Divisible Groups in the K-theory Completion of $SU(n)$

by

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Abstract

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I use the results of Bendersky and Thompson for the $E(1)$-based E_2 -term of S^{2n+1}, ${}^{K}E_2^{s,t}S^{2n+1}$, and the results of Bendersky and Davis concerning the v_1-periodic groups of $SU(n)$ to compute the $E(1)$-based E_2-term for $X = SU(n)$ for all primes p. This computation is performed using the Bendersky Thompson spectral sequence for $SU(n)$. For spaces like $SU(n)$ this spectral sequence converges to homotopy groups of the K-theory completion of $SU(n)$, denoted $\pi_*\widehat{SU}(n)$. Of particular interest is the existence of infinitely many divisible groups in the homotopy groups of the K-theory completion of SU which offers an example of how E-completion does not commute with direct limits.

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Introduction

The main goal of this paper is to present new information regarding the homotopy groups of the K-theory completion of the simple compact Lie group $SU(n)$ localized at odd-primes. In order to obtain information about the homotopy of a space, it is a standard practice to use an unstable Adams spectral sequence. An unstable Adams spectral sequence is a sequence of functors which take as input the homology of a space,X,and under certain conditions, to be discussed below, converges to the homotopy groups of a space. In the current situation, the spectral sequence converges to the homotopy groups of a closely related space, the K-theory completion of the space, which will be defined below. The main result of this paper is the calculation of the E_2-term of the Bendersky-Thompson spectral sequence for $SU(n)$, denoted ${}^{K}E_2^{s,t}(SU(n))$ which converges to a K-theory completion of $SU(n)$, denoted $\widehat{SU}(n)$. Most notable are the divisible groups that appear for $3 \leq t \leq n$, due to the presence of divisible groups in ${}^{K}E_2^{s,t}(S^{2m+1})$ as shown in [7].

In order to perform the calculation, one must consider the results of Bendersky [11], and Davis [17] who compute the odd-primary v_1-periodic homotopy groups of $SU(n)$, denoted $v_1^{-1}\pi_{2k}(SU(n))$. These groups have been studied extensively, and are known quite well for k sufficiently large and n within a certain range. In [17] the groups $v_1^{-1}\pi_{2k}(SU(n))$ which correspond to the ${}^{BP}E_2^{1,2k+1}(SU(n))$ are determined to be cyclic of order $p^{e_p(k,n)}$ where the number $e_p(k,n)$ will be defined below. The groups $v_1^{-1}\pi_{2k-1}SU(n)$ corresponding to the 2-line of the unstable Novikov E_2-term are shown to be the same order, but not necessarily cyclic. [20] gives some criteria for determining the structure of these 2-line groups for $n < p^2 - p + 1$. Although these groups only exist for k sufficiently large, and the groups on the 2-line are known only partially, they can be reduced mod-p and used to determine the existence of finite cyclic summands in ${}^{K}E_2^{s,t}(SU(n))$ for $s = 2$, $t < 2n - 1$,where t is the dimension of the sphere in the fibration

$$SU(n-1) \to SU(n) \to S^{2n-1}$$

The fibration above induces a long exact sequence in ${}^{K}E_2$-terms. It is this same long exact sequence and theorem 1.4 of [17] (which relies on [11]), and theorem 5.2 of [7] that allows the computation of the remainder of the E_2-term.

Since the tool of choice for the calculation is an unstable Adams spectral sequence based on a generalized homology theory associated to a ring spectrum E, it is necessary to review the definition, relevent properties, and convergence criteria of such sequences. I will quickly specialize to the main example of [5], namely the unstable Adams spectral sequence based on BP theory (UANSS). This topic, and the calculation of the one-line of the ${}^{BP}E_2$-term for an odd sphere (theorem 9.12[5]) will occupy the bulk of chapter 2 and 3. Chapter 3 will also address the genralization of the UANSS to non-connective generalized homology theories, in particular to the theory $E(1)$, a summand of p-local complex K-theory, the Bendersky Thompson spectral sequence, BTSS. While many aspects of the construction in [5] carry over to the non-connective case, the generalization is not entirely straightforward. The proof of the convergence of the UANSS does not hold for non-connective theories, so the issue of convergence of the BTSS must be addressed. I will then summarize the results presented in [7] in order to use their calculation of the ${}^{K}E_2$-term for an odd sphere (theorem 3.1 in [7]). Chapter 4 gives a summary of the relevant results for the v_1-periodic homotopy groups of $SU(n)$ from [17] and [11] which will be used in this same chapter to compute ${}^{K}E_2$-term of $SU(n)$ localized at odd primes which gives the homotopy groups of the K-theory completion of $SU(n)$.

Chapter 1

Main Results

The following gives a description of the unstable $E(1)$-based E_2-term for $SU(n)$ where p is an odd prime.

Theorem 1 *For each* $3 \leq n$, $t - s \geq 1$, $0 < m < n$, *and* $r \geq p$, $r \geq \frac{2n-1}{2}$, *the* E_2*-term of the* $E(1)$*-BKSS for* $X = SU(n)$ *for an odd prime* p *is given by*

$$E_2^{s,t}(SU(n)) =$$

$$\begin{cases} 0 & \textit{if } s \geq 4, \\ Q/Z_{(p)} & \textit{if } s = 3,\ t-s = 2m-2, \\ Q/Z_{(p)} \oplus Q/Z_{(p)} \oplus G(Z/p^f) & \textit{if } s = 2,\ t-s = 2m-1, \\ G(Z/p^{e_p(r,n)}) & \textit{if } s = 2,\ t-s = 2r-1, k > 0 \\ Z/p^{e_p(r,n)} & \textit{if } s = 1,\ t-s = 2r, k > 0 \\ Z_{(p)} & \textit{if } s = 0,\ t-s = 2m+1, \\ 0 & \textit{otherwise} \end{cases}$$

where $G(Z/p^f)$ *is a finite cyclic group of order* p^f *if* $v_1^{-1}\pi_{2(m-1)-1}SU(n)$ *is non-cyclic and 0 otherwise.* $G(Z/p^{e_p(r,n)})$ *is a finite group, possibly non-cyclic, of order* $e_p(r,n)$*. The* $G(Z/p^f)$ *on the 2-line is known only for certain* n *such that* $n \leq p^2 - p + 1$*.*

$$e_p(r,n) = min\{\nu_p(a(r,j)) : n \leq j \leq r\}$$

and

$$a(r,j) = \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} i^r$$

The details of the proof of this theorem can be found in chapter 4, but the general strategy is easy enough to summarize here. We begin by considering the long exact sequence in the $E(1)$-based E_2-term of the Bousfield-Kan spectral sequence, denoted ${}^K E_2^{s,t}(X)$, for an odd prime p induced by the fiber sequence

$$SU(n-1) \to SU(n) \to S^{2n-1}$$

and proceed by induction on n. The first cases to consider, $SU(3)$ and $SU(4)$, are completely (or almost completely) given by theorem 3.1 in [7]. Beginning already with $SU(4)$, however, the boundary homomorphism

$${}^K E_2^{s,t} S^{2n-1} \overset{d}{\to} {}^K E_2^{s+1,t} SU(n-1),$$

is not obvious, so it becomes necessary to compute the mod-p reduced ${}^K E_2$-terms. Doing this allows us to exploit the feature given in theorem 5.2 of [7] that

$${}^K E_2^{s,t}(SU(n); Z/p) = v_1^{-1 BP} E_2^{s,t}(SU(n); Z/p)$$

which by the proof of theorem 1.4 of [17] makes the results of [8] and [17] relevant. In short, it is in reducing the v_1-periodic homotopy groups of $SU(n)$ given in theorem 1.4 from [17] and corollary 1.10 [20] and comparing them to the mod-p $E(1)$-based E_2-term that gives the above result and leads to the following two corollaries.

Corollary 2 *For $t - s < 2n - 1$, ${}^K E_s^{1,t} SU(n) = 0$.*

Corollary 3

$$^{K}E_2^{s,t}(\lim_{\vec{n}} SU(n)) \neq \lim_{\vec{n}} {}^{K}E_2^{s,t}(SU(n)),$$

which is to say K-theory completion does not commute with direct limits.

Although there is nothing to indicate why completion based on a non-connective theory should commute with direct limits,it is interesting to see a specific example of this failure to commute. It is suspected that this property of the K-theory completion (the failure to commute with direct limits) is the result of K-theory being a non-connective theory, and as such gives rise to homotopy groups that are not finitely generated.

Chapter 2

Stable Beginnings

Since the main result here is the E_2-term of an unstable Adams spectral sequence it is both desirable and appropriate to begin with a brief exposition of the construction and properties of the specific spectral sequence used. For completeness sake, this will be preceded by a discussion of relevent predecessors based on [15], [5],and [7] which will lead naturally to a comparsion (and contrasting) of the target and the convergence of these sequences. Unless otherwise stated, a space will be a pointed Hausdorff topological space. For a space X, ΣX denotes the reduced suspension of X, while ΩX denotes the loopspace of X. The homotopy category of pointed topological spaces will be denoted Ho. Other convenient and/or standard notation will be given over the course of the paper.

2.1 Preliminaries

In [16] A.K. Boufield and D.M. Kan constructed an unstable Adams spectral sequence (UASS) for ordinary homology. This construction was expanded by M. Bendersky, E.Curtis and H. Miller in [5] to construct an UASS for a generalized homology theory. This in turn was generalized further to a spectral sequence for the non-connective theory $E(1)$. It is this final spectral sequence that will occupy the discussion in chapter 3. For now, Brown's representability theorem says that each generalized homology theory can be given by a spectrum and vice verse. A common starting point for a presentation of the spectral sequences mentioned above is the notion of spectra.

Definition 4 *A spectrum E is a sequence of spaces $\{E_n\}$ together with maps $\varepsilon_n : \Sigma E_n \to E_{n+1}$. The adjoints of these maps are $\widetilde{\varepsilon}_n : E_n \to \Omega E_{n+1}$. If the $\widetilde{\varepsilon}_n$'s are weak homotopy equivalences, then E is said to be an Ω-spectrum.*

An example of a spectrum is the *suspension spectrum* of a space X. This is defined by $E_n = \Sigma^n X$ with the obvious structure map, and denoted $\Sigma^\infty X$. If $X = S^0$, then $\Sigma^\infty X$ is called the *sphere spectrum* and denoted by S.

If we let $F_n = \lim_{\overrightarrow{k}} \Omega^k E_{n+k}$, then any spectrum E is homotopy equivalent to an Ω-spectrum F which is the collection of spaces $\{F_n\}$ with structure maps from the direct system. Notice that Σ^∞ is a functor from the category of topological spaces to the category of spectra, and its adjoint, Ω^∞ is, of course, a functor that takes spectra to spaces. We will see this functor again. Certain spectra are endowed with more stucture than just $\widetilde{\varepsilon}$.

Definition 5 *A ring spectrum E is a spectrum with a multiplication*

$$m : E \wedge E \to E$$

and a unit

$$\eta : S \to E$$

such that the following associativity and unitary diagrams commute up to homotopy

$$\begin{tikzcd}
E \wedge E \wedge E \arrow[r, "m \wedge E"] \arrow[d, "E \wedge m"'] & E \arrow[d, "m"] \\
E \wedge E \arrow[r, "m"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \wedge S \arrow[r, "E \wedge \eta"] \arrow[dr, "="] & E \wedge E \arrow[d, "m"'] & S \wedge E \arrow[l, "\eta \wedge E"'] \arrow[dl, "="'] \\
 & E &
\end{tikzcd}$$

Such a thing, an E, an m and a η taken together in a more general setting will be known as a *triple*. For X a spectrum, and E a ring spectrum, the unit η of E induces a natural transformation

$$\eta_X = \eta \wedge X : S \wedge X \approx X \to E \wedge X.$$

Let X_1 denote the fibre of η_x. More generally let X_i be the fibre of the map η_{X_i}. This gives rise to a tower of fibrations *over* the spectrum X which is constructed in the next section.

Definition 6 *A module spectrum F over a ring spectrum E is a spectrum* *together with a map* $\rho : E \wedge F \to F$ *such that the following diagrams commute*

$$\begin{tikzcd}
E \wedge E \wedge F \arrow[r, "m\wedge id"] \arrow[d, "id\wedge\rho"'] & E \wedge F \arrow[d, "\rho"] \\
E \wedge F \arrow[r, "\rho"] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
F \wedge S \arrow[r, "id\wedge\eta"] \arrow[dr, "="] & E \wedge F \arrow[d, "\rho"] & S \wedge F \arrow[l, "\eta\wedge id"'] \arrow[dl, "="'] \\
& F &
\end{tikzcd}$$

2.2 The Stable Adams Spectral Sequence based on a Generalized Homology Theory

We note that if E is a ring spectrum, then for any spectrum X, we have the following commutative diagram

$$\begin{tikzcd}
E \wedge X \arrow[r, "E\wedge id"] \arrow[dr, "id"] & E \wedge E \wedge X \arrow[d, "m\wedge I"] \\
& E \wedge X
\end{tikzcd}$$

Furthermore, there is a map

$$X = X_0 \approx S^0 \wedge X \to E \wedge X$$

Let X_1 denote the fiber of this map, and in general let X_{i+1} denote the fiber of the map

$$X_i \to E \wedge X_i$$

then for X a spectrum and E a ring spectrum we have

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X_2 \arrow[r, "\eta_{X2}"] \arrow[d] & E \wedge X_2 \\
X_1 \arrow[r, "\eta_{X1}"] \arrow[d] & E \wedge X_1 \\
X_0 = X \arrow[r, "\eta_X"] & E \wedge X
\end{tikzcd} \tag{2.2.1}$$

Applying $\pi_*(-)$ to the above gives a sequence of long exact sequences which in turn gives a *stable* Adams spectral sequence for which the E_1-term is given by

$$E_1^{s,t} = \pi_{t-s}(X_s \wedge E)$$

with differentials

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

A ring spectrum E determines a cohomology theory $E^n(X) = [X, E_n]$. This in turn gives a multiplicative homology theory $E_n(X) = \lim_{\overrightarrow{k}} \pi_{n+k}(X \wedge E_k)$ with coefficients in the ring E_*, where E_* is defined to be $\pi_*(E)$, and $\pi_r(E) = \lim_{\rightarrow} \pi_{n+r}(E_n)$ [2].

Remark 7 *This is fine since E_* is functorial in X, but we will need E_* to be functorial in a functor of X. In order for this to be the case, we assume E is an associative, commutative ring spectrum, flat as an E_* module, i.e. We assume $E_*E = \pi_*(E \wedge E)$ is free as a left E_*-module.*

Applying $E_*(-)$ to 2.2.1 we obtain a sequence of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & E_*X \arrow[r] & E_*(X \wedge E) \arrow[r] & E_*(\Sigma X_1) \arrow[r] & 0 \\
0 \arrow[r] & E_*\Sigma X_1 \arrow[r] & E_*(\Sigma X_1 \wedge E) \arrow[r] & E_*(\Sigma^2 X_2) \arrow[r] & 0
\end{tikzcd}$$

$$\vdots$$

Splicing together these short exact sequences gives a resolution of $E_*(X)$ (by cofree E_*E comodules)

$$0 \to E_*(X) \to E_*(X_0 \wedge E) \to E_*(\Sigma X_1 \wedge E) \to ...$$

We can apply $Hom_{E_*E}(E_*, -)$ to the above resolution and conclude that

$$E_2^{s,t} = Ext_{E_*E}(E_*, E_*(X))$$

More concisely, let $A = \pi_*(E) = E_* = E^{-*}$, and $\Gamma = E_*E = \pi_*(E \wedge E)$

Theorem 8 *Let* $A = \pi_*(E) = E_* = E^{-*}$*, and* $\Gamma = E_*E = \pi_*(E \wedge E)$*. If* E *satisfies Remark 7, the* E_2*-term of the Adams spectral sequence based on* E *is given by*

$$E_2^{s,t} = Ext_{\Gamma}(A, E_*(X))$$

.

We can consider $\Gamma = E_*E$ as a sort of coalgebra:

$$\begin{aligned}
\Gamma = E_*E &\cong \pi_*(E \wedge E) \\
&\to \pi_*(E \wedge E \wedge E) \\
&\cong \pi_*((E \wedge E) \wedge (E \wedge E)) \\
&\cong \pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge E) \\
&\cong E_*(E) \otimes_{E_*} E_*(E) = \Gamma \otimes_A \Gamma
\end{aligned}$$

Before, however, we allow ourselves to think of Γ as a Hopf algebra, we must admit, the world being so various, that it is possible that the left E_*-module stucture of Γ is not necessarily the same as the right module structure.

Definition 9 *Let E be an Ω-spectrum satisfying the hypotheses of Remark 7, and let A and Γ be as above. The pair (A, Γ) is a Hopf algebroid if there are two unit maps, a left and a right, denoted $\eta_L, \eta_R : A \to \Gamma$ and there is a counit $\epsilon : \Gamma \to \Gamma$, a diagnol map $\psi : \Gamma \to \Gamma \otimes_A \Gamma$, and a canonical antiautomorphism $c : \Gamma \to \Gamma$ satisfying various properties. Further details can be found in [38].*

2.3 Convergence and other Considerations

We take a moment to comment on the convergence of the above spectral sequence, introduce a computational tool, and define some general notions motivated by an example that will assist in these and subsequent applications.

Definition 10 *The E-localization of a space X with respect to a spectrum E is a space X_E with a localization map $l_E : X \to X_E$ which induces an isomorphism in E-homology and is terminal among such maps, that is if there exists another $f : X \to Y$ which induces an ismorphism in E-homology, then there is a unique $g : Y \to X_E$ such that $gf = l_E$.*

Definition 11 *A spectrum $E = E_n$ is said to be* connective *if there is an integer N such that $\pi_k(E) = 0$ for all $k < N$.*

Theorem 12 *If X is a connective spectrum, and E is a connective ring spectrum such that $\pi_0 E$ is a solid ring, that is $E \otimes E \cong E$, then if $\pi_0 E = \mathbb{Q}$, $Z_{(p)}$, or Z/p, then the spectral sequence defined above converges to $\pi_*(X_{\mathbb{Q}})$, $\pi_*(X_{(p)})$, or $\pi_*(X_{HZ/p})$, respectively.*

The significance of the connective assumption will become clear when we discuss unstable theories in the next chapter.

To facilitate computations now and later, let $A = E_*$, $\Gamma = E_*E_*$, then

$$\Omega^n(M) = \Gamma \otimes_A ... \otimes_A \Gamma \otimes_A M$$

is the (stable) cobar resolution with differential given by

$$\begin{aligned} d([\gamma_1 \mid \gamma_2 \mid ... \mid \gamma_s]a) =& [1 \mid \gamma_1 \mid ... \mid \gamma_s]a \\ &+ \sum_{j=1}^{s} (-1)^j [\gamma_1 \mid ... \mid \gamma_j' \mid \gamma_j'' \mid ... \mid \gamma_s]a \\ &+ (-1)^{s+1} \sum [\gamma_1 \mid ... \mid \gamma_s \mid a']a'', \end{aligned}$$

where

$$\psi(\gamma) = \sum \gamma' \otimes \gamma'' \text{and } \psi(a) = \sum a' \otimes a'',$$

that is we have a chain complex known as the (stable) cobar complex,

$$0 \to E_* \xrightarrow{d} E_*E \otimes_{E_*} E_* \to E_*E \otimes_{E_*} E_*E \otimes_{E_*} E_* \to ...$$

and we see that

$$d(a) = 1|a - a|1 = \eta_R - \eta_L.$$

From the above construction we have

$$Ext^1 = P(E_*E)/d(a)$$

where $P(E_*E)$ is the polynomial algebra over E_*E, and we apply this to two specific examples.

2.4 Examples

Example 13 *For the* stable *Adams spectral sequence (ASS), let* $E = HZ/p$ *be the Eilenberg-Mac Lane spectrum, and* $A_* = E_*E$ *be the dual of the Steenrod algebra. Then for a spectrum* X *the spectral sequence converges to* $\pi_*(X_{HZ/p})$ *and it is shown in [28] that*

- *for the prime* $p = 2$, $A_* = P(\epsilon_1, \epsilon_2, ...)$, *where* $|\epsilon_n| = 2^n - 1$ *with coproduct* $\Delta(\epsilon_n) = \sum_{i=0}^{n} \epsilon_{n-i}^{2^i} \otimes \epsilon_i$ *where* $\epsilon_0 = 1$.
- *for* $p > 2$, $A_* = P(\epsilon_1, \epsilon_2, ...) \otimes \Lambda(\tau_0, \tau_1, ...)$ *where* $|\epsilon_n| = 2(p^n - 1)$ *and* $|\tau_n| = 2p^n - 1$ *with coproduct* $\Delta(\epsilon_n) = \sum_{i=0}^{n} \epsilon_{n-i}^{p^i} \otimes \epsilon_i$; $\Delta(\tau_n) =$

$\sum_{i=0}^{n} \epsilon_{n-i}^{p^i} \otimes \tau_i + \tau_n \otimes 1$.

Using the cobar complex, we see that Ext $= P(A_*)$ *since* $\eta_R - \eta_L = 0$

We counter the above example with one in which $\eta_R - \eta_L \neq 0$.

Example 14 *For the* stable *Adams-Novikov spectral sequence (ANSS), let* $E = BP$ *which is a p-local ring spectrum satisfying the hypothesis of 12 then* $A_* = BP_*$, $\Gamma = BP_*BP$. *In [39] it is shown that* $BP_* = Z_(p)[v_1, v_2, ...]$ *where* $|v_n| = 2(p^n - 1)$ *and* $BP_*BP = BP_*[t_1, t_2, ...]$ *where* $|t_n| = 2(p^n - 1)$. *There is a coproduct* $\Delta : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP$ *which is a longer story, and unlike the previous example there exist two distinct unit maps,* η_R *and* η_L. *This means that Ext* $= P(BP_*BP)/(\eta_R - \eta_L)(BP_*)$. *Nevertheless,* BP *is a connective theory and the spectral sequence converges to* $\pi_*(X_{BP}) \simeq \pi_*(X) \otimes Z_(p)$. *With 8 in mind it would be very helpful to know* $Ext_\Gamma^{*,*}(A, M) = E_2^{s,t}(M)$. *The following calculation [42] gives this object:*

Let M *be a* Γ*-comodule,* $\Gamma = BP_*BP$, $A = BP_*$. *We have the cobar complex*

$$0 \to M \xrightarrow{d=\eta_R-\eta_L} \Gamma \otimes_A M \to \Gamma \otimes_A \Gamma \otimes_A M \to ...$$

and the short exact sequence

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/p \to 0$$

.

Applying Ext to this short exact sequence gives the "Bockstein" long exact sequence

$$0 \to Ext^0 BP_* \to Ext^0 BP_* \to Ext^0(BP_*/p) \xrightarrow{\delta} Ext^1 BP_* \to ...$$

An element in BP_*/p $(Ext^0(BP_*/p))$ *is of the form* $v_1^{sp^i}$ *as is an element in* BP_**, except not mod* p*.*

Let $I_n = (p, v_1, v_2, ..., v_{n-1})$ *then we obtain the following formulas due to [33] and [36]*

- $\eta_R(v_n) = v_n$ *mod* I_n*. In particular,*

  $$\eta_R(v_1) = v_1 \ modp = v_1 + pt_1$$

- $\eta_R(v_n) = v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1$ *mod* I_{n-2} *for* $n \geq 2$*. In particular,*

  $$\eta_R(v_2) = v_2 + v_1 t_1^p - v_1^p t_1$$

- $\delta_{n-1}(v_n^k) = (v_{n-1})^{-1}(\eta_R(v_n^k) - \eta_L(v_n^k))$ *mod* I_{n-1}*. For* $n = 1$*,*

  $$\delta(v_1^k) = (1/p)(\eta_R(v_1^k) - \eta_L(v_1^k)$$

With these formulas we compute $Ext^{1,sp^iq}(BP_*)$ *as in [42]:*

In

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/p \to 0$$

take α *in* BP_*/p, *then* $\alpha = m/p^i$ *where* $m = v_1^k$ *in* BP_*. *We pull-back* m/p^i *to* BP_* *and apply* d *from*

$$0 \to M \xrightarrow{d=\eta_R-\eta_L} \Gamma \otimes_A M \to \Gamma \otimes_A \Gamma \otimes_A M \to ...$$

We now have $d(m/p^i)$ *which we pull-back to* BP_*, *so we have* $d(v_1^k)/p^i$, *and so we see from 14*

$$\begin{aligned}\delta(v_1^k) &= (1/p)(\eta_R(v_1^k) - \eta_L(v_1^k)\\ &= (1/p)d(v_1^k)/p^i\\ &= d(v_1^k)/p^{i+1}\end{aligned}$$

Let $k = sp^i$ *and* $\alpha_{sp^i/n} = d(v_1^k)/p^n$ *then for any* $1 \leq n \leq i+1$

$$\alpha_{sp^i/n} = d(v_1^k)/p^n \in Ext^{1,sp^iq}(BP_*), where\ q = 2(p-1)$$

and it follows that the one-line of the E_2*-term of the stable ANSS for* BP_* *is given by*

$$E_2^{1,sp^iq}(BP_*) = Z/p^{i+1}.$$

A stable non-connective theory and the main example of this paper is the following

Example 15 *Let $E = E(1)$ the summand of p-local complex periodic K-theory, and $X = S^{2n+1}$, then $E(1)_* = Z_{(p)}[v_1, v_1^{-1}]$, where $|v_1| = q = 2(p-1)$ as in the previous example. For $k \in Z$, let $\nu(k)$ be the exponent in the highest power of p that divides k, that is $k = ap^{\nu(k)}$ such that $p \nmid a$.*

Using calculations from [31] it is shown that the E_2-term of the $E(1)$-based stable Adams spectral sequence for the sphere spectrum is given by

$$E_2^{s,t} = Ext^{s,t}_{E(1)_*E(1)}(E(1)_*, E(1)_*)$$

such that

$$E_2^{s,t}(S) = \begin{cases} 0 & \text{if } s \geq 3, \\ Q/Z_{(p)} & \text{if } s = 2,\ t-s = -2, \\ Z/p^{\nu(k)+1} & \text{if } s = 1,\ t-s = qk-1, \\ Z_(p) & \text{if } s = 0,\ t-s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 3

The Unstable Condition

In order to define a similar but unstable spectral sequence where X is a space rather than a spectrum, we follow [15] as in [5] to construct an unstable Adams spectral sequence (UASS). Since our main example is $E(1)$ we proceed in great enough generality to cover at least this special case. However, the $E(1)$-based Bousfield Kan spectral sequence (BTSS) has a great deal in common with BP-based Bousfield-Kan spectral sequence (UNSS) constructed in [5]. In what follows we exploit these similarities as much as possible, while at the same time, address the important differences between these two Bousfield-Kan spectral sequences. The most significant difference between BTSS and UNSS is that the latter is in general based on a connective spectrum while the former is based on a non-connective spectrum. This difference has serious implications when it comes to the convergence of these spectral sequences. Convergence will be addressed in section 3.2. We assume that E is a homotopy-associative ring spectrum and begin by considering the functor Ω^∞ mentioned above, so that we may enter an unstable setting.

3.1 The Bousfield-Kan Spectral Sequence

Definition 16 *For a space X, and a ring spectrum, E, let $E(X)$ denote the 0^{th} space in the Ω-spectrum associated to the spectrum $E \wedge \Sigma^\infty X$*

$$E(X) = \Omega^\infty(E \wedge \Sigma^\infty(X)) \tag{3.1.1}$$

Thus $E(X)$ is a functor from Ho to Ho, and for $i \geq 0$, $E_i(X) \approx \pi_i(E(X))$, where $E_*(X)$ denotes reduced E-homology. Notice how this parallels $E_i(X) \approx \pi_i(E \wedge X)$, the analogous statement for the stable Adams spectral sequence, where E is a ring spectrum, but X is a spectrum not a space. Statements and constructions made above in the stable case can now be made in the unstable construction. Specifically, the unitary property of the ring spectrum E, $\eta : S \to E$ gives an *Hurewicz map*

$$\eta_X : X = S_0 \wedge E \xrightarrow{j} S(X) \xrightarrow{\eta} E(X)$$

which is so called because it induces a map

$$\eta_{X*} : \pi_*(X) \to E_*(X).$$

Furthermore, there is a pairing $E_i \wedge E_j \to E_{i+j}$ induced by the multiplication of the ring spectrum E, and this induces a map

$$\Omega^i(E_i \wedge \Omega^j(E_j \wedge X)) \to \Omega^{i+j}(R_{i+j} \wedge X)$$

which upon passing to the limit gives a natural transformation in Ho

$$\mu_X : E(E(X)) \to E(X)$$

Definition 17 *A triple (G, μ, η) on a category $\mathcal{C}$ consists of a functor $G : \mathcal{C} \to \mathcal{C}$ and natural transformations $\mu : G^2 \to G$ and $\eta : I \to G$ such that the following diagrams commute*

$$\begin{tikzcd}
G \arrow[r, "G\eta"] \arrow[dr, "="] & G^2 \arrow[d, "\mu"] & G \arrow[l, "\eta G"'] \arrow[dl, "="] \\
 & G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G^3 \arrow[r, "G\mu"] \arrow[d, "\mu G"'] & G^2 \arrow[d, "\mu"] \\
G^2 \arrow[r, "\mu"] & G
\end{tikzcd}$$

It is clear that our functor E is the functor of a triple (E, μ, η) on the homotopy category of spaces Ho.

The first derived functor of X is defined to be the fibre of the Hurewicz map, and is denoted $D_1(X)$. more precisely, $D_1(X)$ is defined to be the pull-back of the path-space fibration over $E(X)$ via the Hurewicz map. This begins a tower very similar to (2.2.1).

$$\begin{tikzcd}
D_1(X) \arrow[d] & \\
X \arrow[r, "\eta"] & E(X)
\end{tikzcd} \tag{3.1.2}$$

This is continued with $D_i(X)$ defined to be the pull-back of the path-space fibration over $D_i(E(X))$ via $D_i\eta$ giving a tower of fibrations

$$\begin{tikzcd}
\vdots \arrow[d] & \\
D_2(X) \arrow[r, "D_2\eta"] \arrow[d] & D_2(E(X)) \\
D_1(X) \arrow[r, "D_1\eta"] \arrow[d] & D_1(E(X)) \\
X \arrow[r, "\eta"] & E(X)
\end{tikzcd} \tag{3.1.3}$$

The spectral sequence associated to this tower of fibrations is the Unstable Adams Spectral Sequence (UASS) based on E-homology known as the Bousfield-Kan spectral sequence (BKSS).

Definition 18 *The homotopy exact couple of the tower* (3.1.3),

$$\ldots \xrightarrow{\delta} \pi_* D_*(X) \to \pi_* D_*(X) \xrightarrow{D_*\eta_X} \pi_* D_*(E(X)) \xrightarrow{\delta} \pi_* D_*(X) \to \ldots$$

is called the homotopy spectral sequence of X with coefficients in E (UASS), *with*

$$E_1^{s,t} = \begin{cases} \pi_{t-s} D_s(E(X)), & \textit{for } t > s \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$

3.2 Convergence

3.2.1 Connective

In the stable case, we mentioned that the connectivity of the spectrum E will play an impotant role the convergence of Bousfield Kan spectral sequences. We illustrate this in the following. In [5], Bendersky, Curtis, and Miller construct a BKSS based on E for which the spectrum E is required to be unital, connective, and admit a *Thom* map, $\tau : E \to H$ where H is the integral Eilenberg-Mac Lane spectrum, $H = H_n$ with $H_n = K(Z, n)$, and τ preserves the unit map, that is if $\eta : S \to E$ is the unit map of E, then $\tau\eta$ is the unit map of H. For any E satisfying these conditions, or any sufficiently 'nice' ring spectrum and any simply connected space X, [5] shows that this spectral sequence converges to either the homotopy groups of X or to the homotopy groups of the *unstable* the E-localization of X, X_E, as defined in [13]. The primary example of [5] was the calculation of the E_2-term of the BKSS based on BP (UNSS), for which it is shown that for a fixed prime p and a simply connected space X the UNSS converges to the homotopy groups of X localized at p.

3.2.2 Non-connective

The main example and main result of this paper is the E_2-term of a BKSS based on a non-connective E, $E(1)$. For non-connective E, X_E is close, but not exactly the right target of the spectral sequence. It turns out that the right target is a more general space called the E-completion of X, denoted $\widehat{E}(X)$. This space is constructed in [16] as the inverse limit of a tower *under* X. Specifically, suppose we have a tower *under* X:

$$\begin{tikzcd}
 & X \arrow[d] \\
 & \vdots \arrow[d] \\
F_2 \arrow[r] & X_2 \arrow[d] \\
F_1 \arrow[r] & X_1 \arrow[d] \\
F_0 \arrow[r, "="] & X_0
\end{tikzcd}$$

We can apply $\pi_*(-)$ to the tower to get a homotopy spectral sequence with $E_1^{s,t} = \pi_{t-s}F_s$. If X is the homotopy inverse limit of this tower then we have

Proposition 19 *[16] Given a tower as above, where* $X = \underleftarrow{\operatorname{holim}} X_n$ *and*

$i \geq 1$. *Suppose that*

$$\lim_{\overleftarrow{r}}{}^1 E_r^{s,s+i} = 0 = \lim_{\overleftarrow{r}}{}^1 E_r^{s,s+i+1}$$

for all $s \geq 0$. *Then* $\{E_r\}$ *converges completely to* $\pi_i X$.

Definition 20 *A spectral sequence is said to have a horizontal vanishing line if there exists an* N *and* r *such that* $E_r^{s,t} = 0$ *for all* $s \geq N$

Remark 21 *If a spectral sequence has a horizontal vanishing line, then Proposition19 holds.*

The problem we are left with, however, is that our spectral sequence (BKSS), at the moment, comes from a tower *over* X while our convergence criterion holds for X, the homotopy inverse limit of a tower *under* X.

We need to remedy this situation. First, we assume we are given a tower under X and see that every tower under X gives a tower over X as follows. As before, let

$$\begin{tikzcd}
 & X \arrow[d] \\
 & \vdots \arrow[d] \\
F_2 \arrow[r] & X_2 \arrow[d] \\
F_1 \arrow[r] & X_1 \arrow[d] \\
F_0 \arrow[r, "="] & X_0
\end{tikzcd}$$

be our tower under X, and define X^{s+1} to be the homotopy fiber of the map $X \to X_s$ and $D^s = F_s$, then we have the tower over X

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X^2 \arrow[r] \arrow[d] & D^2 \\
 & X^1 \arrow[r] \arrow[d] & D^1 \\
X \arrow[r, "="] & X^0 \arrow[r] & D^0
\end{tikzcd}$$

Splicing together these two towers

$$\begin{tikzcd}
X^{s+1} \arrow[r] \arrow[d, "="'] & X^{s} \arrow[r] \arrow[d] & F_s = D^s \arrow[d] \\
X^{s+1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X_s \arrow[d] \\
* \arrow[r] & X_{s-1} \arrow[r, "="] & X_{s-1}
\end{tikzcd}$$

we can see that the two towers give the same spectral sequence. In this construction we are given a tower under X which determines a tower over X. Thus a tower over X comes, almost for free, from a tower under X. What we really would like, since our situation calls for it, is to be given a tower over X and have that be enough to determine a tower under X. In other words, we would like to know when a tower over X comes from an associated tower under X. While this is possible in the the stable situation in which fibrations and cofibrations are essentially the same, in an unstable setting a tower over X, at best, comes from an associated tower under ΩX.

In [16] Bousfield and Kan show that for their UASS in which $E = HR$, ordinary homology with coefficients in a ring R, [15] shows that a tower over X comes from a tower under $\widehat{E}(X)$, the E-completion of X. However, their tower relies on a cosimplicial space $\mathbf{E}(X)$ associated to the functor $E(X)$. Furthermore their $E(X)$ is represented by a functor on the category of topological spaces, not on Ho. In order to use these results of [15] we must proceed cosimplicially and make sure that our $\mathbf{E}(X)$ is a cosimplicial object in the category of topological spaces. Once the issue of convergence is settled we will also use the following notions to describe the E_2-term of the BKSS.

Definition 22

- *A cosimplicial object* $\boldsymbol{X}$ *over a category* C *is a collection of objects* $X_i \in \mathcal{C}$ *such that for each* $n \geq 0$ *there are co-face and co-degeneracy maps*

  $$d^i : X_n \to X_{n+1}$$

  and

  $$s^i : X_{n+1} \to X_n$$

  for which $0 \leq i \leq n$ *satisfying the following cosimplicial identities*

  $$\begin{aligned} d^j d^i &= d^i d^{j-1}, && i < j, \\ s^j d^i &= d^i s^{j-1}, && i < j, \\ &= id, && i = j, j+1, \\ &= d^{i-1} s^j, && i > j+1, \\ s^j s^i &= s^{i-1} s^j, && i > j. \end{aligned}$$

- *A cosimplicial space* $\boldsymbol{X}$ *is a cosimplicial object over the category of spaces.*

Example 23 *The cosimplicial standard simplex,* Δ *consists of the standard* n*-simplex* $\Delta[n]$ *in each dimesion* n *with the usual coface and codegeneracy* *maps.*

Remark 24 *Any triple* (G, μ, η) *on the category of topological spaces induces a functor from the category of spaces, denoted* $\mathcal{C}$*, to the category of cosimplicial objects over* $\mathcal{C}$ *as follows.*

Using η, for an object C in $\mathcal{C}$, we define the map

$$d^i = G^i \eta G^{n-i} : G^n(C) \to G^{n+1},\ 0 \leq i \leq n.$$

while μ allows us to define

$$s^i = G^i \mu G^{n-i} : G^{n+2}(C) \to G^{n+1}(C),\ 0 \leq i \leq n.$$

We let $\mathbf{G}(C)_n = G^{n+1}(C)$, and obtain a $\boldsymbol{G}$-resolution of C:

$$\begin{tikzcd}
\mathbf{G}C^0 \arrow[r, shift left=3, "d^0"] \arrow[r, shift right=3, "d^1"'] & \mathbf{G}C^1 \arrow[l, "s^0"'] \arrow[r, shift left=6, "d^0"] \arrow[r, shift right=0, "d^1"] \arrow[r, shift right=6, "d^2"'] & \cdots \arrow[l, shift left=3, "s^0"'] \arrow[l, shift right=3, "s^1"]
\end{tikzcd} \tag{3.2.1}$$

Applying these general definitions to 3.1.1 gives a functor $\mathbf{E}$ of a triple from the homotopy category of spaces to the category of cosimplicial objects over the homotopy category of spaces.

This is a problem, however, since in order to apply the methods of [15] we need 3.1.1 to be a functor on the category of spaces not just on the homotopy category of spaces. [21] develops the notion of the category of S-modules, $\mathcal{M}_{\mathcal{S}}$ which is a category of spectra which possess a smash product that has associativity, commutativity, and unitary diagrams which commute "on the nose" rather than just up to homotopy. The category $\mathcal{M}_{\mathcal{S}}$ has a subcategory of S-algebra that consists of those spectra which not only have a strict smash product, but also have a ring spectrum multiplication, with strictly commuting unitary, and associativity diagrams.

In particular, it has been shown that the spectra MU, BP, and $E(1)$ can be represented as S-algebras.

Proposition 25 *If E can be represented by an S-algebra, then $E(X)$ is the functor of a triple on the category of topological spaces.*

Definition 26 *Let $\boldsymbol{X}$ be a cosimplicial space then the total space $Tot_\infty \mathbf{E}X$ of $\mathbf{X}$ is the function space $Hom(\boldsymbol{\Delta}, \mathbf{X})$.*

As in [15], we can let $\Delta^{[s]}$ denote the s-skeleton of $\boldsymbol{\Delta}$, and let $Tot_s(\mathbf{X}) = Hom(\Delta^{[s]}, \mathbf{X})$, then we see that

$$Tot_\infty \mathbf{E}X = \underleftarrow{\operatorname{holim}}\, \mathrm{Tot}_s \mathbf{E}X$$

where $\underleftarrow{\operatorname{holim}}\, \mathrm{Tot}_s \mathbf{E}X$ can be seen as the homotopy inverse limit of a tower of fibrations consisiting of the individual $Tot_s \mathbf{E}X$, i.e. we have a tower under $\underleftarrow{\operatorname{holim}}\, \mathrm{Tot}_s \mathbf{E}X$.

Definition 27 *Let (E, μ, η) be a triple on the category of spaces, the E-completion of a space X is given by*

$$\widehat{E}(X) = Tot_{\infty}\mathbf{E}X$$

From the above we have

Theorem 28 *[7] If E is an S-algebra, i.e. E is the functor of a triple on the category of spaces, and X is a simply connected space, then the spectral sequence induced by the tower over X is equivalent to the spectral sequence induced by the tower under $\widehat{E}(X)$.*

In summary, for a spectral sequence based on a non-connective spectrum E, like $E(1)$, we have the following.

- A spectral sequence based on non-connective E converges if the spectral sequence has a vanishing-line.
- If such a spectral sequence converges, then it converges to $\widehat{E}(X)$.

We can now apply these facts to our main example.

The $E(1)$-based BKSS for $SU(n)$ has a horizontal vanishing line, and so the $E(1)$-based spectral sequence for $SU(n)$ converges to the homotopy groups of ${}^{K}\widehat{SU}(n)$, the K-theory completion of $SU(n)$, and these homotopy groups, in positive dimensions, are given by the E_2-term.

The above is clear from the the ${}^{K}E_2$-term presented in chapter 1.

3.3 A Cosimplicial Description of the E_2-term

In order to make the calculations of [7] possible, we now move on to describe the E_2-term of both types of Bousfield-Kan spectral sequences, connective and non-connective, to facilitate calculations. This E_2-term has a few characterizations and also depends, of course, on the spectrum E.

In this section we review the construction given in [5] of the Bousfield-Kan spectral sequence based on a genralized homology theory E. As above, denote the Hopf algebroid (E_*, E_*E) as (A, Γ). In order for this spectral sequence to be useful in general, i.e. in the non-connective case, we make the following assumptions about the ring spectrum E which is represented by the Ω spectrum E_k.

Hypothesis 3.3.1 *(i) E is a homotopy associative, homotopy commutative CW ring spectrum with unit.*

(ii) For each $k \geq 0$, $E_*(\boldsymbol{E}_k)$ is a free A-module.

(iii) Let $PE_*(\boldsymbol{E}_k)$ denote the primitives in the coalgebra $E_*(\boldsymbol{E}_k)$ and σ_* denote homology suspension. Then $PE_*(\boldsymbol{E}_k)$ is free as an A-module and following composition is injective.

$$PE_*(\boldsymbol{E}_k) \to E_*(\boldsymbol{E}_k) \xrightarrow{\sigma_*} \Gamma$$

In the previous section we defined the notion of a cosimplicial object over a category, and saw that the existence of a triple gave rise to a cosimplicial object and an Suppose we are given a cosimplicial object $\mathbf{X}$ over a category $\mathcal{C}$, we can apply a functor $T : \mathcal{C} \to \mathcal{A}$ from where $\mathcal{A}$ is an abelian category to the above resolution where $\mathcal{A}$ is an abelian category. In this way we obtain the so-called *G-derived functors of T*. The category we have in mind for $\mathcal{A}$ is the that of abelian groups, with $\pi_*(-)$ the functor. Regardless, as long as we have a cosimplicial object, say $\mathbf{X}$, over an abelian category we get a chain complex, ch$X$ by defining chX^n=X^n and $\delta = \sum(-1)^i d^i : \text{ch}X^n \to \text{ch}X^{n+1}$, so the right G-derived functors of T applied to the object C are given by

$$R^i_G T(C) = H^i(chT(G)C)$$

Remark 29 *For $T = \pi_*(-)$ and $\mathcal{C} = Ho$, $H^i(chT(G)C) = H^i(ch\pi_*(\boldsymbol{G}C)) = \pi^*\pi_*\boldsymbol{G}C$ and is called the cohomotopy of the cosimplicial group $\pi_*\boldsymbol{G}C$. [15]*

Theorem 30 *[15] Let X be a topological space then $E_2^{s,t}(X) = \pi^s\pi_t\boldsymbol{E}(X)$*

3.4 Cotriples, Coalgebras, and the category $\mathcal{M}(G)$

The triple of our main example is the adjoint of a cotriple.

Definition 31 *A cotriple on a category $\mathcal{C}$ is a functor $G : \mathcal{C} \to \mathcal{C}$ and natural transformations $\delta : G \to G^2$ and $\varepsilon : G \to I$ such that the following diagrams commute*

$$\begin{tikzcd}
G & G^2 \arrow[l, "G\varepsilon"'] \arrow[r, "\varepsilon G"] & G & G^3 & G^2 \arrow[l, "G\delta"'] \\
& G \arrow[ul, "="] \arrow[u, "\delta"'] \arrow[ur, "="'] & & G^2 \arrow[u, "\delta G"] & G \arrow[l, "\delta"'] \arrow[u, "\delta"']
\end{tikzcd}$$

Definition 32 *Given a cotriple G, we define a G-coalgebra as an object C in $\mathcal{C}$ together with a map $\psi : C \to GC$ making the following diagrams commute*

$$\begin{tikzcd}
C \arrow[r, "\psi"] & GC \arrow[d, "\varepsilon"] & G^2C & GC \arrow[l, "\delta"'] \\
& C \arrow[ul, "="] & GC \arrow[u, "G\psi"] & C \arrow[l, "\psi"'] \arrow[u, "\psi"']
\end{tikzcd}$$

Let $\mathcal{C}(\mathcal{G})$ denote the category of G-coalgebras, the category whose objects are G-coalgebras, and whose morphims are maps of G-coalgebras as follows:

A map $f : C \to D$ is a map of G-coalgebras if the following diagram commutes

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\psi"'] & D \arrow[d, "\psi"] \\
GC \arrow[r, "Gf"] & GD
\end{tikzcd}$$

Suppose we have an object C in $\mathcal{C}$, and a functor of a cotriple G, then $G(C)$ is naturally a G-coalgebra with $\psi = \delta$. Thus we can consider G to be a functor from the category $\mathcal{C}$ to category $\mathcal{C}(\mathcal{G})$ of G-coalgebras. This makes the forgetful functor from $C(G)$ to C left adjoint to the functor G, that is for $X \in \mathcal{C}(\mathcal{G})$ and $Y \in \mathcal{C}$

$$Hom_{\mathcal{C}}(X,Y) \approx Hom_{\mathcal{C}(\mathcal{G})}(X,GY)$$

We stated above that our triple is the adjoint of a cotriple, we are now in a position to show this is indeed the case. As noted above, a cotriple (G,δ,ε) determines a G-coalgebra, say (F,ψ). Let

$$\mu = G(\varepsilon) : G^2(F) \to G(F)$$

and

$$\eta = \psi : F \to G(F)$$

then $(G, G(\varepsilon), \psi)$ is a triple on the category $\mathcal{C}(\mathcal{G})$.

We have established enough of the general underpinnings of the theory to move onto our specific examples: The BP-based unstable Adams spectral sequence from [5] and the $E(1)$-based UASS as in [7].

We begin by recalling a construction from [5]. Let E be an Ω CW ring spectrum that satisfies 3.3.1 (i) and (ii), and let $\mathcal{M}$ be the category of graded free $A = E_*$-modules. We note here that there are further conditions placed on E in [5] than those listed in 3.3.1. We will address these omissions below. For now, let F be a spectrum such that for an $M \in \mathcal{M}$, $\pi_*(F) = M$. We define a functor G from $\mathcal{M}$ to $\mathcal{M}$ by

$$G(M) = E_*(\Omega^\infty(F))$$

As in [5], G is the functor of a cotriple on $\mathcal{M}$, and so G is the functor of a triple on $\mathcal{M}(G)$, the category of G-coalgebras over $\mathcal{M}$. Since G is the functor of a triple on $\mathcal{M}(G)$, for $M \in \mathcal{M}(G)$ there is a cosimplicial G-resolution of M as in 3.2.1

$$\begin{tikzcd} \mathbf{G}(M) \arrow[r, shift left=3, "d^0"] \arrow[r, shift right=3, "d^1"'] & \mathbf{G}^2(M) \arrow[l, "s^0"'] \arrow[r, shift left=6, "d^0"] \arrow[r, "d^1"] \arrow[r, shift right=6, "d^2"'] & \cdots \arrow[l, shift right=3, "s^0"'] \arrow[l, shift left=3, "s^1"] \end{tikzcd} \tag{3.4.1}$$

Let $A[t] = E_*S^t \in \mathcal{M}(G)$ and just like before we can apply an appropriate abelian group-valued Hom functor, $Hom_{\mathcal{M}(G)}(E_*(S^t), -)$ in this case, to the above resolution to get a chain complex. Taking the homology of the chain complex gives the G-derived functors of $Hom_{\mathcal{M}(G)}(A[t], M)$ which we denote $Ext_{\mathcal{M}(G)}(A[t], M)$.

Theorem 33 *[7] theorem 6.17 [5] If X is a CW-complex such that $E_*X \in \mathcal{M}$ and E satisfies at least (i) and (ii) of 3.3.1, then*

$$E_2^{s,t}(X) = Ext^{s,t}_{\mathcal{M}(G)}(A[t], E_*X), \ \textit{for } t - s > 0$$

In order to perform calculations we need to move from the category $\mathcal{M}(G)$ to an abelian category.

Suppose $G(M) \in \mathcal{M}(G)$ then we have the composition

$$M \xrightarrow{\psi} G(M) \xrightarrow{\psi=\delta} G^2(M) = G(M) \otimes_{E_*} G(M) \xrightarrow{\varepsilon\otimes\varepsilon} M \otimes_{E_*} M$$

which allows us to consider $G(M)$ as a coalgebra over $E_* = A$. Let $P(M)$ be the module of primitives with respect to this coalgebra structure. Then P is a functor from $\mathcal{M}(G)$ to the category of A-modules. By 3.3.1 (iii) it follows that

$$PG(M) \to G(M) \to \Gamma \otimes_A M$$

is injective, and so $PG(M)$ is a free A-module. Thus

Lemma 34 *([5],lemma 7.5) $U = PG$ is a functor of a cotriple on $\mathcal{M}$ and a subcotriple of the cotriple determined by G.*

U can be extended to a functor of a cotriple over the category $\mathcal{A}$ of all graded A modules. Let $M \in \mathcal{A}$ and let

$$F_1 \xrightarrow{d} F_0 \to M \to 0$$

be a resolution such that F_1 and F_2 are free modules then we define

$$U(M) = coker(U(d) : U(F_1) \to U(F_0)).$$

Thus U is an exact functor on $\mathcal{A}$. Let $A(U)$ denote the category of U-coalgebras. Since U is exact, $A(U)$ is an abelian category.

Definition 35 $A(U)$ *is the category of* U*-coalgebras called the category of unstable* Γ*-comodules.*

Let $M \in \mathcal{A}(U)$ then we have a cosimplicial resolution $\mathbf{U}(M)$ to which we can apply $Hom_{\mathcal{A}(U)}(A[t], -)$ and obtain the chain complex denoted in [7] as

$$C^{s,t}(M) = Hom_{\mathcal{A}(U)}(A[t], (\mathbf{U}(M))^s)$$

which is the *unstable* cobar complex for M.

Adjointness gives

$$Hom_{\mathcal{A}(U)}(A[t], U^s(M)) \approx Hom_{\mathcal{A}}(A[t], M)$$

which in turn shows that

$$C^{s,t}(M) = U^s(M)_t.$$

[5] then shows that there is a diagram of functors

$$\mathcal{M}(G) \xrightarrow{P} \mathcal{A}(U) \xrightarrow{Hom_{\mathcal{A}(U)}(A[t],-)} \mathcal{A}b$$

for which $\mathcal{A}b$ is the category of abelian groups, and as such an abelian category, as is $\mathcal{A}(U)$ as previously noted. For such a diagram, and since $U(M)$ is injective in $\mathcal{A}(U)$, there is a composite functor spectral sequence

$$E_2^{r,s} = Ext^r_{\mathcal{A}(U)}(A[t], R^s_G PM) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{M}(G)}(A[t], M)$$

In order to use this spectral sequence, we need to compute the derived functors of the primitives P, $R^s_G PM$. This calculation requires a further restriction on E that holds in the connective setting of [5], but fails to be the case for spectra like $E(1)$ in [7]. In [5], the cofree cocommutative coalgebra functor is denoted S, and it is assumed that for each $k \in Z^+$, $E_*(\mathbf{E_k}) \cong SP(E_*(\mathbf{E_k}))$. This imples that the G-derived functors of P are the same as the S-derived functors of P. In particular, if $E_*(X)$ is indeed cofree as a coalgebra, which is the case when $M = M(2n+1) = E_*(S^{2n+1})$, then $R^s_G PM = 0$ for $s > 0$ and the composite functor spectral sequences collapses giving

$$E_2^{s,t}(E_*X) = Ext_{\mathcal{A}(U)}(A[t], PM),$$

Specifically, the E_2-term for the sphere $X = S^{2n+1}$ is given by the homology of the unstable cobar complex.

For non-connective spectra, the conclusion that the E_2-term is the homology of the unstable cobar complex is slightly more difficult to come by. In this setting there is no clear notion of cofree E_*-coalgebras, so we need another way to show that the E_2-term of the $E(1)$-based unstable Adams is the homology of the unstable cobar complex.

We begin by fixing a prime p. Let $E = BP$, let (A, Γ) be the Hopf algebroid as in chapter 2 where $A = BP_* = Z_{(p)}[v_1, v_2, ...]$, where the v_i are the Hazelwinkel generators, $\Gamma = A_*[h_1, h_2, ...]$, where $h_i = c(t_i)$, and c is the canonical antiautomorphism of Γ. Recall that $|v_i| = |h_i| = 2(p^i - 1)$ and that there are two unit maps $\eta_R, \eta_L : A \to \Gamma$.

Definition 36 *[26] Let E_* be the homology theory associated to the p-local spectrum E. E_* is said to be Landweber exact if*

$$E_*(X) = E_* \otimes_{BP_*} BR_*(X)$$

From this, it follows that

$$E_*(E) = E_* \otimes_{BP_*} BP_*(BP)_{BP_*} E_*$$

Example 37 *Let $E = E(1)$. The spectrum $E(1)$ is obtained from BP by killing off v_i, for $i > 1$ and inverting v_1 on the right and on the left giving*

$$A = E(1)_* = Z_{(p)}[v_1, v_1^{-1}]$$

and

$$\Gamma = E(1)_* E(1) = E_(1) * \otimes_{BP_*} BP_*(BP)_{BP_*} E(1)_*$$

The Hopf ring $E_*(\mathbf{E}_*)$ was computed in [24].

Proposition 38 *[24] Let E_* represent a p-local Landweber exact multiplicative homology theory. If the coefficients E_* are concentrated in even dimensions and a a free R-module of countable rank for some subring of Q, then*

$$E_*(\mathbf{E}_*) = E_* \otimes_{BP_*} BP_*(\mathbf{BP}_*) \otimes_{BP_*[BP^*]} BP_*[E^*]. \tag{3.4.2}$$

where $F_*[G^*]$ denotes the sub-Hopf ring of $F_*(\mathbf{G}_*)$ obtained by applying F_* to elements in $\pi_* G = G^{-*}$.

Clearly Proposition 38 applies to $E(1)$.

Proposition 39 *[7]*

Let $I = (i_1, i_2, ...)$ be a sequence of non-negative integers. Let h^I denote $h_1^{i_1} h_2^{i_2} \ldots$.

(i) If M is a free left A-module, then $U(M)$ is the A-span of

$$\{h^I \otimes m \mid 2(i_1 + i_2 + ...) < |m|\} \subset \Gamma \otimes_A M.$$

(ii) Suppose M is an unstable Γ-comodule, free as an A-module, with coaction $\psi : M \to U(M)$. Then the unstable cobar complex is the chain

complex $C^{s,t}(M) = U^s(M)_t$ *with differential given by*

$$\begin{aligned} d([\gamma_1 \mid \gamma_2 \mid \ldots \mid \gamma_s]m) = &[1 \mid \gamma_1 \mid \ldots \mid \gamma_s]m \\ &+ \sum_{j=1}^{s} (-1)^j [\gamma_1 \mid \ldots \mid \gamma_j' \mid \gamma_j'' \mid \ldots \mid \gamma_s]m \\ &+ (-1)^{s+1} \sum [\gamma_1 \mid \ldots \mid \gamma_s \mid \gamma']m'', \end{aligned}$$

where $\gamma_j \in \Gamma$, $\psi(\gamma_j) = \sum \gamma_j' \otimes \gamma_j''$ *and* $\psi(m) = \sum \gamma' \otimes m''$.

Theorem 40 *[7] Let E be a Landweber exact ring spectrum satisfying 3.3.1. Let $M = M(2n_1+1, 2n_2+1, \ldots, 2n_k+1)$ be a free A-module on a sequence of odd dimensional generators $\{x_{2n_1+1}, \ldots, x_{2n_k+1}\}$. Let X be an H-space such that $E_*(X) \cong A[x_{2n_1+1}, \ldots, x_{2n_k+1}]$ the free commutative algebra. Then*

$$\mathrm{Ext}^s_{\mathcal{M}(G)}(A[t], E_*(X)) = \mathrm{Ext}^s_{\mathcal{A}(U)}(A[t], M).$$

We now have enough to compute using either the BP-based BKSS or the $E(1)$-based BKSS.

Chapter 4

Divisible groups in the ${}^K\widehat{SU}(n)$

4.1 A Calculation in the BP-based BKSS

The primary method for computing the E_2-term of a BKSS for a given space X is to begin with an element in the *stable* cobar complex, and try to desuspend this element as much as possible. This method is exemplified in the following computation of the unstable 1-line of the BP-based UASS, presented first in [5], again in [7], and again here as it is the foundation for subsequent calculations of BKSS E_2-terms.

We begin with the stable setup from chapter 1. Let M be a free BP_*-comodule, $\Gamma = BP_*BP$, and $A = BP_*$. Then we have the stable cobar complex

$$0 \to M \xrightarrow{d} \Gamma \otimes_A M \to \Gamma \otimes_A \otimes_A M \to ...$$

with

$$d = \eta_R - \eta_L$$

Based on considerations from the previous chapter and since M is free there is an injection

$$U(M) \to \Gamma \otimes_A M$$

and a factorization

$$\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[dr] & \Gamma \otimes_A M \\
& U(M) \arrow[u]
\end{tikzcd}$$

Where $U(M)$ has a basis of monomials $h^J \otimes v^I m$, where $h^J \in \Gamma$, $m \in M$, and so $v^I m \in M$ such that (I, J) are *allowable with respect to the dimension of* M as described in [5]. However, instead of writing down a specific basis Bendersky showed it is equivalent to describe $U(M)$ as in the previous chapter, that is

$$U(M) = A \text{ span of} \{h^J \otimes m | 2l(J) < |m|\}$$

In other words, (I, J) is allowable if and only if $2l(J) < |m|$

For example, let $M = A[2n + 1]$ with generator ι_{2n+1}. Then

$$U(M) = BP_*(S^{2n+1}) = U(A[2n+1])$$

Here $h_1^n \otimes \iota_{2n+1} \in U(M)$ with $l(J) = n$ and $|m| = 2n+1$, and so (I, J) is allowable.

Theorem 41

$$p^j h_1^{n+j} \equiv v_1^j h_1^n \ mod\ E_1(S^{2n-1})$$

Proof. We proceed by induction on J. We will actually just present $J = 1, 2$ which suffice to give the idea for the general calculation.

Lemma 42 $ph_1^{n+1} \equiv v_1 h_1^n \ mod\ E_1(S^{2n+1})$

Proof. Since $v_1 h_1^n$ lives on the $(2n+1)$-sphere and h_1^{n+1} must live on S^{2n+3}, this lemma says that $ph_1^{n+1} \in S^{2n+3}$. Let $j \geq n$. We need to show that $v_1 h_1^n \otimes \iota - ph_1^{n+1} \otimes \iota \in E_1(S^{2n-1})$. We have

$$h_1^{j-1} \otimes v_1^{j-n} \in E_1(S^{2n-1})$$

Let $j = n+1$. Then we have $h_1^n \otimes v_1 \in E_1(S^{2n-1})$. Also recall that $h_1^n \otimes v_1 \in \Gamma \otimes_A M$. This means that v_1 is in M and M is a left A-module. However, we can view $v_1 \in M$ as a *right* A-module by applying η_R to v_1:

$$h_1^n \otimes v_1 \iota = \eta_R(v_1) h_1^n \otimes \iota$$

Recall $\eta_R(v_1) = v_1 - ph_1$, so

$$\begin{aligned} h_1^n \otimes v_1\iota &= (v_1 - ph_1)h_1^n \otimes \iota \\ &= (v_1h_1^n - ph_1^{n+1}) \otimes \iota \\ &= v_1h_1^n \otimes \iota - ph_1^{n+1} \otimes \iota \end{aligned}$$

Thus

$$ph_1^{n+1} = v_1h_1^n - h_1^n \otimes v_1$$

and this gives the result ■

Lemma 43

$$p^2h_1^{n+2} \equiv v_1^2h_1^n \ modE_1(S^{2n-1})$$

Proof. Using previous lemma, let $n = n+1$, multiply by p, and the result follows. ■

■

Theorem 44 *[5] In filtration $s = 1$, in positive stems, the element of order p^n in the stable $qk-1$ stem desuspends to S^{2n+1}, but not to S^{2n-1}, that is if $n \leq \nu(k)+1$ then $d(v_1^k)/p^n \equiv -v_1^{k-n}h_1^n$ mod $E_1(S^{2n-1})$.*

Proof. Recall from chapter 2 that $\alpha_{sp^i/n}$ is represented in the cobar complex as $d(v_1^{sp^i})/p^n$. Let $k = sp^i$

$$\begin{aligned}
d(v_1^k)/p^n &\equiv \frac{1}{p^n}\left[\eta_R(v_1^k) - v_1^k\right] \\
&\equiv \frac{1}{p^n}\left[\eta_R(v_1)^k - v_1^k\right] \\
&\equiv \frac{1}{p^n}\left[(v_1 - ph_1)^k - v_1^k\right] \\
&\equiv \frac{1}{p^n}\left[(v_1 - ph_1)^k - v_1^k\right] \\
&= \sum_{j=1}^{k}(-1)^j\binom{k}{j}v_1^{k-j}p^{j-n}h_1^j \\
&\equiv \sum_{j=n}^{k}(-1)^j\binom{k}{j}v_1^{k-j}p^{j-n}h_1j \\
&\equiv \sum_{j=n}^{k}(-1)^j\binom{k}{j}v_1^{k-n}h_1^n \\
&\equiv -\sum_{j=0}^{n-1}(-1)^j\binom{k}{j}v_1^{k-n}h_1^n \\
&\equiv -v_1^{k-n}h_1^n.
\end{aligned}$$

This last class is in $E_1^{1,*}(S^{2n+1})$ and not in $E_1^{1,*}(S^{2n-1})$.

■

The previous theorem gives the finite groups on the 1-line of the $E(1)$-based E_2-term for S^{2n+1}. The entire E_2-term is computed in [7] and given by

Theorem 45 *For each $n \geq 1$, $t-s \geq 1$, the E_2-term of the $E(1)$-BKSS for $X = S^{2n+1}$ is given by*

$$E_2^{s,t}(S^{2n+1}) = \begin{cases} 0 & \textit{if } s \geq 4, \\ Q/Z_{(p)} & \textit{if } s = 3,\ t-s = 2n-2, \\ Q/Z_{(p)} \oplus Q/Z_{(p)} & \textit{if } s = 2,\ t-s = 2n-1, \\ Z/p^{\min(\nu(k)+1,n)} & \textit{if } s = 2,\ t-s = 2n+qk-1,\ k > 0, \\ Z/p^{\min(\nu(k)+1,n+k(p-1))} & \textit{if } s = 2,\ t-s = 2n+qk-1,\ k < 0, \\ Z/p^{\min(\nu(k)+1,n)} & \textit{if } s = 1,\ t-s = 2n+qk,\ k > 0, \\ Z/p^{\min(\nu(k)+1,n+k(p-1))} & \textit{if } s = 1,\ t-s = 2n+qk,\ k < 0, \\ Z_{(p)} & \textit{if } s = 0,\ t-s = 2n+1, \\ 0 & \textit{otherwise.} \end{cases}$$

the divisible groups in the negative stems are also determined using cobar calculations, but their determination requires the use of the method of numerical polynomials.

4.2 v_1-Periodic Homotopy

The computation of the $E(1)$-based E_2-term for $SU(n)$ relies on Theorem 45 and results of Davis and Bendersky concerning the v_1-periodic homotopy groups of $SU(n)$. In this section we present the definition and properties of v_1-periodic homotopy groups as given in [17]. The standard way for succinctly capturing the essence of these groups is to say that the v_1-periodic groups of a space X are the periodic version of the part of the actual homotopy of X that is seen by periodic K-theory.

$v_1^{-1}\pi_*(SU(n))$ and more generally $v_1^{-1}\pi_*(X)$ are defined in stages. We begin as in [18] by letting $M^n(k)$ denote the Moore space $S^{n-1} \cup_k e^n$. We have in mind $k = p^e$. The mod p^e homotopy group $\pi_n(X; Z/p^e)$ is the set of homotopy classes of maps $[M^n(p^e), X]$. We let A denote the Adams map as defined in [1],

$$A : M^{n-p^{e-1}q}(p^e) \to M^n(p^e)$$

which induces an isomorphism in K-theory. We note that such a map exists for $n > 2e + 3$ [18]. This fact will need to be addressed in the main calculation of this paper, since we will need to use v_1-periodic homotopy groups to say something about the E_2-term of the BTSS for $SU(n)$ for $n < 2e+3$ in the cases of interest. The maps A are used to define a direct system, and we have

$$v_1^{-1}\pi_i(X; Z/p^e) = \lim_{\vec{N}} \left[M^{i+N(p^{e-1}q}(p^e), X\right]$$

Furthermore, any two such maps are shown to be homotopic after finitely many suspensions, and so the group does not depend on the choice of A. We next need a map from $M^n(p^{e+1})$ to $M^n(p^e)$. There is a canonical map $\rho : M^n(p^{e+1}) \to M^n(p^e)$ which has degree p on the top cell and degree 1 on the bottom cell. In [22] it is shown that if $A : M^{n+p^{e-1}q}(p^e) \to M^n(p^e)$ and $A' : M^{n+p^e q}(p^{e+1}) \to M^n(p^{e+1})$ induce isomorphisms in K-theory, then there exists a k such that ρ makes the following diagram commute:

$$\begin{tikzcd}
M^{n+k(p^e q)}(p^{e+1}) \arrow[r, "\rho"] \arrow[d, "{A'}^k"] & M^{n+k(p^e q)}(p^e) \arrow[d, "A^{kp'}"] \\
M^n(p^{e+1}) \arrow[r, "\rho"] & M^n(p^e)
\end{tikzcd}$$

This means that after sufficiently many iterations of the Adams map there will be morphisms, ρ^* between the individual direct systems defined above for the $v_1^{-1}\pi_*(X; Z/p^e)$ and upon passing to direct limits we obtain a direct system(of direct systems).

$$\begin{tikzcd}
v_1^{-1}\pi_*(X, Z/p^e)(p^{e+1}) \arrow[r, "\rho^*"] & v_1^{-1}\pi_*(X, Z/p^e) \arrow[r, "\rho^*"] & \cdots
\end{tikzcd}$$

this leads to the following definition of v_1-periodic homotopy groups

Definition 46 *[18][19] For any space X and any integer i,*

$$v_1^{-1}\pi_i(X) = \lim_{\overrightarrow{e}} v_1^{-1}\pi_{i+1}(X; Z/p^e)$$

Lemma 47 *If a map $X \to Y$ induces an isomorphism in $v_1^{-1}\pi_*(-; Z/p)$, then it induces an isomophism in $v_1^{-1}\pi_*(-; p)$.*

Theorem 1.4 from [17] states

Theorem 48 *If $k \geq n$, then $v_1^{-1}\pi_{2k}(SU(n))$ is cyclic of order $p^{e_p(k,n)}$ and $v_1^{-1}\pi_{2k-1}(SU(n))$ is an abelian group of the same order, but not always cyclic.*

Since knowing whether or not $v_1^{-1}\pi_{2k-1}(SU(n))$ is cyclic is important in the following calculation we state the following due to [20] which is the corrollary of a result that gives a tractable way of computing the order of the groups in Theorem 48 and the stucture of the groups $v_1^{-1}\pi_{2k-1}(SU(n))$.

Theorem 49 *Let $\bar{k}$ be defined by $1 \leq \bar{k} \leq p-1$ and $\bar{k} \equiv k \bmod p-1$. Then $v_1^{-1}\pi_{2k-1}(SU(n))$ is*

(i) cyclic if $n < (\bar{k}+1)p$

(ii) the direct sum of two cyclic summands if $(\bar{k}+1)p \leq n \leq p^2 - p + 1$.

4.3 ${}^{K}\widehat{SU}(n)$

Here we present the proof of the main results stated in chapter 1. Before the calculation can begin we need to establish the fact analogous to that for BP in [8] that the fiber sequence

$$SU(n) \to SU(n+1) \to S^{2n+1}$$

induces a long exact sequence in $E(1)$-based E_2-terms. Let $PE_*(SU(n)) = M(n)$ be the submodule of primitives. Then $M(n)$ is a free A-module on odd dimiensional generators $x_3, x_5, ..., x_{2n-1}$. As mentioned above, $E(1)$ is a Landweber exact ring spectrum and so by 40

$${}^{K}E_2^{s,t}(SU(n)) = Ext_{\mathcal{A}(U)}(A[t], M(n))$$

Thus the groups $Ext_{\mathcal{A}(U)}(A[t], M(n))$ are the homology of the unstable cobar complex. As in the BP-case [8], sparseness gives that $E_2^{s,t}(SU(n)) = 0$ for all even t. Now apply E_* to the above fiber sequence and take primitives to obtain a short exact sequence in E_1-terms

$$0 \to M(n) \to M(n+1) \to A[2n+1] \to 0$$

this induces a long-exact sequence in Ext groups which gives a long exact sequence in E_2-terms

$$\ldots \to E_2^{s,t}(SU(n)) \to E_2^{s,t}(SU(n+1)) \to E_2^{s,t}(S^{2n+1}) \xrightarrow{\delta} \ldots$$

where δ has bi-degree $(1,0)$ and $t-s$ gives homotopy dimension as opposed to stem dimension.

In order to exploit the results of Bendersky, Davis and Yang, we need to establish a relationship among the BP-based BKSS, the $E(1)$-based BKSS, and v_1-periodic homotopy groups.

This relationship is given by the following theorem, theorem 5.2 from [7] which states that the mod-p E_2-term of the $E(1)$-based BKSS, ${}^KE_2^{s,t}(-;Z/p)$ is the same as the mod-p v_1^{-1}-periodic BP-based BKSS, $v_1^{-1BP}E_2^{s,t}(-;Z/p)$

Theorem 50 *$Ext_{U_\Gamma}(BP_*, v_1^{-1}M/pM) \cong Ext_{U_\Sigma}(E(1)_*, \bar{M}/p\bar{M})$ where*

- $(A,\Gamma) = (BP_*, BP_*BP)$
- $\tilde{\Gamma} = \mathrm{Ker}(\epsilon : \Gamma \to A)$
- *$U_\Gamma(M)$ is the A-span of* $\{h^I \otimes_A m \mid 2(i_1 + i_2 \cdots) < |m|\} \subset \Gamma \otimes_A M$
- $\tilde{U}_\Gamma(M) = U_\Gamma(M) \cap (\tilde{\Gamma} \otimes_A M)$
- $(B,\Sigma) = (E(1)_*, E(1)_* \otimes_A \Gamma \otimes_A E(1)_*)$
- $\tilde{\Sigma} = E(1)_* \otimes_A \tilde{\Gamma} \otimes_A E(1)_*$
- $\bar{M} = B \otimes_A M$

- *$U_\Sigma(\bar{M})$ is the B span of*

$$\begin{aligned}\{h^I \otimes_B \bar{m} \mid 2(i_1 + i_2 \cdots) < |\bar{m}|\} \subset \Sigma \otimes_B \bar{M} \\ = B \otimes_A \Gamma \otimes_A \bar{M}\end{aligned}$$

- *$\tilde{U}_\Sigma(\bar{M}) = U_\Sigma(\bar{M}) \cap (\tilde{\Sigma} \otimes_B \bar{M})$*

At the prime $p = 3$, the ${}^K E_2$-terms for $SU(3)$, $SU(4)$, and $SU(5)$ are calculated according to the method outlined in chapter 1. While $SU(3)$ will provide the basis for our induction for any p, it is rather unilluminating as to what exactly is going on, and so we present ${}^K E_2^{s,t} SU(4)$, and ${}^K E_2^{s,t} SU(5)$ to indicate the complications that will exist in the inductive assumption.

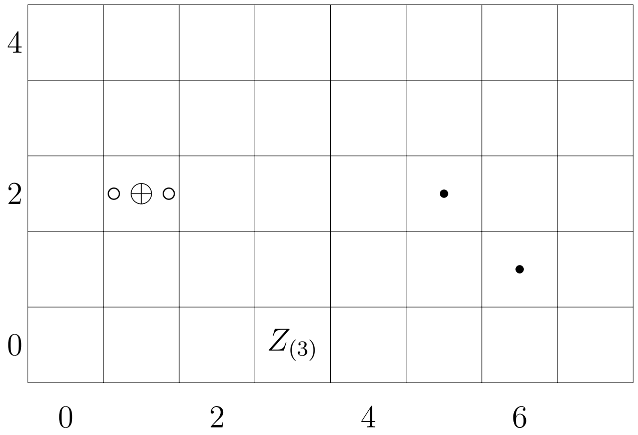
Theorem 51 *Suppose p is an odd prime, where $0 < m < 3$, then for $t - s > 0$*

$$E_2^{s,t}(SU(3)) =$$

$$\begin{cases} 0 & \textit{if } s \geq 4, \\ Q/Z_{(p)} & \textit{if } s = 3,\ t-s = 2m-2, \\ Q/Z_{(p)} \oplus Q/Z_{(p)} & \textit{if } s = 2,\ t-s = 2m-1, \\ G(Z/p^{e_p(r,n)}) & \textit{if } s = 2,\ r \geq 3, \\ Z/p^{e_p(r,n)} & \textit{if } s = 1,\ r \geq 3, \\ Z_{(p)} & \textit{if } s = 0,\ t-s = 2m+1, \\ 0 & \textit{otherwise} \end{cases}$$

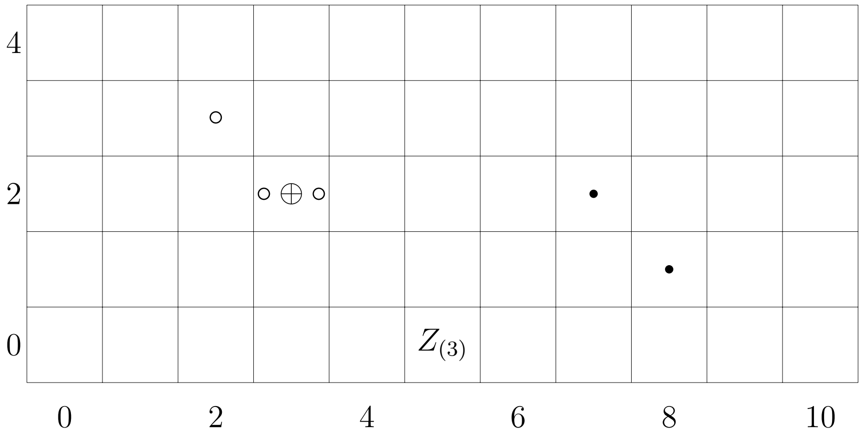
Proof.

The finite groups on the 1-, and 2-lines for $k > 0$ follow from [8] and [17]. The $k < 0$ groups of the E_2-term of $SU(2) \simeq S^3$ at the prime 3 can, for the sake of concreteness, be represented by the diagram



The vertical axis represents s filtration, and the horizontal axis represents homotopy degree $t - s$. Let ○ denote a single $Q/Z_{(3)}$, ○ ⊕ ○ denotes $Q/Z_{(3)} \oplus Q/Z_{(3)}$, and • represents a finite abelian group, a $Z/3$ in this case.

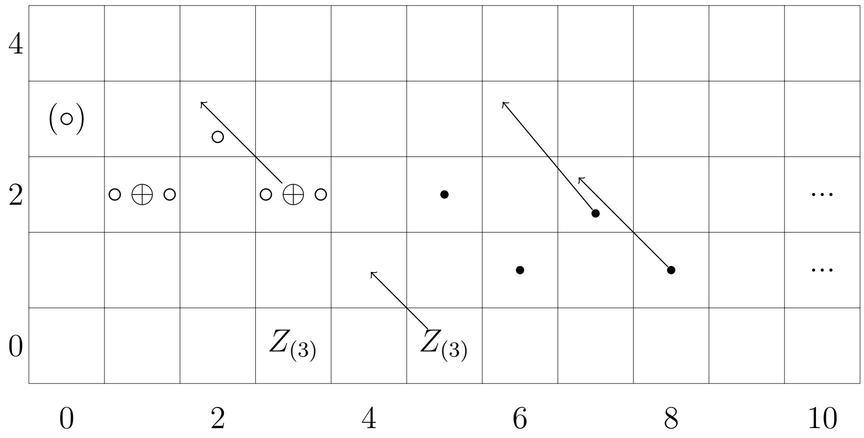
A similar diagram is used to represent $E_2^{s,t}(S^5)$ at the prime $p = 3$:



The fibre sequence

$$SU(2) \approx S^3 \to SU(3) \to S^5$$

induces the spectral sequence described by the diagram below for $p = 3$. This is easily generalized for any odd prime resulting in a sparser diagram.



The groups that could support a differential correspond to the groups in the E_2-term of S^5, while the groups potentially being hit by a differential correspond to groups in the E_2-term of $SU(2) \simeq S^3$. However, since no non-trivial differentials are possible the result follows. ■

The E_2-terms for $SU(4)$, $SU(5)$, and $SU(6)$ are given below in order to highlight the strategy employed to compute the E_2-term of $SU(n)$. $SU(4)$ is the first case in which finite groups in the negative stems of the ${}^K E_2$-term of an odd sphere appear in the long exact sequence of ${}^K E_2$-terms induced by the fiber sequence $SU(n-1) \to SU(n) \to S^{2n-1}$.

Theorem 52 *Suppose* p *is an odd prime, and* $0 < m < 4$, *then for* $t - s > 0$ *and* $r > 3$

$$E_2^{s,t}(SU(4)) = \begin{cases} 0 & \text{if } s \geq 4, \\ Q/Z_{(p)} & \text{if } s = 3,\ t-s = 2m-2, \\ Q/Z_{(p)} \oplus Q/Z_{(p)} & \text{if } s = 2,\ t-s = 2m-1, \\ G(Z/p^{e_p(r,n)}) & \text{if } s = 2,\ t-s = 2r-1, \\ Z/p^{e_p(r,n)} & \text{if } s = 1,\ t-s = 2r, \\ Z_{(p)} & \text{if } s = 0,\ t-s = 2m+1, \\ 0 & \text{otherwise} \end{cases}$$

where $G(Z/p^A)$ *is an abelian group of order* p^A.

$$e_p(r,n) = min\{\nu_p(a(r,j)) : n \leq j \leq r\}$$

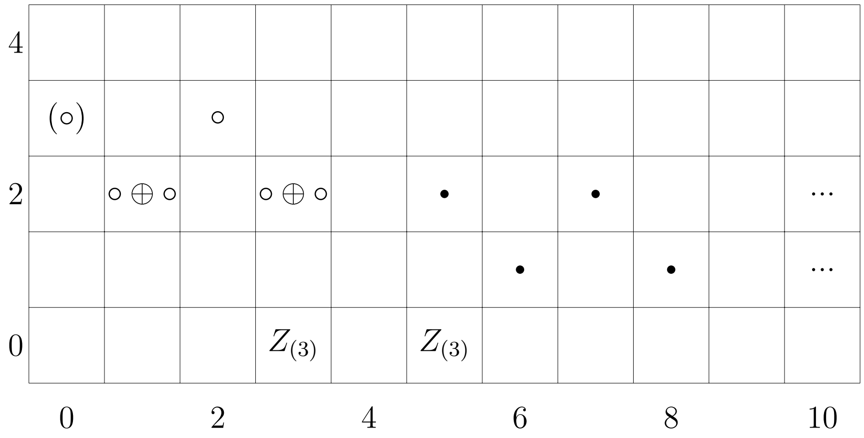
and

$$a(r,j) = \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} i^r$$

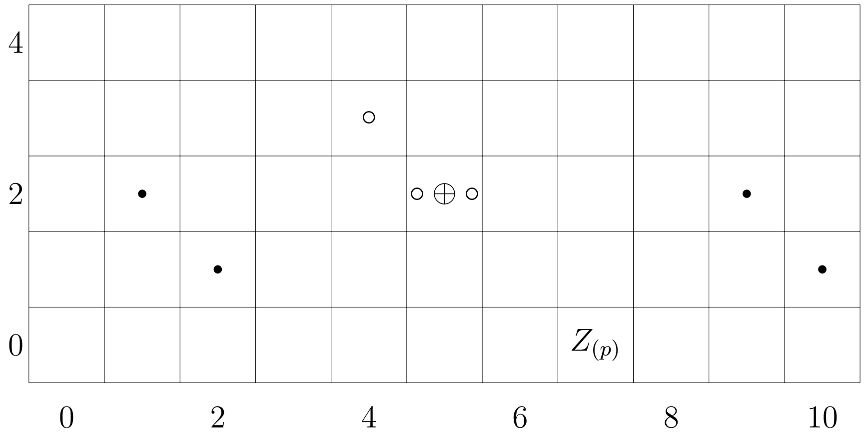
as defined in [17].

Proof. The groups corresponding to $k > 0$ follow by theorem 3.1 [7] and the five lemma.

Using the E_2-term of $SU(3)$ (at the prime $p = 3$ for illustrative purposes),

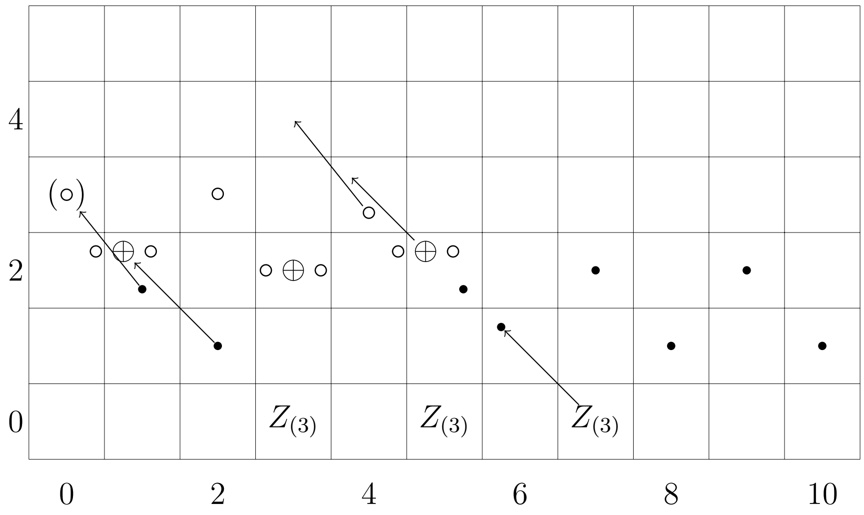


the E_2-term for S^7 at $p = 3$ from [7],



and the long exact sequence in E_2-terms induced by the fibre sequence,

$SU(3) \to SU(4) \to S^7$, summarized for $p = 3$ in the following diagram.



We now consider two possibilities for the ${}^{K}E_2$-term for $SU(4)$ which correspond to the two possibilities for $d_1 : E_2^{s,t}(S^7) \to E_2^{s+1,t}(SU(3))$, the boundary homomorphism in the long exact sequence of E_2-terms.

(i) If $d_1 = 0$, then $E_2^{1,t}(SU(4)) = Z/3$ with $E_2^{2,t}(SU(4)) = 2Q/Z_{(3)}$ for $t = 3, 7$.

(ii) If $d_1 \neq 0$, then $E_2^{1,t}(SU(4)) = 0$ with $E_2^{2,t}(SU(4)) = 2Q/Z_{(3)}$ for $t = 3, 7$.

Suppose $E_2^{1,7}(SU(4)) = Z/3$ with $E_2^{0,7}(SU(4)) = Z_{(3)}$. Then

$$E_2^{0,7}(SU(4); Z/3) = Z/3 \oplus Z/3.$$

If, on the other hand, $E_2^{1,7}(SU(4)) = 0$ with $E_2^{0,7}(SU(4)) = Z_{(3)}$. Then

$$E_2^{0,7}(SU(4); Z/3) = Z/3.$$

Both of these situations are algebraically consistent. We need some extra information in order to tell which is actually the case. We look to the v_1-periodic homotopy groups of $SU(n)$. By the proof of theorem 1.4 in [17] and [11],

$$v_1^{-1}\pi_{2(k)}SU(n) = v_1^{-1BP}E_2^{1,2k+1}SU(n)$$

and

$$v_1^{-1}\pi_{2(k)-1}SU(n) \approx v_1^{-1BP}E_2^{2,2k+1}SU(n)$$

Ideally we would like to compute $v_1^{-1}\pi_{2(3)}SU(4)$ to get at ${}^{K}E_2^{1,7}(SU(4))$. We cannot, however, compute this group directly using [17] since theorem 1.4 only holds for $k > n$. To satisfy this condition we instead compute $v_1^{-1}\pi_{2(5)}SU(4)$ since mod-3 the two groups are isomorphic. In fact,

$$v_1^{-1BP}E_2^{1,2k+1}(SU(n); Z/3) \equiv v_1^{-1BP}E_2^{1,2k+qi+1}(SU(n); Z/3)$$

for $q = 2(p-1)$ and $i \in Z$ which by theorem 5.2 [7] 40

$$v_1^{-1BP}E_2^{s,2k+1}(SU(n); Z/3) \cong^{K} E_2^{s,2k+1}(SU(n); Z/3).$$

The relevant mod-3 v_1-periodic homotopy groups of $SU(4)$ are

(i) $v_1^{-1}\pi_{2(k)}SU(4) =^{BP} E_2^{0,11}(SU(4); Z/3) = Z/3$

(ii) ${}^{BP}E_2^{1,11}(SU(4); Z/3) = Z/3 \oplus Z/3$

(iii) ${}^{BP}E_2^{2,11}(SU(4); Z/3) = Z/3$

Thus

$$^{K}E_2^{1,7}(SU(4)) = 0$$

For ${}^{K}E_2^{2,7}(SU(4))$ where there are two divisible groups and a finite group depicted in 4.3, we have the exact sequence

$$0 \to Z/3 \to ** \to 2Z/3^{\infty} \to 0$$

There are two possibilites for **, either

$$^{K}E_2^{2,7}SU(4) = Z/3 \oplus 2Z/3^{\infty}$$

or

$$^{K}E_2^{2,7}SU(4) = 2Z/3^{\infty} = Z/3^{\infty} \oplus Z/3^{\infty \cdot}$$

However, since $v_1^{-1}\pi_{2(5)}(SU(4)) = Z/3 \oplus Z/3$ mod-3, it follows that there is an extension and ${}^{K}E_2^{2,7}SU(4) = 2Z/3^{\infty}$, and clearly ${}^{K}E_2^{3,7}SU(4) = Z/3^{\infty}$.

Extending using periodicity of the mod-3 v_1-periodic homotopy groups and 40 gives

$$^{K}E_2^{0,3}(SU(4); Z/3) = Z/3$$

$$^{K}E_2^{1,3}(SU(4); Z/3) = Z/3 \oplus Z/3$$

$$^{K}E_2^{2,3}(SU(4); Z/3) = Z/3$$

and this implies

$$^{K}E_2^{0,3}(SU(4)) = Z_{(3)}$$

$$^{K}E_2^{1,3}(SU(4)) = 0$$

$$^{K}E_2^{2,3}(SU(4)) = 2Z/3^{\infty}$$

■

$SU(5)$ is the first case in which, for $p = 3$, the order of the v_1-periodic homotopy group that corresponds to the 1-line of the ${}^{BP}E_2$-term for $SU(5)$ is greater than 3. Specifically, the order of $v_1^{-1}\pi_{2(6)}SU(5)$ is 3^2. This means that the group on the 2-line could be non-cyclic. Applying corollary 1.10 from [20] to this situation, however, shows the group to be cyclic. It follows that the ${}^{K}E_2^{s,t}SU(5)$ is given by

Theorem 53 *Suppose* p *is an odd prime, and* $0 < m < 5$*, then for* $t-s > 0$*,* *and* $r \geq p$, $r \geq \frac{9}{2}$

$$E_2^{s,t}(SU(5)) =$$

$$\begin{cases} 0 & \textit{if } s \geq 4, \\ Q/Z_{(p)} & \textit{if } s = 3,\ t-s = 2m-2, \\ Q/Z_{(p)} \oplus Q/Z_{(p)} & \textit{if } s = 2,\ t-s = 2m-1, \\ G(Z/p^{e_p(r,5)}) & \textit{if } s = 2,\ t-s = 2r-1, \\ Z/p^{e_p(r,5)} & \textit{if } s = 1,\ t-s = 2r, \\ Z_{(p)} & \textit{if } s = 0,\ t-s = 2m+1, \\ 0 & \textit{otherwise} \end{cases}$$

We now come to the first case in which we have a non-cyclic v_1-periodic homotopy group. Using theorem 1.4 of [17], the order of the groups

$$v_1^{-1}\pi_{2(7)-i}SU(6)$$

for $i = 0, 1$ is 3^2 and by corollary 1.10 of [20] we know that $v_1^{-1}\pi_{2(7)-1}SU(6)$ is non-cyclic, specifically

$$v_1^{-1}\pi_{2(7)-1}SU(6) = Z/3 \oplus Z/3.$$

It follows that

$$v_1^{-1}\ {}^{BP}E_2^{1,15}SU(6) = Z/3^2$$

and

$$v_1^{-1}\ {}^{BP}E_2^{2,15}SU(6)) = Z/3 \oplus Z/3.$$

Reducing mod-3 gives

$$v_1^{-1}\ {}^{BP}E_2^{0,15}(SU(6); Z/3) = Z/3$$

$$v_1^{-1}\ {}^{BP}E_2^{1,15}(SU(6); Z/3) = 3Z/3$$

$$v_1^{-1}\ {}^{BP}E_2^{2,15}(SU(6); Z/3) = 2Z/3$$

Applying theorem 5.2 [7] and periodicity, we have

$$\begin{aligned}{}^{K}E_2^{0,15}(SU(6); Z/3) &= Z/3\\ {}^{K}E_2^{1,15}(SU(6); Z/3) &= 3Z/3\\ {}^{K}E_2^{2,15}(SU(6); Z/3) &= 2Z/3\end{aligned}$$

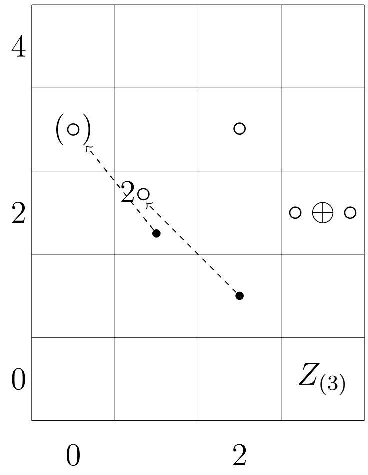
and

$${}^{K}E_2^{s,t}(SU(6); Z/3) =^{K} E_2^{s,t+qi}(SU(6); Z/3)$$

where $q = 4$ and $i \in Z$.

The following diagrams represent the long exact sequence in integal ${}^{K}E_2$-terms induced by the usual fiber sequence in the dimensions of interest. We use these in conjuction with the above mod-3 groups to determine the integral

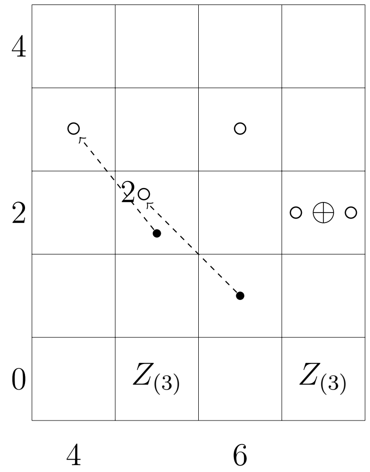
E_2-term where any finite summand in $t-s<2(n-1)+1$ is known only mod-p. Working from "left to right" in homotopy dimension, $t-s$, consider the following diagram.

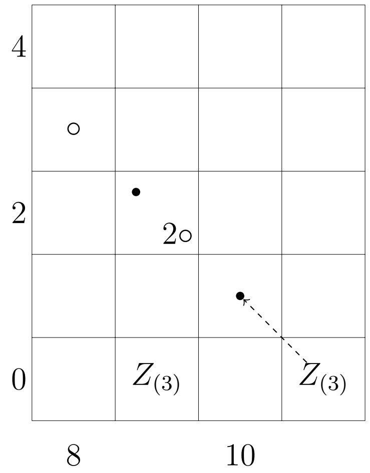


By 4.3 and assuming ${}^{K}E_2^{3,3}S^3 = Q/Z_{(p)}$, it follows that

$$
\begin{aligned}
{}^{K}E_2^{0,3}(SU(6)) &= Z_{(3)}\\
{}^{K}E_2^{1,3}(SU(6)) &= 0\\
{}^{K}E_2^{2,3}(SU(6)) &= 2Z/3^{\infty} \oplus Z/3\\
{}^{K}E_2^{3,3}(SU(6)) &= Z/3^{\infty}
\end{aligned}
$$

and the following diagrams





indicate the same is true for $^{K}E_2^{s,t}SU(6)$ for $t = 7, 11$.

The specific examples above suggest the general argument which proceeds by induction on n, and which begins, and is trivially true, for $SU(3)$. We assume the result 1 holds for $SU(n)$.

In order to determine $^{K}E_2^{s,t}SU(n+1)$, as in our examples, we look to the known v_1-periodic homotopy group of $SU(n+1)$ in the minimum periodic $k = n+p-1$ for k in $v_1^{-1}\pi_{2k+i}(SU(n+1))$ for $i = -1, 0, 1$. As long as n is less than $p^2 - p + 1$ we can "compute" whether this homotopy group is cyclic or not. Once cyclicity is determined it can be extended mod-p to lower and higher dimensions by periodicity. The mod-p condition means that we can detect if the group is cyclic or a direct sum of two cyclic summands, but due to extensions we can no longer be sure of the order of the group. We use the v_1-periodic homotopy which is isomorphic to the E_2-term of the v_1^{-1} periodic unstable Novikov spectral sequence. Reducing mod-p gives the isomorphism between the mod-p v_1^{-1}-periodic ${}^{BP}E_2$-term and the mod-p ${}^{K}E_2$-term from theorem 5.2 of [7]. Finally we reconcile the mod-p version with the possible maps and differentials in the long exact sequence of E_2-tems induced by the fiber sequence

$$SU(n) \to SU(n+1) \to S^{2n+1}$$

to determine the integral ${}^{K}E_2$-term for $SU(n+1)$.

There are four cases, or more pecisely there are two cases, and then two cases within those two cases.

First, suppose that for $0 < t-s < 2n+1$ i.e. for $0 < t-s$ and $k < 0$

$$ {}^{K}E_2^{s,2n+qk+1}SU(n) = \begin{cases} 0 & \text{for } s>3 \\ Z/p^\infty & \text{for } s=3 \\ 2Z/p^\infty & \text{for } s=2 \\ Z_{(p)} & \text{for } s=0 \\ 0 & \text{otherwise} \end{cases} $$

and suppose that $v_1^{-1}\pi_{2(n+p-1)-1}SU(n+1)$ is cyclic, specifically, suppose

$$ v_1^{-1BP}E_2^{s,2n+q+1}SU(n+1) = \begin{cases} 0 & \text{for } s>2 \\ Z/p^e & \text{for } s=2 \\ Z/p^e & \text{for } s=1 \\ 0 & \text{otherwise} \end{cases} $$

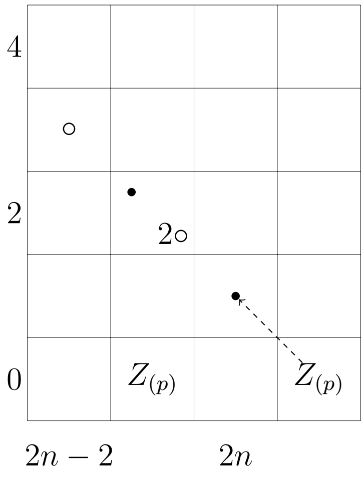
where $e = e_p(k,n)$ as defined above.

Then

$$ {}^{K}E_2^{s,t}(SU(n+1);Z/p) \begin{cases} 0 & \text{for } s>2 \\ Z/p & \text{for } s=2 \\ 2Z/p & \text{for } s=1 \\ Z/p & \text{for } s=0 \end{cases} $$

Using the long-exact sequence in integral E_2-terms, in conjunction with mod-p reduced E_2-term we determine ${}^{K}E_2^{s,t}SU(n+1)$ under these conditions.

From the long exact sequence in integral E_2-terms induced by the fiber sequence $SU(n) \to SU(n+1) \to S^{2n+1}$, we have the diagram representing the situation at $t - s = 2n + 1$



In order for the mod-p ${}^{K}E_2$-term to be as above 4.3 the differential

$$d_0 : E_2^{0,2n+1} S^{2n+1} \to E_2^{1,2n+1} SU(n)$$

must be non-zero, to give

$$E_2^{1,2n+1} SU(n+1) = 0,$$

If this were not the case, that is if this differential were zero, then the group on the 1-line of integral K-theory E_2-term would be a finite cyclic of some order, and reducing mod-p would give a non-cyclic group with two cyclic summands on the 0-line of the mod-p ${}^{K}E_2$-term for $SU(n+1)$ which is not the case.

There are no differentials

$$d_1 : E_2^{1,2n+1} S^{2n+1} \to E_2^{2,2n+1} SU(n)$$

or

$$d_2 : E_2^{2,2n+1} S^{2n+1} \to E_2^{3,2n+1} SU(n),$$

It follows that in the long exact sequence above we have the short exact sequence

$$0 \to Z/p^* \xrightarrow{f} E_2^{2,2n+1} SU(n+1) \to 2Q/Z_{(p)} \to 0.$$

Evidently, the map f must be an extension in order for the mod-p reduction to be consistent.

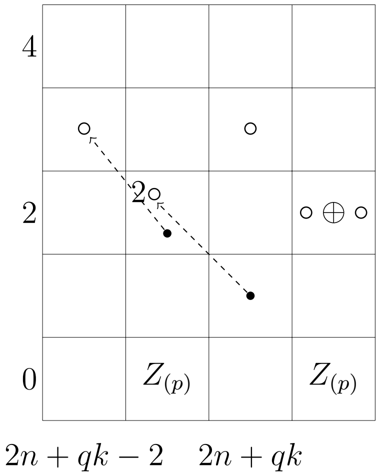
Thus,

$$E_2^{2,2n+1} SU(n+1) = 2Q/Z_{(p)},$$

and

$$E_2^{3,2n+1} SU(n+1) = Q/Z_{(p)}.$$

Proceeding from “right to left”, we have the diagram for $k < 0$



From this we see that

$$d_1 : E_2^{1,2n+1+qk} S^{2n+1} \to E_2^{2,2n+1+qk} SU(n)$$

must be injective, and it follows that

$$E_2^{1,2n+qk+1} SU(n+1) = 0.$$

If this map were not as indicated, then it would have a non-zero kernel giving a finite cyclic group of some order on the 1-line. Reducing mod-p would give a $Z/p \oplus Z/p$ on the mod-p 0-line which is not possible. Similarly the differential

$$d_2 : E_2^{2,2n+1+qk} S^{2n+1} \to E_2^{2,2n+1+qk} SU(n)$$

must be injective, or else we would have a $2Q/Z_{(p)} \oplus Z/p$ on the 2-line leading to a $3Z/p$ on the mod-p 1-line which is not the case. Thus,

$$E_2^{2,2n+1+qk} SU(n+1) = 2Q/Z_{(p)}$$

and

$$E_2^{3,2n+1+qk} SU(n+1) = Q/Z_{(p)}.$$

On the other hand, if we suppose again that

for $0 < t - s < 2n+1$ i.e. for $0 < t - s$ and $k < 0$

$$^{K}E_2^{s,2n+qk+1} SU(n) = \begin{cases} 0 & \text{for } s > 3 \\ Z/p^\infty & \text{for } s = 3 \\ 2Z/p^\infty & \text{for } s = 2 \\ Z_{(p)} & \text{for } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

and instead that $v_1^{-1}\pi_{2(n+p-1)-1} SU(n+1)$ is non-cyclic, that is

$$v_1^{-1BP}E_2^{s,2n+q+1}SU(n+1) = \begin{cases} 0 & \text{for } s > 2 \\ Z/p^a \oplus Z/p^c & \text{for } s = 2 \\ Z/p^e & \text{for } s = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $a + c = e$ and e is as defined above.

Reducing this mod-p

$${}^{K}E_2^{s,t}(SU(n+1); Z/p) = \begin{cases} 0 & \text{for } s > 2 \\ 2Z/p & \text{for } s = 2 \\ 3Z/p & \text{for } s = 1 \\ Z/P & \text{for } s = 0 \end{cases}$$

We now reexamine the differentials and maps in this case.

In our present situation

$$d_0 : E_2^{0,2n+1}S^{2n+1} \to E_2^{1,2n+1}SU(n)$$

again must be non-zero, giving $E_2^{1,2n+1}SU(n+1) = 0$. However, here we must have

$$E_2^{2,2n+1}SU(n+1) = Z/p^* \oplus 2Q/Z_{(p)}$$

which means that in the short exact sequence

$$0 \to Z/p^i \oplus Z/p^j \xrightarrow{f} Z/p^* \oplus 2Q/Z_{(p)} \to 2Q/Z_{(p)} \to 0$$

the map f takes one cyclic summand into a $Q/Z_{(p)}$ and is zero on the other cyclic summand.

In this degree and subsequent, we leave for future investigation the determination of which cyclic summand is included into which divisible group. For $t = 2n+1$, d_1 is trivial, as is d_2. Thus,

$$E_2^{3,2n+1} SU(n+1) = Q/Z_{(p)}$$

as this is what is required for the E_2-term to be consistent with the mod-p reduction.

Now suppose that for $0 < t - s < 2n+1$ i.e. for $0 < t - s$ and $k < 0$

$${}^K E_2^{s,2n+qk+1} SU(n) = \begin{cases} 0 & \text{for } s > 3 \\ Z/p^\infty & \text{for } s = 3 \\ 2Z/p^\infty \oplus Z/p^* & \text{for } s = 2 \\ Z_{(p)} & \text{for } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

and that $v_1^{-1}\pi_{2(n+p-1)-1} SU(n+1)$ is cyclic.

Then as above the mod-p reduction of the v_1-periodic UNSS is

$$ {}^{K}E_2^{s,t}(SU(n+1); Z/p) = \begin{cases} 0 & \text{for } s > 2 \\ Z/p & \text{for } s = 2 \\ 2Z/p & \text{for } s = 1 \\ Z/P & \text{for } s = 0 \end{cases}. $$

For $t-s = 2n+1$, the differential, d_0 is the same as in the case considered above where $v_1^{-1}\pi_{2(n+p-1)-1}SU(n+1)$ was assumed to be cyclic. Specifically,

$$ d_0 : E_2^{0,2n+1}S^{2n+1} \to E_2^{1,2n+1}SU(n) $$

must be non-zero and "kill" the cyclic group $E_2^{1,2n+1}SU(n)$. The differential d_1 must be trivial. Thus,

$$ E_2^{0,2n+1}(SU(n+1)) = Z_{(p)} $$

and

$$ E_2^{1,2n+1}(SU(n+1)) = 0 $$

However, since we are assuming that

$$ {}^{K}E_2^{2,2n+1+qk}(SU(n)) = 2Q/Z_{(p)} \oplus Z/p^{*} $$

it follows that ${}^{K}E_2^{2,2n+1}(SU(n))$ is non-cyclic with two cyclic summands, say $Z/p^i \oplus Z/p^j$. This means that in order for the v_1 and $E(1)$ mod-p E_2-terms to be isomorphic, the map f in the short exact sequence

$$0 \to Z/p^i \oplus Z/p^j \xrightarrow{f} 2Q/Z_{(p)} \to 2Q/Z_{(p)} \to 0$$

must be injective on both summands, and give

$$E_2^{2,2n+1}(SU(n+1)) = 2Q/Z_{(p)}$$

For $t - s = 2n + 1 + qk$ and $k < 0$, we can see that the differential d_0 is trivial, leaving the exact sequence

$$0 \to Z_{(p)} \to E_2^{0,2n+1+qk}(SU(n+1)) \to 0$$

which means

$$E_2^{0,2n+1+qk}(SU(n+1)) = Z_{(p)}.$$

In order for theorem 5.2 of [7] to hold, the differentials

$$d_1 : E_2^{1,2n+1}S^{2n+1} = Z/p^* \to E_2^{2,2n+1}SU(n) = 2Q/Z_{(p)} \oplus Z/p^*$$

$$d_2 : E_2^{2,2n+1}S^{2n+1} = Z/p^* \to E_2^{3,2n+1}SU(n) = Q/Z_{(p)}$$

must be non-zero and injective. In fact, d_1 must be such that it maps the Z/p^i on the 1-line onto the Z/p^j summand on the 2-line which implies that $i = j$. If either differential failed to satisfy the properties mentioned, there would be an "extra" Z/p^* on the 1-line and/or an extra summand on the 2-line that would cause the mod-p $v_1^{BP}E_2$-term and the mod-p ${}^{K}E_2$-term to be non-isomorphic. Thus,

$$E_2^{1,2n+1+qk}(SU(n+1)) = 0.$$

$$E_2^{2,2n+1+qk}(SU(n+1)) = 2Q/Z_{(p)}.$$

and

$$E_2^{3,2n+1+qk}(SU(n+1)) = Q/Z_{(p)}.$$

Finally we assume

$${}^{K}E_2^{s,2n+qk+1}SU(n) = \begin{cases} 0 & \text{for } s > 3 \\ Z/p^\infty & \text{for } s = 3 \\ 2Z/p^\infty \oplus Z/p^* & \text{for } s = 2 \\ Z_{(p)} & \text{for } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_1^{-1BP}E_2^{s,2n+q+1}SU(n+1) = \begin{cases} 0 & \text{for } s>2 \\ Z/p^a \oplus Z/p^c & \text{for } s=2 \\ Z/p^e & \text{for } s=1 \\ 0 & \text{otherwise} \end{cases}$$

where $a+c=e$

Theorem 1.5 and corollary 1.10 of [20] imply that if we suppose that we have non-cyclic groups in $v_1^{-1BP}E_2^{s,2n+q+1}SU(n+1)$ then these groups are of the form $Z/p^i \oplus Z/p^j$. This means, as above,

$${}^{K}E_2^{s,t}(SU(n+1);Z/p) \begin{cases} 0 & \text{for } s>2 \\ 2Z/p & \text{for } s=2 \\ 3Z/p & \text{for } s=1 \\ Z/p & \text{for } s=0 \end{cases}$$

Again, as in the above cases, a Z/p on the mod-p 0-line means that for $t-s=2n+qk+1$ for $k \le 0$, there is a $Z_{(p)}$ on the integral 0-line and that the 1-line is 0.

In order for the v_1-UNSS and $E(1)$ mod-p E_2-terms to be isomorphic, we must have

$$E_2^{2,2n+1+qk}(SU(n+1)) = 2Q/Z_{(p)} \oplus Z/p^*,$$

and

$$E_2^{3,2n+1+qk}(SU(n+1)) = Q/Z_{(p)}.$$

Proposition 54

$${}^K E_2^{s,*}(SU) = \begin{cases} \pi_*(\widehat{SU}) & s = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. This follows from theorem 5.2 in [7] and theorem 3.1 in [8]. ■

This proposition and theorem 1 gives

Corollary 55

$${}^K E_2^{s,t}(\lim_{\vec{n}} SU(n)) \neq \lim_{\vec{n}} {}^K E_2^{s,t}(SU(n))$$

4.4 Concluding Remarks

There are obvious questions that remain to be investigated upon completion of this paper. First, it is expected that a computation in the cobar complex would lead to information regarding the order of the cyclic groups that appear as summands of the two divisible groups on the two-line. However, this extra information will still be limited by what is known for v_1-periodic homotopy.

It would also be interesting to look into how, if at all, knowledge of the ${}^KE_2$-term,the divisible groups (and summands of divisible groups, in particular) could give information about the structure of the cyclic summands in the v_1-periodic homotopy groups of $SU(n)$. Beyond $SU(n)$, it seems reasonable to pursue the divisible groups present in the K-theory completion of other spherically resolved spaces, in particular the remaining simple compact Lie groups and some exceptional Lie groups, a program that could parallel, but rely very much on the program completed by Davis and Bendersky for v_1-periodic homotopy groups.

Bibliography

[1] J. F. Adams, *On the groups J(X), IV*, Topology **5** (1966), 21-71.

[2] J.F. Adams, *Stable Homotopy and Generalised Homology*, Chicago Lecture Notes in Mathematics. The University of Chicago Press. 1974.

[3] J. F. Adams, A. S. Harris, and R. M. Switzer, *Hopf algebras of cooperations for real and complex K-theory*, Proc. London Math. Soc. (3) **23** (1971), 385–408.

[4] Andrew Baker, Francis Clarke, Nigel Ray, and Lionel Schwartz, *On the Kummer congruences and the stable homotopy of BU*, Trans. Amer. Math. Soc. **316** (1989), 385–432.

[5] M. Bendersky, E. B. Curtis, and H. R. Miller, *The unstable Adams spectral sequence for generalized homology*, Topology **17** (1978), 229–248.

[6] M. Bendersky, E. B. Curtis, and D. Ravenel, *EHP sequences in BP theory*, Topology **21** (1982), no. 4, 373–391.

[7] M. Bendersky and R.D. Thompson, *The Bousfield-Kan spectral sequence for periodic homology theories.*, Amer. J. of Math. **122** (2000) 599-635.

[8] Martin Bendersky, *Some Calculations In the Unstable Adams-Novikov Spectral Sequence1*, RIMS, Kyoto Univ. **16** (1980), 739 – 766.

[9] Martin Bendersky, *The derived functors of the primitives for* $BP_*(\Omega S^{2n+1})$, Trans. Amer. Math. Soc. **276** (1983), no. 2, 599–619.

[10] Martin Bendersky, *Unstable towers in the odd primary homotopy groups of spheres*, Trans. Amer. Math. Soc. **287** (1985), no. 2, 529–542.

[11] Martin Bendersky, *The v_1-periodic unstable Novikov spectral sequence*, Topology **31** (1992), no. 1, 47–64.

[12] A. K. Bousfield, *Nice homology coalgebras*, Trans. Amer. Math. Soc. **148** (1970), 473–489.

[13] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150.

[14] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281.

[15] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, vol. 304, Springer-Verlag, 1972.

[16] A. K. Bousfield and D. M. Kan, *The homotopy spectral sequence of a space with coefficients in a ring*, Topology **11** (1972), 79–106.

[17] D. M. Davis, *v_1-periodic homotopy groups of SU(n) at an odd prime*, Proc. London Math Soc. **43** (1991) 529-544.

[18] D. M. Davis, *Computing v_1-periodic homotopy groups of spheres and certain Lie groups*, Handbook of Algebraic Topology, Elsevier (1995) 993-1049

[19] D. M. Davis and M. Mahowald,*v_1-periodicity in the unstable Adams spectral sequence*, Math. Zeit **204** (1990) 319-339.

[20] D.M. Davis, H. Yang *Tractable formulas for v_1-periodic homotopy groups of $SU(n)$ when $n < p^2 - p + 2$*, Forum Math **8** (1996) 585-620.

[21] A.D. Elmendorf, I. Kriz, M.A. Mandell, and J.P. May, *Modern foundations for stable homotopy theory*, Handbook of Algebraic Topology (I.M. James, ed.), North-Holland, Amsterdam, 1995, pp. 213–253.

[22] P. Hoffman, *Relations in the stable homotopy rings of Moore spaces*, Proc London Math Society **18** (1968), 621-634.

[23] J. R. Harper and H. R. Miller, *On the double suspension homomorphism at odd primes*, Trans. Amer. Math. Soc. **273** (1982), no. 1, 319–331.

[24] M. J. Hopkins and J. R. Hunton, *The structure of spaces representing a Landweber exact cohomology theory*, Topology **34** (1995), 29–36.

[25] I. Kriz, *Towers of E_∞ ring spectra with an application to BP*, preprint.

[26] P. S. Landweber, *Homological properties of comodules over MU_*MU and BP_*BP*, Amer. J. Math. **98** (1976), 591–610.

[27] Mark Mahowald, *On the double suspension homomorphism*, Trans. Amer. Math. Soc. **214** (1975), 169–178.

[28] J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. **67** (1958), 150171.

[29] Mark Mahowald and Robert Thompson, *The K-theory localization of an unstable sphere*, Topology **31** (1992), no. 1, 133–141.

[30] J.P. May, N. Ray, F. Quinn, and J. Tornehave, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Math., vol. 577, Springer-Verlag, 1977.

[31] H. R. Miller, D. C. Ravenel, and W. S. Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. **106** (1977), 469–516.

[32] Haynes R. Miller and Douglas C. Ravenel, *Morava stabilizer algebras and the localization of Novikov's E_2-term*, Duke Math. J. **44** (1977), no. 2, 433–447.

[33] H. R. Miller and W.S. Wilson, *On Novikov's Ext^1 modulo an invariant prime ideal*, Topology **15** (1976), 131 - 141.

[34] G. Mislin, *Localizations with respect to K-theory*, J. Pure Appl. Algebra **10** (1977), 201–213.

[35] D. C. Ravenel and W. S. Wilson, *The Hopf ring for complex cobordism*, J. Pure Appl. Algebra **9** (1977), 241–280.

[36] D. C. Ravenel, *The structure of BP_*BP modulo an invariant prime ideal*, Topology **15** (1976), 149 - 153.

[37] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351–414.

[38] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, second edition, AMS Chelsea Publishing, (2004)

[39] R.M. Switzer, *Algebraic topology: homotopy and homology*, Springer-Verlag, New York, 1975.

[40] W. S. Wilson, *The Ω-spectrum for Brown-Peterson cohomology, part I*, Comment. Math. Helv. **48** (1973), 45–55.

[41] W. S. Wilson, *The Ω-spectrum for Brown-Peterson cohomology, part II*, Amer. J. Math. **97** (1975), 101–123.

[42] W. Stephen Wilson, *Brown-peterson homology, an introduction and sampler*, Regional Conference Series in Math, vol. 48, AMS, Providence RI, 1980.