Weakly Measurable Cardinals and Partial Near Supercompactness

by

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Abstract

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I will introduce a few new large cardinal concepts. A weakly measurable cardinal is a new large cardinal concept obtained by weakening the familiar concept of a measurable cardinal. Specifically, a cardinal κ is weakly measurable if for every collection $\mathcal{A}$ containing at most κ^+ many subsets of κ, there exists a nonprincipal κ-complete filter on κ measuring all sets in $\mathcal{A}$. Every measurable cardinal is weakly measurable, but a weakly measurable cardinal need not be measurable. Moreover, while the GCH cannot fail first at a measurable cardinal, I will show that it can fail first at a weakly measurable cardinal. More generally, if κ is measurable, then we can make its weak measurability indestructible by the forcing $\mathrm{Add}(\kappa, \eta)$ for all η while forcing the GCH to hold below κ. Nevertheless, I shall prove that weakly measurable

cardinals and measurable cardinals are equiconsistent.

A cardinal κ is nearly θ-supercompact if for every $A \subseteq \theta$, there exists a transitive $M \models \mathrm{ZFC}^-$ closed under $<\kappa$ sequences with $A, \kappa, \theta \in M$, a transitive N, and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''\theta \in N$. This concept strictly refines the θ-supercompactness hierarchy as every θ-supercompact cardinal is nearly θ-supercompact, and every nearly $2^{\theta^{<\kappa}}$-supercompact cardinal κ is θ-supercompact. Moreover, if κ is a θ-supercompact cardinal for some θ such that $\theta^{<\kappa} = \theta$, we can move to a forcing extension preserving all cardinals below θ^{++} where κ remains θ-supercompact but is not nearly θ^+-supercompact. I will also show that if κ is nearly θ-supercompact for some $\theta \geq 2^\kappa$ such that $\theta^{<\theta} = \theta$, then there exists a forcing extension preserving all cardinals at or above κ where κ is nearly θ-supercompact but not measurable. These types of large cardinals also come equipped with a nontrivial indestructibility result, and I will prove that if κ is nearly θ-supercompact for some $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$, then there is a forcing extension where its near θ-supercompactness is preserved and indestructible by any further $<\kappa$-directed closed θ-c.c. forcing of size at most θ. Finally, these cardinals have high consistency strength. Specifically, I will show that if κ is nearly θ-supercompact for some $\theta \geq \kappa^+$ for which $\theta^{<\theta} = \theta$, then AD holds in $\mathbf{L}(\mathbb{R})$. In particular, if κ is nearly κ^+-supercompact and

$2^\kappa = \kappa^+$, then AD holds in $\mathbf{L}(\mathbb{R})$.

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Contents

Introduction

Large cardinals play an important role in set theory. Gödel's groundbreaking incompleteness theorems of 1931 showed that our axiomatic system of proof is not sufficient for deciding the truth values of all statements in our set-theoretic universe (assuming that there even is one, i.e., that the theory of ZFC is not self-contradictory). Nevertheless, one can still consider theories resulting from the addition of "natural" axioms to the axioms of ZFC. By increasingly augmenting our list of axioms, we begin to reveal the complexity of our set-theoretic universe. However, we cannot effectively do this indefinitely as we would eventually add an axiom that contradicts our previous ones. Large cardinal axioms are meant to strengthen ZFC in such a way as to reveal the richness of our set-theoretic universe without causing an inconsistency to arise. However, since we can never hope to prove the existence of large cardinals or even their relative consistency with ZFC from ZFC alone, we study a hierarchy of such axioms, balancing the tradeoff between richness and the possibility for inconsistency.

Our large cardinal axioms cannot settle the relative consistency of all statements with ZFC, but they can for the relative consistency of many of the ones we care about. In fact, even the most mild of our large cardinal assumptions (e.g., the existence of an inaccessible cardinal) imply the relative consistency of an entire hierarchy of statements. Specifically, letting ZFC_0 be the theory ZFC, $\mathrm{ZFC}_{\alpha+1}$ be the theory $\mathrm{ZFC}_\alpha +$ "ZFC_α is consistent.", and $\mathrm{ZFC}_\gamma = \bigcup_{\alpha<\gamma} \mathrm{ZFC}_\alpha$ for all computable limit ordinals γ, an inaccessible cardinal implies the relative consistency of all such ZFC_α with ZFC. By Gödel's second incompletness theorem, we know that every consistent computable extension of ZFC cannot prove its own consistency so the large cardinal assumptions are quite strong. In fact, much weaker assumptions would prove the relative consistency of all such ZFC_α with ZFC, illustrating the strength of such large cardinal assumptions. [14] At higher levels, the existence of large cardinals settles questions such as whether it is possible to extend the Lebesgue measure to the full powerset of the Real numbers or whether we can have various generalizations of the Compactness Theorem.

But what has proven remarkable about these large cardinals is how they all tie together despite their seemingly different formulations. Specifically, for a number of pairs of our "natural" large cardinal axioms LC_1 and LC_2, either the relative consistency of LC_1 with ZFC implies the relative consistency of LC_2 with ZFC or visa versa. In this way, the large cardinal hierarchy as we understand it to date appears to be mostly linear in terms of consistency strength. Because of this feature, we now attempt to evaluate almost every set-theoretic statement in the lens of its large cardinal consistency strength. [15] [9] [18] In particular, whenever we ask whether it is possible for a new type of large cardinal to exist or exhibit certain properties, we want to know the weakest large cardinal assumption necessary to prove its existence in some definable ZFC model.

A wide variety of large cardinals can be characterized by the existence of certain types of elementary embeddings. Some examples with well-known embedding characterizations include the measurable, strong, strongly compact, and supercompact cardinals. Another example is the class of tall cardinals, which were recently introduced in [13]. These large cardinals give rise to a family of embeddings that collectively map the critical point arbitrarily high but without any sort of presumed codomain containment or closure conditions provided by strong and supercompactness embeddings. All of these large cardinals are witnessed by definable proper class embeddings from the set-theoretic universe into inner models. However, there are also large cardinals that can be defined by asserting the existence of certain *set* elementary embeddings, and older and more recent research alike have exhibited the fruitfulness of exploring large cardinals in this way.

The weakly compact and Ramsey cardinals, which are better known for their combinatorial characterizations, are two such examples fitting this description. By considering the embedding classifications of these types of cardinals, Gitman and Gitman and Welch in [4] and [5] were able to introduce an infinite hierarchy of cardinals of increasing consistency strength strictly below the Ramsey cardinals.

Less recent but more well-known examples are the unfoldable and strongly unfoldable cardinals introduced by Villaveces in [26]. Both of these versatile types of cardinals are witnessed by set embeddings emulating tallness and strongness proper class embeddings, respectively. However, unlike the tall and strong cardinals, unfoldable and strongly unfoldable cardinals are relatively consistent with $\mathbf{V} = \mathbf{L}$. Furthermore, what was historically remarkable about the strongly unfoldable cardinals was that they automatically give rise to embeddings emulating supercompactness despite the fact that actual strong cardinals have consistency strength strictly weaker than actual supercompact cardinals [9].

Other useful more historical examples are the extendible and I3 cardinals, which are witnessed by elementary embeddings between cuts of the universe.

Unlike the aforementioned smaller large cardinals relatively consistent with $\mathbf{V} = \mathbf{L}$, these large cardinals are witnessed by embeddings whose size strictly exceeds their critical points and whose consistency strength strictly exceeds supercompactness. What makes the I3 cardinal so useful is that its consistency strength supersedes most of the other considered large cardinals known to be consistent with ZFC, and yet there is no known proof of its own inconsistency from ZFC. The analogous Reinhardt cardinal, which asserts the existence of a nontrivial elementary embedding from the entire set-theoretic universe into itself instead of one from $\mathbf{V}_\lambda$ into $\mathbf{V}_\lambda$, falls prey to the Kunen inconsistency theorem [15] [18].

An example that provided the inspiration for the cardinals that I introduce in the next couple of chapters is the $\mathbf{H}_{\lambda^+}$-reflecting cardinals introduced by Miyamoto in [23]. These cardinals, which generalize the very mild Σ_1-reflecting cardinals from [6], are witnessed by set elementary embeddings with arbitrary domains of size λ and codomains that are as closed as we want. These cardinals form a hierarchy that approach the fully supercompact cardinals, and they were prompted by Miyamoto's desire to gain more insight into the consistency strength of fragments of PFA [23].

In Chapter 1, I will be introducing the weakly measurable cardinals, which are a weakening of the familiar measurable cardinals. Despite the fact that

I will present the definition in terms of the existence of certain filters, I will show that there is an equivalent characterization in terms of elementary embeddings. When κ is weakly measurable, it will have witnessing embeddings of size κ^+. In the case that the GCH holds at κ, there will be a witnessing embedding containing the full powerset of κ in the domain thereby witnessing full measurability. However, when the GCH does not hold at κ, these cardinals can be remarkably different [25].

In Chapter 2, I will introduce the nearly θ-supercompact cardinals, which form a hierarchy converging to full supercompactness. They generalize the weak compactness embedding characterizations along θ-supercompactness lines, and they are distinctively different from the $\mathbf{H}_{\theta^+}$-reflecting cardinals in that they are not witnessed by embeddings with arbitrarily closed codomains. Specifically, if κ is nearly θ-supercompact and $\theta^{<\kappa} = \theta$, then the witnessing embeddings will all have size θ. The actual closure of the codomains of these embeddings will thus be limited by their respective sizes of θ. Nevertheless the codomains will be θ-closed under the elementary embeddings j, and this will give the nearly θ-supercompact cardinals their consistency strength.

We conclude Chapter 2 by introducing the λ-nearly θ-supercompact cardinals that form a hybrid of varying domain sizes and closure under j. Specifically, when κ is λ-nearly θ-supercompact for $\lambda^{<\kappa} = \lambda$ and $\lambda \geq \theta$, these cardinals will have witnessing embeddings j of size λ and have the same θ-closure under j exhibited by the nearly θ-supercompact cardinals κ. Under this definition, the nearly θ-supercompact cardinals will be the θ-nearly θ-supercompact cardinals κ while the weakly measurable cardinals will be the κ^+-nearly κ-supercompact cardinals κ.

What we really aim to explore with this paper is the extent to which we are able to push the separation between these new types of cardinals and the ones that are already established. As will be shown through forcing and embedding lifting arguments, these boundaries turn out to be very loose. Specifically, we will force over suitable posets so that we only need to partially lift our elementary embeddings through the notions in the entire generic extension in order to preserve weak measurability or near θ-supercompactness of a given cardinal possessing these properties in the ground model. But we will also design our forcing so that a full lift of a ground model ultrapower embedding would be required for the cardinal to remain measurable in the extension. The separations that will be exhibited will show that these cardinals actually form a refining hierarchy of supercompactness that is off to the side of the θ-supercompact cardinals from the standpoint of size.

Finally, in order to whet the appetite of the reader, I should mention some enticing results and promising future possibilities for these new large cardinals. It turns out that adjusting the embedding characterizations of the weak compactness of a cardinal κ by increasing the allowed sizes of the domains from κ to κ^+ or requiring embeddings with domains that cover all subsets of κ^+ instead of κ as we do for various characterizations of weak measurability increases the consistency strength of the cardinals considerably. Specifically, unlike even the strongly unfoldable cardinals, which are witnessed by elementary embeddings into arbitrarily closed codomains, weakly measurable cardinals will not be consistent with $\mathbf{V} = \mathbf{L}$. In fact, they will be shown to be equiconsistent with measurable cardinals themselves. If we pump up the size of the allowable domains for which we assume witnessing embeddings j to 2^{2^κ} relative to their critical points κ and also assume the closure of the codomain under j suitably so that κ becomes nearly 2^{2^κ}-supercompact, then κ will also be 2^κ-supercompact. The consistency strength of these cardinals thus transcends Woodin cardinals [1]. In fact, we will show that if the GCH holds at κ, then the consistency strength of a nearly κ^+-supercompact cardinal κ already transcends Woodin cardinals. But despite being left as an open question in this paper, there are promising relative consistency proofs of the possibilities that the least weakly compact cardinal could also be weakly measurable. Assuming even stronger large cardinal hypotheses allows us to make the same conclusions for nearly θ-supercompact cardinals κ so long as

$\theta < 2^\kappa$ (even after pumping up 2^κ as much as we want). That is, it seems possible that relative to suitably strong large cardinal hypotheses, it is consistent that κ is the least weakly compact cardinal and also nearly θ-supercompact for every $\theta < 2^\kappa$, regardless of how large we push up 2^κ. Therefore, it may be possible to have large cardinals with potentially high consistency strength that nevertheless are also the least weakly compact cardinals. There are currently promising developments that are being undertaken jointly with Gitik and Hamkins, which are aimed at this possibility for these large cardinals.

Chapter 1

Weakly measurable cardinals

In this chapter, I introduce the weakly measurable cardinal, a new large cardinal concept obtained by weakening the familiar notion of a measurable cardinal, and I show some of the features that can be exhibited by it but not by the measurable cardinal. We know that a cardinal κ is measurable if there exists a nonprincipal κ-complete (ultra)filter on κ measuring all of its subsets. I propose a weakening of this notion by insisting only that for every $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ of size at most κ^+, we can find a nonprincipal κ-complete filter on κ measuring all subsets in $\mathcal{A}$. Of course, if $2^\kappa = \kappa^+$, then these large cardinal concepts are equivalent assertions for κ because we can set $\mathcal{A} = \mathcal{P}(\kappa)$. However, my analysis reveals the fact that a measurable cardinal κ can become nonmeasurable and yet still be weakly measurable in a forcing extension (where the GCH fails *first* at κ).

Main Definition. *A cardinal κ is weakly measurable if for every collection $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ containing at most κ^+ many subsets of κ, there exists a nonprincipal κ-complete filter on κ measuring every set in $\mathcal{A}$. (i.e. For every subset $A \in \mathcal{A}$, either A or $\kappa \setminus A$ is in the filter.)*

Main Theorem.

1. *If κ is measurable, then there exists a forcing extension where κ is weakly measurable, but not measurable.*
2. *In fact, if κ is measurable, then there exists a forcing extension where κ is weakly measurable (but not measurable), and the GCH fails first at κ.*
3. *Indeed, if κ is measurable, then there exists a forcing extension where the GCH holds, the measurability of κ is preserved, and the weak measurability of κ is indestructible by further forcing $Add(\kappa, \eta)$ for all η.*
4. *Finally, if κ is weakly measurable, then κ is measurable in an inner model, and so the two concepts of measurability are equiconsistent.*

While the consistency strength of weakly measurable cardinals is settled by the Main Theorem, Statement 4, many open questions remain. If we had only required such filters for $\mathcal{A}$ containing at most κ many subsets of κ, then the Main Definition would be a characterization of the weak compactness for κ. Weakly measurable cardinals therefore offer a refinement between the filter characterizations of the weakly compact and measurable cardinals. Moreover, by the Main Theorem Statement 4, we know that the gap in consistency strength between weakly compact and weakly measurable cardinals is quite sharp. Nevertheless, while there are a number of large cardinal notions that bridge the consistency divide between them in the large cardinal hierarchy, the central open question about weakly measurable cardinals (analogous to the theorem that the least strongly compact cardinal could coincide with the least measurable cardinal in [21]) is the following:

Question. *Is it possible for the least weakly measurable cardinal to also be the least weakly compact cardinal?*

In pursuit of new related large cardinal axioms, the inquisitive reader may wonder what would happen if we strengthened the Main Definition to require such filters for families $\mathcal{A}$ of size at most κ^{++}, at most κ^{+++}, or more. In the next chapter and in a planned subsequent follow-up paper [24], I explore these and other notions inspired by [23], including the concept of λ-near θ-supercompactness of which λ-near measurability (λ-near κ-supercompactness) forms a special case. Under this terminology, κ is weakly measurable if and only if it is κ^+-nearly measurable. The strengthening of the Main Definition to require filters on κ measuring arbitrary families $\mathcal{A}$ of size at most κ^{++} or κ^{+++} are the concepts of κ^{++}-near measurability or κ^{+++}-near measurability, respectively. As a preview, it is possible (relative to stronger large cardinal axioms) to show a separation between measurable cardinals and some of these strengthenings of weak measurability, but it is *not* possible to make the GCH fail first at such large cardinals.

1.1 Weak measurability

In this section, we introduce alternative characterizations of weak measurability and make some observations about these types of large cardinals. In particular, we will show they are unaffected by small forcing and certain types of distributive forcing. We also show modulo the proof of Statement 1 of the Main theorem how it is possible to have weakly measurable but nonmeasurable cardinals that cannot become measurable by a certain class of forcing posets.

Throughout this chapter, we will use ZFC^- to mean the theory of ZFC without the powerset axiom, but where ZFC is understood to be axiomatized with Collection instead of Replacement. The work of Gitman, Hamkins and Johnstone shows that it is important to use the Collection axiom rather than merely Replacement when omitting the Powerset axiom in order to be able to prove consequences of ZFC such as Łoś's theorem for ultrapowers or even the fact that ω_1 is regular. Since for all η, the set $\mathbf{H}_{\eta^+}$ will model this stronger ZFC$^-$ theory axiomatized with Collection, we can still consider elementary substructures to obtain models of this theory, making the substitution innocuous.

Every weakly measurable cardinal is inaccessible. The argument that a weakly measurable cardinal is regular and a strong limit is easily adaptable from proofs showing that a measurable cardinal exhibits these properties. Specifically to show that a cardinal κ is inaccessible, we only required that every collection $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ of size at most κ has a κ-complete filter on κ measuring every set in $\mathcal{A}$, and a weakly measurable cardinal κ has this property for such $\mathcal{A}$ of size κ^+. Indeed, every weakly measurable cardinal is weakly compact since the filter characterization for families of size κ is a possible definition for κ being weakly compact. Similarly, every measurable cardinal is weakly measurable because a nonprincipal κ-complete ultrafilter on κ will measure all subsets in every family of subsets of κ of size at most κ^+. Finally, we observe that under the GCH, a cardinal is weakly measurable if and only if it is measurable. In particular, the existence of a weakly measurable cardinal implies that $\mathbf{V} \neq \mathbf{L}$. Specifically, if κ is weakly measurable and

$2^\kappa = \kappa^+$, then we can find a nonprincipal κ-complete filter on κ measuring all sets in the collection $\mathcal{P}(\kappa)$. As a result, if there exists a measurable cardinal, then after the canonical forcing of the GCH, the weakly measurable cardinals will coincide precisely with the measurable cardinals. In particular, the least weakly measurable cardinal will also be the least measurable cardinal.

Theorem 1.1.1 (Characterizations of Weak Measurability). *The following are equivalent:*

1. *(Weak Measurability) κ is uncountable, and for every collection $\mathcal{A}$ of at most κ^+ many subsets of κ, there exists a nonprincipal κ-complete filter on κ measuring all sets in $\mathcal{A}$.*

2. *(Weak Embedding) $(\kappa^+)^{<\kappa} = \kappa^+$, and for every $A \subseteq \kappa^+$, there exists a transitive $M \models ZFC^-$ with $A, \kappa \in M$, a transitive N, and an elementary embedding $j : M \longrightarrow N$ with critical point κ.*

3. *(Embedding) $(\kappa^+)^{<\kappa} = \kappa^+$, and for every transitive set M of size κ^+ with $\kappa \in M$, there exists a transitive N and an elementary embedding $j : M \longrightarrow N$ with critical point κ.*

4. *(Normal Embedding) $(\kappa^+)^{<\kappa} = \kappa^+$, and for every transitive $M \models ZFC^-$ of size κ^+ closed under $<\kappa$ sequences with $\kappa \in M$, there exists a tran-*

sitive N of size κ^+ closed under $<\kappa$ sequences and a cofinal elementary embedding $j : M \longrightarrow N$ with critical point κ such that the codomain $N = \{j(f)(\kappa) | f \in M; f : \kappa \longrightarrow M\}$.

5. *(Normal ZFC Embedding) For every $A \subseteq \boldsymbol{H}_{\kappa^+}$ of size κ^+, there exists a transitive $M \models$ ZFC of size κ^+ closed under $<\kappa$ sequences with $A \subseteq M$ and $\kappa \in M$, a transitive N of size κ^+ closed under $<\kappa$ sequences, and a cofinal elementary embedding $j : M \longrightarrow N$ with critical point κ such that $N = \{j(f)(\kappa) | f \in M; f : \kappa \longrightarrow M\}$.*

6. *(Weak Measurability with normality) κ is uncountable, and for every collection $\mathcal{A}$ of at most κ^+ many subsets of κ and collection $\mathcal{F}$ of at most κ^+ many functions from κ into κ, there exists a nonprincipal κ-complete filter F on κ measuring all sets in $\mathcal{A}$, which is $\mathcal{F}$-normal in the sense that for every $f \in \mathcal{F}$ regressive on some set in F, there exists $\beta_f < \kappa$ for which $\{\alpha < \kappa | f(\alpha) = \beta_f\} \in F$.*

Proof. (Weak Measurability $\Rightarrow$ Weak Embedding): If κ is weakly measurable, then it is inaccessible so $\kappa^{<\kappa} = \kappa$ whereby $(\kappa^+)^{<\kappa} = \kappa^+$. Fix $A \subseteq \kappa^+$, and let M be a transitive ZFC$^-$ model of size κ^+ containing κ and A. Then by the weak measurability of κ, it follows that we can find a nonprincipal κ-complete filter F on κ measuring all subsets of κ in M. By restricting such a filter if necessary, we may assume that this filter is an M-measure (i.e. $F \cap \mathcal{P}(\kappa)^M$). We may then let $j : M \longrightarrow N$ be the induced ultrapower embedding where $N = \mathrm{Ult}(M, F)$ is the Mostowski Collapse of the reduced ultrapower structure only using functions in M. The critical point of j will be κ since F is nonprincipal and κ-complete.

(Weak Embedding $\Rightarrow$ Embedding): If M is a transitive set of size κ^+, then we may code it as $A \subseteq \kappa^+$. It then follows by the weak embedding characterization of weak measurability that there exists a transitive $\overline{M} \models \mathrm{ZFC}^-$ with $A \in \overline{M}$, a transitive $\overline{N}$, and an elementary embedding $\overline{j} : \overline{M} \longrightarrow \overline{N}$ with critical point κ. Consequently, M is an element of (and hence a subset of) $\overline{M}$ so we may let $j : M \longrightarrow N$ be the restricted elementary embedding $\overline{j} \restriction M$ where $N = j(M)$.

(Embedding $\Rightarrow$ Normal Embedding): Fix a transitive ZFC^- model M of size κ^+ closed under $<\!\kappa$ sequences with $\kappa \in M$ (which is possible since $(\kappa^+)^{<\kappa} = \kappa^+$ by the embedding characterization). Then (also) by the embedding characterization of weak measurability, there exists an elementary embedding $\overline{j} : M \longrightarrow \overline{N}$ with critical point κ. Let $X_\kappa = \{\overline{j}(f)(\kappa) | f \in M \cap {}^\kappa M\}$ be the seed hull of the embedding generated by using κ as a seed so that $\mathcal{X}_\kappa \prec \overline{N}$, and then let $\pi : X_\kappa \overset{\cong}{\longrightarrow} N$ be the Mostowski Collapse. By seed, we simply mean any element s of $j(D)$ for some set $D \in M$ that induces a filter F on D defined by $A \in F \Leftrightarrow A \subseteq D$ and $s \in j(A)$, for every subset A of D in M. The filter measures every set in M, and thus s is "a seed for the M-measure F." In this case, s and D are both κ. Now we may define $j : M \longrightarrow N$ to be the induced factor embedding where $\overline{j} = \pi^{-1} \circ j$. I claim that $j : M \longrightarrow N$ is the desired embedding.

Such an embedding j exists since $\text{range}(\overline{j}) = \{\overline{j}(c_m)(\kappa) | m \in M\} \subseteq X_\kappa$ where $c_m : \kappa \longrightarrow M$ is the constant function assuming m at every input value. N has size κ^+ because its size is bounded below by $|M| = \kappa^+$ by virtue of $j''M \subseteq N$ and above by $|M| = \kappa^+$ by virtue of it being the Mostowski Collapse of a set only containing elements of the form $\overline{j}(f)(\kappa)$ for $f \in M$. We also have $\kappa + 1 \subseteq X_\kappa$ since every α below the critical point is in $\text{range}(\overline{j}) \subseteq X_\kappa$, and $\kappa \in X_\kappa$ ($\overline{j}$ of the identity function evaluated at κ). Consequently, the Mostowski Collapse of X_κ fixes all ordinals below $\kappa + 1$ so that the critical point of $j = \pi \circ \overline{j}$ is κ. Note then that this means that all elements are of the form $\pi(\overline{j}(f)(\kappa)) = \pi \circ \overline{j}(f)(\pi(\kappa)) = \pi \circ \overline{j}(f)(\kappa) = j(f)(\kappa)$ for some function $f \in M$ with domain κ.

Consequently, the embedding is cofinal since all elements of N are in $\text{range}(j(f)) = j(\text{range}(f))$ for a respective function f in M. Finally, to prove that $N^{<\kappa} \subseteq N$, suppose $\beta < \kappa$, and let $\vec{x} = \langle x_\alpha | \alpha < \beta \rangle \in N^\eta$. Then there is a sequence of functions $\langle f_\alpha | \alpha < \beta \rangle \in M^\beta \subseteq M$ with $j(f_\alpha)(\kappa) = x_\alpha$ for all

$\alpha < \beta$. Therefore, $j(\langle f_\alpha | \alpha < \beta \rangle) = \langle j(f_\alpha) | \alpha < \beta \rangle \in N$. The model N can then construct the sequence $\vec{x}$ by evaluating each of the coordinates of this sequence at κ. Therefore $N^\beta \subseteq N$ for all $\beta < \kappa$ so N is closed under $<\kappa$ sequences.

(Normal Embedding $\Rightarrow$ Normal ZFC Embedding): Fix $A \subseteq \mathbf{H}_{\kappa^+}$ of size κ^+, and let $\overline{M}$ be a transitive ZFC$^-$ model of size κ^+ closed under $<\kappa$ sequences with a κ-enumeration of the transitive closure of each element of A and $\kappa \in \overline{M}$. Note the $<\kappa$-closure is possible since by the normal embedding characterization, we have $(\kappa^+)^{<\kappa} = \kappa^+$. Now also by the normal embedding characterization of weak measurability, there exists a transitive $\overline{N}$ of size κ^+ closed under $<\kappa$ sequences, and an elementary embedding $\overline{j} : \overline{M} \longrightarrow \overline{N}$ with critical point κ. Since κ is inaccessible, it is inaccessible in $\overline{M}$ whereby $\overline{j}(\kappa)$ is inaccessible in $\overline{N}$. Consequently, because $\overline{N}$ is a transitive ZFC$^-$ model, $M = \mathbf{V}^{\overline{N}}_{\overline{j}(\kappa)}$ will be a transitive ZFC model. Note now that since $\overline{M}$ knows that $\mathbf{V}_\kappa$ is closed under $<\kappa$ sequences, $\overline{N}$ will think that M is closed under $<\overline{j}(\kappa)$ (and specifically κ sequences) by elementarity. In particular, M will contain $\overline{N}$'s powerset of κ and hence its $\mathbf{H}_{\kappa^+}$ and will actually be closed under $<\kappa$ sequences by virtue of $\overline{N}^{<\kappa} \subseteq \overline{N}$. But now note also that $A \subseteq \mathbf{H}^{\overline{M}}_{\kappa^+} \subseteq \mathbf{H}^{\overline{N}}_{\kappa^+}$ since $\overline{j}(B) \upharpoonright \kappa = B$ for every $B \subseteq \kappa$ in $\overline{M}$. Consequently, $A \subseteq M$ as well. Since $M \subseteq \overline{N}$ must clearly have size at most $|\overline{N}| = \kappa^+$, we may now apply the normal embedding characterization of weak measurability to M to get a desired elementary embedding $j : M \longrightarrow N$.

(Normal ZFC Embedding $\Rightarrow$ Weak Measurability with normality): Let $\mathcal{A}$ be a collection of at most κ^+ many subsets of κ and $\mathcal{F}$ be a collection of at most κ^+ many functions from κ into κ. Now by the normal ZFC embedding property, let M be a $<\kappa$-closed transitive ZFC model of size κ^+ containing κ and all elements of $\mathcal{A}$ and $\mathcal{F}$ ($\mathcal{A} \cup \mathcal{F} \subseteq \mathbf{H}_{\kappa^+}$ and has size at most κ^+), and let $j : M \longrightarrow N$ be an elementary embedding with critical point κ such that N is also transitive. We may then let F be the nonprincipal κ-complete filter on κ generated by using κ as a seed for j. This F will then measure all subsets of κ in M, and for every function $f : \kappa \longrightarrow \kappa$ in M that's regressive on a set in F, there exists $\beta_f < \kappa$ with $\{\alpha < \kappa | f(\alpha) = \beta_f\} \in F$. Note that F is indeed κ-complete since M is $<\kappa$-closed. Also, since $\mathcal{A}, \mathcal{F} \subseteq M$, F will be a desired κ-complete filter on κ.

(Weak Measurability with normality $\Rightarrow$ Weak Measurability): Immediate.

□

As is the case with the fully measurable cardinals and most other types of large cardinals, weakly measurable cardinals are unaffected by small forcing. For transitive ZFC^- models M and N, a ground model elementary embedding $j : M \longrightarrow N$, a (hereditarily) small forcing notion $\mathbb{P} \in M$ relative to the critical point, and a filter $G \subseteq \mathbb{P}$ that's M-generic and N-generic, we can uniquely lift the embedding to $j : M[G] \longrightarrow N[G]$. Additionally, if κ is weakly measurable in a small forcing extension, then it was weakly measurable in the ground model. Before proving the following result for weakly measurable cardinals, I should make a small terminology note and an observation about the equivalent characterizations for the weak measurability of a cardinal κ. We use ZFC* (say ZFC$\cap\Sigma_{100}$) to denote a large finite fragment of ZFC sufficient for satisfying the prerequisites in the Main Theorem from [12]. In the proof of the equivalences of the weak measurability of a cardinal κ, we assumed that M was a ZFC$^-$ model for the normal embedding characterization, but we could have also assumed that M were a ZFC* model. In the proof of the direct implication of the Lévy-Solovay theorem for weakly measurable cardinals presented below, we will use the ZFC* replacement of the normal embedding characterization.

Theorem 1.1.2. *After forcing of size less than κ, the cardinal κ is weakly measurable in the extension if and only if it was weakly measurable in the ground model.*

Proof. ($\Leftarrow$) Let $\mathbb{P}$ be a partial order such that $|\mathbb{P}| < \kappa$. By associating $\mathbb{P}$

with an isomorphic copy in $\mathbf{H}_\kappa$ if necessary, we may assume that $\mathbb{P} \in \mathbf{H}_\kappa$ is a partial order on an ordinal less than κ. Let $G \subseteq \mathbb{P}$ be $\mathbf{V}$-generic, and fix any $A \subseteq \kappa^+$ in $\mathbf{V}[G]$. Then since $\check{\kappa^+} \times \mathbb{P} \in \mathbf{H}_{\kappa^{++}}$, we may find a $\mathbb{P}$-name $\dot{A} \in \mathbf{H}_{\kappa^{++}}$ such that $\dot{A}_G = A$. Let M be a transitive ZFC$^-$ model of size κ^+ containing $\dot{A}$, the cardinal κ, and $\mathbb{P}$. Then by the embedding characterization of κ's weak measurability, we may find a transitive N an elementary embedding $j : M \longrightarrow N$ with critical point κ. Since $\mathbb{P}$ is a partial order on an ordinal less than κ, it has hereditary size below the critical point whereby $\mathbb{P}$ and all of its elements will be fixed by j. We may therefore lift the embedding to $j : M[G] \longrightarrow N[G]$. The lifted embedding will have critical point κ and be between transitive ZFC$^-$ models $M[G]$ and $N[G]$. Furthermore, since $(\kappa^+)^{<\kappa} = \kappa^+$ in $\mathbf{V}$ with $\mathbb{P}$ having size less than κ, we have $(\kappa^+)^{<\kappa} = \kappa^+$ in $\mathbf{V}[G]$ as well. Consequently, since $A \in M[G]$ by virtue of the fact that $A = \dot{A}_G$, this embedding witnesses the weak embedding characterization of the weak measurability of κ for an arbitrary subset of κ^+ in $\mathbf{V}[G]$. Therefore, κ remains weakly measurable in $\mathbf{V}[G]$.

($\Rightarrow$) Now suppose we have a partial order $\mathbb{P} \in \mathbf{H}_\lambda$ for some $\lambda < \kappa$. Let G be a $\mathbf{V}$-generic filter for $\mathbb{P}$, and suppose that κ is weakly measurable in $\overline{\mathbf{V}} = \mathbf{V}[G]$. Also let A be a subset of κ^+ in the ground model $\mathbf{V}$. By Lemma 13 of [12], $\mathbf{V} \subseteq \mathbf{V}[G]$ satisfies the λ approximation and cover properties. (i.e., For each $C \subseteq \mathbf{V}$ that's a set in $\mathbf{V}[G]$ having the property that $C \cap c \in \mathbf{V}$ for every $c \in \mathbf{V}$ of size less than λ in $\mathbf{V}$, we have $C \in \mathbf{V}$, and for each $D \subseteq \mathbf{V}$ that's a set having size less than λ in $\mathbf{V}[G]$, we have an $E \supseteq D$ of size less than λ in $\mathbf{V}$.)

By the Lévy Reflection Theorem, let η be a sufficiently large ordinal above κ^+ so that $A, \kappa^+ \in \mathbf{V}_\eta$ and that every ZFC* statement reflects down from the structure $\langle \overline{\mathbf{V}}, \mathbf{V}, \in \rangle$ to $\langle \overline{\mathbf{V}}_\eta, \mathbf{V}_\eta, \in \rangle$. In particular, $\overline{\mathbf{V}}_\eta$ and $\mathbf{V}_\eta$ will be ZFC* models. Now in $\overline{\mathbf{V}}$, let $\overline{X}$ be a $<\kappa$-closed elementary substructure of $\overline{\mathbf{V}}_\eta$ in the language with a predicate for $\mathbf{V}$ of size κ^+ having A as an element and κ^+ as a subset. In particular, we have $\langle \overline{X}, X, \in \rangle \prec \langle \overline{\mathbf{V}}_\eta, \mathbf{V}_\eta, \in \rangle$ where $X = \overline{X} \cap \mathbf{V}$. Letting $\overline{M}$ and M be the Mostowski Collapses of $\overline{X}$ and X, respectively, it then follows by Lemma 15 of [12] that $X \in \mathbf{V}$ and $M = \overline{M} \cap \mathbf{V}$. In particular, M will be a model of ZFC* in $\mathbf{V}$ since $X \in \mathbf{V}$ is a model of ZFC*, and M is its elementarily equivalent Mostowski collapse. Note that $\overline{M}$ will have κ and the subset A as elements since they are fixed by the Mostowski Collapse by virtue of the fact that $\kappa^+ \subseteq \overline{X}$. Also, observe that $\overline{M}$ will be closed under $<\kappa$ sequences in $\overline{\mathbf{V}}$ because $\overline{X}$ is $<\kappa$-closed in $\overline{\mathbf{V}}$.

Now since in $\overline{\mathbf{V}}$, the set $\overline{M}$ is the Mostowski Collapse of a ZFC* model of size κ^+, the set $\overline{M}$ will be a ZFC* model having size κ^+ in $\overline{\mathbf{V}}$ as well.

Furthermore, since $\overline{M}^{<\kappa} \subseteq \overline{M}$ in $\overline{\mathbf{V}}$ and κ is an element in $\overline{M}$, it follows by the normal embedding characterization of the weak measurability of κ in $\overline{\mathbf{V}}$ that there exists a $<\kappa$-closed transitive $\overline{N}$ and a cofinal elementary embedding $\bar{j} : \overline{M} \longrightarrow \overline{N}$ with critical point κ. One can now easily verify that the conditions of the Main Theorem 3 from [12] are met so that the elementary embedding $j = (\bar{j} \upharpoonright M) : M \longrightarrow N$, where $N = \bigcup \bar{j}'' M$, is a set embedding in $\mathbf{V}$ by its Statement (3). Furthermore, observe that κ is inaccessible in $\mathbf{V}$ since it is inaccessible in $\overline{\mathbf{V}}$ by virtue of being weakly measurable there. Hence, $(\kappa^+)^{<\kappa} = \kappa^+$ in $\mathbf{V}$. Finally, because $A \in \overline{M}$ is a set in the ground model and $M = \overline{M} \cap \mathbf{V}$, we will have $A \in M$. Therefore, the elementary embedding $j : M \to N$ witnesses the (weak embedding characterization of the) weak measurability of κ in $\mathbf{V}$ for the arbitrarily selected A from the ground model. $\square$

Notice that in the above proof of the direct implication, we only used the smallness of $\mathbb{P}$ to show that $\mathbb{P}$ satisfies the λ approximation and cover properties for some $\lambda < \kappa$. Therefore, we actually proved the following stronger theorem:

Theorem 1.1.3. *Suppose κ is a cardinal in a forcing extension $\boldsymbol{V}[G]$. Suppose further that $\boldsymbol{V} \subseteq \boldsymbol{V}[G]$ satisfies the δ approximation and cover properties* *for some* $\delta < \kappa$ *and that* κ *is weakly measurable in* $\boldsymbol{V}[G]$. *Then* κ *is weakly measurable in* $\boldsymbol{V}$.

Meanwhile, since the weak measurability of a cardinal κ is witnessed by objects having hereditary size κ^+, it follows that $\leq\kappa^+$-distributive forcing will introduce no new κ^+ families of subsets of κ to be measured. More generally, the weak measurability of κ will be preserved by $\leq\kappa$-distributive forcing so long as each κ^+-sized family of subsets of κ in the forcing extension can be covered by a κ^+-sized collection in the ground model. In this case, for each κ^+-sized collection $\mathcal{A}$ of subsets of κ in the forcing extension, we will have a nonprincipal κ-complete filter on κ measuring all subsets in some $\bar{\mathcal{A}} \supseteq \mathcal{A}$ in the ground model. Also by the $\leq\kappa$-distributivity of the forcing, no new γ sequence of subsets of κ can be added for all $\gamma < \kappa$ so this filter will remain κ-complete in the forcing extension. Thus we have the following result:

Theorem 1.1.4. *If* κ *is weakly measurable, then it is weakly measurable in all forcing extensions over any* $\leq\kappa^+$*-distributive partial order* $\mathbb{P}$. *Indeed, the weak measurability of* κ *will be preserved by all* $\leq\kappa$*-distributive forcing with the* κ^{++} *cover property. (i.e. Every set of ordinals having size less than* κ^{++} *in the forcing extension is a subset of a set having size less than* κ^{++} *in the ground model.)*

Since every $\leq\kappa$-distributive forcing introducing new κ^+ sequences of subsets of κ that cannot be covered by sequences of size κ^+ in the ground model necessarily collapses κ^{++} when $2^\kappa = \kappa^{++}$, we get the following result as a corollary.

Corollary 1.1.5. *If κ is weakly measurable and $2^\kappa = \kappa^{++}$, then it is weakly measurable in all forcing extensions over $\leq\kappa$-distributive κ^{++}-preserving forcing.*

Proof. Suppose κ is weakly measurable and $2^\kappa = \kappa^{++}$, and let $\mathbf{V}[G]$ be a forcing extension over a $\leq\kappa$-distributive κ^{++}-preserving partial order $\mathbb{P}$. Fix a one-to-one enumeration $\vec{A} = \langle A_\alpha | \alpha < \kappa^{++}\rangle \in \mathbf{V}$ of all of the subsets of κ in the ground model, and let $\vec{s} = \langle s_\beta | \beta < \kappa^+\rangle$ be an arbitrary κ^+-sequence of subsets of κ in the forcing extension. Since $\mathbb{P}$ is $\leq\kappa$-distributive, $\vec{s}$ is a subsequence of $\vec{A}$. Then since the enumeration $\vec{A}$ lists each subset only once, $I = \{\alpha < \kappa^{++} | \exists\beta < \kappa^+(s_\beta = A_\alpha)\}$ is a set of size κ^+ in $\mathbf{V}[G]$. Then because $\mathbb{P}$ is κ^{++}-preserving, there must exist a $\gamma < \kappa^{++}$ such that $\sup I = \gamma$ whereby $\vec{A} \restriction (\gamma + 1)$ will be a κ^+-sized sequence covering $\vec{s}$ in the ground model. $\square$

If the κ^{++}-preserving requirement could be removed, then we could always force a weakly measurable cardinal κ to become measurable by forcing the GCH to hold at κ with the forcing $\text{Add}(\kappa^{++}, 1) * \text{Add}(\dot{\kappa}^+, 1)$. Specifically, forcing over the $\leq\kappa^+$-distributive partial order $\mathrm{Add}(\kappa^{++},1)$ preserves the weak measurability of κ by Theorem 1.1.4 and ensures that $2^\kappa = \kappa^{++}$ if the GCH did not already hold at κ. This seemingly unlikely possibility is ruled out by the following more general result (in conjunction with the Main Theorem which shows that relative to measurable cardinals, we can have weakly measurable cardinals that are not measurable):

Theorem 1.1.6. *If κ is weakly measurable but not measurable, then after small (nontrivial) forcing of size $\lambda < \kappa$, the cardinal κ remains weakly measurable, but no further $\leq\lambda$-closed forcing can make κ measurable. In particular, forcing with Add($\kappa^+, 1$) or any other $\leq\lambda$-closed poset that collapses 2^κ to κ^+ in the extension will destroy the weak measurability of κ.*

Proof. Suppose κ is weakly measurable but not measurable, and let $\mathbb{P}$ be a partial order of size $\lambda < \kappa$. By the Lévy-Solovay Theorem 2.1.6 for weakly measurable cardinals, κ remains weakly measurable in the extension $\mathbf{V}[g]$ after forcing to add a $\mathbf{V}$-generic $g \subseteq \mathbb{P}$. Now suppose $\mathbb{Q}$ is a $\leq\lambda$-closed partial order in $\mathbf{V}[g]$, and let $G \subseteq \mathbb{Q}$ be $\mathbf{V}[g]$-generic. Then the partial order $\mathbb{P} * \dot{\mathbb{Q}}$ admits a closure point at λ so by Lemma 13 of [12], $\mathbf{V} \subseteq \mathbf{V}[g][G]$ satisfies the λ^+ approximation and cover properties. Then by Theorem 10 of [12], if κ were measurable in $\mathbf{V}[g][G]$, then it would have had to been measurable in the ground model so κ is not measurable in $\mathbf{V}[g][G]$. In particular, since forcing the GCH to hold at κ with $\leq\lambda$-closed forcing would preserve the weak measurability of κ only if κ becomes measurable and were thus measurable in the ground model contrary to assumption, κ cannot be weakly measurable in $\mathbf{V}[g][G]$. $\square$

Consequently, when $2^\kappa = \kappa^{++}$, Corollary 1.1.5 offers for weakly measurable cardinals the optimal analogue to the $\leq\kappa$-distributive indestructibility that we have for measurable and weakly compact cardinals. Even when $2^\kappa > \kappa^{++}$, we have already seen that we can force to make $2^\kappa = \kappa^{++}$ while preserving the weak measurability of κ. When combined with the Main Theorem, Theorem 1.1.6 also offers a corollary that helps further separate the notions of weak measurability and measurability.

Corollary 1.1.7. *If κ is measurable, then there exists a forcing extension where κ is weakly measurable, but its measurability can never be restored by countably closed forcing.*

Proof. By the Main Theorem, we can force so that κ remains weakly measurable but is not measurable. By Theorem 1.1.6, we can then force with any nontrivial countable partial order such as the notion adding a Cohen real to achieve the desired extension where κ remains weakly measurable but no countably closed forcing can restore its original measurability. □

1.2 Separating weak measurability from full measurability and the first failure of the GCH at a weakly measurable cardinal

Let us begin this section by proving Statement 3 of the Main Theorem, using ideas from [7], [13]. Later in this section, I will show how Statements 1 and 2 follow. First, however, I want to mention a standard lemma used in the first step of the lifting argument below that is referred to as the Diagonalization Criterion (51) in [9]. The lemma states that if we are given a θ-closed set-theoretic model M having at most θ^+ many antichains for a poset $\mathbb{P}$ in M that M thinks is $\leq\theta$-closed, then for any $p \in \mathbb{P}$, we can (in $\mathbf{V}$) diagonalize against all θ^+ many antichains to produce an M-generic filter for $\mathbb{P}$ below p.

Theorem 1.2.1. *If κ is measurable, then there exists a forcing extension where the GCH holds, the measurability of κ is preserved, and the weak measurability of κ is indestructible by further forcing $Add(\kappa, \eta)$ for all η.*

Proof. Suppose κ is measurable. Fix any ultrapower embedding $j : \mathbf{V} \longrightarrow M$ generated by a nonprincipal κ-complete measure on κ. By performing the canonical forcing $\mathbb{S}$ of the GCH, which is the Easton support forcing iteration that forces with $\mathrm{Add}(\gamma^+, 1)^{\mathbf{V}^{\mathbb{S}_\gamma}}$ at every stage γ that is a cardinal in $\mathbf{V}^{\mathbb{S}_\gamma}$, we may assume that the GCH holds and in particular that $2^\kappa = \kappa^+$. Now let $\mathbb{P}$ be the κ-stage Easton support iteration where at every stage γ that's regular in $\mathbf{V}^{\mathbb{P}_\gamma}$, we force with $\mathrm{Add}(\gamma, \gamma^+)^{\mathbf{V}^{\mathbb{P}_\gamma}}$ and perform trivial forcing otherwise. We may assume that $\mathbb{P}_\alpha \in \mathbf{V}_\kappa$ for all $\alpha < \kappa$. Since each $\mathbb{P}_\alpha$ of the entire iteration $\mathbb{P}$ is constructed in $\mathbf{V}_\kappa$, we may factor the forcing $j(\mathbb{P})$ as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$ where $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for $\mathrm{Add}(\kappa, \kappa^+)$, and $\mathbb{P}_{\text{tail}}$ is the forcing beyond stage κ. Note that since M is closed under κ sequences and $\mathbb{P}$ is trivially κ^+-c.c., $\mathrm{Add}(\kappa, \kappa^+)^{\mathbf{V}^{\mathbb{P}}} = \mathrm{Add}(\kappa, \kappa^+)^{M^{\mathbb{P}}}$ so that $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for the forcing at stage κ for $j(\mathbb{P})$ is indeed an actual $\mathbb{P}$-name for the $\mathrm{Add}(\kappa, \kappa^+)$ of $\mathbf{V}^{\mathbb{P}}$. Let $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be a $\mathbf{V}$-generic filter whereby it will be M-generic. We now verify that the requirements of the Diagonalization Criterion can be met in $\mathbf{V}[G][g]$ so that we will be able to find an $M[G][g]$-generic filter G_{tail} in the forcing extension $\mathbf{V}[G][g]$. Since all conditions in G have bounded support below the critical point, we would then have $j''G \subseteq G * g * G_{\text{tail}}$ enabling us to form a partial lift $j : \mathbf{V}[G] \longrightarrow M[j(G)]$ where $j(G) \cong G * g * G_{\text{tail}} \subseteq j(\mathbb{P})$ in $\mathbf{V}[G][g]$.

First we verify that $M[G][g]$ is κ-closed in $\mathbf{V}[G][g]$. To see this, note that $M^\kappa \subseteq M$ in $\mathbf{V}$ and $\mathbb{P} * \dot{\mathbb{Q}}$ is κ^+-c.c. Next we confirm the $\leq\kappa$ closure of $\mathbb{P}_{\text{tail}}$ in $M[G][g]$. Because $j(\mathbb{P})$ only forces at regular stages γ, the tail forcing $\mathbb{P}_{\text{tail}}$ performs trivial forcing below stage κ^+. Then since in M, the poset $j(\mathbb{P})$ will be an Easton support iteration performing $\leq\kappa$-closed forcing at all stages at or above κ^+ (i.e. either $\mathrm{Add}(\gamma, \gamma^+)$ for regular γ or trivial forcing otherwise), the forcing $\mathbb{P}_{\mathrm{tail}}$ will be $\leq\kappa$-closed in $M[G][g]$. Finally we check that there are at most κ^+ many antichains for $\mathbb{P}_{\mathrm{tail}}$ in $M[G][g]$. Since $\mathbb{P}$ has at most 2^κ many (dense) subsets, $j(\mathbb{P})$ and hence $\mathbb{P}_{\mathrm{tail}}$ will have at most $|j(2^\kappa)|^{\mathbf{V}} = |j(\kappa^+)|^{\mathbf{V}} = |(\kappa^+)^\kappa|^{\mathbf{V}} = |(2^\kappa)^\kappa|^{\mathbf{V}} = |2^\kappa|^{\mathbf{V}} = \kappa^+$ antichains in $M[G][g]$ when counted in $\mathbf{V}[G][g]$.

Let $\mathbb{Q} = \dot{\mathbb{Q}}_G$. To fully lift the embedding to have domain $\mathbf{V}[G][g]$, we will construct an $M[j(G)]$-generic filter $h \subseteq j(\mathbb{Q})$ in $\mathbf{V}[G][g]$ satisfying the necessary pull-back criterion $j''g \subseteq h$. First however, we show that a few conditions hold. These are sufficient for applying the Diagonalization Criterion to justify the existence of an $M[j(G)]$-generic filter for $j(\mathbb{Q})$ although we will not be directly using this lemma since the filter must also be constructed so that it extends $j''g$. Now let's verify the κ closure of $M[j(G)]$ in $\mathbf{V}[G][g]$. We have $M[G][g][G_{\mathrm{tail}}]^\kappa = M[j(G)]^\kappa \subseteq M[j(G)]$ in $\mathbf{V}[G][g]$ since the $M[G][g]$-generic filter $G_{\mathrm{tail}} \in \mathbf{V}[G][g]$, and $M[G][g]$ is closed under κ sequences in $\mathbf{V}[G][g]$. Next we verify that $j(\mathbb{Q})$ is closed under $\leq\kappa$ unions of compatible conditions (in $M[j(G)]$). Because κ remains regular in $\mathbf{V}[G]$, we know that $j(\kappa)$ remains regular in $M[j(G)]$ by elementarity. Consequently, $j(\mathbb{Q})$, which adds Cohen subsets to $j(\kappa)$, will be closed under $<j(\kappa)$ unions of compatible conditions in $M[j(G)]$. In particular, $j(\mathbb{Q})$ will be closed under

κ unions of compatible conditions in $M[j(G)]$. Finally, we check that there are at most κ^+ many antichains for $j(\mathbb{Q})$ in $M[j(G)]$. Since $\mathbf{V}[G]$ thinks that $\mathrm{Add}(\kappa, \kappa^+)^{\mathbf{V}[G]}$ has at most $(\kappa^{+<\kappa})^\kappa = \kappa^+$ many antichains in $\mathbf{V}[G]$, the model $M[j(G)]$ will have at most $|j(\kappa^+)|^{\mathbf{V}[G][g]} = \kappa^+$ many antichains for $j(\mathbb{Q})$.

The $M[j(G)]$-generic filter h will be built by induction similar to the construction in the proof of the Diagonalization Criterion but will be done in such a way as to ensure that all conditions agree with $j''g$ on their respective domains. Specifically, let $\langle A_\alpha | \alpha < \kappa^+ \rangle \in \mathbf{V}[G][g]$ be an enumeration of the antichains of $j(\mathbb{Q})$ in $M[j(G)]$. We define a descending sequence of conditions $\vec{r} = \langle r_\alpha | \alpha < \kappa^+ \rangle$ such that each $r_{\beta+1}$ extends a condition in A_β and is compatible with every condition in $j''g$. The upward closure of the range of this sequence will then be the desired filter h below $j''g$ meeting each A_α.

We construct $\vec{r}$ by transfinite recursion and verify by transfinite induction that such an $\vec{r}$ is a descending sequence of conditions in $j(\mathbb{Q})$ meeting the specified criteria. Start with $r_0 = \emptyset \in j(\mathbb{Q})$, which is trivially compatible with every condition in $j''g$. At every limit stage $\alpha < \kappa^+$, let $r_\alpha = \bigcup\{r_\beta | \beta < \alpha\}$, and assume that $\langle r_\beta | \beta < \alpha \rangle$ is a descending sequence such that $r_\beta \in j(\mathbb{Q})$ is compatible with all conditions in $j''g$ for each $\beta < \alpha$. Since α has size at most κ in $\mathbf{V}[G][g]$, we may temporarily rearrange the sequence $\langle r_\beta | \beta < \alpha \rangle$ if necessary so that it becomes a $\leq\kappa$ sequence of compatible conditions. Then since each r_β is a condition in $j(\mathbb{Q}) \in M[j(G)]$, the reordered sequence is a $\leq\kappa$ sequence of compatible conditions in $M[j(G)]^\kappa \subseteq M[j(G)]$. Consequently, by the closure of $j(\mathbb{Q})$ under compatible $\leq\kappa$ unions, $\bigcup\{r_\beta | \beta < \alpha\}$ will be a condition in $j(\mathbb{Q})$. Also, r_α is still compatible with every condition in $j''g$. Otherwise r_α would disagree with one of the conditions in $j''g$ on a coordinate in their mutual domains whereby there would be an r_β for some $\beta < \alpha$ disagreeing with this condition in $j''g$ on a coordinate in their mutual domains. In that case, r_β would be incompatible with this condition in $j''g$, contrary to assumption. Finally the sequence $\langle r_\beta | \beta \leq \alpha\rangle$ will clearly be a descending one since $\langle r_\beta | \beta < \alpha\rangle$ is a descending sequence by hypothesis, and the element listed at α will be below all conditions listed in $\langle r_\beta | \beta < \alpha\rangle$.

For successor stages $\alpha = \beta + 1$, first assume that $r_\beta \in j(\mathbb{Q})$ is compatible with all conditions in $j''g$. I claim that there exists a condition in A_β compatible with r_β that is also compatible with every condition in $j''g$ so that we can choose such an $s_\beta \in A_\beta$. Then we can define $r_{\beta+1} = r_\beta \cup s_\beta$, which would be a condition in $j(\mathbb{Q})$ extending r_β and compatible with every one in $j''g$. The key to proving the existence of such a condition is to show that the union of the domains of all conditions in A_β, which will be denoted $d_\beta = \mathrm{dom}(\bigcup A_\beta)$, meets the domain of the function determined by $j''g$ on at most κ many coordinates. In this case, we simply take the function s with this κ-sized domain agreeing with the function determined by $j''g$ on all of its coordinates. Having size κ, the function s will be in $M[j(G)]^\kappa \subseteq M[j(G)]$ so it will be in $j(\mathbb{Q})$. Because $r_\beta \in j(\mathbb{Q})$ also agrees with the function determined by $j''g$ on all of its coordinates, r_β will be compatible with s so that $r_\beta \cup s \in j(\mathbb{Q})$. Then by virtue of A_β being a maximal antichain, there will exist an $s_\beta \in A_\beta$ compatible with $r_\beta \cup s$. Consequently, because $\mathrm{dom}(s) = d_\beta \cap \mathrm{dom}(\bigcup j''g)$ is a superset of $\mathrm{dom}(s_\beta) \cap \mathrm{dom}(\bigcup j''g)$, the condition s_β must be compatible with all conditions in $j''g$. Also, the condition s_β is clearly compatible with r_β since it's compatible with the stronger condition $r_\beta \cup s$. It therefore suffices to verify the key observation that d_β meets the domain of the function determined by $j''g$ on at most κ many coordinates.

Since $j(\mathbb{Q})$ is $j(\kappa)^+$-c.c. in $M[j(G)]$ with A_β an antichain of $j(\mathbb{Q})$ in $M[j(G)]$, we know that $|A_\beta|^{M[j(G)]} \leq j(\kappa)$. Then since each condition has size less than $j(\kappa)$ in $M[j(G)]$, the union of the domains of all conditions in A_β, i.e., $d_\beta = \mathrm{dom}(\bigcup A_\beta)$, has size at most $j(\kappa)$ in $M[j(G)]$. Note then that because $\sup j''\kappa^+ = j(\kappa^+)$ by virtue of j being an ultrapower embedding generated by a nonprincipal κ-complete measure on κ, the domain d_β can make use of only κ many columns of $j''\kappa^+$. Specifically, if the set of columns that d_β used from $j''\kappa^+$ had size κ^+ in $\mathbf{V}[G][g]$, then it would have to use unboundedly many columns from $j''\kappa^+$. But then the set of these column indices would form an unbounded set in $\sup j''\kappa^+ = j(\kappa^+) = (j(\kappa)^+)^{M[j(G)]}$ of size at most $j(\kappa)$ in $M[j(G)]$, contradicting the fact that $(j(\kappa)^+)^{M[j(G)]}$ is regular in $M[j(G)]$. Therefore d_β can only use κ many columns from $j''\kappa^+$ and hence only κ many coordinates in $\kappa \times j''\kappa^+$. But because $g \subseteq \mathrm{Add}(\kappa, \kappa^+)$ only has conditions with domains which are subsets of $\kappa \times \kappa^+$ and have size bounded below the critical point κ, all of the conditions in $j''g$ will have a domain that's a subset of $j''\kappa \times j''\kappa^+ = \kappa \times j''\kappa^+$. Therefore, d_β meets the domain of the function determined by $j''g$ on at most κ many coordinates.

Now that we have shown that we can construct an $M[j(G)]$-generic filter h for $j(\mathbb{Q})$ below $j''g$ in $\mathbf{V}[G][g]$, we can fully lift the embedding to $j : \mathbf{V}[G][g] \longrightarrow M[j(G)][j(g)]$ where $j(G) \cong G * g * G_{\mathrm{tail}}$ and $j(g) = h$ in the extension $\mathbf{V}[G][g]$. Therefore κ is measurable in $\mathbf{V}[G][g]$. Furthermore, if we force over $\mathbf{V}[G][g]$ with a generic filter H_0 for the poset $\mathrm{Add}(\kappa, \gamma)^{\mathbf{V}[G][g]}$ for some $\gamma \leq \kappa^+$, then $g * H_0$ will be a $\mathbf{V}[G]$-generic filter for the poset $\mathrm{Add}(\kappa, \kappa^+)^{\mathbf{V}[G]} * \mathrm{Add}\check{(\kappa, \gamma)}^{\mathbf{V}[G]}$, which is isomorphic to $\mathrm{Add}(\kappa, \kappa^+)^{\mathbf{V}[G]}$. Consequently, we can use such an isomorphism in $\mathbf{V}[G]$ to associate $g * H_0$ with a corresponding $\mathbf{V}[G]$-generic filter g^* for $\mathrm{Add}(\kappa, \kappa^+)^{\mathbf{V}[G]}$ in $\mathbf{V}[G][g][H_0]$. But then the same lifting argument would apply for this new filter g^*, showing that actually κ is also measurable in $\mathbf{V}[G][g^*] = \mathbf{V}[G][g * H_0] = \mathbf{V}[G][g][H_0]$.

Then since κ^+ is not collapsed in $\mathbf{V}[G][g]$, the forcing extension $\mathbf{V}[G][g]$ will be a model where κ is measurable, and the measurability of κ is indestructible by the forcing $\mathrm{Add}(\kappa, \gamma)$ for all $\gamma \leq \kappa^+$. Finally, we note that the GCH still holds in $\mathbf{V}[G][g]$ because for all $\gamma < \kappa$, the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ can be factored as $\mathbb{P}_{\gamma^+} * \mathrm{Add}(\gamma^+, 1) * \mathbb{P}_{(\gamma^+, \kappa)} * \dot{\mathbb{Q}}$ where the forcing $\mathrm{Add}(\gamma^+, 1) * \mathbb{P}_{(\gamma^+, \kappa)} * \dot{\mathbb{Q}}$ adds no new subsets of γ, and the forcing $\mathrm{Add}(\gamma^+, 1)$ codes all subsets of γ of $\mathbf{V}[G \restriction \mathbb{P}_{\gamma^+}]$ as a γ^+ sequence.

In summary, we have constructed a model $\mathbf{V}[G][g]$ where the GCH holds, κ is measurable, and its (weak) measurability is indestructible by the forcing to add γ many Cohen subsets to κ for all $\gamma \leq \kappa^+$. We now show in the following lemma that since the weak measurability of κ is indestructible by the forcing to add $\leq \kappa^+$ many Cohen subsets to κ, it is actually indestructible by the forcing $\mathrm{Add}(\kappa, \eta)$ for every ordinal η. This would show that $\mathbf{V}[G * g] = \mathbf{V}[G][g]$ is the desired forcing extension, completing this proof. $\square$

Lemma 1.2.2. *Suppose κ is weakly measurable. Then if the weak measurability of κ is indestructible by the forcing to add γ many Cohen subsets of κ for all $\gamma \leq \kappa^+$, then its weak measurability is indestructible by the forcing to add η many Cohen subsets of κ for all η.*

Proof of Lemma. Let $H \subseteq \mathrm{Add}(\kappa, \eta)$ be $\mathbf{V}[G * g]$-generic and then let $\mathcal{A}$ be an arbitrary collection of at most κ^+ many subsets of κ in $\mathbf{V}[G * g][H]$. Without loss of generality, we may assume $\eta \geq \kappa^+$. Now let $A \subseteq \kappa^+ \times \kappa$ in $\mathbf{V}[G * g][H]$ be such that every subset of $\mathcal{A}$ is listed on at least one of the columns of A.

We argue in $\mathbf{V}[G * g]$. Since A is a subset of a set of cardinality κ^+, there exists a κ^+ collection of antichains $\{A_s | s \in \kappa^+ \times \kappa\}$ such that the name $\dot{A} \equiv \bigcup_{s \in \kappa^+ \times \kappa} \{\check{s}\} \times A_s$ is a nice name for A. Now because $\mathrm{Add}(\kappa, \eta)$ is κ^+-c.c. ($\kappa^{<\kappa} = \kappa$ by virtue of weakly measurable cardinals being inaccessible), $\dot{A}$ uses at most κ^+ many conditions. But then since the domain of each condition has cardinality less than κ, the name $\dot{A}$ uses at most κ^+ many coordinates. Therefore, we may apply an automorphism to $\mathrm{Add}(\kappa, \eta)$ so that the columns of $\mathrm{Add}(\kappa, \eta)$ are reordered in such a way that the induced name for $\dot{A}$ becomes an $\mathrm{Add}(\kappa, \kappa^+)$-name.

By associating H with its image under the automorphism, the generic extension remains the same. Also, the intermediate filter $H_0 = H \cap \mathrm{Add}(\kappa, \kappa^+)$ is $\mathrm{Add}(\kappa, \kappa^+)$-generic. Furthermore, under this association, all of the conditions used in $\dot{A}$ will be in H if and only if they're in H_0. Consequently, $\dot{A}_{H_0} = \dot{A}_H = A$. Therefore $A \in \mathbf{V}[G * g][H_0]$ so $\mathcal{A} \in \mathbf{V}[G * g][H_0]$. Furthermore, because the forcing to add η many Cohen subsets of κ is κ^+-c.c.,

$\mathbf{V}[G*g]$ and $\mathbf{V}[G*g][H]$ (and hence $\mathbf{V}[G*g][H_0]$ and $\mathbf{V}[G*g][H]$) agree on the ordinal that's κ^+. Consequently because the weak measurability of κ is indestructible by the forcing to add κ^+ many subsets of κ, there exists a nonprincipal κ-complete filter $F \in \mathbf{V}[G*g][H_0]$ measuring all the subsets of κ in $\mathcal{A}$. Since $H_0 \in \mathbf{V}[G*g][H]$, we know that $\mathbf{V}[G*g][H_0] \subseteq \mathbf{V}[G*g][H]$ whereby this $F \in \mathbf{V}[G*g][H]$. This F will continue to be κ-complete in $\mathbf{V}[G*g][H]$ since no new $<\kappa$ sequences of F are added from $\mathbf{V}[G*g][H_0]$ to $\mathbf{V}[G*g][H]$ by virtue of the forcing to add Cohen subsets of κ being $<\kappa$-closed. Therefore, the weak measurability of κ is preserved in the forcing extension $\mathbf{V}[G*g][H]$. □

The two-step lifting argument in the proof of the Main Theorem, Statement (3) is similar to the proof of the theorem in [9] showing that a measurable cardinal can be made indestructible by the forcing to add one subset of κ. The preparatory forcing is different only in that the nontrivial stages of forcing γ are $\mathrm{Add}(\gamma, \gamma^+)$ instead of $\mathrm{Add}(\gamma, 1)$, but the arguments for the ability to lift through this forcing in $\mathbf{V}[G][g]$ are virtually identical. The divergence of these proofs is when we fully lift the embedding. Specifically, if we had only needed to lift through $\mathrm{Add}(\kappa, 1)$, then we'd have had a master condition below the filter $j''g$ since $j''g$ would only have size κ and $M[j(G)]$

is closed under κ sequences in $\mathbf{V}[G][g]$. Alternatively, we'd have $j''g = g$ so $M[j(G)]$, which has g, can construct the master condition $\bigcup g$. When $j''g$ has size κ^+, it is no longer the case that $j''g = g$, and the master condition may be missing in action. We therefore needed to build a family of master conditions below $j''g$ in $M[j(G)]$. In this case, we could not simply appeal to the Diagonalization Criterion, and we needed to build the $M[j(G)]$-generic filter h for $j(\mathrm{Add}(\kappa, \kappa^+))$ more carefully to ensure satisfaction of the necessary pull-back criterion (i.e. $j''g \subseteq h$). Note that if we had assumed the stronger hypothesis that κ were κ^+-supercompact, then we could have carried out the proof preserving κ^+-supercompactness and had a master condition at this stage. However, then we would need to appeal to a stronger hypothesis to get the relative consistency result that the GCH can fail first at a weakly measurable cardinal.

The lemma is analogous to the result that a weakly compact cardinal κ can be made indestructible by the forcing to add any number of Cohen subsets of κ if its weak compactness can be made indestructible by the forcing to add κ many Cohen subsets of κ (or equivalently, just one). The underlying theme here is the size of the witnessing embeddings for the large cardinal notions. The weak compactness of κ is witnessed by embeddings of hereditary size κ while the weak measurability of κ is witnessed by embeddings of hereditary size κ^+. Indeed, the proof of Statement (3) was adapted from the proof of the well-known theorem that if κ is weakly compact, then there exists a forcing extension where the GCH holds, and the weak compactness of κ is preserved and indestructible by the further forcing $\text{Add}(\kappa, \eta)$ for all η. (See [9].)

Starting from a weakly compact cardinal κ, the strategy to achieve indestructibility would be to make its weak compactness indestructible by the forcing to add one Cohen subset of κ whereby its weak compactness would become indestructible by the forcing to add any number of Cohen subsets of κ. Because the witnessing weak compactness embeddings can be made to be between $<\kappa$-closed transitive models of size κ, we can apply the Diagonalization Criterion by virtue of there being at most κ many (dense) subsets (counted in $\mathbf{V}$) of every poset in the target model. Consequently, the same Silver iteration used to achieve the forcing extension where the measurability of κ becomes indestructible by the forcing to add one Cohen subset of κ can be used to achieve the forcing extension where the weak compactness of κ becomes indestructible by the forcing to add one Cohen subset of κ. With weakly measurable cardinals, we needed to first make the weak measurability indestructible by the forcing to add κ^+ many Cohen subsets to κ whereby its weak measurability would become indestructible by the forcing to add any number of Cohen subsets of κ. Unfortunately, it seems that we needed to start with a measurable cardinal to do this because the κ^+-sized transitive target models of weakly measurable witnessing embeddings could not be better than $<\kappa$-closed if the GCH fails at κ. In this case, we would not be able to appeal to the Diagonalization Criterion because we would potentially have to diagonalize against the κ^+ many subsets in the $<\kappa$-closed target models.

Returning to Statements 1 and 2, Dana Scott proved that the GCH could not fail first at a measurable cardinal κ by using a simple reflection argument. Specifically, fix any elementary embedding $j : \mathbf{V} \longrightarrow M$ into some inner model M with critical point κ. Then if the GCH holds below κ in $\mathbf{V}$, then in M it must hold below $j(\kappa)$, and hence at κ, by elementarity. But M has the actual powerset of κ so if M has a κ^+-enumeration of $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{\mathbf{V}}$, then $\mathbf{V} \supseteq M$ will have this κ^+-enumeration of the true powerset of κ as well. (See [18].) The fact then that the GCH cannot fail first at a measurable cardinal shows that Statement 1 of the Main Theorem follows from Statement 2 because in a forcing extension where κ is weakly measurable and the GCH fails first at this cardinal, κ will not be measurable. To see that the Main Theorem, Statement 2 follows from Statement 3, assume κ is measurable, and move to the forcing extension $\overline{\mathbf{V}}$ where the the GCH holds, and the weak measurability of κ is preserved and indestructible by the further forcing $\mathrm{Add}(\kappa, \eta)$ for all η. Then in $\overline{\mathbf{V}}$, force with $\mathrm{Add}(\kappa, \kappa^{++})$, and let $\overline{\mathbf{V}}[H]$

be such an extension. In this case, the GCH fails at κ in $\overline{\mathbf{V}}[H]$, but since the GCH holds below κ in $\overline{\mathbf{V}}$ and $\mathrm{Add}(\kappa, \kappa^{++})$ is $<\kappa$-closed, it will hold below κ in $\overline{\mathbf{V}}[H]$. Consequently, $\overline{\mathbf{V}}[H]$ will be a forcing extension as in Main Theorem, Statement 2 where the GCH fails first at κ, and κ remains weakly measurable.

We close this section with a few results that follow from the Main Theorem. Statement 1 of the Main Theorem actually implies that all weakly measurable cardinals κ can be made nonmeasurable in a forcing extension: either a trivial forcing extension if κ is not already measurable in the ground model or a forcing extension constructed in the proof of Statement 2 where the GCH fails first at κ if κ is measurable. As mentioned earlier, forcing to make the GCH fail first at a cardinal κ makes it impossible for κ to be measurable in that extension. Consequently, it is relatively consistent that a weakly measurable cardinal is not measurable. We can also conclude the following:

Corollary 1.2.3. *If κ is weakly measurable, then there is an inner model where κ is measurable and a forcing extension of this inner model where κ is weakly measurable, and the GCH fails first at κ. Consequently, it is consistent, relative to a weakly measurable cardinal, that the GCH fails first at a weakly measurable cardinal.*

Proof. Suppose κ is weakly measurable. By Theorem 1.3.3 (stated and proved in the next section), κ is measurable in an inner model. Then by the Main Theorem, Statement 2, there is a forcing extension of this inner model where κ is weakly measurable, and the GCH fails first at κ. $\square$

1.3 Consistency strength of weakly measurable cardinals

In this section, I prove that if κ is weakly measurable, then there is an inner model in which κ is a measurable cardinal. Thus, the consistency strength of weakly measurable and measurable cardinals is the same. To prove this result, it is natural to want to build $\mathbf{L}[\mu]$ by starting with a family $\mathcal{A}$ containing κ^+ many subsets of κ and then appealing to the existence of a nonprincipal κ-complete filter F on κ that measures all subsets of κ in $\mathcal{A}$ such that $\mu = F \cap \mathbf{L}[F]$ measures all subsets of κ in $\mathbf{L}[\mu]$. However, finding an appropriate family of subsets $\mathcal{A}$ does not seem to be so easy as one might think at first. If we try to take too much of a minimalistic approach to selecting our $\mathcal{A}$, we may find out that our $\mathcal{A}$ only has size κ. In this case if we were to choose too small of a collection, such as $\mathcal{P}(\kappa)^{\mathbf{L}}$, then $\mathbf{L}[\mu]$ can contain new subsets of κ not measured by F because of the woeful lack of richness of $\mathcal{A}$. For example, if we were to restrict F to be a measure on $\mathcal{P}(\kappa)^{\mathbf{L}}$, then $\mathbf{L}$ would be able to compute the sets $A_\alpha \in F$ for any of its κ-sequences $\langle A_\alpha | \alpha < \kappa \rangle$ ($\{\alpha < \kappa | A_\alpha \in F\} \in \mathbf{L}$). However, because of $\mathbf{L}$'s misconception about κ^+ (when κ is weakly measurable), $\mathbf{L}[\mu]$ would even add subsets of ω not in $\mathbf{L}$ such as $0^\sharp$. On the other hand, if we choose an $\mathcal{A}$ with a relatively rich powerset of κ such as $\mathcal{A} = \mathcal{P}(\kappa)^M$ for an M agreeing with $\mathbf{V}$ on the ordinal corresponding to κ^+, then F may be so complex that M may not even contain all of its κ approximations to F, and unmeasured subsets of κ may be added by $\mathbf{L}[\mu]$ simply because of this complexity. Of course, both of these problems can be remedied by selecting an $\mathcal{A}$ containing the full powerset of κ, but if the GCH fails at κ, then the weak measurability of κ alone would not guarantee the existence of the corresponding F.

Instead of proceeding along these lines, I shall circumvent the aforementioned problems by making use of an old implicit result of Kunen in [19] mentioned by Jech, Magidor, Mitchell and Prikry in [17]. Before stating this lemma, I should introduce Kunen's weak amenability definition that is a weakening of Jensen's amenability condition.

Definition 1.3.1 (Weakly Amenable). *Let $j : M \longrightarrow N$ be the ultrapower embedding with critical point κ generated by a well-founded nonprincipal filter $F \subseteq \mathcal{P}(\kappa)^M$ on κ that measures all subsets in $\mathcal{P}(\kappa)^M$ and is closed under $<\kappa$ intersections in M. Then the filter F or embedding j is said to be weakly* *amenable if for every* κ*-sequence* $\langle s_\xi | \xi < \kappa \rangle$ *in* M *of subsets of* κ*, the set* $\{\xi < \kappa | s_\xi \in F\}$ *is also in* M*.*

The condition of weak amenability is what I was referring to when I talked about M having all of its κ approximations to F. Jensen's amenability condition would dictate that $F \cap M = F$ be in M, but Kunen realized that such a requirement in its entirety was not necessary for iterating the measure. This is significant because Kunen used an ultrapower iteration construction (which requires the weak amenability hypothesis) to prove the lemma. [19] [18] As an aside, I would like to thank Gunter Fuchs [3] for helping me with the below proof of the Main Theorem, Statement 4, which follows the key lemma from [17].

Lemma 1.3.2 (Kunen, Jech, Magidor, Mitchell, and Prikry). *Let* C *be the set of strong limit cardinals with cofinality greater than* κ*, and let* $A = \{\gamma_n | n < \omega\}$ *be the set containing the first* ω *least ordinals* α *of* C *such that* $ot(\alpha \cap C) = \alpha$*. If there exists an* $\boldsymbol{L}[A]$*-normal well-founded weakly amenable measure on* κ*, then* κ *is measurable in an inner model.*

In the aforementioned lemma, the terminology $\mathbf{L}[A]$-normal measure on κ is used to mean that the filter measures precisely the subsets of κ in $\mathbf{L}[A]$ and that every function $f : \kappa \longrightarrow \kappa$ in $\mathbf{L}[A]$ that's regressive on a set in the filter is also constant on a set in the filter. The weak amenability condition permits the construction of the iterated ultrapower for $\mathbf{L}[A]$, and the filter is said to be well-founded if each iterate of the ultrapower structure of $\mathbf{L}[A]$ is well-founded. Also, in the proof below, κ-complete is used to mean closed under $<\kappa$-intersections in $\mathbf{L}[A]$, which need not mean actual closure under all $<\kappa$-intersections.

Theorem 1.3.3. *If κ is weakly measurable, then κ is measurable in an inner model, and so the two concepts of measurability are equiconsistent.*

Proof. Suppose κ is weakly measurable. I claim that for the A of Lemma 1.3.2, we can find an $\mathbf{L}[A]$-normal well-founded weakly amenable measure on κ so that κ is measurable in an inner model as per the lemma.

Since κ is weakly measurable, it is inaccessible. This is downward absolute to $\mathbf{L}[A]$ so in particular $\kappa^\omega = \kappa$ in this inner model. From this and the fact that A has order type ω, we can show by standard methods that $\mathbf{L}[A] \models 2^\kappa = \kappa^+$. Consequently, by the weak measurability of κ, we may find a nonprincipal κ-complete $\mathbf{L}[A]$-normal measure F on κ. In particular, F will be well-founded by its countable completeness.

Let $j : \mathbf{L}[A] \longrightarrow \mathbf{L}[j(A)]$ be the ultrapower embedding with critical point κ induced by F. We verify that this embedding is weakly amenable. First note that since A has size $\omega < \kappa$, we can conclude that $j(A) = j''A$. Now fix $\alpha \in A$. Since α has cofinality greater than κ with j an ultrapower embedding induced by a measure on κ, it follows that $j(\alpha) = \sup j''\alpha$. But now because α is a strong limit cardinal above κ with j an ultrapower embedding induced by a measure on κ, we know that $j(\beta) < \alpha$ for all $\beta < \alpha$. Consequently, $j(\alpha) = \alpha$ so j fixes A pointwise. It then follows that $j(A) = j''A = A$ so $j : \mathbf{L}[A] \longrightarrow \mathbf{L}[A]$. The weak amenability of F then follows immediately from the fact that j is generated by this nonprincipal $\mathbf{L}[A]$-normal κ-complete measure F on κ. Specifically, if $\vec{s} = \langle s_\xi | \xi < \kappa \rangle$ is a κ sequence of subsets of κ, then $\{\xi < \kappa | s_\xi \in F\} = \{\xi < \kappa | \kappa \in j(s_\xi)\} = \{\xi < \kappa | \kappa \in j(\vec{s})_\xi\} \in \mathbf{L}[j(A)] = \mathbf{L}[A]$.

□

I would like to close this section by describing an alternative way to prove the equiconsistency of weakly measurable and measurable cardinals using the core model K below $0^\dagger$, insightfully recommended to me by a referee. Moreover, we can achieve an inner model, mainly K, where κ is measurable if we assume the anti-large cardinal hypothesis that $0^\dagger$ does not exist. Assume that we have a weakly measurable cardinal κ. Without loss of generality, we may assume that $0^\dagger$ does not exist, for if it does, then we already have a measurable cardinal in an inner model. In this nonexistence case, it will follow from the weak covering lemma that if U is a K-normal measure on κ that generates a well-founded ultrapower of K, then $U \in K$. (See [22].) But K has only κ^+ many subsets of κ so that we can find such a U by the weak measurability of κ. Then $U \in K$ and so κ is measurable there.

1.4 Toward stronger notions and open questions

In the introduction, we asked whether it was possible for the least weakly measurable cardinal to also be the least weakly compact cardinal. In section 1.3, we appealed to an old lemma to be able to show that we could find an inner model of the form $\mathbf{L}[\mu]$ where κ would be measurable. In this section, we consider properties that weakly measurable cardinals may or may not already necessarily possess in an attempt to better understand the possibilities along these lines. The first property that we consider for a weakly measurable cardinal κ is one of weak amenability which strengthens the (weak embedding characterization of a) weakly measurable cardinal in an analogous way to how a weakly Ramsey cardinal (see [5]) strengthens the corresponding characterization of weak compactness.

Question 1.4.1. *If κ is weakly measurable, can we find for every $A \subseteq \kappa^+$ a* *transitive* $M \models ZFC^-$ *with* $A, \kappa \in M$*, a transitive* N *with* $\mathcal{P}(\kappa)^N = \mathcal{P}(\kappa)^M$*, and an elementary embedding* $j : M \longrightarrow N$ *with critical point* κ*?*

Notice that what is asked for does indeed strengthen the weak embedding characterization of weak measurability (assuming $(\kappa^+)^{<\kappa} = \kappa^+$) by requiring that M and N agree on the powersets of κ. The reader recalling Definition 1.3.1 of weak amenability from section 1.3 may wonder how this concept relates. The following lemma answers this question.

Lemma 1.4.2. *Let* $j : M \longrightarrow N$ *be the ultrapower embedding with critical point* κ *generated by a well-founded nonprincipal filter* $F \subseteq \mathcal{P}(\kappa)^M$ *on* κ *that measures all subsets in* $\mathcal{P}(\kappa)^M$ *and is closed under* $<\kappa$ *intersections in* M*. Then* j *is weakly amenable if and only if* M *and* N *have the same powerset of* κ*.*

Proof. ($\Rightarrow$): Let $f \in M \cap {}^\kappa M$, and suppose that $[f]_F \in N$ is a subset of κ. For all $\alpha < \kappa$, the ordinal $\alpha \in [f]_F$ if and only if $A_\alpha = \{\beta < \kappa | \alpha \in f(\beta)\}$ is in F so that $[f]_F = \{\alpha < \kappa | A_\alpha \in F\}$. Because the function $f \in M$, the sequence $\langle A_\alpha | \alpha < \kappa\rangle$ is in M. It then follows by the weak amenability of j that $[f]_F = \{\alpha < \kappa | A_\alpha \in F\} \in M$. Consequently, M has all subsets of κ that N does. N will have all subsets of κ that M does even without the weak amenability condition since for all $A \subseteq \kappa$ in M, we have $A = j(A) \restriction \kappa \in N$.

Therefore, M and N have the same powerset of κ.

($\Leftarrow$): Suppose $A = \langle A_\alpha | \alpha < \kappa \rangle \in M$ is a κ-sequence of subsets of κ in M. Then $A_\alpha \in F$ if and only if $[\mathrm{id}]_F \in j(A_\alpha) = j(A)_\alpha$. Consequently, we will have $\{\alpha < \kappa | A_\alpha \in F\} = \{\alpha < \kappa | [\mathrm{id}]_F \in j(A)_\alpha\}$, which will be in N's powerset of κ. But because M and N have the same powerset of κ, it will also be in M, as desired. □

Consequently, Question 1.4.1 could have instead asked that the embeddings j were weakly amenable in terms of Definition 1.3.1. If the answer to this question is "yes," then the question of whether the least weakly measurable cardinal can also be the least weakly compact cardinal is resolved with a negative answer in a very strong way.

Theorem 1.4.3. *Suppose $j : M \longrightarrow N$ is a weakly amenable ultrapower embedding with critical point κ generated by a well-founded nonprincipal filter $F \subseteq \mathcal{P}(\kappa)^M$ on κ that measures all subsets in $\mathcal{P}(\kappa)^M$ and is closed under $<\kappa$ intersections in M. Then κ is weakly compact in both M and N, and consequently κ will be a weakly compact limit of weakly compact cardinals in M.*

Proof. Let $\vec{s} = \langle s_\xi | \xi < \kappa \rangle \in N$ be a κ-sequence of subsets of κ in N. Without loss of generality, we may assume that all $\alpha < \kappa$ are mentioned by $\vec{s}$. Since all $\leq\kappa$-enumerations of subsets of κ can be encoded as subsets of κ, we may take Lemma 1.4.2 to also mean that M and N agree on the $\leq\kappa$-sequences of subsets of κ. In particular, $\vec{s}$ will only list subsets of κ in M and will be in M. Then by weak amenability, $\mathcal{I} = \{\xi < \kappa | s_\xi \in F\}$ is also in M. By Lemma 1.4.2, $\mathcal{I}$ will be in N.

Now let $E \in N$ be the collection of subsets of κ that results by closing $B = \{s_\xi | \xi \in \mathcal{I}\} \cup \{\kappa \setminus s_\xi | \xi \in \kappa \setminus \mathcal{I}\}$ under supersets and $<\kappa$-intersections in N. By construction, E is a filter closed under $<\kappa$-intersections in N measuring all subsets listed by $\vec{s}$. Because $B \subseteq \mathcal{P}(\kappa)^M$ is constructed to agree with a filter F on κ that measures all subsets of κ in M, it follows that $B \subseteq F$. But then since F is closed under supersets and $<\kappa$-intersections in M, and M and N agree on their mutual powerset of κ and its $<\kappa$-sequences, we have $E \subseteq F$. This will ensure that E is proper, and since E measures all ordinals, it also means that E is nonprincipal.

Consequently, κ is weakly compact in N. Therefore, by reflection, M thinks that the set of weakly compact cardinals below κ is unbounded in M. Then by elementarity, N will agree with M on the weakly compact cardinals below κ so it will think that κ is a weakly compact limit of weakly compact cardinals. Note that if we had generated E in M rather than in N, this would show that κ is weakly compact in M. In fact, we would be generating the same E since M and N agree on the powerset of κ and the $\leq\kappa$-sequences of subsets of κ. Therefore M will also think that κ is a weakly compact limit of weakly compact cardinals. $\square$

I should mention that despite the fact that we only ask for arbitrary powerset-preserving elementary embeddings in Question 1.4.1, we may assume without loss of generality that these are in fact ultrapower embeddings. This is because every elementary embedding $j : M \to N$ between ZFC^- models induces a factor ultrapower embedding $j_0 : M \to N_0$ for some transitive N_0 as in the embedding implies normal embedding part of the proof of Theorem 1.1.1. The induced elementary embedding $k : N_0 \to N$ fixing all ordinals below the critical point κ of j would tell us that N_0's powerset of κ must be a subset of N's powerset of κ because for every $A \subseteq \kappa$ in N_0, we have $A = k(A) \restriction \kappa$. Therefore M's powerset of κ, which is the same as N's powerset of κ by assumption, must be the same as N_0's powerset of κ.

Corollary 1.4.4. *If the answer to Question 1.4.1 is "yes," then every weakly measurable cardinal is a limit of weakly compact cardinals so the answer to Question 1.4.6 posed in the introduction is "no."*

Proof. Suppose κ is a weakly measurable cardinal, and the implication in Question 1.4.1 holds. Let $A \subseteq \kappa$ be an encoding of $\mathbf{V}_\kappa$. Then there exists a transitive $M \models \text{ZFC}^-$ with $A, \kappa \in M$, a transitive N, and a weakly amenable elementary embedding $j : M \longrightarrow N$ with critical point κ. By Theorem 1.4.3, κ is a limit of weakly compact cardinals in M. But then because M contains $\mathbf{V}_\kappa$, the model M is correct in this belief. $\square$

As the proof of Corollary 1.4.4 indicated, we do not need to posit such a strong hypothesis to show that a weakly measurable cardinal is a limit of weakly compact cardinals. Having only one such ZFC^- model M containing $\mathbf{V}_\kappa$ and one weakly amenable embedding $j : M \longrightarrow N$ with critical point κ would have been sufficient. In fact, from [5], we know that if the implication were to hold, then every weakly measurable cardinal would be weakly ineffable and more.

If we replace weakly measurable in Question 1.4.1 with measurable, then the implication holds. In this case, we have a fully amenable embedding $j : \mathbf{V} \longrightarrow N$ having critical point κ with $\mathbf{V}$ containing all subsets of κ^+. If we also wanted to ensure that the domain of the embedding is a transitive model of size κ^+, we could do this by selecting a ZFC^- model M of size κ^+ containing the desired $A \subseteq \kappa^+$ where κ is measurable and taking the induced M-ultrapower embedding $j : M \longrightarrow N$ by the measure on κ in M. However, the next result shows that κ need not be measurable in order to have a family of ZFC$^-$ models M covering every $A \subseteq \kappa^+$ and a corresponding family of weakly amenable or even fully amenable elementary embeddings $j : M \longrightarrow N$.

Theorem 1.4.5. *In the forcing extension* $\boldsymbol{V}[G][g][H]$ *constructed in the proof of Lemma 1.2.2 of Theorem 1.2.1, where* κ *is weakly measurable and* $H \subseteq Add(\kappa, \eta)$ *is* $\boldsymbol{V}[G][g]$*-generic for some* η*, the set of ZFC*$^-$ *models* M *where* κ *is measurable forms a covering of* $\mathcal{P}(\kappa^+)^{\boldsymbol{V}[G][g][H]}$.

Proof. The argument is similar to the one given in Lemma 1.2.2. Letting $A \subseteq \kappa^+$ in $\mathbf{V}[G][g][H]$, we may transform an $\mathrm{Add}(\kappa, \eta)$-name for A into an $\mathrm{Add}(\kappa, \kappa^+)$-name $\dot{A} \in \mathbf{H}_{\kappa^{++}}^{\mathbf{V}[G][g]}$ by reordering the forcing. Then because κ remains measurable after forcing with the $\mathbf{V}[G][g]$-generic $H_0 \subseteq \mathrm{Add}(\kappa, \kappa^+)$, the cardinal κ is measurable in $\mathbf{V}[G][g][H_0]$. Consequently, in $\mathbf{V}[G][g][H_0]$, we can find a transitive ZFC$^-$ model M containing $\dot{A}_{H_0} = A$ and κ where κ is measurable. This will remain true in $\mathbf{V}[G][g][H] \supseteq \mathbf{V}[G][g][H_0]$. $\square$

Despite the aforementioned results, the possibility that the least weakly measurable cardinal can also be the least weakly compact cardinal still remains. Measurable cardinals κ are limits of weakly compact cardinals because the M of every ultrapower embedding $j : \mathbf{V} \longrightarrow M$ will contain the restricted elementary embedding $j \restriction M^* : M^* \longrightarrow j(M^*)$ for every $M^* \in \mathbf{H}_{\kappa^+}^M = \mathbf{H}_{\kappa^+}$.

Consequently, κ is weakly compact in the ultrapower so that κ is a limit of weakly compact cardinals in $\mathbf{V}$ by reflection. But if the GCH fails at κ, we can potentially witness the weak measurability of κ with a family of elementary embeddings $j : M \longrightarrow N$, all of which satisfy $\mathcal{P}(\kappa)^M \subsetneq \mathcal{P}(\kappa)^N \subsetneq \mathcal{P}(\kappa)$. In this case, it is conceivable for all such N to think that κ is *not* weakly compact. Assuming weakly compact cardinals are relatively consistent, this must certainly be the case for the witnessing embeddings $j : M \longrightarrow N$ for the least weakly compact cardinal, and because the target models N for the witnessing embeddings of size κ^+ cannot be κ-closed when the GCH fails at κ, this may be the case for the least weakly measurable cardinal as well. Accordingly, we re-iterate a central question left open by this chapter:

Question 1.4.6. *Is it relatively consistent with any large cardinal notion that the least weakly measurable cardinal is also the least weakly compact cardinal?*

Before I continue this section along a different strengthening property, I would like to take this time to make a few more remarks regarding Question 1.4.1. As alluded to in Section 1.3, a weakly measurable cardinal guarantees that for some $A \subseteq \kappa^+$, we will be able to find a transitive ZFC$^-$ model M containing A and a weakly amenable elementary embedding $j : M \longrightarrow N$ with critical point κ where N is also transitive. For example, this will be true for every constructible $A \subseteq \kappa^+$. If κ is weakly measurable, then we will have an elementary embedding $j : \mathbf{L} \longrightarrow \mathbf{L}$ with critical point κ, and this embedding will clearly be weakly amenable since $\mathbf{L}$ has the same powerset of κ as itself. However, for more complex subsets of κ^+ that would necessitate a ZFC$^-$ model to contain the true $\mathbf{V}_\kappa$, this may not be the case. As mentioned earlier, a weakly compact cardinal is sufficiently weaker than a weakly Ramsey cardinal in consistency strength. Since weakly measurable cardinals are equiconsistent with measurable ones, we cannot hope to prove an analogous result for weakly measurable and weakly measurable cardinals with the added amenability condition. Nevertheless, the relationship between weakly compact and weakly Ramsey cardinals provides a precedent for the possibility that these notions are different.

We now turn to a second type of property that potentially strengthens weak measurability motivated by our difficulty in selecting an appropriate filter F where κ becomes measurable in $\mathbf{L}[\mu]$. Recall in the discussion before the proof of the equiconsistency of weakly measurable and measurable cardinals, we talked about the problem of "chasing our own tail" trying to find an appropriate κ^+-sized collection $\mathcal{A}$ of subsets of κ. Specifically, we wanted to choose an $\mathcal{A}$ of size κ^+ so that a filter F on κ that measured all subsets of $\mathcal{A}$ wouldn't be such that $\mathbf{L}[F]$ would contain subsets of κ that were not

not already in $\mathcal{A}$. At first glance, it may seem that we can immediately find such $\mathcal{A}$ if our weakly measurable cardinal κ possessed the weak amenability property already discussed. A proposed strategy might be to use a weakly amenable embedding with a domain of a ZFC$^-$ model M that agrees with $\mathbf{V}$ on κ^+. We could do this by encoding a κ^+-sequence $\langle b_\gamma | \kappa < \gamma < \kappa^+ \rangle$ as an $A \subseteq \kappa^+$ where each b_γ is a bijection from κ into γ. The argument would note that all $C \subseteq \kappa$ of the associated $\mathbf{L}[F]$ are built by some stage $\gamma < \kappa^+$ and therefore only make use of at most κ many subsets of F. The *fallacious* conclusion would then be that since every such C is definable from this $\leq\kappa$-sized subcollection $D \subseteq F$, the model M will know that D has size at most κ by virtue of agreeing with $\mathbf{V}$ on κ^+. If this were indeed the case, it would have D by weak amenability and then have the D-definable A. However, the problem with this conclusion is that weak amenability does not require all $\leq\kappa$-sized subsets of F to be in M, which is actually impossible if M is to have size κ^+ and $2^\kappa > \kappa^+$, but rather only the ones for which M can cover with a κ-sequence.

What we want then is for all subsets of F in $\mathbf{L}[F]$ that have size at most κ in $\mathbf{L}[F]$ to be in M. Note that even when the GCH fails at κ, the model $\mathbf{L}[F]$ will still only have at most κ^+ many such approximations. Consequently, we cannot rule out κ^+-sized transitive M satisfying this condition (and there will be such κ^+-sized transitive M if κ is measurable). We can formalize this proposition and justification as follows:

Proposition 1.4.7. *Let $M \models ZFC^-$ and $F \subseteq \mathcal{P}(\kappa)^M$ be a nonprincipal filter on a cardinal κ that measures all subsets in $\mathcal{P}(\kappa)^M$ and is closed under $<\kappa$ intersections in M. If $(\kappa^+)^{\mathbf{L}[F]} \in M$, and for every $F_0 \subseteq F$ of size at most κ in $\mathbf{L}[F]$, we have $F_0 \in M$, then $\mathbf{L}[F] \models$ "κ is measurable."*

Proof. I claim that $\mu = F \cap \mathbf{L}[F]$ is a nonprincipal filter in $\mathbf{L}[F]$ on κ that measures all subsets in $\mathcal{P}(\kappa)^{\mathbf{L}[F]}$ and is closed under $<\kappa$ intersections in $\mathbf{L}[F]$ so that κ is measurable in $\mathbf{L}[F]$. Given some β-sequence $\vec{a} = \langle A_\alpha | \alpha < \beta \rangle$ of subsets of κ from μ in $\mathbf{L}[F]$ with $\beta < \kappa$, we will have range$(\vec{a}) \in M$ by the assumption that M contains all subcollections of F of size at most κ in $\mathbf{L}[F]$. Then because $\beta < \kappa$ and κ is a cardinal, M can list range$(\vec{a})$ with a $<\kappa$-sequence. Consequently, $\bigcap$range$(\vec{a}) \in F$ by the closure of F under $<\kappa$ intersections in M. Thus $\bigcap$range$(\vec{a}) \in \mu$ so the nonprincipal filter μ is indeed closed under $<\kappa$ intersections in $\mathbf{L}[F]$. Let's therefore prove that all subsets of κ in $\mathbf{L}[F] = \mathbf{L}[\mu]$ are measured by μ.

To do this, first fix a regular θ such that $\mu \in \mathbf{L}_\theta[\mu]$, and suppose $A \subseteq \kappa$ is in $\mathbf{L}[\mu]$. We will show that this A is already a subset of κ in M so that it is measured by F and hence μ. Working in $\mathbf{L}[F]$, let X be an elementary substructure of $\mathbf{L}_\theta[\mu]$ of size κ such that $A, \mu \in X$ and $\kappa \subseteq X$. Then let $\pi : X \overset{\cong}{\longrightarrow} \mathbf{L}_\gamma[\pi(\mu)]$ be the Mostowski Collapse. Since $\kappa \subseteq X$, all ordinals below κ are fixed by π. In particular, $\pi(A) = A \in \mathbf{L}_\gamma[\pi(\mu)]$ and $\pi(\mu) \subseteq \mu \subseteq F$. Also since X has size κ in $\mathbf{L}[F]$, the ordinal $\gamma < (\kappa^+)^{\mathbf{L}[F]}$ and $\pi(\mu)$, which is a subset of $\mathbf{L}_\gamma[\pi(\mu)]$, will have size at most κ in $\mathbf{L}[F]$. Consequently, $\pi(\mu)$ will be a subset of F of size at most κ in $\mathbf{L}[F]$ whereby it will be in M. Then because we also have $\gamma < (\kappa^+)^{\mathbf{L}[F]} \in M$, the subset $A \in \mathbf{L}_\gamma[\pi(\mu)] \subseteq M$. Therefore, $A \in M$, as desired. □

This discussion and proposition motivate the question of whether a weakly measurable cardinal already possesses a certain property:

Question 1.4.8. *If κ is weakly measurable, can we for every $A \subseteq \kappa^+$ find a transitive $M \models ZFC^-$ with $A, \kappa \in M$ and a nonprincipal well-founded filter $F \subseteq \mathcal{P}(\kappa)^M$ on κ that measures all subsets in $\mathcal{P}(\kappa)^M$ and is closed under $<\kappa$ intersections in M such that for every $F_0 \subseteq F$ of size at most κ in $\boldsymbol{L}[F]$, we have $F_0 \in M$?*

Note that this question is asking for a potential strengthening of the weak embedding characterization of weak measurability (when $(\kappa^+)^{<\kappa} = \kappa^+$) since we can get an elementary embedding $j : M \longrightarrow N$ with critical point κ by taking the M-ultrapower embedding induced by F. Also note that we ask that every $F_0 \subseteq F$ of size at most κ in $\mathbf{L}[F]$ be in M. If we choose an M with the correct κ^+, then it may seem that if the implication of Question 1.4.1 holds, then the implication of Question 1.4.8 will as well so that the former implication potentially insists on more. Specifically, an affirmative answer to Question 1.4.1 requires that for every collection in M of subsets of κ that has size at most κ in M (and hence in $\mathbf{V}$), M must be able to identify which of them are in F as opposed to an affirmative answer to Question 1.4.8 which would require that M be able to identify the $\leq\kappa$ approximations in $\mathbf{L}[F]$, some actual κ collections of which may be excluded by virtue of not being definable from F or being too big in $\mathbf{L}[F]$. However, as mentioned earlier, this is not necessarily the case as there may very well be κ sequences of subsets of κ in $\mathbf{L}[F]$ that are not in the corresponding M due to M's inability to use F as a predicate for forming subsets of κ. This line of thought leads to a question of the comparability of the two properties:

Question 1.4.9. *If a weakly measurable cardinal possesses the condition of weak amenability from Question 1.4.1 or the condition from Question 1.4.8, will it possess the other property as well?*

The significance of Question 1.4.8 is in part derived from Proposition 1.4.7, which shows that from each such M-ultrafilter F for an M of height at least κ^+, the cardinal κ becomes measurable in the inner model $\mathbf{L}[F]$. Of course, if F is a nonprincipal κ-complete ultrafilter on κ, then κ will always be measurable in $\mathbf{L}[F]$. Moreover, it is clear that if we replace weakly measurable in Question 1.4.8 with measurable, then the implication will hold as every transitive ZFC$^-$ model M with height at least κ^+ having an F which it thinks is a nonprincipal κ-complete ultrafilter on κ will actually have $\mathbf{L}_{\kappa^+}[F] \subseteq M$. In particular, it will contain all subsets of F of size at most κ in $\mathbf{L}[F]$. However, by Theorem 1.4.5, we know that κ need not be measurable in order to have such transitive ZFC$^-$ models M covering every $A \subseteq \kappa^+$.

1.5 Interaction of weakly measurable cardinals and forcing and more open questions

We conclude this chapter by considering the interaction of weakly measurable cardinals and forcing. In the lifting argument for the proof of the Main Theorem, Statement 3, I started with the assumption of a measurable cardinal κ instead of a weakly measurable one. The reason for this was because of the ease of which we could apply the Diagonalization Criterion from [9] in order to diagonalize against the κ^+ many maximal antichains of the poset contained in the κ-closed ultrapower. Unfortunately, I have been unable to obtain a substitute for this Criterion for weakly measurable cardinals when the target models of the weak measurability embeddings have κ^+ many maximal antichains of the poset but are only closed under $<\kappa$-sequences. Consequently, a question that is left open is the following:

Question 1.5.1. *Can we make the weak measurability of κ indestructible by Add$(\kappa, 1)$ for arbitrary weakly measurable cardinals κ?*

Another question about weakly measurable cardinals is the extent to which the notions of weakly measurable and measurable cardinals can differ. In Section 1.3, we showed that a weakly measurable cardinal is measurable in an inner model of the form $\mathbf{L}[A]$. In the proof of the Main Theorem, we start with a measurable cardinal and force to achieve an extension where we have destroyed its measurability while retaining its weak measurability. We now show a somewhat surprising result:

Theorem 1.5.2. *In the forcing extension $\mathbf{V}[G][g][H]$ from Lemma 1.2.2 where the GCH fails first at the weakly measurable cardinal κ (if $\eta \geq \kappa^{++}$), we can force over a partial order of the form Coll$(\kappa^+, \lambda)^{\mathbf{V}}$ to restore the measurability of κ.*

Proof. Choose any $\lambda \geq (2^\kappa)^{\mathbf{V}[G][g][H]}$, and let $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathrm{Add}}(\kappa, \eta)$ be the poset that we force over to get from $\mathbf{V}$ to $\mathbf{V}[G][g][H]$. Now if we force with $\text{Coll}(\kappa^+, \lambda)^{\mathbf{V}}$ after forcing with $\mathbb{R}$ over the ground model, then the forcing factors as $\mathbb{R} * \text{Coll}(\check{\kappa^+}, \lambda)^{\mathbf{V}} \cong \text{Coll}(\kappa^+, \lambda)^{\mathbf{V}} \times \mathbb{R}$. But $\text{Coll}(\kappa^+, \lambda)^{\mathbf{V}}$ is $\leq\kappa$-distributive (in $\mathbf{V}$) so it preserves the original measurability of κ in the ground model, and the forcing $\mathbb{R}$ will always preserve the weak measurability of κ if κ is a measurable cardinal. Consequently, κ remains weakly measurable after the entire forcing iteration. Also since $\text{Coll}(\kappa^+, \lambda)^{\mathbf{V}}$ is $\leq\kappa$-closed in the ground model, it is easy to verify that $\mathbb{R}$ retains the same meaning modulo the size of η after forcing with $\text{Coll}(\kappa^+, \lambda)^{\mathbf{V}}$ (i.e. the modified Silver κ-iteration of $\text{Add}(\gamma, \gamma^+)$ followed by the adding of Cohen subsets to κ). Then since the GCH was assumed to hold in $\mathbf{V}$, and the GCH is preserved after collapse forcing, we will still have $2^\kappa \leq \max\{\kappa^+, \eta\}$ in every forcing extension over $\text{Coll}(\kappa^+, \lambda)^{\mathbf{V}} \times \mathbb{R}$. But since $\mathbb{R}$ collapses λ, which is greater than or equal to η, to κ^+, this then means that the GCH will hold at κ in this extension whereby the weakly measurable cardinal κ will be measurable. $\square$

This result is a little peculiar because its proof shows that we can switch measurability back on by simply forcing to turn the GCH back on at κ through the addition of a $\mathbf{V}[G][g][H]$-generic surjection from κ^+ onto 2^κ of the extension. It means that this surjection is somehow sufficient for generating a nonprincipal κ-complete ultrafilter that can measure all of the freshly added

η subsets to $\mathbf{V}[G][g]$. The collections $\mathcal{A}$ of subsets of κ of sizes between κ^{++} and 2^κ, inclusively, in $\mathbf{V}[G][g][H]$, which need not have measuring filters F by the weak measurability of κ before adding the surjection, afterward would be required to have such filters when their sizes are collapsed to κ^+.

The above Theorem provides an example of an extension where we have the ability to reinstate measurability by forcing, but Theorem 1.1.6 suggests that this may not be possible in all universes of set theory. This leads to the following open question:

Question 1.5.3. *Can we always resurrect the measurability of a weakly measurable cardinal whose measurability was destroyed? More generally, can we always force a weakly measurable cardinal to become measurable in a forcing extension?*

Theorem 1.1.6 only rules out the possibility that a $<\kappa$-closed forcing notion can always make a weakly measurable cardinal κ measurable in the extension for *arbitrary* set-theoretical universes. However, for posets that need not be countably closed, we are left without an answer.

Question 1.5.4. *If κ is weakly measurable, can we always force to achieve an extension where κ remains weakly measurable but can never become measurable by any further forcing?*

Many other open questions regarding weak measurability remain.

Chapter 2

Nearly θ-supercompact cardinals

In this chapter, we introduce the nearly θ-supercompact cardinal hierarchy, a large cardinal concept that stratifies the θ-supercompact cardinal hierarchy and can be made indestructible by a wide variety of forcing notions. Nearly θ-supercompact cardinals generalize embedding characterizations of weak compactness in a strong way, allowing the domains to have size θ while requiring the codomains to exhibit θ-closure under their respective elementary mappings. The θ-supercompact cardinals are always nearly θ-supercompact, but the converse is not necessarily true. Analogous to how a weakly compact cardinal κ can be the residual of a measurable cardinal after forcing to add too many subsets of κ, a nearly θ-supercompact cardinal can be what's left after we force to add too many subsets of θ to a θ-supercompact cardinal. But the nearly θ-supercompact cardinals, like the weakly compact cardinals that they generalize, can also be preserved by other forcing posets as well and may even fail to be measurable. Despite these facts, the indestructibility we receive with nearly θ-supercompact cardinals is very much like the indestructibility we can get for θ-supercompact cardinals. More generally, nearly θ-supercompact cardinals share many of the characteristics of their θ-supercompact counterparts, and they can sometimes serve as weaker substitutes in hypotheses necessary for proving certain theorems.

When the GCH holds at a θ for which $\theta^{<\kappa} = \theta$, nearly θ^+-supercompact cardinals κ fall directly in between θ-supercompact and θ^+-supercompact cardinals. More generally, a θ-supercompact cardinal will always be nearly θ-supercompact, and a nearly $2^{\theta^{<\kappa}}$-supercompact cardinal κ will always be θ-supercompact. However, relative to a supercompact cardinal κ, we can move to a forcing extension preserving all cardinals above κ where the GCH can fail at κ as badly as desired, and κ is nearly θ-supercompact for all $\theta < 2^\kappa$ but not even measurable! Nevertheless, if κ is θ-supercompact for some θ such that $\theta^{<\kappa} = \theta$, we can also move to a forcing extension preserving all cardinals below θ^{++} where κ remains θ-supercompact but is not nearly θ^+-supercompact. Let us now provide a formal definition of what it means for a cardinal κ to be nearly θ-supercompact.

Main Definition. *A cardinal κ is nearly θ-supercompact if for every $A \subseteq \theta$,* *there exists a transitive* $M \models ZFC^-$ *closed under* $<\kappa$ *sequences having the subset* A *and the cardinals* κ *and* θ *as elements, a transitive* N*, and an elementary embedding* $j : M \to N$ *with critical point* κ *such that* $j(\kappa) > \theta$ *and* $j''\theta \in N$*.*

As mentioned before, near θ-supercompactness generalizes embedding characterizations of weak compactness to allow for larger domains and closure under witnessing elementary embeddings. Indeed, using one of these embedding characterizations, it is easy to verify that κ is weakly compact if and only if κ is nearly κ-supercompact. To illuminate the $j''\theta$ condition in the definition, it should be noted that if such an embedding is generated by a κ-complete M-normal fine measure μ on $P_\kappa\theta$, then $j''\theta$ is simply the equivalence class of the identity function $[id]_\mu$. Such a filter characterization and other equivalent embedding characterizations of near θ-supercompactness will be given in the first section.

Now let us formally state the main results of this chapter.

Main Theorem.

1. *If* κ *is nearly* θ*-supercompact for some* $\theta \geq \kappa$ *such that* $\theta^{<\theta} = \theta$*, then there is a forcing extension where its near* θ*-supercompactness is preserved and indestructible by any further* $<\kappa$*-directed closed* θ*-c.c. forc-*

ing of size at most θ.

2. *If κ is a θ-supercompact cardinal for some θ such that $\theta^{<\kappa} = \theta$, then there exists a forcing extension where κ remains θ-supercompact but is not nearly θ^+-supercompact. Moreover, this forcing extension will preserve the cofinalities of all ordinals with cofinalities below θ^{++} or above 2^θ. Thus, θ-supercompactness does not necessarily imply any greater degree of near supercompactness. (Note that if κ is a nearly $2^{\theta^{<\kappa}}$-supercompact cardinal, then it is θ-supercompact, and if κ is a θ-supercompact cardinal, then it is nearly θ-supercompact.)*

3. *If κ is nearly θ-supercompact for some $\theta \geq 2^\kappa$ such that $\theta^{<\theta} = \theta$, then there exists a forcing extension preserving all cardinals and cofinalities above κ where κ is nearly θ-supercompact but not measurable.*

4. *If κ is nearly θ-supercompact for some $\theta \geq \kappa^+$ for which $\theta^{<\theta} = \theta$, then AD holds in $\boldsymbol{L}(\mathbb{R})$. In particular, if κ is nearly κ^+-supercompact and $2^\kappa = \kappa^+$, then AD holds in $\boldsymbol{L}(\mathbb{R})$.*

Let us now elaborate on how the nearly θ-supercompact cardinals refine the level-by-level θ-supercompactness hierarchy. If we restrict an elementary embedding $j : \mathbf{V} \to N$ witnessing the θ-supercompactness of κ to $M = \mathbf{H}_{\max\{\kappa^+,\theta^+\}}$, then the induced elementary mapping $j \restriction M : M \to j(M)$

will simultaneously witness the near θ-supercompactness of κ for all $A \subseteq \theta$. Consequently, the notion of near θ-supercompactness is indeed a weakening of θ-supercompactness. However, more is true. A θ-supercompact cardinal κ will be nearly θ-supercompact in the codomain of every θ-supercompactness ultrapower embedding $j : \mathbf{V} \to N$ generated by a κ-complete normal fine measure on $P_\kappa\theta$ because it will contain $j \restriction M$ for every transitive $<\kappa$-closed M of size $\theta^{<\kappa}$ and in particular for every such $<\kappa$-closed ZFC$^-$ model containing the desired subset of θ. Consequently, by a standard reflection argument, we could for example show that if κ is κ^+-supercompact, then it is a limit of nearly δ^+-supercompact cardinals δ. But also if κ is nearly θ-supercompact and $2^{\gamma^{<\kappa}} \leq \theta$, then we can encode the full powerset of $P_\kappa\gamma$ and all of the functions from γ into $P_\kappa\gamma$ as a subset A of θ, and find an elementary embedding j witnessing the near θ-supercompactness of κ for A (i.e., A is in its domain). The induced filter on $P_\kappa\gamma$ generated by using $j''\gamma$ as a seed will then be an actual κ-complete normal fine measure on $P_\kappa\gamma$. Since this filter has size $2^{\gamma^{<\kappa}} \leq \theta$, we can encode it as a subset of θ, and identify any elementary embedding $h : M^* \to N^*$ witnessing the near θ-supercompactness of κ for A. Any such N^* will then be able to also construct this subset of θ and will therefore have this κ-complete normal fine measure on $P_\kappa\gamma$. It follows that κ will be γ-supercompact in all such codomains N^*. In particular, κ

will be γ-supercompact in the codomains of all "miniature ultrapower embeddings" generated by κ-complete M^*-normal fine ultrafilters on $P_\kappa\theta$ when using $<\kappa$-closed ZFC$^-$ models M^* having A and θ as elements. Note that when $\theta^{<\theta} = \theta$, we'll be able to do this for all such γ for which $\gamma^{<\kappa} < \theta$. Therefore, like the θ-supercompact cardinals, which exhibit the definitively stronger nature than their nearly θ-supercompact cardinal counterparts, the nearly θ-supercompact cardinals exhibit an analogous strength over the γ-supercompact cardinals for all γ such that $2^{\gamma^{<\kappa}} \leq \theta$, or for all γ such that $\gamma^{<\kappa} < \theta$ in the special case that $\theta^{<\theta} = \theta$.

Prior to the presented indestructibility result of this Main Theorem, one would most likely appeal to the existence of a θ-supercompact cardinal κ to achieve $<\kappa$-directed closed indestructibility preserving partial supercompactness for arbitrary partial orders of size θ. For example, the Level by Level Preparation Theorem 4.8 of [10] starts with the assumption that κ is a θ-supercompact cardinal for which $2^{\theta^{<\kappa}} = \theta^+$ in order to assert that there exists a forcing extension where the θ-supercompactness of κ is made indestructible by all $<\kappa$-directed closed forcing of size at most θ. The main indestructibility result of this chapter shows that under the GCH or for sufficiently closed θ (i.e., $\theta^{<\theta} = \theta$), one need only appeal to the weaker notion of near θ-supercompactness for many types of posets (satisfying the θ-c.c.) of size θ. In fact, when applicable, Statement 1 of the Main Theorem strengthens the indestructibility that we receive by the Level by Level Preparation Theorem for all $\gamma < \theta$. Specifically, if $2^{\gamma^{<\kappa}} = \gamma^+$ as the theorem from [10] posits and κ is nearly θ-supercompact for some cardinal $\theta > \gamma$, then κ will be γ-supercompact. Consequently, in the forcing extension where the near θ-supercompactness of κ becomes indestructible by all $<\kappa$-directed closed θ-c.c. forcing of size at most θ, the γ-supercompactness of κ will also become indestructible by this forcing class and in particular by the class of $<\kappa$-directed closed partial orders of size at most $\gamma < \theta$. In this way, Statement 1 of the Main Theorem demonstrates the fact that even if we can only assert that κ is γ-supercompact for all $\gamma < \theta$, it may still be possible to squeeze out more indestructibility preserving the γ-supercompactness of κ for all such γ than $<\theta$-sized $<\kappa$-directed closed forcing. Finally, to achieve the indestructibility of Statement 1 of the Main Theorem, it is never necessary to collapse any cardinals above θ unlike the standard lifting techniques that preserve the θ-supercompactness of κ by first forcing to make the GCH hold at θ. (i.e., The Level by Level Preparation Theorem requires that $2^{\theta^{<\kappa}} = \theta^+$.)

The striking separation between how much θ-supercompact cardinals and nearly θ-supercompact cardinals can differ from a size standpoint is in the third statement of the theorem where we can achieve nearly θ-supercompact cardinals κ that are not even measurable without collapsing cardinals above κ. For example, this means that is realizable relative to large cardinal notions that we can have a nearly θ-supercompact cardinal κ where θ is a weakly inaccessible cardinal above κ, and κ is not even measurable!

In particular, Statement 3 of the Main Theorem generalizes the separation stated in the Main Theorem from Chapter 1 and [25]. Recall that a cardinal κ is defined to be weakly measurable if for every collection $\mathcal{A}$ containing at most κ^+ many subsets of κ, there exists a nonprincipal κ-complete filter on κ such that each subset of κ in $\mathcal{A}$ or its complement is in this filter. We will generalize this concept to a notion of η-near measurability where a cardinal κ such that $\eta^{<\kappa} = \eta$ can be characterized to be η-nearly measurable provided that for every collection $\mathcal{A}$ containing at most η many subsets of κ, there exists a nonprincipal κ-complete filter on κ such that each subset of κ in $\mathcal{A}$ or its complement is in this filter. Since every nearly η-supercompact cardinal will be η-nearly measurable, a corollary of Statement 3 is that if κ is a nearly θ-supercompact cardinal for some $\theta \geq 2^\kappa$ for which $\theta^{<\theta} = \theta$, then we can move to a forcing extension preserving all cardinals and cofinalities above κ where κ is η-nearly measurable for all $\eta \leq \theta$ but not measurable. Since $2^\kappa = \theta^+$ in this forcing extension, κ will have the largest degree of near measurability that it can possess there without being outright measurable.

2.1 Near θ-Supercompactness

In this section, we introduce alternative characterizations of the near θ-supercompactness of cardinals and definitions that are used later. We also make some observations about these types of large cardinals and show they are unaffected by small forcing. Let's start with some observations having short proofs.

Observation 2.1.1.

1. *If κ is θ-supercompact, then it is nearly θ-supercompact.*
2. *If κ is nearly θ-supercompact for some $\theta \geq \kappa$, then κ is weakly compact and in particular inaccessible.*
3. *κ is nearly θ-supercompact if and only if it is nearly ξ-supercompact for all $\xi < \theta^+$.*
4. *If κ is nearly θ-supercompact for some $\theta \geq 2^{\eta^{<\kappa}}$, then κ will also be η-supercompact.*
5. *κ is nearly θ-supercompact for unboundedly many θ if and only if κ is supercompact.*

Proof. (1) was already verified in the introduction. For (2), first note that if κ is nearly θ-supercompact, then κ is indeed inaccessible since it is the critical point of an elementary embedding $j : M \to N$ for some $<\kappa$-closed ZFC$^-$ model M. Specifically, if there were a cofinal γ-sequence of κ for some γ below the critical point κ, then it would be in M, which is impossible since this sequence would then also have to be cofinal in $j(\kappa) > \kappa$. Consequently, κ is regular. Similarly, if $|\mathcal{P}(\gamma)| \geq \kappa$ for some $\gamma < \kappa$, then M would have had to have contained a κ sequence $\vec{s}$ of $\mathcal{P}(\gamma)$ that did not repeat subsets, which is impossible since the subset listed at $j(\vec{s})(\kappa)$ would have been listed unboundedly often by $\vec{s}$. Consequently, κ is a strong limit. It then follows that κ is weakly compact because for $\theta \geq \kappa$, the definition of near θ-supercompactness strengthens the weak embedding characterization of weak compactness. For the direct implication of (3), code an arbitrarily selected $A \subseteq \xi$ and a surjective function $f : \theta \to \xi$ together as a subset of θ. Then by the near θ-supercompactness of κ, we have an elementary embedding $j : M \to N$ between transitive ZFC$^-$ models with critical point κ such that θ and this subset will be in M, the model N contains $j''\theta$, and $j(\kappa) > \theta$. In this case, A and f will be in M because M will have the subset of θ coding these two objects, and N will be able to construct $j''\xi$ from $j(f)$ and $j''\theta$ since $j''\xi = \text{range}(j(f) \restriction j''\theta)$. Also, because N will be able to construct this subset of θ from j of the subset and $j''\theta$, it too will have the surjection f. It will therefore know that ξ is an ordinal of size θ so that it must be less than $j(\kappa)$, which it thinks is a cardinal greater than θ. Consequently, j will witness the near ξ-supercompactness of κ for A. For (4), if $\theta \geq 2^{\eta^{<\kappa}}$, then we can encode the full powerset of $P_\kappa\eta$ and all (regressive) functions from $P_\kappa\eta$ into η as a subset of θ. Then by the near θ-supercompactness of κ, we can find an elementary embedding $j : M \to N$ with critical point κ between transitive ZFC$^-$ models such that M is a $<\kappa$-closed model containing θ and this subset, $j(\kappa) > \theta$, and $j''\theta \in N$. The induced filter on $P_\kappa\theta$ generated by using $j''\theta$ as a seed will then be an actual κ-complete normal fine measure on $P_\kappa\theta$. Finally for the direct implication of (5), we appeal to (4) to get that κ will be supercompact. Specifically, if κ is nearly θ-supercompact for unboundedly many θ, then for all η, we can find $\theta \geq 2^{\eta^{<\kappa}}$ for which κ is nearly θ-supercompact and hence η-supercompact. $\square$

As was the case in Chapter 1, we will use ZFC$^-$ throughout this chapter, to mean the theory of ZFC without the powerset axiom, but where ZFC is understood to be axiomatized with Collection instead of Replacement. The work of Gitman, Hamkins and Johnstone shows that it is important to use the Collection axiom rather than merely Replacement when omitting the Powerset axiom in order to be able to prove consequences of ZFC such as Łoś's theorem for ultrapowers or even the fact that ω_1 is regular. Since for all η, the set $\mathbf{H}_{\eta^+}$ will model this stronger ZFC$^-$ theory axiomatized with Collection, we can still consider elementary substructures to obtain models of this theory, making the substitution innocuous.

Definition 2.1.2 (Weakly θ-Closed). *A set of size* $\lambda \geq \theta$ *is weakly closed if it exhibits as much closure as a set of size* λ *can exhibit (i.e.* $<\delta$*-closed where* δ *is the least cardinal such that* $\lambda^\delta > \lambda$*). A set is weakly* θ*-closed if it is either weakly closed or* $\leq\theta$*-closed.*

Theorem 2.1.3 (Characterizations of Near θ-Supercompactness). *If* $\theta^{<\kappa} = \theta$*, then the following are equivalent:*

(Near θ*-Supercompactness) For every* $A \subseteq \theta$*, there exists a transitive* $M \models ZFC^-$ *closed under* $<\kappa$ *sequences with* $A, \theta \in M$*, a transitive* N*, and an elementary embedding* $j : M \to N$ *with critical point* κ *such that* $j(\kappa) > \theta$ *and* $j''\theta \in N$*.*

(Embedding) For every transitive set M *of size* θ *having* θ *as an element and* $M^{<\kappa} \subseteq M$*, there exists a transitive* N *and an elementary embedding* $j : M \to N$ *with critical point* κ *such that* $j(\kappa) > \theta$ *and* $j''\theta \in N$*.*

(Normal Embedding) For all $\delta \geq \kappa$ *and every transitive* $M \models ZFC^-$ *of size* θ *closed under* $<\delta$ *sequences with* $\theta \in M$*, there exists a transitive* N *of size* θ *closed under* $<\delta$ *sequences containing all subsets of* θ *in* M*, and*

a cofinal elementary embedding $j : M \to N$ *with critical point* κ *such that* $N = \{j(f)(j''\theta) | f \in M; f : P_\kappa\theta \to M\}$*, the ordinal* $j(\kappa) > \theta$*, and* $j''\theta \in N$*.*

(Normal ZFC Embedding) For every $A \in \boldsymbol{H}_{\theta^+}$*, there is a transitive weakly closed* $M \models$ *ZFC of size* θ *having* A *and* θ *as elements, a transitive weakly closed* N *of size* θ *containing all subsets of* θ *in* M*, and a cofinal elementary embedding* $j : M \to N$ *with critical point* κ *such that the codomain* $N = \{j(f)(j''\theta) | f \in M; f : P_\kappa\theta \to M\}$*, the ordinal* $j(\kappa) > \theta$*, and* $j''\theta \in N$*.*

(Normal Fine Filter) For every collection $\mathcal{A}$ *of at most* θ *many subsets of* $P_\kappa\theta$ *and collection* $\mathcal{F}$ *of at most* θ *many functions from* $P_\kappa\theta$ *into* θ*, there exists a* κ*-complete fine filter* F *on* $P_\kappa\theta$ *measuring all sets in* $\mathcal{A}$*, which is* $\mathcal{F}$*-normal in the sense that for every* $f \in \mathcal{F}$ *regressive on some set in* F*, there exists* $\alpha_f < \theta$ *for which* $\{\sigma \in P_\kappa\theta | f(\sigma) = \alpha_f\} \in F$*.*

(Hauser Embedding) For every transitive ZFC$^-$ *model* $M \in \boldsymbol{H}_{\theta^+}$ *such that* $M^{<\kappa} \subseteq M$ *with* $\theta \in M$*, there exists a transitive weakly closed* $N \in \boldsymbol{H}_{\theta^+}$ *and an elementary embedding* $j : M \to N$ *with critical point* κ *having the Hauser property that* $M, j \in N$*. Moreover,* $j(\kappa) > \theta$*, and* $N \models M \in \boldsymbol{H}_{\theta^+}$*.*

Proof. (Near θ-Supercompactness $\Rightarrow$ Embedding) Let M be a transitive set of size θ closed under $<\kappa$ sequences with $\theta \in M$, and let $A \subseteq \theta$ be an encoding of it. Then by the near θ-supercompactness of κ, there exists a transitive ZFC$^-$ model $\overline{M}$ closed under $<\kappa$ sequences containing A (and hence M), a transitive $\overline{N}$, and an elementary embedding $\overline{j} : \overline{M} \to \overline{N}$ with critical point κ such that $\overline{j}(\kappa) > \theta$, and $\overline{j}''\theta \in \overline{N}$. Letting $N = \overline{j}(M)$ and $j = \overline{j} \restriction M$, it is easily verified that $j : M \to N$ witnesses the embedding property for M. Specifically, since $\overline{M}$ knows that $M^{<\kappa} \subseteq M$, the model $\overline{N}$ will think that $\overline{j}(M)$ is $<\overline{j}(\kappa)$-closed. Then because $\overline{j}(\kappa) > \theta$ and $\overline{j}''\theta$ has size θ in $\overline{N}$, we have $\overline{j}''\theta = j''\theta \in N$. Also, $N = \overline{j}(M)$ is transitive in $\overline{N}$ by elementarity and hence really transitive by the transitivity of $\overline{N}$. The fact that κ is the critical point of this embedding and that $j(\kappa) > \theta$ are both immediate by virtue of being the restriction of an embedding with these properties.

(Embedding $\Rightarrow$ Normal Embedding) Fix $\delta \geq \kappa$, and let $M \preceq \mathbf{H}_{\theta^+}$ be a transitive ZFC$^-$ model of size θ closed under $<\delta$ sequences with $\theta \in M$. Then by the embedding property, there exists a transitive $\overline{N}$ and an elementary embedding $\overline{j} : M \to \overline{N}$ with critical point κ such that $\overline{j}(\kappa) > \theta$ and $\overline{j}''\theta \in \overline{N}$. Let $s = \overline{j}''\theta$, and then let $X_s = \{\overline{j}(f)(s) | f \in M; f : P_\kappa\theta \to M\} \preceq \overline{N}$ be the seed hull of the embedding generated by using $\overline{j}''\theta$ as a seed. By seed, we simply mean any element s^* of $\overline{j}(D)$ for some set $D \in M$ that induces a filter F on D defined by $A \in F \Leftrightarrow A \subseteq D$ and $s^* \in \overline{j}(A)$, for every subset A of D in M. The filter measures every set in M, and thus s^* is "a seed for the M-measure F." In this case, the seed s^* is $s = \overline{j}''\theta$ and D is $P_\kappa\theta$ so that $\overline{j}(D) = \overline{j}(P_\kappa\theta) = (P_{\overline{j}(\kappa)}\overline{j}(\theta))^{\overline{N}}$. Note that it is a simple application of the Tarski-Vaught test to verify that $X_s \preceq \overline{N}$. Specifically, suppose $\overline{N} \models \exists x(\varphi(x, \overline{j}(f_1)(s), \ldots, \overline{j}(f_n)(s)))$ for some $f_1, \ldots, f_n \in M$, and let g be a function with domain $\bigcap_{i=1}^n \mathrm{dom}(f_i)$ constructed in M that sends t to some x for which $M \models \varphi(x, f_1(t), \ldots, f_n(t))$ if there is such an $x \in M$ or to the empty set otherwise. Then by elementarity, $\overline{j}(g)$ is a function with domain $\bigcap_{i=1}^n \mathrm{dom}(j(f_i))$, which includes s, that sends t to some x for which $\overline{N} \models \varphi(x, \overline{j}(f_1)(t), \ldots, \overline{j}(f_n)(t))$ if there is such an $x \in \overline{N}$. But since there is such an $x \in \overline{N}$ for which $\overline{N} \models \varphi(x, \overline{j}(f_1)(s), \ldots, \overline{j}(f_n)(s))$, it follows that $\overline{N} \models \varphi(\overline{j}(g)(s), \overline{j}(f_1)(s), \ldots, \overline{j}(f_n)(s))$, whereby we have a witness in X_s, as desired. Returning to the main argument, let $\pi : X_s \overset{\cong}{\to} N$ be the Mostowski Collapse of X_s, and define $j : M \to N$ to be the induced factor embedding where $j = \pi \circ \overline{j}$. I claim that $j : M \to N$ is the desired embedding.

Such a j exists since the range of $\overline{j}$ is contained in X_s. This can be seen by considering the constant functions on $P_\kappa\theta$ in M. The model N has size θ because its size is bounded below by $|M| = \theta$ by virtue of $j''M \subseteq N$ and above by $|M| = \theta$ by virtue of it being the Mostowski Collapse of a set only containing elements of the form $\overline{j}(f)(s)$ for $f \in M$. The embedding is cofinal since it is a reduced ultrapower embedding. Assuming $j''\theta \in N$, which we will show to be true in a moment, the model N can construct A from $j(A)$ and $j \restriction \theta$ for every $A \subseteq \theta$ in M. We have $\theta \subseteq X_s$ since every $\alpha \leq \theta$ is the order type of $\overline{j}(p_\alpha)(s)$ where p_α is the function sending an element to its intersection with α. Consequently, the Mostowski Collapse of X_s fixes all ordinals less than $\theta+1$, which includes κ and θ. In particular, the critical point of j will be κ with $j(\kappa)>\theta$, and since we also have $s\in X_s$ ($\overline{j}$ of the identity function evaluated at s), it follows that $\pi(s)=(\pi\circ\overline{j})''\theta=j''\theta\in N$. Consequently, we also have $N=\pi''X_s=\{(\pi\circ\overline{j})(f)(\pi(s))|f\in M; f:P_\kappa\theta\to M\}$, which is equal to $\{j(f)(j''\theta)|f\in M; f:P_\kappa\theta\to M\}$, as desired.

Finally, to prove that $N^{<\delta}\subseteq N$, suppose $\eta<\delta$, and let $\vec{x}=\langle x_\alpha|\alpha<\eta\rangle$ be an η-sequence of elements from N. Then there is a sequence of functions $\langle f_\alpha|\alpha<\eta\rangle\in M^\eta\subseteq M$ with $j(f_\alpha)(j''\theta)=x_\alpha$ for all $\alpha<\eta$. Now $j(\langle f_\alpha|\alpha<\eta\rangle)$ is a $j(\eta)$ sequence in N whose restriction to $j''\eta$ enumerates all elements of $\{j(f_\alpha)|\alpha<\eta\}$. But $\eta\leq\delta\leq\theta$ so we may use $j''\theta$ and $j(\langle f_\alpha|\alpha<\eta\rangle)$ to construct the sequence $\langle j(f_\alpha)|\alpha<\eta\rangle$ in N. It follows that N can then construct the sequence $\vec{x}$ by evaluating each of the coordinates of the aforementioned sequence at $j''\theta$. Therefore $N^{<\eta}\subseteq N$ for all $\eta<\delta$.

(Normal Embedding $\Rightarrow$ Normal ZFC Embedding): Let δ be the least cardinal such that $\theta^\delta>\theta$, and let $A\in\mathbf{H}_{\theta^+}$. Note that $\delta\geq\kappa$ since $\theta^{<\kappa}=\theta$. Then let $\overline{M}\preceq\mathbf{H}_{\theta^+}$ be a transitive ZFC$^-$ model of size θ closed under $<\delta$ sequences that has A and a θ-enumeration of the transitive closure of A. Now by the normal embedding characterization of the near θ-supercompactness of κ, there exists a transitive $\overline{N}$ of size θ closed under $<\delta$ sequences and an elementary embedding $\overline{j} : \overline{M} \to \overline{N}$ with critical point κ such that $\overline{j}(\kappa) > \theta$ and $\overline{j}''\theta \in \overline{N}$. Note that since $\overline{M}$ is closed under $<\kappa$ sequences (or even $<\delta$ sequences), we may apply the same justification as the one that was used at the beginning of this section to show that the existence of such a $\overline{j}$ implies that κ is inaccessible. Then κ is inaccessible in $\overline{M}$ whereby $\overline{j}(\kappa)$ is inaccessible in $\overline{N}$. Consequently, because $\overline{N}$ is a transitive ZFC$^-$ model, $M = \mathbf{V}^{\overline{N}}_{\overline{j}(\kappa)}$ will be a transitive ZFC model. Note now that since $\overline{M}$ knows that $\mathbf{V}_\kappa$ is closed under $<\kappa$ sequences, $\overline{N}$ will think that M is closed under $<\overline{j}(\kappa)$ sequences, and hence θ sequences, by elementarity. In particular, M will contain $\overline{N}$'s powerset of θ and hence its $\mathbf{H}_{\theta^+}$ and will actually be closed under $<\delta$ sequences by virtue of $\overline{N}^{<\delta} \subseteq \overline{N}$ and $\delta \leq \theta$. But now also note that $\mathbf{H}^{\overline{M}}_{\theta^+} \subseteq \mathbf{H}^{\overline{N}}_{\theta^+}$ since $\overline{j}^{-1} \restriction \theta''(\overline{j}(B) \cap \overline{j}''\theta) = B$ for every $B \subseteq \theta$ in $\overline{M}$. Consequently, we have $A, \theta \in \mathbf{H}^{\overline{M}}_{\theta^+} \subseteq \mathbf{H}^{\overline{N}}_{\theta^+} = \mathbf{H}^{M}_{\theta^+}$ so that $A, \theta \in M$ as well. Since $M \subseteq \overline{N}$ must clearly have size at most $|\overline{N}| = \theta$, we may now apply the normal embedding characterization of the near θ-supercompactness of κ to M to get a desired elementary embedding $j : M \to N$.

(Normal ZFC Embedding $\Rightarrow$ Normal Fine Filter): Let $\mathcal{A}$ be a collection of at most θ many subsets of $P_\kappa\theta$ and $\mathcal{F}$ be a collection of at most θ many functions from $P_\kappa\theta$ into θ. Note that both $\mathcal{A}$ and $\mathcal{F}$ will have hereditary size $\theta = \theta^{<\kappa}$. It follows then by the Normal ZFC embedding property that we can find a $<\kappa$-closed transitive ZFC model M containing θ with $\mathcal{A} \times \mathcal{F} \subseteq M$ (whereby $\mathcal{A} \subseteq (\mathcal{P}(P_\kappa\theta))^M$ and $\mathcal{F} \subseteq ({}^{P_\kappa\theta}\theta)^M$), a transitive N, and an elementary embedding $j : M \to N$ such that $j''\theta \in N$ and $j(\kappa) > \theta$. Since M is closed under $<\kappa$-sequences, it contains the true $P_\kappa\theta$, and because $j(\kappa) > \theta$, we have $j''\theta \in (P_{j(\kappa)}j(\theta))^N = j(P_\kappa\theta)$. We may therefore generate a filter F on $P_\kappa\theta$ by using $j''\theta$ as a seed ($F = \{S \in (\mathcal{P}(P_\kappa\theta))^M | j''\theta \in j(S)\}$). This filter will indeed be κ-complete because the critical point of j is κ, and M is closed under $<\kappa$ sequences. It will measure all sets in $\mathcal{A}$ because $\mathcal{A} \subseteq (\mathcal{P}(P_\kappa\theta))^M$. Since for all $\alpha < \theta$, we have $j''\theta \in \{\sigma \in j(P_\kappa\theta) | j(\alpha) \in \sigma\}$, which equals $j(\{\sigma \in P_\kappa\theta | \alpha \in \sigma\})$, it follows that $\{\sigma \in P_\kappa\theta | \alpha \in \sigma\} \in F$ for every $\alpha < \theta$. Therefore, F is fine. Also if f is a function in $\mathcal{F}$ that is regressive on a set in F, then $j''\theta \in \{\sigma \in j(P_\kappa\theta) | j(f)(\sigma) \in \sigma\}$ whereby $j(f)(j''\theta) \in j''\theta$. Then $j(f)(j''\theta)$ will equal $j(\alpha)$ for some $\alpha < \theta$ so that $j''\theta$ will be an element of $\{\sigma \in j(P_\kappa\theta) | j(f)(\sigma) = j(\alpha)\}$. Consequently, $\{\sigma \in P_\kappa\theta | f(\sigma) = \alpha\} \in F$ so that f is constant on a set in F. It follows that F is also $\mathcal{F}$-normal. Therefore F is a κ-complete $\mathcal{F}$-normal fine filter on $P_\kappa\theta$ measuring every subset of $P_\kappa\theta$ in $\mathcal{A}$, as desired.

(Normal Fine Filter $\Rightarrow$ Near θ-Supercompactness): Let $A \subseteq \theta$ and M be a transitive ZFC$^-$ model of size θ closed under $<\kappa$ sequences containing both

A and θ. In particular, M will agree with $\mathbf{V}$ on $P_\kappa\theta$ by virtue of being a transitive ZFC$^-$ model containing θ and all of its less than κ sequences. Then since M's powerset of $P_\kappa\theta$ and set of functions from $P_\kappa\theta$ into θ both have size at most θ in $\mathbf{V}$ by virtue of M being a transitive set of size θ, we may find an $({}^\theta P_\kappa\theta)^M$-normal κ-complete fine filter on $P_\kappa\theta$ measuring all subsets of $P_\kappa\theta$ in M by the normal fine filter property of κ. Consequently, by restricting such a filter if necessary, we may assume that F measures precisely the sets in M. We may then let $j : M \to N$ be the induced ultrapower embedding where $N = \mathrm{Ult}(M, F)$ is the Mostowski Collapse of the reduced ultrapower structure only using functions in M. One could now verify that j has critical point κ, the model N has $j''\theta$, and $j(\kappa) > \theta$ so that j is an elementary embedding with the desired properties.

(Normal Embedding $\Rightarrow$ Hauser Embedding): Suppose $M \in \mathbf{H}_{\theta^+}$ is a ZFC$^-$ model closed under $<\kappa$ sequences with $\theta \in M$. Let $\overline{M} \in \mathbf{H}_{\theta^+}$ be a transitive weakly closed ZFC$^-$ model containing M and a θ-enumeration of its elements. Then in $\overline{M}$, we may find a well-founded relation E on θ such that $\langle M, \in\rangle \cong \langle \theta, E\rangle$. Note then that M is the unique Mostowski Collapse of E. By the normal embedding characterization of the near θ-supercompactness of κ, fix an elementary embedding $\overline{j} : \overline{M} \to \overline{N}$ with critical point κ such that $\overline{j}''\theta \in \overline{N}$, the ordinal $\overline{j}(\kappa) > \theta$, and $\overline{N} \in \mathbf{H}_{\theta^+}$ is transitive and weakly closed.

Now let $j = \overline{j} \upharpoonright M$ and $N = \overline{j}(M)$ so that $j : M \to N$ is an elementary embedding. Since $E = \{((\overline{j} \upharpoonright \theta)^{-1}(\alpha), (\overline{j} \upharpoonright \theta)^{-1}(\beta)) | (\alpha, \beta) \in \overline{j}(E) \cap \overline{j}''\theta \times \overline{j}''\theta\}$, the model $\overline{N}$ can construct E from $\overline{j}(E)$ and $\overline{j}''\theta$. Then $\overline{N}$ can construct M by taking the Mostowski Collapse π of E and will know it has size θ since E is a relation on θ. Also, since for all $m \in M$, we have $m = \{\pi(\alpha) | \alpha E \pi^{-1}(m)\}$, it follows by elementarity that $j(m) = \{j(\pi)(\alpha) | \alpha j(E) j(\pi^{-1}(m))\}$. Because we also have range$(\pi^{-1}) = \theta$, the model $\overline{N}$ can construct the embedding j from M into $\overline{j}(M)$ from M, the relation $j(E)$, the isomorphisms π and $j(\pi)$, and $j''\theta$. Therefore, $j \in \overline{N}$.

Now since M is transitive and closed under $<\kappa$ sequences, $N = \overline{j}(M)$ will be transitive and closed in $\overline{N}$ under $<j(\kappa)$ sequences and hence θ sequences, by elementarity. Thus, N will contain $\overline{j}''\theta = j''\theta$ and will actually be weakly closed because $\overline{N}$ exhibits this closure. It will also actually be transitive because $\overline{N}$ is a transitive set. Furthermore, by the θ closure of N in $\overline{N}$, we have $M \in N$ as well as a θ enumeration of its elements found in $\overline{N}$. By transitivity, we also have $M \subseteq N$, so viewed as a function from M into range(j), the model $\overline{N} \models j \in N^\theta \subseteq N$. Therefore, we have an elementary embedding $j : M \to N$ with critical point κ and $j(\kappa) > \theta$ where N is a transitive weakly closed ZFC$^-$ model with $M, j \in N$, and N knows that M has (hereditary) size θ, as desired.

(Hauser Embedding $\Rightarrow$ Near θ-Supercompactness): Fix $A \subseteq \theta$, and let $M \in \mathbf{H}_{\theta^+}$ be a transitive ZFC$^-$ model closed under $<\kappa$ sequences with $\theta \in M$ and $A \in M$. Then by the Hauser embedding property, there exists an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and j is an element of N. In particular, $j''\theta \in N$ so that $j : M \to N$ witnesses the near θ-supercompactness of κ for the arbitrarily selected $A \subseteq \theta$. $\square$

Throughout this chapter, we will be referring to near θ-supercompactness embeddings. In order to avoid ambiguity, it seems prudent to establish the exact definition of this shorthand.

Definition 2.1.4 (Near θ-Supercompactness Embedding). *An elementary embedding $j : M \to N$ with critical point κ is said to be a near θ-supercompactness embedding if it is between transitive models M and N such that $\theta \in M$, the set M is $<\kappa$-closed, $j(\kappa) > \theta$, and $j''\theta \in N$. If left unspecified, we assume that M and N are models of ZFC$^-$.*

We can potentially squeeze out more from θ-supercompactness embeddings, and an analogous result is true for near θ-supercompactness embeddings.

Lemma 2.1.5. *If κ is a nearly θ-supercompact cardinal for some $\theta \geq \kappa$, then every near θ-supercompactness embedding $j : M \to N$ is actually a near*

$\theta^{<\kappa}$*-supercompactness embedding.*

Proof. Fix $\theta \geq \kappa$ and a near θ-supercompactness embedding $j : M \to N$ between transitive ZFC$^-$ models with $M^{<\kappa} \subseteq M$. Note that M has the true $P_\kappa\theta$ since it contains θ and is closed under $<\kappa$ sequences. Then N will have the true $P_\kappa\theta$ since it will be able to construct all $A \subseteq \theta$ in M from $j(A)$ and $j''\theta$. Then since any $\sigma \in P_\kappa\theta$ has size less than the critical point κ, it follows that $j''\sigma = j(\sigma)$ whereby N will also be able to construct $j''P_\kappa\theta$ from $P_\kappa\theta$ and $j''\theta$. Letting $\pi : P_\kappa\theta \xrightarrow{\sim} (\theta^{<\kappa})^M$ be a bijection in M, the function $j(\pi) \restriction j''P_\kappa\theta$ will be a bijection between $j''P_\kappa\theta$ and $j''(\theta^{<\kappa})^M$ in N. Consequently, $j''(\theta^{<\kappa})^M = \text{range}(j(\pi) \restriction j''P_\kappa\theta) \in N$. But then since $M \subseteq \mathbf{V}$ and M contains the true $P_\kappa\theta$, we have $\theta^{<\kappa} \leq (\theta^{<\kappa})^M$ whereby $j''\theta^{<\kappa} \in N$. Also, M knows that κ is inaccessible, so by elementarity, N thinks that $j(\kappa)$ is inaccessible. Consequently, because $j(\kappa)$ is greater than κ and θ, we have $j(\kappa) > (\theta^{<\kappa})^N$. But then since $N \subseteq \mathbf{V}$ and N also contains the true $P_\kappa\theta$, we have $j(\kappa) > (\theta^{<\kappa})^N \geq \theta^{<\kappa}$. Therefore, j is actually a near $\theta^{<\kappa}$-supercompactness embedding, as desired. □

Next I show that nearly θ-supercompact cardinals are unaffected by small forcing. Given a near θ-supercompactness embedding $j : M \to N$ from the ground model, a (hereditarily) small forcing notion $\mathbb{P} \in M$ relative to the critical point, and a filter $G \subseteq \mathbb{P}$ that's both M-generic and N-generic, we can uniquely lift the embedding to $j : M[G] \to N[G]$. Also, if κ is nearly θ-supercompact in a small forcing extension, then it was nearly θ-supercompact in the ground model. Before proving the Lévy-Solovay Theorem for nearly θ-supercompact cardinals, I should make a small terminology note and an observation about the equivalent characterizations for the near θ-supercompactness of a cardinal κ. We use $\mathrm{ZFC}^* = \mathrm{ZFC} \cap \Sigma_{100}$ to denote the sufficiently large fixed fragment of ZFC referenced in [12] for the Main Theorem from this paper to be applicable. In the normal embedding characterization of the near θ-supercompactness of κ, we assumed that M was a ZFC^- model to obtain our near θ-supercompactness embedding $j : M \to N$ with critical point κ, but we could have also assumed that M were a ZFC^* model, and the proof of its equivalence with the other characterizations would have still gone through (in a different way). In the proof of the direct implication of the theorem below, we will use this ZFC^* replacement of the normal embedding characterization.

Theorem 2.1.6. *Fix a cardinal κ and a cardinal θ such that $\theta^{<\kappa} = \theta$. After forcing of size less than κ, the cardinal κ is nearly θ-supercompact in the extension if and only if it was nearly θ-supercompact in the ground model.*

Proof. ($\Leftarrow$) Let $\mathbb{P}$ be a partial order such that $|\mathbb{P}| < \kappa$. By associating $\mathbb{P}$ with an isomorphic copy in $\mathbf{H}_\kappa$ if necessary, we may assume that $\mathbb{P} \in \mathbf{H}_\kappa$ is a partial order on an ordinal less than κ. Let $G \subseteq \mathbb{P}$ be $\mathbf{V}$-generic, and fix any $A \subseteq \theta$ in $\mathbf{V}[G]$. Then since $\check{\theta} \times \mathbb{P} \in \mathbf{H}_{\theta^+}$, we may find a $\mathbb{P}$-name $\dot{A} \in \mathbf{H}_{\theta^+}$ such that $\dot{A}_G = A$. Let M be a transitive $<\kappa$-closed ZFC$^-$ model of size θ containing $\dot{A}$, the cardinal θ, and $\mathbb{P}$. Then by the embedding characterization of the near θ-supercompactness of κ, we may find a transitive N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''\theta \in N$. Since $\mathbb{P}$ is a partial order on an ordinal less than κ, it has hereditary size below the critical point whereby $\mathbb{P}$ and all of its elements will be fixed by j. We may therefore lift the embedding to $j : M[G] \to N[G]$. The lifted embedding will have critical point κ and be between transitive ZFC$^-$ models $M[G]$ and $N[G]$. Also, $A \in M[G]$ since $A = \dot{A}_G$, the ordinal $j(\kappa) > \theta$, and $j''\theta \in N \subseteq N[G]$. Furthermore, since M is $<\kappa$-closed in $\mathbf{V}$, and $\mathbb{P}$ is trivially κ-c.c., $M[G]$ will be $<\kappa$-closed in $\mathbf{V}[G]$. Consequently, this embedding witnesses the weak embedding characterization of the near θ-supercompactness of κ for an arbitrary subset A of θ in $\mathbf{V}[G]$. Therefore, κ remains nearly θ-supercompact in $\mathbf{V}[G]$.

($\Rightarrow$) Now suppose we have a partial order $\mathbb{P} \in \mathbf{H}_\mu$ for some $\mu < \kappa$. Let G be a $\mathbf{V}$-generic filter for $\mathbb{P}$, and suppose that κ is nearly θ-supercompact in

$\overline{\mathbf{V}} = \mathbf{V}[G]$. Also, let A be a subset of θ in the ground model $\mathbf{V}$. By Lemma 13 of [12], $\mathbf{V} \subseteq \mathbf{V}[G]$ satisfies the μ approximation and cover properties. (i.e., For each $C \subseteq \mathbf{V}$ that's a set in $\mathbf{V}[G]$ having the property that $C \cap c \in \mathbf{V}$ for every $c \in \mathbf{V}$ of size less than μ in $\mathbf{V}$, we have $C \in \mathbf{V}$, and for each $D \subseteq \mathbf{V}$ that's a set having size less than μ in $\mathbf{V}[G]$, we have an $E \supseteq D$ of size less than μ in $\mathbf{V}$.)

By the Lévy Reflection Theorem, let η be a sufficiently large ordinal above θ so that $A, \theta \in \mathbf{V}_\eta$ and that every ZFC* statement reflects down from the structure $\langle \overline{\mathbf{V}}, \mathbf{V}, \in \rangle$ to $\langle \overline{\mathbf{V}}_\eta, \mathbf{V}_\eta, \in \rangle$. In particular, $\overline{\mathbf{V}}_\eta$ and $\mathbf{V}_\eta$ will be ZFC* models. Now in $\overline{\mathbf{V}}$, let $\overline{X}$ be a $<\kappa$-closed elementary substructure of $\overline{\mathbf{V}}_\eta$ in the language with a predicate for $\mathbf{V}$ of size θ having A as an element and $\theta + 1$ as a subset. In particular, we have $\langle \overline{X}, X, \in \rangle \prec \langle \overline{\mathbf{V}}_\eta, \mathbf{V}_\eta, \in \rangle$, where $X = \overline{X} \cap \mathbf{V}$. Note that $\theta^{<\kappa} = \theta$ in the forcing extension because this equality holds in the ground model with $|\mathbb{P}| < \kappa$ so finding a model having size θ closed under $<\kappa$ sequences is indeed possible. Letting $\overline{M}$ and M be the Mostowski Collapses of $\overline{X}$ and X, respectively, it then follows by Lemma 15 of [12] that $X \in \mathbf{V}$ and $M = \overline{M} \cap \mathbf{V}$. In particular, M will be a model of ZFC* in $\mathbf{V}$ since $X \in \mathbf{V}$ is a model of ZFC*, and M is its elementarily equivalent Mostowski collapse. Note that $\overline{M}$ will have θ and the subset A as elements since they are fixed by the Mostowski Collapse by virtue of the fact that $\theta + 1 \subseteq X$.

Also, observe that $\overline{M}$ will be closed under $<\kappa$ sequences in $\overline{\mathbf{V}}$ because $\overline{X}$ is $<\kappa$-closed in $\overline{\mathbf{V}}$.

Now since in $\overline{\mathbf{V}}$, the set $\overline{M}$ is the Mostowski Collapse of a ZFC* model of size θ, the set $\overline{M}$ will be a ZFC* model having size θ in $\overline{\mathbf{V}}$ as well. Furthermore, since $\overline{M}^{<\kappa} \subseteq \overline{M}$ in $\overline{\mathbf{V}}$ and θ is an element in $\overline{M}$, it follows by the normal embedding characterization of the near θ-supercompactness of κ in $\overline{\mathbf{V}}$ that there exists a $<\kappa$-closed transitive $\overline{N}$ and a cofinal elementary embedding $\overline{j} : \overline{M} \to \overline{N}$ with critical point κ such that $\overline{j}(\kappa) > \theta$ and $\overline{j}''\theta \in \overline{N}$. One can now easily verify that the conditions of the Main Theorem 3 from [12] are met so that the elementary embedding $j = (\overline{j} \restriction M) : M \to N$ where $N = \bigcup \overline{j}''M$ is a set in $\mathbf{V}$ by its Statement (3). Also, by Statement (2) of this Main Theorem, $N = \overline{N} \cap \mathbf{V}$, so since $j \in \mathbf{V}$ with $\theta \in M$ and $j''\theta = \overline{j}''\theta \in \overline{N}$, we will have $j''\theta \in N$. Compound this with the observations that $j(\kappa) = \overline{j}(\kappa) > \theta$ and that the set M in $\mathbf{V}$, which equals $\overline{M} \cap \mathbf{V}$, will be closed under $<\kappa$ sequences in $\mathbf{V}$ by virtue of $\overline{M}$ being closed under $<\kappa$ sequences in $\overline{\mathbf{V}}$, and we have now shown that j is a near θ-supercompactness embedding in the ground model with domain M. Because $A \in \overline{M}$ is a set in the ground model and $M = \overline{M} \cap \mathbf{V}$, we will have $A \in M$. Therefore, the near θ-supercompactness embedding $j : M \to N$ witnesses the near θ-supercompactness of κ in $\mathbf{V}$ for the arbitrarily selected A from the ground model. $\square$

Notice that in the above proof of the direct implication, we only used the smallness of $\mathbb{P}$ to show that $\mathbb{P}$ satisfies the μ approximation and cover properties for some $\mu < \kappa$ and to show that $\theta^{<\kappa} = \theta$ is preserved in the forcing extension. Therefore, we actually proved the following stronger theorem:

Theorem 2.1.7. *Suppose κ and θ are cardinals for which $\theta^{<\kappa} = \theta$ in a forcing extension $\boldsymbol{V}[G]$. Suppose further that $\boldsymbol{V} \subseteq \boldsymbol{V}[G]$ satisfies the δ approximation and cover properties for some $\delta < \kappa$ and that κ is nearly θ-supercompact in $\boldsymbol{V}[G]$. Then κ is nearly θ-supercompact in $\boldsymbol{V}$.*

As is the case with θ-supercompact cardinals κ, nearly θ-supercompact cardinals κ are indestructible by $\leq\theta^{<\kappa}$-distributive forcing. Since $\leq\theta^{<\kappa}$-distributive forcing adds no new elements or subsets of $P_\kappa\theta$ nor does it add new $<\kappa$-sequences of subsets of $P_\kappa\theta$ or functions from $P_\kappa\theta$ into θ, a κ-complete normal fine measure on $P_\kappa\theta$ in the ground model will continue to be such a measure in every $\leq\theta^{<\kappa}$-distributive forcing extension. In the case of nearly θ-supercompact cardinals κ, we also note that we introduce no new $\theta^{<\kappa}$-sized collections $\mathcal{A}$ of subsets from $P_\kappa\theta$ or $\theta^{<\kappa}$-sized collections $\mathcal{F}$ of functions from $P_\kappa\theta$ into θ by $\leq\theta^{<\kappa}$-distributive forcing. Consequently, when κ is nearly θ-supercompact, the covering collection of κ-complete fine filters on $P_\kappa\theta$ witnessing the near θ-supercompactness of κ in the ground model will continue to be such a collection witnessing the near θ-supercompactness of κ in every $\leq\theta^{<\kappa}$-distributive forcing extension.

Theorem 2.1.8. *If a cardinal κ is nearly θ-supercompact for some $\theta \geq \kappa$, then this is preserved by all $\leq\theta^{<\kappa}$-distributive forcing.*

We now close this section with a characterization of λ-nearly measurable cardinals that most closely mirrors the definition of nearly θ-supercompact cardinals. The embedding definition given below makes it immediate that nearly θ-supercompact cardinals are always λ-nearly measurable. The weakly measurable cardinals κ of [25] will coincide exactly with the κ^+-nearly measurable cardinals. More generally, if $\lambda^{<\kappa} = \lambda$, then κ will be λ-nearly measurable according to the below characterization if and only if for every family $\mathcal{A}$ containing at most λ many subsets of κ, there exits a nonprincipal κ-complete filter on κ measuring all subsets in $\mathcal{A}$.

Definition 2.1.9 (λ-Nearly Measurable). *A cardinal κ is λ-nearly measurable if for every $A \subseteq \lambda$, there exists a $<\kappa$-closed transitive $M \models ZFC^-$ with $A, \kappa \in M$, a transitive N, and an elementary embedding $j : M \to N$ with critical point κ.*

The assertion that κ is λ-nearly measurable for some $\lambda \geq 2^\kappa$ is equivalent to saying that κ is measurable. This follows from the fact that we can encode the full powerset of κ in the domain thereby inducing a nonprincipal κ-complete measure on κ. In particular, when the GCH holds at κ, we get nothing new by considering intermediate levels of near measurability between weak compactness and full measurability. Nevertheless, we later show that relative to sufficiently large amounts of near θ-supercompactness for κ, we can pump up 2^κ without collapsing cardinals above κ to achieve arbitrarily large amounts of λ-near measurability without full measurability in these forcing extensions.

2.2 Generalized Laver functions

In this section, we introduce a sort of generalized Laver function for nearly θ-supercompact cardinals, show conditions sufficient for already having these functions or forcing the existence of such functions, and provide a partial answer to an open question regarding these types of functions for θ-supercompact cardinals. Let us begin with a definition of our Laver diamond principle for nearly θ-supercompact cardinals following [11]:

Definition 2.2.1 ($\Ldiamond_\kappa^{\text{nearly }\theta\text{-sc}}$)**.** *A function $\ell{:}\kappa \to \boldsymbol{V}_\kappa$ is said to anticipate a set $A \in \boldsymbol{H}_{\theta^+}$ for near θ-supercompactness if for every transitive set M of size $\theta^{<\kappa}$ with $A, \ell, \kappa, \theta \in M$ and $M^{<\kappa} \subseteq M$, there exists a transitive N and a near θ-supercompactness embedding $j : M \to N$ with critical point κ such* *that* $j(\ell)(\kappa) = A$. *Define* $◺_\kappa^{\textit{nearly } \theta\textit{-sc}}$ *to hold if there exists a single function* $\ell{:}\kappa \to \boldsymbol{V}_\kappa$ *anticipating all sets* $A \in \boldsymbol{H}_{\theta^+}$ *for near* θ*-supercompactness.*

Now let's show that $◺_\kappa^{\text{nearly } \theta\text{-sc}}$ holds if κ is θ-supercompact.

Theorem 2.2.2 (Laver Functions for Nearly θ-Supercompact Cardinals). *If* κ *is a* θ*-supercompact cardinal for some* θ *such that* $\theta^{<\kappa} = \theta$*, then* $◺_\kappa^{\textit{nearly } \theta\textit{-sc}}$ *is true.*

Proof. Define ℓ by transfinite recursion as follows: Given that $\ell \upharpoonright \gamma$ is defined for some $\gamma < \kappa$ and cardinals $\lambda, \eta < \kappa$ for which there is a set in $\mathbf{H}_\lambda$ not anticipated for near η-supercompactness, define $\ell(\gamma)$ to be such an unanticipated set for which $\max\{\lambda, \eta\}$ is least. Otherwise, leave $\ell(\gamma)$ undefined, or define it to be any element from $\mathbf{V}_\kappa$.

We can now verify that ℓ anticipates every element of $\mathbf{H}_{\theta^+}$ for near θ-supercompactness. Suppose not, and let $j : \mathbf{V} \to N$ be any ultrapower embedding generated by a normal fine κ-complete measure on $P_\kappa\theta$. Then by its closure, the model N will agree with $\mathbf{V}$ on the sets having hereditary size at most $\theta^{<\kappa}$. In particular, it has the same sets in $\mathbf{H}_{\theta^+}$, and the same near θ-supercompactness embeddings having hereditary size $\theta^{<\kappa}$. Consequently, it will agree with $\mathbf{V}$ on the sets in $\mathbf{H}_{\theta^+}$ that are not anticipated by ℓ for near θ-supercompactness whereby it will also know that there is a set in $\mathbf{H}_{\theta^+}$ that is not anticipated by ℓ for near θ-supercompactness. Then since $j(\ell) \restriction \kappa = \ell$, it follows by elementarity that $j(\ell)(\kappa) = A$ for some set $A \in \mathbf{H}_{\theta^+}$ not anticipated by ℓ for near θ-supercompactness. Note that this is indeed the case since $j(\ell)(\kappa)$ must assume an element whose transitive closure has size at most θ that is not anticipated by ℓ for near η-supercompactness for some $\eta \leq \theta$, and near θ-supercompactness embeddings are automatically near ξ-supercompactness embeddings for all $\xi \leq \theta$. But then for every $M \in \mathbf{H}_{(\theta^{<\kappa})^+}$ with $\ell, \kappa, \theta \in M$ and $M^{<\kappa} \subseteq M$, the elementary embedding $j \restriction M : M \to j(M)$ will be a near θ-supercompactness embedding with $j(\ell)(\kappa) = A$ so ℓ anticipates A for near θ-supercompactness, a contradiction. $\square$

If κ is nearly θ-supercompact but not necessarily θ-supercompact for some $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$, then we can still force $◺_{\kappa}^{\text{nearly } \theta\text{-sc}}$ to hold in a forcing extension.

Theorem 2.2.3 (Laver Functions for Nearly θ-Supercompact Cardinals). *If κ is a nearly θ-supercompact cardinal for some $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$, then in every fast function forcing extension $\boldsymbol{V}[f]$ for κ, we have $◺_{\kappa}^{\textit{nearly } \theta\textit{-sc}}$ true in $\boldsymbol{V}[f]$.*

Proof. Suppose κ is nearly θ-supercompact for some $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$

and $\mathbf{V}[f]$ is an extension over the fast function forcing notion $\mathbb{F}_\kappa$ for κ. Let $\vec{x} = \langle x_\alpha | \alpha < \kappa \rangle$ be an enumeration of $\mathbf{V}_\kappa$ in the ground model such that each element is mentioned unboundedly often below κ. Define a partial function $\ell{:}\kappa \to \mathbf{V}[f]_\kappa$ in $\mathbf{V}[f]$ by $\alpha \mapsto (\vec{x} \circ f)(\alpha)_f$ whenever such a mapping makes sense (i.e., whenever $(\vec{x} \circ f)(\alpha)$ is an $\mathbb{F}_\kappa$-name). I claim that the partial function ℓ is the desired near θ-supercompactness Laver function for κ anticipating every element of $\mathbf{H}_{\theta^+}^{\mathbf{V}[f]}$.

First note that $\theta^{<\kappa} = \theta$ in $\mathbf{V}[f]$ because this holds in the ground model and $\mathbb{F}_\kappa$ adds no new $<\kappa$ sequences to θ by virtue of being $<\kappa$-closed. Now let A be any element of $\mathbf{H}_{\theta^+}$, and let M be any transitive $<\kappa$-closed set of size θ in $\mathbf{V}[f]$ with $A, f, \vec{x}, \theta \in M$. Because $\mathbb{F}_\kappa$ has size κ and $\kappa \leq \theta$, we can then let $\dot{A}, \dot{M} \in \mathbf{H}_{\theta^+}$ be $\mathbb{F}_\kappa$-names for A and M, respectively, in the ground model. Now let C be a subset of θ coding $\dot{A}$ and $\dot{M}$ and θ enumerations of their transitive closures in the ground model. Then by the near θ-supercompactness of κ and the fact that $\theta^{<\theta} = \theta$, we may find transitive ZFC$^-$ models $\overline{M}$ and N of size θ that are closed under $<\theta$ sequences and a near θ-supercompactness embedding $j : \overline{M} \to N$ with critical point κ such that $C \in \overline{M}$. Now $\overline{M}$ knows that $\vec{x}$ is a κ sequence enumerating all elements of $\mathbf{H}_\kappa$ unboundedly often below κ. It then follows from elementarity that N knows that $j(\vec{x})$ is a $j(\kappa)$ sequence enumerating all elements of $\mathbf{H}^N_{j(\kappa)}$ unboundedly often below $j(\kappa)$.

But since κ is nearly θ-supercompact for some $\theta \geq \kappa$, we know that κ is inaccessible (in $\overline{M}$) and so $j(\kappa)$, which is greater than θ, will be inaccessible in N. In particular, $j(\kappa) > (\theta^+)^N$ so N thinks that $j(\vec{x})$ lists all elements from $\mathbf{H}^N_{\theta^+}$, which includes $\dot{A}$, by virtue of N containing the powerset of θ from $\overline{M}$ and hence the subset C that reveals a θ enumeration of the transitive closure of $\dot{A}$. It therefore follows that there exists an $\alpha^* > \theta$ such that $j(\vec{x})(\alpha^*) = \dot{A}$.

Let p be the condition $\{\langle \kappa, \alpha^* \rangle\}$, so that below p, the forcing $j(\mathbb{F}_\kappa)$ factors as $\mathbb{F}_\kappa \times \mathbb{F}_{[\lambda, j(\kappa))} \restriction p$ where λ is the next inaccessible beyond α^* in N. Now $\mathbb{F}_{[\lambda, j(\kappa))}$ is ${<}\theta$-closed in N by virtue of $\alpha^* > \theta$, the model N is closed under ${<}\theta$-sequences, and there are at most $|N| = \theta$ many dense subsets of $\mathbb{F}_{[\lambda, j(\kappa))}$ from N in $\mathbf{V}$. Consequently, by the Diagonalization Criterion, we can construct an N-generic filter f_{tail} for $\mathbb{F}_{[\lambda, j(\kappa))}$ in $\mathbf{V}$. Consequently in $\mathbf{V}[f]$, the function $f \cup p \cup f_{\text{tail}}$ will be an N-generic filter for $j(\mathbb{F}_\kappa)$. Therefore, we may lift the embedding in $\mathbf{V}[f]$ to $j : \overline{M}[f] \to N[j(f)]$ where $j(f) = f \cup p \cup f_{\text{tail}}$. Then we have $j(\ell)(\kappa) = j(\vec{x} \circ f)(\kappa)_{j(f)}$, which equals $j(\vec{x})(j(f)(\kappa))_{j(f)} = j(\vec{x})(\alpha^*)_{j(f)} = \dot{A}_{j(f)} = \dot{A}_f = A$. Also since $\dot{M}_f = M$ and $\dot{M}_f \in \overline{M}[f]$, we will have $M \in \overline{M}[f]$. Then by the transitivity of $\overline{M}[f]$, we will have $M \subseteq \overline{M}[f]$ so that we may restrict to a near θ-supercompactness embedding $j \restriction M : M \to j(M)$ with critical point κ such that $j(\ell)(\kappa) = A$, as desired. $\square$

In [20], Laver shows that if κ is supercompact, then $\LaverDiamond_\kappa^{\text{sc}}$ holds. But in this proof, he effectively shows that if κ is $2^{\theta^{<\kappa}}$-supercompact, then $\LaverDiamond_\kappa^{\theta\text{-sc}}$ holds. In [11], Hamkins isolates this idea as a theorem and asks if it is sufficient for κ to be θ-supercompact in order for $\LaverDiamond_\kappa^{\theta\text{-sc}}$ to hold in general. The following theorem gives a partial answer to this open question, showing that we can reduce the hypothesis from κ is $2^{\theta^{<\kappa}}$-supercompact to κ is nearly $2^{\theta^{<\kappa}}$-supercompact. But first let us explicitly define what it means for $\LaverDiamond_\kappa^{\theta\text{-sc}}$ to hold.

Definition 2.2.4 **($\LaverDiamond_\kappa^{\theta\text{-sc}}$).** *A function $\ell\colon\kappa \to \boldsymbol{V}_\kappa$ anticipates an A for θ-supercompactness provided that there exits a θ-supercompactness embedding $j : \boldsymbol{V} \to N$ with critical point κ such that $j(\ell)(\kappa) = A$. We say that the Laver Diamond Principle for the θ-supercompactness of κ holds, denoted $\LaverDiamond_\kappa^{\theta\text{-sc}}$, provided that there exists an $\ell\colon\kappa \to \boldsymbol{V}_\kappa$ simultaneously anticipating every $A \in \boldsymbol{H}_{\theta^+}$ for θ-supercompactness. In this case, we'll call ℓ a θ-supercompactness Laver function for κ.*

Therefore, the following theorem will improve upon the hypothesis of Laver's theorem for θ-supercompact Laver functions for κ by showing that the existence of such functions for κ will be guaranteed by virtue of κ being nearly $2^{\theta^{<\kappa}}$-supercompact rather than requiring κ to be $2^{\theta^{<\kappa}}$-supercompact.

Theorem 2.2.5 (Laver functions for θ-supercompact cardinals). *If κ is a nearly $2^{\theta^{<\kappa}}$-supercompact cardinal, then $\LD^{\theta\text{-sc}}_{\kappa}$ is true.*

Proof. Define $\ell\colon\kappa\to\mathbf{V}_\kappa$ by transfinite recursion as follows: Given that $\ell\restriction\gamma$ is defined for some $\gamma<\kappa$ and cardinals $\lambda,\eta<\kappa$ for which there is a set in $\mathbf{H}_\lambda$ not anticipated for η-supercompactness, define $\ell(\gamma)$ to be such an unanticipated set for which $\max\{\lambda,\eta\}$ is least. Otherwise, leave $\ell(\gamma)$ undefined, or define it to be any element from $\mathbf{V}_\kappa$.

Now let $\mathcal{B}$ be the collection of all pairs $\langle A,\mu_A\rangle$ for which $A\in\mathbf{H}_{\theta^+}$ is anticipated by ℓ for some θ-supercompactness ultrapower embedding $j:\mathbf{V}\to\overline{N}$ generated by a normal fine measure μ_A on $P_\kappa\theta$. Since $\mathbf{H}_{\theta^+}$ has hereditary size 2^θ and μ_A has hereditary size $2^{\theta^{<\kappa}}$, we know that $\mathcal{B}\in\mathbf{H}_{(2^{\theta^{<\kappa}})^+}$. Since $\mathbf{H}_{(\theta^{<\kappa})^+}$ also has hereditary size $2^{\theta^{<\kappa}}$, we can find a transitive $\theta^{<\kappa}$-closed $M\models\mathrm{ZFC}^-$ of size $2^{\theta^{<\kappa}}$ with $\mathcal{B},\mathbf{H}_{(\theta^{<\kappa})^+}\in M$. Then by the near $2^{\theta^{<\kappa}}$-supercompactness of κ, there exists a near $2^{\theta^{<\kappa}}$-supercompactness embedding $j:M\to N$ for some transitive ZFC^- model N.

Now suppose there exists an element in $\mathbf{H}_{\theta^+}$ not anticipated by any θ-supercompactness embedding for κ. Because M has the true $\mathbf{H}_\kappa$, it agrees with $\mathbf{V}$ on the definition of ℓ. Also since M has the same $\mathbf{H}_{(\theta^{<\kappa})^+}$ as $\mathbf{V}$, the model N will too by virtue of having $j''\theta^{<\kappa}$. In particular, N has all of the ${<}\kappa$-sequences of subsets of $P_\kappa\theta$ and all of the functions from $P_\kappa\theta$ into θ that

$\mathbf{V}$ does so it will agree with $\mathbf{V}$ on the objects that are κ-complete normal fine measures on $P_\kappa\theta$. It also contains all of the functions from $P_\kappa\theta$ into $\kappa \times \mathbf{V}_\kappa$ that $\mathbf{V}$ does so if n and n^* are the ultrapower embeddings generated in $\mathbf{V}$ and in N from the same κ-complete normal fine measure on $P_\kappa\theta$ in N, then $n(\ell)(\kappa) = n^*(\ell)(\kappa)$. Consequently, N agrees with $\mathbf{V}$ that there is an $A \in \mathbf{H}_{\theta^+}$ that is not anticipated by any θ-supercompactness embedding for κ. Then by elementarity, $j(\ell)(\kappa) = A$ for some A in $\mathbf{H}_{\theta^+}$ that N thinks cannot be anticipated by ℓ by any θ-supercompactness embedding for κ. Now consider the factor embedding $j_0 : M \to N_0$ generated by using $j''\theta$ as a seed, which is $\pi \circ j$ where $\pi : X_{j''\theta} \overset{\cong}{\to} N_0$ is the Mostowski Collapse of $X_{j''\theta} = \{j(f)(j''\theta) | f \in M^{P_\kappa\theta} \cap M\}$. This elementary embedding is equal to the one generated by the κ-complete normal fine measure μ on $P_\kappa\theta$ via j defined by $S \in \mu \Leftrightarrow S \subseteq P_\kappa\theta \wedge j''\theta \in j(S)$. Note that μ is indeed a true κ-complete normal fine measure on $P_\kappa\theta$ since M has all of the $<\kappa$-sequences of subsets of $P_\kappa\theta$ and all of the functions from $P_\kappa\theta$ into θ that $\mathbf{V}$ does. Also note that because $\theta \subseteq X_{j''\theta}$, the model $X_{j''\theta}$ contains $\mathbf{H}^N_{\theta^+}$ whereby the Mostowski Collapse fixes all elements of $\mathbf{H}^N_{\theta^+}$. In particular, $j_0(\ell)(\kappa) = \pi(j(\ell)(\kappa)) = \pi(A) = A$. But now we have a θ-supercompactness ultrapower embedding $h : \mathbf{V} \to \overline{N}$ with critical point κ induced by μ with $h(\ell)(\kappa) = j_0(\ell)(\kappa) = A$ since M contains all of the functions from $P_\kappa\theta$ into

$\kappa \times \mathbf{V}_\kappa$ that $\mathbf{V}$ does. Therefore, there will be a pair $\langle A, \mu_A \rangle \in \mathcal{B}$ and hence in N so N will know that A is anticipated by ℓ for a θ-supercompactness embedding for κ, a contradiction. □

2.3 Main Indestructibility and Consistency Results

The original proof of Laver's celebrated result that a supercompact cardinal could be made indestructible in a forcing extension by all $<\kappa$-directed closed forcing appeals to the existence of a Laver function, a generalized $\Diamond$ sequence anticipating all objects in the universe. Laver's proof technique has since been generalized to work with a much broader class of large cardinal notions. In order to prove Statement 1 of the Main Theorem, I make use of a relatively recent generalization of Laver's idea known as the Lottery Preparation that was introduced in [10]. Instead of appealing to the existence of a modified Laver function for use with this preparation, I will show that if κ is nearly θ-supercompact, then we can find a function $f : \kappa \to \kappa$ that can reach up high with respect to a family of elementary embeddings j with critical point κ (i.e., $j(f)(\kappa) > \theta$). Indeed, the need to be able to find such functions that can reach up high with respect to an embedding or family of embeddings witnessing a large cardinal property is common in the use of such preparations. We formally define what this means for nearly θ-supercompact cardinals below.

Definition 2.3.1. *A function $f : \kappa \to \kappa$ has the Menas Property for a nearly θ-supercompact cardinal κ if for every $A \subseteq \theta$, there exist weakly closed transitive ZFC$^-$ models M and N of size $\theta^{<\kappa}$ with $A, f \in M$ and a near θ-supercompactness embedding $j : M \to N$ having critical point κ such that $j(f)(\kappa) > \theta$.*

We say that such a function has the Menas Property in honor of Menas, who was concerned with the existence of analogous functions for strongly compact cardinals. It turns out that these types of functions, introduced before Laver's seminal result, can be used to achieve indestructibility results instead of the Laver functions $\ell\colon\kappa \to \mathbf{V}_\kappa$ for which there are a family of embeddings such that every element in some cut of the universe $\mathbf{V}_\gamma$ is assumed by $j(\ell)(\kappa)$ for some elementary mapping j. We now state and prove the preliminary lemmas for Statement 1 of the Main Theorem.

Lemma 2.3.2. *If κ is a nearly θ-supercompact cardinal, then for every $<\kappa$-closed transitive $M \models$ ZFC$^-$ of size θ with $\theta \in M$, there exists a weakly closed transitive N of size θ, and a Hauser near θ-supercompactness embedding $j : M \to N$ with critical point κ, where κ is not nearly θ-supercompact in N.*

Proof. If $\theta^{<\kappa} > \theta$, then this result is vacuously true so assume that $\theta^{<\kappa} = \theta$, and let M be a transitive $<\kappa$-closed transitive ZFC$^-$ model of size θ with $\theta \in M$. Then by the Hauser embedding characterization of the near θ-supercompactness of κ, let N be a transitive weakly closed ZFC$^-$ model of size θ and $j : M \to N$ be a near θ-supercompactness embedding with critical point κ, with $M, j \in \mathbf{H}^N_{\theta+}$ such that $j(\kappa)$ is smallest among all such Hauser near θ-supercompactness embeddings. Now suppose for contradiction that κ were nearly θ-supercompact in N. Then since N knows that M is a transitive $<\kappa$-closed ZFC$^-$ model of size θ with $\theta \in M$, the model N would think that we have a Hauser near θ-supercompactness embedding $h : M \to M^*$ with critical point κ in N by the Hauser embedding characterization of the near θ-supercompactness of κ. But M^* has size $\theta < j(\kappa)$ in N and so $N \models h(\kappa) < \theta^+ < j(\kappa)$ by the inaccessibility of $j(\kappa)$ in N, contradicting the minimality of $j(\kappa)$ among all such embeddings. As an additional clarification, it should be noted that if N thinks that $h : M \to M^*$ is a Hauser near θ-supercompactness embedding, then it is correct about this. In particular, M^* will actually be weakly closed because N exhibits this closure. $\square$

Lemma 2.3.3 (Menas Property for Nearly θ-Supercompact Cardinals). *Every nearly θ-supercompact cardinal κ for which $\kappa \leq \theta$ and $\theta^{<\kappa} = \theta$ has a function with the Menas property for its near θ-supercompactness.*

Proof. Suppose κ is nearly θ-supercompact, and let $A \subseteq \theta$. Also define $f : \kappa \to \kappa$ to be the function $\gamma \mapsto (2^{\sigma^{<\gamma}})^+$ where σ is the least cardinal above γ such that γ is not nearly σ-supercompact if such a $\sigma < \kappa$ can be found and 0 otherwise. I will show that f is a function with the Menas property for the near θ-supercompactness of κ.

Let δ be the least cardinal at or above κ such that $2^{\delta^{<\kappa}} > \theta$. Then κ is β-supercompact for all $\beta < \delta$ at or above κ. For each such $\beta < \delta$ at or above κ, let μ_β be a κ-complete normal fine measure on $P_\kappa\beta$. Then let $D \subseteq \theta$ code the collection of all such μ_β as well as a θ-sized collection of subsets of $\delta^{<\kappa}$. Now let $M \in \mathbf{H}_{\theta^+}$ be a weakly closed (and in particular $<\kappa$-closed) transitive ZFC$^-$ model with $A, D, \theta \in M$. By Lemma 2.3.2, let N be a transitive weakly closed ZFC$^-$ model of size θ where κ is not nearly θ-supercompact and $j : M \to N$ be a near θ-supercompactness embedding with critical point κ. Since M is closed under less than κ sequences, the definition of f is absolute to M. Consequently, by elementarity, N will think that $j(f)(\kappa) = (2^{\sigma^{<\kappa}})^+$ where σ is the least cardinal above κ such that κ is not nearly σ-supercompact if such a $\sigma < j(\kappa)$ can be found and 0 otherwise. Because M contains $D \subseteq \theta$, the model N will be able to construct D from $j(D)$ and $j''\theta$ so that it will know that κ is β-supercompact and hence nearly β-supercompact for all $\beta < \delta$ at or above κ. Because N contains D, it will also know that $2^{\delta^{<\kappa}} \geq \theta$. Finally, because κ is not nearly θ-supercompact in N with $\theta < j(\kappa)$, the model $N \models j(f)(\kappa) \geq (2^{\delta^{<\kappa}})^+ \geq \theta^+ > \theta$, as desired. $\square$

The proof of Statement 1 of the Main Theorem also makes use of a standard lemma that is referred to as the Diagonalization Criterion (51) in [9]. The lemma states that if we are given a θ-closed set-theoretic model M having at most θ^+ many antichains for a poset $\mathbb{P}$ in M that M thinks is $\leq\theta$-closed, then for any $p \in \mathbb{P}$, we can (in $\mathbf{V}$) diagonalize against all θ^+ many antichains to produce an M-generic filter for $\mathbb{P}$ below p.

Armed with the assurance of the existence of a function with the Menas property for the near θ-supercompactness of every nearly θ-supercompact cardinal κ for which $\theta^{<\kappa} = \theta$ and a Lemma for constructing sufficiently generic filters, we can now prove Statement 1 of the Main Theorem, the Lottery Preparation for Nearly θ-Supercompact Cardinals.

Theorem 2.3.4. *If κ is nearly θ-supercompact for some $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$, then there is a forcing extension where its near θ-supercompactness is preserved and indestructible by any further $<\kappa$-directed closed θ-c.c. forcing of size at most θ.*

Proof. Suppose κ is nearly θ-supercompact, and let $f : \kappa \to \kappa$ be a function with the Menas property for its near θ-supercompactness as in Lemma 2.3.3.

Now define $\mathbb{P}$ to be the relevant simplified lottery preparation of κ relative to the function f. Specifically, let $\mathbb{P}$ be the κ-stage Easton support iteration where we force nontrivially precisely at every inaccessible stage $\gamma < \kappa$ such that $f''\gamma \subseteq \gamma$, by the lottery sum of all $<\gamma$-directed closed posets having hereditary size of at most $|f(\gamma)|^{\mathbf{V}^{\mathbb{P}_\gamma}}$ in $\mathbf{V}^{\mathbb{P}_\gamma}$. We may assume that $\mathbb{P}_\alpha \in \mathbf{V}_\kappa$ for all $\alpha < \kappa$.

Let G be a $\mathbf{V}$-generic filter for $\mathbb{P}$, and suppose that $\mathbf{V}[G]$ thinks that $\mathbb{Q}$ is a $<\kappa$-directed closed θ-c.c. partial order of size θ. Without loss of generality, we may assume that $\mathbb{Q} \in \mathbf{H}_{\theta^+}^{\mathbf{V}[G]}$ is a partial order on θ by associating it with an isomorphic one if necessary. I claim that $\mathbf{V}[G]$ is the desired forcing extension where κ is a nearly θ-supercompact cardinal, and its near θ-supercompactness is indestructible by any $<\kappa$-directed closed θ-c.c. forcing of size at most θ. To verify this assertion, let g be a $\mathbf{V}[G]$-generic filter for $\mathbb{Q}$. I will show that κ remains nearly θ-supercompact in $\mathbf{V}[G][g]$.

Because $\mathbb{P}$ is in $\mathbf{H}_{\kappa^+} \subseteq \mathbf{H}_{\theta^+}$ and $\mathbb{Q} \in \mathbf{H}_{\theta^+}^{\mathbf{V}[G]}$ is a partial order on θ, we can find a $\mathbb{P}$-name $\dot{\mathbb{Q}} \in \mathbf{H}_{\theta^+}$ necessarily yielding a $<\kappa$-directed closed θ-c.c. poset on θ such that $\dot{\mathbb{Q}}_G = \mathbb{Q}$. Then $\mathbb{P} * \dot{\mathbb{Q}} \in \mathbf{H}_{\theta^+}$ so by letting A be an arbitrary subset of θ in $\mathbf{V}[G][g]$, we can find a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{A} \in \mathbf{H}_{\theta^+}$ such that $\dot{A}_{G*g} = A$. Code $\dot{\mathbb{Q}}$ and $\dot{A}$ as a single subset of θ. Then since f is a function with the Menas property for the near θ-supercompactness of κ as in Definition 2.3.1, we may fix a near θ-supercompactness embedding $j : M \to N$ between transitive ZFC^{-} models of size θ closed under $<\theta$ sequences such that $j(f)(\kappa) > \theta$, and M has θ, the function f, and this subset coding $\dot{A}$ and the $\mathbb{P}$-name $\dot{\mathbb{Q}}$.

Because each $\mathbb{P}_\alpha$ is constructed within $\mathbf{H}_\kappa$ for all $\alpha < \kappa$, and $\mathbb{P}$ is a direct limit of these posets, M will be able to construct $\mathbb{P}$ and will agree on its definition with $\mathbf{V}$ by virtue of being closed under $<\kappa$-sequences. Then because $\mathbb{P}$ is constructed below the critical point, $j(\mathbb{P}) \restriction \kappa = \mathbb{P}$. Also, by elementarity, N will think that $j(\mathbb{P})$ is a $j(\kappa)$-stage Easton support iteration forcing nontrivially with the lottery sum of all $<\gamma$-directed closed posets having hereditary size of at most $|j(f)(\gamma)|^{N^{j(\mathbb{P})_\gamma}}$ in $N^{j(\mathbb{P})_\gamma}$ at all stages γ for which $j(f)''\gamma \subseteq \gamma$ and γ is inaccessible. Since $j(f) \restriction \kappa = f$, we will have $j(f)''\kappa = f''\kappa \subseteq \kappa$. Consequently, because κ is inaccessible (and hence inaccessible in N), nontrivial forcing will take place at stage κ of $j(\mathbb{P})$. Furthermore, this stage κ of forcing will be the lottery sum of all $<\kappa$-directed closed posets having hereditary size of at most $|j(f)(\kappa)|^{N^{\mathbb{P}}} \geq |\theta|^{N^{\mathbb{P}}}$ in $N^{j(\mathbb{P})_\kappa} = N^{\mathbb{P}}$. Note that because N has all subsets of θ that M does, it will have the one coding $\dot{\mathbb{Q}}$ so that $\dot{\mathbb{Q}}$ will be one of its elements. Also, since N is $<\kappa$-closed and $\mathbb{P}$ is κ-c.c. in $\mathbf{V}$, the model N will have all of the maximal antichains of $\mathbb{P}$ that $\mathbf{V}$ does so they will share the same generic filters for $\mathbb{P}$. Consequently, $\dot{\mathbb{Q}}$ will also be an allowed poset at stage κ for the forcing $j(\mathbb{P})$ by virtue of being $<\kappa$-directed closed and trivially having hereditary size $|\theta|^{N^{\mathbb{P}}}$ in $N^{\mathbb{P}}$. Therefore, below a condition opting for the poset named by $\dot{\mathbb{Q}}$ at stage κ, the poset $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$ where $\mathbb{P}_{\text{tail}}$ is the forcing of $j(\mathbb{P})$ beyond stage κ.

By elementarity, the forcing beyond stage κ of $j(\mathbb{P})$ will be trivial through all stages up to and including θ. Furthermore, since for any stage $\gamma < \kappa$, any collection of compatible conditions (not including the maximum element) in $\mathbb{P}$ opt for the same $<\gamma$-directed closed poset in $M^{\mathbb{P}_\gamma}$, any collection of compatible conditions (not including the maximum element) in $j(\mathbb{P})$ opt for the same $<\gamma$-directed closed poset in $N^{j(\mathbb{P})_\gamma}$ for any stage $\gamma < j(\kappa)$ in N. Combining these two observations, the forcing $\mathbb{P}_{\text{tail}}$ will be $\leq\theta$-(directed) closed in $N[G][g]$. Now since $\mathbb{P} * \dot{\mathbb{Q}}$ is θ-c.c. by virtue of $\mathbb{P}$ being κ-c.c. and $\dot{\mathbb{Q}}$ being necessarily θ-c.c., and $N^{<\theta} \subseteq N$, the model $N[G][g]$ will be closed under $<\theta$ sequences in $\mathbf{V}[G][g]$. Also, there are at most $|N[G][g]|^{\mathbf{V}[G][g]} = |N| = \theta$ many dense subsets for $\mathbb{P}_{\text{tail}}$ in $N[G][g]$. Therefore, we may apply the Diagonalization Criterion to show that there is an $N[G][g]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$ in $\mathbf{V}[G][g]$ and then lift the embedding to $j : M[G] \to N[j(G)]$ in $\mathbf{V}[G][g]$ where $j(G) = G * g * G_{\text{tail}}$. Note then that since $N[G][g]$ is closed under $<\theta$ sequences in $\mathbf{V}[G][g]$ and $G_{\text{tail}} \in \mathbf{V}[G][g]$, the model $N[j(G)]$ will actually be closed under $<\theta$ sequences in $\mathbf{V}[G][g]$ as well.

Now $M[G]$ can construct $\mathbb{Q} = \dot{\mathbb{Q}}_G$, and since $\mathbb{Q}$ is a $<\kappa$-directed closed partial order on θ in $\mathbf{V}[G]$, it will be a $<\kappa$-directed closed partial order on θ in $M[G]$. Then by elementarity, $N[j(G)]$ will think that $j(\mathbb{Q})$ is a $<j(\kappa)$-directed closed (and hence $\leq\theta$-directed closed) partial order on $j(\theta)$. Also, because $N[j(G)]$ contains the filter $g \subseteq \theta$ and $j''\theta$ from which it can construct $j \restriction \theta$, it will contain the directed collection of θ conditions $j''g = (j \restriction \theta)''g$ from $j(\mathbb{Q})$. Combining these two observations, we may find a master condition $q_* \in j(\mathbb{Q})$ below all conditions in $j''g$. Now since $j(\mathbb{Q})$ is $<\theta$-closed in $N[j(G)]$, the model $N[j(G)]$ is closed under $<\theta$ sequences in $\mathbf{V}[G][g]$, and $N[j(G)]$ has at most $|N[j(G)]|^{\mathbf{V}[G][g]} = |N| = \theta$ many dense subsets for $j(\mathbb{Q})$, as counted in $\mathbf{V}[G][g]$, we may again appeal to the Diagonalization Criterion, this time to find an $N[j(G)]$-generic filter g^* for $j(\mathbb{Q})$ containing q_* in $\mathbf{V}[G][g]$. This then allows us to lift the near θ-supercompactness embedding to $j : M[G][g] \to N[j(G)][j(g)]$ where $j(g) = g^*$ in $\mathbf{V}[G][g]$. Note that $M[G][g]$ is $<\theta$-closed (and hence $<\kappa$-closed) in $\mathbf{V}[G][g]$ because in the ground model, M is closed under $<\theta$ sequences and $\mathbb{P} * \dot{\mathbb{Q}}$ is θ-c.c. Then because $M[G][g]$ can construct $A = \dot{A}_{G*g}$ with A an arbitrary subset of θ in $\mathbf{V}[G][g]$, we get an embedding witnessing the near θ-supercompactness for the arbitrary $A \subseteq \theta$ in $\mathbf{V}[G][g]$. Therefore, κ is nearly θ-supercompact in $\mathbf{V}[G][g]$, as desired. □

From the proof of this theorem, we also have the following result that will be used in the proof of the Main Theorem, Statement 3.

Theorem 2.3.5. *If κ is nearly θ-supercompact for some $\theta \geq 2^\kappa$ such that $\theta^{<\theta} = \theta$, then there exists a forcing extension preserving all cardinals and cofinalities above κ where κ remains nearly θ-supercompact, $\theta^{<\theta} = \theta$, and $2^\kappa = \theta$. Furthermore, if the SCH holds below κ, then this forcing extension preserves all cardinals and cofinalities.*

Proof. Suppose κ is nearly θ-supercompact for some cardinal $\theta \geq 2^\kappa$ such that $\theta^{<\theta} = \theta$. Then by Theorem 2.3.4, we can move to a forcing extension $\mathbf{V}[G]$ where κ remains nearly θ-supercompact, and its near θ-supercompactness is indestructible by every $<\kappa$-directed closed θ-c.c. forcing notion of size at most θ. Since G is $\mathbf{V}$-generic for a poset of size κ, all cofinalities (and cardinals) above κ are preserved. Because it is also true that $\theta > \kappa$, the equality $\theta^{<\theta} = \theta$ still holds in $\mathbf{V}[G]$. In particular, $\theta \geq 2^\kappa$ and $\theta^{<\kappa} = \theta$ in $\mathbf{V}[G]$. Now consider the forcing extension $\mathbf{V}[G][g]$ where g is $\mathbf{V}[G]$-generic for $\mathrm{Add}(\kappa, \theta)^{\mathbf{V}[G]}$. Since κ is nearly θ-supercompact in $\mathbf{V}[G]$, we have $\kappa^{<\kappa} = \kappa$ so no cofinalities (or cardinals) were collapsed between $\mathbf{V}[G]$ and $\mathbf{V}[G][g]$. Also, in $\mathbf{V}[G]$, this forcing poset is $<\kappa$-directed closed and θ-c.c. (κ^+-c.c.) and has size $\theta^{<\kappa} = \theta$. Consequently, κ remains nearly θ-supercompact in

$\mathbf{V}[G][g]$. Furthermore, since $\theta^{<\theta} = \theta$ and $\mathbb{P}$ is a θ-c.c. partial order of size θ in $\mathbf{V}[G]$, we will still have $\theta^{<\theta} = \theta$ holding true in $\mathbf{V}[G][g]$. But also $2^\kappa \geq \theta$ after adding θ many Cohen subsets of κ so $\mathbf{V}[G][g] \models 2^\kappa = \theta$. Therefore $\mathbf{V}[G][g]$ is the desired forcing extension of $\mathbf{V}$ preserving all cofinalities (and also cardinals) above κ where κ remains nearly θ-supercompact, $\theta^{<\theta} = \theta$, and $2^\kappa = \theta$.

Also, if the SCH holds below κ, we could instead consider the lottery preparation from Theorem 2.3.4 where we restrict ourselves to the lottery sum of all $<\gamma$-directed closed *cofinality-preserving* posets having hereditary size of at most $|f(\gamma)|^{\mathbf{V}^{\mathbb{P}_\gamma}}$ in $\mathbf{V}^{\mathbb{P}_\gamma}$ at every nontrivial stage of forcing γ. The argument that all cardinals and cofinalities are preserved in this case is as in the conclusion of the proof of the Exact Preservation Theorem 4.3 from [8]. Suppose for contradiction that some cardinal $\eta < \kappa$ is collapsed to have cofinality δ for some $\delta < \operatorname{cof}(\eta)$. We may factor $\mathbb{P}$ as $\mathbb{P}_\delta * \dot{\mathbb{S}} * \mathbb{P}_{(\delta+1,\kappa)}$ where $\mathbb{S}$ is the forcing at stage δ. The forcing $\mathbb{P}_{(\delta+1,\kappa)}$ beyond stage δ is $\leq\delta$-closed so it cannot add a δ sequence to η, and the forcing $\mathbb{S}$ cannot collapse any cofinalities or cardinals since we only permit cofinality-preserving posets at every nontrivial stage of forcing. Therefore, $\mathbb{P}_\delta$ must be responsible for collapsing the cofinality of η. Let $\beta \leq \delta$ be the least such that $\mathbb{P}_\beta$ collapses the cofinality of η to δ. Then because the lottery preparation only forces with cofinality-preserving posets, β must be a limit ordinal. Moreover, β must actually be a limit of nontrivial forcing stages because otherwise $\mathbb{P}_{\xi+1}$ would be forcing equivalent to $\mathbb{P}_\beta$ for the maximum stage $\xi < \beta$ where we force nontrivially, contradicting the minimality of β. Since we only force nontrivially at inaccessible stages γ and do so only when the nontrivial posets we forced with before had size less than γ in $\mathbf{V}^{\mathbb{P}_\gamma}$, it then follows that β is a limit of inaccessible cardinals and $\mathbb{P}_\gamma$ will have size less than β for all $\gamma < \beta$ (or at least be forcing equivalent to such a poset). Consequently, if β is regular and hence inaccessible, then $\mathbb{P}_\beta$ must have size $\beta^{<\beta} = \beta$ (or at least be forcing equivalent to such a poset) by virtue of the fact that we take direct limits at inaccessible stages. This would make it impossible for $\mathbb{P}_\beta$ to collapse η since $\beta \leq \delta < \mathrm{cof}(\eta)$. Otherwise, β must be a singular strong limit whereby $\mathbb{P}_\beta$ will have (be forcing equivalent to a poset that has) size at most $\beta^{\mathrm{cof}(\beta)}$, which equals β^+, by the assumed SCH. But this is also impossible since δ must be regular whereby $\beta < \delta < \mathrm{cof}(\eta)$ so that $|\mathbb{P}_\beta| = \beta^+ < \mathrm{cof}(\eta)$. Therefore, no cardinals were collapsed when moving from $\mathbf{V}$ to $\mathbf{V}[G]$ after all, in the case that SCH holds below κ in the ground model, as desired. $\square$

As stated in the introduction, Statement 1 of the Main Theorem is a strengthening of the indestructibility result found in the Level by Level

Preparation Theorem 4.8 of [10] for η-supercompact cardinals for all $\eta < \theta$ when applicable. Recall that the Level by Level Preparation Theorem states that if there exists an η for which κ is η-supercompact and $2^{\eta^{<\kappa}} = \eta^+$, then there exists a forcing extension where the η-supercompactness of κ is indestructible by all $<\kappa$-directed closed forcing of size at most η. If κ is additionally nearly θ-supercompact for some $\theta > \eta$, then we can squeeze out some more indestructibility.

Corollary 2.3.6. *If κ is nearly θ-supercompact for some $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$, and η is a cardinal such that $\eta^{<\kappa} < \theta$, which will be true if $2^{\eta^{<\kappa}} = \eta^+$ and $\eta < \theta$, then there is a forcing extension where its η-supercompactness is preserved and indestructible by any further $<\kappa$-directed closed θ-c.c. forcing of size at most θ. In particular, the η-supercompactness of κ will be preserved by any $<\kappa$-directed closed forcing of size at most η.*

Proof. The forcing extension $\mathbf{V}[G]$ over the κ-c.c. lottery preparation $\mathbb{P}$ of size κ from the proof of Statement 1 of the Main Theorem where the near θ-supercompactness of κ is preserved and indestructible by $<\kappa$-directed closed θ-c.c. forcing of size at most θ has the desired property. To see this, first fix any $<\kappa$-directed closed and θ-c.c. partial order $\mathbb{Q}$ of size θ in $\mathbf{V}[G]$, and let $g \subseteq \mathbb{Q}$ be $\mathbf{V}[G]$-generic. Now note that $\theta^{<\theta} = \theta$ will still hold in $\mathbf{V}[G][g]$. One can see this by observing that $\mathbb{P}$ is a κ-c.c. partial order having size $\kappa \leq \theta$ so that $\theta^{<\theta} = \theta$ is still true in $\mathbf{V}[G]$ whereby it will also be true in $\mathbf{V}[G][g]$ because $\mathbb{Q}$ is θ-c.c. and has size at most θ in $\mathbf{V}[G]$. Also note that if $\eta^{<\kappa} < \theta$, then this will also hold true in $\mathbf{V}[G][g]$ as well. Specifically, in this case, $\kappa < \theta$ so since $\kappa^{<\kappa} = \kappa$, we have $\eta^{<\kappa} < \theta$ holding in $\mathbf{V}[G]$ by virtue of $\mathbb{P}$ having size κ and being κ-c.c. This inequality will then be preserved in $\mathbf{V}[G][g]$ since the $<\kappa$-directed closed forcing $\mathbb{Q}$ adds no new $<\kappa$ sequences to η. Combining these two observations, $\mathbf{V}[G][g] \models 2^{\eta^{<\kappa}} \leq 2^{<\theta} = \theta$ for every η such that $\eta^{<\kappa} < \theta$. Consequently, since κ remains nearly θ-supercompact in $\mathbf{V}[G][g]$ by the indestructibility result from Statement 1 of the Main Theorem, it remains η-supercompact there as well. Finally if $\eta^{<\kappa} < \theta$, then clearly $\eta < \theta$ as well, so any $<\kappa$-directed closed forcing $\mathbb{Q}$ of size at most η will trivially also satisfy the θ-c.c. condition. $\square$

When κ is η-supercompact, the condition requiring $2^{\eta^{<\kappa}} = \eta^+$ in the Level by Level Preparation Theorem 4.8 of [10] to obtain the indestructibility of the η-supercompactness of κ by $<\kappa$-directed closed forcing of size at most η is used so that the Diagonalization Criterion can be applied. Similarly, to preserve the near θ-supercompactness of a nearly θ-supercompact cardinal κ, we make use of the condition that $\theta^{<\theta} = \theta$ so that we can find a collection of embeddings witnessing the near θ-supercompactness of κ with $<\theta$-closed codomains of size θ. This then allows us to diagonalize against the at most θ many dense subsets that these target models can have via the Diagonalization Criterion. The additional restriction that the partial order be θ-c.c. is also used to apply the Diagonalization Criterion because it ensures that the forcing extensions $N[G][g]$ of the $<\theta$-closed codomains N, are $<\theta$-closed in $\mathbf{V}[G][g]$.

The corollary provides us with alternative heuristical reasons for the restriction on the class of θ that can be used and on the indestructibility we might be able to receive. If κ is η-supercompact and $2^{\eta^{<\kappa}} = \eta^+$, then by the Level by Level Preparation Theorem 4.8 of [10], we can force to an extension where the η-supercompactness of κ is preserved and indestructible by $<\kappa$-directed closed forcing of size at most η. If $\theta = 2^{\eta^{<\kappa}} = \eta^+$, and κ is nearly θ-supercompact, then Corollary 2.3.6 tells us that we can force to an extension where the η-supercompactness of κ is indestructible by $<\kappa$-directed closed θ-c.c. forcing of size at most θ. But as shown later in the proof of Theorem 2.4.5, nearly θ-supercompact cardinals are not necessarily θ-supercompact so we would not expect to be able to force to an extension where the η-supercompactness of κ is indestructible by arbitrary $<\kappa$-directed closed forcing notions of size θ, which is what the Level by Level Preparation

Theorem 4.8 would allow us to do when κ is θ-supercompact and $2^{\theta^{<\kappa}} = \theta^+$. Therefore, if the indestructibility received for the η-supercompactness of κ is to improve upon that provided to us by the Level by Level Preparation Theorem 4.8 by virtue of κ being η-supercompact but to also be weaker than the indestructibility we would receive by this theorem when κ is additionally θ-supercompact, then we need to add a restriction to the posets preserving near θ-supercompactness such as satisfying the θ-c.c. condition. Similarly, the $\theta^{<\theta} = \theta$ restriction translates the GCH penalty. Specifically, the substitute of supposing $2^{\theta^{<\kappa}} = \theta^+$ is the requirement that $\theta^{<\theta} = \theta$.

There is also a definitive restriction on the indestructibility we can obtain in general. If κ is a nearly θ-supercompact cardinal and we force over a partial order of size at most θ having a closure point below κ (i.e., forcing of size at most δ for some $\delta < \kappa$ followed by necessarily $\leq\delta$-strategically closed forcing), then it will not necessarily become indestructible by all $<\kappa$-directed closed forcing notions of size θ^+ in the extension. In fact, if $2^{\theta^{<\kappa}} = \theta^+$ and κ is not θ-supercompact, then there are $<\kappa$-directed closed posets of size θ^+ in every forcing extension over a poset of size at most θ having a closure point below κ that necessarily destroy the near θ-supercompactness of κ. Specifically, we have the following limitation:

Theorem 2.3.7. *Suppose κ is nearly θ-supercompact for some $\theta \geq \kappa$, and*

$\boldsymbol{V}[G]$ is a forcing extension over a poset of size at most θ having a closure point below κ, which includes any forcing extension over a lottery preparation of size at most θ such as the one from Theorem 2.3.4. Then if the near θ-supercompactness of κ is indestructible by a $<\kappa$-strategically closed forcing notion in $\boldsymbol{V}[G]$ collapsing $(2^{\theta^{<\kappa}})^{\boldsymbol{V}}$ to κ, then κ was θ-supercompact in $\boldsymbol{V}$. In particular, if $2^{\theta^{<\kappa}} = \theta^+$ and κ is not θ-supercompact in $\boldsymbol{V}$, then the near θ-supercompactness of κ cannot be made indestructible by all $<\kappa$-directed closed forcing notions of size θ^+ in an extension over a forcing notion having size at most θ and a closure point below κ.

Proof. This follows Theorem 28 from [12], which shows that if $\mathbf{V}[G][g]$ is a forcing extension satisfying the δ^+ approximation and cover properties for some $\delta^+ < \kappa$ and collapsing $(2^{\theta^{<\kappa}})^{\mathbf{V}}$ to have size κ, then κ cannot be weakly compact in $\mathbf{V}[G][g]$ without κ having been θ-supercompact in $\mathbf{V}$. Specifically, if $\mathbb{P}$ is a forcing notion having a closure point at δ for some $\delta < \kappa$, then $\mathbb{P} * \dot{\mathbb{Q}}$ will also have a closure point at this δ below κ if $\dot{\mathbb{Q}}$ is necessarily $<\kappa$-strategically closed. But by Lemma 13 of [12], forcing having a closure point at δ satisfies the δ^+ approximation and cover properties. Then since $\delta^+ < \kappa$ by virtue of $\delta < \kappa$ and κ being inaccessible, it follows that $\mathbb{P}*\dot{\mathbb{Q}}$ has the δ^+ approximation and cover properties provided that $\dot{\mathbb{Q}}$ is forced by all conditions in $\mathbb{P}$ to be $<\kappa$-strategically closed. Consequently, if $\dot{\mathbb{Q}}_G$ also collapses $(2^{\theta^{<\kappa}})^{\mathbf{V}}$

to have size κ and κ remains nearly θ-supercompact in $\mathbf{V}[G][g]$ for some g that's $\mathbf{V}[G]$-generic over $\dot{\mathbb{Q}}_G$, then κ must have been θ-supercompact in $\mathbf{V}$. Note here that if κ is nearly θ-supercompact for some $\theta \geq \kappa$, the cardinal κ will automatically be weakly compact. But by forthcoming Theorem 2.4.5, κ can be nearly θ-supercompact without being θ-supercompact. Also, forcing of size θ preserves θ^+, and if $2^{\theta^{<\kappa}} = \theta^+$, then it will also preserve the GCH at θ so that the collapse forcing of the extension $\mathrm{Coll}(\kappa, (2^{\theta^{<\kappa}})^{\mathbf{V}})^{\mathbf{V}[G]}$ will have size $(\theta^+)^{<\kappa} = \theta^+$ in $\mathbf{V}[G]$. But this forcing, which is also $<\kappa$-directed closed in $\mathbf{V}[G]$, collapses $(2^{\theta^{<\kappa}})^{\mathbf{V}}$ to have size κ. Consequently, if $2^{\theta^{<\kappa}} = \theta^+$ and κ is not θ-supercompact, then we cannot make the near θ-supercompactness of κ indestructible by all forcing notions of size θ^+ by first forcing over some $\leq\theta$-sized poset having a closure point below κ. $\square$

Nevertheless, provided that κ is nearly θ-supercompact for some $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$, we can still force to an extension where the near θ-supercompactness of κ is preserved by any $<\kappa$-directed closed forcing notion collapsing θ to have size κ. In fact, the near θ-supercompactness of κ will be indestructible by any such forcing notion in the forcing extension constructed in Theorem 2.3.4 where the near θ-supercompactness of κ was made to be indestructible by any $<\kappa$-directed closed θ-c.c. forcing notion of size at most

θ. Thus, the class of forcing notions for which the near θ-supercompactness of κ becomes indestructible is actually greater than what was stated. In particular, it includes the collapse forcing $\mathrm{Coll}(\kappa, \theta)$ of the extension, which has size θ but will not be θ-c.c.

Theorem 2.3.8. *If κ is nearly θ-supercompact for some $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$, then in the forcing extension $\mathbf{V}[G]$ from Theorem 2.3.4, the near θ-supercompactness of κ is also indestructible by any $<\kappa$-directed closed forcing of size θ collapsing θ to have size κ. In particular, the near θ-supercompactness of κ is indestructible by the collapse forcing $\mathrm{Coll}(\kappa, \theta)^{\mathbf{V}[G]}$.*

Proof. The only places where we used the fact that $\dot{\mathbb{Q}}$ was necessarily θ-c.c. were in the justifications that $M[G][g]$ and $N[G][g]$ would be $<\theta$-closed in $\mathbf{V}[G][g]$ where g was a $\mathbf{V}[G]$-generic filter for $\mathbb{Q}$. But since M and N are both $<\theta$-closed in $\mathbf{V}$ and $\mathbb{P}$ is κ-c.c. with $\kappa \leq \theta$, we still know that $M[G]$ and $N[G]$ are $<\theta$-closed in $\mathbf{V}[G]$. Then since $\mathbb{Q}$ is $<\kappa$-directed closed in $\mathbf{V}[G]$, we know that $M[G][g]$ and $N[G][g]$ will still be closed under $<\kappa$ sequences in $\mathbf{V}[G][g]$ even without $\mathbb{Q}$ being θ-c.c. If θ is collapsed to have size κ in $\mathbf{V}[G][g]$, then $N[G][g]$, which is a forcing extension of a set of size θ in $\mathbf{V}$, will only have size κ in $\mathbf{V}[G][g]$. In this case, $N[G][g]$ will have at most κ many dense sets for $\mathbb{P}_{\mathrm{tail}}$ as counted in $\mathbf{V}[G][g]$ so its $<\kappa$-closure in $\mathbf{V}[G][g]$

is sufficient for applying the Diagonalization Criterion in $\mathbf{V}[G][g]$ to find an $N[G][g]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$ in $\mathbf{V}[G][g]$. This then allows us to form the partial lift $j : M[G] \to N[j(G)]$ in $\mathbf{V}[G][g]$. Also, it allows us to show that $N[j(G)] = N[G][g][G_{\text{tail}}]$ will be $<\kappa$-closed in $\mathbf{V}[G][g]$ because $N[G][g]$ is $<\kappa$-closed in $\mathbf{V}[G][g]$ and $G_{\text{tail}} \in V[G][g]$. But then we can similarly argue for the $N[j(G)]$-generic filter g^* for $j(\mathbb{Q})$ in $\mathbf{V}[G][g]$ containing the condition q_* below all conditions in $j''g$ as $N[j(G)]$ will also have size κ in $\mathbf{V}[G][g]$. Therefore, we will be able to fully lift the embedding to $j : M[G][g] \to N[j(G)][j(g)]$ as in the proof. Since $M[G][g]$ is also closed under $<\kappa$-sequences in $\mathbf{V}[G][g]$ with the subset $A \in M[G][g]$, the lifted near θ-supercompactness embedding will witness the near θ-supercompactness of κ for A. Since A was an arbitrary subset of θ in $\mathbf{V}[G][g]$, the cardinal κ remains nearly θ-supercompact in $\mathbf{V}[G][g]$, as desired. Finally, note that because $\mathbb{P}$ is a κ-c.c. poset of size κ and $\theta^{<\kappa} = \theta$ in $\mathbf{V}$ for some $\theta \geq \kappa$, we will have $\theta^{<\kappa} = \theta$ holding in $\mathbf{V}[G]$ as well. Therefore, $\text{Coll}(\kappa, \theta)^{\mathbf{V}[G]}$ is a $<\kappa$-directed closed poset having size θ in $\mathbf{V}[G]$ that collapses θ to have size κ, thereby satisfying the aforementioned criterion for indestructibility. $\square$

As a corollary, we have the following two lemmas showing the high consistency strength of nearly θ-supercompact cardinals κ for $\theta \geq \kappa^+$ for which

$\theta^{<\theta} = \theta$.

Corollary 2.3.9. *If κ is nearly θ-supercompact for some $\theta \geq \kappa$ for which $\theta^{<\theta} = \theta$, then there exists a countably closed forcing extension $\boldsymbol{V}[G]$ preserving all cardinals and cofinalities above κ where κ is weakly compact and its weak compactness is indestructible by the collapse forcing Coll$(\kappa, \theta)^{\boldsymbol{V}[G]}$.*

Proof. The forcing poset $\mathbb{P}$ from the lottery preparation from Theorem 2.3.4 is an Easton support iteration forcing nontrivially only at specific inaccessible stages γ with $<\gamma$-directed closed forcing in $\mathbf{V}^{\mathbb{P}_\gamma}$ so it is countably closed (and actually $<\lambda$-closed for the least inaccessible cardinal λ). Since $\mathbb{P}$ has size κ, it also preserves all cardinals and cofinalities above κ. By Theorem 2.3.8, the near θ-supercompactness of κ is indestructible by $\mathrm{Coll}(\kappa, \theta)^{\mathbf{V}[G]}$ where G is $\mathbf{V}$-generic for $\mathbb{P}$. But if κ is nearly θ-supercompact for some $\theta \geq \kappa$, then it is weakly compact. Therefore, $\mathbf{V}[G]$ is a countably closed forcing extension preserving all cardinals and cofinalities above κ where κ is weakly compact and its weak compactness is indestructible by the collapse forcing $\mathrm{Coll}(\kappa, \theta)^{\mathbf{V}[G]}$, as desired. □

Corollary 2.3.10. *If κ is nearly θ-supercompact for some $\theta \geq \kappa^+$ for which $\theta^{<\theta} = \theta$, then AD holds in $\boldsymbol{L}(\mathbb{R})$. In particular, if κ is nearly κ^+-supercompact and $2^\kappa = \kappa^+$, then AD holds in $\boldsymbol{L}(\mathbb{R})$.*

Proof. By Corollary 2.3.9, there exists a countably closed forcing extension $\mathbf{V}[G]$ preserving all cardinals above κ where κ is weakly compact and its weak compactness is indestructible by the collapse forcing $\mathrm{Coll}(\kappa, \theta)^{\mathbf{V}[G]}$. But $\mathbf{V}[G]$ preserves cardinals above κ and $\theta \geq \kappa^+$ so in $\mathbf{V}[G]$, the forcing $\mathrm{Coll}(\kappa, \theta)^{\mathbf{V}[G]}$ is a $<\kappa$-closed poset collapsing κ^+. Consequently, in $\mathbf{V}[G]$, Jensen's Square Principle $\square_\kappa$ fails at a weakly compact cardinal κ, which implies that AD holds in $\mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$. [16] But $\mathbf{V}[G]$ is a countably closed forcing extension so it agrees with $\mathbf{V}$ on the set of Reals. Consequently, $\mathbf{L}(\mathbb{R}) = \mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$ so the axiom of determinacy holds in $\mathbf{L}(\mathbb{R})$. $\square$

Although we showed that a nearly θ-supercompact cardinal κ cannot be made indestructible by all forcing notions of size θ^+ in general after preparatory forcing resembling the lottery preparation, the question of whether we can remove the θ-c.c. condition remains open.

Question 2.3.11. *If κ is nearly θ-supercompact for some $\theta \geq \kappa$ for which $\theta^{<\theta} = \theta$, does there exist a forcing extension preserving its near θ-supercompactness and all cardinals above κ where the near θ-supercompactness of κ becomes indestructible by all $<\kappa$-directed closed forcing notions of size at most θ?*

We now close this section with a brief discussion about the preservation of partially supercompact and nearly supercompact cardinals, and questions about consistency strength. In the following section, we will be taking care not to collapse cardinals from κ up to and including θ when asserting results about what can happen to nearly θ-supercompact cardinals κ in various forcing extensions. This is because the consistency strength of a nearly θ-supercompact cardinal κ is very much connected to the relationship between κ and θ. Indeed, a number of the later separation theorems would become easily proven and of little consequence without these assumptions of preservation of cardinals. However, we do change the continuum function, and this is where one of the primary open questions arises.

Question 2.3.12. *For which θ are the theories ZFC + "κ is nearly θ-supercompact" equiconsistent with ZFC + GCH + "κ is nearly θ-supercompact"?*

For a number of results, we want to force the GCH while preserving a degree of partial (near) supercompactness. For one thing, most of the separation theorems involving nearly θ-supercompact cardinals mentioned later will yield cardinal-preserving forcing extensions in the presence of the GCH. More importantly, we often need to at least have some form of the GCH for theorems to be applicable. For example, the GCH holding at $\theta^{<\kappa}$ is required in order to appeal to the the Level by Level Preparation Theorem 4.8 of [10]. In order to appeal to the indestructibility result of Theorem 2.3.4 for nearly θ-supercompact cardinals κ, we require that $\theta^{<\theta} = \theta$. This is more of a hinderance for the nearly θ-supercompact cardinals as forcing the “internal” GCH below the degree of (near) supercompactness does not seem so easy without either appealing to larger degrees of near supercompactness or destroying degrees of (near) supercompactness. Forcing “external” GCH at or above the degree of (near) supercompactness does not require one of these alternatives. The question of self-contained indestructibility results resting on “internal” or “external” GCH considerations exhibit the similarities between these two large cardinal notions, and begs the following related question:

Question 2.3.13. *What is the relationship between the consistency strength of ZFC + “κ is nearly θ-supercompact” and ZFC + “κ is λ-supercompact” when $2^{\lambda^{<\kappa}} > \theta > \lambda$ for various λ and θ?*

Because we also show that relative to increasingly sufficient amounts of partial near supercompactness that it is possible to get arbitrarily large degrees of near supercompactness without even having measurability for a cardinal κ so long as we pump up 2^κ accordingly, we also have the following open question:

Question 2.3.14. *Does any degree of near θ-supercompactness below 2^κ for a nearly θ-supercompact cardinal κ have the consistency strength of a 2^κ-supercompact cardinal κ?*

Note that if a cardinal κ is nearly κ^+-supercompact, then it is weakly measurable. Consequently, by the results of Chapter 1 and [25], the consistency strength of such a cardinal κ is at least that of a measurable cardinal. Of course, the other question related to this is the following:

Question 2.3.15. *If κ is supercompact and $\theta > \kappa$, does there exist a forcing extension preserving all cardinals above κ where κ is simultaneously the least weakly compact and least nearly θ-supercompact cardinal?*

The final question is to what degree full θ-supercompactness is a consistency strengthening of the near θ-supercompactness of a cardinal κ when $\theta \gg \kappa$ becomes too large to get the reflection scenario as in Theorem 2.4.4. Of course, whenever θ is an inaccessible cardinal above κ, nearly θ-supercompact and θ-supercompact cardinals carry the same consistency strength: that of a fully supercompact cardinal.

Question 2.3.16. *For which θ are the notions of near θ-supercompactness and θ-supercompactness for a cardinal κ equiconsistent?*

2.4 Separating the Hierarchy

We now look to separate the nearly θ-supercompact cardinals from the rest of the large cardinal hierarchy. In this section, I will prove a number of such separations showing just how different these types of large cardinals can be. For starters, every θ-supercompact cardinal is nearly θ-supercompact but not necessarily nearly θ^+-supercompact. Moreover, whenever κ is nearly θ^+-supercompact for some θ for which $2^{\theta^{<\kappa}} = \theta^+$, the cardinal κ will be θ-supercompact but not necessarily θ^+-supercompact. More generally, every nearly $2^{\theta^{<\kappa}}$-supercompact cardinal κ is θ-supercompact but not necessarily $2^{\theta^{<\kappa}}$-supercompact. A nearly θ-supercompact cardinal need not be strongly unfoldable, and from a supercompact cardinal, we can accomplish this separation without collapsing cardinals (or cofinalities) when the GCH holds. Moreover, for all θ, we can even force to make a supercompact cardinal κ (or a nearly θ-supercompact cardinal κ for some $\theta \geq 2^\kappa$ for which $\theta^{<\theta} = \theta$) nonmeasurable while retaining its near θ-supercompactness without collapsing any cardinals above κ! Also, if κ is nearly θ-supercompact for some $\theta \geq 2^\kappa$ for which $\theta^{<\theta} = \theta$, then there will be a forcing extension preserving all cardinals and cofinalities above κ where κ is not measurable but is nearly λ-supercompact for all λ less than the cardinality of the powerset of κ in the extension. In particular, this result generalizes the possible separation between weakly measurable (κ^+-nearly measurable) and measurable cardinals shown in Chapter 1 and [25].

We begin this discussion by showing how for certain θ, nearly $2^{\theta^{<\kappa}}$-supercompact cardinals κ can have greater consistency strength than θ-supercompact cardinals in a certain sense. For specific instances of what is proven below, a nearly 2^κ-supercompact cardinal κ is a limit of measurable cardinals, and a nearly 2^{κ^+}-supercompact cardinal κ is a limit of λ^+-supercompact cardinals λ. (Note that $\kappa^{<\kappa} = \kappa$ and $(\kappa^+)^{<\kappa} = \kappa^+$ when κ is nearly θ-supercompact for some $\theta \geq \kappa$ by its inaccessibility.) More generally, the below theorem informally says that if κ is nearly $2^{\theta^{<\kappa}}$-supercompact for some θ definable from κ and parameters from $\mathbf{V}_\kappa$ in every θ-closed transitive ZFC$^-$ model via the same formula, then κ is a limit of β-supercompact cardinals α where the relationship between α and β is the same definable one that exists between κ and θ.

Theorem 2.4.1. *Suppose that κ is nearly $2^{\theta^{<\kappa}}$-supercompact for some $\kappa \leq \theta$. Also let $F : ORD \to ORD$ be a class function definable from parameters in $\boldsymbol{V}_\kappa$ such that $F(\kappa) = \theta$ and $F \upharpoonright (\kappa + 1)$ is absolute to every transitive ZFC$^-$ model closed under θ sequences. Then κ is a limit of $F(\alpha)$-supercompact cardinals α for $\alpha < \kappa$.*

Proof. If κ is nearly $2^{\theta^{<\kappa}}$-supercompact, then κ is θ-supercompact so we may find a normal fine κ-complete measure μ on $P_\kappa\theta$. Encoding μ as $A \subseteq 2^{\theta^{<\kappa}}$, we may by the near $2^{\theta^{<\kappa}}$-supercompactness of κ, find θ-closed transitive ZFC$^-$ models M and N with $A, 2^{\theta^{<\kappa}} \in M$ and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''2^{\theta^{<\kappa}} \in N$. Then N will have A and hence μ so it will know that κ is θ-supercompact.

Also, because N is a θ-closed model of ZFC$^-$, $N \models F(\kappa) = \theta$. Consequently, since $\kappa < j(\kappa)$ and $\theta < j(\kappa)$, it follows that for all $\gamma < \kappa$, we have $N \models \exists\alpha \in (\gamma, j(\kappa))[F(\alpha) < j(\kappa) \wedge$ "α is $F(\alpha)$-supercompact."]. Then, by elementarity and the fact that all parameters in $\mathbf{V}_\kappa$ are fixed, the model $M \models \exists\alpha \in (\gamma, \kappa)[F(\alpha) < \kappa \wedge$ "α is $F(\alpha)$-supercompact."] for all $\gamma < \kappa$. Because M is closed under $<\kappa$ sequences, it will be correct whenever it thinks that a given $\alpha < \kappa$ is β-supercompact for some $\beta < \kappa$. Then because M is closed under θ sequences, it will also be correct about the value of $F(\alpha)$ for all $\alpha < \kappa$ by the absoluteness of $F \upharpoonright \kappa$ to such models. Therefore, $\mathbf{V} \models \exists\alpha \in (\gamma, \kappa)[$"$\alpha$ is $F(\alpha)$-supercompact."] for all $\gamma < \kappa$, as desired. □

This theorem provides us with a formally proven indication that nearly $2^{\theta^{<\kappa}}$-supercompact cardinals can be different than θ-supercompact cardinals and are in a sense stronger. For example, the least nearly 2^κ-supercompact cardinal κ can never be the least measurable cardinal nor can the least 2^{κ^+}-supercompact cardinal κ ever be the least λ^+-supercompact cardinal λ because they both must be limits of these types of cardinals respectively. To demonstrate a definitive and completely general separation however, we will prove a more direct theorem.

However, before doing this, recall from the proof of the Lévy-Solovay Theorem 2.1.6 for nearly θ-supercompact cardinals that $\mathrm{ZFC}^* = \mathrm{ZFC} \cap \Sigma_{100}$ denoted the sufficiently large fixed fragment of ZFC referenced in [12] for the Main Theorem to be applicable. Before the proof, we noted that we could have assumed that M were a ZFC^* model instead of a ZFC^- model in the normal embedding characterization of the near θ-supercompactness of κ. In the below proof of the Main Theorem, Statement 2, we will again use the ZFC^* replacement of the normal embedding characterization.

Theorem 2.4.2. *If κ is a θ-supercompact cardinal for some θ such that $\theta^{<\kappa} = \theta$, then there exists a forcing extension where κ remains θ-supercompact but is not nearly θ^+-supercompact. Moreover, this forcing extension will preserve the cofinalities of all ordinals with cofinalities below θ^{++} or above 2^θ.*

Proof. Let $\mathbb{P} = \mathrm{Add}(\omega, 1)$ and $\lambda = \theta^+$. Since $\mathbb{P}$ is tiny, it won't collapse any cardinals, affect 2^θ, or change the fact that $\theta^{<\kappa} = \theta$. Also, for every set in $\mathbf{V}^{\mathbb{P}}$, we will be able to find a $\mathbb{P}$-name for it having the same hereditary size in $\mathbf{V}$ that the corresponding set named by it has in $\mathbf{V}^{\mathbb{P}}$. In particular, $\mathrm{Add}(\lambda, 1)^{\mathbf{V}^{\mathbb{P}}}$ will have size 2^θ in $\mathbf{V}^{\mathbb{P}}$, and we will be able to find a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for it in $\mathbf{H}_{(2^\theta)^+}$. Now force with $\mathbb{P} * \dot{\mathbb{Q}} = \mathrm{Add}(\omega, 1) * \mathrm{Add}\dot{(\lambda, 1)}$, and let y be a $\mathbf{V}$-generic Cohen real and A be a $\mathbf{V}[y]$-generic Cohen subset of λ. Since the first stage of forcing is small with respect to κ, and the second stage is $\leq\theta^{<\kappa}$-distributive in $\mathbf{V}[y]$, the cardinal κ remains θ-supercompact in the extension $\mathbf{V}[y][A]$, which we'll denote $\overline{\mathbf{V}}$. Note that the forcing extension preserves the cofinalities of all ordinals with cofinalities below θ^{++} or above 2^θ and that it forces the GCH to hold at θ. In particular, $\dot{\mathbb{Q}}$ will have hereditary size λ in $\overline{\mathbf{V}}$. Furthermore, since $\mathbb{P} * \dot{\mathbb{Q}}$ admits a closure point at ω, it follows by Lemma 13 of [12] that the ω_1 approximation and cover properties are satisfied by $\mathbf{V} \subseteq \mathbf{V}[y][A]$.

Now suppose for contradiction that κ is nearly θ^+-supercompact (nearly 2^θ-supercompact) in $\overline{\mathbf{V}} = \mathbf{V}[y][A]$. By the Lévy Reflection Theorem, let η be sufficiently large so that every ZFC* statement and "$\overline{\mathbf{V}}$ is a forcing extension of the ZFC* model $\mathbf{V}$ by y followed by A" reflects down from the structure $\langle \overline{\mathbf{V}}, \mathbf{V}, \in \rangle$ to $\langle \overline{\mathbf{V}}_\eta, \mathbf{V}_\eta, \in \rangle$. More formally, we let η be sufficiently large so that $\mathbb{P} * \dot{\mathbb{Q}}$, the Cohen real y, and the subset A are all elements of $\overline{\mathbf{V}}_\eta$ and that the finite set of statements from ZFC* $+ \forall s \exists \sigma \in {\mathbf{V}_\eta}^{\mathbb{P}*\dot{\mathbb{Q}}}(\sigma_{y*A} = s)$ are true in the structure $\langle \overline{\mathbf{V}}_\eta, \mathbf{V}_\eta, \in \rangle$. In particular, $\overline{\mathbf{V}}_\eta$ and $\mathbf{V}_\eta$ will be ZFC* models, and $\overline{\mathbf{V}}_\eta$ will be a forcing extension of $\mathbf{V}_\eta$, which equals $\mathbf{V} \cap \overline{\mathbf{V}}_\eta$, by $y * A$. Now in $\overline{\mathbf{V}}$, let $\overline{X}$ be a θ-closed elementary substructure of $\overline{\mathbf{V}}_\eta$ in the language with a predicate for $\mathbf{V}$ of size λ having A and $\mathbb{P} * \dot{\mathbb{Q}}$ as elements and $\lambda + 1$ as a subset. In particular, we have $\langle \overline{X}, X, \in \rangle \prec \langle \overline{\mathbf{V}}_\eta, \mathbf{V}_\eta, \in \rangle$ where $X = \overline{X} \cap \mathbf{V}$. Letting $\overline{M}$ and M be the Mostowski Collapses of $\overline{X}$ and X, respectively, it then follows by Lemma 15 of [12] that $X \in \mathbf{V}$ and $M = \overline{M} \cap \mathbf{V}$. In particular, M will be a model of ZFC* in $\mathbf{V}$ since $X \in \mathbf{V}$ is a model of ZFC*, and M is its elementarily equivalent Mostowski collapse. Note that $\overline{M}$ will be closed under θ sequences in $\overline{\mathbf{V}}$ because $\overline{X}$ is θ-closed in $\overline{\mathbf{V}}$. Also since $\lambda + 1 \subseteq \overline{X}$, and $\overline{X}$ is a ZFC* model agreeing with $\mathbf{V}$ that there are subsets of λ coding $\mathbb{P} * \dot{\mathbb{Q}}$ and $y * A$ that can be decoded via functions definable only from λ in any ZFC* model (and being correct about such subsets), it follows that $\mathbb{P} * \dot{\mathbb{Q}}$ and $y * A$ will both be fixed by the Mostowski Collapse. Consequently, because $\overline{X} \prec \overline{\mathbf{V}}_\eta$ and the isomorphism induced by the Mostowski Collapse of $\overline{X}$ is $\overline{M}$, we have $\overline{M} = M[y][A]$ by virtue of $\overline{\mathbf{V}}_\eta$ knowing that it's a forcing extension over $\mathbf{V}_\eta$ by $y * A \subseteq \mathbb{P} * \dot{\mathbb{Q}}$.

Now since in $\overline{\mathbf{V}}$, the set $\overline{M}$ is the Mostowski Collapse of a ZFC* model of size λ, the set $\overline{M}$ will be a ZFC* model having size λ in $\overline{\mathbf{V}}$ as well. Furthermore, since $\overline{M}^{\theta} \subseteq \overline{M}$ and λ is an element in $\overline{M}$, it follows by the normal embedding characterization of the near λ-supercompactness of κ in $\overline{\mathbf{V}}$ that there exists a θ-closed transitive $\overline{N}$ of size λ containing $\mathcal{P}(\lambda)^{\overline{M}}$ and a cofinal elementary embedding $j : \overline{M} \to \overline{N}$ with critical point κ such that $j(\kappa) > \lambda$. Consequently, since $\overline{M} = M[y][A]$, which contains $A \subseteq \lambda$, we have the equality $\overline{N} = N[y][j(A)]$, and $\overline{N}$ will contain A. But because the forcing to add $j(A)$ is $\leq j(\lambda)$-closed in $N[y]$ and $j(\lambda) \geq j(\kappa) > \lambda$, no subset of λ could have been added to $N[y]$. Therefore $A \in N[y]$. We can now show that this situation is impossible by showing that the Main Theorem of [12] implies that $N \subseteq \mathbf{V}$ and hence that the $\mathbf{V}[y]$-generic object $A \in N[y] \subseteq \mathbf{V}[y]$.

To do this, we merely summarize the results showing that the requirements of the Main Theorem are met so that $N \subseteq \mathbf{V}$ follows from Statement (2). We have $\mathbf{V} \subseteq \overline{\mathbf{V}}$ satisfying the ω_1 approximation and cover properties with ω_1 being a regular cardinal below the critical point of j, which is κ. We also know that $\overline{M}$, which is the Mostowski collapse of the elementary substructure $\overline{X}$ of $\overline{\mathbf{V}}_{\eta}$, is surely a transitive ZFC* submodel of $\overline{\mathbf{V}}$. The set $M = \overline{M} \cap \mathbf{V}$ is a ZFC* model, the elementary embedding $j : \overline{M} \to \overline{N}$ is a cofinal one from $\overline{M}$ into a transitive class $\overline{N} \subseteq \overline{\mathbf{V}}$. Also, $\overline{M}$ and $\overline{N}$ are both sufficiently closed in $\overline{\mathbf{V}}$ (both closed under θ sequences) so that $\mathcal{P}(\omega_1)^{\overline{\mathbf{V}}} \subseteq \overline{M}$, the model $\overline{M}$ will be closed under $<\omega_1$ sequences in $\overline{\mathbf{V}}$, and $\overline{N}^{\omega_1} \subseteq \overline{N}$ in $\overline{\mathbf{V}}$

as well. $\square$

When the GCH holds at θ, the aforementioned theorem translates to the following corollary.

Corollary 2.4.3. *If κ is a θ-supercompact cardinal for some θ such that $2^{\theta^{<\kappa}} = \theta^+$, then there exists a cofinality-preserving forcing extension where κ remains θ-supercompact but is not nearly θ^+-supercompact.*

We now turn the discussion toward showing that nearly θ-supercompact cardinals need not be fully θ-supercompact. First, analogous to Theorem 2.4.1, we demonstrate how for certain θ, the θ-supercompact cardinals can actually have greater consistency strength than nearly θ-supercompact cardinals in a certain sense. For specific instances of what is proven below, a κ^{++}-supercompact cardinal κ is a limit of nearly γ^{++}-supercompact cardinals γ, and a 2^κ-supercompact cardinal κ is a limit of nearly 2^γ-supercompact cardinals γ. Another more familiar example includes a measurable (κ-supercompact) cardinal κ being a limit of weakly compact cardinals γ (nearly γ-supercompact cardinals γ). Analogous to Theorem 2.4.1, the below theorem informally says that if κ is θ-supercompact for some θ definable from κ and parameters from $\mathbf{V}_\kappa$ in every θ-closed inner model via the same formula, then κ is actually a limit of nearly β-supercompact cardinals α where the relationship between α and β is the same definable one that exists between κ and θ.

Theorem 2.4.4. *Suppose that κ is θ-supercompact for some $\kappa \leq \theta$. Also let $F : ORD \to ORD$ be a class function definable from parameters in $\boldsymbol{V}_\kappa$ such that $F(\kappa) = \theta$ and $F(\kappa)$ is absolute to every inner model of ZFC closed under θ sequences. Then κ is a limit of nearly $F(\alpha)$-supercompact cardinals α for $\alpha < \kappa$.*

Proof. Suppose κ is θ-supercompact, and fix any θ-supercompactness embedding $j : \mathbf{V} \to N$ generated by a normal fine κ-complete measure on $P_\kappa\theta$. This embedding will be a $\theta^{<\kappa}$-supercompactness embedding. Consequently, for every transitive $<\kappa$-closed set $M \in \mathbf{H}_{(\theta^{<\kappa})^+}$ with $\theta^{<\kappa} \in M$, the near $\theta^{<\kappa}$-supercompactness embedding $j \restriction M : M \to j(M)$ will be in N by virtue of N being closed under $\theta^{<\kappa}$ sequences. Therefore, κ will be nearly $\theta^{<\kappa}$-supercompact in N and hence nearly θ-supercompact in N.

Then because N is an inner model of ZFC closed under θ sequences, $N \models F(\kappa) = \theta$. Consequently, since $\kappa < j(\kappa)$ and $\theta < j(\kappa)$, it follows that the model $N \models \exists\alpha \in (\gamma, j(\kappa))$[“$\alpha$ is nearly $F(\alpha)$-supercompact.”] for all $\gamma < \kappa$. Then, by elementarity and the fact that all parameters in $\mathbf{V}_\kappa$ are fixed, we know that $\mathbf{V} \models \exists\alpha \in (\gamma, \kappa)$[“$\alpha$ is nearly $F(\alpha)$-supercompact.”] for all $\gamma < \kappa$, as desired. □

This result provides us with a formally proven indication that fully θ-supercompact cardinals can be different than nearly θ-supercompact cardinals and are in a sense stronger. For example, the least measurable (κ-supercompact) cardinal κ can never be the least weakly compact cardinal nor can the least 2^κ-supercompact cardinal κ ever be the least nearly 2^κ-supercompact cardinal because they both must be limits of these types of cardinals respectively. To demonstrate a definitive separation, we prove the following theorem.

Theorem 2.4.5. *If κ is nearly θ-supercompact and $\theta^{<\kappa} = \theta$, then there exists a θ-closed inner model preserving this fact where κ is not θ-supercompact. In particular, this inner model will be correct about all cardinals below θ^{++}.*

Proof. Suppose κ is nearly θ-supercompact and $\theta^{<\kappa} = \theta$. If κ is not θ-supercompact in $\mathbf{V}$, then $\mathbf{V}$ is such an inner model. Otherwise, if κ is θ-supercompact, fix a θ-supercompactness ultrapower embedding $j : \mathbf{V} \to N$ where κ is not θ-supercompact in N. Because N is closed under θ sequences, it will have all witnessing embeddings of hereditary size θ to the fact that κ is nearly θ-supercompact. Therefore, N is an inner model closed under θ-sequences where κ is nearly θ-supercompact but not θ-supercompact. □

It is worth noticing the discrepancy in the proof techniques. To show that a θ-supercompact cardinal κ need not be nearly θ^+-supercompact for some θ such that $\theta^{<\kappa} = \theta$, we moved to an outer model forcing extension preserving all cardinals below θ^{++}. For the above proof where we showed that a nearly θ-supercompact cardinal κ need not be θ-supercompact for some θ such that $\theta^{<\kappa} = \theta$, we moved to a θ-closed inner model, which must agree with $\mathbf{V}$ on all cardinals below θ^{++}. From a nearly $2^{\theta^{<\kappa}}$-supercompact cardinal κ, we can find a near $2^{\theta^{<\kappa}}$-supercompactness embedding $j : M \to N$ with critical point κ between $\theta^{<\kappa}$-closed transitive ZFC models where M and hence N can have a normal fine κ-complete measure on $P_\kappa\theta$, and yet, κ will not be nearly $2^{\theta^{<\kappa}}$-supercompact in N. Therefore, we could have used a similar argument to show that there are transitive $\theta^{<\kappa}$-closed ZFC models N where κ is θ-supercompact but not nearly $2^{\theta^{<\kappa}}$-supercompact. But such a proof would not necessarily readily yield a transitive model with the same ordinals as $\mathbf{V}$ where κ is θ-supercompact but not nearly $2^{\theta^{<\kappa}}$-supercompact and so it does not seem as satisfying. We can also show that a nearly θ-supercompact cardinal need not be θ-supercompact in a forcing extension but without some GCH considerations, the forcing may collapse some cardinals between κ and θ, destroying the important relationship between κ and θ for the θ-supercompact cardinals κ. Also, we may need to appeal to notions with greater consistency strength.

Nevertheless, we would like to generalize the ideas from the first three statements of the Main Theorem from Chapter 1 and [25] to the nearly θ-supercompact cardinals. Rather than having the GCH fail first at a nearly θ^+-supercompact cardinal κ, we make it fail first at θ and yet still preserve the near θ^+-supercompactness of κ. It follows that such a cardinal necessarily cannot remain θ-supercompact so we will have actually achieved the separation of having a nearly θ^+-supercompact cardinal not be θ-supercompact in a forcing extension. This is analogous to the case where we made the GCH fail first at a weakly measurable cardinal κ so that κ could not remain measurable. We begin by introducing a generalized analogue of the lemma used in the proof of Statement (3) of the Main Theorem from Chapter 1 and [25].

Lemma 2.4.6. *Suppose κ is a nearly θ-supercompact cardinal for some $\theta \geq \kappa$ and $\kappa \leq 2^{<\sigma} \leq \theta$ for some cardinal σ. Then if the near θ-supercompactness of κ is indestructible by the forcing to add α many Cohen subsets of σ for all $\alpha \leq \theta$, then its near θ-supercompactness is indestructible by the forcing $Add(\sigma, \eta)$ for all η.*

Proof. Suppose κ is a nearly θ-supercompact cardinal and that its near θ-supercompactness is indestructible by the forcing to add α many Cohen subsets of σ for every $\alpha \leq \theta$ where σ is some cardinal at or above above κ such that $2^{<\sigma} \leq \theta$. Since the indestructibility result is assumed to be true for $\eta \leq \theta$, we may assume that $\eta > \theta$. Now let $G \subseteq \mathrm{Add}(\sigma, \eta)$ be $\mathbf{V}$-generic and $A \in \mathbf{V}[G]$ be a subset of θ. Since $\mathrm{Add}(\sigma, \eta)$ is a $(2^{<\sigma})^{+}$-c.c. (and hence a θ^{+}-c.c.) poset with all conditions having size less than σ for $\sigma \leq \theta$, there exists a nice name $\dot{A}$ for A with domain $\check{\theta}$ referencing at most θ many columns $C \subseteq \eta$. We may therefore apply an automorphism to $\mathrm{Add}(\sigma, \eta)$ so that the columns of the partial order are reordered in such a way that the induced name for $\dot{A}$ is an $\mathrm{Add}(\sigma, \theta)$-name. Associating G with the induced generic filter under this automorphism, $G_\theta = G \cap \mathrm{Add}(\sigma, \theta) \subseteq \mathrm{Add}(\sigma, \theta)$ will be $\mathbf{V}$-generic. Then associating $\dot{A}$ with its induced name under the automorphism, $\dot{A} \in \mathbf{H}_{\theta^+}$ so we will have $A = \dot{A}_G = \dot{A}_{G_\theta} \in \mathbf{H}_{\theta^+}^{\mathbf{V}[G_\theta]}$. Now because the near θ-supercompactness of κ is indestructible by the forcing $\mathrm{Add}(\sigma, \theta)$ by hypothesis, the cardinal κ is nearly θ-supercompact in $\mathbf{V}[G_\theta]$. Consequently, in $\mathbf{V}[G_\theta]$, there exists a transitive $<\kappa$-closed ZFC^- model M with $A, \theta \in M$, a transitive N, and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''\theta \in N$. This embedding will then be in $\mathbf{V}[G]$, and because the forcing to add Cohen subsets of σ is $<\sigma$-closed (and hence $<\kappa$-closed), no new $<\kappa$ sequences of M are added so that M remains $<\kappa$-closed in $\mathbf{V}[G]$. Therefore, this elementary embedding continues to be a near θ-supercompactness embedding in $\mathbf{V}[G]$. Since A was arbitrary, we can find such a near θ-supercompactness embedding $j : M \to N$ with critical point κ and $A \in M$, in $\mathbf{V}[G]$, for each of the $A \subseteq \theta$ in $\mathbf{V}[G]$, so κ is nearly θ-supercompact in $\mathbf{V}[G]$. $\square$

Since we will be starting with a fully supercompact cardinal, the proof of the analogue of Statement (3) of the Main Theorem from the first chapter and [25] rests on known standard techniques and therefore is relatively short.

Theorem 2.4.7. *Let κ and θ be regular cardinals such that $\kappa \leq \theta$. If κ is supercompact and the GCH holds, then there exists a cofinality-preserving (and hence cardinal-preserving) forcing extension where the GCH continues to hold, the supercompactness of κ is preserved, and the near θ^+-supercompactness of κ is indestructible by the further forcing $Add(\theta, \eta)$ for all η.*

Proof. Let $\mathbb{P}$ be the $(\theta + 1)$-stage Easton support iteration where we force nontrivially precisely at stages γ that are regular in $\mathbf{V}^{\mathbb{P}_\gamma}$ and do so with $\mathrm{Add}(\gamma, \gamma^+)^{\mathbf{V}^{\mathbb{P}_\gamma}}$. Now let $G \subseteq \mathbb{P}$ be $\mathbf{V}$-generic. By known standard results, κ remains supercompact in $\mathbf{V}[G]$, the GCH persists in $\mathbf{V}[G]$, and no cofinalities (or cardinals) are collapsed. In particular, θ remains regular in $\mathbf{V}[G]$. Consequently, the final stage θ of forcing is $\mathrm{Add}(\theta, \theta^+)^{\mathbf{V}^{\mathbb{P}_\theta}}$. Then because the forcing to add θ^+ Cohen subsets of θ is isomorphic to the forcing to add θ^+

such subsets followed by the forcing to add α such subsets for all $\alpha \leq \theta^+$, the supercompactness of κ and hence its near θ^+-supercompactness is indestructible by the further forcing $\mathrm{Add}(\theta, \alpha)^{\mathbf{V}[G]}$ for all $\alpha \leq \theta^+$ in $\mathbf{V}[G]$. Therefore, in this forcing extension, the near θ^+-supercompactness of κ is indestructible by $\mathrm{Add}(\theta, \eta)$ for all η by Lemma 2.4.6. $\square$

Note that although the aforementioned theorem requires the GCH to hold in order to be applicable, it is not necessary to appeal to any stronger properties of κ. Specifically, if κ is supercompact, then we can always first force the GCH to hold while retaining its supercompactness. This will not necessarily be true if we were to replace supercompact with (nearly) λ-supercompact for certain λ. Therefore, while we do not actually need κ to be fully supercompact in the above theorem in order to declare the existence of a forcing extension preserving the near θ^+-supercompactness of κ where the GCH holds, we may still need to appeal to higher degrees of supercompactness of κ than mere θ^+-supercompactness. Similar considerations are true, even in the presence of the GCH, in order for the desired degree of partial near supercompactness to be preserved after forcing with the Easton support iteration $\mathbb{P}$, which forces nontrivially precisely at every stage that was regular in $\mathbf{V}^{\mathbb{P}_\gamma}$ with $\mathrm{Add}(\gamma, \gamma^+)^{\mathbf{V}^{\mathbb{P}_\gamma}}$.

Theorem 2.4.7 allows us to prove a claim analogous to the Main Theorem Statement (2) of the first chapter and [25].

Corollary 2.4.8. *Let κ and θ be regular cardinals such that $\kappa \leq \theta$. If κ is supercompact and the GCH holds, then there exists a cofinality-preserving (and hence cardinal-preserving) forcing extension where κ is nearly θ^+-supercompact, and the GCH fails first at θ.*

Proof. Suppose κ is supercompact, θ is a regular cardinal greater than or equal to κ, and the GCH holds. By Theorem 2.4.7, we may move to the cofinality-preserving forcing extension $\mathbf{V}[G]$ where the GCH persists, κ remains supercompact, and the near θ^+-supercompactness of κ is indestructible by the forcing $\mathrm{Add}(\theta, \eta)$ for all η. Now in $\mathbf{V}[G]$, force with $\mathrm{Add}(\theta, \theta^{++})^{\mathbf{V}[G]}$, and move to a forcing extension $\overline{\mathbf{V}}$ over this poset. Then the near θ^+-supercompactness of κ continues to be preserved in $\overline{\mathbf{V}}$. Furthermore, since the forcing poset $\mathrm{Add}(\theta, \theta^{++})^{\mathbf{V}[G]}$ is $<\theta$-closed in $\mathbf{V}[G]$, the GCH continues to hold below θ. But this forcing poset adds θ^{++} many Cohen subsets of θ, thereby forcing the GCH to fail at θ in $\overline{\mathbf{V}}$. Finally, since the GCH held in $\mathbf{V}[G]$ (and in particular, $V[G] \models 2^{<\theta} = \theta$), no cofinalities were collapsed when moving from $\mathbf{V}[G]$ to $\overline{\mathbf{V}}$ and hence from $\mathbf{V}$ to $\overline{\mathbf{V}}$. Thus, $\overline{\mathbf{V}}$ is the desired cofinality-preserving forcing extension where κ remains nearly

θ^+-supercompact, and the GCH fails first at θ. $\square$

Finally, we arrive at the point of separation, the analogue to Main Theorem Statement (1) of the first chapter and [25].

Corollary 2.4.9. *Let κ and θ be regular cardinals such that $\kappa \leq \theta$. If κ is supercompact and the GCH holds, then there exists a cofinality-preserving (and hence cardinal-preserving) forcing extension where κ is nearly θ^+-supercompact but neither θ-supercompact nor strongly unfoldable. In fact, κ will not even be θ-superunfoldable.*

Proof. Suppose κ were θ-supercompact in the cofinality-preserving forcing extension $\overline{\mathbf{V}}$ from Corollary 2.4.8 where κ is nearly θ^+-supercompact, and the GCH fails first at θ. Then taking any θ-supercompactness ultrapower embedding $j : \overline{\mathbf{V}} \to \overline{N}$ generated by what $\overline{\mathbf{V}}$ thinks is a normal κ-complete fine measure on $(P_\kappa\theta)^{\overline{\mathbf{V}}}$, the GCH would have to hold below $j(\theta)$ (and hence below $j(\kappa)$ since $\kappa \leq \theta$) in $\overline{N}$. Then because $j(\kappa) > \theta$, the GCH would have to hold at θ in $\overline{N}$. This means that $\overline{N}$, which is an inner model of $\overline{\mathbf{V}}$, would have a bijection from its θ^+ onto its powerset of θ so that $\overline{\mathbf{V}}$ would have this function too. But this is impossible since $\overline{N}$ agrees with $\overline{\mathbf{V}}$ on θ^+ and the powerset of θ so that this function would be a bijection in $\overline{\mathbf{V}}$ from its θ^+ to its powerset of θ, contradicting the fact that the GCH fails at θ in $\overline{\mathbf{V}}$. Similarly, fix what $\overline{\mathbf{V}}$ believes to be a transitive $<\kappa$-closed ZFC$^-$ model M of size κ containing κ as an element and $\overline{\mathbf{V}}_\kappa$ as a subset. Suppose we were able to accordingly find what $\overline{\mathbf{V}}$ thinks is a θ-closed N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$, per the supposed strong unfoldability or θ-superunfoldability of κ. Then the same phenomenon would occur as M would know that the GCH holds below κ whereby N would have to think that it holds at θ, providing us with a problematic bijection between $\overline{\mathbf{V}}$'s true θ^+ and powerset of θ. Therefore, in the cofinality-preserving forcing extension $\overline{\mathbf{V}}$ from Corollary 2.4.8, κ is nearly θ^+-supercompact but not θ-supercompact, strongly unfoldable, or even θ-superunfoldable. $\square$

At this point, one may wonder whether it is possible to make a measurable and nearly θ-supercompact cardinal κ nonmeasurable while still retaining its near θ-supercompactness if we were to instead make the GCH fail first at κ. It turns out that the answer to this question is yes only when $\theta < \kappa^{++}$ is true in the resulting model (which we'll sometimes assume is $\mathbf{V}$ by changing our perspective to this model in the informal discussion to follow). In fact, we cannot even retain the κ^{++}-near measurability of κ by making the GCH fail first at κ if $\theta \geq \kappa^{++}$. This is due to the observation that if we allow arbitrary ZFC$^-$ model domains of size κ^{++}, then those that contain $(\kappa^{++})^{\mathbf{V}}$

many subsets of κ and $\mathbf{V}_\kappa$ will be “too knowledgable” about $\mathbf{V}$. Specifically, their powersets of κ will be too large for them to think that the GCH holds at κ, and yet they will know that the GCH holds below κ. Consequently, the target models of embeddings with these domains, which will each contain the respective domain's subsets of κ, would also have to know that the GCH does not hold at κ, contradicting the fact that they must think that the GCH fails first at some ordinal above κ.

Theorem 2.4.10.

1. *If κ is nearly κ-supercompact, i.e., weakly compact, then there exists a forcing extension where κ remains nearly κ-supercompact, and the GCH fails first at κ. Therefore, κ is nearly κ-supercompact but not measurable in this extension.*

2. *If κ is nearly κ^+-supercompact and $2^\kappa = \kappa^+$, then there exists a forcing extension where κ is nearly κ^+-supercompact, and the GCH fails first at κ. Therefore, κ is nearly κ^+-supercompact but not measurable in this extension.*

3. *It is impossible for the GCH to fail first at a κ^{++}-nearly measurable cardinal and hence at any θ-nearly measurable or nearly θ-supercompact cardinal for all $\theta \geq \kappa^{++}$.*

Proof. (1) A cardinal κ is nearly κ-supercompact if and only if it is weakly compact. The fact then that we can preserve the weak compactness of a cardinal κ in a forcing extension where the GCH fails first is due to known standard arguments.

(2) Suppose κ is nearly κ^+-supercompact, and the GCH holds at κ. Let $\mathbb{P}$ be the κ-stage Easton support iteration forcing nontrivially exactly at the stages γ for which γ is regular in $\mathbf{V}^{\mathbb{P}_\gamma}$ and doing so with $\mathrm{Add}(\gamma,\gamma^+)^{\mathbf{V}^{\mathbb{P}_\gamma}}$. Without loss of generality, we may assume that each $\mathbb{P}_\alpha \in \mathbf{V}_\kappa$ and that $\mathbb{P} \in \mathbf{H}_{\kappa^+}$. Now let $\dot{\mathbb{Q}} \in \mathbf{H}_{\kappa^{++}}$ be a $\mathbb{P}$-name for $\mathrm{Add}(\kappa,\kappa^+)^{\mathbf{V}^{\mathbb{P}}}$, which is possible since $(\kappa^+)^\kappa = 2^\kappa = \kappa^+$, and $\mathbb{P}$ has hereditary size κ. Let $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be a $\mathbf{V}$-generic filter, and $A \in \mathbf{V}[G][g]$ be a subset of κ^+. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is κ^+-c.c., we may find a $(\mathbb{P} * \dot{\mathbb{Q}})$-name $\dot{A} \in \mathbf{H}_{\kappa^{++}}$ such that $\dot{A}_{G*g} = A$. Then by the near κ^+-supercompactness of κ, we may find κ-closed transitive ZFC^- models M and N of size κ^+ with $\dot{A}, \dot{\mathbb{Q}}, \kappa^+ \in M$ and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \kappa^+$ and $j''\kappa^+ \in N$. Because each $\mathbb{P}_\alpha$ is constructed within $\mathbf{H}_\kappa$ for all $\alpha < \kappa$ and $\mathbb{P}$ is a direct limit of these posets, M will be able to construct $\mathbb{P}$ and will agree on its definition with $\mathbf{V}$ by virtue of being closed under $<\kappa$-sequences. Then because $\mathbb{P}$ is constructed below the critical point, $j(\mathbb{P}) \restriction \kappa = \mathbb{P}$. Also, by elementarity, N will think that $j(\mathbb{P})$ is a $j(\kappa)$-stage Easton support iteration forcing nontrivially with

$\mathrm{Add}(\gamma, \gamma^+)^{N^{\mathbb{P}_\gamma}}$ at all stages γ for which γ is regular in $N^{\mathbb{P}_\gamma}$. Since $\mathbb{P}$ has size κ and N is closed under κ sequences, $\mathrm{Add}(\kappa, \kappa^+)^{N^{\mathbb{P}}} = \mathrm{Add}(\kappa, \kappa^+)^{\mathbf{V}^{\mathbb{P}}}$ and so $\dot{\mathbb{Q}}$ is indeed a $\mathbb{P}$-name for the stage κ of forcing of $j(\mathbb{P})$. Therefore, the forcing $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the forcing beyond stage κ of $j(\mathbb{P})$.

Now because $N^\kappa \subseteq N$, and $\mathbb{P} * \dot{\mathbb{Q}}$ is κ^+-c.c., we have $N[G][g]^\kappa \subseteq N[G][g]$ in $\mathbf{V}[G][g]$. Also since N thinks that the next stage of nontrivial forcing beyond κ is κ^+ and the stage γ forcing of $j(\mathbb{P})$ is $<\gamma$-closed in $N^{\mathbb{P}_\gamma}$, the forcing $\mathbb{P}_{\text{tail}}$ will be $\leq\kappa$-closed in $N[G][g]$. Because $N[G][g]$ can have at most κ^+ many (dense) subsets for $\mathbb{P}_{\text{tail}}$ by virtue of having size $|N| = \kappa^+$ in $\mathbf{V}[G][g]$, we may by the Diagonalization Criterion construct an $N[G][g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ in $\mathbf{V}[G][g]$. Consequently, since all conditions in G have bounded support below the critical point, we have $j''G \subseteq G * g * G_{\text{tail}}$ enabling us to form a partial lift of the embedding to $j : M[G] \to N[j(G)]$ in $\mathbf{V}[G][g]$ where $j(G) \cong G * g * G_{\text{tail}} \subseteq j(\mathbb{P})$.

Note now that $N[j(G)]^\kappa \subseteq N[j(G)] = N[G][g][G_{\text{tail}}]$ in $\mathbf{V}[G][g]$ since the model $N[G][g]$ is closed under κ sequences in $\mathbf{V}[G][g]$, and $G_{\text{tail}} \in \mathbf{V}[G][g]$. Furthermore, because $N[j(G)]$ thinks that $j(\kappa)$ is a regular cardinal above κ and that the forcing $j(\dot{\mathbb{Q}}_G)$ is the partial order $\mathrm{Add}(j(\kappa), j(\kappa^+))$, it will think that $j(\dot{\mathbb{Q}}_G)$ is $\leq\kappa$-closed. Also, since $\mathbf{V}[G][g]$ thinks that the size of $N[j(G)]$ is $|N| = \kappa^+$, it will also think that the model $N[j(G)]$ can have at most κ^+ many (dense) subsets for $j(\dot{\mathbb{Q}}_G)$. Note that $N[j(G)]$ can construct the function $\bigcup j''g : \kappa \times j''\kappa^+ \to 2$ from $j \restriction \kappa^+$ and g since the evaluation of it at $(\alpha, j(\beta))$ is $(\bigcup g)(\alpha, \beta)$. Then because such a function has size $\kappa^+ < j(\kappa)$ in $N[j(G)]$, it will be in $j(\dot{\mathbb{Q}}_G)$, which equals $\mathrm{Add}(j(\kappa), j(\kappa^+))^{N[j(G)]}$. Consequently, we may again appeal to the Diagonalization Criterion, this time to find an $N[j(G)]$-generic filter $h \subseteq j(\dot{\mathbb{Q}}_G)$ below the master condition $\bigcup j''g$ and fully lift the embedding to $j : M[G][g] \to N[j(G)]$ in $\mathbf{V}[G][g]$ where $j(g) = h$. Because $M[G][g]$ will contain $A = \dot{A}_{G*g}$, and $M[G][g]$ will be closed under ${\leq}\kappa$ sequences in $\mathbf{V}[G][g]$ by virtue of M being closed under ${\leq}\kappa$ sequences and $\mathbb{P} * \dot{\mathbb{Q}}$ being κ^+-c.c., this embedding witnesses the near κ^+-supercompactness of κ for A. Since $A \subseteq \kappa^+$ in $\mathbf{V}[G][g]$ was arbitrary, κ remains nearly κ^+-supercompact in $\mathbf{V}[G][g]$.

Now because the forcing to add κ^+ many Cohen subsets of κ once is isomorphic to the forcing to add κ^+ many Cohen subsets of κ followed by the forcing to add α many Cohen subsets of κ for all $\alpha \leq \kappa^+$, the near κ^+-supercompactness of κ will be indestructible by the forcing $\mathrm{Add}(\kappa, \alpha)^{\mathbf{V}[G][g]}$ for all $\alpha \leq \kappa^+$ in $\mathbf{V}[G][g]$. It then follows by Lemma 2.4.6, that in $\mathbf{V}[G][g]$, the near κ^+-supercompactness of κ will be indestructible by the forcing $\mathrm{Add}(\kappa, \eta)^{\mathbf{V}[G][g]}$ for all ordinals η. Furthermore, the GCH holds below κ in $\mathbf{V}[G][g]$ because at any successor stage γ^+ below κ, we forced to add

Cohen subsets of γ^+ and never added any subsets of γ at any later stage. Specifically, all subsets of γ in $\mathbf{V}[G_{\gamma^+}]$ became encoded as a subset of γ^+ in $\mathbf{V}[G_{\gamma^++1}]$, and the forcing beyond stage γ of the $(\kappa+1)$-stage Easton support iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is $\leq\gamma$-closed in $\mathbf{V}^{\mathbb{P}_{\gamma+1}}$. Therefore, if we now force with the poset $\mathrm{Add}(\kappa, \kappa^{++})^{\mathbf{V}[G][g]}$ in $\mathbf{V}[G][g]$, we will have an extension where κ remains nearly κ^+-supercompact, and the GCH fails first at κ.

(3) If κ is κ^{++}-nearly measurable, and $2^\kappa \geq \kappa^{++}$, then we may encode a collection of κ^{++} many subsets of κ as an $A \subseteq \kappa^{++}$. Then by the κ^{++}-near measurability of κ, we can find a $<\kappa$-closed transitive ZFC^- model M with $\kappa, A \in M$, a transitive N, and an elementary embedding $j : M \to N$ with critical point κ. If the GCH were to fail first at κ, then M would know this, so by elementarity, N would think that the GCH fails first at $j(\kappa)$, which is strictly greater than κ. But N has all $S \subseteq \kappa$ in M since $S = j(S) \cap \kappa$, so in particular, N would know that $2^\kappa \geq (\kappa^{++})^{\mathbf{V}} \geq (\kappa^{++})^N$, contradicting the fact that N thinks that the GCH fails first at $j(\kappa)$. $\square$

We could also ask for a generalization of Corollary 2.4.8, which shows us that relative to a supercompact cardinal κ in the presence of the GCH and any regular $\theta \geq \kappa$, it is possible to simultaneously preserve the near θ^+-supercompactness of κ and make the GCH fail first at θ. Specifically, we could ask if it is possible to preserve a greater degree of near supercompactness while making the GCH fail first at θ. However, Corollary 2.4.8 offers the optimal degree of near supercompactness in this scenario since if κ remained nearly θ^{++}-supercompact and the GCH did fail first at θ, then we would be able to find a near θ^{++}-supercompactness embedding $j : M \to N$ where M is stuffed with an actual θ^{++} family of subsets of θ and knows that the GCH holds below κ. In this case, N would have to think that the GCH holds below $j(\kappa)$ and at θ in particular, which is impossible since it could construct each of the θ^{++} subsets $A \subseteq \theta$ that M has, from $j''\theta$ and $j(A)$.

However, if we allow cardinals to be collapsed, then we can preserve any "ordinal degree" of near supercompactness of a cardinal κ while making the GCH fail first at κ. Specifically, if κ is nearly θ-supercompact for some cardinal $\theta \geq \kappa$ for which $\theta^{<\theta} = \theta$, then we can preserve the near θ-supercompactness of κ while making the GCH fail first at κ in a forcing extension. Alternatively, if κ is nearly θ^+-supercompact for some cardinal θ for which $2^{\theta^{<\kappa}} = \theta^+$, then we can make it so that κ remains nearly θ^+-supercompact, and the GCH fails first at κ in a forcing extension preserving θ^+. In both cases, the cardinal κ will not be measurable in the forcing extension. Notice that we will be able to accomplish these preservation results for partially nearly supercompact cardinals without appealing to larger degrees of (near) supercompactness for κ. This is because we will be able to force these results using $(\kappa+1)$-stage iterations.

Theorem 2.4.11.

1. *If κ is nearly θ-supercompact for some cardinal $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$, then there exists a forcing extension where the near θ-supercompactness of κ is preserved, and the GCH fails first at κ. Therefore, κ is nearly θ-supercompact but not measurable in this forcing extension.*

2. *If κ is nearly θ^+-supercompact for some cardinal θ such that $2^{\theta^{<\kappa}} = \theta^+$, then there exists a forcing extension preserving θ^+ where κ remains nearly θ^+-supercompact, and the GCH fails first at κ. Therefore, θ^+ is not collapsed, and κ is nearly θ^+-supercompact but not measurable in this forcing extension.*

Proof. (1) Suppose κ is nearly θ-supercompact for some cardinal $\theta \geq \kappa$ such that $\theta^{<\theta} = \theta$. Then by Theorem 2.3.8, we may move to a forcing extension where θ has been collapsed to have size κ, and κ remains nearly θ-supercompact (weakly compact). The conclusion then follows from the well-known result that if κ is weakly compact, then we can force to an extension where κ remains weakly compact, and the GCH fails first at κ (Theorem 2.4.10, Statement 1). Specifically, if we move to such a forcing extension where κ is weakly compact (nearly κ-supercompact) and the GCH fails first at κ, then since θ will continue to have size κ here, it also follows that κ is nearly θ-supercompact here as well.

(2) Suppose κ is nearly θ^+-supercompact for some cardinal θ such that $2^{\theta^{<\kappa}} = \theta^+$. Then by Theorem 2.3.4, we may force to an extension where the near θ^+-supercompactness of κ becomes indestructible by the $<\kappa$-directed closed collapse forcing $\mathrm{Coll}(\kappa, \theta)$ of the extension. Note that because the forcing extension from this theorem is over a κ-c.c. partial order of size κ, and both $\theta^{<\kappa} = \theta$ and $2^\theta = \theta^+$ are true in the ground model, these equalities will remain true in the forcing extension over the lottery preparation as well. Therefore, the $\mathrm{Coll}(\kappa, \theta)$ of the forcing extension will indeed have size θ there and will thus be trivially θ^+-c.c. there as well. This ensures that it was indeed covered by the indestructibility given to us by Theorem 2.3.4, and it also ensures that forcing over it will not collapse θ^+ nor change the fact that $2^\theta = \theta^+$. Now force to an extension over this poset, and move to this outer model. In this model, where κ remains nearly θ^+-supercompact, θ will be collapsed to have size κ, and therefore θ^+ will be the new κ^+ of this model. The rest of the proof then follows from Theorem 2.4.10, Statement (2) because it now allows us to make the GCH fail first at the nearly κ^+-supercompact (nearly θ^+-supercompact) cardinal κ. □

The above theorem and accompanying proof illustrate why it is important that we preserve the cardinals above κ up to and including θ. Otherwise, the preservation results can become trivial because the notion of the near θ-supercompactness of a cardinal κ can collapse to something like weak compactness or near κ^+-supercompactness.

Now that we have ruled out the possibility of making the GCH fail first at κ^{++}-nearly measurable cardinals κ (without collapsing κ^{++}), one may ask if it is possible to at least obtain a separation between nearly θ-supercompact cardinals and measurable ones. In other words, can we for arbitrarily large θ, apply a technique similar to the proof of Theorem 2.4.9 to find a forcing extension preserving cardinals above κ where a cardinal that's both measurable and nearly θ-supercompact becomes nonmeasurable while still retaining its near θ-supercompactness? In particular, this would generalize Main Theorem, Statement (1) from Chapter 1 and [25] by showing that it is possible relative to a large cardinal notion to make a θ-nearly measurable cardinal κ nonmeasurable without collapsing cardinals in between κ and θ. We show that this is indeed the case by proving Statement 3 of the Main Theorem.

Theorem 2.4.12. *If κ is nearly θ-supercompact for some $\theta \geq 2^\kappa$ such that $\theta^{<\theta} = \theta$, then there exists a forcing extension preserving all cardinals and cofinalities above κ where κ is nearly θ-supercompact but not measurable.*

Furthermore, in this extension $2^\kappa = \theta^+$*, and if the SCH held below* κ *in the ground model, then no cardinals or cofinalities were collapsed.*

Proof. First, allow me to outline the idea of the proof in very broad terms. By Theorem 2.3.5, we may assume that $2^\kappa = \theta$ by moving to a forcing extension preserving all cardinals and cofinalities above κ (and below κ as well if the SCH holds below κ) where κ remains nearly θ-supercompact, $\theta^{<\theta} = \theta$, and $2^\kappa = \theta$. Next, we force with a $(\kappa + 1)$-stage Easton support iteration that adds 2^γ many Cohen subsets of γ at some inaccessible stages γ including stage κ to arrive at a forcing extension $\mathbf{V}[G][g]$. Doing this will not disrupt the continuum function on the set I of inaccessible cardinals in $\mathbf{V}$ below κ, but it will make the near θ-supercompactness indestructible by the forcing to add at most $2^\kappa = \theta$ many Cohen subsets of κ. We can then appeal to Lemma 2.4.6 to be assured that the near θ-supercompactness of κ will be indestructible by the forcing to add any number of Cohen subsets of κ. We then disrupt the continuum function at the inaccessible cardinal κ by forcing to add θ^+ many Cohen subsets of κ over $\mathbf{V}[G][g]$ to arrive at the desired forcing extension $\overline{\mathbf{V}}$. The cardinal κ remains nearly θ-supercompact in $\overline{\mathbf{V}}$ by the aforementioned indestructibility, but because any ultrapower embedding $h : \overline{\mathbf{V}} \to \overline{N}$ having critical point κ that's definable in this forcing extension would have to extend an elementary embedding $h \upharpoonright \mathbf{V} : \mathbf{V} \to N$ definable in the ground model $\mathbf{V}$, the cardinal κ can no longer be measurable in this extension. Specifically, since the continuum function remains undistributed at all inaccessible cardinals below κ in $\mathbf{V}$ when moving from $\mathbf{V}$ to $\overline{\mathbf{V}}$, destroying the pattern of the continuum function at the inaccessible cardinal κ in such an extension $\overline{\mathbf{V}}$ is impossible without destroying the measurability of κ.

Suppose κ is nearly θ-supercompact for some $\theta \geq 2^\kappa$ such that $\theta^{<\theta} = \theta$, and let $f : \kappa \to \kappa$ be a function with the Menas property for the near θ-supercompactness of κ. By performing the lottery preparation and then forcing with the $\mathrm{Add}(\kappa, \theta)$ of the extension as in Theorem 2.3.5, we may assume that $2^\kappa = \theta$. Then let $\mathbb{P}$ be the κ-stage Easton support iteration where at every inaccessible stage γ such that $f''\gamma \subseteq \gamma$, we force with $\mathrm{Add}(\gamma, 2^\gamma)^{\mathbf{V}^{\mathbb{P}_\gamma}}$ and perform trivial forcing otherwise. Because every stage of forcing stays below the size of the next inaccessible cardinal, we may assume that $\mathbb{P}_\alpha \in \mathbf{V}_\delta$ for all $\alpha < \delta$ whenever δ is inaccessible and in particular for $\delta = \kappa$. Also let $\dot{\mathbb{Q}} \in \mathbf{H}_{\theta^+}$ be a $\mathbb{P}$-name for $\mathrm{Add}(\kappa, \theta)^{\mathbf{V}^{\mathbb{P}}}$, which is possible since the relationship $\theta^{<\theta} = \theta$ holds in $\mathbf{V}$, and $\mathbb{P}$ has hereditary size $\kappa < \theta$. Let $G * g$ be a $\mathbf{V}$-generic filter over $\mathbb{P} * \dot{\mathbb{Q}}$ and A be a subset of θ in $\mathbf{V}[G][g]$. Since $\mathbb{P} * \dot{\mathbb{Q}}$ is κ^+-c.c., we may find a $(\mathbb{P} * \dot{\mathbb{Q}})$-name $\dot{A} \in \mathbf{H}_{\theta^+}$ such that $\dot{A}_{G*g} = A$. Also let E be a subset of θ coding a bijection from θ onto $\mathcal{P}(\kappa)$. Then coding E, the $(\mathbb{P} * \dot{\mathbb{Q}})$-

name $\dot{A}$, and $\dot{\mathbb{Q}}$ as a single subset of θ, we may appeal to the Menas property of f for the near θ-supercompactness of κ to find ${<}\theta$-closed transitive ZFC$^-$ models M and N of size θ with $E, \dot{A}, \dot{\mathbb{Q}}, \theta \in M$ and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$, the ordinal $j(f)(\kappa) > \theta$, and $j''\theta \in N$. Because each $\mathbb{P}_\alpha$ is constructed within $\mathbf{H}_\kappa$ for all $\alpha < \kappa$ and $\mathbb{P}$ is a direct limit of these posets, M will be able to construct $\mathbb{P}$ and will agree on its definition with $\mathbf{V}$ by virtue of being closed under ${<}\kappa$-sequences. Since each $\mathbb{P}_\alpha$ is constructed below the critical point, $j(\mathbb{P}) \restriction \kappa = \mathbb{P}$. Then by elementarity, N will think that $j(\mathbb{P})$ is a $j(\kappa)$-stage Easton support iteration forcing nontrivially with $\text{Add}(\gamma, 2^\gamma)^{N^{\mathbb{P}_\gamma}}$ at all stages γ for which $j(f)''\gamma \subseteq \gamma$ and γ is inaccessible. Consequently, N will think that the stage κ forcing of $j(\mathbb{P})$ is $\text{Add}(\kappa, 2^\kappa)^{N^{\mathbb{P}}}$. But N contains E because it can construct all the subsets of θ in M and so it contains a θ-enumeration of the true powerset of κ. Also, since $\mathbb{P}$ has size κ in N and N is closed under κ-sequences, $\text{Add}(\kappa, \theta)^{N^{\mathbb{P}}} = \text{Add}(\kappa, \theta)^{\mathbf{V}^{\mathbb{P}}}$. Consequently, it follows from this and the fact that $\mathbb{P}$ does not inflate 2^κ, that $\dot{\mathbb{Q}}$ is indeed a $\mathbb{P}$-name for the stage κ forcing of $j(\mathbb{P})$. Therefore, the forcing $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the forcing beyond stage κ of $j(\mathbb{P})$.

Now because $N^{<\theta} \subseteq N$ and $\mathbb{P} * \dot{\mathbb{Q}}$ is κ^+-c.c. (and hence θ-c.c.), we have $N[G][g]^{<\theta} \subseteq N[G][g]$ in $\mathbf{V}[G][g]$. Also since N thinks that the next stage of nontrivial forcing after κ is beyond $j(f)(\kappa)$, which is greater than θ, and the stage γ forcing of $j(\mathbb{P})$ is $<\gamma$-closed in $N^{\mathbb{P}_\gamma}$, the forcing $\mathbb{P}_{\text{tail}}$ will be $<\theta$-closed in $N[G][g]$. Because $N[G][g]$ can have at most θ many (dense) subsets for $\mathbb{P}_{\text{tail}}$, as counted in $\mathbf{V}[G][g]$, by virtue of having size $|N| = \theta$ in $\mathbf{V}[G][g]$, we may by the Diagonalization Criterion construct an $N[G][g]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$ in $\mathbf{V}[G][g]$. Consequently, since all conditions in G have bounded support below the critical point, we have $j''G \subseteq G * g * G_{\text{tail}}$ enabling us to form a partial lift of the embedding to $j : M[G] \to N[j(G)]$ in $\mathbf{V}[G][g]$ where $j(G) \cong G * g * G_{\text{tail}} \subseteq j(\mathbb{P})$.

Note now that $N[j(G)]^{<\theta} \subseteq N[j(G)] = N[G][g][G_{\text{tail}}]$ in $\mathbf{V}[G][g]$ since the model $N[G][g]$ is closed under $<\theta$ sequences in $\mathbf{V}[G][g]$, and $G_{\text{tail}} \in \mathbf{V}[G][g]$. Furthermore because $N[j(G)]$ thinks that the forcing $j(\dot{\mathbb{Q}}_G)$ is $\text{Add}(j(\kappa), j(\theta))$ with $j(\kappa) > \theta$, it will think that $j(\dot{\mathbb{Q}}_G)$ is $<\theta$-closed. Also, since $\mathbf{V}[G][g]$ thinks that $N[j(G)]$ has size $|N| = \theta$, it will also think that the model $N[j(G)]$ can have at most θ many (dense) subsets for $j(\dot{\mathbb{Q}}_G)$. Note that $N[j(G)]$ can construct $\bigcup j''g : \kappa \times j''\theta \to \{0,1\}$ from $j \restriction \theta$ and g since the evaluation of it at $(\alpha, j(\beta))$ is $(\bigcup g)(\alpha, \beta)$. Then because such a function has size $\theta < j(\kappa)$ in $N[j(G)]$, it will be in $j(\dot{\mathbb{Q}}_G) = \text{Add}(j(\kappa), j(\theta))^{N[j(G)]}$. Consequently, we may again appeal to the Diagonalization Criterion, this time to find an $N[j(G)]$-generic filter h for $j(\dot{\mathbb{Q}}_G)$ below the master condition $\bigcup j''g$ and fully lift the embedding to $j : M[G][g] \to N[j(G)][j(g)]$ in $\mathbf{V}[G][g]$ where $j(g) = h$. Because $M[G][g]$ will contain $A = \dot{A}_{G*g}$, and $M[G][g]$ will be closed under κ sequences in $\mathbf{V}[G][g]$ by virtue of M being closed under κ sequences and $\mathbb{P} * \dot{\mathbb{Q}}$ being κ^+-c.c., this embedding witnesses the near θ-supercompactness of κ for A. Since $A \subseteq \theta$ in $\mathbf{V}[G][g]$ was arbitrary, κ remains nearly θ-supercompact in $\mathbf{V}[G][g]$.

Because the forcing to add θ many Cohen subsets of κ once is isomorphic to the forcing to add θ many Cohen subsets of κ followed by the forcing to add α many Cohen subsets of κ for all $\alpha \leq \theta$, the near θ-supercompactness of κ will be indestructible by the further forcing $\mathrm{Add}(\kappa, \alpha)^{\mathbf{V}[G][g]}$ for all $\alpha \leq \theta$ in $\mathbf{V}[G][g]$. It then follows by Lemma 2.4.6 that in $\mathbf{V}[G][g]$, the near θ-supercompactness of κ will be indestructible by the forcing $\mathrm{Add}(\kappa, \eta)^{\mathbf{V}[G][g]}$ for all ordinals η. Also, since the first nontrivial stage γ_0 of forcing, $\mathrm{Add}(\gamma_0, 2^{\gamma_0})$, has size 2^{γ_0}, which is strictly less than the next stage of nontrivial forcing occurring at an inaccessible cardinal above γ_0, and all nontrivial stages of forcing $\alpha > 2^{\gamma_0}$ are adding Cohen subsets to α only when α is a regular cardinal in $\mathbf{V}^{\mathbb{P}_\alpha}$, the subsequent forcing will be $\leq 2^{\gamma_0}$-closed whereby $\mathbb{P} * \dot{\mathbb{Q}}$ admits a closure point at 2^{γ_0} below κ.

Now, observe that the continuum function restricted to the set of inaccessible cardinals at or below κ from the ground model is preserved by the forcing $\mathbb{P} * \dot{\mathbb{Q}}$. To see this, first fix an inaccessible cardinal $\delta \leq \kappa$. Then by construction, every $\mathbb{P}_\gamma$ is constructed in $\mathbf{V}_\delta$ for all $\gamma < \delta$ so that since we take bounded support at inaccessible stages, $|\mathbb{P}_\delta| = \delta$. Note that δ remains inaccessible in $\mathbf{V}^{\mathbb{P}_\delta}$ because for all $\gamma < \delta$, we have $|\mathbb{P}_{\gamma+1}| < \delta$ so that $\mathbb{P}_{\gamma+1}$ is too small to add a cofinal γ-sequence to δ or δ many subsets of γ, and the forcing beyond stage γ is $\leq\gamma$-closed in $\mathbf{V}^{\mathbb{P}_{\gamma+1}}$ so that it does not add any γ sequences of δ or subsets of γ. Now at stage δ, we will either perform trivial forcing or force with $\mathrm{Add}(\delta, 2^\delta)^{\mathbf{V}^\delta}$, which will necessarily be δ^+-c.c. by virtue of δ remaining inaccessible in $\mathbf{V}^\delta$. Either way, $\mathbb{P}_{\delta+1}$ (or $\mathbb{P} * \dot{\mathbb{Q}}$ if $\delta = \kappa$) will have size at most 2^δ and will be δ^+-c.c. Consequently, it cannot change 2^δ. Then since all later forcing is $\leq\delta$-closed, the forcing beyond stage δ cannot add any subsets of δ. Therefore, 2^δ is preserved in $\mathbf{V}[G][g]$ for all inaccessible cardinals $\delta < \kappa$ in $\mathbf{V}$.

Now in $\mathbf{V}[G][g]$, force with $\mathrm{Add}(\kappa, \theta^+)^{\mathbf{V}[G][g]}$ to arrive at a forcing extension $\overline{\mathbf{V}}$. I claim that $\overline{\mathbf{V}}$ is the desired forcing extension preserving cofinalities and cardinals above κ (and also below κ if the SCH held below κ in $\mathbf{V}$) where κ remains nearly θ-supercompact, $2^\kappa = \theta^+$, and κ is not measurable. We already verified that $\overline{\mathbf{V}}$ will think that κ is nearly θ-supercompact since in $\mathbf{V}[G][g]$, the near θ-supercompactness of κ is indestructible by the forcing $\mathrm{Add}(\kappa, \eta)$ for all ordinals η. Furthermore, no cardinals above κ are collapsed when moving from $\mathbf{V}$ to $\mathbf{V}[G]$ by virtue of $\mathbb{P}$ having size κ. Additionally, when the SCH holds below κ in $\mathbf{V}$, we have an almost identical justification to the corresponding part of the proof for Theorem 2.3.5 showing that all cardinals below κ were also preserved. Also, no cardinals or cofinalities are collapsed when moving from $\mathbf{V}[G]$ to $\overline{\mathbf{V}}$ by virtue of the facts that the forcing to add Cohen subsets to the regular cardinal κ is $<\kappa$-closed and will be κ^+-c.c. when $\kappa^{<\kappa} = \kappa$, which is the case in $\mathbf{V}[G]$. Consequently, $\overline{\mathbf{V}}$ is a forcing extension of $\mathbf{V}$ preserving all cofinalities and cardinals above κ, and all cofinalities and cardinals when the SCH holds below κ in $\mathbf{V}$. Note then that because θ^+ is not collapsed when moving from $\mathbf{V}[G][g]$ to $\overline{\mathbf{V}}$, the forcing extension $\overline{\mathbf{V}}$ will think that $2^\kappa \geq \theta^+$. But also because $\mathbf{V}[G][g]$ thinks that $2^\kappa = \theta$, there can be at most $(\theta^+)^\kappa = \theta^+$ many nice names in $\mathbf{V}[G][g]$ for subsets of κ in $\overline{\mathbf{V}}$. Consequently, $\overline{\mathbf{V}}$ will think that $2^\kappa = \theta^+$.

Finally, we will show that κ cannot remain measurable in $\overline{\mathbf{V}}$, completing the proof that $\overline{\mathbf{V}}$ has all of the desired properties. Suppose for contradiction that κ remains measurable in $\overline{\mathbf{V}}$. Then there exists an elementary embedding $h : \overline{\mathbf{V}} \to \overline{N}$ with critical point κ definable in $\overline{\mathbf{V}}$ such that $\overline{N}$ is closed under κ sequences in $\overline{\mathbf{V}}$. Because the forcing $\mathbb{P} * \dot{\mathbb{Q}}$ admits a closure point below κ and $\mathbf{V}[G][g]$ thinks that $\mathrm{Add}(\kappa, \theta^+)^{\mathbf{V}[G][g]}$ is $<\kappa$-closed, $\overline{\mathbf{V}}$ is also a forcing extension of $\mathbf{V}$ over a partial order admitting a closure point below

κ. Consequently, it follows by Lemma 13 of [12] that $\mathbf{V} \subseteq \overline{\mathbf{V}}$ satisfies the δ approximation and cover properties for some $\delta < \kappa$. It therefore follows from Corollary 8 of [12] that h lifts an elementary embedding $h \restriction \mathbf{V} : \mathbf{V} \to N^*$ definable in $\mathbf{V}$. Now letting I be the set of inaccessible cardinals below κ in $\mathbf{V}$ and then defining $\mathfrak{C}$ to be the continuum function on I in $\mathbf{V}$, the model N^* will think that the set $h(I)$ is the collection of inaccessible cardinals below $h(\kappa)$ and that the function $h(\mathfrak{C})$ is the continuum function on the set $h(I)$, by elementarity. Then because $h \restriction \mathbf{V} : \mathbf{V} \to N^*$ is definable in $\mathbf{V}$, the inaccessible cardinal κ will also be inaccessible in N^* whereby $\kappa \in h(I)$, and $N^* \subseteq \mathbf{V}$ whereby $h(\mathfrak{C})(\kappa) < \theta^+$. But note also that since $\mathbb{P} * \dot{\mathbb{Q}}$ preserves the continuum function on I in $\mathbf{V}$ and $\mathrm{Add}(\kappa, \theta^+)$ is ${<}\kappa$-closed, $\mathfrak{C}$ will also be the continuum function on the set I in $\overline{\mathbf{V}}$. It then follows by the elementarity of h in $\overline{\mathbf{V}}$ that $h(\mathfrak{C})$ is the continuum function on $h(I)$, which includes κ, in $\overline{N}$. But $\overline{\mathbf{V}} \models 2^\kappa = \theta^+$, and $\mathcal{P}(\kappa)^{\overline{\mathbf{V}}} \subseteq \overline{N} \subseteq \overline{\mathbf{V}}$ so that $\overline{N} \models h(\mathfrak{C})(\kappa) \geq (\theta^+)^{\overline{\mathbf{V}}} = (\theta^+)^{\mathbf{V}}$, contradicting the fact that $h(\mathfrak{C})(\kappa) < \theta^+$. $\square$

The above theorem provides the ideal separation between near supercompactness and measurability obtainable in this forcing extension. Specifically, since $2^\kappa = \theta^+$ in the constructed forcing extension $\overline{\mathbf{V}}$, the cardinal κ could not be nearly θ^+-supercompact there without being measurable. In addition, since κ is θ-nearly measurable by virtue of being nearly θ-supercompact, we also get the optimal degree of near measurability without outright measurability that κ can possess when $2^\kappa = \theta^+$.

Corollary 2.4.13. *Suppose κ is nearly θ-supercompact for some $\theta \geq 2^\kappa$ such that $\theta^{<\theta} = \theta$. Then there exists a forcing extension preserving all cofinalities and cardinals above κ where κ is λ-nearly measurable for all $\lambda < 2^\kappa$ but is not measurable. Additionally, if the SCH holds below κ, then all cardinals and cofinalities are preserved.*

If κ is supercompact, then Theorem 2.4.12 shows that for arbitrarily large θ, there will be a forcing extension where κ is nearly θ-supercompact but not measurable. However, in each of these forcing extensions, 2^κ will be a successor cardinal. The following corollary allows for the possibility that 2^κ is weakly inaccessible, and κ is nearly η-supercompact for all $\eta < 2^\kappa$ but not measurable.

Corollary 2.4.14. *If κ is nearly θ-supercompact for some inaccessible cardinal $\theta > \kappa$, then there is a forcing extension preserving all cofinalities and cardinals from κ up to θ where $\theta^{<\theta} = \theta$, and κ is nearly η-supercompact for all $\eta < \theta$ but not measurable. In this forcing extension, θ will be weakly inaccessible, $2^\kappa = \theta$, and $\eta^{<\kappa} < \theta$ for all $\eta < \theta$.*

Proof. In the forcing extension $\overline{\mathbf{V}}$ from Theorem 2.4.12, κ is nearly θ-supercompact but not measurable, $2^\kappa = \theta^+$, and no cardinals above κ have been collapsed from $\mathbf{V}$. I claim that the further forcing to add a Cohen real x followed by the collapse forcing $\mathrm{Coll}(\theta, \theta^+)^{\overline{\mathbf{V}}[x]}$ results in the desired forcing extension.

First notice that no cofinalities or cardinals from κ up to θ are collapsed because they aren't when moving from $\mathbf{V}$ to $\overline{\mathbf{V}}$, the forcing to add a Cohen real preserves all cofinalities and cardinals, and the forcing $\mathrm{Coll}(\theta, \theta^+)^{\overline{\mathbf{V}}[x]}$ adds no $<\theta$ sequences of ordinals that weren't already present in $\overline{\mathbf{V}}[x]$. In particular, θ will remain weakly inaccessible.

In the final forcing extension, $\theta^{<\theta} = \theta$ and $\eta^{<\kappa} < \theta$ for all $\eta < \theta$. To see this, first note that the (strong) inaccessibility of θ is preserved in the intermediate forcing extension $\mathbf{V}[G]$ over the poset $\mathbb{P}$ of size κ from Theorem 2.4.12 since $\mathbb{P}$ is too small to destroy it. Then the forcing $\mathrm{Add}(\kappa, \theta)^{\mathbf{V}[G]}$ is $<\kappa$-closed so it adds no new $<\kappa$ sequences to any ordinal (less than θ), and it is a κ^+-c.c. (and hence a θ-c.c.) poset of size θ in $\mathbf{V}[G]$. Thus, $\eta^{<\kappa} < \theta$ for all $\eta < \theta$ and $\theta^{<\theta} = \theta$ in $\mathbf{V}[G][g]$. The forcing $\mathrm{Add}(\kappa, \theta^+)^{\mathbf{V}[G][g]}$ also adds no new $<\kappa$ sequences so that $\overline{\mathbf{V}}$ still thinks that $\eta^{<\kappa} < \theta$ for all $\eta < \theta$. Since this forcing is κ^+-c.c. and of size θ^+ in $\mathbf{V}[G][g]$, we know that $\overline{\mathbf{V}} \models \theta^{<\theta} = \theta^+$. The forcing to add a Cohen real x is too small to change the facts true in

$\overline{\mathbf{V}}$ that $\eta^{<\kappa} < \theta$ for all $\eta < \theta$ or $\theta^{<\theta} = \theta^+$. Finally, the subsequent forcing $\mathrm{Coll}(\theta, \theta^+)^{\overline{\mathbf{V}}[x]}$ cannot add any new $<\kappa$ sequences to all $\eta < \theta$ so $\eta^{<\kappa} = \eta$ for all $\eta < \theta$ in the desired forcing extension. But it also collapses θ^+ to have size θ while not adding any new $<\theta$ sequences to θ so that $\theta^{<\theta} = \theta$ in the final forcing extension. In particular, $2^\kappa = \theta$ in this forcing extension because this held in $\overline{\mathbf{V}}$, and θ was not collapsed.

Also, if $\eta < \theta$, then κ will be nearly η-supercompact in the desired forcing extension. To see this, fix an arbitrary $\eta < \theta$, and first note that κ is nearly θ-supercompact in $\overline{\mathbf{V}}$. Then by the Lévy-Solovay Theorem 2.1.6, it will remain so in the forcing extension $\overline{\mathbf{V}}[x]$ over the small poset adding a Cohen real. Then because $\eta^{<\kappa} < \theta$ in $\overline{\mathbf{V}}[x]$, and the forcing $\mathrm{Coll}(\theta, \theta^+)^{\overline{\mathbf{V}}[x]}$ is $<\theta$-distributive in $\overline{\mathbf{V}}[x]$, the near η-supercompactness is preserved in the desired forcing extension by Theorem 2.1.8.

Finally, κ is not measurable in the desired forcing extension because any ultrapower embedding from the extension would have to be the lift of an elementary embedding from $\overline{\mathbf{V}}$ into an inner model definable in $\overline{\mathbf{V}}$ by virtue of admitting a closure point below κ. But this is impossible since κ is not measurable in $\overline{\mathbf{V}}$. $\square$

2.5 λ-Nearly θ-Supercompact Cardinals

We now consolidate the notions of λ-near measurability with near θ-supercompactness and generalize some of the lemmas and theorems that held for the nearly θ-supercompact cardinals to these more general types of cardinals. More generally, the λ-nearly θ-supercompact cardinals are an attempt to factor the power of an elementary embedding j witnessing the θ-supercompactness of a cardinal into the components of domain size and closure under j. First, let us begin with the definition.

Definition 2.5.1 (λ-Nearly θ-Supercompact Cardinals). *A cardinal κ is λ-nearly θ-supercompact if for every $A \subseteq \lambda$, there exists a transitive $M \models ZFC^-$ closed under $<\kappa$ sequences having the subset A and the cardinals κ and θ as elements, a transitive N, and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''\theta \in N$.*

Under this definition, the nearly θ-supercompact cardinals are the θ-nearly θ-supercompact cardinals while the λ-nearly measurable cardinals are the λ-nearly 0-supercompact cardinals. In particular, κ is weakly measurable if and only if it is κ^+-nearly 0-supercompact. Alternatively, we can associate a λ-nearly measurable cardinal κ with a λ-nearly κ-supercompact one since if κ is the critical point of an elementary embedding j, then $j''\kappa = \kappa$. In particular, κ is weakly measurable if and only if it is κ^+-nearly κ-supercompact. Note that we will be concentrating on the instances when $\lambda \geq \theta$ in this definition. The reason for this is so that all of the standard characterizations for near θ-supercompactness and λ-near measurability, including the filter property, generalize.

Now, let us generalize the preliminary observations that we made earlier for the nearly θ-supercompact cardinals.

Observation 2.5.2.

1. *If κ is θ-supercompact, then it is λ-nearly θ-supercompact for every λ.*

2. *If κ is λ-nearly θ-supercompact for some $\lambda \geq \kappa$, then κ is weakly compact and in particular inaccessible.*

3. *κ is λ-nearly θ-supercompact for some $\lambda \geq \theta$ if and only if it is γ-nearly ξ-supercompact for all $\gamma < \lambda^+$ and $\xi < \theta^+$.*

4. *If κ is λ-nearly θ-supercompact for some $\lambda \geq 2^{\eta^{<\kappa}}$ and $\theta \geq \eta$, then κ will also be η-supercompact.*

5. *κ is supercompact if and only if for every γ, we can find $\lambda, \theta \geq \gamma$ such that κ is λ-nearly θ-supercompact.*

Proof. For (1), note that we may restrict any θ-supercompactness embedding $j : \mathbf{V} \to N$ with critical point κ to the near θ-supercompactness embedding $j \restriction \mathbf{H}_{\eta^+} : \mathbf{H}_{\eta^+} \to j(\mathbf{H}_{\eta^+})$ where $\eta = \max\{\kappa^+, \theta^+, \lambda^+\}$, simultaneously witnessing the λ-near θ-supercompactness of κ for all subsets of λ. The justification for (2) is the same as the corresponding one for θ-supercompact cardinals when $\theta \geq \kappa$. For Statement (3), we can code an arbitrarily selected $A \subseteq \gamma$ and a surjective function $f : \theta \to \xi$ together as a subset of λ whenever $\theta \leq \lambda$, the ordinal $\gamma < \lambda^+$, and $\xi < \theta^+$. We may then appeal to the λ-near θ-supercompactness of κ to obtain a near θ-supercompactness embedding $j : M \to N$ with critical point κ between transitive ZFC$^-$ models with M having θ and the subset of λ as elements and hence also both A and f. Note then that N can construct $j''\xi$ from $j(f)$ and $j''\theta$ since $j''\xi = \text{range}(j(f) \restriction j''\theta)$. Furthermore, because $j(f) \restriction j''\theta$ enumerates all elements of $j''\xi$, the model N will also know that ξ is an ordinal of size at most $|\theta|^N$, which is at most θ. Then since N thinks that $j(\kappa)$ is a cardinal greater than θ, we must have $j(\kappa) > \xi$. Consequently, j will witness the γ-near ξ-supercompactness of κ for A. For Statement (4), if $\lambda \geq 2^{\eta^{<\kappa}}$, then we can encode the full powerset of $P_\kappa\eta$ and all (regressive) functions from $P_\kappa\eta$ into η as a subset A of λ. By the λ-near θ-supercompactness of κ, we can then find a near θ-supercompactness embedding $j : M \to N$ between transitive ZFC$^-$

models with critical point κ such that M is $<\kappa$-closed and has the ordinals κ and θ and the subset A as elements. Then M has the full powerset of $P_\kappa\eta$ and all (regressive) functions from $P_\kappa\eta$ into η, and if $\eta \leq \theta$, then $j''\eta \in N$. Consequently, in this case, the induced filter on $P_\kappa\eta$ by using $j''\eta$ as a seed will be an actual κ-complete normal fine measure on $P_\kappa\eta$. The direct implication of (5) follows from the specific case of (1) that every supercompact cardinal is γ-nearly γ-supercompact for all γ. Finally, for the indirect implication of (5), note that if for every η, we can find $\lambda, \theta \geq 2^{\eta^{<\kappa}}$ such that κ is λ-nearly θ-supercompact, then κ will be η-supercompact for every η by (4). $\square$

From the aforementioned observations, we also know that κ is a $2^{\theta^{<\kappa}}$-nearly θ-supercompact cardinal for some $\theta \geq \kappa$ if and only if it is a θ-supercompact cardinal. Thus, the λ-nearly θ-supercompact cardinals unify the nearly θ-supercompact and θ-supercompact cardinal hierarchies and provide exact equivalences for which the more specific nearly θ-supercompact cardinals could not. We now state characterizations of λ-near θ-supercompactness.

Theorem 2.5.3 (Characterizations of λ-Near θ-Supercompactness). *If $\lambda \geq \theta$ and $\lambda^{<\kappa} = \lambda$, then the following are equivalent:*

(λ-Near θ-Supercompactness) For every $A \subseteq \lambda$, there exists a transitive

$M \models ZFC^{-}$ closed under $<\kappa$ sequences with $A, \kappa, \theta \in M$, a transitive N, and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''\theta \in N$.

(Embedding) For every transitive $<\kappa$-closed M of size λ with $\kappa, \theta \in M$, there exists a transitive N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''\theta \in N$.

(Normal Embedding) For all $\delta \geq \kappa$ and every transitive $M \models ZFC^{-}$ of size λ closed under $<\delta$ sequences with $\kappa, \theta \in M$, there exists a transitive N of size λ closed under $<\delta$ sequences containing all subsets of θ in M, and a cofinal elementary embedding $j : M \to N$ with critical point κ such that $N = \{j(f)(j''\theta) | f \in M; f : P_\kappa\theta \to M\}$, the ordinal $j(\kappa) > \theta$, and $j''\theta \in N$.

(Normal ZFC Embedding) For every $A \subseteq \boldsymbol{H}_{\theta^+}$ of size λ, there is a transitive weakly θ-closed $M \models ZFC$ of size λ with $A \subseteq M$ and $\kappa, \theta \in M$, a transitive weakly θ-closed N of size λ containing all subsets of θ in M, and a cofinal elementary embedding $j : M \to N$ with critical point κ such that $N = \{j(f)(j''\theta) | f \in M; f : P_\kappa\theta \to M\}$, the ordinal $j(\kappa) > \theta$, and $j''\theta \in N$. Furthermore, if $\lambda = \theta$, then $A \in M$.

(Normal Fine Filter) For every collection $\mathcal{A}$ of at most λ many subsets of $P_\kappa\theta$ and collection $\mathcal{F}$ of at most λ many functions from $P_\kappa\theta$ into θ, there exists a κ-complete fine filter F on $P_\kappa\theta$ measuring all sets in $\mathcal{A}$, which is

$\mathcal{F}$-normal in the sense that for every $f \in \mathcal{F}$ regressive on some set in F, there exists $\alpha_f < \theta$ for which $\{\sigma \in P_\kappa\theta | f(\sigma) = \alpha_f\} \in F$.

Proof. The proof of the implications is very similar to the proof of the equivalent characterizations of the special case of this Theorem when $\lambda = \theta$ from Theorem 2.1.3. Except for one implication, which we prove below, all other ones basically follow by some substitutions of λ for θ in the proof of Theorem 2.1.3. Even the implication proven below is very similar to the proof for the nearly θ-supercompact cardinals. However, for the proof of the normal ZFC embedding characterization from the normal embedding characterization, the proof from Theorem 2.1.3 does not directly generalize to dictate that $A \in M$, but only that $A \subseteq M$ since $\overline{N}$ and hence $\mathbf{V}^{\overline{N}}_{\bar{j}(\kappa)}$ need only have what $\overline{M}$ views as θ-sized subsets of $\mathbf{H}^{\overline{M}}_{\theta^+}$ as elements. If $\lambda > \theta$, then $\overline{N}$ may not be able to construct A because it may not have $\bar{j}''\lambda$.

(Normal Embedding $\Rightarrow$ Normal ZFC Embedding): Let δ be the smaller of θ^+ or the least cardinal such that $\lambda^\delta > \lambda$, and let A be a subset of $\mathbf{H}_{\theta^+}$ of size λ. Note that δ is at least as big as κ since $\lambda^{<\kappa} = \lambda$. Then let $\overline{M} \preceq \mathbf{H}_{\lambda^+}$ be a transitive ZFC$^-$ model of size λ closed under $<\delta$ sequences that has each element of A, a θ-enumeration of the transitive closure of each element of A, and a λ-enumeration of A. Now by the normal embedding characterization of the λ-near θ-supercompactness of κ, there exists a transitive $\overline{N}$ of size λ

closed under $<\delta$ sequences and an elementary embedding $\bar{j} : \overline{M} \to \overline{N}$ with critical point κ such that $\bar{j}(\kappa) > \theta$ and $\bar{j}''\theta \in \overline{N}$. Since κ is inaccessible, it is inaccessible in $\overline{M}$ whereby $\bar{j}(\kappa)$ is inaccessible in $\overline{N}$. Consequently, because $\overline{N}$ is a transitive ZFC$^-$ model, $M = \mathbf{V}^{\overline{N}}_{\bar{j}(\kappa)}$ will be a transitive ZFC model. Note now that since $\overline{M}$ knows that $\mathbf{V}_\kappa$ is closed under $<\kappa$ sequences, $\overline{N}$ will think that M is closed under $<\bar{j}(\kappa)$ sequences and hence θ sequences, by elementarity. In particular, M will contain $\overline{N}$'s powerset of θ and hence its $\mathbf{H}_{\theta^+}$ and will actually be closed under $<\delta$ sequences by virtue of $\overline{N}^{<\delta} \subseteq \overline{N}$. But now also note that $\mathbf{H}^{\overline{M}}_{\theta^+} \subseteq \mathbf{H}^{\overline{N}}_{\theta^+}$ since $\bar{j}^{-1} \restriction \theta''(\bar{j}(B) \cap \bar{j}''\theta) = B$ for all $B \subseteq \theta$ in $\overline{M}$. Consequently, we have $A \subseteq \mathbf{H}^{\overline{M}}_{\theta^+} \subseteq \mathbf{H}^{\overline{N}}_{\theta^+} = \mathbf{H}^{M}_{\theta^+}$ so that $A \subseteq M$ as well. We also have $\kappa, \theta \in M$ as well. Then since M, which is a subset of $\overline{N}$, must clearly have size at most $|\overline{N}| = \lambda$, we may now apply the normal embedding characterization of the λ-near θ-supercompactness of κ to M to get a desired elementary embedding $j : M \to N$. Note also that if $\lambda = \theta$, then $A \in \mathbf{H}^{\overline{M}}_{\theta^+}$ so that $A \in M$. $\square$

Note that the Hauser embedding characterization is absent from this list of characterizations. The reason for this is similar to why the Normal ZFC embedding characterization only has $A \subseteq M$ instead of $A \in M$, mainly that when $\lambda > \theta$, the target model need not contain the image of λ under the embedding so that we cannot use this line of reasoning to show that it can construct a transitive model of size λ from the domain of the embedding.

Next, we state a simple corollary of Lemma 2.1.5 and a generalization of the Lévy-Solovay Theorem for nearly θ-supercompact cardinals, for λ-nearly θ-supercompact cardinals. The generalization of the Lévy-Solovay Theorem to the more general setting of λ-nearly θ-supercompact cardinals from the specific case when $\lambda = \theta$ has an almost identical proof.

Lemma 2.5.4. *If κ is λ-nearly θ-supercompact for some $\theta \geq \kappa$, and $\lambda \geq \theta$, then it is λ-nearly $\theta^{<\kappa}$-supercompact.*

Proof. For every $A \subseteq \lambda$, we can find a near θ-supercompactness embedding $j : M \to N$ with critical point κ between transitive ZFC$^-$ models with $A \in M$ by the λ-near θ-supercompactness of κ. By Lemma 2.1.5, it then follows that j is actually a near $\theta^{<\kappa}$-supercompactness embedding. Thus, we have an elementary embedding witnessing the λ-near $\theta^{<\kappa}$-supercompactness of κ for every $A \subseteq \lambda$ so κ is λ-nearly $\theta^{<\kappa}$-supercompact. $\square$

Theorem 2.5.5. *Fix cardinals $\lambda, \theta,$ and κ such that $\lambda \geq \theta \geq \kappa$ and $\lambda^{<\kappa} = \lambda$. After forcing of size less than κ, the cardinal κ is λ-nearly θ-supercompact in the extension if and only if it was λ-nearly θ-supercompact in the ground model.*

Proof. This proof of the Lévy-Solovay theorem for the λ-nearly θ-supercompact cardinals is almost identical to the corresponding proof of Theorem 2.1.6 for the nearly θ-supercompact cardinals. $\square$

Similarly, we have the following theorem:

Theorem 2.5.6. *Suppose κ, λ, and θ are cardinals for which $\lambda \geq \theta \geq \kappa$ and $\lambda^{<\kappa} = \lambda$ in a forcing extension $\boldsymbol{V}[G]$. Suppose further that $\boldsymbol{V} \subseteq \boldsymbol{V}[G]$ satisfies the δ approximation and cover properties for some $\delta < \kappa$ and that κ is λ-nearly θ-supercompact in $\boldsymbol{V}[G]$. Then κ is λ-nearly θ-supercompact in $\boldsymbol{V}$.*

Although the proof of the indestructibility result from Statement 1 of the Main Theorem does not readily generalize to λ-nearly θ-supercompact cardinals for arbitrary pairs (λ, θ), we can still prove the existence of functions with the Menas property for such cardinals. Specifically, given a λ-nearly θ-supercompact cardinal and a natural generalization of what it means to have a mapping with the Menas property for its λ-near θ-supercompactness, we can prove the existence of such a function.

Definition 2.5.7. *A function $f : \kappa \to \kappa$ has the Menas Property for a λ-nearly θ-supercompact cardinal κ if for every $A \subseteq \lambda$, there exist weakly θ-closed transitive ZFC$^-$ models M and N of size $\lambda^{<\kappa}$ with $A, f \in M$ and a* *near θ-supercompactness embedding $j : M \to N$ with critical point κ such that $j(f)(\kappa) > \theta$.*

Lemma 2.5.8. *If κ is a λ-nearly θ-supercompact cardinal and $\kappa \leq \delta \leq \theta^+$, then for every $<\delta$-closed transitive $M \models ZFC^-$ of size λ with $\theta \in M$, there exists a $<\delta$-closed transitive N of size λ, and a near θ-supercompactness embedding $j : M \to N$ with critical point κ where κ is not θ-supercompact in N.*

Proof. If $\lambda^{<\kappa} > \lambda$, then this result is vacuously true so assume that $\lambda^{<\kappa} = \lambda$, and suppose M is a $<\delta$-closed transitive ZFC$^-$ model of size λ with $\theta \in M$ for some δ such that $\kappa \leq \delta \leq \theta^+$. Also assume that κ is λ-nearly θ-supercompact. Then by the λ-near θ-supercompactness of κ, let N be a transitive $<\delta$-closed ZFC$^-$ model of size λ and $j : M \to N$ be a near θ-supercompactness embedding with critical point κ such that $j(\kappa)$ is smallest among all near θ-supercompactness embeddings $h^* : M \to M^*$ with critical point κ where M^* is a transitive $<\delta$-closed ZFC$^-$ model of size λ. Now suppose for contradiction that κ were θ-supercompact in N. Then we would have an elementary embedding $h : N \to N^*$ generated by what N thinks is a normal κ-complete fine measure μ_N on $P_\kappa\theta$. Because j is actually a near $\theta^{<\kappa}$-supercompactness embedding by Lemma 2.1.5, N can construct all subsets of $P_\kappa\theta$ contained in M, and all (regressive) functions from $P_\kappa\theta$ into θ contained in M. Consequently, we may restrict μ_N to $\mu_M \equiv \mu_N \cap M$ so that μ_M will generate a near θ-supercompactness embedding $h_M : M \to N_0^*$ with critical point κ where $N_0^* = \mathrm{Ult}(M, \mu_M)$ is the Mostowski collapse of the reduced ultrapower structure only using functions from M. In this case, N_0^* will be a $<\delta$-closed transitive ZFC$^-$ model of size λ.

Now, I claim that $h_M(\kappa) \leq h(\kappa)$. To see this, first observe that N contains all functions from $P_\kappa\theta$ into κ that M does, and for every function f with domain $P_\kappa\theta$ in M such that $\{\sigma \in P_\kappa\theta | f(\sigma) \in \kappa\} \in \mu_M$, the function $f' : P_\kappa\theta \to \kappa$ defined by setting $f'(\sigma)$ to $f(\sigma)$ when $f(\sigma) \in \kappa$ and 0 otherwise will be in M. Moreover, the equivalence class $(f)_{\mu_M}$ will equal $(f')_{\mu_M}$ in the reduced ultrapower structure. Therefore, it suffices to restrict our attention to functions $f : P_\kappa\theta \to \kappa$ from M when determining the $[f]_{\mu_M}$ that are in $h_M(\kappa)$. But then because $\mu_M \subseteq \mu_N$, we will have $[f]_{\mu_M} \in h_M(\kappa)$ only when $[f]_{\mu_N} \in h(\kappa)$ for every such function f in M (and hence in N). Furthermore, for all such $f : P_\kappa\theta \to \kappa$ and $g : P_\kappa\theta \to \kappa$ in M, we will have $\{\sigma \in P_\kappa\theta | f(\sigma) = g(\sigma)\} \in M$ so in this case $[f]_{\mu_M} = [g]_{\mu_M}$ if and only if $[f]_{\mu_N} = [g]_{\mu_N}$ by virtue of the fact that μ_M and μ_N agree on the measure 1 sets in M. Therefore, $h_M(\kappa) \leq h(\kappa)$.

However, now since $j(\kappa)$ is inaccessible in N with $j(\kappa) > \theta \geq \kappa$, and h is generated by what N thinks is a normal κ-complete fine measure μ_N on $P_\kappa\theta$, we have $h(\kappa) < ((\kappa^{\theta^{<\kappa}})^+)^N < j(\kappa)$. This contradicts the minimality of $j(\kappa)$ because then $h_M(\kappa) < j(\kappa)$. $\square$

Lemma 2.5.9 (Menas Property for λ-Nearly θ-Supercompact Cardinals). *Every λ-nearly θ-supercompact cardinal κ for which $\lambda \geq \theta \geq \kappa$ and $\lambda^{<\kappa} = \lambda$ has a function with the Menas property for its λ-near θ-supercompactness.*

Proof. Suppose κ is λ-nearly θ-supercompact, and let $A \subseteq \lambda$. Also, define $f : \kappa \to \kappa$ to be the function $\gamma \mapsto (2^{\sigma^{<\gamma}})^+$ where σ is the least cardinal above γ such that γ is not σ-supercompact if such a $\sigma < \kappa$ can be found and 0 otherwise. I will show that f is a function with the Menas property for the λ-near θ-supercompactness of κ.

Let δ be the least cardinal at or above κ such that $2^{\delta^{<\kappa}} > \theta$. Then κ is β-supercompact for all $\beta < \delta$ at or above κ. For each such $\beta < \delta$ at or above κ, let μ_β be a κ-complete normal fine measure on $P_\kappa\beta$. Then let $D \subseteq \theta$ code the collection of all such μ_β as well as a θ-sized collection of subsets of $\delta^{<\kappa}$. Now let $M \in \mathbf{H}_{\lambda^+}$ be a weakly θ-closed (and in particular $<\kappa$-closed) transitive ZFC$^-$ model with $A, D, \theta \in M$. By Lemma 2.5.8, let N be a transitive weakly θ-closed ZFC$^-$ model of size λ where κ is not θ-supercompact and $j : M \to N$

be a near θ-supercompactness embedding with critical point κ. Since M is closed under less than κ sequences, the definition of f is absolute to M. Consequently, by elementarity, N will think that $j(f)(\kappa) = (2^{\sigma^{<\kappa}})^+$ where σ is the least cardinal above κ such that κ is not σ-supercompact if such a $\sigma < j(\kappa)$ can be found and 0 otherwise. Because M contains $D \subseteq \theta$, the model N will be able to construct D from $j(D)$ and $j''\theta$ so that it will know that κ is β-supercompact for all $\beta < \delta$ at or above κ. Because N contains D, it will also know that $2^{\delta^{<\kappa}} \geq \theta$. Finally, because κ is not θ-supercompact in N with $\theta < j(\kappa)$, the model $N \models j(f)(\kappa) \geq (2^{\delta^{<\kappa}})^+ \geq (\theta^+) > \theta$, as desired. $\square$

Finally, we state the generalization of Lemma 2.4.6 to λ-nearly θ-supercompact cardinals.

Lemma 2.5.10. *Suppose κ is a λ-nearly θ-supercompact cardinal for some $\lambda \geq \theta \geq \kappa$ and $\kappa \leq 2^{<\sigma} \leq \lambda$ for some cardinal σ. Then if the λ-near θ-supercompactness of κ is indestructible by the forcing to add α many Cohen subsets of σ for all $\alpha \leq \lambda$, then its λ-near θ-supercompactness is indestructible by the forcing $Add(\sigma, \eta)$ for all η.*

Proof. This proof of this lemma is almost identical to the corresponding proof of Lemma 2.4.6 for the nearly θ-supercompact cardinals. $\square$

There is still much to explore with these new large cardinals, and I plan to continue investigating them further. As referenced in the introduction, I plan to pursue the questions about whether we can make a weakly measurable cardinal or nearly θ-supercompact cardinal the least weakly compact cardinal in a forcing extension with Gitik and Hamkins. There are also questions about the consistency strength of nearly θ-supercompact cardinals κ when $\theta < 2^\kappa$ that I would like to help answer. More generally, what further insights about our large cardinal hierarchy might these notions provide? I look forward to my future study on these questions.

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