | 257.50     | 43.95        | 13.55           | 315     |
|                     | multiple crimes | 85.01      | 14.51        | 4.47            | 103.99  |
|                     |                 | 2033       | 347          | 106.99          | 2486.99 |

**Partially Classified Data (Hypothetical Contingency Table)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 812.55     | 138.69       | 42.77           | 994  |
|                     | single crime    | 144.69     | 24.70        | 7.62            | 177  |
|                     | multiple crimes | 62.13      | 10.60        | 3.27            | 76   |
|                     |                 | 1019.37    | 173.99       | 53.66           | 1247 |

$$\beta = (3.21, -1.29, .44, -1.47, .30, .58, -.21, -.05)$$

Likelihood Ratio fit statistic = 59.3404

Degrees of freedom = 4

Uniform Association Model - Data Set 1

Type I: uniform association; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 1   | -1  | 1   | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 1   | 1   | 1   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | -1  | 1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | 0   | 0   | 0   | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 0   | 0   | 0   | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the linear row variable term and the linear column variable term, the seventh column, the missing data indicator, and the eighth and ninth columns, the missing data indicator by column variable interactions. The nine parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the independence models considered above, we have  $E(Y) = C \exp(X\beta)$ .

Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model)

matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 24) is the same as for the independence models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $\beta$ 's) of the model.

As discussed earlier for the independence models (page 31), for the portion of the vector  $Y$  corresponding to the completely classified data (and the first nine rows of  $X$ , the model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or  $\ln(E(Y)) = X\beta$ .

Let us assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

)



$$b_1 - b_2 + b_3 - b_4 + b_5 + b_6 + b_7 - b_8 + b_9 = \ln f_1 \quad (2.a.1)$$

$$b_1 - b_2 + b_3 + b_7 = \ln f_2 \quad (2.a.2)$$

$$b_1 - b_2 + b_3 + b_4 + b_5 - b_6 + b_7 + b_8 + b_9 = \ln f_3 \quad (2.a.3)$$

$$b_1 - b_4 + b_5 + b_7 - b_8 + b_9 = \ln f_4 \quad (2.a.4)$$

$$b_1 + b_7 = \ln f_5 \quad (2.a.5)$$

$$b_1 + b_4 + b_5 + b_7 + b_8 + b_9 = \ln f_6 \quad (2.a.6)$$

$$b_1 + b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - b_8 + b_9 = \ln f_7 \quad (2.a.7)$$

$$b_1 + b_2 + b_3 + b_7 = \ln f_8 \quad (2.a.8)$$

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 = \ln f_9 \quad (2.a.9)$$

Here,  $b_1, b_2, \dots, b_9$  are the nine parameters of the matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's.

$$\text{From (2.a.5), } b_1 + b_7 = \ln f_5 \quad (2.a.10)$$

Substituting (2.a.10) into (2.a.2) and into (2.a.8), we have, respectively,

$$-b_2 + b_3 = \ln f_2 - \ln f_5$$

$$b_2 + b_3 = \ln f_8 - \ln f_5$$

$$\text{Thus, } b_2 = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (2.a.11)$$

$$\text{and } b_3 = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (2.a.12)$$

Subtracting (2.a.5) from (2.a.4), and subtracting (2.a.5) from (2.a.6), we have, respectively,

$$-b_4 + b_5 - b_8 + b_9 = \ln f_4 - \ln f_5$$

$$b_4 + b_5 + b_8 + b_9 = \ln f_6 - \ln f_5$$

$$\text{Thus, } b_5 + b_9 = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (2.a.13)$$

$$\text{Also, } b_4 + b_8 = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (2.a.14)$$

Finally, subtracting the sum of (2.a.2) and (2.a.4) from the sum of (2.a.1) and (2.a.5), we have

$$b_6 = \ln \frac{f_1 f_5}{f_2 f_4} \quad (2.a.15)$$

Note that  $b_1$  and  $b_7$  cannot be determined uniquely from equations (2.a.1) - (2.a.9), since whenever  $b_1$  appears in one of the equations, so does  $b_7$ ; that is  $b_1 + b_7$  appears as a unit in each of the equations. Similarly,  $b_5$  and  $b_9$  cannot be determined uniquely, nor can  $b_4$  and  $b_8$ . Equations (2.a.10) - (2.a.15) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (2.a.1) - (2.a.9).

The last three elements of vector  $Y$  correspond to the partially classified data (and the last nine rows of the  $X$  matrix). Each of these last three elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f_{10}, \dots, f_{18}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b_1 - b_2 + b_3 - b_4 + b_5 + b_6) + \exp(b_1 - b_2 + b_3) + \exp(b_1 - b_2 + b_3 + b_4 + b_5 - b_6) = s_1 \quad (2.a.16)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (2.a.17)$$

$$\exp(b_1 + b_2 + b_3 - b_4 + b_5 - b_6) + \exp(b_1 + b_2 + b_3) + \exp(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) = s_3 \quad (2.a.18)$$

Here,  $s_1 = f_{10} + f_{11} + f_{12}$ , the first row marginal for the partially classified data,  
 $s_2 = f_{13} + f_{14} + f_{15}$ , the second row marginal for the partially classified data, and  
 $s_3 = f_{16} + f_{17} + f_{18}$ , the third row marginal for the partially classified data.

We assume that we can obtain estimates  $s1$ ,  $s2$ , and  $s3$ , for the last three elements of vector  $Y$ .

Let  $k = \exp(b6)$ ,  $k1 = \exp(-b2+b3)$  and  $k2 = \exp(b2+b3)$ .

Then equations (2.a.16) - (2.a.18) become

$$kk1\exp(b1-b4+b5) + k1\exp(b1) + \frac{k1}{k}\exp(b1+b4+b5) = s1 \quad (2.a.19)$$

$$\exp(b1-b4+b5) + \exp(b1) + \exp(b1+b4+b5) = s2 \quad (2.a.20)$$

$$\frac{k2}{k}\exp(b1-b4+b5) + k2\exp(b1) + kk2\exp(b1+b4+b5) = s3 \quad (2.a.21)$$

Equations (2.a.19) - (2.a.21) are three linear equations in  $\exp(b1-b4+b5)$ ,  $\exp(b1)$ , and  $\exp(b1+b4+b5)$ . We solve these equations for  $\exp(b1-b4+b5)$ ,  $\exp(b1)$ , and  $\exp(b1+b4+b5)$  using Cramer's Rule (Brown and Robbins, 1989).

$$\exp(b1-b4+b5) = \frac{\begin{vmatrix} s1 & k1 & \frac{k1}{k} \\ s2 & 1 & 1 \\ s3 & k2 & kk2 \end{vmatrix}}{\begin{vmatrix} kk1 & k1 & \frac{k1}{k} \\ 1 & 1 & 1 \\ \frac{k2}{k} & k2 & kk2 \end{vmatrix}} = \frac{s1(k2(k-1)) + s2\left(\frac{1}{k} - k\right)(k1k2) + s3\left(1 - \frac{1}{k}\right)k1}{k1k2\left(k - \frac{1}{k}\right)\left(k + \frac{1}{k} - 2\right)} \quad (2.a.22)$$

$$\exp(b1) = \frac{\begin{vmatrix} kk1 & s1 & \frac{k1}{k} \\ 1 & s2 & 1 \\ \frac{k2}{k} & s3 & kk2 \end{vmatrix}}{\begin{vmatrix} kk1 & k1 & \frac{k1}{k} \\ 1 & 1 & 1 \\ \frac{k2}{k} & k2 & kk2 \end{vmatrix}} = \frac{s1\left(\frac{1}{k} - k\right)k2 + s2\left(k^2 - \frac{1}{k^2}\right)k1k2 + s3\left(\frac{1}{k} - k\right)k1}{k1k2\left(k - \frac{1}{k}\right)\left(k + \frac{1}{k} - 2\right)} \quad (2.a.23)$$

$$\exp(b1 + b4 + b5) = \frac{\begin{vmatrix} kk1 & k1 & s1 \\ 1 & 1 & s2 \\ \frac{k2}{k} & k2 & s3 \end{vmatrix}}{\begin{vmatrix} kk1 & k1 & \frac{k1}{k} \\ 1 & 1 & 1 \\ \frac{k2}{k} & k2 & kk2 \end{vmatrix}} = \frac{s1\left(1 - \frac{1}{k}\right)k2 + s2\left(\frac{1}{k} - k\right)k1k2 + s3(k1(k-1))}{k1k2\left(k - \frac{1}{k}\right)\left(k + \frac{1}{k} - 2\right)} \quad (2.a.24)$$

From (2.a.23),

$$b1 = \ln \frac{k2\left(\frac{1}{k} - k\right)s1 + k1k2\left(k^2 - \frac{1}{k^2}\right)s2 + k1\left(\frac{1}{k} - k\right)s3}{k1k2\left(k - \frac{1}{k}\right)\left(k + \frac{1}{k} - 2\right)} \quad (2.a.25)$$

Further, dividing (2.a.22) by (2.a.23), and dividing (2.a.24) by (2.a.23), we have, respectively,

$$\exp(-b4 + b5) = \frac{s1(k2(k-1)) + s2\left(\frac{1}{k} - k\right)k1k2 + s3\left(1 - \frac{1}{k}\right)k1}{s1\left(\frac{1}{k} - k\right)k2 + s2\left(k^2 - \frac{1}{k^2}\right)k1k2 + s3\left(\frac{1}{k} - k\right)k1} \quad (2.a.26)$$

and

$$\exp(b4 + b5) = \frac{s1\left(1 - \frac{1}{k}\right)k2 + s2\left(\frac{1}{k} - k\right)k1k2 + s3(k1(k-1))}{s1\left(\frac{1}{k} - 1\right)k2 + s2\left(k^2 - \frac{1}{k^2}\right)k1k2 + s3\left(\frac{1}{k} - k\right)k1} \quad (2.a.27)$$

Equivalently,

$$-b4 + b5 = \ln \frac{s1(k2(k-1)) + s2\left(\frac{1}{k} - k\right)k1k2 + s3\left(1 - \frac{1}{k}\right)k1}{s1\left(\frac{1}{k} - k\right)k2 + s2\left(k^2 - \frac{1}{k^2}\right)k1k2 + s3\left(\frac{1}{k} - k\right)k1} \quad (2.a.28)$$

and

$$b4 + b5 = \ln \frac{s1\left(1 - \frac{1}{k}\right)k2 + s2\left(\frac{1}{k} - k\right)k1k2 + s3(k1(k-1))}{s1\left(\frac{1}{k} - 1\right)k2 + s2\left(k^2 - \frac{1}{k^2}\right)k1k2 + s3\left(\frac{1}{k} - k\right)k1} \quad (2.a.29)$$

Thus,

$$b4 = \frac{1}{2} \ln \frac{s1\left(1 - \frac{1}{k}\right)k2 + s2\left(\frac{1}{k} - k\right)k1k2 + s3(k-1)k1}{s1(k-1)k2 + s2\left(\frac{1}{k} - k\right)k1k2 + s3\left(1 - \frac{1}{k}\right)k1} \quad (2.a.30)$$

and

$$b5 = \frac{1}{2} \ln \frac{\left[ s1(k2(k-1)) + s2\left(\frac{1}{k} - k\right)k1k2 + s3\left(1 - \frac{1}{k}\right)k1 \right] \left[ s1\left(1 - \frac{1}{k}\right)k2 + s2\left(\frac{1}{k} - k\right)k1k2 + s3(k-1)k1 \right]}{\left[ s1\left(\frac{1}{k} - 1\right)k2 + s2\left(k^2 - \frac{1}{k^2}\right)k1k2 + s3\left(1 - \frac{1}{k}\right)k1 \right]^2} \quad (2.a.31)$$

We have determined  $b1$ ,  $b4$  and  $b5$  in equations (2.a.25), (2.a.30) and (2.a.31), respectively. We can now substitute these values into equations (2.a.10), (2.a.14) and (2.a.13) respectively, to find  $b7$ ,  $b8$  and  $b9$ . Thus, all of the  $b$ 's can be uniquely determined. This model, therefore, is identified.

Type II: uniform association; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | 1   | -1  | 1   | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | -1   | 1    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 1   | -1   | 1    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 1   | -1  | 1   | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 1   | 1   | 1   | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | -1  | 1   | 1    | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 1    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 0   | 0   | 0   | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the linear row variable term and the linear column variable term, the seventh column, the missing data indicator, the eighth and ninth columns, the interactions of the missing data indicator and the column variable, and the tenth and eleventh columns, the interactions of the missing data indicator and the row variable. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I case, that estimates are available for the frequency counts for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1, s2$ , and  $s3$ , for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 + b7 - b8 + b9 - b10 + b11 = \ln f1 \quad (2.b.1)$$

$$b1 - b2 + b3 + b7 - b10 + b11 = \ln f2 \quad (2.b.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 + b9 - b10 + b11 = \ln f3 \quad (2.b.3)$$

$$b1 - b4 + b5 + b7 - b8 + b9 = \ln f4 \quad (2.b.4)$$

$$b1 + b7 = \ln f5 \quad (2.b.5)$$

$$b1 + b4 + b5 + b7 + b8 + b9 = \ln f6 \quad (2.b.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 - b8 + b9 + b10 + b11 = \ln f7 \quad (2.b.7)$$

$$b1 + b2 + b3 + b7 + b10 + b11 = \ln f8 \quad (2.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 + b11 = \ln f9 \quad (2.b.9)$$

Here,  $b1, b2, \dots, b11$  are the eleven parameters of matrix  $\beta$ , which are to be determined.

We solve the equations for the  $b$ 's. Using the same procedure as that for the uniform association model with nonignorable nonresponse Type I (pages 57 and 58), we have

$$b1 + b7 = \ln f5 \quad (2.b.10)$$

$$b_2 + b_{10} = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (2.b.11)$$

$$b_3 + b_{11} = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (2.b.12)$$

$$b_5 + b_9 = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (2.b.13)$$

$$b_4 + b_8 = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (2.b.14)$$

$$b_6 = \ln \frac{f_1 f_5}{f_2 f_4} \quad (2.b.15)$$

Note that  $b_1$  and  $b_7$  cannot be determined uniquely from equations (2.b.1) - (2.b.9), nor can  $b_5$  and  $b_9$ ,  $b_4$  and  $b_8$ ,  $b_2$  and  $b_{10}$ , and  $b_3$  and  $b_{11}$ . Only one of the eleven model parameters ( $b_6$ ) has been uniquely determined. Equations (2.b.10) - (2.b.15) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (2.b.1) - (2.b.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type I (page 58).

They are given by

$$\exp(b_1 - b_2 + b_3 - b_4 + b_5 + b_6) + \exp(b_1 - b_2 + b_3) + \exp(b_1 - b_2 + b_3 + b_4 + b_5 - b_6) = s_1 \quad (2.b.16)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (2.b.17)$$

$$\exp(b_1 + b_2 + b_3 - b_4 + b_5 - b_6) + \exp(b_1 + b_2 + b_3) + \exp(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) = s_3 \quad (2.b.18)$$

Equations (2.b.10) - (2.b.14) and (2.b.16) - (2.b.18) provide eight equations to determine the remaining ten parameters ( $b$ 's) of the model. These eight equations are insufficient to uniquely determine the ten unknowns. This model, therefore, is not identified.

While the uniform association model for this data set with nonignorable nonresponse of Type II is not identified, it may be that the elimination of the interaction term of the



missing data indicator and the quadratic term for the column variable, and the elimination of the interaction term of the missing data indicator and the quadratic term for the row variable will result in an identified model. This modified nonignorable nonresponse Type II model is considered next.

Type II: uniform association; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, and the missing data indicator and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | -1  | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 1   | -1  | 0   | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 1   | 1   | 0   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | -1  | 1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 1   | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | 0   | 0   | 0   | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 0   | 0   | 0   | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the linear row variable term and the linear column variable term, the seventh column, the missing data indicator, the eighth column, the interaction of the missing data indicator and the linear column variable term, and the ninth column, the interaction of the missing data indicator and the linear row variable term. The nine parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Let us assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 + b7 - b8 - b9 = \ln f1 \quad (2.c.1)$$

$$b1 - b2 + b3 + b7 - b9 = \ln f2 \quad (2.c.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 - b9 = \ln f3 \quad (2.c.3)$$

$$b1 - b4 + b5 + b7 - b8 = \ln f4 \quad (2.c.4)$$

$$b1 + b7 = \ln f5 \quad (2.c.5)$$

$$b1 + b4 + b5 + b7 + b8 = \ln f6 \quad (2.c.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 - b8 + b9 = \ln f7 \quad (2.c.7)$$

$$b1 + b2 + b3 + b7 + b9 = \ln f8 \quad (2.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 = \ln f9 \quad (2.c.9)$$

Here,  $b1, b2, \dots, b9$  are the nine parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the uniform association model with nonignorable nonresponse Type I (page 57), we have

$$b1 + b7 = \ln f5 \quad (2.c.10)$$

$$b_2 + b_9 = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (2.c.11)$$

$$b_3 = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (2.c.12)$$

$$b_5 = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (2.c.13)$$

$$b_4 + b_8 = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (2.c.14)$$

$$b_6 = \ln \frac{f_1 f_5}{f_2 f_4} \quad (2.c.15)$$

Note that  $b_1$  and  $b_7$  cannot be determined uniquely from equations (2.c.1) - (2.c.9), nor can  $b_2$  and  $b_9$ , and  $b_4$  and  $b_8$ . Only three of the nine model parameters ( $b_3$ ,  $b_5$  and  $b_6$ ) have been uniquely determined. Equations (2.c.10) - (2.c.15) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (2.c.1) - (2.c.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Type I and Type II. They are given by

$$\exp(b_1 - b_2 + b_3 - b_4 + b_5 + b_6) + \exp(b_1 - b_2 + b_3) + \exp(b_1 - b_2 + b_3 + b_4 + b_5 - b_6) = s_1 \quad (2.c.16)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (2.c.17)$$

$$\exp(b_1 + b_2 + b_3 - b_4 + b_5 - b_6) + \exp(b_1 + b_2 + b_3) + \exp(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) = s_3 \quad (2.c.18)$$

Let  $k_1 = \exp(b_6)$ ,  $k_2 = \exp(b_3)$ , and  $k_3 = \exp(b_5)$ .

Then equations (2.c.16) - (2.c.18) can be rewritten as

$$k_1 k_3 \exp(b_1 - b_2 - b_4) + \exp(b_1 - b_2) + \frac{k_3}{k_1} \exp(b_1 - b_2 + b_4) = \frac{s_1}{k_2} \quad (2.c.19)$$

$$k_3 \exp(b_1 - b_4) + \exp(b_1) + k_3 \exp(b_1 + b_4) = s_2 \quad (2.c.20)$$

$$\frac{k_3}{k_1} \exp(b_1 + b_2 - b_4) + \exp(b_1 + b_2) + k_1 k_3 \exp(b_1 + b_2 + b_4) = \frac{s_3}{k_2} \quad (2.c.21)$$

$$\text{From (2.c.20), } \exp(b1) = \frac{s2}{k3 \exp(-b4) + 1 + k3 \exp(b4)} \quad (2.c.22)$$

Multiplying equations (2.c.19) and (2.c.21), we have

$$\exp(2b1) \left( k1 k3 \exp(-b4) + 1 + \frac{k3}{k1} \exp(b4) \right) \left( \frac{k3}{k1} \exp(-b4) + 1 + k1 k3 \exp(b4) \right) = \frac{s1 s3}{(k2)^2} \quad (2.c.23)$$

Substituting (2.c.22) into (2.c.23),

$$\begin{aligned} & \frac{(s2)^2}{(k3 \exp(-b4) + 1 + k3 \exp(b4))^2} \left( k1 k3 \exp(-b4) + 1 + \frac{k3}{k1} \exp(b4) \right) \left( \frac{k3}{k1} \exp(-b4) + 1 + k1 k3 \exp(b4) \right) \\ & = \frac{s1 s3}{(k2)^2} \end{aligned} \quad (2.c.24)$$

Equation (2.c.24) contains only one unknown,  $b4$ . If there is no solution for  $b4$ , the model is not identified. Furthermore, equation (2.c.24) is symmetric in  $b4$ : if  $b4 = m$  is a solution, so is  $b4 = -m$ . Since the solution for  $b4$  would then be nonunique, the model also would not be identified.

**Type III:**

As was discussed in the case of the independence model (page 47), if all of the interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable, and the row variable are included in the model, there would be too many parameters for the model, given the data. Thus, Type III nonignorable nonresponse with uniform association for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

**Type III: uniform association; 3x3 table with one supplemental margin**

- modified (with only the linear interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interaction of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | -1  | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 1   | -1  | 0   | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 1   | 1   | 0   | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 1   | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | 0   | 0   | 0   | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 0   | 0   | 0   | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the linear row variable term and the linear column variable term, the seventh column, the missing data indicator, the eighth column, the interaction of the missing data indicator and the linear column variable term, the ninth column, the interaction of the missing data indicator and the linear row variable term, and the tenth

column, the interaction of the missing data indicator, the linear column variable term and the linear row variable term. The ten parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 + b7 - b8 - b9 + b10 = \ln f1 \quad (2.d.1)$$

$$b1 - b2 + b3 + b7 - b9 = \ln f2 \quad (2.d.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 - b9 - b10 = \ln f3 \quad (2.d.3)$$

$$b1 - b4 + b5 + b7 - b8 = \ln f4 \quad (2.d.4)$$

$$b1 + b7 = \ln f5 \quad (2.d.5)$$

$$b1 + b4 + b5 + b7 + b8 = \ln f6 \quad (2.d.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 - b8 + b9 - b10 = \ln f7 \quad (2.d.7)$$

$$b1 + b2 + b3 + b7 + b9 = \ln f8 \quad (2.d.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 = \ln f9 \quad (2.d.9)$$



Here,  $b1, b2, \dots, b10$  are the ten parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the uniform association model with nonignorable nonresponse Type I (pages 57 and 58), we have

$$b1 + b7 = \ln f5 \quad (2.d.10)$$

$$b2 + b9 = \frac{1}{2} \ln \frac{f8}{f2} \quad (2.d.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2f8}{(f5)^2} \quad (2.d.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4f6}{(f5)^2} \quad (2.d.13)$$

$$b4 + b8 = \frac{1}{2} \ln \frac{f6}{f4} \quad (2.d.14)$$

$$b6 + b10 = \ln \frac{f1f5}{f2f4} \quad (2.d.15)$$

Note that  $b1$  and  $b7$  cannot be determined uniquely from equations (2.d.1) - (2.d.9), nor can  $b2$  and  $b9$ ,  $b4$  and  $b8$ , and  $b6$  and  $b10$ . Only two of the ten model parameters ( $b3$  and  $b5$ ) have been uniquely determined. Equations (2.d.10) - (2.d.15) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (2.d.1) - (2.d.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type II. They are given by

$$\exp(b1 - b2 + b3 - b4 + b5 + b6) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5 - b6) = s1 \quad (2.d.16)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (2.d.17)$$

$$\exp(b1 + b2 + b3 - b4 + b5 - b6) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5 + b6) = s3 \quad (2.d.18)$$

Equations (2.d.10), (2.d.11), (2.d.14), (2.d.15) and (2.d.16) - (2.d.18) provide seven equations to determine the remaining eight parameters ( $b$ 's) of the model. These seven equations are insufficient to uniquely determine the eight unknowns. This model, therefore, is not identified.

In summary, if the row and column variables are related by uniform association, and nonignorable nonresponse is to be modeled, only the Type I model is identified.

### Empirical Analysis for Data Set 1

Parameter and contingency table estimates and likelihood ratio chi-squares are now presented for the uniform association models hypothesized to fit Data Set 1 (page 27). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I.

#### Data Set 1

##### Uniform Association Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1736.35         | 264.60       | 67.06           | 2068.01 |
|                     | single crime    | 233.95          | 57.53        | 23.52           | 315     |
|                     | multiple crimes | 62.70           | 24.88        | 16.42           | 104     |
|                     |                 | 2033            | 347.01       | 107             | 2487.01 |

$$\beta = (4.05, -1.18, .34, -1.15, .25, .48)$$

Likelihood Ratio fit statistic = 6.78264

Degrees of freedom = 3

**Data Set 1**

Uniform Association Model - One Supplemental Margin

Missing Completely at Random (MCAR)

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1712.35    | 260.94       | 66.13           | 2039.42 |
|                     | single crime    | 243.38     | 59.84        | 24.47           | 327.69  |
|                     | multiple crimes | 72.28      | 28.68        | 18.92           | 119.88  |
|                     |                 | 2028.01    | 349.46       | 109.52          | 2486.99 |

**Partially Classified Data (Hypothetical Contingency Table)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 858.59     | 130.84       | 33.16           | 1022.58 |
|                     | single crime    | 122.03     | 30.01        | 12.27           | 164.31  |
|                     | multiple crimes | 36.24      | 14.38        | 9.49            | 60.11   |
|                     |                 | 1016.86    | 175.23       | 54.92           | 1247    |

$$\beta = (3.40, -1.10, .37, -1.15, .25, .48, .69)$$

Likelihood Ratio fit statistic = 15.5184

Degrees of freedom = 5

**Data Set 1**

Uniform Association Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1736.35         | 264.60       | 67.06           | 2068.01 |
|                     | single crime    | 233.95          | 57.53        | 23.52           | 315     |
|                     | multiple crimes | 62.70           | 24.88        | 16.42           | 104     |
|                     |                 | 2033            | 347.01       | 107             | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 834.59          | 127.18       | 32.23           | 994  |
|                     | single crime    | 131.46          | 32.32        | 13.22           | 177  |
|                     | multiple crimes | 45.82           | 18.18        | 12.00           | 76   |
|                     |                 | 1011.87         | 177.68       | 57.45           | 1247 |

$$\beta = (3.48, -.97, .40, -1.15, .25, .48, .58, -.21, -.05)$$

Likelihood Ratio fit statistic = 6.78264

Degrees of freedom = 3

**Data Set 1**

Uniform Association Model - One Supplemental Margin

Nonignorable Nonresponse - Type I

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1736.35    | 264.60       | 67.06           | 2068.01 |
|                     | single crime    | 233.95     | 57.53        | 23.52           | 315     |
|                     | multiple crimes | 62.70      | 24.88        | 16.42           | 104     |
|                     |                 | 2033       | 347.01       | 107             | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 752.51     | 68.28        | 173.21          | 994  |
|                     | single crime    | 101.39     | 14.85        | 60.76           | 177  |
|                     | multiple crimes | 27.18      | 6.42         | 42.40           | 76   |
|                     |                 | 881.08     | 89.55        | 276.37          | 1247 |

$$\beta = (2.70, -1.18, .34, -.26, 1.67, .48, 1.35, -.89, -1.41)$$

Likelihood Ratio fit statistic = 6.78264

Degrees of freedom = 3

Row Effects Model - Data Set 1

Type I: row effects; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 1   | 1    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | -1  | 1    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 1    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | -1  | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 0   | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the row variable with the linear column variable term, the eighth column, the missing data indicator, and the ninth and tenth columns, the missing data indicator by column variable interaction. The ten parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the independence and uniform association models considered above, we have  $E(Y) = C \exp(X\beta)$ .  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the

design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 24) is the same as for the independence and uniform association models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for the independence and uniform association models, for the portion of vector  $Y$  corresponding to the completely classified data (and the first nine rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Let us assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f1, f2, \dots, f9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 + b10 = \ln f1 \quad (3.a.1)$$

$$b1 - b2 + b3 + b8 = \ln f2 \quad (3.a.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 + b9 + b10 = \ln f3 \quad (3.a.3)$$

$$b1 - b4 + b5 + b8 - b9 + b10 = \ln f4 \quad (3.a.4)$$

$$b1 + b8 = \ln f5 \quad (3.a.5)$$

$$b1 + b4 + b5 + b8 + b9 + b10 = \ln f6 \quad (3.a.6)$$



$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 - b9 + b10 = \ln f7 \quad (3.a.7)$$

$$b1 + b2 + b3 + b8 = \ln f8 \quad (3.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 = \ln f9 \quad (3.a.9)$$

Here,  $b1, b2, \dots, b10$  are the ten parameters of the matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's.

$$\text{From (3.a.5), } b1 + b8 = \ln f5 \quad (3.a.10)$$

Substituting (3.a.10) into (3.a.2) and into (3.a.8), we have, respectively,

$$-b2 + b3 = \ln f2 - \ln f5$$

$$b2 + b3 = \ln f8 - \ln f5$$

$$\text{Thus, } b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (3.a.11)$$

$$\text{and } b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (3.a.12)$$

Subtracting (3.a.5) from (3.a.4), and subtracting (3.a.5) from (3.a.6), we have, respectively,

$$-b4 + b5 - b9 + b10 = \ln f4 - \ln f5$$

$$b4 + b5 + b9 + b10 = \ln f6 - \ln f5$$

$$\text{Thus, } b5 + b10 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (3.a.13)$$

$$\text{Also, } b4 + b9 = \frac{1}{2} \ln \frac{f6}{f4} \quad (3.a.14)$$

Subtracting (3.a.3) from (3.a.1), and substituting (3.a.14), we obtain

$$2b6 - 2b7 = \ln f1 - \ln f3 + \ln f6 - \ln f4$$

Subtracting (3.a.9) from (3.a.7), and substituting (3.a.14), we have

$$-2b6 - 2b7 = \ln f7 - \ln f9 + \ln f6 - \ln f4$$

$$\text{Thus, } b7 = \frac{1}{4} \ln \frac{f3f9(f4)^2}{f1f7(f6)^2} \quad (3.a.15)$$

$$\text{and } b6 = \frac{1}{4} \ln \frac{f1f9}{f3f7} \quad (3.a.16)$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (3.a.1) - (3.a.9), since whenever  $b1$  appears in one of the equations, so does  $b8$ ; that is  $b1 + b8$  appears as a unit in each of the equations. Similarly,  $b5$  and  $b10$  cannot be determined uniquely, nor can  $b4$  and  $b9$ . Only four of the model parameters ( $b2$ ,  $b3$ ,  $b6$  and  $b7$ ) have been uniquely determined. Equations (3.a.10) - (3.a.16) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (3.a.1) - (3.a.9).

The last three elements of vector  $Y$  correspond to the partially classified data (and the last nine rows of the  $X$  matrix). Each of these last three elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f10, \dots, f18$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b1 - b2 + b3) [\exp(-b4 + b5 + b6 - b7) + 1 + \exp(b4 + b5 - b6 + b7)] = s1 \quad (3.a.17)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (3.a.18)$$

$$\exp(b1 + b2 + b3) [\exp(-b4 + b5 - b6 - b7) + 1 + \exp(b4 + b5 + b6 + b7)] = s3 \quad (3.a.19)$$

Here,  $s1 = f10 + f11 + f12$ , the first row marginal for the partially classified data,  $s2 = f13 + f14 + f15$ , the second row marginal for the partially classified data, and

$s3 = f16 + f17 + f18$ , the third row marginal for the partially classified data.

We assume that we can obtain estimates  $s1$ ,  $s2$ , and  $s3$ , for the last three elements of vector  $Y$ .

Let  $k1 = \exp(b6 - b7)$ ,  $k2 = \exp(b6 + b7)$ ,  $k3 = \exp(b2 - b3)$ , and  $k4 = \exp(-b2 - b3)$ .

Then equations (3.a.17) - (3.a.19) can be rewritten as

$$k1 \exp(b1 - b4 + b5) + \exp(b1) + \frac{1}{k1} \exp(b1 + b4 + b5) = k3s1 \quad (3.a.20)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (3.a.21)$$

$$\frac{1}{k2} \exp(b1 - b4 + b5) + \exp(b1) + k2 \exp(b1 + b4 + b5) = k4s3 \quad (3.a.22)$$

Equations (3.a.20) - (3.a.22) are three linear equations in  $\exp(b1 - b4 + b5)$ ,  $\exp(b1)$ , and  $\exp(b1 + b4 + b5)$ . We solve these equations for  $\exp(b1 - b4 + b5)$ ,  $\exp(b1)$ , and  $\exp(b1 + b4 + b5)$  using Cramer's Rule.

$$\exp(b1 - b4 + b5) = \frac{\begin{vmatrix} k3s1 & 1 & \frac{1}{k1} \\ s2 & 1 & 1 \\ k4s3 & 1 & k2 \end{vmatrix}}{\begin{vmatrix} k1 & 1 & \frac{1}{k1} \\ 1 & 1 & 1 \\ \frac{1}{k2} & 1 & k2 \end{vmatrix}} = \frac{s1(k3(k2 - 1)) + s2\left(\frac{1}{k1} - k2\right) + s3\left(1 - \frac{1}{k1}\right)k4}{k1(k2 - 1) + \left(\frac{1}{k1} - k2\right) + \frac{1}{k2}\left(1 - \frac{1}{k1}\right)} \quad (3.a.23)$$

$$\exp(b_1) = \frac{\begin{vmatrix} k_1 & k_3 s_1 & \frac{1}{k_1} \\ \frac{1}{k_2} & s_2 & 1 \\ \frac{1}{k_2} & k_4 s_3 & k_2 \end{vmatrix}}{\begin{vmatrix} k_1 & 1 & \frac{1}{k_1} \\ 1 & 1 & 1 \\ \frac{1}{k_2} & 1 & k_2 \end{vmatrix}} = \frac{s_1 \left( \frac{1}{k_2} - k_2 \right) k_3 + s_2 \left( k_1 k_2 - \frac{1}{k_1 k_2} \right) + s_3 \left( \frac{1}{k_1} - k_1 \right) k_4}{k_1(k_2 - 1) + \left( \frac{1}{k_1} - k_2 \right) + \frac{1}{k_2} \left( 1 - \frac{1}{k_1} \right)} \quad (3.a.24)$$

$$\exp(b_1 + b_4 + b_5) = \frac{\begin{vmatrix} k_1 & 1 & k_3 s_1 \\ \frac{1}{k_2} & 1 & s_2 \\ \frac{1}{k_2} & 1 & k_4 s_3 \end{vmatrix}}{\begin{vmatrix} k_1 & 1 & \frac{1}{k_1} \\ 1 & 1 & 1 \\ \frac{1}{k_2} & 1 & k_2 \end{vmatrix}} = \frac{s_1 \left( 1 - \frac{1}{k_2} \right) k_3 + s_2 \left( \frac{1}{k_2} - k_1 \right) + s_3 (k_4(k_1 - 1))}{k_1(k_2 - 1) + \left( \frac{1}{k_1} - k_2 \right) + \frac{1}{k_2} \left( 1 - \frac{1}{k_1} \right)} \quad (3.a.25)$$

From (3.a.24),

$$b_1 = \ln \frac{k_3 \left( \frac{1}{k_2} - k_2 \right) s_1 + \left( k_1 k_2 - \frac{1}{k_1 k_2} \right) s_2 + k_4 \left( \frac{1}{k_1} - k_1 \right) s_3}{k_1(k_2 - 1) + \left( \frac{1}{k_1} - k_2 \right) + \frac{1}{k_2} \left( 1 - \frac{1}{k_1} \right)} \quad (3.a.26)$$

Further, dividing (3.a.23) by (3.a.24), and dividing (3.a.25) by (3.a.24), we have, respectively,

$$\exp(-b_4 + b_5) = \frac{s_1(k_3(k_2 - 1)) + s_2 \left( \frac{1}{k_1} - k_2 \right) + s_3 \left( 1 - \frac{1}{k_1} \right) k_4}{s_1 \left( \frac{1}{k_2} - k_2 \right) k_3 + s_2 \left( k_1 k_2 - \frac{1}{k_1 k_2} \right) + s_3 \left( \frac{1}{k_1} - k_1 \right) k_4} \quad (3.a.27)$$

and

$$\exp(b_4 + b_5) = \frac{s_1 \left( 1 - \frac{1}{k_2} \right) k_3 + s_2 \left( \frac{1}{k_2} - k_1 \right) + s_3 (k_4(k_1 - 1))}{s_1 \left( \frac{1}{k_2} - k_2 \right) k_3 + s_2 \left( k_1 k_2 - \frac{1}{k_1 k_2} \right) + s_3 \left( \frac{1}{k_1} - k_1 \right) k_4} \quad (3.a.28)$$

Equivalently,

$$-b4 + b5 = \ln \frac{s1(k3(k2-1)) + s2\left(\frac{1}{k1} - k2\right) + s3\left(1 - \frac{1}{k1}\right)k4}{s1\left(\frac{1}{k2} - k2\right)k3 + s2\left(k1k2 - \frac{1}{k1k2}\right) + s3\left(\frac{1}{k1} - k1\right)k4} \quad (3.a.29)$$

and

$$b4 + b5 = \ln \frac{s1\left(1 - \frac{1}{k2}\right)k3 + s2\left(\frac{1}{k2} - k1\right) + s3(k4(k1-1))}{s1\left(\frac{1}{k2} - k2\right)k3 + s2\left(k1k2 - \frac{1}{k1k2}\right) + s3\left(\frac{1}{k1} - k1\right)k4} \quad (3.a.30)$$

Thus,

$$b4 = \frac{1}{2} \ln \frac{s1\left(1 - \frac{1}{k2}\right)k3 + s2\left(\frac{1}{k2} - k1\right) + s3(k1-1)k4}{s1(k2-1)k3 + s2\left(\frac{1}{k1} - k2\right) + s3\left(1 - \frac{1}{k1}\right)k4} \quad (3.a.31)$$

and

$$b5 = \frac{1}{2} \ln \frac{\left[ s1(k3(k2-1)) + s2\left(\frac{1}{k1} - k2\right) + s3\left(1 - \frac{1}{k1}\right)k4 \right] \left[ s1\left(1 - \frac{1}{k2}\right)k3 + s2\left(\frac{1}{k2} - k1\right) + s3(k1-1)k4 \right]}{\left[ s1\left(\frac{1}{k2} - k2\right)k3 + s2\left(k1k2 - \frac{1}{k1k2}\right) + s3\left(\frac{1}{k1} - k1\right)k4 \right]^2} \quad (3.a.32)$$

Having determined  $b1$ ,  $b4$  and  $b5$ , we can now substitute these values into equations (3.a.10), (3.a.14) and (3.a.13) respectively, to find  $b8$ ,  $b9$  and  $b10$ . Thus, all of the  $b$ 's can be uniquely determined. The model, therefore, is identified.

Type II: row effects; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | -1   | 1    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 1   | 1    | -1   | 1    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | -1  | 1    | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 1    | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | -1  | 1    | 1    | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 1    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the row variable with the linear column term, the eighth column, the missing data indicator, the ninth and tenth columns, the interaction of the missing data indicator and the column variable, and the eleventh and twelfth columns, the interaction of the missing data indicator and the row variable. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I case, that estimates are available for the frequency counts for the completely classified data and estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 + b10 - b11 + b12 = \ln f1 \quad (3.b.1)$$

$$b1 - b2 + b3 + b8 - b11 + b12 = \ln f2 \quad (3.b.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 + b9 + b10 - b11 + b12 = \ln f3 \quad (3.b.3)$$

$$b1 - b4 + b5 + b8 - b9 + b10 = \ln f4 \quad (3.b.4)$$

$$b1 + b8 = \ln f5 \quad (3.b.5)$$

$$b1 + b4 + b5 + b8 + b9 + b10 = \ln f6 \quad (3.b.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 - b9 + b10 + b11 + b12 = \ln f7 \quad (3.b.7)$$

$$b1 + b2 + b3 + b8 + b11 + b12 = \ln f8 \quad (3.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 + b11 + b12 = \ln f9 \quad (3.b.9)$$

Here,  $b1, b2, \dots, b12$  are the twelve parameters of matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the row effects model with nonignorable nonresponse Type I (pages 81 - 82), we have

$$b1 + b8 = \ln f5 \quad (3.b.10)$$

$$b_2 + b_{11} = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (3.b.11)$$

$$b_3 + b_{12} = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (3.b.12)$$

$$b_5 + b_{10} = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (3.b.13)$$

$$b_4 + b_9 = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (3.b.14)$$

$$b_6 = \frac{1}{4} \ln \frac{f_1 f_9}{f_3 f_7} \quad (3.b.15)$$

$$b_7 = \frac{1}{4} \ln \frac{f_3 f_9 (f_4)^2}{f_1 f_7 (f_6)^2} \quad (3.b.16)$$

Note that  $b_1$  and  $b_8$  cannot be determined uniquely from equations (3.b.1) - (3.b.9), nor can  $b_5$  and  $b_{10}$ ,  $b_4$  and  $b_9$ ,  $b_2$  and  $b_{11}$ , and  $b_3$  and  $b_{12}$ . Only two of the twelve model parameters ( $b_6$  and  $b_7$ ) have been uniquely determined. Equations (3.b.10) - (3.b.16) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (3.b.1) - (3.b.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type I (page 82). They are given by

$$\exp(b_1 - b_2 + b_3) [\exp(-b_4 + b_5 + b_6 - b_7) + 1 + \exp(b_4 + b_5 - b_6 + b_7)] = s_1 \quad (3.b.17)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (3.b.18)$$

$$\exp(b_1 + b_2 + b_3) [\exp(-b_4 + b_5 - b_6 - b_7) + 1 + \exp(b_4 + b_5 + b_6 + b_7)] = s_3 \quad (3.b.19)$$



Equations (3.b.10) - (3.b.14) and (3.b.17) - (3.b.19) provide eight equations to determine the remaining ten parameters ( $b$ 's) of the model. These nine equations are insufficient to uniquely determine the ten unknowns. This model, therefore, is not identified.

While the row effects model for this data set with nonignorable nonresponse of Type II is not identified, it may be that the elimination of the interaction term involving the missing data indicator and the quadratic column variable term, and the interaction term involving the missing data indicator and the quadratic row variable term will result in an identified model. This modified nonignorable nonresponse model is considered next.

Type II: row effects; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, and the missing data indicator and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | -1   | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 1   | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | -1  | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | -1  | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 0   | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the row variable with the linear column variable term, the eighth column, the missing data indicator, the ninth column, the interaction of the missing data indicator and the linear column variable term, and the tenth column, the interaction of the missing data indicator and the linear row variable term. The ten parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 - b10 = \ln f1 \quad (3.c.1)$$

$$b1 - b2 + b3 + b8 - b10 = \ln f2 \quad (3.c.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 + b9 - b10 = \ln f3 \quad (3.c.3)$$

$$b1 - b4 + b5 + b8 - b9 = \ln f4 \quad (3.c.4)$$

$$b1 + b8 = \ln f5 \quad (3.c.5)$$

$$b1 + b4 + b5 + b8 + b9 = \ln f6 \quad (3.c.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 - b9 + b10 = \ln f7 \quad (3.c.7)$$

$$b1 + b2 + b3 + b8 + b10 = \ln f8 \quad (3.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 = \ln f9 \quad (3.c.9)$$

Here,  $b1, b2, \dots, b10$  are the ten parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the row effects model with nonignorable nonresponse Type I (pages 81 - 82), we have

$$b1 + b8 = \ln f5 \quad (3.c.10)$$

$$b_2 + b_{10} = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (3.c.11)$$

$$b_3 = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (3.c.12)$$

$$b_5 = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (3.c.13)$$

$$b_4 + b_9 = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (3.c.14)$$

$$b_6 = \frac{1}{4} \ln \frac{f_1 f_9}{f_3 f_7} \quad (3.c.15)$$

$$b_7 = \frac{1}{4} \ln \frac{f_3 f_9 (f_4)^2}{f_1 f_7 (f_6)^2} \quad (3.c.16)$$

Note that  $b_1$  and  $b_8$  cannot be determined uniquely from equations (3.c.1) - (3.c.9), nor can  $b_2$  and  $b_{10}$ , and  $b_4$  and  $b_9$ . Only four of the ten model parameters ( $b_3$ ,  $b_5$ ,  $b_6$  and  $b_7$ ) have been uniquely determined. Equations (3.c.10) - (3.c.16) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (3.c.1) - (3.c.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Type I and Type II. They are given by

$$\exp(b_1 - b_2 + b_3) [\exp(-b_4 + b_5 + b_6 - b_7) + 1 + \exp(b_4 + b_5 - b_6 + b_7)] = s_1 \quad (3.c.17)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (3.c.18)$$

$$\exp(b_1 + b_2 + b_3) [\exp(-b_4 + b_5 - b_6 - b_7) + 1 + \exp(b_4 + b_5 + b_6 + b_7)] = s_3 \quad (3.c.19)$$

Let  $k_1 = \exp(b_3)$ ,  $k_2 = \exp(b_5)$ ,  $k_3 = \exp(b_6)$  and  $k_4 = \exp(b_7)$ . Then these equations can be rewritten as

$$\frac{k_2 k_3}{k_4} \exp(b_1 - b_2 - b_4) + \exp(b_1 - b_2) + \frac{k_2 k_4}{k_3} \exp(b_1 - b_2 + b_4) = \frac{s_1}{k_1} \quad (3.c.20)$$

$$k_2 \exp(b_1 - b_4) + \exp(b_1) + k_2 \exp(b_1 + b_4) = s_2 \quad (3.c.21)$$

$$\frac{k_2}{k_3 k_4} \exp(b_1 + b_2 - b_4) + \exp(b_1 + b_2) + k_2 k_3 k_4 \exp(b_1 + b_2 + b_4) = \frac{s_3}{k_1} \quad (3.c.22)$$

$$\text{From (3.c.21), } \exp(b_1) = \frac{s_2}{k_2 \exp(-b_4) + 1 + k_2 \exp(b_4)} \quad (3.c.23)$$

Thus, given  $b_4$ ,  $b_1$  is also uniquely determined.

Dividing (3.c.20) by (3.c.22), we have

$$\exp(2b_2) = \frac{\frac{k_2 k_3}{k_4 \exp(b_4)} + 1 + \frac{k_2 k_4 \exp(b_4)}{k_3}}{\frac{s_1}{s_3} \left( \frac{k_2}{k_3 k_4 \exp(b_4)} + 1 + k_2 k_3 k_4 \exp(b_4) \right)} \quad (3.c.24)$$

Since  $\exp(b_2)$  must be nonnegative, given  $b_4$ ,  $b_2$  is uniquely determined.

As noted above, given  $b_4$ ,  $b_1$  and  $b_2$  will be uniquely determined. Using equations (3.c.10), (3.c.14) and (3.c.11),  $b_8$ ,  $b_9$  and  $b_{10}$  will then also be uniquely determined. Thus, if  $b_4$  is uniquely determined, all of the model parameters will be found uniquely, and so the model will be identified. Conversely, if  $b_4$  is not uniquely determined, as can be seen from equations (3.c.23) and (3.c.24), neither will  $b_1$  and  $b_2$  be uniquely determined. In this case, the model will not be identified. Thus, the identification status of this model depends on the uniqueness (or nonuniqueness) of the solution for  $b_4$ . We attempt to find  $b_4$ .

Multiplying equations (3.c.20) and (3.c.22), we obtain

$$\exp(2b1) \left( \frac{k2k3}{k4} \exp(-b4) + 1 + \frac{k2k4}{k3} \exp(b4) \right) \left( \frac{k2}{k3k4} \exp(-b4) + 1 + k2k3k4 \exp(b4) \right) = \frac{s1s3}{(k1)^2} \quad (3.c.25)$$

Substituting (3.c.23) into (3.c.25), equation (3.c.25) can be rewritten as

$$\begin{aligned} & \exp(-2b4) \left[ (k2)^2 \left( \frac{1}{(k4)^2} - \frac{s1s3}{(s2)^2 (k1)^2} \right) \right] + \exp(-b4) \left[ k2 \left( \frac{1}{k4} \left( k3 + \frac{1}{k3} \right) - 2 \frac{s1s3}{(s2)^2 (k1)^2} \right) \right] \\ & + \left[ (k2)^2 (k3)^2 + 1 + \frac{(k2)^2}{(k3)^2} - \frac{s1s3}{(s2)^2 (k1)^2} (2(k2)^2 + 1) \right] \\ & + \exp(b4) \left[ k2 \left( k4 \left( k3 + \frac{1}{k3} \right) - 2 \frac{s1s3}{(s2)^2 (k1)^2} \right) \right] + \exp(2b4) \left[ (k2)^2 \left( (k4)^2 - \frac{s1s3}{(s2)^2 (k1)^2} \right) \right] = 0 \end{aligned}$$

If the above equation is multiplied by  $\exp(2b4)$ , and if the substitution  $x = \exp(b4)$  is made, we have

$$\begin{aligned} & \left[ (k2)^2 \left( \frac{1}{(k4)^2} - \frac{s1s3}{(s2)^2 (k1)^2} \right) \right] + x \left[ k2 \left( \frac{1}{k4} \left( k3 + \frac{1}{k3} \right) - 2 \frac{s1s3}{(s2)^2 (k1)^2} \right) \right] \\ & + x^2 \left[ (k2)^2 (k3)^2 + 1 + \frac{(k2)^2}{(k3)^2} - \frac{s1s3}{(s2)^2 (k1)^2} (2(k2)^2 + 1) \right] \\ & + x^3 \left[ k2 \left( k4 \left( k3 + \frac{1}{k3} \right) - 2 \frac{s1s3}{(s2)^2 (k1)^2} \right) \right] + x^4 \left[ (k2)^2 \left( (k4)^2 - \frac{s1s3}{(s2)^2 (k1)^2} \right) \right] = 0 \end{aligned} \quad (3.c.26)$$

Equation (3.c.26) is a fourth degree polynomial equation in  $x = \exp(b4)$ . Since  $\exp(b4) > 0$ , we seek only positive solutions to (3.c.26).

Since the exponential function is a 1:1 function, if there is exactly one positive solution for  $x = \exp(b_4)$  in equation (3.c.26),  $b_4$  will be uniquely determined and the model will be identified. If equation (3.c.26) has more than one positive solution or no positive solutions,  $b_4$  will not be uniquely determined and the model will not be identified. As can be seen from equation (3.c.26), the number of positive solutions for  $x = \exp(b_4)$  depends on the expected frequencies and row marginals estimated by the model. The uniqueness (or lack of uniqueness) of  $b_4$  will be determined empirically for a given data set.

**Type III:**

As was discussed for the independence and uniform association models, if all the interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable, and the row variable are included in the model, there would be too many parameters for the model, given the data. Thus, the row effects Type III nonignorable nonresponse model cannot be considered for this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.



**Type III: row effects; 3x3 table with one supplemental margin**

- modified (with only the linear interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interaction of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 1   | -1   | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | -1  | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | -1  | 1    | -1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 0   | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the row variable and the linear column variable term, the eighth column, the missing data indicator, the ninth column, the interaction of the missing data indicator and the linear column variable term, the tenth column, the interaction of the missing data indicator and the linear row variable term, and the eleventh column, the

interaction of the missing data indicator, the linear column variable term and the linear row variable term. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 - b10 + b11 = \ln f1 \quad (3.d.1)$$

$$b1 - b2 + b3 + b8 - b10 = \ln f2 \quad (3.d.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 + b9 - b10 - b11 = \ln f3 \quad (3.d.3)$$

$$b1 - b4 + b5 + b8 - b9 = \ln f4 \quad (3.d.4)$$

$$b1 + b8 = \ln f5 \quad (3.d.5)$$

$$b1 + b4 + b5 + b8 + b9 = \ln f6 \quad (3.d.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 - b9 + b10 - b11 = \ln f7 \quad (3.d.7)$$

$$b1 + b2 + b3 + b8 + b10 = \ln f8 \quad (3.d.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 + b11 = \ln f9 \quad (3.d.9)$$

Here,  $b1, b2, \dots, b11$  are the eleven parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the row effects model with nonignorable nonresponse Type I (pages 81 - 82), we have

$$b1 + b8 = \ln f5 \quad (3.d.10)$$

$$b2 + b10 = \frac{1}{2} \ln \frac{f8}{f2} \quad (3.d.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2f8}{(f5)^2} \quad (3.d.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4f6}{(f5)^2} \quad (3.d.13)$$

$$b4 + b9 = \frac{1}{2} \ln \frac{f6}{f4} \quad (3.d.14)$$

$$b7 = \frac{1}{4} \ln \frac{f3f9(f4)^2}{f1f7(f6)^2} \quad (3.d.15)$$

$$b6 + b11 = \frac{1}{4} \ln \frac{f1f9}{f3f7} \quad (3.d.16)$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (3.d.1) - (3.d.9), nor can  $b2$  and  $b10$ ,  $b4$  and  $b9$ , and  $b6$  and  $b11$ . Only three of the eleven model parameters ( $b3, b5$  and  $b7$ ) have been uniquely determined. Equations (3.d.10) - (3.d.16) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (3.d.1) - (3.d.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Types I and II. They are given by

$$\exp(b1 - b2 + b3) [\exp(-b4 + b5 + b6 - b7) + 1 + \exp(b4 + b5 - b6 + b7)] = s1 \quad (3.d.17)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (3.d.18)$$

$$\exp(b1 + b2 + b3) [\exp(-b4 + b5 - b6 - b7) + 1 + \exp(b4 + b5 + b6 + b7)] = s3 \quad (3.d.19)$$

The seven equations, (3.d.10), (3.d.11), (3.d.14), (3.d.16) and (3.d.17) - (3.d.19) are available to determine the remaining eight unknown model parameters. These seven equations are insufficient to uniquely solve for these eight unknowns. As the  $b$ 's of this model cannot, therefore, be uniquely determined, the model is not identified.

In summary, for the row effects hypothesis, the nonignorable nonresponse, Type I model is identified, while the Type III model, (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicator and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 1

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the row effects models hypothesized to fit Data Set 1 (page 27). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I, and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicator).

#### Data Set 1

##### Row Effects Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1741.16         | 261.68       | 65.16           | 2068    |
|                     | single crime    | 225.10          | 61.79        | 28.10           | 314.99  |
|                     | multiple crimes | 66.74           | 23.52        | 13.74           | 104     |
|                     |                 | 2033            | 346.99       | 107             | 2486.99 |

$$\beta = (4.12, -1.20, .24, -1.04, .25, .43, -.18)$$

Likelihood Ratio fit statistic = 4.40433

Degrees of freedom = 2

**Data Set 1**

Row Effects Model - One Supplemental Margin

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1717.10         | 258.07       | 64.26           | 2039.43 |
|                     | single crime    | 234.17          | 64.28        | 29.24           | 327.69  |
|                     | multiple crimes | 76.93           | 27.12        | 15.84           | 119.89  |
|                     |                 | 2028.20         | 349.47       | 109.34          | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 860.96          | 129.40       | 32.22           | 1022.58 |
|                     | single crime    | 117.42          | 32.23        | 14.66           | 164.31  |
|                     | multiple crimes | 38.57           | 13.60        | 7.94            | 60.11   |
|                     |                 | 1016.95         | 175.23       | 54.82           | 1247    |

$$\beta = (3.47, -1.13, .26, -1.04, .25, .43, -.18, .69)$$

Likelihood Ratio fit statistic = 13.1401

Degrees of freedom = 4

**Data Set 1**

Row Effects Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1741.16    | 261.68       | 65.16           | 2068    |
|                     | single crime    | 225.10     | 61.79        | 28.10           | 314.99  |
|                     | multiple crimes | 66.74      | 23.52        | 13.74           | 104     |
|                     |                 | 2033       | 346.99       | 107             | 2486.99 |

**Partially Classified Data (Hypothetical Contingency Table)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 836.90     | 125.78       | 31.32           | 994  |
|                     | single crime    | 126.49     | 34.72        | 15.79           | 177  |
|                     | multiple crimes | 48.77      | 17.19        | 10.04           | 76   |
|                     |                 | 1012.16    | 177.69       | 57.15           | 1247 |

 $\beta = (3.55, -1.00, .29, -1.04, .25, .43, -.18, .58, -.21, -.05)$ 

Likelihood Ratio fit statistic = 4.40433

Degrees of freedom = 2

**Data Set 1**

Row Effects Model - One Supplemental Margin

Nonignorable Nonresponse - Type I

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1742.24    | 262.12       | 65.26           | 2069.62 |
|                     | single crime    | 223.37     | 60.37        | 27.00           | 310.74  |
|                     | multiple crimes | 67.39      | 24.50        | 14.74           | 106.63  |
|                     |                 | 2033       | 346.99       | 107             | 2486.99 |

**Partially Classified Data (Hypothetical Contingency Table)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |        |
|---------------------|-----------------|------------|--------------|-----------------|--------|
| Second<br>Interview | crime free      | 779.81     | 36.39        | 176.18          | 992.38 |
|                     | single crime    | 99.98      | 8.38         | 72.90           | 181.26 |
|                     | multiple crimes | 30.17      | 3.40         | 39.80           | 73.36  |
|                     |                 | 909.96     | 48.17        | 288.88          | 1247   |

$$\beta = (2.13, -1.19, .28, -.16, 2.32, .44, -.14, 1.97, -.90, -2.07)$$

Likelihood Ratio fit statistic = 4.80974

Degrees of freedom = 2



### **Data Set 1**

Row Effects Model - One Supplemental Margin

Nonignorable Nonresponse - Type II (modified)

As indicated on page 95, the identification status of the row effects nonignorable nonresponse Type II model, modified to include only the linear interaction terms involving the missing data indicator, depends on the number of solutions to equation (3.c.26). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (3.c.26) giving the following fourth degree polynomial equation:

$$.05 - .27x - .54x^2 - 1.31x^3 - 1.20x^4 = 0$$

By Descartes' Rule of Signs (Brown and Robbins, 1989), since there is only one change of sign in this fourth degree polynomial equation, there is exactly one positive solution for  $x = \exp(b_4)$ . As discussed on page 95, the model is therefore identified for the given data set. Given below are the parameter and contingency table estimates, together with the likelihood ratio fit statistic for this identified model.

**Data Set 1**

Row Effects Model - One Supplemental Margin

Nonignorable Nonresponse - Type II (modified)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1742.14         | 261.77       | 65.16           | 2069.07 |
|                     | single crime    | 223.22          | 61.54        | 28.11           | 312.87  |
|                     | multiple crimes | 67.64           | 23.69        | 13.74           | 105.07  |
|                     |                 | 2033            | 347          | 107.01          | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 937.28          | 51.05        | 4.61            | 992.93  |
|                     | single crime    | 160.45          | 16.03        | 2.65            | 179.13  |
|                     | multiple crimes | 64.95           | 8.25         | 1.73            | 74.93   |
|                     |                 | 1162.68         | 75.33        | 8.99            | 1246.99 |

$$\beta = (2.775, -.912, .247, -2.051, .252, .423, -.184, 1.345, 1.015, -.290)$$

Likelihood Ratio fit statistic = 4.47678

Degrees of freedom = 2

Column Effects Model - Data Set 1

Type I: column effects; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 1   | 1   | 1    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | -1  | 1    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 1    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 0   | 0   | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the column variable and the linear row variable term, the eighth column, the missing data indicator, and the ninth and tenth columns, the missing data indicator by column variable interaction. The ten parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the independence, uniform association and row effects models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell

frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 24) is the same as for the independence, uniform association and row effects models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for the independence, uniform association and row effects models, for the portion of vector  $Y$  corresponding to the completely classified data (and the first nine rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f1, f2, \dots, f9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 + b10 = \ln f1 \quad (4.a.1)$$

$$b1 - b2 + b3 + b8 = \ln f2 \quad (4.a.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 - b7 + b8 + b9 + b10 = \ln f3 \quad (4.a.3)$$

$$b1 - b4 + b5 + b8 - b9 + b10 = \ln f4 \quad (4.a.4)$$

$$b1 + b8 = \ln f5 \quad (4.a.5)$$

$$b1 + b4 + b5 + b8 + b9 + b10 = \ln f6 \quad (4.a.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 + b8 - b9 + b10 = \ln f7 \quad (4.a.7)$$

$$b1 + b2 + b3 + b8 = \ln f8 \quad (4.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 = \ln f9 \quad (4.a.9)$$

Here,  $b1, b2, \dots, b10$  are the ten parameters of the matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's.

$$\text{From (4.a.5), } b1 + b8 = \ln f5 \quad (4.a.10)$$

Substituting (4.a.10) into (4.a.2) and into (4.a.8), we have, respectively,

$$-b2 + b3 = \ln f2 - \ln f5$$

$$b2 + b3 = \ln f8 - \ln f5$$

$$\text{Thus, } b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (4.a.11)$$

$$\text{and } b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (4.a.12)$$

Subtracting (4.a.5) from (4.a.4), and subtracting (4.a.5) from (4.a.6), we have, respectively,

$$-b4 + b5 - b9 + b10 = \ln f4 - \ln f5$$

$$b4 + b5 + b9 + b10 = \ln f6 - \ln f5$$

$$\text{Thus, } b4 + b9 = \frac{1}{2} \ln \frac{f6}{f4} \quad (4.a.13)$$

$$\text{Also, } b4 + b9 = \frac{1}{2} \ln \frac{f6}{f4} \quad (4.a.14)$$

Subtracting (4.a.3) from (4.a.1), and substituting (4.a.14), we obtain

$$b6 = \frac{1}{2} \ln \frac{f1 f6}{f3 f4} \quad (4.a.15)$$

Subtracting (4.a.7) from (4.a.3), and substituting (4.a.11) and (4.a.14), we have

$$b7 = \frac{1}{2} \ln \frac{f2 f6 f7}{f3 f4 f8} \quad (4.a.16)$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (4.a.1) - (4.a.9), since whenever  $b1$  appears in one of the equations, so does  $b8$ ; that is  $b1 + b8$  appears as a unit

in each of the equations. Similarly,  $b5$  and  $b10$  cannot be determined uniquely, nor can  $b4$  and  $b9$ . Only four of the ten model parameters ( $b2$ ,  $b3$ ,  $b6$  and  $b7$ ) have been uniquely determined. Equations (4.a.10) - (4.a.16) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (4.a.1) - (4.a.9).

The last three elements of vector  $Y$  correspond to the partially classified data (and the last nine rows of the  $X$  matrix). Each of these last three elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f10, \dots, f18$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b1 - b2 + b3) [\exp(-b4 + b5 + b6 - b7) + 1 + \exp(b4 + b5 - b6 - b7)] = s1 \quad (4.a.17)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (4.a.18)$$

$$\exp(b1 + b2 + b3) [\exp(-b4 + b5 - b6 + b7) + 1 + \exp(b4 + b5 + b6 + b7)] = s3 \quad (4.a.19)$$

Here,  $s1 = f10 + f11 + f12$ , the first row marginal for the partially classified data,  $s2 = f13 + f14 + f15$ , the second row marginal for the partially classified data, and  $s3 = f16 + f17 + f18$ , the third row marginal for the partially classified data. It is assumed that we can obtain estimates  $s1$ ,  $s2$ , and  $s3$ , for the last three elements of vector  $Y$ .

Let  $k1 = \exp(b6 - b7)$ ,  $k2 = \exp(b6 + b7)$ ,  $k3 = \exp(b2 - b3)$ , and  $k4 = \exp(-b2 - b3)$ .

Then equations (4.a.17) - (4.a.19) can be rewritten as

$$k1 \exp(b1 - b4 + b5) + \exp(b1) + \frac{1}{k2} \exp(b1 + b4 + b5) = k3s1 \quad (4.a.20)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (4.a.21)$$

$$\frac{1}{k1} \exp(b1 - b4 + b5) + \exp(b1) + k2 \exp(b1 + b4 + b5) = k4s3 \quad (4.a.22)$$

Equations (4.a.20) - (4.a.22) are three linear equations in  $\exp(b1-b4+b5)$ ,  $\exp(b1)$ , and  $\exp(b1+b4+b5)$ . We solve these equations for  $\exp(b1-b4+b5)$ ,  $\exp(b1)$ , and  $\exp(b1+b4+b5)$  using Cramer's Rule.

$$\exp(b1 - b4 + b5) = \frac{\begin{vmatrix} k3s1 & 1 & \frac{1}{k2} \\ s2 & 1 & 1 \\ k4s3 & 1 & k2 \end{vmatrix}}{\begin{vmatrix} k1 & 1 & \frac{1}{k2} \\ 1 & 1 & 1 \\ \frac{1}{k1} & 1 & k2 \end{vmatrix}} = \frac{s1(k3(k2-1)) + s2\left(\frac{1}{k2} - k2\right) + s3\left(1 - \frac{1}{k2}\right)k4}{k1(k2-1) + \left(\frac{1}{k2} - k2\right) + \frac{1}{k1}\left(1 - \frac{1}{k2}\right)} \quad (4.a.23)$$

$$\exp(b1) = \frac{\begin{vmatrix} k1 & k3s1 & \frac{1}{k2} \\ \frac{1}{k1} & s2 & 1 \\ \frac{1}{k1} & k4s3 & k2 \end{vmatrix}}{\begin{vmatrix} k1 & 1 & \frac{1}{k2} \\ 1 & 1 & 1 \\ \frac{1}{k1} & 1 & k2 \end{vmatrix}} = \frac{s1\left(\frac{1}{k1} - k2\right)k3 + s2\left(k1k2 - \frac{1}{k1k2}\right) + s3\left(\frac{1}{k2} - k1\right)k4}{k1(k2-1) + \left(\frac{1}{k2} - k2\right) + \frac{1}{k1}\left(1 - \frac{1}{k2}\right)} \quad (4.a.24)$$

$$\exp(b1 + b4 + b5) = \frac{\begin{vmatrix} k1 & 1 & k3s1 \\ \frac{1}{k1} & 1 & s2 \\ \frac{1}{k1} & 1 & k4s3 \end{vmatrix}}{\begin{vmatrix} k1 & 1 & \frac{1}{k2} \\ 1 & 1 & 1 \\ \frac{1}{k1} & 1 & k2 \end{vmatrix}} = \frac{s1\left(1 - \frac{1}{k1}\right)k3 + s2\left(\frac{1}{k1} - k1\right) + s3(k4(k1-1))}{k1(k2-1) + \left(\frac{1}{k2} - k2\right) + \frac{1}{k1}\left(1 - \frac{1}{k2}\right)} \quad (4.a.25)$$

From (4.a.24),

$$b1 = \ln \frac{k3\left(\frac{1}{k1} - k2\right)s1 + \left(k1k2 - \frac{1}{k1k2}\right)s2 + k4\left(\frac{1}{k2} - k1\right)s3}{k1(k2-1) + \left(\frac{1}{k2} - k2\right) + \frac{1}{k1}\left(1 - \frac{1}{k2}\right)} \quad (4.a.26)$$

Further, dividing (4.a.23) by (4.a.24), and dividing (4.a.25) by (4.a.24), we have, respectively,

$$\exp(-b_4 + b_5) = \frac{s_1(k_3(k_2 - 1)) + s_2\left(\frac{1}{k_2} - k_2\right) + s_3\left(1 - \frac{1}{k_2}\right)k_4}{s_1\left(\frac{1}{k_1} - k_2\right)k_3 + s_2\left(k_1k_2 - \frac{1}{k_1k_2}\right) + s_3\left(\frac{1}{k_2} - k_1\right)k_4} \quad (4.a.27)$$

and

$$\exp(b_4 + b_5) = \frac{s_1\left(1 - \frac{1}{k_1}\right)k_3 + s_2\left(\frac{1}{k_1} - k_1\right) + s_3(k_4(k_1 - 1))}{s_1\left(\frac{1}{k_1} - k_2\right)k_3 + s_2\left(k_1k_2 - \frac{1}{k_1k_2}\right) + s_3\left(\frac{1}{k_2} - k_1\right)k_4} \quad (4.a.28)$$

Equivalently,

$$-b_4 + b_5 = \ln \frac{s_1(k_3(k_2 - 1)) + s_2\left(\frac{1}{k_2} - k_2\right) + s_3\left(1 - \frac{1}{k_2}\right)k_4}{s_1\left(\frac{1}{k_1} - k_2\right)k_3 + s_2\left(k_1k_2 - \frac{1}{k_1k_2}\right) + s_3\left(\frac{1}{k_2} - k_1\right)k_4} \quad (4.a.29)$$

and

$$b_4 + b_5 = \ln \frac{s_1\left(1 - \frac{1}{k_1}\right)k_3 + s_2\left(\frac{1}{k_1} - k_1\right) + s_3(k_4(k_1 - 1))}{s_1\left(\frac{1}{k_1} - k_2\right)k_3 + s_2\left(k_1k_2 - \frac{1}{k_1k_2}\right) + s_3\left(\frac{1}{k_2} - k_1\right)k_4} \quad (4.a.30)$$

Thus,

$$b_4 = \frac{1}{2} \ln \frac{s_1\left(1 - \frac{1}{k_1}\right)k_3 + s_2\left(\frac{1}{k_1} - k_1\right) + s_3(k_1 - 1)k_4}{s_1(k_2 - 1)k_3 + s_2\left(\frac{1}{k_2} - k_2\right) + s_3\left(1 - \frac{1}{k_2}\right)k_4} \quad (4.a.31)$$

and

$$b_5 = \frac{1}{2} \ln \frac{\left[ s_1(k_3(k_2 - 1)) + s_2\left(\frac{1}{k_2} - k_2\right) + s_3\left(1 - \frac{1}{k_2}\right)k_4 \right] \left[ s_1\left(1 - \frac{1}{k_1}\right)k_3 + s_2\left(\frac{1}{k_1} - k_1\right) + s_3(k_1 - 1)k_4 \right]}{\left[ s_1\left(\frac{1}{k_1} - k_2\right)k_3 + s_2\left(k_1k_2 - \frac{1}{k_1k_2}\right) + s_3\left(\frac{1}{k_2} - k_1\right)k_4 \right]^2} \quad (4.a.32)$$



Having determined  $b1$ ,  $b4$  and  $b5$ , we can now substitute these values into equations (4.a.10), (4.a.14) and (4.a.13) respectively, to find  $b8$ ,  $b9$  and  $b10$ . Thus, all of the  $b$ 's can be uniquely determined. The model, therefore, is identified.

Type II: column effects; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | -1   | 1    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 1   | 1   | 1    | -1   | 1    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | -1  | 1    | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 1    | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1    | 1    | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 1    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the column variable and the linear row variable term, the eighth column, the missing data indicator, the ninth and tenth columns, the interaction of the missing data indicator and the column variable, and the eleventh and twelfth columns, the interaction of the missing data indicator and the row variable. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I case, that estimates are available for the frequency counts for the completely classified data and estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 + b10 - b11 + b12 = \ln f1 \quad (4.b.1)$$

$$b1 - b2 + b3 + b8 - b11 + b12 = \ln f2 \quad (4.b.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 - b7 + b8 + b9 + b10 - b11 + b12 = \ln f3 \quad (4.b.3)$$

$$b1 - b4 + b5 + b8 - b9 + b10 = \ln f4 \quad (4.b.4)$$

$$b1 + b8 = \ln f5 \quad (4.b.5)$$

$$b1 + b4 + b5 + b8 + b9 + b10 = \ln f6 \quad (4.b.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 + b8 - b9 + b10 + b11 + b12 = \ln f7 \quad (4.b.7)$$

$$b1 + b2 + b3 + b8 + b11 + b12 = \ln f8 \quad (4.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 + b11 + b12 = \ln f9 \quad (4.b.9)$$

Here,  $b1, b2, \dots, b12$  are the twelve parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the column effects model with nonignorable nonresponse Type I (page 109), we have

$$b1 + b8 = \ln f5 \quad (4.b.10)$$

$$b_2 + b_{11} = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (4.b.11)$$

$$b_3 + b_{12} = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (4.b.12)$$

$$b_5 + b_{10} = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (4.b.13)$$

$$b_4 + b_9 = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (4.b.14)$$

$$b_6 = \frac{1}{2} \ln \frac{f_1 f_6}{f_3 f_4} \quad (4.b.15)$$

$$b_7 = \frac{1}{2} \ln \frac{f_2 f_6 f_7}{f_3 f_4 f_8} \quad (4.b.16)$$

Note that  $b_1$  and  $b_8$  cannot be determined uniquely from equations (4.b.1) - (4.b.9), nor can  $b_5$  and  $b_{10}$ ,  $b_4$  and  $b_9$ ,  $b_2$  and  $b_{11}$ , and  $b_3$  and  $b_{12}$ . Only two of the twelve model parameters ( $b_6$  and  $b_7$ ) have been uniquely determined. Equations (4.b.10) - (4.b.16) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (4.b.1) - (4.b.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type I. They are given by

$$\exp(b_1 - b_2 + b_3) [\exp(-b_4 + b_5 + b_6 - b_7) + 1 + \exp(b_4 + b_5 - b_6 - b_7)] = s_1 \quad (4.b.17)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (4.b.18)$$

$$\exp(b_1 + b_2 + b_3) [\exp(-b_4 + b_5 - b_6 + b_7) + 1 + \exp(b_4 + b_5 + b_6 + b_7)] = s_3 \quad (4.b.19)$$

Equations (4.b.10) - (4.b.14) and (4.b.17) - (4.b.19) provide eight equations to determine the remaining ten parameters ( $b$ 's) of the model. These eight equations are insufficient to uniquely determine the ten unknowns. This model, therefore, is not identified.

While the column effects model for this data set with nonignorable nonresponse of Type II is not identified, it may be that the elimination of the interaction term involving the missing data indicator and the quadratic term for the column variable, and the interaction term involving the missing data indicator and the quadratic term for the row variable will result in an identified model. This modified Type II nonignorable nonresponse model is considered next.

Type II: column effects; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, and the missing data indicator and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | -1   | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 1   | 1   | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | -1  | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 0   | 0   | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the column variable and the linear row variable term, the eighth column, the missing data indicator, the ninth column, the interaction of the missing data indicator and the linear column variable term, and the tenth column, the interaction of the missing data indicator and the linear row variable term. The ten parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 - b10 = \ln f1 \quad (4.c.1)$$

$$b1 - b2 + b3 + b8 - b10 = \ln f2 \quad (4.c.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 - b7 + b8 + b9 - b10 = \ln f3 \quad (4.c.3)$$

$$b1 - b4 + b5 + b8 - b9 = \ln f4 \quad (4.c.4)$$

$$b1 + b8 = \ln f5 \quad (4.c.5)$$

$$b1 + b4 + b5 + b8 + b9 = \ln f6 \quad (4.c.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 + b8 - b9 + b10 = \ln f7 \quad (4.c.7)$$

$$b1 + b2 + b3 + b8 + b10 = \ln f8 \quad (4.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 = \ln f9 \quad (4.c.9)$$

Here,  $b1, b2, \dots, b10$  are the ten parameters of matrix  $\beta$ , which are to be determined. We solve the equations for the  $b$ 's. Using the same procedure as that for the column effects model with nonignorable nonresponse Type I (page 109), we have

$$b1 + b8 = \ln f5 \quad (4.c.10)$$

$$b2 + b10 = \frac{1}{2} \ln \frac{f8}{f2} \quad (4.c.11)$$

$$b_3 = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (4.c.12)$$

$$b_5 = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (4.c.13)$$

$$b_4 + b_9 = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (4.c.14)$$

$$b_6 = \frac{1}{2} \ln \frac{f_1 f_6}{f_3 f_4} \quad (4.c.15)$$

$$b_7 = \frac{1}{2} \ln \frac{f_2 f_6 f_7}{f_3 f_4 f_8} \quad (4.c.16)$$

Note that  $b_1$  and  $b_8$  cannot be determined uniquely from equations (4.c.1) - (4.c.9), nor can  $b_2$  and  $b_{10}$ , and  $b_4$  and  $b_9$ . Only four of the ten model parameters ( $b_3$ ,  $b_5$ ,  $b_6$  and  $b_7$ ) have been uniquely determined. Equations (4.c.10) - (4.c.16) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (4.c.1) - (4.c.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Type I and Type II. They are given by

$$\exp(b_1 - b_2 + b_3) [\exp(-b_4 + b_5 + b_6 - b_7) + 1 + \exp(b_4 + b_5 - b_6 - b_7)] = s_1 \quad (4.c.17)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (4.c.18)$$

$$\exp(b_1 + b_2 + b_3) [\exp(-b_4 + b_5 - b_6 + b_7) + 1 + \exp(b_4 + b_5 + b_6 + b_7)] = s_3 \quad (4.c.19)$$

Let  $k_1 = \exp(b_3)$ ,  $k_2 = \exp(b_5)$ ,  $k_3 = \exp(b_6)$  and  $k_4 = \exp(b_7)$ . Then equations (4.c.17) - (4.c.19) can be rewritten as

$$\frac{k_2 k_3}{k_4} \exp(b_1 - b_2 - b_4) + \exp(b_1 - b_2) + \frac{k_2}{k_3 k_4} \exp(b_1 - b_2 + b_4) = \frac{s_1}{k_1} \quad (4.c.20)$$

$$k_2 \exp(b_1 - b_4) + \exp(b_1) + k_2 \exp(b_1 + b_4) = s_2 \quad (4.c.21)$$

$$\frac{k_2 k_4}{k_3} \exp(b_1 + b_2 - b_4) + \exp(b_1 + b_2) + k_2 k_3 k_4 \exp(b_1 + b_2 + b_4) = \frac{s_3}{k_1} \quad (4.c.22)$$



From (4.c.21), 
$$\exp(b1) = \frac{s2}{k2 \exp(-b4) + 1 + k2 \exp(b4)} \quad (4.c.23)$$

Thus, given  $b4$ ,  $b1$  is also uniquely determined.

Dividing (4.c.20) by (4.c.22), we have

$$\exp(2b2) = \frac{\frac{k2k3}{k4 \exp(b4)} + 1 + \frac{k2 \exp(b4)}{k3k4}}{\frac{s1}{s3} \left( \frac{k2k4}{k3 \exp(b4)} + 1 + k2k3k4 \exp(b4) \right)} \quad (4.c.24)$$

Since  $\exp(b2)$  must be nonnegative, given  $b4$ ,  $b2$  is uniquely determined.

As noted above, given  $b4$ ,  $b1$  and  $b2$  will be uniquely determined. Using equations (4.c.10), (4.c.14) and (4.c.11),  $b8$ ,  $b9$  and  $b10$  will then also be uniquely determined. Thus, if  $b4$  is uniquely determined, all of the model parameters will be found uniquely, and so the model will be identified. Conversely, if  $b4$  is not uniquely determined, as can be seen from equations (4.c.23) and (4.c.24), neither will  $b1$  and  $b2$  be uniquely determined. In this case, the model will not be identified. Thus, the identification status of this model depends on the uniqueness (or nonuniqueness) of the solution for  $b4$ . We attempt to find  $b4$ .

Multiplying equations (4.c.20) and (4.c.22), we obtain

$$\exp(2b1) \left( \frac{k2k3}{k4} \exp(-b4) + 1 + \frac{k2}{k3k4} \exp(b4) \right) \left( \frac{k2k4}{k3} \exp(-b4) + 1 + k2k3k4 \exp(b4) \right) = \frac{s1s3}{(k1)^2} \quad (4.c.25)$$

Substituting (4.c.21) into (4.c.25), equation (4.c.25) can be rewritten as

$$\begin{aligned}
& \exp(-2b_4) \left[ (k_2)^2 \left( 1 - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] + \exp(-b_4) \left[ k_2 \left( \frac{k_3}{k_4} + \frac{k_4}{k_3} - 2 \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] \\
& + \left[ (k_2)^2 (k_3)^2 + 1 + \frac{(k_2)^2}{(k_3)^2} - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} (2(k_2)^2 + 1) \right] \\
& + \exp(b_4) \left[ k_2 \left( k_3 k_4 + \frac{1}{k_3 k_4} - 2 \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] + \exp(2b_4) \left[ (k_2)^2 \left( 1 - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] = 0
\end{aligned}$$

If the above equation is multiplied by  $\exp(2b_4)$ , and if the substitution  $x = \exp(b_4)$  is made, we have

$$\begin{aligned}
& \left[ (k_2)^2 \left( 1 - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] + x \left[ k_2 \left( \frac{k_3}{k_4} + \frac{k_4}{k_3} - 2 \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] \\
& + x^2 \left[ (k_2)^2 (k_3)^2 + 1 + \frac{(k_2)^2}{(k_3)^2} - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} (2(k_2)^2 + 1) \right] \\
& + x^3 \left[ k_2 \left( k_3 k_4 + \frac{1}{k_3 k_4} - 2 \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] + x^4 \left[ (k_2)^2 \left( 1 - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] = 0 \quad (4.c.26)
\end{aligned}$$

Equation (4.c.26) is a fourth degree polynomial equation in  $x = \exp(b_4)$ . Since  $\exp(b_4) > 0$ , we seek only positive solutions to (4.c.26).

Since the exponential function is a 1:1 function, if there is exactly one positive solution for  $x = \exp(b_4)$  in equation (4.c.26),  $b_4$  will be uniquely determined and the model will be identified. If equation (4.c.26) has more than one positive solution or no positive solutions,  $b_4$  will not be uniquely determined and the model will not be identified. As can be seen from equation (4.c.26), the number of positive solutions for  $x = \exp(b_4)$  depends on the expected frequencies and row marginals estimated by the model. The uniqueness (or lack of uniqueness) of  $b_4$  will be determined empirically for a given data set.

**Type III:**

As was discussed in the case of the independence, uniform association and row effects models, if all the interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable, and the row variable are included in the model, there would be too many parameters for the model, given the data. Thus, the column effects Type III nonignorable nonresponse model cannot be considered for this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

Type III: column effects; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interaction of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | -1  | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 1   | 1   | -1   | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | -1  | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1    | -1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | -1  | 0   | 0   | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the column variable and the linear row variable term, the eighth column, the missing data indicator, the ninth column, the interaction of the missing data indicator and the linear column variable term, the tenth column, the interaction of the missing data indicator and the linear row variable term, and the eleventh column, the

interaction of the missing data indicator, the linear column variable term and the linear row variable term. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b9 - b10 + b11 = \ln f1 \quad (4.d.1)$$

$$b1 - b2 + b3 + b8 - b10 = \ln f2 \quad (4.d.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 - b7 + b8 + b9 - b10 - b11 = \ln f3 \quad (4.d.3)$$

$$b1 - b4 + b5 + b8 - b9 = \ln f4 \quad (4.d.4)$$

$$b1 + b8 = \ln f5 \quad (4.d.5)$$

$$b1 + b4 + b5 + b8 + b9 = \ln f6 \quad (4.d.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 + b8 - b9 + b10 - b11 = \ln f7 \quad (4.d.7)$$

$$b1 + b2 + b3 + b8 + b10 = \ln f8 \quad (4.d.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 + b11 = \ln f9 \quad (4.d.9)$$

Here,  $b_1, b_2, \dots, b_{11}$  are the eleven parameters of matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the column effects model with nonignorable nonresponse Type I (page 109), we have

$$b_1 + b_8 = \ln f_5 \quad (4.d.10)$$

$$b_2 + b_{10} = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (4.d.11)$$

$$b_3 = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (4.d.12)$$

$$b_5 = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (4.d.13)$$

$$b_4 + b_9 = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (4.d.14)$$

$$b_7 = \frac{1}{2} \ln \frac{f_2 f_6 f_7}{f_3 f_4 f_8} \quad (4.d.15)$$

$$b_6 + b_{11} = \frac{1}{2} \ln \frac{f_1 f_6}{f_3 f_4} \quad (4.d.16)$$

Note that  $b_1$  and  $b_8$  cannot be determined uniquely from equations (4.d.1) - (4.d.9), nor can  $b_2$  and  $b_{10}$ ,  $b_4$  and  $b_9$ , and  $b_6$  and  $b_{11}$ . Only three of the eleven model parameters ( $b_3$ ,  $b_5$  and  $b_7$ ) have been uniquely determined. Equations (4.d.10) - (4.d.16) provide the most complete solutions for the  $b$ 's which can be determined from equations (4.d.1) - (4.d.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Types I and II. They are given by

$$\exp(b_1 - b_2 + b_3) [\exp(-b_4 + b_5 + b_6 - b_7) + 1 + \exp(b_4 + b_5 - b_6 - b_7)] = s_1 \quad (4.d.17)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (4.d.18)$$

$$\exp(b_1 + b_2 + b_3) [\exp(-b_4 + b_5 - b_6 + b_7) + 1 + \exp(b_4 + b_5 + b_6 + b_7)] = s_3 \quad (4.d.19)$$

The seven equations, (4.d.10), (4.d.11), (4.d.14), (4.d.16) and (4.d.17) - (4.d.19) are available to determine the remaining eight unknown model parameters. These seven equations are insufficient to uniquely solve for these eight unknowns. As the  $b$ 's of this model cannot, therefore, be uniquely determined, the model is not identified.

In summary, for the column effects hypothesis, the nonignorable nonresponse, Type I model is identified, while the Type III model (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicator and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 1

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the column effects models hypothesized to fit Data Set 1 (page 27). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicator).

#### Data Set 1

##### Column Effects Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 1740.18         | 257.56       | 70.26           | 2068 |
|                     | single crime    | 231.64          | 60.89        | 22.47           | 315  |
|                     | multiple crimes | 61.18           | 28.56        | 14.26           | 104  |
|                     |                 | 2033            | 347.01       | 106.99          | 2487 |

$$\beta = (4.11, -1.10, .34, -1.17, .17, .44, -.14)$$

Likelihood Ratio fit statistic = 5.31944

Degrees of freedom = 2



**Data Set 1**

Column Effects Model - One Supplemental Margin

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1716.13         | 254.00       | 69.29           | 2039.42 |
|                     | single crime    | 240.97          | 63.34        | 23.38           | 327.69  |
|                     | multiple crimes | 70.53           | 32.92        | 16.44           | 119.89  |
|                     |                 | 2027.63         | 350.26       | 109.11          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 860.48          | 127.36       | 34.74           | 1022.58 |
|                     | single crime    | 120.83          | 31.76        | 11.72           | 164.31  |
|                     | multiple crimes | 35.36           | 16.51        | 8.24            | 60.11   |
|                     |                 | 1016.67         | 175.63       | 54.7            | 1247    |

$$\beta = (3.46, -1.02, .37, -1.17, .17, .44, -.14, .69)$$

Likelihood Ratio fit statistic = 14.0552

Degrees of freedom = 4

**Data Set 1**

Column Effects Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1740.17    | 257.56       | 70.27           | 2068    |
|                     | single crime    | 231.64     | 60.89        | 22.48           | 315.01  |
|                     | multiple crimes | 61.18      | 28.56        | 14.27           | 104.01  |
|                     |                 | 2032.99    | 347.01       | 107.02          | 2487.02 |

**Partially Classified Data (Hypothetical Contingency Table)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 836.43     | 123.80       | 33.77           | 994  |
|                     | single crime    | 130.16     | 34.21        | 12.63           | 177  |
|                     | multiple crimes | 44.71      | 20.87        | 10.42           | 76   |
|                     |                 | 1011.3     | 178.88       | 56.82           | 1247 |

$$\beta = (3.53, -.89, .40, -1.17, .17, .44, -.14, .58, -.21, -.05)$$

Likelihood Ratio fit statistic = 5.31944

Degrees of freedom = 2

**Data Set 1**

Column Effects Model - One Supplemental Margin

Nonignorable Nonresponse - Type I

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1741.09         | 257.72       | 70.21           | 2069.02 |
|                     | single crime    | 230.18          | 60.48        | 22.35           | 313.01  |
|                     | multiple crimes | 61.73           | 28.79        | 14.44           | 104.96  |
|                     |                 | 2033            | 346.99       | 107             | 2486.99 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 639.46          | 216.50       | 137.03          | 992.98  |
|                     | single crime    | 84.54           | 50.81        | 43.63           | 178.98  |
|                     | multiple crimes | 22.67           | 24.19        | 28.18           | 75.04   |
|                     |                 | 746.67          | 291.5        | 208.84          | 1247.01 |

$$\beta = (3.93, -1.10, .35, -.33, .18, .44, -.13, .17, -.84, -.01)$$

Likelihood Ratio fit statistic = 5.37683

Degrees of freedom = 2

### **Data Set 1**

#### **Column Effects Model - One Supplemental Margin**

#### **Nonignorable Nonresponse - Type II (modified)**

As indicated on page 122, the identification status of the column effects nonignorable nonresponse Type II model, modified to include only the linear interaction terms involving the missing data indicator, depends on the number of solutions to equation (4.c.26). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (4.c.26) giving the following fourth degree polynomial equation:

$$-.19+.08x+.63x^2-.21x^3-19x^4 = 0$$

By Descartes' Rule of Signs, since there are two changes of sign in this fourth degree polynomial equation, there are either two or no positive solutions for  $x = \exp(b_4)$ . Using Newton's Method (Anton, 1992), we find that there are two positive solutions for  $x = \exp(b_4)$ . As discussed on page 122, the model is therefore not identified for the given data set.

R + C - Data Set 1

Type I: R + C model; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 1   | 1    | 1    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 1   | -1   | 1    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 1   | 1    | 1    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 1   | -1   | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 0   | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 0   | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the quadratic row variable term and the linear column variable term, the seventh and eighth columns, the interactions of the column variable and the linear row variable term, the ninth column, the missing data indicator, and the tenth and eleventh columns, the missing data indicator by column variable interaction. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the independence, uniform association, row effects and column effects models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 24) is the same as for the independence, uniform association and row effects models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for the independence, uniform association, row effects and column effects models, for the portion of vector  $Y$  corresponding to the completely classified data (and the first nine rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f1, f2, \dots, f9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 - b6 + b7 - b8 + b9 - b10 + b11 = \ln f1 \quad (5.a.1)$$

$$b1 - b2 + b3 + b9 = \ln f2 \quad (5.a.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 - b7 - b8 + b9 + b10 + b11 = \ln f3 \quad (5.a.3)$$

$$b_1 - b_4 + b_5 + b_9 - b_{10} + b_{11} = \ln f_4 \quad (5.a.4)$$

$$b_1 + b_9 = \ln f_5 \quad (5.a.5)$$

$$b_1 + b_4 + b_5 + b_9 + b_{10} + b_{11} = \ln f_6 \quad (5.a.6)$$

$$b_1 + b_2 + b_3 - b_4 + b_5 - b_6 - b_7 + b_8 + b_9 - b_{10} + b_{11} = \ln f_7 \quad (5.a.7)$$

$$b_1 + b_2 + b_3 + b_9 = \ln f_8 \quad (5.a.8)$$

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} + b_{11} = \ln f_9 \quad (5.a.9)$$

Here,  $b_1, b_2, \dots, b_{11}$  are the eleven parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's.

$$\text{From (5.a.5), } b_1 + b_9 = \ln f_5 \quad (5.a.10)$$

Substituting (5.a.10) into (5.a.2) and into (5.a.8), we have

$$-b_2 + b_3 = \ln f_2 - \ln f_5$$

$$b_2 + b_3 = \ln f_8 - \ln f_5$$

$$\text{Thus, } b_2 = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (5.a.11)$$

$$\text{and } b_3 = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (5.a.12)$$

Subtracting (5.a.5) from (5.a.4), and subtracting (5.a.5) from (5.a.6), we have

$$-b_4 + b_5 - b_{10} + b_{11} = \ln f_4 - \ln f_5$$

$$b_4 + b_5 + b_{10} + b_{11} = \ln f_6 - \ln f_5$$

$$\text{Thus, } b_5 + b_{11} = \frac{1}{2} \ln \frac{f_4 f_6}{(f_5)^2} \quad (5.a.13)$$

$$\text{Also, } b_4 + b_{10} = \frac{1}{2} \ln \frac{f_6}{f_4} \quad (5.a.14)$$

Subtracting (5.a.9) from (5.a.7) and substituting (5.a.14) we have

$$2b_6 + 2b_7 = \ln f_9 - \ln f_7 - \ln f_6 + \ln f_4 \quad (5.a.15)$$

Subtracting (5.a.1) from (5.a.3) and substituting (5.a.14), we obtain

$$2b_6 - 2b_7 = \ln f_3 - \ln f_1 - \ln f_6 + \ln f_4 \quad (5.a.16)$$

From (5.a.15) and (5.a.16), we have

$$b6 = \frac{1}{4} \ln \frac{f3f9(f4)^2}{f1f7(f6)^2} \quad (5.a.17)$$

$$\text{and } b7 = \frac{1}{4} \ln \frac{f1f9}{f3f7} \quad (5.a.18).$$

Finally, subtracting (5.a.3) from (5.a.7) and substituting (5.a.11), (5.a.14) and (5.a.17), we

$$\text{have } b8 = \frac{1}{4} \ln \frac{f7f9(f2)^2}{f1f3(f8)^2} \quad (5.a.19)$$

Note that  $b1$  and  $b9$  cannot be determined uniquely from equations (5.a.1) - (5.a.9), since whenever  $b1$  appears in an equation, so does  $b9$ ; that is  $b1 + b9$  appears as a unit in each of the equations. Similarly,  $b5$  and  $b11$  cannot be determined uniquely, nor can  $b4$  and  $b10$ . Only five of the eleven model parameters ( $b2$ ,  $b3$ ,  $b6$ ,  $b7$  and  $b8$ ) have been uniquely determined. Equations (5.a.10) - (5.a.19) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (5.a.1) - (5.a.9).

The last three elements of vector  $Y$  correspond to the partially classified data (and the last nine rows of the  $X$  matrix). Each of these last three elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f10, \dots, f18$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b1 - b2 + b3) [\exp(-b4 + b5 - b6 + b7 - b8) + 1 + \exp(b4 + b5 + b6 - b7 - b8)] = s1 \quad (5.a.20)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (5.a.21)$$

$$\exp(b1 + b2 + b3) [\exp(-b4 + b5 - b6 - b7 + b8) + 1 + \exp(b4 + b5 + b6 + b7 + b8)] = s3 \quad (5.a.22)$$



Here,  $s1 = f10 + f11 + f12$ , the first row marginal for the partially classified data,  $s2 = f13 + f14 + f15$ , the second row marginal for the partially classified data, and  $s3 = f16 + f17 + f18$ , the third row marginal for the partially classified data. It is assumed that we can obtain estimates  $s1$ ,  $s2$ , and  $s3$ , for the last three elements of vector  $Y$ .

Let  $k1 = \exp(-b2+b3)$ ,  $k2 = \exp(b6)$ ,  $k3 = \exp(b7)$ ,  $k4 = \exp(b8)$ , and  $k5 = \exp(b2+b3)$ .

Then equations (5.a.20) - (5.a.22) become

$$\frac{k1k3}{k2k4} \exp(b1 - b4 + b5) + k1 \exp(b1) + \frac{k1k2}{k3k4} \exp(b1 + b4 + b5) = s1 \quad (5.a.23)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (5.a.24)$$

$$\frac{k4k5}{k2k3} \exp(b1 - b4 + b5) + k5 \exp(b1) + k2k3k4k5 \exp(b1 + b4 + b5) = s3 \quad (5.a.25)$$

Equations (5.a.23) - (5.a.25) are three linear equations in  $\exp(b1-b4+b5)$ ,  $\exp(b1)$ , and  $\exp(b1+b4+b5)$ . We solve these equations for  $\exp(b1-b4+b5)$ ,  $\exp(b1)$ , and  $\exp(b1+b4+b5)$  using Cramer's Rule.

$$\exp(b1 - b4 + b5) = \frac{\begin{vmatrix} s1 & k1 & \frac{k1k2}{k3k4} \\ s2 & 1 & 1 \\ s3 & k5 & k2k3k4k5 \end{vmatrix}}{\begin{vmatrix} \frac{k1k3}{k2k4} & k1 & \frac{k1k2}{k3k4} \\ 1 & 1 & 1 \\ \frac{k4k5}{k2k3} & k5 & k2k3k4k5 \end{vmatrix}} =$$

$$\frac{s1(k2k3k4k5 - k5) + s2\left(\frac{k1k2k5}{k3k4} - k1k2k3k4k5\right) + s3\left(k1 - \frac{k1k2}{k3k4}\right)}{k1(k3)^2 k5 + \frac{k1k2k5}{k3k4} + \frac{k1k4k5}{k2k3} - \frac{k1k5}{(k3)^2} - k1k2k3k4k5 - \frac{k1k3k5}{k2k4}} \quad (5.a.26)$$

$$\exp(b_1) = \frac{\begin{vmatrix} \frac{k_1 k_3}{k_2 k_4} & s_1 & \frac{k_1 k_2}{k_3 k_4} \\ 1 & s_2 & 1 \\ \frac{k_4 k_5}{k_2 k_3} & s_3 & k_2 k_3 k_4 k_5 \end{vmatrix}}{\begin{vmatrix} \frac{k_1 k_3}{k_2 k_4} & k_1 & \frac{k_1 k_2}{k_3 k_4} \\ 1 & 1 & 1 \\ \frac{k_4 k_5}{k_2 k_3} & k_5 & k_2 k_3 k_4 k_5 \end{vmatrix}} = \frac{s_1 \left( \frac{k_4 k_5}{k_2 k_3} - k_2 k_3 k_4 k_5 \right) + s_2 \left( k_1 (k_3)^2 k_5 - \frac{k_1 k_5}{(k_3)^2} \right) + s_3 \left( \frac{k_1 k_2}{k_3 k_4} - \frac{k_1 k_3}{k_2 k_4} \right)}{k_1 (k_3)^2 k_5 + \frac{k_1 k_2 k_5}{k_3 k_4} + \frac{k_1 k_4 k_5}{k_2 k_3} - \frac{k_1 k_5}{(k_3)^2} - k_1 k_2 k_3 k_4 k_5 - \frac{k_1 k_3 k_5}{k_2 k_4}} \quad (5.a.27)$$

$$\exp(b_1 + b_4 + b_5) = \frac{\begin{vmatrix} \frac{k_1 k_3}{k_2 k_4} & k_1 & s_1 \\ 1 & 1 & s_2 \\ \frac{k_4 k_5}{k_2 k_3} & k_5 & s_3 \end{vmatrix}}{\begin{vmatrix} \frac{k_1 k_3}{k_2 k_4} & k_1 & \frac{k_1 k_2}{k_3 k_4} \\ 1 & 1 & 1 \\ \frac{k_4 k_5}{k_2 k_3} & k_5 & k_2 k_3 k_4 k_5 \end{vmatrix}} = \frac{s_1 \left( k_5 - \frac{k_4 k_5}{k_2 k_3} \right) + s_2 \left( \frac{k_1 k_4 k_5}{k_2 k_3} - \frac{k_1 k_3 k_5}{k_2 k_4} \right) + s_3 \left( \frac{k_1 k_3}{k_2 k_4} - k_1 \right)}{k_1 (k_3)^2 k_5 + \frac{k_1 k_2 k_5}{k_3 k_4} + \frac{k_1 k_4 k_5}{k_2 k_3} - \frac{k_1 k_5}{(k_3)^2} - k_1 k_2 k_3 k_4 k_5 - \frac{k_1 k_3 k_5}{k_2 k_4}} \quad (5.a.28)$$

From (5.a.27),

$$b_1 = \ln \frac{s_1 \left( \frac{k_4 k_5}{k_2 k_3} - k_2 k_3 k_4 k_5 \right) + s_2 \left( k_1 (k_3)^2 k_5 - \frac{k_1 k_5}{(k_3)^2} \right) + s_3 \left( \frac{k_1 k_2}{k_3 k_4} - \frac{k_1 k_3}{k_2 k_4} \right)}{k_1 (k_3)^2 k_5 + \frac{k_1 k_2 k_5}{k_3 k_4} + \frac{k_1 k_4 k_5}{k_2 k_3} - \frac{k_1 k_5}{(k_3)^2} - k_1 k_2 k_3 k_4 k_5 - \frac{k_1 k_3 k_5}{k_2 k_4}} \quad (5.a.29)$$

Further, dividing (5.a.26) by (5.a.27), and dividing (5.a.28) by (5.a.27), we have

$$\exp(-b_4 + b_5) = \frac{s_1 (k_2 k_3 k_4 k_5 - k_5) + s_2 \left( \frac{k_1 k_2 k_5}{k_3 k_4} - k_1 k_2 k_3 k_4 k_5 \right) + s_3 \left( k_1 - \frac{k_1 k_2}{k_3 k_4} \right)}{s_1 \left( \frac{k_4 k_5}{k_2 k_3} - k_2 k_3 k_4 k_5 \right) + s_2 \left( k_1 (k_3)^2 k_5 - \frac{k_1 k_5}{(k_3)^2} \right) + s_3 \left( \frac{k_1 k_2}{k_3 k_4} - \frac{k_1 k_3}{k_2 k_4} \right)} \quad (5.a.30)$$

and

$$\exp(b_4 + b_5) = \frac{s_1 \left( k_5 - \frac{k_4 k_5}{k_2 k_3} \right) + s_2 \left( \frac{k_1 k_4 k_5}{k_2 k_3} - \frac{k_1 k_3 k_5}{k_2 k_4} \right) + s_3 \left( \frac{k_1 k_3}{k_2 k_4} - k_1 \right)}{s_1 \left( \frac{k_4 k_5}{k_2 k_3} - k_2 k_3 k_4 k_5 \right) + s_2 \left( k_1 (k_3)^2 k_5 - \frac{k_1 k_5}{(k_3)^2} \right) + s_3 \left( \frac{k_1 k_2}{k_3 k_4} - \frac{k_1 k_3}{k_2 k_4} \right)} \quad (5.a.31)$$

Equivalently,

$$-b_4 + b_5 = \ln \frac{s_1 (k_2 k_3 k_4 k_5 - k_5) + s_2 \left( \frac{k_1 k_2 k_5}{k_3 k_4} - k_1 k_2 k_3 k_4 k_5 \right) + s_3 \left( k_1 - \frac{k_1 k_2}{k_3 k_4} \right)}{s_1 \left( \frac{k_4 k_5}{k_2 k_3} - k_2 k_3 k_4 k_5 \right) + s_2 \left( k_1 (k_3)^2 k_5 - \frac{k_1 k_5}{(k_3)^2} \right) + s_3 \left( \frac{k_1 k_2}{k_3 k_4} - \frac{k_1 k_3}{k_2 k_4} \right)} \quad (5.a.32)$$

and

$$b_4 + b_5 = \ln \frac{s_1 \left( k_5 - \frac{k_4 k_5}{k_2 k_3} \right) + s_2 \left( \frac{k_1 k_4 k_5}{k_2 k_3} - \frac{k_1 k_3 k_5}{k_2 k_4} \right) + s_3 \left( \frac{k_1 k_3}{k_2 k_4} - k_1 \right)}{s_1 \left( \frac{k_4 k_5}{k_2 k_3} - k_2 k_3 k_4 k_5 \right) + s_2 \left( k_1 (k_3)^2 k_5 - \frac{k_1 k_5}{(k_3)^2} \right) + s_3 \left( \frac{k_1 k_2}{k_3 k_4} - \frac{k_1 k_3}{k_2 k_4} \right)} \quad (5.a.33)$$

Thus,

$$b_4 = \frac{1}{2} \ln \frac{s_1 \left( k_5 - \frac{k_4 k_5}{k_2 k_3} \right) + s_2 \left( \frac{k_1 k_4 k_5}{k_2 k_3} - \frac{k_1 k_3 k_5}{k_2 k_4} \right) + s_3 \left( \frac{k_1 k_3}{k_2 k_4} - k_1 \right)}{s_1 (k_2 k_3 k_4 k_5 - k_5) + s_2 \left( \frac{k_1 k_2 k_5}{k_3 k_4} - k_1 k_2 k_3 k_4 k_5 \right) + s_3 \left( k_1 - \frac{k_1 k_2}{k_3 k_4} \right)} \quad (5.a.34)$$

$$b_5 = \frac{1}{2} \ln \frac{AB}{C}, \quad (5.a.35)$$

where

$$\begin{aligned} \text{and } A &= \left[ s_1 k_5 (k_2 k_3 k_4 - 1) + s_2 k_2 k_5 \left( \frac{k_1}{k_3 k_4} - k_1 k_3 k_4 \right) + s_3 k_1 \left( 1 - \frac{k_2}{k_3 k_4} \right) \right], \\ B &= \left[ s_1 k_5 \left( 1 - \frac{k_4}{k_2 k_3} \right) + s_2 \frac{k_1 k_5}{k_2} \left( \frac{k_4}{k_3} - \frac{k_3}{k_4} \right) + s_3 k_1 \left( \frac{k_3}{k_2 k_4} - 1 \right) \right], \text{ and} \\ C &= \left[ s_1 \left( \frac{k_4 k_5}{k_2 k_3} - k_2 k_3 k_4 k_5 \right) + s_2 \left( k_1 (k_3)^2 k_5 - \frac{k_1 k_5}{(k_3)^2} \right) + s_3 \left( \frac{k_1 k_2}{k_3 k_4} - \frac{k_1 k_3}{k_2 k_4} \right) \right]^2 \end{aligned}$$

Having determined  $b_1$ ,  $b_4$  and  $b_5$ , we can now substitute these values into equations (5.a.10), (5.a.14) and (5.a.13) respectively, to find  $b_9$ ,  $b_{10}$  and  $b_{11}$ . Thus, all of the  $b$ 's can be uniquely determined. The model, therefore, is identified.

**Type II:**

If all of the interaction terms involving the missing data indicator and the column variable and the missing data indicator and the row variable are included in the model, thirteen parameters would be required. This data set allows for no more than twelve parameters. Thus, Type II nonignorable response in this  $R + C$  model cannot be considered. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type II model is examined below.

Type II: R + C model; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, and the missing data indicator and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1   | -1   | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | -1   | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 1   | 1    | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 1   | -1   | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 1   | 1    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 1   | -1   | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 0   | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 0   | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the quadratic row variable term and the linear column variable term, the seventh and eighth columns, the interactions of the column variable and the linear row variable term, the ninth column, the missing data indicator, and the tenth and eleventh columns, the interactions of the missing data indicator and the linear column variable term and the linear row variable term, respectively. The eleven parameters of the model,

comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I case, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 - b6 + b7 - b8 + b9 - b10 - b11 = \ln f1 \quad (5.b.1)$$

$$b1 - b2 + b3 + b9 - b11 = \ln f2 \quad (5.b.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 - b7 - b8 + b9 + b10 - b11 = \ln f3 \quad (5.b.3)$$

$$b1 - b4 + b5 + b9 - b10 = \ln f4 \quad (5.b.4)$$

$$b1 + b9 = \ln f5 \quad (5.b.5)$$

$$b1 + b4 + b5 + b9 + b10 = \ln f6 \quad (5.b.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 + b9 - b10 + b11 = \ln f7 \quad (5.b.7)$$

$$b1 + b2 + b3 + b9 + b11 = \ln f8 \quad (5.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 + b11 = \ln f9 \quad (5.b.9)$$

Here,  $b1, b2, \dots, b11$  are the eleven parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. In a manner identical to that

used for the R + C model with nonignorable nonresponse Type I (pages 135 -136), we have

$$b1 + b9 = \ln f5 \quad (5.b.10)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (5.b.11)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (5.b.12)$$

$$b4 + b10 = \frac{1}{2} \ln \frac{f6}{f4} \quad (5.b.13)$$

$$b6 = \frac{1}{4} \ln \frac{f3 f9 (f4)^2}{f1 f7 (f6)^2} \quad (5.b.14)$$

$$b7 = \frac{1}{4} \ln \frac{f1 f9}{f3 f7} \quad (5.b.15)$$

$$b7 = \frac{1}{4} \ln \frac{f1 f9}{f3 f7} \quad (5.b.16)$$

$$b8 = \frac{1}{4} \ln \frac{f7 f9 (f2)^2}{f1 f3 (f8)^2} \quad (5.b.17)$$

Note that  $b1$  and  $b9$  cannot be determined uniquely from equations (5.a.1) - (5.a.9), since whenever  $b1$  appears in an equation, so does  $b9$ ; that is  $b1 + b9$  appears as a unit in each of the equations. Similarly,  $b2$  and  $b11$  cannot be determined uniquely, nor can  $b4$  and  $b10$ . Only five of the eleven model parameters ( $b3$ ,  $b5$ ,  $b6$ ,  $b7$  and  $b8$ ) have been uniquely determined. Equations (5.b.10) - (5.b.17) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (5.b.1) - (5.b.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type I. They are given by

$$\exp(b1 - b2 + b3) [\exp(-b4 + b5 - b6 + b7 - b8) + 1 + \exp(b4 + b5 + b6 - b7 - b8)] = s1 \quad (5.b.18)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (5.b.19)$$

$$\exp(b1 + b2 + b3) [\exp(-b4 + b5 - b6 - b7 + b8) + 1 + \exp(b4 + b5 + b6 + b7 + b8)] = s3 \quad (5.b.20)$$



Let  $k1 = \exp(b3)$ ,  $k2 = \exp(b5)$ ,  $k3 = \exp(b6)$ ,  $k4 = \exp(b7)$ , and  $k5 = \exp(b8)$ . Then equations (5.b.18) - (5.b.20) become

$$\frac{k1k2k4}{k3k5} \exp(b1 - b2 - b4) + k1 \exp(b1 - b2) + \frac{k1k2k3}{k4k5} \exp(b1 - b2 + b4) = s1 \quad (5.b.21)$$

$$k2 \exp(b1 - b4) + \exp(b1) + k2 \exp(b1 + b4) = s2 \quad (5.b.22)$$

$$\frac{k1k2k5}{k3k4} \exp(b1 + b2 - b4) + k1 \exp(b1 + b2) + k1k2k3k4k5 \exp(b1 + b2 + b4) = s3 \quad (5.b.23)$$

$$\text{From (5.b.22),} \quad \exp(b1) = \frac{s2}{k2 \exp(-b4) + 1 + k2 \exp(b4)} \quad (5.b.24)$$

Thus, given  $b4$ ,  $b1$  is also uniquely determined.

Dividing (5.b.21) by (5.b.23), we have

$$\exp(2b2) = \frac{\frac{k2k4}{k3k5 \exp(b4)} + 1 + \frac{k2k3 \exp(b4)}{k4k5}}{\frac{s1}{s3} \left( \frac{k2k5}{k3k4 \exp(b4)} + 1 + k2k3k4k5 \exp(b4) \right)} \quad (5.b.25)$$

Since  $\exp(b2)$  must be nonnegative, given  $b4$ ,  $b2$  is uniquely determined.

As noted above, given  $b4$ ,  $b1$  and  $b2$  will be uniquely determined. Using equations (5.b.10), (5.b.14) and (5.b.11),  $b9$ ,  $b10$  and  $b11$  will then also be uniquely determined. Thus, if  $b4$  is uniquely determined, all of the model parameters will be found uniquely, and so the model will be identified. Conversely, if  $b4$  is not uniquely determined, as can be seen from equations (5.b.24) and (5.b.25), neither will  $b1$  and  $b2$  be uniquely determined. In this case, the model will not be identified. Thus, the identification status of this model depends on the uniqueness (or nonuniqueness) of the solution for  $b4$ . We attempt to find  $b4$ .

Multiplying equations (5.b.21) and (5.b.23), we obtain

$$\exp(2b_1) \left( \frac{k_2 k_4}{k_3 k_5} \exp(-b_4) + 1 + \frac{k_2 k_3}{k_4 k_5} \exp(b_4) \right) \left( \frac{k_2 k_5}{k_3 k_4} \exp(-b_4) + 1 + k_2 k_3 k_4 k_5 \exp(b_4) \right) = \frac{s_1 s_3}{(k_1)^2} \quad (5.b.26)$$

Substituting (5.b.24) into (5.b.26), equation (5.b.26) can be rewritten as

$$\begin{aligned} & \exp(-2b_4) \left[ (k_2)^2 \left( \frac{1}{(k_3)^2} - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] + \exp(-b_4) \left[ k_2 \left( \frac{k_4}{k_3 k_5} + \frac{k_5}{k_3 k_4} - 2 \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] \\ & + \left[ (k_2)^2 (k_4)^2 + 1 + \frac{(k_2)^2}{(k_4)^2} - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} (2(k_2)^2 + 1) \right] \\ & + \exp(b_4) \left[ k_2 \left( k_3 k_4 k_5 + \frac{k_3}{k_4 k_5} - 2 \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] + \exp(2b_4) \left[ (k_2)^2 \left( (k_3)^2 - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] = 0 \end{aligned}$$

If the above equation is multiplied by  $\exp(2b_4)$ , and if the substitution  $x = \exp(b_4)$  is made, we have

$$\begin{aligned} & \left[ (k_2)^2 \left( \frac{1}{(k_3)^2} - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] + x \left[ k_2 \left( \frac{k_4}{k_3 k_5} + \frac{k_5}{k_3 k_4} - 2 \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] \\ & + x^2 \left[ (k_2)^2 (k_4)^2 + 1 + \frac{(k_2)^2}{(k_4)^2} - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} (2(k_2)^2 + 1) \right] \\ & + x^3 \left[ k_2 \left( k_3 k_4 k_5 + \frac{k_3}{k_4 k_5} - 2 \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] + x^4 \left[ (k_2)^2 \left( (k_3)^2 - \frac{s_1 s_3}{(s_2)^2 (k_1)^2} \right) \right] = 0 \quad (5.b.27) \end{aligned}$$

Equation (5.b.27) is a fourth degree polynomial equation in  $x = \exp(b_4)$ . Since  $\exp(b_4) > 0$ , we seek only positive solutions to (5.b.27).

Since the exponential function is a 1:1 function, if there is exactly one positive solution for  $x = \exp(b_4)$  in equation (5.b.27),  $b_4$  will be uniquely determined and the model will be identified. If equation (5.b.27) has more than one positive solution or no positive solutions,  $b_4$  will not be uniquely determined and the model will not be identified. As can

be seen from equation (5.b.27), the number of positive solutions for  $x = \exp(b_4)$  depends on the expected frequencies and row marginals estimated by the model. The uniqueness (or lack of uniqueness) of  $b_4$  will be determined empirically for a given data set.

**Type III:**

If all of the interaction terms involving the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable, and the row variable are included in the model, seventeen parameters would be required. This data set allows for no more than twelve parameters. Thus, Type III nonignorable response in this  $R + C$  model cannot be considered. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

Type III: R + C model; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interaction of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1   | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 1   | 1    | -1   | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 1   | -1   | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 1   | 1    | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 1   | -1   | 1    | -1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the quadratic row variable term and the linear column variable term, the seventh and eighth columns, the interactions of the column variable and the linear row variable term, the ninth column, the missing data indicator, and the tenth through twelfth columns, the linear interactions of the missing data indicator and the linear column variable term, the missing data indicator and the linear row variable term, and the three

way interaction of the missing data indicator, the linear column variable term and the linear row variable term, respectively. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 - b6 + b7 - b8 + b9 - b10 - b11 + b12 = \ln f1 \quad (5.c.1)$$

$$b1 - b2 + b3 + b9 - b11 = \ln f2 \quad (5.c.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 - b7 - b8 + b9 + b10 - b11 - b12 = \ln f3 \quad (5.c.3)$$

$$b1 - b4 + b5 + b9 - b10 = \ln f4 \quad (5.c.4)$$

$$b1 + b9 = \ln f5 \quad (5.c.5)$$

$$b1 + b4 + b5 + b9 + b10 = \ln f6 \quad (5.c.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 + b9 - b10 + b11 - b12 = \ln f7 \quad (5.c.7)$$

$$b1 + b2 + b3 + b9 + b11 = \ln f8 \quad (5.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b10 + b11 + b12 = \ln f9 \quad (5.c.9)$$

Here,  $b1$ ,  $b2$ , ... ,  $b12$  are the twelve parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. In a manner identical to that used for the R + C model with nonignorable nonresponse Type I (pages 135 - 136), we have

$$b1 + b9 = \ln f5 \quad (5.c.10)$$

$$b2 + b11 = \frac{1}{2} \ln \frac{f8}{f2} \quad (5.c.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (5.c.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (5.c.13)$$

$$b4 + b10 = \frac{1}{2} \ln \frac{f6}{f4} \quad (5.c.14)$$

$$b6 = \frac{1}{4} \ln \frac{f3 f9 (f4)^2}{f1 f7 (f6)^2} \quad (5.c.15)$$

$$b7 + b12 = \frac{1}{4} \ln \frac{f1 f9}{f3 f7} \quad (5.c.16)$$

$$b8 = \frac{1}{4} \ln \frac{f7 f9 (f2)^2}{f1 f3 (f8)^2} \quad (5.c.17)$$

Note that  $b1$  and  $b9$  cannot be determined uniquely from equations (5.c.1) - (5.c.9), since whenever  $b1$  appears in an equation, so does  $b9$ ; that is  $b1 + b9$  appears as a unit in each of the equations. Similarly,  $b2$  and  $b11$  cannot be determined uniquely, nor can  $b4$  and  $b10$  and  $b7$  and  $b12$ . Only four of the model parameters ( $b3$ ,  $b5$ ,  $b6$  and  $b8$ ) have been uniquely determined. Equations (5.c.10) - (5.c.17) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (5.c.1) - (5.c.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type I and Type II. They are given by

$$\exp(b_1 - b_2 + b_3)[\exp(-b_4 + b_5 - b_6 + b_7 - b_8) + 1 + \exp(b_4 + b_5 + b_6 - b_7 - b_8)] = s_1 \quad (5.c.18)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (5.c.19)$$

$$\exp(b_1 + b_2 + b_3)[\exp(-b_4 + b_5 - b_6 - b_7 + b_8) + 1 + \exp(b_4 + b_5 + b_6 + b_7 + b_8)] = s_3 \quad (5.c.20)$$

The seven equations, (5.c.10), (5.c.11), (5.c.14), (5.c.16) and (5.c.18) - (5.c.20) are available to determine the remaining eight unknown model parameters. These seven equations are insufficient to uniquely solve for these eight unknowns. As the  $b$ 's of this model cannot, therefore, be uniquely determined, the model is not identified.

In summary, for the R + C model hypothesis, the nonignorable nonresponse Type I model is identified, while the Type III model, (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicator and each of the row and column variables will be determined empirically for a given data set.



### Empirical Analysis for Data Set 1

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the R + C models hypothesized to fit Data Set 1 (page 27). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I, and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicator).

#### Data Set 1

##### R + C Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 1744.17         | 255.66       | 68.17           | 2068 |
|                     | single crime    | 223.66          | 64.68        | 26.66           | 315  |
|                     | multiple crimes | 65.17           | 26.66        | 12.17           | 104  |
|                     |                 | 2033            | 347          | 107             | 2487 |

$$\beta = (4.17, -1.13, .24, -1.06, .18, -.17, .39, -.12)$$

Likelihood Ratio fit statistic = 3.28098

Degrees of freedom = 1

**Data Set 1**

R + C Model - One Supplemental Margin

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1720.06         | 252.13       | 67.23           | 2039.42 |
|                     | single crime    | 232.67          | 67.29        | 27.73           | 327.69  |
|                     | multiple crimes | 75.13           | 30.73        | 14.03           | 119.89  |
|                     |                 | 2027.86         | 350.15       | 108.99          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 862.45          | 126.42       | 33.71           | 1022.58 |
|                     | single crime    | 116.66          | 33.74        | 13.91           | 164.31  |
|                     | multiple crimes | 37.67           | 15.41        | 7.04            | 60.11   |
|                     |                 | 1016.78         | 175.57       | 54.66           | 1247    |

$$\beta = (3.52, -1.05, .27, -1.06, .18, -.17, .39, -.12, .69)$$

Likelihood Ratio fit statistic = 12.0167

Degrees of freedom = 3

**Data Set 1**

R + C Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 1744.16         | 255.66       | 68.18           | 2068 |
|                     | single crime    | 223.66          | 64.68        | 26.66           | 315  |
|                     | multiple crimes | 65.17           | 26.66        | 12.17           | 104  |
|                     |                 | 2032.99         | 347          | 107.01          | 2487 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 838.34          | 122.89       | 32.77           | 994  |
|                     | single crime    | 125.68          | 36.35        | 14.98           | 177  |
|                     | multiple crimes | 47.63           | 19.48        | 8.89            | 76   |
|                     |                 | 1011.65         | 178.72       | 56.64           | 1247 |

$$\beta = (3.59, -.92, .30, -1.06, .18, -.17, .39, -.12, .58, -.21, -.05)$$

Likelihood Ratio fit statistic = 3.28097

Degrees of freedom = 1

**Data Set 1**

R + C Model - One Supplemental Margin

Nonignorable Nonresponse - Type I

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1745.42         | 256.24       | 68.38           | 2070.04 |
|                     | single crime    | 221.31          | 62.82        | 25.43           | 309.56  |
|                     | multiple crimes | 66.27           | 27.95        | 13.19           | 107.41  |
|                     |                 | 2033            | 347.01       | 107             | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 637.48          | 238.49       | 116.00          | 991.97 |
|                     | single crime    | 80.83           | 58.47        | 43.15           | 182.44 |
|                     | multiple crimes | 24.21           | 26.01        | 22.37           | 72.59  |
|                     |                 | 742.52          | 322.97       | 181.52          | 1247   |

$$\beta = (4.07, -1.11, .30, -.31, .01, -.13, .41, -.12, .07, -.77, .17)$$

Likelihood Ratio fit statistic = 3.90687

Degrees of freedom = 1

### **Data Set 1**

#### **R + C Model - One Supplemental Margin**

#### **Nonignorable Nonresponse - Type II (modified)**

As indicated on pages 146 and 147, the identification status of the R + C nonignorable nonresponse Type II model, modified to include only the linear interaction terms involving the missing data indicator, depends on the number of solutions to equation (5.b.27). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (5.b.27) giving the following fourth degree polynomial equation:

$$.03 - .11x - .61x^2 - 1.25x^3 - .98x^4 = 0$$

By Descartes' Rule of Signs, since there is only one change of sign in this fourth degree polynomial equation, there is exactly one positive solution for  $x = \exp(b_4)$ . As discussed on pages 146 and 147, the model is therefore identified for the given data set. Given below are the parameter and contingency table estimates, together with the likelihood ratio fit statistic for this identified model.

**Data Set 1**

R + C Model - One Supplemental Margin

Nonignorable Nonresponse - Type II (linear interactions only)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1745.23         | 255.77       | 68.16           | 2069.16 |
|                     | single crime    | 221.65          | 64.37        | 26.65           | 312.67  |
|                     | multiple crimes | 66.12           | 26.86        | 12.19           | 105.17  |
|                     |                 | 2033            | 347          | 107             | 2487    |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 929.30          | 57.14        | 6.39            | 992.84 |
|                     | single crime    | 156.89          | 19.12        | 3.32            | 179.33 |
|                     | multiple crimes | 62.21           | 10.60        | 2.02            | 74.84  |
|                     |                 | 1148.4          | 86.86        | 11.73           | 1247   |

$$\beta = (2.951, -.842, .253, -1.928, .177, -.174, .388, -.122, 1.214, .869, -.285)$$

Likelihood Ratio fit statistic = 3.36627

Degrees of freedom = 1

Symmetry Model - Data Set 1

Type I: symmetry; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | -1  | 1   | 1 | 1 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1 | 2 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 1   | 1   | 1 | 3 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | -1  | 1   | 2 | 1 | 1 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 2 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 1   | 1   | 2 | 3 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | -1  | 1   | 3 | 1 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 3 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 3 | 3 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1 | 1 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1 | 2 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1 | 3 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2 | 1 | 2 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 2 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 2 | 3 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 3 | 1 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 3 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 3 | 3 | 2 |

In this matrix, the first six columns model the symmetry of the cell frequencies, the seventh column, the missing data indicator, and the eighth and ninth columns, the interaction of the missing data indicator and the column variable. The nine parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for all of the association models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model)

matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 24) is the same as for the association models. The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for association models, for the portion of vector  $Y$  corresponding to the completely classified data (and the first nine rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b_1 + b_7 - b_8 + b_9 = \ln f_1 \quad (6.a.1)$$

$$b_2 + b_7 = \ln f_2 \quad (6.a.2)$$

$$b_3 + b_7 + b_8 + b_9 = \ln f_3 \quad (6.a.3)$$

$$b_2 + b_7 - b_8 + b_9 = \ln f_4 \quad (6.a.4)$$

$$b_4 + b_7 = \ln f_5 \quad (6.a.5)$$

$$b_5 + b_7 + b_8 + b_9 = \ln f_6 \quad (6.a.6)$$

$$b_3 + b_7 - b_8 + b_9 = \ln f_7 \quad (6.a.7)$$

$$b_5 + b_7 = \ln f_8 \quad (6.a.8)$$

$$b_6 + b_7 + b_8 + b_9 = \ln f_9 \quad (6.a.9)$$



Here,  $b_1, b_2, \dots, b_9$  are the nine parameters of the matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's.

Substituting (6.a.2) into (6.a.4) and (6.a.8) into (6.a.6), we have

$$-b_8 + b_9 = \ln f_4 - \ln f_2$$

$$b_8 + b_9 = \ln f_6 - \ln f_8$$

$$\text{Thus, } b_8 = \frac{1}{2} \ln \frac{f_2 f_6}{f_4 f_8} \quad (6.a.10)$$

$$\text{and } b_9 = \frac{1}{2} \ln \frac{f_4 f_6}{f_2 f_8} \quad (6.a.11)$$

Substituting (6.a.10) and (6.a.11) into (6.a.1), (6.a.3), (6.a.9), respectively, gives

$$b_1 + b_7 = \ln f_1 - \ln f_4 + \ln f_2 \quad (6.a.12)$$

$$b_3 + b_7 = \ln f_3 - \ln f_6 + \ln f_8 \quad (6.a.13)$$

$$b_6 + b_7 = \ln f_9 - \ln f_6 + \ln f_8 \quad (6.a.14)$$

The last three elements of vector  $Y$  correspond to the partially classified data (and the last nine rows of the  $X$  matrix). Each of these last three elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f_{10}, \dots, f_{18}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b_1) + \exp(b_2) + \exp(b_3) = s_1 \quad (6.a.15)$$

$$\exp(b_2) + \exp(b_4) + \exp(b_5) = s_2 \quad (6.a.16)$$

$$\exp(b_3) + \exp(b_5) + \exp(b_6) = s_3 \quad (6.a.17)$$

Multiplying (6.a.15) by  $\exp(b7)$ , and substituting (6.a.12), (6.a.2) and (6.a.13) into (6.a.15) gives

$$\frac{f1f2}{f4} + f2 + \frac{f3f8}{f6} = s1 \exp(b7) \quad (6.a.18)$$

$$\text{Thus, } \exp(b7) = \frac{f1f2f6 + f2f4f6 + f3f4f8}{s1f4f6}$$

$$\text{or } b7 = \ln\left(\frac{f1f2f6 + f2f4f6 + f3f4f8}{s1f4f6}\right) \quad (6.a.19)$$

The remaining  $b$ 's can now be determined:

Substituting (6.a.19) into (6.a.12) gives

$$b1 = \ln\left(\frac{s1f1f2f6}{f1f2f6 + f2f4f6 + f3f4f8}\right) \quad (6.a.20)$$

Substituting (6.a.19) into (6.a.2) gives

$$b2 = \ln\left(\frac{s1f2f4f6}{f1f2f6 + f2f4f6 + f3f4f8}\right) \quad (6.a.21)$$

Substituting (6.a.19) into (6.a.13) gives

$$b3 = \ln\left(\frac{s1f3f4f8}{f1f2f6 + f2f4f6 + f3f4f8}\right) \quad (6.a.22)$$

Substituting (6.a.19) into (6.a.5) gives

$$b4 = \ln\left(\frac{s1f4f5f6}{f1f2f6 + f2f4f6 + f3f4f8}\right) \quad (6.a.23)$$

Substituting (6.a.19) into (6.a.8) gives

$$b5 = \ln\left(\frac{s1f4f6f8}{f1f2f6 + f2f4f6 + f3f4f8}\right) \quad (6.a.24)$$

Finally, substituting (6.a.19) into (6.a.14) gives

$$b6 = \ln\left(\frac{s1f4f8f9}{f1f2f6 + f2f4f6 + f3f4f8}\right) \quad (6.a.25)$$

Since unique solutions can be found for all of the  $b$ 's in the model, it is identified.

Type II: symmetry; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | -1  | 1   | -1   | 1    | 1 | 1 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | -1   | 1    | 1 | 2 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 1   | 1   | -1   | 1    | 1 | 3 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | -1  | 1   | 0    | 0    | 2 | 1 | 1 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 2 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 1   | 1   | 0    | 0    | 2 | 3 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | -1  | 1   | 1    | 1    | 3 | 1 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 1    | 1    | 3 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 1    | 1    | 3 | 3 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 1 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 3 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 3 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 1 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first six columns model the symmetry of the cell frequencies, the seventh column, the missing data indicator, the eighth and ninth columns, the interaction of the missing data indicator and the column variable, and the tenth and eleventh columns, the interaction of the missing data indicator and the row variable. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I case, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified

data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 + b7 - b8 + b9 - b10 + b11 = \ln f1 \quad (6.b.1)$$

$$b2 + b7 - b10 + b11 = \ln f2 \quad (6.b.2)$$

$$b3 + b7 + b8 + b9 - b10 + b11 = \ln f3 \quad (6.b.3)$$

$$b2 + b7 - b8 + b9 = \ln f4 \quad (6.b.4)$$

$$b4 + b7 = \ln f5 \quad (6.b.5)$$

$$b5 + b7 + b8 + b9 = \ln f6 \quad (6.b.6)$$

$$b3 + b7 - b8 + b9 + b10 + b11 = \ln f7 \quad (6.b.7)$$

$$b5 + b7 + b10 + b11 = \ln f8 \quad (6.b.8)$$

$$b6 + b7 + b8 + b9 + b10 + b11 = \ln f9 \quad (6.b.9)$$

The algebraic equations which correspond to the partially classified data are given by

$$\exp(b1) + \exp(b2) + \exp(b3) = s1 \quad (6.b.10)$$

$$\exp(b2) + \exp(b4) + \exp(b5) = s2 \quad (6.b.11)$$

$$\exp(b3) + \exp(b5) + \exp(b6) = s3 \quad (6.b.12)$$

Here,  $b1, b2, \dots, b11$  are the eleven parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's.

Multiplying (6.b.10) by  $\exp(b7 - b10 + b11)$  and substituting (6.b.1) - (6.b.3) gives

$$f1\exp(b8 - b9) + f2 + f3\exp(-b8 - b9) = s1\exp(b7 - b10 + b11) \quad (6.b.13)$$

Multiplying (6.b.11) by  $\exp(b7)$  and substituting (6.b.4) - (6.b.6) gives

$$f4\exp(b8 - b9) + f5 + f6\exp(-b8 - b9) = s2\exp(b7) \quad (6.b.14)$$

Multiplying (6.b.12) by  $\exp(b7 + b10 + b11)$  and substituting (6.b.7) - (6.b.9) gives

$$f7\exp(b8 - b9) + f8 + f9\exp(-b8 - b9) = s3\exp(b7 - b10 + b11) \quad (6.b.15)$$

From (6.b.2) and (6.b.4),  $b7 - b10 + b11 = \ln f2 - \ln f4 + b7 - b8 + b9$

From (6.b.8) and (6.b.6),  $b7 + b10 + b11 = \ln f8 - \ln f6 + b7 + b8 + b9$ .

Thus, equations (6.b.13) - (6.b.15) become

$$f1\exp(b8 - b9) + f2 + f3\exp(-b8 - b9) = s1 \frac{f2}{f4} \exp(b7 - b8 + b9) \quad (6.b.16)$$

$$f4\exp(b8 - b9) + f5 + f6\exp(-b8 - b9) = s2\exp(b7) \quad (6.b.17)$$

$$f7\exp(b8 - b9) + f8 + f9\exp(-b8 - b9) = s3 \frac{f8}{f6} \exp(b7 + b8 + b9) \quad (6.b.18)$$

Dividing each of (6.b.16) and (6.b.18) by (6.b.17) gives

$$f1\exp(b8 - b9) + f2 + f3\exp(-b8 - b9) = \frac{s1f2}{s2f4} \exp(-b8 + b9) [f4\exp(b8 - b9) + f5 + f6\exp(-b8 - b9)]$$

$$f7\exp(b8 - b9) + f8 + f9\exp(-b8 - b9) = \frac{s3f8}{s2f6} \exp(b8 + b9) [f4\exp(b8 - b9) + f5 + f6\exp(-b8 - b9)]$$

If each of these equations is multiplied by  $\exp(b9)$ , we obtain

$$f1 \exp(b8) + f2 \exp(b9) + f3 \exp(-b8) = \frac{s1 f2}{s2 f4} \exp(-b8 + b9) [f4 \exp(b8) + f5 \exp(b9) + f6 \exp(-b8)] \quad (6.b.19)$$

$$f7 \exp(b8) + f8 \exp(b9) + f9 \exp(-b8) = \frac{s3 f8}{s2 f6} \exp(b8 + b9) [f4 \exp(b8) + f5 \exp(b9) + f6 \exp(-b8)] \quad (6.b.20)$$

Dividing (6.b.19) by (6.b.20), we have

$$f1 \exp(b8) + f2 \exp(b9) + f3 \exp(-b8) = \frac{s1 f2 f6}{s3 f4 f8} \exp(-2b8) [f7 \exp(b8) + f8 \exp(b9) + f9 \exp(-b8)]$$

We can isolate  $\exp(b9)$  from the above equation to obtain

$$\exp(b9) = \frac{-f1 \exp(b8) - \left( f3 - \frac{s1 f2 f6 f7}{s3 f4 f8} \right) \exp(-b8) + \frac{s1 f2 f6 f9}{s3 f4 f8} \exp(-3b8)}{f2 - \frac{s1 f2 f6}{s3 f4} \exp(-2b8)}$$

Thus,

$$b9 = \ln \left( \frac{-f1 \exp(4b8) - \left( f3 - \frac{s1 f2 f6 f7}{s3 f4 f8} \right) \exp(2b8) + \frac{s1 f2 f6 f9}{s3 f4 f8}}{f2 \exp(3b8) - \frac{s1 f2 f6}{s3 f4} \exp(b8)} \right) \quad (6.b.21)$$

As can be seen from (6.b.21), if  $b8$  is determined uniquely, so is  $b9$ .

We now demonstrate that if  $b8$  and  $b9$  are determined uniquely, so will  $b10$  and  $b11$  be uniquely determined.

Subtracting (6.b.4) from (6.b.2) we have

$$b8 - b9 - b10 + b11 = \ln f2 - \ln f4 \quad (6.b.22)$$

Subtracting (6.b.8) from (6.b.6) we have

$$b8 + b9 - b10 - b11 = \ln f6 - \ln f8 \quad (6.b.23)$$

$$\text{Thus, } b8 - b10 = \frac{1}{2} \ln \frac{f2 f6}{f4 f8} \quad (6.b.24)$$

$$\text{and } b9 - b11 = \frac{1}{2} \ln \frac{f4 f6}{f2 f8} \quad (6.b.25)$$

As can be seen, if  $b8$  and  $b9$  are known,  $b10$  and  $b11$  can be determined from (6.b.24) and (6.b.25), respectively. Further, if  $b8$  and  $b9$  are known,  $b7$  can be determined from (6.b.17). Then, if each of  $b7$ ,  $b8$ ,  $b9$ ,  $b10$ , and  $b11$  are known,  $b1$ ,  $b2$ , ...,  $b6$  can be determined from (6.b.1), (6.b.2), (6.b.3), (6.b.5), (6.b.6) and (6.b.9), respectively. Thus, the identification status of this model hinges on the number of solutions for  $b8$ . If there is a unique solution for  $b8$  the model is identified; if there are no solutions or several solutions for  $b8$ , it is not.

We attempt to solve for  $b8$ . Substituting (6.b.21) into (6.b.20), we have

$$f7 \exp(b8) + \frac{-f1f8 \exp(4b8) + \exp(2b8)(Af7f8 - f3f8) + Af8f9}{\exp(b8)(f2 \exp(2b8) - Af8)} + f9 \exp(-b8) = QR$$

where  $Q = \frac{s1f2}{As2f4} \left( \frac{-f1 \exp(4b8) + \exp(2b8)(Af7 - f3) + Af9}{f2 \exp(2b8) - Af8} \right)$  and

$$R = \left( f4 \exp(b8) + \frac{-f1f5 \exp(4b8) + \exp(2b8)(Af5f7 - f3f5) + Af5f9}{\exp(b8)(f2 \exp(2b8) - Af8)} + f6 \exp(-b8) \right).$$

(6.b.26)

Here,  $A = \frac{s1f2f6}{s3f4f8}$ .

When simplified, the above equation can be expressed as a fourth degree polynomial equation in  $x = \exp(2b8)$ :

$$\begin{aligned}
& x^4 \left( \frac{s1f2}{As2f4} \right) [f1f24f4 - (f1)^2 f5] \\
& + x^3 \left[ \begin{aligned} & (f2)^2 f7 - f1f2f8 + \\ & \left( \frac{s1f2}{As2f4} \right) \{ A(2f1f5f7 - f1f4f8 - f2f4f7) + f1f2f6 - 2f1f3f5 + f2f3f4 \} \end{aligned} \right] \\
& + x^2 \left[ \begin{aligned} & A(f1(f8)^2 - f2f7f8) + (f2)^2 f9 - f2f3f8 \\ & + \left( \frac{s1f2}{As2f4} \right) \left\{ \begin{aligned} & A^2(f4f7f8 - f5(f7)^2) + \\ & A(2f1f5f9 - f1f6f8 - f2f4f9 - f2f6f7 + 2f3f5f7 - f3f4f8) \\ & + f2f3f6 - (f3)^2 f5 \end{aligned} \right\} \end{aligned} \right] \\
& + x \left[ \begin{aligned} & A(f3(f8)^2 - f2f8f9) + \\ & \left( \frac{s1f2}{As2f4} \right) \{ A^2(f4f8f9 - 2f5f7f9 + f6f7f8) + A(2f3f5f9 - f2f6f9 - f3f6f8) \} \end{aligned} \right] \\
& + \left( \frac{s1f2}{As2f4} \right) [A^2(f6f8f9 - f5(f9)^2)] = 0
\end{aligned}
\tag{6.b.27}$$

For the model to be identified, we thus seek one positive solution to this fourth degree polynomial equation. The identification status of this model will be determined empirically for a given data set.



Type II: symmetry; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, and the missing data indicator and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | -1  | -1  | 1 | 1 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 1 | 2 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 1   | -1  | 1 | 3 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | -1  | 0   | 2 | 1 | 1 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 2 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 1   | 0   | 2 | 3 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | -1  | 1   | 3 | 1 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 1   | 3 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 3 | 3 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1 | 1 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1 | 2 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1 | 3 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2 | 1 | 2 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 2 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 2 | 3 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 3 | 1 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 3 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 3 | 3 | 2 |

In this matrix, the first six columns model the symmetry of the cell frequencies, the seventh column, the missing data indicator, and the eighth and ninth columns, the interaction of the missing data indicator and the linear column variable term and the missing data indicator and the linear row variable term, respectively. The nine parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 + b7 - b8 - b9 = \ln f1 \quad (6.c.1)$$

$$b2 + b7 - b9 = \ln f2 \quad (6.c.2)$$

$$b3 + b7 + b8 - b9 = \ln f3 \quad (6.c.3)$$

$$b2 + b7 - b8 = \ln f4 \quad (6.c.4)$$

$$b4 + b7 = \ln f5 \quad (6.c.5)$$

$$b5 + b7 + b8 = \ln f6 \quad (6.c.6)$$

$$b3 + b7 - b8 + b9 = \ln f7 \quad (6.c.7)$$

$$b5 + b7 + b9 = \ln f8 \quad (6.c.8)$$

$$b6 + b7 + b8 + b9 = \ln f9 \quad (6.c.9)$$

The partially classified data give

$$\exp(b1) + \exp(b2) + \exp(b3) = s1 \quad (6.c.10)$$

$$\exp(b2) + \exp(b4) + \exp(b5) = s2 \quad (6.c.11)$$

$$\exp(b3) + \exp(b5) + \exp(b6) = s3 \quad (6.c.12)$$

Here,  $b_1, \dots, b_9$  are the nine parameters of matrix  $\beta$ , to be determined. We attempt to solve the equations for the  $b$ 's.

Multiplying (6.c.10) by  $\exp(b_7 - b_9)$  and substituting (6.c.1) - (6.c.3) gives

$$f_1 \exp(b_8) + f_2 + f_3 \exp(-b_8) = s_1 \exp(b_7 - b_9) \quad (6.c.13)$$

Multiplying (6.c.11) by  $\exp(b_7)$  and substituting (6.c.4) - (6.c.6) gives

$$f_4 \exp(b_8) + f_5 + f_6 \exp(-b_8) = s_2 \exp(b_7) \quad (6.c.14)$$

Multiplying (6.c.12) by  $\exp(b_7 + b_9)$  and substituting (6.c.7) - (6.c.9) gives

$$f_7 \exp(b_8) + f_8 + f_9 \exp(-b_8) = s_3 \exp(b_7 + b_9) \quad (6.b.15)$$

From (6.c.2) and (6.c.4),  $b_7 - b_9 = \ln f_2 - \ln f_4 + b_7 - b_8$

From (6.c.8) and (6.c.6),  $b_7 + b_9 = \ln f_8 - \ln f_6 + b_7 + b_8$ .

Thus, equations (6.c.13) - (6.c.15) become

$$f_1 \exp(b_8) + f_2 + f_3 \exp(-b_8) = s_1 \frac{f_2}{f_4} \exp(b_7 - b_8) \quad (6.c.16)$$

$$f_4 \exp(b_8) + f_5 + f_6 \exp(-b_8) = s_2 \exp(b_7) \quad (6.c.17)$$

$$f_7 \exp(b_8) + f_8 + f_9 \exp(-b_8) = s_3 \frac{f_8}{f_6} \exp(b_7 + b_8) \quad (6.c.18)$$

Dividing (6.c.16) by (6.c.17) and dividing (6.c.18) by (6.c.17), respectively, gives

$$f_1 \exp(b_8) + f_2 + f_3 \exp(-b_8) = \frac{s_1 f_2}{s_2 f_4} (f_4 + f_5 \exp(-b_8) + f_6 \exp(-2b_8)) \quad (6.c.19)$$

$$f_7 \exp(b_8) + f_8 + f_9 \exp(-b_8) = \frac{s_3 f_8}{s_2 f_6} (f_4 \exp(2b_8) + f_5 \exp(b_8) + f_6) \quad (6.c.20)$$

Multiplying (6.c.19) by  $\exp(2b8)$  and dividing the resulting equation by (6.c.20), we obtain

$$f1 \exp(3b8) + f2 \exp(2b8) + f3 \exp(b8) = \frac{s1 f2 f6}{s3 f4 f8} (f7 \exp(b8) + f8 + f9 \exp(-b8))$$

If the above equation is multiplied by  $\exp(b8)$ , and the terms rearranged, we have

$$f1 \exp(4b8) + f2 \exp(3b8) + \left( f3 - \frac{s1 f2 f6 f7}{s3 f4 f8} \right) \exp(2b8) - \frac{s1 f2 f6}{s3 f4} \exp(b8) - \frac{s1 f2 f6 f9}{s3 f4 f8} = 0 \quad (6.c.21)$$

Equation (6.c.21) is a fourth degree polynomial equation in  $x = \exp(b8)$ . Since  $\exp(b8) > 0$ , we seek positive real solutions to equation (6.c.21). It can be seen that the coefficients of  $\exp(4b8)$  and  $\exp(3b8)$  are positive, while the coefficient of  $\exp(b8)$  and the constant term are negative. Thus, equation (6.c.21) has exactly one change of sign. By Descartes' Rule of Signs, there must be exactly one positive solution for  $\exp(b8)$ . Since the exponential function is monotonically increasing, and thus a one to one function,  $b8$  is uniquely determined.

From (6.c.17), 
$$\exp(b7) = \frac{f4 \exp(b8) + f5 + f6 \exp(-b8)}{s2}.$$

Since  $b8$  is uniquely determined, so is  $\exp(b7)$ , and thus  $b7$ , itself.

Further, subtracting (6.c.7) from (6.c.3),

$$b8 - b9 = \frac{1}{2} \ln \frac{f3}{f7}$$

Since  $b8$  is uniquely determined, we can thus obtain  $b9$ .

Having found  $b7$ ,  $b8$  and  $b9$ , we can use equations (6.c.1), (6.c.2), (6.c.3), (6.c.5), (6.c.6) and (6.c.9) to find  $b1$ ,  $b2$ , ...,  $b6$ . respectively.

Since all of the  $b$ 's can be uniquely determined, this model is identified.

**Type III:**

As was discussed in the case of the association models, if all the interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable, and the row variable are included in the model, there would be too many parameters for the model, given the data. Thus, Type III nonignorable nonresponse with symmetry modeled for the row and column variables cannot be considered for this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

Type III: symmetry; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interaction of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|---|---|---|
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | -1  | -1  | 1    | 1 | 1 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 1 | 2 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 1   | -1  | -1   | 1 | 3 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | -1  | 0   | 0    | 2 | 1 | 1 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 0    | 2 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 1   | 0   | 0    | 2 | 3 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | -1  | 1   | -1   | 3 | 1 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 1   | 0    | 3 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 1    | 3 | 3 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 1 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 2 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 3 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 1 | 2 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 3 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 3 | 1 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 3 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 3 | 3 | 2 |

In this matrix, the first six columns model symmetry in the cell frequencies, the seventh column, the missing data indicator, and the eighth through tenth columns the interactions of the missing data indicator and the linear column variable term, the missing data indicator and the linear row variable term, and the missing data indicator, the linear row variable term and the linear column variable term, respectively. The ten parameters of the

model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 + b7 - b8 - b9 + b10 = \ln f1 \quad (6.d.1)$$

$$b2 + b7 - b9 = \ln f2 \quad (6.d.2)$$

$$b3 + b7 + b8 - b9 - b10 = \ln f3 \quad (6.d.3)$$

$$b2 + b7 - b8 = \ln f4 \quad (6.d.4)$$

$$b4 + b7 = \ln f5 \quad (6.d.5)$$

$$b5 + b7 + b8 = \ln f6 \quad (6.d.6)$$

$$b3 + b7 - b8 + b9 - b10 = \ln f7 \quad (6.d.7)$$

$$b5 + b7 + b9 = \ln f8 \quad (6.d.8)$$

$$b6 + b7 + b8 + b9 + b10 = \ln f9 \quad (6.d.9)$$

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type I and Type II. They are given by

$$\exp(b1) + \exp(b2) + \exp(b3) = s1 \quad (6.d.10)$$

$$\exp(b2) + \exp(b4) + \exp(b5) = s2 \quad (6.d.11)$$

$$\exp(b3) + \exp(b5) + \exp(b6) = s3 \quad (6.d.12)$$

Here,  $b1, b2, \dots, b10$  are the ten parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's.

Multiplying (6.d.10) by  $\exp(b7 - b9)$  and substituting (6.d.1) - (6.d.3) gives

$$f1\exp(b8 - b10) + f2 + f3\exp(-b8 + b10) = s1\exp(b7 - b9) \quad (6.d.13)$$

Multiplying (6.d.11) by  $\exp(b7)$  and substituting (6.d.4) - (6.d.6) gives

$$f4\exp(b8) + f5 + f6\exp(-b8) = s2\exp(b7) \quad (6.d.14)$$

Multiplying (6.d.12) by  $\exp(b7 + b9)$  and substituting (6.d.7) - (6.d.9) gives

$$f7\exp(b8+b10) + f8 + f9\exp(-b8-b10) = s3\exp(b7+b9) \quad (6.d.15)$$

From (6.d.2) and (6.d.4),  $b7 - b9 = \ln f2 - \ln f4 + b7 - b8$

From (6.d.8) and (6.d.6),  $b7 + b9 = \ln f8 - \ln f6 + b7 + b8$ .

Thus, equations (6.d.13) - (6.d.15) become



$$f_1 \exp(b_8 - b_{10}) + f_2 + f_3 \exp(-b_8 + b_{10}) = s_1 \frac{f_2}{f_4} \exp(b_7 - b_8) \quad (6.d.16)$$

$$f_4 \exp(b_8) + f_5 + f_6 \exp(-b_8) = s_2 \exp(b_7) \quad (6.d.17)$$

$$f_7 \exp(b_8 + b_{10}) + f_8 + f_9 \exp(-b_8 - b_{10}) = s_3 \frac{f_8}{f_6} \exp(b_7 + b_8) \quad (6.d.18)$$

Dividing (6.d.16) by (6.d.17) and (6.d.18) by (6.d.17) gives, respectively,

$$f_1 \exp(b_8 - b_{10}) + f_2 + f_3 \exp(-b_8 + b_{10}) = \frac{s_1 f_2}{s_2 f_4} (f_4 + f_5 \exp(-b_8) + f_6 \exp(-2b_8))$$

$$f_7 \exp(b_8 + b_{10}) + f_8 + f_9 \exp(-b_8 - b_{10}) = \frac{s_3 f_8}{s_2 f_6} (f_4 \exp(2b_8) + f_5 \exp(b_8) + f_6)$$

Multiplying equation (6.d.19) by  $\exp(2b_8)$ , we have

$$f_1 \exp(3b_8 - b_{10}) + f_2 \exp(2b_8) + f_3 \exp(b_8 + b_{10}) = \frac{s_1 f_2}{s_2 f_4} (f_4 \exp(2b_8) + f_5 \exp(b_8) + f_6) \quad (6.d.21)$$

Dividing (6.d.21) by (6.d.20), we obtain

$$\begin{aligned} & \frac{s_3 f_8}{f_6} (f_1 \exp(3b_8 - b_{10}) + f_2 \exp(2b_8) + f_3 \exp(b_8 + b_{10})) \\ &= \frac{s_1 f_2}{f_4} (f_7 \exp(b_8 + b_{10}) + f_8 + f_9 \exp(-b_8 - b_{10})) \end{aligned} \quad (6.d.22)$$

Multiplying (6.d.22) by  $\exp(b_8 + b_{10})$  and rearranging terms, we have

$$\begin{aligned} & \exp(2b_{10}) \left[ \left( \frac{s_3 f_3 f_8}{f_6} - \frac{s_1 f_2 f_7}{f_4} \right) \exp(2b_8) \right] + \\ & \exp(b_{10}) \left[ \frac{s_3 f_2 f_8}{f_6} \exp(3b_8) - \frac{s_1 f_2 f_8}{f_4} \exp(b_8) \right] + \left[ \frac{s_3 f_1 f_8}{f_6} \exp(4b_8) - \frac{s_1 f_2 f_9}{f_4} \right] = 0 \end{aligned} \quad (6.d.23)$$

Equation (6.d.23) is a quadratic equation in  $\exp(b10)$ , expressed in terms of  $b8$ . For the model to be identified, there must be a unique solution for  $b10$  and a unique solution for  $b8$ . For if  $b8$  and  $b10$  are each uniquely determined,  $b7$  could then also be determined from equation (6.d.14). Using these values for  $b7$ ,  $b8$  and  $b10$ ,  $b9$  could then be found from equation (6.d.15) and  $b1$ ,  $b2$ , ... ,  $b6$  could be determined from equations (6.d.1), (6.d.2), (6.d.3), (6.d.5), (6.d.6), and (6.d.9), respectively. With all of the  $b$ 's having been found, the model would then be identified. If  $b8$  and  $b10$  cannot be determined uniquely, the model is not identified. The identification status of this model will be determined empirically for a particular set of data.

In summary, if symmetry of the row and column variables is modeled for this data set, the Type I nonignorable nonresponse model and a modified Type II nonignorable nonresponse model (with only linear interactions concerning the missing data indicator) are identified. The Type III nonignorable nonresponse model is not identified. The identification status of the Type II nonignorable nonresponse model and a modified Type III nonignorable nonresponse model (with only linear interactions involving the missing data indicator) will be determined empirically for a given data set.

### Empirical Analysis for Data Set 1

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the symmetry models hypothesized to fit Data Set 1 (page 27). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I, one hypothesizing nonignorable nonresponse Type II, one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicator) and one hypothesizing nonignorable nonresponse, Type III (modified to include only the linear interaction terms involving the missing data indicator).

#### Data Set 1

##### Symmetry Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 1742            | 244          | 64.5            | 2050.5 |
|                     | single crime    | 244             | 56           | 31              | 331    |
|                     | multiple crimes | 64.5            | 31           | 10              | 105.5  |
|                     |                 | 2050.5          | 331          | 105.5           | 2487   |

$$\beta = (7.46, 5.50, 4.17, 4.03, 3.43, 2.30)$$

Likelihood Ratio fit statistic = 2.16964

Degrees of freedom = 3

**Data Set 1**

Symmetry Model - One Supplemental Margin

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1719.84         | 245.37       | 69.50           | 2034.71 |
|                     | single crime    | 245.37          | 57.38        | 34.09           | 336.84  |
|                     | multiple crimes | 69.50           | 34.09        | 11.86           | 115.45  |
|                     |                 | 2034.71         | 336.84       | 115.45          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 862.34          | 123.03       | 34.85           | 1020.22 |
|                     | single crime    | 123.03          | 28.77        | 17.09           | 168.89  |
|                     | multiple crimes | 34.85           | 17.09        | 5.95            | 57.89   |
|                     |                 | 1020.22         | 168.89       | 57.89           | 1247    |

$$\beta = (6.76, 4.81, 3.55, 3.36, 2.84, 1.78, .69)$$

Likelihood Ratio fit statistic = 10.3431

Degrees of freedom = 5

**Data Set 1**

Symmetry Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 1731.64         | 260.89       | 75.47           | 2068 |
|                     | single crime    | 225.02          | 55.87        | 34.11           | 315  |
|                     | multiple crimes | 60.73           | 31.83        | 11.44           | 104  |
|                     |                 | 2017.39         | 348.59       | 121.02          | 2487 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 842.69          | 126.96       | 36.73           | 1006.38 |
|                     | single crime    | 126.96          | 31.52        | 19.25           | 177.73  |
|                     | multiple crimes | 36.73           | 19.25        | 6.92            | 62.89   |
|                     |                 | 1006.38         | 177.73       | 62.89           | 1247    |

 $\beta = (6.74, 4.84, 3.60, 3.45, 2.96, 1.93, .57, -.11, .04)$ 

Likelihood Ratio fit statistic = 4.64926

Degrees of freedom = 3

**Data Set 1**

Symmetry Model - One Supplemental Margin

Nonignorable Nonresponse - Type I

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1719.68         | 252.47       | 64.42           | 2036.57 |
|                     | single crime    | 240.49          | 57.85        | 30.91           | 329.25  |
|                     | multiple crimes | 72.83           | 36.68        | 11.67           | 121.18  |
|                     |                 | 2033            | 347          | 107             | 2487    |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 863.13          | 120.70       | 36.55           | 1020.39 |
|                     | single crime    | 120.70          | 27.66        | 17.54           | 165.9   |
|                     | multiple crimes | 36.55           | 17.54        | 6.62            | 60.71   |
|                     |                 | 1020.38         | 165.9        | 60.71           | 1247    |

$$\beta = (6.76, 4.79, 3.60, 3.32, 2.86, 1.89, .74, -.06, -.11)$$

Likelihood Ratio fit statistic = 8.81721

Degrees of freedom = 3

### **Data Set 1**

#### **Symmetry Model - One Supplemental Margin**

#### **Nonignorable Nonresponse - Type II**

As indicated on page 168, the identification status of the symmetry model with nonignorable nonresponse Type II depends on the number of solutions to equation (6.b.27). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (6.b.27) giving the following fourth degree polynomial equation in  $x = \exp(2b\delta)$ :

$$-30157124.16x^4 - 35090597x^3 + 33705104.94x^2 + 4982818.47x + 357892.00 = 0$$

By Descartes' Rule of Signs, since there is exactly one change of sign in this fourth degree polynomial equation, there is exactly one positive solution for  $x = \exp(2b\delta)$ . Since  $\exp(2b\delta) > 0$ , there is one solution for  $\exp(2b\delta)$ , and so for  $\exp(b\delta)$ , and so for  $b\delta$ , itself. As is indicated on page 168, the model is therefore identified for the given data set. Given below are the parameter and contingency table estimates, together with the likelihood ratio fit statistic for this identified model.

**Data Set 1**

Symmetry Model - One Supplemental Margin

Nonignorable Nonresponse - Type II

**Completely Classified Data**

|                     |                 | First Interview |              |                 |
|---------------------|-----------------|-----------------|--------------|-----------------|
|                     |                 | crime free      | single crime | multiple crimes |
| Second<br>Interview | crime free      | 1742            | 259.18       | 66.81           |
|                     | single crime    | 228.82          | 56.00        | 30.18           |
|                     | multiple crimes | 62.18           | 31.82        | 10.00           |
|                     |                 | 2033            | 347          | 106.99          |
|                     |                 |                 |              | 2486.99         |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |
|---------------------|-----------------|-----------------|--------------|-----------------|
|                     |                 | crime free      | single crime | multiple crimes |
| Second<br>Interview | crime free      | 826.37          | 123.89       | 43.74           |
|                     | single crime    | 123.89          | 30.55        | 22.55           |
|                     | multiple crimes | 43.74           | 22.55        | 9.71            |
|                     |                 | 994             | 177          | 76              |
|                     |                 |                 |              | 1247            |

$$\beta = (6.717, 4.819, 3.778, 3.419, 3.116, 2.273, .606, -.161, -.153, -.197, -.065)$$

Likelihood Ratio fit statistic = .0693144

Degrees of freedom = 1



**Data Set 1**

Symmetry Model - One Supplemental Margin

Nonignorable Nonresponse - Type II (linear interactions only)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1745.52         | 249.76       | 70.54           | 2065.82 |
|                     | single crime    | 232.65          | 54.41        | 32.29           | 319.35  |
|                     | multiple crimes | 61.21           | 30.07        | 10.55           | 101.83  |
|                     |                 | 2039.38         | 334.24       | 113.38          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 819.12          | 129.94       | 40.69           | 989.75  |
|                     | single crime    | 129.94          | 33.69        | 22.16           | 185.79  |
|                     | multiple crimes | 40.69           | 22.16        | 8.62            | 71.47   |
|                     |                 | 989.75          | 185.79       | 71.47           | 1247.01 |

 $\beta' = (6.71, 4.87, 3.71, 3.52, 3.10, 2.15, .48, -.10, -.17)$ 

Likelihood Ratio fit statistic = 1.74343

Degrees of freedom = 3

### Data Set 1

#### Symmetry Model - One Supplemental Margin

#### Nonignorable Nonresponse - Type III (linear interactions only)

As indicated on page 178, the identification status of the modified nonignorable nonresponse Type III model with symmetry depends on the number of solutions to equations (6.d.23) and the uniqueness or nonuniqueness of  $b_8$ . The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (6.d.23) giving the following quadratic equation in  $x = \exp(b_{10})$  in terms of  $\exp(b_8)$ :

$$[-59072.19 \exp(2b_8)]x^2 + [17872.43 \exp(3b_8) - 31910.79(\exp(b_8))]x + [123212.75 \exp(4b_8) - 10680.52] = 0$$

Since  $\exp(2b_8) > 0$ , the coefficient of  $x^2$  must be negative. By Descartes' Rule of Signs, for there to be two positive solutions for  $x = \exp(b_{10})$ , the coefficient of  $x$ , and the "constant" terms must be positive and negative, respectively (this would result in two sign changes in the quadratic equation). This would require that

$$\exp(2b_8) > 1.7855 \text{ and } \exp(4b_8) < .0867.$$

Clearly, this is impossible. Thus, there is a single solution for  $\exp(b_{10})$  and so for  $b_{10}$ , itself. (We know that there is at least one solution for  $b_{10}$  since the computer program found one.)

When the quadratic equation in  $x = \exp(b_{10})$  is rewritten as a polynomial in  $y = \exp(b_8)$ , we have

$$[123212.75]y^4 + [17872.43 \exp(b10)]y^3 - [59072.19 \exp(2b10)]y^2 - [31910.79 \exp(b10)]y - [10680.52] = 0$$

Since  $\exp(b10) > 0$ , there is exactly one sign change in the above equation for  $y = \exp(b8)$ . Thus, by Descartes' Rule of Signs, there is exactly one positive solution for  $\exp(b8)$ . Therefore,  $b8$  is uniquely determined. As noted on page 178, all of the remaining model parameters can be determined uniquely, and so the model is identified for the given data set. Given below are the parameter and contingency table estimates, together with the likelihood ratio fit statistic for this identified model.

**Data Set 1**

Symmetry Model - One Supplemental Margin

Nonignorable Nonresponse - Type III (linear interactions only)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1742.02         | 252.69       | 69.11           | 2063.82 |
|                     | single crime    | 235.28          | 55.98        | 32.10           | 323.36  |
|                     | multiple crimes | 59.92           | 29.89        | 10.01           | 99.82   |
|                     |                 | 2037.22         | 338.56       | 111.22          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 854.10          | 112.11       | 27.75           | 993.96 |
|                     | single crime    | 112.11          | 36.43        | 28.53           | 177.08 |
|                     | multiple crimes | 27.75           | 28.53        | 19.69           | 75.96  |
|                     |                 | 993.96          | 177.08       | 75.96           | 1247   |

 $\beta = (6.750, 4.720, 3.323, 3.595, 3.351, 2.980, .430, -.312, -.383, -.412)$ 

Likelihood Ratio fit statistic = .814409

Degrees of freedom = 2

### Quasi-Symmetry Model - Data Set 1

As noted previously, Rindskopf (1990) showed that the quasi-symmetry and quasi-independence models were equivalent for the 3x3 table.

Type I: quasi-symmetry; 3x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 1   | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 1   | 0    | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1    | 1    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 1   | -1   | 1    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 1    | 1    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 1   | -1   | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 1   | 0    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth through eighth columns, the symmetry of the off diagonal elements, the ninth column, the missing data indicator, and the tenth and eleventh columns, the missing data indicator by column variable interaction. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the association and symmetry models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 24) is the same as for the association and symmetry models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for the association and symmetry models, for the portion of vector  $Y$  corresponding to the completely classified data (and the first nine rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f1, f2, \dots, f9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b9 - b10 + b11 = \ln f1 \quad (7.a.1)$$

$$b1 - b2 + b3 + b6 + b9 = \ln f2 \quad (7.a.2)$$

$$b1 - b2 + b3 + b4 + b5 + b7 + b9 + b10 + b11 = \ln f3 \quad (7.a.3)$$

$$b1 - b4 + b5 + b6 + b9 - b10 + b11 = \ln f4 \quad (7.a.4)$$

$$b_1 + b_9 = \ln f_5 \quad (7.a.5)$$

$$b_1 + b_4 + b_5 + b_8 + b_9 + b_{10} + b_{11} = \ln f_6 \quad (7.a.6)$$

$$b_1 + b_2 + b_3 - b_4 + b_5 + b_7 + b_9 - b_{10} + b_{11} = \ln f_7 \quad (7.a.7)$$

$$b_1 + b_2 + b_3 + b_8 + b_9 = \ln f_8 \quad (7.a.8)$$

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_9 + b_{10} + b_{11} = \ln f_9 \quad (7.a.9)$$

Here,  $b_1, b_2, \dots, b_{11}$  are the eleven parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's.

$$\text{From (7.a.5), } b_1 + b_9 = \ln f_5 \quad (7.a.10)$$

Adding the difference of (7.a.1) and (7.a.5) to the difference of (7.a.4) and (7.a.2), we have

$$2b_2 - 2b_3 = \ln f_4 - \ln f_2 - \ln f_1 + \ln f_5 \quad (7.a.11)$$

Adding the difference of (7.a.6) and (7.a.8) to the difference of (7.a.9) and (7.a.5), we have

$$2b_2 + 2b_3 = \ln f_9 - \ln f_5 - \ln f_6 + \ln f_8 \quad (7.a.12)$$

$$\text{Thus, } b_2 = \frac{1}{4} \ln \frac{f_4 f_8 f_9}{f_1 f_2 f_6} \quad (7.a.13)$$

$$\text{and } b_3 = \frac{1}{4} \ln \frac{f_1 f_2 f_8 f_9}{f_4 f_6 (f_5)^2} \quad (7.a.14)$$

Subtracting (7.a.2) from (7.a.4) and substituting (7.a.13) and (7.a.14), we obtain

$$-(b_4 + b_{10}) + (b_5 + b_{11}) = \frac{1}{2} \ln \frac{f_1 f_4}{f_2 f_5} \quad (7.a.15)$$

Subtracting (7.a.8) from (7.a.6) and substituting (7.a.13) and (7.a.14), we obtain

$$(b_4 + b_{10}) + (b_5 + b_{11}) = \frac{1}{2} \ln \frac{f_6 f_9}{f_5 f_8} \quad (7.a.16)$$

$$\text{Thus, } b_5 + b_{11} = \frac{1}{4} \ln \frac{f_1 f_4 f_6 f_9}{f_2 f_8 (f_5)^2} \quad (7.a.17)$$

$$\text{and } b_4 + b_{10} = \frac{1}{4} \ln \frac{f_2 f_6 f_9}{f_1 f_4 f_8} \quad (7.a.18)$$

Subtracting (7.a.5) from (7.a.2) and substituting (7.a.13) and (7.a.14), we have

$$b_6 = \frac{1}{2} \ln \frac{f_2 f_4}{f_1 f_5} \quad (7.a.19)$$

Subtracting (7.a.5) from (7.a.8) and substituting (7.a.13) and (7.a.14), we obtain

$$b_8 = \frac{1}{2} \ln \frac{f_6 f_8}{f_5 f_9} \quad (7.a.20)$$

Finally, subtracting (7.a.5) from (7.a.3) and substituting (7.a.13), (7.a.14), (7.a.17) and (7.a.18) gives

$$b_7 = \frac{1}{2} \ln \frac{f_4 f_8 (f_3)^2}{f_1 f_2 f_6 f_9} \quad (7.a.21)$$

Note that  $b_1$  and  $b_9$  cannot be determined uniquely from equations (7.a.1) - (7.a.9), since whenever  $b_1$  appears in an equation, so does  $b_9$ ; that is  $b_1 + b_9$  appears as a unit in each of the equations. Similarly,  $b_5$  and  $b_{11}$  cannot be determined uniquely, nor can  $b_4$  and  $b_{10}$ . Only five of the model parameters ( $b_2$ ,  $b_3$ ,  $b_6$ ,  $b_7$  and  $b_8$ ) have been uniquely determined. Equations (7.a.10) - (7.a.21) provide the most complete solution for the  $b$ 's which can be determined solely from equations (7.a.1) - (7.a.9).

The last three elements of vector  $Y$  correspond to the partially classified data (and the last nine rows of the  $X$  matrix). Each of these last three elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f_{10}$ , ...,  $f_{18}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of



matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3 + b6) + \exp(b1 - b2 + b3 + b4 + b5 + b7) = s1 \quad (7.a.22)$$

$$\exp(b1 - b4 + b5 + b6) + \exp(b1) + \exp(b1 + b4 + b5 + b8) = s2 \quad (7.a.23)$$

$$\exp(b1 + b2 + b3 - b4 + b5 + b7) + \exp(b1 + b2 + b3 + b8) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (7.a.24)$$

Here,  $s1 = f10 + f11 + f12$ , the first row marginal for the partially classified data,  $s2 = f13 + f14 + f15$ , the second row marginal for the partially classified data, and  $s3 = f16 + f17 + f18$ , the third row marginal for the partially classified data. It is assumed that we can obtain estimates  $s1$ ,  $s2$ , and  $s3$ , for the last three elements of vector  $Y$ .

Let  $k1 = \exp(b2 - b3)$ ,  $k2 = \exp(-b2 - b3)$ ,  $k3 = \exp(b6)$ ,  $k4 = \exp(b7)$ , and  $k5 = \exp(b8)$ .

Then equations (7.a.22) - (7.a.24) become

$$\exp(b1 - b4 + b5) + k3 \exp(b1) + k4 \exp(b1 + b4 + b5) = k1s1 \quad (7.a.25)$$

$$k3 \exp(b1 - b4 + b5) + \exp(b1) + k5 \exp(b1 + b4 + b5) = s2 \quad (7.a.26)$$

$$k4 \exp(b1 - b4 + b5) + k5 \exp(b1) + \exp(b1 + b4 + b5) = k2s3 \quad (7.a.27)$$

Equations (7.a.25) - (7.a.27) are three linear equations in  $\exp(b1 - b4 + b5)$ ,  $\exp(b1)$ , and  $\exp(b1 + b4 + b5)$ . We solve these equations for  $\exp(b1 - b4 + b5)$ ,  $\exp(b1)$ , and  $\exp(b1 + b4 + b5)$  using Cramer's Rule.

$$\exp(b1 - b4 + b5) = \frac{\begin{vmatrix} k1s1 & k3 & k4 \\ s2 & 1 & k5 \\ k2s3 & k5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & k3 & k4 \\ k3 & 1 & k5 \\ k4 & k5 & 1 \end{vmatrix}} = \frac{s1(k1 - k1(k5)^2) + s2(k4k5 - k3) + s3(k2k3k5 - k2k4)}{1 + 2k3k4k5 - (k3)^2 - (k4)^2 - (k5)^2} \quad (7.a.28)$$

$$\exp(b_1) = \frac{\begin{vmatrix} 1 & k_1 s_1 & k_4 \\ k_3 & s_2 & k_5 \\ k_4 & k_2 s_3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & k_3 & k_4 \\ k_3 & 1 & k_5 \\ k_4 & k_5 & 1 \end{vmatrix}} = \frac{s_1(k_1 k_4 k_5 - k_1 k_3) + s_2(1 - (k_4)^2) + s_3(k_2 k_3 k_4 - k_2 k_5)}{1 + 2k_3 k_4 k_5 - (k_3)^2 - (k_4)^2 - (k_5)^2} \quad (7.a.29)$$

$$\exp(b_1 + b_4 + b_5) = \frac{\begin{vmatrix} 1 & k_3 & k_1 s_1 \\ k_3 & 1 & s_2 \\ k_4 & k_5 & k_2 s_3 \end{vmatrix}}{\begin{vmatrix} 1 & k_3 & k_4 \\ k_3 & 1 & k_5 \\ k_4 & k_5 & 1 \end{vmatrix}} = \frac{s_1(k_1 k_3 k_5 - k_1 k_4) + s_2(k_3 k_4 - k_5) + s_3(k_2 - k_2(k_3)^2)}{1 + 2k_3 k_4 k_5 - (k_3)^2 - (k_4)^2 - (k_5)^2} \quad (7.a.30)$$

From (7.a.29),

$$b_1 = \ln \frac{s_1(k_1 k_4 k_5 - k_1 k_3) + s_2(1 - (k_4)^2) + s_3(k_2 k_3 k_4 - k_2 k_5)}{1 + 2k_3 k_4 k_5 - (k_3)^2 - (k_4)^2 - (k_5)^2} \quad (7.a.31)$$

Further, dividing (7.a.28) by (7.a.29), and dividing (7.a.30) by (7.a.29), we have

$$\exp(-b_4 + b_5) = \frac{s_1(k_1 - k_1(k_5)^2) + s_2(k_4 k_5 - k_3) + s_3(k_2 k_3 k_5 - k_2 k_4)}{s_1(k_1 k_4 k_5 - k_1 k_3) + s_2(1 - (k_4)^2) + s_3(k_2 k_3 k_4 - k_2 k_5)} \quad (7.a.32)$$

and

$$\exp(b_4 + b_5) = \frac{s_1(k_1 k_3 k_5 - k_1 k_4) + s_2(k_3 k_4 - k_5) + s_3(k_2 - k_2(k_3)^2)}{s_1(k_1 k_4 k_5 - k_1 k_3) + s_2(1 - (k_4)^2) + s_3(k_2 k_3 k_4 - k_2 k_5)} \quad (7.a.33)$$

Equivalently,

$$-b_4 + b_5 = \ln \frac{s_1(k_1 - k_1(k_5)^2) + s_2(k_4 k_5 - k_3) + s_3(k_2 k_3 k_5 - k_2 k_4)}{s_1(k_1 k_4 k_5 - k_1 k_3) + s_2(1 - (k_4)^2) + s_3(k_2 k_3 k_4 - k_2 k_5)}$$

and

$$b_4 + b_5 = \ln \frac{s_1(k_1 k_3 k_5 - k_1 k_4) + s_2(k_3 k_4 - k_5) + s_3(k_2 - k_2(k_3)^2)}{s_1(k_1 k_4 k_5 - k_1 k_3) + s_2(1 - (k_4)^2) + s_3(k_2 k_3 k_4 - k_2 k_5)}$$

Thus,

$$b_4 = \frac{1}{2} \ln \frac{s_1(k_1 k_3 k_5 - k_1 k_4) + s_2(k_3 k_4 - k_5) + s_3(k_2 - k_2(k_3)^2)}{s_1(k_1 - k_1(k_5)^2) + s_2(k_4 k_5 - k_3) + s_3(k_2 k_3 k_5 - k_2 k_4)} \quad (7.a.34)$$

and

$$b_5 = \frac{1}{2} \ln \frac{AB}{C}$$

$$\text{where } A = s_1 k_1 (1 - (k_5)^2) + s_2 (k_4 k_5 - k_3) + s_3 k_2 (k_3 k_5 - k_4)$$

$$B = [s_1 k_1 (k_3 k_5 - k_4) + s_2 (k_3 k_4 - k_5) + s_3 k_2 (1 - (k_3)^2)]$$

$$\text{and } C = [s_1 (k_1 k_4 k_5 - k_1 k_3) + s_2 (1 - (k_4)^2) + s_3 (k_2 k_3 k_4 - k_2 k_5)]^2 \quad (7.a.35)$$

Having determined  $b_1$ ,  $b_4$  and  $b_5$ , we can now substitute these values into equations (7.a.10), (7.a.18) and (7.a.17) respectively, to find  $b_9$ ,  $b_{10}$  and  $b_{11}$ . Thus, all of the  $b$ 's can be uniquely determined. The model, therefore, is identified.

**Type II:**

If all of the interaction terms involving the missing data indicator and the column variable and the missing data indicator and the row variable are included in the model, thirteen parameters would be required. This data set allows for no more than twelve parameters. Thus, Type II nonignorable response for this quasi-symmetry model cannot be considered. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type II model is examined below.

Type II: quasi-symmetry; 3x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the column variable, and the missing data indicator and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 1   | -1   | -1   | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 1   | 0    | -1   | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1    | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 1   | -1   | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 1    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 1   | -1   | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 1   | 0    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth through eighth columns, the symmetry of the off diagonal elements, the ninth column, the missing data indicator, and the tenth and eleventh columns, the interactions of the missing data indicator and the linear column variable term, and the linear row variable term, respectively. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I case, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b9 - b10 - b11 = \ln f1 \quad (7.b.1)$$

$$b1 - b2 + b3 + b6 + b9 - b11 = \ln f2 \quad (7.b.2)$$

$$b1 - b2 + b3 + b4 + b5 + b7 + b9 + b10 - b11 = \ln f3 \quad (7.b.3)$$

$$b1 - b4 + b5 + b6 + b9 - b10 = \ln f4 \quad (7.b.4)$$

$$b1 + b9 = \ln f5 \quad (7.b.5)$$

$$b1 + b4 + b5 + b8 + b9 + b10 = \ln f6 \quad (7.b.6)$$

$$b1 + b2 + b3 - b4 + b5 + b7 + b9 - b10 + b11 = \ln f7 \quad (7.b.7)$$

$$b1 + b2 + b3 + b8 + b9 + b11 = \ln f8 \quad (7.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b9 + b10 + b11 = \ln f9 \quad (7.b.9)$$

Here,  $b1, b2, \dots, b11$  are the eleven parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. In a manner identical to that used for the quasi - symmetry model with nonignorable nonresponse Type I (pages 191 - 192), we have

$$b1 + b9 = \ln f5 \quad (7.b.10)$$

$$b2 + b11 = \frac{1}{4} \ln \frac{f4f8f9}{f1f2f6} \quad (7.b.11)$$

$$b3 = \frac{1}{4} \ln \frac{f1f2f8f9}{f4f6(f5)^2} \quad (7.b.12)$$

$$b5 = \frac{1}{4} \ln \frac{f1f4f6f9}{f2f8(f5)^2} \quad (7.b.13)$$

$$b4 + b10 = \frac{1}{4} \ln \frac{f2f6f9}{f1f4f8} \quad (7.b.14)$$

$$b6 = \frac{1}{2} \ln \frac{f2f4}{f1f5} \quad (7.b.15)$$

$$b7 = \frac{1}{2} \ln \frac{f4f8(f3)^2}{f1f2f6f9} \quad (7.b.16)$$

$$b8 = \frac{1}{2} \ln \frac{f6f8}{f5f9} \quad (7.b.17)$$

Note that  $b1$  and  $b9$  cannot be determined uniquely from equations (7.b.1) - (7.b.9), since whenever  $b1$  appears in an equation, so does  $b9$ ; that is  $b1 + b9$  appears as a unit in each of the equations. Similarly,  $b2$  and  $b11$  cannot be determined uniquely, nor can  $b4$  and  $b10$ . Only five of the eleven model parameters ( $b3$ ,  $b5$ ,  $b6$ ,  $b7$  and  $b8$ ) have been uniquely determined. Equations (7.b.10) - (7.b.17) provide the most complete solution for the  $b$ 's which can be determined solely from equations (7.b.1) - (7.b.9).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type I. They are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3 + b6) + \exp(b1 - b2 + b3 + b4 + b5 + b7) = s1 \quad (7.b.18)$$

$$\exp(b1 - b4 + b5 + b6) + \exp(b1) + \exp(b1 + b4 + b5 + b8) = s2 \quad (7.b.19)$$

$$\exp(b1 + b2 + b3 - b4 + b5 + b7) + \exp(b1 + b2 + b3 + b8) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (7.b.20)$$

Let  $k1 = \exp(b3)$ ,  $k2 = \exp(b5)$ ,  $k3 = \exp(b6)$ ,  $k4 = \exp(b7)$ , and  $k5 = \exp(b8)$ . Then equations (7.b.18) - (7.b.20) become

$$k_2 \exp(b_1 - b_2 - b_4) + k_3 \exp(b_1 - b_2) + k_2 k_4 \exp(b_1 - b_2 + b_4) = \frac{s_1}{k_1} \quad (7.b.21)$$

$$k_2 k_3 \exp(b_1 - b_4) + \exp(b_1) + k_2 k_5 \exp(b_1 + b_4) = s_2 \quad (7.b.22)$$

$$k_2 k_4 \exp(b_1 + b_2 - b_4) + k_5 \exp(b_1 + b_2) + k_2 \exp(b_1 + b_2 + b_4) = \frac{s_3}{k_1} \quad (7.b.23)$$

$$\text{From (7.b.22),} \quad \exp(b_1) = \frac{s_2}{k_2 k_3 \exp(-b_4) + 1 + k_2 k_5 \exp(b_4)} \quad (7.b.24)$$

Thus, given  $b_4$ ,  $b_1$  is also uniquely determined.

Dividing (7.b.21) by (7.b.23), we have

$$\exp(2b_2) = \frac{\frac{k_2}{\exp(b_4)} + k_3 + k_2 k_4 \exp(b_4)}{\frac{s_1}{s_3} \left( \frac{k_2 k_4}{\exp(b_4)} + k_5 + k_2 \exp(b_4) \right)} \quad (7.b.25)$$

Since  $\exp(b_2)$  must be nonnegative, given  $b_4$ ,  $b_2$  is uniquely determined.

As noted above, given  $b_4$ ,  $b_1$  and  $b_2$  will be uniquely determined. Using equations (7.b.10), (7.b.14) and (7.b.11),  $b_9$ ,  $b_{10}$  and  $b_{11}$  will then also be uniquely determined. Thus, if  $b_4$  is uniquely determined, all of the model parameters will be found uniquely, and so the model will be identified. Conversely, if  $b_4$  is not uniquely determined, as can be seen from equations (7.b.24) and (7.b.25), neither will  $b_1$  and  $b_2$  be uniquely determined. In this case, the model will not be identified. Thus, the identification status of this model depends on the uniqueness (or nonuniqueness) of the solution for  $b_4$ . We attempt to find  $b_4$ .

Multiplying equations (7.b.21) and (7.b.23), we obtain

$$\exp(2b_1) (k_2 \exp(-b_4) + k_3 + k_2 k_4 \exp(b_4)) (k_2 k_4 \exp(-b_4) + k_5 + k_2 \exp(b_4)) = \frac{s_1 s_3}{(k_1)^2} \quad (7.b.26)$$



Substituting (7.b.22) into (7.b.26), equation (7.b.26) can be rewritten as

$$\begin{aligned} & \exp(-2b_4) \left[ (k_2)^2 k_4 - \frac{s_1 s_3 (k_2)^2 (k_3)^2}{(k_1)^2 (s_2)^2} \right] + \exp(-b_4) \left[ k_2 k_3 k_4 + k_2 k_5 - 2 \frac{s_1 s_3 k_2 k_3}{(k_1)^2 (s_2)^2} \right] \\ & + \left[ (k_2)^2 (k_4)^2 + k_3 k_5 + (k_2)^2 - \frac{s_1 s_3}{(k_1)^2 (s_2)^2} (2(k_2)^2 k_3 k_5 + 1) \right] \\ & + \exp(b_4) \left[ k_2 k_4 k_5 + k_2 k_3 - 2 \frac{s_1 s_3 k_2 k_5}{(k_1)^2 (s_2)^2} \right] + \exp(2b_4) \left[ (k_2)^2 k_4 - \frac{s_1 s_3 (k_2)^2 (k_5)^2}{(k_1)^2 (s_2)^2} \right] = 0 \end{aligned}$$

If the above equation is multiplied by  $\exp(2b_4)$ , and if the substitution  $x = \exp(b_4)$  is made, we have

$$\begin{aligned} & \left[ (k_2)^2 k_4 - \frac{s_1 s_3 (k_2)^2 (k_3)^2}{(k_1)^2 (s_2)^2} \right] + x \left[ k_2 k_3 k_4 + k_2 k_5 - 2 \frac{s_1 s_3 k_2 k_3}{(k_1)^2 (s_2)^2} \right] \\ & + x^2 \left[ (k_2)^2 (k_4)^2 + k_3 k_5 + (k_2)^2 - \frac{s_1 s_3}{(k_1)^2 (s_2)^2} (2(k_2)^2 k_3 k_5 + 1) \right] \\ & + x^3 \left[ k_2 k_4 k_5 + k_2 k_3 - 2 \frac{s_1 s_3 k_2 k_5}{(k_1)^2 (s_2)^2} \right] + x^4 \left[ (k_2)^2 k_4 - \frac{s_1 s_3 (k_2)^2 (k_5)^2}{(k_1)^2 (s_2)^2} \right] = 0 \quad (7.b.27) \end{aligned}$$

Equation (7.b.27) is a fourth degree polynomial equation in  $x = \exp(b_4)$ . Since  $\exp(b_4) > 0$ , we seek only positive solutions to (7.b.27).

Since the exponential function is a 1:1 function, if there is exactly one positive solution for  $x = \exp(b_4)$  in equation (7.b.27),  $b_4$  will be uniquely determined (and the model will be identified). If equation (7.b.27) has more than one positive solution or no positive solutions,  $b_4$  will not be uniquely determined and the model will not be identified. As can be seen from equation (7.b.27), the number of positive solutions for  $x = \exp(b_4)$  depends on the expected frequencies and row marginals estimated by the model. The uniqueness (or lack of uniqueness) of  $b_4$  will be determined empirically for a given data set.

**Type III:**

If all of the interaction terms involving the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable, and the row variable are included in the model, seventeen parameters would be required. This data set allows for no more than twelve parameters. Thus, Type III nonignorable response in this quasi - symmetry model cannot be considered. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

**Type III: quasi-symmetry; 3x3 table with one supplemental margin**

- modified (with only the linear interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interaction of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 1   | -1   | -1   | 1    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 1   | 0    | -1   | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1    | -1   | -1   | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 1   | -1   | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 1    | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 1   | -1   | 1    | -1   | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 1   | 0    | 1    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 1    | 1    | 1    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth through eighth columns, the symmetry of the off diagonal elements, the ninth column, the missing data indicator, and the tenth through twelfth columns, the linear interactions of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interaction of the missing data indicator, the column variable

and the row variable, respectively. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and estimates  $s1, s2$ , and  $s3$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b9 - b10 - b11 + b12 = \ln f1 \quad (7.c.1)$$

$$b1 - b2 + b3 + b6 + b9 - b11 = \ln f2 \quad (7.c.2)$$

$$b1 - b2 + b3 + b4 + b5 + b7 + b9 + b10 - b11 - b12 = \ln f3 \quad (7.c.3)$$

$$b1 - b4 + b5 + b6 + b9 - b10 = \ln f4 \quad (7.c.4)$$

$$b1 + b9 = \ln f5 \quad (7.c.5)$$

$$b1 + b4 + b5 + b8 + b9 + b10 = \ln f6 \quad (7.c.6)$$

$$b1 + b2 + b3 - b4 + b5 + b7 + b9 - b10 + b11 - b12 = \ln f7 \quad (7.c.7)$$

$$b1 + b2 + b3 + b8 + b9 + b11 = \ln f8 \quad (7.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b9 + b10 + b11 + b12 = \ln f9 \quad (7.c.9)$$

Here,  $b1, b2, \dots, b12$  are the twelve parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's.

From (7.c.5),  $b1 + b9 = \ln f5$  (7.c.10)

Subtracting (7.c.1) from (7.c.9), we have

$$2b2 + 2b4 + 2b10 + 2b11 = \ln f9 - \ln f1$$

Subtracting (7.c.3) from (7.c.7), we have

$$2b2 - 2b4 - 2b10 + 2b11 = \ln f7 - \ln f3$$

Thus,  $b2 + b11 = \frac{1}{4} \ln \frac{f7f9}{f1f3}$  (7.c.11)

and  $b4 + b10 = \frac{1}{4} \ln \frac{f3f9}{f1f7}$  (7.c.12)

If (7.c.10), (7.c.11) and (7.c.12) are substituted into equations (7.c.1) - (7.c.9), six non-redundant equations result. They are as follows:

$$b3 + b5 + b12 = \frac{1}{2} \ln \frac{f1f9}{(f5)^2} \quad (7.c.13)$$

$$b3 + b6 = \frac{1}{4} \ln \frac{(f2)^4 f7f9}{(f5)^4 f1f3} \quad (7.c.14)$$

$$b3 + b5 + b7 - b12 = \frac{1}{2} \ln \frac{f3f7}{(f5)^2} \quad (7.c.15)$$

$$b5 + b6 = \frac{1}{4} \ln \frac{(f4)^4 f3f9}{(f5)^4 f1f7} \quad (7.c.16)$$

$$b5 + b8 + b10 = \frac{1}{4} \ln \frac{(f6)^4 f1f7}{(f5)^4 f3f9} \quad (7.c.17)$$

$$b3 + b8 = \frac{1}{4} \ln \frac{(f8)^4 f1f3}{(f5)^4 f7f9} \quad (7.c.18)$$

Subtracting (7.c.13) from (7.c.15), we have

$$b7 - 2b12 = \frac{1}{2} \ln \frac{f3f7}{f1f9} \quad (7.c.19)$$

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Types I and II. They are given by

$$\exp(b_1 - b_2 + b_3 - b_4 + b_5) + \exp(b_1 - b_2 + b_3 + b_6) + \exp(b_1 - b_2 + b_3 + b_4 + b_5 + b_7) = s_1 \quad (7.c.20)$$

$$\exp(b_1 - b_4 + b_5 + b_6) + \exp(b_1) + \exp(b_1 + b_4 + b_5 + b_8) = s_2 \quad (7.c.21)$$

$$\exp(b_1 + b_2 + b_3 - b_4 + b_5 + b_7) + \exp(b_1 + b_2 + b_3 + b_8) + \exp(b_1 + b_2 + b_3 + b_4 + b_5) = s_3 \quad (7.c.22)$$

The eleven equations (7.c.10) - (7.c.12), (7.c.14), and (7.c.16) - (7.c.22) are available to solve for the twelve unknown model parameters ( $b$ 's). These equations are insufficient to solve for the twelve unknowns. This model, therefore, is not identified.

In summary, if the row and column variables are related by quasi - symmetry, the nonignorable nonresponse Type I model is identified, while the Type III model, (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicator and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 1

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the quasi-symmetry models hypothesized to fit Data Set 1 (page 27). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I, and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicator).

#### Data Set 1

##### Quasi-Symmetry Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 1742            | 259.18       | 66.82           | 2068 |
|                     | single crime    | 228.82          | 56           | 30.18           | 315  |
|                     | multiple crimes | 62.18           | 31.82        | 10              | 104  |
|                     |                 | 2033            | 347          | 107             | 2487 |

$$\beta = (4.03, -1.31, .47, -1.27, .38, -.25, -.72, .27)$$

Likelihood Ratio fit statistic = .0693133

Degrees of freedom = 1

**Data Set 1**

Quasi-Symmetry - One Supplemental Margin

Missing Completely at Random (MCAR).

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1717.92         | 255.61       | 65.89           | 2039.42 |
|                     | single crime    | 238.04          | 58.26        | 31.40           | 327.70  |
|                     | multiple crimes | 71.68           | 36.68        | 11.53           | 119.89  |
|                     |                 | 2027.64         | 350.55       | 108.82          | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 861.38          | 128.16       | 33.04           | 1022.58 |
|                     | single crime    | 119.35          | 29.21        | 15.74           | 164.31  |
|                     | multiple crimes | 35.94           | 18.39        | 5.78            | 60.11   |
|                     |                 | 1016.67         | 175.76       | 54.56           | 1247    |

$$\beta = (3.37, -1.23, .50, -1.27, .38, -.25, -.72, .27, .69)$$

Likelihood Ratio fit statistic = 8.80506

Degrees of freedom = 3



**Data Set 1**

Quasi-Symmetry Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1742.00         | 259.19       | 66.82           | 2068.01 |
|                     | single crime    | 228.82          | 56.00        | 30.18           | 315     |
|                     | multiple crimes | 62.18           | 31.82        | 10.00           | 104     |
|                     |                 | 2033            | 347.01       | 107             | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table)**

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 837.30          | 124.58       | 32.12           | 994  |
|                     | single crime    | 128.57          | 31.47        | 16.96           | 177  |
|                     | multiple crimes | 45.44           | 23.25        | 7.31            | 76   |
|                     |                 | 1011.31         | 179.3        | 56.39           | 1247 |

 $\beta = (3.45, -1.10, .53, -1.27, .38, -.25, -.72, .27, .58, -.21, -.05)$ 

Likelihood Ratio fit statistic = .0693128

Degrees of freedom = 1

**Data Set 1**

Quasi - Symmetry Model - One Supplemental Margin

Nonignorable Nonresponse - Type I

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1742.10    | 259.27       | 66.79           | 2068.16 |
|                     | single crime    | 228.62     | 55.80        | 30.19           | 314.61  |
|                     | multiple crimes | 62.29      | 31.93        | 10.02           | 104.24  |
|                     |                 | 2033.01    | 347          | 107             | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 545.67     | 408.80       | 39.37           | 993.85  |
|                     | single crime    | 71.61      | 87.98        | 17.80           | 177.39  |
|                     | multiple crimes | 19.51      | 50.35        | 5.91            | 75.76   |
|                     |                 | 636.79     | 547.13       | 63.08           | 1246.99 |

$$\beta = (4.48, -1.31, .48, -.96, -.91, -.25, -.72, .27, -.46, -.32, 1.30)$$

Likelihood Ratio fit statistic = .0724859

Degrees of freedom = 1

**Data Set 1**

Quasi - Symmetry Model - One Supplemental Margin

Nonignorable Nonresponse - Type II (modified)

As indicated on page 201, the identification status of the quasi-symmetry nonignorable nonresponse Type II model, modified to include only the linear interaction terms involving the missing data indicator, depends on the number of solutions to equation (7.b.27). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (7.c.27) giving the following fourth degree polynomial equation:

$$-.06+.54x-.90x^2-1.19x^3-2.09x^4=0$$

By Descartes' Rule of Signs, since there are two changes of sign in the polynomial equation, there are exactly two or no positive solutions for  $x = \exp(b_4)$ . Using Newton's Method, there are two positive solutions. As discussed on page 201, since  $b_4$  is not uniquely determined, the model is therefore not identified for the given data set.

### Summary of Results - Data Set 1

Presented below is a summary of the determination of the identification status of nonignorable nonresponse models for Data Set 1, a 3x3 table with one supplemental margin, as well as the empirical analyses for a specific data set of this structure (page 27).

#### Identification Status of Nonignorable Nonresponse Models for Data Set 1

|                     | Type I | Type I<br>(modified) | Type II | Type II<br>(modified) | Type III<br>(modified) |
|---------------------|--------|----------------------|---------|-----------------------|------------------------|
| Independence        | X      | X                    | X       | X                     | X                      |
| Uniform Association | *      | -                    | X       | X                     | X                      |
| Row Effects         | *      | -                    | X       | ?                     | X                      |
| Column Effects      | *      | -                    | X       | ?                     | X                      |
| R + C               | *      | -                    | X       | ?                     | X                      |
| Symmetry            | *      | -                    | ?       | *                     | ?                      |
| Quasi - symmetry    | *      | -                    | X       | ?                     | X                      |

x = not identified

\* = identified

- If a full model is identified, the modified model is not explored for identification (It can be shown, however, that the modified Type I model is identified, except in the case of the independence model)

? The identification status of the model will be examined empirically for a given data set

**Chi Square Goodness of Fit for Tested Models**  
**Data Set 1 (page 27)**

|                        | Complete<br>Data Only | MCAR           | MAR             | NN Type<br>I    | NN I<br>(mo-<br>di-<br>fied) | NN<br>TYPE II | NN<br>TYPE II<br>(modi-<br>fied) | NN<br>Type III<br>(modi-<br>fied) |
|------------------------|-----------------------|----------------|-----------------|-----------------|------------------------------|---------------|----------------------------------|-----------------------------------|
| Independence           | 59.3404<br>(4)        | 68.0761<br>(6) | 59.3404<br>(4)  | X               | X                            | X             | X                                | X                                 |
| Uniform<br>Association | 6.78264<br>(3)        | 15.5184<br>(5) | 6.78264<br>(3)  | 6.78264<br>(3)  | -                            | X             | X                                | X                                 |
| Row Effects            | 4.40433<br>(2)        | 13.1401<br>(4) | 4.40433<br>(2)  | 4.80974<br>(2)  | -                            | X             |                                  | X                                 |
| Column Effects         | 5.31944<br>(2)        | 14.0552<br>(4) | 5.31944<br>(2)  | 5.37683<br>(2)  | -                            | X             |                                  | X                                 |
| R + C                  | 3.28098<br>(1)        | 12.0167<br>(3) | 3.28097<br>(1)  | 3.90687<br>(1)  | -                            | X             |                                  | X                                 |
| Symmetry               | 2.16964<br>(3)        | 10.3431<br>(5) | 4.64926<br>(3)  | 8.81721<br>(3)  | -                            |               | 1.74343<br>(3)                   |                                   |
| Quasi -<br>symmetry    | .0693133<br>(1)       | 8.80506<br>(3) | .0693128<br>(1) | .0724859<br>(1) | -                            | X             |                                  | X                                 |

- Convergence criterion: difference in successive chi square values is <.00001
- Those chi square values in bold type are not significant at the .05 level. Thus, the models for which bold type appears fit the data.
- Models whose identification status was empirically determined have shaded entries in the table. Note that the column effects nognignorable nonresponse

modified Type II model and the quasi-symmetry nonignorable nonresponse modified Type II model were each determined, empirically, to be not identified for this data set.

## DATA SET 2

The second kind of data set explored in this work contains two ordinal variables, each having three categories. In addition to the fully classified data, there are two supplemental margins.

### Determination of the Identification Status of Nonignorable Nonresponse Models

As discussed earlier (page 17), we use the extension of the generalized linear model with composite links as conceptualized by Thompson and Baker (1981) and as applied by Rindskopf (1992) to the loglinear model. We have,  $E(Y) = C \exp(X\beta)$ .

Here,  $Y$  is a column vector of fifteen elements. The first nine are the cell frequencies in the 3x3 completely classified contingency table. The next three are the supplemental row marginals for the partially classified data. The last three are the supplemental column marginals for the partially classified data.

$\beta$  is a vector of parameters whose values are unknown and must be estimated from the data.

$C$  is a matrix of 1's and 0's. For all models for Data Set 2,  $C$  is the same 15 by 27 matrix, whose elements indicate how the observed and unobserved cell counts are to be added to make use of the observed information. More specifically, for these data the matrix  $C$  has the following form:

$$\begin{pmatrix} I_{9 \times 9} & 0_{9 \times 9} & 0_{9 \times 9} \\ 0_{3 \times 9} & A & 0_{3 \times 9} \\ 0_{3 \times 9} & 0_{3 \times 9} & D \end{pmatrix}, \text{ where } I \text{ and } 0 \text{ are the identity and zero matrices, respectively,}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The rows of the  $C$  matrix correspond to the pattern of the given cell counts and the supplemental margins provided. The first nine rows of  $C$  correspond to the completely classified data. The tenth row of  $C$  indicates that three specific cell counts need to be added in order to produce the first observed supplemental row margin. The 1's in the tenth row appear in columns 10, 11, and 12. These correspond to the (unobserved) frequency counts in the first row of the contingency table in which only the supplementary row margin is observed. Thus, the tenth row of  $C$  indicates that these three (unobserved) cell counts must be added to produce the first observed supplementary row margin. Similarly, the eleventh and twelfth rows of  $C$  indicate the need to add the three (unobserved) frequency counts in the second and third rows, respectively, of the contingency table in which only the supplemental row margins are observed. The thirteenth row of  $C$  indicates that three specific cell counts need to be added in order to produce the first observed supplemental column margin. The 1's in the thirteenth row appear in columns 19, 22, and 25. These correspond to the (unobserved) frequency counts in the first column of the contingency table in which only the supplementary column margin is observed. Thus, the thirteenth row of  $C$  indicates that these three (unobserved) cell counts must be added to produce the first observed supplementary column margin. Similarly, the fourteenth and fifteenth rows of  $C$  indicate the need to add the three (unobserved) frequency counts in the second and third columns, respectively, of the contingency table in which only the supplemental column margins are observed.

Finally,  $X$  is the design (or model) matrix. The  $X$  matrix has 27 rows. The first nine of these rows correspond to each of the nine cells in the fully classified 3x3 contingency table. The next nine rows correspond to each of the nine cells in the (hypothetical) 3x3 contingency table for the partially classified data in which only the row marginals are



observed. Finally, the last nine rows of the  $X$  matrix correspond to each of the nine cells in the (hypothetical)  $3 \times 3$  contingency table for the partially classified data in which only the column marginals are observed. The columns of the  $X$  matrix reflect hypotheses concerning the relationships among the row and column variables and the missing data mechanism in the model.

As the data are categorized by ordinal variables, integer coding will be used as follows:

row variable:  $r = -1, 0, 1$

column variable:  $c = -1, 0, 1$

As is usual for ordinal variables which assume three values and which have integer coding, when the row (or column) variable appears in any of the loglinear models, there will be both a linear and a quadratic term.

Two missing data indicators are required. The first will assume the value '1' if the data are completely classified and '0' if partially classified. The second will assume the value '1' if only the column marginal is observed, and '0' otherwise. Data cells corresponding to the completely classified data will be designated by  $k = 1$ , those for which only the supplemental row marginals are observed will be designated by  $k = 2$ , and those for which only the supplemental column marginals are observed will be designated by  $k = 3$ .

The identification status of models for each of the three types of nonignorable nonresponse given on page 21 will be explored for this first data set. Initially, relationships among the data categorized by the two ordinal variables will be described by each of seven models: the independence, uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry models. To the terms of each of these initial models will be added terms which represent the missing data indicators and nonignorable nonresponse for the partially categorized data. The terms representing

nonignorable nonresponse may be of three types (page 21). Thus, for each of the seven models describing relationships among the row and column variables, three nonignorable nonresponse models will be considered for identification. Further, if a model is determined to be not identified, a modified version of this model will be evaluated for identification. This modified model will include only the linear interaction terms involving the missing data indicators. Thus, for each of the seven models describing relationships among the row and column variables, up to six nonignorable nonresponse models may be considered for identification.

As noted above, the columns of the  $X$  matrix reflect hypotheses concerning the relationships among the row and column variables and the missing data mechanism in the model. For each nonignorable nonresponse model considered for identification, the model matrix (page 17) will first be presented. Cell indices will be displayed to the right of this matrix, with  $i$  = row of cell,  $j$  = column of cell, and  $k$  = classification of contingency table ('1' if completely classified, '2' if only row marginals observed, '3' if only column marginals observed). An analysis to determine the identification status of the model will then follow.

### Empirical Analyses for Data Set 2

Empirical model fitting of a specific data set containing two ordinal variables, each having three categories and two supplemental margins will then be performed. This data set is an enlargement of Data Set 1 (page 27) and is the one which was presented by Stasny (1990) from the National Crime Survey. In particular, reported victimization classifications for households sampled in 1977 are given. These classifications include crime free, single crime (one victimization) and multiple crime (two or more victimizations). Most of the sampled households were interviewed twice during 1977,

though some were only interviewed once. The data set includes frequency counts for those who were either interviewed twice in 1977, only at the first interview or only at the second interview. Given below is the contingency table for the distribution of responses at each of the interviews:

|                     |                 | First Interview |              |                 | missing | total |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|-------|
|                     |                 | crime free      | single crime | multiple crimes |         |       |
| Second<br>Interview | crime free      | 1742            | 260          | 66              | 994     | 3062  |
|                     | single crime    | 228             | 56           | 31              | 177     | 492   |
|                     | multiple crimes | 63              | 31           | 10              | 76      | 180   |
|                     | missing         | 716             | 194          | 79              | -       | 989   |
|                     | total           | 2749            | 541          | 186             | 1247    | 4723  |

Using the computer program written in SAS (page 22), likelihood ratio fit statistics and estimates for parameter and expected frequencies will be determined for models hypothesized to fit this data set. Estimates will be obtained for the independence, uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry models using the both the fully categorized data and the supplemental margins, as well as only the completely classified data. This former group of models will include those which hypothesize that the partially classified data are missing completely at random (MCAR), missing at random (MAR), and identified and subject to nonignorable nonresponse.

### Organization of Presentation of Analyses for Data Set 2

For each of the seven types of relationships among the variables considered for Data Set 2 (independence, uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry), the identification status of the nonignorable nonresponse models will first be established. For each type of relationship, this will be followed by the determination of likelihood ratio fit statistics and parameter and frequency estimates for those models involving the complete data only, as well as for those involving the partially classified data. In particular, this latter group will include those nonignorable nonresponse models determined to be identified.

### Independence Model - Data Set 2

Type I: independence; 3x3 table with two supplemental margins

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | 0   | -1  | 1   | 0   | 0    | 0    | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 0    | 0    | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 1   | 0   | -1  | 1   | 0   | 0    | 0    | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 0    | 0    | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | 1   | 0   | -1  | 1   | 0   | 0    | 0    | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 0    | 0    | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 0   | 1   | 0   | 0   | -1  | 1    | 1    | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1   | 1    | 1    | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 1   | 0   | 0   | -1  | 1    | 1    | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 0   | 1   | 1    | 1    | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | -1  | 1    | 1    | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1   | 1    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the missing data indicators, and the eighth through eleventh columns, the

interactions of each of the missing data indicators and the column variable. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order. The row and column variables are each coded to reflect linear and quadratic terms in the model.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $\beta$ 's) of the model.

As discussed earlier (page 17), we use the extension of the generalized linear model with composite links as conceptualized by Thompson and Baker (1981) and as applied by Rindskopf (1992) to the loglinear model. We have,  $E(Y) = C \exp(X\beta)$ . Recall that  $Y$  is a vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a vector of parameters whose values are unknown and must be estimated from the data. The upper left hand  $9 \times 9$  section of matrix  $C$  is an identity matrix (page 216). It can therefore be seen that for the portion of vector  $Y$  corresponding to the completely classified data (and the first nine rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Let us assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b8 + b9 = \ln f1 \quad (8.a.1)$$

$$b1 - b2 + b3 + b6 = \ln f2 \quad (8.a.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b8 + b9 = \ln f3 \quad (8.a.3)$$

$$b1 - b4 + b5 + b6 - b8 + b9 = \ln f4 \quad (8.a.4)$$

$$b1 + b6 = \ln f5 \quad (8.a.5)$$

$$b1 + b4 + b5 + b6 + b8 + b9 = \ln f6 \quad (8.a.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b8 + b9 = \ln f7 \quad (8.a.7)$$

$$b1 + b2 + b3 + b6 = \ln f8 \quad (8.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b8 + b9 = \ln f9 \quad (8.a.9)$$

Here,  $b1 - b6$ ,  $b8$  and  $b9$  are eight of the eleven parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (8.a.1) - (8.a.9) are exactly the same as equations (1.a.1) - (1.a.9) (page 32), but with  $b7$  and  $b8$  of equations (1.a.1) - (1.a.9) replaced by  $b8$  and  $b9$ , respectively, in equations (8.a.1) - (8.a.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (8.a.1) - (8.a.9), as follows:

$$b1 + b9 = \ln f5 \quad (8.a.10)$$

$$b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (8.a.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (8.a.12)$$

$$b5 + b9 = \frac{1}{2} \ln \frac{f1 f3}{(f2)^2} \quad (8.a.13)$$

$$b4 + b8 = \frac{1}{2} \ln \frac{f3}{f1} \quad (8.a.14)$$

Note that  $b1$  and  $b6$  cannot be determined uniquely from equations (8.a.1) - (8.a.9), since whenever  $b1$  appears in an equation, so does  $b6$ ; that is  $b1 + b6$  appears as a unit in each of the equations. Similarly,  $b5$  and  $b9$  cannot be determined uniquely, nor can  $b4$  and  $b8$ . Only two of the eleven model parameters ( $b2$  and  $b3$ ) have been uniquely determined.

The last six elements of vector  $Y$  correspond to the partially classified data (and the last eighteen rows of the  $X$  matrix). Each of these last six elements of vector  $Y$  represents a sum of cell frequencies in a given row of one of the two (hypothetical) contingency tables for the partially classified data. Let  $f_{10}, \dots, f_{18}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with supplemental row marginals, and let  $f_{19}, \dots, f_{27}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with supplemental column marginals. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b_1 - b_2 + b_3 - b_4 + b_5) + \exp(b_1 - b_2 + b_3) + \exp(b_1 - b_2 + b_3 + b_4 + b_5) = s_1 \quad (8.a.15)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (8.a.16)$$

$$\exp(b_1 + b_2 + b_3 - b_4 + b_5) + \exp(b_1 + b_2 + b_3) + \exp(b_1 + b_2 + b_3 + b_4 + b_5) = s_3 \quad (8.a.17)$$

$$\exp(b_1 - b_4 + b_5 + b_7 - b_{10} + b_{11}) [\exp(-b_2 + b_3) + 1 + \exp(b_2 + b_3)] = s_4 \quad (8.a.18)$$

$$\exp(b_1 + b_7) [\exp(-b_2 + b_3) + 1 + \exp(b_2 + b_3)] = s_5 \quad (8.a.19)$$

$$\exp(b_1 + b_4 + b_5 + b_7 + b_{10} + b_{11}) [\exp(-b_2 + b_3) + 1 + \exp(b_2 + b_3)] = s_6 \quad (8.a.20)$$

Here,  $s_1 = f_{10} + f_{11} + f_{12}$ , the first row marginal for the partially classified data,  
 $s_2 = f_{13} + f_{14} + f_{15}$ , the second row marginal for the partially classified data,  
 $s_3 = f_{16} + f_{17} + f_{18}$ , the third row marginal for the partially classified data,  
 $s_4 = f_{19} + f_{22} + f_{25}$ , the first column marginal for the partially classified data,  
 $s_5 = f_{20} + f_{23} + f_{26}$ , the second column marginal for the partially classified data, and  
 $s_6 = f_{21} + f_{24} + f_{27}$ , the third column marginal for the partially classified data.

We assume that we can obtain estimates  $s_1 - s_6$  for the last six elements of vector  $Y$ .



Equations (8.a.15) - (8.a.17) are identical to equations (1.a.15) - (1.a.17). As was shown on page 34, only one relationship between  $b1$ ,  $b4$  and  $b5$  can be determined from equations (8.a.15) - (8.a.17). This relationship, together with equations (8.a.10), (8.a.13), (8.a.14) and (8.a.18) - (8.a.20) provide seven equations to determine the remaining nine unknown model parameters. These seven equations are insufficient to determine the nine unknowns. Thus, the independence model for this data set, with nonignorable nonresponse of Type I is not identified.

While the independence model for this data set with nonignorable nonresponse of Type I is not identified, it may be that the elimination of the quadratic interaction terms of the missing data indicators and the column variable will result in an identified model. This modified nonignorable nonresponse model is considered next.

Type I: independence; 3x3 table with two supplemental margins

- modified (with only the linear interaction terms of the missing data indicators and the column variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | 0   | -1  | 0   |     | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   |     | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | 1   | 0   | 1   | 0   |     | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 1   | 0   | -1  | 0   |     | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   |     | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 1   | 0   | 1   | 0   |     | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | 1   | 0   | -1  | 0   |     | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 0   |     | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   |     | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   |     | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   |     | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 0   |     | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   |     | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |     | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   |     | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 0   |     | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   |     | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   |     | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 0   | 1   | 0   | -1  |     | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   |     | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 1   |     | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 1   | 0   | -1  |     | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   |     | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 1   |     | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | 0   | 1   | 0   | -1  |     | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   |     | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   |     | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh

columns, the missing data indicators, and the eighth and ninth columns, the interactions of the missing data indicators and the linear column variable term. The nine parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

Using the same conceptual rationale and procedures as for Type I nonignorable nonresponse with all missing data indicator and column variable interaction terms, the identification status of this model is now explored. The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Let us assume that we can obtain estimates  $f1, f2, \dots, f9$  for the first nine elements of vector  $Y$ , and estimates  $s1 - s6$ , for the row and column marginals. Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b8 = \ln f1 \quad (8.b.1)$$

$$b1 - b2 + b3 + b6 = \ln f2 \quad (8.b.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b8 = \ln f3 \quad (8.b.3)$$

$$b1 - b4 + b5 + b6 - b8 = \ln f4 \quad (8.b.4)$$

$$b1 + b6 = \ln f5 \quad (8.b.5)$$

$$b1 + b4 + b5 + b6 + b8 = \ln f6 \quad (8.b.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b8 = \ln f7 \quad (8.b.7)$$

$$b1 + b2 + b3 + b6 = \ln f8 \quad (8.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b8 = \ln f9 \quad (8.b.9)$$

Here,  $b1$  -  $b6$ , and  $b8$  are seven of the nine parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (8.b.1) - (8.b.9) are exactly the same as equations (1.b.1) - (1.b.9) (page 36), but with  $b7$  of equations (1.b.1) - (1.b.9) replaced by  $b8$  in equations (8.b.1) - (8.b.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (8.b.1) - (8.b.9), as follows:

$$b1 + b6 = \ln f5 \quad (8.b.10)$$

$$b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (8.b.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (8.b.12)$$

$$b5 = \frac{1}{2} \ln \frac{f1 f3}{(f2)^2} \quad (8.b.13)$$

$$b4 + b8 = \frac{1}{2} \ln \frac{f3}{f1} \quad (8.b.14)$$

Note that  $b1$  and  $b6$  cannot be determined uniquely from equations (8.b.1) - (8.b.9), nor can  $b4$  and  $b8$ . Only three of the nine model parameters ( $b2$ ,  $b3$  and  $b5$ ) have been uniquely determined. Equations (8.b.10) - (8.b.14) provide the most complete solution for the  $b$ 's which can be determined solely from equations (8.b.1) - (8.b.9).

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5) = s1 \quad (8.b.15)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (8.b.16)$$

$$\exp(b1 + b2 + b3 - b4 + b5) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (8.b.17)$$

$$\exp(b1 - b4 + b5 + b7 - b9)[\exp(-b2 + b3) + 1 + \exp(b2 + b3)] = s4 \quad (8.b.18)$$

$$\exp(b1 + b7)[\exp(-b2 + b3) + 1 + \exp(b2 + b3)] = s5 \quad (8.b.19)$$

$$\exp(b1 + b4 + b5 + b7 + b9)[\exp(-b2 + b3) + 1 + \exp(b2 + b3)] = s6 \quad (8.b.20)$$

Equations (8.b.15) - (8.b.17) are identical to equations (1.b.15) - (1.b.17). As was shown on page 37, these three equations produce only one relationship between  $b1$  and  $b4$ .

Further, when the uniquely determined values for  $b2$ ,  $b3$  and  $b5$  from equations (8.b.11), (8.b.12) and (8.b.13) are substituted into equations (8.b.18) - (8.b.20), we have

$$\exp(b1 - b4 + b7 - b9) = \frac{s4 \frac{f2}{\sqrt{f1f3}}}{\frac{f2}{f5} + 1 + \frac{f8}{f5}} \quad (8.b.21)$$

$$\exp(b1 + b7) = \frac{s5}{\frac{f2}{f5} + 1 + \frac{f8}{f5}} \quad (8.b.22)$$

$$\exp(b1 + b4 + b7 + b9) = \frac{s6 \frac{f2}{\sqrt{f1f3}}}{\frac{f2}{f5} + 1 + \frac{f8}{f5}} \quad (8.b.23)$$

Dividing (8.b.21) by (8.b.22) and dividing (8.b.23) by (8.b.22), respectively, we have

$$\exp(-b4 - b9) = \frac{s4 f2}{s5 \sqrt{f1f3}}$$

$$\exp(b4 + b9) = \frac{s6 f2}{s5 \sqrt{f1f3}}$$

These two equations give one relationship between  $b4$  and  $b9$ . This, together with the one relationship between  $b1$  and  $b4$  from equations (8.b.15) - (8.b.17) and equations (8.b.10) and (8.b.14) are insufficient to uniquely determine the remaining six unknown model parameters. This model is therefore not identified.

We next consider the identification status of the independence model with nonignorable nonresponse Type II.

Type II: independence; 3x3 table with two supplemental margins

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | 0   | -1  | 1   | 0    | 0    | -1   | 1    | 0    | 0    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | -1   | 1    | 0    | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0    | 0    | -1   | 1    | 0    | 0    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | -1  | 1   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 1   | 0   | 1   | 1   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 1   | 0   | -1  | 1   | 0    | 0    | 1    | 1    | 0    | 0    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 1    | 1    | 0    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0    | 0    | 1    | 1    | 0    | 0    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 1   | 0   | 0   | -1   | 1    | 0    | 0    | -1   | 1    | 1 | 1 | 3 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | -1   | 1    | 1 | 2 | 3 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1    | 1    | 0    | 0    | -1   | 1    | 1 | 3 | 3 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 1   | 0   | 0   | -1   | 1    | 0    | 0    | 0    | 0    | 2 | 1 | 3 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 3 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 0   | 1    | 1    | 0    | 0    | 0    | 0    | 2 | 3 | 3 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | -1   | 1    | 0    | 0    | 1    | 1    | 3 | 1 | 3 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 1    | 1    | 3 | 2 | 3 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1    | 1    | 0    | 0    | 1    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the missing data indicators, the eighth through eleventh columns, the interactions of the missing data indicators and the column variable, and the twelfth

through fifteenth columns, the interactions of the missing data indicators and the row variable. The fifteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b8 + b9 - b12 + b13 = \ln f1 \quad (8.c.1)$$

$$b1 - b2 + b3 + b6 - b12 + b13 = \ln f2 \quad (8.c.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b8 + b9 - b12 + b13 = \ln f3 \quad (8.c.3)$$

$$b1 - b4 + b5 + b6 - b8 + b9 = \ln f4 \quad (8.c.4)$$

$$b1 + b6 = \ln f5 \quad (8.c.5)$$

$$b1 + b4 + b5 + b6 + b8 + b9 = \ln f6 \quad (8.c.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b8 + b9 + b12 + b13 = \ln f7 \quad (8.c.7)$$

$$b1 + b2 + b3 + b6 + b12 + b13 = \ln f8 \quad (8.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b8 + b9 + b12 + b13 = \ln f9 \quad (8.c.9)$$

Here,  $b1 - b6, b8, b9, b12$  and  $b13$  are ten of the fifteen parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (8.c.1) - (8.c.9) are exactly the same as equations (1.c.1) - (1.c.9) (page 40), but with  $b7, b8, b9$  and  $b10$  of equations (1.c.1) -

(1.c.9) replaced by  $b_8$ ,  $b_9$ ,  $b_{12}$  and  $b_{13}$ , respectively, in equations (8.c.1) - (8.c.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (8.c.1) - (8.c.9), as follows:

$$b_1 + b_6 = \ln f_5 \quad (8.c.10)$$

$$b_2 + b_{12} = \frac{1}{2} \ln \frac{f_8}{f_2} \quad (8.c.11)$$

$$b_3 + b_{13} = \frac{1}{2} \ln \frac{f_2 f_8}{(f_5)^2} \quad (8.c.12)$$

$$b_5 + b_9 = \frac{1}{2} \ln \frac{f_1 f_3}{(f_2)^2} \quad (8.c.13)$$

$$b_4 + b_8 = \frac{1}{2} \ln \frac{f_3}{f_1} \quad (8.c.14)$$

Note that  $b_1$  and  $b_6$  cannot be determined uniquely from equations (8.c.1) - (8.c.9), nor can  $b_2$  and  $b_{12}$ ,  $b_3$  and  $b_{13}$ ,  $b_4$  and  $b_8$ , and  $b_5$  and  $b_9$ . None of the fifteen model parameters has been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b_1 - b_2 + b_3 - b_4 + b_5) + \exp(b_1 - b_2 + b_3) + \exp(b_1 - b_2 + b_3 + b_4 + b_5) = s_1 \quad (8.c.15)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (8.c.16)$$

$$\exp(b_1 + b_2 + b_3 - b_4 + b_5) + \exp(b_1 + b_2 + b_3) + \exp(b_1 + b_2 + b_3 + b_4 + b_5) = s_3 \quad (8.c.17)$$

$$\exp(b_1 - b_4 + b_5 + b_7 - b_{10} + b_{11}) [\exp(-b_2 + b_3 - b_{14} + b_{15}) + 1 + \exp(b_2 + b_3 + b_{14} + b_{15})] = s_4 \quad (8.c.18)$$

$$\exp(b_1 + b_7) [\exp(-b_2 + b_3 - b_{14} + b_{15}) + 1 + \exp(b_2 + b_3 + b_{14} + b_{15})] = s_5 \quad (8.c.19)$$

$$\exp(b_1 + b_4 + b_5 + b_7 + b_{10} + b_{11}) [\exp(-b_2 + b_3 - b_{14} + b_{15}) + 1 + \exp(b_2 + b_3 + b_{14} + b_{15})] = s_6 \quad (8.c.20)$$

Equations (8.c.15) - (8.c.17) are identical to equations (1.c.15) - (1.c.17). As was shown on pages 41 and 42, these three equations allow for the unique determination of  $b_2$  and



$b_3$ , but not for  $b_1$ ,  $b_4$  and  $b_5$ . In fact, when the values for  $b_2$  and  $b_3$  are substituted into equations (8.c.15) - (8.c.17), only one relationship between  $b_1$ ,  $b_4$  and  $b_5$  is obtained.

Further, the unknowns  $b_7$ ,  $b_{10}$ ,  $b_{11}$ ,  $b_{14}$  and  $b_{15}$  appear only in equations (8.c.18) - (8.c.20). Three equations are inadequate to solve for five unknowns, so these five  $b$ 's cannot all be uniquely determined. Thus, this model is not identified.

While the independence model for this data set with nonignorable nonresponse of Type II is not identified, it may be that the elimination of the quadratic interaction terms of the missing data indicators and the column variable, and the quadratic interaction terms of the missing data indicators and the row variable will result in an identified model. This modified nonignorable nonresponse model is considered next.

Type II: independence; 3x3 table with two supplemental margins

- modified (with only the linear interaction terms of the missing data indicator and the column variable, and the missing data indicator and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | 0   | -1  | 0   | -1  | 0    |      | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | -1  | 0    |      | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | 1   | 0   | 1   | 0   | -1  | 0    |      | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 1   | 0   | -1  | 0   | 0   | 0    |      | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    |      | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    |      | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | 1   | 0   | -1  | 0   | 1   | 0    |      | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0    |      | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0    |      | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    |      | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    |      | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    |      | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    |      | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    |      | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    |      | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    |      | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    |      | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    |      | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 0   | 1   | 0   | -1  | 0   | -1   |      | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | -1   |      | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | -1   |      | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 1   | 0   | -1  | 0   | 0    |      | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    |      | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 0    |      | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | 0   | 1   | 0   | -1  | 0   | 1    |      | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1    |      | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 1    |      | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh

columns, the missing data indicators, the eighth and ninth columns, the interaction terms of the missing data indicators and the linear column variable term, and the tenth and eleventh columns, the interaction terms of the missing data indicators and the linear row variable term. The eleven parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type II case with all missing data indicator and column variable interaction terms, and all missing data indicator and row variable interaction terms, that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b8 - b10 = \ln f1 \quad (8.d.1)$$

$$b1 - b2 + b3 + b6 - b10 = \ln f2 \quad (8.d.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b8 - b10 = \ln f3 \quad (8.d.3)$$

$$b1 - b4 + b5 + b6 - b8 = \ln f4 \quad (8.d.4)$$

$$b1 + b6 = \ln f5 \quad (8.d.5)$$

$$b1 + b4 + b5 + b6 + b8 = \ln f6 \quad (8.d.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b8 + b10 = \ln f7 \quad (8.d.7)$$

$$b1 + b2 + b3 + b6 + b10 = \ln f8 \quad (8.d.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b8 + b10 = \ln f9 \quad (8.d.9)$$

Here,  $b1 - b6$ ,  $b8$  and  $b10$  are eight of the eleven parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (8.d.1) - (8.d.9) are exactly the same as equations (1.d.1) - (1.d.9) (page 44), but with  $b7$  and  $b8$  of equations (1.d.1) - (1.d.9) replaced by  $b8$  and  $b10$ , respectively, in equations (8.d.1) - (8.d.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (8.d.1) - (8.d.9), as follows:

$$b1 + b6 = \ln f5 \quad (8.d.10)$$

$$b2 + b10 = \frac{1}{2} \ln \frac{f8}{f2} \quad (8.d.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (8.d.12)$$

$$b5 = \frac{1}{2} \ln \frac{f1 f3}{(f2)^2} \quad (8.d.13)$$

$$b4 + b8 = \frac{1}{2} \ln \frac{f3}{f1} \quad (8.d.14)$$

Note that  $b1$  and  $b6$  cannot be determined uniquely from equations (8.d.1) - (8.d.9), nor can  $b2$  and  $b10$  and  $b4$  and  $b8$ . Only two of the eleven model parameters ( $b3$  and  $b5$ ) have been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5) = s1 \quad (8.d.15)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (8.d.16)$$

$$\exp(b1 + b2 + b3 - b4 + b5) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (8.d.17)$$

$$\exp(b1 - b4 + b5 + b7 - b9)[\exp(-b2 + b3 - b11) + 1 + \exp(b2 + b3 + b11)] = s4 \quad (8.d.18)$$

$$\exp(b1 + b7)[\exp(-b2 + b3 - b11) + 1 + \exp(b2 + b3 + b11)] = s5 \quad (8.d.19)$$

$$\exp(b1 + b4 + b5 + b7 + b9)[\exp(-b2 + b3 - b11) + 1 + \exp(b2 + b3 + b11)] = s6 \quad (8.d.20)$$

Equations (8.d.15) - (8.d.17) are identical to equations (1.d.15) - (1.d.17). As was indicated on page 46, these three equations allow for the unique determination of  $b2$  (and  $b10$ ). Thus, four of the eleven model parameters ( $b2$ ,  $b3$ ,  $b5$  and  $b10$ ) can be uniquely determined. Equations (8.d.15) - (8.d.17) produce only one relationship between  $b1$  and  $b4$ , however. This relationship, together with equations (8.d.10), (8.d.14) and (8.d.18) - (8.d.20) provide six equations with which to determine the remaining seven unknown model parameters. These six equations are insufficient to solve for the seven unknowns. Thus, this model is not identified.

**Type III:**

A total of fifteen observed data frequencies (nine from the completely classified data, three from the row marginals, and three from the column marginals) are assumed to be estimable. Thus, a model for this data set can have no more than fifteen parameters. If all the interaction terms of the missing data indicators and the column variable, the missing data indicators and the row variable, and the three way interactions of the missing data indicators, the column variable, and the row variable are included in the loglinear model, twenty three parameters are required. Thus, Type III nonignorable nonresponse with independence for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type III model is examined below.

Type III: independence; 3x3 table with two supplemental margins

- modified (with only the linear interaction terms of the missing data indicators and the column variable, the missing data indicators and the row variable, and the missing data indicators, the row variable and the column variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | 0   | -1  | 0   | -1  | 0    | 1    | 0    | 1    | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | -1  | 0    | 0    | 0    | 0    | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | 1   | 0   | 1   | 0   | -1  | 0    | -1   | 0    | 0    | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 1   | 0   | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | 1   | 0   | -1  | 0   | 1   | 0    | -1   | 0    | 0    | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0    | 1    | 0    | 0    | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 0   | 1   | 0   | -1  | 0   | -1   | 0    | 1    | 0    | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | -1   | 0    | 0    | 0    | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | -1   | 0    | -1   | 0    | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 1   | 0   | -1  | 0   | 0    | 0    | 0    | 0    | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | 0   | 1   | 0   | -1  | 0   | 1    | 0    | -1   | 0    | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1    | 0    | 0    | 0    | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 0    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh

columns, the missing data indicators, the eighth and ninth columns, the interaction terms of the missing data indicators with the linear column variable term, the tenth and eleventh columns, the interaction terms of the missing data indicators and the linear row variable term, and the twelfth and thirteenth columns, the three way interaction terms of the missing data indicators with the linear column variable term and the linear row variable term. The thirteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I and Type II cases that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b8 - b10 + b12 = \ln f1 \quad (8.e.1)$$

$$b1 - b2 + b3 + b6 - b10 = \ln f2 \quad (8.e.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 + b8 - b10 - b12 = \ln f3 \quad (8.e.3)$$

$$b1 - b4 + b5 + b6 - b8 = \ln f4 \quad (8.e.4)$$

$$b1 + b6 = \ln f5 \quad (8.e.5)$$



$$b1 + b4 + b5 + b6 + b8 = \ln f6 \quad (8.e.6)$$

$$b1 + b2 + b3 - b4 + b5 + b6 - b8 + b10 - b12 = \ln f7 \quad (8.e.7)$$

$$b1 + b2 + b3 + b6 + b10 = \ln f8 \quad (8.e.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b8 + b10 + b12 = \ln f9 \quad (8.e.9)$$

Here,  $b1 - b6$ ,  $b8$ ,  $b10$  and  $b12$  are nine of the thirteen parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (8.e.1) - (8.e.9) are exactly the same as equations (1.e.1) - (1.e.9) (page 49), but with  $b7$ ,  $b8$  and  $b9$  of equations (1.e.1) - (1.e.9) replaced by  $b8$ ,  $b10$  and  $b12$ , respectively, in equations (8.e.1) - (8.e.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (8.e.1) - (8.e.9), as follows:

$$b1 + b6 = \ln f5 \quad (8.e.10)$$

$$b2 + b10 = \frac{1}{2} \ln \frac{f8}{f2} \quad (8.e.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (8.e.12)$$

$$b5 = \frac{1}{2} \ln \frac{f1 f3}{(f2)^2} \quad (8.e.13)$$

$$b4 + b8 = \frac{1}{2} \ln \frac{f6}{f4} \quad (8.e.14)$$

$$b12 = \frac{1}{2} \ln \frac{f1 f6}{f3 f4} \quad (8.e.15)$$

Note that  $b1$  and  $b6$  cannot be determined uniquely from equations (8.e.1) - (8.e.9), nor can  $b2$  and  $b10$ , and  $b4$  and  $b8$ . Only three of the thirteen model parameters ( $b3$ ,  $b5$  and  $b12$ ) have been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5) = s1 \quad (8.e.16)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (8.e.17)$$

$$\exp(b1 + b2 + b3 - b4 + b5) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (8.e.18)$$

$$\exp(b1 - b4 + b5 + b7 - b9) \{ \exp(-b2 + b3 - b11 + b13) + 1 + \exp(b2 + b3 + b11 - b13) \} = s4 \quad (8.e.19)$$

$$\exp(b1 + b7) \{ \exp(-b2 + b3 - b11) + 1 + \exp(b2 + b3 + b11) \} = s5 \quad (8.e.20)$$

$$\exp(b1 + b4 + b5 + b7 + b9) \{ \exp(-b2 + b3 - b11 - b13) + 1 + \exp(b2 + b3 + b11 + b13) \} = s6 \quad (8.e.21)$$

Equations (8.e.16) - (8.e.18) are identical to equations (1.e.16) - (1.e.18). As was shown on page 51, these three equations can be used to find  $b2$  (and  $b10$ ) uniquely. Thus, five of the thirteen model parameters ( $b2$ ,  $b3$ ,  $b5$ ,  $b10$  and  $b12$ ) can be uniquely determined. Equations (8.e.16) - (8.e.18) produce only one relationship between  $b1$  and  $b4$ , however. This relationship, together with equations (8.e.10), (8.e.14) and (8.e.19) - (8.e.21) provide six equations with which to determine the remaining eight unknown model parameters. These six equations are insufficient to uniquely determine the eight unknowns. Thus, this model is not identified.

In summary, none of the five models considered above which postulate independence among the row and column variables and which model nonignorable nonresponse is identified.

### Empirical Analysis for Data Set 2

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the independence models hypothesized to fit Data Set 2 (page 219). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). (Note that none of the nonignorable nonresponse models was found to be identified.)

#### Data Set 2

##### Independence Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1690.49         | 288.54       | 88.97           | 2068    |
|                     | single crime    | 257.50          | 43.95        | 13.55           | 315     |
|                     | multiple crimes | 85.01           | 14.51        | 4.47            | 103.99  |
|                     |                 | 2033            | 347          | 106.99          | 2486.99 |

$$\beta = (3.78, -1.49, .39, -1.47, .30)$$

Likelihood Ratio fit statistic = 59.3404

Degrees of freedom = 4

**Data Set 2**

Independence Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1612.87         | 317.41       | 109.13          | 2039.41 |
|                     | single crime    | 259.16          | 51.00        | 17.54           | 327.70  |
|                     | multiple crimes | 94.81           | 18.66        | 6.42            | 119.89  |
|                     |                 | 1966.84         | 387.07       | 133.09          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 808.71          | 159.15       | 54.72           | 1022.58 |
|                     | single crime    | 129.94          | 25.57        | 8.79            | 164.31  |
|                     | multiple crimes | 47.54           | 9.36         | 3.22            | 60.11   |
|                     |                 | 986.19          | 194.08       | 66.73           | 1247    |

**Data Set 2**

Independence - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 641.39          | 126.22       | 43.40           | 811.01 |
|                     | single crime    | 103.06          | 20.28        | 6.97            | 130.31 |
|                     | multiple crimes | 37.70           | 7.42         | 2.55            | 47.67  |
|                     |                 | 782.15          | 153.93       | 52.92           | 989    |

$$\beta = (3.24, -1.42, .41, -1.35, .28, .69, -.23)$$

Likelihood Ratio fit statistic = 106.591

Degrees of freedom = 8

**Data Set 2****Independence Model - Two Supplemental Margins****Missing at Random (MAR)****Completely Classified Data**

|                             |                        | <b>First Interview</b> |                     |                        |         |
|-----------------------------|------------------------|------------------------|---------------------|------------------------|---------|
|                             |                        | <b>crime free</b>      | <b>single crime</b> | <b>multiple crimes</b> |         |
| <b>Second<br/>Interview</b> | <b>crime free</b>      | 1635.48                | 321.86              | 110.66                 | 2068    |
|                             | <b>single crime</b>    | 249.12                 | 49.03               | 16.86                  | 315     |
|                             | <b>multiple crimes</b> | 82.25                  | 16.19               | 5.57                   | 104.01  |
|                             |                        | 1966.85                | 387.08              | 133.09                 | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                             |                        | <b>First Interview</b> |                     |                        |      |
|-----------------------------|------------------------|------------------------|---------------------|------------------------|------|
|                             |                        | <b>crime free</b>      | <b>single crime</b> | <b>multiple crimes</b> |      |
| <b>Second<br/>Interview</b> | <b>crime free</b>      | 786.10                 | 154.71              | 53.20                  | 994  |
|                             | <b>single crime</b>    | 139.98                 | 27.55               | 9.47                   | 177  |
|                             | <b>multiple crimes</b> | 60.10                  | 11.83               | 4.07                   | 76   |
|                             |                        | 986.18                 | 194.09              | 66.74                  | 1247 |

**Data Set 2**

Independence Model - Two Supplemental Margins

Missing at Random (MAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 530.03          | 104.31       | 35.86           | 670.20 |
|                     | single crime    | 164.50          | 32.37        | 11.13           | 208    |
|                     | multiple crimes | 87.61           | 17.24        | 5.93            | 110.78 |
|                     |                 | 782.15          | 153.93       | 52.92           | 988.99 |

$$\beta = (3.32, -1.29, .44, -1.35, .28, .58, .16, -.21, -.05, .39, -.17)$$

Likelihood Ratio fit statistic = 97.8555

Degrees of freedom = 4

Uniform Association - Data Set 2

Type I: uniform association; 3x3 table with two supplemental margins

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | 1   | 0   | -1  | 0   | 1    | 0    | 1    | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 0   | 1   | 0   | -1  | 0   | 1    | 0    | 1    | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 1    | 0    | 1    | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | -1  | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 1   | 0   | 1   | 0   | -1  | 0    | 1    | 1    | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | -1  | 0   | 1   | 0   | 1   | 0    | 1    | 1    | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 1   | 0   | -1  | 0    | 1    | 1    | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1   | 0    | 1    | 1    | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 0    | 1    | 1    | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0    | 1    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the



interaction of the linear row variable term and the linear column variable term, the seventh and eighth columns, the missing data indicators, and the ninth through twelfth columns, the interactions of the missing data indicators and the column variable. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the independence models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 215) is the same as for the independence models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for the independence models (page 222), for the portion of the vector  $Y$  corresponding to the completely classified data (and the first nine rows of  $X$ , the model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or  $\ln(E(Y)) = X\beta$ .

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b6 + b7 - b9 + b11 = \ln f1 \quad (9.a.1)$$

$$b1 - b2 + b3 + b7 = \ln f2 \quad (9.a.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b9 + b11 = \ln f3 \quad (9.a.3)$$

$$b1 - b4 + b5 + b7 - b9 + b11 = \ln f4 \quad (9.a.4)$$

$$b1 + b7 = \ln f5 \quad (9.a.5)$$

$$b1 + b4 + b5 + b7 + b9 + b11 = \ln f6 \quad (9.a.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 - b9 + b11 = \ln f7 \quad (9.a.7)$$

$$b1 + b2 + b3 + b7 = \ln f8 \quad (9.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b9 + b11 = \ln f9 \quad (9.a.9)$$

Here,  $b1 - b7$ ,  $b9$  and  $b11$  are nine of the twelve parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (9.a.1) - (9.a.9) are exactly the same as equations (2.a.1) - (2.a.9) (page 57), but with  $b8$  and  $b9$  of equations (2.a.1) - (2.a.9) replaced by  $b9$  and  $b11$ , respectively, in equations (9.a.1) - (9.a.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (9.a.1) - (9.a.9), as follows:

$$b1 + b7 = \ln f5 \quad (9.a.10)$$

$$b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (9.a.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (9.a.12)$$

$$b5 + b11 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (9.a.13)$$

$$b4 + b9 = \frac{1}{2} \ln \frac{f6}{f4} \quad (9.a.14)$$

$$b6 = \ln \frac{f1 f5}{f2 f4} \quad (9.a.15)$$

Note that  $b1$  and  $b7$  cannot be determined uniquely from equations (9.a.1) - (9.a.9), since whenever  $b1$  appears in the equation, so does  $b7$ ; that is  $b1 + b7$  appears as a unit in each of the equations. Similarly,  $b5$  and  $b11$  cannot be determined uniquely, nor can  $b4$  and

*b9*. Only three of the twelve model parameters (*b2*, *b3* and *b6*) have been uniquely determined.

The last six elements of vector *Y* correspond to the partially classified data (and the last eighteen rows of the *X* matrix). Each of these last six elements of vector *Y* represents a sum of cell frequencies in a given row of one of the two (hypothetical) contingency tables for the partially classified data. Let *f10*, ... , *f18* be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with a supplemental row margin, and let *f19*, ... , *f27* be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with a supplemental column margin. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices *C* and *X*, the algebraic equations which correspond to the partially classified data are given by

$$\exp(b1 - b2 + b3 - b4 + b5 + b6) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5 - b6) = s1 \quad (9.a.16)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (9.a.17)$$

$$\exp(b1 + b2 + b3 - b4 + b5 - b6) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5 + b6) = s3 \quad (9.a.18)$$

$$\exp(b1 - b4 + b5 + b8 - b10 + b12) [\exp(-b2 + b3 + b6) + 1 + \exp(b2 + b3 - b6)] = s4 \quad (9.a.19)$$

$$\exp(b1 + b8) [\exp(-b2 + b3) + 1 + \exp(b2 + b3)] = s5 \quad (9.a.20)$$

$$\exp(b1 + b4 + b5 + b8 + b10 + b12) [\exp(-b2 + b3 - b6) + 1 + \exp(b2 + b3 + b6)] = s6 \quad (9.a.21)$$

Here,  $s1 = f10 + f11 + f12$ , the first row marginal for the partially classified data,

$s2 = f13 + f14 + f15$ , the second row marginal for the partially classified data,

$s3 = f16 + f17 + f18$ , the third row marginal for the partially classified data,

$s4 = f19 + f22 + f25$ , the first column marginal for the partially classified data,

$s5 = f20 + f23 + f26$ , the second column marginal for the partially classified data, and

$s6 = f21 + f24 + f27$ , the third column marginal for the partially classified data.

Assume that we can obtain estimates  $s1 - s6$ , for the last six elements of vector *Y*.

Equations (9.a.16) - (9.a.18) are identical to equations (2.a.16) - (2.a.18). As was shown on pages 59-61, these equations are sufficient to solve uniquely for  $b1$ ,  $b4$  and  $b5$ . These, together with equations (9.a.10), (9.a.14) and (9.a.13), also allow for the unique solution of  $b7$ ,  $b9$  and  $b11$ , respectively, as indicated on page 61. The only remaining unknown  $b$ 's are thus  $b8$ ,  $b10$  and  $b12$ . They appear only in equations (9.a.19) - (9.a.21). We will attempt to find these remaining  $b$ 's.

Let  $k = \exp(b6)$ ,  $k1 = \exp(-b4+b5)$ ,  $k2 = \exp(b4+b5)$ ,  $k3 = \exp(b1-b2+b3)$ ,  $k4 = \exp(b1+b2+b3)$ , and  $k5 = \exp(b1)$ .

Then equations (9.a.19) - (9.a.21) can be rewritten as

$$kk1k3 \exp(b8 - b10 + b12) + k1k5 \exp(b8 - b10 + b12) + \frac{k1k4}{k} \exp(b8 - b10 + b12) = s4 \quad (9.a.22)$$

$$k3 \exp(b8) + k5 \exp(b8) + k4 \exp(b8) = s5 \quad (9.a.23)$$

$$\frac{k2k3}{k} \exp(b8 + b10 + b12) + k2k5 \exp(b8 + b10 + b12) + kk2k4 \exp(b8 + b10 + b12) = s6 \quad (9.a.24)$$

From equation (9.a.23),  $\exp(b8) = \frac{s5}{k3 + k4 + k5}$ , or

$$b8 = \ln\left(\frac{s5}{k3 + k4 + k5}\right) \quad (9.a.25)$$

Dividing (9.a.22) by (9.a.23), and dividing (9.a.24) by (9.a.23), we have

$$\exp(-b10 + b12) = \frac{s4(k3 + k4 + k5)}{s5\left(kk1k3 + k1k5 + \frac{k1k4}{k}\right)}$$

$$\exp(b10 + b12) = \frac{s6(k3 + k4 + k5)}{s5\left(\frac{k2k3}{k} + k2k5 + kk2k4\right)}$$

or

$$-b_{10} + b_{12} = \ln \frac{s^4(k_3 + k_4 + k_5)}{s^5 \left( k_1 k_3 + k_1 k_5 + \frac{k_1 k_4}{k} \right)} \quad (9.a.26)$$

$$b_{10} + b_{12} = \ln \frac{s^6(k_3 + k_4 + k_5)}{s^5 \left( \frac{k_2 k_3}{k} + k_2 k_5 + k_2 k_4 \right)} \quad (9.a.27)$$

Thus,

$$b_{10} = \frac{1}{2} \ln \frac{s^6 \left( k_1 k_3 + k_1 k_5 + \frac{k_1 k_4}{k} \right)}{s^4 \left( \frac{k_2 k_3}{k} + k_2 k_5 + k_2 k_4 \right)} \quad (9.a.28)$$

$$b_{12} = \frac{1}{2} \ln \frac{s^4 s^6 (k_3 + k_4 + k_5)^2}{(s^5)^2 \left( k_1 k_3 + k_1 k_5 + \frac{k_1 k_4}{k} \right) \left( \frac{k_2 k_3}{k} + k_2 k_5 + k_2 k_4 \right)} \quad (9.a.29)$$

As has been shown, we can solve uniquely for all of the  $b$ 's of the model. This model, therefore, is identified.

**Type II:**

A total of fifteen observed data frequencies (nine from the completely classified data, three from the row marginals, and three from the column marginals) are assumed to be estimable. Thus, a model for this data set can have no more than fifteen parameters. If all the interaction terms of the missing data indicators and the column variable and the missing data indicators and the row variable are included in the loglinear model, sixteen parameters are required. Thus, Type II nonignorable nonresponse with uniform association for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type II model is examined below.

Type II: uniform association; 3x3 table with two supplemental margins

- modified (only linear interaction terms involving the missing data indicators and the column variable and involving the missing data indicators and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | 1   | 0   | -1  | 0    | -1   | 0    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | -1   | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0    | -1   | 0    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 1   | 0   | -1  | 0    | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 1   | 0    | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0    | 1    | 0    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 1    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | 0   | 1   | 0   | -1   | 0    | -1   | 1 | 1 | 3 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | -1   | 1 | 2 | 3 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 0   | 1   | 0   | 1    | 0    | -1   | 1 | 3 | 3 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | 0   | -1   | 0    | 0    | 2 | 1 | 3 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 2 | 2 | 3 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1    | 0    | 0    | 2 | 3 | 3 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1   | 0    | 1    | 3 | 1 | 3 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 1    | 3 | 2 | 3 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the linear row variable term and the linear column variable term, the seventh and eighth columns, the missing data indicators, the ninth and tenth columns, the interaction terms of the missing data indicators and the linear column variable term, and the last eleventh and twelfth columns, the interaction terms of the missing data indicators and the linear row variable term. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have



$$\begin{aligned}
b1 - b2 + b3 - b4 + b5 + b6 + b7 - b9 - b11 &= \ln f1 & (9.b.1) \\
b1 - b2 + b3 + b7 - b11 &= \ln f2 & (9.b.2) \\
b1 - b2 + b3 + b4 + b5 - b6 + b7 + b9 - b11 &= \ln f3 & (9.b.3) \\
b1 - b4 + b5 + b7 - b9 &= \ln f4 & (9.b.4) \\
b1 + b7 &= \ln f5 & (9.b.5) \\
b1 + b4 + b5 + b7 + b9 &= \ln f6 & (9.b.6) \\
b1 + b2 + b3 - b4 + b5 - b6 + b7 - b9 + b11 &= \ln f7 & (9.b.7) \\
b1 + b2 + b3 + b7 + b11 &= \ln f8 & (9.b.8) \\
b1 + b2 + b3 + b4 + b5 + b6 + b7 + b9 + b11 &= \ln f9 & (9.b.9)
\end{aligned}$$

Here,  $b1 - b7$ ,  $b9$  and  $b11$  are nine of the twelve parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (9.b.1) - (9.b.9) are exactly the same as equations (2.c.1) - (2.c.9) (page 67), but with  $b8$  and  $b9$  of equations (2.c.1) - (2.c.9) replaced by  $b9$  and  $b11$ , respectively in equations (9.b.1) - (9.b.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (9.b.1) - (9.b.9), as follows:

$$b1 + b7 = \ln f5 \quad (9.b.10)$$

$$b2 + b11 = \frac{1}{2} \ln \frac{f8}{f2} \quad (9.b.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (9.b.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (9.b.13)$$

$$b4 + b9 = \frac{1}{2} \ln \frac{f6}{f4} \quad (9.b.14)$$

$$b6 = \ln \frac{f1 f5}{f2 f4} \quad (9.b.15)$$

Note that  $b1$  and  $b7$  cannot be determined uniquely from equations (9.b.1) - (9.b.9), nor can  $b2$  and  $b11$  and  $b4$  and  $b9$ . Only three of the twelve model parameters ( $b3$ ,  $b5$  and  $b6$ ) have been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3 - b4 + b5 + b6) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5 - b6) = s1 \quad (9.b.16)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (9.b.17)$$

$$\exp(b1 + b2 + b3 - b4 + b5 - b6) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5 + b6) = s3 \quad (9.b.18)$$

$$\exp(b1 - b4 + b5 + b8 - b10) [\exp(-b2 + b3 + b6 - b12) + 1 + \exp(b2 + b3 - b6 + b12)] = s4 \quad (9.b.19)$$

$$\exp(b1 + b8) [\exp(-b2 + b3) + 1 + \exp(b2 + b3)] = s5 \quad (9.b.20)$$

$$\exp(b1 + b4 + b5 + b8 + b10) [\exp(-b2 + b3 - b6 - b12) + 1 + \exp(b2 + b3 + b6 + b12)] = s6 \quad (9.b.21)$$

Equations (9.b.16) - (9.b.18) are identical to equations (2.c.16) - (2.c.18). As was shown on page 69, these three equations produce equation (2.c.24) which is symmetric in  $b4$ . This symmetry results in the nonuniqueness of a solution for  $b4$  (and so for  $b1$  and  $b2$ ), using equations (9.c.16) - (9.c.18).

Equations (9.b.19) - (9.b.21) are the only equations which contain the model parameters  $b8$ ,  $b10$ , and  $b12$ . At best, these three equations will allow for the unique determination of these three unknown model parameters. They cannot, in addition, uniquely determine  $b4$ . Thus, this model is not identified.

**Type III:**

As was true in the case of the independence model, if all of the interaction terms of the missing data indicators and the column variable, the missing data indicators and the row variable, and the three way interactions of the missing data indicators, the column variable, and the row variable are included in the loglinear model, too many parameters are required, given the observed data. Thus, Type III nonignorable nonresponse with uniform association for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type III model is examined below.

Type III: uniform association; 3x3 table with two supplemental margins

- modified (only linear interaction terms involving the missing data indicators and the column variable, involving the missing data indicators and the row variable, and involving the missing data indicators, the column variable and the row variable.

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | 1   | 0   | -1  | 0   | -1   | 0    | 1    | 0    |      | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | -1   | 0    | 0    | 0    |      | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | -1   | 0    | -1   | 0    |      | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 0   | 1   | 0   | -1  | 0   | 0    | 0    | 0    | 0    |      | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 0    | 0    | 0    | 0    |      | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 1    | 0    | -1   | 0    |      | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1    | 0    | 0    | 0    |      | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 0    |      | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | -1  | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 1   | 0   | 1   | 0   | -1  | 0    | -1   | 0    | 1    |      | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | -1   | 0    | 0    |      | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | -1  | 0   | 1   | 0   | 1   | 0    | -1   | 0    | -1   |      | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 1   | 0   | -1  | 0    | 0    | 0    | 0    |      | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    |      | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1   | 0    | 0    | 0    | 0    |      | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 0    | 1    | 0    | -1   |      | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 1    | 0    | 0    |      | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0    | 1    | 0    | 1    |      | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the linear row variable term and the linear column variable term, the seventh and eighth columns, the missing data indicators, the ninth and tenth columns, the interaction terms of the missing data indicators and the linear column variable term, the eleventh and twelfth columns, the interactions of the missing data indicators and the linear row variable term, and the thirteenth and fourteenth columns, the interactions of the missing data indicators, the linear column variable term, and the linear row variable term. The fourteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the modified Type II case that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 + b7 - b9 - b11 + b13 = \ln f1 \quad (9.c.1)$$

$$b1 - b2 + b3 + b7 - b11 = \ln f2 \quad (9.c.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b9 - b11 - b13 = \ln f3 \quad (9.c.3)$$

$$b1 - b4 + b5 + b7 - b9 = \ln f4 \quad (9.c.4)$$

$$b1 + b7 = \ln f5 \quad (9.c.5)$$

$$b1 + b4 + b5 + b7 + b9 = \ln f6 \quad (9.c.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 - b9 + b11 - b13 = \ln f7 \quad (9.c.7)$$

$$b1 + b2 + b3 + b7 + b11 = \ln f8 \quad (9.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b9 + b11 + b13 = \ln f9 \quad (9.c.9)$$

Here,  $b1 - b7$ ,  $b9$ ,  $b11$  and  $b13$  are ten of the fourteen parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (9.c.1) - (9.c.9) are exactly the same as equations (2.d.1) - (2.d.9) (page 72), but with  $b8$ ,  $b9$  and  $b10$  of equations (2.d.1) - (2.d.9) replaced by  $b9$ ,  $b11$  and  $b13$ , respectively, in equations (9.c.1) - (9.c.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (9.c.1) - (9.c.9), as follows:

$$b1 + b7 = \ln f5 \quad (9.c.10)$$

$$b2 + b11 = \frac{1}{2} \ln \frac{f8}{f2} \quad (9.c.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (9.c.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (9.c.13)$$

$$b4 + b9 = \frac{1}{2} \ln \frac{f6}{f4} \quad (9.c.14)$$

$$b6 + b13 = \ln \frac{f1 f5}{f2 f4} \quad (9.c.15)$$

Note that  $b1$  and  $b7$  cannot be determined uniquely from equations (9.c.1) - (9.c.9), nor can  $b2$  and  $b11$ ,  $b4$  and  $b9$ , and  $b6$  and  $b13$ . Only two of the fourteen model parameters ( $b3$  and  $b5$ ) have been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3 - b4 + b5 + b6) + \exp(b1 - b2 + b3) + \exp(b1 - b2 + b3 + b4 + b5 - b6) = s1 \quad (9.c.16)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (9.c.17)$$

$$\exp(b1 + b2 + b3 - b4 + b5 - b6) + \exp(b1 + b2 + b3) + \exp(b1 + b2 + b3 + b4 + b5 + b6) = s3 \quad (9.c.18)$$

$$\exp(b1 - b4 + b5 + b8 - b10) [\exp(-b2 + b3 + b6 - b12 + b14) + 1 + \exp(b2 + b3 - b6 + b12 - b14)] = s4 \quad (9.c.19)$$

$$\exp(b1 + b8) [\exp(-b2 + b3 - b12) + 1 + \exp(b2 + b3 + b12)] = s5 \quad (9.c.20)$$

$$\exp(b1 + b4 + b5 + b8 + b10) [\exp(-b2 + b3 - b6 - b12 - b14) + 1 + \exp(b2 + b3 + b6 + b12 + b14)] = s6 \quad (9.c.21)$$

Equations (9.c.16) - (9.c.18) are identical to equations (2.d.16) - (2.d.18). As was indicated on page 74, these three equations are insufficient to find  $b1$ ,  $b2$ ,  $b4$ , and  $b6$ . Further, equations (9.c.19) - (9.c.21) are the only equations which contain  $b10$ ,  $b12$  and  $b14$  and so, at best, are sufficient to determine these three unknowns. Since not all of the  $b$ 's of this model can be uniquely determined, the model is not identified.

In summary, if the row and column variables are related by uniform association, and nonignorable nonresponse is to be modeled, only the Type I model is identified.

### Empirical Analysis for Data Set 2

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the uniform association models hypothesized to fit Data Set 2 (page 219). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I.

#### Data Set 2

##### Uniform Association Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1736.35         | 264.60       | 67.06           | 2068.01 |
|                     | single crime    | 233.95          | 57.53        | 23.52           | 315     |
|                     | multiple crimes | 62.70           | 24.88        | 16.42           | 104     |
|                     |                 | 2033            | 347.01       | 107             | 2487.01 |

$$\beta = (4.05, -1.18, .34, -1.15, .25, .48)$$

Likelihood Ratio fit statistic = 6.78264

Degrees of freedom = 3



**Data Set 2**

Uniform Association Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1661.03         | 290.45       | 81.24           | 2032.72 |
|                     | single crime    | 233.95          | 66.58        | 30.31           | 330.84  |
|                     | multiple crimes | 68.34           | 31.66        | 23.46           | 123.46  |
|                     |                 | 1963.32         | 388.69       | 135.01          | 2487.02 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 832.85          | 145.63       | 40.73           | 1019.21 |
|                     | single crime    | 117.30          | 33.38        | 15.20           | 165.88  |
|                     | multiple crimes | 34.27           | 15.87        | 11.76           | 61.90   |
|                     |                 | 984.42          | 194.88       | 67.69           | 1246.99 |

**Data Set 2**

Uniform Association - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 660.54          | 115.50       | 32.31           | 808.35 |
|                     | single crime    | 93.03           | 26.48        | 12.05           | 131.56 |
|                     | multiple crimes | 27.18           | 12.59        | 9.33            | 49.10  |
|                     |                 | 780.75          | 154.57       | 53.69           | 989.01 |

$$\beta = (3.51, -1.11, .36, -1.02, .23, .49, .69, -.23)$$

Likelihood Ratio fit statistic = 52.0794

Degrees of freedom = 7

**Data Set 2****Uniform Association Model - Two Supplemental Margins****Missing at Random (MAR)****Completely Classified Data**

|                             |                        | <b>First Interview</b> |                     |                        |         |
|-----------------------------|------------------------|------------------------|---------------------|------------------------|---------|
|                             |                        | <b>crime free</b>      | <b>single crime</b> | <b>multiple crimes</b> |         |
| <b>Second<br/>Interview</b> | <b>crime free</b>      | 1734.44                | 268.04              | 65.52                  | 2068    |
|                             | <b>single crime</b>    | 233.37                 | 58.47               | 23.17                  | 315.01  |
|                             | <b>multiple crimes</b> | 62.39                  | 25.34               | 16.27                  | 104     |
|                             |                        | 2030.20                | 351.85              | 104.96                 | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                             |                        | <b>First Interview</b> |                     |                        |      |
|-----------------------------|------------------------|------------------------|---------------------|------------------------|------|
|                             |                        | <b>crime free</b>      | <b>single crime</b> | <b>multiple crimes</b> |      |
| <b>Second<br/>Interview</b> | <b>crime free</b>      | 833.67                 | 128.84              | 31.49                  | 994  |
|                             | <b>single crime</b>    | 131.13                 | 32.85               | 13.02                  | 177  |
|                             | <b>multiple crimes</b> | 45.59                  | 18.52               | 11.89                  | 76   |
|                             |                        | 1010.39                | 180.21              | 56.40                  | 1247 |

**Data Set 2**

Uniform Association Model - Two Supplemental Margins

Missing at Random (MAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 6.99            | 1.08         | .26             | 8.33   |
|                     | single crime    | 649.17          | 162.63       | 64.44           | 876.24 |
|                     | multiple crimes | 62.64           | 25.44        | 16.34           | 104.42 |
|                     |                 | 718.80          | 189.15       | 81.04           | 988.99 |

 $\beta = (3.49, -.97, .40, -1.15, .23, .48, .58, 1.60, -.21, 2.55, -.05, -3.83)$ 

Likelihood Ratio fit statistic = 7.08474

Degrees of freedom = 3

**Data Set 2****Uniform Association Model - Two Supplemental Margins****Nonignorable Nonresponse.- Type I****Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1736.89    | 264.63       | 67.03           | 2068.55 |
|                     | single crime    | 233.11     | 57.34        | 23.45           | 313.9   |
|                     | multiple crimes | 63.00      | 25.02        | 16.52           | 104.54  |
|                     |                 | 2033       | 346.99       | 107             | 2486.99 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 667.96     | 191.00       | 134.50          | 993.46  |
|                     | single crime    | 89.65      | 41.39        | 47.06           | 178.10  |
|                     | multiple crimes | 24.23      | 18.06        | 33.16           | 75.45   |
|                     |                 | 781.84     | 250.45       | 214.72          | 1247.01 |

**Data Set 2**

Uniform Association Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 611.71          | 147.95       | 49.49           | 809.15 |
|                     | single crime    | 82.10           | 32.06        | 17.31           | 131.47 |
|                     | multiple crimes | 22.19           | 13.99        | 12.20           | 48.38  |
|                     |                 | 716             | 194          | 79              | 989    |

 $\beta = (3.72, -1.18, .35, -.32, .45, .48, .33, -.26, -.83, -.46, -.20, -.29)$ 

Likelihood Ratio fit statistic = 6.80069

Degrees of freedom = 3

### Row Effects Model - Data Set 2

Type I: row effects; 3x3 table with two supplemental margins

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | 0   | -1   | 0    | 1    | 0    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 0   | 1    | 0    | 1    | 0    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | 0   | -1   | 0    | 1    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1    | 0    | 1    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | -1   | 0    | 1    | 0    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1    | 0    | 1    | 0    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 1   | 0    | -1   | 0    | 1    | 1 | 1 | 3 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 1 | 2 | 3 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0    | 1    | 0    | 1    | 1 | 3 | 3 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 1    | 2 | 1 | 3 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 2 | 2 | 3 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 1    | 2 | 3 | 3 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 1   | 0    | -1   | 0    | 1    | 3 | 1 | 3 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 3 | 2 | 3 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the row variable and the linear column variable term, the

eighth and ninth columns, the missing data indicators, and the tenth through thirteenth columns, the interactions of the missing data indicators and the column variable. The thirteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the independence and uniform association models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 215) is the same as for the independence and uniform association models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $\beta$ 's) of the model.

As discussed earlier for the independence and uniform association models, for the portion of the vector  $Y$  corresponding to the completely classified data (and the first nine rows of  $X$ , the model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or  $\ln(E(Y)) = X\beta$ .

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have



$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b10 + b12 = \ln f1 \quad (10.a.1)$$

$$b1 - b2 + b3 + b8 = \ln f2 \quad (10.a.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 + b10 + b12 = \ln f3 \quad (10.a.3)$$

$$b1 - b4 + b5 + b8 - b10 + b12 = \ln f4 \quad (10.a.4)$$

$$b1 + b8 = \ln f5 \quad (10.a.5)$$

$$b1 + b4 + b5 + b8 + b10 + b12 = \ln f6 \quad (10.a.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 - b10 + b12 = \ln f7 \quad (10.a.7)$$

$$b1 + b2 + b3 + b8 = \ln f8 \quad (10.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b10 + b12 = \ln f9 \quad (10.a.9)$$

Here,  $b1 - b8$ ,  $b10$  and  $b12$  are ten of the thirteen parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (10.a.1) - (10.a.9) are exactly the same as equations (3.a.1) - (3.a.9) (pages 80 - 81), but with  $b9$  and  $b10$  of equations (3.a.1) - (3.a.9) replaced by  $b10$  and  $b12$ , respectively, in equations (10.a.1) - (10.a.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (10.a.1) - (10.a.9), as follows:

$$b1 + b8 = \ln f5 \quad (10.a.10)$$

$$b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (10.a.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (10.a.12)$$

$$b5 + b12 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (10.a.13)$$

$$b4 + b10 = \frac{1}{2} \ln \frac{f6}{f4} \quad (10.a.14)$$

$$b6 = \frac{1}{4} \ln \frac{f1 f9}{f3 f7} \quad (10.a.15)$$

$$b7 = \frac{1}{4} \ln \frac{f3 f9 (f4)^2}{f1 f7 (f6)^2} \quad (10.a.16)$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (10.a.1) - (10.a.9), since whenever  $b1$  appears in the equation, so does  $b8$ ; that is  $b1 + b8$  appears as a unit in

each of the equations. Similarly,  $b_5$  and  $b_{12}$  cannot be determined uniquely, nor can  $b_4$  and  $b_{10}$ . Only four of the thirteen model parameters ( $b_2$ ,  $b_3$ ,  $b_6$  and  $b_7$ ) have been uniquely determined.

The last six elements of vector  $Y$  correspond to the partially classified data (and the last eighteen rows of the  $X$  matrix). Each of these last six elements of vector  $Y$  represents a sum of cell frequencies in a given row of one of two (hypothetical) contingency tables for the partially classified data. Let  $f_{10}, \dots, f_{18}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with supplemental row marginals, and  $f_{19}, \dots, f_{27}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with supplemental column marginals. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b_1 - b_2 + b_3) [\exp(-b_4 + b_5 + b_6 - b_7) + 1 + \exp(b_4 + b_5 - b_6 + b_7)] = s_1 \quad (10.a.17)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (10.a.18)$$

$$\exp(b_1 + b_2 + b_3) [\exp(-b_4 + b_5 - b_6 - b_7) + 1 + \exp(b_4 + b_5 + b_6 + b_7)] = s_3 \quad (10.a.19)$$

$$\exp(b_1 - b_4 + b_5 + b_9 - b_{11} + b_{13}) [\exp(-b_2 + b_3 + b_6 - b_7) + 1 + \exp(b_2 + b_3 - b_6 - b_7)] = s_4 \quad (10.a.20)$$

$$\exp(b_1 + b_9) [\exp(-b_2 + b_3) + 1 + \exp(b_2 + b_3)] = s_5 \quad (10.a.21)$$

$$\exp(b_1 + b_4 + b_5 + b_9 + b_{11} + b_{13}) [\exp(-b_2 + b_3 - b_6 + b_7) + 1 + \exp(b_2 + b_3 + b_6 + b_7)] = s_6 \quad (10.a.22)$$

Here,  $s_1 = f_{10} + f_{11} + f_{12}$ , the first row marginal for the partially classified data,

$s_2 = f_{13} + f_{14} + f_{15}$ , the second row marginal for the partially classified data,

$s_3 = f_{16} + f_{17} + f_{18}$ , the third row marginal for the partially classified data,

$s_4 = f_{19} + f_{22} + f_{25}$ , the first column marginal for the partially classified data,

$s_5 = f_{20} + f_{23} + f_{26}$ , the second column marginal for the partially classified data, and

$s_6 = f_{21} + f_{24} + f_{27}$ , the third column marginal for the partially classified data.

Assume that we can obtain estimates  $s1 - s6$ , for the last six elements of vector  $Y$ .

Equations (10.a.17) - (10.a.19) are identical to equations (3.a.17) - (3.a.19). As was shown on pages 83 - 85, these equations are sufficient to solve uniquely for  $b1$ ,  $b4$  and  $b5$ . As noted on page 85, these, together with equations (10.a.10), (10.a.14), and (10.a.13), also allow for the unique solution of  $b8$ ,  $b10$  and  $b12$ , respectively. The only remaining unknown  $b$ 's are thus  $b9$ ,  $b11$  and  $b13$ . They appear only in equations (10.a.19) - (10.a.21). We will attempt to find these remaining  $b$ 's.

Let  $k1 = \exp(b1)$ ,  $k2 = \exp(-b2 + b3)$ ,  $k3 = \exp(-b4 + b5)$ ,  $k4 = \exp(b6 - b7)$ ,  
 $k5 = \exp(b4 + b5)$ ,  $k6 = \exp(b2 + b3)$ , and  $k7 = \exp(b6 + b7)$ .

Then equations (10.a.20) - (10.a.22) can be rewritten as

$$k1k2k3k4 \exp(b9 - b11 + b13) + k1k3 \exp(b9 - b11 + b13) + \frac{k1k3k6}{k7} \exp(b9 - b11 + b13) = s4 \quad (10.a.23)$$

$$k1k2 \exp(b9) + k1 \exp(b9) + k1k6 \exp(b9) = s5 \quad (10.a.24)$$

$$\frac{k1k2k5}{k4} \exp(b9 + b11 + b13) + k1k5 \exp(b9 + b11 + b13) + k1k5k6k7 \exp(b9 + b11 + b13) = s6 \quad (10.a.25)$$

From equation (10.a.24),  $\exp(b9) = \frac{s5}{k1k2 + k1 + k1k6}$ , or

$$b9 = \ln \frac{s5}{k1(k2 + 1 + k6)} \quad (10.a.26)$$

Dividing (10.a.23) by (10.a.24), and (10.a.25) by (10.a.24), we have

$$\exp(-b11 + b13) = \frac{s4(k2 + 1 + k6)}{s5 \left( k2k3k4 + k3 + \frac{k3k6}{k7} \right)}$$

$$\exp(b11 + b13) = \frac{s6(k2 + 1 + k6)}{s5 \left( \frac{k2k5}{k4} + k5 + k5k6k7 \right)}$$

or

$$-b_{11} + b_{13} = \ln \frac{s^4(k_2 + 1 + k_6)}{s^5 \left( k_2 k_3 k_4 + k_3 + \frac{k_3 k_6}{k_7} \right)} \quad (10.a.27)$$

and

$$b_{11} + b_{13} = \frac{s^6(k_2 + 1 + k_6)}{s^5 \left( \frac{k_2 k_5}{k_4} + k_5 + k_5 k_6 k_7 \right)} \quad (10.a.28)$$

$$\text{Thus, } b_{11} = \frac{1}{2} \ln \frac{s^6 \left( k_2 k_3 k_4 + k_3 + \frac{k_3 k_6}{k_7} \right)}{s^4 \left( \frac{k_2 k_5}{k_4} + k_5 + k_5 k_6 k_7 \right)} \quad (10.a.29)$$

$$\text{and } b_{13} = \frac{1}{2} \ln \frac{s^4 s^6 (k_2 + 1 + k_6)^2}{(s^5)^2 \left( k_2 k_3 k_4 + k_3 + \frac{k_3 k_6}{k_7} \right) \left( \frac{k_2 k_5}{k_4} + k_5 + k_5 k_6 k_7 \right)} \quad (10.a.30)$$

Having found all of the  $b$ 's of the model, we have determined that it is identified.

**Type II:**

As noted for the uniform association model (page 254), a model for this data set can contain no more than fifteen parameters. If all of the interaction terms of the missing data indicators and the column variable and the missing data indicators and the row variable are included in this loglinear model, seventeen parameters are required. Thus, the row effects model with Type II nonignorable nonresponse cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type II model is examined below.

Type II: row effects; 3x3 table with two supplemental margins

- modified (with only the linear interaction terms involving the missing data indicators and the column variable, and involving the missing data indicators and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | 0   | -1   | 0    | -1   | 0    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | -1   | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 0   | 1    | 0    | -1   | 0    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | 0   | -1   | 0    | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1    | 0    | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | -1   | 0    | 1    | 0    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 1    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1    | 0    | 1    | 0    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 1   | 0    | -1   | 0    | -1   | 1 | 1 | 3 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | -1   | 1 | 2 | 3 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0    | 1    | 0    | -1   | 1 | 3 | 3 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 0    | 2 | 1 | 3 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 2 | 2 | 3 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 0    | 2 | 3 | 3 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 1   | 0    | -1   | 0    | 1    | 3 | 1 | 3 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 1    | 3 | 2 | 3 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh

columns, the interactions of the row variable and the linear column variable term, the eighth and ninth columns, the missing data indicators, the tenth and eleventh columns, the interactions of the missing data indicators and the linear column variable term, and the twelfth and thirteenth columns, the interactions of the missing data indicators and the linear row variable term. The thirteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b10 - b12 = \ln f1 \quad (10.b.1)$$

$$b1 - b2 + b3 + b8 - b12 = \ln f2 \quad (10.b.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 + b7 + b8 + b10 - b12 = \ln f3 \quad (10.b.3)$$

$$b1 - b4 + b5 + b8 - b10 = \ln f4 \quad (10.b.4)$$

$$b1 + b8 = \ln f5 \quad (10.b.5)$$

$$b1 + b4 + b5 + b8 + b10 = \ln f6 \quad (10.b.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 - b10 + b12 = \ln f7 \quad (10.b.7)$$

$$b1 + b2 + b3 + b8 + b12 = \ln f8 \quad (10.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b10 + b12 = \ln f9 \quad (10.b.9)$$

Here,  $b1$ , ...,  $b8$ ,  $b10$  and  $b12$  are ten of the thirteen parameters of the matrix  $\beta$ , which are to be determined. It is to be noted that equations (10.b.1) - (10.b.9) are exactly the same as equations (3.c.1) - (3.c.9) (page 91), but with  $b9$  and  $b10$  of equations (3.c.1) - (3.c.9) replaced, respectively, by  $b10$  and  $b12$  in equations (10.b.1) - (10.b.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (10.b.1) - (10.b.9), as follows:

$$b1 + b8 = \ln f5 \quad (10.b.10)$$

$$b2 + b12 = \frac{1}{2} \ln \frac{f8}{f2} \quad (10.b.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (10.b.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (10.b.13)$$

$$b4 + b10 = \frac{1}{2} \ln \frac{f6}{f4} \quad (10.b.14)$$

$$b7 = \frac{1}{4} \ln \frac{f3 f9 (f4)^2}{f1 f7 (f6)^2} \quad (10.b.15)$$

$$b6 = \frac{1}{4} \ln \frac{f1 f9}{f3 f7} \quad (10.b.16)$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (10.b.1) - (10.b.9), nor can  $b2$  and  $b12$ , and  $b4$  and  $b10$ . Only four of the thirteen model parameters ( $b3$ ,  $b5$ ,  $b6$  and  $b7$ ) have been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by



$$\exp(b1 - b2 + b3)[\exp(-b4 + b5 + b6 - b7) + 1 + \exp(b4 + b5 - b6 + b7)] = s1 \quad (10.b.17)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (10.b.18)$$

$$\exp(b1 + b2 + b3)[\exp(-b4 + b5 - b6 - b7) + 1 + \exp(b4 + b5 + b6 + b7)] = s3 \quad (10.b.19)$$

$$\exp(b1 - b4 + b5 + b9 - b11)[\exp(-b2 + b3 + b6 - b7 - b13) + 1 + \exp(b2 + b3 - b6 - b7 + b13)] = s4 \quad (10.b.20)$$

$$\exp(b1 + b9)[\exp(-b2 + b3 - b13) + 1 + \exp(b2 + b3 + b13)] = s5 \quad (10.b.21)$$

$$\exp(b1 + b4 + b5 + b9 + b11)[\exp(-b2 + b3 - b6 + b7 - b13) + 1 + \exp(b2 + b3 + b6 + b7 + b13)] = s6 \quad (10.b.22)$$

Equations (10.b.17) - (10.b.19) are identical to equations (3.c.17) - (3.c.19). As was shown on page 94, these three equations produce a fourth degree polynomial in  $\exp(b4)$ . While there may be a single solution for  $b4$ , given a particular data set, there may also be no solutions for  $b4$ , or several solutions. As was shown on page 93, if there is a unique solution for  $b4$ , there will also be unique solutions for  $b1$ ,  $b2$ ,  $b8$ ,  $b10$  and  $b12$ . If no such unique solution exists, the model is not identified.

Let us assume that there is a unique solution for  $b4$  (and thus also for  $b1$ ,  $b2$ ,  $b8$ ,  $b10$  and  $b12$ ).

The only remaining unknown  $b$ 's are then  $b9$ ,  $b11$  and  $b13$ . They appear only in equations (10.b.20) - (10.b.22). We will attempt to find these remaining  $b$ 's.

Let  $m1 = \exp(b1)$ ,  $m2 = \exp(-b2 + b3)$ ,  $m3 = \exp(-b4 + b5)$ ,  $m4 = \exp(b6 - b7)$ ,  
 $m5 = \exp(b2 + b3)$ ,  $m6 = \exp(b4 + b5)$ , and  $m7 = \exp(b6 + b7)$ .

Then equations (10.b.20) - (10.b.22) can be expressed as

$$m_1 m_2 m_3 m_4 \exp(b_9 - b_{11} - b_{13}) + m_1 m_3 \exp(b_9 - b_{11}) + \frac{m_1 m_3 m_5}{m_7} \exp(b_9 - b_{11} + b_{13}) = s_4 \quad (10.b.23)$$

$$m_1 m_2 \exp(b_9 - b_{13}) + m_1 \exp(b_9) + m_1 m_5 \exp(b_9 + b_{13}) = s_5 \quad (10.b.24)$$

$$\frac{m_1 m_2 m_6}{m_4} \exp(b_9 + b_{11} - b_{13}) + m_1 m_6 \exp(b_9 + b_{11}) + m_1 m_5 m_6 m_7 \exp(b_9 + b_{11} + b_{13}) = s_6 \quad (10.b.25)$$

From equation (10.b.24),

$$\exp(b_9) = \frac{s_5}{m_1 m_2 \exp(-b_{13}) + m_1 + m_1 m_5 \exp(b_{13})} \quad (10.b.26)$$

Thus, given  $b_{13}$ ,  $b_9$  is also uniquely determined.

Dividing (10.b.23) by (10.b.24), and (10.b.25) by (10.b.24), we have

$$\begin{aligned} m_2 m_3 m_4 \exp(-b_{11} - b_{13}) + m_3 \exp(-b_{11}) + \frac{m_3 m_5}{m_7} \exp(-b_{11} + b_{13}) \\ = \frac{s_4}{s_5} (m_2 \exp(-b_{13}) + 1 + m_5 \exp(b_{13})) \end{aligned} \quad (10.b.27)$$

$$\begin{aligned} \frac{m_2 m_6}{m_4} \exp(b_{11} - b_{13}) + m_6 \exp(b_{11}) + m_5 m_6 m_7 \exp(b_{11} + b_{13}) \\ = \frac{s_6}{s_5} (m_2 \exp(-b_{13}) + 1 + m_5 \exp(b_{13})) \end{aligned} \quad (10.b.28)$$

From (10.b.27),

$$\exp(b_{11}) = \frac{s_5 \left( m_2 m_3 m_4 \exp(-b_{13}) + m_3 + \frac{m_3 m_5}{m_7} \exp(b_{13}) \right)}{s_4 (m_2 \exp(-b_{13}) + 1 + m_5 \exp(b_{13}))} \quad (10.b.29)$$

From (10.b.28),

$$\exp(b_{11}) = \frac{s_6 (m_2 \exp(-b_{13}) + 1 + m_5 \exp(b_{13}))}{s_5 \left( \frac{m_2 m_6}{m_4} \exp(-b_{13}) + m_6 + m_5 m_6 m_7 \exp(b_{13}) \right)} \quad (10.b.30)$$

As can be seen from each of equations (10.b.29) and (10.b.30), given  $b_{13}$ ,  $b_{11}$  is also uniquely determined.

Further, equating the two expressions for  $\exp(b11)$  in equations (10.b.29) and (10.b.30), we have

$$\begin{aligned} & \exp(-2b13) \left[ s4s6(m2)^2 - (s5)^2(m2)^2 m3m6 \right] + \exp(-b13) \left[ 2m2s4s6 - m2m3 \left( \frac{m6}{m4} + m4m6 \right) (s5)^2 \right] \\ & + \left[ 2m2m5s4s6 + s4s6 - m3m6(s5)^2 \left( \frac{m2m5}{m4m7} + 1 + m2m4m5m7 \right) \right] \\ & + \exp(b13) \left[ 2m5s4s6 - m3m5 \left( \frac{m6}{m7} + m6m7 \right) (s5)^2 \right] + \exp(2b13) \left[ s4s6(m5)^2 - m3(m5)^2 m6(s5)^2 \right] = 0 \end{aligned}$$

If the above equation is multiplied by  $\exp(2b13)$  and if the substitution  $x = \exp(b13)$  is made, we have

$$\begin{aligned} & \left[ s4s6(m2)^2 - (s5)^2(m2)^2 m3m6 \right] + x \left[ 2m2s4s6 - m2m3 \left( \frac{m6}{m4} + m4m6 \right) (s5)^2 \right] \\ & + x^2 \left[ 2m2m5s4s6 + s4s6 - m3m6(s5)^2 \left( \frac{m2m5}{m4m7} + 1 + m2m4m5m7 \right) \right] \\ & + x^3 \left[ 2m5s4s6 - m3m5 \left( \frac{m6}{m7} + m6m7 \right) (s5)^2 \right] + x^4 \left[ s4s6(m5)^2 - m3(m5)^2 m6(s5)^2 \right] = 0 \end{aligned}$$

(10.b.31)

Thus, equations (10.b.20) - (10.b.22) generate a fourth degree polynomial equation in  $\exp(b13)$ . Since  $\exp(b13) > 0$ , we seek only positive solutions to (10.b.31). As was noted on page 282, if  $b13$  is uniquely determined, then so will be  $b9$  and  $b11$ , and then the model will be identified. The uniqueness, or lack of uniqueness, of a solution for  $b13$  will be determined empirically for a given data set.

In summary, the identification status of this model first hinges on the uniqueness (or nonuniqueness) of a solution for  $b4$ . If  $b4$  is not uniquely determined, the model is not identified. If  $b4$  is uniquely determined, the identification status of the model then

depends on the uniqueness (or nonuniqueness) of a solution for  $b_{13}$ . If  $b_{13}$  is also uniquely determined, then so are  $b_9$  and  $b_{11}$ , and the model is identified. On the other hand, if  $b_{13}$  is not uniquely determined, then neither are  $b_9$  and  $b_{11}$ , and the model is not identified.

The identification status of this model will be determined empirically for a given data set.

### Type III:

As was discussed in connection with the independence and uniform association models, if all the interaction terms of the missing data indicators and the column variable, the missing data indicators and the row variable, and three way interactions of the missing data indicators, the column variable, and the row variable are included in the model, there would be too many parameters, given the data. Thus, Type III nonignorable nonresponse with row effects for the row and column variables cannot be modeled for this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type III model is examined below.

Type III: row effects; 3x3 table with two supplemental margins

- modified (with only linear interaction terms involving the missing data

indicators and the column variable, involving the missing data indicators and the

row variable, and involving the missing data indicators, the column variable and

the row variable.

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 1   | 0   | -1   | 0    | -1   | 0    | 1    | 0    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | -1   | 0    | 0    | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 0   | 1    | 0    | -1   | 0    | -1   | 0    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 1   | 0   | -1   | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1    | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | -1   | 0    | 1    | 0    | -1   | 0    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 1    | 0    | 0    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1    | 0    | 1    | 0    | 1    | 0    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 1   | 0    | -1   | 0    | -1   | 0    | 1    | 1 | 1 | 3 |
|        | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | -1   | 0    | 0    | 1 | 2 | 3 |
|        | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0    | 1    | 0    | -1   | 0    | -1   | 1 | 3 | 3 |
|        | 1   | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 0    | 0    | 0    | 2 | 1 | 3 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 3 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 0    | 0    | 0    | 2 | 3 | 3 |
|        | 1   | 1   | 1   | -1  | 1   | -1  | -1  | 0   | 1   | 0    | -1   | 0    | 1    | 0    | -1   | 3 | 1 | 3 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 1    | 0    | 0    | 3 | 2 | 3 |
|        | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh

columns, the interaction of the linear row variable term and the linear column variable term, the eighth and ninth columns, the missing data indicators, the tenth and eleventh columns, the interaction terms of the missing data indicators with the linear column variable term, the twelfth and thirteenth columns, the interactions of the missing data indicators and the linear row variable term, and the fourteenth and fifteenth columns, the three way interactions of the missing data indicators with the linear row variable term and the linear column variable term. The fifteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the modified Type II case that estimates  $f_1, f_2, \dots, f_9$ , are available for the first nine elements of vector  $Y$ , and  $s_1 - s_6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b_1 - b_2 + b_3 - b_4 + b_5 + b_6 - b_7 + b_8 - b_{10} - b_{12} + b_{14} = \ln f_1 \quad (10.c.1)$$

$$b_1 - b_2 + b_3 + b_8 - b_{12} = \ln f_2 \quad (10.c.2)$$

$$b_1 - b_2 + b_3 + b_4 + b_5 - b_6 + b_7 + b_8 + b_{10} - b_{12} - b_{14} = \ln f_3 \quad (10.c.3)$$

$$b1 - b4 + b5 + b8 - b10 = \ln f4 \quad (10.c.4)$$

$$b1 + b8 = \ln f5 \quad (10.c.5)$$

$$b1 + b4 + b5 + b8 + b10 = \ln f6 \quad (10.c.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 - b10 + b12 - b14 = \ln f7 \quad (10.c.7)$$

$$b1 + b2 + b3 + b8 + b12 = \ln f8 \quad (10.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b10 + b12 + b14 = \ln f9 \quad (10.c.9)$$

Here,  $b1 - b8$ ,  $b10$ ,  $b12$ , and  $b14$  are eleven of the fifteen parameters of the matrix  $\beta$ , which are to be determined. It is to be noted that equations (10.c.1) - (10.c.9) are exactly the same as equations (3.d.1) - (3.d.9) (page 98), but with  $b9$ ,  $b10$  and  $b11$  of equations (3.c.1) - (3.c.9) replaced, respectively, by  $b10$ ,  $b12$  and  $b14$  in equations (10.c.1) - (10.c.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (10.c.1) - (10.c.9), as follows:

$$b1 + b8 = \ln f5 \quad (10.c.10)$$

$$b2 + b12 = \frac{1}{2} \ln \frac{f8}{f2} \quad (10.c.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (10.c.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (10.c.13)$$

$$b4 + b10 = \frac{1}{2} \ln \frac{f6}{f4} \quad (10.c.14)$$

$$b7 = \frac{1}{4} \ln \frac{f3 f9 (f4)^2}{f1 f7 (f6)^2} \quad (10.c.15)$$

$$b6 + b14 = \frac{1}{4} \ln \frac{f1 f9}{f3 f7} \quad (10.c.16)$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (10.c.1) - (10.c.9), nor can  $b2$  and  $b12$ ,  $b4$  and  $b10$ , and  $b6$  and  $b14$ . Only three of the fifteen model parameters ( $b3$ ,  $b5$  and  $b7$ ) have been uniquely determined.



The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3)[\exp(-b4 + b5 + b6 - b7) + 1 + \exp(b4 + b5 - b6 + b7)] = s1 \quad (10.c.17)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (10.c.18)$$

$$\exp(b1 + b2 + b3)[\exp(-b4 + b5 - b6 - b7) + 1 + \exp(b4 + b5 + b6 + b7)] = s3 \quad (10.c.19)$$

$$\exp(b1 - b4 + b5 + b9 - b11)[\exp(-b2 + b3 + b6 - b7 - b13 + b15) + 1 + \exp(b2 + b3 - b6 - b7 + b13 - b15)] = s4 \quad (10.c.20)$$

$$\exp(b1 + b9)[\exp(-b2 + b3 - b13) + 1 + \exp(b2 + b3 + b13)] = s5 \quad (10.c.21)$$

$$\exp(b1 + b4 + b5 + b9 + b11)[\exp(-b2 + b3 - b6 + b7 - b13 - b15) + 1 + \exp(b2 + b3 + b6 + b7 + b13 + b15)] = s6 \quad (10.c.22)$$

Equations (10.c.17) - (10.c.19) are identical to equations (3.d.17) - (3.d.19). As was indicated on page 100, these equations contain the four unknowns  $b1$ ,  $b2$ ,  $b4$ , and  $b6$ . These unknowns cannot, therefore, each be determined uniquely from these three equations. Further, equations (10.c.20) - (10.c.22) are the only equations which contain the unknowns  $b9$ ,  $b11$ ,  $b13$ , and  $b15$ . The three equations (10.c.20) - (10.c.22) are insufficient to solve uniquely for the four unknowns  $b9$ ,  $b11$ ,  $b13$ , and  $b15$ . Thus, this model is not identified.

In summary, if the row and column variables are related by row effects, and nonignorable nonresponse is to be modeled, the Type I model is identified, while the Type III model, (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear

interaction terms involving the missing data indicator and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 2

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the row effects models hypothesized to fit Data Set 2 (page 219). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicator).

#### Data Set 2

##### Row Effects Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1741.16         | 261.68       | 65.16           | 2068    |
|                     | single crime    | 225.10          | 61.79        | 28.10           | 314.99  |
|                     | multiple crimes | 66.74           | 23.52        | 13.74           | 104     |
|                     |                 | 2033            | 346.99       | 107             | 2486.99 |

$$\beta' = (4.12, -1.20, .24, -1.04, .25, .43, -.18)$$

Likelihood Ratio statistic = 4.40433

Degrees of freedom = 2

**Data Set 2**

Row Effects Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1665.74         | 287.35       | 79.04           | 2032.13 |
|                     | single crime    | 224.80          | 71.26        | 36.02           | 332.08  |
|                     | multiple crimes | 72.90           | 30.09        | 19.81           | 122.80  |
|                     |                 | 1963.44         | 388.70       | 134.87          | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 835.21          | 144.08       | 39.63           | 1018.92 |
|                     | single crime    | 112.72          | 35.73        | 18.06           | 166.51  |
|                     | multiple crimes | 36.55           | 15.09        | 9.93            | 61.57   |
|                     |                 | 984.48          | 194.90       | 67.62           | 1247    |

**Data Set 2**

Row Effects Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 662.41          | 114.27       | 31.43           | 808.11 |
|                     | single crime    | 89.40           | 28.34        | 14.32           | 132.06 |
|                     | multiple crimes | 28.99           | 11.97        | 7.88            | 48.84  |
|                     |                 | 780.80          | 154.57       | 53.63           | 989.01 |

$$\beta = (3.58, -1.13, .27, -.92, .23, .44, -.17, .69, -.23)$$

Likelihood Ratio fit statistic = 49.7935

Degrees of freedom = 6

**Data Set 2**

Row Effects Model - Two Supplemental Margins

Missing at Random (MAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 1739.59         | 264.73       | 63.68           | 2068 |
|                     | single crime    | 225.22          | 62.43        | 27.35           | 315  |
|                     | multiple crimes | 66.59           | 23.88        | 13.53           | 104  |
|                     |                 | 2031.40         | 351.04       | 104.56          | 2487 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 836.15          | 127.24       | 30.61           | 994  |
|                     | single crime    | 126.55          | 35.08        | 15.37           | 177  |
|                     | multiple crimes | 48.66           | 17.45        | 9.89            | 76   |
|                     |                 | 1011.36         | 179.77       | 55.87           | 1247 |

**Data Set 2**

Row Effects Model - Two Supplemental Margins

Missing at Random (MAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 80.27           | 12.22        | 2.94            | 95.43  |
|                     | single crime    | 623.82          | 172.91       | 75.76           | 872.49 |
|                     | multiple crimes | 13.51           | 4.84         | 2.74            | 21.09  |
|                     |                 | 717.59          | 189.97       | 81.44           | 989.01 |

$$\beta = (3.56, -.99, .30, -1.05, .23, .43, -.17, .58, 1.60, -.21, -.05, .53, -3.41)$$

Likelihood Ratio fit statistic = 4.6731

Degrees of freedom = 2

**Data Set 2**

Row Effects Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1742.24    | 262.12       | 65.26           | 2069.62 |
|                     | single crime    | 223.37     | 60.37        | 27.00           | 310.74  |
|                     | multiple crimes | 67.39      | 24.50        | 14.74           | 106.63  |
|                     |                 | 2033       | 346.99       | 107             | 2486.99 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 779.80     | 36.40        | 176.17          | 992.38  |
|                     | single crime    | 99.98      | 8.38         | 72.90           | 181.26  |
|                     | multiple crimes | 30.17      | 3.40         | 39.79           | 73.36   |
|                     |                 | 909.95     | 48.18        | 288.86          | 1246.99 |



**Data Set 2**

Row Effects Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 613.60          | 146.55       | 48.18           | 808.33 |
|                     | single crime    | 78.67           | 33.75        | 19.94           | 132.36 |
|                     | multiple crimes | 23.74           | 13.70        | 10.88           | 48.32  |
|                     |                 | 716             | 194          | 79              | 989.01 |

 $\beta = (2.13, -1.19, .28, -.16, 2.32, .44, -.14, 1.97, 1.39, -2.07, -.90, -.53, -2.16)$ 

Likelihood Ratio fit statistic = 4.80975

Degrees of freedom = 2

## Data Set 2

### Row Effects Model - Two Supplemental Margins

#### Nonignorable Nonresponse - Type II (linear interaction terms only)

As indicated on page 283, the identification status of the modified nonignorable nonresponse Type II model with row effects depends on the number of solutions to equation (3.c.26) and (10.b.31). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (3.c.26) giving the following fourth degree polynomial equation:

$$.03 - .28x - .56x^2 - 1.31x^3 - 1.17x^4 = 0$$

By Descartes' Rule of Signs, since there is one change of sign in this fourth degree polynomial equation, there is exactly one positive solution for  $x = \exp(b4)$ .

Further, the estimated frequencies and column marginals generated were substituted into equation (10.b.31) giving the following fourth degree polynomial equation:

$$670.69 - 69289.35x - 72239.01x^2 - 1676.29x^3 + 18.4817x^4 = 0$$

By Descartes' Rule of Signs, since there are two changes of sign in this fourth degree polynomial equation, there are two or no positive solutions for  $x = \exp(b13)$ . Using Newton's Method, we find that there are two positive solutions for  $x = \exp(b13)$ . As discussed on page 283, the model is therefore not identified for the given data set.

Column Effects Model - Data Set 2

Type I: column effects; 3x3 table with two supplemental margins

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | -1  | 1   | 0   | -1  | 0    | 1    | 0    | 1    | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 1   | 0   | 1    | 0    | 1    | 0    | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 1   | 0   | -1  | 0    | 1    | 0    | 0    | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1   | 0    | 1    | 0    | 0    | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | -1  | 1   | 1   | 0   | -1  | 0    | 1    | 0    | 0    | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 0    | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | -1  | -1  | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 1   | 0   | -1   | 0    | 1    | 0    | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | -1  | -1  | 0   | 1   | 0   | 1    | 0    | 1    | 0    | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 1    | 0    | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | 1    | 0    | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | -1  | 1   | 0   | 1   | 0   | -1   | 0    | 1    | 0    | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 0    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interaction of the column variable and the linear row variable term, the

eighth and ninth columns, the missing data indicators, and the tenth through thirteenth columns, the interactions of the missing data indicators and the column variable. The thirteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the independence, uniform association and row effects models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 215) is the same as for the independence, uniform association and row effects models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for the independence, uniform association and row effects models, for the portion of the vector  $Y$  corresponding to the completely classified data (and the first nine rows of  $X$ , the model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or  $\ln(E(Y)) = X\beta$ .

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b10 + b12 = \ln f1 \quad (11.a.1)$$

$$b1 - b2 + b3 + b8 = \ln f2 \quad (11.a.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 - b7 + b8 + b10 + b12 = \ln f3 \quad (11.a.3)$$

$$b1 - b4 + b5 + b8 - b10 + b12 = \ln f4 \quad (11.a.4)$$

$$b1 + b8 = \ln f5 \quad (11.a.5)$$

$$b1 + b4 + b5 + b8 + b10 + b12 = \ln f6 \quad (11.a.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 + b8 - b10 + b12 = \ln f7 \quad (11.a.7)$$

$$b1 + b2 + b3 + b8 = \ln f8 \quad (11.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b10 + b12 = \ln f9 \quad (11.a.9)$$

Here,  $b1 - b8$ ,  $b10$  and  $b12$  are ten of the thirteen parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (11.a.1) - (11.a.9) are exactly the same as equations (4.a.1) - (4.a.9) (pages 108 - 109), but with  $b9$  and  $b10$  of equations (4.a.1) - (4.a.9) replaced by  $b10$  and  $b12$ , respectively, in equations (11.a.1) - (11.a.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (11.a.1) - (11.a.9), as follows:

$$b1 + b8 = \ln f5 \quad (11.a.10)$$

$$b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (11.a.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (11.a.12)$$

$$b5 + b12 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (11.a.13)$$

$$b4 + b10 = \frac{1}{2} \ln \frac{f6}{f4} \quad (11.a.14)$$

$$b6 = \frac{1}{2} \ln \frac{f1 f6}{f3 f4} \quad (11.a.15)$$

$$b7 = \frac{1}{2} \ln \frac{f2 f6 f7}{f3 f4 f8} \quad (11.a.16)$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (11.a.1) - (11.a.9), since whenever  $b1$  appears in the equation, so does  $b8$ ; that is  $b1 + b8$  appears as a unit in

each of the equations. Similarly,  $b_5$  and  $b_{12}$  cannot be determined uniquely, nor can  $b_4$  and  $b_{10}$ . Only four of the thirteen model parameters ( $b_2$ ,  $b_3$ ,  $b_6$  and  $b_7$ ) have been uniquely determined.

The last six elements of vector  $Y$  correspond to the partially classified data (and the last eighteen rows of the  $X$  matrix). Each of these last six elements of vector  $Y$  represents a sum of cell frequencies in a given row of one of two (hypothetical) contingency tables for the partially classified data. Let  $f_{10}, \dots, f_{18}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with supplemental row marginals, and  $f_{19}, \dots, f_{27}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with supplemental column marginals. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b_1 - b_2 + b_3) [\exp(-b_4 + b_5 + b_6 - b_7) + 1 + \exp(b_4 + b_5 - b_6 - b_7)] = s_1 \quad (11.a.17)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (11.a.18)$$

$$\exp(b_1 + b_2 + b_3) [\exp(-b_4 + b_5 - b_6 + b_7) + 1 + \exp(b_4 + b_5 + b_6 + b_7)] = s_3 \quad (11.a.19)$$

$$\exp(b_1 - b_4 + b_5 + b_9 - b_{11} + b_{13}) [\exp(-b_2 + b_3 + b_6 - b_7) + 1 + \exp(b_2 + b_3 - b_6 + b_7)] = s_4 \quad (11.a.20)$$

$$\exp(b_1 + b_9) [\exp(-b_2 + b_3) + 1 + \exp(b_2 + b_3)] = s_5 \quad (11.a.21)$$

$$\exp(b_1 + b_4 + b_5 + b_9 + b_{11} + b_{13}) [\exp(-b_2 + b_3 - b_6 - b_7) + 1 + \exp(b_2 + b_3 + b_6 + b_7)] = s_6 \quad (11.a.22)$$

Here,  $s_1 = f_{10} + f_{11} + f_{12}$ , the first row marginal for the partially classified data,

$s_2 = f_{13} + f_{14} + f_{15}$ , the second row marginal for the partially classified data,

$s_3 = f_{16} + f_{17} + f_{18}$ , the third row marginal for the partially classified data,

$s_4 = f_{19} + f_{22} + f_{25}$ , the first column marginal for the partially classified data,

$s_5 = f_{20} + f_{23} + f_{26}$ , the second column marginal for the partially classified data, and

$s_6 = f_{21} + f_{24} + f_{27}$ , the third column marginal for the partially classified data.

It is assumed that we can obtain estimates  $s_1 - s_6$ , for the last six elements of vector  $Y$ .

Equations (11.a.17) - (11.a.19) are identical to equations (4.a.17) - (4.a.19). As was shown on pages 110 - 112, these equations are sufficient to solve uniquely for  $b1$ ,  $b4$  and  $b5$ . As noted on page 113, these, together with equations (11.a.10), (11.a.14), and (11.a.13), also allow for the unique solution of  $b8$ ,  $b10$  and  $b12$ , respectively. The only remaining unknown  $b$ 's are thus  $b9$ ,  $b11$  and  $b13$ . They appear only in equations (11.a.20) - (11.a.22). We will attempt to find these remaining  $b$ 's.

Let  $k1 = \exp(b1)$ ,  $k2 = \exp(-b2 + b3)$ ,  $k3 = \exp(-b4 + b5)$ ,  $k4 = \exp(b6 - b7)$ ,  
 $k5 = \exp(b4 + b5)$ ,  $k6 = \exp(b2 + b3)$ , and  $k7 = \exp(b6 + b7)$ .

Then equations (11.a.20) - (11.a.22) can be rewritten as

$$k1k2k3k4 \exp(b9 - b11 + b13) + k1k3 \exp(b9 - b11 + b13) + \frac{k1k3k6}{k4} \exp(b9 - b11 + b13) = s4 \quad (11.a.23)$$

$$k1k2 \exp(b9) + k1 \exp(b9) + k1k6 \exp(b9) = s5 \quad (11.a.24)$$

$$\frac{k1k2k5}{k7} \exp(b9 + b11 + b13) + k1k5 \exp(b9 + b11 + b13) + k1k5k6k7 \exp(b9 + b11 + b13) = s6 \quad (11.a.25)$$

From equation (11.a.24),  $\exp(b9) = \frac{s5}{k1k2 + k1 + k1k6}$ , or

$$b9 = \ln \frac{s5}{k1(k2 + 1 + k6)} \quad (11.a.26)$$

Dividing (11.a.23) by (11.a.24), and (11.a.25) by (11.a.24), we have

$$\exp(-b11 + b13) = \frac{s4(k2 + 1 + k6)}{s5 \left( k2k3k4 + k3 + \frac{k3k6}{k4} \right)}$$

$$\exp(b11 + b13) = \frac{s6(k2 + 1 + k6)}{s5 \left( \frac{k2k5}{k7} + k5 + k5k6k7 \right)}$$

or

$$-b_{11}+b_{13}=\ln \frac{s^4(k_2+1+k_6)}{s^5\left(k_2k_3k_4+k_3+\frac{k_3k_6}{k_4}\right)} \quad (11.a.27)$$

and

$$b_{11}+b_{13}=\frac{s^6(k_2+1+k_6)}{s^5\left(\frac{k_2k_5}{k_7}+k_5+k_5k_6k_7\right)} \quad (11.a.28)$$

$$\text{Thus, } b_{11}=\frac{1}{2}\ln \frac{s^6\left(k_2k_3k_4+k_3+\frac{k_3k_6}{k_4}\right)}{s^4\left(\frac{k_2k_5}{k_7}+k_5+k_5k_6k_7\right)} \quad (11.a.29)$$

$$\text{and } b_{13}=\frac{1}{2}\ln \frac{s^4s^6(k_2+1+k_6)^2}{(s^5)^2\left(k_2k_3k_4+k_3+\frac{k_3k_6}{k_4}\right)\left(\frac{k_2k_5}{k_7}+k_5+k_5k_6k_7\right)} \quad (11.a.30)$$

Having found all of the  $b$ 's of the model, we have determined that it is identified.



**Type II:**

As noted for the uniform association and row effects models, a model for this data set can contain no more than fifteen parameters. If all of the interaction terms of the missing data indicators and the column variable and the missing data indicators and the row variable are included in this loglinear model, seventeen parameters are required. Thus, Type II nonignorable nonresponse with column effects for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type II model is examined below.

Type II: column effects; 3x3 table with two supplemental margins

- modified (only the linear interaction terms involving the missing data indicators and the column variable, and involving the missing data indicators and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 1   | -1  | 1   | 0   | -1  | 0    | -1   | 0    | 1    | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | -1   | 0    | 1    | 2 | 1 | 1 |
| 1      | -1  | 1   | 1   | 1   | -1  | -1  | 1   | 0   | 1   | 0    | -1   | 0    | 1    | 3 | 1 | 1 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 1   | 0   | -1  | 0    | 0    | 0    | 2    | 1 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 2    | 2 | 1 | 1 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1   | 0    | 0    | 0    | 2    | 3 | 1 | 1 |
| 1      | 1   | 1   | -1  | 1   | -1  | 1   | 1   | 0   | -1  | 0    | 1    | 0    | 3    | 1 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 1    | 0    | 3    | 2 | 1 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 3    | 3 | 1 | 1 |
| 1      | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 0   | 0   | 0    | 0    | 0    | 1    | 1 | 2 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1    | 2 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | -1  | -1  | 0   | 0   | 0   | 0    | 0    | 0    | 1    | 3 | 2 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2    | 1 | 2 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2    | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2    | 3 | 2 | 2 |
| 1      | 1   | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 3    | 1 | 2 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3    | 2 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 3    | 3 | 2 | 2 |
| 1      | -1  | 1   | -1  | 1   | 1   | -1  | 0   | 1   | 0   | -1   | 0    | -1   | 1    | 1 | 3 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | -1   | 1    | 2 | 3 | 3 |
| 1      | -1  | 1   | 1   | 1   | -1  | -1  | 0   | 1   | 0   | 1    | 0    | -1   | 1    | 3 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 0    | 2    | 1 | 3 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 2    | 2 | 3 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | 0    | 2    | 3 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | -1  | 1   | 0   | 1   | 0   | -1   | 0    | 1    | 3    | 1 | 3 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 1    | 3    | 2 | 3 | 3 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 3    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh

columns, the interaction of the column variable and the linear row variable term, the eighth and ninth columns, the missing data indicators, the tenth and eleventh columns, the interactions of the missing data indicators and the linear column variable term, and the twelfth and thirteenth columns, the interactions of the missing data indicators and the linear row variable term. The thirteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f_1, f_2, \dots, f_9$ , are available for the first nine elements of vector  $Y$ , and  $s_1 - s_6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b_1 - b_2 + b_3 - b_4 + b_5 + b_6 - b_7 + b_8 - b_{10} - b_{12} = \ln f_1 \quad (11.b.1)$$

$$b_1 - b_2 + b_3 + b_8 - b_{12} = \ln f_2 \quad (11.b.2)$$

$$b_1 - b_2 + b_3 + b_4 + b_5 - b_6 - b_7 + b_8 + b_{10} - b_{12} = \ln f_3 \quad (11.b.3)$$

$$b_1 - b_4 + b_5 + b_8 - b_{10} = \ln f_4 \quad (11.b.4)$$

$$b_1 + b_8 = \ln f_5 \quad (11.b.5)$$

$$b1 + b4 + b5 + b8 + b10 = \ln f6 \quad (11.b.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 + b8 - b10 + b12 = \ln f7 \quad (11.b.7)$$

$$b1 + b2 + b3 + b8 + b12 = \ln f8 \quad (11.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b10 + b12 = \ln f9 \quad (11.b.9)$$

Here,  $b1$ , ... ,  $b8$ ,  $b10$  and  $b12$  are ten of the thirteen parameters of the matrix  $\beta$ , which are to be determined. It is to be noted that equations (11.b.1) - (11.b.9) are exactly the same as equations (4.c.1) - (4.c.9) (page 119), but with  $b9$  and  $b10$  of equations (4.c.1) - (4.c.9) replaced, respectively, by  $b10$  and  $b12$  in equations (11.b.1) - (11.b.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (11.b.1) - (11.b.9), as follows:

$$b1 + b8 = \ln f5 \quad (11.b.10)$$

$$b2 + b12 = \frac{1}{2} \ln \frac{f8}{f2} \quad (11.b.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (11.b.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (11.b.13)$$

$$b4 + b10 = \frac{1}{2} \ln \frac{f6}{f4} \quad (11.b.14)$$

$$b7 = \frac{1}{2} \ln \frac{f2 f6 f7}{f3 f4 f8} \quad (11.b.15)$$

$$b6 = \frac{1}{2} \ln \frac{f1 f6}{f3 f4} \quad (11.b.16)$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (11.b.1) - (11.b.9), nor can  $b2$  and  $b12$ , and  $b4$  and  $b10$ . Only four of the thirteen model parameters ( $b3$ ,  $b5$ ,  $b6$  and  $b7$ ) have been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3)[\exp(-b4 + b5 + b6 - b7) + 1 + \exp(b4 + b5 - b6 - b7)] = s1 \quad (11.b.17)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (11.b.18)$$

$$\exp(b1 + b2 + b3)[\exp(-b4 + b5 - b6 + b7) + 1 + \exp(b4 + b5 + b6 + b7)] = s3 \quad (11.b.19)$$

$$\exp(b1 - b4 + b5 + b9 - b11)[\exp(-b2 + b3 + b6 - b7 - b13) + 1 + \exp(b2 + b3 - b6 + b7 + b13)] = s4 \quad (11.b.20)$$

$$\exp(b1 + b9)[\exp(-b2 + b3 - b13) + 1 + \exp(b2 + b3 + b13)] = s5 \quad (11.b.21)$$

$$\exp(b1 + b4 + b5 + b9 + b11)[\exp(-b2 + b3 - b6 - b7 - b13) + 1 + \exp(b2 + b3 + b6 + b7 + b13)] = s6 \quad (11.b.22)$$

Equations (11.b.17) - (11.b.19) are identical to equations (4.c.17) - (4.c.19). As was shown on pages 120 - 122, these three equations produce a fourth degree polynomial in  $\exp(b4)$ . While there may be a single solution for  $b4$ , given a particular data set, there may also be no solutions for  $b4$ , or several solutions. As was noted on page 121, if there is a unique solution for  $b4$ , there will also be unique solutions for  $b1$  and  $b2$  (and so for  $b8$ ,  $b10$  and  $b12$ ). If no such unique solution exists, the model is not identified.

Let us assume that there is a unique solution for  $b4$  (and thus for  $b1$ ,  $b2$ ,  $b8$ ,  $b10$  and  $b12$ ).

The only remaining unknown  $b$ 's are then  $b9$ ,  $b11$  and  $b13$ . They appear only in equations (11.b.20) - (11.b.22). We will attempt to find these remaining  $b$ 's.

Let  $m1 = \exp(b1)$ ,  $m2 = \exp(-b2 + b3)$ ,  $m3 = \exp(-b4 + b5)$ ,  $m4 = \exp(b6 - b7)$ ,  
 $m5 = \exp(b2 + b3)$ ,  $m6 = \exp(b4 + b5)$ , and  $m7 = \exp(b6 + b7)$ .

Then equations (11.b.20) - (11.b.22) can be expressed as

$$m1m2m3m4 \exp(b9 - b11 - b13) + m1m3 \exp(b9 - b11) + \frac{m1m3m5}{m4} \exp(b9 - b11 + b13) = s4 \quad (11.b.23)$$

$$m1m2 \exp(b9 - b13) + m1 \exp(b9) + m1m5 \exp(b9 + b13) = s5 \quad (11.b.24)$$

$$\frac{m1m2m6}{m7} \exp(b9 + b11 - b13) + m1m6 \exp(b9 + b11) + m1m5m6m7 \exp(b9 + b11 + b13) = s6 \quad (11.b.25)$$

From equation (11.b.24),

$$\exp(b_9) = \frac{s^5}{m_1 m_2 \exp(-b_{13}) + m_1 + m_1 m_5 \exp(b_{13})} \quad (11.b.26)$$

Thus, given  $b_{13}$ ,  $b_9$  is also uniquely determined.

Dividing (11.b.23) by (11.b.24), and (11.b.25) by (11.b.24), we have

$$\begin{aligned} m_2 m_3 m_4 \exp(-b_{11} - b_{13}) + m_3 \exp(-b_{11}) + \frac{m_3 m_5}{m_4} \exp(-b_{11} + b_{13}) \\ = \frac{s^4}{s^5} (m_2 \exp(-b_{13}) + 1 + m_5 \exp(b_{13})) \end{aligned} \quad (11.b.27)$$

$$\begin{aligned} \frac{m_2 m_6}{m_7} \exp(b_{11} - b_{13}) + m_6 \exp(b_{11}) + m_5 m_6 m_7 \exp(b_{11} + b_{13}) \\ = \frac{s^6}{s^5} (m_2 \exp(-b_{13}) + 1 + m_5 \exp(b_{13})) \end{aligned} \quad (11.b.28)$$

From (11.b.27),

$$\exp(b_{11}) = \frac{s^5 \left( m_2 m_3 m_4 \exp(-b_{13}) + m_3 + \frac{m_3 m_5}{m_4} \exp(b_{13}) \right)}{s^4 (m_2 \exp(-b_{13}) + 1 + m_5 \exp(b_{13}))} \quad (11.b.29)$$

From (11.b.28),

$$\exp(b_{11}) = \frac{s^6 (m_2 \exp(-b_{13}) + 1 + m_5 \exp(b_{13}))}{s^5 \left( \frac{m_2 m_6}{m_7} \exp(-b_{13}) + m_6 + m_5 m_6 m_7 \exp(b_{13}) \right)} \quad (11.b.30)$$

As can be seen from each of equations (11.b.29) and (11.b.30), given  $b_{13}$ ,  $b_{11}$  is also uniquely determined.

Further, equating the two expressions for  $\exp(b_{11})$  in equations (11.b.29) and (11.b.30) and collecting terms, we have

$$\begin{aligned}
& \exp(-2b13) \left[ s4s6(m2)^2 - (s5)^2(m2)^2 m3m4 \frac{m6}{m7} \right] + \exp(-b13) \left[ 2m2s4s6 - m2m3 \left( \frac{m6}{m7} + m4m6 \right) (s5)^2 \right] \\
& + \left[ 2m2m5s4s6 + s4s6 - m3m6(s5)^2 \left( \frac{m2m5}{m4m7} + 1 + m2m4m5m7 \right) \right] \\
& + \exp(b13) \left[ 2m5s4s6 - m3m5 \left( \frac{m6}{m4} + m6m7 \right) (s5)^2 \right] + \exp(2b13) \left[ s4s6(m5)^2 - m3(m5)^2 m6 \frac{m7}{m4} (s5)^2 \right] = 0
\end{aligned}$$

If the above equation is multiplied by  $\exp(2b13)$  and if the substitution  $x = \exp(b13)$  is made, we have

$$\begin{aligned}
& \left[ s4s6(m2)^2 - (s5)^2(m2)^2 m3m4 \frac{m6}{m7} \right] + x \left[ 2m2s4s6 - m2m3 \left( \frac{m6}{m7} + m4m6 \right) (s5)^2 \right] \\
& + x^2 \left[ 2m2m5s4s6 + s4s6 - m3m6(s5)^2 \left( \frac{m2m5}{m4m7} + 1 + m2m4m5m7 \right) \right] \\
& + x^3 \left[ 2m5s4s6 - m3m5 \left( \frac{m6}{m4} + m6m7 \right) (s5)^2 \right] + x^4 \left[ s4s6(m5)^2 - m3(m5)^2 m6 \frac{m7}{m4} (s5)^2 \right] = 0
\end{aligned}$$

(11.b.31)

Thus, equations (11.b.20) - (11.b.22) generate a fourth degree polynomial in  $\exp(b13)$ . Since  $\exp(b13) > 0$ , we seek only positive solutions to (11.b.31). As noted above, if  $b13$  is uniquely determined, then so will be  $b9$  and  $b11$ , and the model will be identified. The uniqueness, or lack of uniqueness, of a solution for  $b13$  will be determined empirically for a given data set.

In summary, the identification status of this model first hinges on the uniqueness (or nonuniqueness) of a solution for  $b4$ . If  $b4$  is not uniquely determined, the model is not identified. If  $b4$  is uniquely determined, the identification status of the model then depends on the uniqueness (or nonuniqueness) of a solution for  $b13$ . If  $b13$  is also uniquely determined, then so are  $b9$  and  $b11$ , and the model is identified. On the other hand, if  $b13$  is not uniquely determined, then neither are  $b9$  and  $b11$ , and the model is not identified. The identification status of this model will be determined empirically for a given data set.

**Type III:**

As was discussed in connection with the independence, uniform association and row effects models, if all the interaction terms of the missing data indicators and the column variable, the missing data indicators and the row variable, and the three way interactions of the missing data indicators, the column variable, and the row variable are included in the model, there would be too many parameters, given the data. Thus, Type III nonignorable nonresponse with column effects for the row and column variables cannot be modeled for this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type III model is examined below.





In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth and seventh columns, the interactions of the column variable and the linear row variable term, the eighth and ninth columns, the missing data indicators, the tenth and eleventh columns, the interactions of the missing data indicators and the linear column variable term, the twelfth and thirteenth columns, the interactions of the missing data indicators and the linear row variable term, and the fourteenth and fifteenth columns, the three way interactions of the missing data indicators with the linear row variable term and the linear column variable term. The fifteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the modified Type II case that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b6 - b7 + b8 - b10 - b12 + b14 = \ln f1 \quad (11.c.1)$$

$$b1 - b2 + b3 + b8 - b12 = \ln f2 \quad (11.c.2)$$

$$b1 - b2 + b3 + b4 + b5 - b6 - b7 + b8 + b10 - b12 - b14 = \ln f3 \quad (11.c.3)$$

$$b1 - b4 + b5 + b8 - b10 = \ln f4 \quad (11.c.4)$$

$$b1 + b8 = \ln f5 \quad (11.c.5)$$

$$b1 + b4 + b5 + b8 + b10 = \ln f6 \quad (11.c.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 + b7 + b8 - b10 + b12 - b14 = \ln f7 \quad (11.c.7)$$

$$b1 + b2 + b3 + b8 + b12 = \ln f8 \quad (11.c.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b10 + b12 + b14 = \ln f9 \quad (11.c.9)$$

Here,  $b1 - b8$ ,  $b10$ ,  $b12$ , and  $b14$  are eleven of the fifteen parameters of the matrix  $\beta$ , which are to be determined. It is to be noted that equations (11.c.1) - (11.c.9) are exactly the same as equations (4.d.1) - (4.d.9) (page 125), but with  $b9$ ,  $b10$  and  $b11$  of equations (4.d.1) - (4.d.9) replaced, respectively, by  $b10$ ,  $b12$  and  $b14$  in equations (11.c.1) - (11.c.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (11.c.1) - (11.c.9), as follows:

$$b1 + b8 = \ln f5 \quad (11.c.10)$$

$$b2 + b12 = \frac{1}{2} \ln \frac{f8}{f2} \quad (11.c.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (11.c.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (11.c.13)$$

$$b4 + b10 = \frac{1}{2} \ln \frac{f6}{f4} \quad (11.c.14)$$

$$b7 = \frac{1}{2} \ln \frac{f2 f6 f7}{f3 f4 f8} \quad (11.c.15)$$

$$b7 = \frac{1}{2} \ln \frac{f2 f6 f7}{f3 f4 f8} \quad (11.c.16)$$

$$b6 + b14 = \frac{1}{4} \ln \frac{f1 f6}{f3 f4}$$

Note that  $b1$  and  $b8$  cannot be determined uniquely from equations (11.c.1) - (11.c.9), nor can  $b2$  and  $b12$ ,  $b4$  and  $b10$ , and  $b6$  and  $b14$ . Only three of the fifteen model parameters ( $b3$ ,  $b5$  and  $b7$ ) have been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3)[\exp(-b4 + b5 + b6 - b7) + 1 + \exp(b4 + b5 - b6 - b7)] = s1 \quad (11.c.17)$$

$$\exp(b1 - b4 + b5) + \exp(b1) + \exp(b1 + b4 + b5) = s2 \quad (11.c.18)$$

$$\exp(b1 + b2 + b3)[\exp(-b4 + b5 - b6 + b7) + 1 + \exp(b4 + b5 + b6 + b7)] = s3 \quad (11.c.19)$$

$$\exp(b1 - b4 + b5 + b9 - b11)[\exp(-b2 + b3 + b6 - b7 - b13 + b15) + 1 + \exp(b2 + b3 - b6 + b7 + b13 - b15)] = s4 \quad (11.c.20)$$

$$\exp(b1 + b9)[\exp(-b2 + b3 - b13) + 1 + \exp(b2 + b3 + b13)] = s5 \quad (11.c.21)$$

$$\exp(b1 + b4 + b5 + b9 + b11)[\exp(-b2 + b3 - b6 - b7 - b13 - b15) + 1 + \exp(b2 + b3 + b6 + b7 + b13 + b15)] = s6 \quad (11.c.22)$$

Equations (11.c.17) - (11.c.19) are identical to equations (4.d.17) - (4.d.19). These equations contain the four unknowns  $b1$ ,  $b2$ ,  $b4$ , and  $b6$ . These unknowns cannot, therefore, each be determined uniquely from these three equations. Further, equations (11.c.20) - (11.c.22) are the only equations which contain the unknowns  $b9$ ,  $b11$ ,  $b13$ , and  $b15$ . The three equations (11.c.20) - (11.c.22) are insufficient to solve uniquely for the four unknowns  $b9$ ,  $b11$ ,  $b13$ , and  $b15$ . Thus, this model is not identified.

In summary, if the row and column variables are related by column effects, and nonignorable nonresponse is to be modeled, the Type I model is identified, while the Type III model, (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicator and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 2

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the column effects models hypothesized to fit Data Set 2 (page 219). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicator).

#### Data Set 2

##### Column Effects Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1741.16         | 261.68       | 65.16           | 2068    |
|                     | single crime    | 225.10          | 61.79        | 28.10           | 314.99  |
|                     | multiple crimes | 66.74           | 23.52        | 13.74           | 104     |
|                     |                 | 2033            | 346.99       | 107             | 2486.99 |

$\beta = (4.12, -1.20, .24, -1.04, .25, .43, -.18)$

Likelihood Ratio fit statistic = 4.40433

Degrees of freedom = 2

**Data Set 2**

Column Effects Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1664.82         | 282.48       | 85.28           | 2032.58 |
|                     | single crime    | 231.58          | 70.41        | 28.99           | 330.98  |
|                     | multiple crimes | 66.69           | 36.34        | 20.41           | 123.44  |
|                     |                 | 1963.09         | 389.23       | 134.68          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 834.75          | 141.64       | 42.76           | 1019.15 |
|                     | single crime    | 116.12          | 35.31        | 14.54           | 165.96  |
|                     | multiple crimes | 33.44           | 18.22        | 10.23           | 61.89   |
|                     |                 | 984.31          | 195.17       | 67.53           | 1247.01 |

**Data Set 2**

Column Effects - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 662.05          | 112.33       | 33.91           | 808.29 |
|                     | single crime    | 92.09           | 28.00        | 11.53           | 131.62 |
|                     | multiple crimes | 26.52           | 14.45        | 8.11            | 49.08  |
|                     |                 | 780.66          | 154.78       | 53.56           | 988.99 |

$$\beta = (3.56, -1.03, .36, -1.04, .15, .45, -.14, .69, -.23)$$

Likelihood Ratio fit statistic = 50.6083

Degrees of freedom = 6

**Data Set 2****Column Effects Model - Two Supplemental Margins****Missing at Random (MAR)****Completely Classified Data**

|                             |                        | <b>First Interview</b> |                     |                        |         |
|-----------------------------|------------------------|------------------------|---------------------|------------------------|---------|
|                             |                        | <b>crime free</b>      | <b>single crime</b> | <b>multiple crimes</b> |         |
| <b>Second<br/>Interview</b> | <b>crime free</b>      | 1740.26                | 257.31              | 70.43                  | 2068    |
|                             | <b>single crime</b>    | 231.81                 | 60.41               | 22.78                  | 315     |
|                             | <b>multiple crimes</b> | 61.25                  | 28.14               | 14.62                  | 104.01  |
|                             |                        | 2033.32                | 345.86              | 107.83                 | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                             |                        | <b>First Interview</b> |                     |                        |      |
|-----------------------------|------------------------|------------------------|---------------------|------------------------|------|
|                             |                        | <b>crime free</b>      | <b>single crime</b> | <b>multiple crimes</b> |      |
| <b>Second<br/>Interview</b> | <b>crime free</b>      | 836.47                 | 123.68              | 33.85                  | 994  |
|                             | <b>single crime</b>    | 130.25                 | 33.95               | 12.80                  | 177  |
|                             | <b>multiple crimes</b> | 44.76                  | 20.56               | 10.68                  | 76   |
|                             |                        | 1011.48                | 178.19              | 57.33                  | 1247 |



**Data Set 2**

Column Effects Model - Two Supplemental Margins

Missing at Random (MAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 80.42           | 11.89        | 3.25            | 95.56  |
|                     | single crime    | 546.31          | 142.38       | 53.69           | 742.38 |
|                     | multiple crimes | 88.96           | 40.87        | 21.23           | 151.06 |
|                     |                 | 715.69          | 195.14       | 78.17           | 989    |

$$\beta = (3.52, -.90, .40, -1.16, .18, .44, -.12, .58, 1.43, -.21, 1.51, -.05, -2.26)$$

Likelihood Ratio fit statistic = 5.35846

Degrees of freedom = 2

**Data Set 2****Column Effects Model - Two Supplemental Margins****Nonignorable Nonresponse - Type I****Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1741.05         | 257.713      | 70.20           | 2068.96 |
|                     | single crime    | 230.24          | 60.50        | 22.36           | 313.10  |
|                     | multiple crimes | 61.72           | 28.79        | 14.44           | 104.95  |
|                     |                 | 2033.01         | 347          | 107             | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 644.30          | 207.64       | 141.10          | 993.04  |
|                     | single crime    | 85.20           | 48.75        | 44.95           | 178.90  |
|                     | multiple crimes | 22.84           | 23.19        | 29.02           | 75.06   |
|                     |                 | 752.34          | 279.58       | 215.07          | 1246.99 |

**Data Set 2**

Column Effects Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 613.18          | 144.08       | 51.83           | 809.09 |
|                     | single crime    | 81.09           | 33.82        | 16.51           | 131.42 |
|                     | multiple crimes | 21.74           | 16.09        | 10.66           | 48.49  |
|                     |                 | 716             | 194          | 79              | 989    |

$$\beta = (3.89, -1.10, .35, -.32, .24, .44, -.13, .22, -.37, -.85, -.48, -.07, -.16)$$

Likelihood Ratio fit statistic = 5.37314

Degrees of freedom = 2

## Data Set 2

### Column Effects Model - Two Supplemental Margins

#### Nonignorable Nonresponse - Type II (linear interaction terms only)

As indicated on page 311, the identification status of the modified nonignorable nonresponse Type II model with column effects depends on the number of solutions to equations (4.c.26) and (11.b.31). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (4.c.26) giving the following fourth degree polynomial

$$-.20+.07x+.60x^2-.23x^3-.20x^4 = 0$$

By Descartes' Rule of Signs, since there are two changes of sign in this fourth degree polynomial equation, there are two or no positive solutions for  $x = \exp(b_4)$ . Using Newton's Method, there are two positive solutions for  $x = \exp(b_4)$ . Thus, the model is not identified for this data set.

R + C - Data Set 2

Type I: R + C; 3x3 table with two supplemental margins

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 1   | 0   | -1   | 0    | 1    | 0    |      | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    |      | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | 1   | 1   | -1  | -1  | 1   | 0    | 1    | 0    | 1    | 0    | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 1    | 0    | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 1    | 0    | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 1   | 0   | -1   | 0    | 1    | 0    |      | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0    | 1    | 0    | 1    | 0    | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | 1   | 1   | -1  | -1  | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 0   | 1   | 0    | -1   | 0    | 1    |      | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    |      | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | 1   | 1   | -1  | -1  | 0   | 1    | 0    | 1    | 0    | 1    | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 1    |      | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    |      | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 1    |      | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | 1   | 0    | -1   | 0    | 1    |      | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    |      | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1    | 0    | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the interaction of the quadratic row variable term and the linear column variable term, the seventh and eighth columns, the interactions of the column variable and the linear row variable term, the ninth and tenth columns, the missing data indicators, and the eleventh

through fourteenth columns, the interactions of the missing data indicators and the column variable. The fourteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for all of the association models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 215) is the same as for the other association models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $\beta$ 's) of the model.

As discussed earlier for the other association models, for the portion of the vector  $Y$  corresponding to the completely classified data (and the first nine rows of  $X$ , the model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or  $\ln(E(Y)) = X\beta$ .

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 - b6 + b7 - b8 + b9 - b11 + b13 = \ln f1 \quad (12.a.1)$$

$$b1 - b2 + b3 + b9 = \ln f2 \quad (12.a.2)$$

$$b1 - b2 + b3 + b4 + b5 + b6 - b7 - b8 + b9 + b11 + b13 = \ln f3 \quad (12.a.3)$$

$$b1 - b4 + b5 + b9 - b11 + b13 = \ln f4 \quad (12.a.4)$$

$$b1 + b9 = \ln f5 \quad (12.a.5)$$

$$b1 + b4 + b5 + b9 + b11 + b13 = \ln f6 \quad (12.a.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 + b9 - b11 + b13 = \ln f7 \quad (12.a.7)$$

$$b1 + b2 + b3 + b9 = \ln f8 \quad (12.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b11 + b13 = \ln f9 \quad (12.a.9)$$

Here,  $b1 - b9$ ,  $b11$  and  $b13$  are eleven of the fourteen parameters of the matrix  $\beta$ , which are to be determined. It is to be noted that equations (12.a.1) - (12.a.9) are exactly the same as equations (5.a.1) - (5.a.9) (pages 134 - 135), but with  $b10$  and  $b11$  of equations (5.a.1) - (5.a.9) replaced by  $b11$  and  $b13$ , respectively, in equations (12.a.1) - (12.a.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (12.a.1) - (12.a.9), as follows:

$$b1 + b9 = \ln f5 \quad (12.a.10)$$

$$b2 = \frac{1}{2} \ln \frac{f8}{f2} \quad (12.a.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (12.a.12)$$

$$b5 + b13 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (12.a.13)$$

$$b4 + b11 = \frac{1}{2} \ln \frac{f6}{f4} \quad (12.a.14)$$

$$b6 = \frac{1}{4} \ln \frac{f3 f9 (f4)^2}{f1 f7 (f6)^2} \quad (12.a.15)$$

$$b7 = \frac{1}{4} \ln \frac{f1 f9}{f3 f7} \quad (12.a.16)$$

$$b8 = \frac{1}{4} \ln \frac{f7 f9 (f2)^2}{f1 f3 (f8)^2} \quad (12.a.17)$$

Note that  $b_1$  and  $b_9$  cannot be determined uniquely from equations (12.a.1) - (12.a.9), since whenever  $b_1$  appears in an equation, so does  $b_9$ ; that is  $b_1 + b_9$  appears as a unit in each of the equations. Similarly,  $b_5$  and  $b_{13}$  cannot be determined uniquely, nor can  $b_4$  and  $b_{11}$ . Only five of the fourteen model parameters ( $b_2, b_3, b_6, b_7$  and  $b_8$ ) have been uniquely determined. Equations (12.a.10) - (12.a.17) provide the most complete solution for the  $b$ 's which can be determined solely from equations (12.a.1) - (12.a.9).

The last six elements of vector  $Y$  correspond to the partially classified data (and the last eighteen rows of the  $X$  matrix). Each of these last six elements of vector  $Y$  represents a sum of cell frequencies in a given row of one of the two (hypothetical) contingency tables for the partially classified data. Let  $f_{10}, \dots, f_{18}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with supplemental row marginals, and  $f_{19}, \dots, f_{27}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with supplemental column marginals. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b_1 - b_2 + b_3)[\exp(-b_4 + b_5 - b_6 + b_7 - b_8) + 1 + \exp(b_4 + b_5 + b_6 - b_7 - b_8)] = s_1 \quad (12.a.18)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (12.a.19)$$

$$\exp(b_1 + b_2 + b_3)[\exp(-b_4 + b_5 - b_6 - b_7 + b_8) + 1 + \exp(b_4 + b_5 + b_6 + b_7 + b_8)] = s_3 \quad (12.a.20)$$

$$\exp(b_1 - b_4 + b_5 + b_{10} - b_{12} + b_{14})[\exp(-b_2 + b_3 - b_6 + b_7 - b_8) + 1 + \exp(b_2 + b_3 - b_6 - b_7 + b_8)] = s_4 \quad (12.a.21)$$

$$\exp(b_1 + b_{10})[\exp(-b_2 + b_3) + 1 + \exp(b_2 + b_3)] = s_5 \quad (12.a.22)$$

$$\exp(b_1 + b_4 + b_5 + b_{10} + b_{12} + b_{14})[\exp(-b_2 + b_3 + b_6 - b_7 - b_8) + 1 + \exp(b_2 + b_3 + b_6 + b_7 + b_8)] = s_6 \quad (12.a.23)$$

Here,  $s_1 = f_{10} + f_{11} + f_{12}$ , the first row marginal for the partially classified data,



$s_2 = f_{13} + f_{14} + f_{15}$ , the second row marginal for the partially classified data,

$s_3 = f_{16} + f_{17} + f_{18}$ , the third row marginal for the partially classified data,

$s_4 = f_{19} + f_{22} + f_{25}$ , the first column marginal for the partially classified data,

$s_5 = f_{20} + f_{23} + f_{26}$ , the second column marginal for the partially classified data, and

$s_6 = f_{21} + f_{24} + f_{27}$ , the third column marginal for the partially classified data.

It is assumed that we can obtain estimates  $s_1 - s_6$ , for the last six elements of vector  $Y$ .

Equations (12.a.18) - (12.a.20) are identical to equations (5.a.20) - (5.a.22). As was shown on pages 137 - 140, these equations are sufficient to solve uniquely for  $b_1$ ,  $b_4$  and  $b_5$ . As noted on page 140, these, together with equations (12.a.10), (12.a.14), and (12.a.13), also allow for the unique solutions of  $b_9$ ,  $b_{11}$  and  $b_{13}$ , respectively. The only remaining unknown  $b$ 's are thus  $b_{10}$ ,  $b_{12}$  and  $b_{14}$ . They appear only in equations (12.a.21) - (12.a.23). We will attempt to find these remaining  $b$ 's.

Let  $k_1 = \exp(b_1)$ ,  $k_2 = \exp(-b_2 + b_3)$ ,  $k_3 = \exp(-b_4 + b_5)$ ,  $k_4 = \exp(b_6)$ ,  $k_5 = \exp(b_7)$ ,  $k_6 = \exp(b_8)$ ,  $k_7 = \exp(b_4 + b_5)$ , and  $k_8 = \exp(b_2 + b_3)$ .

Then equations (12.a.21) - (12.a.23) can be rewritten as

$$\frac{k_1 k_2 k_3 k_5}{k_4 k_6} \exp(b_{10} - b_{12} + b_{14}) + k_1 k_3 \exp(b_{10} - b_{12} + b_{14}) + \frac{k_1 k_3 k_6 k_8}{k_4 k_5} \exp(b_{10} - b_{12} + b_{14}) = s_4 \quad (12.a.24)$$

$$k_1 k_2 \exp(b_{10}) + k_1 \exp(b_{10}) + k_1 k_8 \exp(b_{10}) = s_5 \quad (12.a.25)$$

$$\frac{k_1 k_2 k_4 k_7}{k_5 k_6} \exp(b_{10} + b_{12} + b_{14}) + k_1 k_7 \exp(b_{10} + b_{12} + b_{14}) + k_1 k_4 k_5 k_6 k_7 k_8 \exp(b_{10} + b_{12} + b_{14}) = s_6 \quad (12.a.26)$$

From equation (12.a.25),  $\exp(b_{10}) = \frac{s_5}{k_1 k_2 + k_1 + k_1 k_8}$ , or

$$b_{10} = \ln \frac{s_5}{k_1(k_2 + 1 + k_8)} \quad (12.a.27)$$

Dividing (12.a.24) by (12.a.25), and (12.a.26) by (12.a.25), we have

$$\exp(-b_{12} + b_{14}) = \frac{s^4(k_2 + 1 + k_8)}{s^5\left(\frac{k_2 k_3 k_5}{k_4 k_6} + k_3 + \frac{k_3 k_6 k_8}{k_4 k_5}\right)}$$

$$\exp(b_{12} + b_{14}) = \frac{s^6(k_2 + 1 + k_8)}{s^5\left(\frac{k_2 k_4 k_7}{k_5 k_6} + k_7 + k_4 k_5 k_6 k_7 k_8\right)}$$

or

$$-b_{12} + b_{14} = \ln \frac{s^4(k_2 + 1 + k_8)}{s^5\left(\frac{k_2 k_3 k_5}{k_4 k_6} + k_3 + \frac{k_3 k_6 k_8}{k_4 k_5}\right)} \quad (12.a.28)$$

and

$$b_{12} + b_{14} = \frac{s^6(k_2 + 1 + k_8)}{s^5\left(\frac{k_2 k_4 k_7}{k_5 k_6} + k_7 + k_4 k_5 k_6 k_7 k_8\right)} \quad (12.a.29)$$

$$\text{Thus, } b_{12} = \frac{1}{2} \ln \frac{s^6\left(\frac{k_2 k_3 k_5}{k_4 k_6} + k_3 + \frac{k_3 k_6 k_8}{k_4 k_5}\right)}{s^4\left(\frac{k_2 k_4 k_7}{k_5 k_6} + k_7 + k_4 k_5 k_6 k_7 k_8\right)} \quad (12.a.30)$$

$$\text{and } b_{14} = \frac{1}{2} \ln \frac{s^4 s^6 (k_2 + 1 + k_8)^2}{(s^5)^2 \left(\frac{k_2 k_3 k_5}{k_4 k_6} + k_3 + \frac{k_3 k_6 k_8}{k_4 k_5}\right) \left(\frac{k_2 k_4 k_7}{k_5 k_6} + k_7 + k_4 k_5 k_6 k_7 k_8\right)} \quad (12.a.31)$$

Having found all of the  $b$ 's of the model, we have determined that it is identified.

**Type II:**

As noted for the association models considered earlier, a model for this data set can contain no more than fifteen parameters. If all of the interaction terms of the missing data indicators and the column variable and the missing data indicators and the row variable are included in this loglinear model, eighteen parameters are required. Thus, Type II nonignorable nonresponse for the  $R + C$  model cannot be explored for this data set. We can, however, investigate whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type II model is examined below.

Type II: R + C; 3x3 table with two supplemental margins

- modified (only the linear interaction terms involving the missing data indicators and the column variable, and involving the missing data indicators and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 1   | 0   | -1   | 0    | -1   | 0    |      | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | -1   | 0    | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | 1   | 1   | -1  | -1  | 1   | 0    | 1    | 0    | -1   | 0    | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 0    | 0    | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 0    | 0    | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 1   | 0   | -1   | 0    | 1    | 0    |      | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 1    | 0    | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0    | 1    | 0    | 1    | 0    | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 0   | 1   | 0    | -1   | 0    | -1   |      | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | -1   |      | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | 1   | -1  | -1  | 0   | 1   | 0    | 1    | 0    | -1   |      | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 0    |      | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    |      | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 0    |      | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | -1  | -1  | 1   | 0   | 1   | 0    | -1   | 0    | 1    |      | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 1    |      | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 1    |      | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth column, the

interaction of the quadratic row variable term and the linear column variable term, the seventh and eighth columns, the interactions of the column variable and the linear row variable term, the ninth and tenth columns, the missing data indicators, the eleventh and twelfth columns, the interactions of the missing data indicators and the linear column variable term, and the thirteenth and fourteenth columns, the interactions of the missing data indicators and the linear row variable term. The fourteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f_1, f_2, \dots, f_9$ , are available for the first nine elements of vector  $Y$ , and  $s_1 - s_6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - b_8 + b_9 - b_{11} - b_{13} = \ln f_1 \quad (12.b.1)$$

$$b_1 - b_2 + b_3 + b_9 - b_{13} = \ln f_2 \quad (12.b.2)$$

$$b_1 - b_2 + b_3 + b_4 + b_5 + b_6 - b_7 - b_8 + b_9 + b_{11} - b_{13} = \ln f_3 \quad (12.b.3)$$

$$b1 - b4 + b5 + b9 - b11 = \ln f4 \quad (12.b.4)$$

$$b1 + b9 = \ln f5 \quad (12.b.5)$$

$$b1 + b4 + b5 + b9 + b11 = \ln f6 \quad (12.b.6)$$

$$b1 + b2 + b3 - b4 + b5 - b6 - b7 + b8 + b9 - b11 + b13 = \ln f7 \quad (12.b.7)$$

$$b1 + b2 + b3 + b9 + b13 = \ln f8 \quad (12.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8 + b9 + b11 + b13 = \ln f9 \quad (12.b.9)$$

Here,  $b1$ , ... ,  $b9$ ,  $b11$  and  $b13$  are eleven of the fourteen parameters of the matrix  $\beta$ , which are to be determined. It is to be noted that equations (12.b.1) - (12.b.9) are exactly the same as equations (5.b.1) - (5.b.9) (page 143), but with  $b10$  and  $b11$  of equations (5.b.1) - (5.b.9) replaced, respectively, by  $b11$  and  $b13$  in equations (12.b.1) - (12.b.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (12.b.1) - (12.b.9), as follows:

$$b1 + b9 = \ln f5 \quad (12.b.10)$$

$$b2 + b13 = \frac{1}{2} \ln \frac{f8}{f2} \quad (12.b.11)$$

$$b3 = \frac{1}{2} \ln \frac{f2 f8}{(f5)^2} \quad (12.b.12)$$

$$b5 = \frac{1}{2} \ln \frac{f4 f6}{(f5)^2} \quad (12.b.13)$$

$$b4 + b11 = \frac{1}{2} \ln \frac{f6}{f4} \quad (12.b.14)$$

$$b7 = \frac{1}{4} \ln \frac{f1 f9}{f3 f7} \quad (12.b.15)$$

$$b6 = \frac{1}{4} \ln \frac{f3 f9 (f4)^2}{f1 f7 (f6)^2} \quad (12.b.16)$$

$$b8 = \frac{1}{4} \ln \frac{f7 f9 (f2)^2}{f1 f3 (f8)^2} \quad (12.b.17)$$

Note that  $b1$  and  $b9$  cannot be determined uniquely from equations (12.b.1) - (12.b.9), nor can  $b2$  and  $b13$ , and  $b4$  and  $b11$ . Only five of the eleven model parameters ( $b3$ ,  $b5$ ,  $b6$ ,  $b7$  and  $b8$ ) have been uniquely determined. Equations (12.b.10) - (12.b.17) provide the most

complete solution for the  $b$ 's which can be determined solely from equations (12.b.1) - (12.b.9).

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b_1 - b_2 + b_3)[\exp(-b_4 + b_5 - b_6 + b_7 - b_8) + 1 + \exp(b_4 + b_5 + b_6 - b_7 - b_8)] = s_1 \quad (12.b.18)$$

$$\exp(b_1 - b_4 + b_5) + \exp(b_1) + \exp(b_1 + b_4 + b_5) = s_2 \quad (12.b.19)$$

$$\exp(b_1 + b_2 + b_3) \exp[-b_4 + b_5 - b_6 - b_7 + b_8) + 1 + \exp(b_4 + b_5 + b_6 + b_7 + b_8)] = s_3 \quad (12.b.20)$$

$$\exp(b_1 - b_4 + b_5 + b_{10} - b_{12})[\exp(-b_2 + b_3 - b_6 + b_7 - b_8 - b_{14}) + 1 + \exp(b_2 + b_3 - b_6 - b_7 + b_8 + b_{14})] = s_4 \quad (12.b.21)$$

$$\exp(b_1 + b_{10})[\exp(-b_2 + b_3 - b_{14}) + 1 + \exp(b_2 + b_3 + b_{14})] = s_5 \quad (12.b.22)$$

$$\exp(b_1 + b_4 + b_5 + b_{10} + b_{12})[\exp(-b_2 + b_3 + b_6 - b_7 - b_8 - b_{14}) + 1 + \exp(b_2 + b_3 + b_6 + b_7 + b_8 + b_{14})] = s_6 \quad (12.b.23)$$

Equations (12.b.18) - (12.b.20) are identical to equations (5.b.18) - (5.b.20). As was shown on pages 145 - 146, these three equations produce a fourth degree polynomial in  $\exp(b_4)$ . While there may be a single solution for  $b_4$ , given a particular data set, there may also be no solutions for  $b_4$ , or several solutions. As was noted on pages 146 and 147, if there is a unique solution for  $b_4$ , there will also be unique solutions for  $b_1$  and  $b_2$  (and so for  $b_9$ ,  $b_{11}$  and  $b_{13}$ ). If no such unique solution exists, the model is not identified.

Let us assume that there is a unique solution for  $b_4$  (and thus for  $b_1$ ,  $b_2$ ,  $b_9$ ,  $b_{11}$  and  $b_{13}$ ). The only remaining unknown  $b$ 's are then  $b_{10}$ ,  $b_{12}$  and  $b_{14}$ . They appear only in equations (12.b.21) - (12.b.23). We will attempt to find these remaining  $b$ 's.

Let  $m1 = \exp(b1)$ ,  $m2 = \exp(-b2 + b3)$ ,  $m3 = \exp(-b4 + b5)$ ,  $m4 = \exp(b6)$ ,  
 $m5 = \exp(b7)$ ,  $m6 = \exp(b8)$ ,  $m7 = \exp(b4 + b5)$  and  $m8 = \exp(b2 + b3)$ .

Then equations (12.b.21) - (12.b.23) can be expressed as

$$\frac{m1m2m3m5}{m4m6} \exp(b10 - b12 - b14) + m1m3 \exp(b10 - b12) + \frac{m1m3m6m8}{m4m5} \exp(b10 - b12 + b14) = s4 \quad (12.b.24)$$

$$m1m2 \exp(b10 - b14) + m1 \exp(b10) + m1m8 \exp(b10 + b14) = s5 \quad (12.b.25)$$

$$\frac{m1m2m4m7}{m5m6} \exp(b10 + b12 - b14) + m1m7 \exp(b10 + b12) + m1m4m5m6m7m8 \exp(b10 + b12 + b14) = s6 \quad (12.b.26)$$

From equation (12.b.25),

$$\exp(b10) = \frac{s5}{m1m2 \exp(-b14) + m1 + m1m8 \exp(b14)} \quad (12.b.27)$$

Thus, given  $b14$ ,  $b10$  is also uniquely determined.

Dividing (12.b.24) by (12.b.25), and (12.b.26) by (12.b.25), we have

$$\begin{aligned} \frac{m2m3m5}{m4m6} \exp(-b12 - b14) + m3 \exp(-b12) + \frac{m3m6m8}{m4m5} \exp(-b12 + b14) \\ = \frac{s4}{s5} (m2 \exp(-b14) + 1 + m8 \exp(b14)) \end{aligned} \quad (12.b.28)$$

$$\begin{aligned} \frac{m2m4m7}{m5m6} \exp(b12 - b14) + m7 \exp(b12) + m4m5m6m7m8 \exp(b12 + b14) \\ = \frac{s6}{s5} (m2 \exp(-b14) + 1 + m8 \exp(b14)) \end{aligned} \quad (12.b.29)$$

From (12.b.28),

$$\exp(b12) = \frac{s5 \left( \frac{m2m3m5}{m4m6} \exp(-b14) + m3 + \frac{m3m6m8}{m4m5} \exp(b14) \right)}{s4 (m2 \exp(-b14) + 1 + m8 \exp(b14))} \quad (12.b.30)$$



From (12.b.29),

$$\exp(b12) = \frac{s6(m2 \exp(-b14) + 1 + m8 \exp(b14))}{s5 \left( \frac{m2m4m7}{m5m6} \exp(-b14) + m7 + m4m5m6m7m8 \exp(b14) \right)} \quad (12.b.31)$$

Thus, equating these two expressions for  $\exp(b12)$  in (12.b.30) and (12.b.31) and collecting terms, we have

$$\begin{aligned} & \exp(-2b14) \left[ (m2)^2 \left\{ s4s6 - (s5)^2 \frac{m3m7}{(m6)^2} \right\} \right] + \exp(-b14) \left[ 2m2s4s6 - m2m3m7 \left( \frac{m5}{m4m6} + \frac{m4}{m5m6} \right) (s5)^2 \right] \\ & + \left[ 2m2m8s4s6 + s4s6 - m3m7(s5)^2 \left( m2(m5)^2 m8 + 1 + \frac{m2m8}{(m5)^2} \right) \right] \\ & + \exp(b14) \left[ 2m8s4s6 - m3m7 \left( \frac{m6m8}{m4m5} + m4m5m6m8 \right) (s5)^2 \right] + \exp(2b14) \left[ (m8)^2 \left\{ s4s6 - m3(m6)^2 m7(s5)^2 \right\} \right] = 0 \end{aligned}$$

If the above equation is multiplied by  $\exp(2b14)$  and if the substitution  $x = \exp(b14)$  is made, we have

$$\begin{aligned} & \left[ s4s6(m2)^2 - (s5)^2 \frac{(m2)^2 m3m7}{(m6)^2} \right] + x \left[ 2m2s4s6 - m2m3m7 \left( \frac{m5}{m4m6} + \frac{m4}{m5m6} \right) (s5)^2 \right] \\ & + x^2 \left[ 2m2m8s4s6 + s4s6 - m3m7(s5)^2 \left( m2(m5)^2 m8 + 1 + \frac{m2m8}{(m5)^2} \right) \right] \\ & + x^3 \left[ 2m8s4s6 - m3m7 \left( \frac{m6m8}{m4m5} + m4m5m6m8 \right) (s5)^2 \right] + x^4 \left[ s4s6(m8)^2 - m3(m6)^2 m7(m8)^2 (s5)^2 \right] = 0 \end{aligned} \quad (12.b.32)$$

Thus, equations (12.b.21) - (12.b.23) generate a fourth degree polynomial in  $\exp(b14)$ . Since  $\exp(b14) > 0$ , we seek only positive solutions to (12.b.32). The uniqueness, or lack of uniqueness, of a solution for  $b14$  will be determined empirically for a given data set. As can be seen from equations (12.b.27) and the expressions for  $\exp(b12)$  generated from (12.b.28) and (12.b.29), if  $b14$  is uniquely determined, so will be  $b10$  and  $b12$ . The

uniqueness, or lack of uniqueness, of a solution for  $b_{14}$  will be determined empirically for a given data set.

In summary, the identification status of this model first hinges on the uniqueness (or nonuniqueness) of a solution for  $b_4$ . If  $b_4$  is not uniquely determined, the model is not identified. If  $b_4$  is uniquely determined, the identification status of the model then depends on the uniqueness (or nonuniqueness) of a solution for  $b_{14}$ . If  $b_{14}$  is also uniquely determined, then so are  $b_{10}$  and  $b_{12}$ , and the model is identified. On the other hand, if  $b_{14}$  is not uniquely determined, the model is not identified.

The identification status of this model will be determined empirically for a given data set.

### Type III:

If all the interaction terms of the missing data indicators and the column variable, the missing data indicators and the row variable, and the three way interactions of the missing data indicators, the column variable and the row variable are included in the model, there would be too many parameters, given the data. Even if only the linear interaction terms involving the missing data indicators appear in the model, sixteen parameters would be required. Since a model for this data set can contain no more than fifteen parameters, even this modified Type III model cannot be considered for this data set. Thus, no Type III nonignorable nonresponse  $R + C$  models can be explored for identification for this data set.

In summary, for the  $R + C$  model with two supplemental margins, the Type I nonignorable nonresponse model is identified, while the Type III model, (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicators and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 2

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the R + C models hypothesized to fit Data Set 2 (page 219). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicators).

#### Data Set 2

##### R + C Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 1744.17         | 255.66       | 68.17           | 2068 |
|                     | single crime    | 223.66          | 64.68        | 26.66           | 315  |
|                     | multiple crimes | 65.17           | 26.66        | 12.17           | 104  |
|                     |                 | 2033            | 347          | 107             | 2487 |

$$\beta' = (4.17, -1.13, .24, -1.06, .18, -.17, .39, -.12)$$

Likelihood Ratio fit statistic = 3.28098

Degrees of freedom = 1

**Data Set 2**

R + C Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1668.72         | 280.51       | 82.84           | 2032.07 |
|                     | single crime    | 223.34          | 74.56        | 34.21           | 332.11  |
|                     | multiple crimes | 71.18           | 34.09        | 17.55           | 122.82  |
|                     |                 | 1963.24         | 389.16       | 134.60          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 836.71          | 140.65       | 41.54           | 1018.89 |
|                     | single crime    | 111.99          | 37.39        | 17.15           | 166.52  |
|                     | multiple crimes | 35.69           | 17.09        | 8.80            | 61.59   |
|                     |                 | 984.39          | 195.13       | 67.49           | 1247.01 |

**Data Set 2**

R + C Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 663.60          | 111.55       | 32.94           | 808.09 |
|                     | single crime    | 88.82           | 29.65        | 13.60           | 132.07 |
|                     | multiple crimes | 28.31           | 13.56        | 6.98            | 48.85  |
|                     |                 | 780.72          | 154.76       | 53.53           | 989.01 |

$$\beta = (3.62, -1.05, .27, -.94, .16, -.16, .40, -.12, .69, -.23)$$

Likelihood Ratio fit statistic = 48.6567

Degrees of freedom = 5

**Data Set 2**

R + C Model - Two Supplemental Margins

Missing at Random (MAR)

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 1744.17    | 255.63       | 68.20           | 2068 |
|                     | single crime    | 223.62     | 64.60        | 26.78           | 315  |
|                     | multiple crimes | 65.13      | 26.59        | 12.28           | 104  |
|                     |                 | 2032.92    | 346.82       | 107.26          | 2487 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 838.35     | 122.87       | 32.78           | 994  |
|                     | single crime    | 125.65     | 36.30        | 15.05           | 177  |
|                     | multiple crimes | 47.60      | 19.43        | 8.97            | 76   |
|                     |                 | 1011.60    | 178.60       | 56.80           | 1247 |

**Data Set 2**

R + C Model - Two Supplemental Margins

Missing at Random (MAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 234.50          | 34.37        | 9.17            | 278.04 |
|                     | single crime    | 308.38          | 89.09        | 36.93           | 434.40 |
|                     | multiple crimes | 173.21          | 70.71        | 32.65           | 276.57 |
|                     |                 | 716.08          | 194.17       | 78.74           | 989.01 |

 $\beta = (3.59, -.92, .30, -1.06, .18, -.17, .39, -.12, .58, .90, -.21, 1.28, -.05, -.89)$ 

Likelihood Ratio fit statistic = 3.28373

Degrees of freedom = 1



**Data Set 2**

R + C Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1745.38    | 256.30       | 68.34           | 2070.02 |
|                     | single crime    | 221.40     | 62.79        | 25.44           | 309.63  |
|                     | multiple crimes | 66.22      | 27.91        | 13.22           | 107.35  |
|                     |                 | 2033       | 347          | 107             | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 651.22     | 212.81       | 127.96          | 991.99  |
|                     | single crime    | 82.61      | 52.13        | 47.63           | 182.37  |
|                     | multiple crimes | 24.71      | 23.18        | 24.76           | 72.64   |
|                     |                 | 758.54     | 288.12       | 200.35          | 1247.01 |

**Data Set 2**

R + C Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 614.70          | 143.29       | 50.46           | 808.45 |
|                     | single crime    | 77.98           | 35.10        | 18.78           | 131.86 |
|                     | multiple crimes | 23.32           | 15.61        | 9.76            | 48.69  |
|                     |                 | 716             | 194          | 79              | 989    |

 $\beta = (3.95, -1.11, .30, -.28, .18, -.13, .41, -.12, .19, -.40, -.81, -.44, -.01, -.10)$ 

Likelihood Ratio fit statistic = 3.89769

Degrees of freedom = 1

## Data Set 2

### R + C Model - Two Supplemental Margins

#### Nonignorable Nonresponse - Type II (linear interaction terms only)

As indicated on pages 337 - 338, the identification status of the modified R + C nonignorable nonresponse Type II model depends on the number of solutions to equations (5.b.27) and (12.b.32). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (5.b.27) giving the following fourth degree polynomial

$$.03 - .11x - .62x^2 - 1.25x^3 - .98x^4 = 0$$

By Descartes' Rule of Signs, since there is one change of sign in this fourth degree polynomial equation, there is exactly one positive solution for  $x = \exp(b4)$ .

Further, the estimated frequencies and column marginals generated were substituted into equation (12.b.32) giving the following fourth degree polynomial

$$-105397 - 82088x - 42949x^2 + 9023.7x^3 + 4518.1x^4 = 0$$

By Descartes' Rule of Signs, since there is one change of sign in this fourth degree polynomial equation, there is exactly one positive solution for  $x = \exp(b14)$ . As discussed on pages 337 - 338, the model is therefore identified for the given data set. Given below are the parameter and contingency table estimates, together with the likelihood ratio fit statistic for this identified model.

**Data Set 2**

R + C Model - Two Supplemental Margins

Nonignorable Nonresponse - Type II (linear interaction terms only)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1745.22         | 255.78       | 68.16           | 2069.16 |
|                     | single crime    | 221.64          | 64.40        | 26.64           | 312.68  |
|                     | multiple crimes | 66.11           | 26.88        | 12.17           | 105.16  |
|                     |                 | 2032.97         | 347.06       | 106.97          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 929.57          | 56.93        | 6.34            | 992.84 |
|                     | single crime    | 156.97          | 19.06        | 3.29            | 179.33 |
|                     | multiple crimes | 62.26           | 10.58        | 2.00            | 74.84  |
|                     |                 | 1148.80         | 86.57        | 11.63           | 1247   |

**Data Set 2**

R + C Model - Two Supplemental Margins

Nonignorable Nonresponse - Type II (linear interactions only)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 315.52          | 47.49        | 13.00           | 376.01 |
|                     | single crime    | 174.31          | 52.02        | 22.10           | 248.43 |
|                     | multiple crimes | 226.19          | 94.44        | 43.93           | 364.56 |
|                     |                 | 716.03          | 193.95       | 79.03           | 989    |

$\beta = (2.948, -.842, .253, -1.932, .177, -.174, .388, -.122, 1.218, 1.004, .873, .899, -.285, 1.185)$

Likelihood Ratio fit statistic = 3.36615

Degrees of freedom = 1

Symmetry Model - Data Set 2

Type I: symmetry; 3x3 table with two supplemental margins

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 1    | 0    | 1 | 1 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 1   | 0    | 1    | 0    | 1 | 3 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 1    | 0    | 2 | 1 | 1 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 1   | 0    | 1    | 0    | 2 | 3 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 1    | 0    | 3 | 1 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 3 | 3 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 1    | 1 | 1 | 3 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 1 | 2 | 3 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | 1    | 1 | 3 | 3 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 1    | 2 | 1 | 3 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 2 | 2 | 3 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 1    | 0    | 1    | 2 | 3 | 3 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 1    | 3 | 1 | 3 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 3 | 2 | 3 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first six columns model the symmetry of the cell frequencies, the seventh and eighth columns, the missing data indicators, and the ninth through twelfth columns, the interactions of the missing data indicators and the column variable. The

twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for all of the association models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 215) is the same as for the association models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $\beta$ 's) of the model.

As discussed earlier for the association models, for the portion of the vector  $Y$  corresponding to the completely classified data (and the first nine rows of  $X$ , the model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or  $\ln(E(Y)) = X\beta$ .

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 + b7 - b9 + b11 = \ln f1 \quad (13.a.1)$$

$$b2 + b7 = \ln f2 \quad (13.a.2)$$

$$b3 + b7 + b9 + b11 = \ln f3 \quad (13.a.3)$$

$$b2 + b7 - b9 + b11 = \ln f4 \quad (13.a.4)$$

$$b4 + b7 = \ln f5 \quad (13.a.5)$$

$$b5 + b7 + b9 + b11 = \ln f6 \quad (13.a.6)$$

$$b3 + b7 - b9 + b11 = \ln f7 \quad (13.a.7)$$

$$b5 + b7 = \ln f8 \quad (13.a.8)$$

$$b6 + b7 + b9 + b11 = \ln f9 \quad (13.a.9)$$

Here,  $b1 - b7$ ,  $b9$  and  $b11$  are nine of the twelve parameters of the matrix  $\beta$ , which are to be determined. It is to be noted that equations (13.a.1) - (13.a.9) are exactly the same as equations (6.a.1) - (6.a.9) (page 160), but with  $b8$  and  $b9$  of equations (6.a.1) - (6.a.9) replaced by  $b9$  and  $b11$ , respectively, in equations (13.a.1) - (13.a.9) given above.

The last six elements of vector  $Y$  correspond to the partially classified data (and the last eighteen rows of the  $X$  matrix). Each of these last six elements of vector  $Y$  represents a sum of cell frequencies in a given row of one of two (hypothetical) contingency tables for the partially classified data. Let  $f10, \dots, f18$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with a supplemental row margin, and  $f19, \dots, f27$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with a supplemental column margin. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b1) + \exp(b2) + \exp(b3) = s1 \quad (13.a.10)$$

$$\exp(b2) + \exp(b4) + \exp(b5) = s2 \quad (13.a.11)$$

$$\exp(b3) + \exp(b5) + \exp(b6) = s3 \quad (13.a.12)$$



$$\exp(b1 + b8 - b10 + b12) + \exp(b2 + b8 - b10 + b12) + \exp(b3 + b8 - b10 + b12) = s4 \quad (13.a.13)$$

$$\exp(b2 + b8) + \exp(b4 + b8) + \exp(b5 + b8) = s5 \quad (13.a.14)$$

$$\exp(b3 + b8 + b10 + b12) + \exp(b5 + b8 + b10 + b12) + \exp(b6 + b8 + b10 + b12) = s6 \quad (13.a.15)$$

Here,  $s1 = f10 + f11 + f12$ , the first row marginal for the partially classified data,

$s2 = f13 + f14 + f15$ , the second row marginal for the partially classified data,

$s3 = f16 + f17 + f18$ , the third row marginal for the partially classified data,

$s4 = f19 + f22 + f25$ , the first column marginal for the partially classified data,

$s5 = f20 + f23 + f26$ , the second column marginal for the partially classified data, and

$s6 = f21 + f24 + f27$ , the third column marginal for the partially classified data.

It is assumed that we can obtain estimates  $s1 - s6$ , for the last six elements of vector  $Y$ .

Equations (13.a.10) - (13.a.12) are identical to equations (6.a.15) - (6.a.17). As was demonstrated on pages 160-162, equations (13.a.1) - (13.a.12) are sufficient to solve for  $b1 - b7$ ,  $b9$  and  $b11$ , as follows:

$$b9 = \frac{1}{2} \ln \frac{f2f6}{f4f8} \quad (13.a.16)$$

$$b11 = \frac{1}{2} \ln \frac{f4f6}{f2f8} \quad (13.a.17)$$

$$b7 = \ln \left( \frac{f1f2f6 + f2f4f6 + f3f4f8}{s1f4f6} \right) \quad (13.a.18)$$

$$b1 = \ln \left( \frac{s1f1f2f6}{f1f2f6 + f2f4f6 + f3f4f8} \right) \quad (13.a.19)$$

$$b2 = \ln \left( \frac{s1f2f4f6}{f1f2f6 + f2f4f6 + f3f4f8} \right) \quad (13.a.20)$$

$$b3 = \ln \left( \frac{s1f3f4f8}{f1f2f6 + f2f4f6 + f3f4f8} \right) \quad (13.a.21)$$

$$b4 = \ln \left( \frac{s1 f4 f5 f6}{f1 f2 f6 + f2 f4 f6 + f3 f4 f8} \right) \quad (13.a.22)$$

$$b5 = \ln \left( \frac{s1 f4 f6 f8}{f1 f2 f6 + f2 f4 f6 + f3 f4 f8} \right) \quad (13.a.23)$$

$$b6 = \ln \left( \frac{s1 f4 f8 f9}{f1 f2 f6 + f2 f4 f6 + f3 f4 f8} \right) \quad (13.a.24)$$

Only  $b8$ ,  $b10$  and  $b12$  remain to be determined. These three unknowns appear only in equations (13.a.13) - (13.a.15). We attempt to find these remaining  $b$ 's.

Let  $k1 = \exp(b1)$ ,  $k2 = \exp(b2)$ ,  $k3 = \exp(b3)$ ,  $k4 = \exp(b4)$ ,  $k5 = \exp(b5)$  and  $k6 = \exp(b6)$ .

Then equations (13.a.13) - (13.a.15) can be written as

$$(k1 + k2 + k3) \exp(b8 - b10 + b12) = s4 \quad (13.a.25)$$

$$(k2 + k4 + k5) \exp(b8) = s5 \quad (13.a.26)$$

$$(k3 + k5 + k6) \exp(b8 + b10 + b12) = s6 \quad (13.a.27)$$

From (13.a.26),  $\exp(b8) = \frac{s5}{k2 + k4 + k5}$  or

$$b8 = \ln \frac{s5}{k2 + k4 + k5} \quad (13.a.28)$$

Substituting (13.a.28) into (13.a.25) and (13.a.27), we have

$$\exp(-b10 + b12) = \frac{s4(k2 + k4 + k5)}{s5(k1 + k2 + k3)}$$

$$\exp(b10 + b12) = \frac{s6(k2 + k4 + k5)}{s5(k3 + k5 + k6)}$$

or

$$\text{and } -b_{10} + b_{12} = \ln \frac{s^4(k_2 + k_4 + k_5)}{s^5(k_1 + k_2 + k_3)} \quad (13.a.29)$$

$$b_{10} + b_{12} = \ln \frac{s^6(k_2 + k_4 + k_5)}{s^5(k_3 + k_5 + k_6)} \quad (13.a.30)$$

Thus,

$$b_{10} = \frac{1}{2} \ln \frac{s^6(k_1 + k_2 + k_3)}{s^4(k_3 + k_5 + k_6)} \quad (13.a.31)$$

$$b_{12} = \frac{1}{2} \ln \frac{s^4 s^6 (k_2 + k_4 + k_5)^2}{(s^5)^2 (k_1 + k_2 + k_3)(k_3 + k_5 + k_6)} \quad (13.a.32)$$

We have determined all of the  $b$ 's in the model. Thus, the model is identified.

**Type II:**

As noted for the association models considered above, a model for this data set can contain no more than fifteen parameters. If all of the interaction terms of the missing data indicators and the column variable and the missing data indicators and the row variable are included in this loglinear model, sixteen parameters are required. Thus, Type II nonignorable nonresponse with symmetry for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type II model is examined below.

Type II: symmetry; 3x3 table with two supplemental margins

- modified (only the linear interaction terms involving the missing data indicators and the column variable, and involving the missing data indicators and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | -1   | 0    | 1 | 1 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | -1   | 0    | 1 | 2 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 1   | 0    | -1   | 0    | 1 | 3 | 1 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 0    | 0    | 2 | 1 | 1 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 1   | 0    | 0    | 0    | 2 | 3 | 1 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 1    | 0    | 3 | 1 | 1 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0    | 1    | 0    | 3 | 2 | 1 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 0   | 1   | 0    | 1    | 0    | 3 | 3 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | -1   | 1 | 1 | 3 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | -1   | 1 | 2 | 3 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | -1   | 1 | 3 | 3 |
|        | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 0    | 2 | 1 | 3 |
|        | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 2 | 2 | 3 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 1    | 0    | 0    | 2 | 3 | 3 |
|        | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 1    | 3 | 1 | 3 |
|        | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0    | 0    | 1    | 3 | 2 | 3 |
|        | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first six columns model the symmetry of the cell frequencies, the seventh and eighth columns, the missing data indicators, the ninth and tenth columns, the interactions of the missing data indicators and the linear column variable term, and the eleventh and twelfth columns, the interactions of the missing data indicators and the linear row variable term. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 + b7 - b9 - b11 = \ln f1 \quad (13.b.1)$$

$$b2 + b7 - b11 = \ln f2 \quad (13.b.2)$$

$$b3 + b7 + b9 - b11 = \ln f3 \quad (13.b.3)$$

$$b2 + b7 - b9 = \ln f4 \quad (13.b.4)$$

$$b4 + b7 = \ln f5 \quad (13.b.5)$$

$$b5 + b7 + b9 = \ln f6 \quad (13.b.6)$$

$$b3 + b7 - b9 + b11 = \ln f7 \quad (13.b.7)$$

$$b5 + b7 + b11 = \ln f8 \quad (13.b.8)$$

$$b6 + b7 + b9 + b11 = \ln f9 \quad (13.b.9)$$

Here,  $b1 - b7$ ,  $b9$  and  $b11$  are nine of the twelve parameters of matrix  $\beta$ , to be determined. It is to be noted that equations (13.b.1) - (13.b.9) are exactly the same as equations (6.c.1) - (6.c.9) (page 170), but with  $b8$  and  $b9$  of equations (6.c.1) - (6.c.9) replaced, respectively, by  $b9$  and  $b11$  in equations (13.b.1) - (13.b.9) given above.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1) + \exp(b2) + \exp(b3) = s1 \quad (13.b.10)$$

$$\exp(b2) + \exp(b4) + \exp(b5) = s2 \quad (13.b.11)$$

$$\exp(b3) + \exp(b5) + \exp(b6) = s3 \quad (13.b.12)$$

$$\exp(b1 + b8 - b10 - b12) + \exp(b2 + b8 - b10) + \exp(b3 + b8 - b10 + b12) = s4 \quad (13.b.13)$$

$$\exp(b2 + b8 - b12) + \exp(b4 + b8) + \exp(b5 + b8 + b12) = s5 \quad (13.b.14)$$

$$\exp(b3 + b8 + b10 - b12) + \exp(b5 + b8 + b10) + \exp(b6 + b8 + b10 + b12) = s6 \quad (13.b.15)$$

Equations (13.b.10) - (13.b.12) are identical to equations (6.c.10) - (6.c.12). As was demonstrated on pages 171 - 172, equations (13.b.1) - (13.b.12) are sufficient to give a unique solution for  $b1 - b7$ ,  $b9$  and  $b11$ .

Only  $b8$ ,  $b10$  and  $b12$  remain to be determined. These three unknowns appear only in equations (13.b.13) - (13.b.15). We attempt to find the remaining  $b$ 's.

Let  $k1 = \exp(b1)$ ,  $k2 = \exp(b2)$ ,  $k3 = \exp(b3)$ ,  $k4 = \exp(b4)$ ,  $k5 = \exp(b5)$  and  $k6 = \exp(b6)$ .

Then equations (13.b.13) - (13.b.15) can be expressed as

$$k1 \exp(b8 - b10 - b12) + k2 \exp(b8 - b10) + k3 \exp(b8 - b10 + b12) = s4 \quad (13.b.16)$$

$$k2 \exp(b8 - b12) + k4 \exp(b8) + k5 \exp(b8 + b12) = s5 \quad (13.b.17)$$

$$k3 \exp(b8 + b10 - b12) + k5 \exp(b8 + b10) + k6 \exp(b8 + b10 + b12) = s6 \quad (13.b.18)$$

From (13.b.17), we have

$$\exp(b8) = \frac{s5}{k2 \exp(-b12) + k4 + k5 \exp(b12)} \quad (13.b.19)$$

Thus, given  $b12$ ,  $b8$  is also uniquely determined.

Dividing (13.b.16) by (13.b.17), and dividing (13.b.18) by (13.b.17), and rearranging terms, we have

$$\exp(b10) = \frac{s5(k1 \exp(-b12) + k2 + k3 \exp(b12))}{s4(k2 \exp(-b12) + k4 + k5 \exp(b12))} \quad (13.b.20)$$

and

$$\exp(b10) = \frac{s6(k2 \exp(-b12) + k4 + k5 \exp(b12))}{s5(k3 \exp(-b12) + k5 + k6 \exp(b12))} \quad (13.b.21)$$

As can be seen from each of equations (13.b.20) and (13.b.21), given  $b12$ ,  $b10$  is also uniquely determined.

Further, equating the two expressions for  $\exp(b10)$  in equations (13.b.20) and (13.b.21) and rearranging terms gives:



$$\begin{aligned}
& \exp(-2b12) \left[ s4s6(k2)^2 - k1k3(s5)^2 \right] + \exp(-b12) \left[ 2k2k4s4s6 - (k1k5 + k2k3)(s5)^2 \right] + \\
& \left[ (2k2k5 + (k4)^2) s4s6 - (s5)^2 (k1k6 + k2k5 + (k3)^2) \right] \\
& + \exp(b12) \left[ 2k4k5s4s6 - (k2k6 + k3k5)(s5)^2 \right] + \exp(2b12) \left[ s4s6(k5)^2 - k3k6(s5)^2 \right] = 0
\end{aligned}
\tag{13.b.22}$$

Multiplying the above equation by  $\exp(2b12)$ , and substituting  $x = \exp(b12)$ , we have

$$\begin{aligned}
& \left[ s4s6(k2)^2 - k1k3(s5)^2 \right] + x \left[ 2k2k4s4s6 - (k1k5 + k2k3)(s5)^2 \right] + \\
& x^2 \left[ (2k2k5 + (k4)^2) s4s6 - (s5)^2 (k1k6 + k2k5 + (k3)^2) \right] \\
& + x^3 \left[ 2k4k5s4s6 - (k2k6 + k3k5)(s5)^2 \right] + x^4 \left[ s4s6(k5)^2 - k3k6(s5)^2 \right] = 0
\end{aligned}
\tag{13.b.23}$$

Thus, equations (13.b.13) - (13.b.15) generate a fourth degree polynomial in  $\exp(b12)$ . Since  $\exp(b12) > 0$ , we seek only positive solutions to (13.b.23). The identification status of the model thus depends on the uniqueness (or nonuniqueness) of a solution for  $b12$ . If  $b12$  is uniquely determined, then so are  $b8$  and  $b10$ , and the model is identified. On the other hand, if  $b12$  is not uniquely determined, the model is not identified.

The identification status of this model will be determined empirically for a given data set.

**Type III:**

As was discussed in the case of the association models, if all the interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and three way interactions of the missing data indicator, the column variable, and the row variable are included in the model, there would be too many parameters for the model, given the data. Thus, Type III nonignorable nonresponse with symmetry modeled for the row and column variables cannot be considered for this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

Type III: symmetry; 3x3 table with two supplemental margins

- modified (only linear interaction terms involving the missing data indicators and the column variable, involving the missing data indicators and the row variable, and involving the missing data indicators, the column variable and the row variable).

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|---|---|---|
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | -1   | 0    | 1    | 0    | 1 | 1 | 1 |
| 0      | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | -1   | 0    | 0    | 0    | 1 | 2 | 1 |
| 0      | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0    | -1   | 0    | -1   | 0    | 1 | 3 | 1 |
| 0      | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 1 |
| 0      | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
| 0      | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 1   | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 1 |
| 0      | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | -1  | 0    | 1    | 0    | -1   | 0    | 3 | 1 | 1 |
| 0      | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0   | 0    | 1    | 0    | 0    | 0    | 3 | 2 | 1 |
| 0      | 0   | 0   | 0   | 0   | 1   | 1   | 0   | 1   | 0   | 1    | 0    | 1    | 0    | 0    | 3 | 3 | 1 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
| 0      | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
| 0      | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
| 0      | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
| 0      | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
| 0      | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
| 0      | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
| 0      | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
| 0      | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | -1   | 0    | 1    | 1 | 1 | 3 |
| 0      | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | -1   | 0    | 0    | 1 | 2 | 3 |
| 0      | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | -1   | 0    | -1   | 1 | 3 | 3 |
| 0      | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 0    | 0    | 0    | 2 | 1 | 3 |
| 0      | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 3 |
| 0      | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | 0    | 0    | 0    | 2 | 3 | 3 |
| 0      | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | 1    | 0    | -1   | 3 | 1 | 3 |
| 0      | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 1    | 0    | 0    | 3 | 2 | 3 |
| 0      | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 1    | 0    | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first six columns model symmetry in the cell frequencies, the seventh and eighth columns, the missing data indicators, the ninth and tenth columns, the interactions of the missing data indicators and the linear column variable term, the eleventh and twelfth columns, the interactions of the missing data indicators and the linear row variable term, and the thirteenth and fourteenth columns, the interactions of the missing data indicators, the linear column variable term, and the linear row variable term. The fourteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the modified Type II case that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 + b7 - b9 - b11 + b13 = \ln f1 \quad (13.c.1)$$

$$b2 + b7 - b11 = \ln f2 \quad (13.c.2)$$

$$b3 + b7 + b9 - b11 - b13 = \ln f3 \quad (13.c.3)$$

$$b2 + b7 - b9 = \ln f4 \quad (13.c.4)$$

$$b4 + b7 = \ln f5 \quad (13.c.5)$$

$$b5 + b7 + b9 = \ln f6 \quad (13.c.6)$$

$$b3 + b7 - b9 + b11 - b13 = \ln f7 \quad (13.c.7)$$

$$b5 + b7 + b11 = \ln f8 \quad (13.c.8)$$

$$b6 + b7 + b9 + b11 + b13 = \ln f9 \quad (13.c.9)$$

Here,  $b1 - b7$ ,  $b9$ ,  $b11$ , and  $b13$  are ten of the fourteen parameters of matrix  $\beta$ , which are to be determined. It is to be noted that equations (13.c.1) - (13.c.9) are exactly the same as equations (6.d.1) - (6.d.9) (page 175), but with  $b8$ ,  $b9$  and  $b10$  of equations (6.d.1) - (6.d.9) replaced, respectively, by  $b9$ ,  $b11$  and  $b13$  in equations (13.c.1) - (13.c.9) given above.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1) + \exp(b2) + \exp(b3) = s1 \quad (13.c.10)$$

$$\exp(b2) + \exp(b4) + \exp(b5) = s2 \quad (13.c.11)$$

$$\exp(b3) + \exp(b5) + \exp(b6) = s3 \quad (13.c.12)$$

$$\exp(b1 + b8 - b10 - b12 + b14) + \exp(b2 + b8 - b10) + \exp(b3 + b8 - b10 + b12 - b14) = s4 \quad (13.c.13)$$

$$\exp(b2 + b8 - b12) + \exp(b4 + b8) + \exp(b5 + b8 + b12) = s5 \quad (13.c.14)$$

$$\exp(b3 + b8 + b10 - b12 - b14) + \exp(b5 + b8 + b10) + \exp(b6 + b8 + b10 + b12 + b14) = s6 \quad (13.c.15)$$

Equations (13.c.10) - (13.c.12) are identical to equations (6.d.10) - (6.d.12). As was demonstrated on pages 176 - 178, there may be more than one set of solutions for  $b1 - b7$ ,  $b9$ ,  $b11$  and  $b13$ .

In addition, equations (13.c.13) - (13.c.15) are the only equations which contain the unknowns  $b_8$ ,  $b_{10}$ ,  $b_{12}$  and  $b_{14}$ . These three equations are insufficient to solve for the four unknowns  $b_8$ ,  $b_{10}$ ,  $b_{12}$  and  $b_{14}$ . Thus, this model is not identified. .

In summary, if symmetry among the cell frequencies is modeled, the Type I nonignorable nonresponse model is identified, and the Type III nonignorable nonresponse model (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicator and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 2

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the symmetry models hypothesized to fit Data Set 2 (page 219). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicators).

#### Data Set 2

##### Symmetry Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 1742            | 244          | 64.5            | 2050.5 |
|                     | single crime    | 244             | 56           | 31              | 331    |
|                     | multiple crimes | 64.5            | 31           | 10              | 105.5  |
|                     |                 | 2050.5          | 331          | 105.5           | 2487   |

$$\beta = (7.46, 5.50, 4.17, 4.03, 3.43, 2.30)$$

Likelihood Ratio fit statistic = 2.16964

Degrees of freedom = 3

**Data Set 2**

Symmetry Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1669.22         | 255.36       | 74.30           | 1998.88 |
|                     | single crime    | 255.36          | 64.55        | 39.74           | 359.65  |
|                     | multiple crimes | 74.30           | 39.74        | 14.43           | 128.47  |
|                     |                 | 1998.88         | 259.65       | 128.47          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 836.96          | 128.04       | 37.26           | 1002.26 |
|                     | single crime    | 128.04          | 32.37        | 19.93           | 180.34  |
|                     | multiple crimes | 37.26           | 19.93        | 7.24            | 64.43   |
|                     |                 | 1002.26         | 180.34       | 64.43           | 1247.03 |



**Data Set 2**

Symmetry Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 663.80          | 101.55       | 29.55           | 794.90 |
|                     | single crime    | 101.55          | 25.67        | 15.80           | 143.02 |
|                     | multiple crimes | 29.55           | 15.80        | 5.74            | 51.09  |
|                     |                 | 794.90          | 143.02       | 51.09           | 989.01 |

 $\beta = (6.73, 4.85, 3.62, 3.48, 2.99, 1.98, .69, -.23)$ 

Likelihood Ratio fit statistic = 55.4829

Degrees of freedom = 7

**Data Set 2**

Symmetry Model - Two Supplemental Margins

Missing at Random (MAR)

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 1731.66    | 260.82       | 75.52           | 2068 |
|                     | single crime    | 225.07     | 55.82        | 34.11           | 315  |
|                     | multiple crimes | 60.68      | 31.76        | 11.56           | 104  |
|                     |                 | 2017.41    | 348.40       | 121.19          | 2487 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 842.70     | 126.93       | 36.75           | 1006.38 |
|                     | single crime    | 126.93     | 31.48        | 19.24           | 177.65  |
|                     | multiple crimes | 36.75      | 19.24        | 7.00            | 62.99   |
|                     |                 | 1006.38    | 177.65       | 62.99           | 1247.02 |

**Data Set 2**

Symmetry Model - Two Supplemental Margins

Missing at Random (MAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 340.97          | 51.36        | 14.87           | 407.20 |
|                     | single crime    | 193.88          | 48.08        | 29.38           | 271.34 |
|                     | multiple crimes | 181.14          | 94.82        | 34.50           | 310.46 |
|                     |                 | 715.99          | 194.26       | 78.75           | 989    |

 $\beta' = (6.74, 4.84, 3.60, 3.45, 2.96, 1.95, .57, .42, -.11, 1.25, .04, -.08)$ 

Likelihood Ratio fit statistic = 4.65182

Degrees of freedom = 3

**Data Set 2**

Symmetry Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 1719.67    | 252.47       | 64.43           | 2036.57 |
|                     | single crime    | 240.49     | 57.85        | 30.91           | 329.25  |
|                     | multiple crimes | 72.84      | 36.68        | 11.66           | 121.18  |
|                     |                 | 2033       | 347          | 107             | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |         |
|---------------------|-----------------|------------|--------------|-----------------|---------|
| Second<br>Interview | crime free      | 863.13     | 120.71       | 36.56           | 1020.40 |
|                     | single crime    | 120.71     | 27.66        | 17.54           | 165.91  |
|                     | multiple crimes | 36.56      | 17.54        | 6.62            | 60.72   |
|                     |                 | 1020.40    | 165.91       | 60.72           | 1247.03 |

**Data Set 2**

Symmetry Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 605.65          | 141.15       | 47.57           | 794.37 |
|                     | single crime    | 84.70           | 32.34        | 22.82           | 139.86 |
|                     | multiple crimes | 25.65           | 20.51        | 8.61            | 54.77  |
|                     |                 | 716             | 194          | 79              | 989    |

$$\beta = (6.76, 4.79, 3.60, 3.32, 2.86, 1.89, .74, .16, -.06, .31, -.11, -.20)$$

Likelihood Ratio fit statistic = 8.81721

Degrees of freedom = 3

## **Data Set 2**

### **Symmetry Model - Two Supplemental Margins**

#### **Nonignorable Nonresponse - Type II (linear interaction terms only)**

As indicated on page 361, the identification status of the symmetry model with modified nonignorable nonresponse Type II depends on the number of solutions to equation (13.b.23). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (13.b.23) giving the following fourth degree polynomial

$$-293422723 - 383974223x - 42351553x^2 + 9170187x^3 + 14837220x^4 = 0$$

By Descartes' Rule of Signs, since there is one change of sign in this fourth degree polynomial equation, there is exactly one positive solution for  $x = \exp(b/4)$ . Thus, the model is identified for this data set. Given below are the parameter and contingency table estimates, together with the likelihood ratio fit statistic for this identified model.

**Data Set 2****Symmetry Model - Two Supplemental Margins****Nonignorable Nonresponse - Type II (linear interaction terms only)****Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1745.50         | 249.77       | 70.52           | 2065.79 |
|                     | single crime    | 232.65          | 54.46        | 32.33           | 319.44  |
|                     | multiple crimes | 61.18           | 30.11        | 10.48           | 101.77  |
|                     |                 | 2039.33         | 334.34       | 113.33          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 819.11          | 129.94       | 40.67           | 989.72 |
|                     | single crime    | 129.94          | 33.72        | 22.19           | 185.85 |
|                     | multiple crimes | 40.67           | 22.19        | 8.57            | 71.43  |
|                     |                 | 989.72          | 185.85       | 71.43           | 1247   |

**Data Set 2**

Symmetry Model - Two Supplemental Margins

Nonignorable Nonresponse - Type II (linear interactions only)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 350.13          | 52.89        | 15.77           | 418.79 |
|                     | single crime    | 181.13          | 44.76        | 28.06           | 253.95 |
|                     | multiple crimes | 184.88          | 96.07        | 35.31           | 316.26 |
|                     |                 | 716.14          | 193.72       | 79.14           | 989    |

 $\beta = (6.708, 4.867, 3.706, 3.518, 3.100, 2.148, .479, .283, -.103, -.049, -.174, 1.182)$ 

Likelihood Ratio fit statistic = 1.74465

Degrees of freedom = 3



Quasi - Symmetry - Data Set 2

Type I: quasi-symmetry; 3x3 table with two supplemental margins

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|---|---|---|
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 1    | 0    | 1 | 1 | 1 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 1 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0    | 1    | 0    | 1    | 0    | 1 | 3 | 1 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 1   | 0    | -1   | 0    | 1    | 0    | 2 | 1 | 1 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 1 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 0    | 1    | 0    | 1    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 1   | 0    | -1   | 0    | 1    | 0    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 1   | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 1    | 0    | 3 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1    | 0    | -1   | 0    | 1    | 1 | 1 | 3 |
|        | 1   | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 1    | 0    | 0    | 0    | 0    | 1 | 2 | 3 |
|        | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1    | 0    | 1    | 0    | 1    | 1 | 3 | 3 |
|        | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 0   | 1    | 0    | -1   | 0    | 1    | 2 | 1 | 3 |
|        | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1    | 0    | 0    | 0    | 0    | 2 | 2 | 3 |
|        | 1   | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1    | 0    | 1    | 0    | 1    | 2 | 3 | 3 |
|        | 1   | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | 1    | 0    | -1   | 0    | 1    | 3 | 1 | 3 |
|        | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | 0    | 0    | 0    | 3 | 2 | 3 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1    | 0    | 1    | 0    | 1    | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth through eighth columns, the symmetry of the off diagonal elements, the ninth and tenth columns, the missing data indicators, and the eleventh through fourteenth columns, the interactions of the missing data indicators and the column variable. The fourteen parameters of the

model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for all of the association models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 215) is the same as for the other models considered for Data Set 2.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for the association and symmetry models, for the portion of the vector  $Y$  corresponding to the completely classified data (and the first nine rows of  $X$ , the model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or  $\ln(E(Y)) = X\beta$ .

Assume that we can obtain estimates for the first nine elements of vector  $Y$ . We will call these estimates  $f_1, f_2, \dots, f_9$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - b2 + b3 - b4 + b5 + b9 - b11 + b13 = \ln f1 \quad (14.a.1)$$

$$b1 - b2 + b3 + b6 + b9 = \ln f2 \quad (14.a.2)$$

$$b1 - b2 + b3 + b4 + b5 + b7 + b9 + b11 + b13 = \ln f3 \quad (14.a.3)$$

$$b1 - b4 + b5 + b6 + b9 - b11 + b13 = \ln f4 \quad (14.a.4)$$

$$b1 + b9 = \ln f5 \quad (14.a.5)$$

$$b1 + b4 + b5 + b8 + b9 + b11 + b13 = \ln f6 \quad (14.a.6)$$

$$b1 + b2 + b3 - b4 + b5 + b7 + b9 - b11 + b13 = \ln f7 \quad (14.a.7)$$

$$b1 + b2 + b3 + b8 + b9 = \ln f8 \quad (14.a.8)$$

$$b1 + b2 + b3 + b4 + b5 + b9 + b11 + b13 = \ln f9 \quad (14.a.9)$$

Here,  $b1 - b9$ ,  $b11$  and  $b13$  are eleven of the fourteen parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. It is to be noted that equations (14.a.1) - (14.a.9) are exactly the same as equations (7.a.1) - (7.a.9) (pages 190 - 191), but with  $b10$  and  $b11$  of equations (7.a.1) - (7.a.9) replaced by  $b11$  and  $b13$ , respectively, in equations (14.a.1) - (14.a.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (14.a.1) - (14.a.9), as follows:

$$b1 + b9 = \ln f5 \quad (14.a.10)$$

$$b2 = \frac{1}{4} \ln \frac{f4 f8 f9}{f1 f2 f6} \quad (14.a.11)$$

$$b3 = \frac{1}{4} \ln \frac{f1 f2 f8 f9}{f4 f6 (f5)^2} \quad (14.a.12)$$

$$b5 + b13 = \frac{1}{4} \ln \frac{f1 f4 f6 f9}{f2 f8 (f5)^2} \quad (14.a.13)$$

$$b4 + b11 = \frac{1}{4} \ln \frac{f2 f6 f9}{f1 f4 f8} \quad (14.a.14)$$

$$b6 = \frac{1}{2} \ln \frac{f2 f4}{f1 f5} \quad (14.a.15)$$

$$b7 = \frac{1}{2} \ln \frac{f4 f8 (f3)^2}{f1 f2 f6 f9} \quad (14.a.16)$$

$$b_8 = \frac{1}{2} \ln \frac{f_6 f_8}{f_5 f_9} \quad (14.a.17)$$

Note that  $b_1$  and  $b_9$  cannot be determined uniquely from equations (14.a.1) - (14.a.9), since whenever  $b_1$  appears in an equation, so does  $b_9$ ; that is  $b_1 + b_9$  appears as a unit in each of the equations. Similarly,  $b_5$  and  $b_{13}$  cannot be determined uniquely, nor can  $b_4$  and  $b_{11}$ . Only five of the fourteen model parameters ( $b_2$ ,  $b_3$ ,  $b_6$ ,  $b_7$  and  $b_8$ ) have been uniquely determined. Equations (14.a.10) - (14.a.17) provide the most complete solution for the  $b$ 's which can be determined solely from equations (14.a.1) - (14.a.9).

The last six elements of vector  $Y$  correspond to the partially classified data (and the last eighteen rows of the  $X$  matrix). Each of these last six elements of vector  $Y$  represents a sum of cell frequencies in a given row of one of the two (hypothetical) contingency tables for the partially classified data. Let  $f_{10}, \dots, f_{18}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with a supplemental row margin, and  $f_{19}, \dots, f_{27}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table with a supplemental column margin. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b_1 - b_2 + b_3 - b_4 + b_5) + \exp(b_1 - b_2 + b_3 + b_6) + \exp(b_1 - b_2 + b_3 + b_4 + b_5 + b_7) = s_1 \quad (14.a.18)$$

$$\exp(b_1 - b_4 + b_5 + b_6) + \exp(b_1) + \exp(b_1 + b_4 + b_5 + b_8) = s_2 \quad (14.a.19)$$

$$\exp(b_1 + b_2 + b_3 - b_4 + b_5 + b_7) + \exp(b_1 + b_2 + b_3 + b_8) + \exp(b_1 + b_2 + b_3 + b_4 + b_5) = s_3 \quad (14.a.20)$$

$$\exp(b_1 - b_4 + b_5 + b_{10} - b_{12} + b_{14}) [\exp(-b_2 + b_3) + \exp(b_6) + \exp(b_2 + b_3 + b_7)] = s_4 \quad (14.a.21)$$

$$\exp(b_1 + b_{10}) [\exp(-b_2 + b_3 + b_6) + 1 + \exp(b_2 + b_3 + b_8)] = s_5 \quad (14.a.22)$$

$$\exp(b_1 + b_4 + b_5 + b_{10} + b_{12} + b_{14}) [\exp(-b_2 + b_3 + b_6 + b_7) + \exp(b_8) + \exp(b_2 + b_3)] = s_6 \quad (14.a.23)$$

Here,  $s_1 = f_{10} + f_{11} + f_{12}$ , the first row marginal for the partially classified data,

$s_2 = f_{13} + f_{14} + f_{15}$ , the second row marginal for the partially classified data, and

$s3 = f16 + f17 + f18$ , the third row marginal for the partially classified data,

$s4 = f19 + f22 + f25$ , the first column marginal for the partially classified data,

$s5 = f20 + f23 + f26$ , the second column marginal for the partially classified data, and

$s6 = f21 + f24 + f27$ , the third column marginal for the partially classified data.

It is assumed that we can obtain estimates  $s1 - s6$ , for the last six elements of vector  $Y$ .

Equations (14.a.18) - (14.a.20) are identical to equations (7.a.22) - (7.a.24). As was shown on pages 193 - 195, these equations are sufficient to solve uniquely for  $b1$ ,  $b4$  and  $b5$ . As noted on page 195, these, together with equations (14.a.10), (14.a.14), and (14.a.13), also allow for the unique solution of  $b9$ ,  $b11$  and  $b13$ . The only remaining unknown  $b$ 's are thus  $b10$ ,  $b12$  and  $b14$ . They appear only in equations (14.a.21) - (14.a.23). We will attempt to find these remaining  $b$ 's.

Let  $k1 = \exp(b1)$ ,  $k2 = \exp(-b2 + b3)$ ,  $k3 = \exp(-b4 + b5)$ ,  $k4 = \exp(b6)$ ,  $k5 = \exp(b7)$ ,  $k6 = \exp(b8)$ ,  $k7 = \exp(b4 + b5)$ , and  $k8 = \exp(b2 + b3)$ .

Then equations (14.a.21) - (14.a.23) can be rewritten as

$$k1k2k3 \exp(b10 - b12 + b14) + k1k3k4 \exp(b10 - b12 + b14) + k1k3k5k8 \exp(b10 - b12 + b14) = s4 \quad (14.a.24)$$

$$k1k2k4 \exp(b10) + k1 \exp(b10) + k1k6k8 \exp(b10) = s5 \quad (14.a.25)$$

$$k1k2k5k7 \exp(b10 + b12 + b14) + k1k6k7 \exp(b10 + b12 + b14) + k1k7k8 \exp(b10 + b12 + b14) = s6 \quad (14.a.26)$$

From equation (14.a.25),  $\exp(b10) = \frac{s5}{k1k2k4 + k1 + k1k6k8}$ , or

$$b10 = \ln \frac{s5}{k1(k2k4 + 1 + k6k8)} \quad (14.a.27)$$

Dividing (14.a.24) by (14.a.25), and (14.a.26) by (14.a.25), we have

$$\exp(-b_{12} + b_{14}) = \frac{s^4(k_2k_4 + 1 + k_6k_8)}{s^5(k_2k_3 + k_3k_4 + k_3k_5k_8)}$$

$$\exp(b_{12} + b_{14}) = \frac{s^6(k_2k_4 + 1 + k_6k_8)}{s^5(k_2k_5k_7 + k_6k_7 + k_7k_8)}$$

or

$$-b_{12} + b_{14} = \ln \frac{s^4(k_2k_4 + 1 + k_6k_8)}{s^5(k_2k_3 + k_3k_4 + k_3k_5k_8)} \quad (14.a.28)$$

and

$$b_{12} + b_{14} = \ln \frac{s^6(k_2k_4 + 1 + k_6k_8)}{s^5(k_2k_5k_7 + k_6k_7 + k_7k_8)} \quad (14.a.29)$$

$$\text{Thus, } b_{12} = \frac{1}{2} \ln \frac{s^6(k_2k_3 + k_3k_4 + k_3k_5k_8)}{s^4(k_2k_5k_7 + k_6k_7 + k_7k_8)} \quad (14.a.30)$$

$$\text{and } b_{14} = \frac{1}{2} \ln \frac{s^4s^6(k_2k_4 + 1 + k_6k_8)^2}{(s^5)^2(k_2k_3 + k_3k_4 + k_3k_5k_8)(k_2k_5k_7 + k_6k_7 + k_7k_8)} \quad (14.a.31)$$

Having found all of the  $b$ 's of the model, we have determined that it is identified.

**Type II:**

As noted for the models considered earlier, a model for this data set can contain no more than fifteen parameters. If all of the interaction terms of the missing data indicators and the column variable and the missing data indicators and the row variable are included in this loglinear model, eighteen parameters are required. Thus, Type II nonignorable nonresponse for the quasi - symmetry model cannot be explored for this data set. We can, however, investigate whether or not such a model is identified if only the linear interaction terms involving the missing data indicators appear in the model. The identification status of this modified Type II model is examined below.

**Type II: quasi-symmetry; 3x3 table with two supplemental margins**

- modified (only the linear interaction terms involving the missing data indicators and the column variable, and involving the missing data indicators and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|---|---|---|
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 1   | 0   | -1   | 0    | -1   | 0    |      | 1 | 1 | 1 |
| 1      | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 1   | 0   | 0    | 0    | -1   | 0    |      | 1 | 2 | 1 |
| 1      | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 1    | 0    | -1   | 0    |      | 1 | 3 | 1 |
| 1      | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 1   | 0   | -1   | 0    | 0    | 0    |      | 2 | 1 | 1 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0   | 0    | 0    | 0    | 0    |      | 2 | 2 | 1 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 1   | 0   | 1    | 0    | 0    | 0    |      | 2 | 3 | 1 |
| 1      | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 1   | 0   | -1   | 0    | 1    | 0    |      | 3 | 1 | 1 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 1   | 0   | 0    | 0    | 1    | 0    |      | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 0   | 1    | 0    | 1    | 0    |      | 3 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 1 | 1 | 2 |
| 1      | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 1 | 2 | 2 |
| 1      | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 1 | 3 | 2 |
| 1      | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 2 | 1 | 2 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 2 | 2 | 2 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    |      | 2 | 3 | 2 |
| 1      | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 3 | 1 | 2 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    |      | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    |      | 3 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | -1   |      | 1 | 1 | 3 |
| 1      | -1  | 1   | 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | -1   |      | 1 | 2 | 3 |
| 1      | -1  | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 1   | 0    | 1    | 0    | -1   |      | 1 | 3 | 3 |
| 1      | 0   | 0   | -1  | 1   | 1   | 0   | 0   | 0   | 1   | 0    | -1   | 0    | 0    |      | 2 | 1 | 3 |
| 1      | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 0    |      | 2 | 2 | 3 |
| 1      | 0   | 0   | 1   | 1   | 0   | 0   | 1   | 0   | 1   | 0    | 1    | 0    | 0    |      | 2 | 3 | 3 |
| 1      | 1   | 1   | -1  | 1   | 0   | 1   | 0   | 0   | 1   | 0    | -1   | 0    | 1    |      | 3 | 1 | 3 |
| 1      | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 1   | 0    | 0    | 0    | 1    |      | 3 | 2 | 3 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0    | 1    | 0    | 1    |      | 3 | 3 | 3 |

In this matrix, the first column reflects the intercept term, the second and third columns, the row variable, the fourth and fifth columns, the column variable, the sixth through



eighth columns, the symmetry of the off diagonal elements, the ninth and tenth columns, the missing data indicators, the eleventh and twelfth columns, the interactions of the missing data indicators and the linear column variable term, and the thirteenth and fourteenth columns, the interactions of the missing data indicators and the linear row variable term. The fourteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row and column marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f1, f2, \dots, f9$ , are available for the first nine elements of vector  $Y$ , and  $s1 - s6$ , for the row and column marginals for the partially classified data. Using the fully classified data, we have

$$b1 - b2 + b3 - b4 + b5 + b9 - b11 - b13 = \ln f1 \quad (14.b.1)$$

$$b1 - b2 + b3 + b6 + b9 - b13 = \ln f2 \quad (14.b.2)$$

$$b1 - b2 + b3 + b4 + b5 + b7 + b9 + b11 - b13 = \ln f3 \quad (14.b.3)$$

$$b1 - b4 + b5 + b6 + b9 - b11 = \ln f4 \quad (14.b.4)$$

$$b1 + b9 = \ln f5 \quad (14.b.5)$$

$$b1 + b4 + b5 + b8 + b9 + b11 = \ln f6 \quad (14.b.6)$$

$$b1 + b2 + b3 - b4 + b5 + b7 + b9 - b11 + b13 = \ln f7 \quad (14.b.7)$$

$$b1 + b2 + b3 + b8 + b9 + b13 = \ln f8 \quad (14.b.8)$$

$$b1 + b2 + b3 + b4 + b5 + b9 + b11 + b13 = \ln f9 \quad (14.b.9)$$

Here,  $b1, \dots, b9, b11$  and  $b13$  are eleven of the fourteen parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. It is to be noted that equations (14.b.1) - (14.b.9) are exactly the same as equations (7.b.1) - (7.b.9) (page 198), but with  $b10$  and  $b11$  of equations (7.b.1) - (7.b.9) replaced, respectively, by  $b11$  and  $b13$  in equations (14.b.1) - (14.b.9) given above. With these replacements made, we can find the most complete solution for the  $b$ 's which can be determined solely from equations (14.b.1) - (14.b.9), as follows:

$$b1 + b9 = \ln f5 \quad (14.b.10)$$

$$b2 + b13 = \frac{1}{4} \ln \frac{f4 f8 f9}{f1 f2 f6} \quad (14.b.11)$$

$$b3 = \frac{1}{4} \ln \frac{f1 f2 f8 f9}{f4 f6 (f5)^2} \quad (14.b.12)$$

$$b5 = \frac{1}{4} \ln \frac{f1 f4 f6 f9}{f2 f8 (f5)^2} \quad (14.b.13)$$

$$b4 + b11 = \frac{1}{4} \ln \frac{f2 f6 f9}{f1 f4 f8} \quad (14.b.14)$$

$$b7 = \frac{1}{2} \ln \frac{f4 f8 (f3)^2}{f1 f2 f6 f9} \quad (14.b.15)$$

$$b6 = \frac{1}{2} \ln \frac{f2 f4}{f1 f5} \quad (14.b.16)$$

$$b8 = \frac{1}{2} \ln \frac{f6 f8}{f5 f9} \quad (14.b.17)$$

Note that  $b1$  and  $b9$  cannot be determined uniquely from equations (14.b.1) - (14.b.9), nor can  $b2$  and  $b13$ , and  $b4$  and  $b11$ . Only five of the fourteen model parameters ( $b3, b5, b6, b7$  and  $b8$ ) have been uniquely determined.

The algebraic equations which correspond to the partially classified data for the current model are given by

$$\exp(b1 - b2 + b3 - b4 + b5) + \exp(b1 - b2 + b3 + b6) + \exp(b1 - b2 + b3 + b4 + b5 + b7) = s1 \quad (14.b.18)$$

$$\exp(b1 - b4 + b5 + b6) + \exp(b1) + \exp(b1 + b4 + b5 + b8) = s2 \quad (14.b.19)$$

$$\exp(b1 + b2 + b3 - b4 + b5 + b7) + \exp(b1 + b2 + b3 + b8) + \exp(b1 + b2 + b3 + b4 + b5) = s3 \quad (14.b.20)$$

$$\exp(b1 - b4 + b5 + b10 - b12) [\exp(-b2 + b3 - b14) + \exp(b6) + \exp(b2 + b3 + b7 + b14)] = s4 \quad (14.b.21)$$

$$\exp(b1 + b10) [\exp(-b2 + b3 + b6 - b14) + 1 + \exp(b2 + b3 + b8 + b14)] = s5 \quad (14.b.22)$$

$$\exp(b1 + b4 + b5 + b10 + b12) [\exp(-b2 + b3 + b7 - b14) + \exp(b8) + \exp(b2 + b3 + b14)] = s6 \quad (14.b.23)$$

Equations (14.b.18) - (14.b.20) are identical to equations (7.b.18) - (7.b.20). As was shown on pages 199 - 201, these three equations produce a fourth degree polynomial in  $\exp(b4)$ . While there may be a single solution for  $b4$ , given a particular data set, there may also be no solutions for  $b4$ , or several solutions. As was noted on page 201, if there is a unique solution for  $b4$ , there will also be unique solutions for  $b1$  and  $b2$  (and so for  $b9$ ,  $b11$  and  $b13$ ). If no such unique solution exists, the model is not identified.

Let us assume that there is a unique solution for  $b4$  (and thus for  $b1$ ,  $b2$ ,  $b9$ ,  $b11$  and  $b13$ ).

The only remaining unknown  $b$ 's are then  $b10$ ,  $b12$  and  $b14$ . They appear only in equations (14.b.21) - (14.b.23). We will attempt to find these remaining  $b$ 's.

Let  $m1 = \exp(b1)$ ,  $m2 = \exp(-b2 + b3)$ ,  $m3 = \exp(-b4 + b5)$ ,  $m4 = \exp(b6)$ ,  
 $m5 = \exp(b7)$ ,  $m6 = \exp(b8)$ ,  $m7 = \exp(b4 + b5)$  and  $m8 = \exp(b2 + b3)$ .

Then equations (14.b.21) - (14.b.23) can be expressed as

$$m_1 m_2 m_3 \exp(b_{10} - b_{12} - b_{14}) + m_1 m_3 m_4 \exp(b_{10} - b_{12}) + m_1 m_3 m_5 m_8 \exp(b_{10} - b_{12} + b_{14}) = s_4 \quad (14. b.24)$$

$$m_1 m_2 m_4 \exp(b_{10} - b_{14}) + m_1 \exp(b_{10}) + m_1 m_6 m_8 \exp(b_{10} + b_{14}) = s_5 \quad (14. b.25)$$

$$m_1 m_2 m_5 m_7 \exp(b_{10} + b_{12} - b_{14}) + m_1 m_6 m_7 \exp(b_{10} + b_{12}) + m_1 m_7 m_8 \exp(b_{10} + b_{12} + b_{14}) = s_6 \quad (14. b.26)$$

From equation (14.b.25),

$$\exp(b_{10}) = \frac{s_5}{m_1 m_2 m_4 \exp(-b_{14}) + m_1 + m_1 m_6 m_8 \exp(b_{14})} \quad (14. b.27)$$

Thus, given  $b_{14}$ ,  $b_{10}$  is also uniquely determined.

Dividing (14.b.24) by (14.b.25), and (14.b.26) by (14.b.25), we have

$$\begin{aligned} m_2 m_3 \exp(-b_{12} - b_{14}) + m_3 m_4 \exp(-b_{12}) + m_3 m_5 m_8 \exp(-b_{12} + b_{14}) \\ = \frac{s_4}{s_5} (m_2 m_4 \exp(-b_{14}) + 1 + m_6 m_8 \exp(b_{14})) \end{aligned} \quad (14. b.28)$$

$$\begin{aligned} m_2 m_5 m_7 \exp(b_{12} - b_{14}) + m_6 m_7 \exp(b_{12}) + m_7 m_8 \exp(b_{12} + b_{14}) \\ = \frac{s_6}{s_5} (m_2 m_4 \exp(-b_{14}) + 1 + m_6 m_8 \exp(b_{14})) \end{aligned} \quad (14. b.29)$$

From (14.b.28),

$$\exp(b_{12}) = \frac{s_5 (m_2 m_3 \exp(-b_{14}) + m_3 m_4 + m_3 m_5 m_8 \exp(b_{14}))}{s_4 (m_2 m_4 \exp(-b_{14}) + 1 + m_6 m_8 \exp(b_{14}))} \quad (14. b.30)$$

From (14.b.29),

$$\exp(b_{12}) = \frac{s_6 (m_2 m_4 \exp(-b_{14}) + 1 + m_6 m_8 \exp(b_{14}))}{s_5 (m_2 m_5 m_7 \exp(-b_{14}) + m_6 m_7 + m_7 m_8 \exp(b_{14}))} \quad (14. b.31)$$

Thus, given  $b_{14}$ ,  $b_{12}$  is also uniquely determined.

Equating these two expressions for  $\exp(b_{12})$  and collecting terms, we have

$$\begin{aligned}
& \exp(-2b14) \left[ s4s6(m2)^2(m4)^2 - (s5)^2(m2)^2 m3m5m7 \right] + \exp(-b14) \left[ 2m2m4s4s6 - m2m3m7(m4m5 + m6)(s5)^2 \right] \\
& + \left[ 2m2m4m6m8s4s6 + s4s6 - m7(s5)^2 (m2m3(m5)^2 m8 + m3m4m6 + m2m3m8) \right] \\
& + \exp(b14) \left[ 2m6m8s4s6 - m7m8(m3m5m6 + m3m4)(s5)^2 \right] \\
& + \exp(2b14) \left[ s4s6(m6)^2(m8)^2 - m3m5m7(m8)^2(s5)^2 \right] = 0
\end{aligned}$$

If the above equation is multiplied by  $\exp(2b14)$  and if the substitution  $x = \exp(b14)$  is made, we have

$$\begin{aligned}
& \left[ s4s6(m2)^2(m4)^2 - (s5)^2(m2)^2 m3m5m7 \right] + x \left[ 2m2m4s4s6 - m2m3m7(m4m5 + m6)(s5)^2 \right] \\
& + x^2 \left[ 2m2m4m6m8s4s6 + s4s6 - m7(s5)^2 (m2m3(m5)^2 m8 + m3m4m6 + m2m3m8) \right] \\
& + x^3 \left[ 2m6m8s4s6 - m7m8(m3m5m6 + m3m4)(s5)^2 \right] + x^4 \left[ s4s6(m6)^2(m8)^2 - m3m5m7(m8)^2(s5)^2 \right] = 0
\end{aligned}$$

(14.b.32)

Thus, equations (14.b.21) - (14.b.23) generate a fourth degree polynomial in  $\exp(b14)$ . Since  $\exp(b14) > 0$ , we seek only positive solutions to (14.b.32). As noted above, if  $b14$  is uniquely determined, so will be  $b10$  and  $b12$ , and the model will be identified. The uniqueness, or lack of uniqueness, of a solution for  $b14$  will be determined empirically for a given data set.

In summary, the identification status of this model first hinges on the uniqueness (or nonuniqueness) of a solution for  $b4$ . If  $b4$  is not uniquely determined, the model is not identified. If  $b4$  is uniquely determined, the identification status of the model then depends on the uniqueness (or nonuniqueness) of a solution for  $b14$ . If  $b14$  is also uniquely determined, then so are  $b10$  and  $b12$ , and the model is identified. On the other hand, if  $b14$  is not uniquely determined, the model is not identified.

The identification status of this model will be determined empirically for a given data set.

### Type III:

If all the interaction terms of the missing data indicators and the column variable, the missing data indicators and the row variable, and the three way interactions of the missing data indicators, the column variable and the row variable are included in the model, there would be too many parameters, given the data. Even if only the linear interaction terms involving the missing data indicators appear in the model, sixteen parameters would be required. Since a model for this data set can contain no more than fifteen parameters, even this modified Type III model cannot be considered for this data set. Thus, no Type III models can be explored for identification involving quasi - symmetry for this data set.

In summary, for the quasi - symmetry model with two supplemental margins, the Type I nonignorable nonresponse model is identified, while the Type III model, (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicators and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 2

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the quasi-symmetry models hypothesized to fit Data Set 2 (page 219). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicators).

#### Data Set 2

##### Quasi-Symmetry Model

##### Completely Classified Data Only

|                     |                 | First Interview |              |                 |      |
|---------------------|-----------------|-----------------|--------------|-----------------|------|
|                     |                 | crime free      | single crime | multiple crimes |      |
| Second<br>Interview | crime free      | 1742            | 259.18       | 66.82           | 2068 |
|                     | single crime    | 228.82          | 56           | 30.18           | 315  |
|                     | multiple crimes | 62.18           | 31.82        | 10              | 104  |
|                     |                 | 2033            | 347          | 107             | 2487 |

$$\beta = (4.03, -1.31, .47, -1.27, .38, -.25, -.72, .27)$$

Likelihood Ratio fit statistic = .0693133

Degrees of freedom = 1

**Data Set 2****Quasi-Symmetry Model - Two Supplemental Margins****Missing Completely at Random (MCAR)****Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1666.64         | 284.36       | 81.15           | 2032.15 |
|                     | single crime    | 228.49          | 64.54        | 38.89           | 331.92  |
|                     | multiple crimes | 67.97           | 40.54        | 14.42           | 122.93  |
|                     |                 | 1963.10         | 389.44       | 134.46          | 2487    |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 835.67          | 142.58       | 40.69           | 1018.94 |
|                     | single crime    | 114.57          | 32.36        | 19.50           | 166.43  |
|                     | multiple crimes | 34.08           | 20.33        | 7.23            | 61.64   |
|                     |                 | 1074.32         | 195.27       | 67.42           | 1247.01 |



**Data Set 2**

Quasi-Symmetry Model - Two Supplemental Margins

Missing Completely at Random (MCAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 662.77          | 113.08       | 32.27           | 808.12 |
|                     | single crime    | 90.86           | 25.67        | 15.46           | 131.99 |
|                     | multiple crimes | 27.03           | 16.12        | 5.73            | 48.88  |
|                     |                 | 780.66          | 154.87       | 53.46           | 988.99 |

 $\beta = (3.48, -1.23, .50, -1.14, .37, -.25, -.74, .26, .69, -.23)$ 

Likelihood Ratio fit statistic = 45.427

Degrees of freedom = 5

**Data Set 2**

Quasi-Symmetry Model - Two Supplemental Margins

Missing at Random (MAR)

**Completely Classified Data****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 1742.01    | 259.08       | 66.91           | 2068 |
|                     | single crime    | 228.84     | 55.90        | 30.26           | 315  |
|                     | multiple crimes | 62.12      | 31.80        | 10.08           | 104  |
|                     |                 | 2032.97    | 346.78       | 107.25          | 2487 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)****First Interview**

|                     |                 | crime free | single crime | multiple crimes |      |
|---------------------|-----------------|------------|--------------|-----------------|------|
| Second<br>Interview | crime free      | 837.31     | 124.53       | 32.16           | 994  |
|                     | single crime    | 128.59     | 31.41        | 17.00           | 177  |
|                     | multiple crimes | 45.39      | 23.24        | 7.37            | 76   |
|                     |                 | 1011.29    | 179.18       | 56.53           | 1247 |

**Data Set 2**

Quasi - Symmetry Model - Two Supplemental Margins

Missing at Random (MAR)

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 217.72          | 32.38        | 8.36            | 258.46 |
|                     | single crime    | 348.48          | 85.12        | 46.08           | 479.68 |
|                     | multiple crimes | 149.83          | 76.71        | 24.31           | 250.85 |
|                     |                 | 716.03          | 194.21       | 78.75           | 988.99 |

 $\beta = (3.45, -1.10, .53, -1.27, .39, -.25, -.72, .27, .58, 1.00, -.21, 1.27, -.05, -1.07)$ 

Likelihood Ratio fit statistic = .0715745

Degrees of freedom = 1

**Data Set 2****Quasi-Symmetry Model - Two Supplemental Margins****Nonignorable Nonresponse - Type I****Completely Classified Data**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 1742.10         | 259.27       | 66.79           | 2068.16 |
|                     | single crime    | 228.62          | 55.80        | 30.19           | 314.61  |
|                     | multiple crimes | 62.29           | 31.93        | 10.02           | 104.24  |
|                     |                 | 2033.01         | 347          | 107             | 2487.01 |

**Partially Classified Data (Hypothetical Contingency Table for Row Marginals)**

|                     |                 | First Interview |              |                 |         |
|---------------------|-----------------|-----------------|--------------|-----------------|---------|
|                     |                 | crime free      | single crime | multiple crimes |         |
| Second<br>Interview | crime free      | 545.68          | 408.78       | 39.38           | 993.84  |
|                     | single crime    | 71.61           | 87.98        | 17.80           | 177.39  |
|                     | multiple crimes | 19.51           | 50.34        | 5.91            | 75.76   |
|                     |                 | 636.80          | 547.10       | 63.09           | 1246.99 |

**Data Set 2**

Quasi-Symmetry Model - Two Supplemental Margins

Nonignorable Nonresponse - Type I

**Partially Classified Data (Hypothetical Contingency Table for Column Marginals)**

|                     |                 | First Interview |              |                 |        |
|---------------------|-----------------|-----------------|--------------|-----------------|--------|
|                     |                 | crime free      | single crime | multiple crimes |        |
| Second<br>Interview | crime free      | 613.55          | 144.95       | 49.31           | 807.81 |
|                     | single crime    | 80.52           | 31.20        | 22.29           | 134.01 |
|                     | multiple crimes | 21.94           | 17.85        | 7.40            | 47.19  |
|                     |                 | 716.01          | 194          | 79              | 989.01 |

 $\beta = (4.48, -1.31, .48, -.96, -.91, -.25, -.72, .27, -.46, -1.03, -.32, .05, 1.30, 1.21)$ 

Likelihood Ratio fit statistic = .0724949

Degrees of freedom = 1

## **Data Set 2**

### **Quasi-Symmetry Model - Two Supplemental Margins**

#### **Nonignorable Nonresponse - Type II (linear interaction terms only)**

As indicated on page 389, the identification status of the modified nonignorable nonresponse quasi - symmetry Type II model depends on the number of solutions to equations (7.b.27) and (14.b.32). The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (7.b.27) giving the following fourth degree polynomial

$$-.06+.54x-.90x^2-1.19x^3-2.09x^4=0$$

By Descartes' Rule of Signs, since there are two changes of sign in this fourth degree polynomial equation, there are two or no positive solutions for  $x = \exp(b_4)$ . By Newton's Method, there are found to be two positive solutions to this equation. As discussed on page 389, this model is therefore not identified for the given data set.

### Summary of Results - Data Set 2

Presented below is a summary of the determination of the identification status of nonignorable nonresponse models for Data Set 2, a 3x3 table with two supplemental margins, as well as the empirical analyses for a specific data set of this structure (page 219).

#### Identification Status of Nonignorable Nonresponse Models for Data Set 2

|                     | Type I | Type I<br>(modified) | Type II | Type II<br>(modified) | Type III<br>(modified) |
|---------------------|--------|----------------------|---------|-----------------------|------------------------|
| Independence        | X      | X                    | X       | X                     | X                      |
| Uniform Association | *      | -                    | X       | X                     | X                      |
| Row Effects         | *      | -                    | X       | ?                     | X                      |
| Column Effects      | *      | -                    | X       | ?                     | X                      |
| R + C               | *      | -                    | X       | ?                     | X                      |
| Symmetry            | *      | -                    | X       | ?                     | X                      |
| Quasi - symmetry    | *      | -                    | X       | ?                     | X                      |

x = not identified

\* = identified

- If a full model is identified, the modified model is not explored for identification (It can be shown, however, that the modified Type I model is identified, except in the case of the independence model)
- ? The identification status of the model will be examined empirically for a given data set

# Chi Square Goodness of Fit for Tested Models

## Data Set 2 (page 219)

|                        | Com-<br>plete<br>Data<br>Only | MCAR           | MAR             | NN Type<br>I    | NN<br>TYPE I<br>(modi-<br>fied) | NN<br>TYPE<br>II | NN<br>TYPE II<br>(modi-<br>fied) | NN<br>Type III<br>(modi-<br>fied) |
|------------------------|-------------------------------|----------------|-----------------|-----------------|---------------------------------|------------------|----------------------------------|-----------------------------------|
| Independence           | 59.3404<br>(4)                | 106.591<br>(8) | 97.8555<br>(4)  | x               | x                               | x                | x                                | x                                 |
| Uniform<br>Association | 6.78264<br>(3)                | 52.0794<br>(7) | 7.08473<br>(3)  | 6.80069<br>(3)  | -                               | x                | x                                | x                                 |
| Row Effects            | 4.40433<br>(2)                | 49.7935<br>(6) | 4.6731<br>(2)   | 4.80975<br>(2)  | -                               | x                |                                  | x                                 |
| Column Effects         | 5.31944<br>(2)                | 50.6083<br>(6) | 5.35846<br>(2)  | 5.37314<br>(2)  | -                               | x                |                                  | x                                 |
| R + C                  | 3.28098<br>(1)                | 48.6567<br>(5) | 3.28373<br>(1)  | 3.89769<br>(1)  | -                               | x                |                                  | x                                 |
| Symmetry               | 2.16964<br>(3)                | 55.4829<br>(7) | 4.65182<br>(3)  | 8.81721<br>(3)  | -                               | x                |                                  | x                                 |
| Quasi -<br>symmetry    | .069313<br>3 (1)              | 45.427<br>(5)  | .0715745<br>(1) | .0724949<br>(1) | -                               | x                |                                  | x                                 |

- Convergence criterion: difference in successive chi square values is <.00001
- Those chi square values in bold type are not significant at the .05 level. Thus, the models for which bold type appears fit the data.
- Models whose identification status was empirically determined have shaded entries in



the table. Note that the row effects, column effects and quasi - symmetry  
nognignorable nonresponse modified Type II models were each determined,  
empirically, to be not identified for this data set.

### DATA SET 3

The third kind of data set explored in this work contains one ordered and one unordered categorical variable. The ordered (row) variable has four categories, while the unordered (column) variable has three categories. In addition to the fully classified data, there is one supplemental margin (for the column variable).

#### Determination of the Identification Status of Nonignorable Nonresponse Models

As discussed earlier (page 17), we use the extension of the generalized linear model with composite links as conceptualized by Thompson and Baker (1981) and as applied by Rindskopf (1992) to the loglinear model. We have,  $E(Y) = C \exp(X\beta)$ .

Here,  $Y$  is a column vector of sixteen elements. The first twelve are the cell frequencies in the 4x3 completely classified contingency table. The last four are the supplemental column marginals for the partially classified data.

$\beta$  is a vector of parameters whose values are unknown and must be estimated from the data.

$C$  is a matrix of 1's and 0's. For all models for Data Set 3,  $C$  is the same 16 by 24 matrix, whose elements indicate how the observed and unobserved cell counts are to be added to make use of the observed information. More specifically, for these data the matrix  $C$  has the following form:

$$\begin{pmatrix} I_{12 \times 12} & 0_{12 \times 12} \\ 0_{4 \times 12} & A \end{pmatrix}, \text{ where } I \text{ and } 0 \text{ are the identity and zero matrices, respectively,}$$

and

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The rows of the  $C$  matrix correspond to the pattern of the given cell counts and the supplemental marginals provided. The first twelve rows of  $C$  correspond to the completely classified data. The thirteenth row of  $C$  indicates that three specific cell counts need to be added in order to produce the first observed supplemental row marginal. The 1's in the thirteenth row appear in columns 13, 14, and 15. These correspond to the (unobserved) frequency counts in the first row of the contingency table in which only the supplementary row marginal is observed. Thus, the thirteenth row of  $C$  indicates that these three (unobserved) cell counts must be added to produce the first observed supplementary row margin. Similarly, the fourteenth, fifteenth and sixteenth rows of  $C$  indicate the need to add the three (unobserved) frequency counts in the second, third and fourth rows, respectively, of the contingency table in which only the supplemental row marginals are observed.

Finally,  $X$  is the design (or model) matrix. The  $X$  matrix has 24 rows. The first twelve of these rows correspond to each of the twelve cells in the fully classified 4x3 contingency table. The last twelve rows correspond to each of the twelve cells in the (hypothetical) 4x3 contingency table for the partially classified data in which only the row marginals are observed. The columns of the  $X$  matrix reflect hypotheses concerning the relationships among the row and column variables and the missing data mechanism in the model.

For the ordered categorical (row) variable, integer coding will be used as follows:

row variable:  $r = -3, -1, 1, 3$

For the unordered categorical variable, two dummy variables,  $c1$  and  $c2$  will be coded as follows: column variable:  $c1 = 1$ , if in column 1; 0 otherwise

$c2 = 1$ , if in column 2; 0 otherwise

As is usual for ordinal variables which assume four values and which have integer coding, when the row (ordinal) variable appears in any of the loglinear models, there will be a linear, a quadratic and a cubic term.

One missing data indicator is required. It will assume the value '1' if the data are completely classified and '0' if partially classified. Data cells corresponding to the completely classified data will be designated by  $k = 1$ , and those for which only the supplemental row marginals are observed will be designated by  $k = 2$ .

The identification status of models for each of the three types of nonignorable nonresponse given on page 21 will be explored for this third data set. Many fewer models can be considered to model this third data set than for the first two data sets explored in this work. The contingency table is not a square one, and only one of the variables is ordinal. Thus, of the eight kinds of models for ordinal variables reviewed on pages 5 - 7, only the independence and column effects models are appropriate. Initially, relationships among the data categorized by the ordinal and nominal variables will be described by each of the independence and column effects models. To the terms of each of these initial models will be added terms which represent the missing data indicator and nonignorable nonresponse for the partially categorized data. The terms representing nonignorable nonresponse may be of three types (page 21). Thus, for each of the independence and column effects models describing relationships among the row and column variables, three nonignorable nonresponse models will be considered for identification. Further, if a model is determined to be not identified, a modified version of this model will be

evaluated for identification. This modified model will include only the linear interaction terms involving the missing data indicator. As the column variable is nominal, no such modification of the Type I nonignorable nonresponse model is possible. Thus, for each of the independence and column effects models describing relationships among the row and column variables, up to five nonignorable nonresponse models may be considered for identification.

As noted above, the columns of the  $X$  matrix reflect hypotheses concerning the relationships among the row and column variables and the missing data mechanism in the model. For each nonignorable nonresponse model considered for identification, the model matrix (page 17) will first be presented, with cell indices displayed to the right of the matrix. An analysis to determine the identification status of the model will then follow.

### Empirical Analyses for Data Set 3

Empirical model fitting of a specific 4x3 data set containing an ordinal (row) variable and a nominal (column) variable, and having one supplemental row margin will then be performed. This data set is part of the Roper polling data for the 1948 Presidential election, as reported and analyzed by Baker and Laird (1988). In particular, the contingency table contains the polling frequencies for the October, 1948 poll. Recorded are those in four (ordered) economic classes, A, B, C and D, who expressed a preference for Truman, Dewey or another candidate. An additional tabulation was made for those who expressed no preference for a candidate. The economic class of each of those in this latter group was recorded, however. Thus, those who did not express a preference comprised the supplemental row margin. Given below is the contingency table for the distribution of frequencies for the October, 1948 Roper poll:

**Presidential Candidate Preference**

|                 |              | <b>Truman</b> | <b>Dewey</b> | <b>other</b> | <b>missing</b> | <b>total</b> |
|-----------------|--------------|---------------|--------------|--------------|----------------|--------------|
|                 | <b>A</b>     | 40            | 175          | 9            | 18             | 242          |
| <b>Economic</b> | <b>B</b>     | 203           | 508          | 45           | 109            | 865          |
| <b>Class</b>    | <b>C</b>     | 552           | 623          | 83           | 315            | 1573         |
|                 | <b>D</b>     | 354           | 211          | 38           | 217            | 820          |
|                 | <b>total</b> | 1149          | 1517         | 175          | 659            | 3500         |

Using the computer program written in SAS (page 22), likelihood ratio chi-squares and estimates for parameter and expected frequencies will be determined for models hypothesized to fit this data set. Estimates will be obtained for the independence and column effects models using the both the fully categorized data and the supplemental margin, as well as only the completely classified data. This former group of models will include those which hypothesize that the partially classified data are missing completely at random (MCAR), missing at random (MAR), and identified and subject to nonignorable nonresponse.

### Organization of Presentation of Analyses for Data Set 3

For each of the two types of relationships among the variables considered for Data Set 3 (independence and column effects), the identification status of the nonignorable nonresponse models will first be established. For each type of relationship, this will be followed by the determination of likelihood ratio fit statistics and parameter and frequency estimates for those models involving the complete data only, as well as for those involving the partially classified data.

### Independence Models - Data Set 3

As noted for Data Set 1, Little and Rubin (1987) explored the identification status of hierarchical loglinear models in which nonignorable nonresponse was hypothesized. In particular, the independence model with nonignorable nonresponse was found to have inestimable parameters, and so was determined to be not identified. As was the case with Data Set 1, for the independence model with hypothesized nonignorable nonresponse, this work should be viewed as a corroboration of the results of Little and Rubin (1987).

Type I: independence; 4x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
| 1      | -3  | 9   | -27 | 1   | 0   | 1   | 1   | 0   |     | 1 | 1 | 1 |
| 1      | -3  | 9   | -27 | 0   | 1   | 1   | 0   | 1   |     | 1 | 2 | 1 |
| 1      | -3  | 9   | -27 | 0   | 0   | 1   | 0   | 0   |     | 1 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 0   | 1   | 1   | 0   |     | 2 | 1 | 1 |
| 1      | -1  | 1   | -1  | 0   | 1   | 1   | 0   | 1   |     | 2 | 2 | 1 |
| 1      | -1  | 1   | -1  | 0   | 0   | 1   | 0   | 0   |     | 2 | 3 | 1 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 3 | 1 | 1 |
| 1      | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 3 | 3 | 1 |
| 1      | 3   | 9   | 27  | 1   | 0   | 1   | 1   | 0   |     | 4 | 1 | 1 |
| 1      | 3   | 9   | 27  | 0   | 1   | 1   | 0   | 1   |     | 4 | 2 | 1 |
| 1      | 3   | 9   | 27  | 0   | 0   | 1   | 0   | 0   |     | 4 | 3 | 1 |
| 1      | -3  | 9   | -27 | 1   | 0   | 0   | 0   | 0   |     | 1 | 1 | 2 |
| 1      | -3  | 9   | -27 | 0   | 1   | 0   | 0   | 0   |     | 1 | 2 | 2 |
| 1      | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 0   |     | 1 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   |     | 2 | 1 | 2 |
| 1      | -1  | 1   | -1  | 0   | 1   | 0   | 0   | 0   |     | 2 | 2 | 2 |
| 1      | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0   |     | 2 | 3 | 2 |
| 1      | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 3 | 1 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 3 | 3 | 2 |
| 1      | 3   | 9   | 27  | 1   | 0   | 0   | 0   | 0   | 0   | 4 | 1 | 2 |
| 1      | 3   | 9   | 27  | 0   | 1   | 0   | 0   | 0   | 0   | 4 | 2 | 2 |
| 1      | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 0   | 0   | 4 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second through fourth columns, the row variable, the fifth and sixth columns, the column variable, the seventh column, the missing data indicator, and the eighth and ninth columns, the interaction of the missing data indicator and the column variable. The nine parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the



columns of the  $X$  matrix, in the same order. The row variable is coded to reflect linear, quadratic and cubic terms in the model.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier (page 17), we use the extension of the generalized linear model with composite links as conceptualized by Thompson and Baker (1981) and as applied by Rindskopf (1992) to the loglinear model. We have,  $E(Y) = C \exp(X\beta)$ . Recall that  $Y$  is a vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a vector of parameters whose values are unknown and must be estimated from the data. The upper left hand  $12 \times 12$  section of matrix  $C$  is an identity matrix (page 402). It can therefore be seen that for the portion of vector  $Y$  corresponding to the completely classified data (and the first twelve rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Let us assume that we can obtain estimates for the first twelve elements of vector  $Y$ . We will call these estimates  $f1, f2, \dots, f12$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b_1 - 3b_2 + 9b_3 - 27b_4 + b_5 + b_7 + b_8 = \ln f_1 \quad (15.a.1)$$

$$b_1 - 3b_2 + 9b_3 - 27b_4 + b_6 + b_7 + b_9 = \ln f_2 \quad (15.a.2)$$

$$b_1 - 3b_2 + 9b_3 - 27b_4 + b_7 = \ln f_3 \quad (15.a.3)$$

$$b_1 - b_2 + b_3 - b_4 + b_5 + b_7 + b_8 = \ln f_4 \quad (15.a.4)$$

$$b_1 - b_2 + b_3 - b_4 + b_6 + b_7 + b_9 = \ln f_5 \quad (15.a.5)$$

$$b_1 - b_2 + b_3 - b_4 + b_7 = \ln f_6 \quad (15.a.6)$$

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_7 + b_8 = \ln f_7 \quad (15.a.7)$$

$$b_1 + b_2 + b_3 + b_4 + b_6 + b_7 + b_9 = \ln f_8 \quad (15.a.8)$$

$$b_1 + b_2 + b_3 + b_4 + b_7 = \ln f_9 \quad (15.a.9)$$

$$b_1 + 3b_2 + 9b_3 + 27b_4 + b_5 + b_7 + b_8 = \ln f_{10} \quad (15.a.10)$$

$$b_1 + 3b_2 + 9b_3 + 27b_4 + b_6 + b_7 + b_9 = \ln f_{11} \quad (15.a.11)$$

$$b_1 + 3b_2 + 9b_3 + 27b_4 + b_7 = \ln f_{12} \quad (15.a.12)$$

Here,  $b_1, b_2, \dots, b_9$  are the nine parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's.

Subtracting (15.a.6) from (15.a.5), we have

$$b_6 + b_9 = \ln f_5 - \ln f_6 \quad (15.a.13)$$

Subtracting (15.a.8) from (15.a.7), and substituting (15.a.13), we obtain

$$b_5 + b_8 = \ln \frac{f_5 f_7}{f_6 f_8} \quad (15.a.14)$$

Subtracting (15.a.6) from (15.a.9), and subtracting (15.a.3) from (15.a.12), we have

$$2b_2 + 2b_4 = \ln f_9 - \ln f_6$$

$$6b_2 + 54b_4 = \ln f_{12} - \ln f_3$$

$$\text{Thus, } b_4 = \frac{1}{48} \ln \frac{f_{12}(f_6)^3}{f_3(f_9)^3} \quad (15.a.15)$$

$$\text{and } b_2 = \frac{1}{48} \ln \frac{f_3(f_9)^{27}}{f_{12}(f_6)^{27}} \quad (15.a.16)$$

Adding (15.a.6) and (15.a.9), and adding (15.a.3) and (15.a.12), we obtain

$$2b_1 + 2b_3 + 2b_7 = \ln f_6 + \ln f_9$$

$$2b_1 + 18b_3 + 2b_7 = \ln f_3 + \ln f_{12}$$

$$\text{Thus, } b_3 = \frac{1}{16} \ln \frac{f_3 f_{12}}{f_6 f_9} \quad (15.a.17)$$

$$\text{and } b_1 + b_7 = \frac{1}{16} \ln \frac{(f_6)^9 (f_9)^9}{f_3 f_{12}} \quad (15.a.18)$$

Note that  $b_1$  and  $b_7$  cannot be determined uniquely from equations (15.a.1) - (15.a.12), since whenever  $b_1$  appears in an equation, so does  $b_7$ ; that is  $b_1 + b_7$  appears as a unit in each of the equations. Similarly,  $b_5$  and  $b_8$  cannot be determined uniquely, nor can  $b_6$  and  $b_9$ . Only three of the nine model parameters ( $b_2$ ,  $b_3$  and  $b_4$ ) have been uniquely determined. Equations (15.a.13) - (15.a.18) provide the most complete solution for the  $b$ 's which can be determined solely from equations (15.a.1) - (15.a.12).

The last four elements of vector  $Y$  correspond to the partially classified data (and the last twelve rows of the  $X$  matrix). Each of these last four elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f_{13}$ , ...,  $f_{24}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b_1 - 3b_2 + 9b_3 - 27b_4)[\exp(b_5) + \exp(b_6) + 1] = s_1 \quad (15.a.19)$$

$$\exp(b_1 - b_2 + b_3 - b_4)[\exp(b_5) + \exp(b_6) + 1] = s_2 \quad (15.a.20)$$

$$\exp(b_1 + b_2 + b_3 + b_4)[\exp(b_5) + \exp(b_6) + 1] = s_3 \quad (15.a.21)$$

$$\exp(b_1 + 3b_2 + 9b_3 + 27b_4)[\exp(b_5) + \exp(b_6) + 1] = s_4 \quad (15.a.22)$$

Here,  $s_1 = f_{13} + f_{14} + f_{15}$ , the first row marginal for the partially classified data,

$s_2 = f_{16} + f_{17} + f_{18}$ , the second row marginal for the partially classified data,

$s_3 = f_{19} + f_{20} + f_{21}$ , the third row marginal for the partially classified data and

$s_4 = f_{22} + f_{23} + f_{24}$ , the fourth row marginal for the partially classified data.

We assume that we can obtain estimates  $s_1$ ,  $s_2$ ,  $s_3$  and  $s_4$ , for the last four elements of vector  $Y$ .

Equations (15a.19) - (15.a.22) can be rewritten as

$$\exp(-3b_2 + 9b_3 - 27b_4)[\exp(b_1 + b_5) + \exp(b_1 + b_6) + \exp(b_1)] = s_1$$

$$\exp(-b_2 + b_3 - b_4)[\exp(b_1 + b_5) + \exp(b_1 + b_6) + \exp(b_1)] = s_2$$

$$\exp(b_2 + b_3 + b_4)[\exp(b_1 + b_5) + \exp(b_1 + b_6) + \exp(b_1)] = s_3$$

$$\exp(3b_2 + 9b_3 + 27b_4)[\exp(b_1 + b_5) + \exp(b_1 + b_6) + \exp(b_1)] = s_4$$

As  $b_2$ ,  $b_3$  and  $b_4$  have been uniquely determined in equations in (15.a.16), (15.a.17) and (15.a.15), respectively, these four equations give only one relationship between  $b_1$ ,  $b_5$  and  $b_6$ . This, together with equations (15.a.13), (15.a.14) and (15.a.18) thus give four relationships among the six remaining unknown parameters. These four relationships are insufficient to uniquely determine the six unknowns. Thus, the independence model for this data set, with nonignorable nonresponse of Type I is not identified.

Type II: independence; 4x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | -3  | 9   | -27 | 1   | 0   | 1   | 1   | 0   | -3   | 9    | -27  | 1 | 1 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 1   | 0   | 1   | -3   | 9    | -27  | 1 | 2 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 1   | 0   | 0   | -3   | 9    | -27  | 1 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 1   | 1   | 0   | -1   | 1    | -1   | 2 | 1 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 1   | 0   | 1   | -1   | 1    | -1   | 2 | 2 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 1   | 0   | 0   | -1   | 1    | -1   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1    | 1    | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 1    | 1    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 1    | 1    | 1    | 3 | 3 | 1 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 1   | 1   | 0   | 3    | 9    | 27   | 4 | 1 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 1   | 0   | 1   | 3    | 9    | 27   | 4 | 2 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 1   | 0   | 0   | 3    | 9    | 27   | 4 | 3 | 1 |
|        | 1   | -3  | 9   | -27 | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 4 | 1 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 4 | 2 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 4 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second through fourth columns, the row variable, the fifth and sixth columns, the column variable, the seventh column, the missing data indicator, the eighth and ninth columns, the interaction of the missing data indicator and the column variable, and the tenth through twelfth columns, the interaction of the missing data indicator and the row variable. The twelve parameters

of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

We assume, as was done for the Type I case, that estimates  $f1, f2, \dots, f12$ , are available for the first twelve elements of vector  $Y$ , and  $s1, s2, s3$  and  $s4$ , for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - 3b2 + 9b3 - 27b4 + b5 + b7 + b8 - 3b10 + 9b11 - 27b12 = \ln f1 \quad (15.b.1)$$

$$b1 - 3b2 + 9b3 - 27b4 + b6 + b7 + b9 - 3b10 + 9b11 - 27b12 = \ln f2 \quad (15.b.2)$$

$$b1 - 3b2 + 9b3 - 27b4 + b7 - 3b10 + 9b11 - 27b12 = \ln f3 \quad (15.b.3)$$

$$b1 - b2 + b3 - b4 + b5 + b7 + b8 - b10 + b11 - b12 = \ln f4 \quad (15.b.4)$$

$$b1 - b2 + b3 - b4 + b6 + b7 + b9 - b10 + b11 - b12 = \ln f5 \quad (15.b.5)$$

$$b1 - b2 + b3 - b4 + b7 - b10 + b11 - b12 = \ln f6 \quad (15.b.6)$$

$$b1 + b2 + b3 + b4 + b5 + b7 + b8 + b10 + b11 + b12 = \ln f7 \quad (15.b.7)$$

$$b1 + b2 + b3 + b4 + b6 + b7 + b9 + b10 + b11 + b12 = \ln f8 \quad (15.b.8)$$

$$b1 + b2 + b3 + b4 + b7 + b10 + b11 + b12 = \ln f9 \quad (15.b.9)$$

$$b1 + 3b2 + 9b3 + 27b4 + b5 + b7 + b8 + 3b10 + 9b11 + 27b12 = \ln f10 \quad (15.b.10)$$

$$b1 + 3b2 + 9b3 + 27b4 + b6 + b7 + b9 + 3b10 + 9b11 + 27b12 = \ln f11 \quad (15.b.11)$$

$$b1 + 3b2 + 9b3 + 27b4 + b7 + 3b10 + 9b11 + 27b12 = \ln f12 \quad (15.b.12)$$

Here,  $b1, b2, \dots, b12$  are the twelve parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as for the independence model with nonignorable nonresponse Type I (pages 410 - 411), we obtain

$$b6 + b9 = \ln f5 - \ln f6 \quad (15.b.13)$$

$$b5 + b8 = \ln \frac{f5f7}{f6f8} \quad (15.b.14)$$

$$b4 + b12 = \frac{1}{48} \ln \frac{f12(f6)^3}{f3(f9)^3} \quad (15.b.15)$$

$$b2 + b10 = \frac{1}{48} \ln \frac{f3(f9)^{27}}{f12(f6)^{27}} \quad (15.b.16)$$

$$b3 + b11 = \frac{1}{16} \ln \frac{f3f12}{f6f9} \quad (15.b.17)$$

$$b1 + b7 = \frac{1}{16} \ln \frac{(f6)^9(f9)^9}{f3f12} \quad (15.b.18)$$

Note that  $b1$  and  $b7$  cannot be determined uniquely from equations (15.b.1) - (15.b.12), since whenever  $b1$  appears in an equation, so does  $b7$ ; that is  $b1 + b7$  appears as a unit in each of the equations. Similarly,  $b5$  and  $b8$  cannot be determined uniquely, nor can  $b6$  and  $b9$ ,  $b4$  and  $b12$ ,  $b2$  and  $b10$ , and  $b3$  and  $b11$ . None of the twelve model parameters has been uniquely determined. Equations (15.b.13) - (15.b.18) provide the most complete solution for the  $b$ 's which can be determined solely from equations (15.b.1) - (15.b.12).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Type I (page 411). They are given by

$$\exp(b1 - 3b2 + 9b3 - 27b4)[\exp(b5) + \exp(b6) + 1] = s1 \quad (15.b.19)$$

$$\exp(b1 - b2 + b3 - b4)[\exp(b5) + \exp(b6) + 1] = s2 \quad (15.b.20)$$

$$\exp(b1 + b2 + b3 + b4)[\exp(b5) + \exp(b6) + 1] = s3 \quad (15.b.21)$$

$$\exp(b1 + 3b2 + 9b3 + 27b4)[\exp(b5) + \exp(b6) + 1] = s4 \quad (15.b.22)$$

Equations (15.b.13) - (15.b.22) give ten equations in the twelve unknown parameters. These ten equations are insufficient to solve for the twelve unknowns. This model, therefore, is not identified.

While the independence model for this data set with nonignorable nonresponse of Type II is not identified, it may be that the elimination of the interaction terms of the missing data indicator and the quadratic and cubic row variable terms will result in an identified model. This modified nonignorable nonresponse model is considered next.



Type II: independence; 4x3 table with one supplemental margin

- modified (with only the linear interaction term of the missing data indicator and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|---|---|---|
|        | 1   | -3  | 9   | -27 | 1   | 0   | 1   | 1   | 0   | -3   | 1 | 1 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 1   | 0   | 1   | -3   | 1 | 2 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 1   | 0   | 0   | -3   | 1 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 1   | 1   | 0   | -1   | 2 | 1 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 1   | 0   | 1   | -1   | 2 | 2 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 1   | 0   | 0   | -1   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 1    | 3 | 3 | 1 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 1   | 1   | 0   | 3    | 4 | 1 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 1   | 0   | 1   | 3    | 4 | 2 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 1   | 0   | 0   | 3    | 4 | 3 | 1 |
|        | 1   | -3  | 9   | -27 | 1   | 0   | 0   | 0   | 0   | 0    | 1 | 1 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | 0   | 0   | 0    | 1 | 2 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 0   | 0    | 1 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 2 | 1 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | 0   | 0   | 0    | 2 | 2 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0   | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 3 | 3 | 2 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 0   | 0   | 0   | 0    | 4 | 1 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 0   | 0   | 0    | 4 | 2 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 0   | 0    | 4 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second through fourth columns, the row variable, the fifth and sixth columns, the column variable, the seventh column, the missing data indicator, the eighth and ninth columns, the interaction of the missing data indicator and the column variable, and the tenth column, the interaction of

the missing data indicator and the linear row variable term. The ten parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

Using the same conceptual rationale and procedures as for Type II nonignorable nonresponse with all missing data indicator and row variable interaction terms, the identification status of this model is now explored. The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Let us assume that estimates  $f1, f2, \dots, f12$ , are available for the first twelve elements of vector  $Y$ , and estimates  $s1, s2, s3$  and  $s4$ , for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - 3b2 + 9b3 - 27b4 + b5 + b7 + b8 - 3b10 = \ln f1 \quad (15.c.1)$$

$$b1 - 3b2 + 9b3 - 27b4 + b6 + b7 + b9 - 3b10 = \ln f2 \quad (15.c.2)$$

$$b1 - 3b2 + 9b3 - 27b4 + b7 - 3b10 = \ln f3 \quad (15.c.3)$$

$$b1 - b2 + b3 - b4 + b5 + b7 + b8 - b10 = \ln f4 \quad (15.c.4)$$

$$b1 - b2 + b3 - b4 + b6 + b7 + b9 - b10 = \ln f5 \quad (15.c.5)$$

$$b1 - b2 + b3 - b4 + b7 - b10 = \ln f6 \quad (15.c.6)$$

$$b1 + b2 + b3 + b4 + b5 + b7 + b8 + b10 = \ln f7 \quad (15.c.7)$$

$$b1 + b2 + b3 + b4 + b6 + b7 + b9 + b10 = \ln f8 \quad (15.c.8)$$

$$b1 + b2 + b3 + b4 + b7 + b10 = \ln f9 \quad (15.c.9)$$

$$b1 + 3b2 + 9b3 + 27b4 + b5 + b7 + b8 + 3b10 = \ln f10 \quad (15.c.10)$$

$$b1 + 3b2 + 9b3 + 27b4 + b6 + b7 + b9 + 3b10 = \ln f11 \quad (15.c.11)$$

$$b1 + 3b2 + 9b3 + 27b4 + b7 + 3b10 = \ln f12 \quad (15.c.12)$$

Here,  $b1, b2, \dots, b10$  are the ten parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. In a procedure identical to that used for the independence model with nonignorable nonresponse Type I (pages 410 - 411), we have

$$b6 + b9 = \ln f5 - \ln f6 \quad (15.c.13)$$

$$b5 + b8 = \ln \frac{f5f7}{f6f8} \quad (15.c.14)$$

$$b4 = \frac{1}{48} \ln \frac{f12(f6)^3}{f3(f9)^3} \quad (15.c.15)$$

$$b2 + b10 = \frac{1}{48} \ln \frac{f3(f9)^{27}}{f12(f6)^{27}} \quad (15.c.16)$$

$$b3 = \frac{1}{16} \ln \frac{f3f12}{f6f9} \quad (15.c.17)$$

$$b1 + b7 = \frac{1}{16} \ln \frac{(f6)^9(f9)^9}{f3f12} \quad (15.c.18)$$

Note that  $b1$  and  $b7$  cannot be determined uniquely from equations (15.c.1) - (15.c.12), since whenever  $b1$  appears in an equation, so does  $b7$ ; that is  $b1 + b7$  appears as a unit in each of the equations. Similarly,  $b5$  and  $b8$  cannot be determined uniquely, nor can  $b6$  and  $b9$  and  $b2$  and  $b10$ . Only two of the ten model parameters ( $b3$  and  $b4$ ) have been uniquely determined. Equations (15.c.13) - (15.c.18) provide the most complete solution for the  $b$ 's which can be determined solely from equations (15.c.1) - (15.c.12).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Type I (page 411). They are given by

$$\exp(b1 - 3b2 + 9b3 - 27b4)[\exp(b5) + \exp(b6) + 1] = s1 \quad (15.c.19)$$

$$\exp(b1 - b2 + b3 - b4)[\exp(b5) + \exp(b6) + 1] = s2 \quad (15.c.20)$$

$$\exp(b1 + b2 + b3 + b4)[\exp(b5) + \exp(b6) + 1] = s3 \quad (15.c.21)$$

$$\exp(b1 + 3b2 + 9b3 + 27b4)[\exp(b5) + \exp(b6) + 1] = s4 \quad (15.c.22)$$

Let  $k1 = \exp(9b3 - 27b4)$ ,  $k2 = \exp(b3 - b4)$ ,  $k3 = \exp(b3 + b4)$ ,  $k4 = \exp(9b3 + 27b4)$ . Then equations (15.c.19) - (15.c.22) can be rewritten as

$$\exp(b1 - 3b2)[\exp(b5) + \exp(b6) + 1] = \frac{s1}{k1} \quad (15.c.23)$$

$$\exp(b1 - b2)[\exp(b5) + \exp(b6) + 1] = \frac{s2}{k2} \quad (15.c.24)$$

$$\exp(b1 + b2)[\exp(b5) + \exp(b6) + 1] = \frac{s3}{k3} \quad (15.c.25)$$

$$\exp(b1 + 3b2)[\exp(b5) + \exp(b6) + 1] = \frac{s4}{k4} \quad (15.c.26)$$

Dividing (15.c.25) by (15.c.24), we have

$$\exp(2b2) = \frac{s3k2}{k3s2}$$

$$\text{or} \quad b2 = \frac{1}{2} \ln \frac{s3k2}{k3s2} \quad (15.c.27)$$

If (15.c.27) is substituted into (15.c.23) - (15.c.26), only one relationship between  $b1$ ,  $b5$  and  $b6$  is created. This, together with (15.c.13), (15.c.14), (15.c.16) and (15.c.18) provide five relationships to determine the remaining seven unknown model parameters. These five equations are insufficient to determine these seven unknowns. This model is therefore not identified.

**Type III:**

A total of sixteen observed data frequencies (twelve from the completely classified data and four from the row marginals) are assumed to be estimable. If all the interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable, and the row variable are included in the loglinear model, eighteen parameters are required. Thus, Type III nonignorable nonresponse with independence for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

Type III: independence; 4x3 table with one supplemental margin

- modified (with only the linear interaction term of the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | -3  | 9   | -27 | 1   | 0   | 1   | 1   | 0   | -3   | -3   | 0    | 1 | 1 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 1   | 0   | 1   | -3   | 0    | -3   | 1 | 2 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 1   | 0   | 0   | -3   | 0    | 0    | 1 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 1   | 1   | 0   | -1   | -1   | 0    | 2 | 1 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 1   | 0   | 1   | -1   | 0    | -1   | 2 | 2 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 1   | 0   | 0   | -1   | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1    | 1    | 0    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 0   | 1   | 1    | 0    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 1   | 0   | 0   | 1    | 0    | 0    | 3 | 3 | 1 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 1   | 1   | 0   | 3    | 3    | 0    | 4 | 1 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 1   | 0   | 1   | 3    | 0    | 3    | 4 | 2 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 1   | 0   | 0   | 3    | 0    | 0    | 4 | 3 | 1 |
|        | 1   | -3  | 9   | -27 | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 4 | 1 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 0   | 0   | 0    | 0    | 0    | 4 | 2 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 4 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second through fourth columns, the row variable, the fifth and sixth columns, the column variable, the seventh column, the missing data indicator, the eighth and ninth columns, the interactions of the

missing data indicator and the column variable, the tenth column, the interaction of the missing data indicator and the linear row variable term, and the eleventh and twelfth columns, the three way interactions of the missing data indicator with the column variable and the linear row variable term. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f12$ , are available for the first twelve elements of vector  $Y$ , and estimates  $s1, s2, s3$  and  $s4$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - 3b2 + 9b3 - 27b4 + b5 + b7 + b8 - 3b10 - 3b11 = \ln f1 \quad (15.d.1)$$

$$b1 - 3b2 + 9b3 - 27b4 + b6 + b7 + b9 - 3b10 - 3b12 = \ln f2 \quad (15.d.2)$$

$$b1 - 3b2 + 9b3 - 27b4 + b7 - 3b10 = \ln f3 \quad (15.d.3)$$

$$b1 - b2 + b3 - b4 + b5 + b7 + b8 - b10 - b11 = \ln f4 \quad (15.d.4)$$

$$b1 - b2 + b3 - b4 + b6 + b7 + b9 - b10 - b12 = \ln f5 \quad (15.d.5)$$

$$b1 - b2 + b3 - b4 + b7 - b10 = \ln f6 \quad (15.d.6)$$

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_7 + b_8 + b_{10} + b_{11} = \ln f_7 \quad (15.d.7)$$

$$b_1 + b_2 + b_3 + b_4 + b_6 + b_7 + b_9 + b_{10} + b_{12} = \ln f_8 \quad (15.d.8)$$

$$b_1 + b_2 + b_3 + b_4 + b_7 + b_{10} = \ln f_9 \quad (15.d.9)$$

$$b_1 + 3b_2 + 9b_3 + 27b_4 + b_5 + b_7 + b_8 + 3b_{10} + 3b_{11} = \ln f_{10} \quad (15.d.10)$$

$$b_1 + 3b_2 + 9b_3 + 27b_4 + b_6 + b_7 + b_9 + 3b_{10} + 3b_{12} = \ln f_{11} \quad (15.d.11)$$

$$b_1 + 3b_2 + 9b_3 + 27b_4 + b_7 + 3b_{10} = \ln f_{12} \quad (15.d.12)$$

Here,  $b_1, b_2, \dots, b_{12}$  are the twelve parameters of matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's.

Subtracting (15.d.6) from (15.d.9), and subtracting (15.d.3) from (15.d.12), we have

$$2b_2 + 2b_4 + 2b_{10} = \ln f_9 - \ln f_6$$

$$6b_2 + 54b_4 + 2b_{10} = \ln f_{12} - \ln f_3$$

$$\text{Thus, } b_4 = \frac{1}{48} \ln \frac{f_{12}(f_6)^3}{f_3(f_9)^3} \quad (15.d.13)$$

$$\text{and } b_2 + b_{10} = \frac{1}{48} \ln \frac{f_3(f_9)^{27}}{f_{12}(f_6)^{27}} \quad (15.d.14)$$

Subtracting (15.d.3) from (15.d.2), and subtracting (15.d.9) from (15.d.8), we have

$$b_6 + b_9 - 3b_{12} = \ln f_2 - \ln f_3$$

$$b_6 + b_9 + b_{12} = \ln f_8 - \ln f_9$$

$$\text{Thus, } b_{12} = \frac{1}{4} \ln \frac{f_3 f_8}{f_2 f_9} \quad (15.d.15)$$

$$\text{and } b_6 + b_9 = \frac{1}{4} \ln \frac{(f_8)^3 f_2}{(f_9)^3 f_3} \quad (15.d.16)$$

Subtracting (15.d.8) from (15.d.7), and subtracting (15.d.2) from (15.d.1), we have

$$b_5 - b_6 + b_8 - b_9 + b_{11} - b_{12} = \ln f_7 - \ln f_8 \quad (15.d.17)$$

$$b_5 - b_6 + b_8 - b_9 - 3b_{11} + 3b_{12} = \ln f_1 - \ln f_2 \quad (15.d.18)$$

Subtracting (15.d.18) from (15.d.17) and substituting (15.d.15), we have



$$b_{11} = \frac{1}{4} \ln \frac{f_3 f_7}{f_1 f_9} \quad (15.d.19)$$

Substituting (15.d.15), (15.d.16), and (15.d.19) into (15.d.17), we obtain

$$b_5 + b_8 = \frac{1}{4} \ln \frac{(f_7)^3 f_1}{(f_9)^3 f_3} \quad (15.d.20)$$

Substituting (15.d.13) and (15.d.14) into (15.d.3), and substituting (15.d.13) and (15.d.14) into (15.d.6), we have

$$b_1 + 9b_3 + b_7 = \frac{1}{2} \ln f_3 f_{12} \quad (15.d.21)$$

$$b_1 + b_3 + b_7 = \frac{1}{2} \ln f_6 f_9 \quad (15.d.22)$$

$$\text{Thus, } b_3 = \frac{1}{16} \ln \frac{f_3 f_{12}}{f_6 f_9} \quad (15.d.23)$$

$$\text{and } b_1 + b_7 = \frac{1}{16} \ln \frac{(f_6)^9 (f_9)^9}{f_3 f_{12}} \quad (15.d.24)$$

Note that  $b_1$  and  $b_7$  cannot be determined uniquely from equations (15.d.1) - (15.d.12), since whenever  $b_1$  appears in an equation, so does  $b_7$ ; that is  $b_1 + b_7$  appears as a unit in each of the equations. Similarly,  $b_5$  and  $b_8$  cannot be determined uniquely, nor can  $b_6$  and  $b_9$ , and  $b_2$  and  $b_{10}$ . Only four of the twelve model parameters ( $b_3$ ,  $b_4$ ,  $b_{11}$  and  $b_{12}$ ) have been uniquely determined. Equations (15.d.13) - (15.d.24) provide the most complete solution for the  $b$ 's which can be determined solely from equations (15.d.1) - (15.d.12).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for nonignorable nonresponse, Type I (page 411). They are given by

$$\exp(b1 - 3b2 + 9b3 - 27b4)[\exp(b5) + \exp(b6) + 1] = s1 \quad (15.d.25)$$

$$\exp(b1 - b2 + b3 - b4)[\exp(b5) + \exp(b6) + 1] = s2 \quad (15.d.26)$$

$$\exp(b1 + b2 + b3 + b4)[\exp(b5) + \exp(b6) + 1] = s3 \quad (15.d.27)$$

$$\exp(b1 + 3b2 + 9b3 + 27b4)[\exp(b5) + \exp(b6) + 1] = s4 \quad (15.d.28)$$

Using the same procedure as on page 420,  $b2$  can be uniquely determined. Using this value for  $b2$ ,  $b10$  can then be found from equation (15.d.14). Thus, six of the model parameters remain to be determined.

If the uniquely determined values for  $b2$ ,  $b3$  and  $b4$  are substituted into (15.d.25) - (15.d.28), only one relationship between  $b1$ ,  $b5$  and  $b6$  is created. This, together with equations (15.d.16), (15.d.20) and (15.d.24) provide four relationships to determine the remaining six unknown model parameters. These four equations are insufficient to determine these six unknowns. This model is therefore not identified.

In summary, none of the five models considered above which postulate independence among the row and column variables and which model nonignorable nonresponse is identified.

### Empirical Analysis for Data Set 3

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the independence models hypothesized to fit Data Set 3 (page 406). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). (Note that none of the nonignorable nonresponse models was found to be identified.)

#### Data Set 3

##### Independence Model

##### Completely Classified Data Only

**Presidential Candidate Preference**

|                 |              | <b>Truman</b> | <b>Dewey</b> | <b>other</b> | <b>total</b> |
|-----------------|--------------|---------------|--------------|--------------|--------------|
|                 | <b>A</b>     | 90.59         | 119.61       | 13.80        | 224          |
| <b>Economic</b> | <b>B</b>     | 305.75        | 403.68       | 46.57        | 756          |
| <b>Class</b>    | <b>C</b>     | 508.78        | 671.73       | 77.49        | 1258         |
|                 | <b>D</b>     | 243.87        | 321.98       | 37.14        | 602.99       |
|                 | <b>total</b> | 1148.99       | 1517         | 175          | 2480.99      |

$$\beta = (4.22, .27, -.12, -.01, 1.88, 2.16)$$

Likelihood Ratio fit statistic = 219.065

Degrees of freedom = 6

**Data Set 3**

Independence Model - One Supplemental Margin

Missing Completely at Random (MCAR)

**Completely Classified Data****Presidential Candidate Preference**

|          |       | Truman  | Dewey   | other | total   |
|----------|-------|---------|---------|-------|---------|
|          | A     | 79.44   | 104.89  | 12.10 | 196.43  |
| Economic | B     | 283.97  | 374.92  | 43.25 | 702.14  |
| Class    | C     | 516.39  | 681.79  | 78.65 | 1276.83 |
|          | D     | 269.19  | 355.41  | 41.00 | 665.60  |
|          | total | 1148.99 | 1517.01 | 175   | 2481    |

**Partially Classified Data (Hypothetical Contingency Table)****Presidential Candidate Preference**

|          |       | Truman | Dewey  | other | total  |
|----------|-------|--------|--------|-------|--------|
|          | A     | 18.43  | 24.33  | 2.81  | 45.57  |
| Economic | B     | 65.87  | 86.97  | 10.03 | 162.87 |
| Class    | C     | 119.78 | 158.15 | 18.24 | 296.17 |
|          | D     | 62.44  | 82.44  | 9.51  | 154.39 |
|          | total | 266.52 | 351.89 | 40.59 | 659    |

$$\beta = (2.73, .31, -.12, -.01, 1.88, 2.16, 1.46)$$

Likelihood Ratio fit statistic = 298.728

Degrees of freedom = 9

**Data Set 3**

Independence Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data****Presidential Candidate Preference**

|          |       | Truman  | Dewey  | other | total   |
|----------|-------|---------|--------|-------|---------|
|          | A     | 90.59   | 119.61 | 13.80 | 224     |
| Economic | B     | 305.75  | 403.68 | 46.57 | 756     |
| Class    | C     | 508.78  | 671.73 | 77.49 | 1258    |
|          | D     | 243.87  | 321.98 | 37.14 | 602.99  |
|          | total | 1148.99 | 1517   | 175   | 2840.99 |

**Partially Classified Data (Hypothetical Contingency Table)****Presidential Candidate Preference**

|          |       | Truman | Dewey  | other | total  |
|----------|-------|--------|--------|-------|--------|
|          | A     | 7.28   | 9.61   | 1.11  | 18     |
| Economic | B     | 44.08  | 58.20  | 6.71  | 108.99 |
| Class    | C     | 127.40 | 168.20 | 19.40 | 315    |
|          | D     | 87.76  | 115.87 | 13.37 | 217    |
|          | total | 266.52 | 351.88 | 40.59 | 658.99 |

$$\beta = (2.57, .55, -.14, -.01, 1.88, 2.16, 1.65, -.28, .01, .003)$$

Likelihood Ratio fit statistic = 219.065

Degrees of freedom = 6

Column Effects Model - Data Set 3

Type I: column effects; 4x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|---|---|---|
| 1      | -3  | 9   | -27 | 1   | 0   | -3  | 0   | 1   | 1   | 0    |      | 1 | 1 | 1 |
| 1      | -3  | 9   | -27 | 0   | 1   | 0   | -3  | 1   | 0   | 1    |      | 1 | 2 | 1 |
| 1      | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 1   | 0   | 0    |      | 1 | 3 | 1 |
| 1      | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 1   | 1   | 0    |      | 2 | 1 | 1 |
| 1      | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 1   | 0   | 1    |      | 2 | 2 | 1 |
| 1      | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 1   | 0   | 0    |      | 2 | 3 | 1 |
| 1      | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1   | 0    |      | 3 | 1 | 1 |
| 1      | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1   | 0   | 1    |      | 3 | 2 | 1 |
| 1      | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0   | 0    |      | 3 | 3 | 1 |
| 1      | 3   | 9   | 27  | 1   | 0   | 3   | 0   | 1   | 1   | 0    |      | 4 | 1 | 1 |
| 1      | 3   | 9   | 27  | 0   | 1   | 0   | 3   | 1   | 0   | 1    |      | 4 | 2 | 1 |
| 1      | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 1   | 0   | 0    |      | 4 | 3 | 1 |
| 1      | -3  | 9   | -27 | 1   | 0   | -3  | 0   | 0   | 0   | 0    |      | 1 | 1 | 2 |
| 1      | -3  | 9   | -27 | 0   | 1   | 0   | -3  | 0   | 0   | 0    |      | 1 | 2 | 2 |
| 1      | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 0   | 0   | 0    |      | 1 | 3 | 2 |
| 1      | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 0   | 0   | 0    |      | 2 | 1 | 2 |
| 1      | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 0   | 0   | 0    |      | 2 | 2 | 2 |
| 1      | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0   | 0   | 0    |      | 2 | 3 | 2 |
| 1      | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0    |      | 3 | 1 | 2 |
| 1      | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 0   | 0    |      | 3 | 2 | 2 |
| 1      | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0    |      | 3 | 3 | 2 |
| 1      | 3   | 9   | 27  | 1   | 0   | 3   | 0   | 0   | 0   | 0    |      | 4 | 1 | 2 |
| 1      | 3   | 9   | 27  | 0   | 1   | 0   | 3   | 0   | 0   | 0    |      | 4 | 2 | 2 |
| 1      | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 0   | 0   | 0    |      | 4 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second through fourth columns, the row variable, the fifth and sixth columns, the column variable, the seventh and eighth columns, the interactions of the column variable and the linear row variable term, the ninth column, the missing data indicator, and the tenth and eleventh columns, the missing data indicator by column variable interaction. The eleven parameters of the

model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

As for the independence models considered above, we have  $E(Y) = C \exp(X\beta)$ . Here,  $Y$  is a column vector of selected sums of cell frequencies,  $X$  is the design (or model) matrix, and  $\beta$  is a column vector of parameters whose values are unknown and must be estimated from the data. The  $C$  matrix (page 402) is the same as for the independence models.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

As discussed earlier for the independence models, for the portion of vector  $Y$  corresponding to the completely classified data (and the first twelve rows of the  $X$ , or model matrix),  $E(Y) = C \exp(X\beta)$  is equivalent to  $E(Y) = \exp(X\beta)$ , or

$$\ln(E(Y)) = X\beta.$$

Assume that we can obtain estimates for the first twelve elements of vector  $Y$ . We will call these estimates  $f1, f2, \dots, f12$ . Then from  $\ln(E(Y)) = X\beta$ , we have

$$b1 - 3b2 + 9b3 - 27b4 + b5 - 3b7 + b9 + b10 = \ln f1 \quad (16.a.1)$$

$$b1 - 3b2 + 9b3 - 27b4 + b6 - 3b8 + b9 + b11 = \ln f2 \quad (16.a.2)$$

$$b1 - 3b2 + 9b3 - 27b4 + b9 = \ln f3 \quad (16.a.3)$$

$$b1 - b2 + b3 - b4 + b5 - b7 + b9 + b10 = \ln f4 \quad (16.a.4)$$

$$b1 - b2 + b3 - b4 + b6 - b8 + b9 + b11 = \ln f5 \quad (16.a.5)$$

$$b1 - b2 + b3 - b4 + b9 = \ln f6 \quad (16.a.6)$$

$$b1 + b2 + b3 + b4 + b5 + b7 + b9 + b10 = \ln f7 \quad (16.a.7)$$

$$b1 + b2 + b3 + b4 + b6 + b8 + b9 + b11 = \ln f8 \quad (16.a.8)$$

$$b1 + b2 + b3 + b4 + b9 = \ln f9 \quad (16.a.9)$$

$$b1 + 3b2 + 9b3 + 27b4 + b5 + 3b7 + b9 + b10 = \ln f10 \quad (16.a.10)$$

$$b1 + 3b2 + 9b3 + 27b4 + b6 + 3b8 + b9 + b11 = \ln f11 \quad (16.a.11)$$

$$b1 + 3b2 + 9b3 + 27b4 + b9 = \ln f12 \quad (16.a.12)$$

Here,  $b1, b2, \dots, b11$  are the eleven parameters of the matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's.

Subtracting (16.a.6) from (16.a.9), and subtracting (16.a.3) from (16.a.12), we have

$$2b2 + 2b4 = \ln f9 - \ln f6$$

$$6b2 + 54b4 = \ln f12 - \ln f3$$

$$\text{Thus, } b4 = \frac{1}{48} \ln \frac{f12(f6)^3}{f3(f9)^3} \quad (16.a.13)$$

$$\text{and } b2 = \frac{1}{48} \ln \frac{(f9)^{27} f3}{(f6)^{27} f12} \quad (16.a.14)$$

Subtracting (16.a.5) from (16.a.8) and substituting (16.a.13) and (16.a.14), we obtain

$$b8 = \frac{1}{2} \ln \frac{f6 f8}{f5 f9} \quad (16.a.15)$$

Subtracting (16.a.6) from (16.a.4), and subtracting (16.a.9) from (16.a.7), we have

$$b5 - b7 + b10 = \ln f4 - \ln f6$$

$$b5 + b7 + b10 = \ln f7 - \ln f9$$

$$\text{Thus, } b7 = \frac{1}{2} \ln \frac{f6 f7}{f4 f9} \quad (16.a.16)$$



$$\text{and } b_5 + b_{10} = \frac{1}{2} \ln \frac{f_4 f_7}{f_6 f_9} \quad (16.a.17)$$

Subtracting (16.a.12) from (16.a.9), and substituting (16.a.13) and (16.a.14), we obtain

$$b_3 = \frac{1}{16} \ln \frac{f_3 f_{12}}{f_6 f_9} \quad (16.a.18)$$

Substituting (16.a.13), (16.a.14) and (16.a.18) into (16.a.9), we have

$$b_1 + b_9 = \frac{1}{48} \ln \frac{(f_6)^{27} (f_9)^{27}}{(f_3)^3 (f_{12})^3} \quad (16.a.19)$$

Finally, substituting (16.a.13), (16.a.14), (16.a.15), (16.a.18) and (16.a.19) into (16.a.8), we obtain

$$b_6 + b_{11} = \frac{1}{2} \ln \frac{f_5 f_8}{f_6 f_9} \quad (16.a.20)$$

Note that  $b_1$  and  $b_9$  cannot be determined uniquely from equations (16.a.1) - (16.a.12), since whenever  $b_1$  appears in one of the equations, so does  $b_9$ ; that is  $b_1 + b_9$  appears as a unit in each of the equations. Similarly,  $b_5$  and  $b_{10}$  cannot be determined uniquely, nor can  $b_6$  and  $b_{11}$ . Only five of the eleven model parameters ( $b_2$ ,  $b_3$ ,  $b_4$ ,  $b_7$  and  $b_8$ ) have been uniquely determined. Equations (16.a.13) - (16.a.20) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (16.a.1) - (16.a.12).

The last four elements of vector  $Y$  correspond to the partially classified data (and the last twelve rows of the  $X$  matrix). Each of these last four elements of vector  $Y$  represents a sum of cell frequencies in a given row of the (hypothetical) contingency table for the partially classified data. Let  $f_{13}$ , ...,  $f_{24}$  be the (hypothetical) cell frequencies in the (hypothetical) partially classified contingency table. Since  $E(Y) = C \exp(X\beta)$ , and given the form of matrices  $C$  and  $X$ , the algebraic equations which correspond to the partially classified data are given by

$$\exp(b1 - 3b2 + 9b3 - 27b4)[\exp(b5 - 3b7) + \exp(b6 - 3b8) + 1] = s1 \quad (16.a.21)$$

$$\exp(b1 - b2 + b3 - b4)[\exp(b5 - b7) + \exp(b6 - b8) + 1] = s2 \quad (16.a.22)$$

$$\exp(b1 + b2 + b3 + b4)[\exp(b5 + b7) + \exp(b6 + b8) + 1] = s3 \quad (16.a.23)$$

$$\exp(b1 + 3b2 + 9b3 + 27b4)[\exp(b5 + 3b7) + \exp(b6 + 3b8) + 1] = s4 \quad (16.a.24)$$

Here,  $s1 = f13 + f14 + f15$ , the first row marginal for the partially classified data,

$s2 = f16 + f17 + f18$ , the second row marginal for the partially classified data,

$s3 = f19 + f20 + f21$ , the third row marginal for the partially classified data and

$s4 = f22 + f23 + f24$ , the fourth row marginal for the partially classified data.

We assume that we can obtain estimates  $s1$ ,  $s2$ ,  $s3$  and  $s4$ , for the last four elements of vector  $Y$ .

Let  $k1 = \exp(-3b2 + 9b3 - 27b4)$ ,  $k2 = \exp(-b2 + b3 - b4)$ ,  $k3 = \exp(b2 + b3 + b4)$ ,

$k4 = \exp(3b2 + 9b3 + 27b4)$ ,  $k5 = \exp(b7)$  and  $k6 = \exp(b8)$ . Then equations (16.a.21) - (16.a.24) can be rewritten as

$$\frac{1}{(k5)^3} \exp(b1 + b5) + \frac{1}{(k6)^3} \exp(b1 + b6) + \exp(b1) = \frac{s1}{k1} \quad (16.a.25)$$

$$\frac{1}{k5} \exp(b1 + b5) + \frac{1}{k6} \exp(b1 + b6) + \exp(b1) = \frac{s2}{k2} \quad (16.a.26)$$

$$k5 \exp(b1 + b5) + k6 \exp(b1 + b6) + \exp(b1) = \frac{s3}{k3} \quad (16.a.27)$$

$$(k5)^3 \exp(b1 + b5) + (k6)^3 \exp(b1 + b6) + \exp(b1) = \frac{s4}{k4} \quad (16.a.28)$$

Dividing (16.a.25) by (16.a.26), and dividing (16.a.27) by (16.a.28), we have

$$\frac{1}{k5} \exp(b5) + \frac{1}{k6} \exp(b6) + 1 = \frac{s2k1}{k2s1} \left( \frac{1}{(k5)^3} \exp(b5) + \frac{1}{(k6)^3} \exp(b6) + 1 \right) \quad (16.a.29)$$

$$k5 \exp(b5) + k6 \exp(b6) + 1 = \frac{s3k4}{k3s4} \left( (k5)^3 \exp(b5) + (k6)^3 \exp(b6) + 1 \right) \quad (16.a.30)$$

Rearranging terms in (16.a.29) and (16.a.30), we have

$$\left[ \frac{1}{k5} - \frac{s2k1}{k2s1(k5)^3} \right] \exp(b5) + \left[ \frac{1}{k6} - \frac{s2k1}{k2s1(k6)^3} \right] \exp(b6) = \frac{s2k1}{k2s1} - 1 \quad (16.a.31)$$

$$\left[ k5 - \frac{s3k4(k5)^3}{k3s4} \right] \exp(b5) + \left[ k6 - \frac{s3k4(k6)^3}{k3s4} \right] \exp(b6) = \frac{s3k4}{k3s4} - 1 \quad (16.a.32)$$

Thus,

$$\exp(b5) = \frac{k6 \left( \frac{s2k1}{k2s1} - 1 \right) \left( 1 - \frac{s3k4(k6)^2}{k3s4} \right) - \frac{1}{k6} \left( \frac{s3k4}{k3s4} - 1 \right) \left( 1 - \frac{s2k1}{k2s1(k6)^2} \right)}{\frac{k6}{k5} \left( 1 - \frac{s2k1}{k2s1(k5)^2} \right) \left( 1 - \frac{s3k4(k6)^2}{k3s4} \right) - \frac{k5}{k6} \left( 1 - \frac{s3k4(k5)^2}{k3s4} \right) \left( 1 - \frac{s2k1}{k2s1(k6)^2} \right)}$$

and

$$\exp(b6) = \frac{\frac{1}{k5} \left( 1 - \frac{s2k1}{k2s1(k5)^2} \right) \left( \frac{s3k4}{k3s4} - 1 \right) - k5 \left( 1 - \frac{s3k4(k5)^2}{k3s4} \right) \left( \frac{s2k1}{k2s1} - 1 \right)}{\frac{k6}{k5} \left( 1 - \frac{s2k1}{k2s1(k5)^2} \right) \left( 1 - \frac{s3k4(k6)^2}{k3s4} \right) - \frac{k5}{k6} \left( 1 - \frac{s3k4(k5)^2}{k3s4} \right) \left( 1 - \frac{s2k1}{k2s1(k6)^2} \right)}$$

or equivalently,

$$b5 = \ln \frac{k6 \left( \frac{s2k1}{k2s1} - 1 \right) \left( 1 - \frac{s3k4(k6)^2}{k3s4} \right) - \frac{1}{k6} \left( \frac{s3k4}{k3s4} - 1 \right) \left( 1 - \frac{s2k1}{k2s1(k6)^2} \right)}{\frac{k6}{k5} \left( 1 - \frac{s2k1}{k2s1(k5)^2} \right) \left( 1 - \frac{s3k4(k6)^2}{k3s4} \right) - \frac{k5}{k6} \left( 1 - \frac{s3k4(k5)^2}{k3s4} \right) \left( 1 - \frac{s2k1}{k2s1(k6)^2} \right)} \quad (16.a.33)$$

and

$$b6 = \ln \frac{\frac{1}{k5} \left( 1 - \frac{s2k1}{k2s1(k5)^2} \right) \left( \frac{s3k4}{k3s4} - 1 \right) - k5 \left( 1 - \frac{s3k4(k5)^2}{k3s4} \right) \left( \frac{s2k1}{k2s1} - 1 \right)}{\frac{k6}{k5} \left( 1 - \frac{s2k1}{k2s1(k5)^2} \right) \left( 1 - \frac{s3k4(k6)^2}{k3s4} \right) - \frac{k5}{k6} \left( 1 - \frac{s3k4(k5)^2}{k3s4} \right) \left( 1 - \frac{s2k1}{k2s1(k6)^2} \right)}$$

(16.a.34)

Having determined  $b5$  and  $b6$ ,  $b1$  can now be determined from equation (16.a.25), and  $b9$ ,  $b10$  and  $b11$  can be found from equations (16.a.19), (16.a.17) and (16.a.20), respectively.

Thus, all of the  $b$ 's can be uniquely determined. The model, therefore, is identified.

## Type II: column effects; 4x3 table with one supplemental margin

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|---|---|---|
|        | 1   | -3  | 9   | -27 | 1   | 0   | -3  | 0   | 1   | 1    | 0    | -3   | 9    | -27  | 1 | 1 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | -3  | 1   | 0    | 1    | -3   | 9    | -27  | 1 | 2 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 1   | 0    | 0    | -3   | 9    | -27  | 1 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 1   | 1    | 0    | -1   | 1    | -1   | 2 | 1 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 1   | 0    | 1    | -1   | 1    | -1   | 2 | 2 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 1   | 0    | 0    | -1   | 1    | -1   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1    | 0    | 1    | 1    | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1   | 0    | 1    | 1    | 1    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 1    | 1    | 1    | 3 | 3 | 1 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 3   | 0   | 1   | 1    | 0    | 3    | 9    | 27   | 4 | 1 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 3   | 1   | 0    | 1    | 3    | 9    | 27   | 4 | 2 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 3    | 9    | 27   | 4 | 3 | 1 |
|        | 1   | -3  | 9   | -27 | 1   | 0   | -3  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | -3  | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 3   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 4 | 1 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 3   | 0   | 0    | 0    | 0    | 0    | 0    | 4 | 2 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 4 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second through fourth columns, the row variable, the fifth and sixth columns, the column variable, the seventh and eighth columns, the interactions of the column variable and the linear row variable term, the ninth column, the missing data indicator, the tenth and eleventh columns, the interaction of the missing data indicator and the column variable, and the twelfth through fourteenth columns, the interaction of the missing data indicator and the row variable.

The fourteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I case, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f12$ , are available for the first twelve elements of vector  $Y$ , and estimates  $s1, s2, s3$  and  $s4$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - 3b2 + 9b3 - 27b4 + b5 - 3b7 + b9 + b10 - 3b12 + 9b13 - 27b14 = \ln f1 \quad (16.b.1)$$

$$b1 - 3b2 + 9b3 - 27b4 + b6 - 3b8 + b9 + b11 - 3b12 + 9b13 - 27b14 = \ln f2 \quad (16.b.2)$$

$$b1 - 3b2 + 9b3 - 27b4 + b9 - 3b12 + 9b13 - 27b14 = \ln f3 \quad (16.b.3)$$

$$b1 - b2 + b3 - b4 + b5 - b7 + b9 + b10 - b12 + b13 - b14 = \ln f4 \quad (16.b.4)$$

$$b1 - b2 + b3 - b4 + b6 - b8 + b9 + b11 - b12 + b13 - b14 = \ln f5 \quad (16.b.5)$$

$$b1 - b2 + b3 - b4 + b9 - b12 + b13 - b14 = \ln f6 \quad (16.b.6)$$

$$b1 + b2 + b3 + b4 + b5 + b7 + b9 + b10 + b12 + b13 + b14 = \ln f7 \quad (16.b.7)$$

$$b1 + b2 + b3 + b4 + b6 + b8 + b9 + b11 + b12 + b13 + b14 = \ln f8 \quad (16.b.8)$$

$$b1 + b2 + b3 + b4 + b9 + b12 + b13 + b14 = \ln f9 \quad (16.b.9)$$

$$b1 + 3b2 + 9b3 + 27b4 + b5 + 3b7 + b9 + b10 + 3b12 + 9b13 + 27b14 = \ln f10 \quad (16.b.10)$$

$$b1 + 3b2 + 9b3 + 27b4 + b6 + 3b8 + b9 + b11 + 3b12 + 9b13 + 27b14 = \ln f11 \quad (16.b.11)$$

$$b1 + 3b2 + 9b3 + 27b4 + b9 + 3b12 + 9b13 + 27b14 = \ln f12 \quad (16.b.12)$$

Here,  $b_1, b_2, \dots, b_{14}$  are the fourteen parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the column effects model with nonignorable nonresponse Type I (pages 432 - 433), we have

$$b_4 + b_{14} = \frac{1}{48} \ln \frac{f_{12}(f_6)^3}{f_3(f_9)^3} \quad (16.b.13)$$

$$b_2 + b_{12} = \frac{1}{48} \ln \frac{(f_9)^{27} f_3}{(f_6)^{27} f_{12}} \quad (16.b.14)$$

$$b_8 = \frac{1}{2} \ln \frac{f_6 f_8}{f_5 f_9} \quad (16.b.15)$$

$$b_7 = \frac{1}{2} \ln \frac{f_6 f_7}{f_4 f_9} \quad (16.b.16)$$

$$b_5 + b_{10} = \frac{1}{2} \ln \frac{f_4 f_7}{f_6 f_9} \quad (16.b.17)$$

$$b_3 + b_{13} = \frac{1}{16} \ln \frac{f_3 f_{12}}{f_6 f_9} \quad (16.b.18)$$

$$b_1 + b_9 = \frac{1}{48} \ln \frac{(f_6)^{27} (f_9)^{27}}{(f_3)^3 (f_{12})^3} \quad (16.b.19)$$

$$b_6 + b_{11} = \frac{1}{2} \ln \frac{f_5 f_8}{f_6 f_9} \quad (16.b.20)$$

Note that  $b_1$  and  $b_9$  cannot be determined uniquely from equations (16.b.1) - (16.b.12), nor can  $b_5$  and  $b_{10}$ ,  $b_4$  and  $b_{14}$ ,  $b_2$  and  $b_{12}$ ,  $b_3$  and  $b_{13}$ , and  $b_6$  and  $b_{11}$ . Only two of the twelve model parameters ( $b_7$  and  $b_8$ ) have been uniquely determined. Equations (16.b.13) - (16.b.20) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (16.b.1) - (16.b.12).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type I. They are given by

$$\exp(b1 - 3b2 + 9b3 - 27b4)[\exp(b5 - 3b7) + \exp(b6 - 3b8) + 1] = s1 \quad (16.b.21)$$

$$\exp(b1 - b2 + b3 - b4)[\exp(b5 - b7) + \exp(b6 - b8) + 1] = s2 \quad (16.b.22)$$

$$\exp(b1 + b2 + b3 + b4)[\exp(b5 + b7) + \exp(b6 + b8) + 1] = s3 \quad (16.b.23)$$

$$\exp(b1 + 3b2 + 9b3 + 27b4)[\exp(b5 + 3b7) + \exp(b6 + 3b8) + 1] = s4 \quad (16.b.24)$$

Equations (16.b.13), (16.b.14) and (16.b.17) - (16.b.24) provide ten equations to determine the remaining twelve unknown model parameters. These ten equations are insufficient to determine the remaining twelve unknowns. This model, therefore, is not identified.

While the column effects model for this data set with nonignorable nonresponse of Type II is not identified, it may be that the elimination of the interaction terms involving the missing data indicator and the quadratic and cubic row variable terms will result in an identified model. This modified Type II nonignorable nonresponse model is considered next.



Type II: column effects; 4x3 table with one supplemental margin

- modified (with only the linear interaction term of the missing data indicator and the row variable):

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|---|---|---|
|        | 1   | -3  | 9   | -27 | 1   | 0   | -3  | 0   | 1   | 1    | 0    | -3   | 1 | 1 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | -3  | 1   | 0    | 1    | -3   | 1 | 2 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 1   | 0    | 0    | -3   | 1 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 1   | 1    | 0    | -1   | 2 | 1 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 1   | 0    | 1    | -1   | 2 | 2 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 1   | 0    | 0    | -1   | 2 | 3 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1    | 0    | 1    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1   | 0    | 1    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 1    | 3 | 3 | 1 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 3   | 0   | 1   | 1    | 0    | 3    | 4 | 1 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 3   | 1   | 0    | 1    | 3    | 4 | 2 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 3    | 4 | 3 | 1 |
|        | 1   | -3  | 9   | -27 | 1   | 0   | -3  | 0   | 0   | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | -3  | 0   | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 0   | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 0   | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 3   | 0   | 0   | 0    | 0    | 0    | 4 | 1 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 3   | 0   | 0    | 0    | 0    | 4 | 2 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 4 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second through fourth columns, the row variable, the fifth and sixth columns, the column variable, the seventh and eighth columns, the interactions of the column variable and the linear row variable

term, the ninth column, the missing data indicator, the tenth and eleventh columns, the interaction of the missing data indicator and the column variable, and the twelfth column, the interaction of the missing data indicator and the linear row variable term. The twelve parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

To explore the identification status of this model, we assume, as was done for the Type I and Type II cases, that estimates are available for the frequency counts for the completely classified data and that estimates can be found for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f12$ , are available for the first twelve elements of vector  $Y$ , and that estimates  $s1, s2, s3$  and  $s4$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - 3b2 + 9b3 - 27b4 + b5 - 3b7 + b9 + b10 - 3b12 = \ln f1 \quad (16.c.1)$$

$$b1 - 3b2 + 9b3 - 27b4 + b6 - 3b8 + b9 + b11 - 3b12 = \ln f2 \quad (16.c.2)$$

$$b1 - 3b2 + 9b3 - 27b4 + b9 - 3b12 = \ln f3 \quad (16.c.3)$$

$$b1 - b2 + b3 - b4 + b5 - b7 + b9 + b10 - b12 = \ln f4 \quad (16.c.4)$$

$$b1 - b2 + b3 - b4 + b6 - b8 + b9 + b11 - b12 = \ln f5 \quad (16.c.5)$$

$$b1 - b2 + b3 - b4 + b9 - b12 = \ln f6 \quad (16.c.6)$$

$$b1 + b2 + b3 + b4 + b5 + b7 + b9 + b10 + b12 = \ln f7 \quad (16.c.7)$$

$$b1 + b2 + b3 + b4 + b6 + b8 + b9 + b11 + b12 = \ln f8 \quad (16.c.8)$$

$$b1 + b2 + b3 + b4 + b9 + b12 = \ln f9 \quad (16.c.9)$$

$$b1 + 3b2 + 9b3 + 27b4 + b5 + 3b7 + b9 + b10 + 3b12 = \ln f10 \quad (16.c.10)$$

$$b1 + 3b2 + 9b3 + 27b4 + b6 + 3b8 + b9 + b11 + 3b12 = \ln f11 \quad (16.c.11)$$

$$b1 + 3b2 + 9b3 + 27b4 + b9 + 3b12 = \ln f12 \quad (16.c.12)$$

Here,  $b1, b2, \dots, b12$  are the twelve parameters of matrix  $\beta$ , which are to be determined.

We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the column effects model with nonignorable nonresponse Type I (pages 432 - 433), we have

$$b4 = \frac{1}{48} \ln \frac{f12(f6)^3}{f3(f9)^3} \quad (16.c.13)$$

$$b2 + b12 = \frac{1}{48} \ln \frac{(f9)^{27} f3}{(f6)^{27} f12} \quad (16.c.14)$$

$$b8 = \frac{1}{2} \ln \frac{f6 f8}{f5 f9} \quad (16.c.15)$$

$$b7 = \frac{1}{2} \ln \frac{f6 f7}{f4 f9} \quad (16.c.16)$$

$$b5 + b10 = \frac{1}{2} \ln \frac{f4 f7}{f6 f9} \quad (16.c.17)$$

$$b3 = \frac{1}{16} \ln \frac{f3 f12}{f6 f9} \quad (16.c.18)$$

$$b1 + b9 = \frac{1}{48} \ln \frac{(f6)^{27} (f9)^{27}}{(f3)^3 (f12)^3} \quad (16.c.19)$$

$$b6 + b11 = \frac{1}{2} \ln \frac{f5 f8}{f6 f9} \quad (16.c.20)$$

Note that  $b1$  and  $b9$  cannot be determined uniquely from equations (16.c.1) - (16.c.12), nor can  $b5$  and  $b10$ ,  $b2$  and  $b12$ , and  $b6$  and  $b11$ . Only four of the twelve model parameters ( $b3, b4, b7$  and  $b8$ ) have been uniquely determined. Equations (16.c.13) - (16.c.20) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (16.c.1) - (16.c.12).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type II. They are given by

$$\exp(b1 - 3b2 + 9b3 - 27b4)[\exp(b5 - 3b7) + \exp(b6 - 3b8) + 1] = s1 \quad (16.c.21)$$

$$\exp(b1 - b2 + b3 - b4)[\exp(b5 - b7) + \exp(b6 - b8) + 1] = s2 \quad (16.c.22)$$

$$\exp(b1 + b2 + b3 + b4)[\exp(b5 + b7) + \exp(b6 + b8) + 1] = s3 \quad (16.c.23)$$

$$\exp(b1 + 3b2 + 9b3 + 27b4)[\exp(b5 + 3b7) + \exp(b6 + 3b8) + 1] = s4 \quad (16.c.24)$$

Let  $k1 = \exp(-9b3 + 27b4)$ ,  $k2 = \exp(-b3 + b4)$ ,  $k3 = \exp(-b3 - b4)$ ,  $k4 = \exp(-9b3 - 27b4)$ ,  $k5 = \exp(b7)$ ,  $k6 = \exp(-b7)$ ,  $k7 = \exp(b8)$ , and  $k8 = \exp(-b8)$ . Then equations (16.c.21) - (16.c.24) can be rewritten as

$$\exp(b1 - 3b2)[(k6)^3 \exp(b5) + (k8)^3 \exp(b6) + 1] = s1k1 \quad (16.c.25)$$

$$\exp(b1 - b2)[k6 \exp(b5) + k8 \exp(b6) + 1] = s2k2 \quad (16.c.26)$$

$$\exp(b1 + b2)[k5 \exp(b5) + k7 \exp(b6) + 1] = s3k3 \quad (16.c.27)$$

$$\exp(b1 + 3b2)[(k5)^3 \exp(b5) + (k7)^3 \exp(b6) + 1] = s4k4 \quad (16.c.28)$$

Dividing (16.c.25) by (16.c.26), and dividing (16.c.27) by (16.c.28), we have

$$\exp(-2b2) = \frac{s1k1}{s2k2} \left( \frac{k6 \exp(b5) + k8 \exp(b6) + 1}{(k6)^3 \exp(b5) + (k8)^3 \exp(b6) + 1} \right) \quad (16.c.29)$$

$$\exp(-2b2) = \frac{s3k3}{s4k4} \left( \frac{(k5)^3 \exp(b5) + (k7)^3 \exp(b6) + 1}{k5 \exp(b5) + k7 \exp(b6) + 1} \right) \quad (16.c.30)$$

Equating the two expressions for  $\exp(-2b2)$  in (16.c.29) and (16.c.30), we obtain

$$\begin{aligned} & s1s4k1k4[k6 \exp(b5) + k8 \exp(b6) + 1][k5 \exp(b5) + k7 \exp(b6) + 1] = \\ & s2s3k2k3[(k6)^3 \exp(b5) + (k8)^3 \exp(b6) + 1][(k5)^3 \exp(b5) + (k7)^3 \exp(b6) + 1] \end{aligned} \quad (16.c.31)$$

Equation (16.c.31) can be expressed as a quadratic equation in  $\exp(b5)$  as follows:

$$\begin{aligned} & \exp(2b5) \left[ s1s4k1k4k5k6 - s2s3k2k3(k5)^3(k6)^3 \right] \\ & + \exp(b5) \left[ s1s4k1k4k6(k7 \exp(b6) + 1) + s1s4k1k4 \{ k5(k8 \exp(b6) + 1) \} \right. \\ & \quad \left. - s2s3k2k3 \left[ (k5)^3(k8)^3 \exp(b6) + 1 \right] + (k6)^3(k7)^3 \exp(b6) + 1 \right] \\ & + \left[ s1s4k1k4(k8 \exp(b6) + 1)(k7 \exp(b6) + 1) - s2s3k2k3(k7)^3 \exp(b6) + 1 \right] (k8)^3 \exp(b6) + 1 = 0 \end{aligned}$$

Equation (16.c.32) is a quadratic equation in  $\exp(b5)$ , expressed in terms of  $b6$ . For the model to be identified, there must be a unique solution for  $b5$  and a unique solution for  $b6$ . For if  $b5$  and  $b6$  are each uniquely determined,  $b2$  could then also be determined from equation (16.c.29). Using these values for  $b2$ ,  $b5$  and  $b6$ ,  $b1$  could then be found from equation (16.c.25) and  $b9, \dots, b12$  could be determined from equations (16.c.19), (16.c.17), (16.c.20) and (16.c.14), respectively. With all of the  $b$ 's having been found, the model would then be identified. If  $b5$  and  $b6$  cannot be determined uniquely, the model is not identified. The identification status of this model will be determined empirically for a particular set of data.

**Type III:**

A total of sixteen observed data frequencies (twelve from the completely classified data and four from the row marginals) are assumed to be estimable. If all the interaction terms of the missing data indicator and the column variable, the missing data indicator and the row variable, and three way interactions of the missing data indicator, the column variable, and the row variable are included in the loglinear model, twenty parameters are required. Thus, Type III nonignorable nonresponse with column effects for the row and column variables cannot be considered to model this data set. We can, however, explore whether or not such a model is identified if only the linear interaction terms involving the missing data indicator appear in the model. The identification status of this modified Type III model is examined below.

Type III: column effects; 4x3 table with one supplemental margin

- modified (with only the linear interaction terms of the missing data indicator and the row variable, and the three way interactions of the missing data indicator, the column variable and the row variable)

| column | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | i | j | k |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|---|---|---|
|        | 1   | -3  | 9   | -27 | 1   | 0   | -3  | 0   | 1   | 1    | 0    | -3   | -3   | 0    | 1 | 1 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | -3  | 1   | 0    | 1    | -3   | 0    | -3   | 1 | 2 | 1 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 1   | 0    | 0    | -3   | 0    | 0    | 1 | 3 | 1 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 1   | 1    | 0    | -1   | -1   | 0    | 2 | 1 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 1   | 0    | 1    | -1   | 0    | -1   | 2 | 2 | 1 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 1   | 0    | 0    | -1   | 0    | 0    | 2 | 3 | 1 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1    | 0    | 1    | 1    | 0    | 3 | 1 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 1   | 0    | 1    | 1    | 0    | 1    | 3 | 2 | 1 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 1    | 0    | 0    | 3 | 3 | 1 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 3   | 0   | 1   | 1    | 0    | 3    | 3    | 0    | 4 | 1 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 3   | 1   | 0    | 1    | 3    | 0    | 3    | 4 | 2 | 1 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 1   | 0    | 0    | 3    | 0    | 0    | 4 | 3 | 1 |
|        | 1   | -3  | 9   | -27 | 1   | 0   | -3  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 1 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 1   | 0   | -3  | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 2 | 2 |
|        | 1   | -3  | 9   | -27 | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 1 | 3 | 2 |
|        | 1   | -1  | 1   | -1  | 1   | 0   | -1  | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 1 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 1   | 0   | -1  | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 2 | 2 |
|        | 1   | -1  | 1   | -1  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 2 | 3 | 2 |
|        | 1   | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 1 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 1   | 0   | 1   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 2 | 2 |
|        | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 3 | 3 | 2 |
|        | 1   | 3   | 9   | 27  | 1   | 0   | 3   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 4 | 1 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 1   | 0   | 3   | 0   | 0    | 0    | 0    | 0    | 0    | 4 | 2 | 2 |
|        | 1   | 3   | 9   | 27  | 0   | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0    | 0    | 4 | 3 | 2 |

In this matrix, the first column reflects the intercept term, the second through fourth columns, the row variable, the fifth and sixth columns, the column variable, the seventh and eighth columns, the interaction of the column variable and the linear row variable term, the ninth column, the missing data indicator, the tenth and eleventh columns, the interactions of the missing data indicator and the column variable, the twelfth column, the interaction of the missing data indicator and the linear row variable term, and the

thirteenth and fourteenth columns, the three way interactions of the missing data indicator, the column variable and the linear row variable term. The fourteen parameters of the model, comprising the  $\beta$  matrix, reflect the same model variables and interactions as do the columns of the  $X$  matrix, in the same order.

The current model will be identified if we can uniquely determine its parameters ( $\beta$  matrix). We will assume that we are given estimates for the frequency counts in the contingency table for the completely classified data, and estimates for the row marginals for the partially classified data. We attempt to find the model parameters in terms of the frequency and marginals estimates. We first consider those equations which correspond to the completely classified data, and then incorporate the equations which correspond to the partially classified data in an attempt to find the parameters ( $b$ 's) of the model.

Assume that estimates  $f1, f2, \dots, f12$  are available for the first twelve elements of vector  $Y$ , and estimates  $s1, s2, s3$  and  $s4$ , are available for the row marginals for the partially classified data. Using the fully classified data, we have

$$b1 - 3b2 + 9b3 - 27b4 + b5 - 3b7 + b9 + b10 - 3b12 - 3b13 = \ln f1 \quad (16.d.1)$$

$$b1 - 3b2 + 9b3 - 27b4 + b6 - 3b8 + b9 + b11 - 3b12 - 3b14 = \ln f2 \quad (16.d.2)$$

$$b1 - 3b2 + 9b3 - 27b4 + b9 - 3b12 = \ln f3 \quad (16.d.3)$$

$$b1 - b2 + b3 - b4 + b5 - b7 + b9 + b10 - b12 - b13 = \ln f4 \quad (16.d.4)$$

$$b1 - b2 + b3 - b4 + b6 - b8 + b9 + b11 - b12 - b14 = \ln f5 \quad (16.d.5)$$

$$b1 - b2 + b3 - b4 + b9 - b12 = \ln f6 \quad (16.d.6)$$

$$b1 + b2 + b3 + b4 + b5 + b7 + b9 + b10 + b12 + b13 = \ln f7 \quad (16.d.7)$$

$$b1 + b2 + b3 + b4 + b6 + b8 + b9 + b11 + b12 + b14 = \ln f8 \quad (16.d.8)$$

$$b1 + b2 + b3 + b4 + b9 + b12 = \ln f9 \quad (16.d.9)$$

$$b1 + 3b2 + 9b3 + 27b4 + b5 + 3b7 + b9 + b10 + 3b12 + 3b13 = \ln f10 \quad (16.d.10)$$

$$b1 + 3b2 + 9b3 + 27b4 + b6 + 3b8 + b9 + b11 + 3b12 + 3b14 = \ln f11 \quad (16.d.11)$$

$$b1 + 3b2 + 9b3 + 27b4 + b9 + 3b12 = \ln f12 \quad (16.d.12)$$



Here,  $b_1, b_2, \dots, b_{14}$  are the fourteen parameters of matrix  $\beta$ , which are to be determined. We attempt to solve the equations for the  $b$ 's. Using the same procedure as that for the column effects model with nonignorable nonresponse Type I (pages 432 - 433), we have

$$b_4 = \frac{1}{48} \ln \frac{f_{12}(f_6)^3}{f_3(f_9)^3} \quad (16.d.13)$$

$$b_2 + b_{12} = \frac{1}{48} \ln \frac{(f_9)^{27} f_3}{(f_6)^{27} f_{12}} \quad (16.d.14)$$

$$b_8 + b_{14} = \frac{1}{2} \ln \frac{f_6 f_8}{f_5 f_9} \quad (16.d.15)$$

$$b_7 + b_{13} = \frac{1}{2} \ln \frac{f_6 f_7}{f_4 f_9} \quad (16.d.16)$$

$$b_5 + b_{10} = \frac{1}{2} \ln \frac{f_4 f_7}{f_6 f_9} \quad (16.d.17)$$

$$b_3 = \frac{1}{16} \ln \frac{f_3 f_{12}}{f_6 f_9} \quad (16.d.18)$$

$$b_1 + b_9 = \frac{1}{48} \ln \frac{(f_6)^{27} (f_9)^{27}}{(f_3)^3 (f_{12})^3} \quad (16.d.19)$$

$$b_6 + b_{11} = \frac{1}{2} \ln \frac{f_5 f_8}{f_6 f_9} \quad (16.d.20)$$

Note that  $b_1$  and  $b_9$  cannot be determined uniquely from equations (16.d.1) - (16.d.12), nor can  $b_5$  and  $b_{10}$ ,  $b_2$  and  $b_{12}$ ,  $b_6$  and  $b_{11}$ ,  $b_8$  and  $b_{14}$ , and  $b_7$  and  $b_{13}$ . Only two of the twelve model parameters ( $b_3$  and  $b_4$ ) have been uniquely determined. Equations (16.d.13) - (16.d.20) provide the most complete solutions for the  $b$ 's which can be determined solely from equations (16.d.1) - (16.d.12).

The algebraic equations which correspond to the partially classified data for the current model are identical to those for that of nonignorable nonresponse, Type II. They are given by

$$\exp(b1 - 3b2 + 9b3 - 27b4)[\exp(b5 - 3b7) + \exp(b6 - 3b8) + 1] = s1 \quad (16.d.21)$$

$$\exp(b1 - b2 + b3 - b4)[\exp(b5 - b7) + \exp(b6 - b8) + 1] = s2 \quad (16.d.22)$$

$$\exp(b1 + b2 + b3 + b4)[\exp(b5 + b7) + \exp(b6 + b8) + 1] = s3 \quad (16.d.23)$$

$$\exp(b1 + 3b2 + 9b3 + 27b4)[\exp(b5 + 3b7) + \exp(b6 + 3b8) + 1] = s4 \quad (16.d.24)$$

The ten equations (16.d.14) - (16.d.17), and (16.d.19) - (16.d.24) are insufficient to uniquely determine the remaining twelve model parameters. This model, therefore, is not identified.

In summary, for the column effects model with one supplemental margin for Data Set 3, the Type I nonignorable nonresponse model is identified, while the Type III model (even with only linear interactions among the row, column and missing data indicator variables) is not. The Type II nonignorable nonresponse model is not an identified model. The identification status of the Type II nonignorable nonresponse model with only linear interaction terms involving the missing data indicators and each of the row and column variables will be determined empirically for a given data set.

### Empirical Analysis for Data Set 3

Parameter and contingency table estimates and likelihood ratio fit statistics are now presented for the column effects models hypothesized to fit Data Set 3 (page 406). These models include the one involving the complete data only, one hypothesizing that the partially classified data are missing completely at random (MCAR), and one hypothesizing that the partially classified data are missing at random (MAR). Also included is the model hypothesizing nonignorable nonresponse, Type I and one hypothesizing nonignorable nonresponse, Type II (modified to include only the linear interaction terms involving the missing data indicator).

#### Data Set 3

##### Column Effects Model

##### Completely Classified Data Only

**Presidential Candidate Preference**

|          |       | Truman | Dewey   | other | total   |
|----------|-------|--------|---------|-------|---------|
|          | A     | 37.05  | 176.56  | 10.39 | 224     |
| Economic | B     | 212.09 | 500.33  | 43.58 | 756     |
| Class    | C     | 542.67 | 633.67  | 81.66 | 1258    |
|          | D     | 357.19 | 206.45  | 39.37 | 603.01  |
|          | total | 1149   | 1517.01 | 175   | 2841.01 |

$$\beta = (4.22, .33, -.14, -.01, 1.74, 2.24, .16, -.20)$$

Likelihood Ratio fit statistic = 1.53521

Degrees of freedom = 4

**Data Set 3**

Column Effects Model - One Supplemental Margin

Missing Completely at Random (MCAR)

**Completely Classified Data****Presidential Candidate Preference**

|          |       | Truman  | Dewey   | other  | total   |
|----------|-------|---------|---------|--------|---------|
|          | A     | 32.49   | 154.83  | 9.11   | 196.43  |
| Economic | B     | 196.98  | 464.68  | 40.47  | 702.13  |
| Class    | C     | 550.79  | 643.15  | 82.88  | 1276.82 |
|          | D     | 394.27  | 227.88  | 43.45  | 665.60  |
|          | total | 1174.53 | 1490.54 | 175.91 | 2840.98 |

**Partially Classified Data (Hypothetical Contingency Table)****Presidential Candidate Preference**

|          |       | Truman | Dewey  | other | total  |
|----------|-------|--------|--------|-------|--------|
|          | A     | 7.54   | 35.91  | 2.11  | 45.56  |
| Economic | B     | 45.69  | 107.79 | 9.39  | 162.87 |
| Class    | C     | 127.76 | 149.19 | 19.23 | 296.18 |
|          | D     | 91.46  | 52.86  | 10.08 | 154.40 |
|          | total | 272.45 | 345.75 | 40.81 | 659.01 |

$$\beta = (2.73, .37, -.13, -.01, 1.74, 2.24, .16, -.20, 1.46)$$

Likelihood Ratio fit statistic = 81.1982

Degrees of freedom = 7

**Data Set 3**

Column Effects Model - One Supplemental Margin

Missing at Random (MAR)

**Completely Classified Data****Presidential Candidate Preference**

|                 |              | Truman | Dewey   | other | total   |
|-----------------|--------------|--------|---------|-------|---------|
|                 | <b>A</b>     | 37.05  | 176.56  | 10.39 | 224     |
| <b>Economic</b> | <b>B</b>     | 212.09 | 500.33  | 43.58 | 756     |
| <b>Class</b>    | <b>C</b>     | 542.67 | 633.67  | 81.66 | 1258    |
|                 | <b>D</b>     | 357.19 | 206.45  | 39.37 | 603.01  |
|                 | <b>total</b> | 1149   | 1517.01 | 175   | 2841.01 |

**Partially Classified Data (Hypothetical Contingency Table)****Presidential Candidate Preference**

|                 |              | Truman | Dewey  | other | total  |
|-----------------|--------------|--------|--------|-------|--------|
|                 | <b>A</b>     | 2.98   | 14.19  | .84   | 18.01  |
| <b>Economic</b> | <b>B</b>     | 30.58  | 72.14  | 6.28  | 109    |
| <b>Class</b>    | <b>C</b>     | 135.88 | 158.67 | 20.45 | 315    |
|                 | <b>D</b>     | 128.54 | 74.29  | 14.17 | 217    |
|                 | <b>total</b> | 297.98 | 319.29 | 41.74 | 659.01 |

$$\beta = (2.58, .60, -.15, -.01, 1.74, 2.24, .16, -.20, 1.65, -.28, .01, .003)$$

Likelihood Ratio fit statistic = 1.5352

Degrees of freedom = 4

**Data Set 3**

Column Effects Model - One Supplemental Margin

Nonignorable Nonresponse Type I

**Completely Classified Data****Presidential Candidate Preference**

|          |       | Truman | Dewey   | other | total   |
|----------|-------|--------|---------|-------|---------|
|          | A     | 34.88  | 176.73  | 10.39 | 222     |
| Economic | B     | 204.97 | 499.02  | 43.44 | 747.43  |
| Class    | C     | 543.53 | 635.78  | 81.95 | 1261.26 |
|          | D     | 365.62 | 205.48  | 39.22 | 610.32  |
|          | total | 1149   | 1517.01 | 175   | 2841.01 |

**Partially Classified Data (Hypothetical Contingency Table)****Presidential Candidate Preference**

|          |       | Truman | Dewey | other | total  |
|----------|-------|--------|-------|-------|--------|
|          | A     | 20.00  | .00   | .01   | 20.01  |
| Economic | B     | 117.52 | .00   | .05   | 117.57 |
| Class    | C     | 311.64 | .00   | .10   | 311.74 |
|          | D     | 209.63 | .00   | .05   | 209.68 |
|          | total | 658.78 | 0.00  | .21   | 659    |

$$\beta = (-2.47, .33, -.14, -.01, 7.86, -7.71, .17, -.20, 6.69, -6.13, 9.95)$$

Likelihood Ratio fit statistic = 3.26372

Degrees of freedom = 5

### Data Set 3

#### Column Effects Model - One Supplemental Margin

#### Nonignorable Nonresponse - Type II (modified)

As indicated on page 445, the identification status of the modified nonignorable nonresponse Type II model with column effects depends on the number of solutions to equations (16.c.32) and the uniqueness or nonuniqueness of  $b_5$ . The maximum likelihood solution for this model was found by using the computer program written in SAS for this purpose (page 22). The estimated frequencies and row marginals generated were substituted into equation (16.c.32) giving the following quadratic equation in  $x = \exp(b_5)$  in terms of  $\exp(b_6)$ :

$$275.009 \exp(2b_5) + \exp(b_5) [-49301.79 \exp(b_6) - 8489.54] \\ + [275.01 \exp(2b_6) - 13833.83 \exp(b_6) + 275.01] = 0$$

Since  $\exp(b_6) > 0$ , the coefficient of  $x$  must be negative. The coefficient of  $x^2$  is positive. By Descartes' Rule of Signs, for there to be exactly one positive solution for  $x = \exp(b_5)$  (we know that there is at least one solution for  $b_6$  since the computer program found one), the "constant" term must be negative, thus resulting in one sign change in the quadratic equation. This would require that

$$275.01 \exp(2b_6) - 13833.83 \exp(b_6) + 275.01 < 0$$

or that  $b_6 < 3.9177$ . As the maximum likelihood solution generated a value for  $b_6$  is 3.95261, the quadratic equation in  $x = \exp(b_5)$  does not have exactly one solution. Thus, there are no solutions for  $b_6$ , or two solutions. There is at least one solution for  $b_6$  since the computer program generated it. Thus, there must be two solutions for  $b_6$ . Thus, the model is not identified.

### Summary of Results - Data Set 3

Presented below is a summary of the determination of the identification status of nonignorable nonresponse models for Data Set 3, a 4x3 table with one supplemental margin, as well as the empirical analyses for a specific data set of this structure (page 406).

#### Identification Status of Nonignorable Nonresponse Models for Data Set 3

|                | Type I | Type II | Type II<br>(modified) | Type III<br>(modified) |
|----------------|--------|---------|-----------------------|------------------------|
| Independence   | X      | X       | X                     | X                      |
| Column Effects | *      | X       | ?                     | X                      |

x = not identified

\* = identified

? The identification status of the model will be examined empirically for a given data set



Chi Square Goodness of Fit for Tested Models  
Data Set 3 (page 406)

|                | Com-<br>plete<br>Data<br>Only | MCAR           | MAR                  | NN Type<br>I          | NN<br>TYPE II | NN<br>TYPE II<br>(modi-<br>fied) | NN<br>Type III<br>(modi-<br>fied) |
|----------------|-------------------------------|----------------|----------------------|-----------------------|---------------|----------------------------------|-----------------------------------|
| Independence   | 219.065<br>(6)                | 298.728<br>(9) | 219.065<br>(6)       | X                     | X             | X                                | X                                 |
| Column Effects | <b>1.53521</b><br>(4)         | 81.1982<br>(7) | <b>1.5352</b><br>(4) | <b>3.26372</b><br>(5) | X             |                                  | X                                 |

- Convergence criterion: difference in successive chi square values is <.00001
- Those chi square values in bold type are not significant at the .05 level. Thus, the models for which bold type appears fit the data.
- The model whose identification status was empirically determined has a shaded entry in the table. (It is not identified for this data set.)

## CHAPTER 4

### DISCUSSION

An introduction and overview was first presented summarizing many of the methods, models and assumptions which have been used to analyze nominal and ordinal data, missing data and missing categorical data. The current work then focused on three kinds of categorical data sets in which at least one of the variables was ordinal and in which some of the data were missing. Models which hypothesized nonignorable nonresponse for each of these three kinds of data sets were first investigated for a determination of their identification status. These models were nonstandard, in the sense that they were not hierarchical, and employed an extension of the generalized linear model with composite links. Empirical model fitting for these three kinds of data sets was then performed using specific examples from the literature. These models were fit to the completely classified data only, as well to both the completely and partially classified data. Explored for goodness of fit were those models which hypothesized that the missing data mechanism was ignorable, as well as those identified models which assumed that this mechanism was nonignorable.

The three kinds of data sets investigated each involved two variables. The first contained two ordinal variables, each having three categories, with a supplementary row margin. The second also contained two ordinal variables, each having three categories, with both a supplementary row margin and a supplementary column margin. The third involved an ordinal variable with four categories, a nominal variable with three categories and a supplementary margin for the nominal variable.

### Identification Status of Nonignorable Nonresponse Models

Each of the three kinds of data sets considered in this work was examined for the identification status of each of three types of nonignorable nonresponse. A first type hypothesized that the missingness of the data was related only to the variable which had missing values, while a second type hypothesized that this missingness was related to each of the variables, but not to an interaction of the variables. A third type assumed that the missing data mechanism was related to each of the variables, as well as to the interaction of the two variables.

A model is identified when all of its parameters are uniquely determined. For each model explored for identification in the current work, it was assumed that maximum likelihood estimates were available for the fully classified data frequencies and for the observed marginals. Parameters were expressed in terms of these estimates. Those models for which all parameters had unique solutions in terms of these estimated data frequencies and observed marginals were determined to be identified. While some of the nonignorable nonresponse models considered in this work were found to be not identified due to the nonuniqueness of the determination of their parameters, others had too many parameters, given the data. If a model was found to be not identified, a modified version of the model was evaluated for identification. This modification involved the elimination of all but the lowest order interaction term(s) involving the missing data indicator(s) and one (or both) of the ordinal variables.

A general approach was taken in this work to determine the identification status of the nonignorable nonresponse models. First, as complete a solution as possible for the model parameters was obtained using a system of equations employing the fully classified data, only. To this was added the system of equations obtained using the supplemental

marginals. For some models, these additional equations led to unique model parameter determination, while for others, it did not. For some of the models, fourth degree polynomial equations in the exponential of one of the model parameters were generated. The identification of the models then hinged on the number of positive solutions to these quartic equations. Generally, the number of such solutions was not possible to determine in the abstract; it was necessary to know the numerical coefficients in these polynomials to establish the number of solutions to the equations. These coefficients were functions of the actual numerical estimates for the completely classified data frequencies and the marginals.

For the first kind of data set having two ordinal variables, each with three categories, as well as a supplemental row margin, the identification status was explored for the independence, uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry nonignorable nonresponse models. Of all the models considered, except for the independence model, those which hypothesized that the missingness of the data was related only to the variable which had missing values, were found to be identified. Also identified was one of the nonignorable nonresponse symmetry models. This model hypothesized that the missingness of the data was related to each of the variables, but not to an interaction of the variables. It was modified to include only the lowest order interaction terms of the missing data indicator and each of the variables. In addition, the identification status of some of the models was found to be dependent on the actual numerical frequency and supplemental margin estimates for a given data set. These included the modified versions of one of each of the row effects, column effects,  $R + C$ , and quasi-symmetry nonignorable nonresponse models. Each of these modified models hypothesized that the missingness of the data was related to each of the variables, but not to an interaction of the variables. Also, the identification status of each of two of the nonignorable nonresponse symmetry models was found to be dependent on the actual

numerical frequencies and marginal estimates for a given data set. One hypothesized that the missingness of the data was related to each of the variables, but not to an interaction of the two variables. The other was the modified version of the model which assumed that the missing data mechanism was related to each of the variables, as well as to the interaction of the two variables.

For the second kind of data set having two ordinal variables, each with three categories, as well as both a supplemental row margin and a supplemental column margin, the identification status was explored for the independence, uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry nonignorable nonresponse models. As for the first kind of data set, of all the models considered, except for the independence model, those which hypothesized that the missingness of the data was related only to the variable which had missing values, were found to be identified. In addition, the identification status of some of the models was found to be dependent on the actual numerical frequencies and supplemental margin estimates for a given data set. These included the modified versions of one of each of the row effects, column effects,  $R + C$ , symmetry and quasi-symmetry nonignorable nonresponse models. Each of these hypothesized that the missingness of the data was related to each of the model variables, but not to an interaction of the two variables.

For the third kind of data set having an ordinal variable with four categories and a nominal variable with three categories, as well as a supplemental margin for the nominal variable, the identification status was explored for the independence and column effects nonignorable nonresponse models. The column effects model which hypothesized that the missingness of the data was related only to the variable which had missing values, was found to be identified. In addition, the identification status of one of the column effects models was found to be dependent on the actual numerical frequencies and

supplemental margin estimates for a given data set. This model was the modified version of the one which hypothesized that the missingness of the data was related to each of the model variables, but not to an interaction of the two variables.

The algebraic manipulations in establishing the identification status of some of these models were sometimes complex. The determination of the identification status of a proposed model, however, is an essential task before an attempt is made to understand relationships imposed by this model among the data. In fact, for each of the three kinds of data sets considered in this work, some, but not all, of the nonignorable nonresponse models were identified and could legitimately be used to model the data.

Future work is needed to explore the possibility of generalizing the results obtained for the data sets in the current work to other kinds of data sets. These sets of data should contain only categorical variables, at least one of which is ordinal, and have missing values on at least one of the variables. Such general results would better classify those models which are identified, and so could be used by the analyst to explore relationships among the data when some data frequencies are unobserved.

### Empirical Analyses

#### Data Set 1

Models were hypothesized to fit Data Set 1 (page 27). This data set was part of the 1977 National Crime Survey in which respondents were asked to report the incidence of their crime victimization (no crime, one crime, multiple crimes). Most participants were interviewed twice during the year, the second time, six months after a first interview, but some were only interviewed at the time of the second interview. Models for this data set

were explored for goodness of fit, and maximum likelihood estimates were obtained for cell frequencies and marginals. Estimates of cell frequency proportions and local odds ratios which characterize the models considered in this work were computed for Data Set 1. These proportions and odds ratio estimates have been evaluated for the uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry models, both for the completely classified data, only, and for the completely classified data with supplemental marginals. For this latter group of models, both ignorable and nonignorable response were considered. Included were three types of identified nonignorable nonresponse models (page 21) considered in this work. Here, the Type I nonignorable nonresponse model proposes that whether or not a person was interviewed at the first interview was related only to that person's crime victimization status at the time of the first interview. The Type II nonignorable nonresponse model suggests that whether or not a person was interviewed at the first interview was related to that person's crime victimization status at the times of both the first and second interviews. The Type III nonignorable nonresponse model hypothesizes that whether or not a participant was interviewed at the time of the first interview was related to that person's crime victimization status at the times of the first and second interviews, as well as to the relationship between this status at the times of the two interviews. The estimates of cell frequency proportions and local odds ratios for those models considered in this work which adequately fit Data Set 1 (at the .05 significance level) are presented below. Here,  $\hat{p}_{ij}$  is the cell frequency proportion estimated by the model, for  $i = 1,2,3$   $j = 1,2,3$ .

## Uniform Association Models - Data Set 1

 $\theta_{ij} = \theta$ , for some constant  $\theta$ , and for  $i=1,2, j=1,2$ 

|                | Complete Data Only | MAR   | NN Type I |
|----------------|--------------------|-------|-----------|
| $\hat{\theta}$ | 1.614              | 1.614 | 1.614     |

## Uniform Association Models - Data Set 1

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type I |
|----------------|-----------------------|------|-----------|
| $\hat{p}_{11}$ | .698                  | .689 | .667      |
| $\hat{p}_{12}$ | .106                  | .105 | .089      |
| $\hat{p}_{13}$ | .027                  | .027 | .064      |
| $\hat{p}_{21}$ | .094                  | .098 | .090      |
| $\hat{p}_{22}$ | .023                  | .024 | .019      |
| $\hat{p}_{23}$ | .009                  | .010 | .023      |
| $\hat{p}_{31}$ | .025                  | .029 | .024      |
| $\hat{p}_{32}$ | .010                  | .012 | .008      |
| $\hat{p}_{33}$ | .007                  | .008 | .016      |



## Row Effects Model - Data Set 1

$$\theta_{ij} = \theta_{i.}, \text{ for } i = 1, 2 \text{ } j = 1, 2$$

|                     | Complete Data<br>Only | MAR   | NN Type I | NN Type II<br>(modified) |
|---------------------|-----------------------|-------|-----------|--------------------------|
| $\hat{\theta}_{1.}$ | 1.827                 | 1.827 | 1.797     | 1.835                    |
| $\hat{\theta}_{2.}$ | 1.284                 | 1.284 | 1.345     | 1.270                    |

## Row Effects Model - Data Set 1

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type I | NN Type II<br>(modified) |
|----------------|-----------------------|------|-----------|--------------------------|
| $\hat{p}_{11}$ | .700                  | .690 | .675      | .718                     |
| $\hat{p}_{12}$ | .105                  | .104 | .080      | .084                     |
| $\hat{p}_{13}$ | .026                  | .026 | .065      | .019                     |
| $\hat{p}_{21}$ | .091                  | .094 | .087      | .103                     |
| $\hat{p}_{22}$ | .025                  | .026 | .018      | .021                     |
| $\hat{p}_{23}$ | .011                  | .012 | .027      | .008                     |
| $\hat{p}_{31}$ | .027                  | .031 | .026      | .036                     |
| $\hat{p}_{32}$ | .009                  | .011 | .007      | .009                     |
| $\hat{p}_{33}$ | .006                  | .006 | .015      | .004                     |

## Column Effects Model - Data Set 1

$$\theta_{ij} = \theta_{.j}, \text{ for } i = 1, 2 \quad j = 1, 2$$

|                     | Complete Data Only | MAR   | NN TYPE I |
|---------------------|--------------------|-------|-----------|
| $\hat{\theta}_{.1}$ | 1.776              | 1.776 | 1.775     |
| $\hat{\theta}_{.2}$ | 1.353              | 1.353 | 1.357     |

## Column Effects Model - Data Set 1

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type I |
|----------------|-----------------------|------|-----------|
| $\hat{p}_{11}$ | .700                  | .690 | .638      |
| $\hat{p}_{12}$ | .104                  | .102 | .127      |
| $\hat{p}_{13}$ | .028                  | .028 | .055      |
| $\hat{p}_{21}$ | .093                  | .097 | .084      |
| $\hat{p}_{22}$ | .024                  | .025 | .030      |
| $\hat{p}_{23}$ | .009                  | .009 | .018      |
| $\hat{p}_{31}$ | .025                  | .028 | .023      |
| $\hat{p}_{32}$ | .011                  | .013 | .014      |
| $\hat{p}_{33}$ | .006                  | .007 | .011      |

## R + C Model - Data Set 1

$$\theta_{ij} = \theta_i \theta_j, \text{ for } i = 1, 2 \text{ } j = 1, 2$$

|                     | Complete Data<br>Only | MAR   | NN Type II<br>(modified) |
|---------------------|-----------------------|-------|--------------------------|
| $\hat{\theta}_{11}$ | 1.973                 | 1.973 | 1.982                    |
| $\hat{\theta}_{12}$ | 1.546                 | 1.546 | 1.553                    |
| $\hat{\theta}_{21}$ | 1.414                 | 1.414 | 1.399                    |
| $\hat{\theta}_{22}$ | 1.108                 | 1.108 | 1.096                    |

## R + C Model - Data Set 1

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type II<br>(modified) |
|----------------|-----------------------|------|--------------------------|
| $\hat{p}_{11}$ | .701                  | .692 | .716                     |
| $\hat{p}_{12}$ | .103                  | .101 | .084                     |
| $\hat{p}_{13}$ | .027                  | .027 | .020                     |
| $\hat{p}_{21}$ | .090                  | .094 | .101                     |
| $\hat{p}_{22}$ | .026                  | .027 | .022                     |
| $\hat{p}_{23}$ | .011                  | .011 | .008                     |
| $\hat{p}_{31}$ | .026                  | .030 | .034                     |
| $\hat{p}_{32}$ | .011                  | .012 | .010                     |
| $\hat{p}_{33}$ | .005                  | .006 | .004                     |

## Symmetry Model - Data Set 1

$$\theta_{ij} = \theta_{ji} \text{ for } i = 1, 2 \text{ } j = 1, 2$$

|                     | Complete<br>Data Only | MCAR  | MAR   | NN Type<br>II | NN Type<br>II (modi-<br>fied) | NN Type<br>III (modi-<br>fied) |
|---------------------|-----------------------|-------|-------|---------------|-------------------------------|--------------------------------|
| $\hat{\theta}_{11}$ | 1.639                 | 1.639 | 1.648 | 1.645         | 1.634                         | 2.476                          |
| $\hat{\theta}_{12}$ | 2.094                 | 2.097 | 2.111 | 2.091         | 2.101                         | 3.164                          |
| $\hat{\theta}_{22}$ | .583                  | .586  | .589  | .583          | .591                          | .881                           |

## Symmetry Model - Data Set 1

## Estimated Cell Proportions

|                | Complete<br>Data Only | MCAR | MAR  | NN Type<br>II | NN Type<br>II<br>(modified) | NN Type<br>III<br>(modified) |
|----------------|-----------------------|------|------|---------------|-----------------------------|------------------------------|
| $\hat{p}_{11}$ | .700                  | .692 | .689 | .687          | .687                        | .695                         |
| $\hat{p}_{12}$ | .098                  | .099 | .104 | .103          | .102                        | .098                         |
| $\hat{p}_{13}$ | .026                  | .028 | .030 | .030          | .030                        | .026                         |
| $\hat{p}_{21}$ | .098                  | .099 | .094 | .094          | .097                        | .093                         |
| $\hat{p}_{22}$ | .023                  | .023 | .023 | .023          | .024                        | .025                         |
| $\hat{p}_{23}$ | .012                  | .014 | .014 | .014          | .015                        | .016                         |
| $\hat{p}_{31}$ | .026                  | .028 | .026 | .028          | .027                        | .023                         |
| $\hat{p}_{32}$ | .012                  | .014 | .014 | .015          | .014                        | .016                         |
| $\hat{p}_{33}$ | .004                  | .005 | .005 | .005          | .005                        | .008                         |

## Quasi-Symmetry Model - Data Set 1

$$\theta_{ij} = \theta_{ji} \text{ for } i = 1, 2 \text{ } j = 1, 2$$

|                     | Complete Data Only | MAR   | NN Type I |
|---------------------|--------------------|-------|-----------|
| $\hat{\theta}_{11}$ | 1.645              | 1.645 | 1.640     |
| $\hat{\theta}_{12}$ | 2.091              | 2.091 | 2.100     |
| $\hat{\theta}_{22}$ | .583               | .583  | .580      |

## Quasi-Symmetry Model - Data Set 1

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type I |
|----------------|-----------------------|------|-----------|
| $\hat{p}_{11}$ | .700                  | .691 | .613      |
| $\hat{p}_{12}$ | .104                  | .103 | .179      |
| $\hat{p}_{13}$ | .027                  | .026 | .028      |
| $\hat{p}_{21}$ | .092                  | .096 | .080      |
| $\hat{p}_{22}$ | .023                  | .023 | .039      |
| $\hat{p}_{23}$ | .012                  | .013 | .013      |
| $\hat{p}_{31}$ | .025                  | .029 | .022      |
| $\hat{p}_{32}$ | .013                  | .015 | .022      |
| $\hat{p}_{33}$ | .004                  | .005 | .004      |

## Data Set 2

Models were hypothesized to fit Data Set 2 (page 219). This data set was an enlargement of Data Set 1, from the 1977 National Crime Survey. In addition to those who were interviewed twice during 1977 regarding their incidence of crime victimization, some were interviewed only at the time of the second interview (as with Data Set 1), but some were also interviewed only at the time of the first interview. Models for this data set were explored for goodness of fit, and maximum likelihood estimates were obtained for cell frequencies and marginals. Estimates of cell frequency proportions and local odds ratios which characterize the models considered in this work were computed for Data Set 2. These proportions and odds ratio estimates have been evaluated for the uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry models, both for the completely classified data, only, and for the completely classified data with supplemental marginals. For this latter group of models, both ignorable and nonignorable response were considered. Included were two of the three types of identified nonignorable nonresponse models (page 21) considered in this work. Here, the Type I nonignorable nonresponse model proposes that whether a person was interviewed once or twice (and if once, at which interview time) was related only to that person's crime victimization status at the time of the first interview. The Type II nonignorable nonresponse model suggests that whether a person was interviewed once or twice (and if once, at which interview time) was related to that person's crime victimization status at the times of both the first and second interviews. The estimates of cell frequency proportions and local odds ratios for those models considered in this work which adequately fit Data Set 2 (at the .05 significance level) are presented below. Here,  $\hat{p}_{ij}$  is the cell frequency proportion estimated by the model, for  $i = 1, 2, 3$   $j = 1, 2, 3$ .

## Uniform Association Models - Data Set 2

 $\theta_{ij} = \theta$ , for some constant  $\theta$ , and for  $i=1,2, j=1,2$ 

|                | Complete Data Only | MAR   | NN Type I |
|----------------|--------------------|-------|-----------|
| $\hat{\theta}$ | 1.614              | 1.621 | 1.615     |

## Uniform Association Models - Data Set 2

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type I |
|----------------|-----------------------|------|-----------|
| $\hat{p}_{11}$ | .698                  | .545 | .639      |
| $\hat{p}_{12}$ | .106                  | .084 | .128      |
| $\hat{p}_{13}$ | .027                  | .021 | .053      |
| $\hat{p}_{21}$ | .094                  | .215 | .086      |
| $\hat{p}_{22}$ | .023                  | .054 | .028      |
| $\hat{p}_{23}$ | .009                  | .021 | .019      |
| $\hat{p}_{31}$ | .025                  | .036 | .023      |
| $\hat{p}_{32}$ | .010                  | .015 | .012      |
| $\hat{p}_{33}$ | .007                  | .009 | .013      |

## Row Effects Model - Data Set 2

$$\theta_{ij} = \theta_{i.}, \text{ for } i = 1, 2 \quad j = 1, 2$$

|                     | Complete Data Only | MAR   | NN Type I |
|---------------------|--------------------|-------|-----------|
| $\hat{\theta}_{1.}$ | 1.827              | 1.821 | 1.797     |
| $\hat{\theta}_{2.}$ | 1.284              | 1.294 | 1.345     |

## Row Effects Model - Data Set 2

## Estimated Cell Proportions

|                | Complete Data Only | MAR  | NN Type I |
|----------------|--------------------|------|-----------|
| $\hat{p}_{11}$ | .700               | .562 | .664      |
| $\hat{p}_{12}$ | .105               | .086 | .094      |
| $\hat{p}_{13}$ | .026               | .021 | .061      |
| $\hat{p}_{21}$ | .091               | .207 | .085      |
| $\hat{p}_{22}$ | .025               | .057 | .022      |
| $\hat{p}_{23}$ | .011               | .025 | .025      |
| $\hat{p}_{31}$ | .027               | .027 | .026      |
| $\hat{p}_{32}$ | .009               | .010 | .009      |
| $\hat{p}_{33}$ | .006               | .006 | .014      |



## Column Effects Model - Data Set 2

$$\theta_{ij} = \theta_j, \text{ for } i = 1, 2 \quad j = 1, 2$$

|                     | Complete Data Only | MAR   | NN TYPE I |
|---------------------|--------------------|-------|-----------|
| $\hat{\theta}_{.1}$ | 1.776              | 1.763 | 1.775     |
| $\hat{\theta}_{.2}$ | 1.353              | 1.378 | 1.357     |

## Column Effects Model - Data Set 2

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type I |
|----------------|-----------------------|------|-----------|
| $\hat{p}_{11}$ | .700                  | .563 | .635      |
| $\hat{p}_{12}$ | .104                  | .083 | .129      |
| $\hat{p}_{13}$ | .028                  | .023 | .056      |
| $\hat{p}_{21}$ | .093                  | .192 | .084      |
| $\hat{p}_{22}$ | .024                  | .050 | .030      |
| $\hat{p}_{23}$ | .009                  | .019 | .018      |
| $\hat{p}_{31}$ | .025                  | .041 | .023      |
| $\hat{p}_{32}$ | .011                  | .019 | .014      |
| $\hat{p}_{33}$ | .006                  | .010 | .011      |

## R + C Model - Data Set 2

$$\theta_{ij} = \theta_i \theta_j, \text{ for } i = 1, 2 \quad j = 1, 2$$

|                     | Complete Data<br>Only | MAR    | NN Type II<br>(modified) |
|---------------------|-----------------------|--------|--------------------------|
| $\hat{\theta}_{11}$ | 1.973                 | 1.9731 | 1.983                    |
| $\hat{\theta}_{12}$ | 1.546                 | 1.554  | 1.552                    |
| $\hat{\theta}_{21}$ | 1.414                 | 1.413  | 1.399                    |
| $\hat{\theta}_{22}$ | 1.108                 | 1.114  | 1.095                    |

## R + C Model - Data Set 2

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type II<br>(modified) |
|----------------|-----------------------|------|--------------------------|
| $\hat{p}_{11}$ | .701                  | .596 | .633                     |
| $\hat{p}_{12}$ | .103                  | .087 | .076                     |
| $\hat{p}_{13}$ | .027                  | .023 | .019                     |
| $\hat{p}_{21}$ | .090                  | .139 | .117                     |
| $\hat{p}_{22}$ | .026                  | .040 | .029                     |
| $\hat{p}_{23}$ | .011                  | .017 | .011                     |
| $\hat{p}_{31}$ | .026                  | .061 | .075                     |
| $\hat{p}_{32}$ | .011                  | .025 | .028                     |
| $\hat{p}_{33}$ | .005                  | .011 | .012                     |

## Symmetry Model - Data Set 2

$$\theta_{ij} = \theta_{ji} \text{ for } i = 1, 2 \text{ } j = 1, 2$$

|                     | Complete Data<br>Only | MAR   | NN Type II (modi-<br>fied) |
|---------------------|-----------------------|-------|----------------------------|
| $\hat{\theta}_{11}$ | 1.639                 | 1.647 | 1.636                      |
| $\hat{\theta}_{12}$ | 2.094                 | 2.104 | 2.103                      |
| $\hat{\theta}_{22}$ | .583                  | .595  | .586                       |

## Symmetry Model - Data Set 2

## Estimated Cell Proportions

|                | Complete Data Only | MAR  | NN Type II<br>(modified) |
|----------------|--------------------|------|--------------------------|
| $\hat{p}_{11}$ | .700               | .617 | .617                     |
| $\hat{p}_{12}$ | .098               | .093 | .092                     |
| $\hat{p}_{13}$ | .026               | .027 | .027                     |
| $\hat{p}_{21}$ | .098               | .116 | .115                     |
| $\hat{p}_{22}$ | .023               | .029 | .028                     |
| $\hat{p}_{23}$ | .012               | .018 | .017                     |
| $\hat{p}_{31}$ | .026               | .059 | .061                     |
| $\hat{p}_{32}$ | .012               | .031 | .031                     |
| $\hat{p}_{33}$ | .004               | .011 | .012                     |

## Quasi-Symmetry Model - Data Set 2

$$\theta_{ij} = \theta_{ji} \text{ for } i = 1, 2 \text{ } j = 1, 2$$

|                     | Complete Data Only | MAR   | NN Type I |
|---------------------|--------------------|-------|-----------|
| $\hat{\theta}_{11}$ | 1.645              | 1.642 | 1.640     |
| $\hat{\theta}_{12}$ | 2.091              | 2.096 | 2.100     |
| $\hat{\theta}_{22}$ | .583               | .585  | .580      |

## Quasi-Symmetry Model - Data Set 2

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type I |
|----------------|-----------------------|------|-----------|
| $\hat{p}_{11}$ | .700                  | .592 | .614      |
| $\hat{p}_{12}$ | .104                  | .088 | .172      |
| $\hat{p}_{13}$ | .027                  | .023 | .033      |
| $\hat{p}_{21}$ | .092                  | .149 | .081      |
| $\hat{p}_{22}$ | .023                  | .037 | .037      |
| $\hat{p}_{23}$ | .012                  | .020 | .015      |
| $\hat{p}_{31}$ | .025                  | .054 | .022      |
| $\hat{p}_{32}$ | .013                  | .028 | .021      |
| $\hat{p}_{33}$ | .004                  | .009 | .005      |

For each of the six kinds of models summarized (uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry) which adequately fit the 1977 National Crime Data, the estimated local odds ratios and estimates for the cell proportions are close in value for the models involving only the completely classified data and for those models hypothesizing ignorable nonresponse for Data Set 1. Some (but not all) of the nonignorable nonresponse models which adequately fit this Data Set predict a somewhat lower proportion of respondents to be crime free, and a somewhat higher proportion to be the victim of multiple crimes at the time of the first interview, than do the models for only the completely classified data, as well as for those models hypothesizing ignorable nonresponse. Other models make different predictions about the crime victimization status at the time of the first interview. Thus, many relationships among the variables are feasible and fit the data for this Data Set. Nonignorable nonresponse should certainly not be ruled out.

For each of the six kinds of models summarized (uniform association, row effects, column effects,  $R + C$ , symmetry and quasi-symmetry) which adequately fit the 1977 National Crime Data, the estimated local odds ratios for those models which hypothesize that data are missing at random differ somewhat from those involving only the completely classified data for Data Set 2. Further, the models which hypothesize that the data are missing at random predict a lower proportion of respondents to be crime free at the times of both interviews than do those involving only the completely classified data. These MAR models also predict a higher proportion of participants to have had at least one crime incident at the time of the second interview than do those models involving only the completely classified data. While the nonignorable nonresponse models also predict a lower proportion of crime free respondents at the time of both of the interviews than do the models for only the completely classified data, the difference is less pronounced than for the MAR models and the models involving only the completely

classified data. Overall, if both supplemental margins are included in the models, the incidence of crime victimization is predicted to be greater than if only the completely classified data are utilized.

Conaway's (1993) approach was to use the symmetry model to reflect relationships concerning first and second interview responses. He used the time ordering of the interviews, and assumed that a person's victimization status at the time of the second interview was related to that person's victimization status at the time of the first interview, but not to whether a person was actually interviewed at the first interview session. Though Conaway's nonignorable nonresponse models were different from those considered in this work, his models also predicted a lower proportion of crime free respondents than if only the completely classified data were used.

### Data Set 3

Data Set 3 (page 406) contains the October, 1948 Roper polling data for the Presidential election. People in four economic classes (A, B, C and D) were asked to state their voting preference (Truman, Dewey or another candidate) for the following month's Presidential race. Those who expressed no preference comprised the supplemental margin. Models were hypothesized to fit Data Set 3. They were explored for goodness of fit, and maximum likelihood estimates were obtained for cell frequencies and marginals. Estimates of cell frequency proportions and local odds ratios which characterize the models considered in this work were computed for Data Set 3. These proportions and odds ratio estimates have been evaluated for the column effects models, both for the completely classified data, only, and for the completely classified data with supplemental marginals. For this latter group of models, both ignorable and nonignorable response were considered. Included was the identified Type I nonignorable nonresponse model. It

proposes that whether a person expressed a Presidential preference was related only to that person's actual leaning for President. The estimates of cell frequency proportions and local odds ratios for those models considered in this work which adequately fit Data Set 3 (at the .05 significance level) are presented below. Here,  $\hat{p}_{ij}$  is the cell frequency proportion estimated by the model, for  $i = 1, 2, 3$   $j = 1, 2, 3$ .

Column Effects Model - Data Set 3

$$\theta_{ij} = \theta_{.j}, \text{ for } i = 1, 2, 3 \quad j = 1, 2$$

|                     | Complete Data Only | MAR   | NN TYPE I |
|---------------------|--------------------|-------|-----------|
| $\hat{\theta}_{.1}$ | .495               | .495  | .480      |
| $\hat{\theta}_{.2}$ | 1.480              | 1.480 | 1.481     |

## Column Effects Model - Data Set 3

## Estimated Cell Proportions

|                | Complete Data<br>Only | MAR  | NN Type I |
|----------------|-----------------------|------|-----------|
| $\hat{p}_{11}$ | .013                  | .011 | .016      |
| $\hat{p}_{12}$ | .062                  | .054 | .050      |
| $\hat{p}_{13}$ | .004                  | .011 | .003      |
| $\hat{p}_{21}$ | .075                  | .069 | .092      |
| $\hat{p}_{22}$ | .176                  | .164 | .143      |
| $\hat{p}_{23}$ | .015                  | .014 | .012      |
| $\hat{p}_{31}$ | .191                  | .194 | .244      |
| $\hat{p}_{32}$ | .223                  | .226 | .102      |
| $\hat{p}_{33}$ | .029                  | .029 | .023      |
| $\hat{p}_{41}$ | .126                  | .139 | .164      |
| $\hat{p}_{42}$ | .073                  | .080 | .059      |
| $\hat{p}_{43}$ | .014                  | .015 | .011      |

The nonignorable nonresponse model predicts that approximately 52% will vote for Truman and 35% will vote for Dewey, while the model which includes only the completely classified data predicts that the vote will be 40% for Truman and 53% for Dewey. The model which hypothesizes that the lack of expression of a Presidential preference is related to economic class (the MAR model) predicts similar candidate percentages to the one which uses only the completely classified data. Though three models adequately fit Data Set 3, only the nonignorable nonresponse model predicts that Truman would emerge victorious, as he did in the actual election.



Baker and Laird (1988) used the saturated model for the Roper polling data from July, August, September and October classified by economic class (A, B, C and D) and voter preference (Truman, Dewey or other) to model the fully classified data. They considered a variety of ignorable and nonignorable nonresponse models to incorporate those who expressed no Presidential preference. Their objective was to make predictions on the election based on the October polling results, and included both the completely and partially classified data. In the current work, only the October polling results were modeled, using the independence and column effects models, with those who expressed no candidate leanings forming a supplemental margin. Using the nonignorable nonresponse model in the current work, the predicted percentage of votes for Truman was consistent with that projected by Baker and Laird's nonignorable nonresponse models.

As with all model fitting, while one can sometimes find models which adequately fit a set of data, one can never establish with absolute certainty that a particular model is the "true" and "only" appropriate one to represent relationships among the data. It is clear, however, that to ignore the possibility that missing data are not missing at random is often inappropriate and fallacious. As was borne out in the statistical analyses concerning the 1948 Presidential election, for example, ignoring missing data can lead to erroneous conclusions. It would be just as inappropriate, however, to attempt to fit unidentified nonignorable nonresponse models to a data set. Thus, in exploring the possibility of nonignorable nonresponse, one must first make certain that the proposed model is a statistically legitimate one. Having done so, the statistical analyst can then evaluate the possible relationships among the variables predicted by a broader variety of models than just those which assume that the missing data mechanism is ignorable. In many cases, the models hypothesizing the nonrandom nature of the missing data mechanism may be truest to "real life".

# Appendix

## Example of SAS Program to Compute Estimates of Cell Frequencies, Marginals, Parameters and Likelihood Ratio Chi Square for Models Considered in This Work

—

```

1      PROC MATRIX;
2      NOTE LINES 3,4 AND 5 GIVE A TITLE TO THE MODEL;
3      NOTE CRIME DATA WITH ONE SUPPLEMENTAL MARGIN;
4      NOTE R + C MODEL;
5      NOTE NONIGNORABLE NONRESPONSE I;
6      NOTE;
7      NOTE LINES 8 - 25 GIVE THE MODEL MATRIX;
8      X=1 -1 1 -1 1 -1 1 -1 1 -1 1/
9          1 -1 1 0 0 0 0 0 1 0 0/
10         1 -1 1 1 1 1 -1 -1 1 1 1/
11         1 0 0 -1 1 0 0 0 1 -1 1/
12         1 0 0 0 0 0 0 0 1 0 0/
13         1 0 0 1 1 0 0 0 1 1 1/
14         1 1 1 -1 1 -1 -1 1 1 -1 1/
15         1 1 1 0 0 0 0 0 1 0 0/
16         1 1 1 1 1 1 1 1 1 1 1/
17         1 -1 1 -1 1 -1 1 -1 0 0 0/
18         1 -1 1 0 0 0 0 0 0 0 0/
19         1 -1 1 1 1 1 -1 -1 0 0 0/
20         1 0 0 -1 1 0 0 0 0 0 0/
21         1 0 0 0 0 0 0 0 0 0 0/
22         1 0 0 1 1 0 0 0 0 0 0/
23         1 1 1 -1 1 -1 -1 1 0 0 0/

```

```

24      1  1  1  0  0  0  0  0  0  0  0/
25      1  1  1  1  1  1  1  1  0  0  0;
26      NOTE LINES 27 - 38 GIVE THE C MATRIX;
27      C=1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0/
28      0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0/
29      0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0/
30      0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0/
31      0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0/
32      0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0/
33      0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0/
34      0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0/
35      0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0/
36      0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0/
37      0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0/
38      0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1;
39      NOTE VECTOR Y LISTS THE OBSERVED FREQUENCIES;
40      NOTE AND MARGINALS;
41      Y=1742/260/66/228/56/31/63/31/10/
42      994/177/76;
43      NOTE VECTOR B CONTAINS INITIAL PARAMETER;
44      NOTE ESTIMATES;
45      B=3.5/-1/.3/-1/.3/.4/- .2/.3/.7/.1/.2;
46      NOTE LINES 50 - 56 SPECIFY INITIAL, MINIMUM AND/;
47      NOTE OR MAXIMUM VALUES FOR SOME OF THE COUNTERS;
48      NOTE AS WELL AS DIVERGENCE/CONVERGENCE TEST;
49      NOTE VARIABLES USED IN THE PROGRAM;
50      MINDIFF=.0001;

```

```
51      MAXDIFF=1000;
52      MINDIV=.000001;
53      CTROUTR=0;
54      STOPOUTR=0;
55      MINCHSQ=.00001;
56      MAXCHSQ=1000;
57      NOTE THE OUTER LOOP OF THE PROGRAM GOES FROM;
58      NOTE LINES 72-131. USING THE OBSERVED FREQUEN-;
59      NOTE CIES, Y, MODEL MATRIX, X, MATRIX, C, AND;
60      NOTE INITIAL ESTIMATES OF THE MODEL PARAMETERS;
61      NOTE B, NEW ESTIMATES OF THE MODEL PARAMETERS;
62      NOTE AND EXPECTED CELL FREQUENCIES ARE;
63      NOTE COMPUTED. THESE ESTIMATES ARE USED IN THE;
64      NOTE NEXT PROGRAM ITERATION. THIS OUTER LOOP;
65      NOTE WILL CONTINUE UNTIL THE CONVERGENCE;
66      NOTE CRITERION IS REACHED, OR WHEN 5000;
67      NOTE ITERATIONS HAVE TAKEN PLACE. THIS OUTER;
68      NOTE LOOP IS MODELED AFTER THE WORK OF ESPELAND;
69      NOTE AND ODOROFF(1985);
70      NOTE IT IS THE EXPECTATION STEP IN THE EM;
71      NOTE ALGORITHM;
72      DO WHILE (STOPOUTR=0);
73      ETA=X*B;
74      H1=EXP(ETA);
75      YHAT=C*H1;
76      ABSYHAT=ABS(YHAT);
77      MINABS=MIN(ABSYHAT);
```

```
78      IF (MINABS<MINDIV) THEN DO;
79      STOPOUTR=2;
80      NOTE STOPPING BECAUSE YHAT TERM TOO SMALL;
81      PRINT YHAT H1 ETA X B;
82      STOP;
83      END;
84      APPFREQ=(C'*(Y#/YHAT))#H1;
85      NOTE LINES 89-112 CONTAIN THE INNER LOOP OF THE;
86      NOTE PROGRAM. THIS IS THE MAXIMIZATION STEP IN;
87      NOTE THE EM ALGORITM AND IS MODELED AFTER THE;
88      NOTE WORK OF RINDSKOPF(1992);
89      CTRINR=0;
90      STOPINR=0;
91      SAVEBINR=B;
92      DO WHILE (STOPINR=0);
93      H=DIAG(H1);
94      W=INV(H);
95      Z1=(H*ETA)+(APPFREQ-H1);
96      XS=H*X;
97      B=INV(XS'*W*XS)*XS'*W*Z1;
98      ETA=X*B;
99      H1=EXP(ETA);
100     CTRINR=CTRINR+1;
101     DIFFB=SAVEBINR-B;
102     DIFFB=ABS(DIFFB);
103     MAXABS=MAX(DIFFB);
104     IF (MAXABS<MINDIFF) THEN STOPINR=1;
```

```
105      IF (MAXABS>MAXDIFF) THEN DO;
106      STOPINR=2;
107      NOTE STOPPING BECAUSE OF DIVERGENCE;
108      PRINT SAVEBINR B;
109      STOP;
110      END;
111      IF (CTRINR>=5) THEN STOPINR=1;
112      END;
113      CTROUTR=CTROUTR+1;
114      IF (CTROUTR>=5000) THEN STOPOUTR=1;
115      IF (CTROUTR>1) THEN CHISQO=CHISQ;
116      MU=C*H1;
117      LIK=SUM(Y'*LOG(MU));
118      LIKO=SUM(Y'*LOG(Y));
119      CHISQ=2*(LIKO-LIK);
120      IF (CTROUTR>1) THEN DO;
121          DIFFCHSQ=CHISQO-CHISQ;
122          DIFFCHSQ=ABS(DIFFCHSQ);
123          IF (DIFFCHSQ<MINCHSQ) THEN STOPOUTR=1;
124          IF (DIFFCHSQ>MAXCHSQ) THEN DO;
125              NOTE STOPPING BECAUSE OF DIVERGENCE;
126              PRINT CHISQO CHISQ;
127              STOPOUTR=2;
128              STOP;
129          END;
130      END;
131      END;
```

```
132     NOTE THE PROGRAM NOW PRINTS THE NUMBER OF PASSES;  
133     NOTE THROUGH THE OUTER LOOP, THE MAXIMUM;  
134     NOTE LIKELIHOOD CHI SQUARE, DEGREES OF FREEDOM;  
135     NOTE FINAL PARAMETER ESTIMATES, AND ESTIMATED;  
136     NOTE CELL FREQUENCIES AND MARGINALS;  
137     NOTE # OF ITERATIONS FOR 'E' STEP;  
138     NOTE PRINT CTROUTR;  
139     DF=NROW(Y)-NROW(B);  
140     PRINT CHISQ DF;  
141     NOTE B MATRIX (FINAL);    PRINT B;  
142     NOTE H1 MATRIX (ESTIMATED CELL PROBABILITIES);  
143     PRINT H1;  
144     NOTE MU MATRIX (ESTIMATED EXPECTED FREQUENCIES);  
145     PRINT MU;
```

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