Order Number 9304652

Proof of universality for critical circle mappings

de Faria, Edson, Ph.D.

City University of New York, 1992

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

PROOF OF UNIVERSALITY FOR CRITICAL CIRCLE MAPPINGS

by

EDSON DE FARIA

A dissertation submitted to the Graduate Faculty in Mathematics

in partial fulfillment of the requirements for the degree of Doctor of Philosophy,

The City University of New York.

1992

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

9/23/92
Date

Dennis Sullivan
Chair of Examining Committee

9/23/92
Date

Executive Officer

Frederick P. Gardiner

John A. Velling

Richard Sacksteder

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

PROOF OF UNIVERSALITY FOR CRITICAL CIRCLE MAPPINGS

by

Edson de Faria

Adviser: Professor Dennis P. Sullivan

In this thesis we employ techniques from quasiconformal mappings and Teichmüller theory to establish a contraction property for the renormalization operator acting on critical circle mappings with a cubic-exponent singularity and rotation number of bounded combinatorial type. As a consequence, we derive the so-called golden mean universality conjecture: the successive scaling ratios of a critical circle mapping as above with rotation number equal to the golden number converge to a universal constant.

ACNOWLEDGEMENTS

This thesis would have never been written if not for the financial support from FAPESP (São Paulo, Brazil).

I thank Fred Gardiner, Richard Sacksteder and John Velling for reading the manuscript, pointing several mistakes and kindly suggesting improvements, in contents as well as in form. I thank Oscar Lanford and Henri Epstein for teaching me about renormalization, as well as for listening with so much patience to what I had to say about fish, bow-ties and holomorphic commuting pairs.

My heartfelt thanks to those who encouraged me the most, among them Charles Tresser, Fred Gardiner, Michael Benedicks, Nils Tongring, Michael Samra, Marina Saadia-Otero, André de Carvalho, Elizabeth Schneider and Sevin Yaraman in New York, as well as Diva Brunieri, Zaqueu Coelho, Raquel de Jesus and José Carlos F. Rodrigues in São Paulo.

I thank also Burton Randol, Jozef Dodziuk, Martin Moskowitz and everybody else at CUNY for their help and support. Special thanks go to Walter V. Fernandes for his excellent typing job.

Throughout this journey, I was extremely fortunate to be guided by Dennis Sullivan, truly one of the greatest *mathemagicians* of our time. This thesis is a very modest attempt, albeit an honest one, at translating a piece of his magic into somewhat more accessible terms.

I dedicate this work to the memory of my grandfather

Alfredo de Faria
(1910 - 1991)

TABLE OF CONTENTS

INTRODUCTION

In this thesis our purpose is to verify certain renormalization conjectures in the context of critical circle mappings.

By critical circle mapping we understand a *smooth*, orientation-preserving self-homeomorphism $f : \mathbf{T}^1 \hookleftarrow$ having a unique critical point $c \in \mathbf{T}^1$ *of cubic type*. The italicized terms are taken to mean that we can write $f = h \circ Q$ where h is a C^{1+z-}-diffeomorphism (Sullivan [S1]) and Q is some standard real analytic model with a unique cubic singularity at c in the usual sense: $Q'(c) = Q''(c) = 0$, $Q'''(c) \neq 0$ (for example, Q could be an element in the Arnold-Herman family $x \mapsto x + \theta - \frac{1}{2\pi}\sin(2\pi x)$ mod. 1).

One conjecture concerns the *asymptotic geometric rigidity* of the forward critical orbit of an f as such. More precisely, for each $n \geq 0$ consider the "scaling" ratio $\mathbf{sr}_n(f) := \mathrm{dist}\,(f^{q_{n+1}}(c), c)/\mathrm{dist}\,(f^{q_n}(c), c)$. Here the rotation number is $\rho(f) = [r_0, r_1, \ldots, r_n, \ldots]$ and $\frac{p_n}{q_n} = [r_0, r_1, \ldots, r_n]$ in its irreducible form. We shall prove the following theorem:

Main Theorem: Let $f, g : \mathbf{T}^1 \hookleftarrow$ be critical circle mappings as above with the *same* rotation number $\rho = [r_0, r_1, \ldots, r_n, \ldots]$ of *bounded combinatorial type* i.e. max $r_n < \infty$. Then $\lim\limits_{n \to \infty} [\mathbf{sr}_n(f) - \mathbf{sr}_n(g)] = 0$.

This theorem will be a corollary to the *renormalization convergence and contraction* results that we proceed to explain. (Theorems A and B.)

Given f as above, it will be more convenient to deal with its (weakly) commuting pair representation $\zeta_f = (\xi_f, \eta_f)$ (see Lanford [L1] or Rand [Ra1]) as explained in

Chapter I. *Renormalization* is well-defined in the class of all commuting pairs by $\mathcal{R}(\xi,\eta) = (\overline{\eta}, \overline{\eta^r} \circ \overline{\xi})$ whenever $\rho(\xi,\eta) = [r, r_1, \ldots, r_n, \ldots]$, where $\overline{\eta}$ and $\overline{\xi}$ are just η and ξ linearly rescaled by the factor $\lambda = \xi(0)/\eta(0) < 0$.

A (strictly) commuting pair (ξ,η) is said to be in the Epstein-Lanford class $\mathcal{EL}$ if and only if $\xi \equiv h_\xi \circ Q$ and $\eta \equiv h_\eta \circ Q$ where Q is the cubic polynomial $z \mapsto z^3$ and h_γ^{-1} is schlicht in the whole upper half-plane, for $\gamma = \xi, \eta$.

We have then:

Theorem A: Given any critical circle mapping f as above, the sequence $\{\mathcal{R}\zeta_f\}_{n\geq 0}$ of successive renormalizations of its associated weakly commuting pair ζ_f converges exponentially to the class $\mathcal{EL}$ in the $C^{1+\beta}$ metric topology, for any $0 < \beta < 1$.

This theorem will be proved in Chapter I.

Theorem B: Given $\zeta_f, \zeta_g \in \mathcal{EL}$ corresponding to critical circle mappings f, g with the same rotation number ρ of *bounded combinatorial type*, there exists a $C^{1+\alpha}$ conjugacy $\zeta_f \sim \zeta_g$ for some $0 < \alpha < 1$ depending only on ρ.

These two theorems combined yield easily the main theorem. In order to prove Theorem B, however, we have to break it into two further statements.

It is at this point that we come in fact to the cornerstone of this work: the notion of a *holomorphic commuting pair*, a complex analytic dynamical system that plays the same role here as do quadratic-like mappings in [S1]. Holomorphic commuting pairs are introduced, and their relevant properties established, in Chapter II. Among then is a pull-back theorem that allows us to transfer the real bounds of Chapter I to the complex domain.

Using Sullivan's powerful sector theorem (Sullivan [S1]) we prove in Chapter III that the class of all holomorphic commuting pairs "contains" the *attractor* of renormalization, i.e.:

Theorem $\mathbf{B_1}$: Given $\zeta_f \in \mathcal{EL}$ such that $\rho(f)$ is of bounded combinatorial type, there exists $N = N(\zeta_f)$ such that, for all $n \geq N, \mathcal{R}^n\zeta_f$ extends to a holomorphic commuting pair (with bounds).

In Chapter IV, on the other hand, by building a *compact Riemann surface lamination* from a given holomorphic commuting pair, considering its Teichmüller space and the natural interplay between the corresponding Teichmüller metric and the quasi-conformal conjugacy metric (on holomorphic commuting pairs), we obtain through the bounds of Chapter III and Sullivan's *almost geodesic lemma* (Sullivan [S1]):

Theorem $\mathbf{B_2}$: The quasi-conformal conjugacy metric on holomorphic commuting pairs is exponentially contracted by renormalization.

Given that the quasi-conformal conjugacy metric in question dominates the quasi-simmetric conjugacy metric between underlying real commuting pairs, and given that the conjugacies between $\mathcal{R}^n\zeta_f$ and $\mathcal{R}^n\zeta_g$ ($n \geq 0$) are obtained from the conjugacy h between ζ_f and ζ_g merely by restriction and affine rescaling, we deduce, combining theorems B_1 and B_2 with a classical theorem of Carleson [Crl], that h is indeed $C^{1+\alpha}$ for some $\alpha = \alpha(\rho) < 1$, whence Theorem B.

As a corollary to the main theorem, we obtain the *golden mean universality conjecture* (Mestel [Mes], Lanford [L1], [L2], Rand [Ra1], [Ra2], Feigenbaum, Kadanoff

& Shenker [FKS], McKay [McK]):

Corollary: If f is a critical circle mapping as above with $\rho(f) = [1, 1, \ldots, 1, \ldots] = \frac{\sqrt{5}-1}{2}$ (the golden mean) then its scaling ratios $\mathbf{sr}_n(f)$ converge to a universal constant $\lambda = 0.7760513\ldots$.

In fact, we obtain more generally the universality of such scaling ratios whenever the rotation number is a quadratic algebraic number, i.e., has an eventually periodic continued-fraction development.

CHAPTER I
The A-priori Real Renormalization Bounds

In this chapter we prove, for C^{1+z} smoothness, certain compactness results due originally for C^3 critical circle mappings to Swiatek [Sw1], [Sw2], Yoccoz [Y] and Lanford [L2]. The results as presented are straight-forward adaptations of certain ideas of Sullivan [S1].

We work in the realm of *generalized* circle mappings or *weakly commuting pairs* in the sense of Lanford [L1]. After certain preliminaries, which include the relevant definition of renormalization operator, we introduce the Epstein-Lanford class $\mathcal{EL}$ and then prove the crucial Theorem A of the introduction, according to which $\mathcal{EL}$ contains all limits of renormalization. In particular, we derive *beau* bounds (in the sense of Sullivan [S1]) on the *scaling ratios* of a smooth critical circle mapping, which are fundamental to the complex bounds of Chapter III (cf. axioms 1 and 2 in III §2).

I.1 Commuting Pairs and Renormalization

Consider $f : \mathbf{T}^1 \to \mathbf{T}^1$ a critical circle homeomorphism as defined at the introduction, let $c \in \mathbf{T}^1$ be its critical point and $\rho(f) = [r_0, r_1, \ldots, r_n, \ldots]$ be its (irrational) rotation number. Write, as usual, $(q_n)_{n\geq 0}$ for the successive closest return times given recursively by $q_{n+1} = r_n q_n + q_{n-1}$ (with $q_0 = 1$, $q_1 = r_0$).

Denote by $I_n(c)$ the closed interval in $\mathbf{T}^1$ with endpoints c and $f^{q_n}(c)$ containing $f^{q_{n+2}}(c)$. The dynamical first return map to the interval $I_n(c) \cup I_{n+1}(c)$ is given by $f^{q_{n+1}}$ on $I_n(c)$ and by f^{q_n} on $I_{n+1}(c)$. The pair $(f^{q_n}, f^{q_{n+1}})$ is an example of what one calls a *commuting pair*, after Lanford [L₁], [L₂] or Rand [Ra1], [Ra2]. Here is a formal definition.

Definition 1.1: A weakly commuting pair $\zeta = (\xi, \eta)$ consists of two orientation preserving differentiable homeomorphisms $\xi : I_\xi \to \xi(I_\xi), \eta : I_\eta \to \eta(I_\eta)$ where: (a) $I_\xi = [\eta(0), 0] \subseteq \mathbb{R}$, $I_\eta = [0, \xi(0)] \subseteq \mathbb{R}$; (b) both ξ and η have homeomorphic extensions to interval neighborhoods of their corresponding domains with the same smoothness properties, and such extensions commute, i.e., $\xi \circ \eta = \eta \circ \xi$, wherever both sides are defined; (c) $\xi \circ \eta(0) \in I_\eta$; (d) $\xi'(0) = 0 = \eta'(0)$, but $\xi'(x) \neq 0 \neq \eta'(y)$, $\forall x \in I_\xi - \{0\}$, $\forall y \in I_\eta - \{0\}$.

In Lanford's terminology, if ξ and η are both *real analytic* then ζ is a *strictly* commuting pair.

If f is as above then $\zeta_f^{(n)} = (\xi_f^{(n)}, \eta_f^{(n)}) := (f^{q_n}, f^{q_{n+1}})$ is a weakly commuting pair up to orientation, for all $n \geq 0$. Notice that, since f has an hQ decomposition (Introduction), the same is true of $\xi_f^{(n)}$ for all $n \geq 0$, i.e., $\xi_f^{(n)} = h_\xi^{(n)} \circ Q_\xi^{(n)}$ with

$h_\xi^{(n)} \in C^{1+z}$ and $Q_\xi^{(n)}$ a cubic polynomial and similarly for $\eta_f^{(n)}$, for all $n \geq 0$.

Conversely, given $\zeta = (\xi, \eta)$ we define a critical circle homeomorphism by regarding I_n as our circle (identifying 0 and $\xi(0)$) and letting $f_\zeta : I_n \hookleftarrow$ be given by:

$$f_\zeta(x) = \begin{cases} \xi \circ \eta(x), & \text{if } x \in [0, n^{-1}(0)) \\ n(x), & \text{if } x \in [n^{-1}(0), \xi(0] \end{cases}.$$

This map becomes a critical circle homeomorphism provided a suitable smooth structure for I_η is chosen and the glueing of 0 to $\xi(0)$ is performed accordingly, and provided ξ and η have hQ decompositions as explained above.

We let $\rho(\zeta) := \rho(f_\zeta)$ be the rotation number of ζ.

We are ready to define the renormalization operator.

Definition 1.2: If $\rho(\zeta) = [r, r_1, r_2, \ldots, r_n, \ldots]$, the weakly commuting pair $\mathcal{R}\zeta := (\overline{\eta}, \overline{\eta}^r \circ \overline{\xi})$, where $\overline{\xi}$ and $\overline{\eta}$ are just ξ and η, respectively, rescaled linearly by the factor $\lambda = \xi(0)/\eta(0) < 0$, is called the (first) renormalization of ζ.

It is easy to see that $\rho(\mathcal{R}\zeta) = [r_1, r_2, \ldots, r_n, \ldots]$, i.e., renormalization acts as the Gauss map on rotation numbers.

Thus, in the notation introduced above, we have $\zeta_f^{(n)} = \mathcal{R}\zeta_f^{(n-1)}$ for all $n \geq 1$. These are therefore the successive renormalizations of f.

I.2 The Epstein-Lanford Class

We define a class of commuting pairs containing the "attractor" of renormalization.

Definition 2.1: A strictly commuting pair $\zeta = (\xi, \eta)$ is said to be in the Epstein-Lanford class $\mathcal{EL}$ if, for $\gamma = \xi, \eta$, we can write $\gamma = h_\gamma \circ Q_\gamma$, where: (a) $h_\gamma : \gamma(I_\gamma) \hookleftarrow$ is an orientation preserving diffeomorphism; (b) $Q_\gamma : I_\gamma \rightarrow \gamma(I_\gamma)$ is the restriction of a cubic polynomial in $\mathbf{C}$ (with real coeficients); (c) h_γ^{-1} extends to a schlicht mapping $\mathbf{C}(\tilde{I}_\gamma) \rightarrow \mathbf{C}$, where $\tilde{I}_\gamma \supset I_\gamma$ is some open interval.

The class $\mathcal{EL}$ is easily seen to be invariant under the renormalization operator $\mathcal{R}$ introduced in §1. This class is akin to the Epstein class in [S1]. Our main goal in this chapter is to imitate Sullivan in showing that (a) limits of renormalization (in $C^{1+\alpha}$ for all $0 < \alpha < 1$) always exist and have good bounds and (b) such limits are in the Epstein-Lanford class.

More precisely, let $\zeta = (\xi, \eta)$ be a weakly commuting pair with $\rho(\zeta) = [r_0, r_1, \ldots, r_n]$ irrational and consider analogous decompositions $\xi = h_\xi \circ Q_\xi$, $\eta = h_\eta \circ Q_\eta$ to the ones considered above, except that now we assume instead that $h_\gamma : \gamma(I_\gamma) \hookleftarrow$ is such that $\varphi_\gamma := \log|h'_\gamma|$ satisfies the little Zygmund condition:

$$|2\varphi_\gamma(x) - \varphi_\gamma(x+t) - \varphi_\gamma(x-t)| = o(t) \ .$$

Write, as before, $(\xi_n, \eta_n) = \mathcal{R}^n\zeta = \zeta_n$, and consider similar decompositions $\xi_n = h_\xi^{(n)} \circ Q_\xi^{(n)}, \eta_n = h_\eta^{(n)} \circ Q_\eta^{(n)}$.

Theorem 2.2: The families $\{\xi_n = h_\xi^{(n)} \circ Q_\xi^{(n)}\}_{n\geq 0}$, $\{\eta_n = h_\eta^{(n)} \circ Q_\eta^{(n)}\}_{n\geq 0}$ are precompact in the sense that the critical values of $Q_\gamma^{(n)}$ are bounded away from zero and $\{h_\gamma^{(n)}\}_{n\geq 0}$ is precompact in the $C^{1+\alpha}$ topology on diffeomorphisms for $\gamma = \xi, \eta$, for any $0 < \alpha < 1$. Any C^0 limit of $\zeta_n = (\xi_n, \eta_n)$ is a strictly commuting pair in the Epstein-Lanford class. □

CHAPTER II
Holomorphic Commuting Pairs and Renormalization

In this chapter we introduce a class of holomorphic dynamical systems together with a suitably defined renormalization operator acting on it. Such dynamical systems - which we baptize holomorphic commuting pairs - arise as natural extensions to the complex plane of certain real analytic commuting pairs (in the sense of Lanford), which in turn correspond to critical circle homeomorphisms with a singularity of cubic exponent-type. The operator acting on them restricts to the usual real renormalization operator of chapter I, as one might expect.

After presenting a few basic properties of holomorphic commuting pairs, as well as some quasiconformal pre-requisites, we state and prove (§3) a pull-back type theorem that allows us, among other things, to transfer the real renormalization bounds developed in chapter I to this complex analytic context.

Subsequently (§4), we define the Teichmüller space of a holomorphic commuting pair. Its elements correspond to quasiconformal deformations of the given object which are later shown (§5) to be entirely supported in the non-recurrent part of the associated dynamical system. This is accomplished by: first, showing that in some sense all renormalizations coming from mappings in the classical Arnold-Herman family (see M. Herman [H]) are holomorphic commuting pairs; second, proving the desired rigidity properties for that family; and third, using the results of §3 to spread this information over to all holomorphic commuting pairs. Notice that the question of existence of such objects is settled *en passant*.

We close these remarks with a few words on notation that pervades this chapter. If $I \subseteq \mathbb{R}$ is an interval, we write $\mathbf{C}(I)$ to denote $\mathbf{C} - (\mathbb{R} - \text{int}(I))$. An expression such as $f : A \twoheadrightarrow B$ means that f is onto B, and is used whenever we want to emphasize surjectivity somehow. All covering mappings appearing in the text, branched or unbranched, will be regular coverings. Finally, if $A \subseteq \mathbf{C}$ then A^+ denotes the set of all $z \in A$ such that $\text{Im}\, z > 0$, and A^- is similarly defined.

II.1 Holomorphic Commuting Pairs

In order to better isolate the defining properties of holomorphic commuting pairs (below) as well as save some space in their statement, we start with an auxiliary definition.

We shall say that a 4-tuple $(\Delta, U_\xi, U_\eta, U_\nu)$ of simply connected domains in the plane is *special* if the following conditions are satisfied: (a) all four domains are symmetric about the real axis; (b) each U_i $(i = \xi, \eta, \nu)$ is a Jordan domain whose closure is contained in Δ; (c) $\overline{U}_\xi \cap \overline{U}_\eta = \{0\} \subset U_\nu$; (d) the differences $U_i - U_\nu$ and $U_\nu - U_i$ are non-empty connected sets for $i = \xi, \eta$; (e) the interval $U_\xi \cap \mathbb{R}$ lies to the left of zero.

It is clear from these conditions that the intervals $J_i := U_i \cap \mathbb{R}$ $(i = \xi, \eta, \nu)$ are such that: $(\alpha)\overline{J}_\xi \cap \overline{J}_\eta = \{0\} \subset J_\nu$; (β) if $x \in J_\xi$ then $x < 0$ while if $x \in J_\eta$ then $x > 0$; $(\gamma) J_\nu \subset \overline{J}_\xi \cup J_\eta$. Moreover, due to condition (d) we know that $U_\xi \cup U_\eta \cup U_\nu$ is a Jordan domain, the same being true of $U_i \cap U_\nu$ $(i = \xi, \eta)$.

In the sequel we adopt the following convention: we reserve the symbols Δ, U_i, J_i $(i = \xi, \eta, \nu)$ for denoting what they do in the above definition whenever the latter is invoked.

We are now ready to introduce our main object of discussion.

Definition 1.1: A *holomorphic commuting pair* consists of a special 4-tuple $(\Delta, U_\xi, U_\eta, U_\nu)$ together with complex analytic mappings ξ, η, ν having U_ξ, U_η, U_ν respectively as their domains and a positive integer m satisfying the following conditions:

(C_1) All three mappings commute with complex conjugation.

(C_2) $\xi : U_\xi \twoheadrightarrow \Delta \cap \mathbf{C}(\xi(J_\xi))$ and $\eta : U_\eta \twoheadrightarrow \Delta \cap \mathbf{C}(\eta(J_\eta))$ are schlicht.

(C_3) $\nu : U_\nu \twoheadrightarrow \Delta \cap \mathbf{C}(\nu(J_\nu))$ is a 3-fold branched covering with a unique critical point at zero.

(C_4) Both ξ and η have analytic extensions to a certain neighborhood of zero where in fact $\xi \circ \eta$ and $\eta \circ \xi$ are well-defined and we have $\xi \circ \eta(z) = \eta \circ \xi(z) = \nu(z)$ for every z in that neighborhood.

(C_5) If $x \in J_\xi$ then $\xi(x) > x$, while if $x \in J_\eta$ then $\eta(x) < x$; moreover, $\xi(0), \nu(0) \in J_\eta$ whereas $\eta(0) \in J_\xi$.

(C_6) If a is the left endpoint of J_ξ then $\xi^m(a)$ is well-defined and $\xi^m(a) = \eta(0)$, while if b is the right endpoint of J_η then $\eta(b) = \xi(0)$.

A rough sketch of the situation we have in mind is shown in Figure 1.

Remarks:

(1) By condition (C_1), the restrictions $i|J_i$ ($i = \xi, \eta, \nu$) are real-valued. Using conditions (C_2), (C_5) and (C_6) we easily deduce that $\xi|J_\xi$ and $\eta|J_\eta$ are both strictly increasing. This fact yields (when combined with (C_3) and (C_4)) that $\nu|J_\nu$ is strictly increasing also.

(2) Due particularly to conditions (C_4) and (C_5), as well as remark (1), it is clear that the restrictions $\xi|[\eta(0), 0]$ and $\eta|[0, \xi(0)]$ determine a real analytic strictly commuting pair in the sense of Lanford, corresponding by condition (C_3) to a critical circle homeomorphism with a cubic singularity.

(3) Accordingly, and quite naturally, we define the rotation number of a holomorphic commuting pair as the rotation number of its underlying real commuting pair.

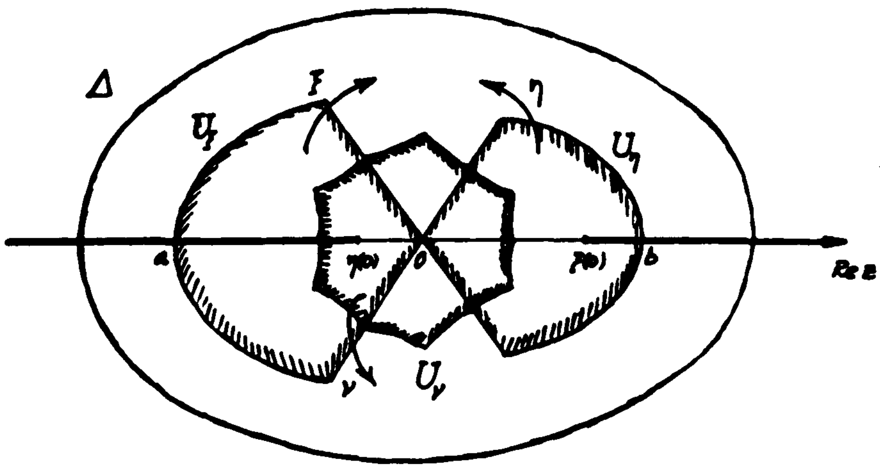

Figure 1

(4) The following comment on condition (C_6) is in order. Since *strictu sensu* $\xi(a)$ (and consequently $\xi^m(a)$) and $\eta(b)$ are not defined (as a, b fall outside the domains of ξ, η respectively), they must be interpreted as limits. In this regard we simply observe that ξ, η, ν all extend continuously to the closures of their original domains by a well-known theorem of C. Carathéodory [C].

(5) The interval $J := \overline{J}_\xi \cup J_\eta$ is clearly forward invariant under the dynamical system generated by ξ and η (after proposition 1.2 it will be clear that J is forward invariant under the action of ξ, η *and* ν). It will be referred to as

the *large* dynamical interval, whereas $I := [\eta(0), \xi(0)]$ will be called the *small* dynamical interval.

(6) The natural, self-posing question of *existence* of an object satisfying the above-stated conditions will be answered by explicit construction in §5.

The following proposition is fundamental.

Proposition 1.2: In any holomorphic commuting pair, the mappings ξ and η have analytic extensions to $U_\xi \cup U_\nu$ and $U_\eta \cup U_\nu$ respectively. Moreover, the restrictions $\xi_* := \xi | U_\nu$ and $\eta_* := \eta | U_\nu$ are 3-fold branched covering maps onto U_η and $U_\xi \cap \mathbf{C}([\xi^{-1} \circ \eta(0), 0])$ respectively, and we have $\eta \circ \xi_* = \xi \circ \eta_* = \nu$.

Proof: Obviously the idea is to use the 3-fold symmetry of U_ν coming from ν in order to extend ξ and η by some kind of Schwarz reflection. At the risk of being pedantic, however, we present a detailed argument.

(i) Over $V := \xi^{-1}(U_\eta)$ the composition $\eta \circ \xi$ is a well-defined schlicht mapping onto $\Delta \cap \mathbf{C}([\eta(0), \eta\xi(0)])$, by condition (C_2).

(ii) Let $Y := \nu^{-1}([\nu(0), +\infty))$. Then by condition (C_3) Y consists of three (analytic) Jordan arcs $\gamma_0, \gamma_1, \gamma_2$ such that $\gamma_i \cap \gamma_j = \{0\}$, each γ_i joining 0 to the boundary of U_ν; by condition (C_1), one of them, say γ_0, is a segment in the real axis, while the other two are mirror images of each other. Accordingly, $U_\nu - Y$ has exactly 3 connected components (Jordan domains), one of which is symmetric about the real axis. Let W be this component; it should be clear that $\partial W \cap U_\nu = \gamma_1 \cup \gamma_2$ (Figure 2).

(iii) Notice also that $V \cap W \neq \emptyset$. We claim that in fact $V = W$. It is enough to show that $V^+ = W^+$, for both V and W are symmetric about the real axis. But ν maps W^+ injectively onto Δ^+ (condition (C_1) and remark (1)); likewise, $\eta \circ \xi$ maps V^+ onto Δ^+ injectively. Hence the composition $\phi := \nu^{-1} \circ (\eta \circ \xi)$ is well-defined in V^+ and maps it onto W^+. Since by condition (C_4) we have $\eta \circ \xi \equiv \nu$ on some neighborhood $\mathcal{O}$ of zero, we must have $\phi(z) = z$ for all $z \in \mathcal{O} \cap V^+$, and so ϕ must be the identity map, whence $V^+ = W^+$. Thus, $V = W$ as claimed.

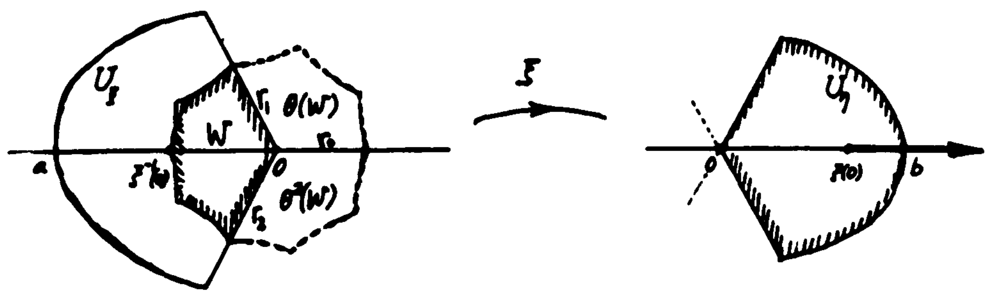

Figure 2

(iv) In particular, $W \cap \mathbb{R} = V \cap \mathbb{R} = (\xi^{-1}(0), 0)$; in other words, $\xi^{-1}(0)$ is the left endpoint of J_ν, and since ν agrees with $\eta \circ \xi$ over all of W, we see that $\nu(\xi^{-1}(0)) = \eta(0)$.

(v) Switching the roles of ξ and η throughout the discussion up to this point, we deduce that $\eta^{-1}(0)$ is the right endpoint of J_ν and that $\nu(\eta^{-1}(0)) = \xi(0)$. Therefore by condition (C_3) the image of U_ν under ν is $\Delta \cap \mathbf{C}([\eta(0), \xi(0)])$ which by condition (C_2) and the last equality of condition (C_6) is precisely the image of U_η under η.

(vi) This in turn implies that $\xi_* := \eta^{-1} \circ \nu : U_\nu \to U_\eta$ is a well-defined (holomorphic) mapping. It is clearly a 3-fold branched covering onto U_η. Since ν agrees with $\eta \circ \xi$ over W, we have $\xi_* \equiv \xi$ there. This takes care of the proposition for ξ; the proof for η is analogous. □

Our first theorem introduces a renormalization operator for holomorphic commuting pairs which exhibits the expected compatibility with the real renormalization operator of chapter I. In the proof we shall use the following elementary set-theoretic remark.

Lemma 1.3: Let $\phi : A \to B$ be a bijection and let $(B_n)_{n\geq 0}$ be the sequence of subsets of B defined as follows: $B_0 = B$ and for $n \geq 0$, $B_{n+1} := \phi(A \cap B_n)$. Then the n-th iterate ϕ^n is well-defined over $A_{n-1} := A - \bigcup_{i=0}^{n-1} \phi^{-i}(B - A)$ and maps A_{n-1} onto B_{n-1} bijectively for each $n \geq 1$. □

Suppose that Γ is a holomorphic commuting pair with rotation number $\rho(\Gamma) = [r, r_1, r_2, \ldots, r_n, \ldots]$. We have:

Theorem 1.4: There exists a holomorphic commuting pair $R(\Gamma)$ whose underlying real commuting pair is the first renormalization of the real commuting pair of Γ.

Proof: Recall from chapter I that the first renormalization of (ξ, η) is the pair $(\eta, \eta^r \circ \xi)$ up to the linear rescaling given by $x \to \lambda x$, where $\lambda = \xi(0)/\eta(0) < 0$. The key combinatorial fact to be employed here is the double inequality $\eta^r \xi(0) > 0 > \eta^{r+1} \xi(0)$ coming from $\rho(\Gamma) = [r, r_1, \ldots, r_n, \ldots]$.

We start by constructing domains $U_{\hat{\xi}}, U_{\hat{\eta}}, U_{\hat{\nu}}$ and corresponding maps $\hat{\xi}, \hat{\eta}, \hat{\nu}$, which will determine the desired $R(\Gamma)$ up to the above-mentioned linear rescaling.

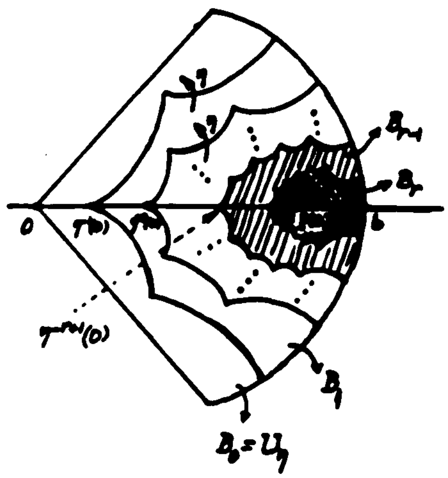

Figure 3

(i) Firstly, take $U_{\hat{\xi}} := U_\eta$ and set $\hat{\xi} := \eta$.

(ii) Let $A := \Delta \cup \mathrm{C}([\eta(0), \xi(0)])$ be the image of $U_\eta =: B$ under η and set $\phi := \eta^{-1} : A \to B$; notice that $B - A = [\xi(0), b) \subset \mathbb{R}$. For each $k \geq 1$ we know by lemma 1.3 that ϕ^k is well-defined over $A_{k-1} = A - \bigcup_{i=0}^{k-1} \phi^{-i}(B - A) = \Delta \cap \mathrm{C}([\eta(0), \eta^{k-1}\xi(0)]$ and maps it bijectively onto its image $B_{k-1} \subset U_\eta$. By the key combinatorial fact at the start of this proof we have $\eta(0) < \eta^{k-1}\xi(0)$ provided $1 \leq k \leq r+1$. Therefore, for such values of k, A_{k-1} is a simply-connected domain (symmetric about the real axis) and since ϕ^k is schlicht we see that the same is true of B_{k-1}.

(iii) But more is true: B_{k-1} is a Jordan domain for each $k \leq r+1$. Indeed, $B_{k-1} -$

$A \subset B - A = [\xi(0), b)$; on the other hand $\eta^k([\xi(0), b)) = [\eta^k(\xi(0)), \eta^{k-1}(\xi(0))) \subset A_{k-1}$, and so $[\xi(0), b) \subset B_{k-1}$. Thus $B_{k-1} - A = [\xi(0), b)$. In other words $A \cap B_{k-1}$ is just B_{k-1} minus the slit $[\xi(0), b)$, which is "opened-up" when $\phi = \eta^{-1}$ is applied. Since $B_k = \phi(A \cap B_{k-1})$ and $B_0 = U_\eta$ is a Jordan domain, an easy inductive argument shows that B_{k-1} is a Jordan domain for $1 \leq k \leq r + 1$, as claimed (Figure 3).

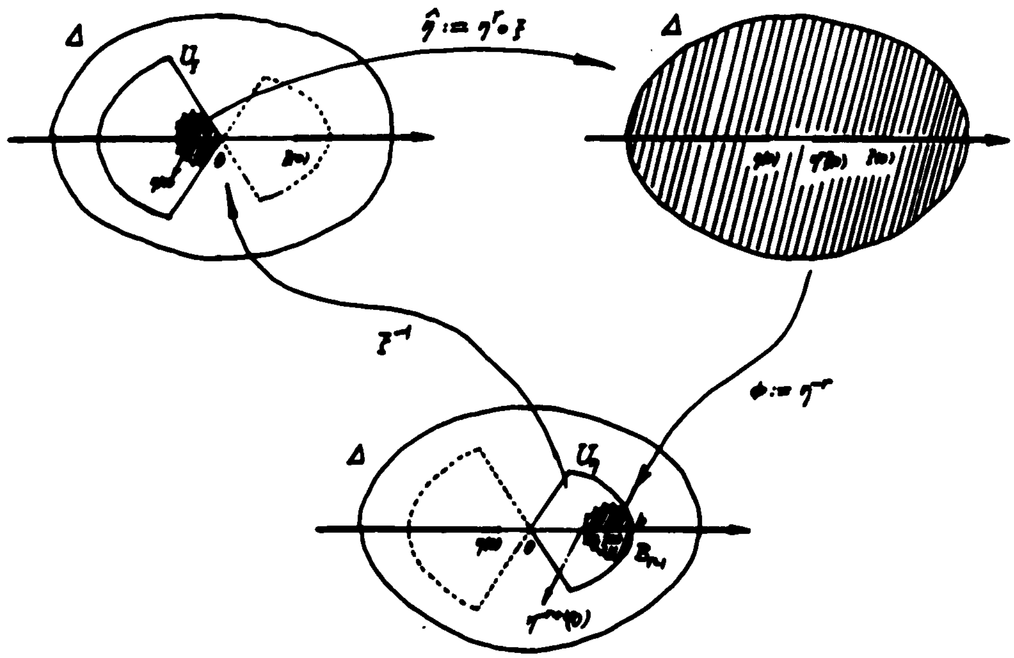

Figure 4

(iv) In particular, $B_{r-1} \subset U_\eta$ is a Jordan domain. Let $\xi_* : U_\nu \to U_\eta$ be the 3-fold branched covering given by proposition 1.2, put $U := \xi_*^{-1}(B_{r-1})$ and define $\hat{\eta}_* : U \to A_{r-1}$ by $\hat{\eta}_* = \eta^r \circ \xi_*$. Since $\xi(0)$ (the critical value of ξ_*) belongs to B_{r-1}, we see that U is a Jordan domain and that $\hat{\eta}_*$ is a 3-fold branched covering onto its image. We then simply take $U_{\hat{\eta}} := U \cap U_\xi$ and set $\hat{\eta} := \hat{\eta}_*|U_{\hat{\eta}}$. [Obs.: $U \cap U_\xi = \xi_*^{-1}(B_{r-1}) \cap U_\xi$ is mapped bijectively by $\hat{\eta}_*$ onto $B_{r-1} - [\xi(0), b)$; thus, $U_{\hat{\eta}}$ is a Jordan domain and $\hat{\eta}$ is schlicht over $U_{\hat{\eta}}$]. See Figure 4.

(v) Finally we define $U_{\hat{\nu}}$ and $\hat{\nu}$ as follows. As we saw in (iii), $B_r \subset U_\eta$ is also a Jordan domain containing $\xi(0)$, the critical value of ξ_*. Hence, if we let $U_{\hat{\nu}} := \xi_*^{-1}(B_r) \subset U_\nu$ and put $\hat{\nu} := \eta^{r+1} \circ \xi_* : U_{\hat{\nu}} \to A_r$, we see at once that $U_{\hat{\nu}}$ is a Jordan domain and that $\hat{\nu}$ is a 3-fold branched covering onto its image. Moreover: $\hat{\nu} = \eta \circ (\eta^r \circ \xi_*) = \hat{\xi} \circ \hat{\eta}_*$.

At last, if we linearly rescale $\Delta, U_{\hat{\xi}}, U_{\hat{\eta}}, U_{\hat{\nu}}, \hat{\xi}, \hat{\eta}, \hat{\nu}$ by the map $z \to \lambda z$ ($\lambda = \xi(0)/\eta(0) < 0$) we get the desired holomorphic commuting pair $R(\Gamma)$. Indeed, it is first of all quite obvious that $(\Delta, U_{\hat{\xi}}, U_{\hat{\eta}}, U_{\hat{\nu}})$ constitutes a special 4-tuple up to linear rescaling. Moreover, conditions (C_1) - (C_3) have all been indirectly checked in the very construction above. Condition (C_4) follows from the last assertion in (v). Condition (C_5) is not satisfied until we do the rescaling (which reverses orientation on the line), when it becomes obvious. As for condition (C_6), we claim it is satisfied if we choose $m = r + 1$, for since b is the right endpoint of $U_{\hat{\xi}} = U_\eta$, we have $\hat{\xi}^{r+1}(b) = \eta^{r+1}(b) = \eta^r \xi(0) = \hat{\eta}(0)$, whereas if we denote the left endpoint of $U_{\hat{\eta}}$ by c then we know that c is mapped by ξ to the left endpoint of B_{r-1} which is easily seen to be $\eta^{-r+1}(0)$ (see Figure 3 once again) and so $\hat{\eta}(c) = \eta^r \circ \xi(c) = \eta^r(\eta^{-r+1}(0)) = \eta(0) = \hat{\xi}(0)$; therefore both equalities in condition (C_6) are taken care of as claimed.

This finishes the proof of the theorem. □

Associated to a given holomorphic commuting pair Γ, the dynamical system which is of interest to us is the one generated by the mappings $\xi|U_\xi, \eta|U_\eta, \nu|U_\nu$ (A "pseudo-semigroup", for lack of a better term). Through an understandable abuse of language we shall identify Γ itself with this dynamical system.

It will be very convenient (§3) to know that the Γ-orbits can be encoded by a single (discontinuous, piecewise holomorphic) transformation. Let $F : (U_\xi \cup U_\eta \cup U_\nu) \to \Delta$ be given by

$$F(z) = \begin{cases} \xi(z) & \text{if } z \in U_\xi \\ \eta(z) & \text{if } z \in U_\eta \\ \nu(z) & \text{if } z \in U_\nu - (U_\xi \cup U_\eta) \end{cases}$$

We call F the *auxiliary transformation* to the holomorphic commuting pair Γ. Its essential features are summarized in the following:

Proposition 1.5: Given a holomorphic commuting pair Γ, consider its auxiliary transformation F and write $D := U_\xi \cup U_\eta \cup U_\nu$ and $X := J \cup F^{-1}(J)$, where J is the large dynamical interval of Γ. Then: (a) $F|(D - X) : (D - X) \twoheadrightarrow \Delta^+ \cup \Delta^-$ is a regular 3-fold covering mapping; (b) F and Γ share the same orbits as sets.

Proof: Part (a) is very easy ($D - X$ consists of six connected components - "patches" - each of which is mapped bijectively onto either Δ^+ or Δ^-; still, one needs proposition 1.2 here).

As for part (b), notice first of all that the F-orbit of any point of D is trivially contained in the corresponding Γ-orbit. For the reverse inclusion, let $z \in D$ and let ω be any finite admissible word in the alphabet $\{\xi, \eta, \nu\}$. If the letter ν does not occur in ω then simply $\omega(z) = F^{|\omega|}(z)$ where $|\omega|$ = length of ω. Otherwise we write

$\omega = \omega_L \nu \omega_R$ for some other words ω_L, ω_R in the same alphabet (possibly empty); setting $x := \omega_R(z)$. we have three possibilities:

(i) $x \in U_\nu - (U_\xi \cup U_\eta)$: in this case $\nu(x) = F(x)$ by definition so we may replace ν by F in ω.

(ii) $x \in U_\xi \cap U_\nu$: here we may write, using proposition 1.2, $\nu(x) = \eta\xi(x) = \eta F(x)$; since by that same proposition the inclusion $\xi(U_\xi \cap U_\nu) \subset U_\eta$ holds, we have $F(x) \in U_\eta$ whence $\eta F(x) = F \circ F(x) = F^2(x)$. Hence in this case we may replace ν by F^2 in ω.

(iii) $x \in U_\eta \cap U_\nu$: same as (ii).

This substitution process applied to all occurences of ν in ω clearly shows that $\omega(z) = F^n(z)$ for some $n \geq |\omega|$, and so part (b) is proved too. □

II.2 Some Quasiconformal Tools

In addition to some of the basic building blocks of quasiconformal theory such as the measurable Riemann mapping theorem, the notion of conformal modulus of an annulus and the compactness principle for quasiconformal mappings (coming from Grötzsch's argument), for which Ahlfors [A1] and Lehto-Virtannen [LV] are standard references, a few auxiliary, more specific results will be needed in §3.

The first of these is a lemma due to L. Bers; we shall refer to it as the *qc-sewing lemma*. For a proof, see Bers [B] or Rickmann [Ric].

Lemma 2.1: Let $\phi : \mathcal{O} \to \phi(\mathcal{O}) \subset \hat{\mathbf{C}}$ be a homeomorphism of an open set $\mathcal{O} \subset \hat{\mathbf{C}}$ onto its image, let $\Lambda \subset \mathcal{O}$ be closed in $\hat{\mathbf{C}}$ and assume that: (a) $\phi|\Lambda$ agrees with the restriction to Λ of a K_1-quasiconformal homeo defined on some neighborhood of Λ; (b) $\phi|(\mathcal{O} - \Lambda)$ is K_2-quasiconformal. Then ϕ is K-quasiconformal with $K \leq \max\{K_1, K_2\}$. □

Recall that a *K-quasidisk* is the image of a round disk in the extended complex plane under a global Kquasiconformal mapping. A *K-quasicircle* is simply the boundary of a K-quasidisk. It is well-known that a Jordan curve $\gamma \subset \hat{\mathbf{C}}$ is a K-quasicircle iff it admits a *K′-quasiconformal reflection* (a sense-reversing K'-qc involution $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$ fixing γ pointwise) where K and K' are bounded in terms of universal functions of each other. Another extremely important characterization in terms of quasisymmetric mappings is provided by the Bers embedding idea. See the above-mentioned references.

Proposition 2.2: Let C_0, C_1 be disjoint K-quasicircles in the Riemann sphere and let Ω be the doubly connected region they determine. Then there exists a quasiconformal homeo $\psi : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ which is conformal on Ω, maps C_0, C_1 onto round circles and whose maximal dilatation depends only on K and $\operatorname{mod}\Omega$.

Proof: Let $A_r := \{z : r < |z| < 1\}$ where $\log(r^{-1}) = 2\pi \operatorname{mod}\Omega$. Then $\operatorname{mod} A_r = \operatorname{mod}\Omega$ and so we know that there exists a conformal equivalence $\rho : A_r \to \Omega$; we may assume that ρ maps $\partial\mathbb{D}$ onto C_0 (where $\mathbb{D} = \{z : |z| < 1\}$).

Call Q the (simply-connected) component of $\hat{\mathbf{C}} - C_0$ containing C_1, and let $z_0 \in Q - \Omega$ be fixed. Let $\phi_{in} : \mathbb{D} \to Q$ be a Riemann mapping partially normalized so that $\phi_{in}(0) = z_0$, and let $\phi_{out} : \hat{\mathbf{C}} - \overline{\mathbb{D}} \to \hat{\mathbf{C}} - \overline{Q}$ be a Riemann mapping between the respective complements.

Claim: $h := (\phi_{in}^{-1} \circ \rho)|\partial\mathbb{D} : \partial\mathbb{D} \to \partial\mathbb{D}$ is quasisymmetric and its quasisymmetric distortion depends only on $\operatorname{mod}\Omega$.

Assuming this claim for a moment, we finish up the proposition's proof as follows. Since C_0 is a K-quasicircle, the boundary composition $\phi_{out}^{-1} \circ \phi_{in} : \partial\mathbb{D} \to \partial\mathbb{D}$ is quasisymmetric, with qs-distortion depending only on K; this is contained in the above-mentioned Bers embedding idea. Thus, $(\phi_{out}^{-1} \circ \phi_{in}) \circ h : \partial\mathbb{D} \to \partial\mathbb{D}$ is quasisymmetric as well, with qs-distortion depending only on K and $\operatorname{mod}\Omega$.

Consider the Ahlfors-Beurling extension $H : \hat{\mathbf{C}} - \mathbb{D} \to \hat{\mathbf{C}} - \mathbb{D}$ of this last map to the outside of the unit disk; its maximal dilatation is still bounded in terms of K and $\operatorname{mod}\Omega$ only. Let ψ map $\hat{\mathbf{C}} - Q$ onto $\hat{\mathbf{C}} - \mathbb{D}$ by setting it equal to $(\phi_{out} \circ H)^{-1}$; then $\psi|\partial Q \equiv \rho^{-1}|\partial Q$ and so ψ fits with the conformal equivalence $\rho^{-1} : \Omega \to A_r$. Defining ψ over $Q - \Omega$ is completely analogous.

We now prove the claim. The composition $\phi_{in}^{-1} \circ \rho$ is well-defined over A_r and maps it conformally onto another ring domain $A \subset \mathbb{D}$ having $\partial\mathbb{D}$ as its outer boundary (its inner boundary being some quasicircle inside $\mathbb{D}$). Therefore $\phi_{in}^{-1} \circ \rho$ can be extended by Schwarz reflection to a conformal mapping between two annular regions symmetric about $\partial\mathbb{D}$; let σ denote such extension.

Set $\delta := \inf\{d(z, \partial\mathbb{D}) : z \in \mathbb{D} - A\} > 0$. Notice that since $\phi_{in}(0) = z_0 \in Q - \Omega$, we have $0 \notin A$. Thus, by a classical result due to O. Teichmüller, we know that $\operatorname{mod}\Omega = \operatorname{mod} A \leq \frac{1}{2\pi}\log\Psi(\frac{\delta}{1-\delta})$ where Ψ is a universal monotone increasing function (that may be explicity computed by means of elliptic functions); once again, see Ahlfors [A1]. Accordingly we deduce that δ is bounded from below by some universal function of $\operatorname{mod}\Omega$. This says that both domain and range of σ are definitely "thick" annuli. Applying Koebe's distortion lemma to disks of a definite size around $z \in \partial\mathbb{D}$ and its image $\sigma(z) = h(z) \in \partial\mathbb{D}$ we get easily that $|h'(z)|$ is bounded above and below by some universal function of $\operatorname{mod}\Omega$ for every $z \in \partial\mathbb{D}$, whence the claim. □

As an easy consequence, we have:

Corollary 2.3: Let $Q_0, Q_1 \subset \mathbf{C}$ be K-quasidisks, symmetric with respect to the real axis and satisfying $\overline{Q}_0 \subset Q_1$. Then the Jordan regions $(Q_1 - Q_0)^{\pm}$, $Q_0 \cup Q_1^{\pm}$ are all K'-quasidisks with K' depending only on K and $\operatorname{mod}(Q_1 - Q_0)$. □

It is only to the extent of the above corollary that we shall use proposition 2.2 in §3, so in this regard it may be viewed as a luxury. We have included a proof mainly because it is not totally trivial. The situation is quite different regarding our final two lemmas, which we state without proof.

Lemma 2.4: Let $I_0, I_1 \subset \mathbb{D} \cap \mathbb{R}$ be closed intervals and let $\phi : I_0 \to I_1$ be a K-quasisymmetric homeomorphism. Then ϕ has a K-quasiconformal extension to a self-mapping of $\mathbb{D}$ which is symmetric about the real axis and whose maximal dilatation K depends only on k, $\mathrm{mod}\,(\mathbb{D} - I_0)$, $\mathrm{mod}\,(\mathbb{D} - I_1)$. □

Lemma 2.5: Let A_0, A_1 be disjoint closed arcs in $\partial\mathbb{D}$ and let Q be the oriented conformal quadrilateral determined by $\mathbb{D}$, A_0 and A_1. If $h : \partial\mathbb{D} \to \partial\mathbb{D}$ is a homeomorphism such that $h|(\partial\mathbb{D} - A_i)$ is k-quasisymmetric $(i = 0, 1)$ then h is k'-quasisymmetric with k' depending only on k and $\mathrm{mod}\,Q$. □

II.3 The Pull-Back Theorem

In this section and throughout the remainder of this chapter the following terminology will be employed. Given a domain $\mathcal{O} \subset \mathbf{C}$ symmetric about the real axis, we say that a homeomorphism $\psi : \mathcal{O} \to \psi(\mathcal{O}) \subset \mathbf{C}$ is *symmetric* if it commutes with complex conjugation and satisfies $\psi(\mathcal{O}^+) = [\psi(\mathcal{O})]^+$. A given holomorphic commuting pair Γ is said to have *geometric boundaries* if its associated special 4-tuple $(\Delta, U_\xi, U_\eta, U_\nu)$ is such that $\partial\Delta$ and ∂D are K-quasicircles for some $K \geq 1$, where $D := U_\xi \cup U_\eta \cup U_\nu$. The smallest such K together with the number $\operatorname{mod}(\Delta - D)$ are referred to as the *geometric parameters* of Γ.

Let Γ_0 and Γ_1 be holomorphic commuting pairs and F_0, F_1 be their corresponding auxiliary transformations (§1). Assume that $h : J_0 \to J_1$ is a conjugacy between the restrictions $F_i|J_i$ $(i = 0, 1)$; notice that in view of condition (C_6) of definition 1.1, $h(0) = 0$ necessarily (as well as $m_0 = m_1$). Then we have the following lemma:

Lemma 3.1: Let $\psi : \Delta_0 \twoheadrightarrow \Delta_1$ be any symmetric homeomorphic extension of h. Then there exists a symmetric homeomorphism $\tilde{\psi} : D_0 \twoheadrightarrow D_1$ such that $F_1 \circ \tilde{\psi} = \psi \circ F_0$ which is still an extension of h.

Proof: Writing, as in §1, $X_i := J_i \cup F_i^{-1}(J_i)$ we know by proposition 1.5 that $F_i|(D_i - X_i) : D_i - X_i \twoheadrightarrow \Delta_i^+ \cup \Delta_i^-$ is a regular 3-fold covering map for $i = 0, 1$. Thus we can lift the restriction $\psi|\Delta_0^+ \cup \Delta_0^-$ through the F_i's to get a homeomorphism $\hat{\psi} : D_0 - X_0 \twoheadrightarrow D_1 - X_1$; such lift is uniquely determined if we require in addition that it be symmetric. Since $F_i(X_i) \subset J_i$, and using the fact that F_i is schlicht when restricted to each of the six components of $D_i - X_i$ $(i = 0, 1)$, we deduce easily that

$\hat{\psi}$ extends to a symmetric homeomorphism $\tilde{\psi} : D_0 \twoheadrightarrow D_1$ satisfying (by continuity) $F_1 \circ \tilde{\psi} = \psi_0 \circ F_0$ everywhere. As $\psi | J_0 \equiv h$ and given that J_i is F_i-forward invariant ($i = 0, 1$), we conclude that $\tilde{\psi} | J_0 \equiv h$ also. □

We are now ready to state and prove the main theorem in this chapter.

Theorem 3.2: Let Γ_0, Γ_1 be holomorphic commuting pairs having geometric boundaries and same irrational rotation number and let $h : J_0 \to J_1$ be a k-quasisymmetric conjugacy between the restrictions of Γ_0, Γ_1 to their respective large dynamical intervals. Then there exists a quasiconformal conjugacy $H : \Delta_0 \to \Delta_1$ between Γ_0 and Γ_1 which extends h and whose maximal dilatation depends only on k and on the geometric parameters of both pairs.

Proof: We divide it into several steps.

(i) Firstly observe that if $\psi : \Delta_0 \to \Delta_1$ satisfying the hypothesis of lemma 3.1 is quasiconformal then so is the corresponding lift $\tilde{\psi}$ and the maximal dilatations are equal: $K(\tilde{\psi}) = K(\psi)$. This happens because the F_i's are holomorphic over $D_i - X_i$ while the X_i's are negligible (have two-dimensional Lebesgue measure zero).

(ii) Lemma 2.4 provides us with a symmetric quasiconformal homeomorphism $G : \Delta_0 \to \Delta_1$ extending h and whose maximal dilatation depends only on k and the geometric parameters. Applying lemma 3.1 to $\psi = G$ yields, in face of (i), a symmetric quasiconformal lift $\tilde{G} : D_0 \to D_1$ with $K(\tilde{G}) = K(G)$, still satisfying $\tilde{G} | J_0 \equiv h$. By corollary 2.3, the Jordan domains $(\Delta_i - D_i)^+$ and $D_i \cup \Delta_i^-$ are K'-quasidisks with K' depending only on the geometric parameters;

combining this fact with lemma 2.5, the Riemann mapping theorem and the Ahlfors-Beurling extension theorem we get a quasiconformal homeo $\hat{G} : (\Delta_0 - D_0)^+ \twoheadrightarrow (\Delta_1 - D_1)^+$ such that $\hat{G} \equiv \tilde{G}$ over $(\partial D_0)^+$ and $\hat{G} \equiv G$ over the remaining part of the boundary of $(\Delta_0 - D_0)^+$. Then, let $H_1 : \Delta_0 \to \Delta_1$ be given by:

$$H_1(x) = \begin{cases} \tilde{G}(z) & \text{if } z \in \overline{D_0} \\ \hat{G}(z) & \text{if } z \in \overline{(\Delta_0 - D_0)^+} \\ \sigma(\hat{G}(\sigma z)) & \text{if } z \in (\Delta_0 - D_0)^- \end{cases}$$

(where $\sigma : \mathbf{C} \to \mathbf{C}$ is complex conjugation). This map is a quasiconformal homeomorphism (once again the boundaries involved are negligible) with $K(H_1) = \max\{K(\hat{G}), K(\tilde{G})\}$, a constant still depending only on the geometric parameters and on $k = k(h)$. Moreover, $H_1|J_0 \equiv h$.

(iii) Now we may start the pull-back construction as such. Define inductively a sequence $\{H_n\}$ of symmetric quasiconformal homeomorphisms as follows: for $n = 1$ use the same H_1 constructed in (ii) while for $n > 1$ let $H_n : \Delta_0 \twoheadrightarrow \Delta_1$ be given by

$$H_n(z) = \begin{cases} H_1(z) & \text{if } z \in (\Delta_0 - D_0) \\ \widetilde{H}_{n-1}(z) & \text{if } z \in D_0 \end{cases}$$

where $\widetilde{H}_{n-1} : D_0 \twoheadrightarrow D_1$ is the lift that we obtain applying lemma 3.1 to $\psi = H_{n-1}$ (it should be clear that $\widetilde{H}_1(z) = H_1(z)$ for each $z \in \partial D_0$; it follows inductively that each H_n is indeed well-defined and a homeomorphism). From (i) we deduce that H_n is a symmetric quasiconformal homeo with $K(H_n) = K(H_1)$ for every n. Also, $H_n|J_0 \equiv h$ for every n.

Therefore $\{H_n\}$ is a normalized sequence of uniformly ($\leq K(H_1)$) quasiconformal mappings and so (by the compactness principle) $\{H_n\}$ is relatively compact in the quasiconformal topology. Accordingly, let $H_\infty : \Delta_0 \to \Delta_1$ be a limit of $\{H_n\}$. Observe that $K(H_\infty) \leq K(H_1)$, and $H_\infty|J_0 \equiv h$ as well.

(iv) Notice that $\{H_n\}$ has the following stabilization property: if $z \in D_0$ then $H_n \circ F_0(z) = F_1 \circ H_n(z)$ if and only if $H_{n+1}(z) = H_n(z)$ (use the fact that F_i is injective (schlicht) on each of the six connected components of $D_i - X_i$, $i = 0, 1$). Let E be the set of all $z \in D_0$ which iterated finitely many times by F_0 either land outside D_0, where $H_n \equiv H_1$ for all n, or land on J_0, which is forward invariant and where $H_n \equiv h$ for all n. Then for every $z \in E$ the sequence $\{H_n(z)\}$ is eventually constant (whence eventually equal to $H_\infty(z)$). By the above-mentioned stabilization property it follows that $H_\infty \circ F_0(z) = F_1 \circ H_\infty(z)$ for all $z \in E$. Since $X_0 \subset E$, we have $(D_0 \cap \overline{E}) - E \subset D_0 - X_0$, where F_0 is continuous, and so for all $z \in D_0 \cap \overline{E}$ we must have $H_\infty \circ F_0(z) = F_1 \circ H_\infty(z)$ also.

(v) If $\Omega \subset D_0 - \overline{E} \subset D_0 - X_0$ is a connected component then the restriction $F_0|\Omega$ is schlicht. From this and the fact that E is backward invariant it follows easily that $F_0(\Omega) \subset D_0 - \overline{E}$ is a connected component as well. By induction, the same is true of $F_0^n(\Omega)$ for all $n > 0$. Notice also that for any such Ω, $\partial\Omega \cap X_0$ consists of at most one point. For if $a, b \in \partial\Omega \cap X_0$ are two distinct points then by mapping Ω forward if necessary we may assume that $a, b \in J_0$. Choose $n > 0$ so that the points $F_0^n(a), F_0^n(b) \in J_0$ lie in opposite sides of zero: this is possible by the combinatorics (of rotations). Then $F_0^n(\Omega)$ is a connected open set with one boundary point on each side of zero, whence $F_0^n(\Omega) \cap X_0 \neq \emptyset$, a contradiction.

(vi) Next, suppose there existed an $n > 0$ such that $F_0^n(\Omega) = \Omega$, for some Ω as in (v); there is no loss of generality in assuming that $\Omega \subset \Delta_0^+$. Then there would be a (necessarily schlicht) inverse branch $\Phi : \Delta_0^+ \to \Delta_0^+$ to F_0^n for which $\Phi(\Omega) = \Omega$. Since Δ_0^+ is a Jordan domain, we known by the Denjoy-Wolff theorem (see for instance Sullivan [S3] or Milnor [Mil]) that either there exists a $z \in \Delta_0^+$ such that $\Phi(z) = z$, necessarily attracting because $\Phi(\Delta_0^+) \subset D_0^+ \neq \Delta_0^+$, or there exists a $z \in \partial\Delta_0^+$ such that $\Phi(z) = z$ (Φ extends continously to $\partial\Delta_0^+$). The first possibility is incompatible with $\Phi(\Omega) = \Omega$, for then the points in $\partial\Omega \cap \Delta_0^+$ cannot converge to z under iteration by Φ, whereas the second implies in fact that $z \in J_0$ (the large dynamical interval of Γ_0), which is impossible because the F_0-dynamics there is without periodic points (the rotation number of Γ_0 is irrational).

(vii) From (v) and (vi) we deduce that each connected component of $D_0 - \overline{E}$ is a *wandering domain*, i.e. its forward images under F_0 are pairwise disjoint. Thus we may rephrase the conclusion of (iv) as follows: H_∞ conjugates F_0 and F_1 everywhere except along the grand-orbits of wandering domains.

(viii) Accordingly, we perform a sequence of quasiconformal sewings in order to finally change H_∞ into a global conjugacy between both pairs. Partitioning the connected components of $D_0 - \overline{E}$ into grand-orbit equivalence classes and selecting one representative from each class yields countably many domains $\{\Omega_n\}_{n\geq 1}$. We change H_∞ along the forward F_0-orbit of Ω_1 first. As already remarked in (v), $F_0|\Omega_1$ is schlicht and extends homeomorphically to the closure $\overline{\Omega}_1$; let $\varphi_0 : \overline{\Omega}_1 \to \overline{F_0(\Omega_1)}$ denote this extension: we know by (v) that $\varphi_0(z) = F_0(z)$ for all $z \in \overline{\Omega}_1$ with at most one exception $z_0 \in \partial\Omega_1$.

Similarly, define $\varphi_1 : \overline{H_\infty(\Omega_1)} \to \overline{F_1 H_\infty(\Omega_1)}$ as the homeomorphic extension of F_1 to the closure of $H_\infty(\Omega_1)$, which is obviously a wandering domain for F_1. Once again $\varphi_1 \equiv F_1$ with at most one exception $z_1 \in \partial H_\infty(\Omega_1)$. Let $\phi : \overline{F_0(\Omega_1)} \to \overline{F_1 H_\infty(\Omega_1)}$ be given by $\phi := \varphi_1 \circ H_\infty \circ \varphi_0^{-1}$, then ϕ is a $K(H_\infty)$-quasiconformal homeomorphism and *a priori* agrees with H_∞ over $\partial F_0(\Omega_1)$ except possibly at one point, so by continuity of both maps $\phi \equiv H_\infty$ everywhere along $\partial F_0(\Omega_1)$. Hence, if we set $\psi^{(1)} \equiv H_\infty$ over $\Delta_0 - F_0(\Omega_1)$ and $\psi^{(1)} \equiv \phi$ over $F_0(\Omega_1)$ we get by the qc-sewing lemma 2.1 a $K(H_\infty)$-quasiconformal homeo $\psi^{(1)} : \Delta_0 \to \Delta_1$ which satisfies the conjugacy equation $\psi^{(1)} \circ F_0(z) = F_1 \circ \psi^{(1)}(z)$ for all $z \in (D_0 \cap \overline{E}) \cup \Omega_1$. Repeating this argument with $\psi^{(1)}$ replacing H_∞ and $F_0(\Omega_1)$ replacing Ω_1 we get $\psi^{(2)} : \Delta_0 \to \Delta_1$ and so on: we obtain inductively a sequence $\psi^{(n)} : \Delta_0 \to \Delta_1$ of uniformly ($\leq K(H_\infty)$) quasiconformal mappings. Again by the compactness principle we extract a limit ψ_1 of $\{\psi^{(n)}\}$; feeding this ψ_1 into step (iii) in place of H_1 and once again going to a limit yields a quasiconformal homeo $H_{1,\infty} : \Delta_0 \to \Delta_1$ with $K(H_{1,\infty}) \leq K(H_\infty)$ which is now (in view of the stabilization property in (iv)) a conjugacy between F_0 and F_1 not only on $D_0 \cap \overline{E}$ but also along the full grand-orbit of Ω_1.

Proceeding as in the above paragraph inductively we take care of the full grand-orbits of $\Omega_2, \Omega_3, \ldots$ through partial quasiconformal conjugacies $H_{2,\infty}, H_{3,\infty}, \ldots$ satisfying $K(H_{n,\infty}) \leq K(H_\infty)$ as well as $H_{n,\infty}|J_0 \equiv h$ for every n. Going to a limit one final time yields $H : \Delta_0 \to \Delta_1$, a global quasiconformal conjugacy with $K(H) \leq K(H_\infty)$ which is still an extension of h. The theorem is proved. □

II.4 Deformation Spaces

In this section, we merely mimic, for holomorphic commuting pairs, the standard development of basic Teichmüller theory of Fuchsian groups (see for instance the book by F. Gardiner [G]). We do it in order to establish an important complement to the pull-back theorem of §3, a property which is best explained through an equivalence between two definitions (theorem 4.2 below).

Let Γ be a fixed holomorphic commuting pair and let Δ be its associated outer disk. Let us denote by Def(Γ) the class of all holomorphic commuting pairs which are conjugate to Γ via a symmetric qc-homeomorphism. In Def(Γ), declare Γ_0 to be equivalent to Γ_1 iff there exists a symmetric conformal mapping $\Delta_0 \to \Delta_1$ conjugating Γ_0 to Γ_1.

Definition 4.1: The Teichmüller space of Γ, henceforth denoted Teich(Γ), is the quotient of Def(Γ) by the above equivalence relation.

If $[\Gamma_0], [\Gamma_1] \in$ Teich(Γ), set $d_T([\Gamma_0],[\Gamma_1]) := \inf_H \log K(H)$ where H ranges over all possible symmetric qc-conjugacies between any two representatives $\tilde{\Gamma}_0 \in [\Gamma], \tilde{\Gamma}_1 \in [\Gamma_1]$, and where as before $K(H)$ denotes the maximal dilatation of H. This defines the so-called Teichmüller metric on Teich(Γ).

As is always the case in this framework, an alternative description of Teich(Γ) as an orbit space is available. We start by observing that if G is a group of qc-selfhomeomorphisms of Δ and B^∞ is the unit ball of $L^\infty_{\mathbb{C}}(\Delta)$ then there is a natural action $G \times B^\infty \to B^\infty$ given by:

$$(h,\mu) \mapsto h^*\mu := \frac{\mu_h + (\mu \circ h).\frac{\overline{h}_z}{h_z}}{1 + \overline{\mu}_h.(\mu \circ h).\frac{\overline{h}_z}{h_z}} \ ,$$

which consists of taking the pull-back under h of $\mu \in B^\infty$ viewed as a Beltrami differential (or measurable conformal structure) on Δ. Each $h^* : B^\infty \hookleftarrow$ is a homeomorphism (in fact a biholomorphism) with $(h^*)^{-1} = (h^{-1})^*$.

Let us agree to call a given $\mu \in B^\infty$: (a) symmetric, if μ commutes with complex conjugation, and (b) Γ-invariant if $\mu(\gamma z).\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z)$ for all $z \in \gamma^{-1}(\Delta)$ [i.e., $\gamma^*\mu = \mu$] for $\gamma = \xi, \eta, \nu$.

If we take G to be the group of all symmetric qc-selfhomeos of Δ which commute with Γ and let $M(\Gamma) := \{\mu \in B^\infty : \mu$ is symmetric and Γ-invariant $\}$ then the above G-action on B^∞ restricts to an action $G \times M(\Gamma) \to M(\Gamma)$, as is easily checked. Let us denote by $\mathcal{O}_G(\Gamma)$ the corresponding orbit space.

Such space can be given by the following metric: put $d([\mu_0], [\mu_1]) := \inf \log K(h^{\tilde{\mu}_0} \circ (h^{\tilde{\mu}_1})^{-1})$, where the infimum is taken over all $\tilde{\mu}_0, \tilde{\mu}_1 \in M(\Gamma)$ in the G-orbits of μ_0 and μ_1, respectively. Here h^μ denotes the unique symmetric qc-homeo $\Delta \to \Delta$ with $h^\mu(0) = 0$ and such that $\mu_{h^\mu} := (h^\mu)^*(0) = \mu$; existence and uniqueness are guaranteed by the Measurable Riemann Mapping Theorem (MRMT).

Theorem 4.2: The orbit space $\mathcal{O}_G(\Gamma)$ with the metric d is naturally isomorphic to Teich(Γ) with its Teichmüller metric d_T.

Proof:

(i) Given $\mu \in M(\Gamma)$ and the corresponding $h^\mu : \Delta \hookleftarrow$, let us consider the following objects: (a) the Jordan domains Δ and $U_{\gamma,\mu} := h^\mu(U_\gamma)$, for $\gamma = \xi, \eta, \nu$; (b) the maps $\gamma^\mu := h^\mu \circ \gamma \circ (h^\mu)^{-1} : U_{\gamma,\mu} \to \gamma(U_{\gamma,\mu})$ for $\gamma = \xi, \eta, \nu$, which are holomorphic because μ is Γ-invariant.

Claim: These objects determine a holomorphic commuting pair.

Indeed, all conditions of definition 1.1 are trivially satisfied, except (C_4): we must check that both ξ^μ and η^μ extend holomorphically across some neighborhood of zero, where they ought to commute. The point is that, by proposition 1.2, the map $\eta^{-1} \circ \nu$ is well-defined over U_ν and agrees with ξ over $U_\xi \cap U_\gamma$; therefore $(\eta^\mu)^{-1} \circ \nu^\mu = h^\mu \circ (\eta^{-1} \circ \nu) \circ (h^\mu)^{-1}$ is well-defined over U_{ν^μ} (which is a neighborhood of zero) and agrees with ξ^μ over $U_{\xi^\mu} \cap U_{\nu^\mu}$. One shows in similar fashion that $(\xi^\mu)^{-1} \circ \nu$ is well-defined over U_{ν^μ} and extends η^μ, and it follows at once that both extensions commute on that common part of their domains. This proves the claim.

If we denote by Γ^μ the resulting holomorphic commuting pair then we have just given ourselves the right to write $\Gamma^\mu = h^\mu \circ \Gamma \circ (h^\mu)^{-1}$.

(ii) Observe that if $\tilde{\mu} \in [\mu]$ then there exists $h \in G$ such that $\tilde{\mu} = h^*\mu$, whence $\tilde{\mu} = h^*(h^\mu)^*(0) = (h^\mu \circ h)^*(0)$. Since $\tilde{\mu} = (h^{\tilde{\mu}})^*(0)$ by definition, we must have $h^{\tilde{\mu}} = h^\mu \circ h$, for both members are normalized and the uniqueness part of the MRMT applies.

(iii) Now we simply let $\Phi : \mathcal{O}_G(\Gamma) \rightarrow \mathrm{Teich}(\Gamma)$ be given by $\Phi([\mu]) = [\Gamma^\mu]$. This map is well-defined, for if $\tilde{\mu} \in [\mu]$ then, by (ii), $\Gamma^{\tilde{\mu}} = (h^\mu \circ h) \circ \Gamma \circ (h^\mu \circ h)^{-1}$ for some $h \in G$, and since $h \circ \Gamma \circ h^{-1} = \Gamma$, we have $\Gamma^{\tilde{\mu}} = \Gamma^\mu$. It should be clear that Φ is onto. Moreover, (ii) also allows us to re-define the d-metric as follows: $d([\mu_0],[\mu_1]) = \inf_{h \in G} \log K(h^{\mu_0} \circ h \circ (h^{\mu_1})^{-1})$.

But for each $h \in G$, the map $h^{\mu_0} \circ h \circ (h^{\mu_1})^{-1}$ conjugates Γ^{μ_1} to Γ^{μ_0}. Conversely, if $H : \Delta \hookleftarrow$ is a symmetric, normalized qc-homeo such that $\Gamma^{\mu_1} = H^{-1} \circ \Gamma^{\mu_0} \circ$

H then we have: $(h^{\mu_0})^{-1} \circ H \circ h^{\mu_1} \circ \Gamma \circ (h^{\mu_1})^{-1} \circ H^{-1} \circ h^{\mu_0} = \Gamma$, whence $(h^{\mu_0})^{-1} \circ H \circ h^{\mu_1} \in G$ and so $H = h^{\mu_0} \circ h \circ (h^{\mu_1})^{-1}$ for some $h \in G$. We deduce that $d([\mu_0], [\mu_1]) = d_T(\Phi([\mu_0]), \Phi([\mu_1]))$, which shows that Φ is an isometry and proves the theorem. □

II.5 Existence and Non-Existence through Examples

For each $\theta \in [0,1)$ we let $E_\theta : \mathbf{C} \to \mathbf{C}$ be the entire mapping given by $E_\theta(z) = z + \theta - \frac{1}{2\pi}\sin(2\pi z)$. Since $E_\theta \circ T = T \circ E_\theta$, where T is the translation $z \mapsto z+1$, E_θ is seen to be the lift to the complex plane of a holomorphic self-mapping of the cylinder, $f_\theta : \mathbf{C}/\mathbb{Z} \cong \mathbf{C}^* \hookleftarrow$. Moreover, the restriction $E_\theta|\mathbb{R}$ maps the real axis onto itself and satisfies $E_\theta'(x) \geq 0$ for all $x \in \mathbb{R}$, with equality holding iff $x \in \mathbb{Z}$ (these constitute all the critical points of E_θ). Therefore the restriction $f_\theta|\mathbf{T}^1(\cong \partial\mathbb{D}) : \mathbf{T}^1 \hookleftarrow$ is a critical circle homeomorphism with rotation number, say, $\rho(\theta)$. It is well-known that $\theta \mapsto \rho(\theta)$ is continuous, non-decreasing, maps $[0,1)$ onto itself and is such that the interval $\rho^{-1}(t) \subseteq [0,1)$ degenerates to a point whenever $t \in [0,1) - \mathbf{Q}$ (see M. Herman [H]).

With the family $\{E_\theta\}$ at hand we shall construct in this section examples of holomorphic commuting pairs exhibiting any prescribed set of combinatorial data, i.e., any given rotation number and any given value of the parameter m appearing in condition (C_6) of definition 1.1 (henceforth referred to as the *height* of the corresponding holomorphic commuting pair). When combined with the results of §3, this construction yields two crucial properties of our complex dynamical systems in the cases of interest to us: (a) they have no wandering domains; (b) they carry no invariant Beltrami differentials supported on a positive measure subset of their limit sets.

We divide the required work into several steps. Let us fix θ for the time being and write $\rho(\theta) = [r_0, r_1, \ldots, r_n, \ldots]$. We conform with the notation established in chapter I: thus, in its irreducible form, $\frac{p_n}{q_n} = [r_0, r_1, \ldots, r_{n-1}]$ satisfies $p_0 = 0$, $q_0 = 1$; $p_1 = 1$, $q_1 = r_0$ and for $n \geq 1$, $p_{n+1} = r_n p_n + p_{n-1}$, $q_{n+1} = r_n q_n + q_{n-1}$; moreover, for all $n \geq 0$ we have $\frac{p_{2n}}{q_{2n}} \leq \rho(\theta) \leq \frac{p_{2n+1}}{q_{2n+1}}$, and equality holds at some point iff $\rho(\theta)$ is rational (only the irrational cases will be of any interest to us, however).

1) The pre-image of the real axis under E_θ consists of $\mathbb{R}$ itself together with the family of analytic curves $\gamma_+^{(k)}, \gamma_-^{(k)}$ ($k \in \mathbb{Z}$) given by the solutions $z = x + iy$ to $\operatorname{Im} E_\theta(x + iy) = 0$ or, more explicitly:

$$[\gamma_\pm^{(k)}] : x = k \pm \frac{1}{2\pi} \arccos\left[\frac{-2\pi|y|}{\sinh(2\pi y)}\right] \;;$$

for each $k \in \mathbb{Z}$, $\gamma_+^{(k)}$ and $\gamma_-^{(k)}$ meet at the critical point $c_k = k$, and are both asymptotic to the vertical lines $x = k \pm \frac{1}{2}$ (see Figure 5). Notice that each c_k is a critical point of cubic type.

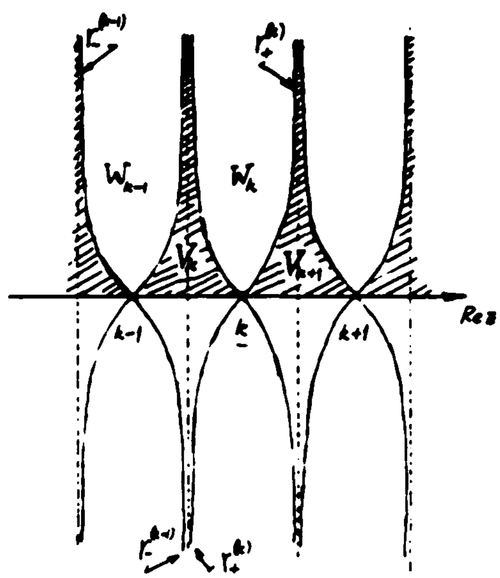

Figure 5

2) In the upper half-plane $\mathbf{C}^+$, let V_k be the simply connected region bounded by the arcs $\gamma_+^{(k-1)} \cap \mathbf{C}^+$ and $\gamma_-^{(k)} \cap \mathbf{C}^+$ plus the interval $[k-1, k] \subseteq \mathbb{R}$. Then $E_\theta | V_k$ is schlicht and maps V_k onto $\mathbf{C}^+$; we let $\phi_k : \mathbf{C}^+ \to V_k$ denote the corresponding inverse mapping, for each $k \in \mathbb{Z}$. Similarly, let $W_k \subseteq \mathbf{C}^+$ be the simply-connected region bounded by $\gamma_-^{(k)} \cap \overline{\mathbf{C}}^+$ and $\gamma_+^{(k)} \cap \mathbf{C}^+$, observe that $E_\theta | W_k$ is schlicht and onto $\mathbf{C}^-$ and let $\psi_k : \mathbf{C}^- \to W_k$ be the corresponding inverse mapping, for each $k \in \mathbb{Z}$.

3) Let $A_n \subseteq \mathbf{C}^+$ be the unique connected component of $(E_\theta^{q_n})^{-1}(\mathbf{C}^+)$ whose closure contains the point $T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}(0) \in \mathbb{R}$. Similarly, let $B_n \subseteq \mathbf{C}^+$ be the unique connected component of $(E_\theta^{q_{n+1}})^{-1}(\mathbf{C}^+)$ such that $T^{-p_n} \circ E_\theta^{q_n}(0) \in \overline{B}_n$. We have either $A_n \subseteq V_0$ and $B_n \subseteq V_1$ or $A_n \subseteq V_1$ and $B_n \subseteq V_0$, depending on whether n is even or odd, respectively (figure 7 illustrates the former case).

Claim # 1: For each $n \geq 0$ there exists a unique q_n-tuple $(k_1, k_2, \ldots, k_{q_n})$ with $0 = k_1 \leq k_2 \leq \cdots \leq k_{q_n} \leq p_n + 1$ such that $A_n = \phi_{k_1} \circ \phi_{k_2} \circ \cdots \circ \phi_{k_{q_n}}(\mathbf{C}^+)$. A similar statement holds for B_n.

Claim # 2: For each $n \geq 1$ we have $\overline{A}_n \cap \mathbb{R} = < \alpha_n, 0 >$ and $\overline{B}_n \cap \mathbb{R} = < 0, \beta_n >$ where the points $\alpha_n, \beta_n \in \mathbb{R}$ are uniquely determined by the requirements: $T^{-p_n} \circ E_\theta^{q_n}(\alpha_n) = T^{-p_{n-1}} \circ E_\theta^{q_{n-1}}(0)$ and $T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}(\beta_n) = T^{-p_n} \circ E_\theta^{q_n}(0)$. On the other hand, for $n = 0$ we have $\overline{A}_0 \cap \mathbb{R} = < \alpha_0, 0 >$, $\overline{B}_0 \cap \mathbb{R} = < 0, \beta_0 >$ where $\alpha_0 = -1$ and $\beta_0 = \alpha_1$ [Obs.: The symbols $< \alpha, \beta >$ represent a closed interval on the line with endpoints α and β, irrespective of order].

The first claim is an easy consequence of the fact that $0 \leq E_\theta^j(0) < p_n + 1$ for $j = 0, 1, \ldots, q_n$, for all $n \geq 0$, which in turn follows from the very definitions of

p_n, q_n. As for the second claim, we have:

Lemma 5.1: Let $f : \mathbf{T}^1 \hookleftarrow$ be a circle homeomorphism with $\rho(f) = [r_0, r_1, \ldots, r_n, \ldots]$ and for $n \geq 1$ let $J_n \subseteq \mathbf{T}^1$ be the closed interval of endpoints c and $f^{q_{n-1}-q_n}(c)$ containing $f^{q_{n-1}}(c)$, where $c \in \mathbf{T}^1$ is given. If $j < q_n$ is such that $f^{-j}(c)$ belongs to J_n, then in fact $j \leq 0$.

Proof: The closed interval $f^{q_n}(J_n)$ contains c (as an interior point) and has $f^{q_{n-1}}(c)$ and $f^{q_n}(c)$ as its endpoints; since these are successive dynamical closest returns to c, there can be no k with both $0 \leq k < q_n$ and $f^k(c) \in f^{q_n}(J_n)$. But the assumptions entail $f^{q_n-j}(c) \in f^{q_n}(J_n)$ with $q_n - j > 0$, whence $q_n - j \geq q_n$ and so $j \leq 0$ as asserted. □

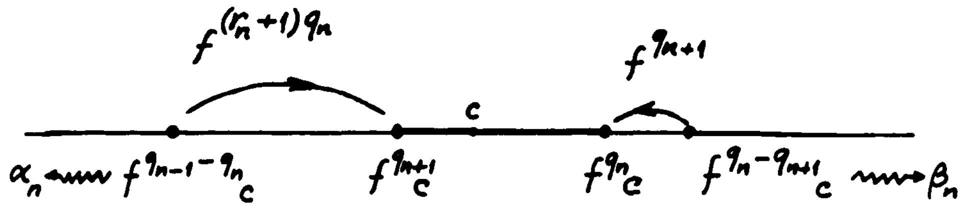

Figure 6

In particular, if $f = f_\theta$ and c happens to be the critical point of f_θ, the lemma says there can be no critical points for $f_\theta^{q_n}$ in the interior of J_n, for by the chain rule these are precisely the pre-images $f_\theta^{-j}(c)$ with $0 \leq j < q_n$. This is the only non-trivial point behind claim # 2.

4) Given $R > 0$, let $\Delta_R = \{z : |z| < R\}$ and let $A_{n,R}$ be the unique connected

component of $(T^{-p_n} \circ E_\theta^{q_n})^{-1}(\Delta_R^+)$ contained in A_n (for each $n \geq 0$). Let $B_{n,R}$ be similarly defined (for each $n \geq 0$). If R is sufficiently large ($R > p_n + 1$ is good enough) we see that $\overline{A}_{n,R} \cap \mathbb{R} = \overline{A}_n \cap \mathbb{R}$ and $\overline{B}_{n,R} \cap \mathbb{R} = \overline{B}_n \cap \mathbb{R}$ for $n \geq 0$. It should be clear that both $A_{n,R}$ and $B_{n,R}$ are Jordan domains in fact quasidisks, and that they are mapped respectively by $T^{-p_n} \circ E_\theta^{q_n}$ and $T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}$ bijectively onto Δ_R^+, for all $n \geq 0$ and all $R > 0$.

Claim # 3: For every sufficiently large R we have $\overline{A}_{n,R} \subseteq \Delta_R \cap \overline{\mathbb{C}^+}$ and $\overline{B}_{n,R} \subseteq \Delta_R \cap \overline{\mathbb{C}^+}$.

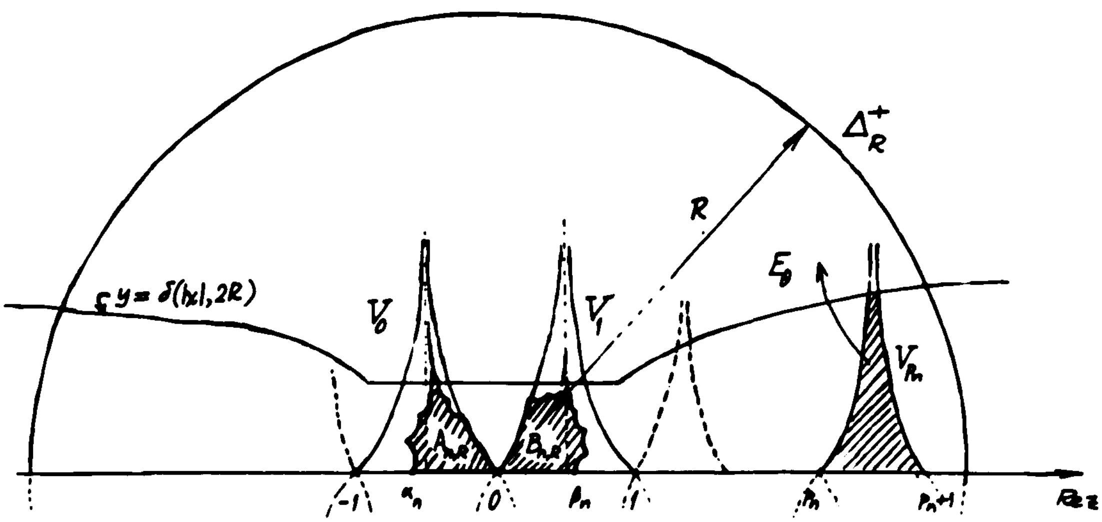

Figure 7

In order to prove this claim we need the following calculus estimate:

Lemma 5.2: There exist a constant $C_0 > 0$ and a positive monotone non-decreasing function $\varphi(s)$ defined for $s \geq 0$ such that if $|y| \geq \varphi(|x|)$ then $|E_\theta(x + iy)| \geq$

$C_0 \exp(\pi|y|)$.

Proof: When $\theta = 0$, a straightforward computation yields:

$$|E_0(x+iy)|^2 = [x^2 + y^2 - \frac{1}{4\pi^2}\cos^2(2\pi x)] - \frac{1}{\pi}[x\sin(2\pi x)\cosh(2\pi y) + y\cos(2\pi x)\sinh(2\pi y)] + \frac{1}{4\pi^2}\cosh^2(2\pi y) .$$

The first expression between brackets is positive as soon as, say, $|y| \geq 1$, while the second is dominated by $(|x| + |y|)\cosh(2\pi y)$. Thus, if $|y| \geq 1$ we have:

$$|E_0(x+iy)|^2 \geq \frac{1}{4\pi^2}[\cosh(2\pi y) - 4\pi(|x| + |y|)]\cosh(2\pi y) \qquad (*)$$

Now, set $\epsilon(t) := \frac{1}{4\pi}\cosh(2\pi t) - t - 1$; this is a strictly convex function having a minimum at $t_0 = \frac{1}{2\pi}\text{arcsinh}(2) > 0$, with $\epsilon(t_0) < 0$. Hence for each $s \geq 0$ there exists a unique $\overline{\varphi}(s) > t_0$ such that $\epsilon(\overline{\varphi}(s)) = s$; given that $\epsilon(t)$ is strictly increasing for $t \geq t_0$, so is $\overline{\varphi}(s)$ for $s \geq 0$, and $t \geq \overline{\varphi}(s) \Rightarrow \epsilon(t) \geq s$. Since the expression between brackets in $(*)$ is equal to $4\pi[\epsilon(|y|)+1-|x|]$, we deduce, setting $\varphi(s) := \max\{1, \overline{\varphi}(s)\}$, that if $|y| \geq \varphi(|x|)$ then:

$$|E_0(x+iy)|^2 \geq \frac{1}{\pi}\cosh(2\pi|y|) \geq \frac{1}{2\pi}\exp(2\pi|y|)$$

or yet:

$$|E_0(x+iy)| \geq \frac{1}{\sqrt{2\pi}}\exp(\pi|y|) . \qquad (**)$$

On the other hand, when $0 < \theta < 1$ we have simply $E_\theta(z) = E_0(z) + \theta$, whence $|E_\theta(z)| \geq |1 - |E_0(z)|^{-1}|.|E_0(z)|$. Therefore, if $|y| \geq \varphi(|x|)$ (which is ≥ 1) then $|E_\theta(x+iy)| \geq \frac{1}{\sqrt{2\pi}}[1 - e^{-\pi}\sqrt{2\pi}]\exp(\pi|y|)$ by $(**)$ above, and so the lemma is proved in all cases if we take $C_0 = \frac{1}{\sqrt{2\pi}}[1 - e^{-\pi}\sqrt{2\pi}]$. □

The proof of claim # 3 now runs as follows. For s, R positive numbers, let $\delta(s,R) := \varphi(s) + \frac{1}{\pi}\log^+(C_0^{-1}R)$ where φ and C_0 are given by lemma 5.2; then $|y| \geq \delta(|x|, R)$ implies $|E_\theta(x+iy)| \geq R$, which in turn means that $E_\theta(x+iy) \in \mathbf{C} - \Delta_R$. Therefore, for each $k \in \mathbb{Z}$ we have

$$\phi_k(\overline{\Delta_R^+}) \subseteq \overline{V}_k \cap \{x+iy : y \leq \delta(|x|, R)\} =: V_{k,R} .$$

Because $\delta(s,R)$ has logarithmic growth in R, every sufficiently large R satisfies the inequality $R > p_n + 1 + \delta(p_n+1, 2R)$; for a given R as such, if $0 \leq k \leq p_n + 1$ and z is any point in $V_{k,2R}$ with $z = x+iy$, then $|z| \leq |x| + \delta(|x|, 2R) \leq p_n + 1 + \delta(p_n+1, 2R) < R$, whence $z \in \Delta_R \cap \overline{\mathbf{C}^+}$. In other words, if $0 \leq k \leq p_n+1$ then $\phi_k(\overline{\Delta_{2R}^+}) \subseteq \Delta_R \cap \overline{\mathbf{C}^+} \subseteq \overline{\Delta_{2R}^+}$. Since $T^{p_n}(\overline{\Delta_R^+}) \subseteq \overline{\Delta_{2R}^+}$ (because $R > p_n$), if we take $(k_1, k_2, \ldots, k_{q_n})$ as in claim # 1 we deduce that $\overline{A}_{n,R} = \phi_{k_1}\circ\phi_{k_2}\circ\cdots\circ\phi_{k_{q_n}}(T^{p_n}\overline{\Delta_R^+}) \subseteq \phi_{k_1}\circ\phi_{k_2}\circ\cdots\circ\phi_{k_{q_n}}(\overline{\Delta_{2R}^+}) \subseteq \Delta_R \cap \overline{\mathbf{C}^+}$, and this proves the first stated inclusion in claim # 3; the second is completely analogous [see figure 7].

Observe that if we define $\mathcal{O}_{n,R} := \phi_{k_2}\circ\phi_{k_3}\circ\cdots\circ\phi_{k_{q_n}}(\Delta_R^+)$ and set $A'_{n,R} := \phi_1(\mathcal{O}_{n,R})$ and $A''_{n,R} := \varphi_0\sigma(\mathcal{O}_{n,R})$, where $\sigma : \mathbf{C} \to \mathbf{C}$ is complex conjugation, then the above argument applies *mutatis mutandis* to yield $\overline{A'_{n,R}} \subseteq \Delta_R \cap \overline{\mathbf{C}^+}$, $\overline{A''_{n,R}} \subseteq \Delta_R \cap \overline{\mathbf{C}^+}$ as well, for every sufficiently large R and all $n \geq 0$. This remark will be used shortly.

5) Given $n \geq 0$, let $R_n > 0$ be chosen so that claim # 3 is satisfied. If we set $\xi_n := T^{-p_n}\circ E_\theta^{q_n}$ and $\eta_n := T^{-p_{n+1}}\circ E_\theta^{q_{n+1}}$ and define $U_{\xi_n}, U_{\eta_n} \subseteq \mathbf{C}$ to be the symmetric Jordan domains (quasidisks) such that $U_{\xi_n}^+ = A_{n,R_n}, U_{\eta_n}^+ = B_{n,R_n}$ then we have: (a) ξ_n and η_n commute (obvious); (b) $\overline{U}_{\xi_n}, \overline{U}_{\eta_n} \subseteq \Delta_{R_n}$, by claim # 3; (c) the restrictions $\xi_n|U_{\xi_n}$ and $\eta_n|U_{\eta_n}$ are schlicht and onto their images, respectively $\Delta_{R_n} \cap \mathbf{C}(<\xi_n(\alpha_n), \xi_n(0)>)$ and $\Delta_{R_n} \cap \mathbf{C}(<\eta_n(0), \eta_n(\beta_n)>)$ by claim # 2 and the

work in step 4 above.

Also, let $U_{\nu_n} \subseteq \mathbf{C}$ be the connected component of $\xi_n^{-1}(U_{\eta_n})$ containing the origin and let $\nu_n := \xi_n \circ \eta_n$. Then the restriction $\nu_n|U_{\nu_n}$ is a holomorphic 3-fold branched covering map onto its image, $\nu_n(U_{\nu_n}) = \Delta_{R_n} \cap \mathbf{C}(<\eta_n(0), \xi_n(0)>)$. Moreover, by the observation made in the last paragraph of 4), we have $\overline{U^+_{\nu_n}} \subseteq \overline{A}_{n,R} \cup \overline{A'_{n,R}} \cup \overline{A''_{n,R}} \subseteq \Delta_{R_n} \cap \overline{\mathbf{C}^+}$, whence $\overline{U}_{\nu_n} \subseteq \Delta_{R_n}$. It follows at once that $(\Delta_{R_n}, U_{\xi_n}, U_{\eta_n}, U_{\nu_n})$ is a special 4-tuple in the sense of §1. At last, we have:

Claim # 4: For each $n \geq 0$ the elements $\Delta_{R_n}, U_{\xi_n}, U_{\eta_n}, U_{\nu_n}, \xi_n, \eta_n, \nu_n$ determine (up to conjugation by $z \to -z$ whenever n is odd) a holomorphic commuting pair $\Gamma_{n,\theta}$ with geometric boundaries, whose rotation number is just $\rho(\Gamma_{n,\theta}) = [r_{n+1}, r_{n+2}, \ldots]$ and whose height is given by $m(\Gamma_{0,\theta}) = r_0$, when $n = 0$, and by $m(\Gamma_{n,\theta}) = r_n + 1$, when $n > 0$.

With the work done so far, we have indirectly verified all items of definition 1.1, except perhaps condition (C_6). We check it for $n > 0$, referring to figure 6, and leave the equally straightforward case $n = 0$ to the reader. Using the commutativity of T and E_θ, claim # 2 and the recurrence relations defining p_{n+1} and q_{n+1}, we get:

$$\begin{aligned} \xi_n^{r_n+1}(\alpha_n) &= (T^{-p_n} \circ E_\theta^{q_n})^{r_n}(T^{-p_n} \circ E_\theta^{q_n}(\alpha_n)) = T^{-(r_n p_n + p_{n-1})} \circ E_\theta^{r_n q_n + q_{n-1}}(0) \\ &= T^{-p_{n+1}} \circ E_\theta^{q_{n+1}}(0) = \eta_n(0) \ . \end{aligned}$$

We deduce in similar fashion that $\eta_n(\beta_n) = \xi_n(0)$. Thus condition ($C_6$) is satisfied too, with $m = r_n + 1$, which is therefore the height of $\Gamma_{n,\theta}$. Finally, the statement on rotation numbers follows immediately from the definition. This establishes claim # 4.

Remark: Because of theorem 1.4, once R_0 is chosen so that the above construction works for $n = 0$, we may in fact take $R_n = R_0$ thereafter. If this is done then, for each $n \geq 0$, $\Gamma_{n+1,\theta}$ becomes the first renormalization of $\Gamma_{n,\theta}$ up to linear rescaling, in the sense of that same theorem.

Summarizing, we have proved the following result.

Theorem 5.3: For each $n \geq 0$ and each $\theta \in [0,1)$ whose corresponding $\rho(\theta)$ has a continued fraction expansion of length at least $n+1$, the real commuting pair determined by $(f_\theta^{q_n}, f_\theta^{q_{n+1}})$ extends to a holomorphic commuting pair $\Gamma_{n,\theta}$ (with geometric boundaries). The family $\{\Gamma_{n,\theta}\}$ runs through all possible pairs of combinatorial invariants at least once, and in fact for each $(m,\rho) \in \mathbb{N} \times [0,1)$ with $m \geq 2$ there exist countably many $(n,\theta) \in \mathbb{N} \times [0,1)$ such that $m(\Gamma_{n,\theta}) = m$ and $\rho(\Gamma_{n,\theta}) = \rho$. $\square$

The construction we've come to perform has much more relevant consequences for our purposes, however, in the form of two rigidity-type properties enjoyed by holomorphic commuting pairs. We shall extract such properties from corresponding ones found naturally in the family $\{f_\theta\}$ introduced above.

Given a holomorphic self-map $f : \mathbf{C}^* \hookleftarrow$ we shall denote by $SV(f)$ the set of singular values of f, i.e., points in $\mathbf{C}^*$ all neighborhoods U of which are such that $f^{-1}(U) \xrightarrow{f} U$ fails to be a covering map. We shall write also $X_f := \mathbf{C}^* - SV(f)$; evidently, $f^{-1}(X_f) \xrightarrow{f} X_f$ is always a covering map. For example, since $1 \in \partial\mathbb{D}$ is the unique critical point of f_θ, it is easy to see that $SV(f_\theta) = \{f_\theta(1)\}$; in this case $f_\theta^{-1}(X_{f_\theta})$ has an (infinite) discrete complement in $\mathbf{C}^*$.

On the other hand, recall that the Julia set of f, denoted by J_f, is the complement in $\mathbf{C}^*$ of the so-called domain of normality of f, which in turn is the set of all points of $\mathbf{C}^*$ for which there exists a neighborhood U where the restrictions $f^n|U, n \geq 0$, constitute a normal family. Thus, J_f is totally invariant and closed as a subset of $\mathbf{C}^*$. Notice also that if $V \subseteq \mathbf{C}^*$ is any neighborhood of a point in J_f then by Montel's theorem we have $\mathbf{C}^* = \bigcup_{n\geq 0} f^n(V)$; in particular, if $\emptyset \neq \Lambda \subseteq \mathbf{C}^*$ then $J_f \subset \overline{\bigcup_{n\geq 0} f^{-n}(\Lambda)}$.

We are ready to state and prove:

Theorem 5.4: For each $\theta \in [0, 1)$ the corresponding map f_θ has no wandering domains. Moreover, whenever $\rho(\theta)$ is irrational, f_θ admits no non-trivial, symmetric, invariant Beltrami differentials entirely supported in its Julia set J_{f_θ}.

Proof: Since $SV(f_\theta)$ is a finite set, the first assertion follows from a theorem due to L. Keen [K]. As for the second, we have:

(i) Let $f : \mathbf{C}^* \hookleftarrow$ be holomorphic and suppose $h : \hat{\mathbf{C}} \hookleftarrow$ is an orientation preserving homeo fixing $\{0, \infty\}$ pointwise and satisfying $h \circ f_\theta = f \circ h$. Let $A \in \mathrm{Aut}(\hat{\mathbf{C}})$ be given by $A(z) = \lambda z$, where $\lambda := h \circ f_\theta(1)/f_\theta(1)$. We have $A \simeq h$ (rel. $SV(f_\theta) \cup \{0, \infty\}$), so the covering homotopy theorem yields $\hat{A} : f_\theta^{-1}(X_{f_\theta}) \to f^{-1}(X_f)$ with $\hat{A} \simeq h$ (rel. $f_\theta^{-1}(SV(f_\theta)) \cup \{0, \infty\}$) such that the diagram:

$$\begin{array}{ccc} f_\theta^{-1}(X_{f_\theta}) & \xrightarrow{\hat{A}} & f^{-1}(X_f) \\ f_\theta \downarrow & & \downarrow f \\ X_{f_\theta} & \xrightarrow{A} & X_f \end{array}$$

commutes. This $\hat{A}$ is clearly holomorphic, and some easy topological considerations combined with Riemann's removable singularity theorem show it to be the restriction of some element in $\mathrm{Aut}(\hat{\mathbf{C}})$ fixing $\{0,\infty\}$ pointwise. In particular, if f is symmetric, i.e. commutes with geometric reflection about $\partial\mathbb{D}$, and is normalized so that its critical point lies at $1 \in \partial\mathbb{D}$ (i.e., $h(1) = 1$) then we have: (a) $\hat{A}$ now fixes $\{0,1,\infty\}$ pointwise and so it must be the identity; (b) $|\lambda| = 1$, say $\lambda = e^{2\pi i\alpha}$. Therefore $f = A \circ f_\theta \circ \hat{A}^{-1} = f_{\theta'}$ with $\theta' \equiv \theta + \alpha$ (mod. 1); in other words, every symmetric, normalized, holomorphic self-map of $\mathbf{C}^*$ which is topologically conjugate to a member of $\{f_\theta\}$ is itself a member.

(ii) Now suppose μ is an f_θ-invariant Beltrami differential in $\hat{\mathbf{C}}$ with $\| \mu \|_\infty < 1$ and $\mathrm{supp}(\mu) \subset J_{f_\theta}$; assume also that μ is symmetric about $\partial\mathbb{D}$ (meaning $\mu(\bar{z}^{-1}) = \overline{\mu(z)}$). For each $t \in (-\| \mu \|_\infty^{-1}, \| \mu \|_\infty^{-1})$ let $h_t : \hat{\mathbf{C}} \hookleftarrow$ be the unique solution to $\bar{\partial} h_t = (t\mu).\partial h_t$ normalized so that it fixes $\{0,1,\infty\}$ pointwise. Set $f_t := h_t \circ f_\theta \circ h_t^{-1}$; since $t\mu$ is symmetric and f_θ-invariant, each $f_t : \mathbf{C}^* \hookleftarrow$ is symmetric and holomorphic, with its unique critical point at $1 \in \partial\mathbb{D}$. Therefore by (i) we must have $f_t = f_{\theta_t}$ for some $\theta_t \in [0,1)$; but then $\rho(\theta_t) = \rho(\theta) \notin \mathbf{Q}$, whence $\theta_t = \theta$ for all t, by the last remark in the very first paragraph of this section. Thus, $f_\theta = h_t \circ f_\theta \circ h_t^{-1}$ for all t; in particular h_t must permute the elements in $Y_n := f_\theta^{-n}(1)$, which is discrete in $\mathbf{C}^*$, for each $n \geq 0$. Since $h_0 = id_{\hat{\mathbf{C}}}$ and for each $z \in \hat{\mathbf{C}}$ the path $t \to h_t(z)$ is continuous by Ahlfors-Bers, we deduce that h_t fixes Y_n pointwise for each $n \geq 0$, for all t. But $\overline{\bigcup_{n\geq 0} Y_n} \supset J_{f_\theta}$ by the observation just preceding the statement of this theorem, whence h_t agrees with the identity over J_{f_θ} for all t. As h_t is conformal off J_{f_θ} (because μ vanishes identically there, by assumption), it follows that $h_t \equiv id_{\hat{\mathbf{C}}}$ for all t, and so, $\mu \equiv 0$ a.e. as was to be proved. □

Remark: This theorem owes its inspiration to a beautiful related argument by D.

Sullivan in [S1].

Finally, combining this result with theorem 5.3, theorem 3.2 and the real bounds of chapter I, we derive:

Corollary 5.5: Let Γ be a holomorphic commuting pair with geometric boundaries and irrational rotation number. Then Γ has no wandering domains and admits no non-trivial, symmetric, invariant line fields entirely supported in its limit set. $\square$.

CHAPTER III
The Complex Bounds

As we have seen in chapter I, in their commuting pair representations, the successive renormalizations of a sufficiently smooth critical circle homeo (with a cubic singularity) converge asymptotically to the Epstein-Lanford class. Restricting ourselves to the bounded combinatorics case, we shall prove in this chapter that, after finitely many renormalizations, every element in the Epstein-Lanford class extends to a holomorphic commuting pair with geometric boundaries and universally bounded geometric parameters (II §3).

The proof of this fundamental compactness result relies heavily on Sullivan's sector theorem (Sullivan [S1]), a slightly extended version of which is proved in §1. For its use in §3, it is required that we break the relevant renormalization compositions up into certain factors having "nice" geometric properties; such properties should in addition depend only on the combinatorics and the a-priori real bounds of chapter I. All this is accomplished in §2.

Needless to say, the notion of holomorphic commuting pair would have no real substance of its own if not for the results in this chapter.

III.1 On Sullivan's Sector Theorem

Given $a, b \in \mathbb{R}$ with $a < b$, let $S(a,b)$ be the class of all schlicht mappings ϕ defined on $\mathbf{C}(I_\phi) := \mathbf{C} - (\mathbb{R} - I_\phi)$, where $I_\phi \supset (a,b)$ is some open interval, which preserve both half-planes $\mathbf{C}^+, \mathbf{C}^-$ and are such that $\phi((a,b)) = (a,b)$. We refer to I_ϕ as the *base* of $\phi \in S(a,b)$: it is the largest interval containing (a,b) restricted to which ϕ is a homeomorphism into the reals.

An element $A \in S(a,b)$ is called a *left α-root* (where $0 < \alpha < 1$) if there exists $a_0 \leq a$ such that $A(z) = u.(z - a_0)^\alpha + v$, where $u, v \in \mathbb{R}$ and the branch of $z \mapsto (z - a_0)^\alpha$ are uniquely determined by the requirements $A(a) = a$, $A(b) = b$. The point $a_0 \in \mathbb{R}$ is called the *pole* of A. Right roots are defined similarly.

Given a bounded interval $J \subseteq \mathbb{R}$ and some $\lambda > 0$, we shall denote by J^λ the (closed) interval centered at the midpoint of J whose length is $(1 + \lambda)$-times the length of J.

We are ready to state our version of Sullivan's sector theorem (compare Sullivan [S]):

Theorem 1.1: Let there be given $A_i, B_i \in S(a,b)$, for $i = 1, 2, \ldots, m$, and constants $\lambda, K, s > 0$ and $0 < \alpha < 1$ satisfying: (a) each A_i is a left α_i-root with $\alpha_i \leq \alpha$ and pole at a_i, where $a_1 = a$ and $a_i < a$ for all $i \geq 2$; (b) there exists a finite sequence of stopping times $1 = i_0 < i_1 < \cdots < i_q = m$ with $i_{n+1} - i_n \leq s$ such that, setting $d_n := \min\{|a_i - a| : i_n \leq i < i_{n+1}\}$, the inequality $\sum_{j \geq n} d_j^{-1} \leq K d_n^{-1}$ holds for all n; (c) the following holds for all $i \geq 2$: if I_i is the base of B_i then $B_i(I_i) \supseteq [a_i, b]$ and setting $J_i := B_i^{-1}([a_i, b])$ then $J_i^\lambda \subseteq I_i$. Under these assumptions, there exists a positive angle $\theta = \theta(\alpha, s, K, \lambda)$ such that the image of the upper half-plane by the composition $A_m B_m \cdots A_i B_i \cdots A_1 B_1$ is contained in the sector $0 \leq \arg(z-a) \leq \pi - \theta$.

Before proving this theorem, we recall some classical geometric properties of conformal mappings. If $\phi : \Omega \to \mathbf{C}$ is a schlicht mapping and $D \subseteq \Omega$ is either a disk or a square, then the distortion of ϕ over D is defined to be $L_\phi(D) := \inf\{\ell_2/\ell_1 : 0 \leq \ell_1|x-y| \leq |\phi(x) - \phi(y)| \leq \ell_2|x-y|, \text{ for all } x, y \in D\}$. If we write $N_\phi(D) := (\sup_{z \in D} |\eta\phi(z)|).\text{diam}(D)$, where $\eta\phi := \phi''/\phi'$ is the non-linearity of ϕ, then a well-known fact is: $L_\phi(D) \leq \exp\{O(N_\phi(D))\}$. In bounding the non-linearity, the main tool at hand is *Köbe's distortion lemma*: $|\eta\phi(z)| \leq 4/\text{dist}(z, \partial\Omega)$. We shall need also *Köbe's one-quarter theorem*: if $D \subseteq \Omega$ is a disk with center z then $\phi(D)$ contains a disk about $\phi(z)$ of diameter $\frac{1}{4}|\phi'(z)|.\text{diam}(D)$. Both are corollaries to a stronger statement known as Köbe's distortion theorem (see Ahlfors [A2]).

Here and throughout we shall write $\theta(z) := \pi - \arg(z-a)$.

Lemma 1.2: If $\phi \in S(a,b)$ and if D is either a disk or a square with $[a,b] \subseteq D \subseteq \mathbf{C}(I_\phi)$ then for all $z \in D$ we have $|\sin\theta(\phi(z))| \geq \exp\{-O(N_\phi(D))\}|\sin\theta(z)|$.

Proof: As ϕ maps $\mathbf{C}^+$ into itself, the Schwarz-Pick lemma yields $\text{Im}\,\phi(z) \geq |\phi'(z)|\text{Im}\,z$, for all $z \in \mathbf{C}^+$ (consider the upper half-plane with its Poincaré metric). Therefore, for all $z \in D \cap \mathbf{C}^+$:

$$
\begin{aligned}
\sin\theta(\phi(z)) &= \frac{\text{Im}\,\phi(z)}{|\phi(z)-a|} \geq \frac{|\phi'(z)|.|z-a|}{|\phi(z)-a|}.\sin\theta(z) \\
&\geq [L_\phi(D)]^{-1}.\sin\theta(z) ,
\end{aligned}
$$

and the proof is obviously the same for $z \in D \cap \mathbf{C}^-$. □

Another well-known fact to be used here concerns invariant Poincaré neighborhoods in the doubly-slit plane. Consider $\mathbf{C}((a,b))$ with its Poincaré metric ρ, and let γ denote the segment (a,b) (a geodesic in this metric). By the Schwarz-Pick lemma, each element in $S(a,b)$ maps the Poincaré neighborhood $\mathcal{N}(r) := \{z \in \mathbf{C}((a,b)) : \rho(z,\gamma) \leq r\}$ into itself, for all $r > 0$. One verifies easily that $\partial\mathcal{N}(r)$ consists of a circular arc with endpoints a, b sitting in the upper half-plane union its mirror image in the lower half-plane; in particular, each $z \in \partial\mathcal{N}(r)$ "views" γ under the same angle $\omega = \omega(r)$, see Figure 1.

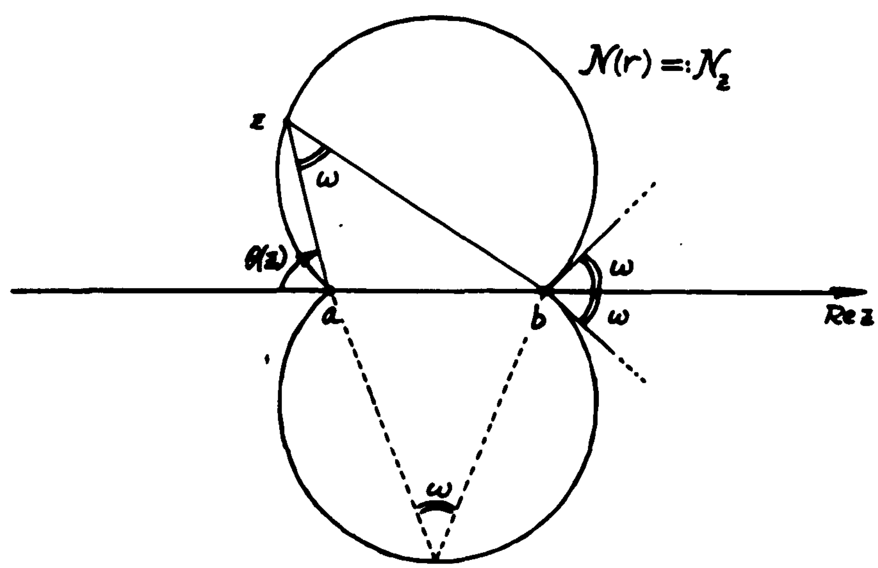

Figure 1

Lemma 1.3: Let $\mathcal{N}_z$ be the smallest of the Poincaré neighborhoods $\mathcal{N}(r)$ containing a given point z in the upper half-plane. Then $\operatorname{diam} \mathcal{N}_z^+ = |z - b| / \sin\theta(z)$.

Proof: Apply the law-of-sines to the triangle azb (Fig. 1). □

Proof of Theorem 1.1: First, let us agree in this proof to reserve the symbols $C_0, C_1, \ldots$ to denote positive constants depending only on the parameters α, s, K, λ of the statement. Regarding assumption (b), on the other hand, let us write j_n to denote the smallest i in the interval $i_n \leq i < i_{n+1}$ such that $|a_i - a| = d_n$.

1) Start with any point $z_1 \in \mathbf{C}^+$ and set $z_{i+1} = A_i B_i(z_i)$ for $i = 1, 2, \ldots, m$. Our goal is to show that $\theta(z_{m+1}) \geq \theta$, with θ as stated. Observe that we always have $\theta(z_2) \geq (1-\alpha)\pi$; thus we may assume $m > 1$, otherwise there is nothing to prove.

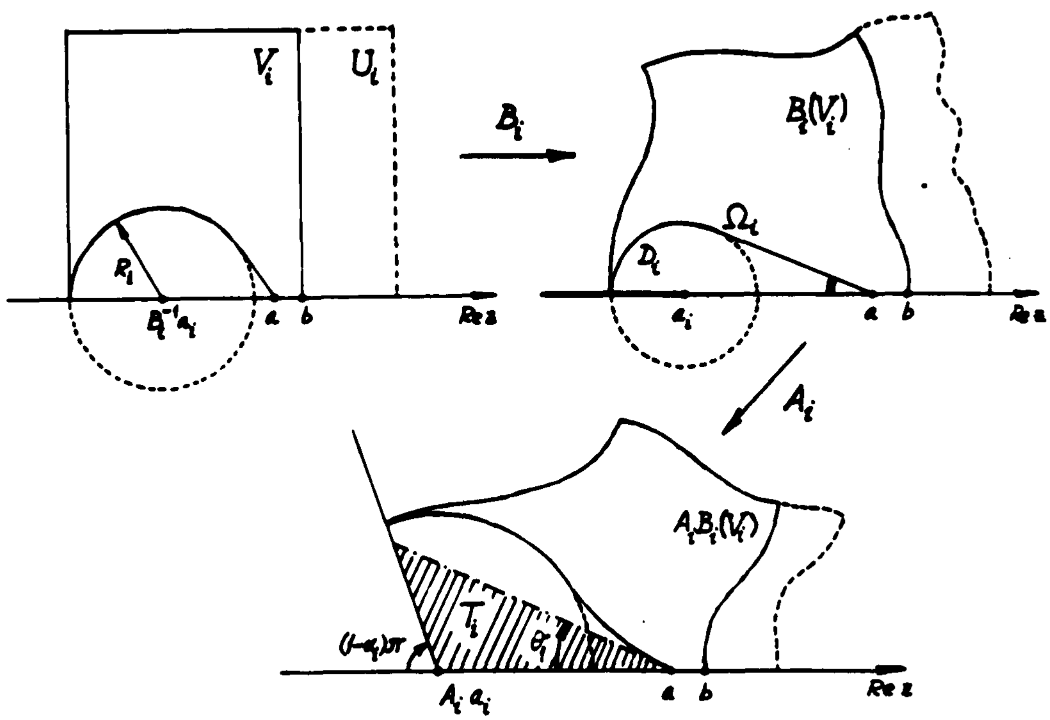

Figure 2

2) Let U_i, $i \geq 2$, be a square sitting in the upper half-plane, one of its sides being the interval $J_i^{\lambda/2} \subseteq J_i^{\lambda}$. Then U_i is a Köbe region for B_i, meaning $N_{B_i}(U_i) \leq C_0$;

here C_0 is in fact a constant depending only on λ. Let V_i denote the rectangle $U_i - \{z : \operatorname{Re} z > b\}$.

Claim #1: There exists $\theta_1 > 0$ depending only on α and λ such that $A_i B_i(V_i)$ contains the triangle T_i in the upper half-plane with base $[A_i a_i, a] \subseteq \mathbb{R}$ and angles $\pi\alpha_i$ and θ_1 respectively at $A_i a_i$ and a (Figure 2).

Here is a sketch of the proof. Since there is a $\xi_i \in (a,b)$ such that $B_i'(\xi_i) = 1$ and the distortion of B_i over V_i (or U_i) is uniformly bounded as we've seen, we have $C_1^{-1} \leq |B_i'(x)| \leq C_1$ for some $C_1 > 1$, for all $x \in J_i$. Hence, letting R_i denote the radius of the largest semi-disk centered at $B_i^{-1}a_i$ and contained in V_i, and observing that $R_i = O(|a_i - a|)$, we deduce from Köbe's $\frac{1}{4}$-theorem that $B_i(V_i)$ contains the convex-hull Ω_i of $\{a\} \cap D_i$, where $D_i \subseteq \mathbf{C}^+$ is a semi-disk centered at a_i of radius $\frac{1}{4}C_1 R_1 \geq C_2|a_i - a|$, see Figure 2. It already follows that the region Ω_i exhibits a definite angle at a; taking its image by the (very explicit) left α_i-root A_i and using some elementary geometry yields the claim.

3) From this claim and the fact that the mappings involved are one-to-one, if $z \in \mathbf{C}^+ - V_i$ then $\theta(A_i B_i(z)) \geq \theta_1$, for all $i \geq 2$.

4) For the sake of what follows we assume, as we may, that $\theta_1 \leq \min\{\frac{\pi}{4}, (1-\alpha)\pi\}$. Let j be the smallest index ≥ 2 such that $z_i \in V_i$ for all $i \geq j$ (if no such j exists then, by 3), $\theta(z_{m+1}) \geq \theta_1$ and we are done). Take the smallest $j_n \geq j$ and notice that $t := j_n - j \leq 2s$, from assumption (b). Our aim is to control the "loss of angle" when we apply to z_j the next t factors $A_j B_j, A_{j+1} B_{j+1}, \ldots, A_{j_n - 1} B_{j_n - 1}$.

As we know thus far, z_j belongs to $V_j^* = V_j - \{z : 0 \leq \theta(z) < \theta_1\}$. Let Q_j be the square of Figure 3 containing V_j^*. We have $\operatorname{diam} Q_j \leq \sqrt{2} \operatorname{diam} V_j^* \leq C_3 |a_j - a|$;

moreover, the base of $\omega_j := A_j B_j$ contains an interval of length $O(|a_j - a|)$ by assumption (c), whence $N_{\phi_j}(Q_j) \le C_4(\sin\theta_1)^{-1}$. Therefore, by lemma 1.2:

$$\sin\theta(z_{j+1}) \ge \exp\{-C_5(\sin\theta_1)^{-1}\}\sin\theta(z_j)\ . \qquad (*)$$

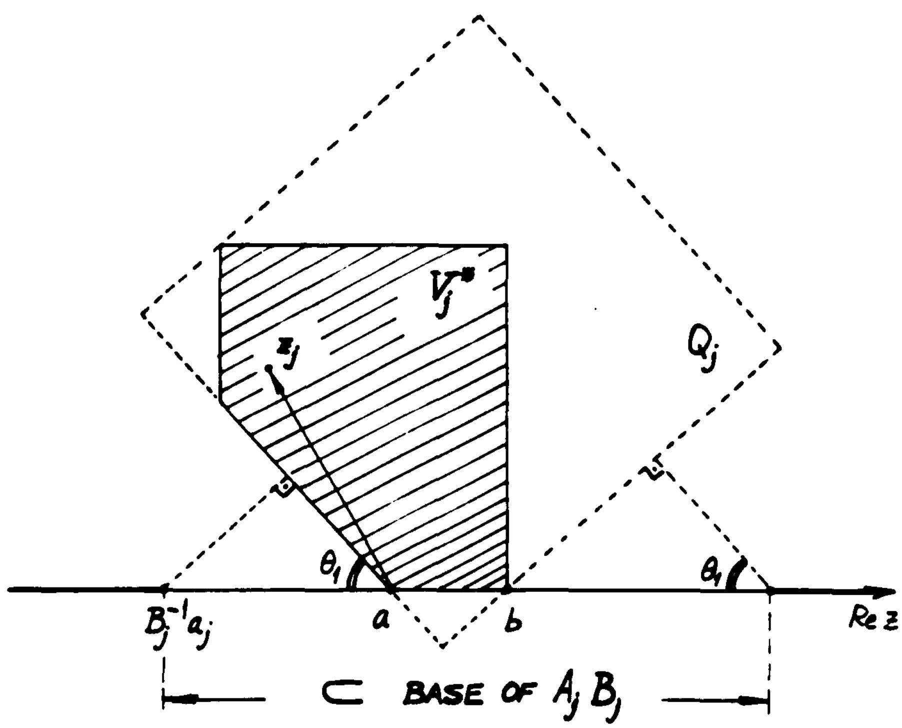

Figure 3

5) Defining $0 < \theta_k < \theta_1$ recursively by $\sin\theta_{k+1} = \exp\{-C_5(\sin\theta_k)^{-1}\}\sin\theta_k$, $k = 2, \ldots, t$, and repeating the argument leading to $(*)$, *mutatis mutandis*, another $t-1$ times yields $\theta(z_{j_n}) \ge \theta_t$, a constant still depending only on the parameters in the statement.

6) At this point we appeal to Sullivan's neat Poincaré-neighborhood trapping idea, in order to control the loss of angle due to the remaining factors. Let $\mathcal{N}$ be

the smallest of the Poincaré neighborhoods $\mathcal{N}(r)$ of (a,b) in $\mathbf{C}((a,b))$ containing $V_{j_n} - \{z : 0 \leq \theta(z) < \theta_t\}$. From lemma 1.3 we know that $2R := \operatorname{diam}\mathcal{N}^+ = O(d_n)$. (We have incorporated the factor $(\sin\theta_t)^{-1}$ into the constants here, yet notice that $\theta_t \leq \frac{\pi}{4}$ entails $R \geq |a-b|/\sqrt{2}$, which is used below).

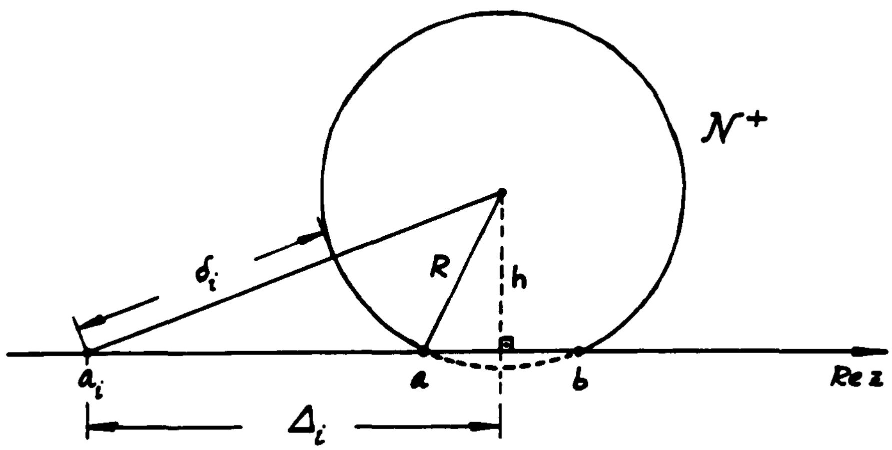

Figure 4

Claim # 2: For each $i \geq j_n$, $\operatorname{dist}(a_i, \mathcal{N}^+) \geq C_6|a_i - a|$.

Indeed, from Figure 4 we have:

$$\frac{\delta_i}{|a_i - a|} \geq \frac{\delta_i}{\Delta_i} = \frac{1 - \left(\frac{|a-b|}{2\Delta_i}\right)^2}{\sqrt{1 + \left(\frac{h}{\Delta_i}\right)^2} + \frac{R}{\Delta_i}} . \tag{$**$}$$

By assumption (b) we have $|a_i - a| \geq K^{-1}d_n \geq C_7 R$ for all $i \geq j_n$. Therefore $\frac{h}{\Delta_i} < \frac{R}{\Delta_i} \leq C_7^{-1}$, and this bounds the denominator of $(**)$ from above. At the same time, $\Delta_i = |a_i - a| + \frac{1}{2}|a-b| \geq (1 + C_7\sqrt{2})\frac{|a-b|}{2}$, which in turn bounds the numerator of $(**)$ from below. The claim follows.

Clearly, claim #2 still holds true if dist $(a_i, \mathcal{N}^+)$ is replaced by dist $(x, \mathcal{N}^+)$ where x is any point on the line at distance at least $O(|a_i - a|)$ from $[a, b]$. Since the base of $\phi_i := A_i B_i$ contains intervals of length $O(|a_i - a|)$ on either side of $[a, b]$, we deduce from this claim and Köbe's distortion lemma that: $N_{\phi_i}(\mathcal{N}^+) \leq C_8 d_n / |a_i - a|$, for all $i \geq j_n$. As each ϕ_i leaves $\mathcal{N}^+$ invariant, iterated application of lemma 1.2 yields:

$$\sin\theta(z_{m+1}) \geq \exp\{-C_9 d_n \sum_{i=j_n}^{m} |a_i - a|^{-1}\} \sin\theta(z_{j_n}) \ . \qquad (***)$$

But now, using the full strength of assumption (b) we have: $\sum_{i=j_n}^{m} |a_i - a|^{-1} \leq s. \sum_{k \geq n} d_k^{-1} \leq K s d_n$. Taking this back to $(***)$ shows us, at last, that $\sin\theta(z_{m+1}) \geq \exp\{-C_9 K s\} \sin\theta_t =: C_{10}$, as was to be proved. □

III.2 The Factoring of Renormalization Compositions

We want the powerful tool of §1 to be brought into our dynamics problem. To be more specific, considering the long renormalization compositions of a strictly commuting pair in the Epstein-Lanford class of chapter I, we would like to break them up into factors that will satisfy the hypotheses of Sullivan's sector theorem after affine rescaling. This is accomplished at the end of this section.

Let $f : \mathbf{T}^1 \hookleftarrow$ be an orientation preserving homeo with irrational rotation number $\rho(f) = [r_0, r_1, \dots, r_n, \dots]$. Given $x \in \mathbf{T}^1$, set $I_0(x) := \mathbf{T}^1$ and, for $k \geq 1$, let $I_k(x) \subseteq \mathbf{T}^1$ be the unique closed interval with endpoints x and $f^{q_k}(x)$ containing $f^{q_{k+2}}(x)$. For a distinguished point $c \in \mathbf{T}^1$, we shall write I_k instead of $I_k(c)$.

Fix some (large) positive integer n, and consider the ordered collection of intervals $\mathcal{B} := \{f^i(I_n) : 1 \leq i \leq q_{n+1} - 1\}$. Recall that these intervals have pairwise disjoint interiors. For $k = 0, 1, \dots, n+1$, let j_k be the largest $j \geq 1$ with the property that $f^i(I_n) \cap \mathring{I}_k = \emptyset$ for $1 \leq i < j$. Observe that $j_0 = 1$.

Lemma 2.1: For $1 \leq k \leq n+1$, we have $j_k = q_k$ if $k \not\equiv n$ (mod. 2), while $j_k = q_k + q_{k+1}$ if $k \equiv n$ (mod.2).

Proof: For all k in the given range, either $I_n \subset I_k$ or I_n is adjacent to I_k, depending on whether $|n-k|$ is even or odd, respectively. The lemma follows, then, from the dynamical interpretation of $\{q_i\}_{i \geq 0}$ as a sequence of return times (see Figure 5). □

Let us consider the "blocks" $\mathcal{B}_k := \{f^i(I_n) \in \mathcal{B} : j_{k-1} \leq i < j_{k+1}\}$, for $k =$

$1, 2, \ldots, n$. Notice that $\mathcal{B} = \bigcup_{k=1}^{n} \mathcal{B}_k$ and that $\mathcal{B}_k \cap \mathcal{B}_{k+2} = \emptyset$ for $1 \leq k \leq n-2$. These blocks correspond roughly to what Sullivan calls *epochs* in [S1].

Definition 2.2: Let the *scale* of $f^i(I_n) \in \mathcal{B}$ be the largest $k \geq 1$, if any, such that $f^i(I_n) \subset I_{k-1} - \mathring{I}_{k+1}$, and let it be equal to zero otherwise.

Definition 2.3: An element of $\mathcal{B}_k$ is called a *k-marked interval* if its scale is equal to k. We also call an element of $\mathcal{B}_1$ a *0-marked interval* if its scale is zero and it precedes all 1-marked intervals in the forward dynamical order of $\mathcal{B}$.

We denote by $\mathcal{M}_k$ the collection of all k-marked intervals, for $k = 0, 1, \ldots, n$. It is not difficult to see that $\mathcal{M}_0$ has either 0 or $r_0 - 1$ elements, depending on whether n is odd or even, respectively. On the other hand:

Proposition 2.4: If $1 \leq k \leq n$, then $r_k \leq \text{card}(\mathcal{M}_k) \leq r_k(r_{k+1}+1)$, and in fact $\text{card}(\mathcal{M}_k) = r_k$ whenever $k \equiv n$ (mod. 2).

Proof: (a) Observe that the intervals $J_0 := f^{q_{k-1}}(I_k)$, $J_s := f^{sq_k}(J_0)$, $s = 1, 2, \ldots, r_k - 1$, constitute a partition of $I_{k-1} - I_{k+1}$ modulo endpoints; see Figure 5.

(b) Suppose $i < j$ are such that the intervals $f^i(I_n), f^j(I_n)$ belong to $\mathcal{B}_k$ and are both in the same J_s. Then they are k-marked by definition, and by lemma 2.1 either: (1) $j - i \leq q_{k+1} - q_{k-1}$ or (2) $j - i \leq r_{k+1}q_{k+1} + r_k q_k$, depending on whether $n - k$ is even or odd, respectively. On the other hand, $f^{j-i}(J_s) \cap \mathring{J}_s \neq \emptyset$, and since $f^{q_{k-1}+sq_k}$ is a homeo, we have $f^{j-i}(I_k) \cap \mathring{I}_k \neq \emptyset$ as well, which implies $j - i \geq q_{k+1}$. Against case (1) this yields a contradiction, and so each J_s contains at most one element of $\mathcal{M}_k$, i.e., $\text{card}(\mathcal{M}_k) \leq r_k$. Against case (2) it shows that the number of elements of $\mathcal{M}_k$ in each J_s is at most the smallest integer greater than $(r_{k+1}q_{k+1} + r_k q_k)/q_{k+1}$, which is $r_{k+1} + 1$, whence $\text{card}(\mathcal{M}_k) \leq r_k(r_{k+1} + 1)$.

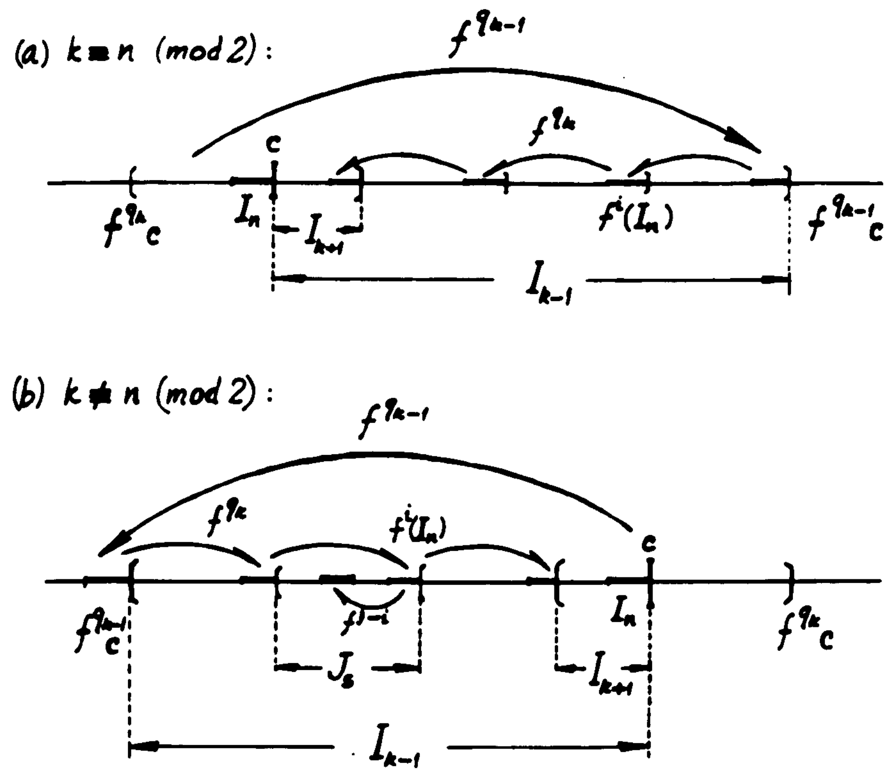

Figure 5

(c) Finally, in either case we have $J_s \supset f^{q_{k-1}+(s+\epsilon_k)q_k}(I_n)$ for $s = 0, 1, \ldots, r_k - 1$, where ϵ_k is the remainder of $n - k$ modulo 2. Since such images of I_n are in $\mathcal{B}_k$ by lemma 2.1, they are in $\mathcal{M}_k$ too, and so the lower bound $\text{card}(\mathcal{M}_k) \geq r_k$ is established. □

Let us assume from this point on that f is a critical circle homeo (with $c \in \mathbf{T}^1$

a critical point of cubic type), smooth enough for the bounded geometry results of chapter I to be valid for f. More precisely, let us assume the following "axioms":

Axiom 1: There exists $K_1 > 1$ such that the inequality $|I_{n-1}(f^i c)| \geq K_1 . |I_n(f^i c)|$ holds for all $n \geq 1$ and all $i \in \mathbb{Z}$.

Axiom 2: There exists $\lambda > 0$ such that the following holds for all $n \geq 1$: if $0 < i < i+j \leq q_{n+1} - 1$ and $J \supseteq f^i(I_n)$ is the largest interval restricted to which f^j is a diffeo onto its image then $[f^i(I_n)]^\lambda \subseteq J$.

In the bounded combinatorial type case ($\bar{r} = \max r_n < \infty$) inequalities and inclusions in the opposite direction are valid also, with suitable constants.

Definition 2.5: Let the *polar-ratio* of a non-degenerate interval J with respect to a point x be the number $P(x, J) := \text{dist}(x, J)/|J|$.

Observe that, under a map with bounded cross-ratio distortion, polar-ratios do not decrease by more than a multiplicative factor depending only on the cross-ratio distortion of the map.

For $1 \leq i \leq q_{n+1} - 1$, let us write $P_{i,n} := P(c, f^i(I_n))$.

Proposition 2.6: There exist constants $C > 0$ and $\mu > 1$, depending only on constant K_1 of axiom 1, such that $P_{i,n} \geq C\mu^{n-k}$ for each interval $f^i(I_n)$ whose scale is equal to k.

Proof: Set $x = f^i c$; all intervals written $[a, b]$ in this proof will be contained in $\mathbf{T}^1 - \{x\}$.

Since f is topologically conjugate to the corresponding rotation, we have: (a)

$f^{-q_j}(x) \in [f^{q_j-1}(x), f^{q_j+1}(x)]$ for all $j \geq 1$; (b) if $f^i(x) \in [f^{q_k-1}(x), f^{q_k+1}(x)]$ then $f^{-i}(x) \in [f^{-q_k-1}(x), f^{-q_k+1}(x)]$.

Putting these two facts together yields $c = f^{-i}(x) \in I_{k-2}(x) - I_{k+2}(x)$. Applying axiom 1 for $x = f^i c$ and using an obvious "telescoping" trick we deduce that $P_{i,n} \geq K_1^{n-k-2}$, which proves the proposition if we take $C = K_1^{-2}$ and $\mu = K_1$. □

With these two propositions at hand, we proceed to exhibit the promised factoring of the n-th renormalization of a strictly commuting pair (ξ, η) in the Epstein-Lanford class.

So we know there exist open intervals $\tilde{I}_\xi \supseteq \xi(I_\xi)$ and $\tilde{I}_\eta \supseteq \eta(I_\eta)$, as well as (symmetric) schlicht mappings $h_\xi^{-1} : \mathbf{C}(\tilde{I}_\xi) \to \mathbf{C}$, $h_\eta^{-1} : \mathbf{C}(\tilde{I}_\eta) \to \mathbf{C}$ such that $\xi \equiv h_\xi \circ Q$ and $\eta \equiv h_\eta \circ Q$, where Q denotes the cubic polynomial $z \mapsto z^3$. If we think of (ξ, η) already renormalized enough times, we may assume also that $\gamma(I_\gamma)$ sits inside $\tilde{I}_\gamma$ with universal space around it, $\gamma = \xi, \eta$; in particular, each restriction $h_\gamma^{-1}|\gamma(I_\gamma)$ has universally bounded cross-ratio distortion (cf. observation preceding proposition 2.6).

Consider the successive renormalizations of (ξ, η) without rescaling: $(\xi_0, \eta_0) := (\xi, \eta)$ and $(\xi_{n+1}, \eta_{n+1}) := \mathcal{R}(\xi_n, \eta_n) = (\eta_n, \eta_n^{r_n} \circ \xi_n)$, for all $n \geq 0$. We have seen in chapter I that the following "hybrid" representation of (ξ_n, η_n) is available for $n \geq 1$:

$$\begin{cases} n \quad \text{even} & \Rightarrow \begin{cases} \xi_n \equiv f^{q_{n-1}} \circ \xi & \text{over } \xi^{-1}(I_n) \\ \eta_n \equiv \eta \circ f^{q_n - 1} & \text{over } I_{n-1} \end{cases} \\ n \quad \text{odd} & \Rightarrow \begin{cases} \xi_n \equiv \eta \circ f^{q_{n-1}-1} & \text{over } I_n \\ \eta_n \equiv \eta \circ f^{q_n - 1} \circ \xi & \text{over } \xi^{-1}(I_{n-1}) \end{cases} \end{cases}$$

where $f : I_n \hookleftarrow$ is the circle mapping associated to (ξ, η), while, as before, the intervals I_j are just $I_j(c)$ for $c = 0$ ($\equiv \xi(0)$), the critical point.

Remark: *Strictu sensu*, f is not a critical circle mapping: it is necessary to consider instead $\overline{f} := \varphi^{-1} \circ f \circ \varphi$ where $\varphi : I_\eta \to I_\eta$ is a smooth change of coordinates with a cubic singularity at $\xi(0)$. Nevertheless, the geometric properties expressed through axioms 1 and 2 above do remain valid for f, as already remarked in chapter I, and all bounds involved are *beau* in the sense of Sullivan [S1].

We henceforth restrict our attention to the case n even (and large), and within it we show how to achieve the desired factoring only for ξ_n; the other cases are handled in similar fashion.

For the time being, only the diffeomorphic part of ξ_n^{-1} concerns us, namely the composition:

$$\hat{\xi}_n := (f^{q_{n-1}})^{-1} : f^{q_{n-1}}(I_n) \to I_n \; ; \qquad (*)$$

the remaining "root factor" ξ^{-1} will come into play only in §3. We work through steps:

1) Consider $(*)$ written as a word in ξ^{-1}, η^{-1}. A factor γ^{-1} in this composition is called a *left* or a *right* factor, according to whether $\gamma = \eta$ or $\gamma = \xi$, respectively. Each such factor, remember, has a further decomposition $\gamma^{-1} \equiv Q^{-1} \circ h_\gamma^{-1}$. A *left root* is the part of a left factor corresponding to Q^{-1}; *right roots* are similarly defined. The factors h_γ^{-1} are called simply *h-factors*.

Definition 2.7: (a) A left root is said to be k-marked (k necessarily even) if the interval domain of its associated left factor is $\eta(J)$ for some $J \in \mathcal{M}_k$; (b) A right root is said to be k-marked (k necessarily odd) if the interval domain of its associated right factor is some $J \in \mathcal{M}_k$.

Here k ranges from 0 up to $n-1$. Proposition 2.4 bounds the number of k-

marked roots for any such k in terms of the combinatorics of the rotation number $\rho(f) = \rho(\xi, \eta)$.

2) Organize all marked left roots in the composition giving $\hat{\xi}_n$ by their order of appearance (from right to left) in that composition, and call them successively $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_m$. In this order, first come the $(n-2)$-marked roots, then come the $(n-4)$-marked roots, and so on. Moreover, $\hat{A}_m$ is the very last factor in the composition in question, as is easily checked. We have also $m = (r_0 - 1) + r_2 + \cdots + r_{n-2}$, after proposition 2.4.

3) In $(*)$, let $\hat{B}_1$ be the sub-compostion going from the first factor on the right up to and including the left-most factor before $\hat{A}_1$, which is precisely the h-factor associated to the left-root $\hat{A}_1$. On the other hand, for $j = 2, 3, \ldots, m$, let $\hat{B}_j$ be the sub-composition running strictly between $\hat{A}_{j-1}$ and $\hat{A}_j$. With this new notation, $(*)$ now reads:

$$\hat{\xi}_n = \hat{A}_m \circ \hat{B}_m \circ \cdots \circ \hat{A}_j \circ \hat{B}_j \circ \cdots \circ \hat{A}_1 \circ \hat{B}_1 \ . \qquad (**)$$

4) Let $T_1 \subset I_\eta$ be the largest interval containing $f(I_n)$ restricted to which $f^{q_{n-1}-1}$ is a diffeo onto its image. Set $(a, b) := T_0 := \eta^{-1}(T_1)$ (we are now remembering that η is defined well to right of $\xi(0)$) and put also $T_i := f^{i-1}(T_1)$ for $i = 2, \ldots, q_{n-1}$. Then each T_i contains the corresponding $f^i(I_n)$ plus definite, *beau* space on both sides, after axiom 2.

5) Accordingly, let all factors $\hat{A}_j, \hat{B}_j$ be rescaled via the affine, orientation preserving maps taking the relevant T_i's back onto (a, b). Call the rescaled mappings A_j, B_j, respectively: these are now elements in the class $S(a, b)$ of §1.

We are ready to state and prove:

Theorem 2.8: If $\rho(\xi,\eta)$ is a rotation number of bounded combinatorial type then the rescaled composition $A_m \circ B_m \circ \cdots \circ A_j \circ B_j \circ \cdots \circ A_1 \circ B_1$ satisfies all the hypotheses of Sullivan's sector theorem with *beau* constants.

Proof: (a) Assumption (a) of theorem 1.1 is satisfied, for each A_j is certainly a left α_j-root with $\alpha_j = \frac{1}{3}$ (i.e., a cubic root), and the pole of A_1 is in dynamical correspondance with a, as is easily checked.

(b) The number of marked left-roots at each scale being uniformly bounded (by the hypothesis on $\rho(\xi,\eta)$ and proposition 2.4), the "bounded gap" condition of assumption (b) is fulfilled if we group the roots together by scales.

Now, let a_j be the pole of A_j as in §1, and i be such that $f^i(I_n)$ is the marked left interval corresponding to A_j. Combining the observation following axiom 2 with steps 4) and 5) above and the definition of polar ratio, we obtain: $|a_j - a| \geq C_0|a-b|P_{i,n}$, for a certain beau constant C_0. Therefore, by proposition 2.6, $|a_j - a|$ grows exponentially with $n-k$, where k is the scale of A_j, and this takes care of the series condition of assumption (b).

(c) It remains to check whether assumption (c) holds true here.

But for all $j \geq 2$ we may write $\hat{B}_j = h_j \circ \xi^{-1} \circ (f^{q_k - 1})^{-1}$, provided k is the scale of $\hat{A}_j$, where h_j is the h-factor associated to $\hat{A}_j$. Thus, we have the situation depicted in Figure 6 [in fact, there are two cases, depending on whether the scale of $\hat{A}_{j-1}$, the preceding marked left-root, is equal to k or $k+2$; Figure 6 illustrates the former].

By axiom 2, $f^{q_k - 1}$ is defined on an interval $J \supseteq [f(I_{k-1})]^\lambda$ as a diffeo onto its image. Hence assumption (c) is indeed verified if we take into account that: (i)

J is in dynamical correspondence with the base (§1) of B_i; (ii) $f(\xi(0))$, the left endpoint of $f(I_{k-1})$, is in dynamical correspondance with the pole of A_j, (iii) ξ^{-1} is defined well to the left of $f(\xi(0))$; and (iv) h_j is a map of beau bounded cross-ratio distortion. The case $j = 1$ is similarly proved.

Since all bounds involved are beau, the theorem is proved too. □

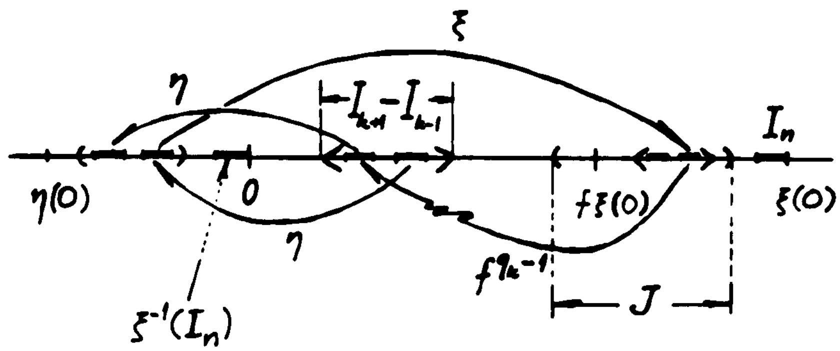

Figure 6

CHAPTER IV
The Teichmüller Contraction Property of Renormalization

Following Sullivan's strategy in [S1], we shall prove renormalization contraction at the level of the Teichmüller space of a compact Riemann surface lamination (§2), suitably constructed from the orbit structure of the relevant holomorphic commuting pair.

The key to contraction at that level is Sullivan's *almost geodesic lemma* (§3). Contraction at the level of commuting pairs follows from this sort of Teichmüller contraction because, at small scales at least, Teichmüller's metric dominates the quasiconformal conjugacy metric (on holomorphic commuting pairs). All the necessary bounds for this argument are already in place since chapter III, given that the opposite inequality on metrics is (almost) a tautology.

IV.1 Holomorphic Repellers and their Riemann Surface Laminations

By *holomorphic repeller* we mean a triple (U, V, f) where $U, V \subseteq \mathbf{C}$ are bounded open sets with $\overline{U} \subseteq V$ and $f : U \twoheadrightarrow V$ is a proper holomorphic branched covering map. Examples include polynomial-like mappings in the sense of Douady-Hubbard [DH], as well as the so-called Cantor repellers discussed below.

The *filled-in limit set* of a holomorphic repeller is, by definition, $\Lambda_f := \bigcap_{n \geq 0} f^{-n}(V)$. Since f is proper, we have $\overline{f^{-n}(V)} \subseteq f^{-(n-1)}(V)$ for all $n \geq 1$, and so Λ_f is compact; it is in fact the largest totally f-invariant compact subset of the domain of f. Throughout this section we make the convenient assumption that the critical points of f (if any) belong to Λ_f; in this way, $U - \Lambda_f \twoheadrightarrow V - \Lambda_f$ becomes a proper, unbranched covering map.

We shall need the following very simple yet extremely important:

Proposition 1.1: Given a holomorphic repeller (U, V, f), we have: (a) if $\mathcal{U}, \mathcal{V}$ are any two neighborhoods of Λ_f, there exists $m \geq 0$ such that $f^{-m}(\mathcal{V}) \subseteq \mathcal{U}$; (b) the multivalued map f^{-1} acts discontinuously on $V - \Lambda_f$, i.e., for each compact $K \subseteq V - \Lambda_f$, $K \cap f^{-n}(K) = \emptyset$ for all but finitely many values of n.

Proof: In (a), it is enough to consider the case $\mathcal{V} = V$; the sequence $(f^{-n}V)_{n \geq 0}$, being properly nested, shrinks down to Λ_f as claimed. As for (b), choose some neighborhood $\mathcal{U} \supseteq \Lambda_f$ such that $\mathcal{U} \cap K = \emptyset$ and apply (a). □

Our goal here is to associate to the germ of a given holomorphic repeller around its limit set Λ_f a compact Riemann surface lamination in the sense of Sullivan [S1], up to the appropriate notion of isomorphism. Roughly speaking, this will be the space of backward branch orbits (or "threads") of points in any deleted neighborhood of Λ_f factored by the equivalence relation determined by the dynamics of f itself.

First we recall Sullivan's definition.

Definition 1.2: A *Riemann surface lamination (or RSL -) structure* on a Hausdorff topological space X consists of an atlas $\{(U_\alpha, \varphi_\alpha)\}$ covering X such that: (a) each φ_α maps the corresponding U_α homeomorphically onto $D_\alpha \times T_\alpha$, where $D_\alpha \subseteq \mathbf{C}$ is a disk and T_α is a Hausdorff space; (b) each ovelapping homeo $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is of the form $(z,t) \mapsto (\psi_t(z), \phi(t))$ with ψ_t holomorphic for each t; (c) the atlas is maximal with respect to these two properties. Provided with such a structure, the space X is called a *Riemann surface lamination.*

As trivial examples, we have the products $S \times T$ where S is a free union of Riemann surfaces and T is some Hausdorff transverse space.

The concept of leaf of a Riemann surface lamination is completely analogous to the corresponding one for foliations; leaves come with obvious intrinsic structures making them into Riemann surfaces in a natural way. Accordingly, an RSL (resp. QCL) - morphism $X \to Y$ between two Riemann surface laminations is a continuous map which sends leaves of X into leaves of Y and is holomorphic (resp. quasiconformal) on leaves.

Next we offer some fairly general tools by means of which several non-trivial examples of Riemann surface laminations may be constructed.

Proposition 1.3: Let (E, p, B) be a locally trivial bundle over a Riemann surface B, with totally disconnected fiber above each element of the base. Then there exists a unique RSL-structure on the total space E making the projection p into an RSL-morphism.

Proof: For $i = 1, 2$, let $p^{-1}(U_i) \xrightarrow{\varphi_i} U_i \times F_i$ be a local trivialization of the given bundle, where $U_i \subseteq B$ is open and $F_i = p^{-1}(b_i)$ with $b_i \in U_i$, and suppose $U_1 \cap U_2 \neq \emptyset$. If $\pi_i : U_i \times F_i \to U_i$ denotes the canonical projection ($i = 1, 2$) then we have $\pi_1 \circ \varphi_1 \equiv p \equiv \pi_2 \circ \varphi_2$ over $p^{-1}(U_1 \cap U_2)$. Thus, for all $(z, t) \in \varphi_1(p^{-1}(U_1 \cap U_2)) = (U_1 \cap U_2) \times F_1$ we may write $\varphi_2 \circ \varphi_1^{-1}(z, t) = (z, \Phi(z, t))$, where $\Phi : (U_1 \cap U_2) \times F_1 \to F_2$ is continuous. Since F_2 is totally disconnected, for each connected component $\mathcal{O} \subset U_1 \cap U_2$ the restriction $\Phi | \mathcal{O} \times F_1$ must be independent of the first variable, compare condition (b) of definition 1.2. It follows that the local trivializations determine an RSL-structure on E. Uniqueness of such structure is proved in similar fashion. □

[This remains unchanged, of course, if more generally B is taken to be a free union of Riemann surfaces].

Proposition 1.4: Given an inverse system $\ldots \to S_n \xrightarrow{\phi_n} S_{n-1} \to \ldots \xrightarrow{\phi_1} S_0$, where each S_n is a free union of Riemann surfaces and each ϕ_n is a proper holomorphic covering map, let S_∞ be its topological inverse limit and consider the canonical projections $\pi_n : S_\infty \to S_n$, $n \geq 0$. Then S_∞ is a locally compact, 2^{nd}-countable space and has a unique RSL-structure making each π_n into an RSL-morphism.

Proof: We write $\hat{\varphi}_n := \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n$ for each $n \geq 1$.

Let $x_0 \in S_0$, set $F_0 := \{x_0\}$ and $F_n := \hat{\varphi}_n^{-1}(F_0)$ for each $n \geq 1$, and consider the inverse system $\ldots \to F_n \xrightarrow{\phi_n} F_{n-1} \to \ldots \xrightarrow{\phi_1} F_0$. Since each ϕ_i is a proper convering map, we have:

1) Each F_n is discrete and compact, whence finite, and so $F_\infty := \varprojlim F_n$ is a compact, non-empty (see Eilenberg-Steenrod [ES] or Dugundji [D]) totally disconnected space having a countable basis.

2) If $D \subseteq S_0$ is a compact, simply-connected neighborhood of x_0 then any connected component $\mathcal{C}$ of $\hat{\phi}_n^{-1}(D) \subseteq S_n$ must be compact and simply-connected as well (use induction!) and $\hat{\phi}_n$ must map it homeomorphically onto D. In particular, there exists a unique $x \in F_n \cap (\text{int}\,\mathcal{C})$, and so we write $D_x := \mathcal{C}$.

Hence, let $\psi : D \times F_\infty \to \prod_{n=0}^{\infty} S_n$ (both spaces with the product topology) be defined as follows: $\psi(z, (x_n)_{n\geq 0}) = (z_n)_{n\geq 0}$, where $\{z_n\} := D_{x_n} \cap \hat{\phi}_n^{-1}(z)$ for all $n \geq 0$. This map is well-defined and injective by 2), and it is clearly continuous. Since its domain is compact and its range Hausdorff, ψ is a homeo onto $\psi(D \times F_\infty) = \pi_0^{-1}(D)$. Setting $\varphi := \psi^{-1} : \pi_0^{-1}(\text{int}\,D) \to (\text{int}\,D) \times F_\infty$, we have $\pi_D \circ \varphi = \pi_0$, where $\pi_D : (\text{int}\,D) \times F_\infty \to \text{int}\,D$ is the canonical projection, whence φ is a local trivialization of (S_∞, π_0, S_0) around x_0; this already shows us that S_∞ is locally compact with a countable basis. Invoking proposition 1.3, there exists a unique RSL-structure on S_∞ which is compatible with π_0; but since $\pi_0 = \hat{\phi}_n \circ \pi_n$ and each $\hat{\phi}_n$ is holomorphic, such structure is in fact compatible with π_n for all $n \geq 0$, and we are done. □

An "analogous" situation in the context of direct limits is provided by the much simpler result that follows.

Proposition 1.5: Let $X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} \cdots \to X_n \xrightarrow{\phi_n} \cdots$ be a direct system where each X_n is a Riemann surface lamination and each ϕ_n is an open, injective RSL-morphism. Then the direct limit space X^∞ has a unique RSL-structure making the canonical maps $\rho_n : X_n \to X^\infty$, $n \geq 0$, as well as the direct limit map $\phi^\infty : X^\infty \to X^\infty$, into open, injective RSL-morphisms. □

Proposition 1.6: Set $X_n = X$ and $\phi_n = \phi$ for each $n \geq 0$ in proposition 1.5, where X is assumed locally compact and first countable and ϕ is taken to act discontinuously on X (cf. proposition 1.1(b)). Then the orbit space $X^\infty / < \phi^\infty >$ has a unique RSL-structure for which the canonical projection $X^\infty \to X^\infty / < \phi^\infty >$ is an RSL-morphism. Moreover, if ϕ has a relatively compact fundamental domain in X then $X^\infty / < \phi^\infty >$ is a compact space.

Proof: In proving the first assertion, the only non-trivial point is to show that the orbit space in question is Hausdorff. Any group of homeomorphisms of a locally compact, first countable Hausdorff space Y that acts discontinuously on Y has a Hausdorff orbit space Y/G (see Beardon [Be] for a proof in the context of "Kleinian" groups). Therefore it suffices to show that ϕ^∞ acts discontinously on X^∞. Let $K \subset X^\infty$ be compact and consider the canonical maps $\rho_n : X \to X^\infty$, $n \geq 0$. Since $\{\rho_n(X)\}_{n \geq 0}$ is an increasing sequence of open subsets of X^∞ whose union is X^∞, there must be a $k \geq 0$ such that $K \subset \rho_k(X)$, and because ρ_k is a homeo onto its image we have that $\widetilde{K} := \rho_k^{-1}(K) \subseteq X$ is compact, whence $\phi^n(\widetilde{K}) \cap \widetilde{K} = \emptyset$ for all but finitely many values of n. But then $(\phi^\infty)^n(K) \cap K = \emptyset$ for those same values, and so ϕ^∞ acts discontinuously on X^∞ as claimed.

As for the second assertion, if $\Omega \subset X$ is a fundamental domain for ϕ with $\overline{\Omega}$ compact then, since each ρ_n is open, $\rho_n(\Omega)$ is a relatively compact fundamental domain for ϕ^∞ (for each $n \geq 0$) and so $X^\infty / < \phi^\infty > \cong \overline{\rho_n(\Omega)}/\phi^\infty$ must be compact.

□

Remark: One may not replace "discontinuously" by "properly discontinuously" (meaning: each $x \in X$ has a neighborhood whose translates by the semi-group generated by ϕ are pairwise disjoint) in the above statement. For example, take $X = (\mathbf{C} \times \mathbb{R}) - \{(0,0)\}$ and let $\phi : X \to X$ be the map $(z,t) \mapsto (\lambda z, \lambda^{-1} t)$ where $\lambda > 1$; then ϕ is a properly discontinuous RSL-isomorphism, but $X/ < \phi >$ is not even Hausdorff.

After such preliminaries, we are ready to state and prove the main result of this section.

Theorem 1.7: Every holomorphic repeller (U, V, f) has a compact Riemann surface lamination $L(U, V, f)$ associated to it in such a way that: (a) if (U, V, f) and $(\tilde{U}, \tilde{V}, \tilde{f})$ represent the same germ (i.e., $\Lambda_f = \Lambda_{\tilde{f}}$ and $f \equiv \tilde{f}$ over $U \cap \tilde{U}$) then we have an RSL-isomorphism $L(U, V, f) \cong L(\tilde{U}, \tilde{V}, \tilde{f})$; (b) every quasiconformal conjugacy $(U_1, V_1, f_1) \sim (U_2, V_2, f_2)$ induces a QCL-isomorphism $L(U_1, V_1, f_1) \cong L(U_2, V_2, f_2)$.

Proof: 1) Construction of $L(U, V, f)$. Consider the inverse system:

$$\cdots \xrightarrow{f} f^{-n}V_0 \xrightarrow{f} f^{-(n-1)}V_0 \to \cdots \to f^{-1}V_0 \xrightarrow{f} V_0 \qquad (*)$$

together with its sub-system:

$$\cdots \xrightarrow{f} f^{-(n+1)}V_0 \xrightarrow{f} f^{-n}V_0 \to \cdots \to f^{-2}V_0 \xrightarrow{f} f^{-1}V_0 \qquad (**)$$

where $V_0 := V - \Lambda_f$, and let V_∞, V'_∞ be their respective inverse limit spaces. Both are Riemann surface laminations by proposition 1.4, and since $(**)$ is cofinal in $(*)$, we have an RSL-isomorphism $\varphi : V_\infty \to V'_\infty$. On the other hand, the inclusions

$f^{-(n+1)}V_0 \subseteq f^{-n}V_0$ yield an open, injective RSL-morphism $\psi : V'_\infty \hookrightarrow V_\infty$; therefore we have an open, injective RSL-morphism $\phi := \psi \circ \varphi : V_\infty \hookrightarrow V_\infty$. We get easily from proposition 1.1(b) that ϕ acts discontinuously on V_∞. Since V_∞ is locally compact and second-countable, proposition 1.5 tells us that the direct limit space of $V_\infty \xrightarrow{\phi} V_\infty \xrightarrow{\phi} \cdots \rightarrow V_\infty \xrightarrow{\phi} \cdots$, call it V_∞^∞, is a Riemann surface lamination with those properties too. If $\phi^\infty : V_\infty^\infty \hookleftarrow$ denotes the corresponding direct limit map (an RSL-isomorphism), we deduce from proposition 1.6 that $L(U, V, f) := V_\infty^\infty / < \phi^\infty >$ is a compact Riemann surface lamination.

2) Property (b) is clear.

Proof of (a): by proposition 1.1(a), there exists $m \geq 0$ such that $\tilde{f}^{-m}(\tilde{V}) \subseteq U \cap \tilde{U} \subseteq V$, and so we have $\tilde{f}^{-(m+k)}(\tilde{V}) = f^{-k}(\tilde{f}^{-m}\tilde{V}) \subseteq f^{-k}(V)$ for all $k \geq 0$. Passing to the inverse limit and then to the direct limit just like in the above construction yields an open, injective RSL-morphism $\beta : \tilde{V}_\infty^\infty \hookrightarrow V_\infty^\infty$ (conforming with the notation in 1)) such that the diagram:

$$\begin{array}{ccc} \tilde{V}_\infty^\infty & \overset{\beta}{\hookrightarrow} & V_\infty^\infty \\ \tilde{\phi}^\infty \downarrow & & \downarrow \phi^\infty \\ \tilde{V}_\infty^\infty & \underset{\beta}{\hookrightarrow} & V_\infty^\infty \end{array}$$

commutes; since once again by proposition 1.1(a) we have $f^{-\ell}(V) \subseteq \tilde{V}$ for some $\ell \geq 0$, β must be onto, i.e., an RSL-isomorphism. Therefore $\tilde{V}_\infty^\infty / < \tilde{\phi}^\infty > \cong V_\infty^\infty / < \phi^\infty >$, as stated. □

The specific holomorphic repellers to which this theorem will be applied belong in the following group of examples.

A Cantor repeller consists of two collections, say $\{D_0, D_1, \ldots, D_{n-1}\}$ and $\{\Delta_0,$

$\Delta_1, \ldots, \Delta_{m-1}\}$, of topological disks in the plane together with a surjective mapping $f : \bigcup_{i=0}^{n-1} D_i \to \bigcup_{j=0}^{m-1} \Delta_j$ satisfying the following conditions: (a) in each collection, any two distinct elements have disjoint closures; (b) for each i there exists $j(i)$ such that $\overline{D}_i \subseteq \Delta_{j(i)}$; (c) each restriction $f|D_i$ is a schlicht mapping onto $\Delta_{k(i)}$ for some $k(i)$. [Observe that we do not require that $j(i) = k(i)$]. It is implicit in this definition that f is a proper covering, so every Cantor repeller is indeed a holomorphic repeller.

We make a few remarks on Cantor repellers, preparing ground for the next sections:

1) If we consider in the union $\bigcup_{j=0}^{m-1} \Delta_j$ the infinitesimal conformal metric that agrees in each disk with the corresponding hyperbolic metric and apply the Schwarz-Pick lemma to the inverse mappings $\Delta_{k(i)} \xrightarrow{f^{-1}} \Delta_{j(i)}$, we obtain the following result:

Proposition 1.8: The mapping f expands this infinitesimal conformal metric by a definite factor. In particular, for some $N \geq 1$ f^N expands the euclidian metric on some neighborhood of Λ_f by a definite factor also. □

For this last statement, simply observe that the conformal metric in question and the euclidian metric are quasi-isometric in any neighborhood of Λ_f which is compactly contained in the range of f.

2) The dynamics of $f|\Lambda_f$ is topologically the same as that of a certain subshift of finite type. This is seen in the usual way (Bowen [Bo]): let G be the oriented graph (in fact a 2-graph) whose vertices are $0, 1, \ldots, n-1$ and whose directed edges are the ordered pairs (i, i') for which $f(D_i) \supseteq D_{i'}$, and let $\Sigma_G \subseteq \{0, 1, \ldots, n-1\}^{\mathbf{N}} =: \Omega_n$ (with the product topology) consist of those sequences $(i_k)_{k \geq 0}$ such that (i_k, i_{k+1}) is an edge of G for all $k \geq 0$. Then Σ_G is a compact σ-invariant sub-space, where

$\sigma : \Omega_n \hookleftarrow$ is the one-sided shift $(i_k)_{k\geq 0} \overset{\sigma}{\mapsto} (i_{k+1})_{k\geq 0}$, and because of the expanding property stated in proposition 1.8, the dynamical system $(\Lambda_f, f|\Lambda_f)$ is topologically conjugate to $(\Sigma_G, \sigma|\Sigma_G)$.

3) It should be pointed out that not every finite oriented 2-graph G is the graph of a Cantor repeller! The so-called Fibonacci graph, whose transition matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, yields the simplest example. However, by suitably increasing the number of vertices (re-coding), one can always obtain a realizable G' such that $(\Sigma_{G'}, \sigma|\Sigma_{G'})$ is topologically conjugate to $(\Sigma_G, \sigma|\Sigma_G)$.

4) A Cantor repeller only deserves its name when Λ_f, or equivalently Σ_G, is a Cantor set. This is always the case if the corresponding transition matrix A_G is aperiodic, i.e., some power of A_G consists of strictly positive entries. When this happens, the associated compact lamination whose existence is asserted by theorem 1.7 exhibits uncountably many dense leaves.

5) Let us agree to call a Cantor repeller *geometric* if the boundaries $\partial D_i, \partial\Delta_j$ are quasicircles for all i, j. Then a fairly simple pull-back argument in the spirit of theorem II.3.2 shows that if two geometric Cantor repellers have isomorphic graphs, they are quasiconformally conjugate.

IV.2 Constructing the appropriate Cantor Repeller

In this section we conform with the notation established in Chapter II.

Let Γ be a holomorphic commuting pair having a connected limit set Λ_Γ [which, as we know after II §5, is the closure of the set of points that never leave the domain of Γ under iteration]. We want to show how to extract from "within" Γ a certain Cantor repeller of fixed combinatorial type that turns out to be conformally conjugate to Γ in some deleted neighborhood of (part of) Λ_Γ.

The construction is easy, but its description is somewhat painful. Some elementary facts about the boundary behavior of conformal mappings (such as Carathéodory's extension theorem) will be implicitly used; we refer the reader to Figure 1.

Here we go:

1) Consider the (simply-connected) regions: $V_0 := U_\xi^+ - \Lambda_\Gamma$; $V_1 := U_\nu^+ - (U_\xi \cup U_\eta \cup \Lambda_\Gamma)$; $V_2 := U_\eta^+ - \Lambda_\Gamma$; $V_3 := U_\xi^- - \Lambda_\Gamma$; $V_4 := U_\nu^- - (U_\xi \cup U_\eta \cup \Lambda_\Gamma)$; $V_5 := U_\eta^- - \Lambda_\Gamma$, as well as: $W_0 := \Delta^+ - \Lambda_\Gamma$; $W_1 := \Delta^- - \Lambda_\Gamma$. [One should look back at Figure II.1].

2) Notice that $\hat{\mathbb{C}} - \Lambda_\Gamma$ is simply-connected; accordingly, let $\Phi : \hat{\mathbb{C}} - \overline{\mathbb{D}} \to \hat{\mathbb{C}} - \Lambda_\Gamma$ be the Riemann-mapping, normalized to be symmetric about the real axis (keeping its orientation) and fixing ∞. Set $\mathcal{O}_i := \Phi^{-1}(V_i)$ for $i = 0, 1, \ldots, 5$ and $\Omega_j := \Phi^{-1}(W_j)$ for $j = 0, 1$.

3) Let $f : \bigcup_{i=0}^{5} \mathcal{O}_i \to \Omega_0 \cup \Omega_1$ be the mapping $\Phi^{-1} \circ F \circ \Phi$ where F is the auxiliary transformation associated to Γ (see II §1). Then each restriction $f|\mathcal{O}_i$ is schlicht and maps $\mathcal{O}_i$ onto either Ω_0 or Ω_1, depending on whether i is even or odd, respectively.

4) If we write $I_i := \partial\mathcal{O}_i \cap \partial\mathbb{D}$, $i = 0, 1, \ldots, 5$, then each I_i is a closed interval, and these intervals are pairwise disjoint. It is easy to see that each schlicht mapping $f|\mathcal{O}_i$ carries the corresponding I_i onto $\overline{(\partial\mathbb{D})^+}$ when i is even, and onto $\overline{(\partial\mathbb{D})^-}$ when i is odd.

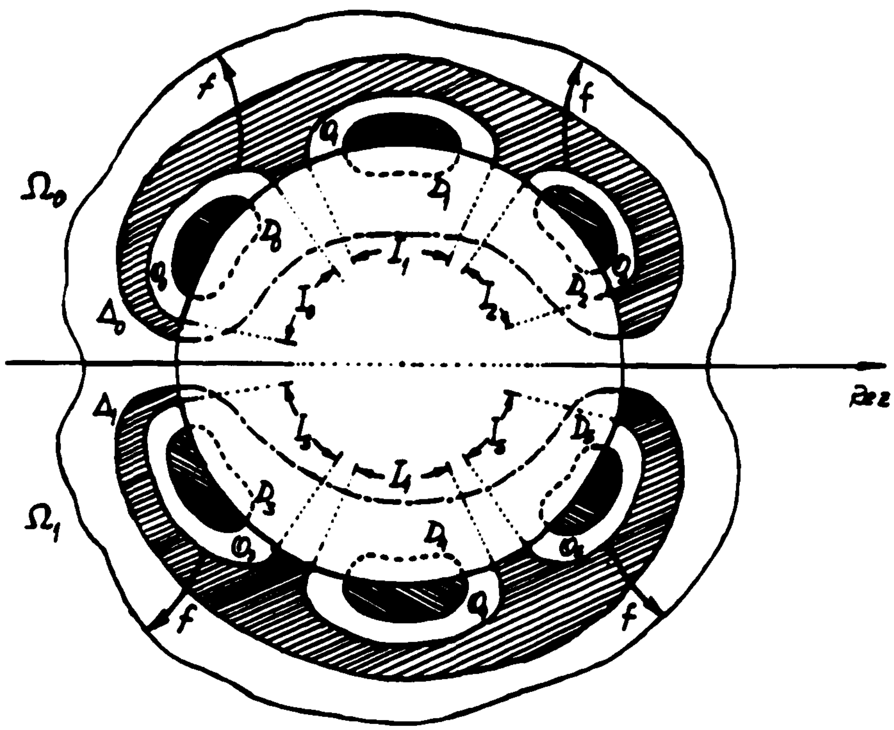

Figure 1

5) Next, let $\Delta_0 \subsetneq \mathbf{C}^+$ be a Jordan domain such that: (a) Δ_0 is symmetric about $\partial\mathbb{D}$ (with respect to geometric inversion); (b) $\overline{\Delta}_0 \cap (\hat{\mathbf{C}} - \overline{\mathbb{D}}) \subsetneq \Omega_0$; (c) $\overline{\mathcal{O}}_0 \cup \overline{\mathcal{O}}_1 \cup \overline{\mathcal{O}}_2 \subsetneq \Delta_0$. Let $\Delta_1 \subsetneq \mathbf{C}^-$ be similarly defined (or equivalently, take it to

be the mirror image of Δ_0 across the real axis).

6) Finally, consider the inverse mappings $f_i := f^{-1} : \Omega_0 \to \mathcal{O}_i$ for i even. Restrict each f_i to $\Delta_0 \cap (\hat{\mathbf{C}} - \overline{\mathbb{D}})$ and then extend the corresponding restriction to Δ_0 by Schwarz's reflection: this is made possible by 4) above. We continue to denote these extensions by the same names, so now set $D_i := f_i(\Delta_0)$, i even. Define D_i for i odd in similar fashion, using Δ_1 (see Figure 1). We have (at last!) constructed $f : (D_0 \cup \cdots \cup D_5) \to \Delta_0 \cup \Delta_1$, a Cantor repeller as defined in §1.

7) It should be clear that a further conjugation of f by a suitable Möbius transformation puts our Cantor repeller in the linear form examined in §3.

8) Since the small dynamical interval of Γ (cf. II §1) lies in the interior of the domain of Γ and is forward invariant under F, the following key property holds true: for each $n \geq 0$, the image of $[f^{-n}(\bigcup_{i=0}^{5} D_i)] \cap (\hat{\mathbf{C}} - \overline{\mathbb{D}})$ under the Riemann map Φ is of the form $\mathcal{O} - \Lambda_\Gamma$, where $\mathcal{O}$ is some neighborhood of the small dynamical interval of Γ. This will be used in full in §4.

9) We remark, merely for the sake of completeness, that the subshift of finite type associated to f has the following transition matrix:

$$A := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

(whose characteristc polynomial is $p_A(\lambda) = \lambda^4(\lambda - 1)(\lambda - 3)$).

IV.3 Around Sullivan's Almost Geodesic Lemma

In [S1], Sullivan defined Beltrami differentials (Beltrami vectors) and quadratic differentials on a compact Riemann surface lamination X as cross-sections of suitable tensor bundles over X. Thus, he defined a Beltrami differential μ locally on each flow-box chart $(D_\alpha \times T_\alpha, \psi_\alpha)$ (see §1) as a Borel measurable function $\mu_\alpha : D_\alpha \times T_\alpha \to$ **C** satisfying: (a) $\mu_\alpha(\cdot, t) \in \mathcal{L}^\infty(D_\alpha)$ for each $t \in T_\alpha$, and the map $t \mapsto \mu_\alpha(\cdot, t)$ is continuous if we provide $\mathcal{L}^\infty(D_\alpha)$ with the weak topology; (b) if $\psi_{\alpha\beta}$ denotes the chart transition $\psi_\beta \circ \psi_\alpha^{-1}$ and we write $\psi_{\alpha\beta} = (\psi^z_{\alpha\beta}, \psi^t_{\alpha\beta})$ then, over the domain of $\psi_{\alpha\beta}$, we have:

$$(*) \qquad \mu_\alpha = \frac{\overline{\partial \psi^z_{\alpha\beta}}}{\partial \psi^z_{\alpha\beta}} \mu_\beta \circ \psi_{\alpha\beta} \ .$$

Sullivan defined also quadratic differentials on X essentially as the corresponding "dual" objects. More precisely, a quadratic differential φ on X is an assignment of a σ-finite measure class $[m_\alpha]$ to the transversal T_α of each flow-box chart satisfying: (a) the transversal components $\psi^t_{\alpha\beta}$ of chart transitions are absolutely continuous as maps $(T_\alpha, [m_\alpha]) \to (T_\beta, [m_\beta])$; (b) for each choice of representative $m_\alpha \in [m_\alpha]$ there exists a measurable function $\varphi_\alpha : D_\alpha \times T_\alpha \to$ **C** such that, on overlappings:

$$(**) \qquad \varphi_\alpha = \varphi_\beta \circ \psi_{\alpha\beta} . \left[\frac{\partial \psi^z_{\alpha\beta}}{\partial z}\right]^2 . J_{\psi^t_{\alpha\beta}} \ ,$$

where $J_{\psi^t_{\alpha\beta}}$ is the Jacobian of $\psi^t_{\alpha\beta}$ with respect to the measures m_α and m_β; (c) each φ_α is integrable with respect to the product measure $dz\, d\bar{z}\, dm_\alpha$ on $D_\alpha \times T_\alpha$. It follows from this definition that there exists a well-defined measure $d|\varphi|$ associated to a quadratic differential on X; its expression on a given chart is $|\varphi_\alpha|\, dz\, d\bar{z}\, dm_\alpha$ for each choice of measure m_α, and any two such choices, by $(**)$, differ by the Jacobian of the identity with respect to both transversal measures, which is nothing but their Radon-Nikodym derivative. If the total mass $|\varphi| := \int_X d|\varphi|$ is finite, we say that φ is an ***integrable*** quadratic differential; we call $|\varphi|$ the ***norm*** of φ.

A quadratic differential is said to be ***holomorphic*** if it is holomorphic on almost all leaves with respect to the transversal measure class that it defines.

Sullivan's definitions are set-up so that the natural pairing $< \nu, \varphi > := \int_X \nu\varphi$ is well-defined, whenever ν is an (essentially bounded) Beltrami vector and φ is an integrable quadratic differential on X. In terms of such pairing, one can formulate the following:

Definition 3.1: The Teichmüller norm of a Beltrami vector ν is $|\nu|_T := \sup |\int_X \nu\varphi|$, where the supremum is taken over all holomorphic, integrable quadratic differentials on X of norm $|\varphi| = 1$.

Definition 3.2: Given $\varepsilon \geq 0$, a Beltrami vector ν on X is called ε-extremal if $|\nu|_\infty \leq (1+\varepsilon)|\nu|_T$.

Unlike the case of ordinary quadratic differentials on Riemann surfaces, a holomorphic quadratic differential on X does not give rise to a metric on the leaves of X, but yields instead, in very rough terms, a "metric up to a multiple" along each leaf. Rather than trying to make sense out of this in general, we explain briefly what we mean in the particular case of interest to us.

Let $\{\psi_t : X \to X\}$ be a leafwise isotopy between $\psi_0 = id_x$ and a QCL-isomorphism (§1) $\psi_1 = \psi$. Let $p \in X$ be a point of differentiability for the horizontal component ψ^z, and let u be a tangent vector at p to the leaf through p. Then the ratio of lenghts between $D\psi^{t}(p).u$ and u, denoted $D_\psi(p,u)$, can be defined as follows: given a flow-box chart at p, a choice of measure m_α on the corresponding transversal T_α yields locally a holomorphic quadratic differential φ_α, and therefore a conformal metric $\sqrt{\varphi_\alpha}\,|dz|$, on the leaf through p. Let φ_α (and hence the metric) be analytically continued along the path $\psi_t(p)$; define $D_\psi(p,u)$ as the ordinary ratio of lengths of the above vectors with respect to this continued metric. Since any two choices for m_α yield φ_α's that differ by a positive multiple along the leaf in question, $D_\psi(p,u)$ is well-defined. Now define a (measurable) function $D_\psi : X \to \mathbf{C}$ by $D_\psi(p) = D_\psi(p,u)$ where u is *any* vector tangent at p to the leaf through p which is also tangent to the corresponding trajectory through p on that leaf.

We are ready, thus, to state Sullivan's *generalized Grötszch inequality*:

Theorem 3.3: If φ is a holomorphic, integrable quadratic differential of norm one on X and if $\{\psi_t\}$ is a leafwise isotopy to the identity whose time-one map $\psi_1 = \psi : X \to X$ is a QCL-isomorphism, then we have $\int_X D_\psi\, d|\varphi| \geq 1$. □

As a corollary to this theorem, Sullivan obtains his *almost geodesic lemma*, or *non-coiling principle*, which we state below.

Just as in the case of ordinary Teichmüller theory, a Beltrami vector ν on X always gives rise to a path of Beltrami differentials μ_t with $\mu_0 = 0$ and $\frac{d}{dt}\mu_t|_{t=0} = \nu$, obtained by "stretching the original RSL-structure on X in the direction of ν". We have in fact a map $t\nu \mapsto \mu_t$, akin to the exponential map in Riemannian geometry. In general, after integration via the MRMT along the leaves of X, $\{\mu_t\}$ does *not* yield a path of RSL-structures on X (the integrated maps obtained in this fashion may fail to be transversally continuous). If it does, however, we call ν a *special*

Beltrami vector, and denote the RSL-structure given by μ_t by $c_t(\nu)$; we also write $c_0 = c_0(\nu)$ for the standard RSL-structure on X.

Example: If (U, V, f) is some holomorphic repeller as in §1 and $X := \mathcal{L}(U, V, f)$ (Theorem 1.7) then, given any f-invariant Beltrami vector on the Riemann surface $V_0 = V - \Lambda_f$, we can pull it back via the natural projection to the inverse limit space V_∞, in the notation of that theorem, and then project it down to the direct limit space V_∞^∞ to get a ϕ^∞-invariant Beltrami vector, and therefore a special Beltrami vector on $X = V_\infty^\infty / < \phi^\infty >$.

The almost geodesic lemma can therefore be stated as follows:

Theorem 3.4: Let ν be an ε-extremal special Beltrami vector on X for some $\varepsilon > 0$, and let $\{\psi_t\}_{0 \leq t \leq 1}$ be a leafwise QC-isotopy between (X, c_0) and $(X, c_l(v))$ for some $l > 0$. Then, if K denotes the maximal dilatation of ψ_1, there exists a universal positive function $\delta = \delta(\varepsilon, l)$ with $\delta(\varepsilon, l) \to 0$ as $\varepsilon \to 0$, such that $l \leq K(1 + \delta)$. □

For detailed proofs of Theorems 3.3 and 3.4, the reader is referred to the forthcoming book by de Melo-van Strien [MS].

The form of Theorem 3.4 which seems most appropriate for renormalization contraction (§4) is the following. Given a special Beltrami vector on X, let the QC-*distance* between c_0 and $c_t(\nu)$ be measured by $\inf \log K(\psi)$ where ψ ranges over all possible QCL-isomorphisms leafwise isotopic to the identity between (X, c_0) and $(X, c_t(\nu))$. Then we have:

Corollary 3.5: Let $l > l'$ be given. Then there exists a constant $0 < \lambda(l, l') < 1$

such that, for every special Beltrami vector ν on X for which c_0 and $c_l(\nu)$ are within a QC-distance $\log l'$, we have: $|\nu|_T \leq \lambda(l, l')|\nu|_\infty$.

Proof: Suppose that for each $\varepsilon > 0$ we can find a special Beltrami vector ν_ε such that c_0 and $c_l(\nu_\varepsilon)$ are within a QC-distance $\log l'$, and yet $|\nu_\varepsilon| \geq (1 - \varepsilon)|\nu_\varepsilon|_\infty$. This last inequality tells us that ν_ε is $\bar{\varepsilon}$-extremal, where $\bar{\varepsilon} = O(\varepsilon)$. By Theorem 3.4, we must have $l \leq l'(1 + \delta(l, \bar{\varepsilon}))$. Choosing ε small enough so that $\delta(l, \bar{\varepsilon}) < \frac{l}{l'} - 1$, we get a contradiction. □

IV.4 Renormalization Contraction

Let Γ be a holomorphic commuting pair with irrational rotation number $\rho(\Gamma) = [r_0, r_1, \ldots, r_n, \ldots]$ of bounded combinatorial type: $\bar{r} = \max r_n < \infty$. Consider the compact Riemann surface lamination L_Γ (up to RSL-isomorphism) constructed in §1, theorem 1.7, for the Cantor repeller of §2 associated to Γ.

A Beltrami differential μ on the annulus $\Omega_\Gamma := \Delta_\Gamma - D_\Gamma$ represents an element $[\Gamma^\mu] \in \text{Teich}(\Gamma)$ (see II §4), for after corollary II.5.5 we know that Ω_Γ is a fundamental domain for Γ. Here μ is of course assumed to be symmetric about the real axis.

Definition 4.1: The geometric parameter $\delta(\mu) = \text{mod}(\Omega_{\Gamma^\mu})$, where $\Omega_{\Gamma^\mu} := \Delta_{\Gamma^\mu} - D_{\Gamma^\mu}$, is called the *conformal type* of μ.

Next, let ν be a Beltrami vector on Ω_Γ^μ tangent to Teich(Γ) at $[\Gamma^\mu] \equiv [\mu]$. Recall that the Teichmüller norm of ν is by definition $|\nu|_T = \sup|\int_{\Omega_{\Gamma^\mu}} \nu\varphi|$, the supremum being taken over all holomorphic quadratic differentials in $L^1(\Omega_{\Gamma^\mu})$ of norm 1, symmetric about the real axis.

Beltrami differentials or vectors on $\Omega_\Gamma(\Omega_{\Gamma^\mu})$ obviously give rise to Γ-invariant (Γ^μ-invariant) Beltrami differentials or (vectors) on $\Delta_\Gamma(\Delta_{\Gamma^\mu})$. These in turn can be lifted to $\tilde{L}_\Gamma(\tilde{L}_{\Gamma^\mu})$ and then projected down to L_Γ. Here $\tilde{L}_\Gamma(\tilde{L}_{\Gamma^\mu})$ denotes the non-compact Riemann surface lamination obtained by the inverse limit construction of Theorem 1.7 just *before* the direct limit is taken to produce $L_\Gamma(L_{\Gamma^\mu})$.

We are ready to state the renormalization contraction theorem.

Theorem 4.2: If μ has conformal type δ, there exist constants $C = C(\delta, \bar{r}) > 0$,

$\lambda = \lambda(\delta, \bar{r})$ with $0 < \lambda < 1$ and a positive integer $n_0 = n_0(\delta, \bar{r})$ such that, for all $n \geq n_0$, $|\mathcal{R}^n \nu|_T \leq C\lambda^n |\nu|_T$ for every Beltrami vector ν at $[\mu]$.

Proof: 1) By the Hahn-Banach theorem and the Riesz representation theorem, there exists a Beltrami vector $\tilde{\nu}$ on Ω_{Γ^μ} such that $|\tilde{\nu}|_\infty = |\tilde{\nu}|_T$ and having the same holomorphic periods as ν, i.e., such that $\int_{\Omega_{\Gamma^\mu}} \tilde{\nu}\varphi = \int_{\Omega_{\Gamma^\mu}} \nu\varphi$ for each L^1-integrable. holomorphic quadratic differential φ in Ω_{Γ^μ}.

2) Observe that $\int_{\Omega_n} (\mathcal{R}^n \tilde{\nu})\varphi = \int_{\Omega_n} (\mathcal{R}^n \nu)\varphi$, where $\Omega_n := \Omega_{\mathcal{R}^n \Gamma^\mu}$, for each holomorphic quadratic differential φ in $L^1(\Omega_n)$. Indeed, the difference $\nu - \tilde{\nu}$ being infinitesimally trivial, we can write it as $\bar{\partial} V$ for some $h \log h^{-1}$-continuous vector field V on Ω_{Γ^μ}. Its n-th renormalization V_n can be extended to all of Ω_n by setting it equal to zero at the intersection of Ω_n with the limit set of Γ^μ, and such extension still has that same modulus of continuity (this is the infinitesimal version of Bers' quasiconformal sewing lemma II.2.1). In this fashion we have $\mathcal{R}^n \nu - \mathcal{R}^n \tilde{\nu} = \bar{\partial} V_n$, whence $\mathcal{R}^n \nu$ and $\mathcal{R}^n \tilde{\nu}$ have the same holomorphic periods as claimed; in particular, their Teichmüller norms are equal, $\forall n \geq 0$.

3) Deforming Γ^μ along $\tilde{\nu}$ a distance l to be chosen below, we get that under any number of renormalizations larger than some n_1 depending on l and on the complex bounds of Chapter III, the endpoints of such deformation path are brought to within a quasiconformal distance $l' = l'(\bar{r})$. Thus, choose $l = 2l'$, say.

4) For each $n \geq 0$, lift the above n-th renormalized path of conformal structures to a corresponding path on the lamination $L_{\mathcal{R}^n \Gamma^\mu}$. Lift also $\mathcal{R}^n \tilde{\nu}$, obtaining the special Beltrami vector $\mathcal{R}^n \hat{\nu}$, on $L_{\mathcal{R}^n \Gamma^\mu}$. Applying the almost geodesic lemma in the

form given by Corollary 3.5, we have:

$$|\mathcal{R}^n\hat{\nu}|_T \leq \hat{\lambda}(l,l').|\mathcal{R}^n\hat{\nu}|_\infty \leq \hat{\lambda}(l,l').|\hat{\nu}|_\infty = \hat{\lambda}(l,l').|\hat{\nu}|_T$$

for all $n \geq n_1$, where $0 < \hat{\lambda}(l,l') < 1$.

5) By Theorem 3.6 we have $|\mathcal{R}^n\tilde{\nu}|_T \leq \overline{C}|\mathcal{R}^n\hat{\nu}|_T$ for some constant $\overline{C} > 0$, for all $n \geq n_2$, where $\overline{C}$ and n_2 depend only on the conformal type of μ. At the same time, $|\hat{\nu}|_T \leq |\tilde{\nu}|_T$, because every holomorphic quadratic differential on L_{Γ^μ} projects down to a Γ^μ-invariant quadratic differential system on Δ_{Γ^μ}.

Therefore, setting $n_0 = \max\{n_1, n_2\}$, we have $|\mathcal{R}^n\tilde{\nu}|_T \leq C.\lambda^n|\tilde{\nu}|_T$ for some $C = C(\delta,\bar{r}) > 0$ and some $\lambda = \lambda(\delta,\bar{r}) \in (0,1)$, for all $n \geq n_0$. □

Integrating the inequalities in this theorem yields:

Corollary 4.3: The quasiconformal conjugacy distance between holomorphic commuting pairs with same rotation number of bounded combinatorial type is contracted exponentially under normalization. In other words, if $\rho(\Gamma_1) = \rho(\Gamma_2) = [r_0, r_1, \ldots, r_n, \ldots]$ with $\bar{r} = \max r_n < \infty$ then $d_T([\mathcal{R}^n\Gamma_1], [\mathcal{R}^n\Gamma_2]) \leq C_0\lambda_0^n d_T([\Gamma_1], [\Gamma_2])$ for all $n \geq 0$ for certain constants $C_0 > 0$ and $0 < \lambda_0 < 1$. □

This is nothing but Theorem B_2 of the introduction.

Observing that, due to the complex bounds of Chapter III, $\mathcal{R}^n\Gamma_1$ and $\mathcal{R}^n\Gamma_2$ are eventually defined on *definite* neighborhoods of their respective dynamical intervals, we get:

Corollary 4.4: Renormalization also contracts the quasi-symmetric conjugacy distance between real commuting pairs in the Epstein-Lanford class. In other words, if $\zeta_f, \zeta_g \in \mathcal{EL}$ and $\rho(f) = \rho(g)$ is of bounded combinatorial type as above then for all $n \geq 0$ we have $d_{QS}(\mathcal{R}^n\zeta_f, \mathcal{R}^n\zeta_g) \leq C_1\lambda_1^n d_{QS}(\zeta_f, \zeta_g)$ for certain constants $C_1 > 0$ and $0 < \lambda_1 < 1$.

As already observed in Chapter I, this last corollary combined with Carleson's theorem in [Crl] yields Theorem B of the Introduction ($C^{1+\alpha}$ conjugacy), and with it, the Main Theorem.

BIBLIOGRAPHY

[A1] Ahlfors, L.V. : *Lectures on Quasiconformal Mappings.* Van-Nostrand, 1966.

[A2] Ahlfors, L.V. : *Conformal Invariants: Topics in Geometric Function Theory.* McGraw-Hill, New-York, 1973.

[B] Bers, L. : *Quasiconformal mappings and Teichmüller's theorem.* In R. Nevanlinna et al. (eds.): Analytic Functions, pp. 89-119. Princeton University Press, Princeton, 1960.

[Be] Beardon, A. : *The Geometry of Discrete Groups.* Springer-Verlag, Berlin and New York, 1983.

[Bo] Bowen, R. : *On Axiom A Diffeomorphisms.* Regional Conference Series in Mathematics, 35, Providence AMS, 1977.

[C] Carathéodory, C. : *Theory of Functions of a Complex Variable.* Chelsea, New York, 1954-64 (in 2 vols.).

[CE] Collet, P., Eckmann, J.-P. : *Iterated maps of the interval as dynamical systems.* Birkhauser, Boston, 1980.

[Crl] Carleson, L. : *On mappings conformal at the boundary.* J. Anal. Math. **19** (1967), 1-13.

[D] Dugundji, J. : *Topology.* Allyn and Bacon, Boston, 1966.

[DH] Douady, A., Hubbard, J. : *On the dynamics of polynomial-like mappings.* Ann. Scient. Ec. Norm. Sup. 4^{eme} série, **18**, 287-343, 1985.

[EE] Eckmann, J.-P., Epstein, H. : *On the existence of fixed points of the composition operator for circle maps.* Commun. Math. Phys. 107, 213-231 (1986).

[EL] Epstein, H., Lascoux, J. : *Analiticity properties of the Feigenbaum function.* Commun. Math. Phys. **81**, 437-53 (1981).

[ES] Eilenberg, S., Steenrod, N.: *Foundations of algebraic topology.* Princeton University Press, Princeton, 1952.

[FKS] Feigenbaum, M., Kadanoff, L., Shenker, S. : *Quasi-Periodicity in dissipative systems. A renormalization group analysis.* Physica **5D**, 370-386 (1982).

[G1] Gardiner, F. : *Teichmüller Theory and Quadratic Differentials.* John Wiley and Sons, New York, 1987.

[G2] Gardiner, F. : *On Teichmüller contraction.* To appear in the Proc. of the Am. Math. Soc. (1993).

[GS] Gardiner, F., Sullivan, D. : *Symmetric and quasi-symmetric structures on a closed curve.* To appear in Am. J. of Math. (1992).

[H] Herman, M. : *Sur la conjugaison differentiable des difféomorphismes du cercle a des rotations.* Publ.Math.I.H.E.S. **49**, 5-234 (1979).

[K] Keen, L. : *Dynamics of holomorphic self-maps of* $\mathbf{C}^*$. In D. Drasin et al (eds.): Holomorphic Functions and Moduli I, Springer-Verlag, New York, 1988.

[L1] Lanford, O. : *Renormalization group methods for circle mappings.* Proceedings of the Conference on Statistical Mechanics and Field Theory: Mathematical aspects, Groningen 1985 (Springer Lecture Notes on Physics, to appear).

[L2] Lanford, O. : *Critical Circle Mappings.* To appear.

[Leh] Lehto, O. : *Univalent functions and Teichmüller spaces.* Springer-Verlag, Berlin and New York, 1986.

[LV] Lehto, O., Virtannen, K. : *Quasiconformal mappings.* Springer-Verlag, Berlin and New York, 1965.

[McK] McKay, R. : *Renormalisation in area-preserving mappings.* Ph.D. Thesis, Princeton University (1982).

[Mes] Mestel, B. : Ph.D. Dissertation. Department of Mathematics, Warwick University (1985).

[Mil] Milnor, J.: *Dynamics in one-complex variable: introductory lectures.* SUNY, Stony Brook, 1990.

[MS] de Melo, W., van Strien, S. : *One-dimensional Dynamics.* In print, Springer-Verlag, Berlin and New York, 1992.

[ORSS] Ostlund, R., Rand, D., Sethna, J., Siggia, E. : *Universal Properties of the Transition from Quasi-Periodicity to Chaos in Dissipative Systems.* Physica **8D**, 303 (1983).

[Ra1] Rand, D. : *Existence, non-existence and universal breakdown of dissipative golden invariant tori. I. Golden critical circle mappings.* Preprint, I.H.E.S., 1989.

[Ra2] Rand, D. : *Universality and renormalisation in dynamical systems.* In T. Bedford and J. W. Swift (eds.): New Directions in Dynamical Systems, Cambridge University Press, Cambridge, 1987.

[Ric] Rickmann, S. : *Removability theorems for quasiconformal mappings.* Ann. Ac. Scient. Fenn., 449, p.1-8 (1969).

[Ru] Rudin, W. : *Real and Complex Analysis*, 2^{nd} edition. McGraw-Hill, New York, 1974.

[S1] Sullivan, D. : *Bounds, quadratic differentials, and renormalization conjectures.* American Mathematical Society Centennial Publications. Volume 2: Mathematics into the Twenty-first Century (1988 Centennial Symposium, August 8-12), American Mathematical Society, Providence, RI., 1991.

[S2] Sullivan, D. : *Bounded structure of infinitely renormalizable mappings.* In P. Cvitanović (ed): Universality in Chaos, 2^{nd} edition, Adam Hilger, Bristol (1989).

[S3] Sullivan, D. : *Differentiable structures on fractal-like sets, determined by intrinsic scaling functions on dual Cantor sets.* Proceedings of Symposia in Pure Mathematics Volume **48** (1988).

[S4] Sullivan, D. : *Quasiconformal homeomorphisms and Dynamics I. Fatou-Julia Problem on Wandering Domains.* Ann. of Math. **122** (1985) 401-418.

[Sh] Shenker, S. : *Scaling behavior of a map of a circle onto itself: empirical results.* Physica **5D**, 405 (1982).

[ST] Sullivan, D., Thurston, W. : *Extending holomorphic motions.* Acta Math. 157 n.3/4 (1986) 243-257.

[Sw1] Świątek, G. : *Rational rotation numbers for maps of the circle.* Commun. Math. Phys. **119**, 109-128 (1988).

[Sw2] Świątek, G. : *One-dimensional maps and Poincaré metric.* Nonlinearity **5** (1992) 81-108.

[Y1] Yoccoz, J.C. : *Conjugaison differentiable des diffeomorphismes du cercle dont le nombre de rotation verifie une condition Diophantienne.* Ann. Sci. de l'Ec. Norm. Sup., **17** (1984) 333-361.

[Y2] Yoccoz, J.C. : *Il n'y a pas de contre-example de Denjoy analytique.* C. R. Acad. Sci. Paris, **298** (1984) série I, 141-144.