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UNRAMIFIED QUADRATIC EXTENSIONS OF PURE CUBIC FIELDS

by

JEW-CHEN (JOHN) HWANG

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

UNRAMIFIED QUADRATIC EXTENSIONS OF PURE CUBIC FIELDS

By

JEW-CHEN (JOHN) HWANG

Advisor: Professor Harvey Cohn

(1) The Galois groups and their subgroups are constructed by the normal extension K_{24} (K_{96}^{*}) of unramified quadratic (biquadratic) extension of pure cubic fields $Q(\sqrt[3]{m})$ and subfields of K_{24} (K_{96}^{*}), respectively.

(2) The prime factorization in the unramified quadratic extensions of pure cubic fields $Q(\sqrt[3]{m})$ over $Q(\sqrt[3]{m})$ is presented to solve the norm equations.

(3) Even class number of $Q(\sqrt[3]{m})$ is investigated by use of congruence and quadratic reciprocity.

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CHAPTER I. INTRODUCTION

§ 1. Historial Remarks

Algebraic Number Theory may be considered as the special branch of number theory where the mysterious factorization of rational primes takes the form of questions about prime ideals. Many major theorems and results deal with the splitting primes, different, discriminant, algebraic integers as well as class number in the higher (extension) fields, viz Hilbert class fields. In particular, for the pure cubic fields $Q(\sqrt[3]{m})$ Dedekind and Cassels investigated many of theorems about prime factorization and class numbers in $Q(\sqrt[3]{m})$ by use of the rational solutions of the diophantine equation

$$Y^2 = X^3 - D.$$

Later, in 1955 and 1967, Selmer completed the table of class number and prime factorization of pure cubic fields, for $m \leq 250$. In 1975, Frey developed the cohomology and some homological results in the quadratic extension of pure cubic fields.

§ 2. This Dissertation

In this dissertation, I do not attempt to add to the "greater structure" of algebraic number fields. I would rather try to investigate many of results and examples of number fields, in particular, the pure cubic field $Q(\sqrt[3]{m})$ and its class fields $Q(\sqrt[3]{m}, \sqrt{\mu})$ and $Q(\sqrt[3]{m}, \sqrt{-3\mu})$. I study this for three reasons: (a) to obtain a better

understanding of these results and thus the theory as a whole, (b) to find the relationships between the groups of finite order and the number theory, and (c) to derive some results which though specialized are of much more interest in themselves.

The main emphasis in this research is the construction of the quadratic extensions of pure cubic fields for class number 2, then 4 (non-cyclic), then even. To start with, we construct the Galois groups, and find their corresponding subfields of the normal field. Later we apply the formulae of Selmer and Cassels for primes which are completely split in the unramified quadratic extension of pure cubic fields if $N(a_i + b_i\theta + c_i\theta^2) = p$ (prime and $\equiv 1 \pmod 6$) for $i = 1, 2, 3$. is solvable in Z. Furthermore, we investigate the rest of the primes which can be partially split in the unramified quadratic extension of pure cubic fields. By use of congruence and quadratic reciprocity, we also find the necessary condition for the class number of pure cubic fields to be even. Along the way to these results, we give a few examples of primes and fields to demonstrate the existence and facts. By the end of each chapter, we give a couple of diagrams or tables as well.

The reader is to presume to have a knowledge of Algebraic Number Theory and Groups of Finite Order, and especially some knowledge of pure cubic fields.

CHAPTER II. THE GALOIS GROUPS AND THEIR CORRESPONDING SUBFIELDS FOR $h = 2$

§ 1. The subgroups of the Galois group $G(K_{24}/Q)$

Let m be a positive cube-free rational integer, and let h be the class number of the pure cubic fields $Q(\sqrt[3]{m})$. When we study a number field, especially, the class number, algebraic integers, norms, as well as the class fields, they are considered as tools to the special results of many branches of mathematics.

Now let $h = 2$ and let μ be a non-square algebraic integer of $Q(\theta)$, where $\theta = \sqrt[3]{m}$ with the norm q^2 (q is a rational prime number and greater than 3) then we can look at their unramified class fields of $Q(\theta)$. Before seeing this unramified class field $Q(\theta, \sqrt{\mu})$, we like to construct the normal extension. As we know all the conjugates of θ and $\sqrt{\mu}$ are contained in the normal field K_{24}. Thus K_{24} must consist of all the elements generated by $\theta, \rho, \sqrt{\mu}, \sqrt{\mu'}, \sqrt{\mu''}$. Furthermore in the latter of this chapter, we like to see the group structure and the relationships of all the subfields of K_{24} and its Galois group $G(K_{24}/Q)$. First, we list all the definitions and results.

Definition 2.1. A finite extension of the rational numbers Q is called a number field K.

Definition 2.2. An algebraic number is a root of a polynomial equation with rational coefficients. An algebraic integer is

an algebraic number which satisfies some monic equation,

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

with integral coefficients.

Definition 2.3. The integral closure of Z in a number field K is called the ring of algebraic integers of that field, and is denoted by $\mathcal{O}_K$, where Z is the ring of ordinary integers.

Definition 2.4. An algebraic integer μ of K is non-square if $\mu \neq \alpha^2$ for any α in K.

Definition 2.5. The norm of an algebraic integer μ (denoted by $N(\mu)$) is defined as the product of itself and its conjugates.

Definition 2.6. The class number of K is the number of ideal classes.

Fact 2.7.(Selmer) When $m \leq 100$, the only fields $Q(\theta)$ which have class number 2 are $m = 11, 15, 47, 83, 89$.

To begin with, the fixed (base) field Q and the automorphisms can be defined as following:

$$G(K_{24}/Q) = G = \langle S, T, U, V \rangle$$

$$S: \rho \longrightarrow \rho^2 ; \quad S^2 = 1 \qquad T: \theta \longrightarrow \rho\theta ; \quad T^3 = 1$$

$$U: \sqrt{\mu} \longrightarrow -\sqrt{\mu} ; \quad U^2 = 1 \qquad V: \sqrt{\mu'} \longrightarrow -\sqrt{\mu'} ; \quad V^2 = 1$$

Since $N(\mu) = \mu\mu'\mu'' = q^2$, then $\sqrt{\mu''} = (q \cdot \sqrt{\mu\mu'})/\mu\mu'$

From the above identity, we obtain the automorphism UV which can take care of the automorphism of the generator $\sqrt{\mu''}$.

In particular, let $R = US$.

$$R: \rho \xrightarrow{S} \rho^2 \xrightarrow{U} \rho^2 ; \qquad \text{and} \quad R^2: \rho \xrightarrow{R} \rho^2 \xrightarrow{R} \rho$$

$$\theta \xrightarrow{S} \theta \xrightarrow{U} \theta \; ; \qquad \theta \xrightarrow{R} \theta \xrightarrow{R} \theta$$

$$\sqrt{\mu} \xrightarrow{S} \sqrt{\mu} \xrightarrow{U} -\sqrt{\mu}; \qquad \sqrt{\mu} \xrightarrow{R} -\sqrt{\mu} \xrightarrow{R} \sqrt{\mu}$$

$$\sqrt{\mu'} \xrightarrow{S} \sqrt{\mu''} \xrightarrow{U} -\sqrt{\mu''}; \qquad \sqrt{\mu'} \xrightarrow{R} -\sqrt{\mu''} \xrightarrow{R} -\sqrt{\mu'}$$

$$\sqrt{\mu''} \xrightarrow{S} \sqrt{\mu'} \xrightarrow{U} \sqrt{\mu'} \; ; \qquad \sqrt{\mu''} \xrightarrow{R} \sqrt{\mu'} \xrightarrow{R} -\sqrt{\mu''}$$

That is $R^2 = V$ and $RS = U$, therefore, we can eliminate V by R^2 and U by RS. Then we have:

$$G = \langle R, S, T: R^4 = S^2 = T^3 = 1 \rangle$$

Moreover, G also satisfies the commutator relations on the following pages. First of all, we should see the mappings $R^i S^j T^n$ (where $0 \leq i \leq 3$; $0 \leq j \leq 1$ and $0 \leq n \leq 2$). They are:

I: The identity map. R and R^2: See above maps.

R^3: $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$

S : $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$

RS : $\rho \longrightarrow \rho$, $\theta \longrightarrow \theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$

R^2S : $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$

R^3S : $\rho \longrightarrow \rho$, $\theta \longrightarrow \theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$

T : $\rho \longrightarrow \rho$, $\theta \longrightarrow \rho\theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$

T^2 : $\rho \longrightarrow \rho$, $\theta \longrightarrow \rho^2\theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$

ST : $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \rho^2\theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$

ST^2 : $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \rho\theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$

RT : $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \rho^2\theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$

R^2T : $\rho \longrightarrow \rho$, $\theta \longrightarrow \rho\theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$

R^3T: $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \rho^2\theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$

RT^2: $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \rho\theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$

R^2T^2: $\rho \longrightarrow \rho$, $\theta \longrightarrow \rho^2\theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$

R^3T^2: $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \rho\theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$

RST: $\rho \longrightarrow \rho$, $\theta \longrightarrow \rho\theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$

R^2ST: $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \rho^2\theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$

R^3ST: $\rho \longrightarrow \rho$, $\theta \longrightarrow \rho\theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$

RST^2: $\rho \longrightarrow \rho$, $\theta \longrightarrow \rho^2\theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$

R^2ST^2: $\rho \longrightarrow \rho^2$, $\theta \longrightarrow \rho\theta$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$

R^3ST^2: $\rho \longrightarrow \rho$, $\theta \longrightarrow \rho^2\theta$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$

Definition 2.8. Let G be a group. The elements of the set C: $C = \{ ABA^{-1}B^{-1} : A \text{ and } B \text{ in } G \}$ are called commutators of G, and C is denoted by $[A, B]$

Fact 2.9. The inverse of the commutator $[A, B] = ABA^{-1}B^{-1}$ is $BAB^{-1}A^{-1} = [B, A]$ and so is a commutator of G also.

In addition, the conjugate of a commutator C is a commutator of G, ($[A, B]^D = [A^D, B^D]$ where D in G). Furthermore, the set of all the commutators is a normal subgroup of G .

Fact 2.10. Let G be a group. A, B and D are elements of G . Then (a) $[AB, D] = [B, D]^A[A, D]$

(b) $[A, BD] = [A, B][A, D]^B$.

Now we like to consider all the commutators for the generators and automorphisms. We gain the order of these elements of the set of automorphisms of K_{24} . From the above mappings we can find the elements of order 2 by observing or applying the same mapping again.
They are: S, R^2, RS, R^2S, R^3S, ST, ST^2, RT and R^3T^2.
Similarly, we obtain the elements of order 3. They are:
T and T^2; RST^2 and R^3ST; R^2T and R^3ST^2; R^2T^2 and RST, and the elements of order 4 are R and R^3; R^3T and R^2ST; RT^2 and R^2ST^2, where each pair of elements are inverses of each other.

By using the basic computation in commutators of the group $G = G(K_{24}/Q)$, we have

$RSRS = R^3SR^3S = R^2SR^2S = 1$; $SR = R^3S$

$R^2 = RSR^3S = R^3SRS = [R, S] = [R^3, S]$; $SR^3 = RS$

$STST = ST^2ST^2 = 1$; $TS = ST^2$

$T = STST^2 = [S, T]$; $T^2S = ST$

$T^2 = ST^2ST = [S, T^2]$; $T^2R^3 = RT$

$RTRT = R^3T^2R^3T^2 = 1$; $TR = R^3T^2$

$(R^2T)(R^3ST^2) = 1$; $TR^3 = R^2ST^2$

$R(R^2ST^2)T^2 = RTR^3T^2 = [R, T]$; $TR^2 = RST$

$R^3T(R^2ST) = 1$ and $R^2T^2(RST) = 1$ $T^2R = R^2ST$

$RT^2(R^2ST^2) = 1$ $T^2R^2 = R^3ST^2$

By applying Fact 2.10., we have

$[R, S] = R^2$; $[R^2, S] = 1$; $[R^3, S] = R^2$;

$[R, T] = R^3ST$; $[R^2, T] = R^3S$; $[R^3, T] = R^2T$;

$[R, T^2] = R^2T^2$; $[R^2, T^2] = RS$; $[R^3, T^2] = RST^2$;

$[S, T] = T$; $[S, T^2] = T^2$.

Therefore, we complete the Galois group of the normal extension. That is,

$$G = \langle R, S, T : R^4 = S^2 = T^3 = 1 \text{ and } [R, S] = R^2, [R, T] = R^3ST, [S, T] = T \rangle$$

Then G consists of 24 elements and G is non-abelian with the trivial center subgroup of G.

Note that the symmetric group of degree 4 has the same properties as $G = G(K_{24}/Q)$.

Proposition 2.1. The symmetric group of degree 4 is isomorphic to G (i.e., $S_4 \cong G(K_{24}/Q)$)

Proof: Let ϕ be a mapping from G into S_4 defined by

$I \longrightarrow I'$; $S \longrightarrow (12)$; $T \longrightarrow (123)$;

$T^2 \longrightarrow (132)$; $R \longrightarrow (1324)$; $R^2 \longrightarrow (12)(34)$;

$R^3 \longrightarrow (1423)$;

Then we have the following homomorphisms:

$ST \longrightarrow (13)$; $ST^2 \longrightarrow (23)$; $RS \longrightarrow (13)(24)$

$R^2S \longrightarrow (34)$; $R^3S \longrightarrow (14)(23)$; $RT \longrightarrow (24)$;

$R^2T \longrightarrow (124)$; $R^3T \longrightarrow (1432)$; $RT^2 \longrightarrow (1243)$;

$R^2T^2 \longrightarrow (234)$; $R^3T^2 \longrightarrow (14)$; $RST \longrightarrow (243)$;

$R^2ST \longrightarrow (1234)$; $R^3ST \longrightarrow (142)$; $RST^2 \longrightarrow (124)$;

$R^2ST^2 \longrightarrow (1342)$; $R^3ST^2 \longrightarrow (143)$;

From the above mapping, ϕ is 1-1, onto, perseving the operation of homomorphisms as well as $|S_4| = |G|$, therefore ϕ is an isomorphism from S_4 into G . //

Now we like to consider the number of the subgroups of G . Since G is non-abelian and containing the only one normal subgroup of order 12. Therefore, we need the special Sylow Theorem.

Theorem 2.11. (Sylow)

(i) Let p^n be the highest power of the prime number p that divides the order of the group G. Then G contains at least one sylow p-subgroup. i.e., a subgroup of order p^n.

(ii) Any two subgroups of order p^n of G are conjugate.

(iii) Each subgroup H of G whose order is a power of p is contained in a Sylow p-subgroup.

(iv) If r denotes the number of Sylow p-subgroups in G then $r \equiv 1 \pmod{p}$.

From the Sylow Theorem, we can figure out the number of subgroups of special orders 2^3 and 3, and the rest of them by applying the commutators and group cosets, thus the subgroups are determined by the existence of the inverse element.

After examining the inverse for each element of the set we find the subgroups of order 2, 3, 4, 6, 8, 12, and 24.

9 subgroups of order 2.

$\{I, S\}$; $\{I, R^2\}$; $\{I, RS\}$; $\{I, R^2S\}$; $\{I, R^3S\}$;
$\{I, RT\}$; $\{I, R^3T^2\}$; $\{I, ST\}$; $\{I, ST^2\}$;

4 subgroups of order 3.

$\{I, T, T^2\}$; $\{I, RST^2, R^3ST\}$; $\{I, R^2T, R^3ST^2\}$;
$\{I, R^2T^2, RST\}$;

7 subgroups of order 4.

$\{I, R, R^2, R^3\}$; $\{I, R^3T, RS, R^2ST\}$;
$\{I, RT^2, R^3S, R^2ST^2\}$; $\{I, S, R^2S, R^2\}$;
$\{I, RT, ST, RS\}$; $\{I, R^3S, ST^2, R^3T^2\}$;
$\{I, RS, R^2, R^3S\} = L_4^*$

4 subgroups of order 6.

$\{I, T, T^2, S, ST, ST^2\}$; $\{I, RST^2, R^3ST, RT, R^3T^2, S\}$;
$\{I, R^2T, R^3ST^2, R^2S, ST, R^3T^2\}$;
$\{I, R^2T^2, RST, R^2S, ST^2, RT\}$;

3 subgroups of order 8.

$\{I, R, R^2, R^3, S, RS, R^2S, R^3S\}$;
$\{I, R^3T, RS, R^2ST, R^3S, R^2, ST, RT\}$;
$\{I, RT^2, R^3S, R^2ST^2, R^2, RS, ST^2, R^3T^2\}$;

only one subgroup of order 12. (normal)

$\{I, R^2, RS, R^3S, T, T^2, RST^2, R^3ST, R^2T, R^3ST^2, R^2T^2, RST\} = L_{12}$

Proposition 2.2. G is solvable.

Proof: Since S_4 is solvable, so is G. Then $[G, G] = L_{12}$
$[L_{12}, L_{12}] = L_4^*$; $[L_4^*, L_4^*] = \{I\}$. //

§2. The corresponding subfields of K_{24} and the structure of S_4

As we know, each subgroup of G will leave a few generators fixed. This implies each subgroup of G will give us some subfield of K_{24} fixed. These fixed subfields are depended on the subgroups of G. So, we have the following corresponding by checking the elements of the subgroup of G as above mappings and generators. That is:

order 2 $\overset{G}{\longleftrightarrow}$ Degree 2 of K_{24}/K_{12}

$\{I,S\} \overset{G}{\longleftrightarrow} Q(\theta, \sqrt{\delta^{o}})$; $\{I,R^2\} \overset{G}{\longleftrightarrow} Q(\rho,\theta,\sqrt{\mu})$;

$\{I,RS\} \overset{G}{\longleftrightarrow} Q(\rho,\theta,\sqrt{\mu'})$; $\{I,R^3S\} \overset{G}{\longleftrightarrow} Q(\rho,\theta,\sqrt{\mu''})$;

$\{I,R^2S\} \overset{G}{\longleftrightarrow} Q(\theta,\sqrt{\mu},\sqrt{\mu'}-\sqrt{\mu''})$; $\{I,ST\} \overset{G}{\longleftrightarrow} Q(\rho\theta,\sqrt{\delta^{o}})$;

$\{I,ST^2\} \overset{G}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\delta^{o}})$;

$\{I,RT\} \overset{G}{\longleftrightarrow} Q(\rho\theta,\sqrt{\mu'},\sqrt{\mu''}-\sqrt{\mu})$

$\{I,R^3T^2\} \overset{G}{\longleftrightarrow} Q(\rho^2\theta,\sqrt{\mu''},\sqrt{\mu}-\sqrt{\mu'})$

order 3 $\overset{G}{\longleftrightarrow}$ Degree 3 of K_{24}/K_8

$\{I,T,T^2\} \overset{G}{\longleftrightarrow} Q(\rho, \sqrt{\delta^{o}})$ $\{I,RST^2,R^3ST\} \overset{G}{\longleftrightarrow} Q(\rho, \sqrt{\delta'''})$;

$\{I,R^2T,R^3ST^2\} \overset{G}{\longleftrightarrow} Q(\rho,\sqrt{\delta'})$; $\{I,R^2T^2,RST\} \overset{G}{\longleftrightarrow} Q(\rho,\sqrt{\delta''})$;

where $\sqrt{\delta^{o}} = \sqrt{\mu}+\sqrt{\mu'}+\sqrt{\mu''}$; $\sqrt{\delta'} = -\sqrt{\mu}+\sqrt{\mu'}-\sqrt{\mu''}$;

$\sqrt{\delta''} = -\sqrt{\mu}-\sqrt{\mu'}+\sqrt{\mu''}$; $\sqrt{\delta'''} = \sqrt{\mu}-\sqrt{\mu'}-\sqrt{\mu''}$;

order 4 $\overset{G}{\longleftrightarrow}$ Degree 4 of K_{24}/K_6

(i) cyclic of type C(4);

$\{I,R,R^2,R^3\} \overset{G}{\longleftrightarrow} Q(\theta,\sqrt{-3\mu})$; $\{I,R^3T,RS,R^2ST\} \overset{G}{\longleftrightarrow} Q(\rho\theta,\sqrt{-3\mu'})$;

$\{I,RT^2,R^3S,R^2ST^2\} \overset{G}{\longleftrightarrow} Q(\rho^2\theta,\sqrt{-3\mu''})$;

(ii) non-cyclic of type $C(2) \oplus C(2)$

$\{I,RS,R^2,R^3S\} \overset{G}{\longleftrightarrow} Q(\rho,\theta)$; $\{I,RT,ST,RS\} \overset{G}{\longleftrightarrow} Q(\rho\theta,\sqrt{\mu'})$;

$\{I, S, R^2, R^2S\} \overset{G}{\longleftrightarrow} Q(\theta, \sqrt{\mu})$

$\{I, R^3S, ST^2, R^3T^2\} \overset{G}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''})$

order 6 $\overset{G}{\longleftrightarrow}$ Degree 6 of K_{24}/K_4 (of type S_3)

$\{I, S, T, T^2, ST, ST^2\} \overset{G}{\longleftrightarrow} Q(\sqrt{\delta^o})$

$\{I, RST^2, R^3ST, RT, S, R^3T^2\} \overset{G}{\longleftrightarrow} Q(\sqrt{\delta'''})$;

$\{I, R^2T, R^3ST^2, ST, R^3T^2, R^2S\} \overset{G}{\longleftrightarrow} Q(\sqrt{\delta'})$;

$\{I, R^2T^2, RST, R^2S, RT, ST^2\} \overset{G}{\longleftrightarrow} Q(\sqrt{\delta''})$;

order 8 $\overset{G}{\longleftrightarrow}$ Degree 8 of K_{24}/K_3 (of type D_4 dihedral gp.)

$\{I, R, R^2, R^3, S, RS, R^2S, R^3S\} \overset{G}{\longleftrightarrow} Q(\theta)$;

$\{I, R^3T, RS, R^2ST, R^3S, R^2, ST, RT\} \overset{G}{\longleftrightarrow} Q(\rho\theta)$;

$\{I, RT^2, R^3S, R^2ST^2, R^2, RS, ST^2, R^3T^2\} \overset{G}{\longleftrightarrow} Q(\rho^2\theta)$;

order 12 $\overset{G}{\longleftrightarrow}$ Degree 12 of K_{24}/K_2 (of type A_4)

$\{I, T, T^2, R^2, RS, R^3S, R^2T, R^2T^2, RST, RST^2, R^3ST, R^3ST^2\}$

$\overset{G}{\longleftrightarrow} Q(\rho)$

This completes the subfields of the normal extension of Q, and we described all the subfields of K_{24} and the subgroups of S_4 in the following pages. We also list the types and diagrams for K_{24} and S_4. First of all, we have to find the subgroups of order 2, 3, 4, 6, 8, and 12.

order 2.

$\bar{H}_2 = \{I', (12)(34)\}$; $\bar{H}_2' = \{I', (13)(24)\}$; $\bar{H}_2'' = \{I', (14)(23)\}$;

$H_2 = \{I', (12)\}$; $H_2' = \{I', (34)\}$; $H_2'' = \{I', (14)\}$;

$H_2''' = \{I', (23)\}$; $H_2^{iv} = \{I', (24)\}$ $H_2^{v} = \{I', (13)\}$;

order 3.

$H_3' = \{I', (123), (132)\}$; $H_3'' = \{I', (134), (143)\}$

$H_3''' = \{I', (124), (142)\}$; $H_3^{iv} = \{I', (234), (243)\}$;

order 4.

$H_4^- = \{I', (12)(34), (14)(23), (13)(24)\}$;

$H_4^* = \{I', (1234), (13)(24), (1432)\}$;

$H_4^{*'} = \{I', (1324), (12)(34), (1423)\}$;

$H_4^{*''} = \{I', (1243), (14)(23), (1342)\}$;

$H_4^o = \{I', (12), (34), (12)(34)\}$;

$H_4' = \{I', (13), (24), (13)(24)\}$;

$H_4'' = \{I', (14), (23), (14)(23)\}$;

order 6.

$H_6^o = \{I', (12), (13), (23), (123), (132)\}$;

$H_6' = \{I', (13), (14), (34), (134), (143)\}$;

$H_6'' = \{I', (12), (14), (24), (124), (142)\}$;

$H_6'^{''} = \{I', (23), (24), (34), (234), (243)\}$;

order 8.

$H_8^o = \{I', (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$;

$H_8' = \{I', (1243), (14)(23), (1342), (12)(34), (13)(24), (14), (23)\}$;

$H_8'' = \{I', (1324), (12)(34), (1423), (13)(24), (14)(23), (12), (34)\}$;

order 12.

$A_4 = \{I', (12)(34), (13)(24), (14)(23), (123), (132), (124),$

$(142), (234), (243), (134), (143)\}$;

order 24.

$$S_4 = \{I', (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432)\}$$

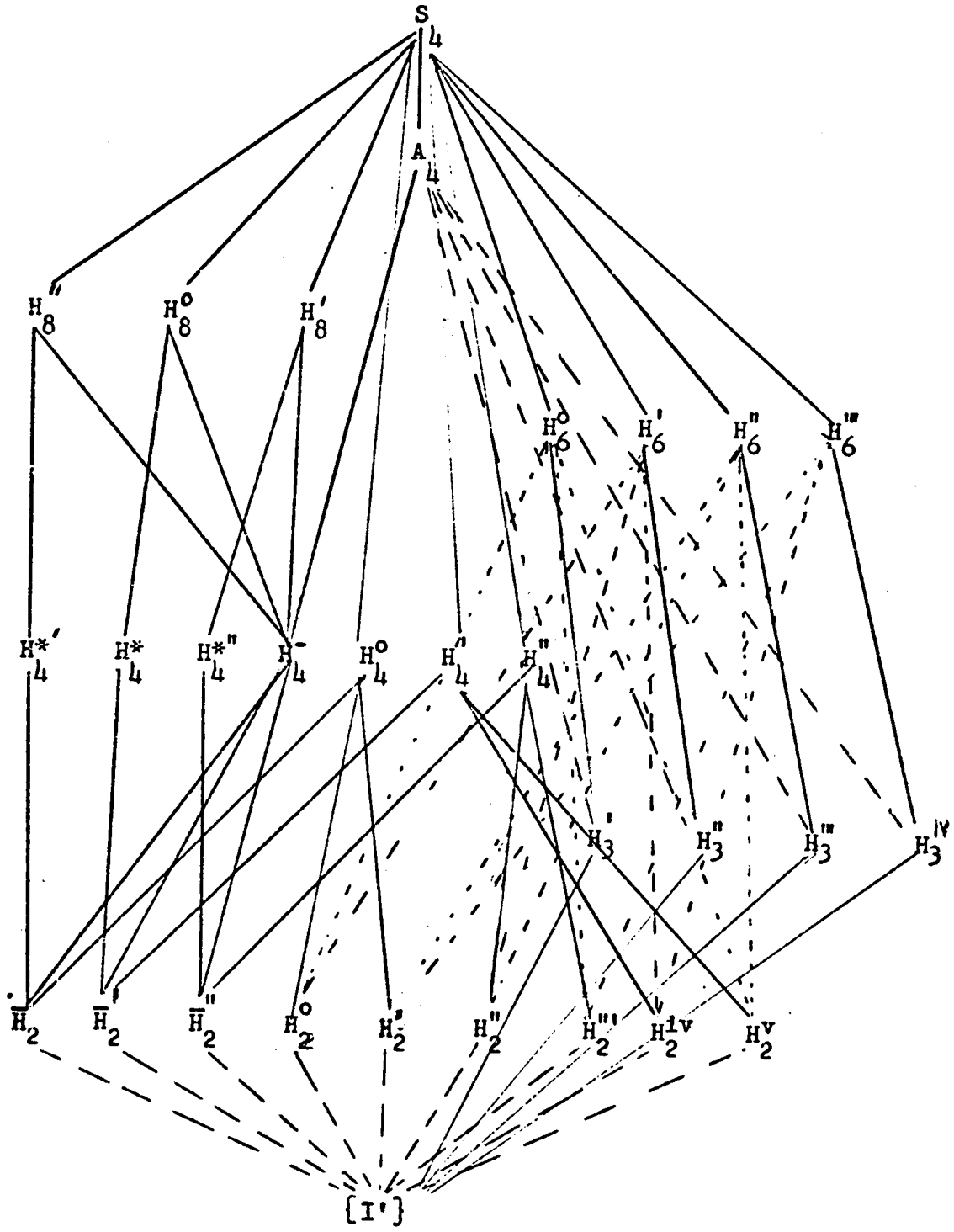
Remark 2.12. S_4 is solvable, because of the following commutator subgroups of S_4:

$$[S_4, S_4] = A_4,$$

$$[A_4, A_4] = H_4^-,$$

$$[H_4^-, H_4^-] = \{I'\}. \qquad \text{So, } S_4 \supset A_4 \supset H_4^- \supset \{I'\}.$$

Moreover, A_4 is the only normal subgroup of order 12 and A_4 has no subgroups of order 6.



30 subgroups

Diagram 2.1.

S_4 — The Symmetric Group of degree 4

1 subgroup — A_4 — The Alternating Group of degree 4

3 subgroups — H_8 —
$H_8^{\circ} = \{I', (1234), (12)(34), (13)(24), (14)(23), (1432), (13), (24)\}$;
$H_8' = \{I', (1243), (12)(34), (13)(24), (14)(23), (1342), (14), (23)\}$;
$H_8'' = \{I', (1324), (12)(34), (13)(24), (14)(23), (1423), (12), (34)\}$;

4 subgroups — H_6 —
$H_6^{\circ} = \{I', (12), (13), (23), (123), (132)\}$;
$H_6' = \{I', (13), (14), (34), (134), (143)\}$;
$H_6'' = \{I', (12), (14), (24), (124), (142)\}$;
$H_6''' = \{I', (23), (24), (34), (234), (243)\}$;

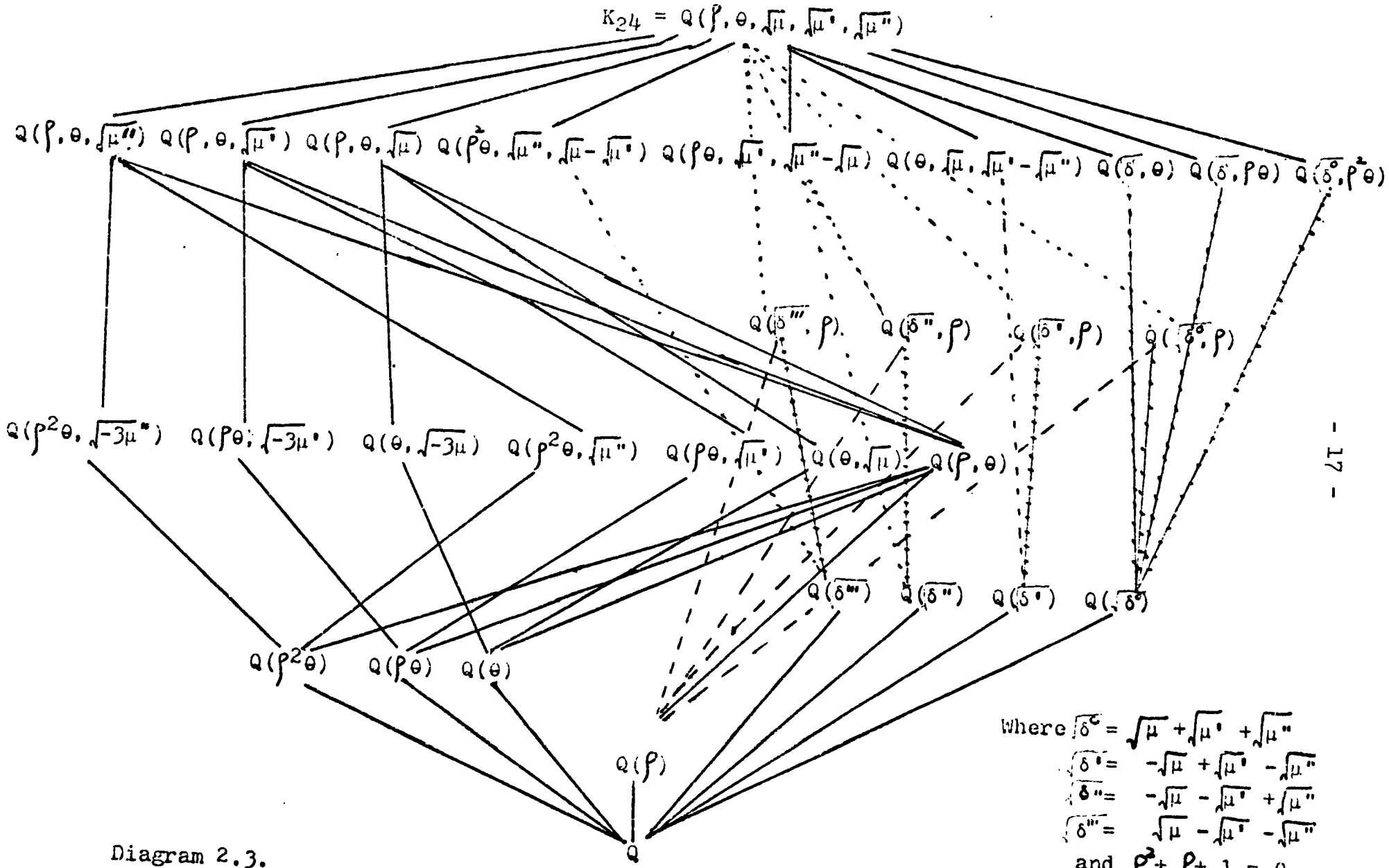
3 types
(1 + 3 + 3) — H_4 — subgroups
$H_4^{-} = \{I', (12)(34), (13)(24), (14)(23)\}$;
$H_4^{\circ} = \{I', (12), (34), (12)(34)\}$;
$H_4' = \{I', (13), (24), (13)(24)\}$;
$H_4'' = \{I', (14), (23), (14)(23)\}$;
$H_4^{*} = \{I', (1234), (13)(24), (1432)\}$;
$H_4^{*'} = \{I', (1324), (12)(34), (1423)\}$;
$H_4^{*''} = \{I', (1243), (14)(23), (1342)\}$;

4 subgroups — H_3 — $\{I', (123), (132)\}$; $\{I', (124), (142)\}$; $\{I', (134), (143)\}$; $\{I', (234), (243)\}$;

3 types
(3 + 3 + 3) — H_2 — subgroups
$\{I', (12)\}$; $\{I', (13)\}$; $\{I', (14)\}$; $\{I', (23)\}$
$\{I', (24)\}$; $\{I', (34)\}$;
$\{I', (12)(34)\}$; $\{I', (13)(24)\}$; $\{I', (14)(23)\}$;

$\{I'\}$ — $\{$The identity element I'$\}$ Total 30 subgps

Diagram 2.2.



Where $\sqrt{\delta^\circ} = \sqrt{\mu} + \sqrt{\mu'} + \sqrt{\mu''}$

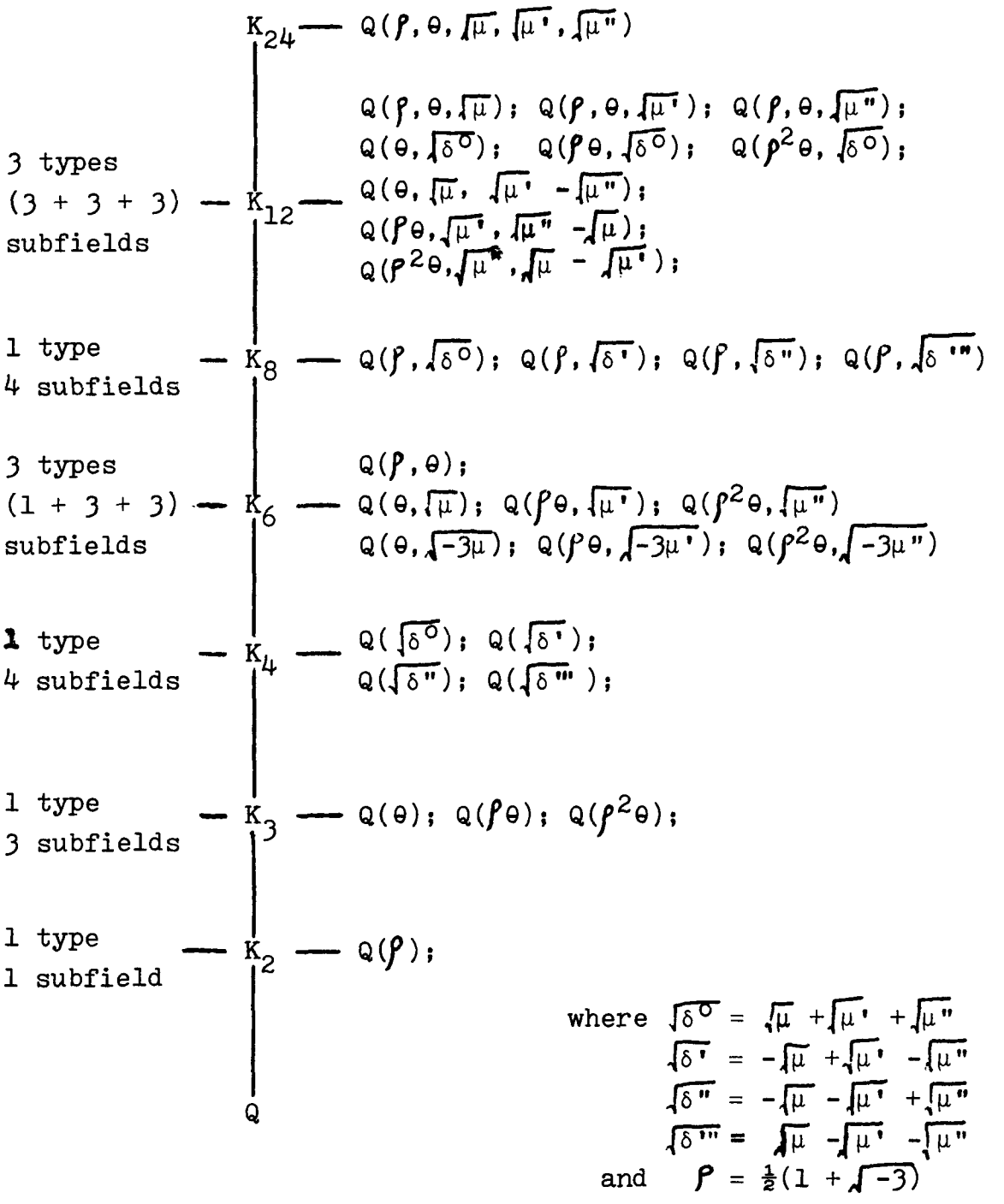
$\sqrt{\delta'} = -\sqrt{\mu} + \sqrt{\mu'} - \sqrt{\mu''}$

$\sqrt{\delta''} = -\sqrt{\mu} - \sqrt{\mu'} + \sqrt{\mu''}$

$\sqrt{\delta'''} = \sqrt{\mu} - \sqrt{\mu'} - \sqrt{\mu''}$

and $\rho^2 + \rho + 1 = 0$

Diagram 2.3.



where $\sqrt{\delta^0} = \sqrt{\mu} + \sqrt{\mu'} + \sqrt{\mu''}$
$\sqrt{\delta'} = -\sqrt{\mu} + \sqrt{\mu'} - \sqrt{\mu''}$
$\sqrt{\delta''} = -\sqrt{\mu} - \sqrt{\mu'} + \sqrt{\mu''}$
$\sqrt{\delta'''} = \sqrt{\mu} - \sqrt{\mu'} - \sqrt{\mu''}$
and $\rho = \frac{1}{2}(1 + \sqrt{-3})$

Total subfields 30

Diagram 2.4.

CHAPTER III. THE GALOIS GROUPS AND THEIR CORRESPONDING SUBFIELDS FOR h = 4 AND 4p (NON-CYCLIC)

§1. The subgroups of the Galois group $G^*(K_{96}^*/Q)$

As we did before, we like to work for $h = 4, 12, 20, \cdots 4p$, (p is an odd rational prime.), the non-cyclic class groups. We take $[\mu, q_1]$ and $[\nu, q_2]$ the two non-equivalent prime ideals: $[\mu, q_1] \neq [\nu, q_2]$, i.e., $[\mu, q_1][\nu, q_2] \neq 1$, where $N(\mu)$ and $N(\nu)$ are equals to q_1^2 and q_2^2, respectively. With μ and ν, we consider the subfields and factorization of the unramified class field, $Q(\theta, \sqrt{\mu}, \sqrt{\nu})$. We investigate most of the properties and results as below. Before observing the class field, we would like to construct the normal field K_{96}^* (normal extension K_{96}^*) of $Q(\theta, \sqrt{\mu}, \sqrt{\nu})$. Therefore the normal field K_{96}^* must consist of all the elements generated by $\theta, \rho, \sqrt{\mu}, \sqrt{\mu'}, \sqrt{\mu''}, \sqrt{\nu}, \sqrt{\nu'}, \sqrt{\nu''}$. Furthermore, we find out the structure and relationships of all subfields of K_{96}^* and its Galois group $G^*(K_{96}^*/Q)$. First, we list all the facts we need.

Fact 3.1. The product of two ideals $[a_1, b_1]$ and $[a_2, b_2]$ is $[a_1b_1, a_1b_2, a_2b_1, a_2b_2]$.

Fact 3.2. (Selmer) When $m \leq 200$, the only field $Q(\theta)$ whose class number is 4 is $m = 113$ and its class group is of type non-cyclic.

Fact 3.3. There are 5 nonisomorphic groups of order 8 $C(8)$, $C(4) \oplus C(2)$, $C(2) \oplus C(2) \oplus C(2)$, Q and D_4.

Fact 3.4. There are 14 non-isomorphic group of order 16. There are 51 non-isomorphic group of order 32.

At first, the base field Q and the automorphisms will be set as following:

$$G^*(K_{96}^*/Q) = G^* = \langle S, T, U_1, U_2, V_1, V_2 \rangle$$

$$S: \rho \longrightarrow \rho^2 \ ; \ S^2 = 1 \qquad T: \ \theta \longrightarrow \rho\theta \ ; \ T^3 = 1$$

$$U_1: \sqrt{\mu} \longrightarrow -\sqrt{\mu}; \ U_1^2 = 1 \qquad U_2: \ \sqrt{\nu} \longrightarrow -\sqrt{\nu}; \ U_2^2 = 1$$

$$V_1: \sqrt{\mu'} \longrightarrow -\sqrt{\mu'}; V_1^2 = 1 \qquad V_2: \ \sqrt{\nu'} \longrightarrow -\sqrt{\nu'}; \ V_2^2 = 1$$

Since $N(\mu) = \mu\mu'\mu'' = q_1^2$, where q_1 ia prime

$N(\nu) = \nu\nu'\nu'' = q_2^2$, where q_2 is prime

So: $\sqrt{\mu''} = q_1 \cdot \sqrt{\mu\mu'}/\mu\mu'$

$\sqrt{\nu''} = q_2 \cdot \sqrt{\nu\nu'}/\nu\nu'$

From the above identities, we obtain the automorphisms U_1V_1 and U_2V_2 which can take care of the automorphisms of the generators $\sqrt{\mu''}$ and $\sqrt{\nu''}$.

In particular, let $R_1 = U_1S$ and $R_2 = U_2S$.

R_1:

$$\rho \xrightarrow{S} \rho^2 \xrightarrow{U_1} \rho^2 \qquad \theta \xrightarrow{S} \theta \xrightarrow{U_1} \theta$$

$$\sqrt{\mu} \xrightarrow{S} \sqrt{\mu} \xrightarrow{U_1} -\sqrt{\mu} \qquad \sqrt{\mu'} \xrightarrow{S} \sqrt{\mu''} \xrightarrow{U_1} -\sqrt{\mu''}$$

$$\sqrt{\mu''} \xrightarrow{S} \sqrt{\mu'} \xrightarrow{U_1} \sqrt{\mu'} \qquad \sqrt{\nu} \xrightarrow{S} \sqrt{\nu} \xrightarrow{U_1} \sqrt{\nu}$$

R_2:

$$\rho \xrightarrow{S} \rho^2 \xrightarrow{U_2} \rho^2 \qquad \theta \xrightarrow{S} \theta \xrightarrow{U_2} \theta$$

$$\sqrt{\mu} \xrightarrow{S} \sqrt{\mu} \xrightarrow{U_2} \sqrt{\mu} \qquad \sqrt{\mu'} \xrightarrow{S} \sqrt{\mu''} \xrightarrow{U_2} \sqrt{\mu''}$$

$$\sqrt{\mu''} \xrightarrow{S} \sqrt{\mu'} \xrightarrow{U_2} \sqrt{\mu'} \qquad \sqrt{\nu} \xrightarrow{S} \sqrt{\nu} \xrightarrow{U_2} -\sqrt{\nu}$$

$$\sqrt{\nu'} \xrightarrow{S} \sqrt{\nu''} \xrightarrow{U_1} \sqrt{\nu''}, \qquad \sqrt{\nu'} \xrightarrow{S} \sqrt{\nu} \xrightarrow{U_2} -\sqrt{\nu''},$$

$$\sqrt{\nu''} \xrightarrow{S} \sqrt{\nu'} \xrightarrow{U_1} \sqrt{\nu'}, \qquad \sqrt{\nu''} \xrightarrow{S} \sqrt{\nu'} \xrightarrow{U_2} \sqrt{\nu'},$$

and moreover,

R_1^2: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,

$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$,

R_2^2: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,

$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

That is, $R_1S = U_1$, $R_2S = U_2$, $R_1^2 = V_1$ and $R_2^2 = V_2$, therefore we can eliminate U_1, U_2, V_1 and V_2 by R_1S, R_2S, R_1^2 and R_2^2, respectively. Then we have:

$$G^* = G^*(K_{96}^*/Q) = \langle R_1, R_2, S, T: R_1^4 = R_2^4 = S^2 = T^3 = 1 \rangle$$

Furthermore, like Chapter II, G^* also satisfies the commutator relations on the following pages. First of all, we should list out all the elements $R_1^i R_2^{i'} S^j T^n$ of G^*. They are: I: The identity map.

R_1, R_2, R_1^2 and R_2^2: See the above maps.

R_1^3: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,

$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu'}$,

R_2^3: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,

$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

S: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,

$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu'}$,

T: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,

$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu'}$,

$$
\begin{array}{lllll}
T^2: & \rho \longrightarrow \rho, & \sqrt{\mu} \longrightarrow \sqrt{\mu''}, & \sqrt{\mu'} \longrightarrow \sqrt{\mu}, & \sqrt{\mu''} \longrightarrow \sqrt{\mu'}, \\
 & \theta \longrightarrow \rho^2\theta, & \sqrt{\nu} \longrightarrow \sqrt{\nu''}, & \sqrt{\nu'} \longrightarrow \sqrt{\nu}, & \sqrt{\nu''} \longrightarrow \sqrt{\nu'}, \\
R_1R_2: & \rho \longrightarrow \rho, & \sqrt{\mu} \longrightarrow -\sqrt{\mu}, & \sqrt{\mu'} \longrightarrow \sqrt{\mu'}, & \sqrt{\mu''} \longrightarrow -\sqrt{\mu''}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow -\sqrt{\nu}, & \sqrt{\nu'} \longrightarrow -\sqrt{\nu'}, & \sqrt{\nu''} \longrightarrow \sqrt{\nu''}, \\
R_1^2R_2: & \rho \longrightarrow \rho^2, & \sqrt{\mu} \longrightarrow \sqrt{\mu}, & \sqrt{\mu'} \longrightarrow -\sqrt{\mu''}, & \sqrt{\mu''} \longrightarrow -\sqrt{\mu'}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow -\sqrt{\nu}, & \sqrt{\nu'} \longrightarrow -\sqrt{\nu''}, & \sqrt{\nu''} \longrightarrow \sqrt{\nu'}, \\
R_1^3R_2: & \rho \longrightarrow \rho, & \sqrt{\mu} \longrightarrow -\sqrt{\mu}, & \sqrt{\mu'} \longrightarrow -\sqrt{\mu'}, & \sqrt{\mu''} \longrightarrow \sqrt{\mu''}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow -\sqrt{\nu}, & \sqrt{\nu'} \longrightarrow -\sqrt{\nu'}, & \sqrt{\nu''} \longrightarrow \sqrt{\nu''}, \\
R_1R_2^2: & \rho \longrightarrow \rho^2, & \sqrt{\mu} \longrightarrow -\sqrt{\mu}, & \sqrt{\mu'} \longrightarrow -\sqrt{\mu''}, & \sqrt{\mu''} \longrightarrow \sqrt{\mu'}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow \sqrt{\nu}, & \sqrt{\nu'} \longrightarrow -\sqrt{\nu''}, & \sqrt{\nu''} \longrightarrow -\sqrt{\nu'}, \\
R_1^2R_2^2: & \rho \longrightarrow \rho, & \sqrt{\mu} \longrightarrow \sqrt{\mu}, & \sqrt{\mu'} \longrightarrow -\sqrt{\mu'}, & \sqrt{\mu''} \longrightarrow -\sqrt{\mu''} \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow \sqrt{\nu}, & \sqrt{\nu'} \longrightarrow -\sqrt{\nu'}, & \sqrt{\nu''} \longrightarrow -\sqrt{\nu''}, \\
R_1^3R_2^2: & \rho \longrightarrow \rho^2, & \sqrt{\mu} \longrightarrow -\sqrt{\mu}, & \sqrt{\mu'} \longrightarrow \sqrt{\mu''}, & \sqrt{\mu''} \longrightarrow -\sqrt{\mu'}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow \sqrt{\nu}, & \sqrt{\nu'} \longrightarrow -\sqrt{\nu''}, & \sqrt{\nu''} \longrightarrow -\sqrt{\nu'}, \\
R_1R_2^3: & \rho \longrightarrow \rho, & \sqrt{\mu} \longrightarrow -\sqrt{\mu}, & \sqrt{\mu'} \longrightarrow \sqrt{\mu'}, & \sqrt{\mu''} \longrightarrow -\sqrt{\mu''}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow -\sqrt{\nu}, & \sqrt{\nu'} \longrightarrow \sqrt{\nu'}, & \sqrt{\nu''} \longrightarrow -\sqrt{\nu''}, \\
R_1^2R_2^3: & \rho \longrightarrow \rho^2, & \sqrt{\mu} \longrightarrow \sqrt{\mu}, & \sqrt{\mu'} \longrightarrow -\sqrt{\mu''}, & \sqrt{\mu''} \longrightarrow -\sqrt{\mu'}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow -\sqrt{\nu}, & \sqrt{\nu'} \longrightarrow \sqrt{\nu''}, & \sqrt{\nu''} \longrightarrow -\sqrt{\nu'}, \\
R_1^3R_2^3: & \rho \longrightarrow \rho, & \sqrt{\mu} \longrightarrow -\sqrt{\mu}, & \sqrt{\mu'} \longrightarrow -\sqrt{\mu'}, & \sqrt{\mu''} \longrightarrow \sqrt{\mu''}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow -\sqrt{\nu}, & \sqrt{\nu'} \longrightarrow \sqrt{\nu'}, & \sqrt{\nu''} \longrightarrow -\sqrt{\nu''}, \\
R_1S: & \rho \longrightarrow \rho, & \sqrt{\mu} \longrightarrow -\sqrt{\mu}, & \sqrt{\mu'} \longrightarrow \sqrt{\mu'}, & \sqrt{\mu''} \longrightarrow -\sqrt{\mu''}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow \sqrt{\nu}, & \sqrt{\nu'} \longrightarrow \sqrt{\nu'}, & \sqrt{\nu''} \longrightarrow \sqrt{\nu''}, \\
R_1^2S: & \rho \longrightarrow \rho^2, & \sqrt{\mu} \longrightarrow \sqrt{\mu}, & \sqrt{\mu'} \longrightarrow -\sqrt{\mu''}, & \sqrt{\mu''} \longrightarrow -\sqrt{\mu'}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow \sqrt{\nu}, & \sqrt{\nu'} \longrightarrow \sqrt{\nu''}, & \sqrt{\nu''} \longrightarrow \sqrt{\nu'}, \\
R_1^3S: & \rho \longrightarrow \rho, & \sqrt{\mu} \longrightarrow -\sqrt{\mu}, & \sqrt{\mu'} \longrightarrow -\sqrt{\mu'}, & \sqrt{\mu''} \longrightarrow \sqrt{\mu''}, \\
 & \theta \longrightarrow \theta, & \sqrt{\nu} \longrightarrow \sqrt{\nu}, & \sqrt{\nu'} \longrightarrow \sqrt{\nu'}, & \sqrt{\nu''} \longrightarrow \sqrt{\nu''},
\end{array}
$$

R_2S: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \theta$. $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

R_2^2S: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

R_2^3S: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$,

R_1T: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$ $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

R_1^2T: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

R_1^3T: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$ $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

R_1T^2: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$ $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$,

$R_1^2T^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu'}$

$R_1^3T^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$

R_2T: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

R_2^2T: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

R_2^3T: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$ $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

R_2T^2: $\rho \longrightarrow \rho^2$ $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_2^2T^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_2^3T^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$,

ST: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

ST^2: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$,

R_1R_2S: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1^2R_2S$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_1^3R_2S$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1R_2^2S$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_1^2R_2^2S$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1^3R_2^2S$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_1R_2^3S$: $\rho \longrightarrow \rho^2$ $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu'}$,

$R_1^2R_2^3S$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$ $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$,

$R_1^3R_2^3S$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu'}$,

R_1R_2T: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1R_2T^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1^2R_2T$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1^2R_2T^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$ $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_1^3R_2T$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1^3R_2T^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1R_2^2T$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

$R_1R_2^2T^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_1^2R_2^2T$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$ $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$ $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

$R_1^2R_2^2T^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1^3R_2^2T$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

$R_1^3R_2^2T^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_1R_2^3T$: $\rho \longrightarrow \rho$ $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1R_2^3T^2$: $\rho \to \rho$, $\sqrt{\mu} \to -\sqrt{\mu''}$, $\sqrt{\mu'} \to -\sqrt{\mu}$, $\sqrt{\mu''} \to \sqrt{\mu'}$,
$\theta \to \rho^2\theta$, $\sqrt{\nu} \to -\sqrt{\nu''}$, $\sqrt{\nu'} \to -\sqrt{\nu}$, $\sqrt{\nu''} \to \sqrt{\nu'}$,

$R_1^2R_2^3T$: $\rho \to \rho^2$, $\sqrt{\mu} \to -\sqrt{\mu''}$, $\sqrt{\mu'} \to -\sqrt{\mu'}$, $\sqrt{\mu''} \to \sqrt{\mu}$,
$\theta \to \rho^2\theta$, $\sqrt{\nu} \to \sqrt{\nu''}$, $\sqrt{\nu'} \to -\sqrt{\nu'}$, $\sqrt{\nu''} \to -\sqrt{\nu}$,

$R_1^2R_2^3T^2$: $\rho \to \rho^2$, $\sqrt{\mu} \to -\sqrt{\mu'}$, $\sqrt{\mu'} \to \sqrt{\mu}$, $\sqrt{\mu''} \to -\sqrt{\mu''}$,
$\theta \to \rho\theta$, $\sqrt{\nu} \to -\sqrt{\nu'}$, $\sqrt{\nu'} \to -\sqrt{\nu}$, $\sqrt{\nu''} \to \sqrt{\nu''}$,

$R_1^3R_2^3T$: $\rho \to \rho$, $\sqrt{\mu} \to -\sqrt{\mu'}$, $\sqrt{\mu'} \to \sqrt{\mu''}$, $\sqrt{\mu''} \to -\sqrt{\mu}$,
$\theta \to \rho\theta$, $\sqrt{\nu} \to \sqrt{\nu'}$, $\sqrt{\nu'} \to -\sqrt{\nu''}$, $\sqrt{\nu''} \to -\sqrt{\nu}$,

$R_1^3R_2^3T^2$: $\rho \to \rho$, $\sqrt{\mu} \to \sqrt{\mu''}$, $\sqrt{\mu'} \to -\sqrt{\mu}$, $\sqrt{\mu''} \to -\sqrt{\mu'}$,
$\theta \to \rho^2\theta$, $\sqrt{\nu} \to -\sqrt{\nu''}$, $\sqrt{\nu'} \to -\sqrt{\nu}$, $\sqrt{\nu''} \to \sqrt{\nu'}$,

R_1ST: $\rho \to \rho$, $\sqrt{\mu} \to \sqrt{\mu'}$, $\sqrt{\mu'} \to -\sqrt{\mu''}$, $\sqrt{\mu''} \to -\sqrt{\mu}$,
$\theta \to \rho\theta$, $\sqrt{\nu} \to \sqrt{\nu'}$, $\sqrt{\nu'} \to \sqrt{\nu''}$, $\sqrt{\nu''} \to \sqrt{\nu}$,

R_1^2ST: $\rho \to \rho^2$, $\sqrt{\mu} \to -\sqrt{\mu''}$, $\sqrt{\mu'} \to -\sqrt{\mu'}$, $\sqrt{\mu''} \to \sqrt{\mu}$,
$\theta \to \rho^2\theta$, $\sqrt{\nu} \to \sqrt{\nu''}$, $\sqrt{\nu'} \to \sqrt{\nu'}$, $\sqrt{\nu''} \to \sqrt{\nu}$,

R_1^3ST: $\rho \to \rho$, $\sqrt{\mu} \to -\sqrt{\mu'}$, $\sqrt{\mu'} \to \sqrt{\mu''}$, $\sqrt{\mu''} \to -\sqrt{\mu}$,
$\theta \to \rho\theta$, $\sqrt{\nu} \to \sqrt{\nu'}$, $\sqrt{\nu'} \to \sqrt{\nu''}$, $\sqrt{\nu''} \to \sqrt{\nu}$,

R_1ST^2: $\rho \to \rho$, $\sqrt{\mu} \to -\sqrt{\mu''}$, $\sqrt{\mu'} \to -\sqrt{\mu}$, $\sqrt{\mu''} \to \sqrt{\mu'}$,
$\theta \to \rho^2\theta$, $\sqrt{\nu} \to \sqrt{\nu''}$, $\sqrt{\nu'} \to \sqrt{\nu}$, $\sqrt{\nu''} \to \sqrt{\nu'}$,

$R_1^2ST^2$: $\rho \to \rho^2$, $\sqrt{\mu} \to -\sqrt{\mu'}$, $\sqrt{\mu'} \to \sqrt{\mu}$, $\sqrt{\mu''} \to -\sqrt{\mu''}$,
$\theta \to \rho\theta$, $\sqrt{\nu} \to \sqrt{\nu'}$, $\sqrt{\nu'} \to \sqrt{\nu}$, $\sqrt{\nu''} \to \sqrt{\nu''}$,

$R_1^3ST^2$: $\rho \to \rho$, $\sqrt{\mu} \to \sqrt{\mu''}$, $\sqrt{\mu'} \to -\sqrt{\mu}$, $\sqrt{\mu''} \to -\sqrt{\mu'}$,
$\theta \to \rho^2\theta$, $\sqrt{\nu} \to \sqrt{\nu''}$, $\sqrt{\nu'} \to \sqrt{\nu}$, $\sqrt{\nu''} \to \sqrt{\nu'}$,

R_2ST: $\rho \to \rho$, $\sqrt{\mu} \to \sqrt{\mu'}$, $\sqrt{\mu'} \to \sqrt{\mu''}$, $\sqrt{\mu''} \to \sqrt{\mu}$,
$\theta \to \rho\theta$, $\sqrt{\nu} \to \sqrt{\nu'}$, $\sqrt{\nu'} \to -\sqrt{\nu''}$, $\sqrt{\nu''} \to -\sqrt{\nu}$,

R_2^2ST: $\rho \to \rho^2$, $\sqrt{\mu} \to \sqrt{\mu''}$, $\sqrt{\mu'} \to \sqrt{\mu'}$, $\sqrt{\mu''} \to \sqrt{\mu}$,
$\theta \to \rho^2\theta$, $\sqrt{\nu} \to -\sqrt{\nu''}$, $\sqrt{\nu'} \to -\sqrt{\nu'}$, $\sqrt{\nu''} \to \sqrt{\nu}$,

R_2^3ST: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

R_2ST^2: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu'}$,

$R_2^2ST^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_2^3ST^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

R_1R_2ST: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1^2R_2ST$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$ $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1^3R_2ST$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1^2R_2^2ST$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

$R_1^3R_2^2ST$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu}$,

$R_1R_2^3ST$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1^2R_2^3ST$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1^3R_2^3ST$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu}$,

$R_1R_2ST^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$,

$R_1^2R_2ST^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu'}$,

$R_1^3R_2ST^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow \sqrt{\nu''}$,

$R_1R_2^2ST^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1^2R_2^2ST^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_1^3R_2^2ST^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow \sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow -\sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow \sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1R_2^3ST^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow \sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

$R_1^2R_2^3ST^2$: $\rho \longrightarrow \rho$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu''}$, $\sqrt{\mu'} \longrightarrow \sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow -\sqrt{\mu'}$,
$\theta \longrightarrow \rho^2\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu''}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu'}$,

$R_1^3R_2^3ST^2$: $\rho \longrightarrow \rho^2$, $\sqrt{\mu} \longrightarrow -\sqrt{\mu'}$, $\sqrt{\mu'} \longrightarrow -\sqrt{\mu}$, $\sqrt{\mu''} \longrightarrow \sqrt{\mu''}$,
$\theta \longrightarrow \rho\theta$, $\sqrt{\nu} \longrightarrow \sqrt{\nu'}$, $\sqrt{\nu'} \longrightarrow -\sqrt{\nu}$, $\sqrt{\nu''} \longrightarrow -\sqrt{\nu''}$,

From the above mappings we can find the order of each element by applying the mapping again and again as in Chapter II. So they are:

<u>order 2</u>: R_1^2, R_2^2, S, R_1R_2, $R_1^3R_2$, $R_1^2R_2^2$, $R_1R_2^3$, $R_1^3R_2^3$, R_1S,
R_1^2S, R_1^3S, R_2S, R_2^2S, R_2^3S, R_1T, $R_1^3T^2$, R_2T, $R_2^3T^2$,
ST, ST^2, $R_1^2R_2S$, $R_1R_2^2S$, $R_1^2R_2^2S$, $R_1^3R_2^2S$, $R_1^2R_2^3S$,

$R_1R_2^3ST$, $R_1^3R_2ST^2$.

order 3: T, T^2; $R_1^2T, R_1^3ST^2$; $R_1^2T^2, R_1ST$; $R_2^2T, R_2^3ST^2$;
$R_2^2T^2, R_2ST$; R_1ST^2, R_1^3ST; R_2ST^2, R_2^3ST;
$R_1R_2T, R_1^2R_2ST^2$; $R_1R_2T^2, R_1^3R_2^2ST$; $R_1^3R_2T, R_1R_2^3T^2$;
$R_1^3R_2T^2, R_1^2R_2^2T$; $R_1^2R_2^2T^2, R_1R_2^3T$; $R_1^3R_2^3T, R_1R_2^2ST^2$;
$R_1^3R_2^3T^2, R_1^2R_2^3ST$; $R_1^2R_2ST, R_1^3R_2^2ST^2$; $R_1R_2^2ST, R_1^2R_2^3ST^2$;

order 4: R_1, R_1^3; R_2, R_2^3; $R_1^2R_2, R_1^2R_2^3$; $R_1R_2^2, R_1^3R_2^2$;
R_1^3T, R_1^2ST; $R_1T^2, R_1^2ST^2$; R_2^3T, R_2^2ST; $R_2T^2, R_2^2ST^2$;
$R_1R_2S, R_1^3R_2^3S$; $R_1^3R_2S, R_1R_2^3S$; $R_1^2R_2T, R_1^3R_2^3ST$;
$R_1^2R_2T^2, R_1R_2^2T^2$; $R_1R_2^2T, R_1R_2ST$; $R_1^3R_2^2T, R_1^2R_2^3T$;
$R_1^3R_2^2T^2, R_1^3R_2^3ST^2$; $R_1^2R_2^3T^2, R_1R_2ST^2$; $R_1^3R_2ST, R_1^2R_2^2ST$;
$R_1^2R_2^2ST^2, R_1R_2^3ST^2$;

where each pair of elements of order 3 or 4 are inverses of each other. By using the same argument as Chapter II, the computation in commutators of Fact 2.10, we obtain the following commutator relations.

$[R_1, R_2] = R_1^2R_2^2$; $[R_1^2, R_2] = 1$; $[R_1^3, R_2] = R_1^2R_2^2$;
$[R_1, R_2^2] = 1$; $[R_1^2, R_2^2] = 1$; $[R_1^3, R_2^2] = 1$;
$[R_1, R_2^3] = R_1^2R_2^2$; $[R_1^2, R_2^3] = 1$; $[R_1^3, R_2^3] = R_1^2R_2^2$;
$[R_1, S] = R_1^2$; $[R_1^2, S] = 1$; $[R_1^3, S] = R_1^2$;
$[R_2, S] = R_2^2$; $[R_2^2, S] = 1$; $[R_2^3, S] = R_2^2$;
$[S, T] = T$; $[S, T^2] = T^2$;

$$[R_1,\ T] = R_1^3ST;\quad [R_1^2,\ T] = R_1^3S;\quad [R_1^3,\ T] = R_1^2T;$$

$$[R_2,\ T^2] = R_1^2T^2;\quad [R_1^2,\ T^2] = R_1S;\quad [R_1^3,\ T^2] = R_1ST^2;$$

$$[R_2,\ T] = R_2^3ST;\quad [R_2^2,\ T] = R_2^3S;\quad [R_2^3,\ T] = R_2^2T;$$

$$[R_2,\ T^2] = R_2^2T^2;\quad [R_2^2,\ T^2] = R_2S;\quad [R_2^3,\ T^2] = R_2ST^2;$$

Therefore, these identities complete all the relators for the Galois group of the normal extension. That is,

$$G^* = \langle R_1,\ R_2,\ S,\ T:\ R_1^4 = R_2^4 = S^2 = T^3 = 1,$$
$$[R_1,\ R_2] = R_1^2R_2^2,\ [R_1,\ S] = R_1^2,\ [R_2,\ S] = R_2^2,$$
$$[R_1,\ T] = R_1^3ST,\ [R_2,\ T] = R_2^3ST,\ [S,\ T] = T \rangle$$

Then G^* consists of 96 elements and G^* is non-abelian with trivial center subgroup of G^*. From Fact 2.10. and Sylow Theorem, we can find all the subgroups of different orders and check the number of subgroups of the special order 2^5 and 3. Hence we have:

<u>27 subgroups of order 2</u>:

$\{I,R_1^2\}$; $\{I,R_2^2\}$; $\{I,S\}$; $\{I,R_1R_2\}$; $\{I,R_1^3R_2\}$; $\{I,R_1^2R_2^2\}$;
$\{I,R_1R_2^3\}$; $\{I,R_1^3R_2^3\}$; $\{I,R_1S\}$; $\{I,R_1^2S\}$; $\{I,R_1^3S\}$; $\{I,R_2S\}$;
$\{I,R_2^2S\}$; $\{I,R_2^3S\}$; $\{I,R_1T\}$; $\{I,R_1^3T^2\}$; $\{I,R_2T\}$; $\{I,R_2^3T^2\}$;
$\{I,ST\}$; $\{I,ST^2\}$; $\{I,R_1^2R_2S\}$; $\{I,R_1R_2^2S\}$; $\{I,R_1^2R_2^2S\}$;
$\{I,R_1^3R_2^2S\}$; $\{I,R_1^2R_2^3S\}$; $\{I,R_1R_2^3ST\}$; $\{I,R_1^3R_2ST^2\}$;

<u>16 subgroups of order 3</u>:

$\{I,T,T^2\}$; $\{I,R_1^2T,R^3ST^2\}$; $\{I,R_1^2T^2,R_1ST\}$; $\{I,R_1ST^2,R_1^3ST\}$;

$\{I,R_2^2T,R_2^3ST^2\}$; $\{I,R_2^2T^2,R_2ST\}$; $\{I,R_2ST^2,R_2^3ST\}$;

$\{I,R_1R_2T,R_1^2R_2ST^2\}$; $\{I,R_1R_2T^2,R_1^3R_2^2ST\}$; $\{I,R_1^3R_2T,R_1R_2^3T^2\}$;

$\{I,R_1^3R_2T^2,R_1^2R_2^2T\}$; $\{I,R_1^2R_2^2T^2,R_1R_2^3T\}$; $\{I,R_1^3R_2^3T,R_1R_2^2ST^2\}$;

$\{I,R_1^3R_2^3T^2,R_1^2R_2^3ST\}$; $\{I,R_1^2R_2ST,R_1^3R_2^2ST^2\}$; $\{I,R_1R_2^2ST,R_1^2R_2^3ST^2\}$;

<u>71 subgroups of order 4:</u>

(i). non-cyclic

$\{I,R_1^2,R_2^2,R_1^2R_2^2\}$; $\{I,R_1^2,S,R_1^2S\}$; $\{I,R_1^2,R_1R_2,R_1^3R_2\}$;

$\{I,R_1^2,R_1R_2^3,R_1^3R_2^3\}$; $\{I,R_1^2,R_1S,R_1^3S\}$; $\{I,R_1^2,R_2S,R_1^2R_2S\}$;

$\{I,R_1^2,R_2^2S,R_1^2R_2^2S\}$; $\{I,R_1^2,R_2^3S,R_1^2R_2^3S\}$; $\{I,R_1^2,R_1R_2^2S,R_1^3R_2^2S\}$;

$\{I,R_2^2,S,R_2^2S\}$; $\{I,R_2^2,R_1R_2,R_1R_2^3\}$; $\{I,R_2^2,R_1^3R_2,R_1^3R_2^3\}$;

$\{I,R_2^2,R_1S,R_1R_2^2S\}$; $\{I,R_2^2,R_1^2S,R_1^2R_2^2S\}$; $\{I,R_2^2,R_1^3S,R_1^3R_2^2S\}$;

$\{I,R_2^2,R_2S,R_2^3S\}$; $\{I,R_2^2,R_1^2R_2S,R_1^2R_2^3S\}$; $\{I,S,R_1^2R_2^2,R_1^2R_2^2S\}$;

$\{I,R_1R_2,R_1^2R_2^2,R_1^3R_2^3\}$; $\{I,R_1R_2,R_1S,R_2^3S\}$; $\{I,R_1R_2,R_1^3S,R_1^2R_2^3S\}$;

$\{I,R_1R_2,R_2S,R_1R_2^2S\}$; $\{I,R_1R_2,R_1^2R_2S,R_1^3R_2^2S\}$;

$\{I,R_1^3R_2,R_1^2R_2^2,R_1R_2^3\}$; $\{I,R_1^3R_2,R_1S,R_1^2R_2^3S\}$;

$\{I,R_1^3R_2,R_1^3S,R_2^3S\}$; $\{I,R_1^3R_2,R_2S,R_1^3R_2^2S\}$;

$\{I,R_1^3R_2,R_1^3T^2,R_2^3T^2\}$; $\{I,R_1^3R_2,ST^2,R_1^3R_2ST^2\}$;

$\{I,R_1^3R_2,R_1^2R_2S,R_1R_2^2S\}$; $\{I,R_1^2R_2^2,R_1S,R_1^3R_2^2S\}$;

$\{I,R_1^2R_2^2,R_1^2S,R_2^2S\}$; $\{I,R_1^2R_2^2,R_1^3S,R_1R_2^2S\}$;

$\{I,R_1^2R_2^2,R_2S,R_1^2R_2^3S\}$; $\{I,R_1^2R_2^2,R_2^3S,R_1^2R_2S\}$;

$\{I,R_1R_2^3,R_1S,R_2S\}$; $\{I,R_1R_2^3,R_1^3S,R_1^2R_2S\}$

$\{I,R_1R_2^3,R_2^3S,R_1R_2^2S\}$; $\{I,R_1R_2^3,R_1T,R_2T\}$;

$\{I,R_1R_2^3,ST,R_1R_2^3ST\}$; $\{I,R_1R_2^3,R_1^3R_2^2S,R_1^2R_2^3S\}$;

$\{I,R_1^3R_2^3,R_1S,R_1^2R_2S\}$; $\{I,R_1^3R_2^3,R_1^3S,R_2S\}$;

$\{I,R_1^3R_2^3,R_2^3S,R_1^3R_2^2S\}$; $\{I,R_1^3R_2^3,R_1R_2^2S,R_1^2R_2^3S\}$;

$\{I,R_1S,R_1T,ST\}$; $\{I,R_1S,R_2T,R_1R_2^3ST\}$;

$\{I,R_1^3S,R_2^3T^2,R_1^3R_2ST^2\}$; $\{I,R_1^3S,ST^2,R_1^3T^2\}$;

$\{I,R_2S,R_1T,R_1R_2^3ST\}$; $\{I,R_2S,R_2T,ST\}$;

$\{I,R_2^3S,R_1^3T^2,R_1^3R_2ST^2,\}$; $\{I,R_2^3S,ST^2,R_2^3T^2\}$;

(ii) cyclic

$\{I,R_1,R_1^2,R_1^3\}$; $\{I,R_2,R_2^2,R_2^3\}$;

$\{I,R_1R_2^2,R_1^2,R_1^3R_2^2\}$; $\{I,\mathbf{R_1^2R_2},R_2^2,R_1^2R_2^3\}$;

$\{I,\mathbf{R_1^3}T,R_1S,R_1^2ST\}$; $\{I,R_1T^2,R_1^3S,R_1^2ST^2\}$;

$\{I,R_2^3T,R_2S,R_2^2ST\}$; $\{I,R_2T^2,R_2^3\mathbf{S},R_2^2ST^2\}$;

$\{I,R_1R_2S,R_1^2R_2^2,R_1^3R_2^3S\}$; $\{I,R_1^3R_2S,R_1^2R_2^2,R_1R_2^3S\}$;

$\{I,R_1^2R_2T,R_1S,R_1^3R_2^3ST\}$; $\{I,R_1^2R_2T^2,R_1^3R_2,R_1R_2^2T^2\}$;

$\{I,R_1R_2^2T,R_2S,R_1R_2ST\}$; $\{I,R_1^3R_2^2T,R_1R_2^3,R_1^2R_2^3T\}$;

$\{I,R_1^3R_2^2T^2,R_2^3S,R_1^3R_2^3ST^2\}$; $\{I,R_1^2R_2^3T^2,R_1^3S,R_1R_2ST^2\}$;

$\{I,R_1^3R_2ST,R_1R_2^3,R_1^2R_2^2ST\}$; $\{I,R_1^2R_2^2ST^2,R_1^3R_2,R_1R_2^3ST^2\}$;

<u>16 subgroups of order 6:</u> (of type S_3)

$\{I,S,T,T^2,ST,ST^2\}$; $\{I,R_1^2T,R_1^3ST^2,R_1^2S,ST,R_1^3T^2\}$;

$\{I,R_1ST^2,R_1^3ST,R_1T,R_1^3T^2,S\}$; $\{I,R_1^2T^2,R_1ST,R_1^2S,ST^2,R_1T\}$;

$\{I,R_2ST^2,R_2^3ST,R_2T,R_2^3T^2,S\}$; $\{I,R_2^2T,R_2^3ST^2,R_2^2S,ST,R_2^3T^2\}$;

$\{I,R_2^2T^2,R_2ST,R_2^2S,ST^2,R_2T\}$; $\{I,R_1R_2T,R_1^2R_2ST^2,R_1^2S,R_2^3T^2,R_1R_2^3ST\}$;

$\{I,R_1R_2T^2,R_1^3R_2^2ST,R_2^2S,R_1T,R_1^3R_2ST^2\}$;

$\{I,R_1^3R_2T,R_1R_2^3T^2,S,R_1R_2^3ST,R_1^3R_2ST^2\}$

$\{I,R_1^3R_2T^2,R_1^2R_2^2T,ST,R_1^2R_2^2S,R_1^3R_2ST^2\}$;

$\{I, R_1^2R_2^2T^2, R_1R_2^3T, ST^2, R_1^2R_2^2S, R_1R_2^3ST\}$;

$\{I, R_1^3R_2^3T, R_1R_2^2ST^2, R_2^2S, R_1^3T^2, R_1R_2^3ST\}$;

$\{I, R_1^3R_2^3T^2, R_1^2R_2^3ST, R_1^2S, R_2T, R_1^3R_2ST^2\}$;

$\{I, R_1^2R_2ST, R_1^3R_2^2ST^2, R_1^3T^2, R_2T, R_1^2R_2^2S\}$;

$\{I, R_1R_2^2ST, R_1^2R_2^3ST^2, R_1T, R_2^3T^2, R_1^2R_2^2S\}$;

<u>63 subgroups of order 8:</u>

(i) non-cyclic of type $C(2) \oplus C(2) \oplus C(2)$

$\{I, R_1^2, R_2^2, S, R_1^2R_2^2, R_2^2S, R_1^2S, R_1^2R_2^2S\}$;

$\{I, R_1^2, R_2^2, R_1R_2, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3R_2^3\}$;

$\{I, R_1^2, R_2^2, R_1^2R_2^2, R_1S, R_1^3S, R_1R_2^2S, R_1^3R_2^2S\}$;

$\{I, R_1^2, R_2^2, R_1^2R_2^2, R_2S, R_1^2R_2S, R_2^3S, R_1^2R_2^3S\}$;

$\{I, R_1^2, R_1R_2, R_1^3R_2, R_1S, R_1^3S, R_2^3S, R_1^2R_2^3S\}$;

$\{I, R_1^2, R_1R_2, R_1^3R_2, R_2S, R_1^2R_2S, R_1R_2^2S, R_1^3R_2^2S\}$;

$\{I, R_1^2, R_1R_2^3, R_1^3R_2^3, R_1S, R_1^3S, R_2S, R_1^2R_2S\}$;

$\{I, R_1^2, R_1R_2^3, R_1^3R_2^3, R_2^3S, R_1R_2^2S, R_1^3R_2^2S, R_1^2R_2^3S\}$;

$\{I, R_2^2, R_1R_2, R_1R_2^3, R_1S, R_2S, R_2^3S, R_1R_2^2S\}$;

$\{I, R_2^2, R_1R_2, R_1R_2^3, R_1^3S, R_1^2R_2S, R_1^3R_2^2S, R_1^2R_2^3S\}$;

$\{I, R_2^2, R_1^3R_2, R_1^3R_2^3, R_1S, R_1^2R_2S, R_1R_2^2S, R_1^2R_2^3S\}$;

$\{I, R_2^2, R_1^3R_2, R_1^3R_2^3, R_1^3S, R_2S, R_2^3S, R_1^3R_2^2S\}$;

$\{I, R_1R_2, R_1^2R_2^2, R_1^3R_2^3, R_1S, R_2^3S, R_1^2R_2S, R_1^3R_2^2S\}$;

$\{I, R_1R_2, R_1^2R_2^2, R_1^3R_2^3, R_1^3S, R_2S, R_1R_2^2S, R_1^2R_2^3S\}$;

$\{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1S, R_2S, R_1^3R_2^2S, R_1^2R_2^3S\}$;

$\{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3S, R_2^3S, R_1^2R_2S, R_1R_2^2S\}$;

$\{I, R_1^3R_2, R_1^3S, R_2^3S, R_1^3T^2, R_2^3T^2, ST^2, R_1^3R_2ST^2\}$;

$\{I, R_1R_2^3, R_1S, R_2S, R_1T, R_2T, ST, R_1R_2^3ST\}$;

(ii) non-cyclic of type $C(4) \oplus C(2)$

$\{I, R_1, R_1^2, R_1^3, R_2^2, R_1R_2^2, R_1^2R_2^2, R_1^3R_2^2\}$;

$\{I, R_2, R_2^2, R_2^3, R_1^2, R_1^2R_2, R_1^2R_2^2, R_1^2R_2^3\}$;

$\{I, R_1^3T, R_1S, R_1^2ST, R_1R_2^3, R_2S, R_1^2R_2T, R_1^3R_2^3ST\}$;

$\{I, R_1T^2, R_1^3S, R_1^2ST^2, R_1^3R_2, R_1^2R_2^3T^2, R_2^3S, R_1R_2ST^2\}$;

$\{I, R_2^3T, R_2S, R_2^2ST, R_1R_2^3, R_1R_2^2T, R_1S, R_1R_2ST\}$;

$\{I, R_2T^2, R_2^3S, R_2^2ST^2, R_1^3R_2, R_1^3R_2^2T^2, R_1^3S, R_1^3R_2^3ST^2\}$;

$\{I, R_1R_2S, R_1^2R_2^2, R_1^3R_2^3S, R_1^2, R_2^2, R_1^3R_2S, R_1R_2^3S\}$;

$\{I, R_1^2R_2T^2, R_1^3R_2, R_1R_2^2T^2, R_1^3S, R_2^3S, R_1^2R_2^2ST^2, R_1R_2^3ST^2\}$;

$\{I, R_1^3R_2^2T, R_1R_2^3, R_1^2R_2^3T, R_1S, R_2S, R_1^2R_2^2ST, R_1^3R_2ST\}$;

(iii) non-abelian of type D_4 (dihedral group)

$\{I, R_1, R_1^2, R_1^3, S, R_1S, R_1^2S, R_1^3S\}$;

$\{I, R_1, R_1^2, R_1^3, R_2^2S, R_1R_2^2S, R_1^2R_2^2S, R_1^3R_2^2S\}$;

$\{I, R_2, R_2^2, R_2^3, S, R_2S, R_2^2S, R_2^3S\}$;

$\{I, R_2, R_2^2, R_2^3, R_1^2S, R_1^2R_2S, R_1^2R_2^2S, R_1^2R_2^3S\}$;

$\{I, R_1R_2^2, R_1^2, R_1^3R_2^2, R_1S, R_1^3S, R_2^2S, R_1^2R_2^2S\}$;

$\{I, R_1R_2^2, R_1^2, R_1^3R_2^2, S, R_1R_2^2S, R_1^2S, R_1^3R_2^2S\}$;

$\{I, R_1^2R_2, R_2^2, R_1^2R_2^3, S, R_1^2R_2S, R_2^2S, R_1^2R_2^3S\}$;

$\{I, R_1^2R_2, R_2^2, R_1^2R_2^3, R_1^2S, R_2S, R_2^3S, R_1^2R_2^2S\}$;

$\{I, R_1^3T, R_1S, R_1^2ST, R_1^2, R_1^3S, R_1T, ST\}$;

$\{I, R_1^3T, R_1S, R_1^2ST, R_1^3R_2^3, R_2T, R_1^2R_2S, R_1R_2^3ST\}$;

$\{I, R_1T^2, R_1^3S, R_1^2ST^2, R_1^2, R_1^3T^2, R_1S, ST^2\}$;

$\{I, R_1T^2, R_1^3S, R_1^2ST^2, R_1R_2, R_2^3T^2, R_1^2R_2^3S, R_1^3R_2ST^2\}$;

$\{I, R_2^3T, R_2S, R_2^2ST, R_2^2, R_2^3S, R_2T, ST\}$;

$\{I, R_2^3T, R_2S, R_2^2ST, R_1R_2, R_1T, R_1R_2^2S, R_1R_2^3ST\}$;

$\{I, R_2T^2, R_2^3S, R_2^2ST^2, R_2^2, R_2^3T^2, R_2S, ST^2\}$;

$\{I, R_2T^2, R_2^3S, R_2^2ST^2, R_1^3R_2^3, R_1^3T^2, R_1^3R_2^2S, R_1^3R_2ST^2\}$;

$\{I, R_1R_2S, R_1^2R_2^2, R_1^3R_2^3S, S, R_1R_2, R_1^2R_2^2S, R_1^3R_2^3\}$;

$\{I, R_1R_2S, R_1^2R_2^2, R_1^3R_2^3S, R_1^3R_2, R_1R_2^3, R_1^2S, R_2^2S\}$;

$\{I, R_1^3R_2S, R_1^2R_2^2, R_1R_2^3S, R_1R_2, R_1^2S, R_1^3R_2^3, R_2^2S\}$;

$\{I, R_1^3R_2S, R_1^2R_2^2, R_1R_2^3S, R_1^3R_2, S, R_1R_2^3, R_1^2R_2^2S\}$;

$\{I, R_1^2R_2T, R_1S, R_1^3R_2^3ST, R_1^2, R_1^3S, R_2T, R_1R_2^3ST\}$;

$\{I, R_1^2R_2T, R_1S, R_1^3R_2^3ST, R_1^3R_2^3, R_1T, ST, R_1^2R_2S\}$;

$\{I, R_1^2R_2T^2, R_1^3R_2, R_1R_2^2T^2, R_1^2R_2^2, R_1^3T^2, R_1R_2^3, R_2^3T^2\}$;

$\{I, R_1^2R_2T^2, R_1^3R_2, R_1R_2^2T^2, ST^2, R_1^2R_2S, R_1^3R_2ST^2, R_1R_2^2S\}$;

$\{I, R_1R_2^2T, R_2S, R_1R_2ST, R_2^2, R_1T, R_2^3S, R_1R_2^3ST\}$;

$\{I, R_1R_2^2T, R_2S, R_1R_2ST, R_1R_2, R_2T, R_1R_2^2S, ST\}$;

$\{I, R_1^3R_2^2T, R_1R_2^3, R_1^2R_2^3T, R_1^3R_2, R_1T, R_1^2R_2^2, R_2T\}$;

$\{I, R_1^3R_2^2T, R_1R_2^3, R_1^2R_2^3T, ST, R_1^3R_2^2S, R_1^2R_2^3S, R_1R_2^3ST\}$;

$\{I, R_1^3R_2^2T^2, R_2^3S, R_1^3R_2^3ST^2, R_2^2, R_1^3T^2, R_2S, R_1^3R_2ST^2\}$;

$\{I, R_1^3R_2^2T^2, R_2^3S, R_1^3R_2^3ST^2, R_1^3R_2^3, R_2^3T^2, R_1^3R_2^2S, ST^2\}$;

$\{I, R_1^2R_2^3T^2, R_1^3S, R_1R_2ST^2, R_1^2, R_1S, R_2^3T^2, R_1^3R_2ST^2\}$;

$\{I, R_1^2R_2^3T^2, R_1^3S, R_1R_2ST^2, R_1R_2, R_1^3T^2, ST^2, R_1^2R_2^3S\}$;

$\{I, R_1^3R_2ST, R_1R_2^3, R_1^2R_2^2ST, R_1^3R_2, ST, R_1^2R_2^2, R_1R_2^3ST\}$;

$\{I, R_1^3R_2ST, R_1R_2^3, R_1^2R_2^2ST, R_1T, R_2T, R_1^2R_2^3S, R_1^3R_2^2S\}$;

$\{I, R_1^2R_2^2ST^2, R_1^3R_2, R_1R_2^3ST^2, R_1R_2^3, R_1^2R_2^2, ST^2, R_1^3R_2ST^2\}$;

$\{I, R_1^2R_2^2ST^2, R_1^3R_2, R_1R_2^3ST^2, R_1^3T^2, R_2^3T^2, R_1R_2^2S, R_1^2R_2S\}$;

<u>20 subgroups of order 12:</u>(of type A_4--Alternating group).

$K_8^{10} = \{I, R_1^2, R_1S, R_1^3S, T, T^2, R_1^2T, R_1^3ST^2, R_1^2T^2, R_1ST, R_1ST^2, R_1^3ST\}$

$K_8^{11} = \{I, R_1^2, R_1S, R_1^3S, R_2ST^2, R_2^3ST, R_1R_2T, R_1^2R_2ST^2, R_1^3R_2T, R_1R_2^3T^2,$

$R_1^3R_2^3T^2, R_1^2R_2^3ST\}$

$$K_8^{12} = \{I, R_1^2, R_1S, R_1^3S, R_2^2T, R_2^3ST^2, R_1R_2T^2, R_1^3R_2^2ST, R_1^2R_2^2T, R_1^3R_2T^2,$$
$$R_1R_2^2ST, R_1^2R_2^3ST^2\};$$

$$K_8^{13} = \{I, R_1^2, R_1S, R_1^3S, R_2^2T^2, R_2ST, R_1^2R_2^2T^2, R_1R_2^3T, R_1^3R_2^3T, R_1R_2^2ST^2,$$
$$R_1^2R_2ST, R_1^3R_2^2ST^2\};$$

$$K_8^{20} = \{I, R_2^2, R_2S, R_2^3S, T, T^2, R_2^2T, R_2^3ST^2, R_2^2T^2, R_2ST, R_2ST^2, R_2^3ST\};$$

$$K_8^{21} = \{I, R_2^2, R_2S, R_2^3S, R_1ST^2, R_1^3ST, R_1R_2T^2, R_1^3R_2^2ST, R_1^3R_2T, R_1R_2^3T^2,$$
$$R_1^3R_2^3T, R_1R_2^2ST^2\};$$

$$K_8^{22} = \{I, R_2^2, R_2S, R_2^3S, R_1^2T, R_1^3ST^2, R_1^3R_2T^2, R_1^2R_2^2T, R_1^3R_2^3T^2, R_1^2R_2^3ST,$$
$$R_1^2R_2ST, R_1^3R_2^2ST^2\};$$

$$K_8^{23} = \{I, R_2^2, R_2S, R_2^3S, R_1^2T^2, R_1ST, R_1R_2T, R_1^2R_2ST^2, R_1^2R_2^2T^2, R_1R_2^3T,$$
$$R_1R_2^2ST, R_1^2R_2^3ST^2\};$$

$$K_8^{30} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, T, T^2, R_1^3R_2T, R_1R_2^3T^2, R_1^3R_2T^2, R_1^2R_2^2T,$$
$$R_1^2R_2^2T^2, R_1R_2^3T\};$$

$$K_8^{31} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1ST^2, R_1^3ST, R_2ST^2, R_2^3ST, R_1^2R_2ST, R_1^3R_2^2ST^2,$$
$$R_1R_2^2ST, R_1^2R_2^3ST^2\};$$

$$K_8^{32} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^2T, R_1^3ST^2, R_2^2T, R_2^3ST^2, R_1R_2T, R_1^2R_2ST^2,$$
$$R_1^3R_2^3T, R_1R_2^2ST^2\};$$

$$K_8^{33} = \{I, R_1^3R_2, R_1^2R_2^2, R_1^3R_2^3, R_1^2T^2, R_1ST, R_2^2T^2, R_2ST, R_1R_2T^2, R_1^3R_2^2ST,$$
$$R_1^3R_2^3T^2, R_1^2R_2^3ST\};$$

$$K_8^{40} = \{I, R_1R_2, R_1^2R_2S, R_1^3R_2^2S, T, T^2, R_1R_2T, R_1^2R_2ST^2, R_1R_2T^2, R_1^3R_2^2ST,$$
$$R_1^2R_2ST, R_1^3R_2^2ST^2\};$$

$$K_8^{41} = \{I, R_1R_2, R_1^2R_2S, R_1^3R_2^2S, R_1ST^2, R_1^3ST, R_2^2T, R_2^3ST^2, R_1^2R_2^2T^2, R_1R_2^3T,$$
$$R_1^3R_2^3T^2, R_1^2R_2^3ST\};$$

$$K_8^{42} = \{I, R_1R_2, R_1^2R_2S, R_1^3R_2^2S, R_1^2T, R_1^3ST^2, R_2^2T^2, R_2ST, R_1^3R_2T, R_1R_2^3T^2,$$
$$R_1R_2^2ST, R_1^2R_2^3ST^2\};$$

$$K_8^{43} = \{I, R_1R_2, R_1^2R_2S, R_1^3R_2^2S, R_1^2T^2, R_1ST, R_2ST^2, R_2^3ST, R_1^3R_2T^2, R_1^2R_2^2T,$$
$$R_1^3R_2^3T, R_1R_2^2ST^2\};$$

$$K_8^{50} = \{I, R_1^3R_2^3, R_1R_2^2S, R_1^2R_2^3S, T, T^2, R_1^3R_2^3T, R_1R_2^2ST^2, R_1^3R_2^3T^2, R_1^2R_2^3ST,$$
$$R_1R_2^2ST, R_1^2R_2^3ST^2\};$$

$$K_8^{51} = \{I, R_1^3R_2^3, R_1R_2^2S, R_1^2R_2^3S, R_1ST^2, R_1^3ST, R_1^2R_2^2T, R_1^3R_2T^2, R_2ST, R_2^2T^2,$$
$$R_1R_2T, R_1^2R_2ST^2\};$$

$$K_8^{52} = \{I, R_1^3R_2^3, R_1R_2^2S, R_1^2R_2^3S, R_1^2T, R_1^3ST^2, R_2^3ST, R_2ST^2, R_1R_2^3T, R_1^2R_2^2T^2,$$
$$R_1R_2T^2, R_1^3R_2^2ST\};$$

$$K_8^{53} = \{I, R_1^3R_2^3, R_1R_2^2S, R_1^2R_2^3S, R_1^2T^2, R_1ST, R_2^2T, R_2^3ST^2, R_1^3R_2T, R_1R_2^3T^2, R_1^2R_2ST, R_1^3R_2^2ST^2\};$$

19 subgroups of order 16:

(i) non-cyclic of type $C(2) \oplus C(2) \oplus C(2) \oplus C(2)$

$$K_6^{oo} = \{I, R_1^2, R_2^2, R_1R_2, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3R_2^3, R_1S, R_1^3S, R_2S, R_2^3S, R_1^2R_2S, R_1R_2^2S, R_1^3R_2^2S, R_1^2R_2^3S\};$$

(ii) non-abelian of type $D_4 \oplus C(2)$

$$K_6^{11} = \{I, R_2, R_2^2, R_2^3, R_1^2, S, R_1^2R_2, R_1^2R_2^2, R_1^2R_2^3, R_1^2S, R_2S, R_2^2S, R_2^3S, R_1^2R_2S, R_1^2R_2^2S, R_1^2R_2^3S\}$$

$$K_6^{12} = \{I, R_1, R_1^2, R_1^3, R_2^2, S, R_1R_2^2, R_1^2R_2^2, R_1^3R_2^2, R_1S, R_1^2S, R_1^3S, R_2^2S, R_1R_2^2S, R_1^2R_2^2S, R_1^3R_2^2S\}$$

$$K_6^{13} = \{I, R_1^2, R_2^2, S, R_1R_2, R_1^3R_2, R_1R_2^3, R_1^2R_2^2, R_1^3R_2^3, R_1^2S, R_2^2S, R_1R_2S, R_1^3R_2S, R_1^2R_2^2S, R_1R_2^3S, R_1^3R_2^3S\}$$

$$K_6^{21} = \{I, R_2^2, R_1R_2, R_1R_2^3, R_1S, R_2S, R_2^3S, R_1T, R_2T, R_2^3T, ST, R_1R_2^2S, R_1R_2^2T, R_2^2ST, R_1R_2ST, R_1R_2^3ST\}$$

$$K_6^{22} = \{I, R_1^2, R_1R_2^3, R_1^3R_2^3, R_1S, R_1^3S, R_2S, R_1T, R_1^3T, R_2T, ST, R_1^2R_2S, R_1^2ST, R_1^2R_2T, R_1R_2^3ST, R_1^3R_2^3ST\};$$

$$K_6^{23} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1S, R_2S, R_1T, R_2T, ST, R_1^3R_2^2S, R_1^2R_2^3S, R_1^3R_2^2T, R_1^2R_2^3T, R_1R_2^3ST, R_1^3R_2ST, R_1^2R_2^2ST\};$$

$$K_6^{31} = \{I, R_2^2, R_1^3R_2, R_1^3R_2^3, R_1^3S, R_2S, R_2^3S, R_1^3T^2, R_2T^2, R_2^3T^2, ST^2,$$
$$R_1^3R_2^2S, R_1^3R_2^2T^2, R_2^2ST^2, R_1^3R_2ST^2, R_1^3R_2^3ST^2\};$$

$$K_6^{32} = \{I, R_1^2, R_1R_2, R_1^3R_2, R_1S, R_1^3S, R_2^3S, R_1T^2, R_1^3T^2, R_2^3T^2, ST^2,$$
$$R_1^2R_2^3S, R_1^2R_2^3T^2, R_1^2ST^2, R_1R_2ST^2, R_1^3R_2ST^2\};$$

$$K_6^{33} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3S, R_2^3S, R_1^3T^2, R_2^3T^2, ST^2, R_1^2R_2S, R_1R_2^2S,$$
$$R_1^2R_2T^2, R_1R_2^2T^2, R_1^3R_2ST^2, R_1^2R_2^2ST^2, R_1R_2^3ST^2\};$$

(iii) non-abelian of type H_{16}^*

$$K_6^{41} = \{I, R_1, R_1^2, R_1^3, R_2, R_2^2, R_2^3, R_1R_2, R_1^2R_2, R_1^3R_2, R_1R_2^2, R_1^2R_2^2, R_1^3R_2^2,$$
$$R_1R_2^3, R_1^2R_2^3, R_1^3R_2^3\};$$

$$K_6^{42} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1S, R_2S, R_1^3T, R_2^3T, R_1^3R_2^2S, R_1^2R_2^3S, R_1^2R_2T,$$
$$R_1R_2^2T, R_1^2ST, R_2^2ST, R_1R_2ST, R_1^3R_2^3ST\};$$

$$K_6^{43} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3S, R_2^3S, R_1T^2, R_2T^2, R_1^2R_2S, R_1R_2^2S,$$
$$R_1^3R_2^2T^2, R_1^2R_2^3T^2, R_1^2ST^2, R_2^2ST^2, R_1R_2ST^2, R_1^3R_2^3ST^2\};$$

$$K_6^{51} = \{I, R_1, R_1^2, R_1^3, R_2^2, R_1R_2^2, R_1^2R_2^2, R_1^3R_2^2, R_2S, R_2^3S, R_1R_2S, R_1^2R_2S,$$
$$R_1^3R_2S, R_1R_2^3S, R_1^2R_2^3S, R_1^3R_2^3S\};$$

$$K_6^{52} = \{I, R_2^2, R_1R_2, R_1R_2^3, R_1S, R_2S, R_2^3S, R_1^3T, R_1R_2^2S, R_1^2R_2T, R_1^3R_2^2T,$$
$$R_1^2R_2^3T, R_1^2ST, R_1^3R_2ST, R_1^2R_2^2ST, R_1^3R_2^3ST\};$$

$$K_6^{53} = \{I, R_2^2, R_1^3R_2, R_1^3R_2^3, R_1^3S, R_2S, R_2^3S, R_1T^2, R_1^3R_2^2S, R_1^2R_2T^2,$$
$$R_1R_2^2T^2, R_1^2R_2^3T^2, R_1^2ST^2, R_1R_2ST^2, R_1^2R_2^2ST^2, R_1R_2^3ST^2\};$$

$$K_6^{61} = \{I, R_1^2, R_2, R_2^2, R_2^3, R_1^2R_2, R_1^2R_2^2, R_1^2R_2^3, R_1S, R_1^3S, R_1R_2S, R_1^3R_2S,$$
$$R_1R_2^2S, R_1^3R_2^2S, R_1R_2^3S, R_1^3R_2^3S\}$$

$$K_6^{62} = \{I, R_1^2, R_1R_2^3, R_1^3R_2^3, R_1S, R_1^3S, R_2S, R_2^3T, R_1^2R_2S, R_1R_2^2T, R_1^3R_2^2T,$$
$$R_1^2R_2^3T, R_2^2ST, R_1R_2ST, R_1^2R_2^2ST, R_1^3R_2ST\};$$

$$K_6^{63} = \{I, R_1^2, R_1R_2, R_1^3R_2, R_1S, R_1^3S, R_2^3S, R_2T^2, R_1^2R_2^3S, R_1^2R_2T^2, R_1R_2^2T^2,$$
$$R_1^3R_2^2T^2, R_2^2ST^2, R_1^2R_2^2ST^2, R_1R_2^3ST^2, R_1^3R_2^3ST^2\};$$

where $H_{16}^* = \langle g_1, g_2 : g_1^4 = g_2^4 = 1, [g_1, g_2] = g_1^2g_2^2 \rangle$

<u>12 subgroups of order 24</u>: (of type S_4 -- the symmetry group)

$$K_4^{10} = \{I, R_1, R_1^2, R_1^3, S, T, T^2, R_1S, R_1^2S, R_1^3S, R_1T, R_1^2T, R_1^3T, R_1T^2, R_1^2T^2,$$
$$R_1^3T^2, ST, ST^2, R_1ST, R_1^2ST, R_1^3ST, R_1ST^2, R_1^2ST^2, R_1^3ST^2\}$$

$$K_4^{11} = \{I, R_1, R_1^2, R_1^3, S, R_1S, R_1^2S, R_1^3S, R_2T, R_2^3T^2, R_1R_2T, R_1^2R_2T, R_1^3R_2T,$$
$$R_1R_2^3T^2, R_1^2R_2^3T^2, R_1^3R_2^3T^2, R_2ST^2, R_2^3ST, R_1R_2ST^2, R_1^2R_2ST^2,$$
$$R_1^3R_2ST^2, R_1R_2^3ST, R_1^2R_2^3ST, R_1^3R_2^3ST\}$$

$$K_4^{12} = \{I, R_1^2, R_1R_2^2, R_1^3R_2^2, R_1S, R_1^3S, R_2^2S, R_1T, R_1^3T, R_2^2T, R_2^3T^2, ST,$$
$$R_1R_2T^2, R_1^3R_2T^2, R_1^2R_2^2T, R_1^2R_2^3T^2, R_1^2R_2S, R_1^2ST, R_2^3ST^2, R_1R_2ST^2,$$
$$R_1^3R_2ST^2, R_1R_2^2ST, R_1^3R_2^2ST, R_1^2R_2^3ST^2\}$$

$$K_4^{13} = \{I, R_1^2, R_1R_2^2, R_1^3R_2^2, R_1S, R_1^3S, R_2^2S, R_1T^2, R_1^3T^2, R_2T, R_2^2T^2, ST^2,$$
$$R_1^2R_2^2S, R_1^2R_2T, R_1^2R_2^2T^2, R_1R_2^3T, R_1^3R_2^3T, R_1^2ST^2, R_2ST, R_1^2R_2ST,$$
$$R_1R_2^2ST^2, R_1^3R_2^2ST^2, R_1R_2^3ST, R_1^3R_2^3ST\}$$

$$K_4^{20} = \{I, R_2, R_2^2, R_2^3, S, T, T^2, R_2S, R_2^2S, R_2^3S, R_2T, R_2^2T, R_2^3T, R_2T^2, R_2^2T_2^2,$$
$$R_2^3T^2, ST, ST^2, R_2ST, R_2^2ST, R_2^3ST, R_2ST^2, R_2^2ST^2, R_2^3ST^2\}$$

$$K_4^{21} = \{I, R_2, R_2^2, R_2^3, S, R_2S, R_2^2S, R_2^3S, R_1T, R_1^3T^2, R_1R_2T^2, R_1^3R_2T,$$
$$R_1R_2^2T, R_1^3R_2^2T^2, R_1R_2^3T^2, R_1^3R_2^3T, R_1ST^2, R_1^3ST, R_1R_2ST, R_1^3R_2ST^2,$$
$$R_1R_2^2ST^2, R_1^3R_2^2ST, R_1R_2^3ST, R_1^3R_2^3ST^2\};$$

$$K_4^{22} = \{I, R_2^2, R_1^2R_2, R_1^2R_2^3, R_1^2S, R_2S, R_2^3S, R_1^2T, R_1^3T^2, R_2T, R_2^3T, ST,$$
$$R_1^2R_2^2S, R_1^3R_2T^2, R_1^2R_2^2T, R_1^3R_2^2T^2, R_1^3R_2^3T^2, R_1^3ST^2, R_2^2ST, R_1^2R_2ST,$$
$$R_1^3R_2ST^2, R_1^3R_2^2ST^2, R_1^2R_2^3ST, R_1^3R_2^3ST^2\};$$

$$K_4^{23} = \{I, R_2^2, R_1^2R_2, R_1^2R_2^3, R_1^2S, R_2S, R_2^3S, R_1T, R_1^2T^2, R_2T^2, R_2^3T^2, ST^2,$$
$$R_1R_2T, R_1R_2^2T, R_1^2R_2^2T^2, R_1R_2^3T, R_1^2R_2^2S, R_1ST, R_2^2ST^2, R_1R_2ST,$$
$$R_1^2R_2ST^2, R_1R_2^2ST, R_1R_2^3ST, R_1^2R_2^3ST^2\}$$

$$K_4^{30} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, S, T, T^2, ST, ST^2, R_1^3R_2S, R_1^2R_2^2S, R_1R_2^3S, R_1^3R_2T,$$
$$R_1^3R_2T^2, R_1^2R_2^2T, R_1^2R_2^2T^2, R_1R_2^3T, R_1R_2^3T^2, R_1^3R_2ST, R_1^3R_2ST^2, R_1^2R_2^2ST,$$
$$R_1^2R_2^2ST^2, R_1R_2^3ST, R_1R_2^3ST^2\}$$

$$K_4^{31} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, S, R_1T, R_1^3T^2, R_2T, R_2^3T^2, R_1^3R_2S, R_1^2R_2^2S, R_1R_2^3S,$$
$$R_1R_2T^2, R_1^2R_2T^2, R_1^3R_2^2T, R_1^2R_2^3T, R_1ST^2, R_1^3ST, R_2ST^2, R_2^3ST, R_1^2R_2ST,$$
$$R_1R_2^2ST, R_1^3R_2^2ST^2, R_1^2R_2^3ST^2\}$$

$$K_4^{32} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^2S, R_2^2S, R_1^2T, R_2^2T, R_1^3T^2, R_2^3T^2, ST, R_1R_2S,$$
$$R_1^3R_2^3S, R_1R_2T, R_1^2R_2T^2, R_1R_2^2T^2, R_1^3R_2^3T, R_1^3ST^2, R_2^3ST^2, R_1^2R_2ST^2,$$
$$R_1^3R_2ST, R_1R_2^2ST^2, R_1^2R_2^2ST, R_1R_2^3ST\};$$

$$K_4^{33} = \{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^2S, R_2^2S, R_1T, R_1^2T^2, R_2T, R_2^2T^2, ST^2, R_1R_2S,$$
$$R_1^3R_2^3S, R_1R_2T^2, R_1^3R_2^2T, R_1^2R_2^3T, R_1^3R_2^3T^2, R_1ST, R_2ST, R_1^3R_2ST^2,$$
$$R_1^2R_2^2ST^2, R_1^3R_2^2ST, R_1R_2^3ST^2, R_1^2R_2^3ST\};$$

3 subgroups of order 32: (of type H^*_{32})

$$K_3^0 = \{I, R_1, R_1^2, R_1^3, R_2, R_2^2, R_2^3, R_1R_2, R_1^2R_2, R_1^3R_2, R_1R_2^2, R_1^2R_2^2, R_1^3R_2^2,$$
$$R_1R_2^3, R_1^2R_2^3, R_1^3R_2^3, S, R_1S, R_1^2S, R_1^3S, R_2S, R_2^2S, R_2^3S, R_1R_2S, R_1^2R_2S,$$
$$R_1^3R_2S, R_1R_2^2S, R_1^2R_2^2S, R_1^3R_2^2S, R_1R_2^3S, R_1^2R_2^3S, R_1^3R_2^3S\};$$

$$K_3^1 = \{I, R_1^2, R_2^2, R_1R_2, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3R_2^3, R_1S, R_1^3S, R_2S, R_2^3S, R_1T,$$
$$R_1^3T, R_2T, R_2^3T, ST, R_1^2R_2S, R_1R_2^2S, R_1^3R_2^2S, R_1^2R_2^3S, R_1^2R_2T, R_1R_2^2T,$$
$$R_1^3R_2^2T, R_1^2R_2^3T, R_1^2ST, R_2^2ST, R_1R_2ST, R_1^3R_2ST, R_1^2R_2^2ST, R_1R_2^3ST,$$
$$R_1^3R_2^3ST\};$$

$$K_3^2 = \{I, R_1^2, R_2^2, R_1R_2, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3R_2^3, R_1S, R_1^3S, R_2S, R_2^3S, R_1T^2,$$
$$R_1^3T^2, R_2T^2, R_2^3T^2, ST^2, R_1^2R_2S, R_1R_2^2S, R_1^3R_2^2S, R_1^2R_2^3S, R_1^2R_2T^2,$$
$$R_1R_2^2T^2, R_1^3R_2^2T^2, R_1^2R_2^3T^2, R_1^2ST^2, R_2^2ST^2, R_1R_2ST^2, R_1^3R_2ST^2,$$
$$R_1^2R_2^2ST^2, R_1R_2^3ST^2, R_1^3R_2^3ST^2\};$$

where $H^*_{32} = \langle h_1, h_2, h_3 \quad h_1^4 = h_2^4 = h_3^2 = 1;$

$$[h_1, h_2] = h_1^2h_2^2; [h_1, h_3] = h_1^2; [h_2, h_3] = h_2^2 \rangle$$

1 subgroup of order 48: (of type H^*_{48})

$$K_2^0 = \{I, R_1^2, R_2^2, R_1R_2, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3R_2^3, T, T^2, R_1S, R_1^3S, R_2S, R_2^3S,$$
$$R_1^2T, R_1^2T^2, R_2^2T, R_2^2T^2, R_1^2R_2S, R_1R_2^2S, R_1^3R_2^2S, R_1^2R_2^3S, R_1ST, R_1ST^2,$$
$$R_1^3ST, R_1^3ST^2, R_2ST, R_2ST^2, R_2^3ST, R_2^3ST^2, R_1R_2T, R_1R_2T^2, R_1^3R_2T,$$
$$R_1^3R_2T^2, R_1^2R_2^2T, R_1^2R_2^2T^2, R_1R_2^3T, R_1R_2^3T^2, R_1^3R_2^3T, R_1^3R_2^3T^2, R_1R_2^2ST,$$
$$R_1R_2^2ST^2, R_1^2R_2ST, R_1^2R_2ST^2, R_1^3R_2^2ST, R_1^3R_2^2ST^2, R_1^2R_2^3ST, R_1^2R_2^3ST^2\};$$

where $H_{48}^{*} = \{ a^{i_1} b^{i_2} c^{j} d^{n} : a^4 = b^4 = c^2 = d^3 = 1$ and

$[a, b] = a^2 b^2$; $[a, c] = a^2$; $[b, c] = b^2$;

$[a, d] = a^3 cd$; $[b, d] = b^3 cd$; $[c, d] = d$

and $i_1 + i_2 + j =$ even integer $\}$

Remark 3.5. $G^*(K_{96}^*/Q)$ is solvable, because of the following commutator subgroups (the derived series).

$G^{*\prime} = [G^*, G^*] = K_2^o$,

$G^{*\prime\prime} = [G^{*\prime}, G^{*\prime}] = [K_2^o, K_2^o] = K_6^{oo}$,

$G^{*\prime\prime\prime} = [G^{*\prime\prime}, G^{*\prime\prime}] = [K_6^{oo}, K_6^{oo}] = \{I\}$

So, $G^* \supset K_2^o \supset K_6^{oo} \supset \{I\}$.

§2. The corresponding subfields of K_{96}^*

We also see the following subfields which are corresponding to the subgroups of the Galois group $G(K_{96}^*/Q)$. We list all of them as the way we did in Chapter II.

Order 2 $\overset{G^*}{\longleftrightarrow}$ Degree 2 of K_{96}^*/K_{48}^*

$\{I, R_1^2\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\nu'}, \sqrt{\nu''})$;

$\{I, R_2^2\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'}, \sqrt{\mu''}, \sqrt{\nu})$;

$\{I, S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'}+\sqrt{\mu''}, \sqrt{\nu'}+\sqrt{\nu''})$;

$\{I, R_1R_2\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu'}, \sqrt{\nu''}, \sqrt{\mu\nu}, \sqrt{\mu''\nu'})$;

$\{I, R_1^3R_2\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\mu\nu}, \sqrt{\mu'\nu'})$;

$\{I, R_1^2R_2^2\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'\nu'}, \sqrt{\mu''\nu''})$;

$\{I, R_1R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu\nu}, \sqrt{\mu''\nu''})$;

$\{I, R_1^3R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu''}, \sqrt{\nu'}, \sqrt{\mu\nu}, \sqrt{\mu'\nu''})$;

$\{I, R_1S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu'}, \sqrt{\nu}, \sqrt{\nu'}, \sqrt{\nu''})$;

$\{I, R_1^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\nu'}+\sqrt{\nu''}, \sqrt{\mu'}-\sqrt{\mu''})$;

$\{I, R_1^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu''}, \sqrt{\nu}, \sqrt{\nu'}, \sqrt{\nu''})$;

$\{I, R_2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'}, \sqrt{\mu''}, \sqrt{\nu'})$;

$\{I, R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'}+\sqrt{\mu''}, \sqrt{\nu'}-\sqrt{\nu''})$;

$\{I, R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'}, \sqrt{\mu''}, \sqrt{\nu''})$;

$\{I, R_1T\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''}-\sqrt{\mu}, \sqrt{\nu''}+\sqrt{\nu})$;

$\{I, R_1^3T^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\mu}-\sqrt{\mu'}, \sqrt{\nu}+\sqrt{\nu'})$

$\{I, R_2T\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''}+\sqrt{\mu}, \sqrt{\nu''}-\sqrt{\nu})$;

$\{I,\ R_2^3T^2\} \xleftrightarrow{G^*} Q(\wp^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\mu} + \sqrt{\mu'}, \sqrt{\nu} - \sqrt{\nu'})$

$\{I,\ ST\} \xleftrightarrow{G^*} Q(\wp\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''} + \sqrt{\mu}, \sqrt{\nu''} + \sqrt{\nu})$;

$\{I,\ ST^2\} \xleftrightarrow{G^*} Q(\wp^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\mu} + \sqrt{\mu'}, \sqrt{\nu} + \sqrt{\nu'})$;

$\{I, R_1^2R_2S\} \xleftrightarrow{G^*} Q(\wp, \theta, \sqrt{\mu}, \sqrt{\nu'}, \sqrt{\mu'\nu}, \sqrt{\mu''\nu''})$;

$\{I, R_1R_2^2S\} \xleftrightarrow{G^*} Q(\wp, \theta, \sqrt{\mu'}, \sqrt{\nu}, \sqrt{\mu\nu'}, \sqrt{\mu''\nu''})$;

$\{I, R_1^2R_2^2S\} \xleftrightarrow{G^*} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'} - \sqrt{\mu''}, \sqrt{\nu'} - \sqrt{\nu''})$;

$\{I, R_1^3R_2^2S\} \xleftrightarrow{G^*} Q(\wp, \theta, \sqrt{\mu''}, \sqrt{\nu}, \sqrt{\mu'\nu'}, \sqrt{\mu\nu''})$;

$\{I, R_1^2R_2^3S\} \xleftrightarrow{G^*} Q(\wp, \theta, \sqrt{\mu}, \sqrt{\nu''}, \sqrt{\mu'\nu'}, \sqrt{\mu''\nu})$;

$\{I, R_1R_2^3ST\} \xleftrightarrow{G^*} Q(\wp\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''} - \sqrt{\mu}, \sqrt{\nu''} - \sqrt{\nu})$;

$\{I, R_1^3R_2ST^2\} \xleftrightarrow{G^*} Q(\wp^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\mu} - \sqrt{\mu'}, \sqrt{\nu} - \sqrt{\nu'})$;

Order 3 $\xleftrightarrow{G^*}$ Degree 3 of K_{96}^*/K_{32}^*

$\{I, T, T^2\} \leftrightarrow Q(\wp, \sqrt{\delta^0}, \sqrt{\gamma^0})$; $\{I, R_1ST^2, R_1^3ST\} \leftrightarrow Q(\wp, \sqrt{\delta'''}, \sqrt{\gamma^0})$;

$\{I, R_1^2T, R_1^3ST^2\} \leftrightarrow Q(\wp, \sqrt{\delta'}, \sqrt{\gamma^0})$; $\{I, R_1^2T^2, R_1ST\} \leftrightarrow Q(\wp, \sqrt{\delta''}, \sqrt{\gamma^0})$;

$\{I, R_2ST^2, R_2^3ST\} \leftrightarrow Q(\wp, \sqrt{\delta^0}, \sqrt{\gamma'''})$; $\{I, R_2^2T, R_2^3ST^2\} \leftrightarrow Q(\wp, \sqrt{\delta^0}, \sqrt{\gamma'})$;

$\{I, R_2^2T^2, R_2ST\} \leftrightarrow Q(\wp, \sqrt{\delta^0}, \sqrt{\gamma''})$; $\{I, R_1R_2T, R_1^2R_2ST^2\} \leftrightarrow Q(\wp, \sqrt{\delta''}, \sqrt{\gamma'''})$;

$\{I, R_1R_2T^2, R_1^3R_2^2ST\} \leftrightarrow Q(\wp, \sqrt{\delta'''}, \sqrt{\gamma'})$; $\{I, R_1^3R_2T, R_1R_2^3T^2\} \leftrightarrow Q(\wp, \sqrt{\delta'''}, \sqrt{\gamma'''})$;

$\{I, R_1^3R_2T^2, R_1^2R_2^2T\} \leftrightarrow Q(\wp, \sqrt{\delta'}, \sqrt{\gamma'})$; $\{I, R_1^2R_2^2T^2, R_1R_2^3T\} \leftrightarrow Q(\wp, \sqrt{\delta''}, \sqrt{\gamma''})$;

$\{I, R_1^3R_2^3T, R_1R_2^2ST^2\} \leftrightarrow Q(\wp, \sqrt{\delta'''}, \sqrt{\gamma''})$; $\{I, R_1^3R_2^3T^2, R_1^2R_2^2ST\} \leftrightarrow Q(\wp, \sqrt{\delta'}, \sqrt{\gamma'''})$;

$\{I, R_1^2R_2ST, R_1^3R_2^2ST^2\} \leftrightarrow Q(\wp, \sqrt{\delta'}, \sqrt{\gamma''})$; $\{I, R_1R_2^2ST, R_1^2R_2^3ST^2\} \leftrightarrow Q(\wp, \sqrt{\delta''}, \sqrt{\gamma'})$;

Order 4 $\xleftrightarrow{G^*}$ Degree 4 of K_{96}^*/K_{24}^*

(i). non-cyclic of type $C(2) \oplus C(2)$

$\{I, R_1^2, R_2^2, R_1^2R_2^2\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu}, \sqrt{\nu})$;

$\{I, R_1^2, S, R_1^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\nu'}+\sqrt{\nu''})$;

$\{I, R_1^2, R_1R_2, R_1^3R_2\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\nu''}, \sqrt{\mu\nu})$;

$\{I, R_1^2, R_1R_2^3, R_1^3R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\nu'}, \sqrt{\mu\nu})$;

$\{I, R_1^2, R_1S, R_1^3S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\nu}, \sqrt{\nu'})$;

$\{I, R_1^2, R_2S, R_1^2R_2S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu}, \sqrt{\nu'})$;

$\{I, R_1^2, R_2^2S, R_1^2R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\nu'}-\sqrt{\nu''})$;

$\{I, R_1^2, R_2^3S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu}, \sqrt{\nu''})$;

$\{I, R_1^2, R_1R_2^2S, R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\nu}, \sqrt{\mu\nu'})$;

$\{I, R_2^2, S, R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'}+\sqrt{\mu''})$;

$\{I, R_2^2, R_1R_2, R_1R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu'}, \sqrt{\mu\nu})$;

$\{I, R_2^2, R_1^3R_2, R_1^3R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu''}, \sqrt{\mu\nu})$;

$\{I, R_2^2, R_1S, R_1R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu'}, \sqrt{\nu})$;

$\{I, R_2^2, R_1^2S, R_1^2R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'}-\sqrt{\mu''})$;

$\{I, R_2^2, R_1^3S, R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu''}, \sqrt{\nu})$;

$\{I, R_2^2, R_2S, R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu}, \sqrt{\mu'})$;

$\{I, R_2^2, R_1^2R_2S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu}, \sqrt{\mu'\nu})$;

$\{I, S, R_1^2R_2^2, R_1^2R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'\nu'}+\sqrt{\mu''\nu''})$;

$\{I, R_1R_2, R_1^2R_2^2, R_1^3R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu\nu}, \sqrt{\mu'\nu''})$;

$\{I, R_1R_2, R_1S, R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\mu'}, \sqrt{\nu''})$;

$\{I, R_1R_2, R_1^3S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\wp, \theta, \sqrt{\nu''}, \sqrt{\mu''\nu'})$;

$\{I, R_1R_2, R_2S, R_1R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu'}, \sqrt{\mu\nu'});$

$\{I, R_1R_2, R_1^2R_2S, R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu\nu'}, \sqrt{\mu'\nu''});$

$\{I, R_1^3R_2, R_1^2R_2^2, R_1R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu\nu}, \sqrt{\mu'\nu'});$

$\{I, R_1^3R_2, R_1S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\nu''}, \sqrt{\mu'\nu'});$

$\{I, R_1^3R_2, R_1^3S, R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu''}, \sqrt{\nu''});$

$\{I, R_1^3R_2, R_2S, R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu''}, \sqrt{\mu'\nu'});$

$\{I, R_1^3R_2, R_1^3T^2, R_2^3T^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\mu\nu} - \sqrt{\mu'\nu'});$

$\{I, R_1^3R_2, ST^2, R_1^3R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\mu\nu} + \sqrt{\mu'\nu'});$

$\{I, R_1^3R_2, R_1^2R_2S, R_1R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu\nu'}, \sqrt{\mu''\nu''});$

$\{I, R_1^2R_2^2, R_1S, R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\nu}, \sqrt{\mu'\nu'});$

$\{I, R_1^2R_2^2, R_1^2S, R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'\nu'} - \sqrt{\mu''\nu''});$

$\{I, R_1^2R_2^2, R_1^3S, R_1R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\nu}, \sqrt{\mu''\nu''});$

$\{I, R_1^2R_2^2, R_2S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'\nu'});$

$\{I, R_1^2R_2^2, R_2^3S, R_1^2R_2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu''\nu''});$

$\{I, R_1R_2^3, R_1S, R_2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu'}, \sqrt{\nu'});$

$\{I, R_1R_2^3, R_1^3S, R_1^2R_2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\nu'}, \sqrt{\mu''\nu''});$

$\{I, R_1R_2^3, R_2^3S, R_1R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu'}, \sqrt{\mu''\nu''});$

$\{I, R_1R_2^3, R_1T, R_2T\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''\nu''} - \sqrt{\mu\nu});$

$\{I, R_1R_2^3, ST, R_1R_2^3ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''\nu''} + \sqrt{\mu\nu});$

$\{I, R_1R_2^3, R_1^3R_2^2S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu\nu''}, \sqrt{\mu'\nu'});$

$\{I, R_1^3R_2^3, R_1S, R_1^2R_2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\nu'}, \sqrt{\mu'\nu});$

$\{I, R_1^3R_2^3, R_1^3S, R_2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu''}, \sqrt{\nu'})$;

$\{I, R_1^3R_2^3, R_2^3S, R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu''}, \sqrt{\mu'\nu''})$;

$\{I, R_1^3R_2^3, R_1R_2^2S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu\nu''}, \sqrt{\mu'\nu})$;

$\{I, R_1S, R_1T, ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\nu''}+\sqrt{\nu})$;

$\{I, R_1S, R_2T, R_1R_2^3ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\nu''}-\sqrt{\nu})$;

$\{I, R_1^3S, R_2^3T^2, R_1^3R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\nu} - \sqrt{\nu'})$;

$\{I, R_1^3S, R_1^3T^2, ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\nu} + \sqrt{\nu'})$;

$\{I, R_2S, R_1T, R_1R_2^3ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''} - \sqrt{\mu})$;

$\{I, R_2S, R_2T, ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''} + \sqrt{\mu})$

$\{I, R_2^3S, R_1^3T^2, R_1^3R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'}, \sqrt{\mu''}- \sqrt{\mu})$;

$\{I, R_2^3S, R_2^3T^2, ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{\nu''}, \sqrt{\mu} + \sqrt{\mu'})$;

(ii). cyclic of type **C**(4)

$\{I, R_1, R_1^2, R_1^3\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{-3\mu}, \sqrt{\nu}, \sqrt{\nu'}+\sqrt{\nu''})$;

$\{I, R_2, R_2^2, R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{-3\nu}, \sqrt{\mu'}+\sqrt{\mu''})$;

$\{I, R_1R_2^2, R_1^2, R_1^3R_2^2\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{-3\mu}, \sqrt{\nu}, \sqrt{\nu'} - \sqrt{\nu''})$;

$\{I, R_1^2R_2, R_2^2, R_1^2R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{-3\nu}, \sqrt{\mu'}-\sqrt{\mu''})$;

$\{I, R_1^3T, R_1S, R_1^2ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{-3\mu'}, \sqrt{\nu'}, \sqrt{\nu''}+\sqrt{\nu})$;

$\{I, R_1T^2, R_1^3S, R_1^2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{-3\mu''}, \sqrt{\nu''}, \sqrt{\nu} + \sqrt{\nu'})$

$\{I, R_2^3T, R_2S, R_2^2ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{-3\nu'}, \sqrt{\mu''}+\sqrt{\mu})$;

$\{I, R_2T^2, R_2^3S, R_2^2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{-3\nu''}, \sqrt{\mu} + \sqrt{\mu'})$;

$\{I, R_1R_2S, R_1^2R_2^2, R_1^3R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{-3\mu}, \sqrt{-3\nu}, \sqrt{\mu'\nu'} - \sqrt{\mu''\nu''})$;

$\{I, R_1^3R_2S, R_1^2R_2^2, R_1R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{-3\mu}, \sqrt{-3\nu}, \sqrt{\mu'\nu'}+\sqrt{\mu''\nu''})$;

$\{I, R_1^2R_2T, R_1S, R_1^3R_2^3ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{-3\mu'}, \sqrt{\nu'}, \sqrt{\nu''}-\sqrt{\nu})$;

$\{I, R_1^2R_2T^2, R_1^3R_2, R_1R_2^2T^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{-3\mu''}, \sqrt{-3\nu''}, \sqrt{\mu\nu}-\sqrt{\mu'\nu'})$;

$\{I, R_1R_2^2T, R_2S, R_1R_2ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{\mu'}, \sqrt{-3\nu'}, \sqrt{\mu''}-\sqrt{\mu})$;

$\{I, R_1^3R_2^2T, R_1R_2^3, R_1^2R_2^3T\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{-3\mu'}, \sqrt{-3\nu'}, \sqrt{\mu''\nu''}-\sqrt{\mu\nu})$;

$\{I, R_1^3R_2^2T^2, R_2^3S, R_1^3R_2^3ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{-3\nu''}, \sqrt{\mu}-\sqrt{\mu'})$;

$\{I, R_1^2R_2^3T^2, R_1^3S, R_1R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{-3\mu''}, \sqrt{\nu''}, \sqrt{\nu}-\sqrt{\nu'})$;

$\{I, R_1^3R_2ST, R_1R_2^3, R_1^2R_2^2ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta, \sqrt{-3\mu'}, \sqrt{-3\nu'}, \sqrt{\mu''\nu''}+\sqrt{\mu\nu})$;

$\{I, R_1^2R_2^2ST^2, R_1^3R_2, R_1R_2^3ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta, \sqrt{-3\mu''}, \sqrt{-3\nu''}, \sqrt{\mu\nu}+\sqrt{\mu'\nu'})$;

Order 6 $\overset{G^*}{\longleftrightarrow}$ Degree 6 of K_{96}/K_{16} (of type S_3)

$\{I, S, T, T^2, ST, ST^2\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta^0}, \sqrt{\gamma^0})$;

$\{I, R_1ST^2, R_1^3ST, S, R_1T, R_1^3T^2\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta'''}, \sqrt{\gamma^0})$;

$\{I, R_1^2T, R_1^3ST^2, R_1^2S, ST, R_1^3T^2\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta'}, \sqrt{\gamma^0})$;

$\{I, R_1^2T^2, R_1ST, R_1^2S, ST^2, R_1T\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta''}, \sqrt{\gamma^0})$;

$\{I, R_2ST^2, R_2^3ST, R_2T, R_2^3T^2, S\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta^0}, \sqrt{\gamma'''})$;

$\{I, R_2^2T, R_2^3ST^2, R_2^2S, ST, R_2^3T^2\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta^0}, \sqrt{\gamma'})$;

$\{I, R_2^2T^2, R_2ST, R_2^2S, ST^2, R_2T\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta^0}, \sqrt{\gamma''})$;

$\{I, R_1R_2T, R_1^2R_2ST^2, R_1^2S, R_2^3T^2, R_1R_2^3ST\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta''}, \sqrt{\gamma'''})$;

$\{I, R_1R_2T^2, R_1^3R_2^2ST, R_2^2S, R_1T, R_1^3R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta'''}, \sqrt{\gamma'})$;

$\{I, R_1^3R_2T, R_1R_2^3T^2, S, R_1R_2^3ST, R_1^3R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta'''}, \sqrt{\gamma'''})$;

$\{I, R_1^3R_2T^2, R_1^2R_2^2T, ST, R_1^2R_2^2S, R_1^3R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta'}, \sqrt{\gamma'})$;

$\{I, R_1^2R_2^2T^2, R_1R_2^3T, ST^2, R_1^2R_2^2S, R_1R_2^3ST\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta"}, \sqrt{\gamma"})$;

$\{I, R_1^3R_2^3T, R_1R_2^2ST^2, R_2^2S, R_1^3T^2, R_1R_2^3ST\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta'''}, \sqrt{\gamma"})$;

$\{I, R_1^3R_2^3T^2, R_1^2R_2^3ST, R_1^2S, R_2T, R_1^3R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta'}, \sqrt{\gamma'''})$;

$\{I, R_1^2R_2ST, R_1^3R_2^2ST^2, R_1^3T^2, R_2T, R_1^2R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta'}, \sqrt{\gamma"})$;

$\{I, R_1R_2^2ST, R_1^2R_2^3ST^2, R_1T, R_2^3T^2, R_1^2R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\sqrt{\delta"}, \sqrt{\gamma'})$;

where $\sqrt{\delta^o} = \sqrt{\mu} + \sqrt{\mu'} + \sqrt{\mu"}$; $\sqrt{\gamma^o} = \sqrt{\nu} + \sqrt{\nu'} + \sqrt{\nu"}$;

$\sqrt{\delta'} = -\sqrt{\mu} + \sqrt{\mu'} - \sqrt{\mu"}$; $\sqrt{\gamma'} = -\sqrt{\nu} + \sqrt{\nu'} - \sqrt{\nu"}$;

$\sqrt{\delta"} = -\sqrt{\mu} - \sqrt{\mu'} + \sqrt{\mu"}$; $\sqrt{\gamma"} = -\sqrt{\nu} - \sqrt{\nu'} + \sqrt{\nu"}$;

$\sqrt{\delta'''} = \sqrt{\mu} - \sqrt{\mu'} - \sqrt{\mu"}$; $\sqrt{\gamma'''} = \sqrt{\nu} - \sqrt{\nu'} - \sqrt{\nu"}$;

Order 8 $\overset{G^*}{\longleftrightarrow}$ Degree 8 of K_{96}^*/K_{12}^*

(i). non-cyclic of type $C(2) \oplus C(2) \oplus C(2)$

$\{I, R_1^2, R_2^2, S, R_1^2R_2^2, R_1^2S, R_2^2S, R_1^2R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\theta, \sqrt{\mu}, \sqrt{\nu})$;

$\{I, R_1^2, R_2^2, R_1R_2, R_1^3R_2, R_1^2R_2^2, R_1R_2^3, R_1^3R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu\nu})$;

$\{I, R_1^2, R_2^2, R_1^2R_2^2, R_1S, R_1^3S, R_1R_2^2S, R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\nu})$;

$\{I, R_1^2, R_2^2, R_1^2R_2^2, R_2S, R_1^2R_2S, R_2^3S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu})$;

$\{I, R_1^2, R_1R_2, R_1^3R_2, R_1S, R_1^3S, R_2^3S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\nu"})$;

$\{I, R_1^2, R_1R_2, R_1^3R_2, R_2S, R_1^2R_2S, R_1R_2^2S, R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu\nu'})$;

$\{I, R_1^2, R_1R_2^3, R_1^3R_2^3, R_1S, R_1^3S, R_2S, R_1^2R_2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\nu'})$;

$\{I, R_1^2, R_1R_2^3, R_1^3R_2^3, R_2^3S, R_1R_2^2S, R_1^3R_2^2S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu\nu"})$;

$\{I, R_2^2, R_1R_2, R_1R_2^3, R_1S, R_2S, R_2^3S, R_1R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu'})$;

$\{I, R_2^2, R_1R_2, R_1R_2^3, R_1^3S, R_1^2R_2S, R_1^3R_2^2S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu"\nu})$;

$\{I, R_2^2, R_1^3R_2, R_1^3R_2^3, R_1S, R_1^2R_2S, R_1R_2^2S, R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho, \theta, \sqrt{\mu'\nu})$;

$\{I,R_2^2,R_1^3R_2,R_1^3R_2^3,R_1^3S,R_2S,R_2^3S,R_1^3R_2^2S\} \quad \overset{G^*}{\longleftrightarrow} Q(\rho,\theta,\sqrt{\mu''});$

$\{I,R_1R_2,R_1^2R_2^2,R_1^3R_2^3,R_1S,R_2^3S,R_1^2R_2S,R_1^3R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho,\theta,\sqrt{\mu'\nu''});$

$\{I,R_1R_2,R_1^2R_2^2,R_1^3R_2^3,R_1^3S,R_2S,R_1R_2^2S,R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho,\theta,\sqrt{\mu''\nu'});$

$\{I,R_1^3R_2,R_1^2R_2^2,R_1R_2^3,R_1S,R_2S,R_1^3R_2^2S,R_1^2R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\rho,\theta,\sqrt{\mu'\nu'});$

$\{I,R_1^3R_2,R_1^2R_2^2,R_1R_2^3,R_1^3S,R_2^3S,R_1^2R_2S,R_1R_2^2S\} \overset{G^*}{\longleftrightarrow} Q(\rho,\theta,\sqrt{\mu''\nu''});$

$\{I,R_1^3R_2,R_1^3S,R_2^3S,R_1^3T^2,R_2^3T^2,ST^2,R_1^3R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta,\sqrt{\mu''},\sqrt{\nu''});$

$\{I,R_1R_2^3,R_1S,R_2S,R_1T,R_2T,ST,R_1R_2^3ST\} \quad \overset{G^*}{\longleftrightarrow} Q(\rho\theta,\sqrt{\mu'},\sqrt{\nu'});$

(ii). non-cyclic of type $C(4)\oplus C(2)$

$\{I,R_1,R_1^2,R_1^3,R_2^2,R_1R_2^2,R_1^2R_2^2,R_1^3R_2^2\} \overset{G^*}{\longleftrightarrow} Q(\theta,\sqrt{-3\mu},\sqrt{\nu});$

$\{I,R_2,R_2^2,R_2^3,R_1^2,R_1^2R_2,R_1^2R_2^2,R_1^2R_2^3\} \overset{G^*}{\longleftrightarrow} Q(\theta,\sqrt{\mu},\sqrt{-3\nu});$

$\{I,R_1^3T,R_1S,R_1^2ST,R_1R_2^3,R_2S,R_1^2R_2T,R_1^3R_2^3ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta,\sqrt{-3\mu'},\sqrt{\nu'});$

$\{I,R_1T^2,R_1^3S,R_1^2ST^2,R_1^3R_2,R_1^2R_2^3T^2,R_2^3S,R_1R_2ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta,\sqrt{-3\mu''},\sqrt{\nu''});$

$\{I,R_2^3T,R_2S,R_2^2ST,R_1R_2^3,R_1R_2^2T,R_1S,R_1R_2ST\} \overset{G^*}{\longleftrightarrow} Q(\rho\theta,\sqrt{\mu'},\sqrt{-3\nu'});$

$\{I,R_2T^2,R_2^3S,R_2^2ST^2,R_1^3R_2,R_1^3R_2^2T^2,R_1^3S,R_1^3R_2^3ST^2\} \overset{G^*}{\longleftrightarrow} Q(\rho^2\theta,\sqrt{\mu''},\sqrt{-3\nu''});$

$\{I,R_1R_2S,R_1^2R_2^2,R_1^3R_2^3S,R_1^2,R_2^2,R_1^3R_2S,R_1R_2^3S\} \overset{G^*}{\longleftrightarrow} Q(\theta,\sqrt{-3\mu},\sqrt{-3\nu});$

$\{I,R_1^2R_2T^2,R_1^3R_2,R_1R_2^2T^2,R_1^3S,R_2^3S,R_1^2R_2^2ST^2,R_1R_2^3ST^2\}$
$\overset{G^*}{\longleftrightarrow} Q(\rho^2\theta,\sqrt{-3\mu''},\sqrt{-3\nu''});$

$\{I,R_1^3R_2^2T,R_1R_2^3,R_1^2R_2^3T,R_1S,R_2S,R_1^2R_2^2ST,R_1^3R_2ST\}$
$\overset{G^*}{\longleftrightarrow} Q(\rho\theta,\sqrt{-3\mu'},\sqrt{-3\nu'});$

(iii). non-abelian of type D_4 (dihedral gp.)

$\{I,R_1,R_1^2,R_1^3,S,R_1S,R_1^2S,R_1^3S\} \overset{G^*}{\longleftrightarrow} Q(\theta,\sqrt{\nu},\sqrt{\nu'}+\sqrt{\nu''});$

$\{I,R_1,R_1^2,R_1^3,R_2^2S,R_1R_2^2S,R_1^2R_2^2S,R_1^3R_2^2S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\nu},\sqrt{\mu\nu'}-\sqrt{\mu\nu''});$

$\{I,R_2,R_2^2,R_2^3,S,R_2S,R_2^2S,R_2^3S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\mu},\sqrt{\mu'}+\sqrt{\mu''});$

$\{I,R_2,R_2^2,R_2^3,R_1^2S,R_1^2R_2S,R_1^2R_2^2S,R_1^2R_2^3S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\mu},\sqrt{\mu'\nu}-\sqrt{\mu''\nu});$

$\{I,R_1R_2^2,R_1^2,R_1^3R_2^2,R_1S,R_1^3S,R_2^2S,R_1^2R_2^2S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\nu},\sqrt{\nu'}-\sqrt{\nu''});$

$\{I,R_1R_2^2,R_1^2,R_1^3R_2^2,S,R_1R_2^2S,R_1^2S,R_1^3R_2^2S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\nu},\sqrt{\mu\nu'}+\sqrt{\mu\nu''});$

$\{I,R_1^2R_2,R_2^2,R_1^2R_2^3,S,R_1^2R_2S,R_2^2S,R_1^2R_2^3S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\mu},\sqrt{\mu'\nu}+\sqrt{\mu''\nu});$

$\{I,R_1^2R_2,R_2^2,R_1^2R_2^3,R_1^2S,R_2S,R_2^3S,R_1^2R_2^2S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\mu},\sqrt{\mu'}-\sqrt{\mu''});$

$\{I,R_1^3T,R_1S,R_1^2ST,R_1^2,R_1^3S,R_1T,ST\} \xleftrightarrow{G^*} Q(\rho\theta,\sqrt{\nu'},\sqrt{\nu''}+\sqrt{\nu});$

$\{I,R_1^3T,R_1S,R_1^2ST,R_1^3R_2^3,R_2T,R_1^2R_2S,R_1R_2^3ST\} \xleftrightarrow{G^*} Q(\rho\theta,\sqrt{\nu'},\sqrt{\mu'\nu''}-\sqrt{\mu'\nu});$

$\{I,R_1T^2,R_1^3S,R_1^2ST^2,R_1^2,R_1^3T^2,R_1S,ST^2\} \xleftrightarrow{G^*} Q(\rho^2\theta,\sqrt{\nu''},\sqrt{\nu}+\sqrt{\nu'});$

$\{I,R_1T^2,R_1^3S,R_1^2ST^2,R_1R_2,R_2^3T^2,R_1^2R_2^3S,R_1^3R_2ST^2\} \xleftrightarrow{} Q(\rho^2\theta,\sqrt{\nu''},\sqrt{\mu''\nu}-\sqrt{\mu''\nu'});$

$\{I,R_2^3T,R_2S,R_2^2ST,R_2^2,R_2^3S,R_2T,ST\} \xleftrightarrow{G^*} Q(\rho\theta,\sqrt{\mu'},\sqrt{\mu''}+\sqrt{\mu});$

$\{I,R_2^3T,R_2S,R_2^2ST,R_1R_2,R_1T,R_1R_2^2S,R_1R_2^3ST\} \xleftrightarrow{G^*} Q(\rho\theta,\sqrt{\mu'},\sqrt{\mu''\nu'}-\sqrt{\mu\nu'});$

$\{I,R_2T^2,R_2^3S,R_2^2ST^2,R_2^2,R_2^3T^2,R_2S,ST^2\} \xleftrightarrow{G^*} Q(\rho^2\theta,\sqrt{\mu''},\sqrt{\mu}+\sqrt{\mu'});$

$\{I,R_2T^2,R_2^3S,R_2^2ST^2,R_1^3R_2^3,R_1^3T^2,R_1^3R_2^2S,R_1^3R_2ST^2\} \xleftrightarrow{G^*} Q(\rho^2\theta,\sqrt{\mu''},\sqrt{\mu\nu''}-\sqrt{\mu'\nu''});$

$\{I,R_1R_2S,R_1^2R_2^2,R_1^3R_2^3S,S,R_1R_2,R_1^2R_2^2,R_1^2R_2^2S,R_1^3R_2^3\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\mu\nu},\sqrt{\mu'\nu''}+\sqrt{\mu''\nu'});$

$\{I,R_1R_2S,R_1^2R_2^2,R_1^3R_2^3S,R_1^3R_2,R_1R_2^3,R_1^2S,R_2^2S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\mu\nu},\sqrt{\mu'\nu'}-\sqrt{\mu''\nu''});$

$\{I,R_1^3R_2S,R_1^2R_2^2,R_1R_2^3S,R_1R_2,R_1^3R_2^3,R_1^2S,R_2^2S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\mu\nu},\sqrt{\mu'\nu''}-\sqrt{\mu''\nu'});$

$\{I,R_1^3R_2S,R_1^2R_2^2,R_1R_2^3S,S,R_1^3R_2,R_1R_2^3,R_1^2R_2^2S\} \xleftrightarrow{G^*} Q(\theta,\sqrt{\mu\nu},\sqrt{\mu'\nu'}+\sqrt{\mu''\nu''});$

$\{I,R_1^2R_2T,R_1S,R_1^3R_2^3ST,R_1^2,R_1^3S,R_2T,R_1R_2^3ST\} \xleftrightarrow{G^*} Q(\rho\theta,\sqrt{\nu'},\sqrt{\nu''}-\sqrt{\nu});$

$\{I,R_1^2R_2T,R_1S,R_1^3R_2^3ST,R_1^3R_2^3,R_1T,ST,R_1^2R_2S\} \xleftrightarrow{G^*} Q(\rho\theta,\sqrt{\nu'},\sqrt{\mu'\nu''}+\sqrt{\mu'\nu});$

$\{I, R_1^2R_2T^2, R_1^3R_2, R_1R_2^2T^2, R_1^2R_2^2, R_1^3T^2, R_1R_2^3, R_2^3T^2\}$

$\xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\mu''\nu''}, \sqrt{\mu\nu} - \sqrt{\mu'\nu'})$;

$\{I, R_1^2R_2T^2, R_1^3R_2, R_1R_2^2T^2, ST^2, R_1^2R_2S, R_1R_2^2S, R_1^3R_2ST^2\}$

$\xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\mu''\nu''}, \sqrt{\mu\nu'} + \sqrt{\mu'\nu})$;

$\{I, R_1R_2^2T, R_2S, R_1R_2ST, R_2^2, R_1T, R_2^3S, R_1R_2^3ST\}$

$\xleftrightarrow{G^*} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\mu''} - \sqrt{\mu})$;

$\{I, R_1R_2^2T, R_2S, R_1R_2ST, R_1R_2, R_2T, ST, R_1R_2^2S\}$

$\xleftrightarrow{G^*} Q(\rho\theta, \sqrt{\mu'}, \sqrt{\mu''\nu'} + \sqrt{\mu\nu'})$;

$\{I, R_1^3R_2^2T, R_1R_2^3, R_1^2R_2^3T, R_1^3R_2, R_1^2R_2^2, R_1T, R_2T\}$

$\xleftrightarrow{G^*} Q(\rho\theta, \sqrt{\mu'\nu'}, \sqrt{\mu''\nu''} - \sqrt{\mu\nu})$;

$\{I, R_1^3R_2^2T, R_1R_2^3, R_1^2R_2^3T, ST, R_1^3R_2^2S, R_1^2R_2^3S, R_1R_2^3ST\}$

$\xleftrightarrow{G^*} Q(\rho\theta, \sqrt{\mu'\nu'}, \sqrt{\mu\nu''} + \sqrt{\mu''\nu})$;

$\{I, R_1^3R_2^2T^2, R_2^3S, R_1^3R_2^3ST^2, R_2^2, R_2S, R_1^3T^2, R_1^3R_2ST^2\}$

$\xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{\mu} - \sqrt{\mu'})$;

$\{I, R_1^3R_2^2T^2, R_2^3S, R_1^3R_2^3ST^2, R_1^3R_2^3, R_2^3T^2, ST^2, R_1^3R_2^2S\}$

$\xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\mu''}, \sqrt{\mu\nu''} + \sqrt{\mu'\nu''})$;

$\{I, R_1^2R_2^3T^2, R_1^3S, R_1R_2ST^2, R_1^2, R_1S, R_2^3T^2, R_1^3R_2ST^2\} \xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\nu''}, \sqrt{\nu} - \sqrt{\nu'})$;

$\{I, R_1^2R_2^3T^2, R_1^3S, R_1R_2ST^2, R_1R_2, R_1^3T^2, ST^2, R_1^2R_2^3S\}$

$\xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\nu''}, \sqrt{\mu''\nu} + \sqrt{\mu''\nu'})$;

$\{I, R_1^3R_2ST, R_1R_2^3, R_1^2R_2^2ST, R_1^3R_2, R_1^2R_2^2, ST, R_1R_2^3ST\}$

$\xleftrightarrow{G^*} Q(\rho\theta, \sqrt{\mu'\nu'}, \sqrt{\mu''\nu''} + \sqrt{\mu\nu})$;

$\{I, R_1^3R_2ST, R_1R_2^3, R_1^2R_2^2ST, R_1T, R_2T, R_1^3R_2^2S, R_1^2R_2^3S\}$

$\longleftrightarrow^{G^*} Q(\rho\theta, \sqrt{\mu'\nu'}, \sqrt{\mu\nu''} - \sqrt{\mu''\nu});$

$\{I, R_1^2R_2^2ST^2, R_1^3R_2, R_1R_2^3ST^2, R_1^2R_2^2, R_1R_2^3, ST^2, R_1^3R_2ST^2\}$

$\longleftrightarrow^{G^*} Q(\rho^2\theta, \sqrt{\mu''\nu''}, \sqrt{\mu\nu} + \sqrt{\mu'\nu'});$

$\{I, R_1^2R_2^2ST^2, R_1^3R_2, R_1R_2^3ST^2, R_1^3T^2, R_2^3T^2, R_1R_2^2S, R_1^2R_2S\}$

$\longleftrightarrow^{G^*} Q(\rho^2\theta, \sqrt{\mu''\nu''}, \sqrt{\mu'\nu} - \sqrt{\mu\nu'});$

<u>Order 12 $\longleftrightarrow^{G^*}$ Degree 12 of</u> K_{96}^*/K_8^* <u>(of type</u> A_4)

$K_8^{10} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\gamma^\sigma});$ $\quad$ $K_8^{20} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\delta^\sigma});$

$K_8^{11} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\gamma''});$ $\quad$ $K_8^{21} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\delta'''});$

$K_8^{12} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\gamma'});$ $\quad$ $K_8^{22} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\delta'});$

$K_8^{13} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\gamma''});$ $\quad$ $K_8^{23} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\delta''});$

$K_8^{30} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\mu\nu} + \sqrt{\mu'\nu'} + \sqrt{\mu''\nu''});$

$K_8^{31} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\mu\nu} - \sqrt{\mu'\nu'} - \sqrt{\mu''\nu''});$

$K_8^{32} \longleftrightarrow^{G^*} Q(\rho, -\sqrt{\mu\nu} + \sqrt{\mu'\nu'} - \sqrt{\mu''\nu''});$

$K_8^{33} \longleftrightarrow^{G^*} Q(\rho, -\sqrt{\mu\nu} - \sqrt{\mu'\nu'} + \sqrt{\mu''\nu''});$

$K_8^{40} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\mu\nu'} + \sqrt{\mu'\nu''} + \sqrt{\mu''\nu});$

$K_8^{41} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\mu\nu'} - \sqrt{\mu'\nu''} - \sqrt{\mu''\nu});$

$K_8^{42} \longleftrightarrow^{G^*} Q(\rho, -\sqrt{\mu\nu'} + \sqrt{\mu'\nu''} - \sqrt{\mu''\nu});$

$K_8^{43} \longleftrightarrow^{G^*} Q(\rho, -\sqrt{\mu\nu'} - \sqrt{\mu'\nu''} + \sqrt{\mu''\nu});$

$K_8^{50} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\mu\nu''} + \sqrt{\mu'\nu} + \sqrt{\mu''\nu'});$

$K_8^{51} \longleftrightarrow^{G^*} Q(\rho, \sqrt{\mu\nu''} - \sqrt{\mu'\nu} - \sqrt{\mu''\nu'});$

$K_8^{52} \xleftrightarrow{G^*} Q(\rho, -\sqrt{\mu\nu''} + \sqrt{\mu'\nu} - \sqrt{\mu''\nu'})$;

$K_8^{53} \xleftrightarrow{G^*} Q(\rho, -\sqrt{\mu\nu''} - \sqrt{\mu'\nu} + \sqrt{\mu''\nu'})$;

Order 16 $\xleftrightarrow{G^*}$ Degree 16 of K_{96}^*/K_6^*

(i). non-cyclic of type $C(2) \oplus C(2) \oplus C(2) \oplus C(2)$.

$K_6^{oo} \xleftrightarrow{G^*} Q(\rho, \theta)$;

(ii). non-abelian of type $D_4 \oplus C(2)$

$K_6^{11} \xleftrightarrow{G^*} Q(\theta, \sqrt{\mu})$; $K_6^{21} \xleftrightarrow{G^*} Q(\rho\theta, \sqrt{\mu'})$;

$K_6^{12} \xleftrightarrow{G^*} Q(\theta, \sqrt{\nu})$; $K_6^{22} \xleftrightarrow{G^*} Q(\rho\theta, \sqrt{\nu'})$;

$K_6^{13} \xleftrightarrow{G^*} Q(\theta, \sqrt{\mu\nu})$; $K_6^{23} \xleftrightarrow{G^*} Q(\rho\theta, \sqrt{\mu'\nu'})$;

$K_6^{31} \xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\mu''})$;

$K_6^{32} \xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\nu''})$;

$K_6^{33} \xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{\mu''\nu''})$;

(iii). non-abelian of type H_{16}^*

$K_6^{41} \xleftrightarrow{G^*} Q(\theta, \sqrt{-3\mu\nu})$; $K_6^{51} \xleftrightarrow{G^*} Q(\theta, \sqrt{-3\mu})$;

$K_6^{42} \xleftrightarrow{G^*} Q(\rho\theta, \sqrt{-3\mu'\nu'})$; $K_6^{52} \xleftrightarrow{G^*} Q(\rho\theta, \sqrt{-3\mu'})$;

$K_6^{43} \xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{-3\mu''\nu''})$; $K_6^{53} \xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{-3\mu''})$;

$K_6^{61} \xleftrightarrow{G^*} Q(\theta, \sqrt{-3\nu})$;

$K_6^{62} \xleftrightarrow{G^*} Q(\rho\theta, \sqrt{-3\nu'})$;

$K_6^{63} \xleftrightarrow{G^*} Q(\rho^2\theta, \sqrt{-3\nu''})$;

Order 24 $\xleftrightarrow{G^*}$ Degree 24 of K_{96}^*/K_4^* (of type H_{32}^*)

$K_4^{10} \xleftrightarrow{G^*} Q(\sqrt{\gamma^o})$;

$$K_4^{11} \xleftrightarrow{G^*} Q(\sqrt{\gamma'''});$$

$$K_4^{12} \xleftrightarrow{G^*} Q(\sqrt{\gamma'});$$

$$K_4^{13} \xleftrightarrow{G^*} Q(\sqrt{\gamma''});$$

$$K_4^{20} \xleftrightarrow{G^*} Q(\sqrt{\delta^{\sigma}});$$

$$K_4^{21} \xleftrightarrow{G^*} Q(\sqrt{\delta'''});$$

$$K_4^{22} \xleftrightarrow{G^*} Q(\sqrt{\delta'});$$

$$K_4^{23} \xleftrightarrow{G^*} Q(\sqrt{\delta''});$$

$$K_4^{30} \xleftrightarrow{G^*} Q(\sqrt{\mu\nu} + \sqrt{\mu'\nu'} + \sqrt{\mu''\nu''});$$

$$K_4^{31} \xleftrightarrow{G^*} Q(\sqrt{\mu\nu} - \sqrt{\mu'\nu'} - \sqrt{\mu''\nu''});$$

$$K_4^{32} \xleftrightarrow{G^*} Q(-\sqrt{\mu\nu} + \sqrt{\mu'\nu'} - \sqrt{\mu''\nu''});$$

$$K_4^{33} \xleftrightarrow{G^*} Q(-\sqrt{\mu\nu} - \sqrt{\mu'\nu'} + \sqrt{\mu''\nu''});$$

Order 32 $\xleftrightarrow{G^*}$ Degree 32 of K_{96}^*/K_3^* (of type H_{32}^*)

$$K_3^o \xleftrightarrow{G^*} Q(\theta);$$

$$K_3^1 \xleftrightarrow{G^*} Q(\rho\theta);$$

$$K_3^2 \xleftrightarrow{G^*} Q(\rho^2\theta);$$

Order 48 $\xleftrightarrow{G^*}$ Degree of K_{96}^*/K_2^* (of type H_{48}^*)

$$K_2^o \xleftrightarrow{G^*} Q(\rho) = Q(\sqrt{-3});$$

This completes all the subfields of the normal extension of Q with corresponding subgroups of its Galois group. Moreover, we described all the subfields of K_{96}^* as we did in Chapter II by the towers only. We only list the symbols

to stand for all the conjugate subfields of K_{96}^*, and we can't list all the subfields in the diagram 3.2., because there are more than two hundred subfields of K_{96}^* .

Proposition 3.1. $G^*(K_{96}^*/Q) = \langle a, b, c, d, \mid a^4 = b^4 = c^2 = d^3 = 1;$ $[a, b] = a^2b^2$, $[a, c] = a^2$, $[a, d] = a^3cd$, $[b, c] = b^2$, $[b, d] = b^3cd$, $[c, d] = d \rangle$.

Proof: See the above group structure. //

Proposition 3.2. $G^*(K_{96}^*/Q(\theta,\sqrt{\mu})) \cong G^*(K_{96}^*/Q(\theta,\sqrt{\nu}))$ $\cong D_4 \oplus C(2)$.

Proof: See the above group structure. //

Proposition 3.3. $G^*(K_{96}^*/Q(\theta,\sqrt{-3\mu})) \cong G^*(K_{96}^*/Q(\theta,\sqrt{-3\nu}))$ $\cong H_{16}^*$.

Proof: See the above group structure. //

Proposition 3.4. $G^*(K_{96}^*/Q(\rho,\theta)) \cong C(2) \oplus C(2) \oplus C(2) \oplus C(2)$.

Proof: See the above group structure. //

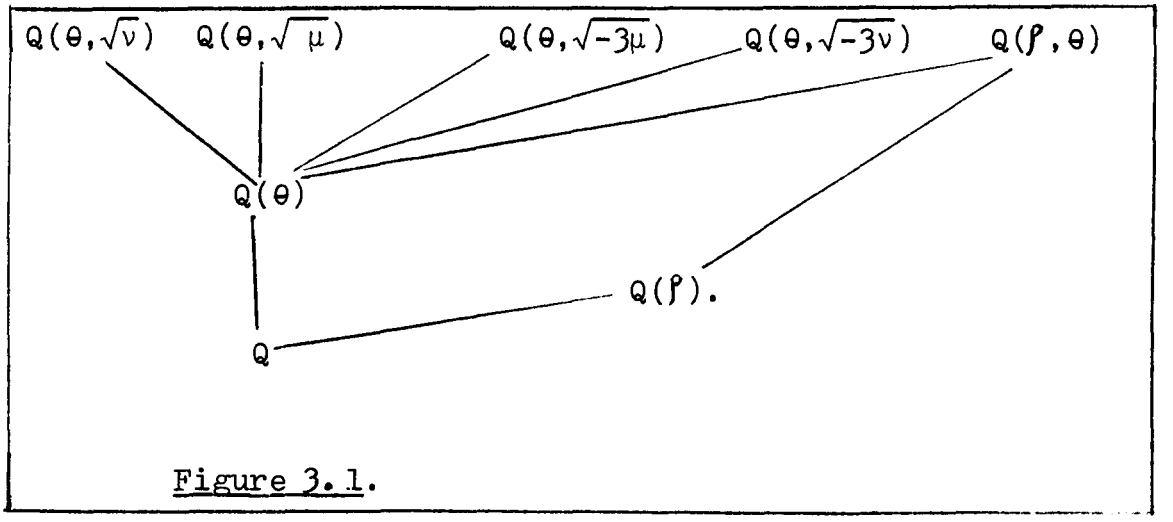


Figure 3.1.

K^*_{96} — $Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'}, \sqrt{\mu''}, \sqrt{\nu}, \sqrt{\nu'}, \sqrt{\nu''})$

3 types - K^*_{48} — $Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'}, \sqrt{\mu''}, \sqrt{\nu})$; $Q(\rho, \theta, \sqrt{\mu'}, \sqrt{\nu}, \sqrt{\nu'}, \sqrt{\nu''})\cdots$
(6+9+12) $Q(\rho, \theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'\nu'}, \sqrt{\mu''\nu''})\cdots$
subfields $Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'}+\sqrt{\mu''}, \sqrt{\nu'}+\sqrt{\nu''})\cdots$

1 type — K^*_{32} — $Q(\rho, \sqrt{\delta^\circ}, \sqrt{\gamma^\circ})$; $Q(\rho, \sqrt{\delta^\circ}, \sqrt{\gamma'})\cdots$
(16)
subfields

7 types - K^*_{24} — $Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'})$; $Q(\rho, \theta, \sqrt{\mu}, \sqrt{\nu})\cdots$
(11+18+6+ $Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'\nu''})$; $Q(\rho, \theta, \sqrt{\mu}, \sqrt{\mu'\nu})\cdots$
12+6+12+6) $Q(\rho, \theta, \sqrt{\mu\nu}, \sqrt{\mu'\nu'})$; $Q(\rho, \theta, \sqrt{\mu\nu}, \sqrt{\mu'\nu''})\cdots$
subfields $Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'}+\sqrt{\mu''})$; $Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'}-\sqrt{\mu''})\cdots$
$Q(\theta, \sqrt{\mu}, \sqrt{\nu}, \sqrt{\mu'\nu'}+\sqrt{\mu''\nu''})\cdots$
$Q(\theta, \sqrt{-3\mu}, \sqrt{\nu}, \sqrt{\nu'}+\sqrt{\nu''})\cdots$
$Q(\theta, \sqrt{-3\mu}, \sqrt{-3\nu}, \sqrt{\mu'\nu'}+\sqrt{\mu''\nu''})\cdots$

1 type — K^*_{16} — $Q(\sqrt{\delta^\circ}, \sqrt{\gamma^\circ})$; $Q(\sqrt{\delta^\circ}, \sqrt{\gamma'})\cdots$
(16)
subfields

7 types - K^*_{12} — $Q(\rho, \theta, \sqrt{\mu})$; $Q(\rho, \theta, \sqrt{\mu'})\cdots$
(6+9+3+9+ $Q(\rho, \theta, \sqrt{\mu\nu})$; $Q(\rho, \theta, \sqrt{\mu\nu'})\cdots$
12+12+12) $Q(\theta, \sqrt{\mu}, \sqrt{\nu})$; $Q(\rho\theta, \sqrt{\mu'}, \sqrt{\nu'})\cdots$
subfields $Q(\theta, \sqrt{-3\mu}, \sqrt{\nu})$; $Q(\theta, \sqrt{\mu}, \sqrt{-3\nu})$; $Q(\theta, \sqrt{-3\mu}, \sqrt{-3\nu})\cdots$
$Q(\theta, \sqrt{\mu}, \sqrt{\mu'}+\sqrt{\mu''})$; $Q(\theta, \sqrt{\mu}, \sqrt{\mu'}-\sqrt{\mu''})\cdots$
$Q(\theta, \sqrt{\mu}, \sqrt{\mu'\nu}+\sqrt{\mu''\nu})$; $Q(\theta, \sqrt{\mu}, \sqrt{\mu'\nu}-\sqrt{\mu''\nu})\cdots$
$Q(\theta, \sqrt{\mu\nu}, \sqrt{\mu'\nu'}+\sqrt{\mu''\nu''})$; $Q(\theta, \sqrt{\mu\nu}, \sqrt{\mu'\nu''}+\sqrt{\mu''\nu'})\cdots$

To be continued

3 types (8+4+8) subfields — K_8^* — $Q(\rho, \sqrt{\delta^o})$; $Q(\rho, \sqrt{\delta'})$; $Q(\rho, \sqrt{\gamma^o})\ldots$
$Q(\rho, \sqrt{\mu\nu} + \sqrt{\mu'\nu'} + \sqrt{\mu''\nu''})$; $Q(\rho, \sqrt{\mu\nu} - \sqrt{\mu'\nu'} - \sqrt{\mu''\nu''})\ldots$
$Q(\rho, \sqrt{\mu\nu'} + \sqrt{\mu'\nu''} + \sqrt{\mu''\nu})$; $Q(\rho, \sqrt{\mu\nu''} + \sqrt{\mu'\nu} + \sqrt{\mu''\nu'})\ldots$

4 types (1+6+3+9) subfields — K_6^* — $Q(\rho, \theta)$
$Q(\theta, \sqrt{\mu})$; $Q(\rho\theta, \sqrt{\mu'})\ldots$
$Q(\theta, \sqrt{\mu\nu})$; $Q(\rho\theta, \sqrt{\mu'\nu'})\ldots$
$Q(\theta, \sqrt{-3\mu})$; $Q(\rho\theta, \sqrt{-3\mu'\nu'})\ldots$

2 types (8+12) subfields — K_4^* — $Q(\sqrt{\delta^o})$; $Q(\sqrt{\delta'})$; $Q(\sqrt{\delta''})\ldots$
$Q(\sqrt{\mu\nu} + \sqrt{\mu'\nu'} + \sqrt{\mu''\nu''})$; $Q(\sqrt{\mu\nu} - \sqrt{\mu'\nu'} - \sqrt{\mu''\nu''})$

1 type (3) subfields — K_3^* — $Q(\theta)$; $Q(\rho\theta)$; $Q(\rho^2\theta)$;

1 type (1) subfield — K_2^* — $Q(\rho)$;

Q

where
$\sqrt{\delta^o} = \sqrt{\mu} + \sqrt{\mu'} + \sqrt{\mu''}$
$\sqrt{\delta'} = -\sqrt{\mu} + \sqrt{\mu'} - \sqrt{\mu''}$
$\sqrt{\delta''} = -\sqrt{\mu} - \sqrt{\mu'} + \sqrt{\mu''}$
$\sqrt{\delta'''} = \sqrt{\mu} - \sqrt{\mu'} - \sqrt{\mu''}$
$\sqrt{\gamma^o} = \sqrt{\nu} + \sqrt{\nu'} + \sqrt{\nu''}$
$\sqrt{\gamma'} = -\sqrt{\nu} + \sqrt{\nu'} - \sqrt{\nu''}$
$\sqrt{\gamma''} = -\sqrt{\nu} - \sqrt{\nu'} + \sqrt{\nu''}$
$\sqrt{\gamma'''} = \sqrt{\nu} - \sqrt{\nu'} - \sqrt{\nu''}$

Total subfields 250

Diagram 3.2.

CHAPTER IV. THE FACTORIZATION AND THE SOLVABILITY OF NORM EQUATIONS

§1. The factorization in the field $Q(\theta)$

Let m be a positive cube-free rational integer; we shall be concerned with the pure cubic field $Q(\sqrt[3]{m}) = Q(\theta)$ again, where θ denotes the real cube root. If m is not square-free,

$$m = m_1 m_2^2, \quad (m_1, m_2) = 1, \quad m_1 \text{ and } m_2 \text{ are square-free,}$$

Furthermore, we search for the equivalent relations between the prime factorization over $Q(\theta)$ and the solvability of norm equations. That is to say, by use of class field theory in solving norm equations, we have following three cases for the unramified quadratic extension fields $Q(\theta, \sqrt{\mu})$ or $Q(\theta, \sqrt{-3\mu})$ over $Q(\theta)$ to consider:

(1) $N(\delta^*) = r$, $\quad r \equiv 5 \pmod 6$

(2) $N(\delta^*) = r^2$ (not r), $\quad r \equiv 5 \pmod 6$

(3) $N(\gamma^*) = p$ $\quad p \equiv 1 \pmod 6$

where p and r are rational primes and greater than 3

γ^* and δ^* are algebraic integers in $Q(\theta)$.

Later in this chapter, by use of quadratic reciprocity, we prove that the necessary and sufficient condition of the defining polynomial of degree 6 for μ in the extension fields (Hilbert class fields) is either solvable or not which is depending on the solvability of norm equations.

By the end of these propositions, we also find the numeral examples for each field of class number 2 together with class number 4 (non-cyclic case). Now we have to list the definitions and results where most of them are referring to Dedekind.

Fact 4.1. The fields $Q(\theta)$ and $Q(\bar{\theta})$ are identical, where $\theta\bar{\theta} = m_1 m_2$.

Fact 4.2. The algebraic integers of the field $Q(\theta)$ are of the form

(i) $\alpha = a + b\theta + c\bar{\theta}$; if $m \not\equiv \pm 1 \pmod 9$

(ii) $\alpha = \dfrac{a + b\theta + c\bar{\theta}}{3}$, $a \equiv m_1 b \equiv m_2 c \pmod 3$

if $m \equiv \pm 1 \pmod 9$.

where a, b and c rational integers).

Fact 4.3. The discriminant of the field $Q(\theta)$ is given by

(i) $\Delta = -27\, m_1^2 m_2^2$, if $m \not\equiv \pm 1 \pmod 9$.

(ii) $\Delta = -3\, m_1^2 m_2^2$, if $m \equiv \pm 1 \pmod 9$.

Fact 4.4. $a + b\theta$ (or $\dfrac{a + b\theta + c\bar{\theta}}{3}$) is an algebraic integer of the field K if and only if a and b (or a, b and c) are (rational) integers.

Definition 4.5. The norm, conjugate, square and cube of an algebraic number of $K = Q(\theta)$:

$\alpha = a + b\theta + c\bar{\theta}$ or $\beta = a + b\theta + c\theta^2$;

$$N(\alpha) = a^3 + m_1 m_2^2 b^3 + m_1^2 m_2 c^3 - 3m_1 m_2 abc$$

$$N(\beta) = a^3 + mb^3 + m^2 c^3 - 3mabc$$

$$\alpha'\alpha'' = N(\alpha)/\alpha = a^2 - m_1 m_2 bc + (m_1 c^2 - ab)\theta + (m_2 b^2 - ac)\bar{\theta}$$

$$\beta'\beta'' = N(\beta)/\beta = a^2 - mbc + (mc^2 - ab)\theta + (b^2 - ac)\theta^2$$

$$\alpha^2 = a^2 + 2m_1 m_2 bc + (m_1 c^2 + 2ab)\theta + (m_2 b^2 + 2ac)\bar{\theta}$$

$$\beta^2 = a^2 + 2mbc + (mc^2 + 2ab)\theta + (b^2 + 2ac)\theta^2$$

$$\alpha^3 = a^3 + m_1 m_2^2 b^3 + m_1^2 m_2 c^3 + 6m_1 m_2 abc$$
$$+ 3(a^2 b + m_1 ac^2 + m_1 m_2 b^2 c)\theta$$
$$+ 3(a^2 c + m_2 ab^2 + m_1 m_2 bc^2)\bar{\theta}$$

$$\beta^3 = a^3 + mb^3 + m^2 c^3 + 6mabc + 3(a^2 b + mac^2 + mb^2 c)\theta$$
$$+ 3(a^2 c + ab^2 + mbc^2)\theta^2$$

Definition 4.6. The field $Q(\theta)$ has one basic unit ϵ, which we will choose such that $0 < \epsilon < 1$.

Definition 4.7. The algebraic integer μ of the field K is called a trunk element if $c = 0$.

In particular, in this dissertation we like to select μ and ν of the field K such that $\mu \equiv \epsilon^2 \pmod 4$ and $\nu \equiv \epsilon^2 \pmod 4$, where $N(\mu) = q_1^2$ and $N(\nu) = q_2^2$, and q_i is a rational prime and greater than 3, $i = 1, 2$.

Fact 4.8. (Dedekind) The primes p, r and s factor in $\mathcal{O}_K$ as follows: (where $\mathcal{O}_K$ is a ring of integers of K).

If $r \nmid m$: $r = \mathfrak{r}_1\bar{\mathfrak{r}}_2$, $N(\mathfrak{r}_1) = r$ and $N(\bar{\mathfrak{r}}_2) = r^2$ if and only if $r \equiv -1 \pmod 3$. (Then $x^3 \equiv m \pmod r$ is always solvable with a unique solution). 4-8-1

If $p \nmid m$: $p = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$, $N(\mathfrak{p}_i) = p$, ($\mathfrak{p}_i$ different) if and only if $p \equiv 1 \pmod 3$ and $x^3 \equiv m \pmod p$ is solvable in Z (three distinct roots) 4-8-2

If $s \nmid m$: $s = s$ (inert) if and only if $s \equiv 1 \pmod 3$ and $x^3 \equiv m \pmod s$ is unsolvable in Z (no solutions) 4-8-3

If $p \mid m$: (except $p = 3$ for type II of Fact 4.3.)

$p = \mathfrak{p}^3$, $N(\mathfrak{p}) = p$. 4-8-4

For type II ($3 \mid m$), $3 = \mathfrak{3}_1\mathfrak{3}_2^2$, $N(\mathfrak{3}_1) = N(\mathfrak{3}_2) = 3$. 4-8-5

Fact 4.9. (Dedekind) The factors of natural primes are given as follows. p, r and s are primes.

(I) $[p] = \mathfrak{p}_p^3 = [p, \theta]^3$ or $[p, \bar{\theta}]^3$ if $p \mid m_1$ or $p \mid m_2$,

(II) $[3] = \mathfrak{p}_3^3 = [3, \theta - m]^3$ if $m \equiv \pm 2$ or $\pm 4 \pmod 9$.

(III) $[3] = \mathfrak{r}^2\mathfrak{s} = [3, \pm\theta - 1, \frac{1}{3}(\theta^2 \pm \theta + 1)]^2$

$[3, \pm\theta - 1, \frac{1}{3}(\theta^2 \pm \theta - 2)]$

if $m = \pm 1 \pmod 9$

(IV) $[r] = r_1 \bar{r}_2 = [r, \theta - d][r, \theta^2 + d\theta + d^2]$;
if $r \equiv -1 \pmod 3$, $r \nmid m$, and $d^3 \equiv m \pmod r$
(a unique root).

(V) $[p] = \sqrt{p}_1 \sqrt{p}_2 \sqrt{p}_3 = [p, \theta - d][p, \theta - d'][p, \theta - d'']$
if $p \equiv 1 \pmod 3$, $p \nmid m$, and $d^3 \equiv d'^3 \equiv d''^3 \equiv m \pmod p$.

(VI) $[s]$ = a prime ideal, if $s \nmid m$, m a cubic non-residue of s and $s \equiv 1 \pmod 3$.

Fact 4.10. In $K_6^{\#} = Q(\sqrt{-3}, \theta) = Q(\rho, \theta)$, by Fact 4.8., we have

(1) $r = r_1 \bar{r}_2$ (in $K_3 = Q(\theta)$)
$= r_1 r_1' r_1''$ (where r_1' in $Q(\rho\theta)$ and r_1'' in $Q(\rho^2\theta)$)

(2) $p = \sqrt{p}_1 \sqrt{p}_2 \sqrt{p}_3$ (in $Q(\theta)$, if p is a cubic residue. See Fact 4.9.)
$= \sqrt{p}_1 \sqrt{p}_1' \sqrt{p}_1''$ (where $\sqrt{p}_1'$ in $Q(\rho\theta)$ and $\sqrt{p}_1''$ in $Q(\rho^2\theta)$)
$= \sqrt{p}_{11}\sqrt{p}_{12}\sqrt{p}_{21}\sqrt{p}_{22}\sqrt{p}_{31}\sqrt{p}_{32}$ ($\sqrt{p}_{ij}$ in $K_6^{\#}$)

Furthermore when the ideals are principal, they can be expressed as the following:

$r_1 = a_o + b_o\theta + c_o\theta^2$; and $\sqrt{p}_1 = a_o^* + b_o^*\theta + c_o^*\theta^2$

$r_1' = a_o + b_o\rho\theta + c_o\rho^2\theta^2$; $\sqrt{p}_1' = a_o^* + b_o^*\rho\theta + c_o^*\rho^2\theta^2$

$r_1'' = a_o + b_o\rho^2\theta + c_o\rho\theta^2$; $\sqrt{p}_1'' = a_o^* + b_o^*\rho^2\theta + c_o^*\rho\theta^2$

§ 2. The factorization and the normed solvability in the fields $Q(\theta,\sqrt{\mu})$, $Q(\theta,\sqrt{-3\mu})$ and $Q(\rho,\theta)$

We consider, as always, the relationships between the normed solvability and the factorization in the extension fields. (See the diagram 2.3. and the right figure). To start with, we have to consider the fields whose class number is 2 and 4 (non-cyclic case), that is to say the unramified extension fields. Therefore we have to return to basic results and definitions of quadratic extensions of Q.

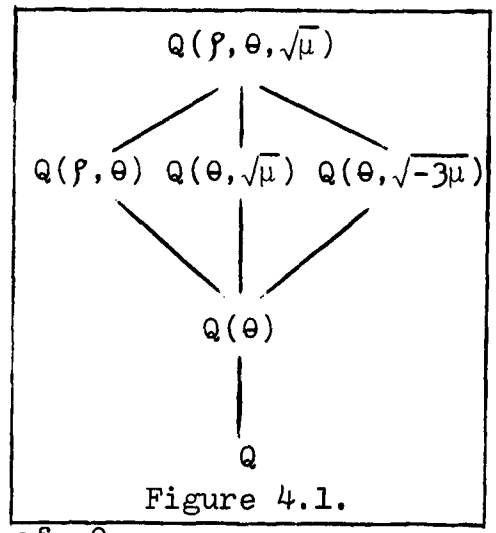


Figure 4.1.

Fact 4.11. (Legendre's Symbol) Let p denote an odd rational prime and let $(MN, p) = 1$.

(I) If $M \equiv N \pmod{p}$, $(M/p) = (N/p)$.

(II) $(MN/p) = (M/p)(N/p)$.

Fact 4.12. (Quadratic reciprocity) Let p denote a (positive) rational prime $\equiv 1 \pmod 4$, and let q denote a (positive) rational prime $\equiv -1 \pmod 4$.

(I) $(p_1/p_2) = (p_2/p_1)$

(II) $(p/q) = (q/p)$

(III) $(q_1/q_2) = -(q_2/q_1)$

(IV) $(-1/p) = (-1)^{\frac{1}{2}(p-1)}$

$(-1/q) = (-1)^{\frac{1}{2}(q-1)}$

(V) $(2/p) = (-1)^{(p^2-1)/8}$

$(2/q) = (-1)^{(q^2-1)/8}$

Fact 4.13. If $(MN/p) = -1$ and $(MN, p) = 1$ then either $(M/p) = -1$ or $(N/p) = -1$.

Fact 4.14. All the pure cubic fields, considered in this chapter, have the class number either 2 or 4 (non-cyclic case), i.e., $m = 11, 15, 47, 83, 89$ and 113.

Proposition 4.1. Let $\mu = A + 4B\theta$ be an algebraic integer of $K_3 = Q(\theta)$ with the norm $N(\mu) = q^2$ (q rational prime). If (I) $A < 2m$ and $4B < m$; $m \not\equiv \pm 1 \pmod 9$

(II) $9A < 2m$ and $9 \cdot 4B < m$; $m \equiv \pm 1 \pmod 9$

then μ is a non-square algebraic integer in K_3. (i.e., $\mu \neq \beta^2$ for any β in K_3).

Proof: (I) If $\mu = \beta^2$ and $m \equiv \pm 1 \pmod 9$ then by Fact 4.2. and Definition 4.5. We set $\beta = a + b\theta + c\theta^2$. Then

(1) $A = a^2 + 2mbc$

(2) $4B = mc^2 + 2ab$

(3) $0 = b^2 + 2ac$

It is solvable for a, b and c in Z. By observing the third equation, we obtain $ac < 0$. Thus, we should consider two cases only.

Case (i) if $bc > 0$, then applying into the first equation, we have $a^2 + 2mbc > 2mbc > 2m > A$, which contradicts the solvability of our assumption.

Case (ii) if $ab > 0$, then applying into the second equation, we have $mc^2 + 2ab > mc^2 > m > 4B$, which contradicts the solvability of our assumption.

(II) Similarly, we have $\mu \neq \beta^2$ for $m \equiv \pm 1 \pmod 9$ since then a, b and c are replaced by $a^*/3$, $b^*/3$ and $c^*/3$, respectively. These complete the proof. (See Table 4.1. on page 68). //

Fact 4.15. (Hilbert) If k is a field, and $(\mu) = \mathfrak{a}^2$ for $\mathfrak{a}$ an ideal, $\mu \equiv \alpha^2 \pmod 4$, where $N(\alpha)$ is odd and $\sqrt{\mu}$ is not in k, then $k(\sqrt{\mu})$ is an unramified extension field over k.

Fact 4.16. (Hilbert) The statement in Fact 4.15. holds if and only if k has even class number. A basic proposition of class field theory is that if K/k is an unramified extension, a prime ideal $\mathfrak{p}$ in k splits completely into $n = [K : k]$ ideal factors in K if and only if $\mathfrak{p}$ is principal. This is in evidence from norm representations.

(1) $\mathfrak{p}$ is of degree 1 in k (say $\mathfrak{p} = [p, \theta + d]$, d in Z and θ in k) $\mathfrak{p}$ is principal if and only if $\mathfrak{p} = (\gamma^*)$ where $N(\gamma^*) = p$ (rational).

(2) $\bar{r}$ is of degree 2 in k (say $\bar{r} = [r, \theta^2 + d_1\theta + d_2]$ d_1 and d_2 in Z, θ in k), $\bar{r}$ is principal if

Table 4.1. $Q(\theta) = Q(\sqrt[3]{m})$. Verification of Proposition 4.1.

m	h	A	B	q	μ	$A < 2m$ and $4B < m$	$9A < 2m$ and $9 \cdot 4B < m$	Type
11	2	9	-1	5	$9 - 4\theta$	$9 < 2 \cdot 11$; $-4 < 11$		I
15	2	1	1	31	$1 + 4\theta$	$1 < 2 \cdot 15$; $4 < 15$		I
47	2	17	1	89	$17 + 4\theta$	$17 < 2 \cdot 47$; $4 < 47$		I
83	2	-47	3	199	$-47 + 12\theta$	$-47 < 2 \cdot 83$; $12 < 83$		I
89	2	17	1	103	$17 + 4\theta$		$9 \cdot 17 < 2 \cdot 89$; $9 \cdot 4 < 89$	II
113	4	-7	1	83	$-7 + 4\theta$	$-7 < 2 \cdot 113$; $4 < 113$		I

Type I : $m \not\equiv \pm 1 \pmod 9$, $A < 2m$ and $4B < m$.

Type II: $m \equiv \pm 1 \pmod 9$, $9A < 2m$ and $9 \cdot 4B < m$.

and only if $\bar{\mathfrak{r}} = (\delta^*)$ where $N(\delta^*) = r^2$ (rational).

Fact 4.17. Since we have $N(\mu) = A^3 + 64B^3m = q^2$ and $N(\mu) = q^2 \equiv A^3 \equiv 1 \pmod 8$, then $A \equiv 1 \pmod 8$ so that Fact 4.15. applies, provided that it can be shown that $(\mu) = \mathfrak{q}^2$ for some ideal $\mathfrak{q}$.

Two cases are used as following:

(1) If $q = r \equiv 5 \pmod 6$ only one ideal $\mathfrak{q}$ has norm r so that $(\mu) = \mathfrak{q}^2$.

(2) If $q = p \equiv 1 \pmod 6$ three distinct ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ have norm p so we exclude the possibility $\mu = \mathfrak{p}_1\mathfrak{p}_2$ (instead of $\mathfrak{p}_1^2$), because p/μ is not an algebraic integer in $Q(\theta)$. (Verified by use of multiplying their conjugates for rationalizing the denominator).

Remark 4.1. For the time being, we define the following:

$K_3 = Q(\theta)$

$K_6 = Q(\theta, \sqrt{\mu})$ with the defining polynomial F_6

$K_6^* = Q(\theta, \sqrt{-3\mu})$ with the defining polynomial F_6^*

$K_6^\# = Q(\theta, \rho) = Q(\theta, \sqrt{-3})$ with the defining polynomial $F_6^\#$

where the defining polynomial for $\sqrt{\mu}$ $(= x)$ or $\alpha^\#$ $(= x)$ is

(1) $F_6(x) = (x^2 - A)^3 - 64B^3m$ (for $\sqrt{\mu}$)

(2) $F_6^*(x) = (x^2 + 3A)^3 - 64(-3B)^3m$ (for $\sqrt{-3\mu}$)

(3) $F_6^\#(x) = (x^3 - 9\hat{a}^2x - \hat{b}^3m)^2 + 27\hat{a}^2(x^2 - \hat{a}^2)^2$

for $\alpha^\# = \hat{a}\cdot\sqrt{-3} + \hat{b}\cdot\theta$, $\hat{a}$, and $\hat{b}$ are in Z.

We take this defining polynomial modulo p or r so that

(1) $F_6(x) \equiv 0 \pmod{p \text{ or } r}$

(2) $F_6^*(x) \equiv 0 \pmod{p \text{ or } r}$

(3) $F_6^\#(x) \equiv 0 \pmod{p \text{ or } r}$

Moreover, all the quadratic extension fields in this section are unramified. (i.e. the class number of K_3 is either 2 or 4 (non-cyclic case). Then $K_6 = K_3(\sqrt{\mu})$ is the class field as well as $K_6^* = K_3(\sqrt{-3\mu})$, where μ is in K_3, since $-3 \equiv 1 \pmod 4$. See Fact 4.15.

Proposition 4.2. $(-3/p) = 1$ when p is prime and $p \equiv 1 \pmod 6$.

Proof: Without loss of generality, we assume that $p \equiv 1$ or $7 \pmod{12}$.

Case (i) $p \equiv 1 \pmod{12}$, then $p = 12t + 1$ and t in Z.

$$(-3/p) = (-3/12t + 1) = (-1/12t + 1)(3/12t + 1)$$
$$= (12t + 1/3) = (1/3) = 1.$$

Case (ii) $p \equiv 7 \pmod{12}$, then $p = 12\bar{t} + 7$ and $\bar{t}$ in Z.

$$(-3/p) = (-1/p)(3/p) = (-1/12\bar{t} + 7)(3/12\bar{t} + 7)$$
$$= (-1)(-1)(12\bar{t} + 7/3) = (7/3) = (1/3) = 1. \quad //$$

Proposition 4.3. $(-3/p) = -1$ when p is prime and $p \equiv 5 \pmod 6$.

Proof: Without loss of generality, we also assume that $p \equiv 5$ or $11 \pmod{12}$.

Case (i) $p \equiv 5 \pmod{12}$, then $p = 12t^* + 5$ and t^* in Z.

$$(-3/p) = (-1/p)(3/p) = (-1/12t^* + 5)(3/12t^* + 5)$$

$= (1)(12t^* + 5/3) = (2/3) = -1.$

Case (ii) $p \equiv 11 \pmod{12}$, then $p = 12\bar{t}^* + 11$ and $\bar{t}^*$ in Z. $(-3/p) = (-1/p)(3/p)$

$= (-1/12\bar{t}^* + 11)(3/12\bar{t}^* + 11)$

$= (-1)(-1)(12\bar{t}^* + 11/3) = (2/3) = -1.$ //

Fact 4.18. (Dedekind) Let $F(x) = 0$ be a defining monic equation for $Q(\theta)$ of degree n with the exception of primes divides the root discriminant Δ of $F(x)$, where $\Delta = \prod_{i \neq j} (\theta_i - \theta_j)^2$ as usual.

If $F(x) \equiv \prod \mathcal{P}_i(x) \pmod{p}$ and $\mathcal{P}_i(x)$ is of degree f_i then $p = \prod \mathcal{P}_i$, where $\mathcal{P}_i = [p, \mathcal{P}_i(\theta)]$ of degree f_i $N(\mathcal{P}_i) = p^{f_i}$ and $\sum f_i = n$.

Proposition 4.4. Let r be a rational prime and $r \equiv 5 \pmod{6}$. If $N(a + b\theta + c\theta^2) = r$ is solvable for a, b and c in Z, then for the field K_6, the defining polynomial $F_6(x) \equiv 0 \pmod{r}$ is solvable in Z. Furthermore,

$$r = [r, d_{11} - \sqrt{\mu}][r, d_{11} + \sqrt{\mu}][r, \mu - \lambda_1\sqrt{\mu} - \lambda_2]$$
$$[r, \mu + \lambda_1\sqrt{\mu} - \lambda_2]$$

where $F_6(x) = (x - d_{11})(x + d_{11})(x^2 - \lambda_1 x - \lambda_2)$
$(x^2 + \lambda_1 x - \lambda_2) \equiv 0 \pmod{r}$

Proof: Let $N(a + b\theta + c\theta^2) = r = \mathfrak{r}_1\bar{\mathfrak{r}}_2$ be solvable for a, b and c in Z. By Fact 4.16. $\mathfrak{r}_1$ principal $\Rightarrow$ $\mathfrak{r}_1$ splits $\Rightarrow \pi^2 \equiv \mu \pmod{\mathfrak{r}_1}$ is solvable for π in K_3.

Then there exists a π such that $\pi^2 \equiv \mu \pmod{\tau_1}$ is solvable in K_3.

$\Rightarrow$ $\pi^2 \equiv A + 4B\theta \pmod{\tau_1}$ is solvable in K_3.

Since we have $T \equiv \pi \pmod{\tau_1}$ for some T in Z.

$\Rightarrow$ $T^2 \equiv A + 4B\theta \pmod{\tau_1}$ is solvable for T in Z.

$\Rightarrow$ $T^2 - A - 4B\theta = \tau_1 \sigma_1$ is solvable for T in Z for some σ_1 in K_3.

We take the norm on both sides, by Fact 4.9., we have

$$(T^2 - A)^3 - 64B^3\theta^3 = \tau_1 \tau_1' \tau_1'' \sigma_1 \sigma_1' \sigma_1''$$

$(T^2 - A)^3 - 64B^3m = r \cdot s^i$ is solvable for T in Z and $N(\sigma_1) = s^i$, where i in Z^+, s in Z.

$\Rightarrow$ $(T^2 - A)^3 - 64B^3m \equiv 0 \pmod{r}$ is solvable in Z.

By Fact 4-8-1, $w^3 \equiv m \pmod{r}$ is solvable for w in Z.

Then there is a unique solution d in Z such that

$$(w - d)(w^2 + dw + d^2) = 0 \pmod{r}.$$

$\Rightarrow$ $\{(T^2 - A) - 4Bd\} \cdot \{(T^2 - A)^2 + 4Bd(T^2 - A) + 16B^2d^2\} \equiv 0 \pmod{r}$ is solvable for T in Z.

$\Rightarrow$ $(T^2 - A - 4Bd)(T^4 - (2A - 4Bd)T^2 + A^2 - 4ABd + 16B^2d^2) \equiv 0 \pmod{r}$ is solvable for T in Z.

Since $N(A + 4Bd) \equiv A^3 + 64B^3m = q^2 \equiv (A^3 + 64B^3d^3) \pmod{r}$.

This implies $(A + 4Bd)(A^2 - 4ABd + 16B^2d^2) \equiv q^2 \pmod{r}$.

i.e., $(A + 4Bd/r)(A^2 - 4ABd + 16B^2d^2/r) = (q^2/r) = 1$.

Then we have to consider two cases for the above equation:

(i) $(A + 4Bd/r) = -1$ and $(A^2 - 4ABd + 16B^2d^2/r) = -1$.

(ii) $(A + 4Bd/r) = 1$ and $(A^2 - 4ABd + 16B^2d^2/r) = 1$.

Let us go back the solvability of the above equation.

(i) If $(T^4 - 2(A - 2Bd)T^2 + A^2 - 4ABd + 16B^2d^2) \equiv 0 \pmod{r}$ is solvable for T in Z. This implies that the discriminant of the quadratic equation is to be perfect square. $((A - 2Bd)^2 - (A^2 - 4ABd + 16B^2d^2)/r) = 1$ $\Rightarrow (-12B^2d^2/r) = 1 \Rightarrow (-3/r) = 1$ which is impossible.

(ii) If $(T^2 - A - 4Bd) \equiv 0 \pmod{r}$ is solvable for T in Z, then there exists a d_{11} such that $d_{11}^2 \equiv A + 4Bd \pmod{r}$. This implies $(T - d_{11})(T + d_{11}) = 0 \pmod{r}$. Since $(A + 4Bd/r) = 1$, then $(A^2 - 4ABd + 16B^2d^2/r) = 1$, and there exists a λ_2 in Z such that $\lambda_2^2 \equiv A^2 - 4ABd + 16B^2d^2 \pmod{r}$

So: $(T^2 - \lambda_2)^2 - (2A - 4Bd - 2\lambda_2)T^2 \equiv 0 \pmod{r}$ or

$(T^2 + \lambda_2)^2 - (2A - 4Bd + 2\lambda_2)T^2 \equiv 0 \pmod{r}$ is solvable for T in Z. Consider

$(2A - 4Bd - 2\lambda_2/r)(2A - 4Bd + 2\lambda_2/r)$

$= ((2A - 4Bd - 2\lambda_2)(2A - 4Bd + 2\lambda_2)/r)$

$= ((2A - 4Bd)^2 - 4\lambda_2^2/r)$

$= (4A^2 - 16ABd + 16B^2d^2 - 4A^2 + 16ABd - 64B^2d^2/r)$

$= (-48B^2d^2/r) = (-3/r) = -1.$

This implies either $w^{*2} \equiv 2A - 4Bd - 2\lambda_2 \pmod{r}$ or $w^{*2} \equiv 2A - 4Bd + 2\lambda_2 \pmod{r}$ is solvable for w^{*2} in Z. (Say) The first one is solvable for w^* in Z and suppose that $\lambda_1^2 \equiv 2A - 4Bd - 2\lambda_2 \pmod{r}$

be solvable for λ_1 in Z. Then we have $(T^2 - \lambda_1 T - \lambda_2)(T^2 + \lambda_1 T - \lambda_2) \equiv 0 \pmod{r}$ is solvable for T in Z. Furthermore, $(T^2 - \lambda_1 T - \lambda_2)$ and $(T^2 + \lambda_1 T - \lambda_2)$ are irreducible over Z because the discriminant of the quadratic equation is not perfect square.

i.e., $(\lambda_1^2 + 4\lambda_2/r) = (2A - 4Bd - 2\lambda_2 + 4\lambda_2/r)$

$(2A - 4Bd + 2\lambda_2/r) = -1.$

Then we have $(T - d_{11})(T + d_{11})(T^2 + \lambda_1 T - \lambda_2)$

$(T^2 - \lambda_1 T - \lambda_2) \equiv 0 \pmod{r}$ is solvable for T in Z.

By Fact 4.18., we also have

$$r = [r, d_{11} - \sqrt{\mu}][r, d_{11} + \sqrt{\mu}][r, \mu - \lambda_1\sqrt{\mu} - \lambda_2]$$
$$[r, \mu + \lambda_1\sqrt{\mu} - \lambda_2]$$

This proves the proposition 4.4. //

<u>Proposition 4.5.</u> Let r be a rational prime and $r \equiv 5 \pmod 6$. If for the field K_6, the defining polynomial $F_6(x) \equiv 0 \pmod{r}$ is solvable for x in Z, then $N(a + b\theta + c\theta^2) = r$ is solvable for a, b and c in Z.

<u>Proof</u>: Since $F_6(x) \equiv 0 \pmod{r}$ is solvable for x in Z, then there exists a g in Z such that $F_6(g) \equiv 0 \pmod{r}$.

$\Rightarrow$ $(x - g)E_5(x) \equiv 0 \pmod{r}$ is solvable for x in Z and for some polynomial E_5 of degree 5.

$\Rightarrow$ $(X - g)E_5(X) \equiv 0 \pmod{\mathfrak{r}_1}$ where $N(\mathfrak{r}_1) = r$ is solvable for X in K_3.

$\Rightarrow$ $X \equiv g \pmod{\mathfrak{r}_1}$ is solvable for X in K_3.

$\Rightarrow$ $X^2 \equiv g^2 \pmod{\mathfrak{r}_1}$ is solvable for X in K_3.

Since $\sqrt{\mu} = X$ satisfies $F_6(X) \equiv 0 \pmod{\mathfrak{r}_1}$

$\Rightarrow$ $\mu \equiv g^2 \pmod{\mathfrak{r}_1}$ is solvable for μ in K_3.

$\Rightarrow$ $\mathfrak{r}_1 \mid \mu - g^2$

$\Rightarrow$ $\mathfrak{r}_1$ splits in $K_3(\sqrt{\mu})/K_3$.

$\Rightarrow$ (By Hilbert) $\mathfrak{r}_1$ = principal ideal

$\Rightarrow$ $r = N(\delta^*)$ is solvable for a, b and c in Z, where $\delta^* = a + b\theta + c\theta^2$.

The proof is completed. //

If we combine these two propositions and apply in the second case of r ($\equiv 5 \pmod 6$) and $N(\delta^*) = r^2$ (not r), then we also obtain the following corollary.

Corollary 4.6. Let r be a rational prime and $r \equiv 5 \pmod 6$. $N(a + b\theta + c\theta^2) = r^2$ is solvable for a, b and c in Z if and only if for the field K_6, the defining polynomial $F_6(x) \equiv 0 \pmod r$ is unsolvable for x in Z. (i.e., there is no linear factors for $F_6(x)$).

To see $F_6(x)$ factors only as (quadratic)·(4th degree polynomial), which is $(x^2 + \cdots)(x^4 + \cdots)$ use the fact that $r = \mathfrak{r}_1\bar{\mathfrak{r}}_2$ so $\mathfrak{r}_1$ is principal only when $\bar{\mathfrak{r}}_2$ is principal. Therefore, the biquadratic factor splits only when the quadratic factor splits.

This is illustrated in Example 4.1.

Example 4.1. P: Principal ideal of K_3

NP: Non-Principal ideal of K_3

m	μ	r	$F_6(x)$ for μ	ideals of K_3	
11	$9-4\theta$	29	$(x-6)(x^2+12x+4)$ $(x+6)(x^2-12x+4)$	$(6+2\theta+\theta^2)$	P
		17	$(x^2+5)(x^4+2x^2+12)$	$[17,\mu+5]$	NP
15	$1+4\theta$	23	$(x-4)(x^2+3x+2)$ $(x+4)(x^2-3x+2)$	$(2+\theta)$	P
		11	$(x^2+1)(x^4+7x^2+7)$	$[11,\mu+1]$	NP
47	$17+4\theta$	17	$(x-4)(x^2+4x+1)$ $(x+4)(x^2-4x+1)$	$(4-\theta)$	P
		23	$(x^2+2)(x^4+16x^2+7)$	$[23,\mu+2]$	NP
83	$-47+12\theta$	281	$(x-15)(x^2+91x-32)$ $(x+15)(x^2-91x-32)$	$(20+4\theta+\theta^2)$	P
		41	$(x^2-6)(x^4+24x^2+6)$	$[41,\mu-6]$	NP
89	$17+4\theta$	17	$(x-1)(x^2-x+1)$ $(x+1)(x^2+x+1)$	$(-9+2\theta)$	P
		5	$(x^2-3)(x^4-3x^2+3)$	$[5,\mu-3]$	NP
113	$-7+4\theta$	887	$(x-337)(x^2+519x+327)$ $(x+337)(x^2-519x+327)$	$(10-\theta)$	P
		41	$(x^2-3)(x^4+4x^2-3)$	$[41,\mu-3]$	NP

Now we like to consider the factorization and the normed solvability in the field $K_6^* = \mathbb{Q}(\theta, \sqrt{-3\mu})$.

Proposition 4.7. Let r be a rational prime as usual and $r \equiv 5 \pmod 6$. If $N(a + b\theta + c\theta^2) = r$ is solvable for a b and c in Z, then for the field K_6^*, the defining polynomial $F_6^*(x) \equiv 0 \pmod r$ is unsolvable for x in Z. (i.e., there is no linear factors for $F_6(x)$).

Proof: By use of Proposition 4.3. and Proposition 4.4., then we have $(-3\mu/r) = (\mu/r)(-3/r) = (\mu/r)(-1) = -(\mu/r)$. Therefore we can use the same argument as in the proof of Proposition 4.4.. //

Proposition 4.8. Let r be a rational prime as usual and $r \equiv 5 \pmod 6$. If $N(a + b\theta + c\theta^2) = r^2$ is solvable for a, b and c in Z, then for the field K_6^*, the defining polynomial $F_6^*(x) \equiv 0 \pmod r$ is solvable for x in Z.

Proof: By use of Proposition 4.3. and Proposition 4.5., then we have $(-3\mu/r) = -(\mu/r)$. Therefore, we can apply the same argument as in the proof of Proposition 4.5.. //

Now we combine these two propositions, we obtain the following corollary.

Corollary 4.9. Let r be a rational prime and $r \equiv 5 \pmod 6$. If $N(a + b\theta + c\theta^2) = r$ is solvable for a, b and c in Z, then for the field K_6^*, the defining polynomial $F_6^*(x) \equiv 0 \pmod r$ is unsolvable for x in Z. (i.e., there is no linear factors for $F_6^*(x)$).

Remark 4.2. There is no further factorization and no further solvability for the extension field $K_6^{\#}$ over the field K_3 because of the non-residue character of r, i.e., $(-3/r) = -1$, where $r \equiv 5 \pmod 6$.

Examples.4.2.

(1) Principal ideals of K_3 in the field K_6^*.

m	-3μ	r	$F_6^*(x)$ for -3μ	Princ. of K_3
11	$-27+12\theta$	29	$(x^2-8)(x^2-14x+12)$ $(x^2+14x+12)$	$(6+2\theta+\theta^2)$
15	$-3-12\theta$	23	$(x^2+8)(x^2-2x-6)$ (x^2+2x-6)	$(2+\theta)$
47	$-51-12\theta$	17	$(x^2+5)(x^2+6x+12)$ $(x^2-6x+12)$	$(4-\theta)$
83	$141-36\theta$	281	$(x^2-168)(x^2+2x-96)$ $(x^2-2x-96)$	$(20+4\theta+\theta^2)$
89	$-51-12\theta$	17	$(x^2+3)(x^2-3x+3)$ (x^2+3x+3)	$(-9+2\theta)$
113	$21-12\theta$	887	$(x^2-788)(x^2-375x+570)$ $(x^2+375x+570)$	$(10-\theta)$

(2) Non-Principal ideals of K_3 in the field K_6^*

m	-3μ	r	$F_6^*(x)$ for -3μ	ideals of K_3
11	$-27+12\theta$	17	$(x+7)(x-7)(x^4-6x^2+6)$	$[17, -3\mu+2]$
15	$-3-12\theta$	11	$(x-5)(x+5)(x^4+12x^2+63)$	$[11, -3\mu-3]$
47	$-51-12\theta$	23	$(x-11)(x+11)(x^4-2x^2+17)$	$[23, -3\mu-6]$

83	$141-36\theta$	41	$(x-8)(x+8)(x^4-31x^2+13)$	$[41,-3\mu-23]$
89	$-51-12\theta$	5	$(x-1)(x+1)(x^4-x^2+2)$	$[5,\ -3\mu-1]$
113	$21-12\theta$	41	$(x-20)(x+20)(x^4+10x^2+3)$	$[41,-3\mu+10]$

Now we turn to the primes which are congruent to 1 modulo 6, the remaining rational primes, $p \equiv 1 \pmod 6$. To start with, we have to consider the factorization in the extension fields K_6 over the fields K_3, then K_6^* and $K_6^\#$ over K_3.

Proposition 4.10. Let p be a rational prime and $p \equiv 1 \pmod 6$. If $N(a + b\theta + c\theta^2) = p$ is solvable for a, b and c in Z, then for the field K_6, the defining polynomial $F_6(x) \equiv 0 \pmod p$ is solvable for x in Z. Moreover,

$$p = \text{(either) } [p,d_{11}-\sqrt{\mu}][p,d_{11}+\sqrt{\mu}][p,\mu-d_2][p,\mu-d_3] \quad \text{(I)}$$

$$\text{(or) } [p,\beta_1^*-\sqrt{\mu}][p,\beta_2^*-\sqrt{\mu}][p,\beta_3^*-\sqrt{\mu}][p,\beta_4^*-\sqrt{\mu}]$$
$$[p,\beta_5^*-\sqrt{\mu}][p,\beta_6^*-\sqrt{\mu}] \quad \text{(II)}$$

where $F_6(x) \equiv \text{(either) } (x-d_{11})(x+d_{11})(x^2-d_2)(x^2-d_3)$ (I)

$$\equiv \text{(or) } (x-\beta_1^*)(x-\beta_2^*)(x-\beta_3^*)(x-\beta_4^*)(x-\beta_5^*)(x-\beta_6^*) \text{ (II)}$$

(mod p) and d_{11}, d_2, d_3, β_i^* in Z.

Proof: Let $N(a + b\theta + c\theta^2) = p = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ be solvable for a, b and c in Z. By Fact 4.16. $\mathfrak{p}_1$ principal $\Rightarrow \mathfrak{p}_1$ splits $\Rightarrow \pi^2 \equiv \mu \pmod{\mathfrak{p}_1}$ is solvable for π in K_3. Then there exists a π such that $\pi^2 \equiv \mu \pmod{\mathfrak{p}_1}$

is solvable for π in K_3.

$\Rightarrow$ $\pi^2 \equiv A + 4B\theta \pmod{\sqrt{p_1}}$ is solvable for π in K_3.

Since we have $T \equiv \pi \pmod{\sqrt{p_1}}$ for some T in Z.

$\Rightarrow$ $T^2 \equiv \pi^2 \equiv A + 4B\theta \pmod{\sqrt{p_1}}$ is solvable for T in K_3 and in Z.

$\Rightarrow$ $(T^2 - A - 4B\theta) \equiv \sqrt{p_1}\, s_1^*$ is solvable for T in Z where s_1^* in K_3.

Take the norm on both sides, we have

$$(T^2 - A)^3 - 64B^3\theta^3 = \sqrt{p_1}\sqrt{p_1'}\sqrt{p_1''}\, s_1^* s_1^{*'} s_1^{*''}$$

$\Rightarrow$ $(T^2 - A)^3 - 64B^3m = p \cdot s_1^{*i}$ is solvable for T in Z and $N(s_1^*) = s_1^{*i}$ where i in Z^+, s_1^* in Z.

$\Rightarrow$ $(T^2 - A)^3 \equiv 64B^3m \pmod{p}$ is solvable for T in Z.

By Fact 4-8-2, $\bar{w}^3 \equiv m \pmod{p}$ is solvable for $\bar{w}$ in Z, and there exist three distinct roots d, d', d" in Z. such that $(\bar{w} - d)(\bar{w} - d')(\bar{w} - d'') \equiv 0 \pmod{p}$.

$\Rightarrow$ $((T^2 - A) - 4Bd)((T^2 - A) - 4Bd')((T^2 - A) - 4Bd'') \equiv 0 \pmod{p}$ is solvable for T in Z.

Since $N(A + 4Bd) \equiv N(A + 4Bd') \equiv N(A + 4Bd'') \equiv A^3 + 64B^3m \equiv q^2 \pmod{p}$. This implies

$$(A + 4Bd)(A + 4Bd')(A + 4Bd'') \equiv q^2 \pmod{p}$$

That is, $(A^3 + 64B^3m/p) = (q^2/p) = 1$

$$= (A + 4Bd/p)(A + 4Bd'/p)(A + 4Bd''/p).$$

Then we must consider two cases for the above equation.

(i) $(A + 4Bd/p) = 1$ (say) and

$(A + 4Bd'/p) = (A + 4Bd''/p) = -1$.

(ii) $(A + 4Bd/p) = (A + 4Bd'/p) = (A + 4Bd''/p) = 1$

Let us go back the solvability of the above congruence.

(i) If $(A + 4Bd/p) = 1$ (say), then there exists a d_{11} in Z such that $d_{11}^2 \equiv A + 4Bd \pmod{p}$. The other two algebraic integers are unsolvable for T in Z.

This follows immediately:

$$F_6(T) \equiv (T - d_{11})(T + d_{11})(T^2 - d_2)(T^2 - d_3) \equiv 0 \pmod{p}$$

where $d_2 \equiv A + 4Bd' \pmod{p}$ and $d_3 \equiv A + 4Bd'' \pmod{p}$.

(ii) If $(A + 4Bd/p) = (A + 4Bd'/p) = (A + 4Bd''/p) = 1$, then there exist $\beta_1^*, \cdots, \beta_5^*$ and β_6^* in Z such that

$$\beta_1^{*2} \equiv \beta_2^{*2} \equiv A + 4Bd \pmod{p}; \quad \beta_1^* \equiv -\beta_2^* \pmod{p}$$

$$\beta_3^{*2} \equiv \beta_4^{*2} \equiv A + 4Bd' \pmod{p}; \quad \beta_3^* \equiv -\beta_4^* \pmod{p}$$

$$\beta_5^{*2} \equiv \beta_6^{*2} \equiv A + 4Bd'' \pmod{p}; \quad \beta_5^* \equiv -\beta_6^* \pmod{p}.$$

This follows at once:

$$F_6(T) \equiv (T - \beta_1^*)(T - \beta_2^*)(T - \beta_3^*)(T - \beta_4^*)(T - \beta_5^*)(T - \beta_6^*) \pmod{p}$$

By Fact 4.18., we have

$$p = [p, \sqrt{\mu} - d_{11}][p, \sqrt{\mu} + d_{11}][p, \mu - d_2][p, \mu - d_3] \quad (I)$$

or $p = [p, \sqrt{\mu} - \beta_1^*][p, \sqrt{\mu} - \beta_2^*][p, \sqrt{\mu} - \beta_3^*][p, \sqrt{\mu} - \beta_4^*]$

$$[p, \sqrt{\mu} - \beta_5^*][p, \sqrt{\mu} - \beta_6^*] \quad (II)$$

The proof is completed. //

<u>Proposition 4.11.</u> Let p be a rational prime and $p \equiv 1 \pmod{6}$. If for the field K_6, the defining polynomial $F_6(x) \equiv 0 \pmod{p}$ is solvable for x in Z, then $N(a + b\theta + c\theta^2) = p$ is solvable for a, b and c in Z.

Proof: Since $F_6(x) \equiv 0 \pmod{p}$ is solvable for x in Z. then there exists a $\bar{g}$ in Z such that $F_6(\bar{g}) \equiv 0 \pmod{p}$.

$\Rightarrow$ $(x - \bar{g})\bar{E}_5(x) \equiv 0 \pmod{p}$ is solvable for x in Z for some polynomial $\bar{E}_5$ of degree 5.

$\Rightarrow$ $(X - \bar{g})\bar{E}_5(X) \equiv 0 \pmod{\sqrt{p_1}}$, where $N(\sqrt{p_1}) = p$, is solvable for X in K_3.

$\Rightarrow$ $X \equiv \bar{g} \pmod{\sqrt{p_1}}$ is solvable for X in K_3.

$\Rightarrow$ $X^2 \equiv \bar{g}^2 \pmod{\sqrt{p_1}}$ is solvable for X in K_3.

Since $\sqrt{\mu} = X$ satisfies $F_6(X) \equiv 0 \pmod{\sqrt{p_1}}$,

$\Rightarrow$ $\mu \equiv \bar{g}^2 \pmod{\sqrt{p_1}}$ is solvable for μ in K_3.

$\Rightarrow$ $\sqrt{p_1} \mid \mu - \bar{g}^2$

$\Rightarrow$ $\sqrt{p_1}$ splits in $K_3(\sqrt{\mu})/K_3$

$\Rightarrow$ (By Hilbert) $\sqrt{p_1}$ = principal ideal

$\Rightarrow$ $p = N(\gamma^*) = N(a + b\theta + c\theta^2)$ is solvable for a, b and c in Z.

The proof is completed. //

Example 4.3.

P: Principal of K_3; NP: Non-Principal of K_3 in the field $K_6 = Q(\theta, \sqrt{\mu})$

m	μ	p	$F_6(x)$ for μ	ideals of K_3	
11	$9 - 4\theta$	19	$(x - 6)$ $(x + 6)$	$(2 + \theta)$	P
			$(x^2 - 2)$	$[19, \mu - 2]$	NP
			$(x^2 - 8)$	$[19, \mu - 8]$	NP

15	$1 + 4\theta$	7	$(x - 3)$ $(x + 3)$	$(-2 + \theta)$	P
			$(x^2 - 5)$	$[7, \mu - 5]$	NP
			$(x^2 - 3)$	$[7, \mu - 3]$	NP
47	$17 + 4\theta$	13	$(x - 5)$ $(x + 5)$	$(4428 + 12270 + 340\theta^2)$	P
			$(x^2 - 11)$	$[13, \mu - 11]$	NP
			$(x^2 - 2)$	$[13, \mu - 2]$	NP
83	$-47 + 12\theta$	937	$(x - 30)$ $(x + 30)$	$(22 + 4\theta + \theta^2)$	P
			$(x^2 - 362)$	$[937, \mu - 362]$	NP
			$(x^2 - 471)$	$[937, \mu - 471]$	NP
89	$17 + 4\theta$	79	$(x - 16)$ $(x + 16)$	$(20 - \theta^2)$	P
			$(x^2 - 63)$	$[79, \mu - 63]$	NP
			$(x^2 - 48)$	$[79, \mu - 48]$	NP
113	$-7 + 4\theta$	103	$(x - 29)$ $(x + 29)$	$(6 - \theta)$	P
			$(x^2 - 101)$	$[103, \mu - 101]$	NP
			$(x^2 - 67)$	$[103, \mu - 67]$	NP

Proposition 4.12. Let p be a rational prime and $p \equiv 1 \pmod 6$. If $N(a + b\theta + c\theta^2) = p$ is solvable for a, b and c, then for the field K_6^*, the defining polynomial $F_6^*(x) \equiv 0 \pmod p$ is solvable for x in Z.

where $F_6^*(x) \equiv$ (either) $(x - \bar{d}_{11})(x + \bar{d}_{11})(x^2 - \bar{d}_2)(x^2 - \bar{d}_3)$

(or) $(x-\bar{\beta}_1^*)(x-\bar{\beta}_2^*)(x-\bar{\beta}_3^*)(x-\bar{\beta}_4^*)(x-\bar{\beta}_5^*)(x-\bar{\beta}_6^*)$
(mod p),

and $\bar{d}_{11}$, $\bar{d}_2$, $\bar{d}_3$, $\bar{\beta}_i^*$ in Z .

Proof: Since $(-3/p) = 1$ for $p \equiv 1 \pmod 6$, then
$(-3\mu/p) = (-3/p)(\mu/p) = (\mu/p) = (A + 4B\theta/p)$.
By the same argument as we did the proof of proposition 4.10.
Therefore, we complete the proof. //

Proposition 4.13. Let p be a rational prime and $p \equiv 1 \pmod 6$. If for the field K_6^*, the defining polynomial $F_6^*(x) \equiv 0 \pmod p$ is solvable for x in Z, then $N(a + b\theta + c\theta^2) = p$ is solvable for a, b and c in Z.

Proof: Consider the same cases as we proved in Proposition 4.11. //

Remark 4.3. After combining two previous propositions, we have the following fact:
For the field K_6^*, the defining polynomial $F_6^*(x) \equiv 0 \pmod p$ is solvable for x in Z if and only if $p \equiv 1 \pmod 6$ and $N(a + b\theta + c\theta^2) = p$ is solvable for a, b and c in Z.

Remark 4.4. Since $(-3/p) = 1$, then p can be expressed as the product of 6 linear factors in the field $K_6^\#$ when p splits in K_3. That is to say, if $N(a + b\theta + c\theta^2) = p$ is solvable for a, b and c in Z in the field $K_6^\#$ then the defining polynomial $F_6^\#(x) \equiv \prod P_{ij}(x) \pmod p$

where $p \equiv 1 \pmod 6$, $\mathfrak{P}_{ij}(x) \equiv (x - \alpha_{ij}) \pmod p$
and α_{ij} in Z for $i = 1, 2, 3$ and $j = 1, 2$.
Furthermore, $p = \mathfrak{P}_{11}\mathfrak{P}_{12}\mathfrak{P}_{21}\mathfrak{P}_{22}\mathfrak{P}_{31}\mathfrak{P}_{32}$

$$= [p,\alpha^{\#} - \alpha_{11}][p,\alpha^{\#} - \alpha_{12}][p,\alpha^{\#} - \alpha_{21}]$$
$$[p,\alpha^{\#} - \alpha_{22}][p,\alpha^{\#} - \alpha_{31}][p,\alpha^{\#} - \alpha_{32}]$$

where $\alpha^{\#}$ in $K_6^{\#}$.
This is illustrated in Example 4.5.

Example 4.4.

P: Principal of K_3

NP: Non-Principal of K_3 in the field K_6^*

m	-3μ	p	$F_6^*(x)$ for -3μ	ideals of K_3	
11	$-27 + 12\theta$	19	$(x - 5)$ $(x + 5)$	$(2 + \theta)$	P
			$(x^2 - 13)$	$[19,-3\mu-13]$	NP
			$(x^2 - 14)$	$[19,-3\mu-14]$	NP
15	$-3 - 12\theta$	7	$(x - 1)(x + 1)$	$(-2 + \theta)$	P
			$(x^2 - 6)$	$[7,-3\mu - 6]$	NP
			$(x^2 - 5)$	$[7,-3\mu - 5]$	NP
47	$-51 - 12\theta$	13	$(x - 2)$ $(x + 2)$	$(4428 + 12270 + 340\theta^2)$	P
			$(x^2 - 6)$	$[13,-3\mu - 6]$	NP
			$(x^2 - 7)$	$[13,-3\mu - 7]$	NP
83	$141 - 36\theta$	937	$(x-327)(x+327)$	$(22 +4\theta + \theta^2)$	P
			$(x^2 - 461)$	$[937,-3\mu-461]$	NP
			$(x^2 - 788)$	$[937,-3\mu-788]$	NP

89	-51 - 12θ	79	(x - 38) (x + 38)	$(20 - \theta^2)$	P
			$(x^2 - 48)$	$[79,-3\mu\ -48]$	NP
			$(x^2 - 14)$	$[79,-3\mu\ -14]$	NP
113	21 - 12θ	103	(x - 19) (x + 19)	(6 - θ)	P
			$(x^2 - 6)$	$[103,-3\mu\ -6]$	NP
			$(x^2 - 5)$	$[103,-3\mu\ -5]$	NP

Example 4.5.

$d^3 \equiv -3 \pmod{p}$; Take $\alpha^{\#} = \theta + \sqrt{-3}$ in $K_6^{\#}$

$e^2 \equiv -3 \pmod{p}$;

m	p	d	e	$F_6^{\#}(x)$ for $\alpha^{\#}$
11	19	5,16,17	± 4	$(x-1)^2(x-2)(x-9)(x-12)(x-13)$
15	7	1, 2, 4,	± 2	$x\,(x-2)(x-3)(x-4)(x-6)^2$
47	13	2, 5, 6,	± 6	$x(x-8)(x-11)(x-12)^2(x-13)$
83	937	893 824 157	± 292	$(x - 179)(x - 248)(x - 449)$ $(x - 532)(x - 601)(x - 802)$
89	79	40 51 67	± 32	$(x - 8)(x - 10)(x - 19)$ $(x - 35)(x - 72)(x - 83)$
113	103	6 27 70	± 10	$(x - 16)(x - 17)(x - 37)$ $(x - 60)(x - 80)(x - 99)$

Note: p is an inessential divisor of the root discriminant. (It can be avoided by a different defining equation). Therefore, the factors are included in the example.

Finally, we obtain the conclusion of this Chapter as describing the corresponding fields with the relationship between the normed equation and the prime factorization in the extension fields $K_6^{\#} = Q(\rho,\theta)$, $K_6 = Q(\theta,\sqrt{\mu})$ and $K_6^{*} = Q(\theta,\sqrt{-3\mu})$ in Table 4.2. (as below). Furthermore, we can apply the above results into their conjugate fields and their conjugate class fields (field extensions), as well. For instance, the equivalence relation between the normed equation and the factorization in the fields $Q(\rho,\theta)$, $Q(\theta,\sqrt{\mu'})$ and $Q(\theta,\sqrt{-3\mu'})$ or in the fields $Q(\rho,\theta)$, $Q(\theta,\sqrt{\mu''})$ and $Q(\theta,\sqrt{-3\mu''})$.

In Table 4.3. and Table 4.4., we list 6 linear factors of the rational prime p ($\equiv 1 \pmod 6$) in the fields $K_6 = Q(\theta, \sqrt{\mu})$ and $K_6^{*} = Q(\theta, \sqrt{-3\mu})$

$p \equiv 1 \pmod 6$; $N(\gamma^*) = N(a + b\theta + c\theta^2) = p = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3$ if solvable.

$r \equiv 5 \pmod 6$; $N(\delta^*) = N(\bar{a} + \bar{b}\theta + \bar{c}\theta^2) = r = \mathfrak{r}_1 \bar{\mathfrak{r}}_2$ if solvable.

$r \equiv 5 \pmod 6$; $N(\delta^*) = N(\bar{\bar{a}} + \bar{\bar{b}}\theta + \bar{\bar{c}}\theta^2) = r^2$ (not r) if solvable.

$K_3 = Q(\theta)$	$K_6^{\#} = Q(\rho, \theta)$	$K_6 = Q(\theta, \sqrt{\mu})$	$K_6^* = Q(\theta, \sqrt{-3\mu})$	$Q(\sqrt[3]{11})$
$(m/p)_3 = 1$, Princ.	$\mathfrak{p}_1 = \mathfrak{P}_{11}^{\#}\mathfrak{P}_{12}^{\#}$	$\mathfrak{p}_1 = \mathfrak{P}_{11}\mathfrak{P}_{12}$	$\mathfrak{p}_1 = \mathfrak{P}_{11}^{*}\mathfrak{P}_{12}^{*}$	19_1
$(m/p)_3 = 1$, Non-Princ.	$\mathfrak{p}_2 = \mathfrak{P}_{21}^{\#}\mathfrak{P}_{22}^{\#}$	$\mathfrak{p}_2 = \mathfrak{p}_2$	$\mathfrak{p}_2 = \mathfrak{p}_2$	19_2
$(m/r)_3 = 1$, Princ.	$\mathfrak{r}_1 = \mathfrak{r}_1$ $\bar{\mathfrak{r}}_2 = \mathfrak{R}_{21}^{\#}\mathfrak{R}_{22}^{\#}$	$\mathfrak{r}_1 = \mathfrak{R}_{11}\mathfrak{R}_{12}$ $\bar{\mathfrak{r}}_2 = \mathfrak{R}_{21}\mathfrak{R}_{22}$	$\mathfrak{r}_1 = \mathfrak{r}_1$ $\bar{\mathfrak{r}}_2 = \mathfrak{R}_{21}^{*}\mathfrak{R}_{22}^{*}$	29_1 $\overline{29}_2$
$(m/r) = 1$, Non-Princ.	$\mathfrak{r}_1 = \mathfrak{r}_1$ $\bar{\mathfrak{r}}_2 = \mathfrak{R}_{21}^{\#}\mathfrak{R}_{22}^{\#}$	$\mathfrak{r}_1 = \mathfrak{r}_1$ $\bar{\mathfrak{r}}_2 = \bar{\mathfrak{r}}_2$	$\mathfrak{r}_1 = \mathfrak{R}_{11}^{*}\mathfrak{R}_{12}^{*}$ $\bar{\mathfrak{r}}_2 = \bar{\mathfrak{r}}_2$	17_1 $\overline{17}_2$
$(m/p)_3 = 1$, and $(\alpha_1/p) = (\alpha_2/p) =$ $(\alpha_3/p) = 1$, Princ.	$\mathfrak{p}_1 = \mathfrak{P}_{11}^{\#}\mathfrak{P}_{12}^{\#}$ $\mathfrak{p}_2 = \mathfrak{P}_{21}^{\#}\mathfrak{P}_{22}^{\#}$ $\mathfrak{p}_3 = \mathfrak{P}_{31}^{\#}\mathfrak{P}_{32}^{\#}$	$\mathfrak{p}_1 = \mathfrak{P}_{11}\mathfrak{P}_{12}$ $\mathfrak{p}_2 = \mathfrak{P}_{21}\mathfrak{P}_{22}$ $\mathfrak{p}_3 = \mathfrak{P}_{31}\mathfrak{P}_{32}$	$\mathfrak{p}_1 = \mathfrak{P}_{11}^{*}\mathfrak{P}_{12}^{*}$ $\mathfrak{p}_2 = \mathfrak{P}_{21}^{*}\mathfrak{P}_{22}^{*}$ $\mathfrak{p}_3 = \mathfrak{P}_{31}^{*}\mathfrak{P}_{32}^{*}$	193_1 193_2 193_3

where $\alpha_1 = A + 4Bd$ and $d^3 \equiv d'^3 \equiv d''^3 \equiv m \pmod p$. Table 4.2.

$\alpha_2 = A + 4Bd'$

$\alpha_3 = A + 4Bd''$

Linear Factorization of the Rational Prime p ($\equiv 1 \pmod 6$) in the field $K_6 = Q(\theta, \sqrt{\mu})$.

m	p	A	B_*	μ	3 rts α_i^*	6 rts β_i^*	ideals of K_6	princ. ideals of K_3
11	193	9	-4	$9-4\theta$	188	-2 2	$[2 - \sqrt{\mu}, 193]$ $[2 + \sqrt{\mu}, 193]$	$(8 + 2\theta + \theta^2)$
					159	-19 19	$[19 - \sqrt{\mu}, 193]$ $[19 + \sqrt{\mu}, 193]$	$(-3 + 6\theta - 2\theta^2)$
					39	-56 56	$[56 - \sqrt{\mu}, 193]$ $[56 + \sqrt{\mu}, 193]$	$(8 + 3\theta + 2\theta^2)$
15	397	1	4	$1+4\theta$	39	-45 45	$[45 - \sqrt{\mu}, 397]$ $[45 + \sqrt{\mu}, 397]$	$(28 + 12\theta + 5\theta^2)$
					135	-70 70	$[70 - \sqrt{\mu}, 397]$ $[70 + \sqrt{\mu}, 397]$	$(-2 + 3\theta)$
					223	-47 47	$[47 - \sqrt{\mu}, 397]$ $[47 + \sqrt{\mu}, 397]$	$(13 - 2\theta^2)$
47	6091	17	4	$17+4\theta$	4868	-1618 1618	$[1618 - \sqrt{\mu}, p]$ $[1618 + \sqrt{\mu}, p]$	$(6 + 5\theta)$
					3738	-266 266	$[266 - \sqrt{\mu}, p]$ $[266 + \sqrt{\mu}, p]$	$(6 + 13\theta - 4\theta^2)$
					3576	-3000 3000	$[3000 - \sqrt{\mu}, p]$ $[3000 + \sqrt{\mu}, p]$	$(100 + 29\theta + 8\theta^2)$

6 roots β_i^* of $(x^2 - A)^3 \equiv B_*^3\theta^3 \pmod p$ $\Leftrightarrow$ $(x-\beta_1^*)(x-\beta_2^*)\cdots(x-\beta_6^*) \equiv 0 \pmod p$

3 roots α_i^* of $y^3 \equiv B_*^3\theta^3 \pmod p$

To be continued

m	p	A	B_*	μ	3 rts α_i^*	6 rts β_i^*	ideals of K_6	Princ. of K_3
83	285079	-47	12	$-47+12\theta$	236273	-13196 13196	$[13196 - \sqrt{\mu}, p]$ $[13196 + \sqrt{\mu}, p]$	$(39 - 24\theta + 4\theta^2)$
					170938	-84468 84468	$[84468 - \sqrt{\mu}, p]$ $[84468 + \sqrt{\mu}, p]$	$(39 + 6\theta + 4\theta^2)$
					162947	-102341 102341	$[102341 - \sqrt{\mu}, p]$ $[102341 + \sqrt{\mu}, p]$	$(39 + 18\theta + 4\theta^2)$
89	74929	17	4	$17+4\theta$	69710	-31529 31529	$[31529 - \sqrt{\mu}, p]$ $[31529 + \sqrt{\mu}, p]$	$(316 + 73\theta + 16\theta^2)$
					43244	-16177 16177	$[16177 - \sqrt{\mu}, p]$ $[16177 + \sqrt{\mu}, p]$	$(-40 + 8\theta + \theta^2)$
					36904	-9641 9641	$[9641 - \sqrt{\mu}, p]$ $[9641 + \sqrt{\mu}, p]$	$(49 + 8\theta - 4\theta^2)$
113	322459	-7	4	$-7+4\theta$	191093	-51175 51175	$[51175 - \sqrt{\mu}, p]$ $[51175 + \sqrt{\mu}, p]$	$(67 + 22\theta + 4\theta^2)$
					284945	-139298 139298	$[139298 - \sqrt{\mu}, p]$ $[139298 + \sqrt{\mu}, p]$	$(67 - 32\theta + 4\theta^2)$
					168880	-89051 89051	$[89051 - \sqrt{\mu}, p]$ $[89051 + \sqrt{\mu}, p]$	$(67 + 10\theta + 4\theta^2)$

Table 4.3.

Linear Factorization of the Rational Prime p ($= 1 \pmod 6$) in the field $K_6^* = Q(\theta, \sqrt{-3\mu})$.

m	p	A*	B*	-3μ	3 rts $\bar{\alpha}_i$	6 rts $\bar{\beta}_i$	ideals of K_6^*	Princ. of K_3
11	193	-27	12	$-27+12\theta$	15	-48 48	$[48 - \sqrt{-3\mu}, p]$ $[48 + \sqrt{-3\mu}, p]$	$(8 + 2\theta + \theta^2)$
					102	-70 70	$[70 - \sqrt{-3\mu}, p]$ $[70 + \sqrt{-3\mu}, p]$	$(-3 + 6\theta - 2\theta^2)$
					76	-7 7	$[7 - \sqrt{-3\mu}, p]$ $[7 + \sqrt{-3\mu}, p]$	$(8 + 3\theta + 2\theta^2)$
15	397	-3	-12	$-3-12\theta$	389	-66 66	$[66 - \sqrt{-3\mu}, p]$ $[66 + \sqrt{-3\mu}, p]$	$(-2 + 3\theta)$
					280	-71 71	$[71 - \sqrt{-3\mu}, p]$ $[71 + \sqrt{-3\mu}, p]$	$(28+12\theta+5\theta^2)$
					125	-67 67	$[67 - \sqrt{-3\mu}, p]$ $[67 + \sqrt{-3\mu}, p]$	$(13 - 2\theta^2)$
47	6091	-51	-12	$-51-12\theta$	968	-159 159	$[159 - \sqrt{-3\mu}, p]$ $[159 + \sqrt{-3\mu}, p]$	$(6 + 13\theta - 4\theta^2)$
					1454	-2297 2297	$[2297 - \sqrt{-3\mu}, p]$ $[2297 + \sqrt{-3\mu}, p]$	$(100+29\theta+8\theta^2)$
					3669	-2834 2834	$[2834 - \sqrt{-3\mu}, p]$ $[2834 + \sqrt{-3\mu}, p]$	$(6 + 5\theta)$

To be continued

m	p	A*	B*	-3μ	3 rts $\bar{\alpha}_i$	6 rts $\bar{\beta}_i$	ideals of K_6^*	Princ. of K_3
83	285079	141	-36	141-36θ	146418	- 40082 40082	$[40082 - \sqrt{-3\mu}, p]$ $[40082 + \sqrt{-3\mu}, p]$	$(39 - 24\theta + 4\theta^2)$
					57344	- 12872 12872	$[12872 - \sqrt{-3\mu}, p]$ $[12872 + \sqrt{-3\mu}, p]$	$(39 + 6\theta + 4\theta^2)$
					81317	-101037 101037	$[101037 - \sqrt{-3\mu}, p]$ $[101037 + \sqrt{-3\mu}, p]$	$(39 + 18\theta + 4\theta^2)$
89	74929	-51	-12	-51-12θ	15657	- 35559 35559	$[35559 - \sqrt{-3\mu}, p]$ $[35559 + \sqrt{-3\mu}, p]$	$(316 + 73\theta + 16\theta^2)$
					20126	- 34029 34029	$[34029 - \sqrt{-3\mu}, p]$ $[34029 + \sqrt{-3\mu}, p]$	$(- 40 + 8\theta + \theta^2)$
					39146	- 3564 3564	$[3564 - \sqrt{-3\mu}, p]$ $[3564 + \sqrt{-3\mu}, p]$	$(49 + 8\theta - 4\theta^2)$
113	322459	21	-12	21-12θ	71639	- 15973 15973	$[15973 - \sqrt{-3\mu}, p]$ $[15973 + \sqrt{-3\mu}, p]$	$(67 + 22\theta + 4\theta^2)$
					112542	- 74316 74316	$[74316 - \sqrt{-3\mu}, p]$ $[74316 + \sqrt{-3\mu}, p]$	$(67 - 32\theta + 4\theta^2)$
					138278	- 59477 59477	$[59477 - \sqrt{-3\mu}, p]$ $[59477 + \sqrt{-3\mu}, p]$	$(67 + 10\theta + 4\theta^2)$

Table 4.4.

CHAPTER V. EVEN CLASS NUMBER OF PURE CUBIC FIELDS $Q(\sqrt[3]{m})$

§ 1. Basic definitions and basic results.

It is well known that there exists an element of order 2 in the finite groups of order 2n. When we apply this fact to the class groups of pure cubic fields $Q(\theta)$, we look for the algebraic integer μ with norm $N(\mu) = q^2$ (q is a rational prime and greater than 3). Furthermore, the class number of the pure cubic field $Q(\theta)$ is even only if some algebraic integer μ (with norm q^2) is a non-square integer. Sufficient conditions are given in Fact 4.15. To start with, we like to list all the facts and the definitions.

Fact 5.1. If G is a finite group whose order is divisible by a prime p, then G contains an element of order p. In particular, $p = 2$, that is, a finite group of even order contains an element of order 2.

Fact 5.2. (Chebotaroff's Theorem) When the class number is even, we can find an algebraic integer μ such that $\mu = A + 4\theta$ with the norm $N(\mu) = N(A + 4\theta) = A^3 + 64\hat{m} = q^2$ (where q is a rational prime and greater than 3).

Fact 5.3. (Mordell) If $q^2 = A^3 + D^3$, then the solution when A is odd and prime to D, is given by

(1) $A = \tilde{a}^4 + 8\tilde{a}\tilde{b}^3$; $D = -4\tilde{a}^3\tilde{b} + 4\tilde{b}^4$

(2) $A = \tilde{a}^4 + 6\tilde{a}^2\tilde{b}^2 - 3\tilde{b}^4$; $D = -\tilde{a}^4 + 6\tilde{a}^2\tilde{b}^2 + 3\tilde{b}^4$

(3) $A = -\tilde{a}^4 + 6\tilde{a}^2\tilde{b}^2 + 3\tilde{b}^4$; $D = \tilde{a}^4 + 6\tilde{a}^2\tilde{b}^2 - 3\tilde{b}^4$

(4) $A = \tilde{a}^4 + 4\tilde{a}^3\tilde{b} - 6\tilde{a}^2\tilde{b}^2 + 4\tilde{a}\tilde{b}^3 + \tilde{b}^4$;

$$D = 2\tilde{a}^4 - 4\tilde{a}^3\tilde{b} - 4\tilde{a}\tilde{b}^3 + 2\tilde{b}^4$$

(5) $A = \tilde{a}^4 + 8\tilde{a}^3\tilde{b} + 24\tilde{a}^2\tilde{b}^2 + 24\tilde{a}\tilde{b}^3$;

$$D = 4\tilde{a}^3\tilde{b} + 24\tilde{a}^2\tilde{b}^2 + 48\tilde{a}\tilde{b}^3 + 36\tilde{b}^4$$

where $(\tilde{a}, \tilde{b}) = 1$. For the other part, if $q^2 \neq A^3 + D^3$, (In particular, $q^2 = A^3 + 64\hat{m}$), then the solution will be none of above.

Fact. 5.4. (Selmer) The pure cubic fields whose class numbers are even are listed as following: (for $m \leq 160$).

m	11	15	39	43	47	57	58	61
h	2	2	6	12	2	6	6	6

m	63	65	66	67	76	79	83	89
h	6	18	6	6	6	6	2	2

m	101	105	106	113	118	122	123	129
h	2	6	6	4	2	12	2	6

m	131	139	141	142	148	149	151	155
h	2	6	8	6	6	2	6	6

Remark 5.1. Given q and A, and the equation $q^2 = A^3 + 64\hat{m}$, if the equation is solvable for $\hat{m}$ in Z, then $q^2 \equiv A^3 \equiv 1 \pmod 8$ and

(I) $$\hat{m} = \frac{q^2 - A^3}{64}$$

(where q is an odd prime and greater than 3).

§ 2. Even class number of pure cubic fields

It is also well known that $x^3 \equiv m \pmod{r}$ has a unique solution in Z and for the unsolvability in Y in Z of $Y^2 \equiv A + 4R \pmod{r}$, we have $(A + 4R/r) = -1$ for some r in Z, when $r \equiv 5 \pmod 6$ and $\hat{m} \equiv \theta^3 \equiv R^3 \pmod{r}$ for some R in Z. (See Fact 4.9.)

In this section, by method of residue class of $A \pmod{r}$ and of $q \pmod{r}$, such that if $q^2 - A^3 = 64\hat{m}$ ($\hat{m}$ in Z, and $\hat{m} = B^3 m$ as usual we define $\mu = A + 4B\theta$ in Chapter 4) is solvable for $\hat{m}$ in Z together with $(\mu^*/r) = (A + 4R/r) = -1$, (i.e., non-quadratic residuacity) where $\mu^* \equiv \mu \equiv A + 4R \pmod{r}$, then μ will be a non-square algebraic integer. This is equivalent to show that y^2 $(= A + 4.\sqrt[3]{\hat{m}})$ is unsolvable for y in K_3. This implies that $Y^2 \equiv A + 4R \pmod{r}$ is unsolvable for Y in Z, where $\hat{m} \equiv R^3 \pmod{r}$ and R in Z. Thus, the class number of $Q(\sqrt[3]{\hat{m}}) = Q(\sqrt[3]{m})$ is even, when $\hat{m} = \tilde{n}^3 \cdot m$, and $\tilde{n}$ in Z. In the tables (as below), we take the first three values (5, 11 and 17) for r. Then we compute the solvability "?" or the unsolvability "x" of Y^2 by use of quadratic residue character. Therefore we obtain the following important results in the tables.

The occurrence of the "x" in the tables leads to even class number of pure cubic fields.

The occurrence of the "?" or "??" in the tables leads to uncertainty of the evenness of class number. The "??" indicates that A or q is divisible by r. If so, we have to check all tables available to have the same occurrence "?" or "??". Then we have to solve for β in K_3 such that $\mu = \beta^2$. Furthermore, for each $q \not\equiv 0 \pmod{r}$, In Table 5.1. we have 2 occurrences of "x" when $r = 5$. In Table 5.2. we have 6 occurrences of "x" when $r = 11$. In Table 5.3. we have 8 occurrences of "x" when $r = 17$. In general, there are $\frac{1}{2}(r - (-1)^{\frac{1}{2}(r-1)})$ occurrences for each q ($\not\equiv 0 \pmod{r}$).

This is illustrated in the following examples and tables:

Illustration 5.1. Let $q = 7$ and $A = 17$. Then we have the field $Q(\sqrt[3]{76})$, because $-76 = (7^2 - 17^3)/64$. First, we take $r = 5$. This implies $q = 7 \equiv 2 \pmod{5}$ and $A = 17 \equiv 2 \pmod{5}$, then by checking Table 5.1. we have an "x". Thus the class number of $Q(\sqrt[3]{76})$ is even, (because of the unsolvability of the equation

$(\mu =)\ A + 4\theta = \beta^2$).

Illustration 5.2. Let $q = 23$ and $A = -15$. Then we have the field $Q(\sqrt[3]{61})$, because $61 = (23^2 - (-15)^3)/64$. First, we take $r = 5$, which is nonconclusive, because of the "??" in Table 5.1. we could possibly have the solvability for $A + 4\theta = \beta^2$. Then we take $r = 11$. This

implies $q \equiv 1$ and $A \equiv 7 \equiv -4 \pmod{11}$, then by checking Table 5.2. we find an "x" and conclude the class number of $Q(\sqrt[3]{61})$ is even, (again, because of the unsolvability of the equation $(\mu =)\ A + 4\theta = \beta^2$).

Illustration 5.3. Let $q = 181$ and $A = -23$, Then we have the field $Q(\sqrt[3]{26})$, because

$$(26)(27) = 702 = \frac{(181^2 - (-23)^3)}{64}.$$

First we take $r = 5$, this implies $q \equiv 1$ and $A \equiv 2 \pmod{5}$ which is possibly solvable in Table 5.1. Then we take $r = 11$. This implies $q \equiv 5$ and $A \equiv -1 \pmod{11}$ which is possibly solvable in Table 5.2. Then we take $r = 17$ which implies $-q \equiv 6$ and $A \equiv -6 \pmod{17}$ which is possibly solvable in Table 5.3. Since we have "?" in three tables, we try straight computation. We find

$$(\mu =)\ -23 + 12\theta = \{\frac{1}{3}(1 + 2\theta - 2\theta^2)\}^2 \ (= \beta^2).$$

So $Q(\sqrt[3]{702})$ $(= Q(\sqrt[3]{26})$ doses not have an unramified extension through $\sqrt{\mu}$. In fact, the tables of Selmer verify that $K_3 = Q(\sqrt[3]{26})$ does have 3 as an odd class number.

The following example illustrates the even class number of pure cubic fields by obtaining the solvability for $\hat{m}$ of Equation (I) of Remark 5.1. and the occurrence of "x" in one of these tables. (See Table 5.1., Table 5.2. and Table 5.3. as below.).

Example 5.1.

q	A	$\hat{m}$	$\tilde{n}$	$Q(\sqrt[3]{m})$	h
7	17	- 76	- 1	76	6
17	97	-14256	- 6	66	6
19	-7	11	1	11	2
23	-15	61	1	61	6
31	1	15	1	15	2
41	-15	79	1	79	6
43	41	-1048	- 2	131	2
59	9	43	1	43	12_c
71	-47	1701	3	63	6
73	49	-1755	- 3	65	18_{nc}
79	33	-464	- 2	58	6
83	- 7	113	1	113	4_{nc}
89	17	47	1	47	2
101	9	148	1	148	6
103	17	89	1	89	2
157	25	141	1	141	8_{nc}
199	-47	2241	3	83	2
229	9	808	2	101	2
257	1	1032	2	129	6
269	-7	1136	2	142	6
277	41	122	1	122	12_c
283	9	1240	2	155	6

To be continued···

571	73	-984	- 2	123	2
607	-63	9664	4	151	6
787	57	6784	4	106	6
1801	49	48843	9	67	6
2719	193	3186	3	118	2
4421	9	305383	13	139	6

$Q(\sqrt[3]{\hat{m}}) = Q(\sqrt[3]{m})$;

where c: cyclic class group

nc: non-cyclic class group.

$\hat{m}$ / θ / A / s / q	- 2	- 1	0	1	2
0	$\hat{m} \equiv 2$ $\theta \equiv 3$??	$\hat{m} \equiv 4$ $\theta \equiv 4$??	$\hat{m} \equiv 0$ $\theta \equiv 0$??	$\hat{m} \equiv 1$ $\theta \equiv 1$??	$\hat{m} \equiv 3$ $\theta \equiv 2$??
1	$\hat{m} \equiv 1$ $\theta \equiv 1$ x	$\hat{m} \equiv 3$ $\theta \equiv 2$ x	$\hat{m} \equiv 4$ $\theta \equiv 4$??	$\hat{m} \equiv 0$ $\theta \equiv 0$?	$\hat{m} \equiv 2$ $\theta \equiv 3$?
2	$\hat{m} \equiv 3$ $\theta \equiv 2$?	$\hat{m} \equiv 0$ $\theta \equiv 0$?	$\hat{m} \equiv 1$ $\theta \equiv 1$??	$\hat{m} \equiv 2$ $\theta \equiv 3$ x	$\hat{m} \equiv 4$ $\theta \equiv 4$ x

where $r = 5$ ($r \equiv 5 \pmod 6$) and $\theta^3 = \hat{m}$. $q^2 - A^3 = 64\hat{m} \pmod 5$;

s: (i) solvable (?) if $y^2 = A + 4\theta \pmod r$ solvable for $r = 5$

(ii) unsolvable (x) if $y^2 = A + 4\theta \pmod r$ unsolvable for $r = 5$

(?): solvable for $r = 5$ only not necessarily in general.

Table 5.1.

(x): unsolvable for $r = 5$ and $\mu = A + 4\theta$ is a non-square algebraic integer.

$\hat{m}$ A / θ / s / q	- 5	- 4	- 3	- 2	- 1	0	1	2	3	4	5
0	$\hat{m} \equiv 9$ $\theta \equiv 4$??	$\hat{m} \equiv 1$ $\theta \equiv 1$??	$\hat{m} \equiv 3$ $\theta \equiv 9$??	$\hat{m} \equiv 7$ $\theta \equiv 6$??	$\hat{m} \equiv 5$ $\theta \equiv 3$??	$\hat{m} \equiv 0$ $\theta \equiv 0$??	$\hat{m} \equiv 6$ $\theta \equiv 8$??	$\hat{m} \equiv 4$ $\theta \equiv 5$??	$\hat{m} \equiv 8$ $\theta \equiv 2$??	$\hat{m} \equiv 10$ $\theta \equiv 10$??	$\hat{m} \equiv 2$ $\theta \equiv 7$??
1	$\hat{m} \equiv 3$ $\theta \equiv 9$?	$\hat{m} \equiv 6$ $\theta \equiv 8$ x	$\hat{m} \equiv 8$ $\theta \equiv 2$?	$\hat{m} \equiv 1$ $\theta \equiv 1$ x	$\hat{m} \equiv 10$ $\theta \equiv 10$ x	$\hat{m} \equiv 5$ $\theta \equiv 3$??	$\hat{m} \equiv 0$ $\theta \equiv 0$?	$\hat{m} \equiv 9$ $\theta \equiv 4$ x	$\hat{m} \equiv 2$ $\theta \equiv 7$?	$\hat{m} \equiv 4$ $\theta \equiv 5$ x	$\hat{m} \equiv 7$ $\theta \equiv 6$ x
2	$\hat{m} \equiv 7$ $\theta \equiv 6$ x	$\hat{m} \equiv 10$ $\theta \equiv 10$?	$\hat{m} \equiv 1$ $\theta \equiv 1$?	$\hat{m} \equiv 5$ $\theta \equiv 3$ x	$\hat{m} \equiv 3$ $\theta \equiv 9$ x	$\hat{m} \equiv 9$ $\theta \equiv 4$??	$\hat{m} \equiv 4$ $\theta \equiv 5$ x	$\hat{m} \equiv 2$ $\theta \equiv 7$ x	$\hat{m} \equiv 6$ $\theta \equiv 8$ x	$\hat{m} \equiv 8$ $\theta \equiv 2$?	$\hat{m} \equiv 0$ $\theta \equiv 0$?
3	$\hat{m} \equiv 10$ $\theta \equiv 10$ x	$\hat{m} \equiv 2$ $\theta \equiv 7$ x	$\hat{m} \equiv 4$ $\theta \equiv 5$ x	$\hat{m} \equiv 8$ $\theta \equiv 2$ x	$\hat{m} \equiv 6$ $\theta \equiv 8$?	$\hat{m} \equiv 1$ $\theta \equiv 1$??	$\hat{m} \equiv 7$ $\theta \equiv 6$?	$\hat{m} \equiv 5$ $\theta \equiv 3$?	$\hat{m} \equiv 9$ $\theta \equiv 4$ x	$\hat{m} \equiv 0$ $\theta \equiv 0$?	$\hat{m} \equiv 3$ $\theta \equiv 9$ x
4	$\hat{m} \equiv 1$ $\theta \equiv 1$ x	$\hat{m} \equiv 4$ $\theta \equiv 5$?	$\hat{m} \equiv 6$ $\theta \equiv 8$ x	$\hat{m} \equiv 10$ $\theta \equiv 10$?	$\hat{m} \equiv 8$ $\theta \equiv 2$ x	$\hat{m} \equiv 3$ $\theta \equiv 9$??	$\hat{m} \equiv 9$ $\theta \equiv 4$ x	$\hat{m} \equiv 7$ $\theta \equiv 6$?	$\hat{m} \equiv 0$ $\theta \equiv 0$?	$\hat{m} \equiv 2$ $\theta \equiv 7$ x	$\hat{m} \equiv 5$ $\theta \equiv 3$ x
5	$\hat{m} \equiv 2$ $\theta \equiv 7$ x	$\hat{m} \equiv 5$ $\theta \equiv 3$?	$\hat{m} \equiv 7$ $\theta \equiv 6$ x	$\hat{m} \equiv 0$ $\theta \equiv 0$?	$\hat{m} \equiv 9$ $\theta \equiv 4$?	$\hat{m} \equiv 4$ $\theta \equiv 5$??	$\hat{m} \equiv 10$ $\theta \equiv 10$ x	$\hat{m} \equiv 8$ $\theta \equiv 2$ x	$\hat{m} \equiv 1$ $\theta \equiv 1$ x	$\hat{m} \equiv 3$ $\theta \equiv 9$ x	$\hat{m} \equiv 6$ $\theta \equiv 8$?

$q^2 - A^3 = 64\hat{m} \pmod{11}$; $r = 11 \ (\equiv 5 \pmod 6)$

Table 5.2.

θ $\tilde{m}$ A / s / q	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0	8 2 ??	12 6 ??	14 10 ??	7 14 ??	1 1 ??	6 5 ??	15 9 ??	4 13 ??	0 0 ??	13 4 ??	2 8 ??	11 12 ??	16 16 ??	10 3 ??	3 7 ??	5 11 ??	9 15 ??
1	12 6 ?	16 16 x	1 1 ?	11 12 ?	5 11 x	10 3 ?	2 8 ?	8 2 x	4 13 ??	0 0 ?	6 5 x	15 9 x	3 7 ?	14 10 x	7 14 x	9 15 ?	13 4 x
2	7 14 x	11 12 x	13 4 x	6 5 ?	0 0 ?	5 11 x	14 10 ?	3 7 x	16 16 ??	12 6 ?	1 1 x	10 3 ?	15 9 x	9 15 x	2 8 ?	4 13 ?	8 2 ?
3	10 3 ?	14 10 ?	16 16 x	9 15 ?	3 7 x	8 2 x	0 0 ?	6 5 ?	2 8 ??	15 9 x	4 13 x	13 4 ?	1 1 ?	12 6 x	5 11 ?	7 14 x	11 12 x
4	4 13 x	8 2 ?	10 3 x	3 7 x	14 10 ?	2 8 x	11 12 x	0 0 ?	13 4 ??	9 15 x	15 9 ?	7 14 ?	12 6 x	6 5 ?	16 16 ?	1 1 x	5 11 ?
5	11 12 x	10 3 x	12 6 ?	5 11 x	16 16 ?	4 13 ?	13 4 x	2 8 x	15 9 ??	6 5 ?	0 0 ?	9 15 x	14 10 x	8 2 ?	1 1 x	3 7 ?	7 14 ?
6	16 16 x	3 7 ?	5 11 ?	15 9 x	9 15 x	14 10 x	6 5 ?	12 6 x	8 2 ??	4 13 ?	10 3 x	2 8 ?	7 14 ?	1 1 ?	11 12 x	13 4 x	0 0 ?
7	0 0 ?	4 13 x	6 5 x	16 16 ?	10 3 ?	15 9 ?	7 14 x	13 4 ?	9 15 ??	5 11 x	11 12 ?	3 7 x	8 2 x	2 8 x	12 6 ?	14 10 ?	1 1 x
8	9 15 ?	13 4 ?	15 9 ?	8 2 x	2 8 x	7 14 ?	16 16 x	5 11 ?	1 1 ??	14 10 x	3 7 ?	12 6 x	0 0 ?	11 12 ?	4 13 x	6 5 x	10 3 x

$q^2 - A^3 \equiv 64\hat{m} \pmod{17}$; $r = 17 \ (\equiv 5 \pmod 6)$; Table 5.3.

BIBLIOGRAPHY

[1] Artin, E. Modern Higher Algebra, Galois Theory, Courant Institute of Mathematical Sciences, New York University, New York, 1947.

[2] Borevich, Z.I. and Shafarevich, I.R. Number Theory, Academic Press, New York, 1966.

[3] Burnside, W. The Theory of Groups of Finite Order, New York, Cambridge U. Press, 1911.

[4] Cohn, Harvey. A Second Course in Number Theory, Wiley & Sons, New York, 1962.

[5] Cohn, Harvey. A Classical Invitation to Algebraic Numbers and Class Fields, Springer-Verlag, New York, 1978.

[6] Dedekind, R. Über Reine Kubische Körper, J.f. Math. 121 (1900), 40-123, (Werke II, 148-234).

[7] Frey, G. Die Klassengruppen Quadratischer Und Kubischer Zahlkörper Und Die Selmergruppen Gewisser Elliptischer Kurven, Manuscripta Math. V.6. (1975) 332-362.

[8] Goldstein, Larry Joel. Analytic Number Theory, Prentice Hall, Englewood Cliffs, New Jersey, 1971

[9] Hilbert, David. Über Die Theorie Des Relativquadratischen Zahlkörpers, Mathematische Annalen Bd 51, S 1-127 (1899)

[10] Lang, Serge. Algebraic Number Theory, Addison-Wesley Reading, Mass., 1970.

[11] Mordell, L.J. Diophantine Equations, Academic Press, New York, 1969.

[12] Ribenboim, Paulo. Algebraic Numbers, Wiley-Interscience, New York, 1972.

[13] Rotman, Joseph J. The Theory of Groups, Allyn and Bacon, Inc. Boston, Mass., 1976.

[14] Selmer, Ernst S. Tables for the Purely Cubic Field $K(\sqrt[3]{m})$, I. Mat. Naturv. Klasse 1955. No.5. 3-38.

[15] Selmer, Ernst S. Tables for the Purely Cubic Field $K(\sqrt[3]{m})$, for $100 < m \leq 250$, Unpublished, University of Bergen, Norway, 1967.

[16] Weiss, E. Algebraic Number Theory, McGraw-Hill, New York, 1963.

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John Hwang (黃兆乾) was born on November 2, 1946 in Taipei City, Taiwan, Republic of China. When he went to Taipei Fu-Shin Elementary School, he met Lian-Ju Liao whom he married in January 1974. Later, Mike and Karl were born in Oct. 1975 and in Feb. 1978, respectively. He entered National Cheng Kung University, Tainan, Taiwan, and received a B.S. in Mathematics in the spring of 1968. In 1969, he served in Chinese Army. Then he attended McNeese State University, Lake Charles, Louisiana, in the fall of 1971 as a graduate student and was awarded a M.S. in Algebra in 1973. In September 1973, he came to the Graduate Center of CUNY as a graduate student and research assistant. During this time he taught as an adjunct lecturer in Mathematics at LaGuardia Community College for two quarters and New York Institute of Technology for two and a half years.