

A COMPARISON OF VALUE AT RISK APPROACHES AND A NEW METHOD
WITH EXTREME VALUE THEORY AND KERNEL ESTIMATOR

by

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Abstract

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This paper provides an overview of developments, methodologies, and applications of Value at Risk (VaR). Various key methodologies of VaR estimation and evaluation are discussed and compared. A new approach with extreme value theory (EVT) and kernel estimator technique is proposed. The empirical study utilizing a sample of portfolios and stocks for more than 13 years data shows that the new EVT with kernel estimator approach outperforms other existing methods.

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I. Introduction

During the past decade, Value at Risk (VaR) has become one of most commonly used tools in risk measurement. VaR was originally used by few major financial companies to measure market risk of trading portfolios in late 1980's. In early 1990's, importance of risk management was widely recognized due to several cases of significant losses by inappropriate use of derivatives, e.g., Orange County (1994, losses of \$1.8B), Metallgesellschaft (1994, \$1.3B), Barings (1995, \$1.3B), and Daiwa (1995, \$1.1B)¹. In 1994, based on a follow-up to the survey in the Group of Thirty's 1993 global derivatives project, 43% of dealers reported that they were using some form of Value at Risk and 37% and indicated that they planned to use VaR by the end of 1995. Today, VaR is used by most major derivatives dealers to measure and manage risk.

In 1994, JP Morgan published its new risk measured system, *RiskMetrics*, which mainly utilized the concept of VaR. The system contains risk measures over 300 financial instruments across 14 countries and organized and developed detail methodologies of VaR. The publication of the system made ideology of VaR much more informative to financial world and consequently promoted larger utilizations and developments of VaR.

In 1996, the Basel Committee on Banking Supervision amended the Basel Capital Accord and later in 1997 obliged their member banks to reserve capital requirements based on the measurement of their market risk using VaR. Banks need to maintain three times of their VaR measurements to cover potential market risks. Other than the standard model, Banks are allowed to develop and choose their own internal VaR methods

¹ Detail descriptions for the cases can be seen in Jorion (2001) and Dowd (1998).

(subjective to approval by Bank for International Settlements) for measurements. Such flexibility implies that two banks with same level of market risk may have different capital requirements only due to different methods of VaR.

While VaR becomes a standard tool for risk management, its technique has gone through significant refinement since it originally appeared. How to generate precise VaR estimate becomes increasingly needed for practitioners and regulators. Researches and studies in the field have developed various methodologies to generate VaR estimates. In addition, many have developed various methods to evaluate the VaR estimates. For the rest of this paper, in section II, we first introduce and discuss current methodologies of VaR and propose a new approach of VaR. We also talk about methodologies of evaluation of VaR estimates. In section III, we present our empirical findings, and in section IV, we conclude our research.

II. Methodologies

The most appealing concept of VaR is that it provides a single number to summarize potential risk of a financial instrument. Such simplicity attracts great attentions from practitioners and academic researchers because it can be easily understood and applied. Even so, different choices of the methodologies of VaR can generate very different results of VaR estimates. Recent researches in the field develop various kinds of methods in both producing VaR estimates and evaluating the estimates. We divide this section into following 4 subsections in discussion of methodologies of VaR: (1) definition of VaR, (2) existing approaches to produce VaR estimates, (3) new approach of VaR estimation by extreme value theory with kernel estimator, and (4) methods to evaluate VaR estimates.

II-1. Definition of Value at Risk

Simply put, VaR is the maximum amount of money a portfolio might lose within certain period given a selected probability. That is, the VaR can be interpreted as the worse scenario loss when extreme event with small chance happens within a time horizon.

Consider a single asset portfolio, whose underlying asset follows the geometric Brownian motion pricing process.

$$\frac{dS}{S} = \mu dt + \sigma dz, \quad (1)$$

where S is the price of a stock, dS is the change of price of the stock, μ is the expected return, dt is a period of time, and σdz is the white noise for the path of the stock price.

Assuming that the percentage return of the stock price in a short period of time is normally distributed. So

$$\frac{\delta S}{S} \sim \phi(\mu \delta t, \sigma \sqrt{\delta t}), \quad (2)$$

where δt is a small time interval, $\mu \delta t$ is the mean of the normal distribution and $\sigma \sqrt{\delta t}$ is the standard deviation. By Itô's lemma, following lognormal property can be derived for stock price, S_T , at a future time T , where S_0 is the stock price at time zero.

$$\ln S_T \sim \phi \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]. \quad (3)$$

We define a continuously compounded annual rate of return for stock price as η , which follows that

$$S_T = S_0 e^{\eta T}. \quad (4)$$

Therefore,

$$\eta = \frac{1}{T} \ln \frac{S_T}{S_0}. \quad (5)$$

Alone with equation (3), we can see that η has following normal distribution property with mean as $\mu - \sigma^2 / 2$ and standard deviation of σ / \sqrt{T} . That is

$$\eta \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T}}\right). \quad (6)$$

We now define the volatility of a stock is a measure of the uncertainty about the returns of stock. Based on equation (6), the annual volatility can be seen as the same of the standard deviation, σ / \sqrt{T} , where $T = 1$, so the annual volatility of stock is σ . It also can be understood that the daily volatility of stock would be $\sigma / \sqrt{252}$, when using 252 trading days per year.

When we discuss the daily returns of stock, it is reasonable to assume that the expected return as zero, because in such short time horizon, the expected return is relatively small. Therefore, based on the normality property in equation (6), we can easily estimate the distribution of returns from daily volatility. For example, we can estimate that 99% of rates of returns will be greater than -2.33 times the daily volatility. If the VaR tells how much actual amount of money might be lose for a portfolio at 1% of chance in 1 day, the VaR is equal to $2.33 \times \sigma / \sqrt{252} \times P$, where the P is the current value in dollars of the portfolio. Thus, the general formula of VaR would be

$$\text{VaR}(P, \alpha, \delta t) = P \cdot k \cdot \sigma \sqrt{\delta t}, \quad (7)$$

where

$$\Pr (\delta P \geq k \cdot \sigma \cdot \sqrt{\delta t}) = \alpha. \quad (8)$$

Here, α is the left tail percentage of the distribution, δP is the absolute value of negative return of P , k is the numbers of standard deviations according to α , and $\Pr(\cdot)$ is the probability based on the distribution. When α is set as 1% or 5%, the k would be 2.33 and 1.645 respectively for a normal distribution. In practice, α needs to be selected as small as possible to catch extreme events, and 1% has been pretty much a standard setting in the financial sector. One key point is that no matter how small the α is, as long as it is finite, there will always be some probabilities that extreme events occurring far away from the VaR estimate.

The basic concept of VaR is appealing because it is not difficult to be estimated comparing to other market risk measured instruments. Most methods use prior data of returns of financial assets to estimate the VaR. However, when using historical data for estimation of current behaviors, there is one key question needs to be clarified: are the patterns of past analytical sufficient to predict current patterns? In the VaR practice, we need to find out if we understand the behaviors of distributions for returns of financial assets and are able to make proper assumptions when needed. We need to see if such distributions are constant over time, if not, can we effectively pattern the changes of behaviors?

Researches have found out two fundamental attributes regarding distributions of returns of financial assets. First, the distributions have been shown, most of times, are

not normal (Hull and White [1998]). The distributions of different financial assets can very likely be different: the Student- t distribution, the stable Paretian distribution (Mittnik, Paolella and Rachev [2002]), the mixture of normal distribution (Venkataraman [1997]), or the generalized error distribution (Nelson [1991]). Some argue that the distribution should have fat tails (Longin [1996] and Neftci [2000]), and some argue that it should not be symmetric (Glosten [1993]).

Second, some researches found that the distributions are not constant over time. Such findings are related to another field of research in finance: the prediction of volatility of financial assets. There have been a lot of debates about the attributes of volatility; whether the volatility is time-varying or constant, whether it should be weighted through time or not, or what time interval from past is relevant for current volatility. Due to inconsistent patterns of distributions, some argue that VaR does not give an appropriate risk measurement, and its estimation is subject to large estimation errors (Krause [2003]).

Chen, Hueng, and Lin (2000)² show that fundamental pricing model cannot explain the volatility of stock price and the volatility is time varying. Andersen (1994), Ghysels and Jasiak (1996) and Sandmann and Koopman (1998) estimate the volatility as non-constant and non-symmetric with fat left tail. Some argue that the true volatility cannot be estimated because there is no relationship between prior, current, and future volatilities for financial assets. If so, approaches utilizing volatility in VaR estimates should be invalid.

² Their estimates are conducted via Monte Carlo maximum likelihood method and conclusions showing volatility follows $\ln(\chi^2)$ distribution.

Another key concept of the VaR, especially for portfolio with nonlinear relationships from its underlying assets, is the so-called sub-additive property. The issue is simply that the actual risk of a nonlinear portfolio should be lower than the sum of risk measurements from its components³. It has some kind of decreased return of scale situation where the actual risk is lower than the sum of individual risks when combining them into a portfolio. Therefore, when calculating VaR for a portfolio, methods need to be applied for its underlying assets first, not for the combined returns of portfolio. In the next section, we introduce key existing methods to estimate VaR and discuss how they address potential problems of VaR and their advantages and drawbacks.

II-2. Existing Approaches in Value at Risk Estimation

In this section, we discuss following 9 existing approaches for VaR estimation; (1) traditional historical simulation, (2) conditional Value at Risk, (3) variance-covariance approach, (4) variance-covariance approach with EWMA, (5) the GARCH(1,1) model building approach, (6) Monte Carlo simulation, (7) model building with Kalman filter, (8) extreme value theory approach, and (9) conditional autoregressive VaR approach.

II-2.1. Traditional Historical Simulation

The idea of traditional historical simulation is to utilize historical data of returns from a financial asset and estimate the N^{th} percentile as the target threshold. For example,

³ Artzner, Delbaen, Eber, and Heath (1997), on other hand, showed that it is possible to have reverse sub-additive condition where risk of a portfolio is higher than combinational risk of member assets, an example can a portfolio with short option position.

suppose we are interested in using the historical worst 1% of daily return as the estimated threshold. We would rank all returns within a certain period in the past, says 500 trading days, from lowest to highest, and then the 1st percentile would be the daily threshold; in the 500 trading days example, it would be the number 5 lowest return. We then multiply the threshold by the current value of the assets as the 1-day VaR estimate.

For a portfolio, we use percentage daily returns of all its underlying assets to multiply the current values of each asset and combine the multiples proportionally as the estimated returns of the portfolio. That will generate numbers of scenarios, same number as the number of historical data being used, and the 1st percentile ranked return would be the threshold. VaR of the portfolio, again, would be the multiple of the threshold and its current market value.

One advantage of this historical simulation approach is its ease of implementation. It does not require many complicated calculations or mathematical manipulations. Another advantage is that no specific distributional assumption is needed. The approach observes directly what happened in the past and applied the outcomes to future. It assumes current distribution of returns, both shape of the distribution curve and magnitude of spreading, is the same as past ones without knowing exactly the type of distribution.

However, as we discuss in section II-1, the assumption of constant pattern of distribution for the returns of financial assets may not be true, or it may be true for only recent time period. It can be the case that more recent data explain the behavior of returns better than older data does. Then how much past data should be used to produce VaR estimates may be a critical issue. However, the method also requires significant

level of sample size because the N^{th} percentile being selected must come from a large discrete data to make sense. Hendricks (1995) found that longer sample periods result in less variability in the VaR estimate and imprecise estimation of VaR when the sampling period is too short. It becomes a dilemma to pick proper sample size when implement the method.

One way to deal with small sample size is to, instead of obtain the percentile directly, find out the density curve of returns and select the left tail 1% density. Note that we are not assuming any existing distribution and apply it to the data but to obtain the distribution based on actual data through proper method. The obtained distribution needs not to be normal or symmetric or be any kind of known distribution. One approach to obtain the density is kernel estimator and its detail descriptions are in section II-3.

In additional, the historical 1% percentile (or any other percent levels) is not able to catch extreme movements beyond the point. Then, the drawback is that it couldn't provide sufficient information for extreme market activities beyond the VaR estimate. If such event occurs, the VaR estimate would lead users such as senior managers to reserve too less to cover the loss. This problem is true for many other methods, and some methods, such as conditional Value at Risk, address this issue directly. We discuss them in more details in following sections.

II-2.2. Conditional Value at Risk (CVaR)

An alternative way to measure the risk of returns of financial assets is the Conditional Value at Risk (CVaR), also known as the expected shortfall (Artzner, Delbaen, Eber, and

Heath [1997], and Rau-Bredow [2003]). Since the traditional VaR is not able to reflect the risk beyond the targeted threshold, why not estimate what would be the loss, in average, only for the losses that are beyond the given percent level (α)? The key concept of CVaR is to tell how much money the portfolio would expect to lose if return falls below the threshold. Based on the same layout of VaR (equation [7] and [8]), the definition of CVaR can be expressed by following equation.

$$CVaR_{\alpha} = E(\delta P \mid \delta P \geq VaR_{\alpha}). \quad (9)$$

The equation can be written as following density formula.

$$CVaR_{\alpha} = \frac{\exp(-\frac{q_{\alpha}^2}{2})}{\alpha\sqrt{2\pi}} \sigma, \quad (10)$$

where q_{α} is the tail 100α percentile of a standard normal distribution⁴. Therefore, in fact, a CVaR giving $\alpha = 0.01$ would have the standard deviation (daily volatility) multiplied by 2.67, and the CVaR becomes the same as VaR with 99.6 confidence level ($\alpha = 0.004$). However, the approach to produce CVaR using equation (10) requires the estimated volatility based on prior data and certain assumption of the distribution. It then can involve many other issues such as accuracy of volatility estimate or reasonable assumption of distribution. One disadvantage of this approach of CVaR is the

⁴ Detail derivations are shown at Yamai and Yoshida (2002).

unreliability of the estimate if either volatility is incorrectly forecasted or assumption of distribution is violated.

An alternative approach of CVaR estimation from historical data is to actually obtain losses that exceed the target VaR (or simply the top 1% losses among the data), and the average of those losses would be the CVaR estimate. This approach, however, also requires significant amount of historical data, maybe more than the historical simulation approach because the CVaR estimate only uses about 1% of whole sample period. Then the old data vs. new data issue strikes again – even more severe than in historical simulation.

Despite its disadvantages, CVaR does explain, even not fully, the extreme market behavior beyond the VaR. It provides a roughly idea about risk of extreme events in addition to VaR. Some studies show that the CVaR out-performs other approaches such as traditional historical simulation and Monte Carlo simulation (Oda and Muranaga [1997] and Uryasev [2000]). Combination usage of VaR and CVaR might be practical for risk management processes.

II-2.3. Variance-Covariance Approach

The variance-covariance approach is probably the most popular adopted method in financial industry. It estimates the VaR by using the estimated volatilities of a portfolio's underlying financial assets. Let begin with a portfolio with only a single financial asset, (with price S), and then we can define the continuously compounded return during day i as u_i . So,

$$u_i = \ln \left(\frac{S_i}{S_{i-1}} \right). \quad (11)$$

An estimate of the daily variance rate (σ_n^2) of u_i based on past m observations can be given as

$$\sigma_n^2 = \frac{1}{m-1} \sum_{i=1}^m (u_{n-i} - \bar{u})^2, \quad (12)$$

where \bar{u} is the mean of u_i 's, so $\bar{u} = (1/m) \sum (u_{n-i})$. When calculating the VaR, we assume \bar{u} as zero, $m-1$ as m , and u_i as the percentage change of S_i 's. So equation (12) can be simplified as

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^m u_{n-i}^2. \quad (13)$$

Then, the square root of obtained variance would be the estimated volatility for a single financial asset. In order to estimate the VaR of a portfolio with multiple assets, the following linear function for estimated volatility of the portfolio, σ_p , need to be applied.

$$\sigma_p^2 = \sum_{i=1}^n \alpha_i^2 \sigma_i^2 + 2 \sum_{i=1}^n \sum_{j<i} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j, \quad (14)$$

where α_n is the proportional amount invested in asset n in the portfolio and ρ_{ij} is the coefficient of correlation between asset i and j . Finally, estimated volatility of either a single asset or portfolio from estimated variance can be used in estimating the VaR by equation (7).

Note that this linear model approach (equation [14]) can only be applied to portfolio containing options for approximation. The model assumes zero gamma relationship (the curvature relationship) between the portfolio and underlying assets. If there are options or mortgages in the portfolio, such assumption is violated. To accurately estimate VaR of portfolio with nonlinear relationships for its components, nonlinear functions such as a quadratic function need to be used⁵.

Ammann and Reich (2000) found that the VaR estimates by the variance-covariance approach sometimes do not differ greatly from other simulations even for some optioned portfolios. However, they concluded that for heavily optioned portfolio, variance-covariance approaches with linear model are not appropriate to be applied. They argued that only simulation model such as historical simulation or Monte Carol simulation can be appropriate.

Two basic issues of the approach in VaR estimation need to be addressed. First, what is the best sampling period to estimate the volatility? The approach does not require as many data as in historical simulation, but the range of data must be determined. Also, many researches indicate that volatility is not constant over time, and it implies that more recent data are more relevant to current volatility than old ones. Then, modifications of this variance-covariance approach are necessary to account the issue, and couple of approaches discussed later address it.

⁵ Examples of such nonlinear function see Hull (2002).

Second, what distribution should be assumed? As discussed in section II-1, various researches provide different conclusions of the distributions of returns. The traditional VaR estimates based on normal distribution can easily miss-estimate the real risk. Many researches have found that the return distributions have fat-tailed, and consequently, VaR estimates based on normal or student- t distributions are subject to both high variation and upward bias (Kupiec and O'Brien [1995], Eberlein, Keller, and Prause [1998], Hull and White [1998], and Johansen and Sornette [1999]).

One way to deal with the fat-tail problem is to directly estimate the tail distribution, and such approach is the extreme value theory (EVT), which will be discussed in following sections in detail. If we consider the extreme argument saying that there is no systematical distribution pattern for returns of financial assets, the variance-covariance approach and its modifications become completely invalid. This is one major drawback of this approach.

Some researches applied small adjustments to the approach. For example, few researches examine VaR estimation by various multivariate volatility models and found robust estimates⁶. Others used so-called delta-gamma approximation approach, which utilizes Taylor expansion in the return modeling function (Zangari [1996]). In addition, in next section, we talk about one of most commonly used modified variance-covariance model, the EWMA approach.

II-2.4. Variance-Covariance Approach with EWMA

⁶ Alexandar and Leigh (1996), Dave and Stahl (1996), and Kroner and Ng (1998).

The above variance-covariance approach assumes equally weight for all rates of return in the past to estimate the current level of volatility. The exponentially weighted moving average (EWMA) model (also called the Hybrid Method) assigns an exponentially weight for the rates of return (Alexander and Leigh [1997] and Boudoukh [1998]). The idea of weighted rates of return is due to the argument that more recent data are more relevant to the current level. An EWMA formula can be written as

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1 - \lambda)u_{n-1}^2, \quad (15)$$

where λ is the weight assigned to one time interval lagged variance and must be a constant between 0 and 1. The approach is designed to track movements of past volatility. Suppose there is a jump of volatility, based on traditional variance-covariance approach, such jump has equally effect on the predicted volatility as long as it stays in the sample. Under the EWMA model, the jump has relatively high effects on predicted volatilities in short period and such effects fade out as time difference between the jump and predicted volatility gets longer.

Another attraction of the approach is that it requires less data than the variance-covariance approach does and further less than historical simulation needs. This attribute can be critical if the target objective does not have available long data. Smaller sample would make estimates less reliable for both historical simulation and variance-covariance approach but the EWMA approach. Also, the value of λ plays big role in the formula; higher λ assigned means that the estimate more heavily depends on recent data. The *RiskMetrics* database uses $\lambda = 0.94$.

The EWMA model is then estimated by the maximum likelihood method to generate variance. The method estimates the parameters by maximizing the likelihood of occurrence of data. After the variance is obtained, the approach follows same steps as in the variance-covariance approach to obtain the volatility of portfolios and then the VaR estimate. Since the approach also relies on volatility for VaR estimation, it has similar disadvantages as the variance-covariance approach caused by the inconsistency and/or unknown patterns of distribution of returns.

II-2.5. GARCH(1,1) Model Building Approach

The generalized autoregressive conditional heteroscedasticity (GARCH) approach is another modified method from variance-covariance framework. The GARCH(1,1) model (Bollerslev, 1986) assumes that there is a long-run average variance rate when predicting the current rate. Moreover, it treats volatility is time-varied and time-dependent. The GARCH(1,1) model has following form.

$$\sigma_n^2 = \lambda_1 V_L + \lambda_2 \sigma_{n-1}^2 + \lambda_3 u_{n-1}^2, \quad (16)$$

where V_L is the long-run variance rate and the sum of λ coefficients is equal to 1.

Comparing to equation (15), when $\lambda_1 = 0$, the GARCH(1,1) model is the same as EWMA model. However, the long-run rate makes GARCH(1,1) account one important character of variance, the mean reverting concept. We can rewrite equation (16) as a general estimation equation of GARCH(1,1) by following form.

$$\sigma_n^2 = w + \alpha\sigma_{n-1}^2 + \beta u_{n-1}^2. \quad (17)$$

So $V_L = w / (1 - \alpha - \beta)$, where $1 - \alpha - \beta$ is the weight assigned to the long-run variance rate. The GRACH(1,1) is equivalent to the model that the variance V follows a stochastic process, so

$$dV = a(V_L - V) dt + \xi V dz, \quad (18)$$

where dt is in days, $a = 1 - \alpha - \beta$, $\xi = \beta\sqrt{2}$, and dz is a Wiener process⁷. So the variance has a drift pulling it toward to V_L with rate of a . It is called a mean reverting model. In practice, variance rates tend to be mean-reverted. It is the improvement of the GARCH(1,1) method from the EWMA method. After obtaining the estimates and estimated volatility, GARCH(1,1) follows same procedures as EWMA method to estimate the VaR. Maximum likelihood method is also used for parameter estimation for GARCH(1,1).

Numbers of researches have utilized this method for VaR estimation (Alexander and Leigh [1997] and Boudoukh, Richardson, and White [1997]) and found robust estimates. Brooks and Persaud (2000) show that GARCH model is the best method for calculation of VaR when dealing with the downward bias effects of correlations between market and assets to the volatilities of assets. There have been different GARCH models proposed for dealing with time-varied covariance (Bollerslev, Engle and Wooldridge

⁷ Appendix I shows detail derivation of equation (18) from (17).

[1988] and Engle and Kroner [1955]), and then by various ARCH models to obtain better VaR estimates (Giot and Laurent [2001]).

II-2.6. Monte Carlo Simulation

The Monte Carlo simulation is a process that generates random price paths to approximate the movements of financial prices. In the application of VaR, by Monte Carlo simulation, we are able to generate the probability distribution for rates of return of portfolios. In order to calculate a 1-day VaR for a portfolio, Monte Carlo simulation follows below procedures:

1. Obtain current market values of all assets of the portfolio.
2. Generate random samples of rates of return for all underlying assets of the portfolio from multivariate normal distribution.
3. Apply the samples of returns for each asset and obtain their sampling values from current market values.
4. Compose the portfolio value based on the sampling values of assets and find the rate of change from current market value of the portfolio.
5. Repeat above steps many times to generate the probability distribution of rates of return of the portfolio.

Here, the VaR of portfolio is then directly gathered from the distribution of simulated portfolio values. One way to implement step 2 is to estimate S_T based on the equation (3) assuming $T = 1$, so the equation can be rewritten as

$$S_T = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) + \sigma\varepsilon\right], \quad (19)$$

where ε is the random sample from a standard normal distribution. A common way to generate the ε 's is from following function.

$$\varepsilon = \sum_{i=1}^{12} R_i - 6, \quad (20)$$

where R_i are independent random numbers between zero and one.

Monte Carlo simulation has become a powerful tool in financial pricing modeling because its flexible application. We can use the approach to generate total random fashion scenarios or scenarios by utilizing historical data. One particular version of Monte Carlo simulation approach for VaR is based on mixtures of normal distributions (Duffie and Pan [1997] and Rouvinez [1997]). Many researches concluded the approach is so far the best in estimating VaR, and some found that VaR estimate incorporating fat tails performed best (Raaji and Raunig [1998]). Some also argue that whether the Monte Carlo simulation is better than historical simulation depends on the type of components within portfolios (Lambadiaris, Papadopoulou, Skiadopoulos, and Zoulis [2003]).

One major advantage of the approach is that no historical data needed, therefore, consistency between old data and estimates regarding distribution is not an issue. The major drawback of the approach is its high requirement of computing resources. It takes tremendous of technologies, time, and human capital to execute proper simulations. Compare to other approaches, the marginal cost of more accurate VaR estimate by the approach might be too high. However, due to today's technology improvement, this disadvantage becomes less critical.

II-2.7. Model Building with Kalman Filter

The Kalman filter approach for VaR can be generated based on the relationship between variance of returns for financial assets and the measurement of beta from the Capital Asset Pricing model (CAPM). The CAPM, also known as the Diagonal Model of Sharpe (Sharpe [1964]), measures the level of risk for a financial asset by only one parameter, the returns of the market index. A general equation of the CAPM can be written as

$$\delta S = \alpha + \beta \cdot \delta M + \varepsilon, \quad (21)$$

where δS is the return of financial asset, δM is the return of market index, α is the constant, ε is the random walk error term, and most importantly, β is the risk measurement also known as the Beta coefficient. The Beta coefficient can be interpreted as relative level of risk for the asset comparing to its financial market. Note from equation (7), we can rewrite the general equation of VaR in following matrix term.

$$\text{VaR}(P, \alpha, \delta t) = P \cdot k \cdot \sqrt{w' \Sigma w} \sqrt{\delta t}, \quad (22)$$

where $w' \Sigma w$ is the variance of the portfolio returns, σ_p^2 . It can be shown that when number of assets in a portfolio increases, the σ_p^2 converges to $w' \beta \beta' w \sigma_m^2$, where σ_m^2 is the variance of market index. Therefore, the VaR can be shown as

$$\text{VaR}(P, \alpha, \delta t) = P \cdot k \cdot \sqrt{w' \beta \beta' w \sigma_m^2} \sqrt{\delta t}. \quad (23)$$

The beta can be estimated by various methods, and Harvey (1994) showed that Kalman filter is the best approach to explain and forecast time-varying movement of beta. Particularly, he used following state space model.

$$\delta S_{i,t} = \alpha_i + (\bar{\beta}_i + \beta_{i,t}) \cdot \delta M_t + \varepsilon_{i,t} \quad (24)$$

$$\beta_{i,t} = T \beta_{i,t-1} + \zeta_{i,t} \quad (25)$$

where equation (24) is the measurement equation and equation (25) is the state equation with T as a state transition matrix and $\zeta_{i,t}$ as an error term independently from $\varepsilon_{i,t}$. We assume that beta follows a first order autoregressive process and fluctuates around its mean, $\bar{\beta}_i$, so $\beta_{i,t}$ is mean reverting towards it⁸. Unknown parameters and matrices of the model are estimated by maximum likelihood method, and their estimates are calculated

⁸ It is actually same mean-reverting concept expressed in equation (19).

recursively through a series of Kalman filter equations⁹. These equations are a series of regression functions estimating the state vector, $\beta_{i,t}$, and its covariance matrix, and consequently, the VaR estimate can be generated based on equation (22).

This application of CAPM and Kalman filter to VaR was first seen in Beradi, Corradin, and Sommacampagna (2002). They found evidences showing that the approach is highly sensitive to market volatility changes. As market volatility goes high, their empirical outcomes of the approach become more robust. However, this approach employing the CAPM can be arguable for many researchers. Through these years that Beta coefficient was introduced and becomes widely used by industries, many empirical evidences show that the method is not a robust risk measurement tool. In its VaR application, by directly entering the market index variance and beta coefficient, the method requires further examinations and studies.

II-2.8. Extreme Value Theory (EVT) Approach

Latest developments of VaR estimation focus on the extreme value theory (EVT)¹⁰. The fundamental concept of EVT is to model the tail directly instead of the whole distribution of returns of financial instruments. Since VaR is generally focusing on the tail behavior, EVT provides better precisely understanding about the extreme behaviors for returns of financial assets or portfolios¹¹. In general, the EVT approaches use two types of

⁹ Detail description of the Kalman filter equations can be found in Kalman (1960) and Hamilton (1994).

¹⁰ See Danielsson and de Vries (1997), Embrechts, Kluppelberg and Mikosch (1997), McNeil (1997), Kellezi and Gilli (2000), McNeil and Frey (2000), and Clemente and Romano (2003).

¹¹ Detail methods see Embrechts, Resnick, and Samorodnitsky (1999) and Neftci (2000).

distribution: the generalized extreme value distribution and generalized Pareto distribution. This paper uses the later one in the discussion.

At first, we can assign a high threshold, U , for all the historical returns (δP) of financial portfolios and define the exceedances, y , as $\delta P - U$. Note that the U needs to be selected lower than the threshold (in absolute term) of VaR estimate, so the extreme activities can be fully examined beyond and below the target VaR estimate. The distribution of y can be written as

$$\Pr(\delta P < U + y \mid \delta P > U) = F_U(y) = \frac{F(U + y) - F(U)}{1 - F(U)}, y > 0. \quad (26)$$

Balkema and de Haan (1974) and Pickands (1975) showed the asymptotic form of above distribution is so called generalized Pareto distribution as following function,

$$H(y) = 1 - \left(1 - \frac{ky}{\sigma}\right)^{1/k}, \quad (27)$$

where k is a parameter from below density function.

$$h(y) = \frac{1}{\sigma} \left(1 - \frac{ky}{\sigma}\right)^{1/k-1}. \quad (28)$$

Since equation (27) is the asymptotic form of probability in equation (26), we know that

$$[1 - F(U + y)] = [1 - F(U)][1 - H(y)]. \quad (29)$$

Then, we can obtain the estimated parameters, \hat{k} and $\hat{\sigma}$, by maximum likelihood method in maximizing the likelihood function transforming from (28). Let L be the y , $1 - F(U + y) = \alpha$, and $1 - F(U) = n/m$, where L is the tail threshold (not the U but the target VaR threshold), n is the number of exceedences, and m is the total number of observations.

Therefore, by substituting α , n , m , and L into equations (28) and (29), we can obtain the estimated threshold.

$$\hat{L} = \frac{\hat{\sigma}}{\hat{k}} \left[1 - \left(\frac{\alpha m}{n} \right)^{\hat{k}} \right]. \quad (30)$$

Then, the VaR estimate would be the multiple of the threshold and value of portfolio.

Many evidences show that EVT approach produces more reliable VaR estimates than other approaches (Nefci [2000], Odening and Hinrich [2001], Darbha [2001], and Bao, Lee, and Saltoglu [2003]). One advantage of the approach is that no assumption of the distribution of returns is needed nor the estimated volatility is required. Another advantage of EVT is the method directly focuses on the behavior of tails, and extreme return activities are accounted. If the distribution of returns of financial asset cannot be predetermined, not symmetric, and likely to have fat tails, theoretically, EVT approach is more appealing than other approaches. One major drawback is that the method requires significant amount of data due to only small portion of distribution is examined. Next, we present an alternative approach utilizing the concept of EVT for VaR estimation.

II-2.9. Conditional Autoregressive VaR Approach

The conditional autoregressive VaR (CAViaR) approach was introduced by Engle and Manganelli (2002). Instead of explaining the distribution of returns, the CAViaR focuses on modeling the quantiles. Comparing to various other VaR estimating methods such as CVaR and EVT methods, Engle and Manganelli showed that the CAViaR models produce best estimates with heavy-tailed distribution. The idea is similar to the GARCH model approach but with following two major types of regression models. A symmetric CaViaR model

$$\sigma_t = a_0 + a_1\sigma_{t-1} + a_2|u_{t-1}|, \quad (31)$$

and an asymmetric model

$$\sigma_t = a_0 + a_1\sigma_{t-1} + a_2|u_{t-1}| + a_3|u_{t-1}| \cdot \mathbf{1}(u_{t-1} < 0). \quad (32)$$

The estimation then can be obtained by quantile regression on either model. In this study, we adopt the symmetric CaViaR model in our empirical comparison. Since the approach is also depending on the estimated volatility from past data, similar drawbacks in the variance-covariance approach are also applied here. In addition, the method needs relatively more data than traditional model building methods such as the EWMA approach or GARCH(1,1) method.

II-3. Extreme Value Theory with Kernel Estimator

We now present our new approach in VaR estimation – the extreme value theory (EVT) with Kernel estimator. First, we introduce kernel estimator. Second, we show how kernel estimator can be used in VaR estimation by historical simulation model. Finally, we explain the methodology of our approach by applying the concept of EVT.

II-3.1. Introduction to Kernel Estimator

Kernel estimator is a class of nonparametric estimators of densities. Simply put, a kernel estimator is a way to generalize a density curve from a histogram constructed with the data by spreading each data point with a kernel and a parameter called the bandwidth (Silverman [1986] and Sheather and Marron [1990]). A density f is the first derivative of the cumulative density function (cdf); so $F(x) = P(X \leq x)$ and $f(x) = F'(x)$. Thus, we can have a simple nonparametric estimator of F as the following empirical cdf,

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^n 1\{X_i \leq x\}, \text{ where } 1\{X_i \leq x\} = \begin{cases} 0, & \text{if } X_i > x \\ 1, & \text{if } X_i \leq x \end{cases}. \quad (33)$$

Then, we can develop a nonparametric estimator of the density as following function.

$$f(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0} \frac{F(x+h/2) - F(x-h/2)}{h}. \quad (34)$$

For a small and positive h , we can define

$$\hat{f}(x) = \frac{\hat{F}(x + h/2) - \hat{F}(x - h/2)}{h} = \frac{1}{nh} \sum_{i=1}^n 1\{x - h/2 < X_i \leq x + h/2\}, \quad (35)$$

which can be written as

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right), \text{ where} \quad (36)$$

$$K(u) = 1\left\{|u| \leq \frac{1}{2}\right\}. \quad (37)$$

Here, equation (36) is the general form of the kernel estimator with a function of K , known as the kernel, and a positive number of h , known as the bandwidth. Equation (37), the kernel, is the density of the uniform distribution over the interval between $-1/2$ and $1/2$ and is a discontinuous function. Instead of equation (37), we can use a smooth density as the kernel; for example, the Gaussian density as following can be applied,

$$K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right). \quad (38)$$

Finally, equation (36) and (38) combine as the normal distribution density function of kernel estimator, where h is the bandwidth, n is the sample size, and x is target variable.

Other than the uniform and normal distribution kernel, we can select different types of kernel, and following four kernels are commonly discussed in prior researches.

$$K(u) = \frac{3}{4\sqrt{5}} \left(1 - \frac{1}{5}u^2\right) \{ |u| \leq \sqrt{5} \} \quad \text{Epanechnikov Kernel}$$

$$K(u) = \frac{3}{4} (1 - u^2) \{ |u| \leq 1 \} \quad \text{Modified Epanechnikov Kernel}$$

$$K(u) = (1 - |u|) \{ |u| \leq 1 \} \quad \text{Triangular Kernel}$$

$$K(u) = \frac{\pi}{4} \cos\left(\frac{\pi u}{2}\right) \{ |u| \leq 1 \} \quad \text{Cosine Kernel}$$

Both theory and research suggest that different kernels have very small effects on the results of the kernel estimator (Epanechnikov [1969]). In our research, we apply different kernels in our data and find no significant differences in the outcome¹², so we use the Gaussian normal distribution function as the kernel in the VaR estimation model for simplicity. The bandwidth parameter, however, is a crucial component of the kernel estimator. It tells levels of different frequencies are used for estimating distribution among each kernel. A small bandwidth would under-smooth the estimation while a large bandwidth would over-smooth the estimation.

One method to select the bandwidth is the so-called plug-in formula of Sheather and Jones (Jones, Marron, and Sheather [1996]). The method solves the following fixed-point equation,

¹² Detail comparison of outcomes from different kernels is presented in section III-3.4.

$$h = \left[\frac{R(z)}{nR(\hat{f}_{g(h)}''[\int x^2 z(x)dx]^2)} \right]^{1/5}, \text{ where } R(z) = \int z^2(x)dx. \quad (39)$$

Three additional methods of bandwidth selection are also popular in prior researches.

$$h = \hat{\sigma}[4/(3n)]^{1/5} \quad \text{Simple Normal Reference}$$

$$h = 0.9 \min[\hat{\sigma}, (Q_3 - Q_1)/1.34]n^{-1/5} \quad \text{Silverman's Rule of Thumb}$$

$$h = 3\hat{\sigma}[1/(70\sqrt{\pi n})]^{1/5} \quad \text{Over-smoothed Bandwidth}$$

Here, $\hat{\sigma}$ is the sample standard deviation, n is sample size, and Q_3 and Q_1 are third and first sample quartiles respectively. Many applications of kernel estimator utilize the simple form of the Silverman's rule of thumb, $0.9\hat{\sigma}n^{-1/5}$, as the bandwidth (Silverman [1986] and Butler and Schachter [1997]). In this paper, we apply all 4 bandwidth selection methods in VaR estimation for comparison and find that the Silverman's rule of thumb method slightly produces better results than other methods.

Figure 1 illustrates how the method generates the smooth density curve from the histogram. Note that the obtained density curve does not follow any existing distribution curve. The movements of the obtained density curve along the tails are very irregular and the example of figure 1 shows a slightly fat left tail. Such illustration implies that no existing model building approaches can precisely estimate the volatility and apply assumed distribution in VaR.

II-3.2. Historical Simulation with Kernel Estimator

One way to apply kernel estimator in VaR estimation is using historical simulation. The kernel estimator can improve traditional historical simulation is by obtaining a smooth density of sample data (Silverman [1986] and Sheather and Marron [1990]). We can apply the estimator to historical data and obtain actual density. Form the density curve, we can then obtain the coordinated threshold with lowest 1% density and produce the VaR estimates same way as the historical simulation model.

Compare to traditional historical simulation approach, the advantage of this method is that less data is required, so it can use smaller sample size to produce the VaR estimate. Since it utilizes more recent data and smoother percentile than the traditional historical simulation does, it improves the estimation. Butler and Schachter (1997) found that VaR estimates by various types of kernel estimators produce more precisely threshold than the traditional historical simulation and other methods.

II-3.3. Apply Kernel Estimator with EVT on VaR Estimation

In this paper, we present a new approach for VaR estimation by combining the concepts of EVT and kernel estimator. Basically, compare to above presented EVT methods (section II-2.8), instead of assuming certain distribution of the exceedences, we use kernel estimator to understand the tail behavior and generate its distribution curve. We use following procedures to generate our VaR estimates for a single asset portfolio.

1. Select a threshold of returns of portfolios where the threshold should be lower than the target VaR confidence level in absolute term.
2. Obtain the losses (in percent form) that exceeded the threshold.
3. Apply kernel estimator to smooth the distribution of these losses and obtain actual densities of each kernel.
4. Find the cut-point whose cumulative density proportionally (the ratio of sample of exceeded losses over the whole sample) equal to the confidence percentage (α).
5. Such selected cut-point is the VaR estimate in percentage form, and the multiple of the cut-point and the market value of portfolio would be the VaR.

In practice, a predetermined percentage of the threshold may not be suitable for all portfolios. In our empirical research, we simply obtain the worse N percent of returns from the sample as the tailed losses. We then obtain the cut-point who has cumulative density is $(\alpha/N)\%$. For example, suppose we have 500 returns in the sample for VaR estimation, and we are looking for 1 day VaR at 99% confidence level. We can obtain the worse 5% (as an example of N) returns (sample size with 25) and find the cut-point with 20% of density on its left tail. That cut-point would be the VaR estimate in percentage form. More descriptions about the empirical application of the approach are discussed in section IV.

There are few major drawbacks of the approach. First, it requires significant level of sample size. Since the approach targets only the tail returns, small sample might produce bias estimate about the tail distribution. Second, it uses past info of tail returns to explain the current one. Therefore, since tail returns are extreme behavior and may

occur randomly, current tail behavior may be unrelated to the past. Third, different levels of threshold might be critical to the estimation outcomes. Lower threshold means larger sample but also inherits less effects of extreme activities. Trade-off between low and high threshold needs extra attention in utilizing the approach.

One advantage of this approach is that it directly focuses on the tail behavior. It provides precise understanding of the tail distribution. Like the EVT method, no specific distribution of returns and estimated volatility are needed in this approach. In addition, unlike other EVT methods, no pre-assumed tail distribution (or the exceedences of tail returns) is necessary. Finally, the approach requires less modeling than variance-covariance approaches and less computing resources than simulation approaches. Our empirical findings show that the approach outperforms other methods, and detail discussion and comparison of empirical applications of the approach with others are presented in section III.

II-4. Methods of Evaluation of Value at Risk Estimates

In this section, we talk about the main methods to evaluate VaR estimates. We will introduce and discuss following three main methods: (1) back testing, (2) likelihood ratio method, and (3) loss function approach. Note that these methods are not mutually exclusive in usage and, most of time, are simultaneously applied.

II-4.1. Back Testing of VaR Estimates

The back testing is the most popular method in testing VaR estimates and yet most straight-forwarded approach. The VaR shows that there is only $\alpha\%$ of probability that the next day return of portfolio will lose more than the 1-day VaR estimate. Therefore, we can then compare actual returns and the VaR estimates in 1 day to see if the portfolio loses more than the VaR. From the out-of-sample, we can then compute the percentage of occurrence of such losses, and compare it to the desired percent, α . As the percentage gets more close to the desired α , the estimating approach performs better.

For example, when 1% desired percentage is set, we can compute the VaR estimates for over a sample of 100 trading days. For each of 100 days, we compute the VaR by a in-sample prior to the day and compare the estimate with the actual return in the next day. Therefore, we can obtain 100 comparisons, and ideally, only one of the actual returns should fall below its 1% VaR estimate.

The approach is simple and easily implemented. However, simple comparison of percentages between actual and desired level is not sufficient for model comparison when sample sizes are different. Also, when applying VaR estimates in different time frame (N days) or different desired percentages, a general indicator is needed for comparisons among the outcomes of different applications. Next method, likelihood ratio, of VaR evaluation provides such indicator in addition to the back testing approach.

II-4.2. Likelihood Ratio Method

Likelihood ratio test is a testing procedure for interval forecasts. Christoffersen (1998) has designed a three-steps procedure for the evaluation of interval forecasts: a test for

unconditional coverage, independence, and conditional coverage. All three tests are performed using likelihood ratio framework. The test of unconditional coverage tests a null hypothesis that the probability of failure or exception is α (against an alternative that the probability differs from assuming failure process). The test has following likelihood ratio formula based on a chi-square(1) probability distribution.

$$LR_{uc} = -2 \ln \left(\frac{\alpha^{n_1} (1-\alpha)^{n_0}}{\hat{\pi}^{n_1} (1-\hat{\pi})^{n_0}} \right) \sim \chi_1^2, \quad (40)$$

where α is the desired significant level of VaR, n_0 is the number of times that VaR forecast is not exceeded, n_1 is the number of times that VaR forecast is exceeded, and $\hat{\pi}$ is the ratio of $n_1/(n_0 + n_1)$. One can obtain the probability, the p -value, of the ratio in chi-square distribution, and the probability indicates the chance that back testing outcomes of VaR estimate differ from desired level. Therefore, higher the p -value means worse the VaR estimating methods. The test of independence tests a null hypothesis that the failure process is independently distributed against the alternative that the process follows a first order Markove process.

$$LR_{ind} = -2 \ln \left(\frac{(1-\hat{\pi}_2)^{(n_{00}+n_{10})} \hat{\pi}_2^{(n_{01}+n_{11})}}{(1-\hat{\pi}_{01})^{n_{00}} \hat{\pi}_{01}^{n_{01}} (1-\hat{\pi}_{11})^{n_{10}} \hat{\pi}_{11}^{n_{11}}} \right) \sim \chi_1^2, \quad (41)$$

where n_{ij} is number of i values followed by a j value in the series (with $i, j = 0, 1$) and

$\pi_{ij} = \Pr(I_t = i | I_{t-1} = j)$. The $\hat{\pi}$'s are defined as

$$\hat{\pi}_{01} = \frac{n_{01}}{n_{00} + n_{01}}, \hat{\pi}_{11} = \frac{n_{11}}{n_{10} + n_{11}} \text{ and } \hat{\pi}_2 = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}}. \quad (42)$$

Finally, a tighter requirement is that a VaR model provides correct conditional coverage.

If a VaR model has the ability to capture the conditional distribution of returns and its dynamic properties such as time varying volatility accurately, then exception should be unpredictable, then the likelihood ratio formula for the test is

$$LR_{cd} = -2 \ln \left(\frac{\alpha^{n_1} (1-\alpha)^{n_0}}{(1-\hat{\pi}_{01})^{n_{00}} \hat{\pi}_{01}^{n_{01}} (1-\hat{\pi}_{11})^{n_{10}} \hat{\pi}_{11}^{n_{11}}} \right) \sim \chi_1^2. \quad (43)$$

II-4.3. Loss Function Approach

Lopez (1999) proposes a regulatory loss function in order to assess the accuracy of the VaR estimates. The general form of the loss function for asset i at time t is

$$L_{i,t+1} = \begin{cases} f(\delta P_{i,t+1}, VaR_{i,t+1}) & \text{if } \delta P_{i,t+1} < VaR_{i,t+1} \\ g(\delta P_{i,t+1}, VaR_{i,t+1}) & \text{if } \delta P_{i,t+1} \geq VaR_{i,t+1} \end{cases}. \quad (44)$$

The typical back testing of VaR treat every loss over the VaR estimate as a single event, and equally weight is assigned for the cases. Then the loss function would be

$$L_{i,t+1} = \begin{cases} 1 & \text{if } \delta P_{i,t+1} < VaR_{i,t+1} \\ 0 & \text{if } \delta P_{i,t+1} \geq VaR_{i,t+1} \end{cases}. \quad (45)$$

However, the loss function can be modified to account the magnitude of each loss by assigning different weight based on level of exceedences. Lopez suggested following quadratic loss function to realize large losses¹³.

$$L_{i,t+1} = \begin{cases} 1 + (\delta P_{i,t+1} - VaR_{i,t+1})^2 & \text{if } \delta P_{i,t+1} < VaR_{i,t+1} \\ 0 & \text{if } \delta P_{i,t+1} \geq VaR_{i,t+1} \end{cases}. \quad (46)$$

In addition to the above three testing methods, other testing is also available such as the reality check test (White [2000]) and normal transform methods (especially in dealing with small data such as banking performance, Berkowitz [2001]). In this study, we report the back testing outcome (count and percent) and the unconditional likelihood ratio testing results.

¹³ Sarma, Thomas, and Shah (2000) agreed that the modified loss function is the best form for testing VaR of financial assets.

III. Empirical Findings

We present our empirical findings by following three sections. First, we explain our data and samples. Second, we discuss the models of VaR estimation we employ. Lastly, we talk about empirical findings of our studies.

III-1. Data and Sample

For this study, we obtain 3,510 trading records for 15 stocks from 5/13/1991 to 4/22/2005. From these 15 stocks, we compose 3 different portfolios containing 5 to 8 stocks and report 4 stocks individually for all VaR estimation. All together, we have 7 financial assets: portfolio 1 ~ 3 and stock 1 ~ 4. Detail descriptions about the portfolios and stocks are shown in table 1.

We obtain only adjusted closing prices for all the stocks. The prices are adjusted to splits and dividends (not real interest rates) for the last day of sample, 4/22/2005. The combinations of portfolio are 1,000 shares of each member stock, so the shared portions of member stocks to the portfolio are the multiple of 1,000 and their market values (closing prices). The returns of portfolio are then calculated by the differences between total values the day and next day.

We apply 500 days of in-samples to generate the VaR estimates for most of methods. Some models require less than 500 days including EWMA, GARCH, and CaViaR models. So we do not have VaR estimates for the first 500 days; the data is simply for estimation for the day 501 and so on. Starting at day 501, we obtain all VaR estimates as well as the actual change of values for 4 period ranges: 1 day, 2 day, 5 day

and 10 day. We do so until day 3,500, so the last 10 days' data are for actual returns calculations for day 3,501 to 3,510 and are not part of out-of-sample.

In sum, we use maximum of 500 days (depending on models) as in-sample size for every out-of-sample days. Then the data size in generating estimates is 500 or less, and the out-of-sample for evaluations has 3,000 days. The total range of data covers about 14 years, and for an out-of-sample size 3,000, we believe can provide sufficient evaluations.

III-2. Model Selection

In the methodology section, we briefly examine most methods of VaR. We then develop following 10 models for empirical research. Table 2 also provides a brief comparison and descriptions among the models.

- (1) HS: historical simulation. We utilize traditional non-weighted historical simulation method as described in section II.2.1. We use 500 as in-sample size.
- (2) CVaR: conditional VaR. A simple historical simulation is used to produce the CVaR estimates (section II.2.3) with 500 days as sample size. Note that the meaning of CVaR is different from general definition of VaR in other methods, so its evaluations are conducted separately.
- (3) HS-KE: kernel estimator. The kernel estimator is applied to the whole 500 data in the sample to generate density curve for the whole distribution of all returns. The bandwidth is selected by $0.9sn^{-0.2}$.

- (4) VC: variance-covariance. It is typical non-weighted variance-covariance approach with normal distribution assumption.
- (5) EWMA: exponentially weighted moving average. As described in section II.2.5, the parameters are estimated and then applied to the variance-covariance model. 400 days of in-sample are used. The weight assigned to one time lagged variance is 0.94 and normal distribution is assumed.
- (6) GARCH: generalized autoregressive conditional heteroscedasticity. Constant volatility is assumed for in-samples with 200 days of data. Normal distribution is assumed.
- (7) MC: Monte Carlo simulation. We generate iteration for 5,000 times as potential paths of returns.
- (8) CaViaR: conditional autoregressive VaR. Following Engle and Manganelli (2002), we use the symmetric model directly on the return series.
- (9) EVT-KE5: extreme value theory with kernel estimator (5% returns). Here, we filter the worse 5% returns for the in-sample 500 data, so we obtain 25 tail returns. Then we applied the kernel estimator to obtain the estimate by selecting the cut-point with bottom 20% of the total density. Figure 2 provides an example of the smoothing process by kernel estimator for day 1 of portfolio 3.
- (10) EVT-KE10: extreme value theory with kernel estimator (10% returns). Similar to above EVT-KE5 model, but instead of 5%, we now filter the worse 10% returns for the in-sample 500 data, so we obtain 50 tail returns. Then we applied the kernel estimator to obtain the estimate by selecting the cut-point with bottom 10% of the

total density. Figure 3 provides an example of the smoothing process by kernel estimator for same tail but bigger range (lower threshold) as in figure 2.

Note that we obtain both 99% and 95% VaR estimates for model 1 to model 8. We like to examine the performances of different confidence levels. The EVT-KE models (9 and 10), however, focus on most extreme activities, and if they need to produce 95% estimates, bigger tails are required and some non-extreme activities would be covered. Therefore, we use EVT-KE models for 99% estimates only. Also, we use normal distribution density function as our kernel and simplified Silverman's rule of thumb ($0.9\hat{\sigma}n^{-1/5}$) as the bandwidth for both EVT-KE models.

III-3. Empirical Outcomes

We present our findings based on 3 dimensions and 2 separately report. First, we compare testing results between various models. Second, we discuss testing results between different desired confidence levels. Third, we like to see will VaR estimates performances change among different time-horizons. For these three comparisons, we present back testing result (count and percentage) and the unconditional likelihood ratio tests. In addition, we also present two separate reports: first, we compare results of the EVT-KE approach based on different kernel selections. Finally, we talk about the back testing outcomes for Conditional VaR separately.

III-3.1. Comparison between Models

This section presents the main findings of our study. Most researches of VaR mainly focus on how to produce best estimates. The desired confidence level and time-horizon are either standardized in financial industry or play less critical role in the field. Improving estimating techniques to properly interpret the risk by VaR can benefit practitioners and regulators for better risk management.

Table 3 presents our first findings about the performances of different VaR estimation methods. For each model, we estimate 1 day VaR by 99% confidence level for all 3,000 data of out-of-sample. Then we present the number of days whose next day losses exceed the VaR estimates (count) and its ratio over the sample size (percent). We also employ the likelihood ratio testing method (equation [40]) and present its p -value, which shows the probability of unfitted estimation method. Therefore, ideally, if a model works perfectly, it should have count as 30, percent as 1, and the LR test as 0.

In table 3, we have outcomes for all models and for all 7 financial instruments. To determine which model performs the best (or worst), we can focus on outcomes by each financial instrument. For portfolio 1, 8 out of 9 models have their counts more than 40. In term of LR test, only both EVT-KE5 and EVT-KE10 models have significant outcomes at 95% C.I. (p -values less than 95%). For portfolio 2, all models produce significant results, and again both EVT-KE models produce best estimates where counts are both 32 (1.06%). The HS and HS-KE models have next best estimates, and ironically, MC model has worse outcomes with count as 40 and LR test as 0.919 (barely significant).

For portfolio 3, similar comparison can be seen. EVT-KE models have best counts both at 31 with LR test 0.145, and HS and HS-KE follow as next best models. All

model-building methods, including VC, EWMA, GARCH, and CaViaR, produce same counts at 42 and LR test 0.962 (non-significant). MC again produces the worst estimate with count of 49.

When look over the performances of models for individual stocks, all 4 stocks tell similar story. The only exception is stock 2, where MC has best estimate at count as 32 and LR test as 0.283. For the same stock, EVT-KE models and model-building models have similar outputs with counts around 24 and 25 and LR test around 0.7. For stock 4, model-building models produce good outcomes with counts around 31 to 32, and so does EVT-KE models with counts both at 32. For stock 1 and 3, comparisons between models are same as portfolios.

When looking over all assets by model, all 7 estimates from EVT-KE models are statistical significant. Both HS and HS-KE models have significant results for 6 assets while VC has 5, and other models have 4. For all assets, both EVT-KE models are within top 3 models based on LR tests. Such results are remarkable for the EVT-KE models. In a large sample of 3,000 over more than 10 years daily data, we are convinced that the EVT-KE models outperform all others. In fact, EVT-KE models produce counts within 1.1% almost for all assets, and this precise and reliable estimating can ensure high quality risk management.

For table 4, we use 1 day VaR estimates for 95% confidence level. As previous discussed, EVT-KE models are not used for 95% VaR estimation. Here, the best testing ideally should have count as 150 and percent as 5%. Among remaining models, HS has estimates significant for all assets, while HS-KE has 6 significant results. The rest of models perform very badly, and MC model generate significant result for only portfolio 1.

All model-building methods fail to produce good estimates; in fact, many of them have counts too small (some of the percents are around only 3.8% which it should be 5%).

In sum, comparison of models in this study shows that EVT-KE (extreme value theory with kernel estimator) models produce the best VaR estimates. HS and HS-KE models perform next best, and MC and model-building methods perform the worst. In general, only EVT-KE models' estimates are reliable across different portfolios and individual assets. At 95% confidence level, less significant results are shown.

III-3.2. Comparison between Confidence Levels

As we already suspect from previous comparison, estimates for 95% VaR seem less robust. Now we look at both outcomes at the same table for detail comparison. Table 5 shows 1 day VaR estimates for both 99% and 95% for all 3 portfolios and for 5 key models. We eliminate two model-building methods because they generally produce similar outcomes. We also found that differences between models are consistent among portfolios and individual stocks, so we focus on only portfolios.

For portfolio 1, all 5 models' VaR estimates of 95% C.I. perform better than 99% based on LR test. For portfolio 2 and 3, only HS still has better result for 95% C.I.; other 4 models have better outcomes for 99% C.I. Combine observations in previous tables, most models have better results in 99% C.I. than in 95% C.I. just like in portfolio 2 and 3. Therefore, we can say that, in general, VaR estimates in 99% C.I. are more reliable than in 95%.

Another interesting finding is that for many models, especially for model-building methods, the numbers of count are higher than 30 (1%) in 99% while numbers of count are lower than 150 (5%). Since normal distribution is assumed for these models (2.33 and 1.645 are used as k for 99% and 95% respectively), the outcomes are consistent with the fat-tail concept. At the 99% level, with normal distribution assumption, VaR estimates tend to underestimate the risk because of the fat tail; however, at 95% level, the estimates generally overestimate the risk because now the tails become smaller than normal (the normal densities between 99% and 95% are higher than densities of return distribution). Figure 4 illustrates such situation.

III-3.3. Comparison between Time Horizons

In practice, VaR estimates are usually used for 1 day time horizon. Theoretically, VaR can be computed for any time horizon, one week, one month, or even one year. However, it is reasonable to assume that longer the time period is the accuracy of VaR estimates drops. Table 6 provides estimates for 4 different time horizons: 1 day, 2 days, 5 days (a week trading days), and 10 days (two weeks). The table shows same outcomes, count, percent, and LR test, for 7 key models and for portfolio 2 and 3.

For portfolio 2, there are very close results between 1 day and 2 days. For 6 out of 7 models, the differences of counts between 1 day and 2 days are with 2, and more importantly, outcomes of 2 day are all better than outcomes in 1 day. For portfolio 3, 5 out of 7 models have better outcomes of 2 day than ones of 1 day. Such interesting findings show that VaR estimates might be more reliable in 2 day time frame. Actually,

one argument that supports the finding can be following. Since 1 day VaR estimates are generally underestimate the risk, they should perform better in 2 day time horizon because the magnitude of loss in two consecutive days cannot be the same as in 1 day.

However, when we move on to compare the outcomes for 5 days and 10 days, we find that VaR estimates become less reliable as time horizon increases. For portfolio 2, none of 7 models produce significant outcome for 10 day estimates and only 2 models have significant outcomes for 5 day estimates. For portfolio 3, there are relatively more significant estimates than in portfolio 2, but the decrease of significance stands when compare estimates from 1 or 2 days to 5 or 10 days.

III-3.4. Comparison between Different Kernels

As we discuss in section II-3.1, there are different kernel selections available when applying kernel estimator. In our study, we employ the most common three kernels to our data and see if different kernels perform differently in the VaR estimation. We select following three kernels: (1) normal distribution density (Gaussian density), (2) Epanechnikov kernel, and (3) triangular kernel. We apply these kernels into our EVT-KE5 model described in section III-2 and see if they produce significantly different VaR estimates and cause different back testing results.

Table 7 shows the actual VaR estimates between three kernels of all 7 financial instruments for selected 5 days of the out-of-sample. The estimates are generated for 1-day horizon and 99% confidence interval. The report values are VaR estimates in return form. For example, the value of 0.9815 in the upper left corner cell (Day 1, Gaussian

kernel, and portfolio 1) shows that there is 1% chance the Day 2 dollar value of portfolio 1 would be less than 98.15% of Day 1's dollar value. The common VaR estimate would be the multiple of return from value and the dollar value.

When focus on the VaR estimates in table 7, there are very small differences between the outcomes from three kernels. For example, for Day 1 and portfolio 2, the VaR estimates for above three kernels are 0.9792, 0.9794, and 0.9798 respectively. The differences among the three values are all less than 0.01%. In fact, all differences of estimates between three kernels by any group of day and financial instrument are within 0.5%, and most of the differences are less than 0.2%.

We also conduct the back testing evaluation of VaR estimates for the three different kernels and compare their results. Table 8 present the back testing results of model EVT-KE5 by 1-day horizon and 99% confidence interval for three kernels in all 7 financial instruments. The reported counts in table 8 show numbers of times that the next-day return falling below the VaR estimate in a sample of 3,000 days. The values are exactly identical between three different kernels for portfolio 2 and 3 and stock 2 to 4. Even for financial instruments with different results, the reported counts are extremely closed. Based on comparison of both exercises, the actual VaR estimates and back testing, we conclude that different kernel selection does not have significant impact on application of kernel estimator on VaR estimation.

III-3.5. Outcome of Conditional VaR Estimation

We now like to evaluate how the CVaR model performs in our study. The CVaR estimates show the expected loss from losses that exceed the desired threshold in N day. For our comparison, we provide CVaR estimates for 4 different time horizons and for all 7 financial instruments.

Table 9 shows the back testing outcomes of the CVaR estimates. Here, we report the following three indicators: mean-errors, number of losses, and standard deviations of the mean-errors. By mean-error, we mean the average differences between the CVaR estimates (at 99% C.I.) and the actual average of loss from losses that exceed the 1% threshold (estimated by HS VaR). For example, in table 9, the mean-error of CVaR 1 day estimate for portfolio 1 is -136. The value is the average (in dollars) of all differences between the CVaR 1 day estimate and actual losses that exceed the bottom 1% threshold.

Theoretically, if the CVaR estimates are accurate, the mean-error should be equal to the average loss and their mean-differences should have been zero. The back testing sees as the mean-error toward to zero (proportionally by the market values of portfolios or individual stocks) better CVaR estimate is. In table 9, for 1 day CVaR, the mean-errors for most of portfolios or stocks are very low comparing to the market values. It implies that the estimates provide valid info about the risk; actual extreme losses are not far away from the expected values.

When we compare CVaR estimates for different time horizons, we can see the mean-errors generally increase for most financial assets. That means the CVaR estimates in longer time horizon provides less reliable expected loss. This trend is consistent with the finding in previous section for VaR estimates. In practice, despite the CVaR may

have more profound meanings than VaR in explaining risk, VaR is way more popular than CVaR. Good CVaR estimates can be a supplement tool of VaR but not a substitute.

IV. Conclusions

IV-1. Main Findings

This paper provides an overview of VaR estimation in financial assets. We briefly discuss major VaR estimating approaches and analyze their advantages and drawbacks. Models based on historical data require extra cautions to select proper sampling period. Models with distribution assumptions need to account unique attributes of returns' behaviors of financial assets such as fat tails, non-symmetric, or extreme activities.

We also present a new approach for estimation of VaR, the extreme value theory with kernel estimator (EVT-KE) approach. The approach utilizes concept of extreme value theory to model the tail pattern of returns directly. It then employs the kernel estimator technique to understand and explain the patterns and consequently produce valid estimate of VaR.

First key improvement of the EVT-KE approach from existing methods is that no pre-assumed distribution necessary for neither the whole sample nor the tails. Second, it does not required estimated volatility of underlying assets; therefore, discussions of volatility behavior such as weighted scheme, time-varying, or time-dependent need not to be addressed. Third, it directly focuses tail activities, which are the target subjects for risk management in VaR estimation.

Based on our empirical research, we found that the EVT-KE approach outperforms other methods. It provides more precise VaR estimates, and its testing results are significantly superior. Future developments of this method can be application to nonlinear portfolios, standardized procedures in selection initial threshold, or better

bandwidth selection in kernel estimator. Regarding the comparison between various VaR time horizons and confidence levels, our findings show that 1-day and 99% VaR estimate is the best appropriate selection.

IV-2. Applications and Future Developments of VaR

In addition to the reviews that are covered in the paper, developments in this field show some other interests or applications of VaR. Some researches investigate the marginal VaR, the contribution from the individual portfolio components to the diversified portfolio VaR. Such interest can also extend to the component VaR, the proportion of the diversified portfolio VaR that can be attributed to each of the individual components; or the incremental VaR, incremental effect on VaR of adding a new asset to the existing portfolio (Hallerbach [2003]).

Other researches have been done to examine the application of VaR by actual banking data. Berkowitz and O'Brien (2002) applied various VaR methods for six large commercial banks and back-tested the VaR models' performances. It has also been applied to credit risk and liquidity risk measurements (Oda and Muranaga [1997]). The VaR estimation can be applied to improve the forecast performance of volatility models such as a covariance matrix within a VaR framework (Hendricks [1996]).

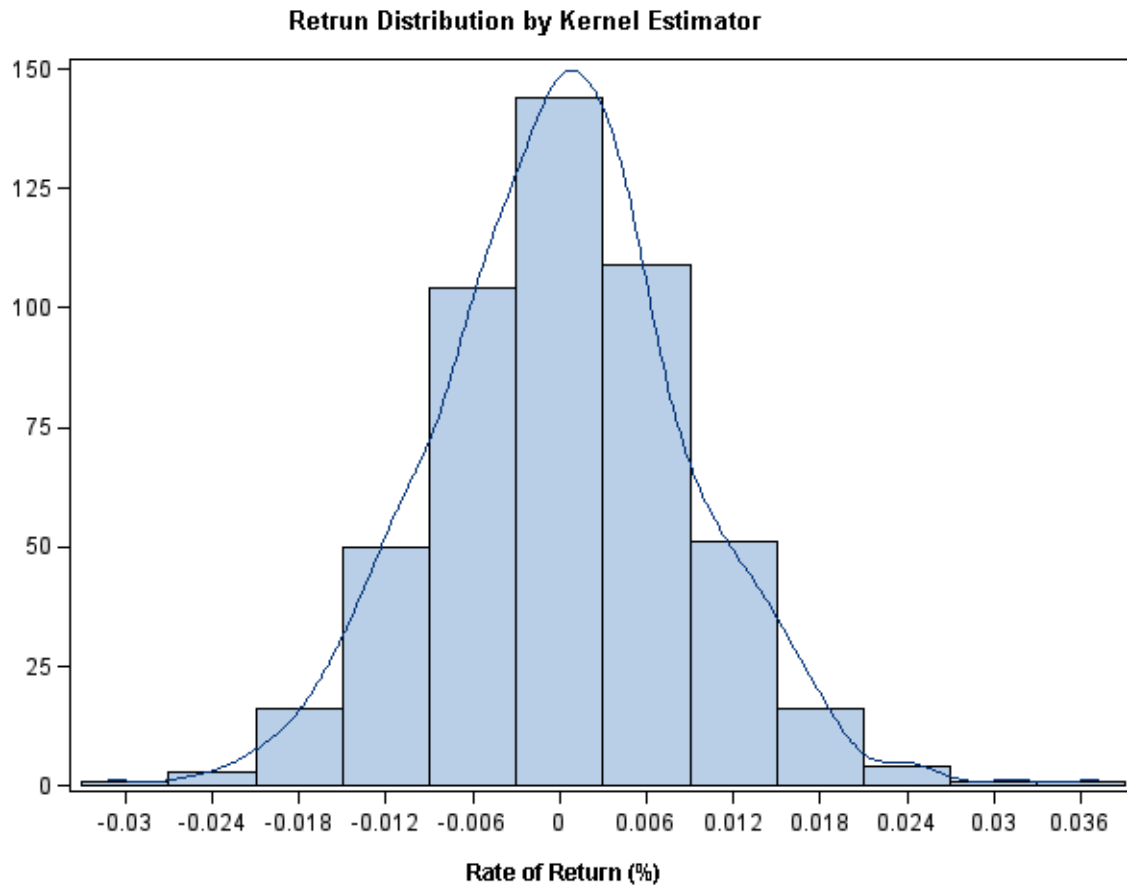
Kaplanski and Kroll (2001) found that VaR can significantly improve the estimation of Capital Assets Pricing model (VaR-Beta approach). Odening and Hinrichs (2001) used VaR method with EVT on agricultural context to forecast the returns in feeder pig production and hog finishing in German hog market. Ho, Abrahamson, and

Abbott (1996) applied both historical simulation and variance-covariance VaR approaches to model banks' structural balance sheet as organizational or structural risk measurement instruments.

Other than market risk, VaR can be used in many risk related areas. Regulator can use the tool to modify abnormal activities in financial market, banking activities, productions, or accounting processes. It can also be applied not only risk management but also quality control, organizational analyses, or operational management. Researcher can utilize it as tool in other finance or economics technologies.

Finally, in term of methodology, developments on how to provide better VaR forecast can never stop. Model selection becomes more critical because different models may perform differently for various underlying subjects and sampling periods. Systematical techniques in model selection are demanded and urgent.

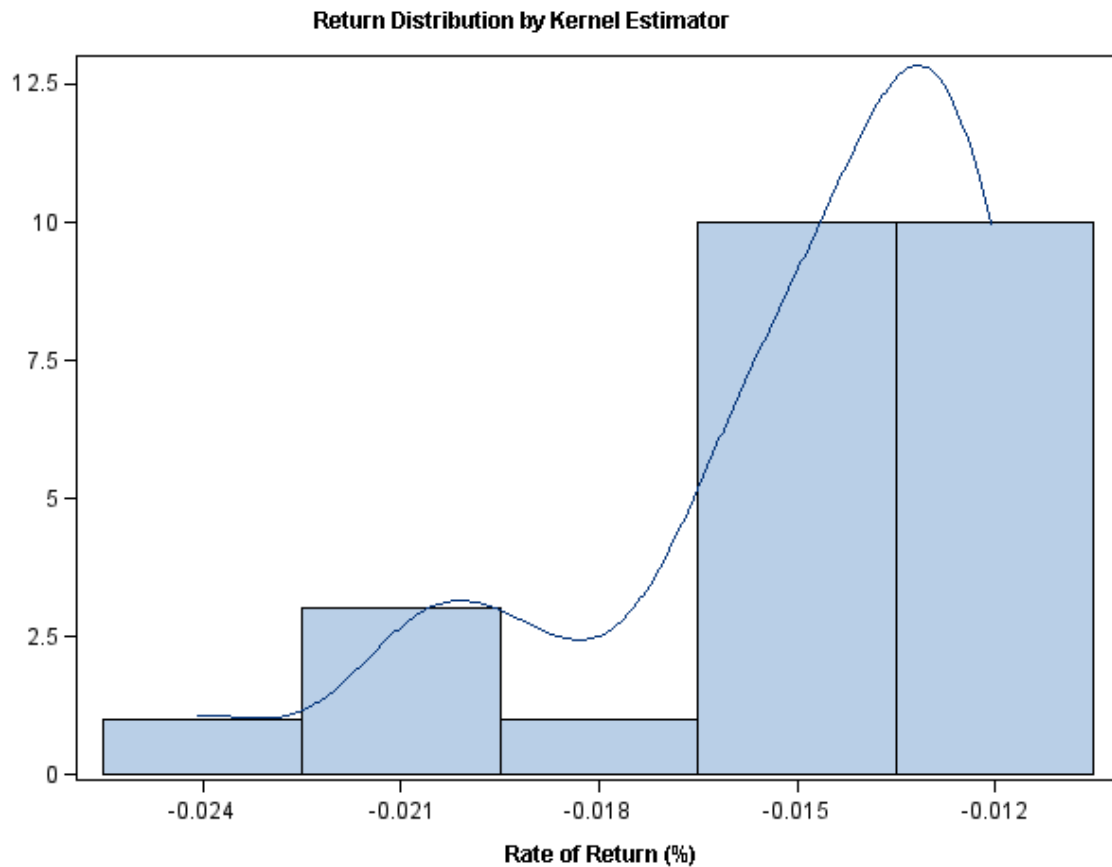
Figure 1, Demonstration of Density Smoothing by Kernel Estimator



Note:

The graph illustrates the smoothed density distribution generated by kernel estimator from a sample. The sample data are 500 rates of return for Portfolio 2 (table 1) from 4/29/2003 to 4/22/2005. The graph is produced by SAS based on KDE procedure and ODS graphing command. The histograms show actual counts of rates of return for groups based on 0.6% intervals, and the curve shows the smooth distribution for the data based on kernel estimator. The bandwidth is selected by the function, $0.9sn^{-0.2}$, where s is the standard deviation of data and n is its sample size. In this case, the bandwidth is 0.0023 ($=0.9*0.0088*500^{-0.2}$).

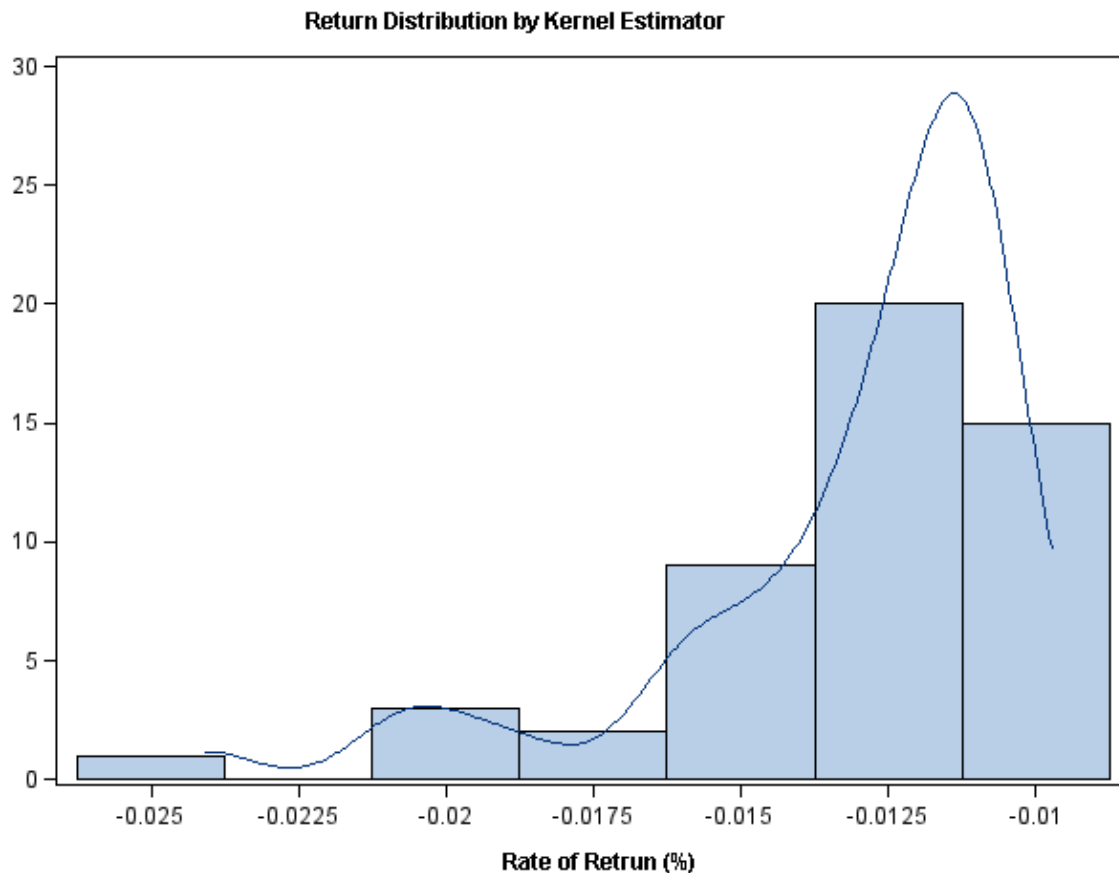
Figure 2, Tail Returns Smoothing by Kernel Estimator (Worse 5% Returns)



Note:

The graph illustrates the smoothed density distribution generated by kernel estimator from a 2 stage sample. The first stage sample data are 500 rates of return for Portfolio 3 (table 1) from 4/29/2003 to 4/22/2005. The second stage sample contains the 5% worse returns from the first stage sample. Then, the graph is produced on second stage sample only (sample size = 25) by SAS based on KDE procedure and ODS graphing command. The histograms show actual counts of rates of return for groups based on 0.3% intervals, and the curve shows the smooth distribution for the data based on kernel estimator. The bandwidth is selected by the function, $0.9sn^{-0.2}$.

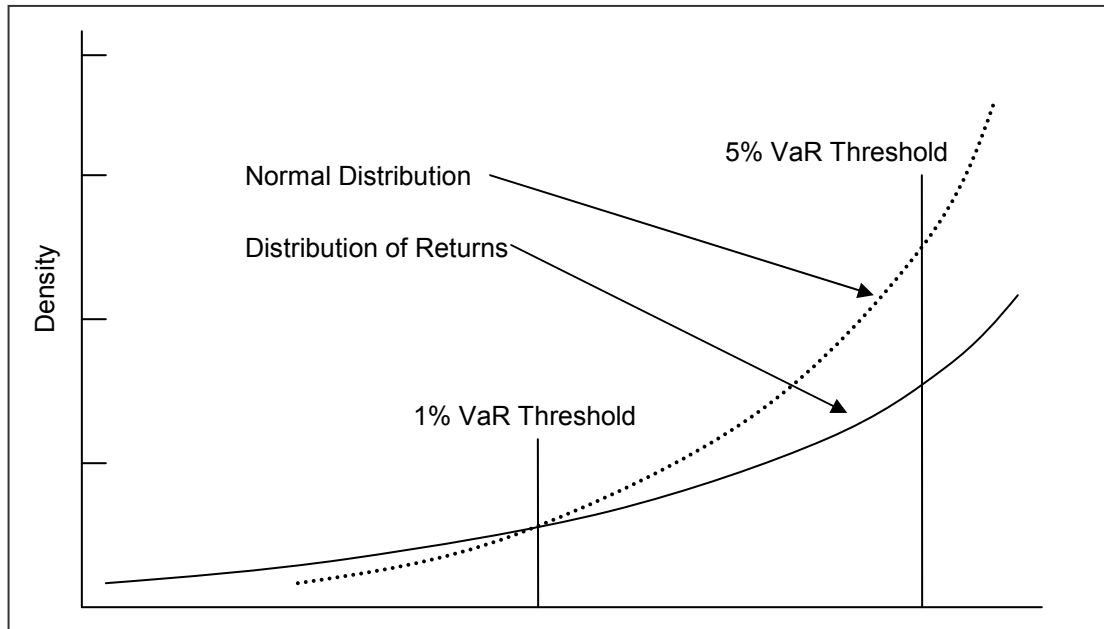
Figure 3, Tail Returns Smoothing by Kernel Estimator (Worse 10% Returns)



Note:

The graph illustrates the smoothed density distribution generated by kernel estimator from a 2 stage sample. The first stage sample data are 500 rates of return for Portfolio 3 (table 1) from 4/29/2003 to 4/22/2005. The second stage sample contains the 10% worse returns from the first stage sample. Then, the graph is produced on second stage sample only (sample size = 50) by SAS based on KDE procedure and ODS graphing command. The histograms show actual counts of rates of return for groups based on 0.25% intervals, and the curve shows the smooth distribution for the data based on kernel estimator. The bandwidth is selected by the function, $0.9sn^{-0.2}$.

Figure 4, Example of Tail Distribution (Left Tail)



Notes:

The graph shows that the difference between normal distribution (dash line) and return distribution (solid line) would make fat tail at $\alpha = 0.01$ but thin tail at $\alpha = 0.05$. As illustrate, the area under the solid line to the left of the 1% VaR threshold is clearly bigger than the area under dash line. So the fat tail exists, and VaR estimates at 99% confidence level based on normal distribution tend to underestimate risk.

On the contrary, the area under solid line to the left of the 5% VaR threshold becomes smaller than the area under dash line (thin tail). It implies that VaR estimates at 99% confidence level based on normal distribution tend to overestimate risk.

Table 1, Descriptions of Portfolios, Stocks, and Sample

Portfolios/Stocks	Components/Descriptions
Portfolio 1	(1) Applied Materials Inc. (2) Citigroup Inc. (3) Exxon Mobil Corp. (4) McDonald's Corp. (5) Merck & Co Inc. (6) Microsoft Corp. (7) Verizon Communications. (8) Wal-Mart Stores Inc.
Portfolio 2	(1) Fannie Mae. (2) Ford Motor Co. (3) General Electric Co. (4) Johnson & Johnson Inc. (5) Microsoft Corp.
Portfolio 3	(1) Citigroup Inc. (2) Coca-Cola Co. (3) Exxon Mobil Corp. (4) International Business Machines Corp. (5) Novell Inc.
Stock 1	The Walt Disney Co.
Stock 2	International Business Machines Corp.
Stock 3	JPMorgan Chase and Co.
Stock 4	Microsoft Corp.

Notes:

1. The sample covers 3,510 trading days from 5/23/1991 to 4/22/2005.
2. Unless indicated otherwise, all VaR estimates are calculated by data of 500 prior trading days, so the in sample size is 500. Beginning at day 501 (5/14/1993), actual returns of portfolios and stocks are obtained for evaluations until day 3,500; therefore, the number of observations for out-of-sample is 3,000. Data of day 3,501 to 3,510 are used for evaluation purpose.
3. All closing prices and returns are adjusted for splits and dividends for the last day of data (4/22/2005).
4. All three portfolios are composed by 1,000 shares of their component stocks, and the dollar-shares for stocks of each portfolio are calculated based on adjusted closing price at day 1.

Table 2, Descriptions of Empirical VaR Estimating Models

Model	Estimating Method	Special Note
HS	Historical Simulation	
CVaR	Conditional Value at Risk	
HS-KE	Kernel Estimator	Bandwidth is selected by $0.9sn^{-0.2}$.
VC	Variance-Covariance	
EWMA	Exponentially Weighted Moving Average	The weight assigned to one-time lagged variance is 0.94 (λ).
GARCH	Generalized Autoregressive Conditional Heteroscedasticity	Constant volatility assumed for in-sample.
MC	Monte Carlo Simulation	5,000 numbers of iteration are used.
CaViaR	Conditional Autoregressive VaR	Symmetric model is applied.
EVT-KE5	Extreme Value Theory with Kernel Estimator (5% left tail)	5% of worst losses are examined
EVT-KE10	Extreme Value Theory with Kernel Estimator (10% left tail)	10% of worst losses are examined

Notes:

1. For model-building approaches including VC, EWMA, GARCH, and CaViaR, normal distribution is assumed. 2.33 and 1.645 are numbers of standard deviation being applied for 99% and 95% confidence levels respectively.
2. In-sample size is 500 for all models except EWMA and GARCH (with sample size 400).
3. Both EVT-KE models employ normal distribution density function as the kernel and simplified Silverman's rule of thumb the bandwidth.

Table 3, Value at Risk Estimates (99%, 1 Day) Back Testing Outcomes

Model	Indicator	Portfolio 1	Portfolio 2	Portfolio 3	Stock 1	Stock 2	Stock 3	Stock 4
HS	Count	47	36	35	35	37	33	38
	Percent	1.57	1.20	1.17	1.17	1.23	1.10	1.27
	LR Test	0.996	0.714	0.629	0.629	0.785	0.412	0.841
HS-KE	Count	43	34	33	35	33	34	35
	Percent	1.43	1.13	1.10	1.17	1.10	1.13	1.17
	LR Test	0.975	0.528	0.412	0.629	0.412	0.528	0.629
VC	Count	46	38	42	41	25	38	31
	Percent	1.53	1.27	1.40	1.37	0.83	1.27	1.03
	LR Test	0.994	0.841	0.962	0.944	0.655	0.841	0.145
EWMA	Count	46	38	42	42	25	38	32
	Percent	1.53	1.27	1.40	1.40	0.83	1.27	1.07
	LR Test	0.994	0.841	0.962	0.962	0.655	0.841	0.283
GARCH	Count	46	38	42	42	24	38	32
	Percent	1.53	1.27	1.40	1.40	0.80	1.27	1.07
	LR Test	0.994	0.841	0.962	0.962	0.746	0.841	0.283
MC	Count	48	40	49	44	32	40	35
	Percent	1.60	1.33	1.63	1.47	1.07	1.33	1.17
	LR Test	0.998	0.919	0.999	0.984	0.283	0.919	0.629
CaViaR	Count	46	38	42	42	24	38	31
	Percent	1.53	1.27	1.40	1.40	0.80	1.27	1.03
	LR Test	0.994	0.841	0.962	0.962	0.746	0.841	0.145
EVT-KE5	Count	41	32	31	32	24	32	32
	Percent	1.37	1.07	1.03	1.07	0.80	1.07	1.07
	LR Test	0.944	0.283	0.145	0.283	0.746	0.283	0.283
EVT-KE10	Count	39	32	31	34	25	32	32
	Percent	1.30	1.06	1.03	1.13	0.83	1.07	1.07
	LR Test	0.886	0.283	0.145	0.528	0.655	0.283	0.283

Notes:

1. Detail descriptions of portfolios and stocks please refer to table 1. Detail descriptions of models please refer to table 2.
2. Out of sample size is 3,000.
3. The VaR estimates are for 1 day time horizon with 99% confidence level ($\alpha = 0.01$).
4. Counts are the actual number of next-day returns exceeding the VaR estimates, and percent is the percentage of the count out of 3,000.
5. LR test is the unconditional likelihood ratio test based on chi-square(1) distribution. The reported number is the p -value, and higher the value indicates worse the VaR estimating method is.

Table 4, Value at Risk Estimates (95%, 1 Day) Back Testing Outcomes

Model	Indicator	Portfolio 1	Portfolio 2	Portfolio 3	Stock 1	Stock 2	Stock 3	Stock 4
HS	Count	163	149	140	157	145	145	142
	Percent	5.43	4.97	4.67	5.23	4.83	4.83	4.73
	LR Test	0.717	0.067	0.603	0.440	0.326	0.326	0.501
HS-KE	Count	145	133	127	140	134	137	130
	Percent	4.83	4.43	4.23	4.67	4.47	4.57	4.33
	LR Test	0.326	0.853	0.952	0.603	0.827	0.731	0.913
VC	Count	127	116	114	111	89	114	96
	Percent	4.23	3.87	3.80	3.70	2.97	3.80	3.20
	LR Test	0.952	0.997	0.998	0.999	1.000	0.998	1.000
EWMA	Count	126	116	114	110	89	114	96
	Percent	4.20	3.87	3.80	3.67	2.97	3.80	3.20
	LR Test	0.961	0.997	0.998	1.000	1.000	0.998	1.000
GARCH	Count	126	116	115	110	89	114	96
	Percent	4.20	3.87	3.83	3.67	2.97	3.80	3.20
	LR Test	0.961	0.997	0.998	1.000	1.000	0.998	1.000
MC	Count	134	120	108	116	96	119	100
	Percent	4.47	4.00	3.60	3.87	3.20	3.97	3.33
	LR Test	0.827	0.991	1.000	0.997	1.000	0.993	1.000
CaViaR	Count	126	116	115	110	89	114	96
	Percent	4.20	3.87	3.83	3.67	2.97	3.80	3.20
	LR Test	0.961	0.997	0.998	1.000	1.000	0.998	1.000

Notes:

1. Detail descriptions of portfolios and stocks please refer to table 1. Detail descriptions of models please refer to table 2.
2. Sample size is 3,000.
3. The VaR estimates are for 1 day time horizon with 95% confidence level ($\alpha = 0.05$).
4. Counts are the actual number of next-day returns exceeding the VaR estimates, and percent is the percentage of the count out of 3,000.
5. LR test is the unconditional likelihood ratio test based on chi-square(1) distribution. The reported number is the p -value, and higher the value indicates worse the VaR estimating method is.

Table 5, Comparison of Value at Risk Estimates (1 Day) between 99% and 95%

Model	Indicator	Portfolio 1		Portfolio 2		Portfolio 3	
		99%	95%	99%	95%	99%	95%
HS	Count	47	163	36	149	35	140
	Percent	1.57	5.43	1.20	4.97	1.17	4.67
	LR Test	0.996	0.717	0.714	0.067	0.629	0.603
HS-KE	Count	43	145	34	133	33	127
	Percent	1.43	4.83	1.13	4.43	1.10	4.23
	LR Test	0.975	0.326	0.528	0.853	0.412	0.952
VC	Count	46	127	38	116	42	114
	Percent	1.53	4.23	1.27	3.87	1.40	3.80
	LR Test	0.994	0.952	0.841	0.997	0.962	0.998
EWMA	Count	46	126	38	116	42	114
	Percent	1.53	4.20	1.27	3.87	1.40	3.80
	LR Test	0.994	0.961	0.841	0.997	0.962	0.998
MC	Count	48	134	40	120	49	108
	Percent	1.60	4.47	1.33	4.00	1.63	3.60
	LR Test	0.998	0.827	0.919	0.991	0.999	1.000

Notes:

1. Detail descriptions of portfolios please refer to table 1. Detail descriptions of models please refer to table 2.
2. Sample size is 3,000.
3. The VaR estimates are for 1 day time horizon, and the 99% and 95% columns shows outcomes by 99% and 95% confidence levels respectively.
4. Counts are the actual number of next-day returns exceeding the VaR estimates, and percent is the percentage of the count out of 3,000.
5. LR test is the unconditional likelihood ratio test based on chi-square(1) distribution. The reported number is the p -value, and higher the value indicates worse the VaR estimating method is.

Table 6, Comparison of Value at Risk Estimates (99%) for Different Time Horizons

Model	Indicator	Portfolio 2				Portfolio 3			
		1 Day	2 Day	5 Day	10 Day	1 Day	2 Day	5 Day	10 Day
HS	Count	36	35	20	4	35	30	21	23
	Percent	1.20	1.17	0.67	0.13	1.17	1.00	0.70	0.77
	LR Test	0.714	0.629	0.949	1.000	0.629	0.000	0.919	0.820
HS-KE	Count	34	32	17	4	33	32	17	22
	Percent	1.13	1.07	0.57	0.13	1.10	1.07	0.57	0.73
	LR Test	0.528	0.283	0.991	1.000	0.412	0.283	0.991	0.877
VC	Count	38	36	18	5	42	38	32	26
	Percent	1.27	1.20	0.60	0.17	1.40	1.27	1.07	0.87
	LR Test	0.841	0.714	0.983	1.000	0.962	0.841	0.283	0.547
EWMA	Count	38	36	18	5	42	38	32	26
	Percent	1.27	1.20	0.60	0.17	1.40	1.27	1.07	0.87
	LR Test	0.841	0.714	0.983	1.000	0.962	0.841	0.283	0.547
MC	Count	40	43	25	7	49	44	37	31
	Percent	1.33	1.43	0.83	0.23	1.63	1.47	1.23	1.03
	LR Test	0.919	0.975	0.655	1.000	0.999	0.984	0.785	0.145
EVT-KE5	Count	32	30	14	3	31	26	16	21
	Percent	1.07	1.00	0.47	0.10	1.03	0.87	0.53	0.7
	LR Test	0.283	0.000	0.999	1.000	0.145	0.547	0.995	0.919
EVT-KE10	Count	32	31	15	4	31	27	17	22
	Percent	1.06	1.03	0.50	0.13	1.03	0.90	0.57	0.73
	LR Test	0.283	0.145	0.998	1.000	0.145	0.424	0.991	0.877

Notes:

1. Detail descriptions of portfolios please refer to table 1. Detail descriptions of models please refer to table 2.
2. Sample size is 3,000.
3. The VaR estimates are 99% confidence level. The N-day VaR estimates are generated based on equation (7), where $\delta t = N$.
4. Counts are the actual number of next N day(s) returns exceeding the VaR estimates, and percent is the percentage of the count out of 3,000.
5. LR test is the unconditional likelihood ratio test based on chi-square(1) distribution. The reported number is the p -value, and higher the value indicates worse the VaR estimating method is.

Table 7, Comparison of VaR Estimates by Different Kernel Selections

Day	Kernel	Portfolio 1	Portfolio 2	Portfolio 3	Stock 1	Stock 2	Stock 3	Stock 4
1	Gaussian	0.9815	0.9792	0.9819	0.9693	0.9751	0.9674	0.9658
	Epanechnikov	0.9814	0.9794	0.9827	0.9687	0.9753	0.9665	0.9668
	Triangular	0.9820	0.9798	0.9844	0.9669	0.9755	0.9671	0.9677
101	Gaussian	0.9742	0.9728	0.9745	0.9575	0.9658	0.9563	0.9601
	Epanechnikov	0.9752	0.9733	0.9744	0.9579	0.9666	0.9570	0.9591
	Triangular	0.9750	0.9735	0.9750	0.9583	0.9666	0.9560	0.9605
201	Gaussian	0.9624	0.9627	0.9623	0.9379	0.9466	0.9326	0.9485
	Epanechnikov	0.9634	0.9630	0.9626	0.9391	0.9468	0.9335	0.9485
	Triangular	0.9630	0.9630	0.9627	0.9385	0.9473	0.9333	0.9486
301	Gaussian	0.9623	0.9629	0.9588	0.9372	0.9395	0.9303	0.9469
	Epanechnikov	0.9637	0.9631	0.9595	0.9379	0.9418	0.9312	0.9464
	Triangular	0.9628	0.9633	0.9588	0.9373	0.9406	0.9306	0.9461
401	Gaussian	0.9629	0.9616	0.9582	0.9310	0.9394	0.9276	0.9398
	Epanechnikov	0.9637	0.9619	0.9588	0.9322	0.9406	0.9273	0.9394
	Triangular	0.9636	0.9612	0.9582	0.9315	0.9405	0.9269	0.9393

Notes:

1. All three kernels are applied to the EVT-KE5 model (table 2) in producing VaR estimates.
2. In-sample size is 500. Estimates are reported for 5 selected days of the out-of-sample of 3,000 days.
3. The VaR estimates are for 1-day time horizon with 99% confidence level ($\alpha = 0.01$).
4. Reported values are VaR estimates in return form. For example, the 0.9815 (Day 1, Gaussian kernel, and portfolio 1) shows that there is 1% chance the next-day dollar value of portfolio 1 would be less than 98.15% of today's dollar value.

Table 8, Back Testing Outcomes of VaR Estimates by Different Kernel Selections

Kernel	Portfolio 1	Portfolio 2	Portfolio 3	Stock 1	Stock 2	Stock 3	Stock 4
Gaussian	41	32	31	32	24	32	32
Epanechnikov	40	32	31	32	24	32	32
Triangular	41	32	31	33	24	32	32

Notes:

1. All three kernels are applied to the EVT-KE5 model (table 2) in producing VaR estimates.
2. Sample size is 3,000.
3. The VaR estimates are for 1-day time horizon with 99% confidence level ($\alpha = 0.01$).
4. Reported values are counts of actual number of next-day returns exceeding the VaR estimates. Since the values are almost identical between different kernel selections, percentages and LR test results are not presented.

Table 9, Conditional Value at Risk Estimate (99%) Back Testing Outcomes

Model	Indicator	Portfolio 1	Portfolio 2	Portfolio 3	Stock 1	Stock 2	Stock 3	Stock 4
Average Value (\$)		210,407	130,180	169,981	26.46	75.34	30.14	23.72
CVaR 1 Day	Mean - Error (\$)	-136	-168	-225	-0.159	0.377	-0.276	-0.077
	N	47	36	35	35	37	33	38
	STD (\$)	2,463	1,199	2,046	0.723	2.615	0.570	0.826
CVaR 2 Day	Mean - Error (\$)	-554	-363	234	-0.153	0.847	-0.233	-0.048
	N	42	35	30	38	40	37	38
	STD (\$)	3,660	1,733	2,761	0.737	2.315	0.811	0.713
CVaR 5 Day	Mean - Error (\$)	-1,946	378	220	-0.149	1.815	-0.455	0.193
	N	29	20	21	28	29	35	30
	STD (\$)	5,788	2,084	3,943	0.911	2.315	0.985	0.767
CVaR 10 Day	Mean - Error (\$)	-2,579	-749	298	0.090	1.671	-1.114	0.686
	N	21	4	23	30	19	26	22
	STD (\$)	10,015	4,318	2,375	0.684	5.594	2.128	1.740

Notes:

1. Detail descriptions of portfolios please refer to table 1.
2. Sample size is 3,000.
3. The CVaR estimates are the expected values of losses that exceed left-tail at 99% confidence level.
4. The mean-error is the average of differences between CVaR estimates and actual losses that exceed the 99% VaR estimate (historical simulation). N is the number of losses exceeding the same threshold, and STD is the standard deviation of those differences.
5. Average value is the average closing prices for each portfolio or stock from the days whose losses exceed the same thresholds by any of 4 models. It provides references about market values of the portfolios and stocks.
6. All prices for portfolios and stocks are adjusted by splits and dividends for the last day of sample (4/22/2005).

Appendix I, Derivation of GARCH(1,1) Time Reverting Stochastic Process

We can first add one time lagged of variance to the function (section II-2.5, equation [17]),

$$\begin{aligned}\sigma_n^2 &= w + \alpha\sigma_{n-1}^2 + \beta u_{n-1}^2, \text{ so} \\ \sigma_n^2 - \sigma_{n-1}^2 &= w + (\alpha - 1)\sigma_{n-1}^2 + \beta u_{n-1}^2.\end{aligned}\quad (1)$$

The u_{n-1}^2 has a mean σ_{n-1}^2 of and a variance of

$$E(u_{n-1})^4 - [E(u_{n-1}^2)]^2 = 2\sigma_{n-1}^4.$$

Assuming that u_i the are generated by a Winener process, dz , and the standard deviation of u_{n-1}^2 should be $\sqrt{2}\sigma_{n-1}^2$, then

$$u_{n-1}^2 = \sigma_{n-1}^2 + \sqrt{2}\sigma_{n-1}^2\varepsilon, \quad (2)$$

where ε is the random walk term from a standard normal distribution. From equation (1) and (2), we have

$$\sigma_n^2 - \sigma_{n-1}^2 = w + (\alpha + \beta - 1)\sigma_{n-1}^2 + \beta\sqrt{2}\sigma_{n-1}^2\varepsilon.$$

Let $\delta V = \sigma_n^2 - \sigma_{n-1}^2$, $V = \sigma_{n-1}^2$, $a = 1 - \alpha - \beta$, $aV_L = w$, and $\zeta = \alpha\sqrt{2}$. We have

$$\delta V = a(V_L - V) + \zeta\varepsilon V.$$

Because time is measured in days, $\delta t = 1$, so the process for V would be

$$dV = a(V_L - V) dt + \zeta V dz.$$

Bibliography

- Alexander, C.O. and C.T. Leigh. (1997) "On the Covariance Models Used in Value at Risk Models." *Journal of Derivatives*, V.4, 50-62
- Ammann, M. and C. Reich. (2001). "Value-at-Risk for Nonlinear Financial Instruments." Working paper.
- Andersen, T.G. (1994) "Stochastic Autoregressive Volatility: A Framework for Volatility Modeling." *Mathematical Finance*, V.4, 75-102.
- Artzner, P., F. Delbaen, J.M. Eber, and D. Heath. (1997) "Thinking Coherently." *Risk*, V.10, 68-71.
- Bao, Y., T.H. Lee, and B. Saltoglu. (2003) "Evaluating Predictive Performance of Value-at-Risk Models in Emerging Markets: A Reality Check." Working paper.
- Basle Committee on Banking Supervision. (1994) "Risk management Guidelines for Derivates." Bank for International Settlements, Basle, Switzerland (July).
- Basle Committee on Banking Supervision. (1996) "Amendment to the Capital Accord to Incorporate Market Risks." Bank for International Settlements, Basle, Switzerland.
- Berardi, A., S. Corradin, and C. Sommacampagna. (2002) "Estimating Value at Risk with the Kalman Filter." Working paper.
- Berkowitz, J. (2001) "Testing Density Forecasts, Applications to Risk Management." *Journal of Business and Economic Statistics*, V.19, 465-474.
- Berkowitz, J. and J. O'Brien (2002) "How Accurate are the Value-at-Risk Models at the Commercial Banks?" *Journal of Finance*, V.57, 1093-1112.
- Bollerslev, T. (1986) "Generalized Autoregressive Conditional Heteroscedasticity." *Journal of Econometrics*, V.31, 307-27.
- Bollerslev, T., R.F. Engle, and J.M. Wooldridge (1988) "A Capital Asset Pricing Model with Time-Varying Covariances." *Journal of Political Economy*, V.96, 116-131.
- Boudoukh, T., M. Richardson and R.F. Whitelaw. (1997) "Investigation of a Class of Volatility Estimates." *Journal of Derivatives*, V.4, 63-71.
- Boudoukh, T., M. Richardson and R.F. Whitelaw. (1998) "The Best of Both Worlds: A Hybrid Approach to Calculating Value at Risk." Working paper.
- Brooks, C. and G. Persaud. (2000) "Value at Risk and Market Crashes." Discussion paper in finance 2000-01, The University of Reading.

Butler, J.S. and B. Schachter. (1997) "Estimating Value-at-Risk with a Precision Measure by Combining Kernel Estimation with Historical Simulation." Economic & policy analysis working paper, Comptroller of the Currency Administrator of National Banks.

Chen, L., C.J. Hueng, and C.J. Lin. (2000) "Do Bubbles and Time-Varying Risk Premiums Affect Stock Prices? A Kalman Filter Approach." Working paper series (No. 00-04-02), The University of Alabama.

Christoffersen, P. (1998) "Evaluating Interval Forecasts." *International Economic Review*, V.39, 841-862.

Clemente, A.D. and C. Romano. (2003) "Measuring Portfolio Value-at-Risk by a Copula-EVT Based Approach." Working paper.

Danielsson, J. and C.G. de Vries (1997) "Value at Risk and Extreme Return." Discussion paper (No. 273), London School of Economic, Financial Market Group.

Darbha, G. (2001) "Value-at-Risk for Fixed Income Portfolios – A Comparison of Alternative Models." Working paper.

Dave, R.D. and G. Stahl. (1996) "On the Accuracy of VaR Estimates Based on the Variance-Covariance Approach." Manuscript, German Federal Banking Supervisory Office.

Dowd, K. (1998) *Beyond Value-at-Risk: The New Science of Risk Management*. John Wiley & Sons, London.

Duffie, D. and J. Pan. (1997) "An Overview of Value at Risk." *Journal of Derivatives*, V.4, 7-49.

Eberlein, E., U. Keller, and K. Prause. (1998) "New Insights into Smile, Mispricing, and Value at Risk: the Hyperbolic Model." *Journal of Business*, V.71, 371-405

Embrechts, P., C. Kluppelberg, and T. Mikosch. (1997) *Modeling Extremal Events for Insurance and Finance*, Springer Verlag, Berlin.

Embrechts, P., S.I. Resnick, and G. Samorodnitsky (1999). "Extreme Value Theory as a Risk Management Tool." *North American Actuarial Journal*, V.3, 30-41

Engle, R.F. (1982) "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation." *Econometrica*, V.50, 987-1008.

Engle, R.F. and K. Kroner. (1995) "Multivariate Simultaneous Generalized ARCH." *Econometric Theory*, V.11, 122-150.

Engle, R.F. and S. Manganelli. (2002) "CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles." Working paper (No. 7341), NBER.

Ghysels, E. and J. Jasiak. (1996) "Stochastic Volatility and Time Deformation: An Application to Trading Volume and Leverage Effects." Working paper, University of Montreal.

Giot, P. and S. Laurent (2001) "Value-at-Risk for Long and Short Trading Positions." Working paper.

Glosten, L. (1993) "On the Relationship between the Expected Value and the Volatility for the normal Excess Return on Stocks." *Journal of Finance*, V.48, 1779-1801.

Hamilton, J. (1994) *Time Series Analysis*. Princeton University Press, Princeton, NJ.

Harvey, A.C. (1994) *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge University Press, Cambridge.

Hallerbach, W.G. (2003) "Decomposing Portfolio Value-at-Risk: A General Analysis." *The Journal of Risk*, V.5, 1-18.

Hendriks, D. (1995) "Evaluating Value-at-Risk Models Using Historical Data." Manuscript, Federal Reserve Bank of New York (July).

Ho, T.S, A.A. Abrahamson, and M.C. Abbott. (1996) "Value at Risk of a Bank's Balance Sheet." Working paper, Global Advanced Technology Corporation.

Hull, J. (2002) *Options, Futures, & Derivatives*, 5th Edition, Prentice Hall, NJ.

Hull, J. and A. White. (1998) "Value at Risk When Daily Changes in Market Variables are not Normally Distributed." *Journal of Risk*, V.1, 47-61.

Johansen, A. and D. Sornette. (1999) "Critical Crashes." *Risk*, V.12, 91-95.

Jorion, P. (2001) *Value-at-Risk: the New Benchmark for Controlling Market Risk*. Chicago: McGraw-Hill.

JP Morgan (1996) *RiskMetrics*, Technical document, 4th Edition, New York.

Kalman, R.E. (1960) "A New Approach to Linear Filtering and Prediction Problems." *Transaction of the ASME - Journal of Basic Engineering*, V.82, 35-45.

Kaplanski, G. and Y. Kroll (2001) "Value-at-Risk Equilibrium Pricing Model." Working paper.

- Kellezi, E. and M. Gilli. (2000) "Extreme Value Theory for Tail-related Risk Measures." Working paper, University of Geneva.
- Keoner, K.F. and V.K. Ng. (1998) "Modeling Asymmetric Comovements of assets Returns." *The Review of Financial Studies*, V.11, 817-844.
- Krause, A. (2003) "Exploring the Limitations of Value at Risk: How Good Is It in Practice?" *The Journal of Risk Finance*, V.4, 19-28.
- Kupiec, P. and J. O'Brien. (1994) "The Use of Bank Risk Measurement Models for Regulatory Capital Purposes." Manuscript, Board of Governors of the Federal Reserve (March).
- Lambadiaris, G., L. Papadopoulou, G. Skiadopoulos, and Y. Zoulis (2003) "VaR: History or Simulation?" *Risk*, V.16, 123-126.
- Longin, F. (1996) "The Asymptotic Distribution of Extreme Stock Market Returns." *Journal of Business*, V.69, 383-408.
- Lopez, J. (1999) "Methods for Evaluating Value-at-Risk Estimates." *Federal Reserve Bank of San Francisco Economic Review*, V.02, 3-17
- McNeil, A.J. (1997) "Estimating the Tails of Loss Severity Distribution Using Extreme Value Theory." *ASTIN Buletin*, V.27, 117-137
- McNeil, A.J. and R. Frey (2000) "Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: an Extreme Value Approach." *Journal of Empirical Finance*, V.7, 271-300.
- Mittnik, S.M., S. Paoletta, and S.T. Rachev. (2002) "Stationary of Stable Power-GARCH Processes." *Journal of Econometrics*, V.106, 97-107.
- Neftci, S.N. (2000) "Value at Risk Calculations, Extreme Events, and Tail Estimation." *The Journal of Derivatives*, V.7, 23-38
- Nelson, D.B. (1991) "Conditional Heteroscedasticity in Asset Returns: A New Approach." *Econometrica*, V.59, 347-370.
- Oda, N. and J. Muranaga. (1997) "A New Framework for Measuring the Credit Risk of a Portfolio: The ExVar Model." *Monetary and Economic Studies*, V.15, 27-62.
- Odening, M. and J. Hinrichs. (2001) "Using Extreme Value Theory to Estimate Value-at-Risk." Working paper.
- Raaji, G. and B. Raunig. (1998) "A Comparison of Value at Risk Approaches and Their Implications for Regulators." *Focus on Austria*, V.4, 57-71.

- Rau-Bredow, H. (2003) "Derivatives of Value at Risk and Expected Shortfall." Working paper.
- Rouvinez, C. (1997) "Going Greek with VaR." *Risk*, V.10, 57-65.
- Sandmann, G. and S.J. Koopman. (1998) "Estimation of Stochastic Volatility Models via Monte Carlo Maximum Likelihood." *Journal of Econometrics*, V.87, 271-301.
- Sarma, M., S. Thomas, and A. Shah. (2000) "Performance Evaluation of Alternative VaR Models." *Indira Gandhi Institute of Development Research* (Mimeo).
- Saunders, A. (1999) *Financial Institutions Management: A Modern Perspective*, 3rd Edition. Irwin Series in Finance, McGraw-Hill, New York.
- Sharpe, W. (1964) "Capital asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance*, V.19, 425-442.
- Sheather, S.J. and J.S. Marron. (1990) "Kernel Quantile Estimators." *Journal of the American Statistical Association*, V.85, 410-416.
- Silverman, B.W. (1986) *Density estimation for Statistics and Data Analysis*. London: Chapman and Hall.
- Uryasev, S. (2000) "Conditional Value-at-Risk: Optimization Algorithms and Applications." *Financial Engineering News*, Issue 14.
- Venkataraman, S. (1997) "Value at Risk for a Mixture of Normal Distributions: The Use of Quasi-Bayesian Estimation Techniques." *Economic Perspectives*, 2-13, Federal Reserve Bank of Chicago
- White, H. (2000) "A Reality Check for Data Snooping." *Econometrica*, V.68, 1097-1126.
- Yamai, Y. and T. Yoshida. (2002) "On the Validity of Value-at-Risk: Comparative Analyses with Expected Shortfall." *Monetary and Economic Studies*, V.20, 57-85.
- Zangari, P. (1996) "A VaR Methodology for Portfolios that Include Options." *RiskMetrics Monitor* (1st quarter), 4-12.