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EQUATIONS FOR BENDING OF ELLIPSOIDS AND SPHERES

by

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Abstract

EQUATIONS FOR BENDING OF ELLIPSOIDS AND SPHERES

by

Michael W. Ecker

Adviser: Professor Harry E. Rauch

This paper deals with the nonlinear bending theory of the thin-shelled ellipsoid of revolution, whose middle surface is described by $(x/a)^2 + (y/a)^2 + (z/b)^2 = 1$. The ellipsoid itself is assumed to be of a material which is elastic, homogeneous, isotropic, and Hookean. The surface is externally loaded uniformly, but only normally, thus including, but not limited to, the important special case of fluid pressure. The Kirchoff hypotheses with other appropriate assumptions are freely accepted. Moreover, upon specializing by setting $b = a$, one obtains the nonlinear spherical theory.

The specific aim here is to develop and obtain equations for the ellipsoid which are analogous to the Foeppl-Von Karman equations for the plate. That is, the goal is

to derive two nonlinear, partial differential equations in two functions. These are: a deflection, w, and a stress function artifice, Φ, often referred to as an Airy stress function.

The special contribution here would be to have equations somewhat more accurate than the very general shallow-shell equations in the literature; in particular, more than those of W. T. Koiter. On the other hand, it would also be desirable to have equations that have the accuracy of the linear parts of equations for the special case of the sphere, as found in Tsuboi and Akino's work. Indeed, their equations virtually serve as a model for this work carried out herein.

While much of the foregoing is obtained, the results for the ellipsoid do not come out as perfectly as those for the special spherical case. Nevertheless, the approximations implicitly involved are sufficiently good, and the errors may be considered to be relatively quite small. (Such an approximation proves unnecessary in the case of the sphere.)

This paper will draw freely upon the general results of Lyell J. Sanders. His equations which will be used are: equilibrium (or deflection) equations (expressing force and moment equilibria), strain-displacement relations, and the constitutive relations ("Hooke's Law"). The latter two sets will yield a compatibility equation, and the former two will yield a single equilibrium equation.

The basic problems in this thesis will be: the elimination of the other two deflection variables, u and v, (of the (u,v,w) triad) to obtain the compatibility equation; and, the representation of the stresses in terms of one stress function, Φ, all, of course, subject to the aforementioned relations. And, while the general results for the ellipsoid are much 'messier' than those for the sphere, the linear parts are both amenable to computer work for solutions, and for further comparison with the elasticity literature already in existence.

PREFACE

In the (previously mentioned) Foeppl-Von Karman bending theory of a plate of thickness h under pressure $p = p(x,y)$, two nonlinear partial differential equations are derived:

$$\frac{D}{h}\Delta^2 w = w_{xx}\phi_{yy} - 2w_{xy}\phi_{xy} + w_{yy}\phi_{xx} + \frac{p}{h}$$

$$\Delta^2\phi = E(w_{xy}{}^2 - w_{xx}w_{yy}).$$

(See nomenclature.)

Suitable boundary conditions are assumed (e.g., clamped edges); none, of course, is involved in the ensuing thesis.

However, note the 'structure' of these equations, such as the presence of the double Laplacians. We will see equations somewhat analogous to these, valid for an ellipsoid of revolution, and we may expect these too to be nonlinear, as we do not make the simplifying, but restrictive, assumptions of "small deflections".

The final versions of these equations for the ellipsoid appear as: equation A) on page 12, and equation B), whose right-hand side appears on pages 16, 17, and left-hand side on pages 17, 18. The special spherical case equations A') and B') appear on pages 13 and 19, respectively.

ACKNOWLEDGEMENT

To my adviser, Professor Harry E. Rauch, I acknowledge gratitude for his assistance throughout my research, especially the beginning.

To my typist, many thanks for a fine job. I am confident that any errors will be mine alone.

Last, but certainly not least, my thanks and love to my wonderful wife, Sue, who ultimately made it all possible.

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Preliminaries: Nomenclature/Notation

Throughout, "ellipsoid" and "ellipsoid of revolution" will be used synonymously to refer to the ellipsoidal surface of revolution described by $(x/a)^2 + (y/a)^2 + (z/b)^2 = 1$ in an (x,y,z) cartesian coordinate system.

θ Angle between the axis of revolution (i.e., positive z axis), and line joining origin (0,0,0) to point (x,y,z) on surface. In the spherical case, θ is the colatitude.

ϕ Azimuth.

With the foregoing, we have a 2 parameter representation of the ellipsoid:

$$x = a\cdot\cos\phi\cdot\sin\theta$$

$$y = a\cdot\sin\phi\cdot\sin\theta$$

$$z = b\cdot\cos\theta$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. We consider θ the first variable, and ϕ the second.

h Shell thickness.

ν Poisson's ratio.

E Young's modulus.

D	$= \dfrac{Eh^3}{12(1-\nu^2)}$ ("flexural rigidity").
p	Pressure normal to surface.
$(\quad)$	$= a^2\cos^2\theta + b^2\sin^2\theta$.
$S; \Delta_1, \Delta_2, \Delta_3$	Defined in body where used.
Δ	Laplacian; for ellipsoid, $\Delta f = \dfrac{1}{(\ )} \dfrac{\partial^2 f}{\partial\theta^2} + \dfrac{\cot\theta}{(\quad)} \dfrac{\partial f}{\partial\theta} + \dfrac{1}{a^2\sin\theta} \dfrac{\partial^2 f}{\partial\phi^2} - \dfrac{(b^2 - a^2)\sin\theta\cdot\cos\theta}{(\qquad)^2} \dfrac{\partial f}{\partial\theta}$ (calculated in appendix).
H_0, H_2	$H_0(f) = a^2\cdot\Delta f$; $H_2(f) = H_0(f) + 2f$. H_0, H_2 are usually applied only to case of sphere: $H_0(f) = \dfrac{\partial^2 f}{\partial\theta^2} + \cot\theta \dfrac{\partial f}{\partial\theta} + \dfrac{1}{\sin^2\theta} \dfrac{\partial^2 f}{\partial\phi^2}$.
R_1, R_2	Principal radii of curvature along θ section (ϕ constant) and ϕ section (θ constant).
G	$= 1/R_1R_2$, Gaussian curvature.
$N_{11}, N_{12}; N_{22}, N_{21}$	Normal and shearing forces per unit length of the sections θ = constant and ϕ = constant, respectively.
$M_{11}, M_{12}; M_{22}, M_{21}$	Bending and twisting moments per unit length of the sections θ = constant and ϕ = constant, respectively.

Q_1, Q_2 — Transverse shearing forces per unit length of the sections θ = constant and ϕ = constant, respectively.

u, v, w — Components of the displacement of a point on the middle surface, in the directions of the three axes of a right-handed XYZ coordinate system at the point, where X and Y are directed to increasing θ and ϕ, respectively.

$\varepsilon_{11}, \varepsilon_{22}, \gamma$ — Normal strains in the θ and ϕ directions, and shearing strain.

$\kappa_{11}, \kappa_{22}, \kappa$ — Bending strains.

Φ — Stress function.

$(g_{\alpha\beta})$ — First fundamental form (matrix). (Calculated in appendix).

$(h_{\alpha\beta})$ — Second fundamental form (matrix). (Calculated in appendix).

α_1, α_2 — $= \sqrt{g_{11}}, \sqrt{g_{22}}$, respectively.

, — Denotes partial differentiation with respect to variable θ or ϕ, corresponding to subscript 1 or 2. (Occasionally, where θ or ϕ appears as subscript alone, this will also indicate partial differentiation, regardless of whether or not comma is used.)

ϕ_1, ϕ_2 — Rotations.

Existing Equations

We will begin by taking for granted the sets of equations to follow, most of which appear in Sanders' paper [3]. Those from Sanders have had originally starred terms dropped (reflecting assumptions in the theory that are not unusual).

In the following, due to our assumptions (e.g., material being isotropic, Hookean, etc., and with z axis equilibrium) we will have, and freely utilize, such symmetry relations as

$$\varepsilon_{12} = \varepsilon_{21}, \quad \kappa_{12} = \kappa_{21}, \quad N_{12} = N_{21}, \quad M_{12} = M_{21}.$$

Sanders points out that the linearized forms of several of his equations coincide with those of small deflection theories.

The equations we will draw freely upon are now listed (in the forms we use):

Force and Moment Equilibria

F 1) $(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{12})_{,2} + \alpha_{1,2} N_{12} - \alpha_{2,1} N_{22} = 0$

F 2) $(\alpha_1 N_{22})_{,2} + (\alpha_2 N_{12})_{,1} + \alpha_{2,1} N_{12} - \alpha_{1,2} N_{11} = 0$

F 3) $(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2} - \alpha_1 \alpha_2 (R_1^{-1} N_{11} + R_2^{-1} N_{22})$

$- (\alpha_2 \phi_1 N_{11} + \alpha_2 \phi_2 N_{12})_{,1} - (\alpha_1 \phi_1 N_{12} + \alpha_1 \phi_2 N_{22})_{,2}$

$+ \alpha_1 \alpha_2 p = 0$

M 1) $(\alpha_2 M_{11})_{,1} + (\alpha_1 M_{12})_{,2} + \alpha_{1,2} M_{12} - \alpha_{2,1} M_{22} - \alpha_1 \alpha_2 Q_1 = 0$

M 2) $(\alpha_1 M_{22})_{,2} + (\alpha_2 M_{12})_{,1} + \alpha_{2,1} M_{12} - \alpha_{1,2} M_{11} - \alpha_1 \alpha_2 Q_2 = 0$

In the foregoing and below, the ϕ_j terms are rotations given by:

$$\phi_1 = -\alpha_1^{-1} w_{,1} \quad \text{and} \quad \phi_2 = -\alpha_2^{-1} w_{,2}.$$

Note also that for our situation, since α_1 and α_2 are independent of ϕ (our second independent parameter for the ellipsoid of revolution) we have $\alpha_{1,2} = 0 = \alpha_{2,2}$. The former ($\alpha_{1,2} = 0$) will be used in the above and below.

We also have:

<u>Strain-Displacement Relations</u>

(Middle Surface Strains and Bending Strains)

MS 1) $\varepsilon_{11} = (\alpha_1\alpha_2)^{-1}(\alpha_2 u_{,1} + \alpha_{1,2} v + \alpha_1 \alpha_2 R_1^{-1} w + \frac{1}{2}\alpha_1\alpha_2\phi_1^2)$

MS 2) $\varepsilon_{22} = (\alpha_1\alpha_2)^{-1}(\alpha_1 v_{,2} + \alpha_{2,1} u + \alpha_1 \alpha_2 R_2^{-1} w + \frac{1}{2}\alpha_1\alpha_2\phi_2^2)$

MS 3) $\gamma = 2\varepsilon_{12} = (\alpha_1\alpha_2)^{-1}(\alpha_2 v_{,1} + \alpha_1 u_{,2} - \alpha_{1,2} u - \alpha_{2,1} v + \alpha_1\alpha_2\phi_1\phi_2)$

and:

BS 1) $\kappa_{11} = (\alpha_1\alpha_2)^{-1}(\alpha_2\phi_{1,1} + \alpha_{1,2}\phi_2)$

BS 2) $\kappa_{22} = (\alpha_1\alpha_2)^{-1}(\alpha_1\phi_{2,2} + \alpha_{2,1}\phi_1)$

BS 3) $\kappa = 2\kappa_{12} = (\alpha_1\alpha_2)^{-1}(\alpha_2\phi_{2,1} + \alpha_1\phi_{1,2} - \alpha_{1,2}\phi_1 - \alpha_{2,1}\phi_2)$

The foregoing all are from Sanders. Middle surface strains are assumed small, and rotations, moderately small.

We will use these <u>moment-bending strain relations</u>:

MB 1) $M_{11} = D(\kappa_{11} + \nu\kappa_{22})$

MB 2) $M_{22} = D(\kappa_{22} + \nu\kappa_{11})$

MB 3) $M_{12} = (1 - \nu)D\kappa \quad (= 2(1 - \nu)D\kappa_{12})$

Later, we will make recourse to:

<u>Constitutive Relations</u>

C 1) $\varepsilon_{11} = \frac{1}{Eh}(N_{11} - \nu N_{22})$

C 2) $\varepsilon_{22} = \frac{1}{Eh}(N_{22} - \nu N_{11})$

C 3) $\gamma = 2\varepsilon_{12} = \frac{2(1 + \nu)}{Eh}N_{12}.$

Note that in each set of equations, where there is a third equation, it is symmetric in the subscripts 1 and 2 (provided we treat u and v as u_1 and u_2). Also, equation 2 of each set may be obtained from equation 1 by this same 1 vs. 2 interchange.

<u>Derivation of Equilibrium/Deflection Equation</u>

To illustrate the natural way in which such operators as Δ and Δ^2 enter into the equations, we will go as far as we can deriving our first equation in abstracto. Only near the end will the various parameters (e.g., α_1, α_2) be replaced by their concrete versions in ordinary notation. (The second equation will be dealt with in concrete form.)

In the process, the most tedious calculations, where appropriate, are either relegated to the appendix, or, where unnecessary, are omitted.

We use equation F 3) and simplify via F 1), F 2):

$$(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2} - \alpha_1\alpha_2(R_1^{-1}N_{11} + R_2^{-1}N_{22}) + \alpha_1\alpha_2 p$$

$$= (\alpha_2\phi_1 N_{11} + \alpha_2\phi_2 N_{12})_{,1} + (\alpha_1\phi_1 N_{12} + \alpha_1\phi_2 N_{22})_{,2}$$

(use of F 1), F 2) yields)

$$= \phi_1(\alpha_{2,1}N_{22} - \overset{0}{\cancel{\alpha_{1,2}N_{12}}}) + \phi_2(\overset{0}{\cancel{\alpha_{1,2}N_{11}}} - \alpha_{2,1}N_{12})$$

$$+ \alpha_2 N_{11}\phi_{1,1} + \alpha_2 N_{12}\phi_{2,1}$$

$$+ \alpha_1 N_{12}\phi_{1,2} + \alpha_1 N_{22}\phi_{2,2}.$$

Using equations M 1) and M 2) to obtain expressions for Q_1 and Q_2, replacing ϕ_1 and ϕ_2, and using equations MB 1), MB 2), MB 3), as well as BS 1), BS 2), BS 3), we get, after performing all the substitutions and differentiations:

$$(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2} = \frac{-D\alpha_2 w_{,1111}}{\alpha_1^3} + \frac{6D\alpha_2\alpha_{1,1}w_{,111}}{\alpha_1^4}$$

$$- \frac{2D\alpha_{2,1}w_{,111}}{\alpha_1^3} + \frac{7D\alpha_{1,1}\alpha_{2,1}w_{,11}}{\alpha_1^4} + \frac{D\nu\alpha_{1,1}\alpha_{2,1}w_{,11}}{\alpha_1^4}$$

$$+ \frac{4D\alpha_2\alpha_{1,11}w_{,11}}{\alpha_1^4} - \frac{15D\alpha_2\alpha_{1,1}^2 w_{,11}}{\alpha_1^5} + \frac{D\alpha_{2,1}^2 w_{,11}}{\alpha_1^3\alpha_2}$$

$$- \frac{D\alpha_{2,11}w_{,11}}{\alpha_1^3} - \frac{D\nu\alpha_{2,11}w_{,11}}{\alpha_1^3} + \frac{2D\alpha_{2,1}\alpha_{2,11}w_{,1}}{\alpha_1^3\alpha_2}$$

$$+ \frac{D\alpha_2\alpha_{1,111}w_{,1}}{\alpha_1^4} + \frac{15D\alpha_2\alpha_{1,1}^3 w_{,1}}{\alpha_1^6} - \frac{D\alpha_{2,1}^3 w_{,1}}{\alpha_1^3\alpha_2^2}$$

$$- \frac{10D\alpha_2\alpha_{1,1}\alpha_{1,11}w_{,1}}{\alpha_1^5} + \frac{2D\alpha_{1,11}\alpha_{2,1}w_{,1}}{\alpha_1^4} + \frac{D\nu\alpha_{1,11}\alpha_{2,1}w_{,1}}{\alpha_1^4}$$

$$- \frac{7D\alpha_{1,1}^2\alpha_{2,1}w_{,1}}{\alpha_1^5} - \frac{3D\alpha_{1,1}\alpha_{2,1}^2 w_{,1}}{\alpha_1^4\alpha_2} + \frac{D\alpha_{1,1}\alpha_{2,11}w_{,1}}{\alpha_1^4}$$

$$- \frac{D\nu\alpha_{2,111}w_{,1}}{\alpha_1^3} + \frac{4D\nu\alpha_{1,1}\alpha_{2,11}w_{,1}}{\alpha_1^4} - \frac{4D\nu\alpha_{1,1}^2\alpha_{2,1}w_{,1}}{\alpha_1^5}$$

$$- \frac{D\alpha_1 w_{,2222}}{\alpha_2^3} - \frac{3D\alpha_{1,1}\alpha_{2,1}w_{,22}}{\alpha_1^2\alpha_2^2} + \frac{D\nu\alpha_{1,1}\alpha_{2,1}w_{,22}}{\alpha_1^2\alpha_2^2}$$

$$+ \frac{3D\alpha_{2,11}w_{,22}}{\alpha_1\alpha_2^2} - \frac{D\nu\alpha_{2,11}w_{,22}}{\alpha_1\alpha_2^2} - \frac{4D\alpha_{2,1}^2 w_{,22}}{\alpha_1\alpha_2^3}$$

$$- \frac{2Dw_{,1122}}{\alpha_1\alpha_2} + \frac{2D\alpha_{2,1}w_{,122}}{\alpha_1\alpha_2^2} + \frac{2D\alpha_{1,1}w_{,122}}{\alpha_1^2\alpha_2} .$$

The details are to be found in the appendix, as well as the computation of Δw and $\Delta\Delta w$ in both abstract and concrete forms. We now obtain, upon initial comparisons:

$$(\alpha_2Q_1)_{,1} + (\alpha_1Q_2)_{,2} = -D\alpha_1\alpha_2\Delta\Delta w$$

$$+ \frac{D\alpha_{2,11}w_{,22}}{\alpha_1\alpha_2^2} - \frac{D\nu\alpha_{2,11}w_{,22}}{\alpha_1\alpha_2^2} - \frac{D\alpha_{1,1}\alpha_{2,1}w_{,11}}{\alpha_1^4} + \frac{D\nu\alpha_{1,1}\alpha_{2,1}w_{,11}}{\alpha_1^4}$$

$$+ \frac{D\alpha_{2,11}w_{,11}}{\alpha_1^3} - \frac{D\nu\alpha_{2,11}w_{,11}}{\alpha_1^3} - \frac{3D\alpha_2\alpha_{1,1}\alpha_{1,11}w_{,1}}{\alpha_1^5} + \frac{3D\alpha_2\alpha_{1,11}^2 w_{,1}}{\alpha_1^5}$$

$$- \frac{D\alpha_{1,11}\alpha_{2,1}w_{,1}}{\alpha_1^{\,4}} + \frac{D\nu\alpha_{1,11}\alpha_{2,1}w_{,1}}{\alpha_1^{\,4}} + \frac{4D\alpha_{1,1}^{\,2}\alpha_{2,1}w_{,1}}{\alpha_1^{\,5}} - \frac{4D\nu\alpha_{1,1}^{\,2}\alpha_{2,1}w_{,1}}{\alpha_1^{\,5}}$$

$$- \frac{4D\alpha_{1,1}\alpha_{2,11}w_{,1}}{\alpha_1^{\,4}} + \frac{4D\nu\alpha_{1,1}\alpha_{2,11}w_{,1}}{\alpha_1^{\,4}} + \frac{D\alpha_{2,111}w_{,1}}{\alpha_1^{\,3}} - \frac{D\nu\alpha_{2,111}w_{,1}}{\alpha_1^{\,3}}$$

$$- \frac{D\alpha_{1,1}\alpha_{2,1}w_{,22}}{\alpha_1^{\,2}\alpha_2^{\,2}} + \frac{D\nu\alpha_{1,1}\alpha_{2,1}w_{,22}}{\alpha_1^{\,2}\alpha_2^{\,2}}.$$

If we now utilize concrete versions

$$\left(\text{e.g., } \alpha_2 = a\cdot\sin\theta,\ \alpha_1 = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\right),$$

particularly, α_2, we obtain Δw and further simplifications. We will divide now by $-\alpha_1\alpha_2$ on both sides: We have for the left-hand side of our first equation:

$$\frac{-1}{\alpha_1\alpha_2}\left[(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2}\right] =$$

$$D\left[\Delta\Delta w + \frac{(1-\nu)b^2 \quad w}{(a^2\cos^2\theta + b^2\sin^2\theta)^2} + S\,\frac{\partial w}{\partial\theta}\right]$$

where S is the following:

$$S = (b^2 - a^2)\left[\frac{-4(1-\nu)\sin\theta\cdot\cos\theta}{(\qquad)^3} - \frac{4(1-\nu)(b^2-a^2)\sin\theta\cdot\cos^3\theta}{(\qquad)^4}\right.$$

$$- \frac{3(b^2-a^2)(\cos^2\theta-\sin^2\theta)^2}{(\qquad)^4} + \frac{3(b^2-a^2)\sin\theta\cdot\cos\theta(\cos^2\theta-\sin^2\theta)}{(\qquad)^4}$$

$$- \frac{3(b^2-a^2)^2\sin^3\theta\cdot\cos^3\theta}{(\qquad)^5} + \frac{6(b^2-a^2)^2\sin^2\theta\cdot\cos^2\theta(\cos^2\theta-\sin^2\theta)}{(\qquad)^5}$$

$$- \frac{3(b^2-a^2)^3 \sin^4\theta \cdot \cos^4\theta}{(\qquad)^6}\Bigg]; \quad \text{and where} \quad (\quad) =$$

$(a^2\cos^2\theta + b^2\sin^2\theta)$ throughout. (When $b = a$, this is just a^2.) Δw is explicitly given by:

$$\frac{1}{(\;)} \frac{\partial^2 w}{\partial \theta^2} + \frac{\cot}{(\;)} \frac{\partial w}{\partial \theta} - \frac{(b^2-a^2)\sin\theta \cdot \cos\theta}{(\qquad)^2} \frac{\partial w}{\partial \theta} + \frac{1}{a^2\sin^2\theta} \frac{\partial^2 w}{\partial \phi^2}.$$

($\Delta\Delta w$ is derived in the appendix.)

Note that the coefficient of Δw in our equation contains, aside from $(1 - \nu)$, the factor $\frac{b^2}{(a^2\cos^2\theta + b^2\sin^2\theta)^2}$.

This latter term is precisely the Gaussian curvature of the ellipsoid of revolution. At this time, also note that when $b = a$, $S = 0$, and the last term, $S \frac{\partial w}{\partial \theta}$, drops out.

To obtain the equation's right-hand side, recall that we had:

$$(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2} - \alpha_1\alpha_2(R_1^{-1}N_{11} + R_2^{-1}N_{22}) + \alpha_1\alpha_2 p$$
$$= \phi_1\alpha_{2,1}N_{22} - \phi_2\alpha_{2,1}N_{12} + \alpha_2 N_{11}\phi_{1,1} + \alpha_2 N_{12}\phi_{2,1}$$
$$+ \alpha_1 N_{12}\phi_{1,2} + \alpha_1 N_{22}\phi_{2,2}.$$

We transpose the third and fourth terms on the left, and recall that we divided by $-\alpha_1\alpha_2$. We replace α_1 and α_2 by their equivalents, and substitute in the expressions for α_1, α_2, R_1, R_2. It remains only to eliminate the (modified) stress variables N_{11}, N_{22}, N_{12}. To do this, we introduce a stress function artifice, Φ. We wish to express each of N_{11}, N_{22}, N_{12} in terms of Φ and its partial derivatives. Based on existing work (plate, sphere), we assume that this will involve derivatives up to second order, and no more. It must be kept in mind that the representations must be consistent with equations F 1), F 2) in the sense that, upon substituting the proposed expressions for N_{11}, N_{22}, N_{12}, equations F 1), F 2) should be satisfied identically. In fact, we will now see that this is the case for the sphere, but for the ellipsoid, an implicit approximation is involved.

One way, which is used here, is to momentarily ignore any possible history or interpretation of Φ, and to treat the problem purely formally; i.e., algebraically. Assume we have a representation of form:

$$N_{11} = f_1\Phi_{\phi\phi} + f_2\Phi_{\theta\phi} + f_3\Phi_{\theta\theta} + f_4\Phi_{\phi} + f_5\Phi_{\theta} + f_6\Phi$$

$$N_{12} = g_1\Phi_{\phi\phi} + g_2\Phi_{\theta\phi} + g_3\Phi_{\theta\theta} + g_4\Phi_{\phi} + g_5\Phi_{\theta} + g_6\Phi$$

$$N_{22} = h_1\Phi_{\phi\phi} + h_2\Phi_{\theta\phi} + h_3\Phi_{\theta\theta} + h_4\Phi_{\phi} + h_5\Phi_{\theta} + h_6\Phi$$

where the coefficients (f's, g's, h's) are functions, <u>a priori</u>, of θ and ϕ. These must satisfy F 1) and F 2):

$$a\cdot\sin\theta\,\frac{\partial N_{11}}{\partial\theta} + a\cdot\cos\theta N_{11} + \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,\frac{\partial N_{12}}{\partial\phi}$$

$$= a\cdot\cos\theta N_{22} = 0$$

and

$$\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,\frac{\partial N_{22}}{\partial\phi} + a\cdot\sin\,\frac{\partial N_{12}}{\partial\theta} + 2a\cdot\cos\theta N_{12} = 0.$$

The idea (some details supplied in appendix) is to substitute into each equation, and to combine like terms involving the same derivative of Φ. The coefficients of each of these is set equal to zero to obtain conditions on the coefficient functions introduced above. (Essentially we are using an "undetermined coefficients" approach.)

Many of the functions above are (immediately) necessarily zero; e.g., f_2, f_3, g_1, g_3, h_1, h_2, h_4. In this case, we have 13 equations which 11 functions must satisfy, an over-determined situation in the general case of the ellipsoid of revolution.

To avoid this, we allow ourselves a small error in the general case, subject to these criteria (partly explained afterwards).

- Only the Φ terms may fail to vanish.
- We may assume that the terms involved have same combinations of derivatives of Φ as already exist for spherical case.
- The representation of N_{11}, N_{22}, N_{12} in terms of Φ and derivatives must be an actual generalization of some existing representation for the sphere (e.g., Tsuboi and Akino [4], Rauch's unpublished results [2]).
- Said representation for the sphere (b = a) must satisfy the equations F 1), F 2) identically.

Relative to the last condition, we may achieve our goal for just the sphere, with:

$$N_{11} = \left(\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \cot\theta\,\frac{\partial\Phi}{\partial\theta} + \Phi\right)/a^2$$

$$N_{22} = \left(\frac{\partial^2\Phi}{\partial\theta^2} + \Phi\right)/a^2$$

$$N_{12} = \left(\frac{-1}{\sin\theta}\frac{\partial^2\Phi}{\partial\theta\partial\phi} + \frac{\cot\theta}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)/a^2.$$

When this is done for the ellipsoid, there is little choice; the following representation is probably the only one to meet all the aforementioned requirements:

$$N_{11} = \frac{1}{a^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\cot\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)}\frac{\partial\Phi}{\partial\theta} + \frac{b^2}{(a^2\cos^2\theta + b^2\sin^2\theta)^2}\Phi$$

$$N_{22} = \frac{1}{(a^2\cos^2\theta + b^2\sin^2\theta)}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{(b^2-a^2)\sin\theta\cdot\cos\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^2}\frac{\partial\Phi}{\partial\theta}$$

$$+ \frac{b^2}{(a^2\cos^2\theta + b^2\sin^2\theta)^2}\Phi$$

$$N_{12} = \frac{-1}{a\cdot\sin\theta\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\frac{\partial^2\Phi}{\partial\theta\partial\phi} + \frac{\cot\theta}{a\cdot\sin\theta\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\frac{\partial\Phi}{\partial\phi}.$$

Note that, in both cases, the Φ terms in N_{11} and N_{22} are the same, each being the Gaussian curvature multiplied by Φ.

In the general case of the ellipsoid of revolution, with the loss of one degree of symmetry, we lose one of the identities; i.e., while F 2) is still satisfied identically, F 1) has a "residue" term. That is, rather than having everything on the left-hand side vanishing, we are left with the term

$$\frac{-4(b^2 - a^2)ab^2\sin\theta\cdot\cos^2\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^3}\Phi .$$

Again, note that this vanishes for the case b = a. This term is relatively small, since such stress function artifices, Φ, are generally known to be relatively small compared to their derivatives. Moreover the numerator of the coefficient is bounded, as is the denominator, since

$$\min(a^2,b^2) \leq (a^2\cos^2\theta + b^2\sin^2\theta) \leq \max(a^2,b^2).$$

With the foregoing, the right-hand member of the equilibrium equation is:

$$\frac{\cot\theta}{(\quad)}\frac{\partial w}{\partial\theta}\left[\frac{1}{(\quad)}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{(b^2 - a^2)\sin\theta\cdot\cos\theta}{(\qquad)^2}\frac{\partial\Phi}{\partial\theta} + \frac{b^2\Phi}{(\quad)^2}\right]$$

$$+ \left[\frac{1}{(\quad)}\frac{\partial^2 w}{\partial\theta^2} - \frac{(b^2-a^2)\sin\theta\cdot\cos\theta}{(\qquad)^2}\frac{\partial w}{\partial\theta}\right]\left[\frac{1}{a^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\cot\theta}{(\quad)}\frac{\partial\Phi}{\partial\theta} + \frac{b^2\Phi}{(\quad)^2}\right]$$

$$+ 2\left[\frac{1}{a\cdot\sin\theta\sqrt{\quad}}\frac{\partial^2 w}{\partial\theta\partial\phi} - \frac{\cot\theta}{a\cdot\sin\theta\sqrt{\quad}}\frac{\partial w}{\partial\phi}\right]\left[\frac{-1}{a\cdot\sin\theta\sqrt{\quad}}\frac{\partial^2\Phi}{\partial\theta\partial\phi} + \frac{\cot\theta}{a\cdot\sin\theta\sqrt{\quad}}\frac{\partial\Phi}{\partial\phi}\right]$$

$$+ \frac{1}{a^2\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2}\left[\frac{1}{(\quad)}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{(b^2 - a^2)\sin\theta\cdot\cos\theta}{(\qquad)^2}\frac{\partial\Phi}{\partial\theta} + \frac{b^2\Phi}{(\quad)^2}\right]$$

$$- \frac{ab}{(\quad)^{3/2}}\left[\frac{1}{a^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\cot\theta}{(\quad)}\frac{\partial\Phi}{\partial\theta} + \frac{b^2\Phi}{(\quad)^2}\right]$$

$$- \frac{b}{a\sqrt{\quad}}\left[\frac{1}{(\quad)}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{(b^2 - a^2)\sin\theta\cdot\cos\theta}{(\qquad)^2}\frac{\partial\Phi}{\partial\theta} + \frac{b^2\Phi}{(\quad)^2}\right] + p.$$

(In the foregoing, all blanks are $a^2\cos^2\theta + b^2\sin^2\theta$.)

Certain expressions occur so repeatedly that it is convenient to introduce some notation. We will use G $\left(= \frac{b^2}{(\ )^2}\right)$ to denote the Gaussian curvature, and we define operators Δ_1, Δ_2, and Δ_3 by:

$$\Delta_1 f = \frac{1}{a^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{\cot\theta}{(\ )} \frac{\partial f}{\partial \theta}$$

$$\Delta_2 f = \frac{1}{(\ )} \frac{\partial^2 f}{\partial \theta^2} - \frac{(b^2 - a^2)\sin\theta \cdot \cos\theta}{(\qquad)^2} \frac{\partial f}{\partial \theta}$$

and $$\Delta_3 f = \frac{1}{a \cdot \sin\theta \sqrt{\ }} \left(- \frac{\partial^2 f}{\partial \theta \partial \phi} + \cot\theta \frac{\partial f}{\partial \phi} \right).$$

These consequent relations are interesting:

$$\Delta_1 + \Delta_2 = \Delta \text{ (i.e., } \Delta_1 f + \Delta_2 f = \Delta f)$$

$$N_{11} = \Delta_1 \Phi + G\Phi$$

$$N_{22} = \Delta_2 \Phi + G\Phi$$

$$N_{12} = \Delta_3 \Phi.$$

With this, we may write our equilibrium equation as:

$$D\left[\Delta\Delta w + (1 - \nu)G\Delta w + S \frac{\partial w}{\partial \theta}\right] = (\Delta_1 w)(\Delta_2 \Phi) + (\Delta_2 w)(\Delta_1 \Phi) - 2(\Delta_3 w)(\Delta_3 \Phi) - \frac{(\Delta_1 \Phi + G\Phi)}{R_1} - \frac{(\Delta_2 \Phi + G\Phi)}{R_2} + p.$$

This is our <u>equation A</u>).

In the case of the sphere of radius a, the operators H_0 and H_2 are frequently used. These are given by $H_0 = a^2\Delta$ and $H_2 = H_0 + 2$ (i.e., $H_2(f) = a^2\Delta f + 2f$). The equation above, after multiplication through by a^4, and setting b = a, becomes:

$$\text{A}') \qquad D\left[H_0H_0(w) + (1-\nu)H_0(w)\right] + aH_2(\Phi) =$$

$$\cot\theta \frac{\partial w}{\partial\theta}\left(\frac{\partial^2\Phi}{\partial\theta^2} + \Phi\right) + \frac{\partial^2 w}{\partial\theta^2}\left(\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \cot\theta\frac{\partial\Phi}{\partial\theta} + \Phi\right)$$

$$-\frac{2}{\sin^2\theta}\left(-\frac{\partial^2 w}{\partial\theta\partial\phi} + \cot\theta\frac{\partial w}{\partial\phi}\right)\left(-\frac{\partial^2\Phi}{\partial\theta\partial\phi} + \cot\theta\frac{\partial\Phi}{\partial\phi}\right)$$

$$+\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2}\left(\frac{\partial^2\Phi}{\partial\theta^2} + \Phi\right) + a^4p.$$

Derivation of Compatibility Equation

In the preceding equation, the primary stumbling block was the problem of representing N_{11}, N_{22}, N_{12} in terms of an Airy stress function. That equation, the equilibrium (or deflection) equation, used all the sets of equations except those for the middle surface strains (MS 1), MS 2), MS 3)) and the constitutive relations (C 1), C 2), C3)). It is precisely these equations which will be utilized to obtain a compatibility equation. Here, the major problem will be the elimination of the deflection variables u and v. (As we will see momentarily, we cannot simply drop these in the equations.) We will again have occasion to make use of our representations for N_{11}, N_{22}, N_{12} in this part of the development.

The equations MS 1), MS 2), MS 3) have this concrete form:

$$\varepsilon_{11} = \frac{1}{\sqrt{\quad}}\frac{\partial u}{\partial \theta} + \frac{ab\cdot w}{(\quad)^{3/2}} + \frac{1}{2(\quad)}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$\varepsilon_{22} = \frac{1}{a\cdot\sin\theta}\frac{\partial v}{\partial \phi} + \frac{\cot\theta\cdot u}{\sqrt{\qquad}} + \frac{bw}{a\sqrt{\quad}} + \frac{1}{2a^2\sin^2\theta}\left(\frac{\partial w}{\partial \phi}\right)^2$$

$$\gamma = \frac{1}{\sqrt{\quad}}\frac{\partial v}{\partial \theta} - \frac{\cot\theta\cdot v}{\sqrt{\qquad}} + \frac{1}{a\cdot\sin\theta}\frac{\partial u}{\partial \phi} + \frac{1}{a\cdot\sin\theta\sqrt{\quad}}\frac{\partial w}{\partial \theta}\frac{w}{\partial \phi}.$$

Note that, while the Kirchoff hypotheses imply u and v are relatively small compared to w, the presence of the factor $\cot\theta$ rules out their omission from these equations.

For the sphere, the concrete forms are just slightly simpler, and the following relation is given in Tsuboi-Akino:

$$\frac{1}{\sin\theta}(\varepsilon_{22}\sin\theta)_{,11} - \frac{1}{\sin\theta}(\varepsilon_{11}\cos\theta)_{,1} + \frac{1}{\sin^2\theta}\varepsilon_{11,22} + \varepsilon_{11} + \varepsilon_{22}$$
$$- \frac{1}{\sin^2\theta}(\gamma\cdot\sin\theta)_{,12} + \frac{1}{a}H_2(w) = 0.$$

(Care must be taken; different authors use different conventions regarding signs; i.e., direction of increasing w.)

It is this last relation we seek to generalize, because once that is done, we may use the constitutive relations to eliminate ε_{11}, ε_{22}, γ, and then our earlier representations for N_{11}, N_{22}, N_{12}, to finally obtain our second equation in w and Φ.

Unfortunately again, no clue is given in Tsuboi and Akino as to how to obtain or deduce such a relation as this last one (even though it is easy to verify). Again, consider an "undetermined coefficient function" approach. We assume we have a relation of the form:

$$f_1\varepsilon_{22,11} + f_2\varepsilon_{22,1} + f_3\varepsilon_{11,1} + f_4\varepsilon_{11,22} + f_5\varepsilon_{11} + f_6\varepsilon_{22} + f_7\varepsilon_{12,2} + f_8\varepsilon_{12,12} = L(w)$$

where the f's are functions, and L(w) is some operator with which we are not too concerned at the moment.

If we now substitute in the concrete forms given for ε_{11}, ε_{22}, γ (by MS 1), MS 2), MS 3)), in this assumed relation, we obtain conditions on the functions $f_1, f_2, \ldots, f_8$, since we assume the sums of all terms involving $\frac{\partial^2 u}{\partial \phi^2}$ vanish, of $\frac{\partial v}{\partial \phi}$ vanish, etc. (Further details appear in appendix.)

Again, we have a slightly over-determined situation for the ellipsoid, so we try the next best thing. We ask whether we can allow one of the aforementioned sums of terms to fail to vanish. There is little choice (partially forced by the equations, and partly resolved by the knowledge that u should be smaller than $\frac{\partial v}{\partial \phi}$.) Our relation is now:

$$\frac{a}{\sqrt{}}\varepsilon_{22,11} + \left[\frac{2a\cdot\cot\theta}{\sqrt{}} - \frac{(b^2 - a^2)a\cdot\sin\theta\cdot\cos\theta}{()^{3/2}}\right]\varepsilon_{22,1}$$

$$- \frac{a\cdot\cot\theta}{\sqrt{}}\varepsilon_{11,1} + \frac{\sqrt{}}{a\cdot\sin^2\theta}\varepsilon_{11,22} + \frac{2ab^2}{()^{3/2}}\varepsilon_{11}$$

$$- \frac{2}{\sin^2\theta}(\gamma_{,2}\sin\theta)_{,1} = L(w) + \frac{4(b^2 - a^2)a\cdot\sin\theta\cdot\cos\theta\cdot u}{(\qquad)^2}$$

$$+ \frac{4(b^2 - a^2)^2 a\cdot\sin\theta\cdot\cos^3\theta\cdot u}{(\qquad)^3}.$$

Despite differences in form, this certainly generalizes the one used for the spheres. Moreover, $L(w)$ is just that obtained by transposing all terms involving w and its derivatives, from the expression which is the left-hand member. We simply drop the terms involving u, noting that the trigonometric functions involved here are bounded (unlike $\cot\theta$, as pointed out earlier).

It remains to write out $L(w)$, and then to eliminate ε_{11}, ε_{22}, γ via the constitutive relations. After the simplifications (tedious, but straightforward), we get:

equation B), right side

$$\frac{b}{(\ )}\frac{\partial^2 w}{\partial\theta^2} + \frac{b}{\sin^2\theta(\ )}\frac{\partial^2 w}{\partial\phi^2} + \frac{2b\cdot\cos\theta}{\sin\theta(\ )}\frac{\partial w}{\partial\theta}$$

$$- \frac{a^2 b\cdot\cos\theta}{\sin\theta(\ )^2}\frac{\partial w}{\partial\theta} + \frac{2a^2b^3}{(\ )^3}w - \frac{3(b^2 - a^2)b\cdot\sin\theta\cdot\cos\theta}{(\qquad)^2}\frac{\partial w}{\partial\theta}$$

$$- \frac{3(b^2 - a^2)b\cdot\cos^2\theta}{(\qquad)^2}w + \frac{(b^2 - a^2)b\cdot\sin^2\theta}{(\qquad)^2}w$$

$$+ \frac{3(b^2 - a^2)a^2b\cdot\cos^2\theta}{(\qquad)^3}w + \frac{4(b^2 - a^2)^2 b\cdot\sin^2\theta\cdot\cos^3\theta}{(\qquad)^3}w$$

$$+ \frac{ab^2}{(\ )^{5/2}}\left(\frac{\partial w}{\partial\theta}\right)^2 - \frac{-1}{a\cdot\sin^2\theta\sqrt{\ }}\frac{\partial^2 w}{\partial\theta^2}\frac{\partial^2 w}{\partial\phi^2} + \frac{1}{a\cdot\sin^4\theta\sqrt{\ }}\left(\frac{\partial w}{\partial\theta}\right)^2$$

$$+ \frac{1}{a\cdot\sin^2\theta\sqrt{\ }}\left(\frac{\partial^2 w}{\partial\theta\partial\phi}\right)^2 - \frac{a\cdot\cos\theta}{\sin\theta(\ )^{3/2}}\frac{\partial w}{\partial\theta}\frac{\partial^2 w}{\partial\theta^2} - \frac{2\cos\theta}{a\cdot\sin^3\theta\sqrt{\ }}\frac{\partial w}{\partial\phi}\frac{\partial^2 w}{\partial\theta\partial\phi}$$

$$+ \frac{(b^2 - a^2)a\cdot\cos^2\theta}{(\qquad)^{5/2}}\left(\frac{\partial w}{\partial\theta}\right)^2 + \frac{(b^2 - a^2)\cos^2\theta}{a\cdot\sin^2\theta(\ )^{3/2}}\left(\frac{\partial w}{\partial\phi}\right)^2$$

$$+ \frac{(b^2 - a^2)\cos\theta}{a\cdot\sin\theta(\ )^{3/2}}\frac{\partial w}{\partial\theta}\frac{\partial^2 w}{\partial\phi^2}\ .$$

(The first 5 terms are linear; the next 5 are also linear but drop out when b = a. The succeeding 6 terms are nonlinear; the remaining 3 terms are also nonlinear, but drop out when b = a.)

It remains only to eliminate ε_{11}, ε_{22}, γ from the other side of this equation. Via the constitutive relations and our earlier expressions for N_{11}, N_{22}, N_{12}, (with details supplied in the appendix), we obtain: <u>equation B), left side</u>

$$\frac{a}{Eh}\left[\sqrt{\ }\Delta\Delta\Phi + \frac{2}{\sqrt{\ }}\left(\frac{b^2}{(\ )}\Delta_1\Phi + \Delta_2\Phi\right) + \frac{(1-\nu)b^2}{(\qquad)^{3/2}}\Delta\Phi\right.$$

$$+ \frac{2(1-\nu)b^4\Phi}{(\qquad)^{7/2}} + \frac{2\nu}{(\ )^{3/2}}\left(1 - \frac{b^2}{(\ )}\right)\frac{\partial^2\Phi}{\partial\theta^2}$$

$$+ \frac{4\nu(b^2 - a^2)\cos^2\theta}{\sqrt{\qquad}}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{4(b^2 - a^2)(\cos^2\theta - \sin^2\theta)}{(\qquad)^{5/2}}\frac{\partial^2\Phi}{\partial\theta^2}$$

$$+ \frac{2(1-\nu)(b^2 - a^2)\cos^2\theta}{(\qquad)^{5/2}}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{6(b^2 - a^2)\sin\theta\cdot\cos\theta}{(\qquad)^{5/2}}\frac{\partial\Phi}{\partial\theta}$$

$$- \frac{3(b^2 - a^2)^2\sin\theta\cdot\cos\theta}{(\qquad)^{7/2}}\frac{\partial\Phi}{\partial\theta} - \frac{3(b^2 - a^2)^3\sin^3\theta\cdot\cos^3\theta}{(\qquad)^{9/2}}\frac{\partial\Phi}{\partial\theta}$$

$$- \frac{8(1-\nu)(b^2 - a^2)b^2\sin\theta\cdot\cos\theta}{(\qquad)^{7/2}}\frac{\partial\Phi}{\partial\theta}$$

$$- \frac{2(b^2 - a^2)(\cos^2\theta - \sin^2\theta)\cos\theta}{\sin\theta(\qquad)^{5/2}} \frac{\partial\Phi}{\partial\theta}$$

$$- \frac{3(b^2 - a^2)^2(\cos^2\theta - \sin^2\theta)^2}{(\qquad)^{7/2}} \frac{\partial\Phi}{\partial\theta}$$

$$+ \frac{6(b^2 - a^2)^3\sin^2\theta\cdot\cos^2\theta(\cos^2\theta - \sin^2\theta)}{(\qquad)^{9/2}} \frac{\partial\Phi}{\partial\theta}$$

$$- \frac{3(b^2 - a^2)^4\sin^4\theta\cdot\cos^4\theta}{(\qquad)^{11/2}} \frac{\partial\Phi}{\partial\theta}$$

$$+ \frac{2(b^2 - a^2)\cos^3\theta}{\sin\theta\sqrt{\qquad}} \frac{\partial\Phi}{\partial\theta} - \frac{3\nu(b^2 - a^2)\cos\theta}{\sin\theta(\qquad)^{5/2}} \frac{\partial\Phi}{\partial\theta}$$

$$- \frac{4\nu(b^2 - a^2)\sin\theta\cdot\cos\theta}{\sqrt{\qquad}} \frac{\partial\Phi}{\partial\theta} + \frac{4\nu(b^2 - a^2)\cos^3\theta}{\sin\theta\sqrt{\qquad}} \frac{\partial\Phi}{\partial\theta}$$

$$- \frac{2\nu(b^2 - a^2)^2\sin\theta\cdot\cos^3\theta}{(\qquad)^{3/2}} \frac{\partial\Phi}{\partial\theta} - \frac{\nu(b^2 - a^2)(\cos^2\theta - \sin^2\theta)\cos\theta}{\sin\theta(\qquad)^{5/2}} \frac{\partial\Phi}{\partial\theta}$$

$$+ \frac{4\nu(b^2 - a^2)^2\sin\theta\cdot\cos^3\theta}{(\qquad)^{7/2}} \frac{\partial\Phi}{\partial\theta} + \frac{2\nu(b^2 - a^2)b^2\sin\theta\cdot\cos\theta}{(\qquad)^{7/2}} \frac{\partial\Phi}{\partial\theta}$$

$$+ \frac{4(1 - \nu)(b^2 - a^2)b^2\sin^2\theta}{(\qquad)^{7.2}}\Phi$$

$$+ \frac{28(1 - \nu)(b^2 - a^2)^2b^2\sin^2\theta\cdot\cos^2\theta}{(\qquad)^{9/2}}\Phi$$

$$- \frac{8(1 - \nu)(b^2 - a^2)b^2\cos^2\theta}{(\qquad)^{7/2}}\Phi\Bigg].$$

This completes the presentation of the second equation, the compatibility equation.

For the sphere of radius a, this equation takes the form (after using the operators H_0, H_2 for this case, where b = a, and multiplying by a^2):

B')

$$\frac{1}{Eh}\left(H_2 H_0(\Phi) + (1 - \nu) H_2(\Phi)\right) - aH_2(w)\Big) =$$

$$= \left(\frac{\partial w}{\partial \theta}\right)^2 - \csc^2\theta \, \frac{\partial^2 w}{\partial \theta^2} \frac{\partial^2 w}{\partial \phi^2} + \csc^4\theta \left(\frac{\partial w}{\partial \phi}\right)^2 + \csc^2\theta \left(\frac{\partial^2 w}{\partial \theta \partial \phi}\right)$$

$$- \cot\theta \, \frac{\partial w}{\partial \theta} \frac{\partial^2 w}{\partial \theta^2} - 2 \cot\theta \cdot \csc^2\theta \, \frac{\partial w}{\partial \phi} \frac{\partial^2 w}{\partial \theta \partial \phi}.$$

Related Efforts In The Literature

The literature is fairly rich in the derivation of equations peculiar to the elasticity theory for particular surfaces. There are also some efforts at obtaining general equations valid for many surfaces; e.g., shallow shells.

Of these, one of the most significant and (to my knowledge) the closest in relation to the two equations just derived, are those of W. T. Koiter [1]. What we will do now is look at those equations, substitute in appropriate parameters for the sphere, and compare those results with mine for the sphere.

Koiter's equations are, in tensor form:

$$D\Delta\Delta w - E^{\alpha\lambda}E^{\beta\mu}(h_{\alpha\beta} + w_{|\alpha\beta})F_{|\lambda\mu} - p = 0$$

$$\frac{1}{Eh}\Delta\Delta F + E^{\alpha\lambda}E^{\beta\mu}(h_{\alpha\beta} + \frac{1}{2} w_{|\alpha\beta})w_{|\lambda\mu} = 0;$$

where: F plays the same role as Φ, indices sum from 1 to 2, $E_{12} = \sqrt{g}$, $E^{12} = \frac{1}{\sqrt{g}}$, $E_{21} = -E_{12}$, $E_\lambda = 0$ for $\lambda = 1,2$, with g being the determinant of the 2 × 2 matrix of the first fundamental form for the surface, and $(h_{\alpha\beta})$ is the (matrix of the) second fundamental form. The notation with the bar denotes covariant surface differentiation (so that index sums are from 1 to 2).

In the case of the sphere, these parameters are:

$$g = a^4 \sin^2\theta$$

$$E_{12} = a^2 \sin\theta \text{ and } E^{12} = \frac{1}{a^2 \sin\theta};\ E^{21} = -E^{12}$$

$$E^{11} = g^{1\alpha} g^{1\beta} E_{\alpha\beta} = g^{11} g^{11} E_{11} = 0;\ E^{22} = 0$$

$$h_{11} = -a;\ h_{22} = -a \sin^2\theta;\ h_{12} = 0 = h_{21}.$$

(Appropriate details supplied in appendix.)

Covariant surface differentiation is performed as usual with $w_{k|\ell} = \frac{\partial w_k}{\partial u^\ell} - \Gamma^i_{\ell k} w_i$, where: the $(\Gamma^i_{\ell k})$ are the Christoffel symbols of the second kind, explicitly given in the appendix; $u^1 = \theta$, $u^2 = \phi$; and we initially take $w_1 = w_{,1} = \frac{\partial w}{\partial \theta}$ and $w_2 = w_{,2} = \frac{\partial w}{\partial \phi}$. When the covariant surface differentiation is performed again, it is applied then to $\frac{\partial w}{\partial \theta}$ and $\frac{\partial w}{\partial \phi}$. Likewise, we obtain the covariant derivatives for F.

We have, for example:

$$w_{|11} = \frac{\partial^2 w}{\partial \theta^2}$$

$$w_{|12} = \frac{\partial^2 w}{\partial \theta \partial \phi} - \cot\theta \frac{\partial w}{\partial \theta} = w_{|21}$$

$$w_{|22} = \frac{\partial^2 w}{\partial\phi^2} + \sin\theta\cdot\cos\theta\,\frac{\partial w}{\partial\theta}.$$

With the foregoing, Koiter's equations become:

$$D\Delta\Delta w - \frac{1}{a^3\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2} - \frac{\cos\theta}{a^3\sin\theta}\frac{\partial F}{\partial\theta} + \frac{1}{a^3}\frac{\partial^2 F}{\partial\theta^2}$$

$$+ \frac{1}{a^4\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{\cos\theta}{a^4\sin\theta}\frac{\partial F}{\partial\theta}\frac{\partial^2 w}{\partial\theta^2}$$

$$- \frac{2}{a^4\sin^2\theta}\frac{\partial^2 w}{\partial\theta\partial\phi}\frac{\partial^2 F}{\partial\theta\partial\phi} + \frac{2\cos\theta}{a^4\sin^3\theta}\frac{\partial F}{\partial\phi}\frac{\partial^2 w}{\partial\theta\partial\phi}$$

$$+ \frac{2\cos\theta}{a^4\sin^3\theta}\frac{\partial^2 F}{\partial\theta\partial\phi}\frac{\partial w}{\partial\phi} - \frac{2\cos^2\theta}{a^4\sin^4\theta}\frac{\partial F}{\partial\phi}\frac{\partial w}{\partial\phi}$$

$$- \frac{1}{a^4\sin^2\theta}\frac{\partial^2 F}{\partial\theta^2}\frac{\partial^2 w}{\partial\phi^2} - \frac{\cos\theta}{a^4\sin\theta}\frac{\partial^2 F}{\partial\theta^2}\frac{\partial w}{\partial\theta} - p = 0; \text{ and,}$$

$$\frac{1}{Eh}\Delta\Delta F - \frac{1}{a^3\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2} - \frac{\cos\theta}{a^3\sin\theta}\frac{\partial w}{\partial\theta} - \frac{1}{a^3}\frac{\partial^2 w}{\partial\theta^2}$$

$$- \frac{1}{a^4\sin^2\theta}\left(\frac{\partial^2 w}{\partial\theta\partial\phi}\right)^2 - \frac{\cos^2\theta}{a^4\sin^4\theta}\left(\frac{\partial w}{\partial\phi}\right)^2$$

$$+ \frac{1}{a^4\sin^2\theta}\frac{\partial^2 w}{\partial\theta^2}\frac{\partial^2 w}{\partial\phi^2} + \frac{\cos\theta}{a^4\sin\theta}\frac{\partial w}{\partial\theta}\frac{\partial^2 w}{\partial\theta^2}$$

$$+ \frac{2\cos\theta}{a^4\sin^3\theta}\frac{\partial w}{\partial\phi}\frac{\partial^2 w}{\partial\theta\partial\phi} = 0.$$

If we take into account differences of form, notation, etc., upon comparing Koiter's equations to mine, note that all his terms for the spherical case are to be found in my results. However, there are other terms not found in Koiter's equations.

Specifically, in comparing compatibility equations, Koiter does not have: certain terms involving the stress function, including a $\Delta\Phi$ term, and some deflection terms, such as 2w, and most significantly, the nonlinear term involving $\left(\frac{\partial w}{\partial \theta}\right)^2$ found in my second equation.

The equilibrium equations are a bit closer, with only certain terms omitted, as is my term involving $(1 - \nu)G\Delta w$. The latter is not a surprising "omission" on the part of Koiter in that his hypotheses explicitly include that the Gaussian curvature be relatively small (in the sense explained there).

If now, the results of Tsuboi and Akino are compared to mine, we find that my work contains essentially the same linear operators as found in theirs. The minor difference is due to a variation in their choice of representation of N_{11}, N_{22}, N_{12} in terms of Φ.

APPENDIX

Calculation of the First Fundamental Form for Ellipsoid:

$$x = a \cdot \cos\phi \cdot \sin\theta$$

(**) $$y = a \cdot \sin\phi \cdot \sin\theta$$

$$z = b \cdot \cos\theta$$

So, $$dx = a \cdot \cos\phi \cdot \cos\theta \cdot d\theta - a \cdot \sin\phi \cdot \sin\theta \cdot d\phi$$

$$dy = a \cdot \sin\phi \cdot \cos\theta \cdot d\theta + a \cdot \cos\phi \cdot \sin\theta \cdot d\phi$$

and $$dz = -b \cdot \sin\theta \cdot d\theta.$$

Squaring and adding gives: $dx^2 + dy^2 + dz^2 = (a^2\cos^2\theta + b^2\sin^2\theta)d\theta^2 + a^2\sin^2\theta \cdot d\phi^2$. Briefly, we will write $g_{11} = a^2\cos^2\theta + b^2\sin^2\theta$, $g_{22} = a^2\sin^2\theta$, and $g_{12} = 0 = g_{21}$ (reflecting orthogonality of our (θ,ϕ) co-ordinates).

Calculation of the Second Fundamental Form for the Sphere:

Using the parametrization(**) of the surface, except $z = a \cdot \cos\theta$ (i.e., $b = a$), consider $X(\theta,\phi) = (x,y,z)$. Then,

$$\frac{\partial X}{\partial \theta} = (a\cdot\cos\phi\cdot\cos\theta,\ a\cdot\sin\phi\cdot\cos\theta,\ -a\cdot\sin\theta)$$

$$\frac{\partial X}{\partial \phi} = (-a\cdot\sin\phi\cdot\sin\theta,\ a\cdot\cos\phi\cdot\sin\theta,\ 0)$$

and $$\frac{\partial^2 X}{\partial \theta^2} = (-a\cdot\cos\phi\cdot\sin\theta,\ -a\cdot\sin\phi\cdot\sin\theta,\ -a\cdot\cos\theta) = X_{11}$$

$$\frac{\partial^2 X}{\partial\theta\partial\phi} = \frac{\partial^2 X}{\partial\phi\partial\theta} = (-a\cdot\sin\phi\cdot\cos\theta,\ a\cdot\cos\phi\cdot\cos\theta, 0) = X_{12} = X_{21}$$

$$\frac{\partial^2 X}{\partial \phi^2} = (-a\cdot\cos\phi\cdot\sin\theta,\ -a\cdot\sin\phi\cdot\sin\theta,\ 0) = X_{22}.$$

Let n denote the (outward) unit surface normal. We compute $h_{ij} = \langle X_{ij,n}\rangle$, the inner product. Then, $n = \frac{1}{a}(x,y,z)$ So,

$$h_{11} = \frac{1}{a}(-a^2\cos^2\phi\cdot\sin^2\theta - a^2\sin^2\phi\cdot\sin^2\theta - a^2\cos^2\theta)$$

$$\frac{1}{a}(-a^2) = -a$$

$$h_{22} = \frac{1}{a}(-a^2\sin^2\theta + 0) = -a\cdot\sin^2\theta$$

and $$h_{12} = \frac{1}{a}(-a^2\sin\phi\cdot\cos\phi\cdot\cos\theta\cdot\sin\theta + a^2\cos\phi\cdot\sin\phi\cdot\cos\theta\cdot\sin\theta + 0)$$

$$= 0 = h_{21}.$$

<u>Calculation of the Principal Radii of Curvature</u>:

For R_1, set $\phi = 0$: $x = a\cdot\sin\theta$, $y = 0$, $z = b\cdot\cos\theta$. The curvature of this section is just

$$\left|\frac{dx}{d\theta}\cdot\frac{d^2z}{d\theta^2} - \frac{dz}{d\theta}\frac{d^2x}{d\theta^2}\right| \Big/ \left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2\right)^{3/2}$$

(The standard curvature formula)

$$= |(a\cdot\cos\theta)(-b\cdot\cos\theta) + (b\cdot\sin\theta)(-a\cdot\sin\theta)|/(\quad)^{3/2}$$

$$= \frac{ab}{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}.$$

Thus,

$$R_1 = \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab}.$$

One can calculate

$$R_2 = \frac{a\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{b}.$$

Thus,

$$G = \frac{1}{R_1R_2} = \frac{b^2}{(a^2\cos^2\theta + b^2\sin^2\theta)^2}.$$

Calculation of the Laplacian:

We use the formula:

$$\Delta w = \frac{1}{\alpha_1\alpha_2}\left(\left(\frac{\alpha_2}{\alpha_1}w_{,1}\right)_{,1} + \left(\frac{\alpha_1}{\alpha_2}w_{,2}\right)_{,2}\right)$$

which $$= \frac{w_{,11}}{\alpha_1^2} + \left(\frac{\alpha_{2,1}}{\alpha_1^2\alpha_2} - \frac{\alpha_{1,1}}{\alpha_1^3}\right)w_{,1} + \frac{w_{,22}}{\alpha_2^2}$$

when we recall $\alpha_{1,2} = 0 = \alpha_{2,2}$ ("abstract" version).

For the ellipsoid of revolution,

$$\Delta w = \frac{1}{(\ )}\,\frac{\partial^2 w}{\partial\theta^2} + \left[\frac{\cot\theta}{(\ \ )} - \frac{(b^2 - a^2)\sin\theta\cdot\cos\theta}{(\qquad\qquad)^2}\right]\frac{\partial w}{\partial\theta} + \frac{1}{a^2\sin^2\theta}\,\frac{\partial^2 w}{\partial\phi^2}.$$

To obtain $\Delta\Delta w$, apply Δ to Δw. This requires the computation of $(\Delta w)_{,1}$ and $(\Delta w)_{,11}$ and $(\Delta w)_{,22}$. After much tedious calculation and simplification,

$$\begin{aligned}
\Delta\Delta w = {} & \frac{w_{,1111}}{\alpha_1^4} - \frac{6\alpha_{1,1}\cdot w_{,111}}{\alpha_1^5} + \frac{2\alpha_{2,1}\cdot w_{,111}}{\alpha_1^4\alpha_2} \\
& - \frac{4\alpha_{1,11}\cdot w_{,11}}{\alpha_1^5} + \frac{15\alpha_{1,1}^2\cdot w_{,11}}{\alpha_1^6} + \frac{2\alpha_{2,11}\cdot w_{,11}}{\alpha_1^4\alpha_2} \\
& - \frac{\alpha_{2,1}^2\cdot w_{,11}}{\alpha_1^4\alpha_2^2} - \frac{8\alpha_{1,1}\,\alpha_{2,1}\cdot w_{,11}}{\alpha_1^5\alpha_2} + \frac{\alpha_{2,111}\cdot w_{,1}}{\alpha_1^4\alpha_2} \\
& + \frac{7\alpha_{1,1}\cdot\alpha_{1,11}\cdot w_{,1}}{\alpha_1^6} - \frac{2\alpha_{2,1}\cdot\alpha_{2,11}\cdot w_{,1}}{\alpha_1^4\alpha_2^2} - \frac{5\alpha_{1,1}\cdot\alpha_{2,11}\cdot w_{,1}}{\alpha_1^5\alpha_2} \\
& + \frac{\alpha_{2,1}^3\cdot w_{,1}}{\alpha_2^3} + \frac{3\alpha_{1,1}\cdot\alpha_{2,1}^2\cdot w_{,1}}{\alpha_1^5\,{}_2^2} - \frac{3\alpha_{1,11}\cdot\alpha_{2,1}\cdot w_{,1}}{\alpha_1^5\,{}_2} \\
& - \frac{\alpha_{1,111}\cdot w_{,1}}{\alpha_1^5} + \frac{11\alpha_{1,1}\cdot\alpha_{2,1}\,w_{,1}}{\alpha_1^6\alpha_2} - \frac{15\alpha_{1,1}^3\cdot w_{,1}}{\alpha_1^7} \\
& + \frac{3\alpha_{1,11}^2\cdot w_{,1}}{\alpha_1^6} + \frac{2w_{,1122}}{\alpha_1^2\alpha_2^2} - \frac{2\alpha_{1,1}\cdot w_{,122}}{\alpha_1^3\alpha_2^2} \\
& - \frac{2\alpha_{2,1}\cdot w_{,122}}{\alpha_1^2\alpha_2^3} - \frac{2\alpha_{2,11}\cdot w_{,22}}{\alpha_1^2\alpha_2^3} + \frac{4\alpha_{2,1}^2\cdot w_{,22}}{\alpha_1^2\alpha_2^4}
\end{aligned}$$

$$+ \frac{2\alpha_{1,1} \cdot \alpha_{2,1} \cdot w_{,22}}{\alpha_1^{\,3} \alpha_2^{\,3}} + \frac{w_{,2222}}{\alpha_2^{\,4}} .$$

The concrete version is no less 'messy'; we will exhibit the result of Δ^2 applied to the function Φ (since that is the calculation used in the body of the thesis). Partial differentiation will be indicated via subscripts.

$$\Delta\Delta\Phi = \frac{\Phi_{\theta\theta\theta\theta}}{(\quad)^2} - \frac{6(b^2 - a^2)\sin\theta \cdot \cos\theta \cdot \Phi_{\theta\theta\theta}}{(\qquad\qquad)^3}$$

$$+ \frac{2\cos\theta \cdot \Phi_{\theta\theta\theta}}{\sin\theta(\quad)^2} + \frac{4(b^2 - a^2)\sin^2\theta \cdot \Phi_{\theta\theta}}{(\qquad\qquad)^3}$$

$$+ \frac{19(b^2 - a^2)^2\sin^2\theta \cdot \cos^2\theta \cdot \Phi_{\theta\theta}}{(\qquad\qquad)^4} - \frac{2\Phi_{\theta\theta}}{(\;)^2} - \frac{\cos^2\theta \cdot \Phi_{\theta\theta}}{\sin^2\theta(\;)^2}$$

$$- \frac{12(b^2 - a^2)\cos^2\theta \cdot \Phi_{\theta\theta}}{(\qquad\qquad)^3} + \frac{\cos\theta \cdot \Phi_{\theta}}{\sin^3\theta(\;)^2}$$

$$+ \frac{12(b^2 - a^2)\sin\theta \cdot \cos\theta \cdot \Phi_{\theta}}{(\qquad\qquad)^3} - \frac{10(b^2 - a^2)^2\sin^3\theta \cdot \cos\theta \cdot \Phi_{\theta}}{(\qquad\qquad)^4}$$

$$+ \frac{24(b^2 - a^2)^2\sin\theta \cdot \cos^3\theta \cdot \Phi_{\theta}}{(\qquad\qquad)^4}$$

$$- \frac{25(b^2 - a^2)^3\sin^3\theta \cdot \cos^3\theta \cdot \Phi_{\theta}}{(\qquad\qquad)^5}$$

$$+ \frac{3(b^2 - a^2)^2(\cos^2\theta - \sin^2\theta)^2 \cdot \Phi_{\theta}}{(\qquad\qquad)^4}$$

$$+ \frac{3(b^2 - a^2)^4 \sin^4\theta \cdot \cos^4\theta \cdot \Phi_\theta}{(\qquad\qquad)^6}$$

$$- \frac{6(b^2 - a^2)^3 \sin^2\theta \cdot \cos^2\theta(\cos^2\theta - \sin^2\theta) \cdot \Phi_\theta}{(\qquad\qquad\qquad)^5}$$

$$+ \frac{2\Phi_{\theta\theta\phi\phi}}{a^2\sin^2\theta(\ )} - \frac{2(b^2 - a^2)\cos\theta \cdot \Phi_{\theta\phi\phi}}{a^2\sin\theta(\qquad)^2} - \frac{2\cos\theta \cdot \Phi_{\theta\phi\phi}}{a^2\sin^3\theta(\ )}$$

$$+ \frac{2\Phi_{\phi\phi}}{a^2\sin^2\theta(\ )} + \frac{4\cos^2\theta \cdot \Phi_{\phi\phi}}{a^2\sin^4\theta(\ )} + \frac{2(b^2 - a^2)\cos^2\theta \cdot \Phi_{\phi\phi}}{a^2\sin^2\theta(\qquad)^2}$$

$$+ \frac{\Phi_{\phi\phi\phi\phi}}{a^4\sin^4\theta} .$$

<u>Calculation of Christoffel Symbols for Sphere</u>:

$\Gamma^i_{\ell k}$: Christoffel symbols of second kind.

$\Gamma_{jk\ell}$: Christoffel symbols of first kind.

I.e., $\Gamma^i_{\ell k} = g^{ij}\Gamma_{j\ell k}$, with

$$\Gamma_{j\ell k} = \tfrac{1}{2}(g_{j\ell,k} + g_{jk,\ell} - g_{\ell k,j}).$$

(Both Christoffel symbols are symmetric in k and ℓ.)

For the sphere, $g_{11} = a^2$, $g_{22} = a^2\sin^2\theta$, and $g_{12} = 0 = g_{21}$.

Hence, we have $g^{11} = \frac{1}{a^2}$, $g^{22} = \frac{1}{a^2\sin^2\theta}$, and $g^{12} = 0 = g^{21}$.

Hence, in this instance, $\Gamma^i_{\ell k} = g^{ii}\Gamma_{i\ell k}$, so:

$$\Gamma^1_{11} = g^{11}\Gamma_{111} = \frac{1}{a^2}\cdot\frac{1}{2}(0 + 0 + 0) = 0.$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{a^2}\cdot\frac{1}{2}(0 + 0 + 0) = 0.$$

$$\Gamma^1_{22} = \frac{1}{a^2}\cdot\frac{1}{2}(0 + 0 - 2a^2\sin\theta\cdot\cos\theta) = -\sin\theta\cdot\cos\theta.$$

$$\Gamma^2_{11} = \frac{1}{a^2\sin^2\theta}\cdot\frac{1}{2}(0 + 0 - 0) = 0.$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{a^2\sin^2\theta}\cdot\frac{1}{2}(2a^2\sin\theta\cdot\cos\theta + 0 - 0) = \cot\theta.$$

$$\Gamma^2_{22} = \frac{1}{a^2\sin^2\theta}\cdot\frac{1}{2}(0 + 0 - 0) = 0.$$

(These, plus the foregoing, provide the parameters needed to compare Koiter's equations in the case of the sphere, with the ones derived in this thesis.)

<u>Details in Obtaining Equilibrium Equation</u>

$$(\alpha_2 Q_1)_{,1} = \left(\frac{\alpha_2}{\alpha_1}\right)M_{11,11} + \frac{(\alpha_1\alpha_{2,1} - \alpha_2\alpha_{1,1})}{\alpha_1{}^2}M_{11,1}$$

$$+ \frac{\alpha_{2,1}}{\alpha_1}(M_{11} - M_{22})_{,1}$$

$$+ \frac{(\alpha_1\alpha_{2,11} - \alpha_{2,1}\alpha_{1,1})}{\alpha_1{}^2}(M_{11} - M_{22})$$

$$+ M_{12,21}.$$

$$(\alpha_1 Q_2)_{,2} = \frac{\alpha_1}{\alpha_2}M_{22,22} + M_{12,12} + \frac{2\alpha_{2,1}}{\alpha_2}M_{12,2}.$$

In turn, M_{11}, M_{22}, M_{12} are eliminable:

$$M_{11} = D(\kappa_{11} + \nu\kappa_{22})$$

$$M_{22} = D(\kappa_{22} + \nu\kappa_{11})$$

$$M_{12} = (1 - \nu)D\kappa.$$

Replace κ_{11}, κ_{22}, κ by the bending strain relations; get:

$$M_{11} = D\left(\frac{\phi_{1,1}}{\alpha_1} + \frac{\nu\phi_{2,2}}{\alpha_2} + \frac{\nu\alpha_{2,1}\phi_1}{\alpha_1\alpha_2}\right)$$

$$M_{22} = D\left(\frac{\phi_{2,2}}{\alpha_2} + \frac{\nu\phi_{1,1}}{\alpha_1} + \frac{\alpha_{2,1}\phi_1}{\alpha_1\alpha_1}\right)$$

and $$M_{12} = (1 - \nu)D\left(\frac{\phi_{2,1}}{\alpha_1} + \frac{\phi_{1,2}}{\alpha_2} - \frac{\alpha_{2,1}\phi_2}{\alpha_1\alpha_2}\right).$$

Again, $\phi_1 = \dfrac{-w_{,1}}{\alpha_1}$ and $\phi_2 = \dfrac{-w_{,2}}{\alpha_2}$, so

$$\phi_{1,1} = \frac{-w_{,11}}{\alpha_1} + \frac{\alpha_{1,1}\cdot w_{,1}}{\alpha_1^2}; \quad \phi_{1,2} = \frac{-w_{,12}}{\alpha_1};$$

$$\phi_{2,1} = \frac{-w_{,12}}{\alpha_2} + \frac{\alpha_{2,1}\cdot w_{,2}}{\alpha_2^2}; \quad \phi_{2,2} = \frac{-w_{,22}}{\alpha_2}.$$

The result is:

$$M_{11} = D\left(\frac{-w_{,11}}{\alpha_1^2} + \frac{\alpha_{1,1}\cdot w_{,1}}{\alpha_1^3} - \frac{\nu w_{,22}}{\alpha_2^2} - \frac{\nu\alpha_{2,1}\cdot w_{,1}}{\alpha_1^2\alpha_2}\right)$$

$$M_{22} = D\left(\frac{-w_{,22}}{\alpha_2^2} - \frac{\nu w_{,11}}{\alpha_1^2} + \frac{\nu\alpha_{1,1}\cdot w_{,1}}{\alpha_1^3} - \frac{\alpha_{2,1}\cdot w_{,1}}{\alpha_1^2\alpha_2}\right)$$

and $M_{12} = (1 - \nu)D\left(\frac{-w_{,12}}{\alpha_1\alpha_2} + \frac{\alpha_{2,1}\cdot w_{,2}}{\alpha_1\alpha_2^{\,2}}\right)$.

Appropriate derivatives are taken, etc., and the result is as stated in the body. The result is compared to $-D\alpha_1\alpha_2\Delta\Delta w$; missing terms are compensated for by adding and subtracting. Finally, substitution of concrete forms for the abstract parameters gives the stated result for the left-hand side of the first equation.

For the right-hand side, it suffices merely to supply further information regarding the representation of N_{11}, N_{22}, N_{12} in terms of Φ.

Following the scheme described in the body of the thesis, after combining $\Phi_{\theta\theta\theta}$ terms, $\Phi_{\theta\theta}$ terms, etc., we have these conditions:

For $\Phi_{\theta\theta\theta}$ terms: $a\cdot\sin\theta\cdot f_3 = 0$, hence $f_3 = 0$.

For $\Phi_{\theta\theta\phi}$: $a\cdot\sin\theta\cdot f_2 + (\surd\overline{})g_3 = 0$.

$\Phi_{\theta\phi\phi}$: $a\cdot\sin\theta\cdot f_1 + (\surd\overline{})g_2 = 0$.

$\Phi_{\phi\phi\phi}$: $(\surd\overline{})g_1 = 0$, hence $g_1 = 0$.

$\Phi_{\theta\theta}$: $a\cdot\sin\theta\cdot f_5 - a\cdot\cos\theta\cdot h_3 + (\surd\overline{})g_{3,\phi} = 0$.

$\Phi_{\theta\phi}$: $a\cdot\sin\theta\cdot f_{2,\theta} + a\cdot\sin\theta\cdot f_4 + a\cdot\cos\theta\cdot f_2 -$
$a\cdot\cos\theta\cdot h_2 + (\surd\overline{})g_{2,\phi} + (\surd\overline{})g_5 = 0$.

$\Phi_{\phi\phi}$: $a\cdot\sin\theta\cdot f_{1,\theta} + a\cdot\cos\theta\cdot f_1 -$
$a\cdot\cos\theta\cdot h_1 + (\surd\overline{})g_4 = 0$.

Φ_θ: $a\cdot\sin\theta\cdot f_{5,\theta} + a\cdot\sin\theta\cdot f_6 + a\cdot\cos\theta\cdot f_5 -$

$a\cdot\cos\theta\cdot h_5 + (\sqrt{})g_{5,\phi} = 0.$

Φ_ϕ: $a\cdot\sin\theta\cdot f_{4,\theta} + a\cdot\cos\theta\cdot f_4 -$

$a\cdot\cos\theta\cdot h_4 + (\sqrt{})g_{4,\phi} + (\sqrt{})g_6 = 0.$

Φ: $a\cdot\sin\theta\cdot f_{6,\theta} + a\cdot\cos\theta\cdot f_6 - a\cdot\cos\theta\cdot h_6 +$

$(\sqrt{})g_{6,\phi} = 0.$

Repeat idea with equation F 2):

$\Phi_{\theta\theta\theta}$: $a\cdot\sin\theta\cdot g_3 = 0$, hence $g_3 = 0$.

$\Phi_{\theta\theta\phi}$: $a\cdot\sin\theta\cdot g_2 + (\sqrt{})h_3 = 0.$

$\Phi_{\theta\phi\phi}$: $(\sqrt{})h_2 = 0$, hence $h_2 = 0$.

$\Phi_{\phi\phi\phi}$: $(\sqrt{})h_1 = 0$, hence $h_1 = 0$.

$\Phi_{\theta\theta}$: $(\sqrt{})h_{3,\phi} + a\cdot\sin\theta\cdot g_5 = 0.$

$\Phi_{\theta\phi}$: $(\sqrt{})h_5 + a\cdot\sin\theta\cdot g_{2,\theta} + a\cdot\sin\theta\cdot g_4 +$

$2a\cdot\cos\theta\cdot g_2 = 0.$

$\Phi_{\phi\phi}$: $(\sqrt{})h_4 = 0$, hence $h_4 = 0$.

Φ_θ: $(\sqrt{})h_{5,\phi} + a\cdot\sin\theta\cdot g_{5,\theta} + a\cdot\sin\theta\cdot g_6 +$

$2a\cdot\cos\theta\cdot g_5 = 0.$

Φ_ϕ: $(\sqrt{})h_6 + a\cdot\sin\theta\cdot g_{4,\theta} + 2a\cdot\cos\theta\cdot g_4 = 0.$

Φ: $(\sqrt{})h_{6,\phi} + a\cdot\sin\theta\cdot g_{6,\theta} + 2a\cdot\cos\theta\cdot g_6 = 0.$

After elimination of coefficient functions which are zero, we have 13 equations in 11 unknowns. What next?

One way out is to set $f_6 = G = h_6$, based on the case for the sphere. When this is done, we obtain the representation ultimately chosen. Just F 2) fails to be satisfied identically, but only in the general ellipsoidal case (with only a Φ term failing to vanish). In the spherical case, both F 1) and F 2) are satisfied.

Details in Obtaining Compatibility Equation:

Assuming the form $f_1\varepsilon_{22,11} + f_2\varepsilon_{22,1} + f_3\varepsilon_{11,1} + f_4\varepsilon_{11,22} + f_5\varepsilon_{11} + f_6\varepsilon_{22} + f_7\gamma_{,2} + f_8\gamma_{,12} = L(w)$, where ε_{11}, ε_{22}, γ are given by the strain-displacement relations SD 1), SD 2), SD 3), we substitute and combine on the left side of the above. Specifically, all terms involving v_ϕ are combined, and sum set equal to zero. Likewise for $v_{\theta\phi}$, and so on.

Results:

$$v_\phi \text{ terms: } \frac{f_6}{a\cdot\sin\theta} - \frac{\cos\theta\cdot f_2}{a\cdot\sin^2\theta} + \frac{f_1}{a\cdot\sin\theta} + \frac{2\cos^2\theta\cdot f_1}{a\cdot\sin^3\theta} - \frac{\cot\theta\cdot f_7}{\sqrt{}} + \frac{\csc^2\theta\cdot f_8}{\sqrt{}} + \frac{(b^2 - a^2)\cos^2\theta\cdot f_8}{()^{3/2}} = 0.$$

$$v_{\theta\phi}: \frac{f_2}{a\cdot\sin\theta} - \frac{2\cos\theta\cdot f_1}{a\cdot\sin^2\theta} + \frac{f}{\sqrt{}} - \frac{\cot\theta\cdot f_8}{\sqrt{}} - \frac{(b^2 - a^2)\sin\theta\cdot\cos\theta\cdot f_8}{()^{3/2}} = 0.$$

$$v_{\theta\theta\phi}: \frac{f_1}{a\cdot\sin\theta} + \frac{f_8}{\sqrt{}} = 0.$$

$$u_{\phi\phi}: \quad \frac{f_7}{a\cdot\sin\theta} - \frac{\cos\theta\cdot f_8}{a\cdot\sin^2\theta} = 0.$$

$$u_{\theta\phi\phi}: \quad \frac{f_8}{a\cdot\sin\theta} + \frac{f_4}{\sqrt{\ }} = 0.$$

$$u_{\theta\theta}: \quad \frac{f_3}{\sqrt{\ }} + \frac{\cot\theta\cdot f_1}{\sqrt{\qquad}} = 0.$$

$$u_{\theta}: \quad \frac{f_5}{\sqrt{\ }} - \frac{(b_2 - a^2)\sin\theta\cdot\cos\theta\cdot f_3}{(\qquad)^{3/2}} + \frac{\cot\theta\cdot f_2}{\sqrt{\qquad}} -$$

$$\frac{2\csc^2\theta\cdot f_1}{\sqrt{\qquad}} - \frac{2(b^2 - a^2)\cos^2\theta\cdot f_1}{(\qquad)^{3/2}} = 0.$$

$$u: \quad \frac{\cot\theta\cdot f_6}{\sqrt{\qquad}} - \left(\frac{\csc^2\theta}{\sqrt{\qquad}} + \frac{(b^2 - a^2)\cos^2\theta}{(\qquad)^{3/2}}\right) f_2 +$$

$$\left(\frac{2\csc^2\theta\cdot\cot\theta}{\sqrt{\qquad}} + \frac{(b^2 - a^2)\csc\theta\cdot\cos\theta}{(\qquad)^{3/2}} + \right.$$

$$\left.\frac{2(b^2 - a^2)\sin\theta\cdot\cos\theta}{(\qquad)^{3/2}} + \frac{3(b^2 - a^2)^2\cos^3\theta\cdot\sin\theta}{(\qquad)^{5/2}}\right) f_1 = 0.$$

At first glance, it appears hopeful: 8 equations in 8 unknowns. In fact, there are only 7 "degrees of freedom". If we assume, say, f_8 is known, we will obtain f_7, f_4, f_1, f_3, f_2, f_6, f_5, except that f_6 will be determined by both the v_ϕ equation and the u equation; i.e., an over-determined situation.

The way out (after much consideration) is to simply allow the terms involving u to fail to sum to zero. The results are then as stated in the text, and the small "error" of the elliposidal case drops out for b = a. The starting point then is to note that for b = a, $f_8 = \frac{-1}{\sin\theta}$. If we accept this also for the ellipsoid, we rapidly get the other coefficient functions (as given in the solution to this problem in the text).

We now have:

$$(*)\quad \frac{a}{\sqrt{}}\varepsilon_{22,11} + \left[\frac{2a\cdot\cot\theta}{\sqrt{}} - \frac{(b^2 - a^2)a\cdot\sin\theta\cdot\cos\theta}{()^{3/2}}\right]\varepsilon_{22,1}$$

$$- \frac{a\cdot\cot\theta}{\sqrt{}}\varepsilon_{11,1} + \frac{\sqrt{}}{a\cdot\sin^2\theta}\varepsilon_{11,22} + \frac{2ab^2}{(\ )^{3/2}}\varepsilon_{11}$$

$$- \frac{2}{\sin^2\theta}(\gamma_{,2}\ \sin\theta)_{,1}$$

as the expression to use. Substitute in the middle surface strain-displacement relation expressions found in SD 1), SD 2), SD 3) for ε_{11}, ε_{22}, γ; i.e.,

$$\varepsilon_{11} = \frac{1}{\sqrt{}}\frac{\partial u}{\partial\theta} + \frac{ab\cdot w}{(\ )^{3/2}} + \frac{1}{2(\ )}\left(\frac{\partial w}{\partial\theta}\right)^2$$

$$\varepsilon_{22} = \frac{1}{a\cdot\sin\theta}\frac{\partial v}{\partial\phi} + \frac{\cot\theta\cdot u}{\sqrt{}} + \frac{b\cdot w}{a\sqrt{}} + \frac{1}{2a^2\sin^2\theta}\left(\frac{\partial w}{\partial\phi}\right)^2$$

$$\gamma = \frac{1}{\sqrt{}}\frac{\partial v}{\partial\theta} - \frac{\cot\theta\cdot v}{\sqrt{}} + \frac{1}{a\cdot\sin\theta}\frac{\partial u}{\partial\phi} + \frac{1}{a\cdot\sin\theta\sqrt{}}\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial\phi}\ .$$

After these are plugged into (*), we have, after simplification, the results given in the text for L(w) and the 'residue' terms involving u.

On the other hand, referring back to (*) again, we may now replace ε_{11}, ε_{22}, γ by their equivalents in terms of N_{11}, N_{22}, N_{12} via the constitutive relations C 1), C 2), C 3). When this is done, we have (*) =:

$$\frac{1}{Eh}\Bigg[\frac{a}{\sqrt{\ }}N_{22,11} - \frac{\nu a}{\sqrt{\ }}N_{11,11} + \frac{2a\cdot\cot\theta}{\sqrt{\qquad}}N_{22,1} - \frac{2\nu a\cdot\cot\theta}{\sqrt{\qquad}}N_{11,1} -$$

$$\frac{(b^2 - a^2)a\cdot\sin\theta\cdot\cos\theta}{(\qquad)^{3/2}}N_{22,1} + \frac{\nu(b^2 - a^2)a\cdot\sin\theta\cdot\cos\theta}{(\qquad)^{3/2}}N_{11,1} -$$

$$\frac{a\cdot\cot\theta}{\sqrt{\qquad}}N_{11,1} + \frac{\nu a\cdot\cot\theta}{\sqrt{\qquad}}N_{22,1} + \frac{\sqrt{\qquad}}{a\cdot\sin^2\theta}N_{11,22} - \frac{\nu\sqrt{\qquad}}{a\cdot\sin^2\theta}N_{22,22}$$

$$\frac{2ab^2}{(\ )^{3/2}}N_{11} - \frac{2\nu ab^2}{(\ )^{3/2}}N_{22} - \frac{2(1+\nu)}{\sin^2\theta}\left(N_{12,2}\cdot\sin\theta\right)_{,1}\Bigg].$$

Now N_{11}, N_{22}, N_{12} are represented in terms of the stress function :

$$N_{11} = \frac{1}{a^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\cot\theta}{(\ )}\frac{\partial\Phi}{\partial\theta} + \frac{b^2}{(\ )^2}\Phi$$

$$N_{22} = \frac{1}{(\ )}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{(b^2 - a^2)\sin\theta\cdot\cos\theta}{(\qquad)^2}\frac{\partial\Phi}{\partial\theta} + \frac{b^2}{(\ )^2}\Phi$$

$$N_{12} = \frac{-1}{a\cdot\sin\theta\sqrt{\ }}\frac{\partial^2\Phi}{\partial\theta\partial\phi} + \frac{\cos\theta}{a\cdot\sin^2\theta\sqrt{\ }}\frac{\partial\Phi}{\partial\phi}.$$

Upon substitution, one obtains, after great diligence, in the order of 70 to 80 terms (approximately) involving Φ and its derivatives. Upon comparison, this last expression is found to contain most of the terms of $\frac{a\sqrt{\ }}{Eh}\Delta\Delta\Phi$; those few missing are compensated for by simultaneous addition and subtraction. In this way, we ultimately obtain the final form of our second equation.

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