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# THE DETERMINATION OF MULTIPERIOD OPTIMAL DEBT STRUCTURE FOR THE FIRM AND THE INTERACTION WITH ITS PRODUCTION AND INVESTMENT DECISIONS

by

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Abstract

THE DETERMINATION OF MULTIPERIOD OPTIMAL DEBT STRUCTURE FOR THE FIRM AND THE INTERACTION WITH ITS PRODUCTION AND INVESTMENT DECISIONS

by

Lucy T. Huffman

Adviser: Professor Stavros B. Thomadakis

This study examines conditions under which an optimal debt structure is determinable for the firm, and relates the optimal debt decisions to the production and investment decisions of the firm in its microeconomic setting. The optimal debt structure is induced by a timing imperfection: there is a gap between the time at which the equity owners make equity maximizing decisions and the time at which the repayment on the debt is made. Since the bondholders cannot legally take control of the firm, investment opportunities valuable to the firm as a whole are lost if the equity owners decide not to produce. Because of the possibility of loss of valuable opportunities, the equity owners receive less in proceeds from the debt flotation. Thus, the optimal debt position is determined by the equality, at the margin, of the tax benefit and the loss in market value of the debt repayment promise.

The study relates this optimal position to the microeconomic setting by examining a particular model of the firm's decision-making process. It is specified to be a quantity

announcer facing a random demand each period with a given monopoly power. The production process is a fixed coefficient one, and there is no second-hand market for capital. Thus, the firm makes production decisions on use of capital stock and investment and use of new capital.

Multiperiod production and investment decisions are first examined under equity financing, with and without (neutral) technological progress. It is found that the non-marketability of capital, a barrier to exit from the industry, induces a barrier to entry. The effect of changing technology is to increase or decrease this barrier, the direction depending on the efficiency of the newer technology relative to the previous technology.

Optimal debt financing decisions are then allowed, and the optimal production, and investment decisions compared with those under equity financing. It is found that production out of previous capital stock is optimally lowered, the degree depending upon the demand risk and the monopoly power. In contrast, the decision on investment in new capacity is not lowered in proportion to the production decision. There is, thus, optimal overinvestment in capacity (or underutilization of capital stock), again depending on demand risk and monopoly power.

The debt position itself, arising from the equality at the margin of the tax benefit and loss in value of investment opportunities, depends also on monopoly power and demand risk. Assuming the same intra- and inter-period risk conponents, the firm with more monopcly power is found to finance optimally with less debt.

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## CHAPTER ONE: INTRODUCTION

### A. The Purpose of the Study

This study has as its purpose the examination of the existence of an optimal debt structure for the firm and the interaction between this and other decision variables of the firm. It proposes to investigate the following questions:

(i) What are the specifics of the equity-maximizing production-investment decisions in a multiperiod setting for the firm financing with debt?

(ii) These decisions, arising out of the maximization of equity claims to the net present value of the firm's operations, are in turn the result of the firm's microeconomic setting. Thus the debt and the accompanying production-investment decisions will depend on relevant parameters of the firm's product and industry environment. Given these and the optimization conditions, how do the debt-financed firm's production and investment decisions differ from those of the all equity-financed firm in the same environment?

### B. The Existence of an Optimal Debt

Under conditions of complete perfect capital markets, the Modigliani-Miller propositions on the optimality of total debt financing under taxes are well known to hold whether or not the debt is risky.[1] In the absence of one or more of these conditions, the proof of the proposition fails and an optimal debt structure of less than one hundred percent debt is indicated, since the value of the levered firm will be lowered from that of the unlevered firm by the cost of the imperfection.

An avenue of investigation has centered on the costs of liquidation. There has been much discussion of these costs (both transactions and loss of asset value to the levered firm and the indications for an optimal capital structure in the financial literature.[2]

A subtle relaxation of the assumptions needed to prove the MM proposition has been examined by Jensen and Meckling, and Myers, separately.[3] They propose that the presence of

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[1]F. Modigliani and M.H. Miller, "Corporate Income Taxes and the Cost of Capital: A Correction," American Economic Review 53 (June 1963): 433-443; R.C. Merton, "On the Pricing of Contingent Claims and the Modigliani-Miller Theorem," Journal of Financial Economics 5 (Nov. 1977): 241-250.

[2]N. Baxter, "Leverage, Risk of Ruin, and the Cost of Capital," The Journal of Finance 22 (Sept. 1977): 395-404; J. Hirshleifer, "Investment Decision Under Uncertainty: Application of the State-Preference Approach," The Quarterly Journal of Economics 80 (May 1966): 252-277; A. Kraus and R. Litzenberger, "A State-Preference Model of Optimal Financial Leverage," The Journal of Finance 28 (Sept. 1973): 911-922; J.H. Scott, Jr., A Theory of Optimal Capital Structure," The Bell Journal of Economics 7 (Spring 1976): 33-53.

[3]Michael Jensen and William Meckling, "Theory of the Firm: Management Behavior, Agency Costs and Ownership Structure," Journal of Financial Economics 3 (Nov. 1976): 305-360; S.C. Myers, "Determinants of Corporate Borrowing," Journal of Financial Economics 5 (Nov. 1977): 147-176. See also D. Galai and R.W. Masulis, "The Option Pricing Model and the Risk Factor of Stock," Journal of Financial Economics 3 (Jan. 1976): 53-82 for an indication of the interdependence between investment policy and capital structure.

outstanding debt induces a sub-optimal (to the firm as a whole) investment strategy by the stockholders who act to maximize the value of their equity claims to the net present value of the firm's operation. They maximize the value of their claims, the difference between the value of the firm (the sum of the net present value of the investment opportunities on which they are deciding) and the value of the debt outstanding when they make their decisions, by making decisions which reduce the value of the outstanding debt relative to the value of the firm. Under the assumption of perfect information, the purchasers of the debt are aware of the change in maximization criterion; thus, when the equity owners first float debt, they receive less in proceeds than if they were forced to make firm-maximization decisions. This loss in value is what Jensen and Meckling call the "agency cost" of debt.

The decisions of the shareholders under outstanding debt differ from firm maximization decisions in that their decisions potentially reduce the value of the debt. The only way this can happen in a perfect capital market setting is that (1) they increase the possibility of bankruptcy and (2) there be a loss in value of the firm under bankruptcy. Myers has examined, in a single period context, equity-maximizing decisions when there are no bankruptcy costs per se, no direct liquidation fees or change in value of existing assets upon liquidation. Instead, the loss arises out of a loss of valuable investment opportunities. The investment opportunities are lost as a result of a timing gap between the decision of the shareholders to go into bankruptcy rather than take on an investment opportunity and the legal time at which the bondholders take over the firm. Since the opportunity may be valuable, this equity-maximizing decision is sub-optimal for the firm as a whole. The cost of this sub-optimal policy then becomes a cost of bankruptcy, thereby inducing an optimal debt policy of zero debt unless the firm is subject to taxes.[4] Under taxation, there exists a trade-off between the tax advantages of debt and the cost of the sub-optimal decisions resulting in an optimal debt structure that is neither zero nor total debt.

## C. Research Methodology

This study will focus on a particular model of the firm's decision-making process in its product and industry setting, to make the optimization calculations less cumbersome. In addition the product and industry setting will be parametrized in a simple way in order to clarify the links between them and the optimization decisions. Thus, the conclusions of the research will be based upon a rather simple model. Wherever possible, any dependence of the conclusions on the specific form of the model will be pointed out. The model will, it is hoped, be flexible enough to permit general

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[1]See M.H. Miller, "Debt and Taxes," The Journal of Finance 32 (May 1977): 261-276 for an optimal debt structure under differing personal bond and stock income taxes.

conclusions to be drawn regarding the interactions of the microeconomic parameters of the firm and its production, investment, and debt decisions.

The firm is specified[5] as a quantity announcer, producing a single output from a fixed coefficient production function, with a (constant) degree of monopoly power in the output market. The price of the output is uncertain over any single period and the distribution of the next period price is dependent on the current outcome. The capacity needed to produce the output is assumed to be non-saleable, once acquired; this lack of ability to resell capacity at cost will make future investment and production decisions dependent on previous ones. The firm will be assumed to make its value-determining decisions and repay outstanding debt according to a time sequence which makes the Myers results applicable.

The firm's production and investment decisions will be formulated in the following way. Because the firm cannot resell capacity, it will have previously purchased capacity on hand as well as having new capacity available for purchase every time it makes a production and investment decision. These decisions may thus be characterized by two variables: production out of previous capacity (or capacity utilization), and purchase of and production from newly pur-

[5]See S.B. Thomadakis, "A Model of Market Power Valuation and the Firm's Returns," The Bell Journal of Economics 7 (Spring 1976): 150-161.

chased capacity (or new investment). The debt variables are formulated as the sequence of promised payments, one due each future period, as determined by optimization of current equity claims. The market value of the debt is the current value of this sequence. Conclusions can then be drawn relating the degree of monopoly power to the firm's capacity utilization and new investment decisions under equity financing, and to the repayment promises in the case of (optimal) debt financing. The effect of debt on the capacity utilization and new investment decisions can be examined by comparing the decisions under debt to those of the all equity financed firm.

D. The Relation Between Optimal Decisions and the Firm's Product Market in the Context of Previous Studies

There is an extensive literature relating optimal decisions, as they are reflected in profits, to monopoly power.[6] Many of these studies use variables intended to measure oligopolistic effects not present in the single firm non-equilibrium model of this study. Many, with exceptions to be noted, do not apparently proceed from a model of the firm within the industry, other than microeconomic models under

[6]See, e.g. W.G. Shepherd, "The Elements of Market Structure," The Review of Economics and Statistics 54 (Feb. 1972): 25-37; H. Michael Mann, "Seller Concentration, Barriers to Entry and Rates of Return in Thirty Industries, 1950 - 1960," The Review of Economics and Statistics 48 (Aug. 1966): 296-307; Marshall Hall and Leonard Weiss, "Firm Size and Profitability," The Review of Economics and Statistics 49 (Aug. 1967): 319-331; Bradley Gale, "Market Share and Rates of Return," The Review of Economics and Statistics 54 (Nov. 1972): 412-423.

certainty. To this extent comparison of their results with those of this study are limited.

The effects of risk (in rates of return) have been examined in the industrial organization literature[7] as well as the more theoretical treatments begun by Leland.[8] The studies of Fisher and Hall, Shepherd, Sherman and Tollison, and Baker use as risk measure the variability of corporate rates of return (or the equivalent, in the cases of Sherman and Tollison and Baker, who actually derive a measure of industry output variability). That of Gale uses the industry equity-to-assets ratio, arguing that low risk industries should support higher leverage for firms in that industry. Hurdle attempts a simultaneous solution, writing leverage as a function of variability of industry output.

According to the capital asset pricing model (CAPM) of capital market equilibrium, the correct measure of risk is the covariance of firm variables (cash flow or value) with total market values. The covariance measures the non-diversifiable risk to the investor-shareholder, the only risk the investor and therefore the firm need consider.

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[7]See, e.g. Shepherd; Gale; R. Sherman and R. Tollison, "Technology, Profit Risk and Assessments of Market Performance," The Quarterly Journal of Economics 86 (Aug. 1972) 448-462; S.H. Baker, "Risk Leverage and Profitability: An Industry Analysis," The Review of Economics and Statistics 55 (Nov. 1973): 503-507; Gloria Hurdle, "Leverage, Risk, Market Structure and Profitability," The Review of Economics and Statistics 56 (Nov. 1974): 478-485.

[8]H.E. Leland, "Theory of the Firm Facing Uncertain Demand," American Economic Review 62 (Jan. 1972): 278-291.

Thus this study uses this measure, rather than the ones used by the previously cited studies. It should be noted that this risk depends on the firm's monopoly power and is thus not an exogenous measure, as the previous studies have assumed (with the exception of Hurdle).[9] In addition, this study makes a distinction between intra- and inter-temporal risk. These two parts of risk enter differently into firm investment and debt decisions, thus into valuation. Therefore, the relationships derived from this study are different from those derived out of the foregoing literature.

Production out of current capacity, or capacity utilization, has been examined in the recent literature.[10] It should be stressed that the production cost assumptions of this research preclude any direct statements on minimal average cost production; thus excess capacity is not defined here in the Chamberlinian-Cassels sense, as use of industry capacity at below the most efficient, the minimum average cost, point. It is simply defined as under-utilization (in an engineering sense) of available means of production.[11]

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[9] See M.G. Subrahmanyam and S. Thomadakis, "Systematic Risk and the Theory of the Firm," Baruch College Working Paper, 1977; S. Thomadakis, "Systematic Risk in a Model of Intertemporal Investment and Monopoly Power," Baruch College Working Paper, 1977.

[10] The literature focuses on industry- and economy-wide capacity utilization. This study, however, is concerned with the firm as the essential unit. Firm capacity utilization studies seem to be non-existent, perhaps due to the lack of data available for empirical measurement on a firm basis.

[11] See Gordon C. Winston, "The Theory of Capital Utilization and Idleness," The Journal of Economic Literature 12 Dec. 1974): 1301-1320 for an extended discussion of these two quite different meanings.

There is, in the literature, as Winston makes clear, a distinction between intended and unintended idle capacity. The intended idleness develops out of expectations of cyclical changes in production in the face of variable period-by-period demand. The unintended idleness in a single period is, of course, the result of what Winston aptly calls "product-demand-adversity". As this study does not detail a specific mode of changes in expectations, no planned extra idle capital develops, since there is no oligopolistic interaction nor is there any fixed cost of idle capital. The capacity utilization decision is, instead, the result of an optimal decision based on a certainty equivalent valuation of current demand expectations. Since the firm may add capital each period, future capacity utilization decisions hinge on current new investment decisions, a function of certainty equivalent of future demand expectations. Thus, capacity utilization is related to inter- and intra-period risk, demand expectations which, of course, incorporate monopoly power, and technological changes. Changes in capacity utilization, providing the "unintended" idleness, arise out of changes in demand expectations, the inter-period risk. As noted before, the presence of debt changes the conclusions on the utilization of current capacity and new investment, thus capacity utilization.

In order to link the decisions of one period to those of the next, it is assumed that the firm cannot resell capacity. Thus, if demand expectations are such that it is advantageous to produce (demand expectations are such that certainty equivalent revenue covers operating costs), firms with capacity will produce while firms without, cannot. Thus, a barrier to entry will arise out of this capacity itself. The barrier to entry can be related to monopoly power since the capacity decision is so related.

There is an extensive literature on the employment of excess capacity as a barrier to entry. Most of the literature has linked it to entry barriers by, essentially, arguments providing the possessor with ability to undercut the entrant through either economies of scale or pricing behavior (limit pricing).[12] In this study, no such behavior on the part of the firm is postulated; there is no interaction between the firm and the rest of the industry. Thus this literature does not bear directly on the conclusions of the study. The study, instead, provides an additional explanation of excess capacity as a barrier to entry.

Of the empirical literature on (industry-wide) capacity utilization,[13] Bain correlates the utilization directly with market structure. His observations lead him to the conclusion that chronic excess capacity is (roughly) inversely correlated with barriers to entry. Meehan and Scherer examine

[12] See F.M. Scherer, Industrial Market Structure and Economic Performance (Chicago: Rand McNalley, 1970); A.M. Spence, "Entry, Capacity, Investment and Oligopolistic Pricing," The Bell Journal of Economics 8 (Autumn 1977): 534-544.

[13] See Frances F. Esposito and Louis Esposito, "Excess Capacity and Market Structure," The Review of Economics and Statistics 56 (May 1974): 188-195.

capacity adjustment to permanent increases in demand and investment instability, respectively. The recent study of Esposito and Esposito, using the minimum average cost definition of excess capacity, finds it correlated to concentration, except for very high concentrations. Thus they conclude that partially oligopolistic industries experience more excess capacity than do tightly oligopolistic ones.

An attempt is made in this study to examine the production-investment decisions under technological progress, in which previous capacity acquired by the firm is less efficient than that currently available. The conclusions are again related to the monopoly power and risk parameters. However, the firm itself does not undertake the production of the innovation; thus the conclusion of this research cannot be directly related to the research and development literature.[14] However, it is relevant to discussions of adoption of process technology. This literature again relies mostly on certainty models of firms in oligopolistic settings; thus the results of this study will supply motivations for (or against) adoption of process innovations in addition to those of the literature. Fellner[15] and Arrow[16] have investigated incentives

[14]See the summary in Markham, "Concentration: A Stimulus or Retardant to Innovation," in Harvey J. Goldschmid, H. Michael Mann, and J. Fred Weston, editors, Industrial Concentration: The New Learning (Boston: Little Brown and Co., 1974).

[15]See Scherer, Chapter 8.

[16]Kenneth J. Arrow, "Economic Welfare and the Allocation of Resources for Invention," The Rate and the Direction of Inventive Activity (National Bureau of Economic Research: Princeton University Press, 1962).

for adoption of process innovation in a single period certainty case for both pure monopolistic and pure competitive industry, facing the same demand conditions. They conclude that the incentive to innovate is greater for the competitive industry as the reduction in cost is the same while the benefits are greater for the competitive industry as a whole. Demsetz,[17] beginning from a different initial assumption, finds that the increase in profit is greater for the monopolist. Hu[18] and Ng[19] have pointed out the effect of the initial assumptions and of using total rather than marginal comparisons. Needham[20] and Scherer[21] argue for an oligopolistic effect; however both consider a product innovation rather than the cost reducing one treated in this study and treat profits expected from innovations as reflected in research and development expenditures.

As this study investigates whether the monopolistic firm will initially turn to the less efficient vintage or the purchase of the most efficient current vintage (occurring when demand prospects are encouraging enough to motivate its purchase), conclusions may be drawn on firm tendencies to operate

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[17]Harold Demsetz, "Information and Efficiency: Another Viewpoint," Journal of Law and Economics 1 (Apr. 1969): 125-140.

[18]S.C. Hu, "On the Incentive to Invent: A Clarificatory Note," Journal of Law and Economics 5 (Apr. 1973): 198-202.

[19]Y.K. Ng, "Competition, Monopoly, and the Incentive to Invent," Australian Economic Papers (June 1971): 330-341

[20]Douglas Needham, "Market Structure Firms' R&D Behavior," The Journal of Industrial Economics 23 (June 1975): 241-255.

[21]See Scherer, Chapter 8.

with combinations of less or more efficient vintages. In this sense, admittedly rather an indirect one, the study has bearing upon the Chamberlinian conclusions on excess capacity in monopolistic competition. At least, a minimum cost point may be defined <u>without</u>, however, the concept of minimum efficient scale (since the minimum cost point will be operation with the latest technology). Thus this study considers only "half" of the influences on average cost. In addition the curve so defined is <u>in no sense</u> a long-run certainty result, but rather a different realization each period of a stochastic "cost possibility" curve whose points depend on realized demand and expectations of future demand that period, since it is these factors which motivate purchase of efficient technology that period.

The presence of debt alters, as previously outlined, the conclusions on investment and production. There seems to have been little examination of the effects of debt on the firm's investment and production decisions or of the effects of microeconomic output market parameters of the firm on its debt decisions.[22] Gale has assumed an inverse correlation between an optimal industry (surrogate for firm) debt to value ratio and industry risk, violating (in a perfect market sense) the MM proposition. Hurdle has attempted to demonstrate the Gale hypothesis using monopoly power and industry return variabil-

[22]See, however, Gale; Hurdle; and expecially Timothy G. Sullivan, "Market Power, Profitability and Financial Leverage," <u>The Journal of Finance</u> 29 (December 1974): 1407-1414.

ity in a simultaneous equation setting. Again, there is no theoretically derived basis for the equation system other than an extension of the Gale conclusions. Recognition is made, however, of the interaction between the firm's parameters and its optimal decisions. The nature of this interaction will, it is hoped, be modelled in this study.

The work of Sullivan, again partial analyses, but specifically documenting an inverse relation between the amount of debt carried by the firm and its monopoly power, has motivated this study. Intuitively, there would be a greater reluctance of firms possessing monopoly power to risk bankruptcy for the benefits of the tax shield. The greater portion of their value is the availability of unusually profitable investment opportunities arising out of their monopolistic power in their market. But these opportunities may be foregone under conditions of financial stress; thus debt based on future investment opportunities (rather than assets in place) stands a greater chance of suffering loss. As this is translated into flotation proceeds, the equity owners suffer a large "agency cost", a large loss, if there is any risk at all in these opportunities causing the risk to the debt layered against them. This would appear to be the intuitive rationale behind the Sullivan results, and the theoretical framework of Myers seems to make this intuitive reasoning concrete.

# CHAPTER TWO: VALUATION OF THE FIRM UNDER ALL-EQUITY FINANCING

## A. The Model of the Firm

### The General Market Environment

The firm is assumed to operate within the following market environment:

(1) The capital market instantaneously takes into account new information affecting the value of the firm. It receives information on the production environment of the firm at the same time as the firm obtains it, and receives information on the firm's investment and financing decisions at the same time that the firm makes them. This information is assumed to be freely available to all participants in the capital market and there are no transactions costs to the participants. Thus, the capital market efficiently values the firm. There is, additionally, a riskless asset yielding a return r constant over time.

(2) It is assumed that information on the production environment is costless and freely available to the firm (and to the industry of which the firm is a member). Also, the investment decision process itself is assumed instantaneous and costless and the production facilities needed to implement the decisions are instantaneously available. If information is assumed to flow in continuously, the above assumptions

would lead to a continuous time investment behavior by the firm and valuation by the market. However, this continuous process will be approximated by a "stylized" discrete time behavior; discrete in the sense that information and investment decisions will be allowed once each period of time, and stylized in the sense that in truth these are continuous processes and the imposition of a particular discrete time framework should not force conclusions on the investment and valuation processes alien to their intrinsic nature. (The reason behind this caveat will appear when the possibility of long term debt financing is introduced).

(3) In the discrete time period framework of the firm's investment decisions the capital asset pricing model (CAPM) is assumed to hold in every period. Then the firm will make its investment decisions as to maximize its value according to this valuation model. The cash flows resulting from the investment decisions of the firm are assumed sufficiently small compared to the existing market cash flows so as to leave the market portfolio of investments the same. The validity of the CAPM as a valuation model is established by a stationarity of distribution assumption for the risky assets comprising the market,[1] by the constancy of the riskless asset and by further assumptions that the

[1] See, however E.F. Fama, "Risk-Adjusted Discount Rates and Capital Budgeting under Uncertainty," Journal of Financial Economics 5 (August 1977): 3-24, for a relaxation of this assumption.

investors have separable constant absolute risk averse utilities of wealth and that the distributions of returns of risky assets are normal in each period. No distribution assumptions will be made regarding the cash flow distributions of the firm. Thus, the valuation of the cash flows of the firm according to the CAPM is approximate since investors are not assumed to have quadratic utility functions. Since the cash flows are "sufficiently" small so as not to affect the market, the CAPM valuation will be "sufficiently" close.

To avoid this equivocation, one must assume that the only uncertainty about future cash flow distributions from investment opportunities possessed by the firm lies in the expectation of cash flows. Thus one may model the expectation at $\tau$ of cash flows to be received at $t > \tau$ as a random realization of the expectation at $\tau-1$:

$$\widetilde{E_\tau(x_t)} = E_{\tau-1}(x_t) + E_{\tau-1}(x_t)\tilde{E}_\tau$$

where $E_{\tau-1}(\tilde{E}_t) = 0$.

The uncertainty about the expectation is then modeled as the distribution of $\tilde{E}_\tau$. To keep the market opportunity set the same each period when one explicitly recognizes that this firm is part of the set requires that one assume that

$$\text{cov}\ (\tilde{E}_\tau,\ \tilde{Rm}_\tau)$$

is constant for all $\tau$ where $\tilde{Rm}_\tau$ is the return on the market portfolio in $\tau$. One thereby assumes that "the degree of association between the reassessments of a firm's earnings and the reassessments of the earnings prospects of all firms" is quite certain.[2]

(4) Production capacity (capital equipment and the requisite labor) is available at a cost of k per unit; the price k is constant over time. The number of units required to produce a unit of output is $q_i$, indicating the period of purchase, i = 1, ..., t. The $q_i$ will be assumed to decrease to account for technological progress in the production of more output per unit of capacity. The operating cost per unit of capacity, c, will be assumed constant (c net of tax effects). Thus, there is assumed to be no choice between capital and labor contributions (i.e., a fixed coefficient production function), and the lack of distinction between these two factors forces the assumption in the next chapter of neutral technological progress.

These costs may be expressed in terms of units of output, an unambiguous measure of capacity for this firm. If c = operating cost and k = investment cost per "unit of capacity" and $q_i$ "units of capacity" of vintage i are needed to produce a unit of output, then the investment cost per unit of output in the current year t (with equipment of vintage t available for

[2]See Fama, 15

purchase) is $kq_t$. The operating cost per unit of output of equipment of vintage i is $cq_i$. Also, if the firm has previously purchased $F_i$ "units of capacity" of vintage $i<t$, it may produce no more than $F_i/q_i$ units of output from this equipment.

(5) The firm is assumed to be unable to sell any part of its previously purchased capacity at any time. There are no storage costs or scrap costs or value associated with the capacity. (Transfer of ownership of the capacity by purchase of all of it, i.e., by change of ownership of the firm, is permitted by perfect capital market assumptions).

The Behavior of the Firm

The firm is described by the following behavior:

(1) its demand curve is, on an after-tax basis

$\tilde{P}_t = \tilde{a}_t x_t^{-n}$

its revenue function is

$\tilde{Y}_t = \tilde{a}_t x_t^{1-n}$

and the marginal revenue curve may be expressed as

$\widetilde{MR}_t = (1-un)\tilde{a}_t x_t^{-n}$

where $\tilde{P}_t$ is the price of a unit of the firm's product in period t

$X_t$ is its output quantity in period t

n is an elasticity constant 0 n 1

$\tilde{a}_t$ is a demand parameter which contains a random element and the effect of the outputs of other firms in the industry, constrained by the constancy of u, and

u is a parameter, $0 \leq u < 1$, describing the firm's ability to influence its demand curve via its production $x_t$. It may be thought of as indicating the firm's "degree of monopoly power" in the sense that, as $u \to 0$, the firm has no influence on demand and prices as a pure competitor, and as $u \to 1$, it prices as a pure monopolist. The parameter u is a function of the firm's market share, the degree to which the products of each firm in the industry are differentiated, the market shares of the other firms, and the reaction functions of the other firms in the industry.

(2) The firm is postulated to be a quantity announcer; its decision is announced as the beginning of period t on the basis of expectations of $a_t$ formed at that time. Any variation of realized demand about this expectation will be absorbed by fluctuation of realized price.

(3) The formation of expectations is as follows: the distribution of the random demand parameter is

$$\tilde{a}_t = Ea_t\ (1 + \tilde{e})$$

where $Ea_t$ is the expected value of $\tilde{a}_t$ at the beginning of period t,

$\tilde{e}$ is a random error with $E(\tilde{e}) = 0$, $var(\tilde{e}) = S_e^2$, $cov(\tilde{e}, \tilde{Ym}_t) = c_{em}$, $E(\tilde{e})$, $S_e^2$, $c_{em}$ constant over time. ($\tilde{Ym}$ is the cash flow of all assets in the market, also with constant distribution over time). The random error $\tilde{e}$ thus implies variation of realized revenue within the period t. In addition, the expectation $Ea_t$ changes over time via the mechanism

$$\tilde{Ea}_{t+1} = Ea_t \ (1 + \tilde{z})$$

where z is a random term with 1 = z a function of the outcome and its expectation in period t, $F\left[\frac{a_t}{Ea_t}\right] = F\ (1 + \tilde{e})$.

## The Decisions of the Firm

The firm faces the environment and market structure described above, except that there is assumed to be no technological progress: $q_i = q_{i+1} = \ldots = q_t = 1$. Thus, the investment cost of capacity sufficient to produce 1 unit of output is k, and the operating cost is c. Under the assumption of no technological progress, capacity purchased in any period is identical and (assuming no depreciation) adds the same capacity to produce output in any future period. The firm must balance the costs of investing in capacity in any period against both revenues that period and possible future revenues from use of that capacity in future periods when demand conditions are such that the purchase of new capacity at that time is not optimal. The firm is assumed to finance the purchase of capacity by selling shares. It is able to do so as long as the value of the firm, equal to the net present value of its current and future production decisions, is greater than 0.

In any period, therefore, the firm optimizes over two production decision variables: the amount it produces out of its current capacity and the amount of new capacity to be added and used for production after its current capacity is fully utilized. Denote the total available capacity at period t by $X_t$. (The amount of capacity sufficient to produce output x is also x by the assumption that $q_t = 1$). Denote the production decision variables by $x_{to}$, the amount of available capacity the firm uses, and $X_{tn}$, the amount of new capacity purchased and used in period t. The firm then maximizes its current value $V_t$ over $(x_{to}, x_{tn})$ subject to the constraint that $x_{to} \leq X_t$. This is a Kuhn-Tucker maximization in each period t, with the dual variable to the constraint $\lambda_t$ measuring the value of having additional current capacity on hand at t. The firm's net present value at t, $V_t$, will be determined by a backward induction from a termination date T, assuming optimal production decisions at each state.

B. The Value of the Optimal Decisions

1. The Single Period Problem

<u>The Firm with Monopoly Power (u > 0)</u>.

The single period or ($T^{th}$ period) problem is thus:

$$\max_{(x_{To},\, x_{Tn})} V_T$$

subject to $x_{To} \leq X_T$ or $g\ (x_{To}) = x_{To} - X_T \leq 0$

$$x_{To},\ x_{Tn} > 0$$

Where $V_T$ = the certainty equivalent, discounted at the risk-free rate r to the beginning of T, of the revenues to be received at the end of T, net of the cost of operation, minus the optimal investment at the beginning of T (all on an after-tax basis). The revenue

$$\tilde{Y}_T = \tilde{P}_T\ (x_{To} + x_{Tn})$$

and cost = $c\ (x_{To} + x_{Tn})$

since total production $x_T = x_{To} + x_{Tn}$, the amount produced from each type of capacity. Then

$$E(\tilde{Y}_T) = E\tilde{a}_T\ (x_{To} + x_{Tn})^{1-n} = Ea_T\ (x_{To} + x_{Tn})^{1-n}$$

$$\text{cov}\ (\tilde{Y}_T, \tilde{Y}m) = Ea_T\ (x_{To} + x_{Tn})\ \text{cov}\ (\tilde{e}, \tilde{Y}m)$$

so that

$$CEQ\ \tilde{Y}_T = Ea_T\ (x_{To} + x_{Tn})^{1-n}\ (1 - m\ c_{em})$$

$$= Ea_T\ (x_{To} + x_{Tn})^{1-n}\ L$$

where $L = 1 - mc_{em}$, m being the market price of systematic risk in T. It is assumed that L is greater than 0.

Investment cost = $k\ x_{Tn}$. Therefore,

$$V_T = \frac{CEQ\tilde{Y}_T - C(X_{To} + X_{Tn})}{1 + r} - kX_{Tn}$$

$$= \frac{Ea_TL\ (X_{To} + X_{Tn})^{1-n} - C(X_{To} + X_{Tn})}{1 + r} - kX_{Tn} \qquad (1)$$

The Kuhn-Tucker maximization conditions are: (2)

(i) $\dfrac{\partial V_T}{\partial X_{To}} - \lambda_t \dfrac{\partial g(X_{To})}{\partial X_{To}} \leq 0$

$\dfrac{\partial V_T}{\partial X_{Tn}} \leq 0$

(ii) $X_{To} \left[ \dfrac{\partial VT}{\partial X_{To}} - \lambda_t \dfrac{\partial g(X_{To})}{\partial X_{To}} \right] = 0$

$$x_{Tn} \frac{\partial Vr}{\partial K_{Tn}} = 0$$

(iii) $\lambda_t g(X_{TO}) = 0$

(iv) $x_{TO}, x_{Tn}, \lambda_t \geq 0$

where $g(x_{TO}) = x_{TO} - X_T, \frac{\partial g(X_{TO})}{\partial X_{TO}} = 1$

$$\frac{\partial V_T}{\partial X_{TO}} = \frac{CEQ(MR_T) - c}{1 + r} = \frac{(1-un)Ea_T L(X_{TO} + X_{Tn})^{-n} - c}{1 + r}$$

$$\frac{\partial V_T}{\partial X_{Tn}} = \frac{(1-un)Ea_T L(X_{TO} + X_{Tn})^{-1} - c}{1 + r} - k$$

The following optimal solutions can occur, depending on the value of $Ea_T$:

Case 1: $x_{Tn}^* = 0, \ 0 < x_{TO}^* < X_T$ (3)

The firm operates at less than full capacity. Then $\lambda_T^* = 0$ and

$$\frac{\partial V_T^*}{\partial X_{TO}} = \frac{(1-un)Ea_T L(X_{TO}^* + X_{Tn}^*)^{-n} - c}{1 + r} = \lambda_T^* = 0$$

$$\frac{\partial V_T^*}{\partial X_{Tn}^*} = \frac{(1-un)Ea_T L(X_{TO}^* + X_{Tn}^*)^{-n} - c}{1 + r} - k < 0$$

Then $(1-un)Ea_T L \ (x_{TO}^*)^{-n} = c$, or

$$x_{TO}^* = \left[\frac{Ea_T L(1-un)}{c}\right]^{1/n} < X_T$$

$$x_{Tn}^* = 0$$

This solution is the optimal one for $0 < x_{TO}^* < X_T$, or

$$0 < Ea_T < \frac{c \ X_T^n}{L(1-un)}$$

Denote this limit by $A_{1T}$.[3]

[3]The case under which there would be no production is that for which $x_{TO}^* = x_{Tn}^* = 0$. Then $_T^* = 0$ and $\frac{\partial V_T^*}{\partial X_{TO}^*} < 0$, or $(1-un)Ea_T L \cdot (x_{TO}^*)^{-n} - c < 0$. This condition cannot be maintained for positive $Ea_T$ as $x_{TO}^* \rightarrow 0$.

Case 2: $x_{Tn}^* = 0,\ x_{To}^* = X_T.$

The firm operates at full capacity but does not add new capacity. Then $\lambda_T^* > 0$ and

$$\frac{\partial V_T^*}{\partial X_{To}^*} = \frac{(1-un)Ea_T L(X_{To}^*)^{-n} - c}{1 + r} = \frac{(1-un)Ea_T L(X_T)^{-n} - c}{1 + r} = \lambda_T^*$$

$$\frac{\partial V_T^*}{\partial X_{Tn}} = \frac{(1-un)Ea_T L(X_{To}^*)^{-n} - c}{1 + r} - k < 0$$

or $0 < \lambda_T^* < k.$

This solution is optimal for $Ea_T$ such that

$$0 < \frac{(1-un)Ea_T L(X_T)^{-n} - c}{1 + r} < k$$

or

$$A_{1T} = \frac{c(X_T)^n}{L(1-un)} < Ea_T < \frac{[c + k(1 + r)](X_T)^n}{L(1-un)}$$

Denote the upper limit by $A_{2T}$.

Case 3: $x_{Tn}^* > 0,\ x_{To}^* = X_T$

The firm operates at full capacity and purchases additional capacity $x_{Tn}^*$. Then $\lambda_T^* > 0$ and

$$\frac{\partial V_T^*}{\partial X_{To}^*} = \frac{(1-un)Ea_T L(K_{To}^* + X_{Tn}^*)^{-n} - c}{1 + r} =$$

$$\frac{(1-un)Ea_T L(X_T + X_{Tn}^*)^{-n} - c}{1 + r} = \lambda_T^*$$

$$\frac{\partial V_T^*}{\partial X_{Tn}^*} = \frac{(1-un)Ea_T L(X_T + X_{Tn}^*)^{-n} - c}{1 + r} - k = 0$$

or

$$\lambda_T^* = k, \text{ and}$$

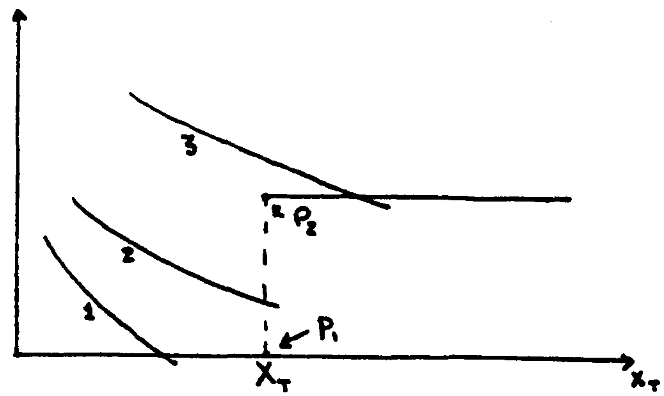
$$x_T^* = X_T + x_{Tn}^* = \left[\frac{Ea_T L(1-un)}{c + k(1+r)}\right]$$

This solution is optimal for $x_{Tn}^* > 0$, or $\left[\frac{Ea_T L(1-un)}{c + k(1+r)}\right] > X_T$, or $Ea_T$ such that $Ea_T > A_{2T}$.

These three solutions may be visualized graphically from a plot of the certainty equivalent of marginal revenue less marginal operating cost, discounted to the beginning of period T, and marginal investment cost, versus total production $x_T^*$:

$$\frac{\text{CEQ } MR_T - c}{1 + r}$$

marginal investment cost $= \begin{cases} 0, & x_T^* \leq X_T \\ k, & x_T^* > X_T \end{cases}$




The optimal solutions for cases 1, 2, and 3 are represented by intersections of curves 1, 2, and 3 respectively, with the marginal investment cost curve. For all $Ea_T$ such that $\frac{\text{CEQ } \tilde{MR}_T - c}{1 + r}$ intersects between $P_1$ and $P_2$, i.e., all $Ea_T$ such that $A_{1T} < Ea_T < A_{2T}$, the case 2 solution is optimal. The firm operates at its previously purchased capacity. The current value of net marginal receipts falls below the marginal cost k of another unit of capacity. For all curves intersecting the marginal cost curve to the right of $P_2$, i.e., $Ea_T > A_{2T}$, the firm purchases additional capacity; the value $Ea_T$ is sufficiently large so that the current value of net marginal receipts exceeds k if the firm operates only at capacity. For all curves intersecting to the left of $P_1$, i.e., all $Ea_T < A_{1T}$, the firm operates at less than capacity; the excess capacity is useless (since this is the terminal period) and the firm would sell it if the opportunity for capacity resale at a price other than 0 were available.

The value of the firm $V_T^*$ in case 1, for which $0 < x_{To}^* < X_T$, $x_{Tn}^* = 0$, $\lambda_T^* = 0$, is, by substitution into (1),

$$V_T^* = \frac{Ea_T L(X_{To}^* + X_{Tn}^*)^{1-n} - c(X_{To}^* + X_{Tn}^*)}{1 + r}$$

$$= X_{To}^* \frac{Ea_T L(X_{To}^*)^{-n} - c}{1 + r}$$

$$= X_{To}^* \frac{unEa_T L(X_{To}^*)^{-n}}{1 + r} \quad \text{from the optimization conditions} \qquad (3)$$

$$= \frac{unEa_T L(X_{To}^*)^{1-n}}{1 + r}$$

where $x_T^* = k_{To}^* + x_{Tn}^* = x_{To}^*$.

In case 2, for which $x_{To}^* = X_T$, $x_{Tn}^* = 0$, $\lambda_T^* = 0$,

$$V_T^* = (X_{To}^* + X_{Tn}^*) \frac{Ea_T L(X_{To}^* + X_{Tn}^*)^{-n} - c}{1 + r}$$

$$= \lambda_T^* X_T + X_T \frac{unEa_T L(X_T)^*}{1 + r} \quad \text{from the optimization conditions} \qquad (4)$$

$$= \lambda_T^* X_T + \frac{unEa_T L(X_T)^{1-n}}{1 + r}$$

where $x_T^* = X_T$.

In case 3, for which $x_{To}^* = X_T$, $x_{Tn}^* > 0$, $\lambda_T^* = k$

$$V_T = (X_{To}^* + X_{Tn}^*) \frac{Ea_T L(X_{To}^* + X_{Tn}^*)^{-n} - c}{1 + r} - kX_{Tn}^*$$

$$= \lambda_T^*(X_{To}^* + X_{Tn}^*) + (X_{To}^* + X_{Tn}^*) \frac{unEa_T L(X_{To}^* + X_{Tn}^*)^{-n}}{1 + r}$$

and from the optimization conditions (5),

$$V_T^* = \lambda_T^* X_T + \frac{unEa_T L(X_T^*)^{1-n}}{1 + r}$$

since $\lambda_T^* = k$ in case 3 and $x_T^* = X_T + x_{Tn}^*$.

In general, then, the value of the firm may be expressed as a function of the variables $(X_T, x_T^*)$, as

$$V_T^* \;=\; \lambda_T^*\, X_T + \frac{unEa_TL(X_T^*)^{1-n}}{1+r} \qquad (6)$$

in which $\lambda_T^*$ is clearly that of the addition to $V_T^*$ of another unit of previously purchased capacity, the shadow price of the constraint on use of previously purchased capacity $X_T$ is not being fully utilized at optimum ($x_T^* < X_T$); $\lambda_T^* = \frac{\partial V_T^*(X_T,\ X_T^*)}{\partial X_T}$ which is less than k if $x_T^* = X_T$, and $\lambda_T^* = k$ if $x_T^* > X_T$ so that the shadow price of a unit of extra capacity equals the external price and new capacity is purchased at optimum.

In (6) $V_T^*$ is divided into a "value of current capacity" term $\lambda_T^* X_T$, plus a term which equals 0 if the firm has no monopoly power. The second term in (6) is the current capitalized value of the future profits arising explicitly from the firm's ability to price non-competitively, since it is the current capitalized value of the difference between marginal and average net revenue multiplied by production.[4] Thus it is the capitalized value of the monopoly rents to be received at the end of the period. Thus (6) may be re-expressed as

$$V_T^* \;=\; \lambda_T^*\, X_T + M_T^* \qquad (7)$$

where

$$M_T^* \;=\; \frac{unEa_TL}{1+r}\,(X_T^*)^{1-n},$$

the capitalized value of future monopoly rents. The first term in (7) is the current value to the firm of having previously purchased capacity $X_T$ on hand. If the firm does not

[4]See Thomadakis.

use all the capacity ($x_T^* < X_T$) so that additional capacity is a "free good" and there is no value to having the amount $X_T$ on hand, this term equals 0. If the firm uses all the capacity ($x_T^* = X_T$) the value is the marginal value of having an extra unit, $\lambda_T^*$, the shadow price, multiplied by the capacity. In this case the shadow price of the capacity is internal, specific to the firm, and is the current capitalized value of the marginal revenue less cost to be received from the use of $X_T$. It is thus dependent on the monopoly power of the firm, and at u = 0, the shadow price becomes the current capitalized value of average revenue. If the firm adds capacity, the shadow price of previous capacity equals the external price k, and the first term is then $kX_T$, the value at the external price of the capacity $X_T$. The first term may thus also be regarded as a "reproduction" value of the capacity which the firm did <u>not</u> pay for at T, with the "reproduction" value being the current capitalized value of the marginal revenue less cost to be received from the use of the current capacity. (The value $V_T^*$ is the net present value. Thus, the reproduction value can only be that of the "free" capacity since the value of any newly purchased capacity has been subtracted). In case 1, this internal value is 0; in case 2, it is the current capitalized value determined at $x_T^* = X_T$; in case 3 the internal value equals the external value k (since the firm did not have to pay k for each unit of previously purchased capacity).

The Firm without Monopoly Power $(u = 0)$

It follows from (7) that, even if the firm were to possess no monopoly power $(u = 0)$, the presence of the current capacity $X_T$ yields a net present value to the firm of $\lambda_T^*(u = 0)\ X_T$, which is greater than zero as long as the firm produces at capacity. The optimal behavior of the firm with $u = 0$ should more properly be derived from an explicit examination of that special case.

The case $u = 0$ implies that the firm has no influence on demand, that its product is undifferentiated and $x_T << x_T$ (industry) = the total production of the industry. Thus it prices as a pure competitor; i.e., price is exogenous to its production. If price = $\tilde{p}_T = \tilde{a}_T\ F[x_T(\text{industry})]$, irrespective of the firm's production decisions (so that $dF/dx_T = 0$), then its value is (8)

$$V_T = \frac{CEQ\ \tilde{P}_T(x_{To} + x_{Tn}) - C(x_{To} + x_{Tn})}{1 + r} - kx_{Tn}$$

and the maximization conditions are (9)

$$x_{To}^*\left[\frac{CEQ\ \tilde{P}_T - c}{1+r} - \lambda_T^*\right] = 0$$

$$x_{Tn}^*\left[\frac{CEQ\ \tilde{P}_T - c}{1+r} - k\right] = 0$$

$$\lambda_T^*\left[x_{To}^* - X_T\right] = 0.$$

The the optimal solutions are:

Case 1: $0 < x_{To}^{*} < X_T$ (10)

$$\text{Then} \quad \lambda_T^{*} = 0 = \frac{CEQ\,\tilde{P}_T - c}{1+r} = \frac{Ea_T LF[X_T(\text{industry})] - c}{1+r}$$

Case 2: $x_{To} = X_T;\ x_{Tn}^{*} = 0$ (11)

$$\text{Then} \quad 0 < \lambda_T^{*} = \frac{CEQ\,\tilde{P}_T - c}{1+r} = \frac{Ea_T LF[X_T(\text{industry})] - c}{1+r} < k$$

Case 3: $x_{To}^{*} = X_T: \quad x_{Tn}^{*} > 0$ (12)

$$\text{Then} \quad \lambda_T^{*} = \frac{Ea_T LF[X_T(\text{industry})] - c}{1+r} = k$$

Thus the case 1 optimum applies only at $Ea_T$ such that

$$Ea_T = \frac{c}{LF[X_T(\text{industry})]},$$

the threshold of application of the case 2 optimum. The firm adds an unspecified amount of new capacity when

$$Ea_T > \frac{c + k(1+r)}{LF[X_T(\text{industry})]}$$

The optimal production decisions are thus: not to produce for

$$Ea_T < \frac{c}{LF[X_T(\text{industry})]}$$

to produce at capacity for

$$\frac{c}{LF[X_T(\text{industry})]} < Ea_T < \frac{c + k(1+r)}{LF[X_T(\text{industry})]}$$

and to add capacity for

$$Ea_T > \frac{c + k(1+r)}{LF[X_T(\text{industry})]}$$

The value of the firm is

$$V_T^* = \frac{[Ea_T LF[x_T(\text{industry})] - c]x_T^*}{1+r} - kx_{Tn}^*$$

$$= 0 \text{ for } Ea_T < \frac{c}{LF[X_T(\text{industry})]} \text{ on substitution of (10);}$$

$$= \lambda_T^* X_T \quad \text{for } \frac{c}{LF[x_T(\text{industry})]} < Ea_T < \frac{c + k(1+r)}{LF[x_T(\text{industry})]}$$

on substitution of (11), with $\lambda_T^*$ as specified in case 2; and

$$V_T^* = \lambda_T^* (X_T + x_{Tn}^*) - k x_{Tn}^* = \lambda_T^* X_T$$

on substitution of (12), with $\lambda_T^*$ as specified in case 3, $\lambda_T^* = k$. Thus in general, $V_T^* = \lambda_T^* X_T$.

In equilibrium, there will be no production from any firm until the variable cost of production, c, is covered by the certainty equivalent of marginal revenue. If it is, the firm produces at capacity until expectations rise to the point that the certainty equivalent of marginal covers variable cost + the cost of adding new production capacity. Since demand for the industry depends on price, total industry production will be set so that the variable cost of production is covered. Thus each firm will produce at capacity, and total production will be the sum of capacities if

$$\frac{c}{LF[\sum_{\text{all firms}} X_T]} \leq Ea_T \leq \frac{c + k(1+r)}{LF[\sum_{\text{all firms}} X_T]}$$

If $Ea_T < \dfrac{c}{LF[\sum_{\text{all firms}} X_T]}$, then industry production will be set below total industry capacity. Since there is no optimal production specification for any firm unless it is at capacity, firms will leave the industry randomly as $Ea_T$ begins to fall below this cutoff value. Thus after these exits, some firms will be producing at capacity (with 0 value) and the others will not be producing. Thus total production = $\sum_{\text{producing firms}} X_T$.

If

$$Ea_T > \frac{c + k(1+r)}{LF[\sum_{\text{all firms}} X_T]}$$

all firms in the industry will add unspecified amounts of capacity and produce, so that

$$Ea_T = \frac{c + k(1+r)}{LF[X_T(\text{industry})]}$$ (each firm exactly covers total costs)

and

$$x_T \text{ (industry)} > \sum_{\text{all firms}} X_T$$

If a firm within the industry at T is compared to a firm with no existing capacity, the industry member is seen from (13) to have a positive net present value when it produces optimally in the range

$$\frac{c}{LF[\sum_{\text{all firms}} X_T]} < Ea_T < \frac{c + k(1+r)}{LF[\sum_{\text{all firms}} X_T]}$$

whereas the firm with no capacity will not produce within this range as the current value of expected net revenues is less than investment cost. The firm with no capacity will not enter and produce until

$$Ea_T > \frac{c + k(1+r)}{LF[\sum_{\text{all firms}} X_T]}$$

and the entry into production has a zero net present value, the usual result under pure competition. Thus there is a barrier to entry of the firm with no capacity within a range of possible outcomes of the random demand parameter, in the sense that firms already "in the industry", i.e., possessing capacity, achieve a positive net present value by producing in this range, within which the firm with no capacity, "waiting to enter", would not produce. This value arises out of the "free" good, capacity (free in the single period sense that its origins are not investigated in this section), and is internal to the firm since the firm is assumed unable to sell and achieve an external measure of the value. If resale were permitted, the firm in the industry could exit without the loss of this value, by selling its capacity to an entering firm. Conversely, the entering firm would pay the external value per unit of capacity, $\lambda_T^*$, and would therefore enter and produce at 0 net present value, satisfying the usual pure competition requirement of no entry barriers. Thus the entry barrier is arising out of the inability to resell capacity at its actual value, the barrier to exit from the industry of the firm having capacity. And therefore, although every firm in the industry (having capacity) behaves as a pure competitor in the pricing sense, the industry does not fulfill the requirements for perfect competition.

## 2. The Multiperiod Problem.

### Introduction

The optimal decisions ($x_{To}$*, $x_{Tn}$*) derived above, and the consequent value $V_T^*(X_T, x_T^*)$, are uncertain in previous periods due to the interperiod uncertainty about the information $Ea_T$ needed to make the decisions. Thus, the value of the firm at any time $t < T$, the value at t of the current investment opportunity plus all the future ones, is the certainty equivalent of sets of decisions, random at t, to be made as uncertainty is resolved. The uncertainty is resolved by the expectations formation mechanism

$$\widetilde{Ea}_{t+1} = Ea_t \cdot (1 + \tilde{z})$$

which represents the knowledge available at t, the realization $Ea_t$, and its implications for future states of nature $Ea_{t+1}$. Coupled with the information available at t are the previous capacity decisions $x_{t-jn}$* which make up the current available capacity $X_t$ at stage t. These stage decisions will also, of course, be functions of the interperiod or "growth" risk, the uncertainty as to future states of nature. The objective of the multiperiod valuation model is the determination of the optimal decisions of the firm in t, given the state of nature and the total capacity at t, hence the relation between these decisions and the interperiod uncertainty resolution. The optimal decision each period is derived from the maximization of current net present value $V_t$ over the two production variables ($x_{to}$, $x_{tn}$) given the previously purchased capacity at t and the information available at the beginning of t, $Ea_t$. Because future decisions are uncertain, the multiperiod model will be built by backward induction assuming the optimal decisions have been made in future periods.

<u>The Valuation at T-1</u>

At the beginning of T-1, the equity holders of the firm, including those who contribute the investment cost at that time, $x_{T-2n}$, own a claim to revenues less cost to be received at the end of T-1, plus a claim to the end-of-period value at optimum $x_T{}^* = x_{To}{}^* + x_{Tn}{}^*$ of their firm, $V_T{}^*$. As before,

$$\tilde{Y}_{T-1} = \tilde{P}_{T-1} x_{T-1} = \tilde{P}_{T-1}(x_{T-1o} + x_{T-1n})$$

$$\text{cost} = cx_{T-1}$$

$$\text{investment cost} = k\, x_{T-1n}$$

Then the current value of these two claims is their certainty equivalents discounted to the beginning of the period. Thus

$$V_{T-1} = \frac{Ea_{T-1} L(x_{T-1o} + x_{T-1n})^{1-n} - c(x_{T-1o} + x_{T-1n})}{1+r} - kx_{T-1n} + \qquad (14)$$

$$+ \text{CEQ}_{T-1} \frac{\tilde{V}_T^*(X_T, x_T^*)}{1+r}$$

where $V_T^*(X_T, x_T^*) = \tilde{\lambda}_T^* X_T + \tilde{M}_T^*(x_T^*)$, with both $\tilde{\lambda}_T^*$ and $\tilde{M}_T^*$ random through their dependence on $Ea_T$.

$$E(\tilde{V}_T^*) = X_T E(\tilde{\lambda}_T^*) + E(\tilde{M}_T^*)$$

$$\text{cov}\ (\tilde{V}_T^*, \tilde{V}m) = X_T \text{cov}\ (\tilde{\lambda}_T^*, \tilde{V}m) + \text{cov}(\tilde{M}_T^*, \tilde{V}m)$$

where Vm = total value of all assets in the market, so that

$$\text{CEQ}_{T-1}\tilde{V}_T^* = E(\tilde{V}_T^*) - m\ \text{cov}(\tilde{V}_T^*, \tilde{V}m) = X_T\ \text{CEQ}_{T-1}\tilde{\lambda}_T^* + \text{CEQ}_{T-1}\tilde{M}_T^*$$

and the state variable $X_T = X_{T-1} + x_{T-1n}^*$.[b]

Note that the contribution of any $\tilde{V}_t^*$ to $V_{t-1}$,

$$\text{CEQ}_{t-1}\tilde{V}_t^* = X_t\left[E(\lambda_t^*) - m\ \text{cov}(\tilde{\lambda}_t^*, \tilde{V}m_t)\right] + E(\tilde{M}_t^*) - m\ \text{cov}(\tilde{M}_t^*, \tilde{V}m_t)$$

$$= X_t\left[E(\lambda_t^*) - \frac{m\ \text{cov}(\tilde{\lambda}_t^*, \tilde{R}m_t)}{Vm_{t-1}}\right] + E(\tilde{M}_t^*) - m\frac{\text{cov}(\tilde{M}_t^*, \tilde{R}m_t)}{Vm_{t-1}}.$$

Express $\tilde{\lambda}_t^* = E(\tilde{\lambda}_t^*)\cdot(1 + \tilde{\varepsilon}_{\lambda_t})$; $M_t^* = E(\tilde{M}_t^*)\cdot(1 + \tilde{\varepsilon}_{M_t})$

so that

$$\text{CEQ}_{t-1}\tilde{V}_t^* = X_t\ E_{t-1}(\tilde{\lambda}_t^*)\left[1 - m\frac{\text{cov}(\varepsilon_{\lambda_t}, \tilde{R}m_t)}{Vm_{t-1}}\right]$$

$$+ E_{t-1}(\tilde{M}_t^*)\left[1 - m\frac{\text{cov}(\tilde{\varepsilon}_{M_t}, \tilde{R}m_t)}{Vm_{t-1}}\right]$$

where $\tilde{\varepsilon}_{\lambda_t}, \tilde{\varepsilon}_{M_t}$ are random components with expectation 0, functionally related to $\tilde{z}$, the random component of $\tilde{E}a_t$. Then, as

[b]Stapleton and Subrahmanyam do show that the market price of risk m will change from period to period in a known fashion as a function of polynomials in the interest rate r; thus m and therefore L as well as any other CEQ should carry a time subscript. This additional notation will be suppressed as it causes no change in the essential conclusions, merely modifying numerical results. See R. Stapleton and M.G. Subrahmanyam, "A Multiperiod Equilibrium Asset Pricing Model," New York University Working Paper, 1975.

Fama has shown, a correct multiperiod valuation of this firm using the CAPM requires the assumption that

$\text{cov}\ (\tilde{\varepsilon}_{\lambda_t},\ \tilde{Rm}_t)$ and $\text{cov}\ (\varepsilon_{M_t},\ \tilde{Rm}_t)$

be constant over time. The constancy requires that the two time-varying sources of these random components, new capacity decisions which change the $X_t$ variable and the demand parameter $\tilde{Ea}_t$, not change their distribution in an unknown way. The constancy of effect of $Ea_t$ has been assumed by letting $c_{em}$ be a constant. Thus the uncertainty in $Ea_t$ must be of a market-wide nature to keep its reassessment relative to the market a constant. The effect of the other will be investigated after the behavior of $\tilde{\lambda}_t$ has been derived.

The problem at T-1 is

$\max V_{T-1}$

$(x_{T-1_o},\ x_{T-1_n})$

subject to $x_{T-1_o} \leq X_{T-1}$ or $g(x_{T-1_o}) = x_{T-1_o} - X_{T-1} \leq 0$

$$x_{T-1_o},\ x_{T-1_n},\ \lambda_{T-1} \geq 0$$

The Kuhn-Tucker optimization conditions are (15)

(i) $\frac{\partial V_{T-1}^*}{\partial x_{T-1_o}^*} - \lambda_{T-1}^* \frac{\partial g(x_{T-1_o}^*)}{\partial x_{T-1_o}^*} \leq 0$

$\frac{\partial V_{T-1}^*}{\partial x_{T-1_n}^*} \leq 0$

(ii) $x_{T-1_o}^* \left[\frac{\partial V_{T-1}^*}{\partial x_{T-1_o}^*} - \lambda_{T-1}^* \frac{\partial g(x_{T-1_o}^*)}{\partial x_{T-1_o}^*}\right] = 0$

$x_{T-1_n}^* \left[\frac{\partial V_{T-1}^*}{\partial x_{T-1_n}^*}\right] = 0$

(iii) $\lambda_{T-1}^* g(x_{T-1_o}^*) = 0$

$x_{T-1_o}^*,\ x_{T-1_n}^*,\ \lambda_{T-1}^* \geq 0$

The partials are:

$$\frac{\partial g(x_{T-1o})}{\partial x_{T-1o}} = 1$$

$$\frac{\partial V_{T-1}}{\partial x_{T-1o}} = \frac{(1-un)\,Ea_{T-1}L\,(X_{T-1o} + x_{T-1n})^{-n} - c}{1+r}$$

$$\frac{\partial V_{T-1}}{\partial x_{T-1n}} = \frac{(1-un)\,Ea_{T-1}L(X_{T-1o}+x_{T-1n})^{-n} - c}{1+r} - k + \frac{1}{1+r}\,\frac{\partial CEQ_{T-1}\tilde{V}_T^*(X_T, x_T^*)}{\partial x_{T-1n}}.$$

Since $X_T = X_{T-1} + x_{T-1_n}$, $\tilde{V}_T^*$ is a function of $x_{T-1_n}$; thus the derivative $\dfrac{\partial CEQ_{T-1}\tilde{V}_T^*}{\partial x_{T-1n}}$

is not equal to 0. Indeed, by definition of $\lambda_T^*$,

$$\frac{\partial \tilde{V}_T^*}{\partial x_{T-1n}} = \frac{\partial \tilde{V}_T^*}{\partial X_T} = \tilde{\lambda}_T^*,$$

the shadow price of an extra unit of capacity at the beginning of T. Since this holds for any $\tilde{Ea}_T$, it follows that

$$\frac{\partial CEQ_{T-1}\tilde{V}_T^*}{\partial x_{T-1n}} = \frac{\partial CEQ_{T-1}\tilde{V}_T^*}{\partial X_T} = CEQ_{T-1}\tilde{\lambda}_T^*.$$

This evaluation of the partial is not obvious from inspection of $CEQ_{T-1}\tilde{V}_T^*$ and will be shown. From the single period results (6)

$$V_T^* = \frac{unEa_tL(X_T^*)^{1-n}}{1+r} \qquad \text{for } Ea_T < A_{1T}$$

$$V_T^* = \frac{Ea_TL(X_T)^{1-n} - cX_T}{1+r} \quad \text{for } A_{1T} < Ea_T < A_{2T}$$

$$= kX_T + \frac{unEa_TL(X_T^*)^{1-n}}{1+r} \qquad \text{for } Ea_T > A_{2T}$$

Since the integrand $V_T^*(\tilde{Ea}_T)$ of the integral form $CEQ_{T-1}\tilde{V}_T^*$ is continuous over the range of $Ea_T$ and in particular is continuous at $A_{1T}$ $(X_T)$ and $A_{2T}$ $(X_T)$, the sum of contributions from the derivatives of the limits $A_{1T}$ and $A_{2T}$ to the derivatives of $CEQ_{T-1}\tilde{V}_T^*$ is zero. Thus, the only non-zero term is the integral of the derivative, $CEQ_{T-1} \frac{\partial V_T^*}{\partial X_T^*}$. From the solution to the single period problem, $x_T^*$ is not a function of $X_T$ for $Ea_T < A_{1T}$ and $Ea_T > A_{2T}$. Thus

$$\frac{\partial V_T^*}{\partial X_T} = 0 \text{ for } Ea_T < A_{1T}$$

$$\frac{\partial V_T^*}{\partial X_T} = \frac{(1-un)Ea_T L(X_T)^{-n} - c}{1+r} \text{ for } A_{1T} < Ea_T < A_{2T}$$

by definition of the marginal revenue function

$$\frac{\partial V_T^*}{\partial X_T} = k \text{ for } Ea_T > A_{2T};$$

i.e., as previously argued,

$$\frac{\partial V_T^*}{\partial X_T} = \lambda_T^*. \text{ Thus,}$$

$$\frac{\partial\, CEQ_{T-1}\tilde{V}_T^*}{\partial X_T} = CEQ_{T-1} \frac{\partial \tilde{V}_T^*}{\partial X_T} = CEQ_{T-1}\tilde{\lambda}_T^* \qquad (16)$$

This result can also be formally derived from the re-expression of $\tilde{V}_T^*$ as $\tilde{\lambda}_T^* X_T + \tilde{M}_T^*$, the sum of the contributions of the firm's previously purchased capacity and its monopoly power to its value at T.

Therefore, the optimization conditions in T-1 are, on substitution of (16) into (15):

$$x_{T-1o}{}^*\left[\frac{(1-un)\,Ea_{T-1}L(x_{T-1o}{}^*+x_{T-1n}{}^*)^{-n}-c}{1+r}-\lambda^*_{T-1}\right]=0$$

$$x_{T-1n}{}^*\left[\frac{(1-un)\,Ea_{T-1}L(x_{T-1o}{}^*+x_{T-1n}{}^*)^{-n}-c}{1+r}-k+CEQ_{T-1}\frac{\tilde{\lambda}^*_T}{1+r}\right]=0$$

$$x^*_{T-1o},\ x^*_{T-1n},\ \lambda^*_{T-1}\ \geq 0$$

As before, the solutions will be derived for the three possible cases:

Case 1: $x_{T-1o}{}^* < X_{T-1},\ x^*_{T-1n} = 0$ (17)

Then $\lambda^*_{T-1} = 0$ and

$$\frac{\partial V_{T-1}{}^*}{\partial x_{T-1o}{}^*}=\frac{(1-un)Ea_{T-1}L(x_{T-1o}{}^*)^{-n}-c}{1+r}=\lambda^*_{T-1}=0,\quad\text{or}$$

$$x_{T-1o}{}^*=\left[\frac{Ea_{T-1}L(1-un)}{c}\right]^{1/n}<X_{T-1}.$$

This solution holds for $Ea_{T-1} < \frac{cX_{T-1}^{\;n}}{L(1-un)}$. Denote this limit by $A1_{T-1}$.

Case 2: $x^*_{T-1o} = X_{T-1},\ x_{T-1n}{}^* = 0$ (18)

Then $\lambda_{T-1}{}^* > 0$ and

$$\frac{\partial V_{T-1}{}^*}{\partial x_{T-1o}{}^*}=\frac{(1-un)Ea_{T-1}L(X_{T-1})^{-n}-c}{1+r}=\lambda^*_{T-1}$$

$$\frac{\partial V_{T-1}{}^*}{\partial x_{T-1n}{}^*}=\frac{(1-un)Ea_{T-1}L(X_{T-1})^{-n}-c}{1+r}-k+CEQ_{T-1}\frac{\tilde{\lambda}_T{}^*}{1+r}<0$$

or $0<\lambda_{T-1}{}^* < k - CEQ_{T-1}\frac{\tilde{\lambda}_T{}^*}{1+r}$

This solution holds for $0 < \frac{(1-un)Ea_{T-1}LX_{T-1}{}^{-n}-c}{1+r} < k - CEQ_{T-1}\frac{\tilde{\lambda}_T{}^*}{1+r}$

Note that the certainty equivalent of $\tilde{\lambda}_T{}^*$ involves integration over the distribution of $Ea_T$ and the mean of this distribution is a function of $Ea_{T-1}$ via the expectations formation mechanism. Thus, $CEQ_{T-1}\,\tilde{\lambda}_T{}^*$ is a function of $Ea_{T-1}$ and the upper limit $A_{2_{T-1}}$ of the range for which $x_{T-1}{}^* = X_{T-1}$ is optimal must be determined by the upper bound of the inequality in $Ea_{T-1}$:

$$Ea_{T-1} \leq X_{T-1}^{n} \left[ \frac{c + k(1+r) - cEQ_{T-1}(Ea_{T-1})\tilde{\lambda}_T^*}{L(1-un)} \right]$$

The lower bound is $A_{1_{T-1}}$. Since $\lambda_T \leq k$, $A_{2_{T-1}} > A_{1_{T-1}}$.

Case 3: $x_{T-1o}{}^* = X_{T-1}$, $x_{T-1n}^* > 0$ (19)

Then $\lambda_{T-1}{}^*$ 0 and

$$\frac{\partial V_{T-1}^*}{\partial x_{T-1o}^*} = \frac{(1-un)Ea_{T-1}L(X_{T-1} + x_{T-1n}^*)^{-n} - c}{1+r} = \lambda_{T-1}^*$$

$$\frac{\partial V_{T-1}^*}{\partial x_{T-1n}^*} = \frac{(1-un)Ea_{T-1}L(X_{T-1} + x_{T-1n}^*)^{-n} - c}{1+r} - k + cEQ_{T-1}\frac{\tilde{\lambda}_T^*}{1+r} = 0, \text{ or}$$

$$x_{T-1}^* = X_{T-1} + x_{T-1n}^* = \left[\frac{Ea_{T-1}L(1-un)}{c + k(1+r) - cEQ_{T-1}\tilde{\lambda}_T^*}\right]^{1/n}$$

$$\lambda_{T-1}^* = k - cEQ_{T-1}\frac{\tilde{\lambda}_T^*}{1+r}.$$

This solution is optimal for $x_{T-1n}{}^* > 0$, or

$$Ea_{T-1} > X_{T-1}^{n} \left[ \frac{c + k(1+r) - cEQ_{T-1}(Ea_{T-1})\lambda_T^*}{L(1-un)} \right]$$

or $Ea_{T-1} > A_{2_{T-1}}$.

Examination of the limits $A_{1_{T-1}}$, $A_{2_{T-1}}$ in (17) and (19) reveals that, although $A_{1_{T-1}}$ is of the same form as $A_{1T}$, the equation for $A_{2_{T-1}}$ involves the subtraction of a positive term on the right hand side of the upper bound inequality. Thus, the point $Ea_{T-1} = A_{2_{T-1}}$ for which the optimal solution is to add capacity is less than the comparable point a period later, $Ea_T = A_{2T}$. The reason can be seen by examination of the criterion for adding capacity $X_{T-1}$ at T-1. For $Ea_{T-1} \ngtr A_{2_{T-1}}$ the shadow price $\lambda_{T-1}{}^*$, the current value of a marginal unit of previously purchased capacity, is less than k, the external price of purchasing the marginal unit, by the factor $CEQ_{T-1}\frac{\tilde{\lambda}_T{}^*}{1+r}$, the current value of the contribution that the marginal unit purchased now will make in the next period, when it will increase the capacity $X_T$ available then. In other words, at $Ea_{T-1} \ngtr A_{2_{T-1}}$, the total current value of net marginal revenue to be received both this period and next is greater than or equal to the external price of another unit of capacity if the firm operates only at $x_{T-1} = X_{T-1}$. Thus the firm will add new capacity this period even though the current value of net marginal revenue this period is less than the cost k. The firm derives the extra benefit from the future optimal operating decisions in T. Note that, if the firm were certain that it would not operate at capacity in the future period, then $\lambda_T = 0$ and there would, of course, be no benefit from next period. If the value $Ea_{T-1}$ were low enough so that the majority of the distribution of $Ea_T$ were below the lower bound for full use of capacity in T, $A_{1T}$, the current value of use of full capacity $CEQ_{T-1}\frac{\tilde{\lambda}_T{}^*}{1+r}$ would be correspondingly small.

In general, then, the difference between $A_{2_t}$ and $A_{2_{t-1}}$, the points at which the firm adds capacity in t and in t-1, lies in the fact that, at t-1, the firm has one more period left in which to cover the cost of the capacity than it has in t. The longer period case will yield more revenue than the other since each period over which the firm has possession of the capacity is, in general, expected to add a non-negative revenue. Thus, there is a "horizon effect" obscuring the relation between previously purchased capacity and its current value. This "horizon effect" is the result of the assumption that capacity lasts, undeteriorated, until the firm shuts down at T. This assumption will be removed in a later section, to see more clearly the origin of the non-negative value of current capacity.

The value of the firm at T-1, $V_{T-1}^*$, is computed by substitution of the optimal decisions in (17), (18), and (19) into (14):

Case 1:
$$V_{T-1}^* = \frac{E a_{T-1} L (X_{T-1}^*)^{1-n} - c X_{T-1}^*}{1+r} + CEQ_{T-1} \frac{\tilde{V}_T^*}{1+r}$$

$$= \frac{un\, E a_{T-1} L (X_{T-1}^*)^{1-n}}{1+r} + CEQ_{T-1} \frac{\tilde{\lambda}_T^* X_T}{1+r} + CEQ_{T-1} \frac{\tilde{M}_T^*}{1+r}$$

$$= CEQ_{T-1} \frac{\tilde{\lambda}_T^*}{1+r} X_{T-1} + \frac{un\, E a_{T-1} L (X_{T-1}^*)^{1-n}}{1+r} + CEQ_{T-1} \frac{\tilde{M}_T^*}{1+r},$$

since $X_T = X_{T-1}$ in this case.

Case 2:
$$V_{T-1}^* = \frac{E a_{T-1} L X_{T-1}^{1-n} - c X_{T-1}}{1+r} + CEQ_{T-1} \frac{\tilde{V}_T^*}{1+r}$$

$$= \frac{(1-un) E a_{T-1} L X_{T-1}^{1-n} - c X_{T-1}}{1+r} + \frac{un\, E a_{T-1} L X_{T-1}^{1-n}}{1+r} + CEQ_{T-1} \frac{\tilde{\lambda}_T^* X_T}{1+r} + CEQ_{T-1} \frac{\tilde{M}_T^*}{1+r}$$

$$= \left[\lambda_{T-1}^* + CEQ_{T-1} \frac{\tilde{\lambda}_T^*}{1+r}\right] X_{T-1} + \frac{un\, E a_{T-1} L X_{T-1}^{1-n}}{1+r} + CEQ_{T-1} \frac{\tilde{M}_T^*}{1+r},$$

since $X_T = X_{T-1}$ in this case.

Case 3: $$V_{T-1}^* = \frac{E a_{T-1} L (X_{T-1} + x_{T-1n}^*)^{1-n} - c(X_{T-1} + x_{T-1n}^*)}{1+r} - k x_{T-1n}^* + CEQ_{T-1} \frac{\tilde{V}_T^*}{1+r}$$

$$= \left[k - CEQ_{T-1} \frac{\tilde{\lambda}_T^*}{1+r}\right]\left[X_{T-1} + x_{T-1n}^*\right] + \frac{un\, E a_{T-1} L (x_{T-1}^*)^{1-n}}{1+r}$$

$$- k x_{T-1n}^* + CEQ_{T-1} \frac{\lambda_T^* [X_T + x_{T-1n}^*]}{1+r} + CEQ_{T-1} \frac{\tilde{M}_T^*}{1+r}$$

since $X_T = X_{T-1} + x_{T-1n}^*$ in this case. Then

$$V_{T-1}^* = k\, X_{T-1} + \frac{un\, E a_{T-1} L (x_{T-1}^*)^{1-n}}{1+r} + CEQ_{T-1} \frac{\tilde{M}_T^*}{1+r}.$$

Thus, in general, if $M_{T-1}^*$ is defined to be = $\frac{un\, E a_{T-1} L (x_{T-1}^*)^{1-n}}{1+r} + CEQ_{T-1} \frac{\tilde{M}_T^*}{1+r}$

$$V_{T-1}^* = \left[\lambda_{T-1}^* + CEQ_{T-1} \frac{\tilde{\lambda}_T^*}{1+r}\right] X_{T-1} + M_{T-1}^* \qquad (20)$$

in which $M_{T-1}^*$ is again, as in T, the current capitalized value of the profits arising in both future periods from the firm's ability to price non-competitively, since it is the current value of the difference between marginal and average net revenue multiplied by production in both future periods. The first term in (20) is, as in the formulation at T, the current shadow price, plus the current value of the future shadow price, of the capacity $X_{T-1}$ multiplied by the capacity. Note that the future shadow price multiplies current capacity $X_{T-1}$ since an increase in previously purchased capacity $X_{T-1}$ will increase $X_T$ and will be of value in the future if the firm operates at $X_T$ in T. As in T, the current shadow price is zero if $x_{T-1}^* < X_{T-1}$, is internally determined in case 2, and is the external price k less the future value of current capacity if the firm adds capacity currently. For the latter to occur, the total current value of an additional unit of capacity at T-1 must equal the external purchase price k. Again, as in T, this current value $\lambda_{T-1}^* + CEQ_{T-1} \frac{\tilde{\lambda}_T^*}{1+r}$ is dependent on the firm's monopoly power u, since a firm with monopoly power will derive greater value out of an additional unit of current capacity.

<u>The Valuation at t</u>

This pattern of optimal production and division of the value of the firm into two parts is repeated as backward induction continues, since the value $V_t$ which the firm maximizes by its production decisions will always consist of current value of net revenue expected at the end of the current period t plus current value of the claims on future income $\tilde{V}_{t+1}^*$. The current optimal value $V_t^*$ will thus always consist of a term representing the current value of previously purchased capacity $X_t$ (which will be the sum of current values of shadow prices of an unit of capacity available in future periods by an increase in $X_t$) plus the sum of current capitalized values of monopoly rents that the firm will receive in future periods. For example, in T-2, the problem is:

$$\max_{(x_{T-2o},\, x_{T-2n})} V_{T-2} = \frac{Ea_{T-2} L (x_{T-2o} + x_{T-2n})^{1-n} - c\,(x_{T-2o} + x_{T-2n})}{1+r} - k x_{T-2n}$$

$$+ \; CEQ_{T-2} \frac{\tilde{V}_{T-1}^{*}}{1+r}$$

This solution is optimal for $A_{1_{T-2}} < Ea_{T-2} < A_{2_{T-2}}$, determined as the upper bound to the inequality.

$$Ea_{T-2} \leq X_{T-2}^{\,n} \left[ \frac{c + k(1+r) - CEQ_{T-2}\left(\tilde{\lambda}_{T-1}^{*} + \widetilde{CEQ}_{T-1} \frac{\lambda_T^{*}}{1+r}\right)}{L(1-un)} \right]$$

in which the certainty equivalent as of T-2 is a function of $Ea_{T-2}$ through the specification of the distribution of $Ea_{T-1}$ on which both $\tilde{\lambda}_{T-1}{}^{*}$ and $\widetilde{CEQ}_{T-1} \frac{\lambda T^{*}}{1+r}$ depend. ($A_{1_{T-2}} < A_{2_{T-2}}$ since $\tilde{\lambda}_{T-1} \leq k - CEQ_{T-1} \frac{\tilde{\lambda} T^{*}}{1+r}$.)

Case 3: $x_{T-2o}{}^{*} = X_{T-2}$, $x_{T-2n}{}^{*} > 0$

$$\frac{(1-un)\, Ea_{T-2} L (x_{T-2}^{*})^{-n} - c}{1+r} = \lambda_{T-2}^{*}$$

$$\frac{(1-un)\, Ea_{T-2} L (x_{T-2}^{*})^{-n} - c}{1+r} = k - CEQ_{T-2} \left[ \frac{\tilde{\lambda}_{T-1}^{*}}{1+r} + \frac{CEQ_{T-1} \tilde{\lambda}_T^{*}}{(1+r)^2} \right]$$

$$x_{T-2}^{*} = \left[ \frac{Ea_{T-2} L (1-un)}{k(1+r) + c - CEQ_{T-2} \left( \tilde{\lambda}_{T-1}^{*} + \frac{\widetilde{CEQ}_{T-1} \lambda_T^{*}}{1+r} \right)} \right]^{1/n}$$

which is optimal for $Ea_{T-2} > A_{2_{T-2}}$.

The value of the firm at optimum, $V_{T-2}^*$, may be derived as before.

Case 1: $$V_{T-2}^* = CEQ_{T-2}\left[\frac{\lambda_{T-1}^*}{1+r} + \frac{CEQ_{T-1}\tilde{\lambda}_T^*}{(1+r)^2}\right]X_{T-2} + \frac{un\,E\lambda_{T-2}L(x_{T-2}^*)^{1-n}}{1+r} + CEQ_{T-2}\frac{\tilde{M}_{T-1}^*}{1+r}$$

Case 2: $$V_{T-2}^* = \left[\lambda_{T-2}^* + CEQ_{T-2}\left(\frac{\tilde{\lambda}_{T-1}^*}{1+r} + \frac{\tilde{CEQ}_{T-1}\lambda_T^*}{(1+r)^2}\right)\right]X_{T-2} + \frac{un\,E\lambda_{T-2}L(x_{T-2}^*)^{1-n}}{1+r} + CEQ_{T-2}\frac{\tilde{M}_{T-1}^*}{1+r}$$

Case 3: $$V_{T-2}^* = kX_{T-2} + \frac{un\,E\lambda_{T-2}L(x_{T-2}^*)^{1-n}}{1+r} + CEQ_{T-2}\frac{\tilde{M}_{T-1}^*}{1+r},$$

using substitutions similar to those used to derive $V_{T-1}^*$. Thus in general

$$V_{T-2}^* = \left[\lambda_{T-2}^* + CEQ_{T-2}\left(\frac{\tilde{\lambda}_{T-1}^*}{1+r} + \frac{\tilde{CEQ}_{T-1}\lambda_T^*}{(1+r)^2}\right)\right]X_{T-2} + M_{T-2}^* \qquad (21)$$

where $M_{T-2}^* = \dfrac{un\,E\lambda_{T-2}L(x_{T-2}^*)^{1-n}}{1+r} + CEQ_{T-2}\dfrac{\tilde{M}_{T-1}^*}{1+r}$.

As in (20), $M_{T-2}^*$ is the current capitalized value of monopoly rents over the current and future periods, and $\lambda_{T-2}^*$ + $CEQ_{T-2}\dfrac{\tilde{\lambda}_{T-1}^*}{1+r} + CEQ_{T-2}\tilde{CEQ}_{T-1}\dfrac{\lambda_T^*}{(1+r)^2}$ is the current value of a unit of previously purchased capacity to be used currently and in future periods if the firm operates at capacity in the current and future periods. Note the deduction for inter-period risk implicit in the form $CEQ_{T-2}\tilde{CEQ}_{T-1}\lambda_T^*$ representing the current value of present capacity two periods hence. This term would reduce to an intraperiod (single) certainty equivalent if there were no uncertainty as to the interperiod change in demand.

For any period t, therefore, the optimal production decisions are:

Case 1: $Ea_t < A_{1_t} = \dfrac{cX_t^{\,n}}{L(1-un)}$ (22)

$$\lambda_t = 0$$

$$X_{t_0}^* = \left[\frac{E\lambda_t L(1-un)}{c}\right]^{1/n} < X_t$$

$$X_{t_n}^* = 0$$

Case 2: $A_{1_t} < Ea_t < A_{2_t}$ where $A_{2_t}$ is the upper bound to the inequality

$$E\lambda_t < X_t^n \left[\frac{k(1+r)+c-\left(\sum_{i=1}^{T-t}\frac{C_{t+i-1}\tilde{\lambda}_{t+i}^*}{(1+r)^{i-1}}\right)}{L(1-un)}\right] \tag{23}$$

in which the summation

$$\sum_{i=1}^{T-t}\frac{C_{t+i-1}\tilde{\lambda}_{t+i}^*}{(1+r)^{i-1}} = CEQ_t\,\tilde{\lambda}_{t+1}^* + CEQ_t\,\frac{CEQ_{t+1}\tilde{\lambda}_{t+2}^*}{1+r} + CEQ_t\,\frac{\tilde{CEQ}_{t+1}\,CEQ_{t+2}\,\lambda_{t+3}^*}{(1+r)^2} + \cdots + \frac{CEQ_t\cdots\tilde{CEQ}_{t+i-1}\,\lambda_{t+i}^*}{(1+r)^{i-1}} + \cdots + \frac{CEQ_t\cdots CEQ_{T-1}\,\tilde{\lambda}_T^*}{(1+r)^{T-t-1}}$$

$$X_{t_0}^* = X_t$$

$$X_{t_n}^* = 0$$

$$\lambda_t^* = \frac{(1-un)\,E\lambda_t L\,(X_t)^{-n} - c}{1+r}$$

and $A_{1_t} < A_{2_t}$ since $\lambda_{t+1} \leq k - \sum_{i=2}^{T-t} \frac{c_{t+i-1}\,\tilde{\lambda}^*_{t+i}}{(1+r)^{i-1}}$

Case 3: $Ea_t > A_{2_t}$ (24)

$$x_{t0}^* = X_t$$

$$x_t^* = X_t + x_{tn}^* = \left[ \frac{Ea_t L(1-un)}{k(1+r)+c - \sum_{i=2}^{T-t} \frac{c_{t+i-1}\,\tilde{\lambda}^*_{t+i}}{(1+r)^{i-1}}} \right]^{1/n}$$

$$\lambda_t^* = \frac{(1-un)Ea_t L(x_t^*)^{-n} - c}{1+r} = k - \sum_{i=2}^{T-t} \frac{c_{t+i-1}\,\tilde{\lambda}^*_{t+i}}{(1+r)^{i-1}}$$

The value of the firm may be written

$$V_t^* = \left[\lambda_t^* + \sum_{i=2}^{T-t} \frac{c_{t+i-1}\,\tilde{\lambda}^*_{t+i}}{(1+r)^{i-1}}\right] X_t + M_t^* \qquad (25)$$

where $M_t^* = \dfrac{un\,Ea_t L(x_t^*)^{1-n}}{1+r} + \text{CEQ}\,\dfrac{\tilde{M}^*_{t+1}}{1+r}$

$$= \frac{un\,Ea_t L(x_t^*)^{1-n}}{1+r} + \sum_{i=1}^{T-t} c_{t+i-1} \frac{\left[un\,Ea_{t+i} L(x^*_{t+i})^{1-n}\right]}{(1+r)^{i+1}}.$$

The first term in (25) is the value of current capacity $X_t$, evaluated at current values of current and future shadow prices of an unit of additional capacity which the increase of an unit of $X_t$ would represent in current and future periods. The second term is the current capitalized value of future monopoly rents.

To complete the statement of value for all time periods for this firm, one should examine the situation at entry, the beginning of the first period for the firm, so that t = 1, and $X_1 = 0$. In this case $x_{1_o} = 0$ and drops out of the maximization problem, thus only case 3 applies. The firm enters when $Ea_1 > A_{2_1}$, where $A_{2_1}$ in this case is any value greater than 0 since $X_1 = 0$. The firm produces at that point at which $V_1$ is maximized: $\frac{\partial V_1}{\partial X_{1_n}} = 0$, or

$$X_1^* = X_{1n}^* = \left[ \frac{Ea_1 L(1-un)}{k(1+r)+c - \sum_{i=1}^{T-1} C_{1+i+1} \frac{\tilde{\lambda}_{1+i}^*}{(1+r)^{i-1}}} \right]^{1/n}$$

There is no shadow price $\lambda_1$ since no variable is constrained. The value of the firm at t = 1, $V_1^*$, is

$$V_1^* = \frac{Ea_1 L(X_1^*)^{1-n} - cX_1^*}{1+r} - kX_1^* + CEQ_1 \frac{V_2^*(X_2)}{1+r}$$

$$= X_1^* \frac{[(1-un) Ea_1 L(X_1^*)^{-n} - c]}{1+r} + \frac{un\, Ea_1 (X_1^*)^{1-n}}{1+r} - kX_1^*$$

$$+ \frac{CEQ_1}{1+r} \left[ \left( \tilde{\lambda}_2^* + \sum_{i=1}^{T-2} \frac{C_{2+i-1} \tilde{\lambda}_{2+i}^*}{(1+r)^i} \right) X_2 + \tilde{M}_2^* \right]$$

$$= X_1^* \left[ \frac{(1-un) Ea_1 L(X_1^*)^{-n} - c}{1+r} - k + \sum_{i=1}^{T-1} \frac{C_{2+i-1} \lambda_{2+i}}{(1+r)^i} \right]$$

$$+ \frac{un\, Ea_1 L(X_1^*)^{1-n}}{1+r} + CEQ_1 \frac{\tilde{M}_2^*}{1+r}$$

$$= \frac{un\, Ea_1 L(X_1^*)^{1-n}}{1+r} + CEQ_1 \frac{\tilde{M}_2^*}{1+r}$$

$$= M_1^*$$

as predicted by the general formula (25). Thus, at entry, the firm's net present value consists only of the current value of future monopoly rents, since it has no capacity at entry (by definition of entry period). If the firm has no monopoly power, it has no net present value if it enters, a familiar result of pure competition pricing.

<u>The Valuation at t of a Firm without Monopoly Power (u = 0)</u>

At a point in time t  1, using (25)

$$V_t^* = [\lambda_t^* + \sum_{i=1}^{T-t} \frac{C_{t+i-1}\tilde{\lambda}_{t+i}^*}{(1+r)^i}] X_t + M_t^*$$

If u = 0,

$$V_t^* = [\lambda_t^*(u=0) + \sum_{i=1}^{T-t} \frac{C_{t+i-1}\tilde{\lambda}_{t+i}^*(u=0)}{(1+r)^i}] X_t(u=0)$$

where $X_t(u=0)$ is the sum of previous investments of the firm with no monopoly power, and $\lambda_{t+i}^*(u = 0)$ are the shadow prices of the capacity constraint for this firm. The values of the shadow prices are derived from backward induction in the special case u = 0, beginning with the result (13), $V_T$ (u = 0) = $\lambda_T^*(u = 0) X_T$. In T-1, the optimization problem is

$$\max_{(x_{T-1o}, x_{T-1n})} \frac{[CEQ\, \tilde{P}_{T-1} - C][x_{T-1o} + x_{T-1n}]}{1+r} - k x_{T-1n} + CEQ_{T-1} \frac{\tilde{V}_T^*}{1+r}$$

with CEQ $\tilde{P}_{T-1} = Ea_{T-1} L\, F[x_{T-1}$ (industry)$]$

subject to $x_{T-1_o} < X_{T-1}$

and the identity $X_T = X_{T-1} + x_{T-1_n}$

The optimization conditions are

$$x_{T-1o}^{*}\left[\frac{Ea_{T-1}LF[x_{T-1}(\text{industry})]-c}{1+r}-\lambda_{T-1}^{*}\right]=0$$

$$x_{T-1n}^{*}\left[\frac{Ea_{T-1}LF[x_{T-1}(\text{industry})]-c}{1+r}-k+CEQ_{T-1}\frac{\tilde{\lambda}_T^{*}}{1+r}\right]=0$$

$$\lambda_{T-1}^{*}\left[x_{T-1o}^{*}-X_{T-1}\right]=0$$

since $\frac{\partial CEQ\tilde{V}_T^{*}}{\partial x_{T-1n}}=\frac{\partial CEQ\tilde{V}_T^{*}}{\partial X_{T}}=CEQ_{T-1}\tilde{\lambda}_T^{*}$ as before.

The solutions are:

Case 1: $0 < x_{T-1_o}{}^{*} < X_{T-1}$.

Then $\frac{Ea_{T-1}LF[x_{T-1}(\text{industry})]-c}{1+r}-\lambda_{T-1}^{*}=0$, or

$$Ea_{T-1}=\frac{c}{LF[x_{T-1}(\text{industry})]},$$

the threshold of case 2.

Case 2: $x_{T-1_o}{}^{*} = X_{T-1}$, $x_{T-1_n}{}^{*} = 0$

Then $0<\frac{Ea_{T-1}LF[x_{T-1}(\text{industry})]-c}{1+r}=\lambda_{T-1}^{*}$,

$$\lambda_{T-1}^{*}<k-\frac{CEQ_{T-1}\tilde{\lambda}_T^{*}}{1+r}$$

This solution is optimal for $\frac{c}{L\ F\ [x_{T-1}\ (\text{industry})]}\leq Ea_{T-1}\leq A_{2_{T-1}}$

where $A_{2_{T-1}}$ is the upper bound to the inequality

$$Ea_{T-1}<\frac{k(1+r)+c-CEQ_{T-1}\tilde{\lambda}_T^{*}}{LF[x_{T-1}(\text{industry})]}$$

and $CEQ_{T-1}\lambda^{*}_{T}$ depends on $Ea_{T-1}$.

Case 3: $x_{T-1_o}{}^{*} = X_{T-1}$, $x_{T-1_n}{}^{*} > 0$

Then $\dfrac{Ea_{T-1}LF[x_{T-1}(\text{industry})]-c}{1+r} = \lambda_{T-1}^* = k - CEQ_{T-1}\dfrac{\tilde{\lambda}_T^*}{1+r}$

and $x_{T-1_n}^*$ is unspecified. Thus, as in the single period case the firm will elect not to produce unless $Ea_{T-1} > \dfrac{c}{LF[x_{T-1}(\text{industry})]}$, will produce at capacity within the range $\dfrac{c}{LF[x_{T-1}(\text{industry})]} < Ea_{T-1} < A_{2_{T-1}}$, and will add capacity for $Ea_{T-1} > A_{2_{T-1}}$, <u>assuming</u> it is able to remain "a member of the industry" if it does not produce for this period. That, it must be able to regain full use of its completely idle capacity $X_{T-1}$ at the beginning of the next decision period T, with no start-up costs. The same comments apply to industry-wide production in each period t as were made in the single period case.

The value of the firm at T-1 is, for $Ea_{T-1} < \dfrac{c}{LF[x_{T-1}(\text{industry})]}$,

$$V_{T-1}^* = CEQ_{T-1}\ \frac{\tilde{V}_T^*}{1+r} \quad \text{since it does not produce.}$$

For $\dfrac{c}{LF[x_{T-1}(\text{industry})]} < Ea_{T-1} < A_{2_{T-1}}$

$$V_{T-1}^* = \frac{[Ea_{T-1}LF[x_{T-1}(\text{industry})]-c]\,X_{T-1}}{1+r} + CEQ_{T-1}\frac{\tilde{\lambda}_T^*X_T}{1+r}$$

$$= \left[\lambda_{T-1}^* + CEQ_{T-1}\frac{\tilde{\lambda}_T^*}{1+r}\right]X_{T-1} \qquad \text{since } X_T = X_{T-1}$$

For $Ea_{T-1} > A_{2_{T-1}}$,

$$V_{T-1}^* = \left[\frac{Ea_{T-1}LF[x_{T-1}(\text{industry})]-c}{1+r}\right](X_{T-1} + x_{T-1_n}^*)$$

$$- k x_{T-1n}^{*} + \frac{c E Q_{T-1} \lambda_T^{*} (X_{T-1} + x_{T-1n}^{*})}{1+r}$$

since $X_T = X_{T-1} + x_{T-1_n}^{*}$

In general $V_{T-1}^{*} = \left[ \lambda_{T-1}^{*} + c E Q_{T-1} \frac{\tilde{\lambda}_T^{*}}{1+r} \right] X_{T-1}$ . (26)

The pattern of this derivation will be repeated for $V_{T-2}, \ldots, V_t$, so that a generalization of (25), the single period case, will apply here. The shadow price $\lambda^{*}_{t+1}$ if the firm produces in t + i will be the certainty equivalent of the price each period less the cost discounted by (1 + r) for $Ea_{t+1} < A_{2_{t+1}}$, will be zero if it does not produce, and will equal k - the discounted certainty equivalents of future shadow prices for $Ea_{t+1} > A_{2_{t+i}}$. The limits $A_{2_{t+i}}$ will fall as the calculation moves backward in time. The lower limits below which the firm will not produce are $Ea_{t+1} < \frac{c}{LF[x_{t+1}(\text{industry})]}$.

If this firm now is compared to an entrant, for which $X_1 = 0$, a result similar to that of the single period case emerges. This firm will produce at capacity for the range $\frac{c}{LF[x_t(\text{industry})]} < Ea_t < A_{2_t}$ and will have a positive net present value $V_t = \left[\lambda_t^{*} + \sum_{i=o}^{T-t-1} \frac{c_{t+i} \tilde{\lambda}^{*}_{t+i+1}}{(1+r)^{1+i}}\right] X_t$. The firm for which $X_t = 0$ will not enter until $Ea_t = A_{2_t}$; i.e., until the current value of all future receipts from purchasing capacity equals its cost. Thus there is a barrier to the entry of the new firm within this range arising out of the barrier to the exit of the old firm.

The question arises as to the firm's behavior when the optimum is not to produce but the firm will then lose all future access to its current capacity $X_t$ if it does not produce. In this case its optimization problem is

$$\max_{(x_{to},\, x_{tn})} V_t = \left[ \frac{Ea_t LF[X_t(\text{industry})] - c}{1+r} \right] (x_{to} + x_{tn}) - k x_{tn} + \frac{CEQ_t \tilde{V}^*_{t+1}}{1+r} D(x_{to})$$

subject to $x_{to} \leq X_t$

$$X_{t+1} = X_t + x_{tn}$$

$$\text{and } D(x_{to}) = \begin{cases} 0 & \text{if } x_{to} = 0 \\ 1 & \text{if } x_{to} > 0 \end{cases}$$

since $V_{t+1} = 0$ if the firm has no capacity in t + 1 and wishes to re-enter. The optimization conditions are

$$x_{to}^* \left[ \frac{Ea_t LF[X_t(\text{industry})] - c}{1+r} + \frac{CEQ_t \tilde{V}^*_{t+1}}{1+r} \delta(x_{to}^*) - \lambda_t^* \right] = 0$$

$$x_{tn}^* \left[ \frac{Ea_t LF[X_t(\text{industry})] - c}{1+r} - k + D(x_{to}^*) \sum_{i=0}^{T-t-1} \frac{C_{t+i}\, \lambda_{t+i+1}}{(1+r)^{1+i}} \right] = 0$$

$$\lambda_t^* \left[ x_{to}^* - X_t \right] = 0$$

where $\delta(x_{to})$ is the Kronecker $\delta$ function of $x_{to}$. Thus $x_{to}^*$ will begin to rise above zero when

$$\frac{Ea_t LF[X_t(\text{industry})] - c}{1+r} + \frac{CEQ_t \tilde{V}^*_{t+1}}{1+r} = 0$$

i.e., when $Ea_t$ is such that the current value of claims to $V_{t+1}$ exceeds the current value of marginal net losses expected from operations this period. This value of $Ea_t$ may be rewritten as the lower bound to the inequality

$$Ea_t < \frac{c - CEQ_t \tilde{V}^*_{t+1}}{LF[x_t(\text{industry})]}$$

for purposes of comparison with the firm's behavior in the previous case when it lost nothing be electing not to produce in the current period. Here the firm will produce, at a current loss, so long as the value of the current loss does not exceed the current value of future production in the industry. Because of the particular situation faced by this firm--that it will not lose future benefits as long as it produces anything, no matter how small--it will minimize its current losses by producing as little as possible. Thus $x_{to}*$ will be very small unless $Ea_t > \frac{c}{LF[x_t(\text{industry})]}$, at which point the solution derived previously becomes optimal.

## C. The Influence of Interperiod Risk on the Capacity Decision

### The "Horizon" Problem

The assumption on the longevity and non-saleability of acquired capital yields the result that the "trigger point" $A_{2_t}$ for investing in new capacity is lower as t moves backward from the horizon T, i.e., that the cost of capital is falling. The capacity lasts until the horizon, after which point it becomes worthless. Thus, the acquisition of a unit of capacity closer to the horizon will occur at a higher "trigger point" or, equivalently, the return to capital required of that unit will be higher.

This effect would be made negligible for t very far from the horizon under an assumption that there is little interperiod uncertainty resolution. Then the extra term which appears in the $A_{2_t}$ expression (23) as compared to $A_{2_{t+1}}$ will be negligible and the $CEQ_t$ of terms known in t+1 but random as of t will not be much less than their expected value in t.

$$A_{2_t} = X_t{}^n \left[ k - \sum_{0}^{T-t} C_{t+i} \frac{\tilde{\lambda}^*_{t+i+1}}{(1+r)^{1+i}} \right]$$

and

$$\lambda_t^* = \frac{(1-un)Ed_t L X_T^{*-n} - c}{1+r} = k - \sum_{0}^{T-t} C_{t+i} \frac{\tilde{\lambda}^*_{t+i+1}}{(1+r)^{1+i}}$$

and at t+1, substituting into (23)

$$A_{2_{t+1}} = (X_t + x_{t_n}^*)^n \left[ k - \sum_{0}^{T-t-1} C_{t+i+1} \frac{\tilde{\lambda}^*_{t+i+2}}{(1+r)^{1+i}} \right]$$

$$\lambda_{t+1}^* = k - \sum_{0}^{T-t-1} C_{t+i+1} \frac{\tilde{\lambda}^*_{t+i+2}}{(1+r)^{1+i}} .$$

The trigger points $A_{2_t}$ and $A_{2+1}$ approach each other as each term in the sum becomes small and as $CEQ_t(Ct+i+1)$ approaches $CEQ_{t+i+1}$; i.e., as the amount of uncertainty resolved at t becomes small. Then, also, $\lambda_t^* \cong \lambda_{t+1}^*$ and the marginal value of the additional capacity, thus also the implicit cost per period of the capital to purchase it, is the same in t and t+1.

Since the cost of capital, or the value of the marginal unit, is the same in every period (as long as these required conditions hold), it must follow that the possession at t of an additional unit of capacity acquired before t cannot confer value to the firm above that conferred by the acquisition at t of this unit. This can be seen mathematically by taking the preceding discussion on trigger points $A_{2_t}$ back to t=1, the entry of the firm. There, the marginal value of new capacity

$$\lambda_1 = k - \sum_{0}^{T-1} C_{1+i} \frac{\lambda_{1+i+1}}{(1+r)^{i+1}}$$

$> 0$ for $Ea_1$ above the trigger point

$= 0$ for $Ea_1 \leq$ the trigger point.

Since all trigger points are the same from t=1 onward, $t \ll T$, all $\lambda_t$ are the same; thus the firm which entered at t=1 will add new capacity in any t as soon as it reaches its capacity limit. There will be no range of $Ea_t$ such that the firm will operate at capacity without adding new capacity, no range for which

$$0 < \lambda_t < k - \sum_{0}^{T-t} C_{t+i} \frac{\tilde{\lambda}^*_{t+i+1}}{(1+r)^{i+1}} ,$$

thus no barrier to entry of new firms.

The Isolation of Uncertainty Resolution

The effect of uncertainty resolution may be examined most easily without extra assumptions on the rate of uncertainty resolution i.e., if the supposition on the longevity of capital is changed to one in which the amount of uncertainty to be resolved obviously changes from one period to the next. This occurs at the end of its life. Thus, specifically, assume the capacity lasts two period (with no change in efficiency during these periods), then becomes unproductive. The capacity available to the firm for production at t,

$$X_t = x_{t-1_n},$$

the capacity purchased last period which still cannot be sold. (Note that the capacity must last at least two periods; otherwise, the link between this period's decisions and next period's value is broken in the present case, in which the market environment is not affected by the firm's production decisions.)

The value-maximizing decisions atT are reached as before,

at

$$x_{To}^* \left[ \frac{\partial V_T^*}{\partial x_{To}^*} - \lambda_T \right] = 0$$

$$x_{Tn}^* \left[ \frac{\partial V_T^*}{\partial x_{Tn}^*} \right] = 0$$

$$\lambda_T^* ( x_{To}^* - X_T ) = 0$$

$$\frac{\partial V_T^*}{\partial x_{To}^*} = \frac{(1-un) E a_T L ( x_{To}^* + x_{Tn}^*)^{-n} - c}{1+r}$$

$$\frac{\partial V_T^*}{\partial x_{Tn}^*} = \frac{(1-un) E a_T L ( x_{To}^* + x_{Tn}^*)^{-n} - c}{1+r} - k$$

and from these conditions, $V_T^* = \lambda_T^* X_T + M_T^*$, where

$$\lambda_T^* = 0, \quad Ea_T < \frac{cX_T^n}{L(1-un)}$$

$$= \frac{(1-un)Ea_T L X_T^{-n} - c}{1+r}, \quad \frac{cX_T^n}{L(1-un)} < Ea_T < \frac{[c+k(1+r)]X_T^n}{L(1-un)}$$

$$= k, \quad Ea_T > \frac{[c+k(1+r)]X_T^n}{L(1-un)}$$

$$X_T = x_{T-1n}^*.$$

(This solution is, of course, the same as the previous one period solution (7).)

The T-1 decisions are again derived similarly:

$$x_{T-1o}^*\left[\frac{\partial V_{T-1}^*}{\partial x_{T-1o}^*} - \lambda_{T-1}^*\right] = 0$$

$$x_{T-1n}^* \frac{\partial V_{T-1}^*}{\partial x_{T-1n}^*} = 0$$

$$\lambda_{T-1}^*(x_{T-1o}^* - X_{T-1}) = 0$$

$$X_{T-1} = x_{T-2n}.$$

$$\frac{\partial V_{T-1}^*}{\partial x_{T-1o}^*} = \frac{(1-un)Ea_{T-1}L(x_{T-1o}^* + x_{T-1n}^*)^{-n} - c}{1+r} \tag{27a}$$

$$\frac{\partial V_{T-1}^*}{\partial x_{T-1n}^*} = \frac{(1-un)Ea_{T-1}L(x_{T-1o}^* + x_{T-1n}^*)^{-n} - c}{1+r} - k + CEQ_{T-1}\frac{\tilde{\lambda}_T^*}{1+r} \tag{27b}$$

Since

$$V_{T-1}^* = \frac{Ea_{T-1}L(x_{T-1o}^* + x_{T-1n})^{1-n} - c(x_{T-1o} + x_{T-1n})}{1+r} + CEQ_{T-1}\frac{V_T^*}{1+r} - kx_{T-1n}^*,$$

for $Ea_{T-1} < \frac{cX_{T-1}^n}{L(1-un)} = A_{1T-1}$: $x_{T-1n}^* = 0$, $x_{T-1o}^* < X_{T-1}$

$$\lambda_{T-1}^* = 0$$

$$V_{T-1}^* = \frac{un\,Ea_{T-1}L(x_{T-1o}^*)^{1-n}}{1+r} + CEQ_{T-1}\frac{\tilde{\lambda}_T^* X_T + \tilde{M}_T^*}{1+r}$$

$$= M_{T-1}^*.$$

For $A_{1_{T-1}} < Ea_{T-1} < \left[\frac{c + k(1+r) - CEQ_{T-1}\tilde{\lambda}_T^*}{L(1-un)}\right] X_{T-1}^n = A_{2_{T-1}}: x_{T-1n}^* = 0, x_{T-1o} = X_{T-1}$

$$\lambda_{T-1}^* = \frac{(1-un)\,Ea_{T-1}LX_{T-1}^{-n} - c}{1+r} < k - CEQ_{T-1}\frac{\tilde{\lambda}_T^*}{1+r}$$

$$V_{T-1}^* = \lambda_{T-1}^* x_{T-2n}^* + \frac{un\,Ea_{T-1}L(x_{T-2n}^*)^{1-n}}{1+r} + CEQ_{T-1}\frac{\tilde{\lambda}_T^* x_{T-1n}^* + \tilde{M}_T^*}{1+r}$$

(since $X_{T-1} = x_{T-2_n}^*$), or

$$V_{T-1}^* = \lambda_{T-1}^* x_{T-2n}^* + M_{T-1}^*.$$

For $Ea_{T-1} > A_{2_{T-1}}: x_{T-1n}^* > 0, \; x_{T-1o}^* = X_{T-1}$

$$\lambda_{T-1}^* = k - CEQ_{T-1}\frac{\tilde{\lambda}_T^*}{1+r}$$

$$V_{T-1}^* = \frac{[(1-un)\,Ea_{T-1}L(x_{T-2n}^* + x_{T-1n}^*)^{-n} - c](x_{T-2n}^* + x_{T-1n}^*)}{1+r} + CEQ_{T-1}\frac{\tilde{\lambda}_T^* x_{T-1n}^* + \tilde{M}_T^*}{1+r} + \frac{un\,Ea_{T-1}L(x_{T-2n}^* + x_{T-1n}^*)^{1-n}}{1+r} - k x_{T-1n}^*$$

$$= \lambda_{T-1}^* x_{T-2n}^* + M_{T-1}^*.$$

In general, therefore, $V_{T-1}^* = \lambda_{T-1}^* X_{T-1} + M_{T-1}^*$ (28)

Note that there is no term for future use of $X_{T-1}$ in (28),

$$CEQ_{T-1}\frac{\tilde{\lambda}_T^* X_{T-1}}{1+r}$$

since capacity $X_{T-1}$ was purchased in T-2 and will not be productive two periods hence, by assumption.

Similarly, in period T-2, the problem is

$$\max_{(x_{T-2o},\, x_{T-2n})} V_{T-2} = \frac{Ea_{T-2}L(x_{T-2o} + x_{T-2n})^{1-n} - c(x_{T-2o} + x_{T-2n})}{1+r} - kx_{T-2n} + CEQ_{T-2}\frac{\tilde{V}_{T-1}^*}{1+r}.$$

The optimization conditions are derived in the same manner and the result is

$$V_{T-2}^{*} = \lambda_{T-2}^{*} X_{T-2} + M_{T-2},$$

$$X_{T-2} = x_{T-3n}^{*}$$

$$\lambda_{T-2}^{*} = 0 \quad \text{for } Ea_{T-2} < \frac{c X_{T-2}^{n}}{L(1-un)} = A_{1T-2}$$

$$= \frac{(1-un) Ea_{T-2} L X_{T-2}^{-n} - c}{1+r}$$

for $$A_{1T-2} < Ea_{T-2} < X_{T-2}^{n} \left[ \frac{c + k(1+r) - CEQ_{T-1} \lambda_{T-1}^{*}}{L(1-un)} \right] = A_{2T-2}$$

and $$\lambda_{T-2}^{*} = k - CEQ_{T-2} \frac{\lambda_{T-1}^{*}}{1+r}, \quad Ea_{T-2} > A_{2T-2}.$$

The process may be similarly carried out for any t.

There exists, thus, a range of values of $Ea_t$ for which firms with existing capacity $X_t = x_{t-1_n}{}^*$ attain net present values $V_t > 0$ by operating at capacity, while firms without existing capacity will not purchase capacity and produce, due as before to the inability to resell capacity.

The effect of uncertainty resolution lies in the fact that the "cost of capital" of the previously purchased capacity is now lower. Since the capacity has produced through one period of uncertainty, the return over the next period compensates the firm for only one (intra) period uncertainty, whereas new capacity, being productive for two periods with the second outcome's depending upon that of the first, must provide return sufficient to compensate, not only for the intra-period uncertainty, but also the interperiod uncertainty resolution.

The return over the first period of the capacity's life, which compensates for inter- and intra-period uncertainty, is just the change in the value of future revenues foreseen for the firm in each period. At t-1, the value of future revenues from the marginal unit is, from (27b),

$$MV_{t-1} = \frac{(1-un)Ea_{t-1}Lx_{t-1}^{-n} - c}{1+r} + CEQ_{t-1}\frac{(1-un)Ea_t L x_t^{-n} - c}{(1+r)^2}$$

and equals k if the unit is purchased so that the net present value is, of course, zero. At t the value of this marginal unit

$$MV_t = \frac{(1-un)Ea_t L x_t^{-n} - c}{1+r} = \lambda_t \quad \text{from (27a)}$$

Thus the change

$$\Delta\tilde{MV}_{t-1} = \frac{(1-un)\tilde{E}a_t L x_t^{-n} - c}{1+r} - CEQ_{t-1}\frac{(1-un)Ea_t L x_t^{-n} - c}{(1+r)^2}$$

$$- \frac{(1-un)Ea_{t-1}Lx_{t-1}^{-n} - c}{1+r}.$$

The cash flow from the marginal unit over t-1 is

$$\tilde{CF}_{t-1} = (1-un)\tilde{a}_{t-1}x_{t-1}^{-n} - c.$$

Thus, the total return over t-1 on the marginal unit is

$$\tilde{CF}_{t-1} + \Delta\tilde{MV}_{t-1} = \frac{(1-un)\tilde{E}a_t L x_t^{-n} - c}{1+r} - E_{t-1}\left[\frac{(1-un)\tilde{E}a_t L x_t^{-n} - c}{(1+r)^2}\right]$$

$$+ \operatorname{cov}\left[\frac{(1-un)Ea_t\, L x_t^{-n} - c}{(1+r)^2}, \frac{R_{mt}}{V_{mt-1}}\right]$$

$$+ \left[(1-un)\tilde{a}_{t-1}\, x_{t-1}^{-n} - c\right]$$

$$- \frac{(1-un)Ea_{t-1}Lx_{t-1}^{-n} - c}{1+r}$$

The expected return is

$$E_{t-1}(\widetilde{CF}_{t-1} + \widetilde{MV}_{t-1}) = \left[\frac{(1-un)Ea_{t-1}X_{t-1}^{-n} - c}{1+r}\right](r + m\,c_{e,m})$$
$$+ r\,E_{t-1}\left[\frac{(1-un)E\tilde{a}_t L X_t^{-n} - c}{1+r}\right]$$
$$+ \frac{m\,\text{cov}_{t-1}}{V_{m_{t-1}}}\left[\frac{(1-un)E\tilde{a}_t L X_t^{-n} - c}{1+r}, \tilde{R}_{m_t}\right] \qquad (29)$$
$$= \left[\frac{(1-un)Ea_{t-1}L X_{t-1}^{-n} - c}{1+r}\right](r + m\,c_{e,m})$$
$$+ E_{t-1}\left[\frac{(1-un)E\tilde{a}_t L X_t^{-n} - c}{1+r}\right]\left[r + m\,\text{cov}(\tilde{\varepsilon}_{\lambda_t}, \tilde{R}_{m_t})\right] \qquad (30)$$

if the Fama condition holds.

The first term in (30) is the expected return on intraperiod risk; the second, the interperiod risk from the uncertainty in $Ea_t$ and the consequent production decision $x_t$*. Over the second period of its life the marginal unit is subject only to intraperiod risk; thus its expected return that period is the (t period equivalent of) the first term in (30). The return to interperiod uncertainty, the positive future value, is expected since the firm cannot sell the marginal unit at k and thereby avoid the uncertainty in future value. It must therefore be compensated in the form of an expected value next period = $\frac{1}{1+r}\,CEQ_{t-1}\lambda_t$ = the value of expected cash flows during the second period discounted at the return required for intraperiod risk cash flows. Thus, the purchase of the marginal unit at cost k signifies that the marginal unit is expected to meet exactly its return requirements both periods, i.e., that the net present value of its investment both periods is zero. The current value of its investment in the following period = $CEQ_{t-1}\frac{\lambda_t}{1+r}$ is an internal value since it is unable to resell capacity, whereas k, of course, is external; i.e., the cost of current capacity usage is implicit in t —the firm does not actually commit funds for it— whereas k is an actual outflow of funds. Thus, the usage of current capacity has a non-negative value whereas the usage of a marginal unit of new capacity has zero net present value. Since (the certainty equivalent of) this value formed part of the required return on the previously purchased capacity, the current value is (the realization of) its "reproduction" cost in t.

The requirement that the value of the marginal unit = the value of its current t-1 expected cash flow plus the current value of its expected cash flows the following period (t) = the cost of the marginal unit obviously requires that the sum of the two contributions = k, requiring a certain level of expectations each period depending on the relative sizes of the two contributions. Thus the firm will purchase a smaller amount of capacity than a firm which does not face the extra uncertainty. Note that this extra return to interperiod uncertainty will be required whenever there is uncertainty as to the future value, even in the case where resale is permitted, as then its resale price would be that uncertain value. The resale price would then appear as extra cash flow and not as an internal value of capacity as it does here.

Whether the required returns in (30) are actually obtained each period is, of course, a question of realizations. So far as the use of t-1 capacity in t is concerned, the marginal unit is expected to generate the required return in t, the second term in (30), as soon as demand expectations are such that the certainty equivalent of marginal revenue = cost c. Thus all of current capacity that period will be used as soon as $\lambda_t \geq 0$, no matter whether or not the marginal unit met its required return in t-1. Its value at t is the discounted value of its expected cash flows, and if this value is less than what was expected in t-1, the firm has simply suffered a capital loss over t-1 which does not affect its optimal decision in t, to produce with the marginal unit as long as it is expected to generate its required return that period. Whether or not the firm purchases new capacity in t depends, as it did in t-1, on whether or not the value of cash flows in t, $\lambda_t$, plus the current value of cash flows in t+1, $\frac{1}{1+r}$ $CEQ_t \lambda_{t+1}$, from the marginal unit = the cost k. This requirement, incorporating the required return to interperiod uncertainty as in t-1, forces the sum of these two to equal k. Whether this requirement is met at the same demand expectation as that which makes $\lambda_t \geq 0$ (the condition for use of previously purchased capacity) depends on the relative sizes of the two contributions.

The derivation of the return to interperiod uncertainty, from (29) to (30) assumed that the Fama conditions on the covariance of future value $\tilde{\lambda}_t$ and the market $Rm_t$ were met, i.e., that $\text{cov}_{t-1}(\tilde{\lambda}_t, \tilde{R}_{m_t}) = E_{t-1}(\tilde{\lambda}_t) \text{ cov }(\tilde{\varepsilon}_{\lambda_t}, \tilde{Rm}_t)$, so that the only interperiod uncertainty concerns the expected value and the discount rate for this uncertainty, r + m cov $(\tilde{\varepsilon}_{\lambda_t}, \tilde{Rm}_t)$, is not uncertain at any t. Whether this assumption actually holds for this model is not immediately obvious.

Two sources of interperiod uncertainty as to future marginal value $\tilde{\lambda}_t$ enter into the model: the uncertainty in the demand parameter $\tilde{Ea}_t$ and in the state variable $X_t$, the latter being the sum of previously optimal decisions to acquire new capacity.

As viewed from t-1, the resolution of uncertainty in $\tilde{Ea}_t$ follows the mechanism

$$\tilde{Ea}_t = Ea_{t-1} (1 + \tilde{e})$$

with $\text{cov}(\tilde{e}, \tilde{Rm}_t) = Vm_{t-1} \text{ cov } (\tilde{e}, \tilde{Ym}_t) = Vm_{t-1} c_{em}$ and $c_{em}$ was assumed constant. Thus any function of $\tilde{Ea}_t$ will yield an expression factorable into an expectation known at t-1 multiplied by 1 + a random term. Any uncertainty in the value of the marginal unit at t from this term thus satisfies the Fama condition. This result demonstrates the constancy of the intraperiod discount rate.

The question of whether the decisions on additional new capacity change the discount rate, thus violating the necessary assumption, will again be answered by examining the t-1 case. In t-1, an optimal decision $x_{T-1_n}^{*}$ is reached. The marginal effect of this decision on the contribution of $\tilde{V}_t$ to $V_{t-1}$ is

$$\frac{\partial CEQ_{t-1} \tilde{V}_t^*}{\partial X_{T-1n}^*} = \frac{\partial \left[ \frac{E_{t-1}(\tilde{V}_t^*)}{1 + \text{required rate}} \right]}{\partial X_{T-1n}^*}$$

where the required rate is determined from equation (29). But

$$\frac{\partial CEQ_{t-1} \tilde{V}_t^*}{\partial X_{t-1n}^*} = CEQ_{t-1} \tilde{\lambda}_t^* \qquad \text{by (16)}$$

$$= \frac{E_{t-1}(\tilde{\lambda}_t^*)}{1 + \text{required rate}} = \frac{\partial E_{t-1}(\tilde{V}_t^*)}{\partial X_{t-1n}^*} \cdot \frac{1}{1 + \text{required rate}} ;$$

i.e., the decision $x_{t.s,n}^*$ does not affect the discount rate for future value, the required return for interperiod uncertainty. What is happening is that the acquisition of capacity causes changes in expected future values (since it reduces future costs), but does not affect the process of taking the certainty equivalent. In essence, it does not affect the "shape" of the distribution; that and the resulting covariance per unit of expected value depend on the realization $Ea_t$.

The only way to remove the capacity effect and retain the uncertainty resolution in expected demand is to remove the restriction of resale of capacity and specifically to have the ability to resell it at cost k. The removal of the resale restriction removes the internal valuation of current capacity (<u>if</u> there is a pre-specified external price) since the actual resale may now be treated as a cash flow realization, just as are net marginal revenues. The second condition removes the possibility of any loss of value. The only return thus required of it is reward for intraperiod uncertainty (the rental cost r + the uncertainty reward m $c_{em}$ ). The purchase of a unit of capacity currently can then not add to next period value nor generate expected rewards above that required for intraperiod uncertainty resolution. These two assumptions thus remove the link between this period's investment decision and next period's value, allowing the firm to "begin again" each period.[6] Since it is required to reproduce its (one-

[6]See Thomadakis

period) capacity cost out of current earnings, it generates a postitive net present value from its ability to earn above this cost, securing for itself a non-zero value of current and future investment opportunities. The firm of this paper, of course, has the same ability; thus the monopoly component of the firm's value is of the same form. It is in the valuation of the capacity returns that they differ: the non-saleability of the capacity exposes it to interperiod uncertainty, forcing an internal future value upon it. Thus there are two components of the current net present value of the firm, its ability to extract monopoly rents and the (internal) current value of its present capacity; and both are functions of its monopoly power (the latter component being internal).

# CHAPTER THREE: THE EFFECT OF TECHNOLOGICAL PROGRESS

## A. Introduction

In the previous chapter, the cost structure was such that the marginal operating (or variable) cost c was constant for $x_t < X_t$ and at $x_t = X_t$ became infinite. The question arises as to whether the results on production and new capacity decisions retain the same structure or whether the possession of previously purchased capacity remains an ex post barrier to entry under a more general cost structure. Hence, this chapter will focus on the optimal production-investment decisions under an increasing marginal cost structure.

In order to retain a relative simplicity, however, there will still be no decisions on factor inputs separately; that is, the assumption of a fixed coefficient production function will be retained. Accordingly, the increasing marginal cost will be admitted through the introduction each period of more efficient production facilities. To maintain the simplicity of eliminating factor input decisions, the technological change will be neutral. The firm possessing production facilities of several vintages, with the most recent being the most efficient, has therefore a marginal cost per unit of output which rises each time it employs a prior vintage to increase production. This way of introducing differing marginal cost implies, of course, that the cost curve changes each time the firm acquires new capacity, and if the firm does so (which it will do, given a particular demand expectation), the resulting cost curve is dependent on the current and future demand expectation that period. Thus it is not an increasing cost curve in the usual sense, either short-run or long-run. The benefit of this approach is that the examination of decisions under an increasing cost curve offered by the introduction of technological progress leads to conclusions on purchase of the new process technology.

## B. The Optimal Decisions of the Firm

Assume that a new technology is introduced in each period such that: $q_i$ units of production facility of period i are necessary to produce a unit of output, where $i = 1, \ldots, t$; and the efficiency of the production facility is improved each period, so that $q_1 > q_2 > \ldots > q_t$. Under the assumption of no depreciation, these efficiencies remain constant over time. The demand environment is the same as described in chapter I.

In any period t, therefore, the firm must now optimize over t production variables: the amounts of all its previously purchased facilities from each type of technology $q_1, \ldots, q_{t-1}$ and the amount of the currently available facility $q_t$. Since no resale of capital equipment is permitted, only technology of type $q_t$, the current technology, is available for purchase. Thus, the firm's use of its previously purchased capital equipment is constrained by the amount it purchased of that type of equipment. Denote the total available stock of production facility i by $F_i$, i = 1, ..., t-1. Each type of facility gives the firm the capacity to produce output $X_i = F_i/q_i$; thus the firm's constraints may be viewed as constraints on capacity produced from each type of facility, similarly to the representation in chapter I (in which all previously purchased capacity was identical). The operating cost of each type of capacity will be different, however, since the cost to operate a unit of any production facility is assumed equal to c but the units required per unit of output increase with earlier vintages. Thus, the operating cost to produce a unit of output from facility of vintage i is $cq_i$. The investment cost per unit of output is, similarly, $kq_t$. The firm therefore maximizes its value $V_t$ over t production decision variables, the amounts of period t output $X_{i_t}$ from each vintage of production facility, subject to t-1 constraints of the form $X_{i_t} \leq X_i$, i = 1,..., t-1, with $X_{t_t}$, the output of the current facility which must be purchased, unconstrained. The determination of the value at any time t will be made by a backward induction from the terminal period, assuming optimal decisions each period.

## The Single Period Problem

At the beginning of the final period T, the firm's problem is

$$\max_{(X_{1_T},\ldots,X_{T_T})} NPV_T = \frac{CEQ\tilde{Y}_T - \sum_{i=1}^{T} Cq_i X_{i_T}}{1+r} - Kq_T X_{T_T} \qquad (1)$$

subject to $X_{i_T} \leq X_i$, $i = 1,\ldots,T-1$ (some of the $X_i$ may be zero if the firm does not have facility of that vintage on hand)

or $g_{i_T} = X_{i_T} - X_i \leq 0$

$X_{i_T} \geq 0$ for all i

As before, $\tilde{Y}_T = \tilde{a}_T(X_{1_T} + X_{2_T} + \ldots + X_{T_T})^{1-n}$

so that $CEQ\ \tilde{Y}_T = Ea_T L\ (X_{1_T} + \ldots + X_{T_T})^{1-n}$

Thus, $$V_T = \frac{Ea_T L\ (\sum_{i=1}^{T} X_{i_T})^{1-n} - C\sum_{i=1}^{T} q_i X_{t_T}}{1+r} - Kq_T X_{T_T}$$

The Kuhn-Tucker maximization conditions are: (2)

(1) $\dfrac{\partial V_T}{\partial X_{i_T}} - \lambda_{i_T}\dfrac{\partial g_{i_T}}{\partial X_{i_T}} \leq 0$, $i = 1,\ldots,T-1$ and $\lambda_{i_T}$ is the shadow price in T of each constraint

$\dfrac{\partial V_T}{\partial X_{T_T}} \leq 0$

(2) $X_{i_T}\left[\dfrac{\partial V_T}{\partial X_{i_T}} - \lambda_{i_T}\dfrac{\partial g_{i_T}}{\partial X_{i_T}}\right] = 0$, $i = 1,\ldots,T-1$

$X_{T_T}\dfrac{\partial V_T}{\partial X_{T_T}} = 0$

(3) $\lambda_{i_T} g_{i_T} = 0$, $i = 1,\ldots,T-1$

(4) $X_{i_T} \geq 0, \; i = 1,\ldots,T$ .

Since $$\frac{\partial V_T}{\partial X_{i_T}} = \frac{(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T})^{-n} - Cq_i}{1+r}, \quad i = 1,\ldots,T-1$$

$$\frac{\partial V_T}{\partial X_{T_T}} = \frac{(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T})^{-n} - Cq_T}{1+r} - Kq_T$$

$$\frac{\partial g_{i_T}}{\partial X_{i_T}} = 1, \quad i=1,\ldots,T-1$$

the optimization conditions are:

$$X_{i_T}^* \left[ \frac{(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T})^{-n} - Cq_i}{1+r} - \lambda_{i_T}^* \right] = 0, \quad i=1,\ldots,T-1$$

$$X_{T_T}^* \left[ \frac{(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T}^*)^{-n} - Cq_T}{1+r} - K \right] = 0$$

$$\lambda_{i_T}^* \, (X_{i_T}^* - X_i) = 0, \quad i=1,\ldots,T-1$$

Because of the assumption on ordering of efficiencies of the vintages, the certainty equivalent of discounted net marginal revenue, $\frac{(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T}^*)^{-n} - Cq_i}{1+r}$ , decreases as i increases.

Therefore the firm will elect to use the newest of its previous vintage facilities first, the T-1 vintage, if it elects to use any of its previous capacity at all. If it purchases any of the current technology, $X_{T_T}^* > 0$, so that

$$(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T}^*)^{-n} = [K(1+r)+C]\, q_T$$

If is uses any, but not all of its T-1 vintage facility, $X_{T-1} > X_{T-1_T}^* > 0$, so that $(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T}^*)^{-n} = Cq_{T-1}$.

The vintage to which the firm first turns is governed by the sizes of the right hand sides of these equalities. If

$$[K(1+r)+C]\, q_T < Cq_{T-1}$$

or the end-of-period total cost per unit of output of the current facility is less than the end-of-period cost per unit of output of the cheapest of its purchases capacities, then the beginning-of-period value of purchasing and operating with the current facility is larger than operating with any of its previous capacity, no matter what the value $Ea_T$. Thus the firm will not use any previously purchased capacity. If $[K(1+r)+C]\, q_T > Cq_{T-1}$, the firm will use its T-1 vintage facility, with the optimum $X_{T-1_T}^*$ chosen to maximize $V_T$. If the two are equal, the firm is indifferent, as then the two vintages are identical.

If the optimum production is greater than $X_{T-1}$, the firm must choose whether to produce any more, and if so, must choose whether to produce using its next most efficient vintage, T-2, or purchase the currently available facility. If it elects to produce $X_{T-2_T}^*$ then $(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T}^*)^{-n} = Cq_{T-2}$.

It if elects to purchase $X_{T_T}^*$ then $(1-un)Ea_T L(\sum_{i=1}^{T} X_{i_T}^*)^{-n} = [K(1+r)+C]q_T$ and its choice is dictated by the same considerations as before: it will not use vintage T-2 or any previous vintage, if $[K(1+r)+C]q_T < Cq_{T-2}$. If the $q_i$ are non-random, the order of choice of vintages will be known in advance, the inverse of the order of end-of-period marginal total costs. Assume the $q_i$ to be such that

$$cq_s < [k(1 + r) + c] \cdot q_T < cq_{s-1} \qquad (3)$$

so that the firm will switch at s--will purchase new facility T after having exhausted its capacity from vintage s.

There are thus s + (s+1) possible solutions: the firm uses $X_{T-1_T}^* < X_{T-1}$; the optimal $X_{T-1_T}^* = X_{T-1}$; the firm uses $X_{T-1_T}^* = X_{T-1}$, $X_{T-2_T}^* < X_{T-2}$; the firm uses $X_{T-1_T}^* = X_{T-1}$, $X_{T-2_T}^* = X_{T-2}$; and the list continues down to the firm's using $X_{s_T}^* = X_{T-k+1}$. The final case is that the firm, having exhausted all previous capacity, adds $X_{T_T}^* > 0$. The solution to case 1 is

$$\frac{(1-un)Ea_T L(X_{T-1_T}^*)^{-n} - Cq_{T-1}}{1+r} = 0$$

$$X_{T-1_T}^* = \left[\frac{Ea_T L(1-un)}{cq_{T-1}}\right]^{1/n} < c\ X_{T-1}$$

which is optimal for $Ea_T < \dfrac{cq_{T-1}(X_{T-1})^n}{L(1-un)}$

The value of the firm $V_T^* = \dfrac{Ea_T L(X_{T-1_T}^*)^{1-n} - cq_{T-1}X_{T-1_T}^*}{1+r}$

$$= \frac{unEa_T L(X_{T-1_T}^*)^{1-n}}{1+r}$$

For case 2, $X_{T-1_T}^* = X_{T-1}$ but all other $Xi_T^* = 0$. Thus

$$\frac{(1-un)Ea_T L(X_{T-1})^{-n} - cq_{T-1}}{1+r} = \lambda_{T-1_T}^* > 0$$

$$\frac{(1-un)Ea_T L(X_{T-1})^{-n} - cq_{T-2}}{1+r} < 0 \text{ since } X_{T-2_T}^* = 0;$$

$$\text{and } \lambda_{T-1_T}^* < \frac{c(q_{T-2} - q_{T-1})}{1+r}.$$

This solution is optimal in the range

$$\frac{cq_{T-1}(X_{T-1})^n}{L(1-un)} < Ea_T < \frac{cq_{T-2}(X_{T-1})^n}{L(1-un)}$$

The value $V_T^* = \lambda_{T-1_T}^* X_{T-1} + \dfrac{unEa_T L(X_{T-1})^{1-n}}{1+r}$

The case 3 solution is $X_{T-1_T}^* = X_{T-1}$, $0 < X_{T-2_T}^* < X_{T-2}$. Then

$$\lambda_{T-1_T}^* = \frac{(1-un)Ea_T L(X_{T-1} + X_{T-2_T}^*)^{-n} - cq_{T-1}}{1+r} > 0$$

$$\frac{(1-un)Ea_T L(X_{T-1} + X_{T-2_T}^*)^{-n} - cq_{T-2}}{1+r} = 0$$

so that $\lambda_{T-1_T}^* = \dfrac{c(q_{T-2} - q_{T-1})}{1+r}$ and

$$x_T^* = \text{total production} = X_{T-1} + X_{T-2_T}^* = \left[\frac{Ea_T L(1-un)}{cq_{T-2}}\right]^{1/n}$$

$$< X_{T-1} + X_{T-2}.$$

This solution is optimal for $\frac{cq_{T-2}X_{T-1}{}^{n}}{L(1-un)} < Ea_T < \frac{cq_{T-2}(X_{T-1}+X_{T-2})^{n}}{L(1-un)}$

The value $V_T^*$

$$= \frac{Ea_T L(X_{T-1}+X_{T-2_T}^{*})^{1-n} - cq_{T-1}X_{T-1} - cq_{T-2}X_{T-2_T}^{*}}{1+r}$$

$$= \frac{X_{T-1}\left[Ea_T L(X_{T-1}+X_{T-2_T}^{*})^{-n} - cq_{T-1}\right] + X_{T-2_T}^{*}\left[Ea_T L(X_{T-1}+X_{T-2_T}^{*})^{-n} - cq_{T-2}\right]}{1+r}$$

$$= \lambda_{T-1_T}^{*}X_{T-1} + \frac{unEa_T L(X_{T-1}+X_{T-2_T}^{*})^{-n}X_{T-1} + unEa_T L(X_{T-1}+X_{T-2_T}^{*})^{-n}X_{T-2_T}^{*}}{1+r}$$

$$= \lambda_{T-1_T}^{*}X_{T-1} + \frac{unEa_T L(X_{T-1} + X_{T-2_T}^{*})^{1-n}}{1+r}$$

The solution for case 4 is $X_{T-1_T}^{*} = X_{T-1}$, $X_{T-2_T}^{*} = X_{T-2}$, all other $Xi_T^{*} = 0$.

Thus $$\lambda_{T-1_T}^{*} = \frac{(1-un)Ea_T L(X_{T-1} + X_{T-2})^{-n} - cq_{T-1}}{1+r} > 0$$

$$\lambda_{T-2}^{*} = \frac{(1-un)Ea_T L(X_{T-1} + X_{T-2})^{-n} - cq_{T-2}}{1+r} > 0$$

$$\frac{(1-un)Ea_T L(X_{T-1} + X_{T-2})^{-n} - cq_{T-3}}{1+r} < 0$$

so that $\frac{c(q_{T-2} - q_{T-1})}{1+r} < \lambda_{T-1_T}^{*} < \frac{c(q_{T-3} - q_{T-1})}{1+r}$; $\lambda_{T-2_T} < \lambda_{T-1_T}^{*}$

This solution is optimal in the range

$$\frac{cq_{T-2}(X_{T-1} + X_{T-2})^{n}}{L(1-un)} < Ea_T < \frac{cq_{T-3}(X_{T-1} + X_{T-2})^{n}}{L(1-un)}$$

Then $V_T^{*} = \lambda_{T-1_T}^{*} X_{T-1} + \lambda_{T-2_T}^{*}X_{T-2} + \frac{unEa_T L(X_{T-1} + X_{T-2})^{1-n}}{1+r}$

This pattern continues until the facility of vintage s is used: here the optimum is $X_{i_T}^* = X_i$ for i = s, s+1,...,T-1; $X_{i_T}^* = X_{T_T}^* = 0$ for $i < s$

Then $$\lambda_{T-1_T^*} = \frac{(1-un)Ea_T L \left(\sum_{i=s}^{T-1} X_i\right)^{-n} - cq_{T-1}}{1+r} > 0 \qquad (4)$$

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$$\lambda s_T^* = \frac{(1-un)Ea_T L\left(\sum_{i=s}^{T-1} X_i\right)^{-n} - cq_s}{1+r} > 0$$

and $$\frac{(1-un)Ea_T L \left(\sum_{i=s}^{T-1} X_i\right)^{-n} - cq_{s-1}}{1+r} \quad 0, \quad \frac{(1-un)Ea_T L \left(\sum_{i=s}^{T-1} X_i\right)^{-n} - cq_T}{1+r} - kq_T < 0$$

so that $$\frac{c(q_s - q_{T-1})}{1+r} < \lambda_{T-1_T}^* < kq_T + \frac{c(q_T - q_{T-1})}{1+r} < \frac{c(q_{s-1} - q_{T-1})}{1+r}$$

This solution is optimal for

$$\ldots_{T} - \frac{\ldots_{i=s}}{1+r} > 0$$

and $$\frac{(1-un)Ea_T L \left(\sum_{i=s}^{T-1} X_i\right)^{-n} - cq_{s-1}}{1+r} \quad 0, \quad \frac{(1-un)Ea_T L \left(\sum_{i=s}^{T-1} X_i\right)^{-n} - cq_T}{1+r} - kq_T < 0$$

so that $$\frac{c(q_s - q_{T-1})}{1+r} < \lambda^*_{T-1_T} < kq_T + \frac{c(q_T - q_{T-1})}{1+r} < \frac{c(q_{s-1} - q_{T-1})}{1+r}$$

This solution is optimal for

$$\frac{cq_s\left(\sum_{i=s}^{T-1} X_i\right)^n}{L(1-un)} < Ea_T < \frac{[k(1+r) + c]\, q_T\left(\sum_{i=s}^{T-1} X_i\right)^n}{L(1-un)}$$

and $$V_T = \sum_{i=s}^{T-1} \lambda^*_{i_T} X_i + \frac{unEa_T L\left(\sum_{i=s}^{T-1} X_i\right)^{1-n}}{1+r}$$

For $Ea_T$ above this range, the firm purchases new facility $X^*_{T_T}$, since the marginal cost of the new facility is less than the marginal cost of facilities from any previous vintage. The optimal conditions then become

$$\lambda_{T-1_T}^{*} = \frac{(1-un)Ea_TL(\sum_{i=s}^{T-1} X_i + X_{T_T}^{*})^{-n} - cq_{T-1}}{1+r} > 0 \quad (5)$$

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$$\lambda_{s_T}^{*} = \frac{(1-un)Ea_TL(\sum_{i=s}^{T-1} X_i + X_{T_T}^{*})^{-n} - cq_s}{1+r} > 0$$

and $$\frac{(1-un)Ea_TL(\sum_{i=s}^{T-1} X_i + X_{T_T}^{*})^{n} - cq_T}{1+r} = kq_T$$

or total production $X_T^{*} = \sum_{T=s}^{T-1} X_i + X_{T_T}^{*} = \left[\frac{Ea_TL(1-un)}{[k(1+r)+c]q_T}\right]^{1/n}$

The firm's value $V_T^{*} = \frac{Ea_TL(\sum_{i=s}^{T-1} X_i + X_{T_T}^{*})^{1-n} - c\sum_{i=s}^{T-1} q_i - cq_TX_{T_T}^{*}}{1+r} - kq_TX_{T_T}^{*}$

$$- \sum^{T-1} \lambda^{*}X + \frac{unEa_TL(\sum_{i=s}^{T-1} X_i + X_{T_T}^{*})^{1-n}}{}$$

or total production $X_T{*} = \sum_{T=s}^{T-1} X_i + X_{T_T}^{*} = \left[\frac{Ea_T L(1-un)}{[k(1+r)+c]q_T}\right]^{1/n}$

The firm's value $V_T{*} = \dfrac{Ea_T L(\sum_{i=s}^{T-1} X_i + X_{T_T}^{*})^{1-n} - c\sum_{i=s}^{T-1} q_i - cq_T X_{T_T}^{*}}{1+r} - kq_T X_{T_T}^{*}$

$$= \sum_{i=s}^{T-1} \lambda_{i_T}^{*} X_i + \frac{unEa_T L(\sum_{i=s}^{T-1} X_i + X_{T_T}^{*})^{1-n}}{1+r}$$

In general, then, $V_T{*} = \sum_{i=1}^{T-1} \lambda_{i_T}^{*} X_i + \dfrac{unEa_T L(X_T{*})^{1-n}}{1+r} = \sum_{i=1}^{T-1} \lambda_{i_T}^{*} X_i + M_T{*}$ (6)

where $X_T{*}$ = total production

$$\lambda_{i_T}^{*} = \frac{(1-un)Ea_T L(X_T{*})^{-n} - cq_i}{1+r} = kq_T + \frac{c(q_T - q_i)}{1+r} \text{ if } X_{i_T}^{*} = X_i$$

$= 0$ if not

and $M_T{*} = \dfrac{unEa_T L(X_T{*})^{1-n}}{1+r}$ .

Thus, as in Chapter 2, the value of the firm may be seen as consisting of two parts: (1) the value of previously purchased capacity, with the value per unit of each vintage equalling the current value of marginal revenue to be received from operation at level $X_T$* less cost of operating a unit of that vintage, and (2) the current value of monopoly rents to be received at the end of the period.

Note that the shadow prices of each vintage decrease with increasing efficiency. As long as all of a facility of vintage $i > s$ is being used

$$\lambda_{i_T}{}^* = \frac{(1-un)Ea_T L(X_T{}^*)^{-n} - cq_i}{1+r} > 0,$$

an internal price = the certainty equivalent of net marginal value of an extra unit of that vintage, since the constraint $x_{i_T}{}^* \leq X_i$ is binding. As soon as the expectation $Ea_T$ rises such that $X_{i-1_T}{}^* > 0$,

$$\lambda_{i_T}{}^* = \frac{c(q_{i-1} - q_i)}{1+r},$$ on substitution of optimization

conditions (4). That is, the value of having an additional unit of facility i is just the marginal cost difference between production with i and the preceding i-1$^{st}$ vintage actually in use (discounted to the beginning of T), an external value. The only internal value will be that of the last, s$^{th}$ vintage, as long as the firm uses $x_{s_T} = X_s$ but does not add new capacity:

$$\lambda_{s_T}{}^* = \frac{(1-un)Ea_T L(X_T{}^*)^{-n} - cq_s}{1+r} > 0$$

If $Ea_T$ is such that the firm adds new capacity, the value of the last vintage becomes external, measuring the value of having an additional unit of facility of vintage s, rather than being constrained, as it is, to purchase new capacity:

$$\lambda_{s_T}{}^* = \frac{(1-un)Ea_T L(X_T{}^*)^{-n} - cq_s}{1+r} = kq_T + \frac{c(q_T-q_s)}{1+r}$$

on substitution of optimization conditions (5). Thus, any previous shadow price $i < s$ may be written in terms of the new purchased capacity cost, as

$$\lambda_{i_T}{}^* = \frac{(1-un)Ea_T L(X_T{}^*)^{-n} - cq_i}{1+r} = kq_T + \frac{c(q_T-q_i)}{1+r} \qquad (7).$$

## The Multiperiod Problem

The multiperiod valuation yields a pattern also similar to the single technology case. For period T-1, the firm's problem is

$$\max_{(X_{1_{T-1}},\ldots,X_{T-2_{T-1}},X_{T-1_{T-1}})} v_{T-1} = \frac{Ea_{T-1}L(\sum_{i=1}^{T} X_{i_{T-1}})^{1-n} - c\sum_{i=1}^{T} q_i X_{i_{T-1}}}{1+r} - kq_{T-1}X_{T-1_{T-1}} + \frac{CEQ_{T-1}\bar{V}_T(X_1,\ldots,X_{T-1})}{1+r}$$

$$\text{subject to } X_{i_{T-1}} \leq X_i,\ i = 1,\ldots,T-2 \qquad (8)$$

and the identity $X_{T-1} = X_{T-1_{T-1}}$; i.e., the amount of facility of vintage T-1 available in T is exactly that amount purchased in period T-1, and available capacities of all previous vintages carry over unchanged from period T-1 to T.

The optimality conditions are: (9)

$$0 = \lambda_{i_{T-1}}^{*} [X_{i_{T-1}}^{*} - X_i] \text{ , } i = 1,\ldots,T-2$$

$$0 = X_{i_{T-1}}^{*} \left[\frac{(1-un)Ea_{T-1}L(\sum_{i=1}^{T} X_{i_{T-1}}^{*})^{-n} - cq_i}{1+r} - \lambda_{i_{T_T}}^{*}\right] \text{for } i=1,\ldots,T-2$$

$$0 = X_{T-1_{T-1}}^{*} \left[\frac{(1-un)Ea_{T-1}L(\sum_{i=1}^{T} X_{i_{T-1}}^{*})^{-n} - cq_{T-1}}{1+r} - kq_{T-1} + \frac{CEQ_{T-1}\tilde{\lambda}_{T-1_T}^{*}}{1+r}\right]$$

since $\dfrac{\partial CEQ_{T-1}\tilde{V}_T^{*}}{\partial X_{T-1_{T-1}}} = \dfrac{\partial CEQ_{T-1}\tilde{V}_T^{*}}{\partial X_{T-1}} = CEQ_{T-1}\lambda_{T-1_T}^{*}$ , analogously to

the single technology case.

Thus, the same pattern of ordered usage of facilities of previous vintages results. However, the effective marginal total cost of the new vintage is reduced by the marginal value of its possible use in the next period. Thus, the condition for non-zero $X_{T-1_{T-1}}^{*}$ is now

$$\frac{(1-un)Ea_{T-1}L(X_{T-1}^{*})^{-n} - cq_T}{1+r} - kq_{T-1} + \frac{CEQ\,\lambda_{T-1}^{*}}{1+r} = 0$$

or current value of marginal revenue $\dfrac{(1-un)Ea_{T-1}L(X_{T-1}^{*})^{-n}}{1+r}$

$= \text{effective marginal cost } k[(1+r) + c]q_{T-1} - CEQ_{T-1}\tilde{\lambda}_{T-1_T}^{*}$, which satisfies $cq_{T-1} < [k(1+r)+c]\, q_{T-1} - \text{CEQ}\, \tilde{\lambda}^*_{T-1_T} < [k(1+r)+c]q_{T-1}$ since $0 < \lambda^*_{T-1_T} < k(1 + r)\cdot q_{T-1}$ (assuming CEQ $\tilde{\lambda}^*_{T-1_T} > 0$). Thus, for sufficiently large $Ea_{T-1}$ the effective marginal cost of the current technology may be lowered sufficiently such that the vintage rank at which the firm switches from use of previously purchased facility to purchase of new facility will be higher than $s_T$, the rank at which it switched in T. Conversely, for sufficiently low $Ea_{T-1}$, there is no reduction from future use and the marginal cost is close to $[k(1 + r) + c]\, q_{T-1}$, a larger figure than the marginal cost of new equipment in T since $q_{T-1} > q_T$. The firm will then elect to use a vintage older than $s_T$ before switching to the new facility, if $Ea_{T-1}$ is high enough to justify this much production. The rank $s_{T-1}$ at which the firm will switch is given, as in T, by the highest vintage rank such that

$$cq_{s_{T-1}} < [k(1 + r) + c]\,.\, q_{T-1} - \text{CEQ}_{T-1}\, \tilde{\lambda}^*_{T-1_T} < cq_{s_{T-1}-1} \qquad (11)$$

Thus, a somewhat different set of solutions will be optimal for each $Ea_{T-1}$ since as $Ea_{T-1}$ changes, the switching vintage will change. For a given $Ea_{T-1}$, assume $S_{T-1}(Ea_{T-1})$ is the switching vintage; the solution is determined by Kuhn-Tucker optimality conditions, by locating the production point at which current value of marginal revenue falls below marginal cost. Thus, assume that $\dfrac{(1-un)Ea_{T-1}L\left(\sum_{i=k}^{T-2} x_i\right)^{-n} - cq_k}{1+r} > 0$ for some k, $s_{T-1}$ k T-1, but $\dfrac{(1-un)Ea_{T-1}L\left(\sum_{i=k-1}^{T-2} x_i\right)^{-n} - cq_{k-1}}{1+r} < 0.$

Then $X_{i_{T-1}}{}^* = X_i$ for $k \le i < T-1$, $X_{i_{T-1}}{}^* = 0$ for $i < k$, if

$$\frac{(1-un)Ea_{T-1}L(\sum_{i=k}^{T-2} X_i)^{-n} - cq_{k-1}}{1+r} < 0.$$ If, however,

$$\frac{(1-un)Ea_{T-1}L(\sum_{i=k}^{T-2} X_i)^{-n} - q_{k-1}c}{1+r} > 0,$$ then $0 < X_{k-1_{T-1}} < X_{k-1}$ and the

optimal amount $X_{k-1_{T-1}}{}^*$ is determined by

$$\sum_{i=k}^{T-2} X_i + X_{k-1_{T-1}}^* = \left[\frac{Ea_{T-1}L(1-un)}{cq_{k-1}}\right]^{1/n} \quad \text{for } s_{T-1} < k.$$

If $k = s_{T-1}$, so that the firm switches to purchase of new capacity, this amount is then determined by

$$X_{T-1}{}^* = \sum_{i=s_{T-1}}^{T-2} X_i + X_{T-1_{T-1}}^* = \left[\frac{Ea_{T-1}L(1-un)}{[k(1+r) + c]q_{T-1} - CEQ_{T-1}\tilde{\lambda}_{T-1_T}^*}\right]^{1/n}$$

The shadow prices are, as in the $T^{th}$ period

$$\lambda_{i_{T-1}}{}^* = \frac{(1-un)Ea_{T-1}L(X_{T-1}{}^*)^{-n} - cq_i}{1+r} = \frac{c(q_{i-1} - q_i)}{1+r}$$

on substitution of (9)

$$= kq_{T-1} + \frac{c(q_{T-1} - q_i) - CEQ\lambda_{T-1_T}^*}{1+r} \quad \text{for } s_{T-1} \le i < T-1$$

$$\lambda_{i_{T-1}}{}^* = 0 \quad \text{for } i < k.$$

The value of the firm is

$$V_{T-1}{}^* = \frac{Ea_{T-1}L(X_{T-1}{}^*)^{-n} - c\sum_{i=s_{T-1}}^{T-2} q_i X_{i_{T-1}}^*}{1+r} - kX_{T-1_{T-1}}^* + \frac{CEQ\,\tilde{V}_T{}^*}{1+r}$$

$$= \sum_{i=s_{T-1}}^{T-2} X_i \left[ \frac{Ea_{T-1}L(X_{T-1}^*)^{-n} - cq_i}{1+r} \right]$$

$$+ X_{T-1_{T-1}}^* \left[ \frac{Ea_{T-1}L(X_{T-1}^*)^{-n} - cq_{T-1}}{1+r} - k \right]$$

$$+ \frac{CEQ_{T-1}}{1+r} \left[ \sum_{i=s_T}^{T-2} \tilde{\lambda}_{i_T} X_i + \tilde{\lambda}_{T-1_T} X_{T-1_{T-1}}^* + M_T^* \right]$$

$$= \sum_{i=s_{T-1}}^{T-2} \lambda_{i_{T-1}}^* X_i + \sum_{i=s_T}^{T-2} \frac{CEQ_{T-1}\tilde{\lambda}_{i_T}^* X_i}{1+r} + \frac{unEa_{T-1}L(X_{T-1}^*)^{1-n}}{1+r}$$

$$+ \frac{CEQ_{T-1}\tilde{M}_T^*}{1+r}$$

or, in general, $V_{T-1}^* = \sum_{i=1}^{T-2} \left[ \lambda_{i_{T-1}}^* + \frac{CEQ\,\tilde{\lambda}_{i_T}^*}{1+r} \right] . X_i + M_{T-1}^*$

where $M_{T-1}^* = \frac{unEa_{T-1}L(X_{T-1}^*)^{1-n}}{1+r} + \frac{CEQ\,\tilde{M}_T^*}{1+r}$

and $\lambda_{i_{T-1}}^* = 0$ if $i < s_{T-1}(Ea_{T-1})$

$CEQ\,\lambda_{i_T}^* = 0$ if $i < s_T$

$$\ldots_{T-1} \quad \ldots \quad \ldots_{T} \quad \frac{\ldots}{1+r} \quad \frac{\ldots}{1+r}$$

$$+ \frac{CEQ_{T-1}\tilde{M}_T^*}{1+r}$$

or, in general, $$V_{T-1}^* = \sum_{i=1}^{T-2} \left[\lambda_{i_{T-1}}^* + \frac{CEQ\,\tilde{\lambda}_{i_T}^*}{1+r}\right] . X_i + M_{T-1}^*$$

where $$M_{T-1}^* = \frac{unEa_{T-1}L(X_{T-1}^*)^{1-n}}{1+r} + \frac{CEQ\,\tilde{M}_T^*}{1+r}$$

and $$\lambda_{i_{T-1}}^* = 0 \text{ if } i < s_{T-1}(Ea_{T-1})$$

$$CEQ\,\lambda_{i_T}^* = 0 \text{ if } i < s_T$$

The solution pattern for any t is determined similarly. Given the realization $Ea_t$, the switching vintage is the first $s_t$ such that the effective marginal cost of purchasing and using the current vintage t is less than the marginal cost of using the preceding vintage $s_t - 1$. At time t there are T-t future periods; thus the end-of-period marginal cost $[k(1+r) + c]q_t$ of the current vintage is reduced by the certainty equivalent at t of the value of this vintage in all future periods, so that the switching vintage $s_t$ $(Ea_t)$ is determined by the inequality

$$cq_{s_t} < [k(1+r) + c]q_t - \sum_{j=1}^{T-t} \frac{c_{t+i-1}\, \lambda^{*t}{}_{t+i}}{(1+r)^{j-1}} < cq_{s_t-1} \quad (12)$$

(where $C_{t+j} = CEQ_t CEQ_{t+1}, \ldots CEQ_{T+j}$). Then, for $k \geq s_t$ $(Ea_t)$ determined by

$$\frac{(1-un)Ea_t(\sum_{i=k}^{t-1} X_i)^{-n} - cq_k}{1+r} > 0, \quad \frac{(1-un)Ea_t(\sum_{i=k-1}^{t-1} X_i)^{-n} - cq_{k-1}}{1+r} < 0$$

$X_{i_t}{}^* = X_i$ for $k \leq i < t$, $0 < X_{k-1}{}^*_t < X_{k-1}$ if $k > s_t$, and

$$\frac{(1-un)Ea_t(\sum_{i=k}^{t-1} X_i)^{-n} - cq_{k-1}}{1+r} > 0$$

and $X_{k-1}{}^*_t$ is determined by $X_t{}^* = \sum_{i=k}^{t-1} X_i + X_{k-1}{}^*_t = \left[\frac{Ea_t L(1-un)}{cq_{k-1}}\right]^{1/n}$

$X_{i_t}{}^* = 0$ for $i < k-1$

$$\frac{(1-un)Ea_t(\sum_{i=k} X_i)^{-n} - cq_k}{1+r} > 0, \quad \frac{(1-un)Ea_t(\sum_{i=k-1} X_i)^{-n} - cq_{k-1}}{1+r} < 0$$

$X_{i_t}{}^* = X_i$ for $k \leq i < t$, $0 < X_{k-1_t}^* < X_{k-1}$ if $k > s_t$, and

$$\frac{(1-un)Ea_t(\sum_{i=k}^{t-1} X_i)^{-n} - cq_{k-1}}{1+r} > 0$$

and $X_{k-1_t}^*$ is determined by $X_t{}^* = \sum_{i=k}^{t-1} X_i + X_{k-1_t}^* = \left[\frac{Ea_t L(1-un)}{cq_{k-1}}\right]^{1/n}$

$X_{i_t}{}^* = 0$ for $i < k-1$

If $k = s_t$ and $X_{t_t}{}^* > 0$, $X_{t_t}{}^*$ is determined by solution of Kuhn-Tucker optimality condition as

$$X_t{}^* = \sum_{i=s_t}^{t-1} X_i + X_{t_t}{}^* = \left[\frac{Ea_t L(1-un)}{[k(1+r)+c]q_t - \sum_{j=1}^{T-t} \frac{c_{t+j-1}\tilde{\lambda}_{t_{t+j}}^*}{(1+r)j-1}}\right]^{1/n}$$

Then $$\lambda_{i_t}{}^* = \frac{(1-un)Ea_t L(X_t{}^*)^{-n} - cq_i}{1+r} \quad \text{for } s_t \leq i < t$$

$$= kq_t + \frac{c(q_t - q_i) - \sum_{j=1}^{T-t} \frac{c_{t+j-1}\lambda^{t}{}^{*}_{t+j}}{(1+r)^{j-1}}}{1+r} \quad \text{for } s_t < i < t \qquad (13)$$

$$= 0 \text{ for } i < s_t$$

$$\text{and } V_t{}^* = \sum_{i-1}^{t-1}\left[\lambda_{i_t}^{*} + \sum_{j-1}^{T-t} \frac{c_{t+j-1}\,\tilde{\lambda}^{i}{}^{*}_{t+j}}{(1+r)^{j}}\right] \cdot X_i + M_t{}^*. \qquad (14)$$

Thus, the value of the firm may be represented as the current value of previously purchased facilities, with the value per unit of each type depending on its vintage, plus the value of monopoly rents to be received.

## C. Conclusions

### The Value of Capacity

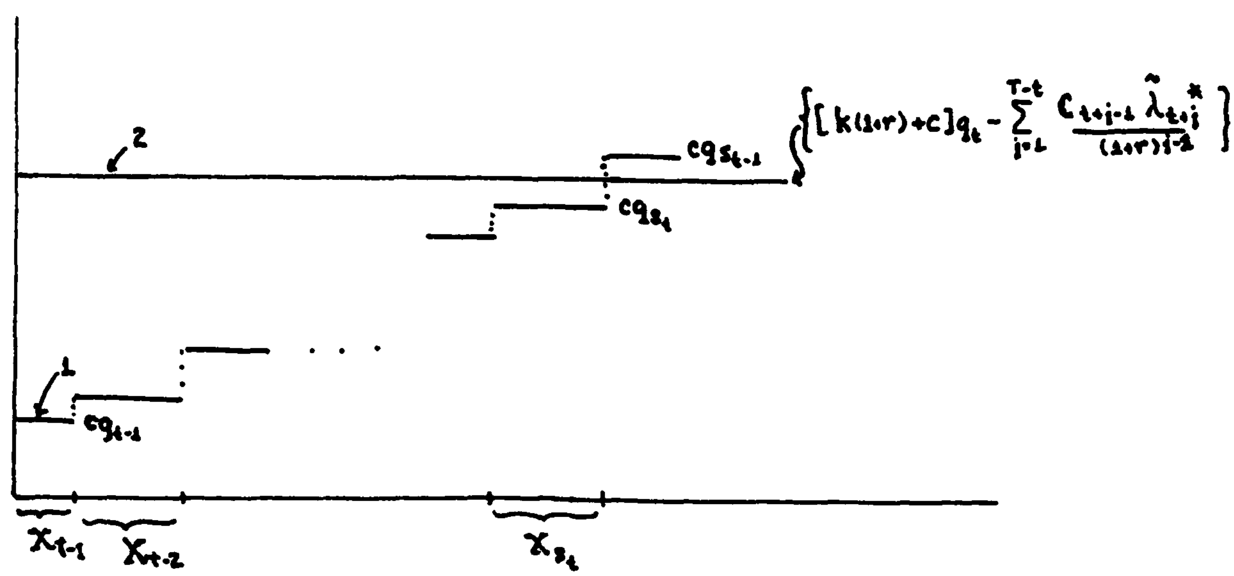
The same valuation pattern for the firm's current capacity and ability to extract monopoly rents holds under the assumptions of technology chánge. The possession of production capacity of a particular vintage confers value to the firm equal to the shadow price of the constraint on the use of that capacity. The shadow price is zero if the constraint on the vintage is not binding. It is the certainty equivalent of net marginal revenue if it is binding. It is equal to the cost savings of using that vintage relative to the previous one (or purchase of use of new capacity) if the firm is producing at a level requiring use of the previous one or purchase of new capacity. (Note that, of course, the production decisions are not the same. As the costs have changed, so have the decisions.)

The reasoning behind this result is the same as that outlined in Chapter 2. The possession of capacity of any vintage has a value to the firm given as the discounted certainty equivalent of future net marginal revenues from production with it. This value is the future net marginal revenues discounted at the required rates of return, the return for waiting plus uncertainty resolution each period. Since the firm cannot sell the capacity, this value is an internal one, the price the firm would ask if the vintage could be sold. The current vintage does not appear in these "reproduction" costs, as before, since it is purchased when its marginal value equals its (external) cost.

Thus, as before, the inability to sell production capacity yields an internal value for it. If there is any chance of using the unit, its value is greater than zero but may be less than the external reproduction cost k, the value the unit must have in order for a new firm to enter. Thus, at demand expectations below those which make the marginal value of a unit of capacity equal to k, there is a barrier to entry of new firms, as in Chapter Two.

Note that not every vintage will, in general, contribute any value to the firm. Examination of equation (11) suggests that, as soon as a sufficiently efficient vintage is introduced such that the marginal cost of purchasing and using it, $[K(1+r) + c]q_t$, is less than the marginal cost of operating the $i^{th}$ vintage ($i<t$), $cq_i$, the $i^{th}$ vintage and all prior ones will be scrapped. As can be seen from (11), this is not the switching vintage $s_t$ if any future value is foreseen for the $t^{th}$ vintage. But the vintages between i and $s_t$ may yet be used in a future period if demand expectations change from those implied by (11). At time t, however, these vintages are not used (their shadow prices are zero). The firm thus has excess inefficient capacity when it is purchasing new capacity. As more and more efficient capacity is offered, more of the previous vintages are scrapped.

What the firm is essentially doing when it selects $s_t$, the switching vintage, is expanding production along its rising marginal cost curve until the curve crosses the marginal cost line of new investment. Line 1 in the diagram illustrates the rising marginal cost of production capacity on had. Line 2 is the effective marginal cost of new investment in t.



If the certainty equivalent of net marginal revenue intersects below the point at which line 1 crosses line 2, the firm does not purchase new capacity, but uses that much of its previously purchased capacity. If it intersects beyond the crossing point (the switching vintage), the firm purchases new capacity. If the firm purchases new capacity, it changes its marginal cost curve for next period. The low point reflects the most efficient technology and the marginal cost of new investment will not in general intersect at the same switching vintage, since the effective cost depends on the demand expectation that period. Thus, each year the firm may face a different rising marginal cost curve. Because the marginal cost rises, the marginal value of each prior vintage declines. Thus, the barrier to entry may be smaller in this Chapter (depending on the change in efficiencies $q_t$), then derived in Chapter Two.

## The Influence of Monopoly Power

The size of the barrier to entry in a competitive market depends on the industry price less cost, thus on the demand elasticity of the product. The barrier to entry into a monopolistic industry would therefore be comparable only if the demand parameters were the same. Under this assumption, the positive net present value which the monopolistic firm derives from producing when entry would produce a negative NPV (even for a firm of similar monopoly power) is the value given in (13) for the case $x_t^* = 0$. Since the value of previous capacity and the monopoly rent term depend on u, this advantage of the existing firm over the entrant depends positively on u. This is, naturally, the same conclusion as in Chapter Two.

An interesting point of inquiry is whether the monopolist will purchase the current technology "sooner" (i.e., at a lower demand expectation $Ea_t$) than the member of the competitive industry. The switching vintage $s_t$ at which the firm purchases new technology is given in general by equation (11) and, for the two-period case, by equation (10). The marginal investment and operating cost of the new technology is reduced by the certainty equivalent of future net marginal revenue at future optimal production decisions. This term depends positively on u since the chance of future net marginal gain from using the additional unit is greater in the monopolist case. Thus, in general the monopolist has a smaller net marginal cost of investment in the new technology at any given demand expectation. The decision to invest is made when certainty equivalent of marginal revenue equals the net marginal investment cost. Since the marginal cost is lower for the monopolist, the equating takes place at a lower expected demand. Thus it is predicted that the monopolist will invest in new process technology at a lower demand expectation. This is due, not to the comparison of current certainty equivalent marginal revenue versus cost for the monopolist and competitive firm, but to a comparison of the future chances of deriving net benefit from the additional new technology unit.[1]

It would appear, then, that the monopolistic firm has a lower cost curve (in the sense of this chapter) since the monopolist will purchase more efficient technology sooner. This conclusion is, of course, qualified by the past purchase decisions of the monopolist relative to the competitive firm. (If the monopolist has purchased fewer amounts of each vintage, the curve will slope up more steeply.)

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[1]See Scherer, 365, for the argument which compares only current marginal revenue to marginal cost, assuming implicitly a long run equilibrium under certainty.

# CHAPTER FOUR: INTRODUCTION OF OPTIMAL DEBT FINANCING

## A. Introduction

### The Existence of an Optimal Level

Jensen and Meckling[1] and Myers[2], separately, have argued that an optimal capital structure (containing debt at a level between 0 and 100%) will be determinable for the firm, even under perfect capital market conditions, as long as the tax advantage of debt exists. The equity-maximizing level is that amount of debt at which the marginal advantage to equity, due to the tax shield, is equal to their marginal cost. The cost of debt, the "agency cost" of Jensen and Meckling, arises out of the decrease in the net present value of the firm (the sum of net present values of the projects) which takes place if the project selection is made with prior debt outstanding. In this case, the selection, made according to the rule of maximizing shareholder equity, results in a value for the firm less than that achieved when the selection is made with no prior debt outstanding. Because this is a perfect market, the debtholders realize that the selection process will operate in this manner and will pay no more than the true worth of their claims to the (lesser) value of the projects to be selected by the firm's managers. Thus the loss in value of the firm will be a cost borne by the shareholders.

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[1]See Jensen and Meckling.

[2]See Myers.

The shareholders' selection decisions change because of, essentially, the nature of the equity claim: their limited liability which places a lower limit of zero on the net present value of their claim, and the fact that the claim is residual, ranking after the prior debt claim to the returns on the investment opportunities selected. The equity claims are thus claims to the net present value of a (European) call option on the set of investment opportunities with "striking price" equal to the current repayment requirement of the prior claim, the outstanding debt. Thus they select investment opportunities so as to maximize, not the net present value of the opportunities, but the net present value of their claim, the net present value of the option they hold on the set.[3]

The firm of this research has a restricted set of investment opportunities available for selection by the owners; specifically, the current production and capacity decisions. (That is, if the owners do not elect to produce currently, the firm leaves the industry; future opportunities are not open to it. This will be amplified in the next section.) When the current decisions are made with prior debt outstanding, the shareholders make their current production-new investment decisions so as to maximize the value of the option they then hold on the returns from the current decisions and the future

[3]Jensen and Meckling note that the Black-Scholes-Merton formulation for the value of the option is not applicable in this case since the underlying value of the assets on which the option is a claim depends on the striking price of the option, the amount of prior debt repayment due.

production-investment decisions. If there is a single repayment due on the prior debt, the owners hold a single option on the current and future net present value of the production-investment opportunities. If, however, the prior debt is long term, with a schedule of repayments due for many periods, then a prior debt repayment must be made in order to gain access to returns from any of the future production-investment opportunities. Then the option they currently own entitles them, if they exercise it, to the value of the returns from the current production-investment decision plus the value of the option on the next period production-investment decisions plus the value of the tax shields, less the exercise price. This option in turn, if exercised, entitles them to returns that period + the option on next period returns, and so on. Thus, the value of the current equity claims = net present value of the firm less the value of the debt may be re-expressed as the value of the current option less its exercise price, since the current option contains the value of the next period option less its exercise price, both contingent upon the acceptance of the current one. Similarly, the net value of the next period option contains the future ones, etc., out to the termination time T. The sum of the value obtainable from the exercise of the option on production-investment decisions plus the (contingent) future ones is the current net present value of the firm and the current value of the debt is the sum of the current exercise price plus the contingent future ones.

The shareholders make their production-investment decisions so as to maximize the current value of equity, the value of the current option (which includes the future contingent values) less the current exercise price plus the value of the future contingent prices. Their decisions maximize the difference in two ways: by increasing the value of the underlying opportunities obtainable if the options are exercised (the value of the firm) and by reducing the value of the exercise prices which must be paid currently and in the future to obtain the underlying value (the value of the outstanding debt). The exercise prices are, of course, the repayments required each period; they cannot be changed, if the same prior debt remains outstanding. Thus, the only way to decrease the current value of the exercise prices of the future options is to decrease the possibility of exercising them, i.e., to increase the possibility of ceasing production (and defaulting on the debt repayment). The owners will therefore make production-investment decisions tending to increase the risk position (a familiar result of the option models) so long as the value of the debt is decreased more than the value of the firm. And they will make decisions increasing return so long as the value of the firm is increased more than the value of the debt. Since the only way to change the value of the debt is to decrease the possibility of the receipt of promised payment, the shareholders will make decisions that increase total risk of the firm, i.e., bankruptcy risk, <u>and</u> consequent loss of value. The trade-off is between changing the value of the exercise prices and changing the underlying value of the future production-investment decisions by both changing their value and changing the chance of reaching them (since future decisions are contingent upon the firm's exercising its options-producing-in prior periods).

These decisions to increase the possibility of bankruptcy represent, however, in the end, a loss to the shareholder. At non-zero debt levels, the value of the firm decreases because the equity-maximizing decisions they will make with the prior debt outstanding increase the possibility of bankruptcy, of non-repayment of the debt. That these decisions do represent a potential loss to the bondholders arises from the existence of a timing difference between the investment decisions and the repayment of the debt implicit in the concept of "prior debt outstanding" at the time the decisions are made. The shareholders decide not to produce when the value of the production-investment opportunities, although possibly positive, is not sufficient to cover the exercise price, the value of the debt. If the debt holders could take over the firm at the time that this investment decision is made, they could undertake the positive-valued opportunities, and there would be no loss, no "agency cost". But, if the investment decisions are made before the time of repyament of the debt, the debt holders cannot gain control of the firm until the declaration of bankruptcy, after the time at which the decision must be made. Thus the value of the current decision plus its future effects are lost to the bondholders; they are left holding, as Myers puts it, "the (empty) bag."

But the bondholders realize the nature of the decisions to be made. They realize that their claims will be worth less as long as the possibility of loss of valuable opportunities exists. They will, therefore pay the true value of their claims and the owners will suffer a net loss, the "agency cost", relative to an all-equity financing position. Thus, for the shareholders to issue debt, there must be a marginal gain, assumed in this paper to be the tax advantage[4], and it must outweigh the marginal loss until the optimum debt is reached, when the marginal gain equals the marginal loss. The trade-off is thus between the loss of valuable investment opportunities as the possibility of bankruptcy is increased by increasing the repayment promises, and the tax shield gain.

## The Environment of the Optimal Decisions

The circumstances in which the shareholders make the optimal production-investment debt decisions are as follows:

(1) The problem of the loss of investment options is, as outlined above, one of timing: the owners, in promising repayment on debt extending over a set number of periods, select a time period framework. The investment opportunities may <u>not</u> necessarily occur at exactly those points in time at which the firm meets payments on its debt. Indeed, as was suggested at the beginning of Chapter 2, investment opportunities appear as soon as

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[4]See Jensen and Meckling for other advantages to retaining full control, which appear to be the result of an imperfect market situation for the owners.

new information is received and if information and decisions on opportunities are costless and information arrives continuously, investment decisions will be implemented continuously throughout the framework of time periods set by the debt repayment schedule. Hence, the caveat of Chapter 2: the stylization of the investment opportunities' appearance into a discrete time framework should not duplicate the framework of the debt repayment schedule, as this would imply a coincidence of occurrences of debt repayment and investment opportunity appearance (and hence, no loss of investment options before the debtholders become owners if the old owners elect not to operate the firm), a situation which does not, in fact, exist. Therefore, a different discrete time framework will be be adopted. Each period of time will be defined as that period extending between two adjacent production investment decisions, as in Chapter 2. The beginning of the $t^{th}$ period is that point at which the decisions ($x_{to}$*, $x_{tn}$*) are made, based on the realization $a_{t-1}$ which yields the information $Ea_t$ for the period. The end of the $t^{th}$ period is the occurrence of the realization $a_t$ which defines the information $Ea_{t+1}$ for the decisions ($x_{t+1_o}$*, $x_{t+1_n}$*), the beginning of the $t+1^{st}$ period. The debt repayment schedule will be set so as not to duplicate this decision framework. Thus, the debt repayment due in period t will be defined as due "during" period t, a finite amount of time <u>after</u> the beginning, the decisions ($x_{to}$*, $x_{tn}$*) and <u>before</u> the end, the realization $a_t$ and the decisions ($x_{t+1_o}$*, $x_{t+1_n}$*). If the debt is callable and a reflotation occurs at the same point within the $t^{th}$ period as the original scheduled repayment. Since no new information is obtained within the $t^{th}$ period (until the end), the new optimal debt decision can (and will) be made at the same point in the period as the production-investment decision ($x_{to}$*, $x_{tn}$*), i.e., at the beginning. However, the old debt will not be recalled and the new debt issued until the point within the $t^{th}$ period at which the original repayment was scheduled.

(2) Given that there will be a possibility each period of the owners' making an equity-maximizing decision not to produce, some assumption must be made as to how much of the value of the firm reamins to the bondholders when they do become legal owners. They will of course lose the current value of the end-of-period net revenue if the firm does not produce in the current period. What remains to be specified is the contribution from the other components of firm value from Chapter 2:

(i) whether the firm preserves its monopoly power for future production decisions by the new owners

(ii) whether the current capacity to produce is available for use in these future period production decisions.

Realistically, the preservation of monopoly power (the constancy of u) seems unlikely under a decision not to produce for a length of time, while the preservation of some future use of current capacity, assets already in place, seems to be reasonable. Thus, realistically, the bondholders would assume control over a firm whose value would be given as the current value at $u = 0$ of the options on future production decisions with current capacity, the value of the firm if it behaved as a pure competitor. For the purpose of examining the case when monopoly power does contribute a large amount to the value of the firm, it will be assumed that the total value of the firm is lost if there is a decision not to produce. This assumption is also chosen for the resulting ease of computation. If the bondholders assume ownership of a continuing firm, their future optimal production-investment and financing decisions will differ from those of the current equity owners because there will, of course, be no outstanding debt the period after the debtholders take over. Had the equity owners retained control, they would have floated new optimal debt and their future period decisions will be affected by the value of this debt next period. Thus, even if there is no change whatever in the market position of the firm's product, the current value of the future cash flows must be computed along both possible paths (that of bankruptcy and of no bankruptcy) open to the firm each period.

(3) The tax regulations are such that

(i) there is a constant tax rate $\tau$

(ii) the tax shield from the debt is lost if the firm goes into bankruptcy

(iii) the tax shield is limited by the amount $\tau RVd_{t-1}$ in any period t, so that the firm is prevented from increasing the shield without limit by increasing the promised payment $P_t$. The rate R will be assumed sufficiently small so that this limit on the tax shield will always be applicable. (This assumption follows the approach of Myers; see his working paper previously cited.)

(4) The sequence of equity and debt flotations and payments takes place as follows. At the beginning of each period, the old shareholders (those of record as of the previous period) make the debt recall and production-investment decisions, financed by retained cash flow and sale of sufficient new equity claims to the future cash flows from the firm to pay for them. The owners then float the equity-maximizing amount of debt (if they are free to do so; i.e., if the call privilege is unlimited), which is determined by the market's valuation of the optimal sequence of payments offered. The funds from this flotation are used to repurchase some, or possibly all, of the new equity; if the funds are greater than the new investment, the remainder is distributed to the old shareholders in the form of dividends or repurchase

of their shares. The owners are assumed able to float new equity to finance the optimal decisions at the beginning of each period as long as the value of the firm (including the optimal new production, investment, and financing decisions) is greater than the value of the outstanding debt at this time, the value of the previously promised sequence of payments. If it is less than the value of the debt at the beginning of the period, the equity owners decide not to produce, and when the first previously promised payment becomes due, in the middle of the current period, the firm declares legal bankruptcy.

The bankruptcy condition at any t is thus that the value of the firm at the point in period t when the payment is due is exactly equal to the payment plus the value of the outstanding future promises. The value of the firm equals the current realized cash flow (net) from the t-1 equity-optimizing production and debt decisions plus the current value of the equity-optimizing production and new debt debt decisions made at the beginning of period t on the basis of information available at the end of period t-1, plus the certainty equivalent of future optimizing decisions.

(5) The optimal debt will be determined in the case that there are no restrictions on either the payment sequence or the maturity of the debt issue. That is, the payments due each period need not be equal and the debt is callable, without penalty, at any time a payment is due. In

this situation, an unrestricted optimum is obtained. Since the firm is able to revise its outstanding debt at t, it will do so. New information $a_{t-1}$ on future demand expectations has been received and the previous debt repayment sequence is in general no longer optimal. At the beginning of each period t, then, the firm's owners make equity-maximizing production decisions $(x_{t_o}{}^*, x_{t_n}{}^*)$ and payment sequence decisions $P^{t*}_{t+1}, \ldots,$ $P^t{}_T{}^*$ (where the superscript t indicates that the payment $P^t_{t+i}{}^*$ was promised in t). Note that, since the risk-free rate of interest r is being held constant, the revaluation of the optimal payment sequence leading to calling of old debt and issuing of new debt is influenced only by revaluation of investment opportunities by the equity holders. There are no interest rate effects.

## B. The Optimal Single Period Decisions

### The Optimization Conditions

As in the previous chapters, the optimal decisions and resulting value at any period are random in earlier period. Thus, the optimal decisions and values of the firm, debt and equity at t will be calculated by backward induction from T, assuming optimal decisions at each future point.

At period T-1, the production decisions $x_{T-1o}$, $x_{T-1n}$, and debt promise $P^{T-1}{}_T$ are determined so as to maximize the value of the equity $Ve_{T-1}$, which is the T-1 value of the firm, $V_{T-1}$, minus the T-1 value of the outstanding debt. The outstanding debt will be that dating from T-2; its T-1 value is denoted $Vd^{T-2}{}_{T-1}$. The value of the firm is the value of all the current and future cash inflows:

$$V_{T-1} = a_{T-2}(x_{T-2o} + x_{T-2n})^{1-n} - c(x_{T-2o} + x_{T-2n})$$
$$+ \frac{Ea_{T-1}L(x_{T-1o} + x_{T-1n})^{1-n} - c(x_{T-1o} + x_{T-1n})}{1+r} - kx_{T-1n}$$
$$+ CEQ_{T-1}\frac{\widetilde{NPV}_T^*}{1+r} + \tau RVd_{T-2} + CEQ\frac{\tau RVd_{T-1}}{1+r}$$

where $CEQ_{T-1}\frac{\widetilde{NPV}_T^*}{1+r}$ is the current value of the optimal production decision in T, $X_T^*$ (the same as in the all equity case, since no debt will be floated in T),

$\tau RVd_{T-2}$ is the tax shield from the debt floated in T-2 to be received this period

$\frac{1}{1+r}CEQ_{T-1}\,\tau RVd_{T-1}$ is the current value of the tax shield expected next period from the current debt decision

$$Vd_{T-2} = \frac{CEQ_{T-2}}{1+r}\left[P^{T-2}_{T-1} + CEQ_{T-1}\frac{P^{T-2}_{T}}{1+r}\right]$$
$$Vd^{T-2}_{T-1} = P^{T-2}_{T-1} + CEQ_{T-1}(Ea_{T-1})\frac{P^{T-2}_{T}}{1+r}$$

and $Vd_{T-1} = CEQ_{T-1}\frac{P^{T-1}_{T}}{1+r}$.

The certainty equivalents are taken over those states $Ea_T \geq Ea_T^B$ where $Ea_T^B$ is the state at which $V_T$ equals the payment to be met in T, $P_T^{T-1}$. The values of both tax shields and firm are zero for states $Ea_T < Ea_T^B$, the occurrence of bankruptcy.

The maximization problem is then

$$\max_{(x_{T-1o},\,x_{T-1n},\,P_T^{T-1})} Ve_{T-1} = V_{T-1} - Vd^{T-2}_{T-1} \qquad (1)$$

subject to

$$x_{T-1o} \le X_{T-1}\ ;\ x_{T-1o},\ x_{T-1n},\ P_T^{T-1} \ge 0$$

and the identity $X_T = X_{T-1} + X_{T-1n}$

where the bankruptcy state $Ea_T^B$ is determined by

the equation

$$P_T^{T-1} = V_T(x_T^*; Ea_T^B, X_T) = NPV_T(x_T^*; Ea_T^B, X_T) + \tau R Vd_{T-1} + a_{T-1}^B (x_{T-1o} + x_{T-1n})^{1-n} - c\,(x_{T-1o} + x_{T-1n}), \quad (2)$$

$$a_{T-1}^B = fcn(Ea_T^B),\ fcn \text{ known}.$$

The Kuhn-Tucker optimum is attained at (3)

(i) $$\left[\frac{\partial V_{T-1}^*}{\partial x_{T-1o}^*} - \frac{\partial Vd_{T-1}^{T-2\,*}}{\partial x_{T-1o}^*} - \lambda_{T-1}^*\right] x_{T-1o}^* = 0$$

$$\left[\frac{\partial V_{T-1}^*}{\partial x_{T-1n}^*} - \frac{\partial Vd_{T-1}^{T-2\,*}}{\partial x_{T-1n}^*}\right] x_{T-1n}^* = 0$$

$$\left[\frac{\partial V_{T-1}^*}{\partial P_T^{T-1\,*}} - \frac{\partial Vd_{T-1}^{T-2\,*}}{\partial P_T^{T-1}}\right] P_T^{T-1\,*} = 0$$

(ii) $$x_{T-1o}^* \le X_{T-1}$$

$$V_T^*(x_{To}^*, x_{Tn}^*; Ea_T^{B*}) = P_T^{T-1\,*}$$

$$X_T = X_{T-1} + x_{T-1n}^*$$

The derivatives are

$$\frac{\partial V_{T-1}}{\partial x_{T-1o}} = \frac{(1-un)\,Ea_{T-1}L\,(x_{T-1o} + x_{T-1n})^{-n} - c}{1+r} + \frac{1}{1+r}\frac{\partial}{\partial x_{T-1o}} CEQ_{T-1}[\widetilde{NPV}_T^* + \tau R Vd_{T-1}]$$

$$\frac{\partial Vd^{T-2}_{T-1}}{\partial x_{T-1o}} = \frac{1}{1+r}\,\frac{\partial}{\partial x_{T-1o}}\,CEQ_{T-1}\,P_T^{T-2}$$

$$\frac{\partial V_{T-1}}{\partial x_{T-1n}} = \frac{(1-un)Ea_{T-1}L\,(x_{T-1o}+x_{T-1n})^{-n}-c}{1+r} - k + \frac{1}{1+r}\,\frac{\partial}{\partial x_{T-1n}}\,CEQ_{T-1}[\tilde{NPV}_T^{*}+\tau R\tilde{V}d_{T-1}]$$

$$\frac{\partial Vd^{T-2}_{T-1}}{\partial x_{T-1n}} = \frac{1}{1+r}\,\frac{\partial}{\partial x_{T-1n}}\,CEQ_{T-1}\,P_T^{T-2}$$

$$\frac{\partial V_{T-1}}{\partial P_T^{T-1}} = \frac{1}{1+r}\,\frac{\partial}{\partial P_T^{T-1}}\,CEQ_{T-1}\,\tau RVd_{T-1} + \frac{1}{1+r}\,\frac{\partial}{\partial P_T^{T-1}}\,CEQ_{T-1}\,\tilde{NPV}_T^{*}$$

$$\frac{\partial Vd^{T-2}_{T-1}}{\partial P_T^{T-1}} = \frac{1}{1+r}\,\frac{\partial}{\partial P_T^{T-1}}\,CEQ_{T-1}\,P_T^{T-2}$$

The lower limit to the certainty equivalent integrals is $Ea_T^B$, a function of the current decisions since it is determined by the future value of the firm plus realized cash flows. Thus, derivatives with respect to it will not vanish when Leibniz' rule is applied. For example,

$$\frac{\partial}{\partial x_{T-1n}}\,CEQ_{T-1}\,\tilde{NPV}_T^{*} = -\,NPV_T^{*}(Ea_T^B)\,\frac{\partial CEQ_{T-1}}{\partial Ea_T^B}\,\frac{\partial Ea_T^B}{\partial x_{T-1n}} + CEQ_{T-1}\,\lambda_T^{*}$$

$$\frac{\partial}{\partial x_{T-1o}}\,CEQ_{T-1}\,\tilde{NPV}_T^{*} = -\,NPV_T^{*}(Ea_T^B)\,\frac{\partial CEQ_{T-1}}{\partial Ea_T^B}\,\frac{\partial Ea_T^B}{\partial x_{T-1o}}$$

where $\frac{\partial CEQ_{T-1}}{\partial Ea_T^B}$ is defined as follows:

$$CEQ_{T-1}\,\tilde{NPV}_T^* = E(\tilde{NPV}_T^*) - m\,\text{cov}(\tilde{NPV}_T^*, \tilde{V}_m)$$

$$= \int_{Ea_T^B}^{\infty} NPV_T^*(Ea_T)\,dg\,(Ea_T | Ea_{T-1}) - m \iint [\tilde{NPV}_T^* - E(\tilde{NPV}_T^*)][\tilde{V}_m - E(\tilde{V}_m)]\,dG$$

in which $g(Ea_T | Ea_{T-1})$ is the density function of $Ea_T$

$G(Ea_T;\ Vm\ |\ Ea_{T-1})$ is the joint density function of $Ea_T$ and Vm, and both are conditioned on $Ea_{T-1}$. The partial derivative with respect to $x_{T-1o}$ is

$$\frac{\partial}{\partial x_{T-1o}} CEQ_{T-1}\,\tilde{NPV}_T^* = -\frac{\partial Ea_T^B}{\partial x_{T-1o}} NPV_T^*(Ea_T^B)\Big[g(Ea_T^B) - m\int [\tilde{V}_m - E(\tilde{V}_m)]\,dG\,(Ea_T^B)\Big]$$

$$= -NPV_T^*(Ea_T^B)\,\frac{\partial Ea_T^B}{\partial x_{T-1o}} \cdot \frac{\partial CEQ_{T-1}}{\partial Ea_T^B}$$

Comparison of the two expressions defines $\frac{\partial CEQ_{T-1}}{\partial Ea_T^B}$ ; it is essentially a probability density of $Ea_T{}^B$ discounted for risk. Similarly

$$\frac{\partial}{\partial x_{T-1n}} CEQ_{T-1}\,\tilde{NPV}_T^* = -NPV_T^*(Ea_T^B)\,\frac{\partial Ea_T^B}{\partial x_{T-1n}}\,\frac{\partial CEQ_{T-1}}{\partial Ea_T^B} + CEQ_{T-1}\,\tilde{\lambda}_T^*$$

since $Ea_T{}^B$ is a function of $x_{T-1o}$ through the realized cash flow term in the bankruptcy equation. The derivatives $\frac{\partial Ea_T^B}{\partial x_{T-1o}}, \frac{\partial Ea_T^B}{\partial x_{T-1n}}$ measuring the effect of the production-investment decisions on bankruptcy are determined by implicit differentiation of the equation determining $Ea_T{}^B$:

$$\frac{\partial Ea_T^B}{\partial x_{T-1o}} = -\left[\frac{\partial V_T^*}{\partial Ea_T^B}\Big|(Ea_T^B)_{x_{T-1o},\,x_{T-1n},\,P_T^{T-1}}\right]^{-1}\left[(1-u\pi)a_{T-1}^{B}\,x_{T-1}^{-\alpha} - c\right] \qquad (42)$$

$$\frac{\partial Ea_T^B}{\partial x_{T-1n}} = - \left[ \frac{\partial V_T^*}{\partial Ea_T^B} \Big|_{x_{T-1o}, x_{T-1n}, P_T^{T-1}} (Ea_T^B) \right]^{-1} \left[ \lambda_T^*(Ea_T^B) + (1-un) a_{T-1}^B x_{T-1}^{-n} - c \right] \quad (4b)$$

where

$$\frac{\partial V_T^*}{\partial Ea_T^B} \Big|_{x_{T-1o}, x_{T-1n}, P_T^{T-1}} (Ea_T^B) = \left[ \frac{\partial NPV_T^*(Ea_T^B)}{\partial Ea_T^B} + (x_{T-1})^{1-n} \frac{\partial a_{T-1}^B}{\partial Ea_T^B} + \tau R \frac{\partial V d_{T-1}}{\partial Ea_T^B} \right]_{x_{T-1o}, x_{T-1n}, P_T^{T-1}}$$

$$= \left[ \frac{\partial NPV_T^*(Ea_T^B)}{\partial Ea_T^B} + (x_{T-1})^{1-n} \frac{\partial a_{T-1}^B}{\partial Ea_T^B} \right]_{x_{T-1o}, x_{T-1n}, P_T^{T-1}} - \tau R \frac{P_T^{T-1}}{1+r} \frac{\partial CEQ_{T-1}}{\partial Ea_T^B}$$

The effect of the decisions variables $(x_{T-1o}, x_{T-1n})$ depends on the sign of $\frac{\partial V_T^*}{\partial Ea_T^B} \Big|_{x_{T-1o}, x_{T-1n}, P_T^{T-1}} (Ea_T^B)$. This in turn is determined by the behavior of the tax shield. As $Ea_T{}^B$ is increased, the increase in $NPV_T$ and realized cash flow is offset by a decrease in the tax shield due to a lower market value for the promised payment $P_T^{T-1}$. Whether it is offset enough to produce a negative value for the marginal change in $V_T^*$ as $Ea_T{}^B$ is increased depends on the parameters $\tau$,R, the distribution of $Ea_T$ and the promised payment $P_T^{T-1}$. For low $P_T^{T-1}$ the bankruptcy state $Ea_T{}^B$ will fall in the left-hand tail of the distribution of $Ea_T$, so that $\frac{\partial CEQ_{T-1}}{\partial Ea_T^B}$ will be small for typical distributions; thus the first two terms will dominate. As $P_T^{T-1}$ increases, the third term increasingly offsets the first two until $Ea_T{}^B$ is pushed so high that $\frac{\partial CEQ_{T-1}}{\partial Ea_T^B}$ begins to decline. It is at this latter stage that $\frac{\partial V_T^*}{\partial Ea_T^B}(Ea_T^B)$ may be less than zero; however, it seems unlikely that the optimal payment would be so high. One would expect that $V_T^*$ evaluated at any $Ea_T$ to rise if $Ea_T$ rises.

The same pattern of non-zero derivatives of $Ea_T^B$ occurs when the partials are taken with respect to $P_T^{T-1}$

$$\frac{\partial}{\partial P_T^{T-1}} CEQ_{T-1} NPV_T^* = - NPV_T^*(Ea_T^B) \frac{\partial CEQ_{T-1}}{\partial Ea_T^B} \frac{\partial Ea_T^B}{\partial P_T^{T-1}}$$

where

$$\left.\frac{\partial V_T^*(Ea_T^B)}{\partial P_T^{T-1}}\right|_{X_{T-1o}, X_{T-1n},} = \left.\frac{\partial V_T^*(Ea_T^B)}{\partial Ea_T^B}\right|_{X_{T-1o}, X_{T-1n}, P_T^{T-1}} \frac{\partial Ea_T^B}{\partial P_T^{T-1}} + \left.\frac{\partial V_T^*}{\partial P_T^{T-1}}(Ea_T^B)\right|_{X_{T-1o}, X_{T-1n}, Ea_T^B} = \frac{\partial P_T^{T-1}}{\partial P_T^{T-1}} = 1$$

and

$$\left.\frac{\partial V_T^*(Ea_T^B)}{\partial P_T^{T-1}}\right|_{X_{T-1o}, X_{T-1n}, Ea_T^B} = \tau R \left.\frac{\partial Vd_{T-1}}{\partial P_T^{T-1}}\right|_{X_{T-1o}, X_{T-1n}, Ea_T} = \tau R \frac{1}{1+r} CEQ_{T-1},$$

$$CEQ_{T-1} = Pr\left\{Ea_T > Ea_T^B\right\} \text{ discounted for risk.}$$

Then

$$\frac{\partial Ea_T^B}{\partial P_T^{T-1}} = \left[\left.\frac{\partial V_T^*}{\partial Ea_T^B}\right|_{X_{T-1o}, X_{T-1n}, P_T^{T-1}} (Ea_T^B)\right]^{-1} \left[1 - \frac{\tau R}{1+r} CEQ_{T-1}\right] > 0 \qquad (4c)$$

for $\left.\frac{\partial V_T^*}{\partial Ea_T^B}\right|_{X_{T-1o}, X_{T-1n}, P_T^{T-1}} (Ea_T^B) > 0$. (Assume.)

Thus, by (3)

$$(i) \quad 0 = \left\{\frac{(1-un)\, Ea_{T-1} L\, (X_{T-1o}^* + X_{T-1n}^*)^{-n} - C}{1+r} - \lambda_{T-1}^* - \frac{1}{1+r}\left[NPV_T^* + \tau R Vd_{T-1}^* + CEQ_{T-1} \frac{\tau R P_T^{T-1*}}{1+r} - P_T^{T-2}\right]_{Ea_T^{B*}} \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}} \frac{\partial Ea_T^{B*}}{\partial X_{T-1o}^*}\right\} X_{T-1o}^* \qquad (5a)$$

$$0 = \left\{\frac{(1-un)\, Ea_{T-1} L\, (X_{T-1o}^* + X_{T-1n}^*)^{-n} - C}{1+r} - k + \frac{CEQ_{T-1}\tilde{\lambda}_T^*}{1+r} - \frac{1}{1+r}\left[NPV_T^* + \tau R Vd_{T-1}^* + CEQ_{T-1} \frac{\tau R P_T^{T-1*}}{1+r} - P_T^{T-2}\right]_{Ea_T^{B*}} \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}} \frac{\partial Ea_T^{B*}}{\partial X_{T-1n}^*}\right\} X_{T-1n}^* \qquad (5b)$$

$$0 = \left\{\frac{\tau R}{(1+r)^2} CEQ_{T-1}^2 - \frac{1}{1+r}\left[NPV_T^* + \tau R Vd_{T-1}^* + CEQ_{T-1} \frac{\tau R P_T^{T-1*}}{1+r} - P_T^{T-2}\right]_{Ea_T^{B*}} \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}} \frac{\partial Ea_T^{B*}}{\partial P_T^{T-1*}}\right\} P_T^{T-1*} \qquad (5c)$$

(ii) $x_{T-1o}^{*} \leq X_{T-1}$

$$V_T^{*}(X_T^{*}; Ea_T^{B*}, X_T) = Vd_T^{T-1}(Ea_T^{B*}) = P_T^{T-1\,*}$$

$$X_T = X_{T-1} + x_{T-1o}^{*}$$

## Results for the Single Period Case

These equations lead to the following single period results for the firm financing optimally with debt:

(1) The determination of the production decision $0 < x_{T-1o}^{*} < X_{T-1}$ is not the same as in the all equity financing case. The current value of net marginal revenue from production out of previous capacity this period is increased by the marginal contribution of the cash flow from this production to the certainty of receiving equity claims to next period's value. The cash flow (or dividend) realized at the end of this period changes the value of the equity at the beginning of the next period, thereby changing the ability of the equity owners to meet bankruptcy. Thus, the optimal $x_{T-1o}^{*}$ is set by (5a) so as to maximize the sum of both contributions to current value:

$$\lambda_{T-1}^{*} = \frac{(1-un)\,Ea_{T-1}\,L(x_{T-1o}^{*} + x_{T-1n}^{*})^{-n} - c}{1+r}$$

$$- \frac{1}{1+r}\left[NPV_T^{*} + \tau RVd_{T+1}^{*} + CEQ_{T-1}\frac{\tau R\,P_T^{T-1\,*}}{1+r} - P_T^{T-2}\right]_{Ea_T^{B*}} \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}}\,\frac{\partial Ea_T^{B*}}{\partial x_{T-1o}^{*}} \qquad (6d)$$

where

$$\frac{\partial Ea_T^{B*}}{\partial x_{T-1o}^{*}} = -\left[\frac{\partial V_T^{*}}{\partial Ea_T^{B*}}\,\middle|\,(Ea_T^{B*})\right]^{-1}_{x_{T-1o},\,x_{T-1n},\,P_T^{T-1}} \left[(1-un)\,a_{T-1}^{B*}(x_{T-1}^{*})^{-n} - c\right]$$

from (4a).

Note that the second term in (6a), the marginal change in the certainty of receiving current equity claims has three sources:

(i) the marginal change at $Ea_T^{B*}$ in the certainty of receiving claims to the value of the firm,

$$\frac{1}{1+r}\left[NPV_T^* + \tau R Vd_T^{T-1}\right]_{Ea_T^{B*}} \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}} \frac{\partial Ea_T^{B*}}{\partial X_{T-1,0}^*}$$

(ii) the marginal change in the tax shield itself because of the change in $Vd_{T-1}^*$ as $Ea_T^{B*}$ is changed,

$$\frac{1}{1+r}\left[CEQ_{T-1}\,\tau R \frac{P_T^{T-1*}}{1+r}\right] \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}} \frac{\partial Ea_T^{B*}}{\partial X_{T-1,0}^*}$$

(This term does not have the appearance of a marginal quantity evaluated at $Ea_T^{B*}$; that it is indeed such will become evident on examination of the T-2 optimization conditions.)

(iii) the marginal change in current equity value because of the change in the current market value of the outstanding debt

$$\frac{1}{1+r} P_T^{T-2} \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}} \frac{\partial Ea_T^{B*}}{\partial X_{T-1,0}^*}.$$

To examine the behavior of $\lambda_{T-1,0}^*$, one should investigate the case $\lambda_{T-1}^* = 0$ — production solely from previous capacity $X_{T-1}$. Then (6a) becomes (6a′)

$$\frac{(1-un)\,Ea_{T-1} L (X_{T-1,0}^*)^{-a}}{1+r} - \frac{c}{1+r} = \frac{1}{1+r}\left[NPV_T^* + \tau R Vd_{T-1}^{T*} + CEQ_{T-1}\,\tau R \frac{P_T^{T-1*}}{1+r} - P_T^{T-2}\right] \cdot \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}} \frac{\partial Ea_T^{B*}}{\partial X_{T-1,0}^*}$$

and the marginal decrease in certainty of receiving equity claims becomes an additional cost if positive, benefit if $< 0$. At $x_{T-10}^*$ the sum is equated to the certainty equivalent of marginal revenue. Whether the left-hand side in (6a') is positive or negative ("underproduction" or "overproduction" relative to the all equity case) depends on the size of $Ea_{T-1}L$ relative to $a_{T-1}^B$. If $Ea_{T-1}L < a_{T-1}^{B*}$, $x_{T-10}^*$ will be such that the left hand side is positive as is the additional cost, and vice versa if $Ea_{T-1}L > a_{T-1}^{B*}$. For $Ea_{T-1}L > a_{T-1}^{B*}$

$$\frac{\partial Ea_T^{B*}}{\partial x_{T-10}^*} > 0 \quad , \quad \frac{(1-un)Ea_{T-1}L(x_{T-10}^*)^{-n} - c}{1+r} > 0 .$$

In maximizing the equity value from current production out of $X_{T-1}$ the shareholders have "overproduced" if they know that the bankruptcy state will be realized (the marginal revenue at bankruptcy, $(1-un)\ a_{T-1}^{B*}(x_{T-1*})^{-n}$ is less than marginal cost c) so that the chance of bankruptcy is increased by increasing $x_{T-10}^*$. But they have "underproduced" relative to all equity financing (the certainty equivalent of marginal revenue $(1-un)Ea_{T-1}\ L(x_{T-10}^*)^{-n}$ is greater than marginal cost c). Thus, they are "hedging": they underproduce for the expected state $Ea_{T-1}$ in order to be closer to, although still greater than, the optimum production should the realized state be the bankruptcy state. (The same result appears out of the case $Ea_{T-1}L < a_{T-1}^{B*}$: they hedge by overproducing for the expected state $Ea_{T-1}$ so that they will have a cash flow closer to optimum for the bankruptcy state.) The hedging thus increases NPV until at the margin the cost in increased bankruptcy just equals the gain. (If $Ea_{T-1}L$ is less than $a_{T-1}^{B*}$ an increase in production decreases the possibility of bankruptcy; thus production is continued until the decrease in NPV equals this gain.) Since one might expect $Ea_{T-1}L$ to be above the realization $a_{T-1}B^*$ which causes bankruptcy next period, one would generally expect to see "under-usage" of capacity relative to all equity firms; production at optimal all equity levels results in too great a cost in bankruptcy possibility. This "under-usage" would appear to be more important for firms with significant monopoly power, as that increases the importance of the right-hand side of (6a') and hence the hedging effect.

Note that this method of hedging, producing sub-optimally relative to all equity financed firms to assure greater certainty of receiving future value, is a consequence of the assumption that the firm is a quantity announcer. Given the uncertainty pattern, the equity owners hedge by means of the decision variables currently available to them, i.e. production decisions. The principle remains, whatever decision variables the model specifies: current NPV from all equity decisions will be sub-optimal for the state $Ea_{T-1}$ with debt financing, traded at the margin for cash flow closer to optimum at the (lower) bankruptcy state realization $a_{T-1}B^*$.

(2) The optimality of an investment-production decision $x_{T-1_n}^*>0$ also occurs at a different level from the all equity case. Specifically, the decision to purchase new capacity is taken when the current value of the net marginal revenue from production this period, plus the

current value of possible use in the future, and plus the changes in the current value of the equity next period from the marginal unit of new capacity, equals the marginal cost k. Thus, at $x_{T-1_n}{}^* > 0$ (5b) can be written

$$\frac{(1-un)\,Ea_{T-1}L(x_{T-1}^{*})^{-n}-c}{1+r} + \frac{CEQ_{T-1}\,\lambda_T^{*}}{1+r} - \frac{1}{1+r}\left[NPV_T^{*} + \tau RVd_{T-1} + TR\,CEQ_{T-1}\frac{P_T^{T-1}}{1+r} - P_T^{T-2}\right]_{Ea_T^{\theta *}} \cdot \frac{\partial CEQ_{T-1}}{\partial Ea_T^{\theta *}}\,\frac{\partial Ea_T^{\theta *}}{\partial x_{T-1n}^{*}} = k \qquad (6b)$$

The left-hand side of (6b) is the value of current and future marginal contributions to equity as in the all equity case, plus the marginal increase in current value from an increase in certainty of receiving next period equity claims as discussed under (1). This increase in the certainty of next period equity claim arises in two ways:

(i) from the changes in marginal cash flows on hand next period from the current period production using $x_{T-1_n}$, exactly parallel to the $x_{T-1_o}{}^*$ optimization condition,

$$-\frac{1}{1+r}\left[NPV_T^{*}(Ea_T^{\theta *}) + \tau RVd_{T-1} + CEQ_{T-1}TR\frac{P_T^{T-1*}}{1+r} - P_T^{T-2}\right]\frac{\partial CEQ_{T-1}}{\partial Ea_T^{\theta *}}\left[\frac{\partial Ea_T^{\theta *}}{\partial x_{T-1n}}\right]_{(1)}$$

where

$$\left[\frac{\partial Ea_T^{\theta}}{\partial x_{T-1n}}\right] = \left[\frac{\partial Ea_T^{\theta}}{\partial x_{T-1n}}\right]_{(1)} + \left[\frac{\partial Ea_T^{\theta}}{\partial x_{T-1n}}\right]_{(2)} \text{ from (4b),}$$

$$\left[\frac{\partial Ea_T^{\theta}}{\partial x_{T-1n}}\right]_{(1)} = -\left[\frac{\partial V_T^{*}}{\partial Ea_T^{\theta *}}\Big|(Ea_T^{\theta *})\right]^{-1}_{x_{T-1o},\,x_{T-1n},\,P_T^{T-1}}\left[(1-un)a_{T-1}^{\theta}(x_{T-1}^{*})^{-n}-c\right];$$

(ii) from the decrease in next period bankruptcy state caused by the possession <u>next</u> period of the extra marginal unit of capacity. The possession of a unit of capacity at any time creates value for the firm at that time because the cost of use of this capital is less than that of new capital, as discussed in the all equity case. Thus, the equity claims to firm value next period will be greater for the firm with previously acquired capacity than for the firm obliged to purchase new capacity for optimal production (assuming the outcome $Ea_T$ to be such that this level of production is optimal). Because this capacity can influence the next period value, it is one of the determinants of the bankruptcy state $Ea_T B^*$. Therefore, it increases current equity value by both the certainty equivalent of its future marginal contribution to firm value, $CEQ_{T-1}\frac{\tilde{\lambda}_T^*}{1+r}$ and the marginal increase in certainty of receiving the equity claim to next period value

$$-\frac{1}{1+r}\left[NPV_T^* + TR\,Vd_{T-1}^* + TR\,CEQ_{T-1}\frac{P_T^{T-1\,*}}{1+r} - P_T^{T-2}\right]\frac{\partial CEQ_{T-1}}{\partial Ea_T B^*}\bigg|_{Ea_T B^*}\left[\frac{\partial Ea_T B^*}{\partial x_{T-1n}^*}\right]_{(2)}$$

$$\text{where}\quad\left[\frac{\partial Ea_T B^*}{\partial x_{T-1n}}\right]_{(2)} = -\left[\frac{\partial V_T^*}{\partial Ea_T B^*}\bigg|_{x_{T-1o},\,x_{T-1n},\,P_T^{T-1}}(Ea_T B^*)\right]^{-1}\lambda_T^*(Ea_T B^*).$$

The question of whether $x_{T-1_n}{}^*$ is greater or less than under all equity financing depends on the sign of $\frac{\partial Ea_T B^*}{\partial x_{T-1n}^*}$, since the third term in (6b) represents an extra marginal advantage (or loss) if it is positive (or negative) to the equity holders. While $\left[\frac{\partial Ea_T^{B*}}{\partial x_{T-1_n}^*}\right]_{(2)}$ is always negative, $\left[\partial Ea_T^{B*}/\partial x_{T-1_n}^*\right]_{(1)}$ described in (i), the contribution to future cash flow from current production, may be positive. Indeed, at levels of $Ea_{T-1}$ such that $x_{T-1_n}^* > 0$, one would expect $x_{T-1}^*$ to be such that $(1-un)a_{T-1}B^*(x_{T-1}^*)^{-n} < c$ assuming $a_{T-1}B^*$ is low as in (i). The firm's owners again "hedge" between the advantage of optimal production at $Ea_{T-1}$ and optimal cash flow at the next period bankruptcy state. Since $\lambda_T^*(Ea_T B^*)$ is probably not large, again assuming $a_{T-1}B^*$ and therefore $Ea_T B^*$ to be low, one would expect to find $\frac{\partial Ea_T^{B*}}{\partial x_{T-1_n}^*} > 0$ in (4b), overall. There is again a marginal decrease in certainty of receiving equity claims to next period value as $x_{T-1_n}$ is increased. Thus the effective current cost of a unit of new capacity is k less the marginal value of future use, but plus the marginal change in the value of equity due to the change in the firm's ability to meet next period debt obligation; at optimum $x_{T-1_n}^* > 0$ this effective cost is equal to the current value of marginal revenue. The level of $Ea_{T-1}$ at which $x_{T-1_n}^* > 0$ becomes optimal is determined by solving the equation at $X_{T-1} = x_{T-1}^*$. As in the all equity case, the effective marginal cost is also a function of $Ea_{T-1}$ since the distribution of $Ea_T$ is conditional on $Ea_{T-1}$.

The owners of the firm financing (optimally) with debt will thus in general optimally invest in less capacity than the all-equity firm (depending on the net result of the two contributions) with the amount depending on the change in effective marginal cost provided by the change in certainty term. Since the marginal value of equity claims to future cash flows increases with monopoly power, one may expect to find firms with more monopoly power "under-invested" relative to firms of equal monopoly power financing without debt.

It should be noted, however, that both the production and addition of capacity decision are determined simultaneously with the debt decisions. If firms with monopoly power carry less debt, there may be little probability that they will find themselves in the low bankruptcy state $Ea_T^{B*}$ (specifically, the probability term $\frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}}$ is very small); thus they have little need of the extra protection afforded by "under-production" and "under-investment". If firms with relatively less monopoly power carry relatively larger amounts of debt, their tendency to "under-produce" and "under-invest" for protection will be counterbalanced by the fact that, with less monopoly power, there is less to protect.

(3) The optimality of $P_T^{T-1*} > 0$ occurs at

$$\frac{\tau R}{(1+r)^2} CEQ_{T-1}^2 = \frac{1}{1+r}\left[NPV_T^* + \tau RV_{d_{T-1}}^* + \tau R\, CEQ_{T-1} \frac{P_T^{T-1*}}{1+r} - P_T^{T-2}\right]_{Ea_T^{B*}} \frac{\partial CEQ_{T-1}}{\partial Ea_T^{B*}} \frac{\partial Ea_T^{B*}}{\partial P_T^{T-1*}}. \quad (6c)$$

That is, the marginal increase in the value of equity from the increase in the tax shield, which comes about from the increase in the value of the debt as $P_T^{T-1*}$ is marginally increased, must equal the marginal decrease in the value of the equity because of the decrease in the certainty of non-zero equity value in the next period due to the marginal increase in the obligation $P_T^{T-1*}$. The decrease in the value of equity due to the change in the certainty equivalent is of the same form as in the optimization condition for $x_{T-1_n}*$, but the source is different; i.e., the partial $\frac{\partial W_T^{e*}}{\partial P_T^{T-1*}}$ (which is greater than zero) enters here. The optimal payment is set so as to equate these two trade-offs, as in the Myers result.

As the equity value depends on the monopoly power, the marginal loss is larger for the firm with the larger monopoly power. Thus, one would expect to see firms with larger monopoly power financing with less debt relative to firms of the same total present value (or value of total assets) with little monopoly power. They have "more to lose" because their value comes from future value which will be lost if bankruptcy occurs. (The amount of the loss has been assumed so far to be the total future value of the firm; in the case that the loss of the investment decision at bankruptcy is not so great, a greater level of debt will be optimal.)

(4) From (3) and (2) and, in actuality, from the original statement of the optimization conditions in equation (1), it is implicit that the decisions $(x_{T-1_o}*, x_{T-1_n}*, P^{T-1}_T*)$ do not maximize the value of the firm:

$\frac{\partial V_{T-1}^*}{\partial x_{T-1o}^*}, \frac{\partial V_{T-1}^*}{\partial x_{T-1n}^*}, \frac{\partial V_{T-1}^*}{\partial P_T^{T-1}} \neq 0$. Also,

$\frac{\partial E\lambda_T^{B*}}{\partial x_{T-1o}^*}, \frac{\partial E\lambda_T^{B*}}{\partial x_{T-1n}^*}, \frac{\partial E\lambda_T^{B*}}{\partial P_T^{T-1*}} > 0$; and

$$\frac{\partial Vd_{T-1}^*}{\partial P_T^{T-1*}} = \frac{CEQ_{T-1}}{1+r} - \frac{P_T^{T-1*}}{1+r} \frac{\partial CEQ_{T-1}}{\partial E\lambda_T^{B*}} \frac{\partial E\lambda_T^{B*}}{\partial P_T^{T-1*}} \neq 0 \quad \text{since}$$

$$\frac{P_T^{T-1*}}{1+r} \frac{\partial CEQ_{T-1}}{\partial E\lambda_T^{B*}} \frac{\partial E\lambda_T^{B*}}{\partial P_T^{T-1*}} = \frac{\tau R}{(1+r)^2} CEQ_{T-1}^2 + P_T^{T-2} \frac{\partial CEQ_{T-1}}{\partial E\lambda_T^{B*}} \frac{\partial E\lambda_T^{B*}}{\partial P_T^{T-1*}}$$

$$\neq \frac{1}{1+r} CEQ_{T-1}$$

except with a particular choice of $P_T^{T-2}$. To see whether $P_T^{T-2*} = 0$, i.e., whether the firm optimally issues one period debt or debt of longer maturity, one must investigate the two period case, optimization at T-2.

## C. The Multiperiod Problem

### <u>The Optimization Conditions</u>

At period T-2, the promised payment sequence $(P_{T-1}^{T-2*}, P_T^{T-2*})$ is determined so as to maximize the value of equity, $Ve_{T-2}$, which is the value of the firm $V_{T-2}$ minus the value of outstanding debt $Vd_{T-2}^{T-3}$, and floats $Vd_{T-1}^*$ as determined in section B. If the optimal payment sequence is one-period, $P_T^{T-2*} = 0$, and the firm pays the promised amount $P_{T-1}^{T-2}$ in T-1 and floats $Vd_{T-1}^*$ (or ceases production).

In general, the optimal payment sequence is $(P_{T-1}^{T-2*}, P_T^{T-2*})$ determined along with $(x_{T-2_o}^*, x_{T-2n})$ so as to

$$\max_{(x_{T-2o},\, x_{T-2n},\, P_{T-1}^{T-2},\, P_T^{T-2})} Ve_{T-2} = V_{T-2} - Vd_{T-2}^{T-3}$$

$$= \frac{\tilde{E}a_{T-2} L(x_{T-2o} + x_{T-2n})^{1-n} - c(x_{T-2o} + x_{T-2n})}{1+r} - kx_{T-2n}$$

$$+ \; CEQ_{T-2} \frac{NPV_{T-1}^*}{1+r} + \tau RVd_{T-3} + \frac{CEQ_{T-2}\, \tau RVd_{T-2}}{1+r}$$

$$+ \; \frac{CEQ_{T-2}\, CEQ_{T-1}\, \tau RVd_{T-1}^*}{(1+r)^2} + a_{T-3}(x_{T-3o} + x_{T-3n})^{1-n} - c(x_{T-3o} + x_{T-3n}) - Vd^{T-3}_{T-2}$$

subject to

$$x_{T-2o} \le X_{T-2}$$
$$V_{T-1}^*(\tilde{E}a_{T-1}^{\theta}) = Vd^{T-2}_{T-1}(\tilde{E}a_{T-1}^{\theta}),$$

where

$$Vd_{T-2} = \frac{CEQ_{T-2}}{1+r}\left[\rho^{T-2}_{T-1} + C\tilde{E}Q_{T-1} \frac{\rho^{T-2}_{T}}{1+r}\right]$$

$CEQ_{T-2} \frac{NPV_{T-1}^*}{1+r}$ is the current value of the optimal production decisions in T-1 and T. The remaining terms containing $\tau R$ are the current tax shield and current values of the future tax shields. The value of the optimal production decisions in the future, $NPV_{T-1}^*$, may be written

$$NPV_{T-1}^* = \frac{\tilde{E}a_{T-1} L(x_{T-1o}^* + x_{T-1n}^*)^{1-n} - c(x_{T-1o}^* + x_{T-1n}^*)}{1+r} - kx_{T-1n}^*$$

$$+ \; CEQ_{T-1} \frac{NPV_T^*}{1+r}$$

$$= \frac{[(1-un)\tilde{E}a_{T-1} L(x_{T-1o}^* + x_{T-1n}^*)^{-n} - c](x_{T-1o}^* + x_{T-1n}^*)}{1+r} - kx_{T-1n}^*$$

$$+ \; \frac{un\, \tilde{E}a_{T-1} L(x_{T-1}^*)^{1-n}}{1+r} + CEQ_{T-1} \frac{\tilde{\lambda}_T^* X_T + \tilde{M}_T^*}{1+r}$$

where $\lambda_T^*$, $M_T^*$ are as in Chapter 2,

$$X_T = X_{T-1} + x_{T-1n}^*.$$

Thus, $$NPV_{T-1}^* = \left[\lambda_{T-1} + \frac{CEQ_{T-1}\lambda_T^*}{1+r}\right] X_{T-1} + M_{T-1}^*$$

$$+ \frac{1}{1+r}\left[NPV_T^* + \tau RVd_{T-1}^* + \tau R\, CEQ_{T-1} \frac{\rho^{T-1*}_{T}}{1+r} - \rho^{T-2}_{T}\right]_{\tilde{E}a_T^{\theta*}} \cdot$$

$$\cdot \left[\frac{\partial \tilde{E}a_T^{\theta*}}{\partial x_{T-1o}^*} x_{T-1o}^* + \frac{\partial \tilde{E}a_T^{\theta*}}{\partial x_{T-1n}^*} x_{T-1n}^*\right]$$

where

$$M_{T-1}{}^{*} = \frac{un\,Ea_{T-1}L(x_{T-1}{}^{*})^{1-n}}{1+r} + CEQ_{T-1}\frac{\tilde{M}_T{}^{*}}{1+r}$$

and

$\frac{\partial Ea_T{}^{B*}}{\partial x_{T-1_o}{}^{*}}$, $\frac{\partial Ea_T{}^{B*}}{\partial x_{T-1n}{}^{*}}$ $> 0$ generally since $(x_{T-1_o}{}^{*}, x_{T-1_n}{}^{*})$ are generally belows the levels which maximize $NPV_{T-1}{}^{*}$. The coefficients of $x_{T-1_o}{}^{*}$ and $x_{T-1_n}{}^{*}$, the value per unit of $(x_{T-1_o}{}^{*}, x_{T-n}{}^{*})$ different from the all equity optimal values, are the marginal increases in T-1 value of equity claims in T at the production level $x_{T-1}{}^{*}$ from the change in bankruptcy state. The current value of the debt sequence promised in T-3, the current outstanding debt, is

$$Vd^{T-3}_{T-2} = P^{T-3}_{T-2} + \frac{CEQ_{T-2}}{1+r}\left[P^{T-3}_{T-1} + CEQ_{T-1}\frac{P^{T-3}_{T}}{1+r}\right]$$

The maximum is attained at (7)

$$x_{T-2o}{}^{*}\left[\frac{\partial Ve_{T-2}{}^{*}}{\partial x_{T-2o}{}^{*}} - \lambda^{*}_{T-2}\right] = 0$$

$$x_{T-2n}{}^{*}\left[\frac{\partial Ve_{T-2}{}^{*}}{\partial x_{T-2n}{}^{*}}\right] = 0$$

$$P^{T-2}_{T-1}{}^{*}\left[\frac{\partial Ve_{T-2}{}^{*}}{\partial P^{T-2}_{T-1}{}^{*}}\right] = 0$$

$$P^{T-2}_{T}{}^{*}\left[\frac{\partial Ve_{T-2}{}^{*}}{\partial P^{T-2}_{T}{}^{*}}\right] = 0, \quad \text{or, on substitution,}$$

(1) $$x_{T-2o}{}^{*}\Bigg\{\frac{(1-un)Ea_{T-2}L(x_{T-2}^{*})^{-n} - c}{1+r} - \lambda_{T-2}{}^{*} - \frac{1}{1+r}\frac{\partial CEQ_{T-2}}{\partial Ea_{T-1}{}^{B*}}\frac{\partial Ea_{T-1}{}^{B*}}{\partial x_{T-2o}{}^{*}}\Big[NPV^{*}_{T-1} + \tau RVd^{*}_{T-2} + CEQ_{T-1}(Ea_T{}^{B*})\frac{\tau RVd^{*}_{T-1}}{1+r} + CEQ_{T-2}\frac{\tau RVd^{T-2}_{T-1}}{1+r}Vd^{T-3}_{T-1}\Big]_{Ea_{T-1}{}^{B*}}\Bigg\} = 0 \qquad (8a)$$

$$x_{T-2n}^* \left\{ \frac{(1-un)\, Ea_{T-2} L\,(x_{T-2}^*)^{-n} - c}{1+r} - k + \frac{CEQ_{T-2}}{1+r}\left[ \lambda_{T-1}^* + CEQ_{T-1} \frac{\lambda_T^*}{1+r} \right] \right.$$

$$- \frac{1}{1+r} \frac{\partial CEQ_{T-2}}{\partial Ea_{T-1}^{B*}} \frac{\partial Ea_{T-1}^{B*}}{\partial x_{T-2n}^*} \Big[ NPV_{T-1}^* + \tau RVd_{T-2}^* + CEQ_{T-1} \frac{\tau RVd_{T-1}^*}{1+r}$$

$$\left. + CEQ_{T-2} \frac{\tau RVd_{T-1}^{T-2*}}{1+r} - Vd_{T-1}^{T-3} \Big]_{Ea_{T-1}^{B*}} \right\} = 0 \qquad (8b)$$

$$P_{T-1}^{T-2*} \left\{ \frac{\tau R}{(1+r)^2} CEQ_{T-2}^2 - \frac{1}{1+r} \frac{\partial CEQ_{T-2}}{\partial Ea_{T-1}^{B*}} \frac{\partial Ea_{T-1}^{B*}}{\partial P_{T-1}^{T-2*}} \Big[ NPV_{T-1}^* + \tau RVd_{T-2}^* + CEQ_{T-1} \frac{\tau RVd_{T-1}^*}{1+r} \right.$$

$$\left. + CEQ_{T-2} \frac{\tau RVd_{T-1}^{T-2*}}{1+r} - Vd_{T-1}^{T-3} \Big]_{Ea_{T-1}^{B*}} \right\} = 0 \qquad (8c)$$

$$P_{T}^{T-2*} \left\{ \frac{\tau R}{(1+r)^2} CEQ_{T-2}^2\, CEQ_{T-1} - \frac{1}{1+r} \frac{\partial CEQ_{T-2}}{\partial Ea_{T-1}^{B*}} \frac{\partial Ea_{T-1}^{B*}}{\partial P_{T}^{T-2*}} \Big[ NPV_{T-1}^* + \tau RVd_{T-2}^* + CEQ_{T-1} \frac{\tau RVd_{T-1}^*}{1+r} \right.$$

$$\left. + CEQ_{T-2} \frac{\tau RVd_{T-1}^{T-2*}}{1+r} - Vd_{T-1}^{T-3} \Big]_{Ea_{T-1}^{B*}} \right\} = 0 \qquad (8d)$$

(2) $x_{T-2o}^* \le X_{T-2}$

$X_{T-1} = X_{T-2} + x_{T-2n}^*$

$$V_{T-1}^*(x_{T-1}^*, Ea_{T-1}^{B*}, X_{T-1}) = Vd_{T-1}^{T-2*}(Ea_{T-1}^{B*}) = P_{T-1}^{T-2*} + CEQ_{T-1}(Ea_{T-1}^{B*}) \frac{P_T^{T-2*}}{1+r} .$$

The implicit derivatives are

$$\frac{\partial Ea_{T-1}^{B*}}{\partial x_{T-2o}^*} = - \left[ \frac{\partial (V_{T-1}^* - Vd_{T-1}^{T-2*})}{\partial Ea_{T-1}^{B*}} \Bigg|_{x_{T-2o},\, x_{T-2o},\, P_{T-1}^{T-2},\, P_{T}^{T-2}} (Ea_{T-1}^{B*}) \right]^{-1} \left[ (1-un)\, a_{T-2}^{B*} (x_{T-2})^{-n} - c \right] \qquad (9a)$$

$$\frac{\partial Ea_{T-1}^{B*}}{\partial x_{T-2n}^*} = - \left[ \frac{\partial (V_{T-1}^* - Vd_{T-1}^{T-2*})}{\partial Ea_{T-1}^{B*}} \Bigg|_{x_{T-2o},\, x_{T-2n},\, P_{T-1}^{T-2},\, P_{T}^{T-2}} (Ea_{T-1}^{B*}) \right]^{-1} \left\{ \left[ \lambda_{T-1}^* + CEQ_{T-1} \frac{\lambda_T^*}{1+r} \right]_{Ea_{T-1}^{B*}} + \left[ (1-un)\, a_{T-2}^{B*} (x_{T-2})^{-n} - c \right] \right\} \qquad (9b)$$

$$\frac{\partial Ea_{T-1}^{B*}}{\partial P_{T-1}^{T-2*}} = \left[ \frac{\partial(V_{T-1}^{*} - Vd_{T-1}^{T-2*})}{\partial Ea_{T-1}^{B*}} \Bigg|_{\substack{(Ea_{T-1}^{B*}) \\ X_{T-2o}, X_{T-2n}, P_{T-1}^{T-2}, P_{T}^{T-2}}} \right]^{-1} \left[ 1 - \frac{\tau R CEQ_{T-2}}{1+r} \right] \qquad (9c)$$

$$\frac{\partial Ea_{T-1}^{B*}}{\partial P_{T}^{T-2*}} = \frac{1}{1+r} \left[ \frac{\partial(V_{T-1}^{*} - Vd_{T-1}^{T-2*})}{\partial Ea_{T-1}^{B*}} \Bigg|_{\substack{(Ea_{T-1}^{B*}) \\ X_{T-2o}, X_{T-2n}, P_{T-1}^{T-2}, P_{T}^{T-2}}} \right]^{-1} \left[ CEQ_{T-1}(Ea_{T-1}^{B*}) - \frac{\tau R CEQ_{T-2} CEQ_{T-1}}{1+r} \right] \qquad (9d)$$

The effect of the decsions on the bankruptcy state $Ea_{T-1}^{B}$ is dependent on, as in section B, the sign of $\frac{\partial(V_{T-1}^{*} - Vd_{T-1}^{T-2*})}{\partial Ea_{T-1}^{B*}}$, the change in equity value in T-1 as $Ea_{T-1}^{B*}$ is changed. Under the usual conditions, as argued for the same derivative in T-1, it will be $> 0$.

<u>Conclusions</u>

These optimality conditions lead to the following conclusions for production decisions and optimal payment next period in the presence of a future of more than one period.

(1) The production decision $0 < x_{T-2_o}^* \leq X_{T-2}$ is again different from the all equity case: the condition is a one-period one similar to that of T-1. That is, under usual conditions ($Ea_{T-2} > a_{T-2}^B$), it is optimal to hedge by producing less out of current capacity than under all equity financing as this hedging reduces the chance of bankruptcy relative to the all equity production level.

(2) The optimal production-new investment decision $x_{T-2_n}^*$ is at a higher $Ea_{T-2}$ than in the all equity case, as was the result in T-1. In this period, however, in order for the hedging effect to dominate in (9b), the negative term $(1-un)\ a_{T-2}^B (x_{T-2}^*)^{-n} - c$ must be greater than the value of future use in both periods, $\lambda_{T-1}(Ea_{T-1}^{B*}) + cEQ_{T-1}(Ea_{T-1}^{B*}) \frac{\lambda_T^*}{1+r}$. The positive effect of increasing $x_{T-2_n}^*$ on the bankruptcy state $Ea_{T-1}^{B*}$, $\frac{\partial Ea_{T-1}^{B*}}{\partial x_{T-2n}^*}$, is counterbalanced by the value that the new capacity will possibly have in the future. Thus, the shareholders are able to increase $x_{T-2n}^*$ more than in T-1 without increasing the chance of next period bankruptcy as much as was the case in T-1.

(3) The optimal next period promise $P_{T-1}^{T-2*}$ is determined, as in T-1, by the equality at the margin of the gain in tax shield due to a marginal increase in $P_{T-1}^{T-2*}$ and the loss

in current value of equity claims to next period value due to a decrease in the certainty of receiving them as $P^{T-2}_{T-1}*$ is marginally increased.

That these conditions (1)-(3) are similar to the T-1 period conditions is to be expected; the trade-offs are single period ones (with the exception of the new capacity decision), equating loss in one period to gain in the next. The difference (in addition to the new capacity decision) arises in the determination of the debt promise of longer than one period.

(4) The optimal promise $P^{T-2}_{T}*$ is determined by a trade-off different from that of $P^{T-2}_{T-1}*$. The equality of increases and decreases in value of equity claims at the margin includes the increase in tax shield next period and the decrease in certainty of receiving equity claims to next period value, both arising because $P^{T-2}_{T}*$ is part of the package which the purchasers of the debt $Vd_{T-2}*$ pay for and is part of the debt $Vd^{T-2}_{T-1}*$ which must be redeemed next period if the firm is to continue to produce.

The first difference between (8c) and (8d) arises in the term reflecting the decrease in certainty of receiving equity claim to firm value next period. This term contains the marginal decrease in equity term exactly as in the equation (8c) for $P^{T-2}_{T-1}*$, the single period promise. However, it is multiplied by $\frac{\partial \varepsilon a_{T-1}^{B*}}{\partial P^{T-2}_{T}*}$ as defined by (9d), the change in bankruptcy as a result of a change in $P^{T-2}_{T}*$ and this term is no longer similar to the $\frac{\partial \varepsilon a_{T-1}^{B*}}{\partial P^{T-2}_{T-1}*}$ term as defined by (9c) appearing in the $P_{T-1}^{T-2}*$ equation, (8c). The difference is (aside from the $1/1+r$ and $CEQ_{T-1}$ factors inherent in the time difference between $P_{T-1}^{T-2}*$ and $P_T^{T-2}*$) the occurrence of the term $CEQ_{T-1}(Ea_{T-1}{}^{B*})$, rather than simply $CEQ_{T-1}$ as would be expected from the time difference between the two promises. That is, the payment $P^{T-2}{}_T*$ affects the bankruptcy state $Ea_{T-1}{}^{B*}$ only through its market value <u>at that state</u> $CEQ_{T-1}(Ea_{T-1}{}^B)P^{T-2}{}_T*/1+r$. This is, of course, the result of the assumption that the current debt in T-1 is called in at market value $CEQ_{T-1}(Ea_{T-1})P^{T-2}{}_T*/1+r$. Its advantage to the equity holders is that this component of the debt payment they must make next period is random, depending on the realization $a_{T-2}$. As the realization moves $Ea_{T-1}$ toward $Ea_{T-1}{}^B$, this component decreases in value, allowing the equity holders more flexibility in meeting debt payments near bankruptcy and thus retaining their claim on the future value of their holdings (although this future value, of course, also shrinks as demand expectations decline). Conversely, if $Ea_{T-1} \gg Ea_{T-1}{}^B$, the market value of the debt rises, limited by the market value where Prob $(Ea_{T-1} > Ea_{T-1}{}^B) = 1$, $\frac{P_T^{T-2}}{1+r}$. Thus there is a downward flexibility inherent in the exercise price of the next period option's equalling the market price of the debt. The flexibility offered by the issue of longer period debt is of value to the equity holders (i.e., $P^{T-2}{}_T* > 0$), but not to the debt holders. Thus the equity holders pay a fair price for the flexibility when the debt is floated.

The second difference arises from the third term in (8d), the current value of the marginal change in optimal equity decisions in the next period. This term arises because $P^{T-2}{}_T{}^*$ constitutes outstanding debt in that period, thus enters into the objective function, the maximization of equity claims in that period, and therefore into the decisions $(x_{T-1_o}{}^*, x_{T-1_n}{}^*, P^{T-1}_T{}^*)$ to be made that period. That is, a marginal change in $P^{T-2}_T{}^*$ changes, not only the tax shield next period and the certainty of receiving the equity claims to the optimal decisions to be made next period, but also changes the decisions themselves, since it changes the value of outstanding debt next period. The optimal $P^{T-2}_T{}^*$ occurs at equality of these three marginal changes. Thus, the issuance of the random claim inherent in longer term debt has an advantage in addition to the flexibility: the equity owners' optimal decisions next period affect its value since their decisions made on the basis of the realization $a_{T-2}$ determine $Ea_T{}^{B*}$, thus the market value of the random claims $CEQ_{T-1}(Ea_{T-1})\ P^{T-2}_T/1+r$.

The seeming advantage lies in the ability of the equity owners to increase the value of their claim at T-1 "at the expense of" the debt holders at T-1 by making equity maximizing decisions that decrease the T-1 market value of the future promised payment $P^{T-2}_T{}^*$, by increasing the possibility of bankruptcy in T. Such decisions are, of course, the "agency cost" of the T-2 debt flotation, since the debt purchasers realize the effect of the equity decisions. The third term in (8d) represents the marginal change in these equity optimizing decisions due to the marginal increase in outstanding debt next period. Any changes in these decisions induced by the greater repayment promise is an increase in the agency cost of the current T-2 debt flotation. Thus the marginal change in value of current equity due to a change in $P^{T-2}{}_{T}*$ will be negative. In (8d), then, the marginal increase in tax shield is balanced by two marginal decreases. Whether the resultant $P^{T-2}_{T}*$ is greater than $P^{T-2}_{T-1}*$ depends on the parameters and on the current outcome.

As noted in Myers, if the equity owners expected no net present value from their claims in T, they would then suffer no marginal loss (assuming no tax shield loss) as $P^{T-1}_{T}$ was increased; i.e., the MM proposition under taxes would hold. Similarly, here, both $P^{T-2}_{T-1}$ and $P^{T-2}_{T}$ are undetermined if the expectation is that there is no net present value to the firm at T-1 (which implies no net present value in T). Suppose, however, the expectation is that there is no net present value at T but there is net present value to the investment opportunities at T-1 (i.e., that the net present value of equity claims decreases greatly between T-1 and T due, e.g., to a loss of monopoly power). Then $P^{T-2}_{T-1}*$ will be set by the equality of marginal gains and losses as before (except that the loss is less). But $P^{T-1}_{T}*$ will be indeterminate and the production decisions will be made as if all equity financed since there is no loss in T. Thus the third marginal change in the $P^{T-2}{}_{T}*$ equation (8d), that arising from the change in optimal T-1 decisions, will be absent, since the optimal T-1 decisions are affected by $P^{T-2}_{T}$ through the loss of equity value in T, and by hypothesis there is none. However, the $P^{T-2}{}_{T}*$ component will retain the flexibility benefit; the equity owners will pay a decreasing price in T-1 for the value being received in T-1. In this case, due to the absence of the third marginal loss term, one would expect to see an increase in $P^{T-2}_{T}*$ relative to the case in which net present value greater than zero is expected both periods. Suppose, on the contrary, that there is no net value in the T-1 investment opportunities but that T investment opportunities have net present value. Then $P^{T-2}_{T-1}*$ would be set high since the large T net present value is discounted in the marginal loss to equity at T-1, which sets the optimal $P^{T-2}_{T-1}*$. But, because the large T value will render the T-1 decisions sensitive to changes in $P^{T-2}_{T}*$, one would expect a reduction in $P^{T-2}_{T}*$.

As has been noted above, the debt holders are aware that part of the value of their claim,

$$CEQ_{T-1}\ (Ea_{T-1})\ \frac{P^{T-2}{}_{T}*}{1+r}$$

will be determined by the equity-maximizing decision at T-1 which sets $Ea_{T}{}^{B}*$. They pay the T-2 equity holders only

$$CEQ_{T-2}\ CEQ_{T-1}\ \frac{P^{T-2}{}_{T}*}{1+r}$$

for this promise, and thence arises the agency cost. $CEQ_{T-2}$ is the certainty equivalent taken over all $a_{T-2}$(i.e., $Ea_{T-1}$) in the distribution conditioned on the current outcome $Ea_{T-2}$.

Since $Ea_T^{B*}$ is determined as a function of the equity optimizing decisions in T-1, which are functions of the outcome at the end of the current period $a_{T-2}$ (or $Ea_{T-1}$), the debt holders have taken the certainty equivalent of the possible market values of their claim in T-1 as determined by the equity holders. What they receive, $\widetilde{CEQ}_{T-1} \frac{P_T^{T-2*}}{1+r}$, determined by the random outcome $Ea_{T-1}$ together with the equity maximizing decisions which depend on it, is thus fairly priced.

D. Summary

By financing with debt previously and currently, the current (T-2) equity holders have as their equity claim an option consisting of: an investment opportunity at T-2 plus an option on the investment opportunities in the future contingent on their exercising their current option. The exercise price of the current option is the current market value of the outstanding debt. The exercise price of the future T-1 option,

$$P_{T-1}^{T-2*} + \widetilde{CEQ}_{T-1} \frac{P_T^{T-2*}}{1+r}$$

is the market value at T-1 of the debt repayment promises they are making currently. (They possess the option on the T-1 investment opportunity rather than the opportunity itself since they are floating debt with these repayment promises due in T-1.) They must pay the T-1 market value of their current promises to take advantage of the T-1 and the T opportunity. Thus they pay $P_{T-1}^{T-2*}$ plus the T-1 value of the future promise $P_T^{T-2*}$ for the T-1 value of the T-1 investment opportunity (the production and financing decisions in T-1), plus the T-1 value of the option on the T investment opportunity (the production decision in T). The T option is clearly contingent on exercising both the current option and the option in T-1. The exercise price of the T option has been set "provisionally" by setting $P_T^{T-2*}$, but will be changed optimally to $P_T^{T-1*}$ in T-1.

The equity holders in T-2 are able to maximize the value of the option they hold at T-2 by (1) optimally selecting the components ($P_{T-1}^{T-2*}$, $P_T^{T-2*}$) of the exercise price on the next period option, and (2) by setting the market value of the random component of the current outstanding debt $Vd_{T-2}^{T-3}$, the price of the current option, through the optimal ($x_{T-2_o}^*$, $x_{T-2_n}^*$, $P_{T-1}^{T-2*}$, $P_T^{T-2*}$) decisions. The equity holders are able to determine optimally the component of the price of the next period option by their T-1 production-investment-debt decisions and to re-specify optimally the price of the T option, $P_T^{T-1*}$, once more information as to their value, the realization $Ea_{T-1}$, becomes available, in T-1.

If the owners were not able to redeem the debt each period so that $P_T^{T-2*}$ would be the promise to be paid in T, the component

$$CEQ_{T-1}(Ea_{T-1})\ \frac{P_T^{T-2**}}{1+r} \quad (P_T^{T-2**} \neq P_T^{T-2*} \text{ above})$$

would be the T-1 value of the future exercise price $P_T^{T-2**}$ on the option to invest in opportunities available in T. In that case the value of the firm must exceed above exercise price for the owners to invest in the opportunity in T-1 and have available to them the option to invest in the opportunity in T, for which they will pay $P^{T-2**}_{T}$ as exercise price. Essentially, for the option to invest at T-1 plus the acquisition of the option to invest in T, they must pay $P^{T-2**}_{T-1}$ and hold "in escrow" the T-1 value of the exercise price $P^{T-2**}_{T}$. This situation constrains them to pay $P^{T-2**}_{T}$ in T for an option whose value has changed. In the case in which the equity holders float new debt each period they still pay the above exercise price for the T-1 opportunity, but will then float new debt according to their re-evaluation of the future opportunity so as to select a new, equity-maximizing price for it, $P^{T-1*}_{T}$.

These prices will be determined as long as the options (the production and investment decisions) have positive net present value. Thus the long term structure of optimal debt arises out of the long term structure of net present value for the firm, coming both from the value of existing capacity and the ability to charge monopoly rents. As long as there is such NPV which may be lost at bankruptcy, there is an optimal capital structure resulting from this agency "cost" of bankruptcy, whose time dimension corresponds exactly with that of the NPV. If one were to remove both these sources of NPV from the model (u=0 and current capacity assumed to have no future value), the NPV of the future options falls to 0. There is then no negative trade-off to increasing the exercise price (debt promises) to counter the tax shield advantage; the future promises become indeterminately large, and the MM proposition under taxes holds.

If one were to remove the monopoly power, the firm would still retain as NPV the value of current capacity (see the all equity case); thus an optimal debt structure would still be dterminable, as long as the assumption that the advantage of this capacity is lost upon bankruptcy still held. If one were to remove only the capacity assumption, the firm would still be left with the ability to charge monopoly rents, thus positive NPV and thus an optimal debt structure, as long as the assumption that the advantage of monopoly power is lost upon bankruptcy still held.

The debt decision is made simultaneously with the production-new investment decisions ($x_o$, $x_n$) in any period. These interact with the debt decision because they are determinants of the ability of the equity owners to meet the debt payments. In fact, the interaction of production-new investment and debt decisions is the source of hedging behavior. The shareholders produce less out of available capacity than under all equity financing in order to have a production level closer to the optimal one in the case that the realized demand expectation is near bankruptcy. Similarly, they purchase, and produce less from, new capacity (relative to the all equity new capacity decision) since a lower level of production is closer to optimality. This advantage to purchasing less new capacity is moderated, however, by the advantage to having more new capacity in the future if there is sufficient chance that the capacity constraint will be binding during the life of the capacity. Thus, the hedging effect on purchase of new capacity is less as one moves away from the terminal period, leading to the conclusion that debt-financed firms will have, generally, less production out of available capacity than all equity ones. The hedging effect is more pronounced if the firm has monopoly power, as the marginal loss of future opportunities is greater. Since the value of additional capacity will be greater also, one would still expect the hedging behavior on new capacity decisions to be moderated by the advantage of having capacity on hand. Thus, there should be less capacity utilization in firms with more monopoly power.

Therefore, in the presence of the ability to finance with debt, the time gap between equity optimizing production-investment decisions and the ability of the debt holders to gain control, and the inability to resell capacity:

(1) there is generally (under the assumptions on expectations above) "under-production" out of existing capacity, the amount positively related to monopoly power u.

(2) there is under-investment in capacity, again dependent on u, but not as much as in (1).

(3) Thus, there is a net under-usage of capacity (relative to all equity financing) dependent upon the monopoly power u.

(4) There is an optimal debt structure of maturity equal to the maturity structure of the firm's positive NPV prospects, with the sizes of the promised payments negatively related to u.

The assumption that the firm is too small to affect market valuation (or makes its decisions as though it is) must be retained here. The valuation of the firm which leads to the above conclusions rests on the predictability of the certainty equivalent process, as shown by Fama. In a market equilibrium analysis, the market valuation would be affected by the optimal decisions <u>if</u> they altered the discount rates, and the equity owners should take this into account. Under the all equity financing, the decisions do not alter the discount rates. Under debt financing, however, the decisions affect the bankruptcy states $Ea_t^B$ . The timing imperfection implies that there is a loss of value for $Ea_t < Ea_t^B$. (In particular, the calculations of this research assume a total loss of value for $Ea_t < Ea_t^B$.) Thus, there is a discontinuity across the bankruptcy state; i.e. $\frac{\partial(EQ_t}{\partial x_{t0}^*}, \frac{\partial(EQ_t}{\partial x_{tm}^*}, \frac{\partial(EQ_t}{\partial p^i_{t+1}} \neq 0$. These decisions at t, random as of an earlier time, thus affect the discount rate at t and in a random way. The Fama conditions for the multiperiod CAPM backward induction valuation procedure to be a general equilibrium valuation of the firm do not hold. Hence, the results obtained hold if

(1) the shareholders' decisions are made under the belief that the market if not affected by them, and

(2) the market consists of firms which can be valued by the multiperiod CAPM; i.e., firms for which the discount rates are non-stochastic.

# CHAPTER FIVE: CONCLUSIONS

## A. The Production and Investment Decisions of the Firm

The optimal production and investment decisions at time t are functions of current and future demand expectations, of previously purchased capacity, and of the firm's ability to internalize the profits from these expectations as a result of its monopoly power. In particular, the decision to produce at a given level is governed by maximizing the certainty equivalent of current net revenue expectations; the optimum is reached at equality of the certainty equivalent of marginal revenue and marginal cost of production, subject to the capacity constraint. The decision to purchase and produce from new capacity is reached by maximization of the net present value of current and future revenues, and the optimum is at the equality of the certainty equivalent of net marginal value and investment cost. The latter condition may be rephrased as the equality of certainty equivalent of current net marginal (from current production with the marginal unit) and the effective marginal investment cost, the cost k less the certainty equivalent of future marginal net revenues. The unit will generate future net revenues only when the certainty equivalent at that time of marginal revenue is equal to marginal cost at a production level sufficiently large to necessitate the use of the unit of capacity under consideration. Thus, the new capacity decision is a function of both current demand expectations and current knowledge of future demand expectations. If these demand expectations are exactly fulfilled, the capacity will be used. If future demand expectations fall sufficiently below what is currently indicated, there will be idle capacity; if above, new capacity will be purchased.

At demand expectations between these two limits --- that at which the capacity constraint becomes binding (the shadow price equals the effective marginal cost of a new unit, the "original" level of demand expectations when the unit was first purchased), firms with previously purchased capacity operate at that capacity level. New firms do not enter, however, since the marginal value to them of the new capacity does not cover the marginal investment cost until the upper limit is reached. Thus there exists a barrier to entry.

There are two sources of this barrier: a different required rate of return for the new capacity than for the old, and the inability to resell capacity. Because of the assumptions on inter- and intra-period risk (that the covariance with the market is constant for all t) the first is precluded for capacity of infinite life. For capacity of a known finite life, it enters at the last period of life of a unit of previously purchased capacity. Since the old unit is expected to last only through the current period, it thus need yield return only for intra-period risk, while newly purchased capacity lasting more than one period should yield a return for inter- as well as intra-temporal risk. Since this age difference will exist between any previously purchased capacity and newly available capacity, the effect of the difference in return expected for the final period of the lifetime will be present as long as previously purchased capacity is not available for purchase.

The second source, present even if capacity lasts an indefinitely long time, again arises from the inability to resell capacity. If demand expectations at the time of purchase of the capacity are not realized, so that the marginal unit purchased then does not currently yield its expected rate, the shadow price of use of that capacity no longer equals the marginal investment cost of a new unit. Thus the firm has suffered a capital loss relative to what it originally expected. However, the certainty equivalent of marginal revenue using the unit equals or exceeds the marginal operating cost. Thus the firm expects to cover operating costs of using its previously purchased capacity and it will therefore maximize its current value by operating at capacity. The use of the capacity that period adds net present value to the firm, (although not as much as was expected when the firm originally purchased it). Note that the current certainty equivalent of net marginal current and future revenue would be the firm's selling price of a unit of this previously purchased capacity if it could sell it.

If it could buy it, this would be the price of the additional unit. Since this marginal value will differ from firm to firm (depending on their relative abilities to capture revenues), the selling price of previously purchased capacity if it were available for sale may be above the purchase price, the marginal value to the entrant. Thus again, there is a barrier to entry, arising out of an inability to sell at the current worth of the previously purchased capacity.

The effect of changing efficiency, essentially, is two-fold: the reduction of the effective investment cost by reducing the current and future operating cost relative to the older capacity and reduction in time of usage of the newly purchased equipment since more efficient vintages are predicted for the future. The second effect raises the effective investment cost. Thus, the barrier to entry, which exists by virtue of the gap in demand expectations between that level at which the shadow price of capacity constraint rises above zero and that at which it equals the effective investment cost may be decreased or increased depending on the relative efficiency of the current technology and the speed at which the technological changes occur.

There are two relative efficiency levels of note. The first is the one referred to in the preceding paragraph, that vintage at which current investment cost and operating cost of the new capacity is less than the operating cost of one of the older vintages. At this point, regardless of future demand expectations, the older vintage (and those preceding it) will be scrapped. If demand expectations are sufficiently large to justify production above capacity (not including the vintages which have been scrapped), the firm will purchase the new production facility.

The second is that vintage at which the effective cost of purchasing and using the new equipment, its operating cost plus effective investment cost (k reduced by the future marginal value of using it as seen by current demand expectation), is less than the operating cost of the older vintage. If the level of demand expectations is sufficiently high to justify more production than is possible from previously purchased capacity (not including this vintage), the firm will purchase new capacity rather than use this vintage, since the effective cost is lower. However, the idleness of this (and older vintages) depends on current demand expectations; this vintage may be used in the future if demand expectations differ. Thus this vintage and all those up to the one being scrapped constitute idle capacity, useless currently but not necessarily in the future.

What the vintage efficiency structure models, is, essentially, an increasing marginal cost curve coming about through employment of vintages which are more and more expensive to operate. The increase in marginal cost does not result from the usual short-run arguments or any assumption of long-term minimum efficient scale production. Thus, conclusions drawn on barriers to entry and capacity utilization are related to arguments on adoption of process technology innovations, in which the firm purchases the innovation (available to any firm at the cost k).

## B. The Interaction with the Debt Decision

If debt financing is allowed, the shareholders will float an optimal amount of debt greater than zero and less than the value of the firm. The existence of this debt will alter their future production and investment decisions, which will be equity-maximizing rather than firm-optimizing. This occurs because the owners may make equity-optimizing decisions not to produce (i.e., to declare bankruptcy), foregoing investment opportunities which the bondholders, if they had control of the firm, would wish to undertake. Due, however, to a timing gap between the decision by the shareholders to forego the opportunity and the legal declaration of bankruptcy, they cannot gain control until after the opportunity, with its repercussions for future decisions, is lost. Thus, there is a "bankruptcy" or "agency" cost to the shareholders which, together with the tax advantage, induces an optimal debt structure, at equality of the marginal gain from the tax shield and loss from the agency cost.

As the current production decisions, out of old and from newly purchased capacity, affect the cash flow to the firm at the end of the period, thus its ability to meet the stated debt payment next period, they interact with the current new debt flotation decision. In particular the production decisions are made so as to hedge the risk of bankruptcy by "underproduction", to be in as nearly optimal position as is possible, given current demand expectations, to meet bankruptcy conditions next period. This enables the shareholders to float more debt thus gaining more tax shield.

Entering into the decision to purchase new capacity, however, is another factor, the Net Present Value that it contributes to the firm next period, helping to avoid bankruptcy then. Thus there are two factors working in opposite directions which determine optimal purchase of new capacity: current production leading to next period cash flow, and next period value. To the extent that the second effect contributes, the equity owners will purchase "excess" capacity, so as to float more debt at optimum. The source of the value of capacity, the barrier to exit from the industry, thus contributes to the debt capacity of the firm, by, essentially, "biasing upwards" the equity owners' decision to produce, rather than not to produce, and thereby lose valuable opportunities.

These optimal decisions are brought about by the assumption of the model that everything is lost upon bankruptcy; all future investment opportunities, the ability of current capacity to produce in the future, and cash flows (if any) from the current production period just ended. Thus, all of these components of the value of the firm affect the bankruptcy state; a marginal change in any one of them at bankruptcy brings about the loss. If any of them were not lost at bankruptcy, there would be no dependence of the state upon them as bankruptcy would then initiate, not a loss, but a change of ownership only. For example, if cash flows from the current production period were not assumed lost, then the bankruptcy state would not depend upon them. In essence, they would appear "on both sides" of the bankruptcy equation as they would be owned by the equity for states above bankruptcy and by the bondholders for states below bankruptcy. Thus the production decisions for the current period would not be affected. Since they are assumed lost in this model, the optimal current decisions on production are altered until, at the margin, the gain in avoiding bankruptcy exactly equals the loss in optimal NPV decision-making (from the point of view of the firm as a whole).

A similar result is found for the investment decision. Since all future value of new capacity is assumed lost upon bankruptcy, but contributes to the value of the firm above bankruptcy, purchase of new capacity can change the bankruptcy state. Thus the investment in new capacity is optimally different from that for the firm financing with equity. (Note that, since the firm produces currently out of new capacity, and current production is optimally below that of a firm financing only with equity, the optimal investment in new capacity depends on the sum of the strengths of these two opposite effects, a typical constrained maximum result). If only the minimum possible loss were assumed upon bankruptcy --- that only the optimal production and new capacity decisions of that period would be lost --- then only the current new capacity investment decision would be affected by the chance of bankruptcy, in that part of its value, the revenues of that period, would be lost. Thus, as long as capacity lasts more than one period and cannot be resold, the optimal decision of the firm financing with debt is to "over-invest" in capacity relative to usage.

The effect of an increase in efficiency is to make the current technology more valuable relative to the previous ones for the periods of time over which it is potentially valuable (i.e., until technology has progressed so much further that it is obsolete and is scrapped). Thus, purchase of the current technology may contribute more or less to the value of the firm, and to its ability to meet the bankruptcy condition (relative to the situation in which there is no technological change), depending on the increase in efficiency over the previous technology and the useful life of the new. If the efficiency of the current technology is sufficiently great and future increases in efficiency come sufficiently slowly, then purchase of the current technology will contribute more, at the margin, to the value of the firm (than in the case of no technology change). Thus the firm financing with debt will purchase optimally more new capacity (relative to the case of no technology change). Conversely, if the efficiency gains will last over so few periods (the technology becomes obsolete rapidly) that the marginal value of the new technology is smaller than the case of constant efficiency, then the over-investment in capacity is relatively less.

## C. The Effect of Monopoly Power

At a given demand expectation, the monopolist, of course, produces less from previous capacity since the optimum occurs at the equating of marginal revenue and marginal cost c. However, the monopolist can "internalize" more of the future benefits of investment in new capacity. That is, the effective marginal cost of new investment is less for the firm with more monopoly power. This occurs, essentially, because this firm is able to realize greater future rents from the marginal unit of capacity, if it uses it in the future, than can the firm with lower u. (Note that the comparison of these two firms takes into account the difference in future capacity positions. The assumption made in the comparison is that both firms face the same industry demand conditions but that marginal revenue conditions are modified by monopoly power). This lower marginal cost is weighed against the lower marginal benefit from current production to determine new investment. If demand expectations are such that the first outweighs the second, the more monopolistic firm will begin positive new investment at this demand level whereas, for the less monopolistic firm, the future benefits are still not enough to outweigh the current marginal loss. Since capacity purchased during period of high expectations cannot be resold, if the second factor is significant, the more monopolistic firm would have lower capacity utilization rates on the average.

The argument above depends on the current versus future advantage of the same technology. The internalization of benefits of a more efficient technology would be greater for the monopolistic firm, according to the same argument as above, while the current marginal gain from production with the new technology would be lower. Again the relative weights of these two factors determine the optimal decision. If technological progress were slow enough and the current gain in efficiency large enough, the monopolistic firm will invest in the new technology at a lower demand expectation than will be the firm with less monopoly power.

The barrier to entry arising out of the "bias" toward production, given the inability to resell capacity, is proportional to the valuation of production at capacity. This valuation is the sum of current and future marginal revenue at capacity multiplied by the (capacity) production level. At a given production level an industry dominated by monopolistic firms will thus present a higher barrier to new entrants since the more monopolistic firms realize greater future benefits from current capacity. Thus, if technological progress of sufficient duration is present, the barrier to entry into this industry will be increased. Conversely, if the industry capacity becomes obsolete too rapidly for the monopolistic firm to realize greater future benefits, technological progress may lower the barrier to entry. The assumption of a given capacity level of production should be noted, however. There would seem to be little logical basis for this convenience, given the different factors entering into new investment decisions of firms with different levels of monopoly power. Although it has been argued that more monopolistic firms may invest in new capacity at a lower demand expectation, their rate of investment as demand expectations rise will generally be lower than that of the less monopolistic firms because of the lower current marginal benefit of a high production level. Thus it seems likely than an industry composed of monopolistic firms will possess less capacity. The size of the barrier to entry will depend on whether the higher total value of a given unit of capacity outweighs the smaller amount of capacity.

If the more monopolistic firm finances optimally with debt, two conflicting effects of the monopoly power are taken into account to generate the optimal financing position. The firm with more monopoly power, having higher value to future production and investment opportunities, thus "more to lose" by bankruptcy, will tend to set lower optimal debt levels (the tax shield gain being the same for every firm). Conversely, (assuming the same loss structure upon bankruptcy as before) this firm is able, from its current production and investment decisions, to realize greater next period cash flow, value of capacity already "in place" and value of future opportunities at the time of possible bankruptcy. Thus the increase in bankruptcy state resulting from a change in debt payment is less than for a firm with lower u. In general, however, the possible loss of all future opportunities upon bankruptcy would outweigh the current ability to avoid bankruptcy and the firm with monopoly power would optimally finance with less debt.

D. Further Investigation

The model reaches conclusions regarding the interrelation of debt, capacity utilization, and monopoly power which are empirically testable, given the risk components. Thus a primary requisite for testing is the effect on intra- and inter-period risk measures, the covariances with market variables at the optimal decision levels. Accordingly, a first task would be the derivation of the covariances. Then, an empirical model derived from the relationships could be formulated and tested, using firm financial and market structure data.

The assumptions of the model raise two questions immediately. The first concerns the effect of intra- and inter-period total demand risk in a world in which the CAPM is not a valid pricing model: is diversification valuable for its own sake? A more general method of approach than the CAPM would be the approach of current option theory.

To see the analogy with option theory, assume first no special restrictions on acquisition or resale of factor inputs. The equity-financed firm with monopoly power has as its underlying asset in any period its monopoly power that period. Cash flows of this asset are obtainable upon production --- that is, upon exercising its option to exploit its ability that period to extract revenues in excess of costs. Thus, to exercise the option, the firm must provide the factors of production. The exercise price of the option that period is then the cost of these factors. The cash flow of the option if exercised in a given period is the revenue from production. The levels of output and the factor inputs will be employed so as to maximize the net present value, the certainty equivalent of net cash flow; these optimal levels are determined by the equality of certainty equivalent of marginal revenue and marginal factor costs. If the net present value at optimum is below zero, the option will not be exercised. Thus the value of the option if exercised is the present value of the cash flows in the given period at optimum output; the striking price is the cost of the factor inputs employed at the optimum levels in the given period. This cost is the sum of the wage of the optimum labor level and the rental price of the optimal capital usage (the rental price per unit being the difference between purchase price and the known resale price at the end of the period discounted at the risk free rate). In the simplified demand and production model of this study, the cost per unit of the facilities necessary to produce one unit of output is simply k (purchase cost) plus c (operating cost). Thus the cash flow is $px^*$; the striking price is $(c+rk)x^*$ (if the capacity unit can be resold at k). Since the firm has monopoly power u and faces an elastic demand curve $p = a\, x^{-n}$, $0<u<1$, $0<n<1$, the option will always be exercised each period. Thus the value of the firm possessing the monopoly power over more than one period is the sum of the current value of the options less the current values of their exercise prices. The values and exercise prices as of t are random since $Ea_{t+i}$ depends on the realization $a_{t+i-1}$. In the case that the dependence of $Ea_{t+i}$ on $a_{t+i-1}$ is time independent, one can place a non-random value on $x^*_{t+i}$ and thus determine the current value of the firm. Otherwise, the future values of the future options --- the intertemporal component of the value of the firm --- are dependent on the changes in the "opportunity set" as presented by the $a_{t+i-1}$. Also, since the exercise prices depend upon the underlying value of the asset (since they both depend on $x^*$), the familiar option theory conclusion on the relation of value to total risk cannot be drawn here.

The possession of the "real asset", capacity to produce for a period, has enabled the firm to exploit its ability to command revenues in excess of the cost of acquiring and using the capacity. Thus the certainty equivalent value of the excess may be considered the value of the possession of the capacity. This valuation clearly depends upon the possessor, as it is the result of the monopoly power u.

The cost of acquisition and use above was derived from a particular assumption on resale. A different resale price, or a random one, will produce a different "rental price", or, in the second case, a random one, much like an unknown wage rate, in the single period case.

The valuation of multiperiod assets depends upon the resale assumption. If the resale price falls below the purchase price of the identical capacity the firm no longer desires to sell the capacity each period. Even if it does not use it in the current period (the cost of operation of the unit exceeds the certainty equivalent of revenue), it will not wish to sell the capacity as long as the current certainty equivalent of future net revenues exceeds the resale price. The "internal" value of the capacity to the firm exceeds its "external" value. In this case the firm will purchase the capacity and possess it, rather than "renting" it.

What the firm then acquires, for price kx* (less resale), is a compound option: production in the current period at operating cost c per unit plus the options to produce in future period, all with exercise prices $cx^*_{t+i}$. Actually, the only difference in structure between the owner and the renter occurs at the initial acquisition and production time: the initial price, the investment cost, occurs all at once. The firm must value the real asset as the certainty equivalent at the time of purchase of the excess cash flows from exercising the future options, purchasing the asset if this is greater than kx* less the current certainty equivalent of future resale possibility. (The undertaking of the purchase on the basis of less knowledge than available in the future to the renter is, of course, the traditional risk of ownership). Once the asset is acquired, however, the decision structure of the owned and rented assets is similar. The difference lies in the lower exercise price of the options to produce with the owned asset. Hence firms with owned assets ("assets in place") and restrictions on resale (the source of the desire to own assets) have assets of higher value than firms without, who must pay investment costs as well as operating costs. The presence of the "asset in place" does not reduce the number of options the possessor has; it reduces their exercise prices.

The firm makes current production and investment decisions on two types of options:

(1) whether or not to exercise its option to produce with its assets in place, incurring an intratemporal risk,

(2) whether or not to exercise its option to purchase (and produce currently with, in the above development) new capacity, an inter- as well as intra-temporal risk. The second option is a compound one, embodying the purchase of a "package" of many simple options of the first type. The value of the firm consists of the value of the total amount of "simple" options available because of its current capacity plus the value of the compound options available each decision period.

Against this value, the owners of the firm may now float an optimal amount of debt, as outlined previously. It has been found that, if the options are to be lost at bankruptcy, decisions on their exercise are optimally altered to avoid bankruptcy. In particular, when the loss structure is as assumed previously, both types of options are lost. The optimal decisions are then (1) to decrease the exercise of the simple (production) option, and (2) to increase the exercise of the compound (investment) option. Since the risk structures of the two are different, an investigation of the difference should lead to the answers to the questions raised by the interpretation of the equity position as an option with the debt repayment as striking price. The value of this option is the sum of the values of the options embodied in the assets in place plus the future decisions on compound options, the investment decisions.

Myers has pointed out that combination of two separate firms, each with its own optimum debt position, into a single merged firm does not unambiguously lead to diversification benefits, since the two firms, once merged, no longer are protected against each other's debt contracts. The equity optimizing decision would probably be to restructure the financing of the purchased firm, as the sum of debt structures of the two separate firms is probably no longer optimal to the merged firm. Thus the constraint that the merged firms support a no-longer-optimal debt level would be removed and the benefits of merger derivable from the answer to the diversification question.

The second question concerns the presence of oligopolistic interaction among the firms in a non-competitive industry.

The assumption that firms retain a constant level of monopoly power in the face of differing investment, production and debt decisions, in a changing demand environment, seems anachronistic. It would seem of interest to investigate changes in monopoly power and attempts to preserve it brought about by these decisions. In particular, a firm may be thought of as preserving monopoly power by changing its investment, production, and debt decisions so as to forestall entry. It has been theorized that firms with monopoly power purchase excess capacity in order to be able to do just that. (Note that the excess capacity decisions in the model above arise out of the inability to resell it and from the resulting ability to carry more debt, not from any such entry-preventing tactics). Such an approach would offer a way to model possible changes in monopoly power and the interaction with production, investment, and debt decisions. The question has been raised as to whether the firm would reserve unused "excess" debt capacity (rather than finance at a higher, optimal debt level) in order to make use of it as excess production capacity available to forestall potential entrants. An investigation of attempts to preserve monopoly power, under the same intra- and inter-period uncertainty conditions, would presumably lead to the answer.

# BIBLIOGRAPHY

## Books

Arrow, Kenneth. "Economic Welfare and the Allocation of Resources for Invention." in The Rate and Direction of Inventive Activity, National Bureau of Economic Research: Princeton University Press, 1962.

Fama, E.F. and M. H. Miller. The Theory of Finance. New York: Holt, Rinehart & Winston, 1972.

Goldschmid, Harvey J., H. Michael Mann, and J. Fred Weston (editors). Industrial Concentration: The New Learning. Boston: Little, Brown and Co., 1974.

Haley, C.W. and L.D. Schall. The Theory of Financial Decisions. New York: McGraw-Hill, 1973.

Hirshleifer, J. Investment, Interest and Capital. Englewood Cliffs: Prenctice-Hall, 1970.

Scherer, F.M. Industrial Market Structure and Economic Performance. Chicago: Rand McNally, 1970.

## Articles

Baker, Samuel H. "Risk, Leverage and Profitability: An Industry Analysis," The Review of Economics and Statistics 55 (November, 1973): 503-507.

Baron, D.P. "Limit Pricing and Models of Potential Entry," The Western Economic Journal 10 (September, 1972): 298-307.

Baxter, Nevins D. "Leverage, Risk of Ruin, and the Cost of Capital." The Journal of Finance 22 (Sept. 1967): 395-404.

Carleton, Willard and Irwin H. Silberman. "Joint Determination of Rate of Return and Capital Structure: An Econometric Analysis." The Journal of Finance 32 (June 1977): 811-821.

Caves, R.E. and B.S. Yamey. "Risk and Corporate Rates of Return: Comment." The Quarterly Journal of Economics 85 (August, 1971): 513-518.

Demsetz, Harold. "Information and Efficiency: Another Viewpoint." Journal of Law and Economics 12 (April, 1969): 1-22.

Duetsch, Larry L. "Elements of Market Structure and the Extent of Suboptimal Capacity." The Southern Economic Journal 40 (October 1973): 216-223.

Esposito, Frances F. and Louis Esposito. "Excess Capacity and Market Structure." The Review of Economics and Statistics 56 (May 1974): 188-195.

Fama, E.F. "Risk-Adjusted Discount Rates and Capital Budgeting Under Uncertainty." Journal of Financial Economics 5 (August 1977): 3-24.

Fisher, I.N. and G.R. Hall. "Risk and Corporate Rates of Return." The Quarterly Journal of Economics 83 (February 1969): 79-92.

Fisher, I.N. and G.R. Hall. "Risk and Corporate Rates of Return: Reply." The Quarterly Journal of Economics 85 (August 1971): 518-522.

Galai, D. and R.W. Masulis. "The Option Pricing Model and the Risk Factor of Stock." Journal of Financial Economics 3 (January 1976): 53-82.

Gale, Bradley T. "Market Share and Rate of Return," The Review of Economics and Statistics 54 (November 1972): 412-423.

Hall, Marshall and Leonard Weiss. "Firm Size and Profitability." The Review of Economics and Statistics 49 (August 1967): 319-331.

Hamada, Robert S. "Portfolio Analysis, Market Equilibrium and Corporation Finance." The Journal of Finance 24 (March 1969): 13-31.

Hirshleifer, J. "Investment Decision Under Uncertainty: Applications of the State-Preference Approach." The Quarterly Journal of Economics 80 (may 1966): 252-277.

Hu, S.C. "On the Incentive to Invent: A Clarificatory Note." Journal of Law and Economics 16 (April 1973): 169-172.

Hurdle, Gloria J. "Leverage, Risk, Market Structure and Profitability." The Review of Economics and Statistics 61 (November 1974): 478-485.

Jensen, Michael C. and William H. Meckling. "Theory of the Firm: Managerial Behavior, Agency Costs and Ownership Structure." Journal of Financial Economics 3 (November 1976): 305-360.

Kamien, Morton I. and Nancy Schwarz. "Market Structure and Innovation: A Survey." Journal of Economic Literature 13 (March 1975): 1-37.

Kraus, Alan and Robert Litzenberger. "A State-Preference Model of Optimal Financial Leverage." The Journal of Finance 28 (September 1973)L 911-922.

Leland, H.E. "Theory of the Firm Facing Uncertain Demand." American Economic Review 62 (June 1972): 278-291.

Lustgarten, Steven H. "The Impact of Buyer Concentration in Manufacturing Industries." The Review of Economics and Statistics 57 (May 1975): 125-132.

Malcomson, James M. "Capacity Utilization, The User Cost of Capital and the Cost of Adjustment." International Economics Review 16 (June 1975): 352-361.

Mann, H. Michael. "Seller Concentration, Barriers to Entry, and Rates of Return in Thirty Industries, 1950-1960." The Review of Economics and Statistics 48 (August 1966): 296-307.

Mann, H. Michael. "The Interaction of Barriers and Concentration: A Reply." The Journal of Industrial Economics 19 (July 1971): 391-393.

Merton, R.C. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates." The Journal of Finance 29 (May 1974): 449-470.

Merton, R. C. "On the Pricing of Contingent Claims and the Modigliani-Miller Theorem." Journal of Financial Economics 5 (NOvember 1977): 241-250.

Miller, M.H. "Debt and Taxes." The Journal of Finance 32 (May 1977): 261-276.

Modigliani, F. and M.H. Miller. "The Cost of Capital, Corporation Finance and the Theory of Investment," American Economic Review 48 (June 1958): 261-297.

Modigliani, F. and M.H. Miller. "Corporate Income Taxes and the Cost of Capital: A Correction." American Economic Review 53 (June 1963): 433-443.

Myers, Stewart C. "Determinants of Corporate Borrowing." Journal of Financial Economics 5 (November 1977): 147-176.

Needham, Douglas. "Market Structure and Firms' R&D Behavior." The Journal of Industrial Economics 23 (June 1975): 241-255.

Ng, Y.K. "Competition, Monopoly and the Incentive to Invent." Australian Economic Paper 10 (June 1971): 330-341.

Ornstein, S.I. "Concentration and Profits." The Journal of Business 43 (October 1972): 519-541.

Pashigian. B.P. "Limit Price and the Market Share of the Leading Firm." The Journal of Industrial Economics 16 (July 1968): 165-177.

Robichek, A.A. and S.C. Myers. "Problems in the Theory of Optimal Capital Structure." Journal of Financial and Quantitative Analysis 1 (March 1965): 1-35.

Ross, S.A. "The Determination of Financial Structure: The Incentive-Signalling Approach." The Bell Journal of Economics 8 (Spring 1977): 23-40.

Scott, James H., Jr. "A Theory of Optimal Capital Structure." The Bell Journal of Economics 7 (Spring 1976): 33-54.

Scott, James H., Jr. "Bankruptcy, Secured Debt, and Optimal Capital Structure." The Journal of Finance 32 (March 1977): 1-19.

Shepherd, William G. "The Elements of Market Structure." The Review of Economics and Statistics 54 (February 1972): 25-37.

Sherman, R. and R. Tollison. "Advertising and Profitability." The Review of Economics and Statistics 53 (November 1971): 397-407.

Sherman, R. and R. Tollison. "Technology, Profit Risk, and Assessments of Market Performance." The Quarterly Journal of Economics 86 (August 1972): 448-462.

Smith, Vernon, "Default Risk, Scale, and the Homemade Leverage Theorem." American Economic Review 62 (March 1972): 66-76.

Spence, A.M. "Entry, Capacity, Investment and Oligopolistic Pricing." The Bell Journal of Economics 8 (Autumn 1977): 534-544

Stapleton, R. and M.G. Subrahmanyam. "A Multiperiod Equilibrium Asset Pricing Model: New York University Working Paper, 1975.

Stiglitz, Joseph. "Some Aspects of the Pure Theory of Corporate Finance: Bankruptcies and Take-Overs." The Bell Journal of Economics 3 (Autumn 1972): 458-482.

Stiglitz, Joseph. "On the Irrelevance of Corporate Financial Policy." American Economic Review 44 (December 1974): 851-866.

Stonebroker, Robert J. "Corporate Profits and the Risk of Entry." The Review of Economics and Statistics 58 (February 1976): 33-39.

Subrahmanyam, M.G. and Stavros Thomadakis. "Systematic Risk and the Theory of the Firm." Baruch College Working Paper 1977.

Sullivan, Timothy G. "Market Power, Profitability and Financial Leverage." The Journal of Finance 29 (December 1974): 1407-1414.

Sullivan, Timothy G. "A Note on Market Power and Returns to Stockholders." The Review of Economics and Statistics 59 (February 1977): 108-113.

Sullivan, Timothy G. "The Cost of Capital and the Market Power of Firms." The Review of Economics and Statistics (Forthcoming 1978).

Thomadakis, Stavros B. "A Model of Market Power, Valuation and the Firm's Returns." The Bell Journal of Economics 7 (Spring 1976): 150-161.

Thomadakis, Stavros B. "Systematic Risk in a Model of Intertemporal Investment and Monopoly Power." Baruch College Working Paper 1977.

Thomadakis, Stavros B. "A Value-Based Test of Profitability and Market Structure." The Review of Economics and Statistics 59 (May 1977): 179-185.

Warner, J.B. "Bankruptcy Costs, Absolute Priority, and the Pricing of Risky Debt Claims." Journal of Financial Economics 4 (May 1977): 239-276.

Wenders, John T. "Excess Capacity as a Barrier to Entry." The Journal of Industrial Economics 20 (November 1971): 14-19.

Winston, Gordon C. "The Theory of Capital Utilization and Idleness." The Journal of Economic Literature 12 (December 1976): 1301-1320.