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# Spectral theory using operator algebra techniques

Paliogiannis, Fotios Constantinou, Ph.D.

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300 N. Zeeb Rd.
Ann Arbor, MI 48106

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# SPECTRAL THEORY USING OPERATOR ALGEBRA TECHNIQUES

by

FOTIOS C. PALIOGIANNIS

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May 10, 1991 — [signature]

date — Chairman of Examining Committee

May 10, 1991 — [signature]

date — Executive Officer

[signature]

[signature]

Adam Koranyi

Supervisory Committee

The City University of New York

## Abstract

SPECTRAL THEORY USING
OPERATOR ALGEBRA TECHNIQUES
by
Fotios Paliogiannis

Adviser: Professor Stanley Kaplan

In this work,we study the Spectral Theorem (the Functional calculus as well) for self adjoint and normal operators, both in the bounded and unbounded cases. The approach, to this structure theorem, is based on the following key theorem:

**Theorem**: The Gelfand (or Structure) space of an abelian von Neumann algebra on a Hilbert space H, is extremely disconnected. The idea goes back to the (1952) paper of M.G.Fell and J.L.Kelley *An algebra of unbounded Operators* Proc.Nat.Acad.Sci. USA 38 592-598. R.V.Kadison and J.R.Ringrose discuss the spectral theorem from this point of view in their (1983) book Chap.5 Vol. I. Our development is strongly suggested by this discussion, although our approach and proofs differ. Our proofs incorporate several ideas tying together function theory and operator theory.

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I consider it my duty to express thanks to the staff of our computer center facilities and to the colleagues Alessandra Carbone and Costas Georgatos for their help in the typing of this work.

May 6, 1991

*To my parents and to my brother Christos*

'Ανèβα... 'Ανèβα... Πàντα ἀνèβαινε
'Ακòμη, ἀκòμη πιὸ ψηλὰ
Στὴν κορφὴ σὲ περιμὲνει ἡ 'Αγὰπη
Μ' ἓνα μπουκèτο τριαντὰφυλλα

'Ανèβα. Ὅλο μπρὸς. Ὅλο ψηλὰ
Κι' ἂν δεν βρεῖς δρὸμο
Φτιὰξε. Στὴν ἀγὰπη -
δὲν ὑπὰρχουν δρὸμοι ἕτοιμοι
Τοὺς φτιὰχνεις ἐσὺ

'Ανèβα... Ἔστω κι' ἂν δεῖς
Πὼς τα λουλοὺδια ἦταν ψεὺτικα
Κι' ἡ ἀγὰπη - ἡ ὁλὸφλογη ἀγὰπη -
Ἓνας καπνὸς
'Εσὺ ἀνèβα

'Ανèβα... Ἔστω κι' ἂν στὴν κορφὴ
- αντὶς γιὰ τὰ τριαντὰφυλλα -
Σὲ περιμὲνει ἓνα μπουκèτο μαχαὶρια
'Εσὺ ἀνèβα

'Ανèβα... καὶ πὲς «εὐχαριστῶ»
- ὄχι στὰ τριαντὰφυλλα, ὄχι στὰ μαχαὶρια -
Πὲς «εὐχαριστῶ» στὴ Δὺναμη
Ποὺ σ' ἒκανε ν' ἀ ν è β ε ι ς

Μενèλαος Λουντèμης
*'Απὸ τα ποιητικὴ συλλογὴ*
*«Κονσèρτο γιὰ δὺο μυδρὰλια κι' ἕνα ἀηδὸνι»*

*Go up ... Reach high ... Go always up*
*On the top Love is waiting for you*
*with a bouquet of roses*

*Go up. Always ahead. Always up.*
*And if you don't find a way*
*Make one. For love the ways are not ready*
*You make them yourself*

*Go up ... Even if you see*
*that the roses were not real*
*And if love - the burning love - were*
*A smoke*
*Go on*

*Go on ... Even if on the top*
*instead of the roses*
*A bouquet of knives is waiting for you*
*Go on.*

*Go up ... And say ' thanks '*
*not to the roses, not to the knives*
*Say ' thanks ' to the Strength*
*that made you to go up*

*Menelaos Lountemis*

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## §.1 INTRODUCTION

*Gentlemen: there is lots of room left in Hilbert space*

Saunders MacLane

Let H be a complex Hilbert space and B(H) be the set of bounded linear operators of H into H. This set possesses the structure of an algebra over the field of complex numbers $\mathbb{C}$. There is an **adjoint** operation in B(H). If $A \in B(H)$ we will always denote the adjoint of A by $A^*$. We have

$$(A+B)^* = A^*+B^*, \quad (\lambda A)^* = \bar{\lambda} A^*$$
$$(AB)^* = B^*A^*, \quad A^{**} = A$$

($\lambda$ denotes a complex number, $\bar{\lambda}$ its complex conjugate )
We thus see that B(H) can be regarded as an involutive or *-algebra.
Every subalgebra of B(H) which is stable with respect to the adjoint operation is called an involutive or *-subalgebra of B(H) or a *-algebra of operators.
For an infinite-dimensional Hilbert space H the operator algebra B(H) has several interesting (vector space) topologies. Besides the (operator) norm topology, the two most important are the strong and the weak operator topology.

**Definition**: The **strong operator topology (s.o.t)** on B(H) is the locally convex vector space topology induced by the family of seminorms

$$\rho_x(A) = \|Ax\| \quad , \; x \in H \quad , A \in B(H) \; .$$

Thus, a net of operators $A_d \to A$ in s.o.t $\iff$ $A_d x \to Ax \quad \forall x \in H$

**Definition**: The **weak operator topology (w.o.t)** on B(H) is the locally convex vector space topology induced by the family of seminorms

$$\sigma_{x,y}(A) = |(Ax,y)| \quad , \; x,y \in H, \; A \in B(H) \; .$$

Thus, a net of operators $A_d \to A$ in w.o.t $\iff$ $(A_d x,y) \to (Ax,y) \quad \forall x,y \in H$

**Definition**: Let S be any subset of B(H). We call the **commutant** of S, to be denoted by S', the set of those elements of B(H) that commute with all the elements of S , i.e.,

$$S' = \{ T \in B(H): \; AT = TA \;\; \forall A \in S \}$$

The set $(S')' = S''$ is called the **bicommutant** of S. $(S'')' = S'''$ , ...

The set $(S')'=S''$ is called the **bicommutant** of S. $(S'')'=S'''$ , ...

It is easy to see that S' is a subalgebra of B(H) which is closed in both s.o.t and w.o.t .

**Definition**: A *-subalgebra $\mathcal{A}$ of B(H) which is closed in the w.o.t is called a **von Neumann algebra** or **W*-algebra.**

The celebrated Double commutant theorem of von Neumann (1929) states that:

**1.1. Theorem: (Double commutant)**
For a *-subalgebra $\mathcal{A}$ of B(H) , $I \in \mathcal{A}$ the following are equivalent:

i) $\mathcal{A} = \mathcal{A}''$

ii) $\mathcal{A}$ is closed in w.o.t

iii) $\mathcal{A}$ is closed in s.o.t. (see [14] p. 326)

It is this fundamental theorem which leads some authors to define a von Neumann algebra to be a *-subalgebra $\mathcal{A}$ of B(H) such that $\mathcal{A} = \mathcal{A}''$.

Note also that a von Neumann algebra $\mathcal{A}$, being closed in the s.o.t. is also closed in the (operator) norm topology (sometimes called the **uniform operator topology (u.o.t.)**). Furthermore,the so-called *C*-condition* $\|A^*A\| = \|A\|^2$ holds $\forall A \in B(H)$.

Thus, we see that $\mathcal{A}$ is a C*-algebra and therefore any result of C*-algebra theory also applies to von Neumann algebras.

We now collect a few results from C*-algebra Theory. For a detailed discussion of C*-algebra theory we refer the reader to R. Doran and V. Belfi [4].

**Definition**: An (abstract) **C*-algebra** is a Banach algebra $\mathcal{U}$ having an involution $*$ (that is, a conjugate-linear map of $\mathcal{U}$ into itself satisfying $x^{**} = x$ and $(xy)^* = y^*x^*$ , $x,y \in \mathcal{U}$ ) which satisfies the condition $\|x^*x\| = \|x\|^2 \quad \forall x \in \mathcal{U}$.

Let $\mathcal{U}$ be a commutative C*-algebra .Then in particular $\mathcal{U}$ is a commutative Banach algebra and the beautiful Gelfand structure theory of commutative Banach algebras applies.

**Definition**: A **multiplicative linear functional** on $\mathcal{U}$ is a nonzero linear functional $\rho$ on $\mathcal{U}$ satisfying $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in \mathcal{U}$ , i.e.,
$\rho: \mathcal{U} \to \mathbb{C}$ is an algebra homomorphism of $\mathcal{U}$ onto $\mathbb{C}$.

The set of all multiplicative linear functionals on $\mathcal{U}$ will be denoted by $\mathbf{X}_{\mathcal{U}}$.

**Definition**:
For each $x \in \mathcal{U}$ we define $\hat{x}: X_{\mathcal{U}} \to \mathbb{C}$ by

$$\hat{x}(\rho) = \rho(x) \quad \forall \rho \in X_{\mathcal{U}}$$

The function $\hat{x}$ is called the **Gelfand transform** of x.

The Gelfand transform has the following properties:
For $x,y \in \mathcal{U}$ and $\lambda \in \mathbb{C}$

i) $(x+y)^\wedge = \hat{x} + \hat{y}$

ii) $(\lambda x)^\wedge = \lambda \hat{x}$

iii) $(xy)^\wedge = \hat{x}\hat{y}$

iv) $|\hat{x}(p)| \leq \|x\| \quad \forall p \in X_{\mathcal{U}}$

v) if $\mathcal{U}$ has an identity e, then $\sigma_{\mathcal{U}}(x) = \hat{x}(X_{\mathcal{U}}) = \text{Range}(\hat{x})$
where, $\sigma_{\mathcal{U}}(x) = \{ \lambda \in \mathbb{C} : x - \lambda e$ does not have a two-sided inverse in $\mathcal{U} \}$
is the **spectrum** of x.

vi) if $\mathcal{U}$ has an identity, $x \in \mathcal{U}$ is invertible iff $\hat{x}(\rho) \neq 0$ for all $\rho \in X_{\mathcal{U}}$

The **Gelfand topology** on $X_{\mathcal{U}}$ is defined to be the weakest topology on $X_{\mathcal{U}}$ under which all $\hat{x}$ are continuous, $x \in \mathcal{U}$. Equivalently, the Gelfand topology is the relative topology which $X_{\mathcal{U}}$ inherits as a subset of the dual space $\mathcal{U}^*$ of $\mathcal{U}$, with the weak *-topology.

**Definition**: The **Gelfand space** of $\mathcal{U}$ is the set $X_{\mathcal{U}}$ with the Gelfand topology (also called the **structure space** or the **maximal ideal space** of $\mathcal{U}$ ).

For an arbitary commutative Banach algebra $\mathcal{U}$, $X_{\mathcal{U}}$ is a locally compact Hausdorff space and if $\mathcal{U}$ has an identity , then $X_{\mathcal{U}}$ is a compact Hausdorff space.

Let $C(X_{\mathcal{U}})$ be the algebra of continuous complex-valued functions on $X_{\mathcal{U}}$.

**Definition**:The mapping $\wedge : \mathcal{U} \to C(X_{\mathcal{U}})$ , $x \mapsto \hat{x}$ called the **Gelfand map** or the **Gelfand representation** is a homomorphism from $\mathcal{U}$ into $C(X_{\mathcal{U}})$.

Moreover, if $\| . \|_\infty$ denotes the sup-norm on $C(X_{\mathcal{U}})$, then $\| \hat{x} \|_\infty \leq \| x \|$ and hence $x \mapsto \hat{x}$ is continuous.
Furthermore, $\| \hat{x} \|_\infty = r(x) = \sup\{ |\lambda| : \lambda \in \sigma_{\mathcal{U}}(x) \}$, the *spectral radius* of x.

One of the most important theorems in the C*-algebra Theory is the Gelfand -Naimark theorem which characterizes all the commutative C*-algebras. (see[4] p.27)

**1.2. Theorem: (Gelfand-Naimark)**
Let $\mathcal{U}$ be a commutative C*-algebra. Then the Gelfand map $x \mapsto \hat{x}$ is an isometric *-isomorphism of $\mathcal{U}$ onto $C(X_{\mathcal{U}})$.

In particular $\widehat{(x^*)} = \overline{\hat{x}}$. We write in this case $\mathcal{U} \cong C(X_{\mathcal{U}})$.

The Gelfand-Naimark theorem provides the basis for a powerful functional calculus in C*-algebras.
In general, the spectrum of an element in a Banach algebra may become larger upon passing to a subalgebra. But if $\mathcal{U}$ is a C*-algebra and $\mathcal{U}_o$ is a C*-subalgebra of $\mathcal{U}$, $x \in \mathcal{U}_o$ then the two spectra are the same, that is, $\sigma_{\mathcal{U}}(x) = \sigma_{\mathcal{U}_o}(x)$ . (see [4] p. 25)

Now, let x be a **normal** element in a C*-algebra $\mathcal{U}$, that is, $\mathbf{xx^* = x^*x}$
Let $\mathcal{U}_o$ be any closed commutative *-subalgebra of $\mathcal{U}$, which contains x and e, (e is the identity in $\mathcal{U}$). For example one can take $\mathcal{U}_o$ to be the closed *-subalgebra generated by x and e, that is, the closure in the u.o.t. of all polynomials in x and x*.
Then $\sigma_{\mathcal{U}}(x) = \sigma_{\mathcal{U}_o}(x)$ and by the Gelfand-Naimark theorem $\mathcal{U}_o \cong C(X_{\mathcal{U}_o})$.
Given now a continuous function $f \in C(\sigma_{\mathcal{U}}(x))$ then $f \circ \hat{x}$ is a continuous function on $X_{\mathcal{U}_o}$ , i.e., $f \circ \hat{x} \in C(X_{\mathcal{U}_o})$.
Hence there exists a unique element $y \in \mathcal{U}_o$ such that $\hat{y} = f \circ \hat{x}$.
It is customary to denote this element y in $\mathcal{U}_o$ by f(x).

The mapping $\mathbf{\Phi}: C(\sigma_{\mathcal{U}}(x)) \longrightarrow \mathcal{U}_o$ defined by $\mathbf{\Phi(f) = (f \circ \hat{x})^{\vee}}$ (= **f(x)**) where $^{\vee}$ denotes the inverse of the Gelfand transform,is an isometric *-isomorphism of $C(\sigma_{\mathcal{U}}(x))$ onto $\mathcal{U}_o$ having the following properties:

i) $\Phi(1) = e$ where 1 denotes the constant function 1 on $\sigma_{\mathcal{U}}(x)$

ii) $\Phi(id) = x$ where id denotes the identity function on $\sigma_{\mathcal{U}}(x)$, $id(\lambda) = \lambda$

iii) $\sigma_{\mathcal{U}}(f(x)) = f(\sigma_{\mathcal{U}}(x))$ **(spectral mapping theorem)**

iv) f(x) is contained in every closed commutative *-subalgebra of $\mathcal{U}$ which contains x and e, thus f(x) is independent of the C*-algebra $\mathcal{U}_o$ used in its definition.

This process of "applying" continuous functions on $\sigma_{\mathcal{U}}(x)$ to x is called
**the (continuous) functional calculus.**

We remark,that if $\varphi : \mathcal{U} \longrightarrow \mathcal{B}$ is a *-homomorphism between the C*-algebras $\mathcal{U}$ and $\mathcal{B}$, then $\varphi$ is continuous,in fact $\|\varphi\| = 1$ and $\forall x \in \mathcal{U}$ $\sigma_{\mathcal{B}}(\varphi(x)) \subseteq \sigma_{\mathcal{U}}(x)$ (see [14] p.242).

**1.1 Proposition**: Let $\mathcal{U}$ and $\mathcal{B}$ be C*-algebras, $x \in \mathcal{U}$ a normal element and $\varphi: \mathcal{U} \to \mathcal{B}$ be a *-homomorphism.Then $\varphi(f(x)) = f(\varphi(x)) \quad \forall f \in C(\sigma_{\mathcal{U}}(x))$.

**Proof.**
First note that $f(\varphi(x))$ makes sense, since $\sigma_{\mathcal{B}}(\varphi(x)) \subseteq \sigma_{\mathcal{U}}(x)$ and $\varphi(x)$ is normal when x is normal.
Now let $\mathcal{F} = \{f \in C(\sigma_{\mathcal{U}}(x)): \varphi(f(x)) = f(\varphi(x))\}$, $\mathcal{F}$ is a closed *-subalgebra of $C(\sigma_{\mathcal{U}}(x))$,contains 1 and id (so separates the points of $\sigma_{\mathcal{U}}(x)$) and so by the Stone-Weierstrass theorem $\mathcal{F} = C(\sigma_{\mathcal{U}}(x))$. ∎

**1.2 Proposition**: Let $\mathcal{U}$ be a C*-algebra and $x \in \mathcal{U}$ be normal.
If $g \in C(\sigma_{\mathcal{U}}(x)$ and $f \in C(\sigma_{\mathcal{U}}(g(x)))$, then $f \circ g \in C(\sigma_{\mathcal{U}}(x))$ and $f \circ g(x) = f(g(x))$.

**Proof.**
By the spectral mapping theorem we have $\sigma_{\mathcal{U}}(g(x)) = g(\sigma_{\mathcal{U}}(x))$, so since f is continuous on $\sigma_{\mathcal{U}}(g(x))$ we conclude that $f \circ g$ is continuous on $\sigma_{\mathcal{U}}(x)$.
Now let $\mathcal{U}_o$ be the C*-subalgebra of $\mathcal{U}$ generated by x; then x, g(x), f(g(x)), all belong to $\mathcal{U}_o$. $\mathcal{U}_o \cong C(X_{\mathcal{U}_o})$ and by the functional calculus, we have

$$\widehat{f(g(x))} = f(\widehat{g(x)}) = f(g(\hat{x})) = \widehat{f \circ g(x)}$$

therefore

$$f(g(x)) = f \circ g(x). \blacksquare$$

Later we will extend propositions 1.1, 1.2, so that they will be valid for functions in a larger algebra.
In fact, they are valid for functions in the algebra of bounded Borel functions (see, propositions 4.4, 4.6) and furthermore for functions in the algebra of unbounded Borel functions(propositions 8.9, 8.11).

The set of all **positive elements** in an abstract C*-algebra $\mathcal{U}$ is defined to be the set of those elements $x \in \mathcal{U}$ such that x is **hermitian** (, i.e., $\mathbf{x = x^*}$) and $\sigma_{\mathcal{U}}(x) \subseteq [0,+\infty)$.
We then write $x \geq 0$. We denote by $\mathcal{U}^+ = \{x \in \mathcal{U} : x \geq 0\}$.
It is well known that $\mathcal{U}^+$ is a closed convex cone in $\mathcal{U}$ such that $\mathcal{U}^+ \cap (-\mathcal{U}^+) = \{0\}$.
(see [4] p.33)

We remark that since Range$(\hat{x}) = \sigma_{\mathcal{U}}(x)$ the Gelfand map $\wedge : \mathcal{U} \to C(X_{\mathcal{U}})$ is order preserving.
Moreover, by the spectral mapping theorem and the fact that $\Phi$ is a *-map, that is,

$\Phi(\bar{f}) = \Phi(f)^*$, the mapping $\Phi: C(\sigma_{\mathcal{U}}(x)) \to \mathcal{U}$ giving the functional calculus is also order preserving, i.e.,

$$\text{if } f \geq 0 \text{ in } C(\sigma_{\mathcal{U}}(x)), \text{ then } \Phi(f) \geq 0 \text{ in } \mathcal{U}.$$

We quote two theorems which are going to be used in various ocassions in the chapters that follow;

**1.3 Theorem**: **(Fuglede)**
If $A \in B(H)$ is a normal operator and B is an operator in B(H) for which AB = BA, then $A^*B = BA^*$ (see [5] p.114).

Fuglede`s theorem is extended and proved here, in the case where A is an unbounded normal operator.(see Theorem 8.10)

**1.4 Theorem**: The weak- and strong-operator closures of a convex subset $\mathcal{K}$ of B(H) coincide (see [14] p.305).

❐

## § 2. THE GELFAND SPACE OF AN ABELIAN VON NEUMANN ALGEBRA.

The topological notions we are going to define play an important role in our study of the Spectral Theory of self adjoint and normal operators on a Hilbert space.

**Definition**: A topological space X is called **extremely disconnected** if whenever G is an open subset of X, $\bar{G}$ is open as well (thus, $\bar{G}$ is **clopen**).

For a topological space X the following are equivalent:
i) X is extremely disconnected
ii) if F is a closed subset of X, then the interior of F, int(F), is clopen

iii) if $G_1, G_2$ are open subsets of X such that $G_1 \cap G_2 = \emptyset$ , then $\bar{G}_1 \cap \bar{G}_2 = \emptyset$
The equivalence of i), ii) and iii) can be easily proved.

**Definition**: A topological space X is called **totally disconnected** if given any two points $x,y \in X$ with $x \neq y$ there are clopen disjoint subsets of X, $C_1$ and $C_2$ such that $x \in C_1$ and $y \in C_2$.

**Definition**: A **Stonean space** is a compact Hausdorff extremely disconnected space.

Note that such a space is totally disconnected. We remark that the converse is not true:

Take for example $X_i = \{0,1\}$ i =1,2,3,... $X = \prod_i^{\infty} X_i$ with the product topology, X is totally disconnected but not extremely disconnected. (see [14] p.222)

However one can see that the following is true:
Let X be a compact Hausdorff space and $\mathcal{C}$ be the family of clopen subsets of X.Then the following are equivalent:
i) X is extremely disconnected.
ii) a) X is totally disconnected and
b) $\mathcal{C}$ is a complete lattice
(i.e., the l.u.b. and the g.l.b. of any family of sets in $\mathcal{C}$ are in $\mathcal{C}$).

Let C(X) as before denote the algebra of continuous complex-valued functions on a compact Hausdorff space X. By the Gelfand-Naimark Theorem, C(X) is the most general example up to *-isometric isomorphism of a commutative C*-algebra. $C_{\mathbb{R}}(X)$ denotes the set of all continuous real-valued functions on X.

If $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ is a collection of functions in $C_{\mathbb{R}}(X)$, we denote by $\bigvee_{\alpha \in A} f_\alpha$ the l.u.b.$\{f_\alpha : \alpha \in A\}$ and by $\bigwedge_{\alpha \in A} f_\alpha$ the g.l.b.$\{f_\alpha : \alpha \in A\}$

**2.1 Theorem**: If each set of functions in $C_{\mathbb{R}}(X)$ that has an upper bound in $C_{\mathbb{R}}(X)$ has a least upper bound in $C_{\mathbb{R}}(X)$ (so that $C_{\mathbb{R}}(X)$ is a boundedly complete lattice) , then X is extremely disconnected, i.e., X is Stonean.

We will see later that the converse of Theorem 2.1 is also true.

**2.2 Theorem**: Let $\mathcal{A}$ be an abelian Von Neumann algebra on a Hilbert space H. Then its Gelfand space $X_{\mathcal{A}}$ is extremely disconnected (thus, Stonean).

The proofs of both of the above Theorems are given in [14] p.223, p.310.

The results that follow are strongly suggested by the developenemt in Chapter 5 of [14].

**Definition**: Let X be a compact Hausdorff space. A **projection** $e \in C(X)$ is the indicator function of a clopen set.A familly $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ of projections in C(X) indexed by $\mathbb{R}$ satisfying:

i) $\bigvee_{\lambda \in \mathbb{R}} e_\lambda = 1 \quad , \quad \bigwedge_{\lambda \in \mathbb{R}} e_\lambda = 0$

ii) $\forall \lambda \in \mathbb{R} \quad e_\lambda = \bigwedge_{\mu > \lambda} e_\mu$

is said to be a **resolution of the identity** in C(X).
If in addition; iii) $\exists$ m,M such that $e_\lambda = 0 \;\; \forall \;\; \lambda < m$ and $e_\lambda = 1 \;\; \forall \;\; \lambda \geq M$ .

Then we say that $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is a **bounded resolution of the identity** with bounds m and M.

**EXAMPLES.**

**2.1**. Let X be a Stonean space and f a real-valued function in C(X). For $\lambda \in \mathbb{R}$, define $X_\lambda = \text{int}\{x \in X: f(x) \leq \lambda\}$. $X_\lambda$ is a clopen subset of X, since f is continuous and X is extremely disconnected.

Now take $e_\lambda = \mathcal{X}_{X_\lambda}$. If $m = \inf\{f(x): x \in X\}$ and $M = \sup\{f(x): x \in X\}$, it can be seen easily that $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is a bounded resolution of the identity with bounds $m$ and $M$.

Note also that $\{x \in X: f(x) < \lambda\} \subseteq X_\lambda \subseteq \{x \in X: f(x) \leq \lambda\}$.

**2.2.** Let X be a Stonean space and let $\{f_a\}_{a \in A}$ be a family of real-valued functions in C(X) bounded below by $m$ and above by $M$. Define $X_\lambda = \text{int}\{\bigcap_{a \in A} \{x \in X: f_a(x) \leq \lambda\}\}$. $X_\lambda$ is clopen $\forall \lambda \in \mathbb{R}$. Take $e_\lambda = \mathcal{X}_{X_\lambda}$. Then $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is a bounded resolution of the identity in C(X) with bounds $m$ and $M$.

**2.1 Proposition**: Let X be a Stonean space and let $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ be a bounded resolution of the identity in C(X).Then there exists a unique real-valued function $\phi$ on X such that

$$(*) \quad \{x \in X: \phi(x) < \lambda\} \subseteq X_\lambda \subseteq \{x \in X: \phi(x) \leq \lambda\},$$

where $X_\lambda = \{x \in X: e_\lambda(x) = 1\} = \{e_\lambda = 1\}$

In fact, for $x \in X$

$$\begin{aligned} \phi(x) &= \sup\{\mu: e_\mu(x) = 0\} && (1) \\ &= \inf\{\kappa: e_\kappa(x) = 1\} && (2) \\ &= \int \lambda \, de_\lambda(x) && (3) \end{aligned}$$

Moreover, $m \leq \phi(x) \leq M \;\; \forall x \in X$, where $m$ and $M$ are the bounds of $\{e_\lambda\}$.

Furthermore, if $\Pi = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is any partition of $[m^-, M]$ ( i.e., $\lambda_0 < m$, $\lambda_0 < \lambda_1 < \ldots < \lambda_n = M$) and $\xi_1, \xi_2 \ldots, \xi_n$ are any numbers chosen so that $\lambda_{j-1} \leq \xi_j \leq \lambda_j \;\; j=1, 2, \ldots n$ and if $\phi_{\pi,\xi} = \sum_{j=1}^{n} \xi_j (e_{\lambda_j} - e_{\lambda_{j-1}})$, then

$$\| \phi - \phi_{\pi,\xi} \|_\infty \leq \|\Pi\|$$

where $\|\Pi\| = \max\{(\lambda_j - \lambda_{j-1}): j = 1,2,..,n\}$. In particular $\phi \in C(X)$.

**Proof.**

Clearly (1), (2), (3) are the same since for fixed x, $e_\lambda(x)$ is nondecreasing in $\lambda$ with values 0 or 1, 0 for $\lambda < m$ and 1 for $\lambda \geq M$.

$\phi(x)$ is the value $\lambda'$ where $e_\lambda(x)$ goes from 0 to 1.

Now, if $\phi(x) < \lambda$, then $e_\lambda(x) = 1$ and so

$$\{x \in X: \phi(x) < \lambda\} \subseteq X_\lambda .$$

If $\phi(x) > \lambda$ , then $e_\lambda(x) = 0$ and so $e_\lambda(x) = 1$ which implies $\phi(x) \le \lambda$ , that is,

$$X_\lambda \subseteq \{x \in X: \phi(x) \le \lambda\}.$$

Thus $(*)$ is proved. It is easy to see that $(*)$ determines a unique $\phi$.
Now let $x \in X$. If $\lambda_{j-1} < \phi(x) < \lambda_j$ , then $\phi_{\pi,\xi}(x) = \xi_j$ ,
therefore

$$|\phi(x) - \phi_{\pi,\xi}(x)| = |\phi(x) - \xi_j| < (\lambda_j - \lambda_{j-1}) \le \|\Pi\|.$$

If $\phi(x) = \lambda_j$ $j < n$ , then $\phi_{\pi,\xi}(x) = \{\xi_j \text{ or } \xi_{j+1}\}$, so $|\phi(x) - \phi_{\pi,\xi}(x)| \le \|\Pi\|$,
hence

$$\sup_{x\in X} |\phi(x) - \phi_{\pi,\xi}(x)| \le \|\Pi\| \quad \text{, i.e.,} \quad \|\phi - \phi_{\pi,\xi}\|_\infty \le \|\Pi\|.$$

If $\phi(x) = M = \lambda_n$ , then $\phi_{\pi,\xi}(x) = \xi_n$ and the same estimate holds. ■

We can now combine example 2.1 and the above proposition to get the following corollary.

**2.1 Corollary**: Let X be a Stonean space and let f be in $C_{\mathbb{R}}(X)$. Then there exists a bounded resolution of the identity $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ in C(X) such that $f = \int \lambda de_\lambda$ , the integral being a Riemann-Stieltjes integral.

Next, we prove the converse of Theorem 2.1.

**2.3 Theorem**: Let X be a Stonean space and let $\mathcal{F}$ be a set of functions in $C_{\mathbb{R}}(X)$ that has an upper bound in $C_{\mathbb{R}}(X)$.Then $\mathcal{F}$ has a least upper bound in $C_{\mathbb{R}}(X)$.

**Proof.**

Let $\mathcal{F} = \{f_a\}_{a\in A}$ be such that $f_a \le M$ $\forall a$.
Choose any $a_0 \in A$ and let $m = \inf\{f_{a_0}(x): x \in X\}$.
Then

$$X_\lambda = \operatorname{int}\{\bigcap_{a\in A}\{x \in X: f_a(x) \le \lambda\} \subseteq \operatorname{int}\{x \in X: f_{a_0}(x) \le \lambda\}.$$

$e_\lambda = \chi_{X_\lambda}$ defines a resolution of the identity bounded by $m$ and $M$.
If $\phi = \int \lambda de_\lambda$ , then it is easily seen that $\phi = \text{l.u.b}(\mathcal{F})$. ■

**2.2 Proposition**: Let X be a Stonean space and let $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ be a bounded resolution of the identity in C(X), $X_\lambda = \{x\in X: e_\lambda(x) = 1\}$, $\phi\in C_{\mathbb{R}}(X)$.
Then the following statements are equivalent:

i) $\phi(x) = \int\lambda de_\lambda(x)$

ii) $\forall\lambda\in\mathbb{R} \quad \{x\in X: \phi(x) < \lambda\}\subseteq X_\lambda \subseteq \{x\in X: \phi(x) \le \lambda\}$

iii) $X_\lambda = \text{int}\{x\in X: \phi(x) \le \lambda\} = \text{int}\{\phi \le \lambda\}$

iv) $\forall\lambda\in\mathbb{R} \quad \phi e_\lambda \le \lambda e_\lambda$ and $\lambda(1-e_\lambda) \le \phi(1-e_\lambda)$.

**Proof.**
(i)$\Leftrightarrow$(ii): is established in proposition 2.1.
(iii)$\Rightarrow$(ii): Obvious
(ii)$\Leftrightarrow$(iv): Obvious
(ii)$\Rightarrow$(iii): Since $X_\lambda\subseteq\{\phi \le\lambda\}$ and $X_\lambda$ is clopen we have that $X_\lambda \subseteq \text{int}\{\phi\le\lambda\}$.
To get equality, suppose $\mu > \lambda$ and let $Y_\lambda = \text{int}\{\phi \le \lambda\}\subseteq\{\phi < \mu\}\subseteq X_\mu$.
Let $g_\lambda = \mathcal{X}_{Y_\lambda}$ ; so $g_\lambda$ is continuous and $g_\lambda < e_\mu \ \forall\mu>\lambda$.
Thus $g_\lambda \le \bigwedge_{\mu>\lambda} e_\mu = e_\lambda$ and so $Y_\lambda\subseteq X_\lambda$ . ■

**2.3 Proposition**: Let X be a Stonean space, $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ a resolution of the identity in C(X), $\phi$ a real-valued function defined by $\phi(x) = \int\lambda de_\lambda(x) \quad \forall x\in X$.
Then the following statements are equivalent:

i) $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ is bounded.

ii) $\phi$ is continuous on X.

iii) $\phi$ is bounded on X.

**Proof.**
(i)$\Rightarrow$(ii): By proposition 2.1, $\phi$ is the uniform limit of continuous functions, so continuous.
(ii)$\Rightarrow$(iii): Since $\phi$ is continuous and X is compact $\phi$ is bounded on X.
(iii)$\Rightarrow$(i): We remark first that,

$$\bigwedge_{\lambda\in\mathbb{R}} e_\lambda = \mathcal{X}_{\text{int}(\bigcap_\lambda X_\lambda)} \quad \text{and} \quad 1 - \bigwedge_{\lambda\in\mathbb{R}}(1-e_\lambda) = \bigvee_{\lambda\in\mathbb{R}} e_\lambda$$

and so

$$\bigvee_{\lambda\in\mathbb{R}} e_\lambda = 1 - \mathcal{X}_{\text{int}(\bigcap_\lambda X_\lambda^c)} = \mathcal{X}_{(\text{int}(\bigcap_\lambda X_\lambda^c))^c} = \mathcal{X}_{\overline{\bigcup_\lambda X_\lambda}}$$

Thus,

$$\bigwedge_{\lambda\in\mathbb{R}} e_\lambda = 0 \quad \text{which implies} \quad \mathcal{X}_{\mathrm{int}(\bigcap_\lambda X_\lambda)} = \emptyset .$$

Now, suppose $m \le \phi(x) \le M \ \ \forall x\in X$ and $\lambda_0 < m$. If $x \in X_{\lambda_0}$, then $e_{\lambda_0}(x) = 1$.

If $\lambda_1 < \lambda_0$ then $e_{\lambda_1}(x) = 1$, for otherwise, i.e., if $e_{\lambda_1}(x) = 0$ then $\lambda_1 \le \phi(x) \le \lambda_0 < m$.

But this is impossible since $m \le \phi(x) \quad \forall x\in X$.

So $\forall\ \lambda_1 < \lambda_0\ \ e_{\lambda_1}(x) = 1$ and since by $e_\lambda = \bigwedge_{\mu>\lambda} e_\mu$ we have that $e_\lambda$ is monotonic,

we conclude that $e_\lambda(x) = 1 \ \ \forall\lambda\in\mathbb{R}$.

Therefore $x \in \bigcap_\lambda X_\lambda$ , so $X_{\lambda_0} \subseteq \bigcap_\lambda X_\lambda$ .

Thus $X_{\lambda_0} \subseteq \mathrm{int}(\bigcap_\lambda X_\lambda) = \emptyset$ , i.e., . $X_{\lambda_0} = \emptyset$ , that is, $e_{\lambda_0} = 0$ for $\lambda_0 < m$.

For $\lambda_0 > M$, let $x \in (X_{\lambda_0})^c$ .So $e_{\lambda_0}(x) = 0$.

If $\lambda_1 > \lambda_0$ , then $e_{\lambda_1}(x) = 0$, for otherwise, i.e., if $e_{\lambda_1}(x) = 1$, then $M < \lambda_0 \le \phi(x) \le \lambda_1$,

which is impossible since $\phi(x) \le M \ \ \forall x\in X$.

So $\forall\ \lambda_1 > \lambda_0\ \ e_{\lambda_1}(x) = 0$ and since $e_\lambda$ is monotonic $e_\lambda(x) = 0 \ \ \forall\lambda\in\mathbb{R}$.

Therefore $x \in \bigcap_\lambda X_\lambda^{\,c}$ so $X_{\lambda_0}^{\,c} \subseteq \bigcap_\lambda X_\lambda^{\,c}$ .

Now since

$$1 = \bigvee_{\lambda\in\mathbb{R}} e_\lambda = \mathcal{X}_{\overline{\bigcup_\lambda X_\lambda}} \quad \text{we have} \quad X_{\lambda_0}^{\,c} \subseteq \mathrm{int}(\bigcap_\lambda X_\lambda^{\,c}) = (\overline{\bigcup X_\lambda})^c = \emptyset .$$

Thus $X_{\lambda_0} = X$ , that is, $e_{\lambda_0} = 1$ for $\lambda_0 > M$.

So for $\lambda > M\ \ e_\lambda = 1$, but $e_M = \bigwedge_{\lambda>M} e_\lambda = 1$ , hence, $e_\lambda = 1\ \ \forall\ \ \lambda \ge M$. ■

**2.4 Proposition**: Let $\mathcal{J} = [a,b]$ be any compact interval in $\mathbb{R}$, f a continuous function on **J**, $\{e_\lambda\}$ a bounded resolution of the identity in C(X), $\phi(x) = \int \lambda de_\lambda(x)$.

If $\Pi = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is any partition of $\mathcal{J}$ , i.e., $a = \lambda_0 < \lambda_1 < \ldots < \lambda_n = b$ and if $\xi_k \in [\lambda_{k-1}, \lambda_k]$ , $k = 1,2,\ldots,n$.

Then

$$\|(e_b - e_a) f\circ\phi - \sum_{k=1}^{n} f(\xi_k)(e_{\lambda_k} - e_{\lambda_{k-1}})\| \le \sup\{|f(\xi) - f(\lambda)| : \xi, \lambda \in \mathcal{J} \text{ with } |\xi - \lambda| \le \|\Pi\|, \lambda \in \mathrm{Range}(\phi)\} ,$$

that is,

$$\int_{\mathcal{J}} f(\lambda) de_\lambda = (e_b - e_a) f\circ\phi .$$

**Proof.**

The proof of this estimate follows the lines of the last paragraph of the proof of proposition 2.1. ■

The set $\mathcal{J}$ of all compact intervals $J$ of $\mathbb{R}$ ordered by inclusion is a directed set.

**Definition**: If f is a continuous function on $\mathbb{R}$ and if the net $\{\int_J f(\lambda)de_\lambda\}_{J\in\mathcal{J}}$ converges in C(X), we call the limit $\int_{\mathbb{R}} f(\lambda)de_\lambda = \int f(\lambda)de_\lambda$

❑

## §3. SPECTRAL THEORY FOR BOUNDED SELF ADJOINT OPERATORS.

**Definition**: A **resolution of the identity** in B(H) is a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of projections in B(H) satisfying:

i) $\bigwedge_{\lambda\in\mathbb{R}} E_\lambda = 0$, $\bigvee_{\lambda\in\mathbb{R}} E_\lambda = I$

ii) $\forall\lambda\in\mathbb{R}$ $E_\lambda = \bigwedge_{\mu>\lambda} E_\mu$

If in addition, there are constants m and M such that $E_\lambda = 0$ $\forall\ \lambda < m$ and $E_\lambda = I$ $\forall\ \lambda \geq M$, we say that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a **bounded resolution of the identity**.

Note that ii) implies $E_\lambda \leq E_\mu$ for $\lambda < \mu$, so $E_\lambda E_\mu = E_\mu E_\lambda = E_{\min(\lambda,\mu)}$ $\forall\ \lambda,\mu\in\mathbb{R}$.

Note also that, if $\mathcal{A}$ is any abelian von Neumann algebra containing $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, $X = X_{\mathcal{A}}$ is the Gelfand space of $\mathcal{A}$ and $\wedge : \mathcal{A} \to C(X)$ is the Gelfand map, then taking $e_\lambda = \hat{E}_\lambda$ $\forall\lambda\in\mathbb{R}$, we have that $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ is a resolution of the identity in C(X).

**Definition:** Let $\mathcal{J} = [a,b]$ be an interval in $\mathbb{R}$ and f a continuous real-valued function on $\mathcal{J}$. We define

$$\int_{\mathcal{J}} f(\lambda)dE_\lambda \in B(H) \quad \text{to be} \quad \left(\int_{\mathcal{J}} f(\lambda)de_\lambda\right)^{\vee}$$

where $^\vee$ is the inverse of the Gelfand map.

**3.1 Proposition**: Let f be a continuous function on $\mathcal{J}$ = [a,b] and $\Pi$ be any partition of [a,b] , $\Pi = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$, $\xi_k \in [\lambda_{k-1}, \lambda_k]$ k=1,2,...,n.
Then

$$\| \int_{\mathcal{J}} f(\lambda)dE_\lambda - \sum_{k=1}^{n} f(\xi_k)(E_{\lambda_k} - E_{\lambda_{k-1}})\| \leq \sup\{|f(\xi) - f(\lambda)|: \xi,\lambda \in \mathcal{J}\}$$

**Proof.**
Since by Gelfand-Naimark Theorem the Gelfand map is an isometry, proposition 2.4 gives the estimate. ■

**Definition**: Let f be a continuous function on $\mathbb{R}$ such that the net $\{\int_{\mathcal{J}} f(\lambda)dE_\lambda\}_{\mathcal{J}\in\mathscr{J}}$ converges. We define

$$\int f(\lambda)dE_\lambda$$

to be the limit in the u.o.t. of the above net of operators, i.e.,

$$\int f(\lambda)dE_\lambda = (\text{u.o.t.})\text{-}\lim \int_{\mathcal{J}} f(\lambda)dE_\lambda .$$

**3.1 Remark**: Given any normal operator A in B(H) we are going to make extensive use of the (abelian) von Neumann algebra generated by A , that is, the s.o.t. (w.o.t.) closure of the C*-algebra, $\mathcal{A}_0$, generated by A.

In fact, if $\mathcal{A}$ is the s.o.t.-closure of $\{p(A,A^*): p(\lambda, \bar{\lambda})$ a polynomial in $\lambda, \bar{\lambda}\}$ , that is , $\mathcal{A}$ = s.o.t.-closure of $\mathcal{A}_0$ , then $\mathcal{A} = \{A\}''$.

In order to justify this we have : $\{A\} \subseteq \mathcal{A} \Rightarrow \{A\}' \supseteq \mathcal{A}' \Rightarrow \{A\}'' \subseteq \mathcal{A}'' = \mathcal{A}$.

On the other hand, since A is normal, $\mathcal{A}_0$ is commutative, so $\mathcal{A}_0 \subseteq \{A\}'$ and so

$$\mathcal{A} \subseteq \{A\}'.$$

Now let $E \in \mathcal{A}$ and $B \in \{A\}'$. Then BA = AB and by Fuglede's theorem BA* = A*B , therefore

$$Bp(A,A^*) = p(A,A^*)B.$$

Since $E \in \mathcal{A}$, E is the limit in s.o.t. of a sequence $\{p_n(A,A^*)\}$.
Let $x \in H$. Then $p_n(A,A^*)Bx = Bp_n(A,A^*)x$ and taking limits we get EBx = BEx , i.e., EB = BE and so $E \in \{A\}''$, that is, $\mathcal{A} \subseteq \{A\}''$.
Thus,

$$\mathcal{A} = \{A\}''. \quad ■$$

Let $A \in B(H)$ be a bounded self adjoint operator on H and let $m = \inf\{(Ax, x): \|x\| = 1\}$, $M = \sup\{(Ax, x): \|x\| = 1\}$ with $x \in H$.
It is well known that $\|A\| = \max\{|m|, |M|\}$, $\sigma(A) \subseteq [m, M]$ and both $m$, $M$ belong in $\sigma(A)$. (see [9] p.220)

Our next theorem is the *Spectral theorem for bounded self adjoint operators.*.

**3.1 Theorem: (Spectral Theorem)**
To each bounded self adjoint operator A in B(H) there corresponds a unique resolution of the identity $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ such that

$$A = \int \lambda dE_\lambda .$$

In fact, $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is bounded by $m$ and $M$ and each $E_\lambda \in \{A\}''$ .

Moreover, if $f \in C(\sigma(A))$ and $\tilde{f}$ is any continuous extension of f to $\mathbb{R}$, then

$$f(A) = \int \tilde{f}(\lambda)dE_\lambda$$

**Proof.**

Take $\mathcal{A} = \{A\}''$ ; $\mathcal{A}$ is an abelian von Neumann algebra.

Let $X = X_{\mathcal{A}}$. By Theorem 2.2 X is Stonean and from the Gelfand-Naimark Theorem $\mathcal{A}$ is isometrically *-isomorphic with C(X).

Since A is self adjoint, $\hat{A}$ the Gelfand transform of A, is in $C_{\mathbb{R}}(X)$.

Let $\phi = \hat{A}$ and let $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ be a resolution of the identity in C(X) such that $\phi = \int \lambda de_\lambda$ (Corolllary 2.1)

Now take $E_\lambda = \check{e}_\lambda \in \mathcal{A}$ ; then $A = \int \lambda dE_\lambda$ . .

Also, since Range$(\phi) = \sigma(A) \subseteq [m, M]$ we get from proposition 2.3 that $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ is bounded with bounds $m$ and $M$ and so $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is bounded with the same bounds.

Now for $f \in C(\sigma(A))$ we have that

$$(f(A))^{\vee} = f \circ \hat{A} = \int \tilde{f}(\lambda)de_\lambda ,$$

therefore

$$f(A) = \int \tilde{f}(\lambda)dE_\lambda .$$

It remains to prove uniqueness. For this suppose that $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a resolution of the identity such that $A = \int \lambda dF_\lambda$ .

Let $\mathcal{B}$ be any abelian von Neumann algebra containing $\{F_\lambda\}$. Take for example the one generated by $\{F_\lambda\}$.

$A = \int \lambda dF_\lambda$ says that A is the limit in the u.o.t.(and so in the s.o.t.) of a net of operators in $\mathcal{B}$, therefore $A \in \mathcal{B}$.
Hence

$$\{A\} \subseteq \mathcal{B} \Rightarrow \{A\}' \supseteq \mathcal{B}' \Rightarrow \{A\}'' \subseteq \mathcal{B}'' = \mathcal{B}.$$

Since $E_\lambda \in \{A\}''$, $E_\lambda \in \mathcal{B}$ as well.

Now, If Y is the Gelfand space of $\mathcal{B}$ and $f_\lambda = \hat{F}_\lambda$, then

$$\phi = \hat{A} = \int \lambda df_\lambda = \int \lambda de_\lambda$$

and since such a representation is unique we have $e_\lambda = f_\lambda \ \forall \lambda \in \mathbb{R}$ which gives $E_\lambda = F_\lambda \ \forall \lambda \in \mathbb{R}$ and the theorem is proved. ∎

**Definition**: Let X be a Stonean space and $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ be a bounded resolution of the identity in C(X). We define $e_{\lambda^-} = \bigvee_{\mu<\lambda} e_\mu$ and $X_{\lambda^-} = \{e_{\lambda^-} = 1\}$.

Note that $e_{\lambda^-}$ is a projection and $e_{\lambda^-} \leq e_\lambda$.

**3.2 Proposition**: $\forall \lambda \in \mathbb{R} \quad X_\lambda = X_{\lambda^-} \cup X_\lambda^\circ$ where $X_\lambda^\circ = \text{int}\{\phi = \lambda\}$, $\phi = \int \lambda de_\lambda$.

**Proof.**

First we show that $X_{\lambda^-} = \overline{\bigcup_{\mu<\lambda} X_\mu}$ : if $\mu < \lambda$, then $X_\mu \subseteq X_{\lambda^-}$ since $e_\mu \leq e_{\lambda^-}$.

So $\bigcup_{\mu<\lambda} X_\mu \subseteq X_{\lambda^-}$ which implies $\overline{\bigcup_{\mu<\lambda} X_\mu} \subseteq X_{\lambda^-}$.

On the other hand, let $Y = \overline{\bigcup_{\mu<\lambda} X_\mu}$ ; Y is clopen.

If $g = \mathcal{X}_Y$, then $g \geq e_\mu \ \forall \mu < \lambda$ , hence $g \geq e_{\lambda^-}$ , i.e., $Y \supseteq X_{\lambda^-}$.

Next we observe that $\bigcup_{\mu<\lambda} X_\mu$ and $X_\lambda^\circ$ are disjoint open sets and since X is extremely disconnected their closures are disjoint, so $X_{\lambda^-} \cap X_\lambda^\circ = \emptyset$.

Now $X_{\lambda^-} \subseteq X_\lambda$ since $e_{\lambda^-} \leq e_\lambda$ and $X_\lambda^\circ = \text{int}\{\phi = \lambda\} \subseteq \text{int}\{\phi \leq \lambda\} = X_\lambda$.

So $X_{\lambda^-} \cup X_\lambda^\circ \subseteq X_\lambda$.

To get equality, suppose $x \in X_\lambda \sim X_\lambda^\circ$ .Then since $X_\lambda$ is open, there exists a net $x_d \rightarrow x$ such that $x_d \in X_\lambda$ and $x_d \notin \{\phi = \lambda\}$, so $\phi(x_d) < \lambda \ \forall d$, but then $x_d \in \bigcup_{\mu<\lambda} X_\mu$.

Thus $x \in \overline{\bigcup_{\mu<\lambda} X_\mu} = X_{\lambda^-}$. ∎

(Note that $e_\lambda - e_{\lambda^-} = \mathcal{X}_{X_\lambda^\circ}$ ).

**3.3 Proposition**: Let A be a bounded self adjoint operator on H, $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ its resolution of the identity and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$.
Then $E_{\beta-} - E_\alpha = 0 \iff \sigma(A) \cap (\alpha, \beta) = \emptyset$ where $E_{\beta-} = \bigvee_{\mu<\beta} E_\mu$

**Proof.**
Let $\mathcal{A} = \{A\}''$, $X = X_{\mathcal{A}}$, $\phi = \hat{A}$, $e_\lambda = \hat{E}_\lambda$, $X_\lambda = \mathrm{int}\{\phi \le \lambda\}$.
($\Rightarrow$): Suppose $E_{\beta-} - E_\alpha = 0$; then $e_{\beta-}(x) - e_\alpha(x) = 0 \ \forall x \in X$.
Recall Range($\phi$) = $\sigma(A)$.
If Range($\phi$)$\cap (\alpha,\beta) \ne \emptyset$, i.e., if there exists $x \in X$ such that $\alpha < \phi(x) < \beta$, then there exist $\lambda, \mu$ with $\alpha < \lambda < \mu < \beta$ such that $e_\lambda(x) = 0$ and $e_\mu(x) = 1$.
So $e_\alpha(x) = 0$ and $e_\mu(x) = 1$ for some $\mu$ with $\alpha < \mu < \beta$, hence

$$e_\alpha(x) = 0 \text{ and } e_{\beta-}(x) = (\bigvee_{\mu<\beta} e_\mu)(x) = 1.$$

Therefore

$$e_{\beta-}(x) - e_\alpha(x) = 1 \quad \text{a contradiction.}$$

($\Leftarrow$): Suppose that Range($\phi$)$\cap (\alpha, \beta) = \emptyset$, then $\{\phi < \beta\} = \{\phi \le \alpha\}$ which implies that $\{\phi < \beta\} = \mathrm{int}\{\phi \le \alpha\} = X_\alpha$, so that closure$\{\phi < \beta\} = X_\alpha$ ($X_\alpha$ is clopen).
Now, let $g = \mathcal{X}_{\mathrm{clos}\{\phi<\beta\}}$. Then $\forall \mu < \beta \quad g \ge e_\mu$ and so $g \ge \bigvee_{\mu<\beta} e_\mu = e_{\beta-}$.
If $h \in C_{\mathbb{R}}(X)$ is such that $e_\mu \le h \ \forall \mu < \beta$, then we have $\{h < 1\} \subseteq X_\mu^{\ c} \quad \forall \mu < \beta$ which implies

$$\{h \ge 1\} \supseteq X_\mu \quad \forall \mu < \beta \quad \text{and so}$$

$$\{\phi < \beta\} = \bigcup_{\mu<\beta} \{\phi < \mu\} \subseteq \bigcup_{\mu<\beta} X_\mu \subseteq \{h \ge 1\}.$$

Hence,

$$\mathrm{clos}\{\phi < \beta\} \subseteq \{h \ge 1\},$$

therefore

$$\mathcal{X}_{\mathrm{clos}\{\phi<\beta\}} = g \le h.$$

Thus,

$$e_{\beta-} = g = \mathcal{X}_{\mathrm{clos}\{\phi<\beta\}} = \mathcal{X}_{X_\alpha} = e_\alpha$$

i.e.,

$$e_{\beta-} - e_\alpha = 0. \quad \blacksquare$$

We recall that $\lambda \in \rho(A)$, the *resolvent set* of A, iff $R(\lambda; A) = (\lambda I - A)^{-1}$ exists (as a bounded operator on H).

By the above proposition we can characterize the points $\lambda$ of $[m, M]$ which belong to $\rho(A)$, as follows: $\lambda_0 \in \rho(A)$ iff there exists an interval $(\alpha, \beta)$ containing $\lambda_0$ on which $E_\lambda$ is constant and in this case

$$R(\lambda_0;A) = \int_m^M \psi(\lambda)dE_\lambda$$

where $\psi(\lambda)$ is any continuous function satisfying

$$\psi(\lambda) = \frac{1}{\lambda_0 - \lambda} \quad \text{if } \lambda \notin (\alpha, \beta).$$

We shall now characterize the eigenvalues of A.

**3.4 Proposition**: Let $A \in B(H)$ be self adjoint and $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be its resolution of the identity.Then $\forall \lambda \in \mathbb{R}$ $\operatorname{Ker}(\lambda I - A) = \operatorname{Range}(E_\lambda - E_{\lambda-})$ where $E_{\lambda-} = \bigvee_{\mu<\lambda} E_\mu$

**Proof.**

Let F be the projection onto $\operatorname{Ker}(\lambda I - A)$. Then $AF = \lambda F$ from which we get $AF = FA$, that is, $F \in \{A\}'$ and so $FE_\lambda = E_\lambda F \ \forall \lambda \in \mathbb{R}$ since $E_\lambda \in \{A\}''$.

Take now $\mathcal{A}$ to be any abelian von Neumann algebra containing A, F, $\{E_\lambda\}$

For example one can take $\mathcal{A}$ to be the von Neumann algebra generated by A, F, $\{E_\lambda\}$.

Let $X = X_{\mathcal{A}}$, $\phi = \hat{A}$, $e_\lambda = \hat{E}_\lambda$, $f = \hat{F}$. Then $\phi f = \lambda f$ which implies $(\phi - \lambda)f = 0$.

$\{f = 1\} \subseteq \{\phi = \lambda\}$ and since $\{f = 1\}$ is clopen we have $\{f = 1\} \subseteq \operatorname{int}\{\phi = \lambda\} = X_\lambda^\circ$

So,

$$f \leq e_\lambda - e_{\lambda-}$$

and by taking the inverse Gelfand transform we have

$$F \leq E_\lambda - E_{\lambda-} \quad .$$

On the other hand, since $\phi(e_\lambda - e_{\lambda-}) = \lambda(e_\lambda - e_{\lambda-})$ we have

$A(E_\lambda - E_{\lambda-}) = \lambda(E_\lambda - E_{\lambda-})$ which implies $(A - \lambda I)(E_\lambda - E_{\lambda-}) = 0$

and so $\operatorname{Range}(E_\lambda - E_{\lambda-}) \subseteq \operatorname{Ker}(A - \lambda I) = \operatorname{Range}(F)$, that is, $E_\lambda - E_{\lambda-} \leq F$. ■

By the above proposition we see that a point $\lambda_0$ in $[m, M]$ is an eigenvalue of A iff $E_\lambda$ is discontinuous at $\lambda = \lambda_0$, i.e., $E_{\lambda_0} \neq E_{\lambda_0^-}$

**3.5 Proposition:** Let $A \in B(H)$ be a self adjoint operator. Then $A$ is the limit in the u.o.t. of finite linear combinations $\sum_{k=1}^{n} \lambda_k E_k$ where $\lambda_k \in \sigma(A)$ and $E_k \leq R(A)$, $R(A)$ is the projection onto the closure of the Range of A.

**Proof.**
Let F be a projection in B(H) such that $F \leq N(A)$, where N(A) is the projection onto Kernel of A. Note that $AF = 0$.

If $\mathcal{a}$ is any abelian von Neumann algebra containing A and F, then $E_\lambda \in \{A\}'' \subseteq \mathcal{a}$.

Let $X = X_{\mathcal{a}}$, $\phi = \hat{A}$, $e_\lambda = \hat{E}_\lambda$, $f = \hat{F}$. f is a projection in C(X) and $\phi f = 0$.

Now, $\phi f = 0$ implies $\{f = 1\} \subseteq \{\phi = 0\}$ which implies $\{f = 1\} \subseteq \text{int}\{\phi = 0\} = X_0^{\circ}$

and therefore $\mathcal{X}_{\{f=1\}} \leq \mathcal{X}_{X_\lambda^\circ} = e_0 - e_{0^-}$, i.e.,. $f \leq e_0 - e_{0^-}$.

If $\Pi = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is any partition of $[m^-, M]$ and $\xi_1, \xi_2, \ldots, \xi_n$ are any numbers chosen so that $\lambda_{k-1} \leq \xi_k \leq \lambda_k$ $k = 1,2,\ldots n$, then we have

$$0 \leq (e_{\lambda_k} - e_{\lambda_{k-1}})f \leq (e_{\lambda_k} - e_{\lambda_{k-1}})(e_0 - e_{0^-}) = e_{\lambda_k}e_0 - e_{\lambda_k}e_{0^-} - e_{\lambda_{k-1}}e_0 + e_{\lambda_{k-1}}e_{0^-} = 0$$

if $\lambda_k > \lambda_{k-1} > 0$ or if $\lambda_{k-1} < \lambda_k < 0$.
So in these two cases $(e_{\lambda_k} - e_{\lambda_{k-1}})f = 0$.

If $\lambda_{k-1} < 0 < \lambda_k$ choose $\xi_k = 0$.
Now, by theorem 3.1 we have that A is the limit in the u.o.t. of finite linear combinations of $(E_{\lambda_k} - E_{\lambda_{k-1}})$ with coefficients $\xi_k \in [\lambda_{k-1}, \lambda_k]$ and by the above argument, we can take the $(E_{\lambda_k} - E_{\lambda_{k-1}}) \leq R(A)$

Furthermore, if $(\lambda_{k-1}, \lambda_k) \cap \sigma(A) \neq \emptyset$,
then

$$E_{\lambda_k^-} - E_{\lambda_{k-1}} \neq 0,$$

so

$$0 < E_{\lambda_k^-} - E_{\lambda_{k-1}} \leq E_{\lambda_k} - E_{\lambda_{k-1}},$$

i.e.,

$$E_{\lambda_k} - E_{\lambda_{k-1}} \neq 0.$$

Choose $\xi_k$ in $(\lambda_{k-1}, \lambda_k) \cap \sigma(A)$.

If $[\lambda_{k-1}, \lambda_k] \cap \sigma(A) = \emptyset$, then $[\lambda_{k-1}, \lambda_k] \subseteq \rho(A)$,

so there exists $\epsilon > 0$ such that $(\lambda_{k-1} - \epsilon, \lambda_k + \epsilon) \subseteq \rho(A)$, hence

$$E_{(\lambda_k + \epsilon)^-} - E_{\lambda_{k-1} - \epsilon} = 0 \quad \text{so} \quad E_{\lambda_k} - E_{\lambda_{k-1}} = 0.$$

Thus we can choose the $\xi_k$ in $\sigma(A)$. ■

**3.2 Theorem:** Let $A \in B(H)$ be self adjoint and $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be a resolution of the identity in $B(H)$. Then $A = \int \lambda dE_\lambda$ iff $AE_\lambda \leq \lambda E_\lambda$ and $\lambda(I - E_\lambda) \leq A(I - E_\lambda)$ $\forall \lambda \in \mathbb{R}$.

**Proof.**

($\Rightarrow$): Let $\mathcal{A} = \{A\}''$, $X = X_{\mathcal{A}}$, $\phi = \hat{A}$, $e_\lambda = \hat{E}_\lambda$ .

Then $\phi = \int \lambda de_\lambda$ which is equivalent to $\phi e_\lambda \leq \lambda e_\lambda$ and $\lambda(1 - e_\lambda) \leq \phi(1 - e_\lambda)$ $\forall \lambda \in \mathbb{R}$.

Taking inverse Gelfand transform we get $AE_\lambda \leq \lambda E_\lambda$ and $\lambda(I - E_\lambda) \leq A(I - E_\lambda)$ $\forall \lambda \in \mathbb{R}$

($\Leftarrow$): $AE_\lambda \leq \lambda E_\lambda$ says that $AE_\lambda$ is self adjoint and so $AE_\lambda = E_\lambda A$ $\forall \lambda \in \mathbb{R}$.

Take now $\mathcal{A}$ to be any abelian von Neumann algebra containing A and $\{E_\lambda\}$ and let $X = X_{\mathcal{A}}$, $\phi = \hat{A}$, $e_\lambda = \hat{E}_\lambda$ . Then $\phi e_\lambda \leq \lambda e_\lambda$ and $\lambda(1 - e_\lambda) \leq \phi(1 - e_\lambda)$ $\forall \lambda \in \mathbb{R}$.

Therefore (by proposition 2.2) $\phi = \int \lambda de_\lambda$ from which $A = \int \lambda dE_\lambda$ . ■

**3.1 Corollary:** Let $A \in B(H)$ be a self adjoint operator and $E \in B(H)$ a projection that commutes with A. Then $AE|_{E(H)}$ has spectral resolution $\{EE_\lambda\}_{\lambda \in \mathbb{R}}$ where $(E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral resolution of A.

**Proof.**

By theorem 3.2 we have $AE_\lambda \leq \lambda E_\lambda$ and $\lambda(I - E_\lambda) \leq A(I - E_\lambda)$ $\forall \lambda \in \mathbb{R}$.

Multiplying both sides of the above inequalities by E we get

$$AEE_\lambda \leq \lambda EE_\lambda \quad \text{and} \quad \lambda(E - EE_\lambda) \leq AE(E - EE_\lambda) \quad \forall \lambda \in \mathbb{R}$$

Now, theorem 3.2 gives that $\{EE_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral resolution of $AE|_{E(H)}$ . ■

The above theorem gives a nice characterization of the resolution of the identity of a bounded self adjoint operator on H. Later on, we will extend this result to apply also to unbounded self adjoint operators (see theorem. 8.4).

We will conclude our discussion on bounded self adjoint operators by considering a very standard example, in fact, the multiplication operator.

**3.1 EXAMPLE.**

Let $(S, \mathcal{S}, \mu)$ be a $\sigma$-finite measure space and f be an essentially bounded measurable function on S , i.e., $f \in L^\infty(S)$.

Let $H = L^2(S)$ and $A \in B(H)$ $A = M_f : H \rightarrow H$ given by multiplication by f , that is,

$\forall g \in L^2(S)$ $\qquad M_f(g) = fg$ .

It is well known that $\|M_f\| = \|f\|$ and $M_f^* = M_{\bar{f}}$ , in fact, the mapping $f \mapsto M_f$ is a *-isometric isomorphism of $L^\infty(S)$. to B(H). (see [5] p. 75)

Hence, $M_f$ is always a normal operator. If f is real-valued, then $M_f$ is self adjoint.

For $\lambda \in \mathbb{R}$, let

$$\varphi_\lambda = \mathcal{X}_{\{f \le \lambda\}} \; (= \mathcal{X}_{f^{-1}((-\infty, \lambda])}) \quad \text{and} \quad E_\lambda = M_{\varphi_\lambda}.$$

$E_\lambda$ is clearly a projection.

We will use theorem 3.2 to show that $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the resolution of the identity for A.

First we show that $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is a resolution of the identity in B(H).

Suppose that $m \le f \le M$ a.e$[\mu]$.

Then $\forall \lambda < m \quad \mu(\{f \le \lambda\}) = 0$ so $\varphi_\lambda = 0$ and so $E_\lambda = 0$.

On the other hand $\forall \lambda \ge M \quad \mu(\{f > \lambda\}) = 0$ so $\varphi_\lambda = 1$ and so $E_\lambda = I$.

Hence

$$\bigvee_{\lambda \in \mathbb{R}} E_\lambda = I \quad \text{and} \quad \bigwedge_{\lambda \in \mathbb{R}} E_\lambda = 0.$$

We prove next that $E_\lambda = \bigwedge_{\mu > \lambda} E_\mu \quad \forall \lambda \in \mathbb{R}$.

Clearly, if $\mu > \lambda$ then $E_\mu \ge E_\lambda$.

We assert that

$$E_\lambda = \text{(s.o.t.)-}\lim E_\mu \text{ as } \mu \downarrow \lambda.$$

For this, let $g \in H$. Then

$$\| E_\mu g - E_\lambda g \|^2 = \int_S |\varphi_\mu - \varphi_\lambda|^2 \, |g|^2 d\mu = \int_S \mathcal{X}_{\{\lambda < f \le \mu\}} |g|^2 d\mu = \int_{\{\lambda < f \le \mu\}} |g|^2 d\mu$$

For $K \in \mathcal{S}$ the relation $\nu(K) = \int_K |g|^2 d\mu$ defines a finite measure on $\mathcal{S}$ since $g \in L^2(S)$.

Also since f is measurable the set $\{\lambda < f < \mu\} \in \mathcal{S} \quad \forall \lambda, \mu \in \mathbb{R}$ with $\lambda < \mu$.

Let $\{\mu_n\}$ $\mu_n \in \mathbb{R}$ be any sequence such that $\mu_n \downarrow \lambda$ and $K_n = \{\lambda < f < \mu_n\}$,
then

$$\bigcap_{n=1}^{\infty} K_n = \emptyset \quad \text{and} \quad 0 = \nu(\emptyset) = \nu(\bigcap_{n=1}^{\infty} K_n) = \lim_{n \to \infty} \nu(K_n).$$

Therefore,

$$E_{\mu_n} g \to E_\lambda g \quad \forall g \in H \text{ as } \mu_n \downarrow \lambda.$$

Now, let E be a projection in B(H) such that $E \le E_\mu \; \forall \mu > \lambda$. Then $E \le E_\lambda$.

So

$$E_\lambda = \bigwedge_{\mu > \lambda} E_\mu.$$

Thus $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is a resolution of the identity in B(H).

Moreover, we have $f\varphi_\lambda \le \lambda\varphi_\lambda$ which implies $AE_\lambda \le \lambda E_\lambda \quad \forall \lambda \in \mathbb{R}$

and $\lambda(1 - \varphi_\lambda) \le f(1 - \varphi_\lambda)$ which implies $\lambda(I - E_\lambda) \le A(I - E_\lambda) \quad \forall \lambda \in \mathbb{R}$.

Therefore $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the resolution of the identity for A. ∎ ❑

## §4. SPECTRAL THEORY FOR BOUNDED NORMAL OPERATORS.

**Definition**: A subset A of a topological space X is called *nowhere dense* if $\mathrm{int}(\bar{A}) = \emptyset$. A subset A is called *meager* (or of the *first category* ) if it is a countable union of nowhere dense sets. (see [15] p.201)

We remark that any nowhere dense set is meager, any subset of a meager set is meager and any countable union of meager sets is meager.

*Baire's category theorem* states that: If X is a locally compact Hausdorff space and A a meager subset of X, then the complement of A, $A^c$, is dense in X. (see [15] p.200)

Let X be any topological space. We denote by B(X) the C*-algebra of all bounded Borel measurable complex-valued functions on X, with norm $\|f\| = \sup_{x\in X}|f(x)|$.

**Definition**: A function $h \in B(X)$ is called a *null-function* if the set $H = \{x \in X: h(x) \neq 0\}$ is meager. We denote by $n(X)$ the set of null-functions on X.
$n(X)$ is in fact a closed ideal in B(X).

Note that Baire's category theorem gives:
On a locally compact Hausdorff space X, $n(X) \cap C_b(X) = \{0\}$, where $C_b(X)$ denotes the algebra of bounded continuous functions on X.

The proofs of the following propositions 4.1 and 4.2 can be found in [14] p.323

**4.1 Proposition**: Let X be a Stonean space and $B_X$ be the σ-algebra of Borel sets in X. Then for each $M \in B_X$ there exists a unique clopen set F such that $M \Delta F$ (= $(M\sim F) \cup (F \sim M)$ ) is meager.

**4.2 Proposition**: If X is a Stonean space, then $B(X) = C(X) \oplus n(X)$.

**Definition**: Let $\mathcal{U}$ be a C*-algebra, $\mathcal{S}$ be a collection of self adjoint elements of $\mathcal{U}$ and $A \in \mathcal{U}$ be self adjoint. We say that A is an *upper bound* of $\mathcal{S}$ if $A \geq B \ \forall B \in \mathcal{S}$.
Moreover, A is called the *least upper bound* of $\mathcal{S}$ if A is an upper bound for $\mathcal{S}$ and whenever C is any upper bound for $\mathcal{S}$ we have $C \geq A$.
We write in this case $A = \bigvee_{B \in \mathcal{S}} B = \mathrm{l.u.b}\{B : B \in \mathcal{S}\}$.

The proof of the following lemma is given in [14] p.307

**4.1 Lemma**: If $\{A_d\}_{d\in D}$ is a monotone increasing net of self adjoint operators in B(H) bounded above, then $A = \bigvee_{d\in D} A_d$ exists (and is self adjoint) and $A_d \uparrow A$ in the w.o.t.. In fact $A_d \uparrow A$ in s.o.t.

**Definition**: Let $\mathcal{U}$ and $\mathcal{B}$ be C*-algebras and $\phi: \mathcal{U} \rightarrow \mathcal{B}$ be a *-homomorphism. We say that $\phi$ is **σ-normal** if whenever $\{A_n\}$ is a non-decreasing sequence in $\mathcal{U}$ such that $A = \bigvee_{n=1}^{\infty} A_n$, then $\phi(A_n) \uparrow \phi(A)$.

Note that, if $\phi$ is a *-isomorphism onto, then $\phi$ is automatically σ-normal. Also, the composition of σ-normal *-homomorphisms is σ-normal.

The proof of the following Lemma is quite elementary and we omit it.

**4.2 Lemma**: Let X be any topological space. If $\{f_n\}$ is any non-decreasing sequence of functions in B(X) which is bounded above, i.e., $\exists\ g \in B(X)$ such that $g \geq f_n \quad \forall n$, then $f = \bigvee_{n=1}^{\infty} f_n$ exists and $f_n(x) \uparrow f(x) \quad \forall x \in X$.

**4.3 Proposition**: Let X be a Stonean space and $\gamma$: $B(X) \rightarrow C(X)$ the mapping given as follows: for $f \in B(X)$, $\gamma(f) = g$ where $g \in C(X)$ is such that $f = g+h$ with $h \in n(X)$. Then $\gamma$ is σ-normal.

**Proof**:
Suppose that $f_n, f \in B(X)$ and $f_n = g_n + h_n$, $f = g+h$ with $h_n, h \in n(X)$, $g_n, g \in C(X)$, i.e., $\gamma(f_n) = g_n$ and $\gamma(f) = g$.
If $f = \bigvee_{n=1}^{\infty} f_n$, then for fixed m,n n >m, we have $\{g_m > g_n\} \subseteq H_m \cup H_n$ where $H_n = \{h_n \neq 0\}$. So $\{g_m > g_n\}$ is meager since $H_m \cup H_n$ is meager. The Baire Category theorem and the continuity of $g_n$ and $g_m$ give that $\{g_m > g_n\} = \emptyset$. Also $\{g_n > g\} \subseteq \{h_n \neq 0\} \cup \{h \neq 0\}$ and similarly $\{g_n > g\} = \emptyset$, that is, $g_n \leq g \ \forall n \in \mathbb{N}$.
Hence $g_n \rightarrow \tilde{f}$ pointwise with $\tilde{f} \in B(X)$ and $\tilde{f} \leq g$.
Now, let $\tilde{g} \in C(X)$ and $\tilde{h} \in n(X)$ be such that $\tilde{f} = \tilde{g} + \tilde{h}$. Then $\{\tilde{f} \neq f\} \subseteq \bigcup_{n=1}^{\infty} H_n$ and arguing as above we get $\tilde{g} = g$.
Moreover, if $g' \in C(X)$ and $g' \geq g_n \ \forall n$, then $g' \geq \tilde{f}$ and therefore $\{g' < g\} \subseteq \{\tilde{h} \neq 0\}$ and so $g \leq g'$. Thus, $\gamma(f) = \bigvee_{n=1}^{\infty} \gamma(f_n)$ and so $\gamma(f_n) \uparrow \gamma(f)$. ■

Next, with the use of the $\sigma$-normal homomorphism $\gamma$ and the Gelfand map, we will extend the (continuous) functional calculus to bounded Borel functions on the spectrum of a bounded normal operator.

We recall that a *-homomorphism between C*-algebras is always continuous. (see [14] p.242)

**4.1 Theorem**: **(Spectral theorem-functional calculus form)**
Let $A \in B(H)$ be a normal operator. Then there exists a unique $\sigma$-normal *-homomorphism $\Phi: B(\sigma(A)) \to \mathcal{A} = \{A\}''$ such that $\Phi(f) = f(A) \quad \forall f \in C(\sigma(A))$.

**Proof.**
If $\mathcal{A} = \{A\}''$, then $X = X_{\mathcal{A}}$ is a Stonean space. If $\wedge$ is the Gelfand map, then $\mathcal{A} \simeq C(X)$ and $\hat{A}: X \to \sigma(A)$ is a continuous onto function. Given $f \in B(\sigma(A))$, then $fo\hat{A} \in B(X)$ and so $\gamma(fo\hat{A}) \in C(X)$ where $\gamma$ is the $\sigma$-normal *-homomorphism defined in proposition 4.3.
Now, the mapping $\Phi: B(\sigma(A)) \to \mathcal{A}$ defined by $\Phi(f) = (\gamma(fo\hat{A}))^{\vee}$ , where $^{\vee}$ denotes the inverse of the Gelfand map, is a $\sigma$-normal *-homomorphism of $B(\sigma(A))$ into $\mathcal{A}$.
Note in particular that $\Phi$ is continuous. Moreover, if f is continuous, then $fo\hat{A}$ is continuous, so $\gamma(fo\hat{A}) = fo\hat{A}$ and so $\Phi(f) = (fo\hat{A})^{\vee} = f(A)$ .
To prove uniqueness, suppose that $\Phi_1$ and $\Phi_2$ both have the properties as stated in the theorem. Let $B_0(\sigma(A)) = \{f \in B(\sigma(A)): \Phi_1(f) = \Phi_2(f)\}$. Then $B_0(\sigma(A))$ is a closed *-subalgebra of $B(\sigma(A))$ containing $C(\sigma(A))$. Also, if $f_n \in B_0(\sigma(A))$ and $f_n \uparrow f$ with $f \in B(\sigma(A))$, then $f \in B_0(\sigma(A))$ since $\Phi_1$ and $\Phi_2$ are $\sigma$-normal. Now, uniqueness will follow from the following lemma. ∎

**4.3 Lemma**: Let $\Lambda$ be a compact metric space (for example $\Lambda = \sigma(A)$, more generally, $\Lambda$ can be any compact Hausdorff space such that any open set is an $F_\sigma$ set ).
If $B_0(\Lambda)$ is a closed *-subalgebra of $B(\Lambda)$ containing 1, id and such that $f_n \in B_0(\Lambda)$ $f_n \uparrow f$ with $f \in B(\Lambda)$ implies $f \in B_0(\Lambda)$, then $B_0(\Lambda) = B(\Lambda)$.

**Proof.**
Let G be an open subset of $\Lambda$. Then $\mathcal{X}_G \in B_0(\Lambda)$. To see this, suppose that $K_n \subseteq G$, $K_n \subseteq K_{n+1}$ , $K_n$ compact for each n=1,2,... and $G = \bigcup_{n=1}^{\infty} K_n$. By Urysohn`s Lemma, choose $g_n \in C(\Lambda)$ with $g_n = 1$ on $K_n$ and $g_n = 0$ on $G^c$. Then $f_n = g_1 \vee g_2 \vee ... \vee g_n$ is continuous on $\Lambda$, i.e., $f_n \in C(\Lambda)$ for all n and $f_n \uparrow \mathcal{X}_G$ as $n \to \infty$. Since $B_0(\Lambda)$ is a closed *-subalgebra (of $B(\Lambda)$) containing 1 and id, the Stone-Weierstrass Theorem gives that $C(\Lambda) \subseteq B_0(\Lambda)$.
So, $f_n \in B_0(\Lambda)$ and $f_n \uparrow \mathcal{X}_G$ , hence $\mathcal{X}_G \in B_0(\Lambda)$.
Now, let $B_\Lambda$ be the $\sigma$-algebra of Borel sets in $\Lambda$ and $\mathcal{M} = \{M \in B_\Lambda : \mathcal{X}_M \in B_0(\Lambda)\}$.

Since the simple functions are (supremum-) norm dense in B(Λ), it will be enough to show that $\mathcal{M}$ is a σ-algebra of subsets of Λ. Clearly, $\mathcal{M}$ is an algebra of subsets of Λ. To see that $\mathcal{M}$ is a σ-algebra, let $M_1, M_2, \ldots$ be pairwise disjoint elements of $\mathcal{M}$, then

$\chi_{\bigcup_{k=1}^{n} M_k} = \sum_{k=1}^{n} \chi_{M_k} \uparrow \sum_{k=1}^{\infty} \chi_{M_k} = \chi_{\bigcup_{k=1}^{\infty} M_k}$ and since $\bigcup_{k=1}^{n} M_k \in \mathcal{M}$, we have $\bigcup_{k=1}^{\infty} M_k \in \mathcal{M}$. ■

A note on the notation: we write f(A) for Φ(f) when $f \in B(\sigma(A))$.

**4.4 Proposition**: Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are Von Neumann algebras and $\varphi: \mathcal{A}_1 \to \mathcal{A}_2$ is a σ-normal *-homomorphism. If $A \in \mathcal{A}_1$ is a normal operator, then $\forall f \in B(\sigma(A))$

$$\varphi(f(A)) = f(\varphi(A)).$$

**Proof.**

Since $\sigma(\varphi(A)) \subseteq \sigma(A)$ (see[2] p.243), $f|_{\sigma(\varphi(A))} \in B(\sigma(\varphi(A)))$ when $f \in B(\sigma(A))$. Let $B_0(\sigma(A)) = \{f \in B(\sigma(A)): \varphi(f(A)) = f(\varphi(A))\}$. Then $B_0(\sigma(A))$ is a closed *-subalgebra of $B(\sigma(A))$ containing $C(\sigma(A))$. Moreover, if $f_n \in B_0(\sigma(A))$ and $f_n \uparrow f$ with $f \in B(\sigma(A))$, then $f \in B_0(\sigma(A))$ since $\varphi \circ \Phi_1$ and $\Phi_2 \circ \psi$ are σ-normal (where $\Phi_1: B(\sigma(A) \to \{A\}'' \subseteq \mathcal{A}_1$, $\Phi_2: B(\sigma(\varphi(A))) \to \{f(A)\}'' \subseteq \mathcal{A}_2$ are the σ-normal *-homomorphisms given by the Borel functional calculus and $\psi: B(\sigma(A)) \to B(\sigma((\varphi(A))$ is the σ-normal *-homomorphism given by $\psi(f) = f|_{\sigma(\varphi(A))}$ ). Now, by Lemma 4.3 we conclude that $B_0(\sigma(A)) = B(\sigma(A))$. ■

**4.4 Lemma**: Let X be a Stonean space and γ the mapping from B(X) into C(X) defined in proposition 4.3. Then $\forall f \in B(X)$ $\text{Range}(\gamma(f)) \subseteq \overline{\text{Range}(f)}$ .

**Proof.**

Let f=g+h, g=γ(f), $g \in C(X)$, $h \in n(X)$. If $\lambda \notin \overline{\text{Range}(f)}$, then $\exists\ \varepsilon > 0$ such that $|\lambda - f(x)| \geq \varepsilon$ $\forall x \in X$. Therefore, $\{x \in X: |\lambda - g(x)| < \varepsilon\} \subseteq \{h \neq 0\}$ which is meager and since $g \in C(X)$, Baire's category theorem gives $|\lambda - g(x)| \geq \varepsilon$ $\forall x \in X$, that is, $\lambda \notin \text{Range}(g) = \text{Range}(\gamma(f))$. ■

**4.5 Proposition**: Let $A \in B(H)$ be a normal operator. Then $\forall f \in B(\sigma(A))$

$$\sigma(f(A)) \subseteq \overline{f(\sigma(A))}\ .$$

**Proof.**

Let $f \in B(\sigma(A))$, by definition $f(\hat{A}) = \gamma(f \circ \hat{A})$, so $\sigma(f(A)) = \text{Range}(f(\hat{A})) = \text{Range}(\gamma(f \circ \hat{A}))$. On the other hand, $f(\sigma(A)) = \text{Range}(f \circ \hat{A})$ and lemma 4.4 gives the result. ■

**4.6 Proposition**: Let $A \in B(H)$ be a normal operator. If $g \in B(\sigma(A))$ and $f \in B(\overline{g(\sigma(A))})$, then $f \circ g(A) = f(g(A))$.

**Proof.**

Let $\Lambda = \overline{g(\sigma(A))}$ and $B_0(\Lambda) = \{f \in B(\Lambda): f \circ g(A) = f(g(A))\}$, $B_0(\Lambda)$ is a closed *-subalgebra of $B(\Lambda)$ containing 1 and id. Moreover, $B_0(\Lambda)$ is closed under monotone sequential limits and so by Lemma 4.3, $B_0(\Lambda) = B(\Lambda)$. ■

We remark that if U is a unitary operator in H, then $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \mathbb{T}$. (see [14] p.184)

We will use the Borel functional calculus to establish the *Spectral theorem for unitary operators..*

**4.7 Proposition**: Let $U \in B(H)$ be a unitary operator (i.e. $UU^* = U^*U = I$). Then $U = e^{iA}$ for some self adjoint A in B(H), with $\sigma(A) \subseteq [-\pi,\pi]$.

**Proof:**

Let $z \in \mathbb{T}$, then $z = e^{i\theta}$ with $-\pi < \theta \leq \pi$. Consider the function $g: \mathbb{T} \to \mathbb{R}$ given by $g(z) = \arg(z)$. $g \in B(\mathbb{T})$. Take $A = g(U)$; since g is real-valued, A is self adjoint. Furthermore, $\sigma(A) = \sigma(g(U)) \subseteq \overline{g(\sigma(U))} = [-\pi,\pi]$. Let $f(t) = e^{it}$.
Then $f \in B([-\pi,\pi])$ and $f \circ g = id$, hence by proposition 4.6 we have $e^{iA} = U$. ■

**4.2 Theorem: (Spectral theorem for unitary operators)**
Let $U \in B(H)$ be a unitary operator. Then there exists a resolution of the identity $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ in B(H) such that $E_\lambda = 0$ for $\lambda < -\pi$, $E_\lambda = I$ for $\lambda \geq \pi$ and $U = \int_{-\pi^-}^{\pi} e^{i\lambda}\, dE_\lambda$.

**Proof.**

Take A to be a self adjoint operator such that $U = e^{iA}$ with $\sigma(A) \subseteq [-\pi,\pi]$.
Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be the resolution of the identity for A. Then, since $\sigma(A) \subseteq [-\pi,\pi]$, $E_\lambda = 0$ when $\lambda < -\pi$ and $E_\lambda = I$ when $\lambda \geq \pi$.
Moreover, from the spectral theorem for self adjoint operators we have that

$$U = e^{iA} = \int_{-\pi^-}^{\pi} e^{i\lambda}\, dE_\lambda . \quad ■$$

Let X be a locally compact space and $B_X$ the $\sigma$-algebra of Borel sets in X.

**Definition**: A **spectral measure** on $B_X$ is an operator-valued function $E: B_X \to B(H)$ such that:

(1) $E(\emptyset) = 0$ and $E(X) = I$

(2) $E(M)$ is a projection for each $M \in B_X$

(3) $E(M_1 \cup M_2) = E(M_1)+E(M_2)$ , if $M_1 \cap M_2 = \emptyset$

(4) $E(M \cap N) = E(M)E(N)$ with $M,N \in B_X$ ,

(5) For $\xi,\eta \in H$ the mapping $\mu_{\xi,\eta}: B_X \to \mathbb{C}$ defined by $\mu_{\xi,\eta}(M) = (E(M)\xi,\eta) =$
$= E_{\xi,\eta}(M)$ is a regular complex Borel measure.

Note that $\mu_\xi(M) = (E(M)\xi,\xi) = E_\xi(M)$ is a positive finite Borel measure whose total variation is $\mu_\xi(X) = \|\xi\|^2$.

Our next theorem is the *Spectral theorem for bounded normal operators.*

**4.4 Theorem: (Spectral theorem)**

Let $A \in B(H)$ be a normal operator. Then there exists a unique spectral measure E on $\sigma(A)$ such that

$$(A\xi,\eta) = \int_{\sigma(A)} \lambda d\mu_{\xi,\eta}(\lambda) = \int_{\sigma(A)} \lambda dE_{\xi,\eta}(\lambda) \quad \text{with } \xi,\eta \in H.$$

Briefly, we write $A = \int_{\sigma(A)} \lambda dE(\lambda)$.

Moreover,

i) For $f \in B(\sigma(A))$ and $\xi \in H$ $\quad (f(A)\xi,\xi) = \int_{\sigma(A)} f(\lambda)\, d\mu_\xi(\lambda)$.

ii) If $T \in B(H)$ commutes with A , then T commutes with each $f(A)$, $f \in B(\sigma(A))$.

**Proof.**

Let $B_{\sigma(A)}$ be the $\sigma$-algebra of Borel sets in $\sigma(A)$. By theorem 4.1, the Borel funtional calculus, the mapping $\Phi: B(\sigma(A)) \to \mathcal{A} = \{A\}''$ is a (unique) $\sigma$-normal *-homomorphism such that $\Phi(1) = I$ and $\Phi(id) = A$.

The operator-valued function $E: B_{\sigma(A)} \to \mathcal{A}$ defined, for $M \in B_{\sigma(A)}$, by

$$E(M) = \mathcal{X}_M(A) = \Phi(\mathcal{X}_M) \quad ,$$

is a spectral measure on $\sigma(A)$.

To see this, we have; (1) $E(\emptyset) = \Phi(\mathcal{X}_\emptyset) = 0$, $E(\sigma(A)) = \Phi(\mathcal{X}_{\sigma(A)}) = \Phi(1) = I$.

(2) For $M \in B_{\sigma(A)}$, $E(M)$ is self adjoint and idempotent since $\Phi$ is a*-homomorphism, $\mathcal{X}_M$ is real-valued and $\mathcal{X}_M = \mathcal{X}_M^2$. (3) and (4) follow from the fact that $\Phi$ is a homomorphism.

(5) If $\{M_j\}$ are pairwise disjoint sets in $B_{\sigma(A)}$ and $M = \bigcup_{j=1}^{\infty} M_j$, let $f_j = \chi_{M_j}$, $g_n = f_1 + f_2 + \ldots + f_n$ and $g = \chi_M$, then

$$g_n = \sum_{j=1}^{n} \chi_{M_j} \uparrow \sum_{j=1}^{\infty} \chi_{M_j} = \chi_{\bigcup_{j=1}^{\infty} M_j} = \chi_M = g \qquad \text{, i.e.,} \quad g_n \uparrow g.$$

Since $\Phi$ is $\sigma$-normal $\Phi(g_n) \uparrow \Phi(g)$ , or, $\bigvee_{n=1}^{\infty} \{ \sum_{j=1}^{n} E(M_j) \} = E(M)$
and so

$$\sum_{j=1}^{n} E(M_j) \to E(M) \text{ in the w.o.t. as } n \to \infty.$$

Therefore, for $\xi, \eta \in H$

$$\begin{aligned} \mu_{\xi,\eta}\left(\bigcup_{j=1}^{\infty} M_j\right) &= \left(E\left(\bigcup_{j=1}^{\infty} M_j\right)\xi,\eta\right) = (E(M)\xi,\eta) = \lim_{n\to\infty} \left(\sum_{j=1}^{n} E(M_j)\xi,\eta\right) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} (E(M_j)\xi,\eta) = \sum_{j=1}^{\infty} (E(M_j)\xi,\eta) = \sum_{j=1}^{\infty} \mu_{\xi,\eta}(M_j). \end{aligned}$$

In particular, $\mu_\xi(M) = (E(M)\xi,\xi)$ is a positive measure on $B_{\sigma(A)}$ with $\mu_\xi(\sigma(A)) = \|\xi\|^2 < \infty$. Moreover, any open set in $\mathbb{C}$ is a countable union of closed disks and since $\sigma(A)$ is compact, any open set in $\sigma(A)$ is $\sigma$-compact. Also for every compact set K, $\mu_\xi(K) < \infty$ since $\mu_\xi$ is a finite measure. Thus $\mu_\xi$ is a regular Borel measure. (see [21] p.50)
Therefore, $\mu_{\xi,\eta}$ is a regular complex measure since $\mu_{\xi,\eta}$ is a linear combination of four positive regular Borel measures of the form $\mu_\xi$ with $\xi \in H$, by polarization.
Hence, E is a spectral measure on $\sigma(A)$.
Now, let $f = \chi_M$ with $M \in B_{\sigma(A)}$.
Then

$$(f(A)\xi,\xi) = (E(M)\xi,\xi) = \mu_\xi(M) = \int_{\sigma(A)} \chi_M(\lambda)\, d\mu_\xi(\lambda) = \int_{\sigma(A)} f(\lambda)\, d\mu_\xi(\lambda) ,$$

so

$$(f(A)\xi,\xi) = \int_{\sigma(A)} f(\lambda)\, d\mu_\xi(\lambda) \quad \text{is valid for simple functions.}$$

If $f \in B(\sigma(A))$, then $\tilde{f} = f \circ \hat{A} \in B(X)$ where $X = X_{\mathcal{a}}$ a Stonean space. So $\tilde{f} = g + h$ with $g \in C(X)$, $h \in n(X)$.
Since Range$(\hat{A}) = \sigma(A)$ we have $\|\tilde{f}\| = \sup_{x\in X} |\tilde{f}(x)| = \sup_{x \in X} |f \circ \hat{A}(x)| = \sup_{\lambda \in \sigma(A)} |f(\lambda)| = \|f\|$.
If, $H = \{h \neq 0\}$, then $H$ is a meager set and $|g| \le |g+h| + |h| = |\tilde{f}| + |h|$ which implies that $|g| \le |\tilde{f}|$ on $H^c$, that is, $|g(x)| \le \|\tilde{f}\|$ $\forall x \in H^c$. But $H^c$ is dense in X, so by the continuity of g $|g(x)| \le \|\tilde{f}\|$ $\forall x \in X$, therefore $\|g\| \le \|\tilde{f}\|$.
Thus,

$$\|f(A)\| = \|(\gamma(\tilde{f}))^{\vee}\| = \|\check{g}\| = \|g\| \leq \|\tilde{f}\| = \|f\|.$$

Now, each $f \in B(\sigma(A))$ is a (supremum-) norm limit of (Borel) simple functions, say $f_n$, i.e., there exist simple functions $f_n$ such that $\| f_n - f \| \to 0$ as $n \to \infty$.
Then,

$$\| f_n(A) - f(A) \| = \| (f_n - f)(A) \| \leq \| f_n - f \| \to 0$$

and so

$$(f(A)\xi,\xi) = \lim_{n\to\infty} (f_n(A)\xi,\xi) \qquad \forall \xi \in H.$$

On the other hand,

$$\left| \int_{\sigma(A)} f_n(\lambda)\, d\mu_\xi - \int_{\sigma(A)} f(\lambda)\, d\mu \right| \leq \int_{\sigma(A)} |f_n(\lambda) - f(\lambda)|\, d\mu_\xi \leq \| f_n - f \| \mu_\xi(\sigma(A)) = \|f_n - f\| \, \|\xi\|^2 \to 0$$

as $n \to \infty$.
Therefore,

$$(f(A)\xi,\xi) = \lim_{n\to\infty} (f_n(A)\xi,\xi) = \lim_{n\to\infty} \int_{\sigma(A)} f_n(\lambda)\, d\mu_\xi = \int_{\sigma(A)} f(\lambda)\, d\mu_\xi .$$

In particular, for $f(\lambda) = \lambda$ we have

$$(A\xi,\xi) = \int_{\sigma(A)} \lambda d\mu_\xi$$

and by polarization

$$(A\xi,\eta) = \int_{\sigma(A)} \lambda \, d\mu_{\xi,\eta} \quad \text{for } \xi,\eta \in H.$$

Next, we show ii):
Let $T \in \{A\}'$, that is, $TA = AT$ (then we remark that by Fuglede`s theorem $TA^* = A^*T$).
Since $\forall f \in B(\sigma(A))$ $f(A) = \Phi(f) \in \mathcal{A} = \{A\}''$ (Theorem 4.1), we have $Tf(A) = f(A)T$.
For the uniqueness of the spectral measure E we refer the reader to Halmos [11] p.65. ■

We remark that if $T \in \{A\}'$, then in particular $TE(M) = E(M)T$ $\forall M \in B_{\sigma(A)}$, that is, T commutes with the spectral projections E(M) of A.

**4.1Remark**: We remark that if A is self adjoint, then its spectral resolution is $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ where

$$E_\lambda = I - E((\lambda,\infty)) = \Phi(\mathcal{X}_{(-\infty,\lambda]}).$$

To see this; let $g = \mathcal{X}_{(\lambda,\infty)}$, $\mathcal{A} = \{A\}''$, $\hat{A} = \phi$, $X = X_{\mathcal{A}}$.
Then $\tilde{g} = g \circ \hat{A} = \mathcal{X}_{(\lambda,\infty)} \circ \phi = \mathcal{X}_{\{\phi > \lambda\}} \in B(X)$.
Since X is Stonean, $B(X) = C(X) \oplus n(X)$ and $\gamma(\tilde{g}) \in C(X)$.
The set $\{\phi > \lambda\}$ is open in X, so $\overline{\{\phi > \lambda\}}$ is clopen and so $\mathcal{X}_{\overline{\{\phi>\lambda\}}}$ is in C(X).
Now $\mathcal{X}_{\overline{\{\phi>\lambda\}}} \neq \tilde{g}$ only on the boundary of $\{\phi > \lambda\}$, but the boundary of any open set (in any topological space) is nowhere dense, hence meager.
Thus,

$$\gamma(\tilde{g}) = \mathcal{X}_{\overline{\{\phi>\lambda\}}} = \mathcal{X}_{(\mathrm{int}\{\phi \le \lambda\})^c} = 1 - \mathcal{X}_{\mathrm{int}\{\phi \le \lambda\}} = 1 - e_\lambda.$$

Therefore,

$$E((\lambda,\infty)) = (\gamma(\tilde{g}))^{\vee} = (1 - e_\lambda)^{\vee} = I - E_\lambda \quad \text{, that is,}$$

$$E_\lambda = I - E((\lambda,\infty)). \quad \blacksquare$$

**4.1 EXAMPLE.**

Let $(S,\mathcal{S},\mu)$ be a $\sigma$-finite measure space and $g \in L^\infty(S,\mathcal{S},\mu)$, $H = L^2(S,\mathcal{S},\mu)$ and $A \in B(H)$ where $A = M_g$, the multiplication operator.
As we already noted in example 3.1, A is a bounded normal operator on H.
Define the *essential range* of g by

$$\mathcal{R}(g) = \{\lambda \in \mathbb{C} : \forall\ \varepsilon > 0\ \ \mu(\{s \in S : |g(s) - \lambda| < \varepsilon\}) > 0\ \}.$$

It is well known that $\sigma(M_g) = \mathcal{R}(g)$ (see [5] p.88).

We are interesting to find the" *spectral projections*" for A.

First, we will describe the Borel functional calculus for A. For this some observations are needed.
Note that if D is a closed disk contained in $\mathbb{C}\backslash\mathcal{R}(g)$ (the complement of $\mathcal{R}(g)$), then $\forall \lambda \in D$ there is $\varepsilon_\lambda > 0$ such that $\mu(g^{-1}(D\,\varepsilon_\lambda(\lambda))) = 0$, where $D\,\varepsilon_\lambda(\lambda)$ is the open disk centered at $\lambda$ with radius $\varepsilon_\lambda$.

Now, $D \subseteq \bigcup_{\lambda \in D} D_{\varepsilon_\lambda}(\lambda)$ and since D is compact $D = \bigcup_{j=1}^{n} D_{\varepsilon_{\lambda_j}}(\lambda_j)$ and so

$\mu(g^{-1}(D)) = \mu(\bigcup_{j=1}^{n} g^{-1}(D_{\varepsilon_{\lambda_j}}(\lambda_j))) \le \sum_{j=1}^{n} \mu(g^{-1}(D_{\varepsilon_{\lambda_j}}(\lambda_j))) = 0$, i.e., $\mu(g^{-1}(D)) = 0$.

Furthermore, since $\mathbb{C}$ is separable, each open subset of $\mathbb{C}$ is a countable union of closed disks. In particular this holds for the open subset $\mathbb{C}\backslash\mathcal{R}(g)$ and hence $\mu(g^{-1}(\mathbb{C}\backslash\mathcal{R}(g))) = 0$.

Now, let $\lambda \in \mathcal{R}(g)$ and define

$$g_o(s) = \begin{cases} g(s) & \text{when } s \in g^{-1}(\mathcal{R}(g)) \\ \lambda & \text{when } s \in g^{-1}(\mathbb{C}\backslash\mathcal{R}(g)) \end{cases}.$$

Then

$$g = g_o \ \text{a.e } [\mu], \ \text{so} \ \ M_g = M_{g_o},$$

hence

$$\sigma(M_{g_o}) = \sigma(M_g) = \sigma(A).$$

Let f be a bounded Borel function on $\sigma(A)$ and consider the mapping $\Psi: B(\sigma(A)) \to \{A\}''$ given by

$$\Psi(f) = M_{f\circ g} \qquad \text{for } f \in B(\sigma(A)).$$

Then $\Psi(1) = I$, $\Psi(id) = A$ and is easy to see that $\Psi$ is a *-homomorphism.
Moreover, $\Psi$ is $\sigma$-normal.
To see this, suppose $f_n \uparrow f$ with $f, f_n \in B(\sigma(A))$. Note that from this we have that
$K = \sup\{|f_n(s)|: s \in S, n=1,2,\ldots\} < \infty$.
We will show that $\Psi(f_n) \uparrow \Psi(f)$ ;

First note that, if $f \geq 0$, then $\Psi(f) = M_{f\circ g} \geq 0$ and so $\{\Psi(f_n)\}$ is increasing (in B(H)).
For $h \in H$ we have

$$\|(\Psi(f_n) - \Psi(f))h\|^2 = \|(f_n\circ g - f\circ g)h\|^2 = \int_S |(f_n\circ g - f\circ g)h|^2 d\mu .$$

But

$|((f_n-f)\circ g)h|^2 \to 0$ as $n \to \infty$ and $|(f_n\circ g - f\circ g)h|^2 \leq 2K|h|^2 \in L^1(S)$, hence the Lebesgue Dominated Convergence theorem gives

$$\|(\Psi(f_n) - \Psi(f))h\|^2 \to 0 \quad \text{as } n \to \infty.$$

Thus, $\Psi$ is a $\sigma$-normal *-homomorphism and by the uniqueness of the Borel functional calculus we have

$$f(M_g) = f(A) = \Phi(f) = \Psi(f) = M_{f\circ g} .$$

Now, it is easy to get the spectral projections for A as follows; For any Borel subset $\Delta$ of $\sigma(A)$, we have

$$E(\Delta) = \Phi(\mathcal{X}_\Delta) = \mathcal{X}_\Delta(A) = \mathcal{X}_\Delta(M_g) = M_{\mathcal{X}_\Delta \circ g} = M_{\mathcal{X}_{g^{-1}(\Delta)}} .$$

**Remark**: A more general spectral theorem exists, often called *the Spectral theorem for commutative C*-algebras*. Here we just state it and we refer the reader to [4] p.228 for its proof.

**4.5 Theorem**: **(General spectral theorem)**
Let H be a Hilbert space and $\mathcal{U}$ a commutative C*-subalgebra of B(H) containing the identity operator I.
Then:

(i) there exists a unique spectral measure E on the Borel subsets of $X = X_{\mathcal{U}}$ such that

$$(A\xi,\eta) = \int_X \hat{A} dE_{\xi,\eta} \qquad \text{for } \xi,\eta \in H,\ A \in \mathcal{U}.$$

(ii) $E(G) \neq 0$ for each nonempty open subset G of X.

(iii) An operator T in B(H) commutes with each A in $\mathcal{U}$ iff T commutes with each projection E(M) $\forall M \in B_X$.

The spectral theorem for a bounded normal operator A , can be easily derived as a special case of the above theorem, when the C*-algebra $\mathcal{U}$ is taken to be the C*-algebra generated by A and I. (see [4] p. 232).

Another approach to the spectral theorem for a normal operator $A \in B(H)$ is given in the classical book of Riesz and Nagy [20] p.286. The spectral theorem there is proved with the use of the spectral theorem for bounded self adjoint operators. In fact, the spectral theorem for self adjoint operators is applied to the real and imaginary part $\frac{A+A^*}{2}$ and $\frac{A-A^*}{2i}$ of A.

Other works on the spectral theorem for self adjoint and normal operators include[1], [3], [7], [9], [11], [16], [17], [18],[19], [20], [23]

□

## § 5. MAXIMAL ABELIAN ALGEBRAS

**Definition**: Let H be a Hilbert Space, a *-subalgebra $\mathcal{A}$ of B(H) is said to be *maximal abelian* if it is abelian and it is not properly contained in any abelian *-subalgebra of B(H) .

Note that, as the condition

$$\mathcal{A} \subseteq \mathcal{A}'$$

characterizes the abelian *-subalgebras of B(H), the condition

$$\mathcal{A} = \mathcal{A}'$$

characterizes the maximal abelian algebras in the class of *-subalgebras of B(H) .
In particular, each maximal abelian algebra is weakly closed and contains I and is, therefore, a von Neumann algebra.

Let $(S, \mathcal{S}, \mu)$ be a σ-finte measure space and $\mathcal{A} = \{M_f : f \in L^\infty(S)\}$ be the so called *multiplication algebra* . Then $\mathcal{A}$ is a maximal abelian algebra of operators in $B(L^2(S))$ (see [14] p.308).

The following proposition is a problem in [14] p.376 .

**5.1 Proposition:** Let S be a locally compact topological space, $\mathcal{S}$ the σ-algebra of Borel sets and μ a σ-finite regular Borel measure on S. Let $\mathcal{U}$ be the algebra of multiplications by bounded continuous functions on $L^2(S,\mu)$ and $\mathcal{A}$ its w.o.t. closure. Then $\mathcal{A}$ is the multiplication algebra.

**Proof.**

It is clear that $\mathcal{U} \subset \mathcal{A}$ and so $\overline{\mathcal{U}}^{w.o.t.} \subset \mathcal{A}$ where $\overline{\mathcal{U}}^{w.o.t.}$ means the w.o.t. closure of $\mathcal{U}$.

To show: $\mathcal{A} \subset \overline{\mathcal{U}}^{w.o.t.}$.

Since $\mathcal{U}$ is an algebra, $\mathcal{U}$ is convex and so $\overline{\mathcal{U}}^{w.o.t.} = \overline{\mathcal{U}}^{s.o.t.}$ (see [2] p.305).

Thus, we will show $\mathcal{A} \subset \overline{\mathcal{U}}^{s.o.t.}$.

Let $\varphi \in L^\infty(S)$ and $f_1, f_2, \ldots, f_n \in H\ (= L^2(S))$ and $\varepsilon > 0$, we must show that there exists $g \in C(S)$ such that

$$\| (M_\varphi - M_g) f_i \| < \varepsilon \qquad \text{for } i = 1, 2, \ldots, n$$

or

$$\| (\varphi - g) f_i \| < \varepsilon \qquad \text{for } i = 1, 2, \ldots, n.$$

For this, first we will show that we can make certain assumptions on the function $\varphi$.

Since the measure $\mu$ is $\sigma$-finite, $S = \bigcup_{n=1}^{\infty} S_n$ with $\mu(S_n) < \infty$ . For every n, we can take $S_n \subseteq S_{n+1}$ and $S = \bigcup_{n=1}^{\infty} S_n$ i.e. $S_n \uparrow S$ with $\mu(S_n) < \infty$.

Let $\varphi_n = \chi_{S_n} \varphi$, $\varphi_n = 0$ off $S_n$.

If $f \in H$, then since $\chi_{S_n} \to 1$ as $n \to \infty$

$$|\chi_{S_n} - 1|^2 |f|^2 \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad |\chi_{S_n} - 1|^2 |f|^2 \le |f|^2 \in L^1(S)$$

and so the Lebesgue's dominated convergence theorem gives that

$$\| (M_{\varphi_n} - M_\varphi) f \|^2 \to 0 \ \text{ as } n \to \infty \ , \text{ i.e., } \ M_{\varphi_n} \to M_\varphi \ \text{(s.o.t)}$$

The above proves that we can assume that $\varphi$ is such that, $\varphi = 0$ off a set of finite measure.

Moreover since $\varphi \in L^\infty(S,\mu)$ we may assume that $0 \le \varphi \le 1$.

Next, we show that $\varphi$ may be taken to be a simple function.

For $n \in \mathbb{N}$, since $0 \le \varphi \le 1$, we define the sequence of functions

$$\psi_n = \sum_{k=1}^{n} \frac{k}{n} \chi_{\varphi^{-1}\left(\frac{k}{n}, \frac{k+1}{n}\right]} \qquad k=1, 2, \ldots, n.$$

Then the $\psi_n$ are simple non-negative functions ( $\psi_n \le \psi_{n+1}$ ) and for all $s \in S$

$$0 \le \varphi(s) - \psi_n(s) \le \frac{1}{n}$$

so $\psi_n \to \varphi$ uniformily and so by Lebesgue`s dominated convergence theorem

$$M_{\psi_n} \to M_\varphi \quad \text{in the s.o.t.}$$

Now, it will be enough to prove our initial statement, with $\varphi$ the characteristic function, $\mathcal{X}_A$, of a subset A of S with $\mu(A) < \infty$ .
So we must find a function $g \in C_B(S)$ such that $\forall \varepsilon > 0$

$$\| (\mathcal{X}_A - g) f_i \| < \varepsilon \qquad \forall i = 1, 2, \dots, n$$

Since the measure $\mu$ is regular, there are an open set $U \supseteq A$ and a compact set $K \subseteq A$ such that $\mu(U \sim K) < \delta$ for any $\delta > 0$.
By Urysohn's lemma, there exists g continuous $0 \leq g \leq 1$ such that $g \equiv 1$ on K and $g \equiv 0$ on $U^c$.
Then

$$\| (\mathcal{X}_A - g) f_i \|^2 = \int_S |\mathcal{X}_A - g|^2 |f_i|^2 d\mu = \int_{U \sim K} |\mathcal{X}_A - g|^2 |f_i|^2 d\mu \leq \int_{U \sim K} |f_i|^2 d\mu$$

for all $i = 1, 2, \dots, n$ .
Now since, the set-function $\lambda(E) = \int_E f \, d\mu$ with f integrable, is absolutely continuous, given any $\varepsilon > 0$ we may choose $\delta > 0$ so small that $\mu(U \sim K) < \delta$ implies $\int_{U \sim K} |f_i|^2 d\mu < \varepsilon$ and the proof is complete. ■

Let H be a Hibert space.

**Definition**: Let $\mathcal{U}$ be a *-subalgebra of B(H), a vector $x \in H$ is called a **cyclic vector** for $\mathcal{U}$ if the subspace $\mathcal{U}x = \{Tx: T \in \mathcal{U}\}$ is dense in H, i.e., if $[\mathcal{U}x] = H$.

**Definition**: We say that a projection P in B(H) is *cyclic relative to* $\mathcal{U}$, where $\mathcal{U}$ is a *-subalgebra of B(H), if $\mathcal{U}x$ is dense in P(H), i.e. if $[\mathcal{U}x] = \text{Range}(P)$.

Note that, every cyclic projection relative to $\mathcal{U}$ belongs to $\mathcal{U}'$.

**Definition**: A vector $\xi \in H$ is called a **separating vector** for $\mathcal{U}$, if $T\xi = 0$ implies $T = 0$ for all $T \in \mathcal{U}$.

**5.1 Lemma**: A vector $x \in H$ is cyclic for a *-subalgebra $\mathcal{U}$ of B(H) iff x is separating for $\mathcal{U}'$. ([5] p. 109)

**Proof.**

($\Longrightarrow$): Suppose $x \in H$ is cyclic for $\mathcal{U}$ i.e. $[\mathcal{U}x] = H$. Let $T' \in \mathcal{U}'$ be such that $T'x = 0$, then

$TT'x = 0 \ \forall T \in \mathcal{U} \Longrightarrow T'Tx = 0 \ \forall T \in \mathcal{U} \Longrightarrow T'(\mathcal{U}x) = 0 \Longrightarrow T'([\mathcal{U}x]) = 0$, i.e., $T' = 0$.

($\Longleftarrow$): Suppose $x$ is separating for $\mathcal{U}'$. If $P'$ is the projection onto $[\mathcal{U}x]$, then $P' \in \mathcal{U}'$ which implies $(I - P') \in \mathcal{U}'$.

Note that $x$ ( $= Ix$ ) belongs to the Range($P'$) so $(I - P')x = 0$ and so $I - P' = 0$, i.e., $P' = I$ or $[\mathcal{U}x] = H$. ■

**5.2 Lemma**: Let $\mathcal{U}$ be a *-subalgebra of $B(H)$. Then there is a family $\{P_j : j \in J\}$ of pairwise orthogonal projections in $\mathcal{U}'$, cyclic relative to $\mathcal{U}$ such that $\sum_j P_j = I$.
Moreover, if $H$ is separable, $J$ is countable.

**5.3 Lemma**: Let $H$ be a separable Hilbert space. Then, every abelian *-subalgebra of $B(H)$ has a separating vector.

For the proofs of Lemmas 5.2, 5.3 see [5] p.109 .

Note, now that by Lemmas 5.1, 5.3 we conclude that any maximal abelian algebra of operators in a separable Hilbert space, has a cyclic vector.
The following theorem is from [22]. Our proof uses the spectral theorem for commutative C*-algebras (Theorem 4.5) .

**5.1 Theorem**: Let $H$ be a separable Hilbert space and $\mathcal{A}$ be a maximal abelian algebra in $B(H)$. Then there exists a measure space $(X, \mathcal{B}_X, \mu)$, with $X$ a compact Hausdorff space, $\mu$ a finite posistive regular Borel measure on $X$, and a unitary operator $U: H \to L^2(X, \mu)$ such that $U\mathcal{A}U^{-1} = \{M_f : f \in L^\infty(X,\mu)\}$

**Proof.**

$\mathcal{A}$ being maximal abelian, has a cyclic vector $z \in H$. Let $H_0 = \mathcal{A}z$.Then $H_0$ is dense in H.

$\mathcal{A}$ in particular is a commutative C*-algebra so $\mathcal{A} \cong C(X_{\mathcal{A}})$, where $X = X_{\mathcal{A}}$ is Stonean.
Also by the spectral theorem for commutative C*-algebras there exists (a unique) spectral measure $E$ on the Borel subsets of $X$ such that

$$(T\xi,\xi) = \int_X \hat{T}\, dE_{\xi,\xi} \qquad \forall T \in \mathcal{A} \quad \forall \xi \in H$$

Moreover, $\forall f \in B(X)$

$$(f(T)\xi,\xi) = \int_X f\, dE_{\xi,\xi} \qquad \forall \xi \in H.$$

So $E_{\xi,\xi}(\cdot) = (E(\cdot)\xi,\xi) = \| E(\cdot)\xi \|^2$, $\xi \in H$ is a positive regular Borel measure on $\mathcal{B}_X$, the $\sigma$-algebra of Borel subsets of $X$.

Furthermore, $E_{\xi,\xi}$ is a finite measure, since $E_{\xi,\xi}(X) = \|\xi\|^2 < \infty$.

Take $\mu = E_{z,z}$ where z is the cyclic vector of $\mathcal{A}$ referred above.
Now, we define a mapping

$$U_0 : H_0 \to C(X) \ (= C_B(X))$$

by $U_0(Tz) \equiv \hat{T}$, where $T \in \mathcal{A}$.

$U_0$ is a well defined linear mapping from $H_0$ onto C(X). The linearity and surjectivity of $U_0$ is clear by the linearity and surjectivity of the Gelfand map $\wedge : \mathcal{A} \to C(X)$.

To see that $U_0$ is well-defined, suppose $Tz = Sz$ $T,S \in \mathcal{A}$. Then $(T - S)z = 0$ which implies $T - S = 0$ since z is also separating for $\mathcal{A}$ (by lemma 5.1).

So $T = S$ and so $\hat{T} = \hat{S}$, i.e., $U_0(Tz) = U_0(Sz)$.

Moreover, for $T \in \mathcal{A}$,

$$\|Tz\|^2 = \int_X |\hat{T}|^2 \, dE_{z,z} = \int_X |\hat{T}|^2 \, d\mu = \|\hat{T}\|^2$$

Thus, $U_0$ is an isometry of $H_0$ onto C(X). Since $H_0$ is dense in H and C(X) is dense in $L^2(X,\mu)$, $U_0$ extends uniquely by continuity to a unitary operator U of H onto $L^2(X,\mu)$.

Now, given $S \in \mathcal{A}$, let $g = \hat{S}$. We will show that $USU^{-1}$ coincides with the bounded operator $M_g$ on a dense subspace of $L^2(X,\mu)$ and thus they will be equal.

Let $f \in C(X)$ and say $f = \hat{T}$, $T \in \mathcal{A}$.

Then $USU^{-1}(f) = U_0SU_0^{-1}(f) = U_0STz = (ST)^{\wedge} = \hat{S}\hat{T} = gf = M_g(f)$.

Hence $USU^{-1} = M_g$.

This shows that U provides a unitary equivalence $S \mapsto USU^{-1}$ between $\mathcal{A}$ and $\{M_g : g \in C(x)\}$, i.e., $U\mathcal{A}U^{-1} = \{M_g : g \in C(X)\} = \mathcal{U}$.

Since $\mathcal{A}$ is maximal abelian so also is $U\mathcal{A}U^{-1}$, i.e., $\mathcal{U} = \mathcal{U}'$

But $\mathcal{U} \subseteq \{M_f : f \in L^\infty(X,\mu)\} = \mathscr{A}$ so $\mathcal{U}' \supseteq \mathscr{A}' = \mathscr{A}$.

Therefore, $\mathcal{U} = \mathscr{A}$, that is, $U\mathcal{A}U^{-1} = \{M_f : f \in L^\infty(X,\mu)\}$. ■

**5.1 Remark:** Let $A \in B(H)$ be a normal operator acting on a separable Hilbert space H. If $\mathcal{A}$ is a maximal abelian von Neumann algebra containing A, then by theorem 5.1 $U\mathcal{A}U^{-1} = \{M_f : f \in L^\infty(X,\mu)\}$. So $A = U^{-1}M_gU$ for some $g \in L^\infty(X,\mu)$, i.e., A is *unitarily equivalent* with a multiplication operator acting on $L^2(X,\mu)$.

Now combining this fact with example 4.1 we see that the functional calculus for A is given by

$$\Phi(f) = f(A) = U^{-1}f(M_g)U = U^{-1}M_{f \circ g}U \quad ■$$

**Definition:** Let m be a regular Borel measure on a locally compact space X. We call m a **standard measure** on X if

$$\int_X f(x)\, dm(x) > 0$$

whenever $f \in C_c(X)$ , $f \geq 0$ and $f \not\equiv 0$ . (see [22] p. 236 )

We remark that the regular Borel measure $\mu$ we got in the Theorem 5.1 is a standard measure. To see this; let $g \in C(X)$ ($X = X_{\mathcal{a}}$ is compact), $g \geq 0$ , $g \neq 0$ .
Then g has a square root $h \in C(X)$ with $h \geq 0$ , $h \neq 0$ and $h^2 = g$ .
Let $T \in \mathcal{a}$ be such that $\hat{T} = h$ ( $T \geq 0$, $T = T^*$ )

$$\int_X g(x)\, d\mu = \int_X (h(x))^2\, d\mu = (T^2 z, z) = \| Tz \|^2 \geq 0$$

If $\| Tz \|^2 = 0$ then $Tz = 0 \Rightarrow T = 0 \Rightarrow h = 0$ contradicting $g \neq 0$ .
Hence

$$\int_X g(x)\, d\mu > 0$$

**Definition**:
A standard (regular) measure m on a locally compact space X is called **perfect** if for every bounded measurable function h on X there is a bounded continuous function g such that h = g a.e.[m] .
A measure space is called **perfect** if its measure is such. (see [22] p. 255)

Next we will show that the measure space $(X, \mathcal{B}_X, \mu)$ in theorem 5.1 is perfect.

**5.2 Proposition**: Let $\mathcal{a}$ be a maximal abelian algebra of operators on a separable Hilbert space and $(X, \mathcal{B}_X, \mu)$ be the measure space associated with it, as in the theorem 5.1. Then $(X, \mathcal{B}_X, \mu)$ is perfect.

**Proof.**
Let $h \in B(X)$. Then $M_h \in \{M_f : f \in L^\infty(X,\mu)\}$ and since $U \mathcal{a} U^{-1} = \{M_f : f \in L^\infty(X,\mu)\}$ there is $T \in \mathcal{a}$ such that

$$UTU^{-1} = M_h$$

On the other hand

$$UTU^{-1} = M_{\hat{T}} \qquad \text{for } \hat{T} \in C(X)$$

Therefore

$$M_h = M_{\hat{T}} \quad \text{which implies} \quad h = \hat{T} \text{ in } L^\infty(X,\mu)$$

or

$$h = \hat{T} \text{ a.e. } [\mu] \qquad \blacksquare$$

**5.3 Proposition**: In a finite perfect measure space $(X, m)$, with $X$ Stonean, a set is of measure zero iff it is meager.

**Proof.**

($\Leftarrow$): First we will show that the measure of any non-empty open set is positive. For this, let $G \neq \emptyset$ be an open set in X and let $x_0 \in G$. Then $\{x_0\}$ and $G^c$ are closed disjoint subsets of X and by Urysohn's lemma $\exists\ f \in C(X),\ 0 \leq f \leq 1$ such that $f(x_0) = 1$ and $f \equiv 0$ on $G^c$.

If $m(G) = 0$, then $\int_X f(x)\, dm(x) = \int_G f(x)\, dm(x) = 0$ but $f \geq 0$ and $f \neq 0$ contradicting the fact that m is a standard measure.

Next, we prove that, if $F$ is closed set in $X$ with $m(F) > 0$ then $\text{int}(F) \neq \emptyset$.

We have $\mathcal{X}_F \in B(X)$, so there exists (a unique) $g \in C_B(X)$ such that

$$\mathcal{X}_F = g \quad \text{a.e.}[m] \quad \text{or} \quad g - \mathcal{X}_F = 0 \quad \text{a.e.}[m].$$

Let $G = \{g \neq 0\}$; $G$ is open since $g$ continuous.

$G \sim F = \{g \neq 0 \text{ and } \mathcal{X}_F = 0\} \subseteq \{g - \mathcal{X}_F \neq 0\}$. But $m(\{g - \mathcal{X}_F \neq 0\}) = 0$, so $m(G \sim F) = 0$ or $m(G \cap F^c) = 0$ and so $G \cap F^c = \emptyset$ since $G \cap F^c$ open. Hence $G \subseteq F$.

Now $G \neq \emptyset$, for if $G = \emptyset$, then $g \equiv 0$ on $X$ so $\mathcal{X}_F = 0$ a.e.[m] which says $m(F) = 0$ .But $m(F) > 0$.

Thus, $\text{int}(F) \neq \emptyset$.

Suppose now, that $F$ is a nowhere dense set . Then by what we just proved $m(\overline{F}) = 0$ and so $m(F) = 0$.

Finally, if F is meager set , then $F = \bigcup_{n=1}^{\infty} F_n$ with $F_n$ nowhere dense for all n and so

$$m(F) = m\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} m(F_n) = 0 \text{ , i.e., } m(F) = 0$$

($\Rightarrow$): Let $F$ be a Borel set with $m(F) = 0$.

Now $\mathcal{X}_F \in B(X)$, so since X is Stonean proposition 4.2 gives $\mathcal{X}_F = g + h$ with $g \in C(X)$, $h \in n(X)$.This implies $\{g \neq 0\} \subseteq \{\mathcal{X}_F \neq 0\} \cup \{h \neq 0\} = F \cup \{h \neq 0\}$

Since $m(F \cup \{h \neq 0\}) \leq m(F) + m(\{h \neq 0\}) = 0$,
we get

$$m(\{g \neq 0\}) = 0 \text{ which implies } \{g \neq 0\} = \emptyset$$

So $g \equiv 0$ on X and therefore $\mathcal{X}_F = h$ , that is, $F = \{h \neq 0\}$ is meager. ■

Note that the assumption that X is Stonean is not needed for the proof of the ($\Leftarrow$) of the above proposition.

**5.4 Proposition**: Let $\mathcal{A}$ be an abelian von Neumann algebra acting on a Hilbert space H and let A be an operator in $\mathcal{A}$. Suppose $\mathcal{A} \cong C(X)$, where $X = X_{\mathcal{A}}$ a Stonean space and f in C(X) represents A, i.e., $f = \hat{A}$. Then, $Ax = \lambda x$ for some unit vector x in H and $\lambda \in \mathbb{C}$ iff $f^{-1}(\{\lambda\})$ contains a non-empty clopen subset of X.

**Proof.**
($\Rightarrow$): If $Ax = \lambda x$ with $\|x\| = 1$ then $(A - \lambda I)x = 0$ so $Ker(A - \lambda I) \neq \{0\}$.
If P is the projection onto $Ker(A - \lambda I)$, then P is in $\mathcal{A}$ (P is the range projection onto the orthogonal complement of $Range(A - \lambda I)^*$, see [14] p.327) and

$$(A - \lambda I)P = 0 \implies AP = \lambda P \implies \hat{A}\hat{P} = \lambda \hat{P}$$

since P is a non-zero projection, $P = \chi_S$ with $S \neq \emptyset$ a clopen subset of X.
Thus $f\chi_S = \lambda \chi_S$ so if $s \in S$, $f(s) = \lambda$, i.e., $s \in f^{-1}(\{\lambda\})$ and so $S \subseteq f^{-1}(\{\lambda\})$.

($\Leftarrow$): Suppose $\emptyset \neq S \subseteq f^{-1}(\{\lambda\})$, S clopen in X. Then $\chi_S \in C(X)$. Let P be the operator (a projection) in $\mathcal{A}$ such that $P = \chi_S$, $P \neq 0$.
Now for all $s \in S$, $f(s) = \lambda$, so $f\chi_S = \lambda \chi_S$.
Therefore $AP = \lambda P \implies (A - \lambda I)P = 0 \implies Range(P) \subseteq Ker(A - \lambda I)$ and since $P \neq 0$, $Ker(A - \lambda I) \neq \{0\}$. Thus there is $x \in Ker(A - \lambda I)$ $x \neq 0$, (in particular one can take x with $\|x\| = 1$) such that $Ax = \lambda x$. ∎

The above proposition characterizes the complex numbers which are eigenvalues for a given operator in the algebra $\mathcal{A}$.

**5.5 Proposition**: Let $\mathcal{A}$ be a maximal abelian algebra of operators. Suppose $\mathcal{A} \cong C(X)$ where $X = X_{\mathcal{A}}$ is Stonean and let $e \in H$, $\|e\| = 1$ be such that $Ae = \lambda e$ for all $A \in \mathcal{A}$ and $\dim(Ker(A - \lambda I)) = 1$. Then $\rho: \mathcal{A} \to \mathbb{C}$ given by $\rho(A) = (Ae, e)$ is a multiplicative linear functional (, i.e., $\rho \in X$) and $\{\rho\}$ is open in X.

**Proof.**
The linearity of $\rho$ is clear; we will show that $\rho$ is multiplicative. Let $A, B \in \mathcal{A}$.
Then

$$Ae = \lambda e \implies BAe = \lambda Be \implies ABe = \lambda Be \implies Be \in Ker(A - \lambda I)$$

and so $Be = \mu e$ for some $\mu \in \mathbb{C}$, since $\dim(Ker(A - \lambda I)) = 1$.
Now, we have

$$\rho(AB) = (ABe, e) = (\lambda\mu e, e) = \lambda\mu = (\lambda e, e)\cdot(\mu e, e) = (Ae, e)\cdot(Be, e) = \rho(A)p(B)$$

Let P be a projection onto Ker(A - $\lambda$ I).

Then $Py = (y,e)e$ $y \in H$, so for $x \in H$, $A \in \mathcal{A}$

$$PAx = (Ax,e)e = (x,A^*e)e = (x, \bar{\lambda}e)e = \lambda(x,e)e = (x,e)Ae .$$

On the other hand,

$$APx = A((x,e)e) = (x,e)Ae.$$

Therefore $AP = PA$ $\forall A \in \mathcal{A} \implies P \in \mathcal{A}' = \mathcal{A}$.

Next we prove that $\{\rho\}$ is open in X. X being extremelly disconnected is totally disconnected.

If $q \in X$, $q \neq \rho$ then there are clopen sets $C_1, C_2$ such that $\rho \in C_1$, $q \in C_2$ and $C_1 \cap C_2 = \emptyset$.

If $f = \chi_{C_1}$ and $g = \chi_{C_2}$ , then $f(\rho) = 1$, $g(q) = 1$ and $F,G \in \mathcal{A}$ with $\hat{F} = f$, $\hat{G} = g$ are projections and $FG = 0$.

Moreover, since $\widehat{PF}(\rho) = (\hat{P}f)(\rho) = \hat{P}(\rho)f(\rho) = \hat{P}(\rho) = \rho(P) = \| e \| = 1$ , $PF \neq 0$;

Also $PF \leq P$ and since P is 1-dim'l we conclude $PF = P \implies P \leq F \implies \hat{P} \leq f$ .

But q was any point of X different from $\rho$ , so $\hat{P}(q) = 0$ or $\hat{P} = \chi_{\{\rho\}}$

So $\chi_{\{\rho\}} \in C(X)$ and so $\{\rho\}$ is open (hence clopen). ■

The following proposition is the converse of Proposition 5.5.

**5.6 Proposition**: Let $\mathcal{A}$ be a maximal abelian algebra of operators. Let $\rho \in X = X_{\mathcal{A}}$ and suppose that $\{\rho\}$ is open in X. Then, there exists a vector $e \in H$, $\| e \| = 1$ such that $\rho(A) = (Ae,e)$ for all $A \in \mathcal{A}$ and e is an eigenvector for every $A \in \mathcal{A}$ with eigenvalue of multiplicity 1.

**Proof.**

Let $P \in \mathcal{A}$ be such that $\hat{P} = \chi_{\{\rho\}}$, P is a non-zero projection in $\mathcal{A}$ and $PA=AP$, for all $A \in \mathcal{A}$.

For each $A \in \mathcal{A}$ let $\lambda = \hat{A}(\rho)$ ; then $(A - \lambda I)P = 0$.

If $e \in \text{Range}(P)$ with $\| e \| = 1$ , then $Ae = \lambda e$ for all $A \in \mathcal{A}$ and so

$$\rho(A) = \hat{A}(\rho) = \lambda = (\lambda e,e) = (Ae,e).$$

Furthermore, let Q be the projection onto span$\{e\}$; then $A(\text{Range}(Q)) \subseteq \text{Range}(Q)$, i.e., $QAQ=AQ$ for all $A \in \mathcal{A}$ so $QA^*Q = A^*Q$ as well, and so by taking adjoints in the last equality $QA = AQ$ for all $A \in \mathcal{A}$ that is $Q \in \mathcal{A}' = \mathcal{A}$ .

Now, let $g = \hat{Q}$ ; since $Q \leq P$ we have $0 \leq g \leq \hat{P} = \chi_{\{\rho\}}$, therefore $g = \chi_{\{\rho\}}$ and so $Q = P$, i.e., Range(P) is 1-dim'l. ■

❑

## § 6. $\beta(\mathbb{N})$

The theory of $\beta(\mathbb{N})$ is developed in [14] in the problem sections see p. 224, p. 374 Here we prove only those facts that are needed for our purposes.

Let $\ell^\infty$ denote the Banach algebra $\ell^\infty(\mathbb{N},\mathbb{C})$ of all bounded complex sequences $x = \{x_n\}_{n=1}^{\infty}$ with norm $\| x \| = \sup_n | x_n |$. The operations on $\ell^\infty$ are as follows:

For $x,y \in \ell^\infty$ $x = \{x_n\}_{n=1}^{\infty}$ , $y = \{ y_n\}_{n=1}^{\infty}$ , $\lambda \in \mathbb{C}$

$$x + y = \{x_n + y_n\} , \lambda x = \{\lambda x_n\}, \; xy = \{ x_n y_n\} , \; x^* = \{ \overline{x_n}\}.$$

In fact , $\ell^\infty$ is a commutative C*-algebra .
Let c and $c_o$ be the linear subspaces of $\ell^\infty$ defined by

$$c = \{ \{x_n\} \in \ell^\infty : \lim_{n\to\infty} x_n \text{ exists}\}$$

$$c_o = \{ \{x_n\} \in \ell^\infty : \lim_{n\to\infty} x_n = 0 \}.$$

Then c and $c_o$ are closed subspaces of $\ell^\infty$.
Moreover, c, provided with pointwise multiplication, is a (commutative) C*-subalgebra of $\ell^\infty$ . Note that , the linear functional m on c given by $m(x) = \lim_{n\to\infty} x_n$ for $x \in c$, is a multiplicative linear functional with kernel $c_o$. Thus $c_o$ is a maximal ideal in c.
We remark that $c_o$ is not a maximal ideal in $\ell^\infty$ . To see this; suppose that $c_o$ were a maximal ideal in $\ell^\infty$ . Then $\dim(\ell^\infty/c_o) = 1$ .
So, if $\pi\colon \ell^\infty \to \ell^\infty/c_o$ is the quotient map , there exists $x \in \ell^\infty$ such that $\{\pi(x)\}$ is a basis for $\ell^\infty/c_o$. Note that $x \notin c_o$.
Thus $x \in c$ or $x \in \ell^\infty \sim c$. Now, if $x \in c$, then for $y \in \ell^\infty \sim c$ there is $\lambda \in \mathbb{C}$ such that $\pi(y) = \lambda\pi(x)$ , so $(y - \lambda x) \in c_o$ and so $y \in c$, a contradiction.
On the other hand, if $x \in \ell^\infty \sim c$, then for $y_o = \{1, 1, 1, ...\} \in \ell^\infty$ there is $\lambda_o \in \mathbb{C}$ such that $\pi(y_o) = \lambda_o\pi(x)$ or $(y_o - \lambda_o x) \in c_o$ , i.e., $1 - \lambda_o x_n \to 0$ as $n \to \infty$. So $x_n \to \frac{1}{\lambda_o}$

(note that $\lambda_o \neq 0$ since $y_o \notin c_o$). Hence, $\lim_{n \to \infty} x_n$ exists and so $x \in c$ which is also a contradiction.

It is often convinient to use function notation for the elements of $\ell^\infty$ . So with this notation

$$\ell^\infty = \{f: \mathbb{N} \to \mathbb{C} \;:\; \|f\| = \sup_n |f(n)| < \infty\}$$

**Definition**: $\beta(\mathbb{N})$ is defined to be the space of all non-zero multiplicative linear functionals on $\ell^\infty$ , topologized with the weak*-topology.

$\beta(\mathbb{N})$ is called " **the $\beta$ -compactification of $\mathbb{N}$** "

Thus $\beta(\mathbb{N})$ is the Gelfand space of the commutative Banach algebra $\ell^\infty$ and as such is a compact Hausdorff space.

Moreover, if we define $\hat{f}(\rho) = \rho(f)$ for $f \in \ell^\infty$ and $\rho \in \beta(\mathbb{N})$, then the Gelfand map $\wedge: \ell^\infty \to C(\beta(\mathbb{N}))$ , $f \mapsto \hat{f}$ is an isometric *-isomorphism of $\ell^\infty$ onto $C(\beta(\mathbb{N}))$, i.e.,

$$(\ell^\infty)^\wedge \cong C(\beta(\mathbb{N})).$$

**6.1 Proposition**: $\beta(\mathbb{N})$ is extremely disconnected (and so Stonean).

**Proof.**

We will show that $C(\beta(\mathbb{N}))$ is a boundedly complete lattice, then (by theorem 2.1) $\beta(\mathbb{N})$ will be extremely disconnected.

Suppose that $\mathcal{F} = \{\hat{f}_a\}_{a \in J}$ is a bounded subset of $C_{\mathbb{R}}(\beta(\mathbb{N}))$, $f_a \in \ell^\infty$ .

So there is $M > 0$ such that $\hat{f}_a \leq M \;\; \forall a \in J$. Define $f \in \ell^\infty$ by $f(n) = \sup_a f_a(n)$.

Then $\hat{f}_a \leq \hat{f} \;\; \forall a$ , since $f_a \leq f \;\; \forall a$ and the Gelfand map is order preserving.

Moreover, if $\hat{f}_a \leq \hat{g} \;\; \forall a \in J$ , $g \in \ell^\infty$ , then $f_a \leq g \;\; \forall a$ which implies

$$f(n) = \sup_a f_a(n) \leq g(n) \quad \forall n \quad \text{, i.e.,} \quad f \leq g \quad \text{and so} \quad \hat{f} \leq \hat{g}.$$

Thus $\hat{f} = \text{lub}\mathcal{F}$ is in $C(\beta(\mathbb{N}))$. ■

**6.2 Proposition**: Let $i: \mathbb{N} \to \beta(\mathbb{N})$ be given by $i(n)(f) = f(n)$ for $f \in \ell^\infty$ .

Then $i(\mathbb{N})$ is dense in $\beta(\mathbb{N})$ and the one-point set $\{i(n)\}$ is open in $\beta(\mathbb{N})$.

Moreover, $\rho(f) = 0$ when $\rho \in \beta(\mathbb{N}) \sim i(\mathbb{N})$ and $f \in c_o$ .

**Proof.**

Suppose that $i(\mathbb{N})$ is not dense in $\beta(\mathbb{N})$ and let $\rho \in \beta(\mathbb{N}) \sim \overline{i(\mathbb{N})}$.

Since $\beta(\mathbb{N})$ is a compact Hausdorff space , $\beta(\mathbb{N})$ is completely regular.

So there is $g \in C(\beta(\mathbb{N}))$ such that $g(\rho) = 1$ and $g \equiv 0$ on $\overline{i(\mathbb{N})}$.

In particular $g(i(n)) = 0 \ \forall n \in \mathbb{N}$.
Since $(\ell^\infty)^\wedge \cong C(\beta(\mathbb{N}))$, let $f \in \ell^\infty$ be such that $\hat{f} = g$.
Then

$$0 = g(i(n)) = \hat{f}(i(n)) = i(n)(f) = f(n) \quad \forall n, \text{ that is, } f = 0 .$$

So

$$g = \hat{f} = 0 \quad \text{a contradiction.}$$

Now, let $n \in \mathbb{N}$ be an arbitrary fixed positive integer and $\rho \in \beta(\mathbb{N})$ with $\rho \notin i(\mathbb{N})$.
If

$$\mathcal{X}_{\{n\}}(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \qquad \text{and} \quad f \in \ell^\infty ,$$

then

$$\mathcal{X}_{\{n\}} f = f(n)\mathcal{X}_{\{n\}} \qquad \text{which implies} \qquad \rho(\mathcal{X}_{\{n\}} f) = f(n)\rho(\mathcal{X}_{\{n\}}).$$

On the other hand,

$$\rho(\mathcal{X}_{\{n\}} f) = \rho(\mathcal{X}_{\{n\}})\rho(f). \text{ So } \rho(\mathcal{X}_{\{n\}})[\rho(f) - f(n)] = 0 .$$

Since $\rho \notin i(N)$, $f \in \ell^\infty$ can be choosen such that $\rho(f) \neq f(n)$.
Thus,

$$\rho(\mathcal{X}_{\{n\}}) = 0 \text{ or } (\mathcal{X}_{\{n\}})^\wedge(\rho) = 0.$$

Note also that $(\mathcal{X}_{\{n\}})^\wedge(i(n)) = i(n)\mathcal{X}_{\{n\}} = \mathcal{X}_{\{n\}}(n) = 1$.
Hence $(\mathcal{X}_{\{i(n)\}})^\wedge = (\mathcal{X}_{\{n\}})^\wedge \in C(\beta(\mathbb{N}))$ and so $\{i(n)\}$ is open (clopen) in $\beta(\mathbb{N})$)
(and $(i(\mathbb{N}) = \bigcup_{n=1}^{\infty} \{i(n)\}$ is open as well ).
Finally, we prove that $\{\rho \in \beta(\mathbb{N}) \colon \hat{f}(\rho) = 0 \ \forall f \in c_o\} = \beta(\mathbb{N})) \sim i(\mathbb{N})$.
Let $\rho \in \beta(\mathbb{N})$ be such that $f^\wedge(\rho) = 0 \quad \forall f \in c_o$ and suppose that $\rho \in i(N)$.
Then there exists $n_o \in \mathbb{N}$ such that $\rho = i(n_o)$ and

$$0 = \hat{f}(\rho) = \rho(f) = i(n_o)(f) = f(n_o) \quad \forall f \in c_o.$$

Taking in particular $f = \mathcal{X}_{\{n_o\}}$ we get $0 = f(n_o) = 1$ which is a contradiction.
On the other hand , if $\rho \in \beta(\mathbb{N}) \sim i(\mathbb{N})$ , then as above we have that $\rho(\mathcal{X}_{\{n\}}) = 0 \quad \forall n \in \mathbb{N}$.
Now, since any $f \in c_o$ can be writen in the form $f = \sum_{K=1}^{\infty} f(k)\, \mathcal{X}_{\{k\}}$ we have
$\rho(f) = \rho(\sum_{K=1}^{\infty} f(k)\mathcal{X}_{\{k\}}) = \sum_{K=1}^{\infty} f(k)\rho(\mathcal{X}_{\{k\}}) = 0$ and the proof is complete. ■

**Remark 6.1.** We recall that $\ell^2 = \{f: \mathbb{N} \to \mathbb{C} \ : \ \sum_{n=1}^{\infty} |f(n)|^2 < \infty\}$. For $f \in \ell^\infty$, let $M_f: \ell^2 \to \ell^2$ be the multiplication operator by f, i.e., $M_f(g) = fg \quad \forall g \in \ell^2$.
It is easy to see that the mapping $f \mapsto M_f$ from $\ell^\infty$ into $B(\ell^2)$ is an isometric *-isomorphism.
The algebra $\mathcal{A} = \{M_f : f \in \ell^\infty\}$ is a maximal abelian algebra of operators in $\ell^2$. To see this; suppose $A \in B(\ell^2)$ is such that $AM_f = M_fA \ \ \forall f \in \ell^\infty$, i.e., $A \in \mathcal{A}'$.
Let $e_n = \mathcal{X}_{\{n\}} \in \ell^2$ and set $g_n = Ae_n$. Then $fg_n = M_fAe_n = AM_fe_n = f(n)Ae_n = f(n)g_n$.
So $\forall k \in \mathbb{N}$ we have $f(k)g_n(k) = f(n)g_n(k)$ or $(f(k) - f(n))g_n(k) = 0 \quad \forall f \in \ell^\infty$
and so $g_n(k) = 0$ for $n \neq k$.

Now

$$|g_n(n)| = \|g_n\|_{\ell^2} = \|Ae_n\| \le \|A\| \qquad \forall n \in \mathbb{N}.$$

Thus, if $g = \{g_n(n)\}_{n=1}^{\infty}$, then $g \in \ell^\infty$ and

$$M_g e_n = ge_n = \{g_k(k)e_n(k)\}\}_{k=1}^{\infty} = \{g_n(k)\}_{k=1}^{\infty} = g_n.$$

Hence

$$M_g e_n = Ae_n$$

Now since elements of the form $h = \{ h(1), h(2), ..., h(m), 0, 0, ...\}$ are dense in $\ell^2$, $h = h(1)e_1 + h(2)e_2 + ... + h(m)e_m$ and A, $M_g$ are bounded operators we conclude that

$$A = M_g. \text{ Thus } \quad \mathcal{A} = \mathcal{A}'. \ \blacksquare$$

Note that $\mathcal{A} = \{M_f : f \in \ell^\infty\}$ can be viewed as a particular case of the multiplication algebra $\{M_f : f \in L^\infty(S, \mu)\}$ by taking $S = \mathbb{N}$ and the measure $\mu$ to be the counting measure on $\mathbb{N}$.
As such $\mathcal{A}$ is maximal abelian.

**6.3 Proposition:** If J is a closed two-sided ideal in a C*-algebra $\mathcal{A}$, $\pi: \mathcal{A} \to \mathcal{A}/J$ the quotient map and $\pi_*: C(X_{\mathcal{A}}) \to C(X_{\mathcal{A}/J})$ is defined by $\pi_*(f) = f\circ\pi^*$ where
$\pi^*: X_{\mathcal{A}/J} \to X_{\mathcal{A}}$ is the map $\pi^*(\tilde{\rho}) = \tilde{\rho}\circ\pi$, then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\wedge} & C(X_{\mathcal{A}}) \\ \pi \downarrow & & \downarrow \pi_* \\ \mathcal{A}/J & \xrightarrow{\wedge} & C(X_{\mathcal{A}/J}) \end{array}$$

**Proof.**

Let $\tilde{\rho} \in X_{\mathcal{a}/J}$ , $A \in \mathcal{a}$ . Then

$(\pi(A))^{\wedge}(\tilde{\rho}) = \tilde{\rho}(\pi(A)) = \tilde{\rho}\circ\pi(A) = \pi^*(\tilde{\rho})(A) = \hat{A}(\pi^*(\tilde{\rho})) = \hat{A}\circ\pi^*(\tilde{\rho}) = \pi_*(\hat{A})(\tilde{\rho})$

Thus $(\pi(A))^{\wedge} = \pi_*(\hat{A})$ and the diagram commutes. ■

**6.4 Proposition**: Let $C_o = \{M_f : f \in c_o\}$. Then the Gelfand space of the quotient C*-algebra $\mathcal{a}/C_o$ is homeomorphic to $\beta(\mathbb{N}) \sim i(\mathbb{N})$.

**Proof.**

Since $c_o$ is a closed ideal in $\ell^\infty$ and the mapping $f \mapsto M_f$ is an isometric *-isomorphism $C_o$ is a closed ideal in $\mathcal{a}$ .So $\mathcal{a}/C_o$ is an abelian C*-algebra .

Now, the mapping $\Psi: X_{\mathcal{a}} \to \beta(\mathbb{N})$ given by $\Psi(\omega) = \omega\circ\Phi$ , with $\omega \in X_{\mathcal{a}}$ and $\Phi: \ell^\infty \to \mathcal{a}$ , $\Phi(f) = M_f$ , is easily seen to be a homeomorphism.

So we may identify $X_{\mathcal{a}}$ with $\beta(\mathbb{N})$.

Furthermore, the mapping $F: X_{\mathcal{a}/C_o} \to \beta(\mathbb{N}) \sim i(\mathbb{N})$ given by $F(\tilde{\omega}) = \tilde{\omega}\circ\pi\circ\Phi$ where $\tilde{\omega} \in X_{\mathcal{a}/C_o}$ , maps into $\beta(\mathbb{N}) \sim i(\mathbb{N})$. To see this; let $f \in c_o$ , $\tilde{\omega} \in X_{\mathcal{a}/C_o}$ .
Then

$$F(\tilde{\omega})(f) = \tilde{\omega}\circ\pi\circ\Phi(f) = \tilde{\omega}(\pi(M_f)) = 0 .$$

So $F(\tilde{\omega})(f) = 0 \ \forall f \in c_o$ and so by proposition 6.2 $F(\tilde{\omega}) \in \beta(\mathbb{N}) \sim i(\mathbb{N})$ .

Also F is a homeomorphism. For this; let $\tilde{\omega}_1, \tilde{\omega}_2 \in X_{\mathcal{a}/C_o}$ and suppose $F(\tilde{\omega}_1) = F(\tilde{\omega}_2)$.
Then

$$F(\tilde{\omega}_1)(f) = F(\tilde{\omega}_2)(f) \quad \forall f \in \ell^\infty \quad \text{or} \quad \tilde{\omega}_1(\pi(M_f)) = \tilde{\omega}_2(\pi(M_f)) \quad \forall f \in \ell^\infty$$

or

$$\tilde{\omega}_1(\pi(A)) = \tilde{\omega}_2(\pi(A)) \quad \forall A \in \mathcal{a}.$$

So $\tilde{\omega}_1 = \tilde{\omega}_2$ and F is one-to-one.

If $\rho \in (\beta(\mathbb{N}) \sim i(\mathbb{N}))$, take $\tilde{\omega} \in X_{\mathcal{a}/C_o}$ defined by $\tilde{\omega}(\pi(A)) = \rho(\Phi^{-1}(A)) \ \ \forall A \in \mathcal{a}$

Then $F(\tilde{\omega}) = \tilde{\omega}\circ\pi\circ\Phi = \rho\circ\Phi^{-1}\circ\Phi = \rho$ and F is onto.

To see that F is continuous , suppose that $\omega_d \to \tilde{\omega}$ in $X_{\mathcal{a}/C_o}$ .
Then

$$\tilde{\omega}_d(\pi(A)) \to \tilde{\omega}(\pi(A)) \quad \forall A \in \mathcal{a} \quad \text{or} \quad (\tilde{\omega}_d\circ\pi\circ\Phi)(f) \to (\tilde{\omega}\circ\pi\circ\Phi)(f) \quad \forall f \in \ell^\infty$$

or

$$F(\tilde{\omega}_d)(f) \to F(\tilde{\omega})(f) \quad \forall f \in \ell^\infty \quad \text{or} \quad F(\tilde{\omega}_d) \to F(\tilde{\omega}) \quad \text{in } \beta(\mathbb{N}) \sim i(\mathbb{N}).$$

Now, since $X_{\mathcal{a}/C_o}$ is compact and $\beta(\mathbb{N}) \sim i(\mathbb{N})$ is Hausdorff, F is a homeomorphism. ■

Note that by proposition 6.3 we have that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\wedge} & C(X_{\mathcal{A}}) \\ \pi \downarrow & & \downarrow \pi_* \\ \mathcal{A}/C_o & \xrightarrow{\wedge} & C(X_{\mathcal{A}/C_o}) \end{array}$$

commutes.

Let r: $C(\beta(\mathbb{N})) \to C(\beta(\mathbb{N}) \sim i(\mathbb{N}))$ be the the restriction mapping and identify $X_{\mathcal{A}/C_o}$ with $\beta(\mathbb{N}) \sim i(\mathbb{N})$. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\wedge} & C(\beta(\mathbb{N})) \\ \pi \downarrow & & \downarrow r \\ \mathcal{A}/C_o & \xrightarrow{\wedge} & C(\beta(\mathbb{N}) \sim i(\mathbb{N})) \end{array} .$$

**6.1 EXAMPLE .**

Let $\mathcal{A} = \{ M_f : f \in \ell^\infty \}$ be the multiplication algebra of operators on $\ell^2$. We noted that $\mathcal{A}$ is a maximal abelian algebra . In this example we are interested to find the regular Borel measure $\mu$ on $X = X_{\mathcal{A}} = \beta(\mathbb{N})$ we get from theorem 5.1.

A vector $g \in \ell^2$ will be a cyclic vector for $\mathcal{A}$ iff g is a separating vector for $\mathcal{A}' = \mathcal{A}$ (lemma 5.1)

So g will be such that $M_f g = 0$ implies $M_f = 0$ $\forall f \in \ell^\infty$ or

$fg = 0$ implies $f = 0$ $\forall f \in \ell^\infty$.

Let $g = \{r^n\}_{n=1}^\infty$ with $0 < | r | < 1$ . Then g is a cyclic vector for $\mathcal{A}$.

From the proof of theorem 5.1, we have that the total variation of the measure $\mu$ is

$$\mu(X) = \mu(\beta(\mathbb{N})) = \| g \|^2 = \sum_{n=1}^{\infty} | g(n) |^2 = \sum_{n=1}^{\infty} (r^n)^2 = \frac{r^2}{1 - r^2} .$$

For $f \in \ell^\infty$ and $\rho \in \beta(\mathbb{N})$, we have

$$(fg,g) = \int_{\beta(\mathbb{N})} \hat{f}(\rho)d\mu = \int_{\beta(\mathbb{N})} \rho(f)d\mu = \int_{i(\mathbb{N})} \rho(f)d\mu + \int_{\beta(\mathbb{N})\sim i(\mathbb{N})} \rho(f)d\mu$$

$$= \int_{i(\mathbb{N})} i(n)(f)d\mu + \int_{\beta(\mathbb{N})\sim i(\mathbb{N})} \rho(f)d\mu = \int_{i(\mathbb{N})} f(n)d\mu + \int_{\beta(\mathbb{N})\sim i(\mathbb{N})} \rho(f)d\mu$$

$$= \sum_{n=1}^{\infty} \int_{i(n)} \{i(n)\} f(n) d\mu + \int_{\beta(\mathbb{N})\sim i(\mathbb{N})} \rho(f) d\mu$$

$$= \sum_{n=1}^{\infty} f(n)\mu(\{i(n)\}) + \int_{\beta(\mathbb{N})\sim i(\mathbb{N})} \rho(f) d\mu$$

On the other hand

$$(fg,g) = \sum_{n=1}^{\infty} f(n) | r^n |^2 = \sum_{n=1}^{\infty} f(n)\mu(\{i(n)\}) + \int_{\beta(\mathbb{N})\sim i(\mathbb{N})} \rho(f) d\mu \qquad (1)$$

For $f = \mathcal{X}_{\{k\}} \in c_o$ ($\rho(f) = 0$ when $\rho \in \beta(\mathbb{N}) \sim i(\mathbb{N})$ ), we have

$$| r^k |^2 = \mu(\{i(k)\})$$

So

$$\mu(\{i(n)\}) = r^{2n} \qquad \forall n \in N$$

Now, if we take $f \equiv 1$ , then $\dfrac{r^2}{1 - r^2} = \dfrac{r^2}{1 - r^2} + \mu(\beta(\mathbb{N}) \sim i(\mathbb{N}))$,

that is, $\mu(\beta(\mathbb{N}) \sim i(\mathbb{N})) = 0$ ∎

**6.2 EXAMPLE.**

Let $H = \ell^2$ and $A = M_h$ where $h(n) = \dfrac{1}{n}$ for $n = 1, 2, \ldots$

Let $\mathcal{A}$ be the abelian von Neumann algebra generated by A .

In this example we show that the Gelfand space $X_{\mathcal{A}}$ of $\mathcal{A}$ is homeomorphic to $\beta(\mathbb{N})$.

First, note that A is a compact operator.

Indeed, if $x^{(n)} = \{h(1), h(2), \ldots, h(n), 0, 0, \ldots\}$ and $A_n = M_{x^{(n)}}$, then

Range($A_n$) is n-dimensional so that $A_n$ is a finite rank operator $\forall n$ (hence compact) .

For $y \in \ell^2$ ( $y = \{y(n)\}_{n=1}^{\infty}$ ) we have

$$\|(A - A_n)y\|^2 = \sum_{k=n+1}^{\infty} \frac{1}{k^2} |y(k)|^2 \leq \frac{1}{(n+1)^2} \| y \|^2$$

so that

$$\| A - A_n \| \leq \frac{1}{(n+1)^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

Hence A is a compact operator.

Moreover, $\Lambda = \sigma(A) = \sigma(M_h) = \overline{\text{Range(h)}} = \{ h(1), h(2), \ldots\} \cup \{0\}$ .

Let $\mathcal{A}_0$ be the C*-algebra generated by A

**Claim 1**: $\mathcal{A}_0 = \{M_a : a \in c\}$ .

By the (continuous) functional calculus the mapping $\Phi: C(\Lambda) \to \mathcal{A}_0$ is an isometric *-isomorphism onto. So given $B \in \mathcal{A}_0$ there is a (unique) continuous function f on $\Lambda$ such that $f(A) = B$ or $f(M_h) = B$

But (by example 4.1) we have that $f(M_h) = M_{foh}$ .

Since f is continuous, if $g(n) = f(h(n))$, then $\lim_{n\to\infty} g(n)$ exists. So that $B = M_g$ with $g \in c$.

Conversely, let $B = M_a$ with $a \in c$. Take $f: \Lambda \to \mathbb{C}$ defined by

$$f(h(n)) = a(n) \text{ and } f(0) = \lim_{n\to\infty} a(n) .$$

Then $f \in C(\Lambda)$ and $B = f(A) \in \mathcal{A}_0$ and our claim is proved.

We recall that $\mathcal{A}$ is the closure in the s.o.t. of $\mathcal{A}$

**Claim 2**: $\mathcal{A} = \{M_a : a \in \ell^\infty\}$

To prove this; let $a \in \ell^\infty$ and $a^{(n)} = \{ a(1), a(2), \ldots a(n), 0, 0, 0, \ldots \} \in c_0$ and $B_n = M_{a^{(n)}}$.

For $x \in \ell^2$ , $x = \{x(n)\}_{n=1}^{\infty}$ we have

$$M_a x - B_n x = \{0, 0, \ldots, 0, a(n+1)x(n+1), a(n+2)x(n+2), \ldots \}$$

so that

$$\| (M_a - B_n)x \|^2 = \sum_{k=n+1}^{\infty} |a(k)|^2 |x(k)|^2 \leq \| a \|^2 \sum_{k=n+1}^{\infty} |x(k)|^2 \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\{M_a : a \in \ell^\infty\} \subseteq \mathcal{A}$.

Since $\{M_a : a \in \ell^\infty\}$ is maximal abelian, it is closed in the s.o.t.

Now, $\mathcal{A}_0 \subseteq \{M_a : a \in \ell^\infty\}$ and so $\mathcal{A} \subseteq \{M_a : a \in \ell^\infty\}$

Thus $\mathcal{A} = \{M_a : a \in \ell^\infty\}$ and our second claim is proved.

Now from our previous discussion ( see proof of proposition 6.4) we conclude that $X_{\mathcal{A}}$ is homeomorhic to $\beta(\mathbb{N})$. ▮

❑

## § 7. SPECTRAL THEORY FOR UNBOUNDED SELF ADJOINT OPERATORS .

**Definition**: Let $H_1$ , $H_2$ be two Hilbert spaces. A *linear operator* A from $H_1$ into $H_2$ is a linear map whose domain of definition is a linear subspace (not necessarely closed) D(A) in $H_1$ .

The unbounded operators A we consider will be densely defined, that is, D(A) is dense in $H_1$ . We denote by $Op(H_1, H_2)$ the unbounded densely defined operators from $H_1$ into $H_2$ $Op(H) = Op(H, H)$.

Note that if $A \in Op(H)$ is bounded, then A can be extended to a bounded linear operator on $\overline{D(A)} = H$. So unless it is specified to the contrary, a bounded operator will always be assumed to be defined on all of H.

For a detailed discussion of the basic definitions and properties of unbounded operators we refer the reader to [20] p. 296.
Next we collect some of these definitions.

**Definition**: Let $A, B \in Op(H)$. We say that B is an *extension* of A , denoted $A \subseteq B$ , if $D(A) \subseteq D(B)$ and $Ax = Bx \;\; \forall x \in D(A)$.

**Definition**: Let $A, B \in Op(H)$. We define the *sum* of A and B, $A + B \in Op(H)$ as follows: $D(A + B) = D(A) \cap D(B)$ and $(A + B)x = Ax + Bx \;\; \forall x \in D(A + B)$.

**Definition**: Let $A, B \in Op(H)$. We define $BA \in Op(H)$ as follows:

$$D(BA) = \{x \in H: x \in D(A) \text{ and } Ax \in D(B)\} = D(A) \cap A^{-1}(D(B))$$
$$(BA)x = B(Ax) \quad \forall x \in D(BA).$$

**Definition:** Let $A \in Op(H)$. The *Graph* of A is the set of pairs $\{ \langle x, Ax \rangle : x \in D(A) \}$.

The Graph of A, denoted by $\Gamma(A)$ is a linear subspace of $H \times H$.

Note, $H \times H$ is a Hilbert space with inner product

$$(\langle x_1 , y_1 \rangle , \langle x_2 , y_2 \rangle) = (x_1 , x_2) + (y_1 , y_2).$$

Note also that for $A, B \in Op(H)$, $A \subseteq B$ iff $\Gamma(A) \subseteq \Gamma(B)$.

**Definition**: Let $A \in Op(H)$. A is called a *closed operator* if $\Gamma(A)$ is a closed subspace of $H \times H$.

**Definition**: $A \in Op(H)$ is called *closable* (or *preclosed*) if the closure of $G(A)$ in $H \times H$ is a graph. In this case, we define $\bar{A} \in Op(H)$ such that $\Gamma(\bar{A}) = \overline{\Gamma(A)}$ and we call $\bar{A}$ the *closure* of A.

The notion of *adjoint* operator can be extended to the unbounded densely defined operators.
We also mention the following facts:
i) If $A,B \in Op(H)$, $BA \in Op(H)$, then $A^*B^* \subseteq (BA)^*$
ii) If $A \in Op(H)$, $B \in B(H)$, then $A^*B^* = (BA)^*$ (see [20] p. 299-300).

**Definition**: Let $A \in Op(H)$. A is called *symmetric* if $A \subseteq A^*$, that is,

$$(Ax,y) = (x,Ay) \quad \forall x,y \in D(A).$$

**Definition**: Let $A \in Op(H)$. A is called **self adjoint** if $A = A^*$, that is, iff A is symmetric and $D(A) = D(A^*)$.

**Definition**: Let $A \in Op(H)$ be a closed operator. We define

$$\{A\}' = \{ T \in B(H) : TA \subseteq AT \}$$

$$\{A\}'' = (\{A\}')' .$$

**Definition**: Let $\mathcal{A}$ be a von Neumann algebra of operators on H and let $A \in Op(H)$ be a closed operator. We say that A is **affiliated** with $\mathcal{A}$ (and we write $A \eta \mathcal{A}$) when $U^*AU = A$ for each unitary operator U commuting with $\mathcal{A}$, i.e,

$$U^*AU = A \qquad \forall U \in \mathcal{A}' \text{ , that is,}$$

$$\forall U \in \mathcal{A}' \qquad U(D(A)) = D(A) \text{ and } U^*AUx = Ax \qquad \forall x \in D(A).$$

Note that $\mathcal{A}'$ is a *-subalgebra of B(H), since $\mathcal{A}$ is a *-algebra. Moreover, since the commutant of any subset of B(H) is always closed in the s.o.t. (and so in the u.o.t.), $\mathcal{A}'$ is a C*-algebra in B(H).
Now, any element of a C*-algebra $\mathcal{U}$ is a finite linear combination of unitary elements of $\mathcal{U}$ (see [14] p. 242).
Thus,

$$A \eta \mathcal{A} \iff BA \subseteq AB \quad \forall B \in \mathcal{A}' \iff \mathcal{A}' \subseteq \{A\}'.$$

We will see later (spectral theorem for unbounded self adjoint operators) that an unbounded self adjoint operator A is affiliated with an abelian von Neumann algebra. The Gelfand space of this algebra will be (by theorem 2.2) extremely disconnected and A will be "represented" there by a function , which as might be expected, will be neither everywhere defined nor bounded.
On the way to this spectral theorem and for the later analysis of unbounded normal operators, the following discussion plays a key role.

**7.1 Theorem**: Let X be an extremely disconnected space and Y be a compact Hausdorff space. Suppose that U is an open dense subset of X. If $f: U \to Y$ is a continuous function, then f has a unique continuous extension $\tilde{f}$ on X , i.e., $\tilde{f}: X \to Y$.

**Proof.**
The uniqueness is clear, since if two continuous functions agree on a dense subset they agree everywhere.
We prove the existance. For each $y \in Y$, let

$A_y = \bigcap_{G \in \mathcal{G}_y} \overline{f^{-1}(G)}$ where $\mathcal{G}_y = \{G: G \text{ is open in } Y, y \in Y\}$. $A_y$ is a closed subset of X $\forall y \in Y$ (possibly empty).

Now, if $x \in X$, then there is a net $\{x_d\}$ in U such that $x_d \to x$. Since Y is compact the net $\{f(x_d)\}$ has a cluster point, say y, in Y.

**Claim**: $x \in A_y$.

If $x \notin Ay$, then there is an open set G in Y such that $y \in G$ and $x \notin \overline{f^{-1}(G)}$ .So there exists $d_0$ such that for $d \geq d_0$ $x_d \notin \overline{f^{-1}(G)}$ . In particular, for $d \geq d_0$ $x_d \notin f^1(G)$, that is, $f(x_d) \notin G$.
Since G is a neighborhood of y and y is a cluster point for $\{f(x_d)\}$ this is a contradiction.
Thus, $x \in A_y$.
We define $\tilde{f}(x) = y$ for $x \in A_y$. This is well defined, for if $y_1 \neq y_2$, then $A_{y_1} \cap A_{y_2} = \emptyset$
To see this, let $y_1 \neq y_2$ $y_1, y_2 \in Y$, Y being a Hausdorff space, there are open sets $G_1, G_2$ in Y with $y_1 \in G_1$, $y_2 \in G_2$ and $G_1 \cap G_2 = \emptyset$. But then $f^1(G_1)$ and $f^1(G_2)$ are open in U (and so in X, since U is open in X) and $f^1(G_1) \cap f^1(G_2) = \emptyset$.
Since X is extremely disconnected we have $\overline{f^{-1}(G_1)} \cap \overline{f^{-1}(G_2)} = \emptyset$ ,
so

$$A_{y_1} \cap A_{y_2} = \emptyset.$$

Now, $\forall x \in U$ $\tilde{f}(x) = f(x)$ , i.e., $f|_U = f$.
For this, let D be any directed set and take $x_d = x$ $\forall d \in D$.
Then $f(x_d) = f(x)$ and $f(x_d) \to f(x)$, so $x \in A_{f(x)}$ and so $\tilde{f}(x) = f(x)$.
It remains to show the continuity of $\tilde{f}$.
Let F be any closed subset of Y and $\Sigma_F = \{G: G \text{ is open in } Y \text{ and } F \subseteq G\}$.

We claim $\tilde{f}^{-1}(F) = \bigcap_{G\in\Sigma_F} \overline{f^{-1}(G)}$ (which immediately gives the continuity of $\tilde{f}$ ).

In fact, if $x\in\tilde{f}^{-1}(F)$, then $\tilde{f}(x) = y\in F$. So $x\in Ay$, i.e., $x\in \bigcap_{G\in\mathcal{G}_y} \overline{f^{-1}(G)} \subseteq \bigcap_{G\in\Sigma_F} \overline{f^{-1}(G)}$ .

Conversely, if $x\notin \tilde{f}^{-1}(F)$, then $\tilde{f}(x) = y\in \notin F$. Choose $G_1$, $G_2$ open sets in Y such that $y\in G_1$, $F \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ . Then $f^{-1}(G_1)$, $f^{-1}(G_2)$ are both open in U (so in X) and disjoint. Again since X is extremely disconnected we have $\overline{f^{-1}(G_1)} \cap \overline{f^{-1}(G_2)} = \emptyset$, so $A_y \cap \bigcap_{G\in\Sigma_F} \overline{f^{-1}(G)} = \emptyset$ . Since $x\in A_y$, $x\notin \bigcap_{G\in\Sigma_F} \overline{f^{-1}(G)}$ . ■

Remark : L.Gillman and M.Jerison [10] p. 96 state the following results:
i) If X is extremely disconnected, then X is the Stone-Čech compactification of every dense subset ; and ii) If X is the Stone-Čech compactification of T and $f: T\to Y$ is continuous (Y compact Hausdorff) , then f extends to $\tilde{f} : X\to Y$
Note that i) and ii) give another proof of theorem 7.1.

We denote by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the one-point compactification of $\mathbb{C}$. $\mathbb{R}^* = [-\infty,\infty]$, denotes the two-point compactification of $\mathbb{R}$.

**Definition**: Let X be a Stonean space . A continuous function $f: X\to \hat{\mathbb{C}}$, such that $U_f = \{f \neq \infty\}$ is (open) dense in X, is called a **normal function** on X.
We denote by N(X) the set of normal functions on X.
A continuous function $f: X\to \mathbb{R}^*$, such that $U_f = \{-\infty < f < \infty\}$ is (open) dense in X , is called a **self adjoint function** on X.
We denote by S(X) the set of self adjoint functions on X.

**7.1 Remark**: We can identify S(X) with $\{g\in N(X)$: g is real whenever it is finite$\}$.
To see this; let $\theta: \mathbb{R}^*\to\hat{\mathbb{C}}$ be the function defined by

$$\theta(x) = \begin{cases} x & \text{if } -\infty<x<\infty \\ \infty & \text{if } x = \pm\infty \end{cases}$$

$\theta$ is continuous. Thus, if $f\in S(X)$, then $\theta o f\in N(X)$.
Moreover, if $f_1, f_2\in S(X)$ and $\theta o f_1 = \theta o f_2$, then $f_1 = f_2$.
In fact, $f_1 = f_2$ on $U_{f_1} \cap U_{f_2}$ which is dense in X.
Now, given $g\in N(X)$ such that, whenever $g(x) \neq \infty$, $g(x)$ is real, then $g|_{U_g}: U_g\to \mathbb{R}^*$ and by theorem 7.1 there is a (unique) $f: X\to \mathbb{R}^*$ which extends $g|_{U_g}$ .
Clearly, g and $\theta o f$ agree on $U_g$ and since $U_g$ is dense $g = \theta o f$. ■

**7.2 Remark**: Note that, if f and g are in N(X), $U_f \cap U_g$ is (open) dense in X and f + g , fg are both defined and continuous functions on $U_f \cap U_g$.
Thus by theorem 7.1 , f + g and fg have unique continuous extensions on X,

$$f \dot{+} g \quad \text{and} \quad f \cdot g \quad \text{respectively.}$$

Similarly, for $f \in N(X)$ $\bar{f}$ is defined first on $U_f$ and then by extension on X.
For $f \in N(X)$,

$$f \in S(X) \iff f = \bar{f}.$$

Thus, N(X) is a *-algebra containing C(X) and S(X) is the set of self adjoint elements of N(X).

**7.3 Remark**: Note also that $f \in N(X)$ is invertible in N(X) $\iff$ $\text{int}\{f = 0\} = \emptyset$ .
To see this we have:

($\Longrightarrow$): If $f \in N(X)$ is invertible in N(X), then there exists $g \in N(X)$ such that $f \cdot g = 1$.
Since $\{g \neq \infty\}$ is dense in X, the set $\{f \neq 0\}$ is also dense in X, equivalently $\text{int}\{f = 0\} = \emptyset$ .

($\Longleftarrow$): Suppose $\text{int}\{f = 0\} = \emptyset$ this implies $\overline{\{f \neq 0\}} = X$ which implies

$$\overline{\{\tfrac{1}{f} \neq \infty\}} = X. \text{ So } \frac{1}{f} \in N(X). \quad ■$$

**7.1 Proposition**: Let $\mathcal{A}$ be an abelian von Neumann algebra, $X = X_{\mathcal{A}}$ its Gelfand space, $\wedge : \mathcal{A} \rightarrow C(X)$ the Gelfand map and $A \in \mathcal{A}$. A is one-to-one $\iff$ $\hat{A}$ is invertible in N(X).

**Proof.**
($\Longrightarrow$): Suppose that $\text{int}\{\hat{A} = 0\} \neq \emptyset$ .
Since X is Stonean, $G = \text{int}\{\hat{A} = 0\}$ is clopen. Let $e = \chi_G \in C(X)$.
If $E \in \mathcal{A}$ is such that $\hat{E} = e$, then $E \neq 0$ and $AE = EA = 0$, since $\hat{A} e = 0$.
Therefore, Range(E) $\subseteq$ Ker(A), hence A is not one-to-one.
($\Longleftarrow$): Suppose that A is not one-to-one. If E be the projection onto Ker(A), then E is a non-zero projection and $E \in \mathcal{A}$, since $\mathcal{A}$ is a von Neumann algebra.( E is the range projection onto the orthogonal complement of Range(A*)) (see [14] p. 327)
Moreover, AE = 0 which implies $\hat{A}\hat{E} = 0$.
Let G is the non-empty clopen set in X such that $\hat{E} = \chi_G$.
Then $\hat{A} \chi_G = 0$ which implies that $G \subseteq \{\hat{A} = 0\}$, so $G \subseteq \text{int}\{\hat{A} = 0\}$.
Hence $\text{int}\{A^\wedge = 0\} \neq \emptyset$ . ■

**7.4 Remark**: Note that, if $f \in N(X)$ and G is a clopen set in X and $g = \mathcal{X}_G$ , then

$$(f\cdot g)(x) = \begin{cases} f(x) & \text{if } x \in G \\ 0 & \text{if } x \notin G \end{cases}.$$

In fact, suppose that $x \in G$ and let $\{x_d\}$ , $x_d \in U_f$ be a net such that $x_d \to x$.
Since G is open may assume $x_d \in G$ $\forall d$.

Furthermore since $f\cdot g$ is continuous, we have $(f\cdot g)(x) = \lim(f\cdot g)(x_d)$.
But

$$(f\cdot g)(x_d) = f(x_d)g(xd) = f(x_d) \to f(x).$$

Thus for $x \in G$ $(f\cdot g)(x) = f(x)$.

A similar argument proves that, if $x \notin G$, then $(f\cdot g)(x) = 0$. ■

**7.2 Proposition.** Let X be a Stonean space and let $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ be a bounded resolution of the identity in C(X), $X_\lambda = \{x \in X: e_\lambda(x) = 1\}$, $\phi \in S(X)$.
Then the following are equivalent:

i) $\forall \lambda \in \mathbb{R}$ $\{x \in X: \phi(x) < \lambda\} \subseteq X_\lambda \subseteq \{x \in X: \phi(x) \leq \lambda\}$

ii) $X_\lambda = \text{int}\{x \in X: \phi(x) \leq \lambda\} = \text{int}\{\phi \leq \lambda\}$

iii) $\forall \lambda \in \mathbb{R}$ $\phi\cdot e_\lambda \leq \lambda e_\lambda$ and $\lambda(1 - e_\lambda) \leq \phi\cdot (1 - e_\lambda)$

Using remark 7.4, the proof of this proposition is on the same lines as the proof of proposition 2.2.

Our next theorem is the *Spectral theorem for unbounded self adjoint operators*.

**7.2 Theorem : (Spectral Theorem )**
Let $A \in \mathcal{O}p(H)$ be a self adjoint (unbounded) operator. Then there exists a unique resolution of the identity $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ such that (in fact, $\mathcal{a} = \{A\}''$):

i) For every compact interval $J=[a,b]$ $a,b \in \mathbb{R}$, if $F_J = E_b - E_a$, then $AF_J$ is a bounded self adjoint operator on H, which leaves $H_J = \text{Range}(F_J)$ invariant and such that $AF_J | H_J$ is a bounded self adjoint operator on $H_J$ with spectral resolution is $\{E_\lambda F_J\}_{\lambda \in \mathbb{R}}$ , i.e.,

$$AF_J = \int_{\mathbb{R}} \lambda d(E_\lambda F_J) = \int_J \lambda dE_\lambda$$

ii) Let $\mathcal{J}$ be the directed set of compact intervals in $\mathbb{R}$ ordered by inclusion.

$$x \in D(A) \iff \text{the net } \{AF_J x\}_{J \in \mathcal{J}} \text{ converges.}$$

For $x \in D(A)$, $Ax = \lim AF_J x = \lim(\int_J \lambda dE_\lambda)x$. Moreover, A is affiliated with $\mathcal{A}$.

**Proof.**

Since A is self adjoint, by the basic criterion for self adjointness (see[14] p.160) we have that $Ker(A \pm iI) = \{0\}$ and $Range(A \pm iI) = H$ and so $(A \pm iI)^{-1}$ exists from H onto $D(A \pm iI) = D(A)$.

Furthermore,

$$\|(A \pm iI)x\|^2 = \|Ax\|^2 + \|x\|^2, \text{ so } \|(A \pm iI)x\|^2 \geq \|x\|^2$$

from which we get that $(A \pm iI)^{-1}$ (exists and ) is bounded , i.e., $(A \pm iI)^{-1} \in B(H)$.

Let $V = (A + iI)^{-1}$.

Then $\| V \| \leq 1$ and $V^* = (A - iI)^{-1}$.

Also

$$(A - iI)V = (A + iI - 2iI)V = (A + iI)V - 2iV = I - 2iV,$$

so

$$V = V^* - 2iV^*V,$$

hence

$$V^*V = \frac{V^*-V}{2i}$$

Similarly,

$$VV^* = \frac{V^*-V}{2i} \quad .$$

So V is normal. Moreover,

$$AVV^* = (A + iI - iI)VV^* = (A + iI)VV^* - iVV^* = V^* - iVV^*$$

$$= V^* - i\frac{V^*-V}{2i} = \frac{V^*+V}{2i} \in B(H) \quad .$$

Let $\mathcal{A} = \{V\}''$ ; $\mathcal{A}$ is an abelian von Neumann algebra.

We claim: $B \in \{A\}'$ iff $B \in \{V\}'$ , i.e., $\{A\}' = \{V\}'$.

To see this; suppose that $B \in \{A\}'$, then $BA \subseteq AB$, that is, $BAx = ABx \ \forall x \in D(BA) = D(A)$.

Now,

$$Bx = B(A + iI)Vx = BAVx + iBVx = ABVx + iBVx = (A + iI)BVx,$$

so

$$VBx = BVx \ \forall x \in D(A). \text{ But } D(A) \text{ is dense in } H,$$

hence

$$VB = BV \text{ since } V, B \in B(H).$$

Conversely, suppose that $BV = VB$ and $x \in D(A)$,

then

$$Bx = BV(A+iI)x = VB(A+iI)x \in D(A), \text{ that is, } D(BA) = D(A) \subseteq D(AB).$$

Moreover, for $x \in D(A)$ we have

$$(A + iI)Bx = (A + iI)BV(A + iI)x = B(A + iI)x,$$

so

$$ABx = BAx.$$

Thus $BA \subseteq AB$ or $B \in \{A\}'$ and the claim is proved.

It follows now that, $\{A\}'' = \{V\}'' = \mathcal{a}$.

By the Gelfand-Naimark theorem and theorem 2.2, $\mathcal{a} \simeq C(X)$ where $X = X_{\mathcal{a}}$ a Stonean space.

Let $v = \hat{V}$, $v \in C(X)$.

Since V is one to one, proposition 7.1 gives that v is invertible in N(X), so $\frac{1}{v} \in N(X)$.

Also, $\hat{V^*} = \overline{\hat{V}} = \bar{v}$.

Define $\phi \in N(X)$ by $\phi = \frac{1}{v} - i$.

Since $V^*V = VV^* = \frac{V^* - V}{2i}$ in $\mathcal{a}$, taking Gelfand transforms we get $\bar{v}v = |v|^2 = \frac{\bar{v} - v}{2i}$

which implies $\phi = \bar{\phi}$. Thus $\phi \in S(X)$. Note that $\phi = \frac{v + \bar{v}}{2|v|^2}$

In a formal sense $\phi$ is the function that corresponds to A . We want to remark at this point that, if A is bounded, then $v = \frac{1}{a+i}$ . So $\phi = a = \hat{A}$,

Now, for $\lambda \in \mathbb{R}$, let $X_\lambda = \mathrm{int}\{\phi \le \lambda\}$ (a clopen subset of X) and $e_\lambda = \chi_{X_\lambda}$.

$\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is a resolution of the identity in C(X), that is,

$$\text{i)} \quad \bigvee_{\lambda \in \mathbb{R}} e_\lambda = 1 \quad , \quad \bigwedge_{\lambda \in \mathbb{R}} e_\lambda = 0$$

$$\text{ii)} \quad \forall \lambda \in \mathbb{R} \quad e_\lambda = \bigwedge_{\mu > \lambda} e_\mu .$$

To see this we have: Clearly $e_\lambda \le 1$ $\forall \lambda \in \mathbb{R}$ and if $f \in C(X)$ is such that $e_\lambda \le f$ $\forall \lambda \in \mathbb{R}$ , then

$$\{f<1\} \subseteq \bigcap_{\lambda \in \mathbb{R}} X_\lambda{}^c = \bigcap_{\lambda \in \mathbb{R}} \overline{\{\phi > \lambda\}} \subseteq \bigcap_{\lambda \in \mathbb{R}} \{\phi \ge \lambda\} = \{\phi = \infty\} ,$$

so

$$\mathrm{int}\{f<1\} = \{f<1\} \subseteq \mathrm{int}\{\phi = \infty\} = \emptyset \quad \text{, i.e., } f \ge 1.$$

Hence

$$\bigvee_{\lambda \in \mathbb{R}} e_\lambda = 1 .$$

On the other hand $e_\lambda \ge 0$ $\forall \lambda \in \mathbb{R}$ and if $g \in C(X)$ is such that $g \le e_\lambda$ $\forall \lambda \in \mathbb{R}$, then

$$\{g>0\} \subseteq \bigcap_{\lambda \in \mathbb{R}} X_\lambda \subseteq \bigcap_{\lambda \in \mathbb{R}} \{\phi \le \lambda\} = \{\phi = -\infty\} ,$$

so

$$\{g>0\} = \mathrm{int}\{g>0\} \subseteq \mathrm{int}\{\phi = -\infty\} = \emptyset \quad \text{, i.e., } g \le 0 .$$

Hence

$$\bigwedge_{\lambda \in \mathbb{R}} e_\lambda = 0.$$

Next we show ii); Clearly $e_\lambda \le e_\mu$ for $\lambda < \mu$. If $h \in C(X)$ is such that $h \le e_\mu \quad \forall \mu \ge \lambda$, then $\{h > 0\} \subseteq X_\mu$, so $h > 0$ implies $\phi \le \mu \; \forall \mu > \lambda$ and so $h > 0$ implies $\phi \le \lambda$.
Thus, $\{h > 0\} \subseteq X_\lambda$ , so $h \le e_\lambda$ .
Hence,

$$e_\lambda = \bigwedge_{\mu > \lambda} e_\mu .$$

Now let $E_\lambda \in \mathcal{A}$ be such that $\hat{E}_\lambda = e_\lambda$ .
Since the Gelfand map is order preserving, $\{E_\lambda\}$ is a resolution of the identity in $\mathcal{A}$.
If $J = [a,b]$ is a compact interval in $\mathbb{R}$ and $F_J = E_b - E_a$ ,
then

$$f_J = e_b - e_a = \chi_{X_b \setminus X_a} \quad \text{, where } f_J = \hat{F}_J .$$

Moreover, if $Z = \{v = 0\}$, then $X_b \cap Z = X_a \cap Z = \{\phi = -\infty\}$,
so that $X_b \setminus X_a \subseteq X \setminus Z$, that is, $v(x) \ne 0$ when $x \in X_b \setminus X_a$.
Let k be the function in C(X) defined by

$$k = \begin{cases} \frac{1}{|v|^2} & \text{when } x \in X_b \setminus X_a \\ 0 & \text{otherwise} \end{cases}$$

Then $k \ge 0$, $k|v|^2 = f_J$ and $k f_J = k$.
If $K \in \mathcal{A}$ is such that $\hat{K} = k$, then $KVV^* = F_J$.
We prove now that $AF_J$ is a bounded self adjoint operator on H.

For $x \in X_b \setminus X_a$ we have $a \le \phi(x) \le b$. Recall that $\phi = \frac{v + \bar{v}}{2|v|^2}$
Then

$$a|v|^2 f_J \le \frac{v+\bar{v}}{2} f_J \le b|v|^2 f_J .$$

Multiplying both sides by k and using $k|v|^2 = f_J$, $k f_J = k$
we get

$$a f_J \le \frac{v+\bar{v}}{2} k \le b f_J .$$

Hence

$$a F_J \le \frac{V+V^*}{2} K \le bF_J$$

or

$$a F_J \le AVV^*K \le b F_J$$

or

$$a F_J \leq AF_J \leq b F_J \qquad \text{since } VV^*K = F_J.$$

Thus $AF_J \in B(H)$ and is self adjoint.

Furthermore, $F_JV = VF_J \Rightarrow F_JA \subseteq AF_J \Rightarrow F_J A F_J \subseteq AF_J$ and since $AF_J \in B(H)$, $F_J A F_J \in B(H)$, so $F_J A F_J = F_J$ which says that $H_J = \text{Range}(F_J)$ is invariant under $AF_J$.

Also $\phi f_J \in C(X)$ and $\phi f_J = \frac{v+\overline{v}}{2|v|^2} f_J = \frac{v+\overline{v}}{2} k$. Taking inverse Gelfand transforms in both sides of the last equality we get

$$\frac{V+V^*}{2} K = AVV^*K = AF_J \quad .$$

Thus $AF_J \in \mathcal{A}$ and $(AF_J)^\wedge = \phi f_J$ .

Next with the use of theorem 3.2 we show that $\{E_\lambda F_J\}_{\lambda \in \mathbb{R}}$ is the spectral resolution for $AF_J|_{H_J}$.

We have,

$$\phi f_J e_\lambda f_J \leq \lambda e_\lambda f_J \text{ and } \lambda(f_J - e_\lambda f_J) \leq \phi f_J (f_J - e_\lambda f_J) \qquad \forall \lambda \in \mathbb{R}$$

and applying the inverse of the Gelfand map we get

$$(AF_J)(E_\lambda F_J) \leq \lambda E_\lambda F_J \quad \text{and} \quad \lambda(F_J - E_\lambda F_J) \leq AF_J(F_J - E_\lambda F_J) \qquad \forall \lambda \in \mathbb{R}.$$

Note that,

$$E_\lambda F_J = \begin{cases} 0 & \text{for } \lambda < a \\ F_J & \text{for } \lambda \geq b \\ E_\lambda - E_a & \text{for } a \leq \lambda < b \end{cases}$$

The spectral theorem for the bounded self adjoint operators $AF_J|_{H_J}$ gives

$$AF_J = \int_{\mathbb{R}} \lambda d(E_\lambda F_J) = \int_J \lambda dE_\lambda.$$

ii): Since $\{E_\lambda\}$ is a resolution of the identity we have that

$$\bigvee_{\lambda \in \mathbb{R}} E_\lambda = I \qquad \text{and} \qquad \bigwedge_{\lambda \in \mathbb{R}} E_\lambda = 0 ,$$

so

$$\lim_{\lambda \to \infty} E_\lambda = I \text{ in the s.o.t.} \quad \text{and} \quad \lim_{\lambda \to -\infty} E_\lambda = 0 \text{ in the s.o.t.}.$$

Hence, the net $\{F_J\}_{J \in \mathcal{J}}$ converges to I in the s.o.t., i.e., $F_J x \to x \ \forall x \in H$.

If $x \in D(A)$, since $F_JA \subseteq AF_J$ , we have $AF_J x = F_J A x \to Ax$ , that is, $\{AF_J x\}_{J \in \mathcal{J}}$ converges.

Suppose now that $\{AF_J x\}_{J \in \mathcal{J}}$ converges . Then $F_J x \to x$ and $\{AF_J x\}_{J \in \mathcal{J}}$ converges.

Since A is self adjoint, A is a closed operator.

Therefore, $x \in D(A)$ and $AF_J x \to Ax$.

Thus for $x \in D(A)$

$$Ax = \lim AF_J x = \lim(\int_J \lambda dE_\lambda)x.$$

Next we show that $A \eta \mathcal{A}$; Let $U \in \mathcal{A}'$ be a unitary operator and $x \in D(A)$.
We have

$$Ux = UV(A + iI)x = VU(A + iI)x$$

which implies

$$(A + iI)Ux = (A + iI)VU(A + iI)x = U(A + iI)x.$$

Thus

$$AUx = UAx \text{ or } U^*AUx = Ax \quad \forall x \in D(A).$$

It remains to prove the uniqueness of the resolution of the identity $\{E_\lambda\}_{\lambda \in \mathbb{R}}$.
We state it and prove it separetely in the following proposition. ■

**7.3 Proposition**: (uniqueness)
If $\{E'_\lambda\}_{\lambda \in \mathbb{R}}$ is a resolution of the identity in B(H), with the following properties:

i) For every compact interval $J = [a,b]$, let $F_J' = E_b' - E_a'$, $H_J' = \text{Range}(F_J')$.
$AF_J' \in B(H_J')$ and is self adjoint with spectral resolution $\{E'_\lambda F_J'\}_{\lambda \in \mathbb{R}}$.

ii) $x \in D(A)$ iff $\{AF_J'x\}_{J \in \mathcal{J}}$ converges and for $x \in D(A)$, $Ax = \lim AF_J' x$

then $E'_\lambda = E_\lambda \quad \forall \lambda \in \mathbb{R}$.

**Proof.**
Since $\{E'_\lambda F_J'\}_{\lambda \in \mathbb{R}}$ is the spectral resolution for $AF_J'$, $AF_J'E'_\lambda F_J' = E'_\lambda F_J' AF_J'$.
Also $E'_\lambda$ and $F_J'$ commute and $F_J'AF_J' = AF_J'$,
so $AF_J'E'_\lambda = AF_J'F_J'E'_\lambda = AF_J'E'_\lambda F_J' = E'_\lambda F_J'AF_J' = E'_\lambda AF_J'$.
For $x \in D(A)$ we have $AF_J'x \to Ax$ which implies $AF_J'E'_\lambda x = E'_\lambda AF_J'x \to E'_\lambda Ax$.
Now $F_J'E'_\lambda x \to E'_\lambda x$ and $AF_J'E'_\lambda x \to E'_\lambda Ax$.
Since A is a closed operator we get that $E'_\lambda x \in D(A)$ and $AE'_\lambda x = E'_\lambda Ax$, that is,
$E'_\lambda A \subseteq AE'_\lambda$ or $E'_\lambda \in \{A\}'$,but then $E'_\lambda \in \{V\}'$.
Let $\mathcal{U}$ be the von Neumann algebra generated by $\{E'_\lambda\}$. $\mathcal{U}$ is an abelian von Neumann algebra.
Since $E'_\lambda F_J' \in \mathcal{U}$, $AF' \in \mathcal{U}$ and so $\mathcal{U}' \subseteq \{AF_J'\}' \ \forall J$.
But, if $B \in \{AF_J'\}' \quad \forall J$, then $BAF_J' = AF_J'B \ \forall J$ and with a similar argument as above
$BA \subseteq AB$, i.e., $B \in \{A\}'$.
Therefore

$$\mathcal{U}' \subseteq \bigcap_J \{AF_J'\}' \subseteq \{A\}',$$

hence $\mathcal{A} = \{A\}'' \subseteq \mathcal{U}$. In particular $E_\lambda \in \mathcal{A} \subseteq \mathcal{U} \quad \forall \lambda \in \mathbb{R}$.

Let $Y = X_{\mathcal{U}}$ and $\wedge$ denote the Gelfand map $\wedge : \mathcal{U} \to C(Y)$.
We pause in our proof at this point to state and prove two lemmas;

Lemma 1: Let $\mathcal{R}$ be an abelian von Neumann algebra on H and A be a self adjoint operator affiliated with $\mathcal{R}$(, i.e.,.$A \,\eta\, \mathcal{R}$). If $B \in R$ and $AB \in B(H)$, then $AB \in \mathcal{R}$.

Proof of lemma 1.
Suppose $C \in \mathcal{R}'$, then $ABC = ACB$. Since $A \eta \mathcal{R}$, $\mathcal{R}' \subseteq \{A\}'$ so $C \in \{A\}'$, therefore $ABC = ACB = CAB$ which implies that $AB \in \mathcal{R}'' = \mathcal{R}$.

Lemma 2: Let $\mathcal{R}$ and A be as in lemma 1. If $B \in \mathcal{R}$ and $AB \in \mathcal{R}$, then $(AB)^\wedge = \phi \cdot \hat{B}$, where $\phi$ is the function on $X_{\mathcal{R}}$ "representing" A ( $\phi = \frac{1}{v} - i$ where $v = \hat{V}$ ).

Proof of lemma 2.
Let $\beta = \hat{B}$, $C = AB$, $c = \hat{C}$. We have $(A + iI)B = AB + iB = C + iB$ which implies $B = V(C + iB)$, so that $\beta = v\,(c + ib)$ and so $c = \beta \cdot (\frac{1}{v} - i) = \beta \cdot \phi = \phi \cdot \beta$, that is , $(AB)^\wedge = \phi \cdot \hat{B}$.

Now back to our proof: $A \,\eta\, \mathcal{A} \subseteq \mathcal{U}$ so $A \,\eta\, \mathcal{U}$, $F_J \in \mathcal{U}$, $AF_J \in B(H)$, so by lemma 1, $AF_J \in \mathcal{U}$ and by lemma 2, $(\hat{AF_J}) = \phi \cdot f_J$.
Also

$$AF_J(F_J E_\lambda) \le \lambda F_J E_\lambda \quad \text{or} \quad AF_J E_\lambda \le \lambda F_J E_\lambda$$

and taking Gelfand transforms we have

$$\phi \cdot f_J\, e_\lambda \le \lambda f_J e_\lambda \quad \forall \lambda \in \mathbb{R}.$$

The set $S = \{y \in Y : \exists\ J \text{ such that } f_J(y) = 1\} = \bigcup_J \{f_J > 0\}$ is an open subset of Y.
Since Y is Stonean, $\bar{S}$ is clopen. Let $h = \chi_{\bar{S}}$.
Then $h \ge f_J\ \forall J$ which implies $h \ge \bigvee_J \{f_J\} = 1$, so $h = 1$ , i.e., $\bar{S} = Y$.
Thus $\phi \cdot e_\lambda \le \lambda e_\lambda$ holds on the dense set S and since $e_\lambda$, $\phi \cdot e_\lambda$ are continuous

$$\phi \cdot e_\lambda \le \lambda e_\lambda \text{ on } Y \qquad \forall \lambda \in \mathbb{R}.$$

Similarly,

$$\lambda(F_J - F_J E_\lambda) \le AF_J(F_J - F_J E_\lambda) \qquad \forall J,$$

implies

$$\lambda(f_J - f_J e_\lambda) \le \phi \cdot f_J (f_J - f_J e_\lambda) \qquad \forall J$$

which implies

$$\lambda(1 - e_\lambda) \le \phi \cdot (1 - e_\lambda) \qquad \forall \lambda \in \mathbb{R}.$$

Now by proposition 7.2, $X_\lambda = \{e_\lambda = 1\} = \text{int}\{y \in Y: \phi(y) \leq \lambda\}$.

On the other hand, by theorem 3.2 we have $AF_J'E'_\lambda \leq \lambda F_J'E'_\lambda \quad \forall \lambda \in \mathbb{R}$.

By lemma 2 we have $(AF_J')^\wedge = \phi \cdot f'_J$ since $A \,\eta\, \mathcal{U}$, $F_J' \in \mathcal{U}$.
Hence, taking Gelfand transforms in the above inequality
we have

$$\phi \cdot f'_J\, e'_\lambda \leq \lambda f'_J e'_\lambda \qquad \forall \lambda \in \mathbb{R}$$

and by the same argument as above we conclude

$$\phi \cdot e'_\lambda \leq \lambda e'_\lambda \qquad \forall \lambda \in \mathbb{R}.$$

Similarly,

$$\lambda(F_J' - F_J'E'_\lambda) \leq AF_J'(F_J' - F_J'E'_\lambda) \qquad \forall \lambda \in \mathbb{R}$$

implies

$$\lambda(f'_J - f'_J\, e'_\lambda) \leq \phi \cdot f'_J\,(f'_J - f'_J\, e'_\lambda) \qquad \forall \lambda \in \mathbb{R}$$

which implies

$$\lambda(1 - e'_\lambda) \leq \phi \cdot (1 - e'_\lambda) \qquad \forall \lambda \in \mathbb{R}.$$

Again by proposition 7.2, $X'_\lambda = \{e'_\lambda = 1\} = \text{int}\{y \in Y: \phi(y) \leq \lambda\} = X_\lambda \quad \forall \lambda \in \mathbb{R}$.
Thus, $e_\lambda = e'_\lambda \ \forall \lambda \in \mathbb{R}$ and so $E_\lambda = E'_\lambda \quad \forall \lambda \in \mathbb{R}$. ■

**7.4 Proposition**:Let $A \in Op(H)$ be a self adjoint operator, $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ its resolution of the identity and $\mathcal{A}$ the the von Neumann algebra generated by $\{E_\lambda\}_{\lambda \in \mathbb{R}}$. Then $\mathcal{A} = \{A\}''$

**Proof.**
Let $\mathcal{A}$ be the (abelian) von Neumann algebra generated by $\{E_\lambda\}$.
Since $E_\lambda \in \mathfrak{a} = \{A\}''$, $\mathcal{A} \subseteq \mathfrak{a}$ and so $\{A\}' \subseteq \mathcal{A}'$.
Since $E_\lambda F_J \in \mathcal{A}$, by the spectral theorem for bounded self adjoint operators, $AF_J \in \mathcal{A}$.
So $\{AF_J\} \subseteq \mathcal{A} \ \forall J$ which implies $\bigcap_J \{AF_J\}' \supseteq \mathcal{A}'$.
If $B \in \bigcap_J \{AF_J\}'$, then $BAF_J = AF_J B \quad \forall J$ . For $x \in D(A)$ we have (theorem 7.1) $AF_J\, x \to Ax$ and so $AF_J\, Bx = BAF_J\, x \to BAx$ which implies (since A is closed) that $Bx \in D(A)$ and $ABx = \lim AF_J\, Bx = BAx$. Therefore $BA \subseteq AB$, i.e., $B \in \{A\}'$.
Thus, $\mathcal{A}' \subseteq \{A\}'$. Hence $\mathcal{A} = \mathfrak{a}$. ■

❐

## § 8. SPECTRAL THEORY FOR UNBOUNDED NORMAL OPERATORS.

**Definition:** A densely defined linear operator A (, i.e., $A \in Op(H)$ ) is called **normal** if A is closed and $AA^* = A^*A$.

Note that the equation $AA^* = A^*A$ in the above definition implicitly carries the condition that $D(AA^*) = D(A^*A)$.

**8.1 Proposition**: Let $A \in Op(H)$ be a closed operator. Then

i) $A^{**} = A$

ii) $A^*A$ is self adjoint.

iii) $A^*A + I$ is bijective from $D(A^*A)$ onto H, so that $(A^*A + I)^{-1} \in B(H)$ with $0 \leq (A^*A + I)^{-1} \leq I$.

(see [14] p. 158)

We also remark that, if $A \in Op(H)$ is a closed operator, then $\{< y , Ay >: y \in D(A^*A)\}$ is dense in $\Gamma(A)$. For this; since A is closed, it suffices to show that no nonzero vector in $\Gamma(A)$ is orthogonal to $\{<y , Ay >: y \in D(A^*A)\}$.

So let $x \in D(A)$ and suppose that $0 = (< x , Ax > , < y , Ay >)$ $\forall y \in D(A^*A)$. Then

$$\begin{aligned} 0 &= (x , y) + (Ax , Ay) \\ &= (x , y) + (x , A^*Ay) \\ &= (x , [I + A^*A]y) \end{aligned}$$

Therefore, x is orthogonal to the $\text{Range}(I + A^*A) = H$, hence $x = 0$.

**8.2 Proposition**: Let $A \in Op(H)$ be a normal operator. Then $D(A) = D(A^*)$ and

$$\| Ax \| = \| A^*x \| \qquad \forall x \in D(A).$$

**Proof.**

First note that, if $x \in D(A^*A) = D(AA^*)$, then $Ax \in D(A^*)$ and $A^*x \in D(A)$.
So

$$\| Ax \|^2 = (A^*Ax , x) = (AA^*x , x) = \| A^*x \|^2.$$

Now, let $x \in D(A)$. Since $\{< y , Ay >: y \in D(A^*A)\}$ is dense in $\Gamma(A)$, there is a sequence $\{y_n\}$ in $D(A^*A)$ such that $< y_n , Ay_n > \rightarrow < x , Ax >$.
So

$$\| Ay_n - Ax \| \rightarrow 0 \text{ as } n \rightarrow \infty .$$

But $\| A^*y_n - A^*y_m \| = \| Ay_n - Ay_m \| \quad \forall y_n, y_m \in D(A^*A)$ and since $\{Ay_n\}$ is a Cauchy sequence $\{A^*y_n\}$ is also Cauchy. Hence $\{A^*y_n\}$ converges to a vector z in H.

Thus, $y_n \rightarrow x$, $y_n \in D(A^*A) \subseteq D(A)$ and $A^*y_n \rightarrow z$.

Since $A^*$ is a closed operator we get $x \in D(A^*)$ and $A^*x = z$.
Therefore,

$$D(A) \subseteq D(A^*) \text{ and } \| Ax \| = \lim_{n\rightarrow\infty} \| Ay_n \| = \lim_{n\rightarrow\infty} \| A^*y_n \| = \| A^*x \|.$$

On the other hand, since $A^{**} = A$, $A^*$ is normal and so $D(A^*) \subseteq D(A^{**}) = D(A)$. ■

**Definition**: Let $A \in Op(H)$ be a closed operator. A dense linear subspace D of D(A) is called a **core** of A if $\overline{A|_D} = A$, equivalently, if $\overline{\Gamma(A|_D)} = \Gamma(A)$.

**8.1 Remark**:
For unbounded operators the following simple facts are easily verified.
(1) If $A \subseteq B$ and $S \subseteq T$, then $A + S \subseteq B + T$.
(2) If $A \subseteq B$, then $TA \subseteq TB$ and $AT \subseteq BT$.
(3) $(A + B)T = AT + BT$ , $TA + TB \subseteq T(A + B)$.

Furthermore, if $T \in B(H)$ and $\{T_d\}$ is a net of operators in B(H) such that $T_d \rightarrow T$ in the s.o.t. and $T_dA \subseteq BT_d \quad \forall d$, where B is a closed operator, then $TA \subseteq BT$.
Moreover, if $A \in Op(H)$ is a closed operator and $FA \subseteq AF$ for each F in a self adjoint subset $\mathcal{F}$ of B(H), then $TA \subseteq AT$ for each T in the von Neumann algebra generated by $\mathcal{F}$.

**Definition**: Let $A \in Op(H)$ be a closed operator.
A projection E in B(H) is called a **bounding projection** for A if $EA \subseteq AE$ and $AE \in B(H)$.
A **bounding sequence** for A is an increasing sequence of projections $\{E_n\}$, each of which is bounding for A with the property that $\bigvee_{n=1}^{\infty} E_n = I$.

**8.1 Lemma**: Let $A \in Op(H)$ be a closed operator and E be a bounding projection for A. Then E is bounding for $A^*$, $A^*A$, $AA^*$ and $(AE)^* = A^*E$.

Moreover, if $\{E_n\}$ is a bounding sequence for A, then $\bigcup_{n=1}^{\infty} E_n(H)$ is a core for each A, $A^*$, $A^*A$, $AA^*$ (see [14] p.351).

Let $\mathcal{A}$ be a von Neumann algebra of operators on a Hilbert space H and $A \in Op(H)$ be a closed operator.
We recall that A is *affiliated* with $\mathcal{A}$ ($A \,\eta\, \mathcal{A}$) if for every unitary operator U in $\mathcal{A}'$,

$$U^*AU = A .$$

Equivalently, $A \,\eta\, \mathcal{A} \iff BA \subseteq AB \quad \forall B \in \mathcal{A}' \iff \mathcal{A}' \subseteq \{A\}'$.

Note that $BA \subseteq AB \;\; \forall B \in \mathcal{A}'$ implies that $BA^* \subseteq A^*B \;\; \forall B \in \mathcal{A}'$, i.e., $A^* \eta\, \mathcal{A}$.

We denote by $N(\mathcal{A})$ the family of operators which are affiliated with an abelian von Neumann algebra $\mathcal{A}$. We will see later (theorem 8.2) that an operator $A \in Op(H)$ is normal iff A is affiliated with an abelian von Neumann algebra.
$S(\mathcal{A})$ denotes the family of self adjoint operators affiliated with $\mathcal{A}$.

**8.1 Theorem**: If $\mathcal{A}$ is an abelian von Neumann algebra acting on the Hilbert space H and $A, B \eta\, \mathcal{A}$, then:

i) each finite set of operators affiliated with a has a common bounding sequence in $\mathcal{A}$.
ii) A + B is densely defined and preclosed and its closure $A \hat{+} B \,\eta\, \mathcal{A}$.
iii) AB is densely defined and preclosed and its closure $A \hat{\cdot} B \eta\, \mathcal{A}$.
iv) $A \hat{\cdot} B = B \hat{\cdot} A$ and $AA^* = A^*A \; ( = A^* \hat{\cdot} A)$
v) $(aA \hat{+} B)^* = \bar{a}A^* \hat{+} B^*$
vi) $(A \hat{\cdot} B)^* = B^* \hat{\cdot} A^*$.
vii) If $A \subseteq B$, then $A = B$ (in particular, if $A \subseteq A^*$, then $A = A^*$)
viii) $N(\mathcal{A})$ forms a commutative *-algebra (with unit I) under the operations of addition $\hat{+}$ and multiplication $\hat{\cdot}$ described in ii) and iii).

For the proof of theorem 8.1 see [14] p. 352

Note that (iv) of the above theorem says that if $A \,\eta\, \mathcal{A}$, then A is normal.

**8.2 Lemma**: If $\{F_n\}$ is a bounding sequence for the closed operator A and $AF_n$ is normal $\forall n$, then A is normal.

**Proof.**
By lemma 8.1 we have $(AF_n)^* = A^*F_n$, so that

$$A^*AF_n = A^*F_nAF_n = (AF_n)^*AF_n = AF_nA^*F_n = AA^*F_n .$$

Thus, the self adjoint operators $A^*A$ and $AA^*$ agree on their common core $\bigcup_{n=1}^{\infty} F_n(H) = D_0$

Hence $\quad A^*A = \overline{A^*A|_{D_0}} = \overline{AA^*|_{D_0}} = AA^* .$ ■

**8.3 Lemma**: Let $A \in Op(H)$ be a self adjoint operator and $B \in Op(H)$ be closed. If $BA \subseteq AB$ and $D(A) \subseteq D(B)$, then $E_\lambda B \subseteq BE_\lambda$ where $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral resolution for A.

**Proof.**
First we prove that $B(A \pm iI) = BA \pm iB$.
For this observe that from Remark 8.1 (3) we have $BA \pm iB \subseteq B(A \pm iI)$.
To get equality let $x \in D(B(A \pm iI))$. Then $x \in D(A) \subseteq D(B)$ and $(Ax \pm ix) \in D(B)$.
So $Ax \in D(B)$, i.e., $x \in D(BA)$. Thus $x \in D(BA) \cap D(B) = D(BA \pm iB)$.
Hence,

$$B(A \pm iI) = BA \pm iB.$$

Now, let $V = (A + iI)^{-1}$, $V^* = (A - iI)^{-1}$.
We have,

$$VB = VB(A + iI)V = V(BA + iB)V \subseteq V(AB + iB)V = V(A + iI)BV \subseteq BV,$$

that is, $VB \subseteq BV$.
Similarly, $V^*B \subseteq BV^*$.
Remark 8.1 gives $TB \subseteq BT$ for every T in the Von Neumann algebra generated by V, V*, I $( = \{V\}'' = \{A\}'')$.
In particular, $E_\lambda B \subseteq BE_\lambda$ $\forall \lambda \in \mathbb{R}$ since $E_\lambda \in \{V\}''$. ■

**8.2 Theorem**: Let $A \in Op(H)$. Then A is normal iff A is affiliated with an abelian von Neumann algebra. If A is normal, there is a smallest (abelian) von Neumann algebra $\mathcal{A}_0$ such that $A \,\eta\, \mathcal{A}_0$. (see [14] p. 354 )

The algebra $\mathcal{A}_0$ is referred to as **the von Neumann algebra generated by the normal operator A.**
$\mathcal{A}_0$ is the von Neumann algebra generated by the family of (commuting) operators $\{F_n, AF_n, A^*F_n : n = 1, 2, ...\}$ where $F_n = E_n - E_{-n}$, with $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ the spectral resolution for the self adjoint operator $A^*A$. $\{F_n\}$, in fact, is a bounding sequence for A.

Note that, since $A \,\eta\, \mathcal{A}_0$ (and so $A^* \,\eta\, \mathcal{A}_0$), $\forall B \in \mathcal{A}_0'$ we have $BA^*A \subseteq A^*BA \subseteq A^*AB$, that is, $B \in \{A^*A\}'$, so $\mathcal{A}_0' \subseteq \{A^*A\}'$ from which $\{A^*A\}'' \subseteq \mathcal{A}_0'' = \mathcal{A}_0$ .

**8.3 Theorem**: Let $\mathcal{A}$ be an abelian von Neumann algebra , $X = X_{\mathcal{A}}$ its Gelfand space, $\wedge : \mathcal{A} \rightarrow C(X)$ the Gelfand map and $A \,\eta\, \mathcal{A}$. There exists a unique mapping $\phi$ of $N(\mathcal{A})$ onto $N(X)$ which extends $\wedge$, identifying $\mathcal{A}$ with the subset of $N(\mathcal{A})$ consisting of bounded operators and C(X) with the subset of N(X) consisting of real-valued functions, such that

$$(AE)^\wedge = \phi(AE) = \phi(A) \cdot \hat{E}$$

when E is a bounding projection in $\mathcal{A}$ for A.
The mapping $\phi$ extending $\wedge$ is a *-isomorphism of $N(\mathcal{A})$ onto N(X).

**Proof.**

We show first the uniqueness of $\phi(A)$. For this suppose that f and g are normal functions with the properties ascribed to $\phi(A)$.

From theorem 8.1 i), A has a bounding sequence $\{E_n\}$ in $\mathcal{a}$.

Let $x \in X$ be such that both f and g are defined at x .

If $e_n = \hat{E}_n$, then

$$f(x)\cdot e_n(x) = \phi(AE_n)(x) = g(x)\cdot e_n(x).$$

Let $S = \{x \in X: \exists n \text{ such that } e_n(x) = 1\}$.

We show that S is dense in X; for this , suppose that G is an open set in X and $G \subseteq \{x \in X: e_n(x) = 0 \quad \forall n\}$; then $e_n = 0$ on G $\forall n$.

So $e_n = 0$ on $\overline{G}$ $\forall n$ and so $\chi_{(\overline{G})^c} \geq e_n \quad \forall n$.

Therefore, $\chi_{(\overline{G})^c} \geq \bigvee_{n=1}^{\infty} e_n = 1$ (since $\bigvee_{n=1}^{\infty} E_n = I$ ).

Thus $\chi_{(\overline{G})^c} = 1$ or $\overline{G} = \emptyset$ which implies $G = \emptyset$ .

Hence, $\text{int}\{x \in X: e_n(x) = 0 \ \forall n\} = \emptyset$ ; equivalently $\overline{S} = X$.

Now since f = g on S and f, g are continuous, we conclude that f = g on X.

Next, we prove the existence of $\phi(A)$.

First we establish $\phi: S(\mathcal{a}) \to S(X)$ and then we extend $\phi: N(\mathcal{a}) \to N(X)$.

For $A \in S(\mathcal{a})$, define $\phi(A) = \frac{1}{v} - i \in S(X)$ (see theorem 7.2) ,

where $v = \hat{V}$ , $V = (A + iI)^{-1}$ .

Then by lemma 7.2 we have $\phi(AE) = (AE)^{\wedge} = \phi(A)\cdot \hat{E}$ .

Let $A, B \in S(\mathcal{a})$ and $\{E_n\}$ be a bounding sequence for both A and B (such an $\{E_n\}$ exists from theorem 8.1 (i) ). Then $AE_n$ , $BE_n$ , $(A + B)E_n$ and $ABE_n$ are in $\mathcal{a}$.

Moreover, $(A \hat{+} B)E_n = (A + B)E_n = AE_n + BE_n$ , since $\bigcup_{n=1}^{\infty} E_n(H)$ is a core for A + B.

Similarly, $(A \hat{\cdot} B)E_n = ABE_n = AE_nBE_n$ .

So that

$$\begin{aligned} \phi((A \hat{+} B)E_n) &= \phi((A + B)E_n) \\ &= ((A + B)E_n)^{\wedge} \\ &= (AE_n + BE_n)^{\wedge} \\ &= \phi(A)\cdot \hat{E}_n + \phi(B)\cdot \hat{E}_n \\ &= (\phi(A) \dot{+} \phi(B))\cdot \hat{E}_n . \end{aligned}$$

On the other hand, from theorem 8.1 (v) we have

$(A \hat{+} B)^* = A^* \hat{+} B^* = A \hat{+} B$ and $A \hat{+} B \ \eta \ \mathcal{a}$, i.e., $A \hat{+} B \in S(\mathcal{a})$.

So $\phi$ is defined at $A \hat{+} B$ and

$$\phi((A \hat{+} B)E_n) = ((A \hat{+} B)E_n)^\wedge = \phi(A \hat{+} B) \cdot \hat{E}_n .$$

Therefore,

$$\phi(A \hat{+} B) \cdot \hat{E}_n = (\phi(A) \dot{+} \phi(B)) \cdot \hat{E}_n$$

and as in the proof of the uniqueness we conclude that

$$\phi(A \hat{+} B) = \phi(A) \dot{+} \phi(B).$$

Similarly,

$$\begin{aligned} \phi(A \hat{\cdot} BE_n) &= \phi(AE_nBE_n) \\ &= (AE_nBE_n)^\wedge \\ &= (AE_n)^\wedge(BE_n)^\wedge \\ &= \phi(A) \cdot \hat{E}_n \ \phi(B) \cdot \hat{E}_n \end{aligned}$$

On the other hand,
$(A \hat{\cdot} B)^* = B^* \hat{\cdot} A^* = B \hat{\cdot} A$ and $A \hat{\cdot} B \,\eta\, \mathcal{A}$ (theorem 8.1), i.e., $A \hat{\cdot} B \in S(\mathcal{A})$
and

$$\phi(A \hat{\cdot} BE_n) = \phi(A \hat{\cdot} B) \cdot \hat{E}_n$$

Therefore

$$\phi(A \hat{\cdot} B) \cdot \hat{E}_n = \phi(A) \cdot \phi(B) \cdot \hat{E}_n .$$

Hence,

$$\phi(A \hat{\cdot} B) = \phi(A) \cdot \phi(B) .$$

Next we show that $\phi$ is a bijection , $S(\mathcal{A}) \rightarrow S(X)$

Let $A, B \in S(\mathcal{A})$ and $\{E_n\}$ a common bounding sequence for A and B.

Suppose that $\phi(A) = \phi(B)$. Then $(AE_n)^\wedge = \phi(A) \cdot \hat{E}_n = \phi(B) \cdot \hat{E}_n = (BE_n)^\wedge$ which implies $AE_n = BE_n$ .

So A and B agree on their common core and so A = B. Thus $\phi$ is one-to-one.

In order to show that $\phi: S(\mathcal{A}) \rightarrow S(X)$ is onto, we need the following lemma.

<< *Lemma*: If $\{E_n\}$ is an increasing sequence of projections on H and $A_0$ is a linear operator with dense domain $\bigcup_{n=1}^{\infty} E_n(H) = D_0$ such that $A_0E_n$ is a bounded self adjoint operator on H, then $A_0$ is preclosed and its closure is self adjoint. If A is closed with core $D_0$ and $AE_n$ is a bounded self adjoint operator, A is self adjoint. (see [14] p.341) >>

Let $h \in S(X)$, $X_\lambda = \mathrm{int}\{h \le \lambda\}$ and $e_\lambda = \chi_{X_\lambda}$ for all $\lambda$ ; $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is a resolution of the identity in C(X).

If $f_n = e_n - e_{-n}$ , then (as in the proof of theorem 7.2) $h \cdot f_n$ is in C(X).

Let $A_n \in \mathcal{A}$ be such that $\hat{A}_n = h \cdot f_n$ .

Note that $(h \cdot f_m) f_n = h \cdot (f_m f_n) = h \cdot f_n$ for $n \le m$, so $A_m F_n = A_n$ , where $F_n = \check{f}_n$

Define $A_0 x = A_n x$ when $x \in F_n(H)$.

Since $\bigvee_{n=1}^{\infty} F_n = I$ , $D_0 = \bigcup_{n=1}^{\infty} F_n(H)$ is dense in H. $A_0 \in \mathcal{Op}(H)$ with $D(A_0) = D_0$ .

Note also that $A_0 F_m = A_m = (h \cdot f_m)^{\vee}$ with $h \cdot f_m$ real-valued, so $A_0 F_n \in B(H)$ and is self adjoint.

Now, by the above lemma we have that $A_0$ is preclosed and its closure A is self adjoint (with core $D_0$ ).

Moreover, if $U \in \mathcal{A}'$ is a unitary operator and $x \in F_n(H)$, then $Ux = UF_n x = F_n Ux \in F_n(H)$.

So $U^{-1}AUx = U^{-1}A_n Ux = U^{-1}UA_n x = A_n x = Ax$ $\forall x \in D_0$ .Since $D_0$ is a core for A $U^{-1}AUx = Ax$ $\forall x \in D(A)$ and $U(D(A) = D(A)$ , that is, $A \,\eta\, \mathcal{A}$, i.e., $A \in S(\mathcal{A})$.

If $g \in S(X)$ represents A , i.e., if $\phi(A) = g$, then $g \cdot f_n$ represents $AF_n$ ( $= A_n$) (theorem 7.2).

Thus $h \cdot f_n = g \cdot f_n$ for all n, so h = g on a dense subset of X and by the continuity of h and g , h = g, that is, $h = \phi(A)$ and $\phi$ is onto.

Next we extend $\phi: S(\mathcal{A}) \to S(X)$ to $\phi: N(\mathcal{A}) \to N(X)$.

If $A \in N(\mathcal{A})$, then $A = A_1 \hat{+} iA_2$ , where $A_1 = \dfrac{A \hat{+} A^*}{2}$ , $A_2 = \dfrac{A \hat{+} -iA^*}{2i}$ .
By theorem 8.1 $A_1$, $A_2$ are self adjoint.

Furthermore, since $A \,\eta\, \mathcal{A}$, $A^* \,\eta\, \mathcal{A}$ and so $A_1, A_2 \,\eta\, \mathcal{A}$, i.e., $A_1, A_2 \in S(\mathcal{A})$.
Define

$$\phi(A) = \phi(A_1 \hat{+} iA_2) = \phi(A_1) \dot{+} i\,\phi(A_2) \in N(X).$$

If E is a bounding projection for A, then (by lemma 8.1) E is a bounding projection for A* and so for $A_1$ , $A_2$ .
We have,

$$\begin{aligned} \phi(AE) &= \phi((A_1 \hat{+} iA_2)E) \\ &= \phi(A_1E \hat{+} iA_2E) \\ &= \phi(A_1E) \dot{+} i\,\phi(A_2E) \\ &= \phi(A_1)\cdot \hat{E} + i\,\phi(A_2)\cdot \hat{E} \\ &= (\phi(A_1) \dot{+} i\,\phi(A_2))\cdot \hat{E} \\ &= \phi(A)\cdot \hat{E} \end{aligned}$$

$\phi$ is clearly a homomorphism from $N(\mathcal{A})$ into $N(X)$.
Moreover, $\phi$ is a *-homomorphism. To see this ; let $A \in N(\mathcal{A})$ and write $A = A_1 \hat{+} \; i A_2$ .
Then

$$A^* = A_1 \hat{+} \; - i A_2$$

and

$$\begin{aligned}\phi(A^*) &= \phi(A_1 \hat{+} \; - i A_2) \\ &= \phi(A_1) \dot{+} - i \phi(A_2) \\ &= \overline{\phi(A_1) \dot{+} \; i \phi(A_2)} \\ &= \overline{\phi(A)} \; .\end{aligned}$$

If $h \in N(X)$, then $h = h_1 \dot{+} \; i h_2$ with $h_1 , h_2 \in S(X)$

$$\left( h_1 = \frac{h \dot{+} \overline{h}}{2} \; , h_2 = \frac{h \dot{+} - \overline{h}}{2i} \right).$$

Since $\phi : S(\mathcal{A}) \rightarrow S(X)$ is onto , there are $A_1$, $A_2$ in $S(\mathcal{A})$ such that

$$\phi(A_1) = h_1 \quad \text{and} \quad \phi(A_2) = h_2 \, .$$

Take $A = A_1 \hat{+} \; i A_2$ . Then $\phi(A) = h$ , i.e., $\phi$ is onto.
That $\phi$ is one to one can be shown as in the self adjoint case.
Thus $\phi$: $N(\mathcal{A}) \rightarrow N(X)$ is a *-isomorphism (onto). ■

We refer to the *-isomorphism $\phi$: $N(\mathcal{A}) \rightarrow N(X)$ as the **the extension of the Gelfand map**. $\phi$ will play a key role in the analysis that follows.

**Definition**: Let $A \in \mathcal{O}p(H)$ be a closed operator. The **spectrum** of A is the set

$$\sigma(A) = \{ z \in \mathbb{C} : A - zI \text{ is not a bijection from } D(A) \text{ to } H. \}.$$

**8.2 Remark**: Note that, if $z_0 \notin \sigma(A)$, then $A - z_0 I$ is a one-to-one linear mapping from $D(A)$ onto H and has a linear inverse $B: H \rightarrow D(A)$ ( $B = (A - z_0 I)^{-1}$).
Let $U: H \times H \rightarrow H \times H$ be the unitary isomorphism $U(< x , y >) = < y , x >$.
Then $\Gamma(B) = U(\Gamma(A - z_0 I)$ (see [20] p. 305).
Since A is a closed operator, $\Gamma(A - z_0 I)$ is closed in $H \times H$. Hence $\Gamma(B)$ is closed in $H \times H$. Thus, B is a closed everywhere defined operator and by the Closed Graph theorem B is bounded.

Note also that , if $A \eta \mathcal{A}$, with $\mathcal{A}$ an abelian von Neumann algebra and $z_0 \notin \sigma(A)$, then $B = (A - z_0 I)^{-1} \in \mathcal{A}$ . To see this; let U be a unitary operator in $\mathcal{A}'$.

Since $A \eta \mathcal{A}$, we have $AU = UA$.
So $(A - z_0 I)U = U(A - z_0 I)$ and so $B(A - z_0 I)UB = BU(A - z_0 I)B$
which implies $BU \subseteq UB$.
But, both $U, B \in B(H)$, so $UB = BU \quad \forall U \in \mathcal{A}'$. Hence $B \in \mathcal{A}'' = \mathcal{A}$. ■

**8.3 Proposition**: Let $A \in Op(H)$ be a normal operator affiliated with an abelian von Neumann algebra $\mathcal{A}$ and $\phi$: $N(\mathcal{A}) \to N(X)$ be the *-isomorphism extending the Gelfand map. Then $\sigma(A) = \text{Range}(\phi(A))$.

**Proof.**
If $z_0 \notin \sigma(A)$, then $B = (A - z_0 I)^{-1} \in \mathcal{A}$.
Now, $I = (A - z_0 I)B = (A - z_0 I) \hat{\cdot} B$ so $1 = (\phi(A) - z_0) \cdot \hat{B}$. Hence $z_0 \notin \text{Range}(\phi(A))$.
On the other hand, if $z_0 \notin \text{Range}(\phi(A))$, then $\frac{1}{\phi(A) - z_0}$ exists in $C(X)$ (where $X = X_{\mathcal{A}}$)
and is such that $(\phi(A) - z_0 I) \cdot \frac{1}{\phi(A) - z_0} = 1$.
Therefore,

$$(A - z_0 I) \hat{\cdot} B = I ,$$

where $B \in \mathcal{A}$ with $\hat{B} = \frac{1}{\phi(A) - z_0}$, that is, $z_0 \notin \sigma(A)$. ■

**Definition**: Let $A \in Op(H)$ be a normal operator on H. A is called **positive** (and we write $A \geq 0$) if $(Ax, x) \geq 0 \quad \forall x \in D(A)$.
For $A, B \in Op(H)$, $A \geq B$ means $A - B \geq 0$

The relation of this condition to the nature of $\sigma(A)$ is given by the following proposition. (see [14] p. 357)
**8.4 Proposition**: A self adjoint operator $A \in Op(H)$ is positive iff $\sigma(A) \subseteq [0, +\infty)$.

Let $\mathcal{A}$ be an abelian von Neumann algebra. The set of positive elements in $S(\mathcal{A})$ forms a (positive) cone $\mathcal{P}$, that is,

i) if $A \in \mathcal{P}$ and $-A \in \mathcal{P}$, then $A = 0$.
ii) if $A \in \mathcal{P}$ and $\lambda \geq 0$, then $\lambda A \in \mathcal{P}$.
iii) if $A, B \in \mathcal{P}$, then $A \hat{+} B \in \mathcal{P}$.

The set $S(\mathcal{A})$ is a partially ordered vector space relative to the partial ordering induced by this cone. The same is true for $S(X)$, where $f \geq 0$ in $S(X)$ iff $f(x) \geq 0 \quad \forall x \in U_f$.

Note that by propositions 8.3 and 8.4, $\phi$: $S(\mathcal{A}) \to S(X)$ is order preserving isomorphism.

Our next theorem is an extension to the unbounded case of theorem 3.2.

**8.4 Theorem**: Let $A \in Op(H)$ be a self adjoint operator and $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be a resolution of the identity in B(H). Then $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral resolution for A iff

$$AE_\lambda \leq \lambda E_\lambda$$

and

$$A(I - E_\lambda) \geq \lambda(I - E_\lambda) \quad \forall \lambda \in \mathbb{R}.$$

**Proof.**

($\Longrightarrow$): With the notation as in Theorem 7.2, let $F_J = E_b - E_a$, $J = [a, b]$. $AF_J$ is a bounded self adjoint operator on $H_J$, $H_J$ = Range($F_J$) is invariant under $AF_J$ and the spectral resolution of $AF_J$ is $\{E_\lambda F_J\}_{\lambda \in \mathbb{R}}$

Let $\mathcal{A}_0 = \{A\}''$ ( $= \{V\}''$ with $V = (A + iI)^{-1}$) the abelian von Neumann algebra generated by A. Since, $E_\lambda \in \mathcal{A}_0$ and $A \eta\, \mathcal{A}_0$, we have $E_\lambda A \subseteq AE_\lambda \ \forall \lambda \in \mathbb{R}$.

Let $\mathcal{A}$ be the (abelian) Von Neumann algebra generated by $\{E_\lambda\}$.

By proposition 7.4 we have $\mathcal{A} = \mathcal{A}_0$.

Let $\phi: S(\mathcal{A}) \longrightarrow S(X)$ $(X = X_{\mathcal{A}})$ be the *-isomorphism extending the Gelfand map and $\phi(A) = \frac{1}{v} - i = h$.

Note that $AE_\lambda$ is a closed operator, since A (= A*) is closed and $E_\lambda$ is bounded.

Moreover, $AE_\lambda$ is self adjoint and $AE_\lambda \,\eta\, \mathcal{A}$.

For this note that; $E_\lambda \hat{\cdot} A = A \hat{\cdot} E_\lambda = AE_\lambda$ and $E_\lambda A \subseteq AE_\lambda$ implies $(AE_\lambda)^* \subseteq A^*E_\lambda = AE_\lambda$.

On the other hand

$$E_\lambda A = E_\lambda^* A^* \subseteq (AE_\lambda)^*$$

Therefore

$$E_\lambda A \subseteq (AE_\lambda)^*.$$

Hence

$$E_\lambda \hat{\cdot} A = AE_\lambda \subseteq (AE_\lambda)^*$$

since $(AE_\lambda)^*$ is a closed extension of $E_\lambda A$.

Thus,

$$AE_\lambda = (AE_\lambda)^*.$$

If $B \in \mathcal{A}'$, then $BE_\lambda = E_\lambda B$ since $E_\lambda \in \mathcal{A}$ and $BA \subseteq AB$ since $A\eta\, \mathcal{A}$.

So $BAE_\lambda \subseteq ABE_\lambda = AE_\lambda B$, that is, $B \in \{AE_\lambda\}'$, i.e., $\mathcal{A}' \subseteq \{AE_\lambda\}'$ which implies $AE_\lambda \eta\, \mathcal{A}$.

Thus,

$$AE_\lambda \in S(\mathcal{A}).$$

Now, from theorem 7.2 we have

$$(AF_J)(E_\lambda F_J) \le \lambda E_\lambda F_J \quad \forall \lambda \in \mathbb{R} \ \forall J$$

or

$$AF_J E_\lambda \le \lambda F_J E_\lambda \quad (*)$$

$AF_J \in \mathcal{A}$ and $(AF_J)^\wedge = \phi(A) \cdot \hat{F}_J = h \cdot f_J$ where $f_J = \hat{F}_J$.
Taking Gelfand transforms in both sides of (∗) we get

$$h \cdot f_J e_\lambda \le \lambda f_J e_\lambda \quad \forall \lambda \in \mathbb{R} \ \forall J$$

The set $S = \{y \in Y : \exists \ J$ such that $f_J(y) = 1\} = \bigcup_J \{f_J > 0\}$ is an open dense subset of Y.
(see proof of proposition 7.3)

So

$$h \cdot e_\lambda \le \lambda e_\lambda \quad \forall \lambda \in \mathbb{R}.$$

Since $\phi : S(\mathcal{A}) \to S(X)$ is order preserving *-isomorphism, we have that

$$AE_\lambda \le \lambda E_\lambda \quad \forall \lambda \in \mathbb{R}.$$

Similarly,

$$A(F_J - F_J E_\lambda) \ge \lambda (F_J - F_J E_\lambda) \quad \forall \lambda \in \mathbb{R}, \forall J$$

implies

$$h \cdot (f_J - f_J e_\lambda) \ge \lambda (f_J - f_J e_\lambda)$$

which implies

$$h \cdot (1 - e_\lambda) \ge \lambda (1 - e_\lambda)$$

and so

$$A(I - E_\lambda) \ge \lambda (I - E_\lambda) \quad \forall \lambda \in \mathbb{R}.$$

(⇐): Suppose $AE_\lambda \le \lambda E_\lambda$ and $A(I - E_\lambda) \ge \lambda(I - E_\lambda)$ hold $\forall \lambda \in \mathbb{R}$.
Then $\lambda E_\lambda - AE_\lambda \ge 0$. In particular $\lambda E_\lambda - AE_\lambda$ is self adjoint.
So

$$\lambda E_\lambda - (AE_\lambda)^* = (\lambda E_\lambda - AE_\lambda)^* = \lambda E_\lambda - AE_\lambda$$

which implies that

$$(AE_\lambda)^* = AE_\lambda \text{ , i.e., } AE_\lambda \text{ is self adjoint.}$$

Moreover,

$$AE_\lambda = (AE_\lambda)^* \supseteq E_\lambda^* A^* = E_\lambda A \text{ , that is, } E_\lambda A \subseteq AE_\lambda \text{ .}$$

Let $\{P_\mu\}_{\mu\in\mathbb{R}}$ be the spectral resolution for A .
Then by lemma 8.3, $E_\lambda P_\mu = P_\mu E_\lambda \quad \forall\lambda,\mu \in \mathbb{R}$ .
If $\mathcal{A}$ is the von Neumann algebra generated by $\{E_\lambda\}$ and $\{P_\mu\}$ , then $\mathcal{A}$ is abelian and $\mathcal{A} \cong C(X)$ with $X = X_{\mathcal{A}}$ a Stonean space.
Let $F_J = P_b - P_a$ , $J = [a, b]$. Then $P_\lambda F_J \in \mathcal{A}$ and $AF_J \in \mathcal{A}$ .
Furthermore, by proposition 7.4 we get $\mathcal{A}' \subseteq \{A\}'$
which implies that $A \,\eta\, \mathcal{A}$, i.e., $A \in S(\mathcal{A})$.
Let $\phi$: $S(\mathcal{A}) \to S(X)$ be the extension of the Gelfand map and $\phi(A) = h$ , $\hat{E}_\lambda = e_\lambda$.
The same argument as before gives that $AE_\lambda \,\eta\, \mathcal{A}$, i.e., $AE_\lambda \in S(\mathcal{A})$.
Now, $\phi(AE_\lambda) = \phi(A)\cdot \hat{E}_\lambda = h\cdot e_\lambda$ and since $\phi$ is order preserving our hypothesis gives

$$h\cdot e_\lambda \le \lambda e_\lambda \text{ and } \lambda(1 - e_\lambda) \le h\cdot(1 - e_\lambda)$$

Let $X_\lambda = \{x \in X: \; e_\lambda(x) = 1\}$.
On $X_\lambda$ $e_\lambda = 1$ , so $h \le \lambda$ and so $X_\lambda \subseteq \{h \le \lambda\}$ which implies $X_\lambda \subseteq \operatorname{int}\{h \le \lambda\}$.
On $X_\lambda^c$ $1 - e_\lambda = 1$ , so $\lambda \le h$ which implies $\{\lambda > h\} \subseteq X_\lambda$ .
Thus,

$$\{h < \lambda\} \subseteq X_\lambda \subseteq \operatorname{int}\{h \le \lambda\}.$$

Since $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ is a resolution of the identity in C(X) ( $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is such in $\mathcal{A}$ ) proposition 7.3 gives that $X_\lambda = \operatorname{int}\{h \le \lambda\}$.
Hence,

$$\hat{E}_\lambda = e_\lambda = \mathcal{X}_{X_\lambda} = \mathcal{X}_{\operatorname{int}\{h \le \lambda\}} = \hat{P}_\lambda \quad \forall\lambda \in \mathbb{R},$$

that is,

$$E_\lambda = P_\lambda \quad \forall\lambda \in \mathbb{R} . \; ■$$

**8.6 Proposition:** If $\mathcal{A}$ is an abelian von Neumann algebra and $\{A_n\}$ is an increasing sequence of operators in $S(\mathcal{A})$ with upper bound $A_0$ in $S(\mathcal{A})$, then $\{A_n\}$ has a least upper bound in $S(\mathcal{A})$. (see [14] p. 359)

Let X be any topological space, we denote by $B_u(X)$ the *-algebra of (unbounded) Borel functions on X.
Recall that when X is a Stonean space N(X) denotes the *-algebra of normal functions on X.

**8.7 Proposition**: Let X be a Stonean space. Then $\forall f \in B_u(X)$ there exists a unique function $g \in N(X)$ such that the set $\{f \neq g\}$ is meager.
The mapping $\gamma: B_u(X) \rightarrow N(X)$ of $B_u(X)$ onto N(X) given by $\gamma(f) = g$ is a *-homomorphism with kernel $\mathfrak{n}(X)$.

**Proof.**

We construct $\gamma$ first for real-valued Borel functions on X onto S(X).
To see that $\gamma$ is onto; let $g \in S(X)$. Take

$$f = \begin{cases} g & \text{when } |g| < \infty \\ 0 & \text{when } |g| = \infty \end{cases}$$

$f \in B_u(X)$ and $\{f \neq g\} = \{|g| = \infty\}$ a nowhere dense set, hence meager.Thus $\gamma(f) = g$.
Now, let f be a real-valued Borel function on X.
Define

$$\tilde{f} = \frac{f+i}{f-i} \quad ; |\tilde{f}| = 1, \text{ that is, } f \in B(X).$$

Note that $\{\tilde{f} = 1\} = \{|f| = \infty\}$ a meager set.
Let $\tilde{g} \in C(X)$ be the (unique) continuous function for which $\{\tilde{f} \neq \tilde{g}\}$ is meager. (proposition 4.2)
We have $\{\tilde{g} = 1\} \subseteq \{\tilde{f} \neq \tilde{g}\} \cup \{\tilde{f} = 1\}$ a meager set. So $\{\tilde{g} = 1\}$ is meager.
Since $U_g = \{g \neq \infty\} = \{\tilde{g} \neq 1\}$ is (open) dense (by Baire's category theorem), the function $g = i\dfrac{\tilde{g}+1}{\tilde{g}-1}$ is normal, i.e., $g \in N(X)$.
Also, since $\{|g| \neq 1\} \subseteq \{\tilde{f} \neq \tilde{g}\}$ and g is real-valued whenever $|\tilde{g}| = 1$, g is real-valued except on a meager set. Hence, $\{-\infty < g < \infty\}$ is dense, i.e., $g \in S(X)$.
Moreover the set $\{f \neq g\} = \{\tilde{f} \neq \tilde{g}\}$ which is meager.
If f is a complex-valued Borel function, then $f = f_1 + if_2$ with $f_1$, $f_2$ real-valued Borel functions and $\gamma(f) = \gamma(f_1) + i\gamma(f_2)$.
For the uniqueness note that, if $g_1, g_2 \in N(X)$ are such that $\{f \neq g_1\}$ and $\{f \neq g_2\}$ are meager sets, then since $\{g_1 \neq g_2\} \subseteq \{f \neq g_1\} \cup \{f \neq g_2\}$, $\{g_1 \neq g_2\}$ is a meager set and so $\{g_1 = g_2\}$ is dense in X. Hence $g_1 = g_2$ since $g_1$ and $g_2$ are continuous.
That $\gamma$ is a *-homomorphism, is quiet obvious.
Finally, $f \in \mathrm{Ker}(\gamma) \Longleftrightarrow \{f \neq 0\}$ is a meager set, i.e., $f \in \mathfrak{n}(X)$. ∎

**Definition**: A *-homomorphism $\varphi: B_u(\mathbb{C}) \rightarrow N(\mathcal{A})$ is called **σ -normal** if whenever $\{f_n\}$ is a non-decreasing sequence in $B_u(\mathbb{C})$ such that $f_n \rightarrow f$ pointwise, $f \in B_u(\mathbb{C})$, then

$$\varphi(f) = \bigvee_{n=1}^{\infty} \varphi(f_n)$$

Note the use of proposition 8.6 in the above definition.

Similarly, a *-homomorphism $\omega: B_u(X) \rightarrow N(X)$ is called **σ -normal** if whenever $\{f_n\}$ is a non-decreasing sequence in $B_u(X)$ such that $f_n \rightarrow f$ pointwise, $f \in B_u(X)$, then

$$\omega(f) = \bigvee_{n=1}^{\infty} \omega(f_n)$$

**8.8 Proposition**: The mapping $\gamma: B_u(X) \rightarrow N(X)$ defined in proposition 8.6 is a σ-normal *-homomorphism.

**Proof.**

Let $\{f_n\}$ be an increasing sequence of Borel functions on X tending pointwise to the (Borel) function f. Let $g_n = \gamma(f_n)$, $g = \gamma(f)$ with $g, g_n \in N(X)$. Fix $m > n$. We have $\{g_n > g_m\} \subseteq \{f_n \neq g_n\} \cup \{f_m \neq g_m\}$ which is a meager set. So $\{g_n \leq g_m\}$ is dense in X and by the continuity of $g_n$, $g_m$ we get that $g_n \leq g_m$.

Also, $\{g_n > g\} \subseteq \{f_n \neq g_n\} \cup \{f \neq g\}$ and as above we get that $g_n \leq g$ $\forall n \in \mathbb{N}$.

Thus $\{g_n\}$ is monotone (increasing) and bounded, hence the pointwise limit of $g_n$ exists on $U = \bigcap_{n=1}^{\infty} U_{g_n}$ . U is dense in X.

Let

$$\tilde{f} = \begin{cases} \lim g_n & \text{on } U \\ 0 & \text{on } U^c \end{cases}$$

$\tilde{f} \in B_u(X)$ and $\{f \neq \tilde{f}\} \subseteq \bigcup_{n=1}^{\infty} \{f_n \neq g_n\}$, a meager set.

Now, if $h \in N(X)$ is such that $g_n \leq h$ $\forall n$, then $\tilde{f} \leq h$ since U is dense in X.

Let $\tilde{g} = \gamma(\tilde{f})$ ; the set $\{\tilde{f} \neq \tilde{g}\}$ is meager. So $\{\tilde{f} = \tilde{g}\}$ is dense in X and so $\tilde{g} \leq h$ on a dense set in X ; again by continuity $\tilde{g} \leq h$ on X.

But $\{\tilde{g} \neq g\} \subseteq \{f \neq \tilde{f}\} \cup \{\tilde{f} \neq \tilde{g}\} \cup \{f \neq g\}$ a meager set,

so $\tilde{g} = g$ on a dense set, hence $\tilde{g} = g$ on X.

Therefore $g \leq h$.

Thus $\gamma(f) = \bigvee_{n=1}^{\infty} \gamma(f_n)$, that is, $\gamma$ is σ-normal. ■

Next with the use of the $\sigma$-normal *-homomorphism $\gamma$ and the extension of the Gelfand map, we will establish the functional calculus for (unbounded) Borel functions on the spectrum of an unbounded normal operator.

**8.5 Theorem: (Spectral theorem - functional calculus form)**

Let $A \in Op(H)$ be a normal operator on H and $\mathcal{A}_0$ be the (abelian) von Neumann algebra generated by A.

Then there exists a unique $\sigma$-normal *-homomorphism $\Phi : B_u(\sigma(A)) \to N(\mathcal{A}_0)$ such that $\Phi(1) = I$, $\Phi(id) = A$. Moreover, if $f \in B(\sigma(A))$, then $\Phi(f) \in \mathcal{A}_0$.

**Proof.**

Let $\phi: N(\mathcal{A}_0) \to N(X)$ be the *-isomorphism extending the Gelfand map; $\phi(A): X \to \sigma(A)$ is a continuous onto function.

Given $f \in B_u(\sigma(A))$, define

$$\tilde{f} = \begin{cases} f \circ \phi(A) & \text{on } U_{\phi(A)} \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{f} \in B_u(X)$ and so $\gamma(\tilde{f}) \in N(X)$.

Now, the mapping $\Phi: B_u(\sigma(A)) \to N(\mathcal{A}_0)$ defined by $\Phi(f) = \phi^{-1}(\gamma(\tilde{f}))$ is a $\sigma$-normal *-homomorphism of $B_u(\sigma(A))$ into $N(\mathcal{A}_0)$. $\Phi(1) = I$ and $\Phi(id) = \phi^{-1}(\phi(A)) = A$.

Moreover, if $f \geq 0$, then $\Phi(f) = \phi^{-1}(\gamma(\tilde{f})) \geq 0$. For $f \in B(\sigma(A))$ there are m, M such that $m \leq f \leq M$, so $mI \leq \Phi(f) \leq MI$, that is, $\Phi(f)$ is bounded and $\Phi(f) \eta \mathcal{A}_0$, hence $\Phi(f) \in \mathcal{A}_0$.

It is customary to denote $\Phi(f)$ by $f(A)$.

The uniqueness of $\Phi$ is a consequence of the following stronger result. ■

**8.6 Theorem**: (uniqueness)

If $A \in Op(H)$ is a normal operator affiliated with an abelian von Neumann algebra $\mathcal{A}$ acting on a Hilbert space H and $\Psi$ is a $\sigma$-normal homomorphism of $B_u(\mathbb{C})$ into $N(\mathcal{A})$ such that $\Psi(1) = I$ and $\Psi(id) = A$, then $\Psi(f) = f_r(A)$ $\forall f \in B_u(\mathbb{C})$, where $f_r$ is the restriction of f to $\sigma(A)$. (see [14] p.362).

**8.9 Proposition:** Let $\mathcal{A}_1$, $\mathcal{A}_2$ be two abelian von Neumann algebras and $\Psi: N(\mathcal{A}_1) \to N(\mathcal{A}_2)$ be a $\sigma$-normal *- homomorphism such that $\Psi(1) = I$.

Then $\Psi(f(A)) = f(\Psi(A))$ $\forall A \in N(\mathcal{A}_1)$ and $f \in B_u(\mathbb{C})$.

**Proof.**

The mapping $\omega: B_u(\mathbb{C}) \to N(\mathcal{A}_2)$ given by $\omega(f) = \Psi(f(A))$ is a $\sigma$-normal *-homomorphism such that $\omega(1) = \Psi(I) = I$ and $\omega(id) = \Psi(A)$.

On the other hand we have the $\sigma$-normal *-homomorphism $\Phi: B_u(\sigma(\Psi(A))) \to N(\mathcal{A}_2)$ given as in theorem 4.1. Thus, by the uniqueness of such a $\sigma$-normal *-homomorphism, we get $\omega(f) = \Psi(f(A)) = f(\Psi(A))$ . ■

**8.10 Proposition**: Let $A \in Op(H)$ be a normal operator . Then $\forall f \in B_u(\sigma(A))$

$$\sigma(f(A)) \subseteq \overline{f(\sigma(A))} .$$

**Proof.**

By definition $f(A) = \phi^{-1}(\gamma(\tilde{f}))$ where $\phi: N(\mathcal{a}_0) \rightarrow N(X)$ is the extension of the Gelfand map and $\tilde{f}$ is as in theorem 8.4

By proposition 8.3 we have

$$\sigma(f(A)) = \text{Range}(\phi(f(A))) = \text{Range}(\gamma(\tilde{f})) .$$

On the other hand $f(\sigma(A)) = f(\text{Range}(\phi(A))) = \text{Range}(f \circ \phi(A))$.

The proof of the following lemma follows the same lines as the proof of Lemma 4.4.

<<*Lemma*:: If X is a Stonean space and $\gamma: B_u(X) \rightarrow N(X)$ is the mapping defined in proposition 8.6, then $\text{Range}(\gamma(f)) \subseteq \overline{\text{Range}(f)}$ $\forall f \in B_u(X)$. >>

Thus,

$$\sigma(f(A)) \subseteq \overline{f(\sigma(A))} . \blacksquare$$

**8.11 Proposition**: Let $A \in Op(H)$ be a normal operator and $f, g \in B_u(\mathbb{C})$ .

Then

$$f \circ g(A) = f(g(A)).$$

**Proof.**

If $\mathcal{a}_0$ is the abelian von Neumann algebra generated by A, then $g(A) \in N(\mathcal{a}_0)$.

Let $\omega: B_u(\mathbb{C}) \rightarrow B_u(\mathbb{C})$ be the σ-normal homomorphism given by $\omega(f) = f \circ g$ .

Composing ω with the σ-normal homomorphism $\Phi: B_u(\mathbb{C}) \rightarrow N(\mathcal{a}_0)$, $\Phi(h) = h(A)$, yields a σ-normal homomorphism $\Psi: B_u(\mathbb{C}) \rightarrow N(\mathcal{a}_0)$ such that $\Psi(1) = I$ and $\Psi(id) = g(A)$ .

On the other hand we have $\Phi_1: B_u(\sigma(g(A))) \rightarrow N(\mathcal{a}_0)$ given by $\Phi_1(f) = f(g(A))$ the σ-normal homomorphism of theorem 8.4.

The uniqueness of such a σ-normal homomorphism gives $f \circ g(A) = f(g(A))$. ■

We recall that, if $\mathcal{U}$ is a commutative C*-algebra of operators on H, then the spectral theorem for commutative C*-algebras (Theorem 4.5) quarantees the existence of a unique spectral measure E on the Borel subsets of $X = X_{\mathcal{U}}$, such that

$$(B\xi,\eta) = \int_X \hat{B}(\rho)dE_{\xi,\eta}(\rho) \qquad \text{for } \xi,\eta \in H,\ B \in \mathcal{U}.$$

If $\nu_\xi(M) = (E(M)\xi, \xi)$ for $\xi \in H$, $M \in B_X$ , then $\nu_\xi$ is a regular Borel measure on X such that

$$(B\xi,\xi) = \int_X \hat{B}(\rho)d\nu_\xi(\rho) \qquad \text{for } \xi \in H,\ B \in \mathcal{U}$$

**8.12 Proposition**: Let $\mathcal{A}$ be an abelian von Neumann algebra on H, $X = X_{\mathcal{A}}$ and $\nu_\xi$ be the regular Borel measure on X such that $(B\xi,\xi) = \int_X \hat{B}(\rho)d\nu_\xi(\rho)$ for $\xi \in H$, $B \in \mathcal{A}$

If $f \in N(X)$, then $\nu_\xi(\{|f| = \infty\}) = 0 \quad \forall \xi \in H$.

**Proof.**

For $n \in \mathbb{N}$, let $X_n = \overline{\{|f| < n\}}$ ; $X_n$ is a clopen subset of X and $X_n \subseteq X_{n+1}$ $\forall n \in \mathbb{N}$.

Let $e_n = \chi_{X_n} \in C(X)$ and $E_n \in \mathcal{A}$ be such that $\hat{E}_n = e_n$ .

We show that $\bigvee_{n=1}^{\infty} e_n = 1$ : Clearly $e_n \leq 1$ $\forall n$. Suppose $g \in C(X)$ is such that $e_n \leq g$ $\forall n$ .

Let $U = \{ g < 1 \}$. U is open and $U \cap X_n = \emptyset$ $\forall n$. So $U \cap \bigcup_{n=1}^{\infty} X_n = \emptyset$ .

But $\{ |f| < n \} \subseteq X_n \subseteq \{ |f| \leq n \}$, hence $\bigcup_{n=1}^{\infty} X_n = \{ |f| < \infty \}$.

Therefore, $U \subseteq \{ |f| = \infty \}$ which is a nowhere dense set (since $f \in N(X)$).

Thus $U = \emptyset$ , that is, $\bigvee_{n=1}^{\infty} e_n = 1$.

Now, $\bigvee_{n=1}^{\infty} e_n = 1$ gives that $\bigvee_{n=1}^{\infty} E_n = I$ which implies that $E_n \to I$ in s.o.t. as $n \to \infty$.

Note that

$$\| E_n\xi \|^2 = (E_n\xi, \xi) = \int_X e_n(\rho)d\nu_\xi(\rho) = \nu_\xi(X_n).$$

So

$$\nu_\xi(X_n^{\,c}) = \nu_\xi(X) - \nu_\xi(X_n) = \| \xi \|^2 - \| E_n\xi \|^2 .$$

Thus,

$$\nu_\xi(\{ |f| = \infty \}) = \nu_\xi( \bigcap_{n=1}^{\infty} X_n^{\,c}) = \lim_{n\to\infty} \nu_\xi(X_n^{\,c}) = \lim_{n\to\infty} ( \| \xi \|^2 - \| E_n\xi \|^2 ) = 0 \ . \blacksquare$$

**8.13 Proposition**: Let $\mathcal{A}$ be an abelian von Neumann algebra on H, $X = X_{\mathcal{A}}$ its Gelfand space, $\phi: N(\mathcal{A}) \to N(X)$ the extension of the Gelfand map and $\nu_\xi$ the regular Borel measure on X such that $(B\xi,\xi) = \int_X \hat{B}(\rho)d\nu_\xi(\rho)$ for $\xi \in H$, $B \in \mathcal{A}$.

Suppose $T \in N(\mathcal{A})$. Then $\xi \in D(T)$ iff $\int_X |\phi(T)(\rho)|^2 d\nu_\xi(\rho) \quad ( = \| T\xi \|^2 ) < \infty$

**Proof.**

($\Longrightarrow$): Set $f = \phi(T) \in N(X)$. Let $X_n = \{\rho \in X: |f(\rho)| \leq n\}$ $n = 1, 2, ...$

Since X is Stonean, $\text{int}(X_n)$ is clopen. So $e_n = \chi_{\text{int}(X_n)} \in C(X)$.

Let $E_n \in \mathcal{A}$ be such that $\hat{E}_n = e_n$ . Set $f_n = f \cdot e_n$ ; note that $f_n \in C(X)$.

Since $T \eta \mathcal{A}$ , $E_nT \subseteq TE_n$ $\forall n$. If $S \in \mathcal{A}'$, then $STE_n \subseteq TSE_n = TE_nS$ .

It follows that $\mathcal{A}' \subseteq \{TE_n\}'$.

Note also that $TE_n$ is closed since T is closed and $E_n$ is bounded. Thus $TE_n \,\eta\, \mathcal{A}$.
Moreover, $\phi(TE_n) = \phi(T)\cdot \hat{E}_n = f\cdot e_n = f_n \in C(X)$. Therefore $TE_n \in \mathcal{A}$ and $(TE_n)^\wedge = f_n$.
Now, for $\xi \in H$ we have

$$\|TE_n\xi\|^2 = \int_X |(TE_n)^\wedge(\rho)|^2 dv_\xi(\rho) = \int_X |f_n(\rho)|^2 dv_\xi(\rho)$$

By a similar argument as in the proof of the previous proposition we get $\bigvee_{n=1}^{\infty} e_n = 1$,
so that $E_n \to I$ in the s.o.t.
For $\xi \in D(T)$, $TE_n\xi = E_nT\xi \to T\xi$
Therefore

$$\|T\xi\|^2 = \lim_{n\to\infty} \|TE_n\xi\|^2 = \lim_{n\to\infty} \int_X |f_n(\rho)|^2 dv_\xi(\rho)$$

Since the sequence $\{f_n\}$ converges monotonically to f a.e $[v_\xi]$ (since $v_\xi(\{|f| = \infty\}) = 0$)
the monotone convergence Theorem gives

$$\|T\xi\|^2 = \int_X |f(\rho)|^2 dv_\xi(\rho) = \int_X |\phi(T)(\rho)|^2 dv_\xi(\rho).$$

($\Leftarrow$): If $\int_X |\phi(T)(\rho)|^2 dv_\xi(\rho)$ converges, then for $\xi \in H$ we have

$$\|TE_n\xi - TE_m\xi\|^2 = \int_X |f_n(\rho) - f_m(\rho)|^2 dv_\xi(\rho) \to 0 \quad \text{as } m, n \to \infty$$

So $\{TE_n\xi\}$ is a Cauchy sequence in H and so $\{TE_n\xi\}$ converges to some $z \in H$.
Now, $TE_n\xi \to z$ and $E_n\xi \to \xi$.
Hence, since T is closed, $\xi \in D(T)$ and $T\xi = z = \lim_{n\to\infty} TE_n\xi$. ∎

**8.3 Remark**: For $\xi,\eta \in H$, $B \in \mathcal{A}$, the relation $\Psi(\xi,\eta) = (B\xi,\eta)$ defines a bounded sesquilinear functional. Let $\tilde{\Psi}(\xi) = \Psi(\xi,\xi) = (B\xi, \xi)$ be the corresponding quadratic form.
The polarization identity gives

$$\Psi(\xi,\eta) = \tilde{\Psi}\left(\tfrac{1}{2}(\xi+\eta)\right) - \tilde{\Psi}\left(\tfrac{1}{2}(\xi-\eta)\right) + i\,\tilde{\Psi}\left(\tfrac{1}{2}(\xi+i\eta)\right) - i\,\tilde{\Psi}\left(\tfrac{1}{2}(\xi-i\eta)\right).$$

Now, if $v_\xi$ is the regular Borel measure of theorem 4.5, then the polarization identity becomes

$$(B\xi,\eta) = \tfrac{1}{4}\left\{\int_X \hat{B}(\rho)dv_{\xi+\eta}(\rho) - \int_X \hat{B}(\rho)dv_{\xi-\eta}(\rho) + i\int_X \hat{B}(\rho)dv_{\xi+i\eta}(\rho) - i\int_X \hat{B}(\rho)dv_{\xi-i\eta}(\rho)\right\}$$

or

$$(B\xi,\eta) = \int_X \hat{B}(\rho)d\sigma_{\xi,\eta}(\rho)$$

where $\sigma_{\xi,\eta} = \tfrac{1}{4}\{v_{\xi+\eta} - v_{\xi-\eta} + iv_{\xi+i\eta} - iv_{\xi-i\eta}\}$

Now, let $T \in N(\mathcal{A})$ and $f = \phi(T) \in N(X)$ , where $\phi$: $N(\mathcal{A}) \to N(X)$ is the extension of the Gelfand map.
By proposition 8.13 , for $\xi \in D(T)$ , we have $\|T\xi\|^2 = \int_X |f(\rho)|^2 dv_\xi(\rho) < \infty$ .
So that $f \in L^2(X, v_\xi)$ and since $v_\xi(X) = \|\xi\|^2 < \infty$ we conclude that $f \in L^1(X, v_\xi)$.
If $E_n$ are as in the proof of proposition 8.13 and $f_n = (TE_n)^\wedge$ , $TE_n \in \mathcal{A}$,
then $f_n \to f$ a.e $[v_\xi]$ and $|f_n| = |f \cdot e_n| \leq |f|$.
The Lebesque`s Dominated convergence Theorem gives

$$(T\xi, \xi) = \lim_{n\to\infty} (TE_n\xi, \xi) = \lim_{n\to\infty} \int_X f_n(\rho)\, dv_\xi(\rho) = \int_X f(\rho)\, dv_\xi(\rho)$$

Furthermore, by polarization we get

$$(T\xi, \eta) = \int_X f(\rho)\, d\sigma_{\xi,\eta}(\rho) \qquad \text{for } \xi, \eta \in D(T)$$ ■

In the following proposition " p(A) " refers to the operator obtained by forming the Borel function p of A.

**8.14 Proposition**: If $A \eta\, \mathcal{A}$ where $\mathcal{A}$ is an abelian von Neumann algebra acting on the Hilbert space H and $p(\lambda) = a_n\lambda^n + ... + a_1\lambda + a_0$ with $a_n \neq 0$, then $a_nA^n + ... + a_1A + a_0I$ is closed and equal to p(A).
(see [14] p. 366)

We now show the existence and uniqueness of the *square root* of a positive (unbounded) operator.

**8.7 Theorem**: Let $A \in Op(H)$ be a self adjoint operator on H. If $A \geq 0$, then there exists a unique self adjoint operator $B \geq 0$ such that $B^2 = A$.

**Proof.**

Since $A \geq 0$ , $\sigma(A) \subseteq [0, +\infty)$. Consider the function g: $\mathbb{R}^+ \to \mathbb{R}^+$ given by $g(\lambda) = \sqrt{\lambda}$
Let $B = g(A)$ $(= \Phi(g))$ ; $\sigma(B) = \sigma(g(A)) \subseteq \overline{g(\sigma(A))} \subseteq [0, +\infty)$ so that $B \geq 0$.
If $f(\lambda) = \lambda^2$, then fog = id on $\sigma(A)$. Hence $(f\circ g)(A) = id(A) = A$ .
On the other hand by proposition 8.14 we have $f(g(A)) = f(B) = B^2 = B \hat{\cdot} B$ .
Now proposition 8.11 gives $A = (f\circ g)(A) = f(g(A)) = B^2$.
It remains only to show that such a B is unique. Suppose Q is a positive operator satisfying $Q^2 = A$. If $f(\lambda) = \lambda^2$ , $\lambda \geq 0$ , then the spectra of Q and B are contained in the domain of the Borel function f and f has a Borel inverse function g , $g(\lambda) = \sqrt{\lambda}$
Now $f(B) = f(Q)$ .
Hence,
$B = id(B) = (g\circ f)(B) = g(f(B)) = g(f(Q)) = (g\circ f)(Q) = id(Q) = Q$, i.e., $B = Q$ . ■

For the proof of the following (spectral) theorem we refer the reader to [14] p. 360

**8.8 Theorem: (Spectral Theorem)**
Let $A \in Op(H)$ be a normal operator and $\mathcal{A}_0$ be the abelian von Neumann algebra generated by A.
The mapping $M \mapsto E(M)$ of Borel subsets M of $\mathbb{C}$ into $\mathcal{A}_0$ is a projection-valued measure on $\mathbb{C}$ (a spectral measure on $\mathbb{C}$ ), where $E(M) = \mathcal{X}_M(A) = \Phi(\mathcal{X}_M)$.
If $\xi \in H$, then for each $h \in B(\mathbb{C})$

$$(h(A)\xi, \xi) = \int_{\mathbb{C}} h(\lambda)\, d\mu_\xi(\lambda). \tag{1}$$

With $f \in B_u(\mathbb{C})$,

$$\xi \in D(f(A)) \iff \int_{\mathbb{C}} |f(\lambda)|^2 d\mu_\xi(\lambda) = \| f(A)\xi \|^2 < \infty \tag{2}$$

and

$$(f(A)\xi, \xi) = \int_{\mathbb{C}} f(\lambda)\, d\mu_\xi(\lambda). \tag{3}$$

If A is a self adjoint operator, its spectral resolution is $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, where $E_\lambda = I - E((\lambda, \infty))$ and

$$\xi \in D(f(A)) \iff \int_{\mathbb{R}} |f(\lambda)|^2 d(E_\lambda \xi, \xi) < \infty$$

if $\xi \in D(f(A))$

$$(f(A)\xi, \xi) = \int_{\mathbb{R}} f(\lambda)\, d(E_\lambda \xi, \xi). \quad \blacksquare$$

Next we wish to prove the analogue for unbounded normal operators of theorem 7.2 , i.e., *the Spectral theorem for unbounded normal operators.*

First we show two lemmas. Our first lemma is a special case of the following proposition.
<< *Proposition*:: If $\mathcal{R}$ is a von Neumann algebra with center $\mathcal{C}$, acting on the Hilbert space H and E' is a projection in $\mathcal{R}'$, then $\mathcal{R}$E acting on E'(H) is a von Neumann algebra with center $\mathcal{C}$E' and commutant E'$\mathcal{R}$'E'. (see [14] p. 335) >>

**8.4 Lemma:** Let $\mathcal{A}$ be an abelian von Neumann algebra, E a projection in $\mathcal{A}$ and

$$\mathcal{A}E = \{ AE|_{E(H)} = A|_{E(H)} : A \in \mathcal{A} \}.$$

Then $\mathcal{A}E$ is an abelian von Neumann algebra with commutant $\mathcal{A}'E$.

**Proof.**
Since $\mathcal{A}$ is abelian $\mathcal{A} \subseteq \mathcal{A}'$ and so by the above proposition we have
$(\mathcal{A}E)' = E\,\mathcal{A}'E = \mathcal{A}'E$ and $\mathcal{A}E$ is an abelian von Neumann algebra. $\blacksquare$

**8.5 Lemma**: Let $A \in B(H)$ and M be a (closed) subspace of H invariant under A. Let $P_M$ be the projection onto M. Then $A|_M$ is invertible in B(M) iff there exists $B \in B(H)$ such that $BP_M = P_M B$ and $ABP_M = BAP_M = P_M$. In fact $(A|_M)^{-1} = B|_M$.

**Proof.**

($\Leftarrow$): For $x \in M$ we have $(A|_M)(B|_M)x = ABP_M x = P_M x = x$.

Similarly, $(B|_M)(A|_M)x = x$.

So $A|_M$ is invertible and $(A|_M)^{-1} = B|_M$

($\Rightarrow$): Let $B_0 \in B(M)$ be the inverse of $A|_M$. Define $B \in B(M)$ by $B = B_0 P_M$.

We have

$$BP_M = B_0 P_M P_M = B_0 P_M = P_M B_0 P_M = P_M B$$

$$ABP_M = AB_0 P_M = (A|_M) B_0 P_M = P_M P_M = P_M$$

and

$$BAP_M = B_0 P_M A P_M = B_0 A P_M = B_0 (A|_M) P_M = P_M P_M = P_M \quad . \blacksquare$$

**8.1 Corollary**: Let $A \in B(H)$ and M be a subspace of H which reduces A. Then

$$\sigma(A) = \sigma(A|_M) \cup \sigma(A|_{M^\perp}).$$

**Proof.**

Let $\lambda \in \rho(A)$ and $B = (\lambda I - A)^{-1}$. Since M reduces A we have $AP_M = P_M A$ which implies $P_M B = BP_M$. Also, $(\lambda I - A)BP_M = B(\lambda I - A)P_M = P_M$.

So, by lemma 8.5, $(\lambda I - A)|_M$ is invertible in B(M), that is , $\lambda P_M - A|_M$ is invertible which says that $\lambda \in \rho(A|_M)$.

Replacing M by $M^\perp$ we get $\lambda \in \rho(A|_{M^\perp})$.

Thus

$$\sigma(A|_M) \cup \sigma(A|_{M^\perp}) \subseteq \sigma(A).$$

To get equality, suppose $\lambda \in \rho(A|_M) \cap \rho(A|_{M^\perp})$.

Then there are B , T in B(H) such that

$BP_M = P_M B$ , $TP_M = P_M T$ and $B(\lambda I - A)P_M = (\lambda I - A)BP_M = P_M$

$T(\lambda I - A)(I - P_M) = (\lambda I - A)T(I - P_M) = I - P_M$

Let $S = BP_M + T(I - P_M)$; $S(\lambda I - A) = BP_M(\lambda I - A) + T(I - P_M)(\lambda I - A) = I$, also $(\lambda I - A)S = I$. Hence $\lambda \in \rho(A)$. $\blacksquare$

**8.9 Theorem: (Spectral Theorem)**

Let $A \in Op(H)$ be a normal operator and $\mathcal{A}_0$ be the abelian von Neumann algebra generated by A. Then there is a unique spectral measure E on the Borel subsets of $\mathbb{C}$ such that:

i) For every closed disk $\Delta$ in $\mathbb{C}$, if $E_\Delta = E(\Delta) = \mathcal{X}_{\Delta \cap \sigma(A)}(A)$, then $AE_\Delta$ is a bounded normal operator on H which leaves $H_\Delta = \text{Range}(E_\Delta)$invariant and such that $AE_\Delta|_{H_\Delta}$ is a bounded normal operator on $H_\Delta$ whose spectral measure is $\tilde{E}(S) = E(S \cap \Delta) = E(S)E_\Delta$ where S is any Borel subset of $\sigma(AE_\Delta|_{H_\Delta})$.

ii) Let $\mathcal{D}$ be the directed set of closed disks in $\mathbb{C}$ ordered by inclusion.

$$x \in D(A) \iff \text{the net } \{AE_\Delta x\}_{\Delta \in \mathcal{D}} \text{converges.}$$

For $x \in D(A)$, $Ax = \lim AE_\Delta x$.

Moreover, $E(S)T = TE(S)$ for every Borel subset S of $\sigma(A)$ and for $T \in B(H)$ with $TA \subseteq AT$ (, i.e., $T \in \{A\}'$)

**Proof.**

i) By theorem 8.4 $\Phi : B_u(\sigma(A)) \to N(\mathcal{A}_0)$ is the unique $\sigma$-normal *-homomorphism such that $\Phi(1) = I$, $\Phi(id) = A$

For $S \in B_{\mathbb{C}}$, a Borel subset of $\mathbb{C}$, let $E(S) = \Phi(\mathcal{X}_{S \cap \sigma(A)}) = \mathcal{X}_{S \cap \sigma(A)}(A)$

As in the proof of theorem 4.3, it can be seen that E is a spectral measure on $\mathbb{C}$.

Now, $AE_\Delta = \Phi(id\mathcal{X}_{\Delta \cap \sigma(A)})$.

Since $id\mathcal{X}_{\Delta \cap \sigma(A)}$ is a bounded function$\Phi(id\mathcal{X}_{\Delta \cap \sigma(A)}) \in \mathcal{A}_0$ and is a normal operator, that is, $AE_\Delta$ is a bounded normal operator and $AE_\Delta \in \mathcal{A}_0$.

Since $A \eta\, \mathcal{A}_0$ and $E_\Delta \in \mathcal{A}_0$, we have $E_\Delta A \subseteq AE_\Delta$.

Hence, if $x = E_\Delta x$, then $E_\Delta Ax = AE_\Delta x = Ax$, that is, A carries $\text{Range}(E_\Delta)$ into $\text{Range}(E_\Delta)$.

So $AE_\Delta \in B(H_\Delta)$. Now, $E_\Delta A^* \subseteq A^*E_\Delta$ since $A^* \eta\, \mathcal{A}_0$, so $(A|_{H_\Delta})^* = A^*|_{H_\Delta}$ and so $AE_\Delta|_{H_\Delta} = A|_{H_\Delta}$ is normal.

Let $\mathcal{A}_0 E_\Delta = \{ TE_\Delta|_{H_\Delta} = T|_{H_\Delta} : T \in \mathcal{A}_0\}$ and $\tilde{\Phi}: B(\sigma(AE_\Delta|_{H_\Delta})) \to \{AE_\Delta|_{H_\Delta}\}''$ be the $\sigma$-normal *-homomorphism of theorem 4.1, such that $\tilde{\Phi}(1) = E_\Delta$ and $\tilde{\Phi}(id) = AE_\Delta|_{H_\Delta}$.

Note that $\{AE_\Delta|_{H_\Delta}\}'' \subseteq \mathcal{A}_0 E_\Delta$. To see this, we have; $AE_\Delta \in \mathcal{A}_0$ so $AE_\Delta|_{H_\Delta} \in \mathcal{A}_0 E_\Delta$ that is $\{AE_\Delta|_{H_\Delta}\} \subseteq \mathcal{A}_0 E_\Delta$ which implies $\{AE_\Delta|_{H_\Delta}\}' \supseteq (\mathcal{A}_0 E_\Delta)' = \mathcal{A}_0' E_\Delta$ (the last equality by lemma 8.4). Therefore $\{AE_\Delta|_{H_\Delta}\}'' \subseteq (\mathcal{A}_0' E_\Delta)' = E_\Delta\, \mathcal{A}_0'' E_\Delta = \mathcal{A}_0'' E_\Delta = \mathcal{A}_0 E_\Delta$.

By the above corollary we have that $\sigma(AE_\Delta|_{H_\Delta}) = \sigma(A|_{H_\Delta}) \subseteq \sigma(A)$.

Hence, if $f \in B(\sigma(A))$, then $f|_{\sigma(AE_\Delta|_{H_\Delta})} \in B(\sigma(AE_\Delta|_{H_\Delta}))$

Define $\omega: N(\mathcal{a}_0) \to N(\mathcal{a}_0 E_\Delta)$ by $\omega(B) = BE_\Delta|_{H_\Delta} = B|_{H_\Delta}$, where $B \in N(\mathcal{a}_0)$.

Note that $BE_\Delta$ is closed, since B is closed and $E_\Delta$ is bounded.

Moreover, $\omega$ maps $N(\mathcal{a}_0)$ into $N(\mathcal{a}_0 E_\Delta)$. To see this, let $Q \in (\mathcal{a}_0 E_\Delta)'$ and $x \in D(\omega(B)Q)$.

Then $Q = \tilde{Q}|_{H_\Delta}$ with $\tilde{Q} \in \mathcal{a}_0'$, $x \in H_\Delta$ and $Qx \in D(\omega(B)) = D(B) \cap H_\Delta$.

So $Qx = \tilde{Q}x$ (since $x \in H_\Delta$) and $\omega(B)Qx = BQx = B\tilde{Q}x$. Therefore $x \in D(B\tilde{Q})$.

But $B\tilde{Q} \subseteq \tilde{Q}B$. Hence $x \in D(\tilde{Q}B) = D(B)$.

Thus $x \in D(B) \cap H_\Delta = D(\omega(B)) = D(Q\omega(B))$ and $B\tilde{Q}x = \tilde{Q}Bx$.

Since $x \in H_\Delta$, this last equality says that $\omega(B)Qx = Q\omega(B)x$.

So $\omega(B)Q \subseteq Q\omega(B)$ or $\omega(B)\ \eta\ \mathcal{a}_0$.

It is easy to see that $\omega$ is an order preserving *-homomorphism.

We show that $\omega$ is a $\sigma$-normal homomorphism: let $B = \bigvee_{n=1}^{\infty} B_n$ in $S(\mathcal{a}_0)$ and suppose that $C \in N(\mathcal{a}_0 E_\Delta)$ and $\omega(B_n) \le C\ \forall n$. Note that $\omega(B_n) \le \omega(B)\ \forall n$.

**Claim:** $B_n \le CE_\Delta \hat{+} B(I - E_\Delta)$.

To see this, first note that in fact $CE_\Delta + B(I - E_\Delta)$ is a closed operator.

Indeed, let $x_n \in D(CE_\Delta) \cap D(B(I - E_\Delta))$ and suppose $x_n \to x$, $[CE_\Delta + B(I - E_\Delta)]x_n \to y$.

Then $E_\Delta x_n \to E_\Delta x$ and $E_\Delta[CE_\Delta + B(I - E_\Delta)]x_n = [E_\Delta CE_\Delta + E_\Delta B(I - E_\Delta D)]x_n = E_\Delta CE_\Delta x_n$ (since $B(\text{Range}(I - E_\Delta)) \subseteq \text{Range}(I - E_\Delta) = CE_\Delta x_n \to E_\Delta y$.

Since C is closed we get that $E_\Delta x \in D(C)$ and $CE_\Delta x = E_\Delta y$.

Applying the same argument with $I - E_\Delta$ in place of $E_\Delta$ we also get that

$(I - E_\Delta)x \in D(B)$ and $B(I - E_\Delta)x = (I - E_\Delta)y$.

Thus, $x \in D(CE_\Delta) \cap D(B(I - E_\Delta))$ and

$[CE_\Delta + B(I - E_\Delta)]x = E_\Delta y + (I - E_\Delta)y = y$, i.e., $CE_\Delta + B(I - E_\Delta)$ is a closed operator.

Hence our claim now is $B_n \le CE_\Delta + B(I - E_\Delta)\ \forall n$.

For this, let $x \in D(B_n)$ and $x \in D(CE_\Delta + B(I - E_\Delta))$.

Since $E_\Delta B_n \subseteq B_n E_\Delta$ and $(I - E_\Delta)B_n \subseteq B_n(I - E_\Delta)$, $E_\Delta x \in D(B_n)$ and $(I - E_\Delta)x \in D(B_n)$.

So

$$\begin{aligned}(B_n x, x) &= (B_n E_\Delta x, E_\Delta x) + (B_n(I - E_\Delta)x, (I - E_\Delta)x) \\ &\le (CE_\Delta x, E_\Delta x) + (B(I - E_\Delta)x, (I - E_\Delta)x) \\ &= ([CE_\Delta + B(I - E_\Delta)]x, x).\end{aligned}$$

Therefore

$$B_n \leq CE_\Delta + B(I - E_\Delta)$$

and the claim is proved.

Now

$$B \leq CE_\Delta + B(I - E_\Delta)$$

or

$BE_\Delta \leq CE_\Delta$, that is , $\omega(B) \leq C$ and $\omega$ is $\sigma$-normal.

Let $\Psi = \omega \circ \Phi: B_u(\sigma(A)) \to N(\mathcal{a}_0 E_\Delta)$ ; $\Psi$ is a $\sigma$-normal *-homomorphism such that $\Psi(1) = \omega(\Phi(1)) = \omega(I) = I|_{H_\Delta} = E_\Delta$ and $\Psi(id) = AE_\Delta|_{H_\Delta}$.

The $\sigma$-normal *-homomorphism $\Psi\,|_{B(\sigma(AE_\Delta|_{H_\Delta}))}$ has the same characteristic propoerties as $\tilde{\Phi}$, so by the uniqueness of such a $\sigma$-normal homomorphism we have

$$\tilde{\Phi}(f) = \Psi(f) = \omega(\Phi(f)) = \Phi(f)E_\Delta \quad \forall f \in B(\sigma(AE_\Delta|_{H_\Delta})).$$

In particular $\tilde{E}(S) = \tilde{\Phi}(\mathcal{X}_S) = \Phi(\mathcal{X}_S)E_\Delta = E(S)E_\Delta$.

ii) ($\Longrightarrow$): From the theorem 8.8 we have

$$x \in D(A) \quad \Longleftrightarrow \quad \int_{\mathbb{C}} |\lambda|^2 d\mu_x(\lambda) = \| Ax \|^2 < \infty$$

For $x \in D(A)$

$$\|AE_\Delta x - Ax\|^2 = ([\,|id|^2(A)(1 - \mathcal{X}_\Delta)(A)]x,x) = \int_{\mathbb{C}} |\lambda|^2 \mathcal{X}_{\Delta^c}(\lambda) d\mu_x(\lambda)$$

Let $\varepsilon > 0$ and $\Delta_n = \{ |\lambda| \leq n\}$ for $n \in \mathbb{N}$. By the monotone convergence theorem there exists $n_o$ such that $\int_{\mathbb{C}} |\lambda|^2 \mathcal{X}_{\Delta_{n_o}{}^c}(\lambda) d\mu_x(\lambda) < \varepsilon$ .

Then for $\Delta \supseteq \Delta_{n_o}$

$$\int_{\mathbb{C}} |\lambda|^2 \mathcal{X}_{\Delta^c}(\lambda) d\mu_x(\lambda) < \varepsilon$$

Therefore the net $\{AE_\Delta x\}_{\Delta \in \mathcal{D}}$ converges and $\lim AE_\Delta x = Ax$ .

($\Longleftarrow$): Since $D(AE_\Delta) = H$, for $x \in H$ we have $E_\Delta x \in D(A)$.

Let $\Delta_n = \{z \in \mathbb{C} : |z| \leq n\}$; $\mathcal{X}_{\Delta_n} \uparrow 1$ as $n \to \infty$.

Since $\Phi$ is $\sigma$-normal $\bigvee_{n=1}^{\infty} E_{\Delta_n} = I$ . So $E_{\Delta_n} \to I$ in the s.o.t.

Now, let $x \in H$ be such that $\{AE_\Delta x\}$ converges.

Then $E_\Delta x \to x$ and $\{AE_\Delta x\}$ converges .

Since A is closed, this gives that $x \in D(A)$ and $Ax = \lim AE_\Delta x$ and (ii) is proved.

Next, suppose $T \in B(H)$ and $TA \subseteq AT$. Note that $E_\Delta A E_\Delta = AE_\Delta$.
We have $(E_\Delta TE_\Delta)(AE_\Delta) = E_\Delta TAE_\Delta \subseteq E_\Delta ATE_\Delta \subseteq AE_\Delta TE_\Delta = (AE_\Delta)(E_\Delta TE_\Delta)$
But, $(E_\Delta TE_\Delta)(AE_\Delta) \in B(H)$. So that $(E_\Delta TE_\Delta)(AE_\Delta) = (AE_\Delta)(E_\Delta TE_\Delta)$
and the spectral theorem for bounded normal operators gives

$$(E_\Delta TE_\Delta)\tilde{E}(S) = \tilde{E}(S)(E_\Delta TE_\Delta) \quad .$$

If S is bounded, take n so large that $S \subseteq \Delta_n$, then

$$E(S) = E(S \cap \Delta_n) = E(\Delta_n)E(S) = E_{\Delta_n} E(S).$$

Therefore

$$E(S) = E_{\Delta_n} E(S) = E_{\Delta_n} E_{\Delta_n} E(S) = E_{\Delta_n} \tilde{E}(S)$$

Hence

$$E_{\Delta_n} TE(S) = E_{\Delta_n} TE_{\Delta_n} \tilde{E}(S) = \tilde{E}(S) E_{\Delta_n} TE_{\Delta_n} = E(S)TE_{\Delta_n}$$

Thus $\forall x \in H$ $E_{\Delta_n} TE(S)x = E(S)TE_{\Delta_n} x$ and letting $n \to \infty$ we get $TE(S) = E(S)T$.
If S is any Borel subset of $\sigma(A)$, writing S as a disjoint countable union of bounded Borel sets and using the bounded case just proved we obtain $TE(S) = E(S)T$.

Finally we prove the uniqueness of the spectral measure E.

Consider the operator $B = A(I + \sqrt{A^*A})^{-1}$ where $\sqrt{A^*A}$ is the unique positive square root of the positive operator $A^*A$.

Let $g(\lambda) = \dfrac{\lambda}{(1 + |\lambda|)}$. Then $g \in B(\mathbb{C})$, $\Phi(g) = g(A) = B \in \mathcal{A}_0$ and is normal.
By theorem 8.8 we have, for $x \in H$

$$(g(A)x, x) = (Bx,x) = \int_{\mathbb{C}} g(\lambda)\, d\mu_x(\lambda).$$

Let $E_B$ be the unique spectral measure corresponding to the bounded normal operator B.
By the spectral theorem 4.3 for $x \in H$, $f \in B(\sigma(B))$ we have

$$(Bx,x) = \int_{\sigma(B)} z\, dm_x(z) \quad \text{and} \quad (f(B)x,x) = \int_{\sigma(B)} f(z)\, dm_x(z)$$

where $m_x(S) = (E_B(S)x,x)$ $\forall$ Borel subset S of $\sigma(B)$.

Note that $\sigma(B) = \sigma(g(A)) \subseteq \overline{g(\sigma(A))}$ (proposition 8.10).
Now, $(f(B)x,x) = (f(g(A))x,x) = (f\circ g(A)x,x)$ the last equality by proposition 8.11.
At the same time we have

$$(f\circ g(A)x,x) = \int_{\mathbb{C}} (f\circ g)(\lambda)\, d\mu_x(\lambda).$$

Thus

$$\int_{\sigma(B)} f(z)\, dm_x(z) = \int_{\mathbb{C}} (f\circ g)(\lambda)\, d\mu_x(\lambda) \qquad \forall\, f \in B(\sigma(B)).$$

Now, since g is continuous $\sigma(B) = \sigma(g(A)) = \overline{g(\sigma(A))}$ ; to prove this all we need to show is $\overline{g(\sigma(A))} \subseteq \sigma(g(A))$.

Suppose that $\lambda \notin \sigma(g(A))$. Then there is $C \in \mathcal{A}_0$ such that $(g(A) - \lambda I)C = C(g(A) - \lambda I) = I$.
So

$$\| x \| \le \| C \| \, \| (g(A) - \lambda I)x \| \quad \forall\, x \in H \quad (*).$$

If $\lambda \in \overline{g(\sigma(A))}$ , choose $\varepsilon > 0$ so small that $\varepsilon \| C \| < 1$ and $\Delta_\varepsilon(\lambda) \cap g(\sigma(A)) \neq \emptyset$ where $\Delta_\varepsilon(\lambda) = \{w \in \mathbb{C} : | w - \lambda | < \varepsilon\}$.

Let $w_0 \in \Delta_\varepsilon(\lambda) \cap g(\sigma(A))$. Then $| g(z_0) - \lambda | < \varepsilon$ with $z_0 \in \sigma(A)$ such that $g(z_0) = w_0$.

Since g is continuous at $z_0$ there is a neighborhood $V \neq \emptyset$ of $z_0$ such that $V \subseteq g^{-1}(\Delta_\varepsilon(\lambda))$.

Note that since $\mathcal{X}_{V\cap\sigma(A)} \circ \phi(A)$ is not a null function (since $\phi^{-1}(V \cap \sigma(A))$ is a non-empty open subset , hence not meager) , $\Phi(\mathcal{X}_{V\cap\sigma(A)}) = E(V) \neq 0$

For $x \in$ Range $(E(V))$ = Range $(\Phi(\mathcal{X}_V))$ with $\| x \| = 1$ we have

$$\| (g(A) - \lambda I)x \|^2 = \int_{\mathbb{C}} | g(z) - \lambda |^2 \, d\mu_x(z) \quad \text{(theorem 8.6).}$$

Since $\mu_x(V^c) = (E(V^c)x,x) = ((I - E(V))x,x) = (0,x) = 0$

$$\int_{\mathbb{C}} | g(z) - \lambda |^2 \, d\mu_x(z) = \int_{V} | g(z) - \lambda |^2 \, d\mu_x(z) \le \varepsilon^2 \mu_x(V) = \varepsilon^2 1 = \varepsilon^2$$

So (*) gives $1 \le \| C \| \varepsilon$ which contradicts the choice of $\varepsilon$.

Thus $\lambda \notin \overline{g(\sigma(A))}$ and our last assertion is proved

Now, given a Borel subset S of $\mathbb{C}$, if S meets $\sigma(A)$, then g(S) meets $\overline{g(\sigma(A))} = \sigma(B)$ and since g is one-to-one we have

$$(E(S)x,x) = \mu_x(S) = \int_{\mathbb{C}} \mathcal{X}_S \, d\mu_x = \int_{\mathbb{C}} \mathcal{X}_{g^{-1}(g(S))} d\mu_x = \int_{\mathbb{C}} \mathcal{X}_{g(S)} \circ g \, d\mu_x$$

$$= \int_{\sigma(B)} \mathcal{X}_{g(S)} dm_x = = m_x(g(S)) = (E_B(g(S))x,x) \qquad \forall x \in H.$$

Therefore $E(S) = E_B(g(S))$ and the uniqueness of E follows from that of $E_B$ . ■

The next theorem is the Fuglede-Putnam theorem for unbounded normal operators. For another proof besides Fuglede's original one, see a proof given by Halmos [13].

**8.10 Theorem: (Fuglede)**
If $A \in \mathcal{O}p(H)$ is an unbounded normal operator and B is a bounded operator on H such that $BA \subseteq AB$, then $BA^* \subseteq A^*B$.

**Proof.**
Let $\Phi: B_u(\sigma(A)) \to N(\mathcal{A}_0)$ be the $\sigma$-normal *-homomorphism such that

$\Phi(1) = I$ and $\Phi(id) = A$

**Claim**: $B\Phi(f) \subseteq \Phi(f)B \quad \forall f \in B_u(\mathbb{C})$

First we will show that for a bounded Borel function f we have $B\Phi(f) \subseteq \Phi(f)B$.
Given $f \in B(\mathbb{C})$ there exists a sequence $\{f_n\}$ of simple functions such that $\| f_n - f \| \to 0$ ( $\|.\|$ denotes the sup-norm)
For $x,y \in H$ we have from theorem 8.8 that $(f(A)x,y) = \int_{\mathbb{C}} f(\lambda)\, d\mu_{x,y}(\lambda)$ and

$(f_n(A)x,y) = \int_{\mathbb{C}} f_n(\lambda)\, d\mu_{x,y}(\lambda)$ where $\mu_{x,y}(S) = (E(S)x,y)$ S a Borel subset of $\mathbb{C}$.

So

$$| (f_n(A)x,y) - (f(A)x,y) | \le \int_{\mathbb{C}} |f_n(\lambda) - f(\lambda)|\, d\mu_{x,y}(\lambda) \le \| f_n - f \| \, |\mu_{x,y}|(\mathbb{C})$$

$$= \|f_n - f\| \, \|x\| \, \|y\| \to 0 \qquad \text{as } n \to \infty.$$

Hence

$$(f(A)x,y) = \lim_{n\to\infty} (f_n(A)x,y)$$

Now

$$(Bf(A)x,y) = (f(A)x,B^*y) = \lim_{n\to\infty} (f_n(A)x,B^*y) =$$

$$= \lim_{n\to\infty} \int_{\mathbb{C}} f_n(\lambda)\, d\mu_{x,B^*y}(\lambda) =$$

$$= \lim_{n\to\infty} \int_{\mathbb{C}} f_n(\lambda)\, d\mu_{Bx,y}(\lambda)$$

since $BE(S) = E(S)B$ (from theorem 8.9).

But

$$\lim_{n\to\infty} \int_{\mathbb{C}} f_n(\lambda)\, d\mu_{Bx,y}(\lambda) = \lim_{n\to\infty} (f_n(A)Bx,y) = (f(A)Bx,y).$$

Thus

$$Bf(A) = f(A)B.$$

Now, let $f \in B_u(\mathbb{C})$ and $S_n = \{z \in \mathbb{C} : |f(z)| \le n\}$.

Note that $f\mathcal{X}_{S_n}$ is bounded and so $H = D(\Phi(f\mathcal{X}_{S_n})B) = D(\Phi(f)\Phi(\mathcal{X}_{S_n})B) = D(f(A)E(S_n)B)$ where $E(S_n) = \Phi(\mathcal{X}_{S_n})$, $f(A) = \Phi(f)$.

Moreover $\mathcal{X}_{S_n} \uparrow 1$ and since $\Phi$ is $\sigma$-normal we get that $E(S_n) \to I$ in the s.o.t.

For $x \in D(\Phi(f))$ $E(S_n)f(A)x \to f(A)x$; this implies $BE(S_n)f(A)x \to Bf(A)x$ since B is bounded or $B\Phi(f\mathcal{X}_{S_n})x \to B\Phi(f)x$.

The above discussion for a bounded function gives $B\Phi(f\mathcal{X}_{S_n}) = \Phi(f\mathcal{X}_{S_n})B$.

So

$$\Phi(f\mathcal{X}_{S_n})Bx \to B\Phi(f)x \quad \text{or} \quad \Phi(f)E(S_n)Bx \to B\Phi(f)x.$$

Since $E(S_n)Bx \to Bx$ and $\Phi(f)$ is a closed operator we conclude $Bx \in D(\Phi(f))$ and $\Phi(f)Bx = B\Phi(f)x$.

Thus,

$$B\Phi(f) \subseteq \Phi(f)B.$$

In particular, then for $f(\lambda) = \overline{\lambda}$ we get $BA^* \subseteq A^*B$ ■

**8.2 Corollary**: If $A \in Op(H)$ is a normal operator and $\mathcal{A}_0$ the von Neumann algebra generated by A , then $\mathcal{A}_0 = \{A\}''$

**Proof.**

If $B \in \{A\}'$, then Fuglede's theorem gives that $BA^* \subseteq A^*B$ , so $B^*A \subseteq AB^*$ , i.e., $\{A\}'$ is *-subalgebra of B(H). Moreover, by remark 8.1 $\{A\}'$ is closed in the s.o.t. and is therefore a von Neumann algebra .

Since any bounded operator B that commutes with A (, i.e, $B \in \{A\}'$), commutes with the operators $F_n$ , $AF_n$, $A^*F_n$ which generate $\mathcal{A}_0$ (see p. 67 ) , we have that

$$\{A\}' \subseteq \mathcal{A}_0' \quad \text{or} \quad \mathcal{A}_0 \subseteq \{A\}''.$$

On the other hand, $\mathcal{A}_0' \subseteq \{A\}'$ since $A\eta\,\mathcal{A}_0$ . Thus $\{A\}'' = \mathcal{A}_0$ . ■

Let $\mathcal{A}$ be an abelian von Neumann algebra acting on a Hilbert space H.

Kadison and Ringrose give an example ([14] p.356) to demonstrate that for A, B $\in N(\mathcal{A})$ $A + B \neq A \hat{+} B$ and $TA \neq T \hat{\cdot} A$, even if T is a bounded operator. At the same time this example provides a nice illustration of the isomorphism of $N(\mathcal{A})$ onto N(X) extending the Gelfand map and the way we work with it. We would like to consider a closely related example here.

**8.1 EXAMPLE.**

Let H be a separable Hilbert space, $\{e_n\}_{n=1,2...}$ an orthonormal basis for H and $\mathcal{A}$ the algebra of operators in B(H) having each $e_n$ as an eigenvector.

For $S \in \mathcal{A}$, let $s_{ij} = (Se_j,e_i) = (\lambda_j e_j,e_i) = \begin{cases} \lambda_j & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$.

Since $S \in B(H)$, $\sup_j |\lambda_j| \leq \|S\|$.

On the other hand, $\forall x \in H$ we have $x = \sum_{j=1}^{\infty}(x,e_j)e_j$ and

$$\|Sx\| \leq \sum_{j=1}^{\infty}|(x,e_j)|^2 |\lambda_j|^2 \leq \sup_j |\lambda_j|^2 \sum_{j=1}^{\infty}|(x,e_j)|^2 = (\sup_j |\lambda_j|)^2 \|x\|^2.$$

Hence

$$\|S\| = \sup_j |\lambda_j|.$$

Thus $\{\lambda_j\}_{n=1}^{\infty} \in \ell^{\infty}$ and $[s_{ij}]$ is a bounded diagonal matrix, corresponding to S (relative to the orthonormal basis $\{e_n\}_{n=1,2...}$)

Clearly the mapping $S \mapsto \{\lambda_j\}$ is an algebra homomorphism and so $\mathcal{A}$ is isometrically *-isomorphic to $\ell^{\infty}$.

In this case $\mathcal{A} \cong C(X)$ and X is $\beta(\mathbb{N})$, the $\beta$-compactification of $\mathbb{N}$.

Let $\rho_n \in X$ be given by $\rho_n(T) = (Te_n,e_n)$, $T \in \mathcal{A}$.

Note that, if $f \in C(X)$ is such that $f(\rho_n) = 0$ $\forall n$, then $f = \hat{T}$ for some $T \in \mathcal{A}$ and $\rho_n(T) = \hat{T}(\rho_n) = 0$ $\forall n$, that is, $0 = (Te_n,e_n) = (\lambda_n e_n,e_n) = \lambda_n$ $\forall n$.

So T = 0 and so f = 0.

Now the set $\{\rho_n : n = 1, 2,...\}$ is a dense subset of X. To see this; suppose $\overline{\{\rho_n : n \in \mathbb{N}.\}} \neq X$.

Let $q \in X \sim \overline{\{\rho_n : n \in \mathbb{N}.\}}$. Since X is a compact Hausdorff space, X is completely regular.

So there exists $f \in C(X)$, $0 \leq f \leq 1$ such that $f(q) = 1$ and $f \equiv 0$ on $\overline{\{\rho_n : n \in \mathbb{N}.\}}$

In particular $f(\rho_n) = 0$ $\forall n$, hence by the above paragraph f = 0 on X.

But f(q) = 1, a contradiction.

Thus, $\{\rho_n : n = 1,2,...\}$ is dense in X.

If $f \in C(X)$ is such that $f(\rho_1) = 1$ and $f(\rho_n) = 0$ for n = 2, 3, ... , then $f = \chi_{\{\rho_1\}}$.

So $\{\rho_1\}$ is open in X, as is each one-point set formed from a $\rho_n$.
We denote by $P_n$ the projection whose range is generated by $e_n$.
Note that $P_n \in \mathcal{A}$ and $\hat{P}_n = \chi_{\{\rho_n\}}$.

Let $x \in H$. Then $x = \sum_{n=1}^{\infty}(x,e_n)e_n = \sum_{n=1}^{\infty} x_n e_n$ and by Parseval's identity the mapping $U: H \to \ell^2$ given by $Ux = \{(x,e_n)\} = \{x_n\}_{n=1}^{\infty}$ is an isometric isomorphism(onto).
Let $\Psi: \mathcal{A} \to B(\ell^2)$ be the mapping $\Psi(T) = UTU^{-1}$ , $T \in \mathcal{A}$.
Then $\Psi(\mathcal{A}) = \{M_g \in B(\ell^2) : g \in \ell^\infty\}$ (the multiplication algebra on $\ell^2$ ).
In fact, if $\{x_n\}_{n=1}^{\infty} \in \ell^2$, $T \in \mathcal{A}$, then there is $x \in H$ such that $\{(x,e_n)\} = \{x_n\}$ and

$$\Psi(T)(\{x_n\}) = UTU^{-1}(\{x_n\}) = UTx = \{\lambda_n x_n\} = M_{\{\lambda_n\}}(\{x_n\})$$

where $\lambda_n$ are the eigenvalues for T ($Te_n = \lambda_n e_n$).

Now, let $f \in N(X)$ be the normal function defined by $f(\rho_n) = b_n$ $\forall n$, where $\{b_n\}_{n=1}^{\infty}$ is any sequence of complex numbers.
Let $\phi: N(\mathcal{A}) \to N(X)$ be the the extension of the Gelfand map and $A \in N(\mathcal{A})$ be such that $\phi(A) = f$.

**Claim:** $\Psi(A) = UAU^{-1} = M_{\{b_n\}}$

Let $\nu_x$ be the regular Borel measure on X such that $(Tx,x) = \int_X \hat{T}(\rho)d\nu_x(\rho)$ , $T \in \mathcal{A}$, $x \in H$.
Then

$$\nu_x(\{\rho_n\}) = \int_X \chi_{\{\rho_n\}}(\rho)d\nu_x(\rho) = \int_X \hat{P}_n(\rho)d\nu_x(\rho) =$$

$$= (P_n x,x) = (x_n e_n, \sum_{k=1}^{\infty} x_k e_k) = |x_n|^2 = (|(x,e_n)|^2).$$

In particular

$$\nu_{e_m}(\{\rho_n\}) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Note that

$$\nu_x(\bigcup_{n=1}^{\infty}\{\rho_n\}) = \sum_{j=1}^{\infty}\nu_x(\{\rho_n\}) = \sum_{j=1}^{\infty}|(x,e_n)|^2 = \|x\|^2 = \nu_x(X).$$

So that $\nu_x(X \sim \{\rho_n : n = 1, 2, \ldots\}) = 0$
Note also that $e_m \in D(A)$ $\forall m = 1, 2, \ldots$

Now, let $x \in D(A)$. Proposition 8.13 gives

$$\infty > \int_X |f(\rho)|^2 dv_x(\rho) = \int_{\bigcup_{n=1}^{\infty}\{\rho_n\}} |f(\rho)|^2 dv_x(\rho) + \int_{X\sim\{\rho_n:\, n\in\mathbb{N}\}} |f(\rho)|^2 dv_x(\rho)$$

$$= \sum_{j=1}^{\infty} |f(\rho_n)|^2 v_x(\{\rho_n\}) = \sum_{j=1}^{\infty} |b_n|^2 |x_n|^2 = \sum_{j=1}^{\infty} |b_n x_n|^2.$$

Hence

$\{b_n x_n\}_{n=1}^{\infty} \in \ell^2$, which implies that $Ux \in D(M_{\{b_n\}})$ . So $U(D(A) \subseteq D(M_{\{b_n\}})$.

On the other hand , if $\{a_n\}_{n=1}^{\infty} \in D(M_{\{b_n\}})$ and $y \in H$ is such that $(y,e_n) = a_n \ \forall n$, then $\{b_n a_n\}_{n=1}^{\infty} \in \ell^2$ and

$$\int_X |f(\rho)|^2 dv_y(\rho) = \sum_{j=1}^{\infty} |f(\rho_n)|^2 v_y(\{\rho_n\}) = \sum_{j=1}^{\infty} |b_n a_n|^2 < \infty \text{ , i.e., } y = U^{-1}(\{a_n\}) \in D(A).$$

Thus

$$U(D(A)) = D(M_{\{b_n\}}).$$

From the Remark 8.3 we have

$$(Ax,y) = \int_X f(\rho) d\,\sigma_{x,y}(\rho) \ \forall x,y \in D(A), \text{ where } \sigma_{x,y} = \frac{1}{4}\{v_{x+y} - v_{x-y} + iv_{x+iy} - iv_{x-iy}\}.$$

Note that

$$\sigma_{x,y}(\{\rho_n\}) = \int_{\{\rho_n\}} d\sigma_{x,y}(\rho) = \int_X \hat{P}_n(\rho) d\sigma_{x,y}(\rho) = (P_n x,y) = (x_n e_n, \sum_{k=1}^{\infty} y_k e_k) = x_n \overline{y_n}$$

In particular

$$\sigma_{x,e_m}(\{\rho_n\}) = \begin{cases} x_m & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and

$$\sigma_{x,y}(\bigcup_{n=1}^{\infty}\{\rho_n\}) = \sum_{j=1}^{\infty} \sigma_{x,y}(\{\rho_n\}) = \sum_{j=1}^{\infty} x_n \overline{y_n} = (x,y) = \sigma_{x,y}(X)$$

So that

$$\sigma_{x,y}(X \sim \{\rho_n : n=1, 2, \ldots\}) = 0$$

Now, for $x \in D(A)$ we have $UAx = \{(Ax,e_n)\}_{n=1}^{\infty}$ and

$$(Ax,e_m) = \int_X f(\rho) d\,\sigma_{x,e_m}(\rho) = \sum_{j=1}^{\infty} f(\rho_n) \sigma_{x,e_m}(\{\rho_n\}) = b_m x_m$$

$\forall m = 1, 2, \ldots$ as we claimed.

In the second example we study the (unbounded) multiplication operator acting on $L^2(S, \mathcal{S}, \mu)$ with $(S, \mathcal{S}, \mu)$ a $\sigma$-finite measure space .

**8.2 EXAMPLE.**

Let $(S, \mathcal{S}, \mu)$ be a $\sigma$-finite measure space , $H = L^2(S)$ and g a (complex) measurable function finite a.e $[\mu]$ on S.

Let $M_g$ be the operator defined as follows :

$$D(M_g) = \{f \in L^2(S): gf \in L^2(S)\} \text{ and for } f \in D(M_g) \quad M_g(f) = gf .$$

$D(M_g)$ is dense in $L^2(S)$. To see this; let $S_n = \{s \in S: \ |g(s)| \le n\}$ and $f \in L^2(S) = H$.

$$\| g\chi_{S_n} f \|^2 = \int_S |g\chi_{S_n} f|^2 d\mu \le n^2 \int_S |f|^2 d\mu < \infty \text{ so that } \chi_{S_n} f \in D(M_g).$$

Now, $|\chi_{S_n} f - f|^2 \to 0$ a.e$[\mu]$ as $n \to \infty$ and $|\chi_{S_n} f - f|^2 \le |f|^2 \in L^1(S)$.

Lebesgue`s dominated convergence theorem gives

$$\lim_{n\to\infty} \|\chi_{S_n} f - f\|^2 = \lim_{n\to\infty} \int_S |\chi_{S_n} f - f|^2 d\mu = 0$$

Thus $M_g$ is a densely defined operator on H.

Moreover, $M_g$ is a closed operator.

For this, let $f_n \in D(M_g)$ be such that $f_n \to f$ in $L^2(S)$.

Suppose $M_g(f_n) = gf_n \to h$ in H , $h \in H$.

Since $f_n \to f$ in $L^2(S)$ and $gf_n \to h$ in $L^2(S)$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$f_{n_k} \to f \quad \text{a.e}[\mu] \text{ and } gf_{n_k} \to h \quad \text{a.e}[\mu].$$

But , then

$$gf_{n_k} \to gf \quad \text{a.e}[\mu] \text{ and so } gf = h.$$

Hence, $f \in D(M_g)$ and $M_g f = h$, that is, $M_g$ is closed.

Next we show that $(M_g)^* = M_{\bar{g}}$ : If $f \in D(M_g)$ and $h \in D(M_{\bar{g}})$, then

$$(M_g f, h) = \int_S gf\,\bar{h} d\mu = (f, \bar{g}h) = (f, M_{\bar{g}} h)$$

So $h \in D((M_g)^*)$ and $(M_g)^* h = M_{\bar{g}} h$ , i.e., $M_{\bar{g}} \subseteq (Mg)^*$ .

Suppose now that $h \in D((M_g)^*)$. Then there is $c \ge 0$ such that

$$|(M_g f,h)| \le c\|f\| \quad \forall f \in D(M_g)$$

or

$$\left|\int_S g f \bar{h} d\mu\right| \le c\|f\| \quad \forall f \in D(M_g)$$

Let $S_n = \{s \in S: |g(s)| \le n\}$. For each $f \in L^2(S)$, $\chi_{S_n} \in D(M_g)$ and $\|\chi_{S_n} f\| \le \|f\|$.
Furthermore,

$$\|\chi_{S_n} \bar{g} h\| = \sup_{\|f\|=1} |(f, \chi_{S_n} \bar{g} h)| = \sup_{\|f\|=1} \left|\int_S g \chi_{S_n} f \bar{h} d\mu\right| \le c.$$

So

$$\int_{S_n} |\bar{g} h|^2 d\mu = \|\chi_{S_n} \bar{g} h\|^2 \le c^2 \quad \forall n.$$

Hence

$$\varliminf_{n\to\infty} \int_{S_n} |\bar{g} h|^2 d\mu \le c^2.$$

Now, $\chi_{S_n} |\bar{g} h|^2 \to |\bar{g} h|^2$ as $n \to \infty$, so by Fatou's lemma, we have

$$\int_S |\bar{g} h|^2 d\mu = \int_S \varliminf_{n\to\infty} (\chi_{S_n} |\bar{g} h|^2) d\mu \le \varliminf_{n\to\infty} \int_S \chi_{S_n} |\bar{g} h|^2 d\mu = \varliminf_{n\to\infty} \int_{S_n} |\bar{g} h|^2 d\mu \le c^2.$$

Thus $\bar{g} h \in L^2(S)$ and so $h \in D(M_{\bar{g}})$.
Hence

$$(M_g)^* = M_{\bar{g}}.$$

Our next objective is to find a bounding sequence for the closed operator $M_g$.
Let $E_n = M_{\chi_{S_n}}$ where $S_n$ is as above. Clearly $\|E_n\| \le 1$, $E_n^* = E_n$ and $E_n^2 = E_n$ $\forall n$.
Since $\chi_{S_n} \le \chi_{S_{n+1}}$ and the mapping $g \mapsto M_g$ from $L^\infty(S)$ into $B(L^2(S))$ is order preserving we get that $E_n \le E_{n+1}$.
Now, since $\mu(\{s \in S: |g(s)| = \infty\}) = 0$, $\chi_{S_n} \to 1$ a.e$[\mu]$ and an application of Lebesgue's dominated convergence theorem gives that $E_n \to I$ in the s.o.t.

Hence $\{E_n\}$ is an increasing sequence of projections and $\bigvee_{n=1}^{\infty} E_n = I$
Furthermore, $\|M_g E_n\| \le n$ and for $f \in D(M_g)$ $E_n f = \chi_{S_n} f \in D(M_g)$ (since $gf \in L^2(S)$) and

$$E_n M_g f = M_g E_n f \quad \text{, i.e.,} \quad E_n M_g \subseteq M_g E_n \quad \text{and} \quad M_g E_n \in B(H) \quad \forall n.$$

Thus, $\{E_n\}$ is a bounding sequence for $M_g$.
Note that $E_n \in \mathcal{A}$ $\forall n$, where $\mathcal{A}$ is the multiplication algebra of operators on $L^2(S)$.

The following theorem characterizes the(closed) operators which are affiliated with the multiplication algebra $\mathcal{A}$. (see [14] p. 343)

**8.2.1 Theorem**: Let $\mathcal{a}$ be the multiplication algebra of operators acting on $H = L^2(S)$ and A be a closed densely defined operator on H. Then A is affiliated with $\mathcal{a}$ iff $A = M_g$ for some measurable function g finite almost everywhere on S.

**8.2.1 Proposition**: With $(S, \mathcal{S}, \mu)$ and a as above , let f, g be measurable functions finite a.e[$\mu$] on S.
Then
i) $M_f = M_g \iff f = g$ a.e[$\mu$]
ii) $M_{af+g} = aM_f \hat{+} M_g$ for each scalar a
iii) $M_{fg} = M_f \hat{\cdot} M_g$
iv) $M_f \geq 0 \iff f \geq 0$ a.e[$\mu$]

**Proof.**
i): Suppose $M_f = M_g$ . If $h \in D(M_f) = D(M_g)$ , then $(f - g)h = 0$.

Let $G_n = \{s \in S: |f(s)| \leq n \text{ and } |g(s)| \leq n\} \bigcap S_n$ where $S_n$ are such that $S = \bigcup_{n=1}^{\infty} S_n$ with $\mu(S_n) < \infty$ (by the $\sigma$-finiteness of the measure $\mu$)

Note that $\mathcal{X}_{G_n} f \in L^2(S)$ . So $\mathcal{X}_{G_n} \in D(M_f)$ and so $(f - g)\mathcal{X}_{G_n} = 0$ .

This implies that $f - g = 0$ a.e [$\mu$] since $\mu(\bigcap_{n=1}^{\infty} \{s \in S: \mathcal{X}_{G_n}(s) = 0\}) = 0$ .
The converse is obvious.
ii): First we will show $aM_f \hat{+} M_g \subseteq M_{af+g}$ .
If $h \in D(aM_f + M_g) = D(aM_f) \bigcap D(M_g)$, then $(af + g)h \in L^2(S)$ (so $h \in D(M_{af+g})$)
and $(aM_f + M_g)(h) = M_{af+g}(h)$.
Hence $aM_f + M_g \subseteq M_{af+g}$. Since $M_{af+g}$ is closed we conclude that $aM_f \hat{+} M_g \subseteq M_{af+g}$
To get equality , let $h \in D(M_{af+g})$ and $G_n = \{ |f| + |g| \leq n\} \bigcap S_n$ .
If $h_n = \mathcal{X}_{G_n} h$, then $afh_n \in L^2(S)$ and $gh_n \in L^2(S)$.
So $h_n \in D(aM_f) \bigcap D(M_g) = D(aM_f + M_g)$
Now, $\mathcal{X}_{G_n} \to 1$ a.e[$\mu$] as $n \to \infty$ (since $\mu(\{|f| + |g| = \infty\}) = 0$).
Therefore $|h_n - h|^2 \to 0$ a.e[$\mu$] and Lebesgue`s dominated convergence theorem gives that $h_n \to h$ in $L^2(S)$ as $n \to \infty$.
Also $|(af + g)(h_n - h)|^2 \to 0$ a.e[$\mu$] and again Lebesgue`s theorem gives that

$$\lim_{n \to \infty} (aM_f + M_g)(h_n) = M_{af+g}(h) \text{ , that is, } h \in D(aM_f \hat{+} M_g) \text{ and}$$

$$(aM_f \hat{+} M_g)(h) = \lim_{n \to \infty} (aM_f + M_g)(h_n) = M_{af+g}(h)$$

iii) As in the first part of the proof of (ii) we can see that $M_f \hat{\cdot} M_g \subseteq M_{fg}$ .
To get equality , let $h \in D(M_{fg})$ and $G_n = \{ |f| + |g| + |fg| \} \cap S_n \ \forall n$ .
If $h_n = \chi_{G_n} h$ ,then $gh_n \in L^2(S)$ and $fgh_n \in L^2(S)$, so that $h_n \in D(M_f M_g)$.
A similar argument as in part (ii) completes the proof of (iii).
iv): Suppose that $f \geq 0$ a.e[$\mu$]. If $h \in D(M_f)$, then

$$(M_f h,h) = \int_S f|h|^2 d\mu \geq 0.$$

So $M_f \geq 0$.
Conversely, suppose $M_f \geq 0$ Clearly f is real-valued.
Let $G_n = \{-n \leq f < 0\} \cap S_n = \{ |f| \leq n\} \cap \{ f < 0\} \cap S_n$.
If $h = \chi_{G_n}$ , then

$$0 \leq (M_f h,h)) = \int_{G_n} f \, d\mu \leq 0 \qquad \text{since } f < 0 \text{ on } G_n .$$

Therefore

$$\int_{G_n} f \, d\mu = 0 .$$

But $f \neq 0$ , so $\mu(G_n) = 0$. Hence $f \geq 0$ a.e[$\mu$] . ■

We remark that our last assertion can be proved in a different way using the square root of a positive operator. In fact , since $M_f \geq 0$ there exists an (unique) operator $B \geq 0$ such that $B^2 = M_f$ ($B \eta \mathcal{A}$).
By the previous theorem (8.2.1) , since $B \eta \mathcal{A}$, $B = M_h$ for some measurable function h finite a.e[$\mu$] on S .(B being self adjoint , h is real-valued)
Now,
$M_f = B^2 = B \hat{\cdot} B = M_h \hat{\cdot} M_h = M_{h^2}$ (by (iii) ). Hence $f = h^2 \geq 0$ a.e[$\mu$] (by (i) ) .

From our previous discussion we see that the operator $M_g$ where g is a measurable function finite a.e[$\mu$] on S, is a normal operator affiliated with the multiplication algebra $\mathcal{A}$ acting on $L^2(S)$.
In the next theorem we describe the *Borel functional calculus* for $M_g$ .

First, we define the essential Range of g by

$\text{essRange}(g) = \{ \lambda \in \mathbb{C} : \forall \varepsilon > 0 \ \mu(g^{-1}(D_\varepsilon(\lambda))) > 0\}$ where $D_\varepsilon(\lambda) = \{ z \in \mathbb{C}: |\lambda - z| < \varepsilon\}$

Now, we show that $\sigma(M_g) = \text{essRange}(g)$:
If $\lambda \notin \text{essRange}(g)$ , then there is $n \in \mathbb{N}$ such that $\mu(g^{-1}(D_{\frac{1}{n}}(\lambda))) = 0$.
This implies $|g(s) - \lambda| \geq \frac{1}{n}$ a.e[$\mu$] , which implies that $\frac{1}{g - \lambda} \in L^\infty(S)$.

So $M_{\frac{1}{g-\lambda}} \in \mathcal{A}$ . Now $(M_g - \lambda I)M_{\frac{1}{g-\lambda}} = M_{g-\lambda}M_{\frac{1}{g-\lambda}} = M_1 = I$ , that is, $\lambda \notin \sigma(M_g)$.
To complete our proof, suppose that $\lambda \notin \sigma(M_g)$. Then there is $B \in \mathcal{A}$ such that $B \hat{\cdot} (M_g - \lambda I) = (M_g - \lambda I) \hat{\cdot} B = (M_g - \lambda I)B = I$ . Therefore $B(M_g - \lambda I) \subseteq I$.
So $\forall f \in D(M_{g-\lambda})$ we have $f = B(M_g - \lambda I)f$ and so $\| f \| \leq \|B\| \, \|(M_g - \lambda I)f\|$.
If $\lambda \in$ essRange(g) , choose $\varepsilon > 0$ so small that $\varepsilon\|B\| < 1$. Let $E_\varepsilon = g^{-1}(D_\varepsilon(\lambda))$; $\mu(E_\varepsilon) > 0$.
Since the measure space $(S, \mathcal{S}, \mu)$ is $\sigma$-finite , choose $Y \subseteq E_\varepsilon$ with $0 < \mu(Y) < \infty$ .
Take $f = \mathcal{X}_Y$ . Then $f \in D(M_{g-\lambda})$ and

$$\| f \| = \sqrt{\mu(Y)} \leq \|B\| \left( \int_Y |g - \lambda|^2 d\mu \right)^{\frac{1}{2}} \leq \varepsilon \|B\| \sqrt{\mu(Y)}$$

which says $1 \leq \varepsilon \|B\|$ contradicting the choice of $\varepsilon$.
Thus $\lambda \notin$ essRange(g) and our last assertion is proved.

It can be seen as in example 4.1, that $\mu(g^{-1}(\mathbb{C}\sim\text{essRange}(g))) = 0$ ($\mathbb{C}\sim$essRange(g) the complement of essRange(g) in $\mathbb{C}$ ) and if

$$g_0 = \begin{cases} g(s) & \text{when } s \in g^{-1}(\text{essRange}(g)) \\ \lambda & \text{when } s \in g^{-1}(\mathbb{C}\sim\text{essRange}(g)) \end{cases}$$

where $\lambda$ is some point in essRange(g), then Range $(g_0) \subseteq$ essRange(g) $= \sigma(M_g)$
Thus , we may assume that Range(g) $\subseteq \sigma(M_g)$.

**8.2.2 Theorem**: With the above notation, let f be a Borel function on $\sigma(M_g)$. Then

$$f(M_g) = M_{fog} \ .$$

**Proof.**
Consider the mapping $\Psi : B_u(\sigma(M_g)) \rightarrow N(\mathcal{A})$ given by $\Psi(f) = M_{fog}$ .
Then $\Psi(1) = I$ and $\Psi(id) = M_g$.
Also

$$\Psi(f_1 + f_2) = M_{(f_1+f_2)og} = M_{f_1og} \hat{+} M_{f_2og} = \Psi(f_1) \hat{+} \Psi(f_2)$$

$$\Psi(f_1f_2) = M_{(f_1f_2)og} = M_{f_1og} \hat{\cdot} M_{f_2og} = \Psi(f_1) \hat{\cdot} \Psi(f_2)$$

$$\Psi(\bar{f}) = M_{\bar{f}og} = M_{\overline{fog}} = (M_{fog})^* = (\Psi(f))^*.$$

So that $\Psi$ is a *-homomorphism from $B_u(\sigma(M_g))$ into $N(\mathcal{A})$

Moreover $\Psi$ is $\sigma$-normal. To see this, let $f, f_n \in B_u(\sigma(M_g))$ with $f = \bigvee_{n=1}^{\infty} f_n$ .

We show $\Psi(f) = \bigvee_{n=1}^{\infty} \Psi(f_n)$:

$\Psi(f) \hat{+} -\Psi(f_n) = M_{fog} \hat{+} (-1)M_{f_n og} = M_{fog - f_n og} = M_{(f - f_n)og}$.

Since $(f - f_n)og \geq 0$ a.e[$\mu$] the previous proposition (vi) gives that $M_{(f - f_n)og} \geq 0$, i.e.,

$$\Psi(f_n) \leq \Psi(f) \quad \forall n.$$

Suppose now that $B \eta\, \mathcal{A}$ is such that $B \hat{+} -\Psi(f_n) \geq 0$ .

Then $B = M_k$ where k is a measurable function finite a.e[$\mu$] on S and

$$B \hat{+} -\Psi(f_n) = M_k \hat{+} M_{-f_n og} = M_{k - f_n og} \geq 0.$$

So $k - f_n og \geq 0$ a.e[$\mu$] or $k \geq f_n og$ a.e[$\mu$] $\forall n$ .

This last inequality implies $k \geq fog$ a.e[$\mu$].

Hence $M_{k - fog} \geq 0$, which says that $B \geq M_{fog} = \Psi(f)$.

Thus $\Psi$ is a $\sigma$-normal *-homomorphism such that $\Psi(1) = I$ and $\Psi(id) = M_g$ and the uniqueness of the Borel functional calculus gives that $\Phi(f) = f(M_g) = M_{fog} = \Psi(f)$ . ■

**8.4 Remark**: Let $A \in$ *Op*(H) be a normal operator on a separable Hilbert space H . There is an abelian von Neuman algebra $\mathcal{A}$ acting on H such that $A \eta\, \mathcal{A}$ . Without loss of generality we may take $\mathcal{A}$ to be maximal abelian.

Theorem 5.1 ; then provides a unitary isomorphism U from H onto $L^2(X, B_X, \mu)$ (where $X = X_{\mathcal{A}}$, $\mu$ a finite positive regular Borel measure,) such that

$U\mathcal{A}U^{-1} = \{M_g : g \in L^{\infty}(X, B_X, \mu)\} = \mathcal{M}$ the multiplication algebra acting on $L^2(X)$

So

$$UAU^{-1} \eta\, \mathcal{M}$$

Theorem 8.2.1 gives that

$$UAU^{-1} = M_g \quad \text{or} \quad A = U^{-1} M_g U$$

with g a Borel measurable function finite a.e [$\mu$] on X

We just proved in in the discussion before theorem 8.2.2 that g may be chosen so that Range(g) $\subseteq \sigma(A)$ .

Now, if $f \in B_u(\sigma(A))$ , then

$$\Phi(f) = f(A) = U^{-1}f(M_g)U = U^{-1}M_{fog}U \ . \ ■$$

❐

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