REAL-WORLD CONTEXTS IN URBAN HIGH SCHOOL MATHEMATICS LESSONS

by

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Abstract

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This study analyzes the uses of real-world contexts in mathematics lessons in the classrooms of four teachers across two school years at an urban high school. Drawing upon a framework of culturally relevant mathematics pedagogy, this dissertation focuses on how real-world contexts are connected to teaching mathematics for understanding, centering mathematics instruction on students' experiences and classroom participation, and developing students' critical consciousness. Analysis of real-world contexts in lessons focuses on the extent to which they are adapted from curricular sources and the role that lessons play within the lesson. For those real-world contexts which are at the center of a mathematics lesson, the nature of the mathematical modeling in which students engage is analyzed. Finally, the extent to which students and the teacher participate in the process of elaborating key features of the context whether in terms of experiences, perceptions, or opinions, is also considered. These different categories for real-world contexts are then used to compare three different measures of the lesson. These include the cognitive demand of the main mathematical task, different ratings of the instructional environment, and the distribution of class time in terms of the participation categories offered to students. Results point at the promise of real-world contexts as the basis for motivating metaphors to explore noncontextualized mathematical procedures and concepts, the need to structure lessons so that students can develop models rather than apply given models, and the importance of elaboration in supporting student understanding and participation.

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Chapter 1

Introduction

Statement of the Problem

Mathematics teachers at the high school level continue to be in high demand across the United States (United States Department of Education, 2012). This demand is especially pressing in urban school districts, and particularly acute at schools which are hard to staff (Ingersoll & Perda, 2010). The challenges faced by hard-to-staff schools include recruitment, retention, and the continued professional development of teachers within professional and collegial environments amid a shifting national policy landscape. In addition, these individual schools are members of urban school districts that have been the focus of multiple neoliberal reform efforts which have been marked by repeated reorganizations and the creeping privatization of public functions, input, governance, and participation (Lipman, 2011).

Meanwhile, what counts as knowledge and teacher expertise has been redefined by the federal No Child Left Behind (NCLB) Act. NCLB has placed new emphasis on student achievement and teachers being “highly qualified”. Schools and districts are held “accountable” for the performance of ethnic, socioeconomic, and linguistic subgroups of students on state standardized exams in mathematics. The aim of NCLB is to have all student subgroups completely proficient by 2014. Simultaneously, schools are required to hire “highly qualified teachers” as defined by largely by their coursework in a subject matter area.

As a result, to fill the shortages created by this requirement, cities such as New York City created alternative certification programs such as the New York City Teaching Fellows. Within this program, the Mathematics Immersion program has come to supply nearly fifty percent of all new certified mathematics teachers in New York City (Boyd et al., 2012). These shifts in student testing and teacher certification privilege traditional forms of mathematical knowledge and understanding which have historically failed to serve poor students of color through acknowledging and affirming their cultures and strengths (Ladson-Billings, 1997).

The Harwood High School[1] is a small high school founded in 2003 as a partnership with a local community-based organization. This school was created as a result of the first wave of schools to be shut down in the new era of mayoral control and is one of four high schools on the campus of a large high school. The mathematics teachers at the school were involved in a multi-school National Science Foundation-funded professional development and research project called *Centering the Teaching of Mathematics on Urban Youth* (CTMUY) (Rubel, 2012). This study is based upon data collected for that larger project. The author of this dissertation served as a research assistant to the principal investigator of the CTMUY project. One goal of CTMUY is to bring to mathematics teaching key elements of culturally relevant pedagogy (Ladson-Billings, 1995). Culturally relevant pedagogy refers to ways of teaching that focus not only on traditional academic achievement, but also support for students to build on their cultural competence while awakening their sociopolitical critical consciousness. The CTMUY project aims to achieve this goal through a variety of professional development activities to improve the teaching of mathematics for urban youth whose backgrounds and positions are significantly different from that of their teachers.

A theory of culturally relevant pedagogy frames this study. Culturally relevant pedagogy as a theoretical framework was developed by observing the exemplary pedagogical practices of teachers nominated by their communities as being particularly effective for African American children (Ladson-Billings, 1995). As usually presented, culturally relevant pedagogy achieves

[1] All proper names are pseudonyms.

three interrelated outcomes for students: academic achievement, cultural competence, and critical consciousness. This dissertation focuses on teachers involved in a larger project, Centering the Teaching of Mathematics on Urban Youth (CTMUY), which aims to support teachers in their development of culturally relevant mathematics pedagogy.[2] Culturally relevant mathematics pedagogy is enacted through a framework developed for the larger project that contains specifications of each of the dimensions of culturally relevant pedagogy to mathematics education (Rubel, 2012). This framework of culturally relevant mathematics pedagogy is referred to as "CureMap".

CureMap's interpretation of academic achievement is informed by the framework of teaching mathematics for understanding (Hiebert & Carpenter, 1992) and is compatible with the Principles and Standards of the National Council of Teachers of Mathematics (NCTM, 2000). The first dimension of CureMap is that understanding emerges from students' ability to communicate about and reflect upon the mathematics that they are learning. At the same time, the NCTM Standards equally emphasize content and process. The Process Standards include students' abilities to generate representations, to make connections from the real-world to mathematics, to use reasoning and proof to construct and critique arguments, to communicate their ideas effectively, and to solve problems. Indeed, the goal of teaching mathematics for understanding focuses on students' abilities not only to become fluent in carrying out procedures but also to connect them to other procedures well as to central mathematical concepts, the real world, and other mathematical representations (Hiebert & Grouws, 2007).

The second dimension of CureMap interprets the dimension of "cultural competence" by asking teachers to center instruction on students in two related ways. First, teachers are

[2] This material is based on work supported by the National Science Foundation under grant no. 0742614. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

encouraged to use local contexts as opportunities for mathematization. In mathematics classrooms, taking local or familiar contexts and finding the mathematics implicit in ordinary activities (Moses & Cobb, 2001; Tate, 1995) provides opportunities for students to legitimately enact their cultural competence to support mathematical understanding. CureMap thus emphasizes conceptions of culture which are dialectical, meaning both dynamic and situated, and not folkloric, meaning fixed as transmitted (Gutstein et al., 1997). Here, "culture" can be taken to include all of the "linguistic and cultural-historical repertoires" which students possess and continually shape through their interaction with their social environments (Gutiérrez & Rogoff, 2003). That is, there may be aspects of being an adolescent or living in an urban environment that may be more salient in students' lived experiences than their membership in linguistic or ethnic groups. While this dissertation most directly addresses this aspect of CureMap, the focus on real-world contexts and on students' lived experiences is broader than popular understandings of "culture". Second, teachers are encouraged to incorporate participation structures that invite students to contribute to the collective development of mathematical understanding in a variety of ways which may further reflect students' lived experiences. Taken together, these two approaches toward the cultural competence component of CureMap emphasize a balance between content or topic and process or participatory structure.

The third component of CureMap applies the notion of critical consciousness to allow students the opportunity for sociopolitical and cultural critique. This component of culturally relevant pedagogy is frequently interpreted as primarily enacted through teaching mathematics for social justice (e.g., González, 2009; Gutstein, 2003). Although social justice pedagogy and culturally relevant pedagogy share many features, there are substantial differences in emphasis and definitions of exemplary practice (Leonard, et al., 2010). As a theoretical framework, CureMap asserts that in addition to, or concurrent with, the central goal of social justice pedagogy of learning mathematics as a tool for analyzing fairness in society, students can also develop the critical tools with which to build mathematical power. That is, students can become critical both *with* and *about* mathematics. Being critical *about* mathematics might include examining the historical and social context for the development of a mathematical concept, and the assumptions and perspectives of the individuals involved in that development.[3]

These three components of CureMap—teaching mathematics for understanding, centering instruction on students, and encouraging them to be critical about and with mathematics—were the basis of a sequence of professional development activities designed and led by the CTMUY principal investigator in which teachers participated across the 2009-10 and 2010-11 school years, as detailed by Rubel (2012).

[3] For example, with the statistics topic of linear regression, students can develop a deeper understanding of the historical origins of that term, the ideas that it encapsulates, and the extent to which use and understanding of that term has shifted over time as values, assumptions, and questions have changed. Specifically, students could contrast how the term "regression" is commonly used to mean the linear model that results from applying a statistical technique rather than the tendency for data, such as heights of parents and children, to "regress" toward the mean.

Purpose of Study

This dissertation complements the CTMUY research project by focusing on how teachers bring real-world contexts into the classroom. Teachers' use of real-world context is potentially related to two of the broader goals of the professional development project: supporting the cultural competence and critical sociopolitical consciousness of students. I use the term "real-world contexts" to refer to objects, situations, and practices that exist independently outside of the classroom setting, in contrast to abstract or formal problems which are posed entirely in mathematical terms. Real-world contexts include the flight of projectiles (quadratic functions), games and gambling (probability), and other experiences, such as working at a job or riding the bus, which exist independent of the classroom setting but may still contain embedded mathematics. The purpose of this study is to investigate how mathematics teachers use real-world contexts as they teach mathematics lessons, with a focus on how the treatment of those real-world contexts is related to and supportive of elements of the framework of culturally relevant mathematics pedagogy.

Research Questions

This dissertation investigates two related research questions:

1) *To what extent* and *in what ways* do mathematics teachers provide opportunities in the classroom for students to connect real-world contexts to mathematics lessons?

    1. With which mathematical topics are real-world contexts associated?
    2. What is the extent and nature of teacher *adaptation* of the presentation of real-world contexts from textbook or other curriculum sources?
    3. What *roles* (*motivating metaphor, incidental or central)* do real-world contexts play in mathematics lessons?
    4. When the role of a real-world context is central to the lesson, do students *develop* mathematical models or solution approaches or are they *given* mathematical models or procedures to apply?
    5. Are real-world contexts *elaborated* or not and what is the nature of elaboration?

2) Are there differences across lessons based upon:

    1. Inclusion of real-world contexts (present or absent)
    2. Level of adaptation (high or low)
    3. Type of role (motivating metaphor, incidental, or central)
    4. Type of modeling (developed or given)
    5. Type of elaboration (elaborated or not)

    in terms of:

    a. level of cognitive demand
    b. mean ratings of instructional environment
    c. distribution of student participation categories

Significance

This dissertation makes contributions in three areas: the focal participants, the topic of real-world contexts, and the methods. First, by focusing on a segment of teachers generally neglected in the literature—high school mathematics teachers of an intermediate level of experience—this study expands the current research base. Second, by sharply focusing on an issue—the inclusion, incorporation, and integration of real-world contexts into mathematics lessons—that lies on the boundaries of multiple components of CureMap, this dissertation details teachers' engagement with multiple aspects of the broader CTMUY project and also examines potential bridges from their existing practices into CureMap. Finally, methodologically, this dissertation draws upon a mixed methods approach to integrate qualitative analysis of classroom observations with quantitative analysis of differences measures of cognitive demand, instructional environment, and student participation distributions. Taken together, this study has direct implications which can inform the current policy environment in mathematics education.

Most accounts of culturally relevant pedagogy have focused on either exemplary teachers (Ladson-Billings, 1997; Tate, 1995; Gutstein, Lipman, Hernandez, & de los Reyes, 1997; Bonner & Adams, 2012) or pre-service teacher preparation (Wager, 2012; Turner et al., 2012), and few are at the high school mathematics level. The exemplary teachers in culturally relevant pedagogy studies tend to be individuals who are very experienced and have become insiders to the communities where they teach.   While these teachers serve as existence proofs for what is possible across diverse local contexts and with different student demographic profiles, they do not necessarily provide guidance for teachers who are new to the field and are still in the early stages of developing many of the fundamental skills of teaching, including classroom management and understanding of the official curriculum.

Indeed, on the other end of the spectrum of teacher experience, local to New York City, multiple studies have focused on development of Math Immersion New York City Teaching Fellows and the impact that they have made on student achievement (Boyd et al., 2012). Although these programs, in contrast to some programs such as Teach for America, also include concurrent graduate coursework, the content and requirements of various alternative mathematics certification programs varies widely, with more variation *within* alternative certification pathways to certification than when contrasted with the traditional "college-recommended" certification route. Further, case studies of the development of new Math Immersion Teaching Fellows have reported a series of tensions or conflicts around mathematical knowledge for teaching, mathematics instruction, and teaching urban youth (Meagher & Brantlinger, 2011; Foote, Smith, & Gellert, 2011). Some of the conflicts that accompany teaching urban youth include having confidence in students' ability to learn, relating to students' cultures, and developing an appropriate sense of personal distance (Foote et al., 2011). The challenges that face mathematics teachers in this environment thus include challenges of content knowledge, pedagogy, and knowledge about students.

This dissertation is about teachers of an intermediate level of experience, with no teacher having more than five years of experience teaching at the high school level, but all having completed their Master's coursework. The teachers are outsiders to the community where they teach in the sense that they do not live there, did not grow up there, and come from different socioeconomic and educational backgrounds. It is expected then that they are in a different position compared to new teachers still taking courses in the evening or compared to accomplished veterans who have developed deep and lasting relationships with the schools, students, and communities where they teach.

Further, the teachers for this study, as more fully described in Chapter 3, are selected from a single school. Within the context of reforms in teacher preparation and retention, the teachers in this study represent a cross-section of experiences, perspectives, and practices while functioning within a single institutional context. In contrast to other studies which have focused on individual teachers involved with a single professional development project across multiple schools (e.g., McCulloch & Marshall, 2011; Wager, 2012), this study has the potential to describe and analyze some of the variation that exists in terms of teaching practices within a single school.

The inclusion of real-world contexts is an ingredient in two of the components of culturally relevant pedagogy as enacted in CureMap, as it could connect to students' lived experiences from a variety of cultural activities, but it could also introduce new contexts which are aligned with the goals of curricular projects which are more concerned with social justice and developing students' critical consciousness. I focus in this dissertation only on the inclusion of real-world contexts, which is broader than the notion of cultural competence, and also broader than what is often described as "out of school experiences" (e.g., Wager, 2012; McCulloch & Marshall, 2011). That is, the real-world contexts which are included in mathematics lessons may not connect at all with students' lived experiences or their other experiences outside of school settings. For example, problems could be set in unfamiliar settings, such as those about historical economic statistics, or literally foreign data such as temperature in world cities. Although the real-world contexts actually observed in classroom lessons may not meet the local, relevant, or socio-politically relevant criteria outlined by CureMap and encouraged by the CTMUY professional development program, they indicate the current status of teacher practice and therefore points of departure for further growth and development.

In addition to focusing solely on real-world contexts, this study seeks to broaden the focus beyond the conceptual or cognitive support that real-world contexts provide to also include the kinds of social interactions and classroom participation structures or discourse norms that accompany real-world contexts. This study specifies the analysis of participation categories (see Rubel & Monroe, 2012) to features related to real-world contexts. Previous studies have not fully integrated analysis of ratings of the general features of instructional environment with specific classroom participation categories for students. That is, many studies have focused on student performance on assessments, qualitative analysis of teacher-reported data (e.g., Wager, 2012; Gainsburg, 2009), qualitative analysis of classroom observations (e.g., McCulloch & Marshall 2011), or quantitative analysis of classroom observation data that has not further been correlated with fine-grained qualitative analysis (e.g. Jackson et al., 2011).

Many of the reform efforts within mathematics education have focused on the content knowledge or pedagogical content knowledge that teachers should have, including such math-specific constructs as mathematics knowledge for teaching (Hill, Rowan, & Ball, 2005). In other cases, reform has focused on aspects of delivering curriculum models such as the cognitive demand of tasks (Stein, Smith, Henningsen, & Silver, 2000) or the orchestration of productive summative mathematical discussions (Stein, Engle, Smith, & Hughes, 2008). Given the popular reputation that mathematics enjoys as a formal, inaccessible, and objective discipline, multicultural perspectives that include culturally relevant pedagogy have not been seen as particularly relevant to mathematics education (Sleeter, 1997).   The focus of this dissertation on real-world contexts thus provides a bridge from mainstream approaches to mathematics education to equity-focused approaches that explicitly address race/ethnicity and the challenges and opportunities posed by urban contexts.

These three elements inform current mathematics education policies. The sustained focus on regular and repeated classroom observations, which incorporate both qualitative and quantitative measures of instructional quality, also contributes to the literature at a time when measuring and rating teachers' instruction is increasingly becoming a policy precondition to qualify for federal funding as provided by new programs such as Race to the Top. Further, the analytical framework developed and applied to the body of lessons in this study has implications for the implementation of the Common Core Standards in Mathematics as well as for the design and delivery of professional development.

Chapter 2

Review of Literature

This review of the literature focuses on how mathematics education research has generally approached and understood real-world contexts in relation to mathematics teaching and learning. This review is organized into three sections which build on each other. First, I consider the different ways that the existing research literature has connected real-world contexts with the school curriculum. Second, I outline research that has used the practice of mathematical modeling as an approach to addressing word problems. Third, I consider specific classroom practices that have been found to support students' understanding of mathematics set in real-world contexts.

Real World Contexts and School Mathematics

This section is divided into five parts. First, I consider how the dilemma of "transfer" in the mathematics education literature has emphasized the difficulty of coordinating real-world practices with school mathematics. Second, I consider different framings and general approaches for relating real-world contexts to school mathematics. Third, I summarize studies that provide negative evidence that real-world contexts support students' development of mathematical understanding of solution of problems. Fourth, I summarize studies that find that real-world contexts can substantially and substantively support students' development of mathematical understanding. Finally, I summarize how real-world contexts have been used as the basis of analogies which support students' mathematical understanding of mathematical concepts, procedures, or rules.

The problem of "transfer". Much of the research on how students connect real-world contexts with school mathematics has been framed by the "transfer" dilemma. Originally, this dilemma of transfer referred to how individuals solving arithmetic problems in real-world situations rarely thought to "transfer" school mathematics algorithms or representations to those settings (Lave, 1988). This dilemma has been one of pillars of the situated and sociocultural approaches to learning which emphasize legitimate peripheral participation, communities of practice, and the negotiation of meaning alongside identity (Lave & Wenger, 1991; Wenger, 1998). Rather than conceiving of knowledge and learning as a cognitive phenomenon in which individuals can engage all by themselves, these situated and sociocultural approaches frame learning as changes in participation within a metaphor of apprenticeship.

These disconnects between situated strategies and school strategies have been studied in a variety of settings, such as carpet-laying and restaurant management. For example, studies have detailed how students in school approached problems significantly differently than individuals who professionally engage in solving those real-world problems and have developed their own strategies, representations, and algorithms (Masingila, Davidenko, & Prus-Wisnowska 1996; Jurdak, 2006). Similarly, Hoyles, Noss, and Pozzi (2001) found that nurses involved in medical dosing calculations used socially-learned scaling strategies rather than the algorithms that they had been taught in their formal coursework.

The sociocultural turn or shift in mathematics education research thus views mathematics knowing as a cultural activity, mathematics learning as a cultural enterprise, and mathematics education as a cultural system (Nasir, Taylor, & Hand, 2008). This perspective posits that what students learn in mathematics classroom, rather than being objective or culture-independent, is constructed out of students' engagement with the cultural enterprise of learning mathematics within the cultural system of schooling and school mathematics. This approach is consistent with the tool of ethnomathematics which provides a means for examining how mathematical are embedded into everyday cultural practices, such as sports statistics and weaving patterns (Barton, 1996). In both cases, there is a tendency for knowledge developed in one cultural milieu to have difficulty transferring into another.

For example, Nasir (2000) considered how African American middle and high school students who were also basketball players approached the issue of percentages differently across contexts. In part, the higher-stakes environment of playing basketball at the high school level in terms of the comparisons of players for determining membership in and status within the school team differentially structured students' abilities to use statistics as a means of self-regulation as well as comparison with others. High school students tended to pay more attention to field goal and free throw percentages, but without the precision associated with school mathematics problems. Rather, estimates sufficient for distinguishing between the performance of players were employed by basketball players given problems set in the context of basketball.

More broadly speaking, Nasir and Hand (2008) asserted that the participation structures and opportunities to be engaged in learning and identity were different across basketball and mathematics classroom settings along three key dimensions. These included: access to the domain; the ability for students to take on integral roles; and to develop forms of personal self-expression within the cultural enterprise. The contrast is between the social structure of the game of basketball, in which there are elements of teamwork as well as well-defined positions, as compared with mathematics classrooms where interactions tend to emphasize individual students' responses and interactions with a teacher who is the dominant authority. The structure of the practice-linked goals and identity which different across these settings has thus been emphasized as a key factor in the difficulty of facilitating transfer across settings.

Approaches to connecting mathematics with students' real-world experiences. With regard to the teaching of mathematics in the classroom, Sierpienska (1995) lays out a continuum of options: learning mathematics in real-life contexts, learning of mathematics with applications, and learning of pure mathematics. The difference between these first two options lies in the explicitness with which mathematics is embedded into a context, and the priority or order in which mathematics and real-world problems are learned. This distinction is aligned with the framework that will be further specified below in terms of modeling.

Civil (2002) analyzes the tension that exists among three understandings and mathematical communities: mathematicians' mathematics, school mathematics, and everyday mathematics. These three different notions come apart because whereas everyday mathematics includes many informal heuristics for solving problems, school mathematics has traditionally been concerned with the transmission of algorithmic knowledge. Mathematicians' mathematics is concerned more with structure, abstraction, and process, emphases which are aligned with many of the reform-oriented approaches to mathematics as a discipline.

Both of these framings highlight the multiple differences and potential obstacles between knowledge and practices developed, used, and valued in different settings. They also emphasize the intermediate position occupied by traditional school mathematics as neither everyday or "real-world" mathematics nor formal or "pure" mathematics. Culturally relevant mathematics pedagogy's focus on teaching mathematics for understanding is more closely aligned with Civil's notion (2002) of bringing mathematicians' mathematics to the classroom than with traditional school mathematics which might focus more on the mastery of procedures than the development of conceptual understanding.

Indeed, Civil (2002) argued that an approach that values the Funds of Knowledge which reside in students' families can be a different basis for developing the mathematics and positioning students as experts. Other frameworks associated with either culturally relevant pedagogy (Ladson-Billings, 1995) or culturally responsive pedagogy (Gay, 2000; Villegas & Lucas, 2002) have emphasized the importance of connecting to students' out-of-school experiences. These pedagogies emphasize that mathematics education is a cultural enterprise that has traditionally marginalized large groups of students based upon race/ethnicity, class, and gender (Nasir, Taylor, & Hand, 2008).

Bishop (1988) hypothesizes that six activities cut across mathematics as implemented across different cultural groups: counting, locating, measuring, designing, playing, and explaining. Although the specific strategies and algorithms for these broader activities may differ across cultural groups, identifying the foundational activity may provide a way to connect and coordinate different algorithms, representations, and practices. This framework suggests that looking beyond surface features to the actual practices may provide more of a basis for making real-world connections. One challenge, however, is the shift that this would imply for what teachers have to know, notice, and do around mathematical practices across settings.

Overall, however, very little is known about teachers' approaches to real-world contexts. Lee (in press) found that prospective elementary teachers had generally positive beliefs about the value of real-life connections, in the form of story problems that emphasized the reality of situations and the utility of mathematics. Teachers' beliefs, however, were generally not specific. Further, their expressed beliefs did not correspond to either how they posed story problems in response to teacher education assignments or how teachers evaluated the relative worth of story problems which they were asked to rate. These findings suggest that the relationships between pre-service teachers' beliefs about story problems and real-world contexts are complex and do not have clear translations or implications for practice. Indeed, other studies of pre-service teachers found that they tended to exclude real-world knowledge and consideration from their solution of word problems (Verschaffel, De Corte, & Borghart, 1997).

Gainsburg (2009) surveyed secondary mathematics teachers about the format, features, and choice of context when making real-world connections. Teachers reported most frequently using the format of word problems for students to solve but also mentioned planned examples or references during a teacher presentation or a more extended project or lab activity. This last format, consisting of extended activities in a real-world setting that required more than algorithmic reasoning, was most commonly observed during classroom visits conducted to corroborate the survey results, but was speculated not to occur more than two or three times per academic year. Further, when asked to evaluate a set of real-world problem tasks drawn from the literature, teachers identified various relevant aspects as influencing their task selection. These included student pre-knowledge of the pertinent mathematics, appropriate cognitive and linguistic demands, task authenticity, and hands-on or manipulative aspects of particular task framings. The nature of the study was exploratory with a small sample size and so is not necessarily representative of how secondary teachers actually approach real-world contexts or real-life connections.

Incongruence, interference, and disregard. Various studies have emphasized potential disconnects when students in school approach mathematics problem set in real-world contexts. These can be characterized as incongruence, where strategies or algorithms do not match across real-world scenarios and classroom problems; interference, where knowledge or lack of knowledge about real-world settings actively interferes with students' abilities to complete a mathematical task; and disregard, where salient or problematic features of real-world contexts are ignored by students who approach the problem as if it were a “bare” number problem.

In other cases involving children, when students were given tasks framed in a “store” setting, they used oral strategies different from written school algorithms (Carraher, Carraher, & Schliemann, 1987). Taylor (2009, in press) studied purchasing practices in liquor stores and found that students employed a variety of mathematical strategies and enlisted the help of clerks and other individuals in solving problems of making purchases and getting change. Many of these strategies may not be fully recognized within the school curriculum.

Leonard et al. (2009) conducted a case study of two teachers who designed an after-school project for students who were recent immigrants. This curriculum unit was designed around an issue that the teachers thought would be of cultural relevance: the caloric content of the McDonald’s menu and the health issues raised by the documentary *Super Size Me*. The researchers found, however, that the students had no experience with fast food restaurants and made choices that did not make sense in the context Thus, the project did not support students’ cultural competence even though it introduced issues which were related to critical consciousness. This confusion on the part of students is an instance of interference.

Verschaffel, De Corte, and Lasure (1994) presented students with an assessment containing pairs of problems, some of which were standard, and others which were problematic given the real-world context. These problems which included real-world situations that were problematic in some way included those such as:

> Steve has bought 4 planks of 2.5 m each. How many planks of 1 m can he get out of these planks? (Kaelen, 1992 as cited in Verschaffel et al. (1994))

Students’ realistic reactions were tallied and compared to their nonrealistic reactions which included simply applying arithmetical operations without considering the context. Reusser and Stebler (1997) replicated this experiment in another setting and hypothesized that part of the social rationality of the institutional logic of schooling is that every problem is solvable and has a solution by some small set of mathematical operations. These results indicate that students who have experienced school mathematics may tend to disregard real-world contexts and considerations that might be problematic from the point of view of directly applying arithmetic operations.

Inoue (2008) extended these studies and investigated two related factors in how students approach story problems: the specificity or ambiguity of the goal stated in the problem and their familiarity or unfamiliarity with the situation. Undergraduate students' responses were characterized in several ways. Responses were categorized by the extent to which they were calculational, by the extent to which they reflected a shared "common sense" understanding of the situation, reflected an idiosyncratic or personal understanding of reality, demonstrated motivation to validate mathematical answers with reality, or recognized realistic constraints. Inoue argued that problem contexts should be less specific with constraints to allow students to associate their own experiences with the problems that they are engaged in solving.

An additional layer of complexity is revealed by considering the performance of different subgroups of students in terms of these problems. Lubienski (2000) studied how within her own reform-oriented, Standards-based mathematics classroom in which students worked on open and contextualized problems, students from lower socioeconomic . Cooper and Harries (2009) conducted a large-scale study of students across different socioeconomic classes and found that students from lower socioeconomic backgrounds were more likely to focus on aspects of the problem solving context, such as the type of vehicle in a problem about speeding rates, which were not intended by the authors of the problems or assessments to be relevant to the application of mathematics.

Transparency, elasticity, and mapping. Van den Heuvel-Panhuizen (2005) outlined several of the possible benefits of including real-world or realistic contexts in mathematics assessments problems. One possible benefit is to increase the accessibility of problems by providing multiple modalities and representations. For example, having a visual representation may contribute to transparency and elasticity of a problem. Transparency refers to… elasticity refers to … Students may have more latitude in solving a problem set in a real-world context than a bare number problems, and suggesting strategies based upon features of the context, such as repeated subtraction, which may not be as salient in a "bare" division problem.

Several studies have found that some real-world practices are more readily compatible with the tasks that students must complete in school. In their attempt to translate the study that Carraher et al. (1987) performed in Brazil, Baranes, Perry, and Stigler (1989) presented United States schoolchildren with problems set in a store setting as well as symbolic arithmetic problems. While they found no significant differences between story and symbolic problems in those settings, they did discover that some numbers were more conducive to the strategy of "mapping", which they defined as the use of "knowledge of a cultural system such as money or time to solve the problem by mapping the numbers in the problem onto that system." Baranes et al. (1989) found that numbers which were recognizable as related to the U. S. currency system were more likely to be "mapped" onto that cultural system (e.g. 7x25 as the value of seven quarters), as compared to arithmetic problems which were compatible to the base-60 system for measuring time. Similarly, Guberman (2004) found that children who had more instrumental experiences with money in home or social settings had more success at solving literal problems involving money compared to exercises with number chips. Working with high school aged students in urban areas, Koedinger and Nathan (2004) gave students numerically equivalent problems across three formats: literal, story, and symbolic. The literal format consisted for purely verbal formulations, while the story problems were set in familiar real-world contexts such as hourly wages. Symbolic problems were written with algebraic equations. There were in addition two other types of problems, those in which some starting value was unknown, and those in which some resulting value was unknown, the product of further calculations. When working on start-unknown problems set in familiar real-world contexts students were more likely to employ arithmetic strategies such as “unwinding” than when given problems in pure-verbal or symbolic algebraic forms.

Domínguez (2011) found that among bilingual students, the posing of problems set in familiar contexts in either English or Spanish made them more than twice as likely to engage in “reinventing” mathematics to solve the problems, as opposed to trying to “reproduce” mathematics that they had learned, often unsuccessfully. An example of a reproducing action was adding two numbers given in a problem when their division was more appropriate in context, similar to the kinds of suspension of sense-making frequently mentioned in the literature. Conversely, when working with unfamiliar contexts, students were more than twice as likely in both languages to “reproduce” as opposed to “reinvent” mathematics. Interestingly, when students worked together in Spanish, they more frequently challenged each others’ reproducing actions by referring to the relationships in the given context.

Multiple studies have emphasized the role of contextual knowledge in students’ development of informal inferential reasoning having to do with statistics. Langrall, Nisbet, Mooney, and Jansem (2011) conducted a study across three countries in which students worked in small groups in which one student was an expert in the real-world context in which a data-based problem was posed. Students positioned as experts tended to use their contextual expertise in a variety of ways. These included bringing new insights into the task, providing explanations for the data, justifying or qualifying their mathematical claims, narrowing focus to useful data for the task at hand, and stating facts that enhance the overall picture of the data but do not contribute to the analysis of the data itself. Diedorp, Bakker, Eijkelhof and van Maanen (2011) similarly provided evidence that extended mathematical problem solving set in authentic practices support students' ability to draw inferences based upon linear correlation. Pfannkuch (2011) found that real-world context assisted students working in a classroom unit to find meaning and real-world interpretations of numerical patterns, but diverted them from constructing concepts and applying new statistical theory.

Real-world contexts and analogies or metaphors. The research reviewed so far in this section reflects the extent to which real-world contexts have traditionally been associated with story problems in which they serve as the setting. An alternative approach is to consider cases in which a real-world context serves as the source of an analogy or metaphor which is then connected to a mathematical idea that may not be as explicitly contextualized. This category is consistent with what in the TIMSS video study was termed the "RLNP" code: a real-life connection that was made during a non-problem segment (Mosvold, 2008; Hiebert et al., 2003).

An analogy consists of a relation mapping from a source to a target (Gentner, 1983), with an emphasis on the relationships between structures. Thus, an analogy not only relates objects, but can also include mappings that relate the properties, relationships, and operations. For example, a frequently used analogy in school mathematics takes a balance-beam mass scale as a real-world context or source, and as a target has open sentences viewed as equations. The two sides of a mass scale are mapped to the two sides of an open sentence. The property of the mass scale being balanced corresponds to the two sides of an open sentence having the same value, or the equation being true or satisfied. Finally, in terms of operations, adding or removing equal masses on both sides of the scale corresponds to performing the same operations on both sides of an open sentence, as when trying to solve an equation. A metaphor can be viewed as a specific species of analogy where the mapping of features beyond the objects themselves may not be as explicit or specific.

Richland, Holyoak, and Stigler (2004) studied the use of analogies in mathematics classrooms using TIMSS data. Real-world contexts were relatively infrequently used as the source of analogies, accounting for 15 out of 103 (15%) instances of analogies. Teachers generally exercised a great deal of control over the introduction and development of analogies, controlling the introduction of the source, the specification of the mathematical target, and the statement of the relational mapping. Students were most frequently involved in the practice of what was termed “elaboration”—providing more information about the source of the analogy than was strictly necessary in order to minimally identify that source.

Mosvold (2008) has meanwhile indicated in a cross-national comparison of TIMSS video data that Japanese lessons tend to develop real-life connections that are metaphorical in nature. For instance, a lesson might draw upon real-world instances of liquor bottles of different but similar sizes in the initial phase of a lesson in order to develop the idea of similarity of geometrical figures. These real-world contexts are thus not the setting of a mathematical problem, but serve as the source of a mapping to noncontextualized mathematical concepts.

Mathematical Modeling as a Perspective

This section is divided into two parts. In the first, I outline the role that mathematical modeling has played in mathematics education research literature on the integration of real-world contexts into classroom practices. In the second, I consider the potential positive effects of supporting students as they develop models instead of giving students models to apply.

Modeling perspectives on mathematics and the real-world. Mathematical modeling was proposed by Greer (1997) as a way to address some of the difficulties that elementary students had with adequately accounting for the real-world contexts of word problems. Specifically, students were seen to extract two numbers from the text of the story problem, select a mathematical operation, perform the computation, and report the result. Mathematical modeling as a pedagogical practice had developed from applications to other disciplines into a process (Pollak, 1968). Further, as suggested by Greer, the modeling process is more conducive toward taking critical stances toward the uses of mathematics. These are critical related to the social expectations of the classroom and other features of the school context, such as curriculum materials and teacher practices.

As a process, modeling involves taking salient features of a real-world problem context and coordinating these with mathematical representations . The motivation for modeling as a process is that it resembles the processes that mathematicians and scientists undertake as they solve problems set in the real-world, while having modelers articulate assumptions and relationships among quantities. Specific to the issues surrounding student work on story problems set in real-world contexts, modeling has been related to the socio-mathematical norms that exist within the classroom (Greer, 1997).

At the same time, there is a great deal of heterogeneity when it comes to perspectives and theories on mathematical modeling (e.g. Larson et al., 2010), including thinking of models as conceptual systems, metaphors, representations, and organizational schema. Each of these perspectives can be viewed as analogous or parallel to the previously described model of analogies, in that they involve mappings from one context (usually a real-world scenario) to different mathematical representations, but which highlight different though processes in which students engage or which differently emphasize different elements as being central to the construction of models.

One common emphasis is on modeling as involving a cycle of transitions between the real-world context and a mathematical tool or representation, such as "expressing, testing, and revising" (Lesh and Doerr, 2003). Many variations on this cycle exist, and in some cases, emphasis has been placed on double-loop learning (Warwick, 2007), in which students explicitly and intentionally develop metacognitive knowledge about their own learning processes as they engage in the process of modeling a specific real-world problem. The cyclical nature of this process which requires feedback from mathematical answers back into the real-world problem and assessments of reasonableness thus flow in both directions, emphasizing modeling as a process rather than product.

Niss, Blum, and Galbraith (2007) articulate three tiers on which problems which involve some kinds of modeling occur. On the simplest level are word problems, sometimes referred to as story problems, which are "dressed up" in a real-world context and which require interpretation, solution and validation. At a higher level are standard applications where appropriate models are readily available, such as problems about maximizing the volume of a cylinder given some constraints. At the highest level are full modeling problems where questions need to be more precisely formulated and solutions need to be reinterpreted and validated within the real-world setting. Lesh and Caylor (2009) further distinguished differences between the inquiry activities characteristic of science classrooms, where data is often actively collected, from case studies typical of professional schools such as law school and business schools. These two approaches are further different from model-eliciting activities as well as the typical routine textbook word problems.

Relative to the framework of Realistic Mathematics Education which frames the Dutch education system, modeling, or in Freudenthal's terms "mathematizating," can be thought of as occurring both horizontally and vertically (van den Heuvel-Paizen & Wijers, 2005). Horizontal mathematization refers to modeling that transfers from the real-world context to a mathematical structure or model. Vertical mathematization involves generalization and greater abstraction within mathematical representations. Both of these processes can be viewed as part of the mathematical modeling process, but at different levels of abstraction and generality. These two related processes have also been described by Gravemeijer (2004) as models-of (representing aspects of real-world contexts) and models-for (using representations in the development of more general mathematical ideas). Modeling, on this view, is the beginning of the development of more sophisticated, higher-order modes of thinking about more abstract objects, such as visual or diagrammatic representations.

Providing or creating models. Some researchers have used the term "representation" interchangeably with "model" (Terwel, van Oers, van Dijk, van den Enden, 2009). The caution here is that the term "modeling" emphasizes its nature as a process, while "model" as a noun emphasizes a product (cf. Gravemeijer, 2002; Niss, Blum, & Galbraith, 2007). Indeed this distinction has been explored by various researchers using different terms, such as providing or generating (Terwel, et al., 2009), providing or designing (van Diijk, van Oers, & Terwel, 2003), model-exploration versus model-eliciting (Lesh & Doerr, 2003), and reproducing versus reinventing actions (Domínguez, 2011). In each case, the first term in each pair refers to cases where the teacher gives students a specified representation or model to apply to a problem set in a real-world context. The second term refers to cases where students create, develop, test and refine models or representations in connection with attempting to solve a problem set in a real-world context.

Of these, the distinction between model-exploration and model-eliciting provides a means to articulate the distinction in a way that is parallel to the other distinctions despite the choice of different but ultimately compatible terms. The framework for model-eliciting activities is perhaps the best developed, and is guided by six principles, which are described briefly below:

1. Model Construction: does the activity require students to actually create a model?
2. Reality: does the activity have the potential to be personally meaningful to students so that they reason with their real-world experiences and knowledge?
3. Self-Assessment: does the activity allow students to assess for themselves whether or not their model works?
4. Model Documentation: does the activity require students to record their thinking and reasoning process in constructing a model?
5. Model Generalizability: does the activity have students reflect on the extent to which their models are shareable and reusable in other contexts?
6. Simplest Prototype: is the activity framed as simply and elegantly as possible while still requiring the process of modeling?

The detailed nature of this framework illustrates the multiple variables which are necessary in creating appropriate mathematical tasks for students which will both require them to create models but also support them in the process. The remainder of this section surveys the advantages found in empirical studies to having students develop, generate, or design their own models and representations of real-world contexts in mathematics problems.

Terwel, et al. (2009) found in an experimental study that primary students who were supported in "generating" their own representations around problems of proportional reasoning were better able to transfer their knowledge to novel and unfamiliar situations. The process of designing their own models and representations shifted the role of the teacher from being the source of transmitted representations to a critical participant in and facilitator of the design and refinement of models and representations. Because the students were engaged in the co-construction and negotiation of representations, while their strategies were still bound to the context of the problem, there was more room and need for abstraction as students collaborated to clarify meanings. Rather than the teacher issuing authoritative statements about which representations or models were useful, students were given the opportunity to select from the other strategies that other students shared in whole-class discussion. These results corroborated an earlier finding in a small-scale, qualitative study by van Dijk, et al. (2003).

Legé (2007) compared the performance of two groups of urban high school students placed in a remedial pre-algebra course in terms of twenty different performance goals around the process of modeling as well as the model structures that reflected a more structured approach to modeling in which examples were given to students. The experimental group was actively engaged in the process of developing models while the control group only read examples. While there were not statistically significant differences in most of the performance goals, the group of students who had been actively engaged in the process of developing models were more likely to use sub-models and multiple iterations of the modeling cycle, more likely to interpret contextual features more directly, and to recognize limitations inherent in a model given by data.

Schwartz and Martin (2004) studied the benefits of allowing students to invent their own approaches before behind taught by the teacher through direct instruction a mainstream approach. This project title reflected this model: Inventing to Prepare for Learning. Explicit comparisons were further made between preparation activities and those activities which were along the traditional problem-solving lines. In design experiments set in statistics, students were given distributions of data in the context of accurate pitching machines and were given the opportunity to invent their own measures of accuracy before being instructed in standard or orthodox representations, approaches, and formulas. The students who were given a chance to invent demonstrated both more adaptive expertise and symbolic insight into the standard formulas.

Iszak (2003) investigated how students generated representations. In this case, students were given a physical problem in which they needed to model relationships among various components of a physical device. As students worked with each other to answer mathematical questions, they subjected their representations to multiple refinements of three types. These included generating, evaluating, and using representations iteratively and dynamically. As students engaged with the data generated by real-world experiments or problem contexts, they modified their representations to match the data.

Other studies have focused on how mathematical modeling as a process promotes high quality instructional environments in terms of students developing, evaluating, and sharing diverse approaches (Doerr & English, 2003). These developed models were more generalizable and transferable to novel and unfamiliar problem settings. These improved outcomes in terms of transfer and student engagement were related to students' active participation in multiple aspects of the development of models, and the higher demands required both in terms of cognitive processing as well as interpersonal communication.

Doerr (2006) on the other hand emphasized three dimensions of teacher knowledge that are essential to facilitating the development of students' emerging models in mathematically productive directions. These dimensions included being able to anticipate the multiple pathways that student reasoning might take, listening and noticing the developmental trajectories of students, and responding appropriately in order to support and guide the development of student ideas. Within a sample of four experienced teachers implementing the same model-eliciting activity about exponential functions given a growth rate of doubling, there was a great deal of variation in terms of what students actually accomplished based upon the interaction between their emergent models (including tabular and symbolic representations and manipulations of concrete objects) and teachers' feedback. Actually achieving the promise of modeling in the classroom is thus contingent on many factors including classroom norms and teacher-facilitated practices.

Classroom Practices that Support Understanding of Real-World Contexts

This section is divided into two parts. In the first part, I survey general influence of school-level practices and approaches toward teaching mathematics and how these correlate with understanding of real-world contexts as well as relationships between real-world contexts and mathematics as measured by student understanding, achievement, and disposition toward mathematics. In the second part, I consider specific classroom practices on the level of the lesson and within the classroom that make connections between mathematics and real-world contexts.

General influences of school practices. Many studies have emphasized that the instructional environment of mathematics classrooms is connected to the success with which real-world contexts are incorporated, connected, and modeled with mathematics. Boaler (1993a) found that students at a school where content and process were fully integrated had more consistent achievement on problems which were either framed purely in arithmetic terms within some real-world context (such as football penalty shots and cutting wood). By contrast, students at the more content-focused school had results on the real-world tasks that were highly variable based on the context. Students at a school with a procedural or "closed" approach to mathematics had "inert" knowledge that was difficult to extend to unfamiliar situations, while students who had been immersed in a more open, project-based environment expressed feeling less of a disconnect between the mathematics they had learned and real-world applications (Boaler, 1998).

Similarly, Molyneux-Hodgson, Rojano, Sutherland, and Ursini (1999) found that at a school which had a more inductive, exploratory approach to classroom instruction, students were more likely to approach mathematical modeling of problems through approximation and graphical representation. Students who had classroom experiences that were “presentational” tended to focus on precision and algebraic representations. Key features of the former, exploratory or “bottom-up” style included students using induction to state general principles from specific examples, and frequent references to familiar “everyday” examples such as ginger beer and pressure cookers. Thus, the general pedagogical environment can influence how students approach connecting real-world contexts to the algorithms and representations they learn in school mathematics.

Other studies have emphasized the importance of “openness”. Boaler (1993b) lists verbs which describe the processes in which students must engage with contexts which offer “open beginningness” in order to develop meaning with mathematics: “discover”, “explore”, “negotiate”, “discuss”, “understand”, and “use”. These correspond to students approaching problems not merely as occasions to apply closed procedures in a determinate fashion (Boaler, 1998). These general approaches to mathematics as a process and solving problems as related to investigations have been linked to deeper levels of student understanding (Boaler & Staples, 2008).

Specific classroom practices. In terms of specific practices and forms of participation which enable the understanding of real-world context or which the real-world contexts themselves enable, Civil (2002) claims that incorporating real-world contexts, such as the tessellations and geometric patterns that students find in everyday life in the American Southwest, can encourage the participation of students who are normally considered low-status academically. In particular, Civil cites the “openness” of the mathematical problems as well as the legitimization of students’ out-of-school knowledge as widening opportunities for classroom participation to students labeled with deficits. Civil also notes that it is not the everyday mathematics per se, but four features of the practices that they are embedded in that are worth bringing into the classroom:

1) learning through apprenticeship,

2) contextualized problems,

3) control to the solver,

4) hidden mathematics that is not automatically at the center of attention.

One caution is that as the class turned to more formal mathematical justification, students with higher academic status again began to dominate the discussion. This challenge was also stated by Nasir et al., (2008) as “keeping the mathematics in sight” while keeping participation open and wide.

Evans (1999) described the practice of “contexting” as a means of unleashing other connections that individuals may have with a real-world context or mathematical representation, by asking: “Does this remind you of anything that you currently do?” and “Does this remind you of any earlier experiences?” These questions serve to first make personal, and possibly affective, connections with a real-world context before trying to apply interpretations specific to school or another discipline.

These practices around contexting are consistent with that Chapman (2006) describes as the paradigmatic and narrative modes of knowing about word problems set in real-world contexts. The focus of these two modes is on how teachers approach explaining or unpacking word problems with their classes. “Paradigmatic” here refers to truth-seeking logico-scientific practices which are focused on verification and proof. Paradigmatic approaches favor impartial, objective mathematical procedures and structures. “Narrative” modes are more humanistic and aligned with understanding the meaning of experience as related to human intentions and motivations.

Chapman (2006) outlined three different paradigmatic approaches:

1) fragmenting or parsing and translating
2) finding independence of mathematical structure from context.
3) interpreting the context mathematically

This last approach focused not on bringing in students' experiences but on articulating the abstract relationships implicit in the problem.

By contrast, there were four narrative approaches based upon what Chapman called "resonance":

1) detached as an aside, not connected to the relevant mathematics
2) accepting non-mathematical interpretations without regard to mathematical validity
3) discussing specific aspects through curiosity or critique
4) entry and exit into a problem referring to real-life experiences.

These two modes were not mutually exclusive, and indeed can support each other, but Chapman found that paradigmatic or logico-scientific modes were more prevalent among mathematics teachers at the secondary level.

Depaepe, De Corte, and Verschaffel (2010) extended this framework and focused on the two "phases" of entry into and exit out of the word problem in order to develop a fine-grained coding instrument to apply to videos to compare elementary teachers' "interventions" in the explanation of word problems. They identified three interventions in the paradigmatic approach for the entry phase:

1) Distinguishing relevant from irrelevant information
2) Applying a prototypical scheme

3) Addressing the underlying mathematical structure (p. 155)

In addition, there were two interventions in the exit phrase:

4) Seeking confirmatory evidence for the solution being obtained
5) Addressing the underlying mathematical structure (p. 155)

For the narrative approach, Depaepe et al. identified four interventions in the entry phase:

1) Rewording the problem
2) Defining notions involved in the problem
3) Building on students' real-life experiences and prior knowledge
4) Taking explicitly into account the realities of the problem context.

There were three additional interventions in the exit phase of the narrative approach:

5) Interpreting the outcome
6) Thinking of corresponding real-life situations
7) Taking explicitly into account the realities of the problem context.

Using this framework, Depaepe et al. (2010) compared two teachers in terms of the relative frequencies during "problem solving sessions" of the different approaches, phases, and interventions, and were able to construct comparative profiles of those teachers' practices. For both of the teachers, however, the paradigmatic approach was mode common. Further, taking up realistic considerations in either entry or the exit phase was very rare.

The entry phrase of Depaepe et al.'s (2010) framework includes what is measured by the ratings rubrics developed by Jackson et al. (2011) focused on the "launch" of a task. The Jackson et al. rubric measures the extent to which relevant features of the context are discussed by the teacher and students to ensure that students will know enough about the context to actually be able to solve the problem. On a four-point scale, ratings of 1 and 2 indicate only “yes” or “no” responses from students, while higher ratings of 3 and 4 indicate that there is substantial exploration of taken-as-shared understandings of what the context means in ways that may be relevant for the mathematical models that students then have to create.

This category of “contextual features” is closely related to ratings of the “mathematical relationships” facilitated by the teacher in the launch of tasks (Jackson et al., 2011). One distinction is that mathematical relationships may be present in any kind of problem encountered in a mathematics classroom, while contextual features would apply only to those “problem solving situations” in which there is some real-world context to be interpreted. The authors found that teachers tended to pay more attention to mathematical relationships than to contextual features, as measured by comparison of mean ratings, among those lessons with “problem solving situations”. Ratings for contextual features were, however, positively associated with ratings for mathematical relationships as measured by linear correlation.

Other studies have explicitly focused on conceptualizing and analyzing teachers’ treatment of out of school experiences. Wager (2012) described four practices observed or reported by teachers involved in a professional development seminar focused on incorporating out of school mathematics. These included using experiences as contexts for problems, linking students’ experiences to school mathematics, and identifying in those experiences salient embedded mathematical practices. A fourth category included shared experiences within the classroom which then served as a basis for mathematics, such as international bazaars or world currencies. Wager identified inherent challenges on teachers having enough time to acquire nuanced knowledge of students’ cultural practices and how to draw out the embedded mathematics and link it to school mathematics.

McCulloch and Marshall (2011) developed three categories for classroom episodes where teachers in grades K-2 connected “out of school” experiences to in-school mathematics learning: cultural connecting, language matching, and relevance making. These categories are largely compatible with Wager’s (2012) framework. Cultural connecting referred to drawing upon students’ out-of-school knowledge in clarifying tasks. Relevance making referred to suggesting applications of school mathematics to real-life contexts or practices. These two practices can be viewed as different ways of what Wager called linking students’ experiences with school mathematics, with cultural connecting in an entry phase and relevance making in the exit phase. Forty-nine teachers in three cohorts participated in a weeklong summer institute and four two-day professional development retreats during the school year. Across 482 classroom observations, however, instances of out-of-school contexts were both infrequent and superficial, present in only 102 lessons. McCulloch and Marshall suggested that teachers’ preexisting and narrow or color-blind understandings of race and culture may have limited the extent to which they were able to incorporate out-of-school experiences into mathematics.

Turner et al. (2012) took the approach of trying to develop learning trajectories along which to describe prospective K-8 mathematics teachers’ learning about how to integrate their students’ multiple mathematical knowledge bases, including those with home, cultural, or community sources, as they complete a methods course in mathematics education. Their tentative trajectory hints at the complexity of this development, consisting of three phases: initial practices, making connections, and regular incorporation of students’ multiple knowledge bases into the classroom. Among these key initial practices are attention, awareness, and eliciting. These three practices are intertwined as teachers first start paying attention to students’ thinking and knowledge. Teachers may have an awareness of how other activity systems and knowledge bases may also be involved. At the same time, teachers can elicit students’ explanations of how other home, cultural, and community knowledge bases are at play as they work on mathematics in the classroom.

Conclusion

This review of the literature has surveyed how the issue of real-world contexts has generally been understood in the existing research on mathematics education. One main focus has been on the use of real-world contexts as the setting of story problems. On the other hand, analogy or metaphor has potential to bring in real-world contexts in ways which may actually support the development of deeper understanding of noncontextualized mathematical procedures and concepts. Further, an emerging body of research has considered what happens within classrooms around developing student understanding of these real-world contexts in order to develop understanding. One practice that is important is that of mathematical modeling, understood to refer to developing models rather than applying given ones. A second practice is more discursive in nature, and concerns elaboration. Elaboration includes both discussions about the key contextual features of a mathematical problem as well as drawing explicit connections to students' personal experiences. This study will contribute to the literature by detailing how these different elements interact with each other.

Chapter 3

Research Design

Setting

Harwood. Harwood, the community in which the school is located, has had a population that is least two-thirds Latino since 1990. As of 2007, the population of Harwood is 69% Latino, 20% Black, 8% White, and 4% Asian (Rodríguez, 2009). Ecuadoreans, Salvadorans, and Mexicans have been rapidly growing Latino subgroups. Puerto Ricans still remain the largest Latino subpopulation in Harwood, but in 2007 they only accounted for 32% of the Latino population compared to 66% in 1990. In Harwood, Dominicans are the second largest Latino group, representing 24% of the Latino population. Dominicans are followed closely by Mexicans (20% of Latinos) and then Ecuadoreans (13%). The more newly arrived groups of Mexicans and Ecuadoreans also have a lower median age (23.0 and 24.5, respectively) than the more established Puerto Rican and Dominican groups (35.0 and 32.0, respectively) (Rodríguez, 2009).

Homeownership among Latinos in Harwood is 17%, whereas it is 31% among non-Hispanic Blacks. Among these, it is the Puerto Ricans and Dominicans who have the highest rates of homeownership (26% and 22%, respectively) compared to more recently arrived groups. Median family income was higher for Blacks ($44,220) than Latinos ($36,833) as of 2007 (Rodríguez, 2009).

According to New York City, 48% of families in Harwood receive some form of public assistance, including Aid for Families with Dependent Children and Medicaid (2008). Educational attainment among Latinos and Blacks, defined as having attained a bachelor's degrees or higher by those aged 25 or older, was lower (7% and 13%, respectively) than for Whites and Asians in the district (Rodríguez, 2009). As of 2007, there were no evident disparities across subgroups in terms of employment rates as defined by the Census. The percentage of foreign-born Latinos in the district has grown steadily since 1990, from 24% to 45% in 2007. Many of these foreign-born Latinos in Harwood become naturalized citizens, although Salvadorans (60% of the foreign-born aged 18 or older) have much higher rates of naturalization than Dominicans (38%), Ecuadoreans (18%), and Mexicans (10%).

This portrait of immigration is also consistent with Harwood's history, as numerous immigrant groups have initially settled in Harwood before moving on to other more suburban neighborhoods. Germans were succeeded by Italians before World War II, and then Italians were succeeded by Puerto Ricans and Blacks in the 1960s (Sullivan, 2006). These groups were targeted by realtors through blockbusting, in which white owners were scared off, selling their houses for low prices. Realtors then sold the same houses to Blacks and Puerto Ricans who could not afford the homes and ended up defaulting. Owners also committed arson on their own properties to collect insurance money (Malanga, 2008). With the fiscal crisis of the 1970s debilitating the city's ability to provide city services, the crime and murder rates soared. The Blackout of 1977 led to widespread looting and the burning of many businesses, the scars of which can still be seen in vacant lots across Harwood. The 1980s brought a crack epidemic that hit Harwood especially hard, with the murder rate soaring until it peaked at 77 in 1990 (Malanga, 2008). More aggressive policing by the NYPD in partnership with community-based organizations has been credited with bringing the crime rate down during the 1990s. Given rising real estate costs in neighboring communities such as Williamsburg, Ridgewood, and Ocean Hill, Harwood has experienced an influx of newer White residents (Sullivan, 2006). Building permits have increased substantially within the district, and signs of gentrification in terms of residents, businesses, and restaurants can already be seen throughout the neighborhood

(Malanga, 2008).

At the same time, Harwood today is host to multiple community-based organizations. Among these is Opportunities for a Better Tomorrow which is a work placement and career development program. The Ridgewood Harwood Senior Citizen Council provides social services to senior citizens as well as youth. The Harwood Making Children Important network meanwhile draws together more than forty social service organizations such as the public library and the Coalition for Hispanic Family Services as part of a Community Partnership Initiative with the city's Administration for Children's Services.

With more of a community organizing orientation, Make the Road New York is partnered with the Harwood School for Social Justice, and runs multiple projects including the Youth Power Project, an LGBT support group, and other initiatives around health, asthma, immigration, workers rights, education, and housing. One recent campaign was organized around the vicious beating of two Ecuadorean brothers in Harwood by men yelling anti-gay and anti-Latino epithets, resulting in one of their deaths (Jaramillo, 2009). In the past, Make the Road has sponsored Legislative Assemblies at the Harwood campus in which lawmakers from across the city and state were invited to listen to the legislative platform which Make the Road sponsored with other community members and stakeholders.

The school. The high school's student population was 420 in the 2009-10 school year. Sixty-one of these students were classified by the Department of Education as "English language learners." The student population was 31% Black and 69% Latino. The gender distribution is 52.4% female and 47.6% male. The average attendance rate is 81.0% and the poverty rate is 83%. Table 1 summarizes some of the characteristics across the two school years included in this study.

Table 1. Summary Demographic Characteristics of Harwood High School		
	2009-10	2010-11
Total enrollment	420	420
Black	129 (31%)	122 (29%)
Latino	282 (67%)	289 (69%)
White, Asian Pacific Islander, Native American, or Other	9 (2%)	9 (2%)
English language learners	61 (15%)	70 (17%)
Free or reduced lunch	348 (83%)	352 (84%)

The school itself is one of four on the campus of what used to be Harwood High School, which was reorganized into smaller schools as part of the Children First Initiative that came with mayoral control and reorganization of the Department of Education. The thirty-five teachers at the school are relatively inexperienced, with only 22.0% having more than five years of experience teaching, and 27.8% with more than two years' experience teaching in the school (NYC Department of Education, 2008). The teaching staff is ethnically diverse, although there are more white teachers than teachers of color and few Latino or Latina teachers on the staff. One assistant principal is Latina and the other administrators are White.

Multiple instruments to measure and document the performance of schools have been developed by the New York City Department of Education.[4] Each of these instruments hinge upon conceptions of academic achievement that are based upon either performance on standardized state tests or performance on coursework, and only secondarily on the perspectives of teachers, students, or parents. From the perspective of NCLB and the New York State

[4] All statistics in this section are from school-level reports. The type and year of each is noted, but to protect the identity of the school these are not included in the bibliography. There are two state-produced documents associated with NCLB status, the Accountability and Overview Report, which is also known as the School Report Card, and the Comprehensive Information Report. The Department of Education produces three reports, though all are referenced by the Progress Report. The Progress Report provides a letter grade in three categories, the Quality Review provides an overall rating and detailed qualitative analysis of data-driven instruction according to a rubric, and the Learning Environment Survey reports detail responses on the question level from parents, teachers, and students in four different domains.

Education Department, Harwood High School was classified in the 2009-10 school year "Improvement/Comprehensive" in mathematics for not meeting Adequate Yearly Progress targets in the previous year (Accountability and Overview Report, 2010). By 2010-11, the school had made its AYP target and was no longer considered a "school in need of improvement" (Accountability and Overview Report, 2011).

New York City Department of Education grades all schools based on its "Progress Report" measures, reported as letter grades that compare schools to each other. Progress reports are divided into three sections: school environment, student performance, and student progress. School environment measures include student attendance and responses from parent, student, and teacher surveys. Student performance combines graduation rates weighting for type of diploma. Student progress is measured in terms of in terms of credit accumulation and weighted Regents pass rates. For the 2009-10 school year Harwood school received "straight A"s in all three categories and was in the 90th percentile of schools citywide, its grades in 2010-11 was lower in terms of school environment (B) and student performance (C) (Progress Report, 2011).

This discrepancy can be located in lower four- and six-year graduation rates as well as taking into account the type of diploma awarded to students. Students also reported lower levels of "Safety and Respect" (Learning Environment Survey, 2011). These responses were centered on students threatening or bullying each other, getting into physical fights with each other, or adults yelling at students. While students reported that teachers generally treated them with respect (80% or more strongly agree or agree across the two school yeras), they did not think that most students treated teachers with respect (approximately evenly split between agreeing or strongly agreeing and disagreeing or strongly disagreeing across both school years) (Learning Environment Survey, 2010, 2011). Evidence that corroborates the need for cultural competence as advocated by culturally relevant pedagogy is contained in student responses to the statement, "Students who get good grades in my school are respected by other students." About 40% of students disagreed or strongly disagreed with this statement in both 2009-10 and 2010-11.

Despite these challenges, the school has done well by Department measures, in particular on the weighted Regents pass rates. These measures account for the previous achievement test history of incoming students by comparing the passing rates of students grouped by their eighth grade achievement levels to those in peer schools as well as citywide. Given that at Harwood, only 30% of students of the 90% of students who had eighth grade test scores were at the performance level of proficient or above in mathematics, however, it must be emphasized that these weighted rates are norm- rather than criterion-referenced.

The study participants. A total of four mathematics teachers are included in this study—two males and two females. Two are White and two are Black. They are all outsiders to the community, because they do not live in the vicinity of the school and come from different geographical, social, and economic backgrounds. None of these teachers had, at the beginning of the study, more than five years of mathematics teaching experience at the high school level. Of the four, only one, teacher A, majored in mathematics as an undergraduate. Two of the other teachers, teachers C and D,participated in New York City Teaching Fellows Mathematics Immersion program and are certified to teach middle childhood, grades 5-9. Teachers A, C, and D, are New York City Teaching Fellows and are therefore alternatively certified. Teachers B, C, and D were middle school teachers who now teach high school classes. Table 2 summarizes some of the demographic information of the teachers included in this study.

Table 2. Demographic Characteristics of Teachers in Study

	Teacher A	Teacher B	Teacher C	Teacher D
Approximate age	Late 20s	Early 30s	Early 40s	Early 30s
Gender	Female	Female	Male	Male
Self-reported detailed ethnic background	White European	Afro-Caribbean	African American	White European
Year of teaching at beginning of 2009-10 school year	3rd	8th	5th	5th
Grade level(s) taught (2009-10 & 2010-11)	10th	9th	12th	11th
Course (2009-10)	Math A	Integrated Algebra	Algebra Prep	Geometry
Course (2010-11)	Geometry	Integrated Algebra	Algebra Prep	Algebra II
Certification	Adolescence Mathematics 7-12	Middle Childhood 5-9	Middle Childhood 5-9	Middle Childhood 5-9

The distribution of teachers in terms of subject areas reflects that they each have a single preparation period per day. Although there was an additional mathematics teacher who was part of the professional development meetings in the second year of the project, this teacher did not fully participate in the data collection for both years and is not included in this study. The distribution of the teachers across different subject areas and the existence of a curriculum which emphasizes preparation for the state Regents examination in algebra effectively narrowed the curriculum students were being taught.

Research Design

The research design for the study is outlined in Table 3 below. All data for this project is derived from instruments, classroom observation schedules, and the research design for the larger CTMUY project created by the principal investigator. As a research assistant on that project, I collected this data. The basic unit of data consists of classroom observation instruments (COI) completed over the course of two school years in the classrooms of the four teachers. The four teachers were observed eighteen times each (ten times in the first year, and eight times in the second) for complete math lessons of 48 minutes. [5] In both years there were four "rounds" of consecutive lessons which were observed at different points during the school year.

The first research question is: "*To what extent* and *in what ways* do mathematics teachers provide opportunities in the classroom for students to connect real-world contexts to mathematics lessons?" To answer this question and its five subquestions, listed below, the lesson descriptions of the classroom observation instruments were analyzed in order to categorize lessons that contained real-world contexts.

1. With which mathematical topics are real-world contexts associated?
2. What is the extent and nature of teacher *adaptation* of the presentation of real-world contexts from textbook or other curriculum sources?
3. What *roles* (*motivating metaphor, incidental or central)* do real-world contexts play in mathematics lessons?
4. When the role of a real-world context is central to the lesson, do students *develop* mathematical models or solution approaches or are they *given* mathematical

[5] One lesson was omitted due to highly atypical circumstances, specifically how the teacher had lost her voice for that lesson. This omission results in a total of 71 lessons.

models or procedures to apply?

5. Are real-world contexts *elaborated* or not and what is the nature of elaboration?

The answers to the first research question define different groups of lessons. For example, lessons can be grouped into:

- lessons which included real-world contexts, and
- lessons which did not include real-world contexts.

Or, among lessons which contained real-world contexts, lessons can be grouped into:

- lessons which elaborated upon the real-world contexts, and
- lessons which did not elaborate.

The second research question is focused on differences across these different groups of lessons on three kinds of measures of classroom instruction:

a. the level of cognitive demand of the main mathematical task,
b. ratings of four different dimensions of the instructional environment, and
c. relative frequencies of different student participation categories.

Each of these three kinds of measures is described in greater detail below. Statistical tests then compared different groups of lessons according to these three measures.

Table 3. Summary Table of Research Design

Research Question	Source of Data	Quantity	Analysis
To what extent and *in what ways* do mathematics teachers provide opportunities in the classroom for students to connect real-world contexts to mathematics lessons?	Classroom Observation Instrument: Lesson Descriptions	18 observations per teacher for each of four teachers.	Qualitative analysis classifying and characterizing real-world contexts by: 1. presence 2. adaptation, 3. role, 4. modeling, and 5. elaboration.
Are there differences across lessons based upon: 1. Inclusion of real-world contexts (present or absent) 2. Level of adaptation (high or low) 3. Type of role (motivating metaphor, incidental, or central) 4. Type of modeling (developed or given) 5. Type of elaboration (elaborated or not) in terms of: a. level of cognitive demand b. mean ratings of instructional environment c. distribution of student participation categories	Classroom Observations Instrument: a. Ratings of level of cognitive demand, b. ratings of instructional environment, and c. relative frequencies of student participation categories.		Quantitative analysis comparing groups of lessons: a. Chi-squared tests or Fisher's exact test for independence on level of cognitive demand, b. Independent samples t-tests on mean instructional environment ratings. c. Independent samples t-test on mean relative frequencies of individual participation categories and combined relative frequency of "active" student participation time.

Data Sources

This section describes the data that was part of the original classroom observation instrument (COI) which was created by the principal investigator of the CTMUY project. The COIs contained four kinds of data: lesson descriptions, cognitive demand ratings, ratings of dimensions of instructional environment, and tallies of the different categories of participation offered to students for the entire class period. Of these, the lesson descriptions were used to sort and classify lessons into groups based upon the presence and treatment of real-world contexts. The other measures, described below, were treated as dependent variables. Across two school years, each of four teachers was observed teaching eighteen times across the school year. Teachers were observed ten times in the first year (three lessons in September, two lessons in December, two lessons in February, and three lessons in April) and eight times in the second year (two lessons in September, December, February, and April).

Lesson descriptions. For each lesson, a detailed classroom observation instrument (COI) was completed (See Appendix I). Each COI contains a detailed narrative description of what happened during the lesson and how the lesson fits into a broader curriculum unit. It was this portion of the COI that was read and analyzed in order to identify answers to the subquestions of the research question:

1. whether or not real-world contexts were included in the lesson,
2. whether or not those real-world contexts were adapted substantially by the teacher,
3. the role of the real-world context within the lesson (motivating metaphor, incidental, or central),
4. among real-world contexts with a central role whether students engaged in developing models or whether models were given to them, and

5. the presence or absence of elaboration.

In addition, each COI contains the following information:

A) the cognitive demand of the main mathematical task,

B) ratings of four aspects of the instructional environment.

C) participation categories tallied for amount of time,

Each of these three components is described in greater detail in the following. These variables were constructed, validated, and reliability was established independent of the analysis of the inclusion and incorporation of real-world contexts (Rubel & Chu, 2012).

Measure A: Levels of cognitive demand. Following the Mathematical Tasks Framework, the main mathematical task of each lesson was identified as the one that took the greatest amount of time (Stein et al., 2000). The cognitive demand of that particular task as implemented by students was rated (Henningsen & Stein, 1997). Four levels of cognitive demand were identified, two of them at a low level (memorization and procedures without connections) and two of them at a higher level (procedures *with* connections and doing mathematics). Tasks at a low level of cognitive demand require very little higher-order reasoning from students, requiring only the recall of previously learned information, as in memorization, or the repetitive execution of given procedures without making connections to other mathematical concepts, real-world contexts, or other mathematical processes or procedures. Procedures *with* connections, a high-demand level, require students to connect different aspects of problems, including making connections across mathematical representations such as tables and graphs. At the level of doing mathematics, students are given a task where there is a substantial amount of ambiguity and not one clear way to proceed with completing that task, with some degree of unpredictability in the process.

Cognitive demand is one measure of the quality of what students are being asked to do in classroom, and subsequently their opportunities to learn mathematics with understanding. As a construct, it has come to be understood as a key measure of students' opportunities to learn important mathematics (Jackson et al., 2011). In contrast to previous work which looked at the implementation of tasks by students after being posed to the class by the teacher (Henningsen & Stein, 1997) or the set-up or launch of tasks as related to the maintenance of cognitive demand (Jackson, et al., 2011), the levels of cognitive demand in this study refer to how students actually implemented the task, not how the task appears in instructional materials, how the teacher planned the task or posed it to students.

Measure B: Instructional environment. Each COI also included ratings of the lesson on four aspects of the instructional environment, using scales designed and validated by Kitchen et al. (2007). The five-point rubric with detailed descriptors for each of the levels developed by Kitchen et al. (2007) was used and a detailed justification was written for why the lesson was rated at that level. The ratings instrument is included in Appendix I as part of the COI. The four categories of instructional environment were:

- mathematical discourse and communication,
- intellectual support,
- depth of student mathematical knowledge, and
- engagement.

Each of these four dimensions is related directly to the framework of teaching mathematics for understanding and the broader goal of supporting the academic achievement of students.

The rating of mathematical discourse and communication evaluated the extent of classroom discussion and the extent to which a wide range of students' ideas were invited and explored seriously in the pursuit of collectively developed shared mathematical understandings. Intellectual support refers to the extent to which students were encouraged by the teacher and their peers to take intellectual risks in suggesting new ideas and making new connections. Ratings of the depth of student knowledge reflected the extent to which students were given opportunities to make connections, generalizations, and synthesize between related concepts and procedures within mathematics. Finally, engagement ratings measured the extent to which students were visibly on task and enthusiastically engaged in activities. Inter-rater reliability on all four of these ratings scales was established as part of the larger CTMUY research project (Rubel & Chu, 2012).

Measure C: Student participation distributions. The total class time of 48 minutes was divided into sections and the duration of each of the class activities was recorded. These activities were then classified according to categories adapted from the Weiss et al. (2003) study, which was a national survey of teachers in mathematics and science on the opportunities that they offered their students to participate. Six categories were taken directly from the Weiss study:

- listening
- discussing,
- investigating,
- writing, reading or reflecting,
- using technology, and
- housekeeping.

Each of these participation categories were divided into more specific participation structures. For instance, “listening” consisted of three categories:

- listening to the teacher,
- listening to other students, or
- listening to an outside guest or visitor.

The COI added an additional participation category to the original Weiss et al. (2003) participation structures. "Practicing" was created as a label used to mark the repetitive application of learned procedures or algorithms to similar exercises. The two specific subcategories within "practicing" were:

- practice on worksheets or textbook exercises and
- practice specifically directed toward preparation for standardized exams.

Thus, the original data collected contained seven participation categories. In subsequent analyses of this data, however, listening to the teacher has been considered a separate category from listening to students, because having students make presentations to the class requires active participation from those students who are presenting and has more potential for the active engagement of students who are also listening (see Rubel & Monroe, 2012). A composite total of the "active" participation categories including listening to students, discussing, investigating, writing/reading or reflecting, and using technology was also computed for each lesson. A complete listing of all of the specific participation structures is provided in the COI template included as Appendix I.

Data Analysis

To answer the first research question, focused upon the quantity and quality of opportunities to engage with real-world contexts in mathematics lessons, the data analyzed are drawn from lesson descriptions of the COI. The overall strategy for data analysis was to conduct a thematic analysis and devise categories for sorting the real-world contexts that arose in the lessons. In each case, the frequency of occurrences of individual types of lessons were tallied and are reported in the results section. To perform the analysis for the first sub-question, real-world contexts were analyzed for:

1. the associated mathematical topic
2. whether the real-world context was substantially adapted from a textual source
3. the role that real-world contexts played in the lesson
4. for real-world contexts in a central role, whether the mathematical models were developed by students or given to them
5. The presence or absence of elaboration.

The purpose of this analysis was to provide a classification system to guide future work and to enable comparisons of the other quantitative measures collected in the COIs and addressed by the second research question. Frequencies answer the question about *if* and *to what extent* lessons with real-world contexts had the above features. The detailed qualitative analysis answers the question of how they actually functioned within the mathematics lessons.

Analysis of lessons containing real-world contexts. The qualitative analysis was conducted in three stages. For each teacher, the eighteen observations were conducted in eight rounds of two or three consecutive lessons over two school years. First, at the end of each of the eight rounds of observations, a summary memo was written about the inclusion of real-world contexts in lessons. These memos were then combined into a cumulative memo that summarized, by teacher, specific uses of real-world contexts in the lessons, as well as any other emerging patterns. Finally, I reviewed the lesson descriptions and sections of the COI which answered the prompts about culturally relevant mathematics pedagogy for the presence of real-world contexts. I then started with two teachers and analyzed their lessons in terms of adaptations to texts presenting real-world contexts, the instructional role of real-world contexts, the nature of modeling for central real-world contexts, and whether students and the teacher engaged in extended elaboration or not upon aspects of the real-world context. I then extended the coding framework generated from analyzing teachers A and B into the lessons in the other two teachers' classrooms.

Adaptation. I identified whether the real-world context was presented to students in an adapted form or whether it was presented without substantial edits or framing by the teacher. This classification emerged largely from reading the lesson descriptions and noticing that a sizeable number of the real-world contexts came in the form of worksheets, lesson plans, or problems which were cut and pasted directly from some other textual source without editing or modification by the teacher. Frequently, the source of these materials was an internet resource. Although these two categories are less nuanced than Remillard's framework (2005) for thinking about teachers' interaction with curriculum materials, this binary classification at the lesson level is focused not on teachers as the unit of analysis but on features of the lesson, and in particular the materials and language provided to students. The extent of adaptation is significant in terms of the research questions because it directly influences the nature of the real-world contexts with which students engage in the classroom.

Instructional role of context. Next, the **role** of the real-world context was identified relative to the entire lesson. Three subcategories emerged:

- sometimes a real-world context served as a form of *motivating metaphor* for introducing or justifying a mathematical procedure or technique which then became the main topic of the lesson.
- In other cases, a real-world context was the setting for a problem, but was

mentioned briefly or *incidentally* and not further explored in greater depth.

- On other occasions, a real-world context was the *central* setting to which mathematical analysis was applied for the lesson. That is, the main task of the lesson was nominally or substantively about a problem set in a real-world context which required students to engage in mathematical modeling.

Each real-world context was classified as either a *metaphor, incidental*, or *central* in terms of the role of the real-world context in the overall lesson. These distinctions depart from previous studies which have tended to focus on cases where the real-world context was central to the mathematics of the lesson (e.g. Jackson et al., 2011), or to focus on individual "word problems" (Verschaffel et al., 2000). The three categories in my study also loosely correspond to the three most frequently self-reported means of making real-life connections reported by Gainsburg (2008): teacher explanations, story problems, and lab or project, respectively.

The *motivating metaphor* role of contexts was compatible with the "Do Now" or "Warm-up" problems which are part of the "Five-Step" or "Workshop" lesson planning template in the New York City Department of Education. In these cases, an introductory prompt made reference to a real-world situation or problem, but focus on this context was not sustained for the remainder of the lesson. For example, in a lesson on translating literal expressions into algebraic expressions, teacher C opened the class by asking them how they would translate words such as "hola" in Spanish and "bonjour" in French (COI, 9/29/09). The rest of the lesson then moved into "translating" literal expressions into algebraic ones, following this general idea of translation with an idea of word-for-word translation cutting across the entire lesson.

For the other two kinds of roles, a real-world context was directly involved in how a mathematical problem was set up or phrased. The difference between *incidental* and *central* roles was how those real-world contexts were subsequently treated within the class. A real-world context was considered to have an *incidental* role in the lesson if it was not directly addressed in the rest of the lesson. For example in teacher C's class there were exercises involving data about a variety of subjects, such as smoking and household income. Students were to calculate cumulative frequencies, but the actual contexts of the problems were not further discussed or mentioned during the class (COI, 9/27/10). While it is possible that these real-world contexts provided some level of support to students in completing the problems, the lack of further exploration or explicit discussion makes the actual real-world context not essential to the completing of the task as implemented by students. The other difference between incidental and central as categories was the number involved: it would be possible to have multiple incidental real-world contexts in a lesson, but for a real-world context to be in a central role, it would, by definition, be the single focus of a lesson.

In other lessons, problems were set in real-world contexts that were explored in greater depth and in which students were asked to answer questions. For example, in teacher B's class there was a consecutive pair of lessons about the motion of projectiles. Because the questions were more closely framed in terms of the problem and there was extended classroom discussion of the problem context, these were categorized as *central* contexts for the two lessons in question (COI, 2/8/10 and 2/9/10). For a real-world context to be central, it had to be the main mathematical focus of the lesson, a situation described in other studies such as Jackson et al., (2011) as a "problem-solving situation."

Modeling for contexts with central role. For each context which was classified as having a *central* role in the lesson, the nature of the mathematics used to analyze the context was classified in terms of:

- whether the model or approach was *developed* by or with students as a means to solve the problem.
- whether the model or approach to the problem was *given* to student as a procedure to apply.

The following two examples illustrate the two different kinds of modeling. In a pair of lessons about projectile motion, students were initially asked in teacher B's class to sketch what they thought the flight path of a thrown object would look like. Most of the sketches that students produced were semicircular in shape. Without further discussion of the differences between a semicircular arc and a parabolic one, students were then given either a graph of a quadratic function describing the motion of that object or the equation of that quadratic function (COI, 2/8/10 and 2/9/10). Because they were given an algebraic model without further discussion of the meaning of the various parameters, students' work on the problems consisted of substituting values or applying the formula for the location of the vertex as instructed by the teacher. While the lesson was indeed set in a real-world and perhaps even relatable context, there was little room to develop mathematical meaning or connections between the model and the situation (COI, 2/8/10 and 2/9/10).

By contrast, in a lesson in teacher C's class, in the context of a problem about the water consumption for the storage tanks of two families, students were given initial values and daily consumption rates in a standard story problem setup. Students then needed to make their own algebraic models and graphs in order to solve the problem for when the two families would have the same amount of water in their tanks (COI, 2/9/10). Students chose to use technology in the form of graphing calculators to create graphical models and compare these to the given data in terms of the slope and y-intercept of the graphs, and in whole-class discussion compared and interpreted the x-intercepts in the context of the problem as when the families would run out of water. These two examples demonstrate the difference between models that are *given* and those that are *developed* by students as they solve a problem.

Elaboration or non-elaboration. Lessons including real-world contexts were also classified by whether or not those contexts were explored in some greater depth within the classroom. Elaboration considers whether a real-world context is explained, interpreted, paraphrased, or expanded upon within the class. As a concept, it is closely related to the kinds of broad participation which Civil (2002) describes accompanying the inclusion of certain geometrical contexts familiar to students' everyday experiences. Elaboration is further compatible with either the paradigmatic or narrative modes of deepening understanding and discussion of a real-world problem context (Chapman, 2006).

- *Elaborated contexts* were, during the course of the lesson, the extended subject of discussion, exploration, explanation, or interpretation.
- *Un-elaborated contexts* are those for which there was no further exploration.

Quantitative analysis of differences in cognitive demand, instructional environment, and student participation distribution. The five subquestions of the first research question produced a classification of all of the lessons in the sample. Thus, the lessons were grouped into:

1. Those with and without real-world contexts.
2. Those with low levels of teacher adaptation of the real-world context and those with high levels.
3. Those where the role of the real-world context was metaphoric, incidental or central.
4. Of those lessons in which the role of the real-world context was central, those in which students developed mathematical models or approaches and those where the

mathematical model was given to students.

5. Those with and without elaboration of the real-world contexts.

To answer the second research question about differences between groups of lessons as outlined above, differences in three kinds of dependent variables were analyzed:

a) Level of cognitive demand,

b) Ratings of instructional environment,

c) Student participation distributions.

Cross-tabulation of cognitive demand with groups of lessons. The original Mathematical Tasks Framework (Stein et al., 2000) has a four-point scale for rating the cognitive demand of a mathematical task, ranging from low-demand levels including *memorization* and *procedures without connections* to high-demand tasks including *procedures with connections* and *doing mathematics*. Among the 71 observed lessons, however, only one lesson was rated as *doing mathematics* and only one lesson was rated as *memorization*. Two additional lessons were rated as *other* but at an overall low level of cognitive demand. Thus, for the purposes of this analysis, the cognitive demand of lessons is characterized as either high or low.

The cognitive demand was then cross-tabulated with the groups of lessons based on presence/absence of real-world contexts, adaptation by teacher, and elaboration (presence or absence), as well as the nature of modeling for central contexts (developed or given). For cross-tabulations that correspond to 2x2 tables, Fisher's exact test is an appropriate measure of the statistical significance of associations, because it corrects for continuity and compensates for both the small sample size and the small number of degrees of freedom. Fisher's exact test thus determined whether or not there was a statistically significant association between cognitive demand and the following dichotomous variables:

- Inclusion of real-world contexts (present versus absent)
- Level of adaptation (high versus low)
- (Among real-world contexts in a central role) Type of modeling (developed versus given)
- Elaboration (elaborated versus unelaborated)

In the case of role, because there were three possible roles (motivating metaphor, incidental, and central), a chi-squared test of independence at the 0.05 significance level was used instead, with a total of two degrees of freedom.

T-tests for instructional environment and participation categories. To compare the extent to which the different groups of lessons, as outlined above, were associated with different ratings of instructional environments and distributions of participation categories, I conducted independent samples t-tests across different groups of lessons to determine if there are statistically significant differences in the means of the ratings of instructional environment and the relative frequencies of participation categories. In each case, Levene's test for the equality of variances was first performed at the 0.05 significance level. In the case that the null hypothesis that the variances of the two groups were equal was rejected, suitable adjustments to the number of degrees of freedom were made to the t-test and are reported in the results. Specifically, the mean instructional environment ratings and the mean relative frequencies of student participation categories were compared for the following groups of lessons:

1. Lessons with real-world contexts compared to those without.

Within lessons with real-world contexts:

2. Level of adaptation: high versus low
3. Pairs of instructional roles (metaphor, incidental, or central). That is:

* Metaphoric versus incidental

* Metaphoric versus central

* Incidental versus central

4. Within lessons where contexts were central, those which had models given to students versus those that students developed.

5. According to elaboration: those lessons which were elaborated versus those which were not.

Table 4, below, summarizes the design for the quantitative comparison of the instructional environment and student participation variables.

<table>
<tr><td colspan="4">Table 4. Summary of Design for Independent-Samples t Tests.</td></tr>
<tr><td rowspan="2">Grouping Dimension</td><td colspan="2">Independent Variables</td><td rowspan="2">Dependent Variables</td></tr>
<tr><td>Group 1</td><td>Group 2</td></tr>
<tr><td>Presence of real-world contexts</td><td>Context present</td><td>Context absent</td><td rowspan="7">Ratings of Instructional Environment
▪ Discourse and Communication
▪ Intellectual Support
▪ Depth of Student Knowledge
▪ Engagement
Relative Frequency of Participation Categories
▪ Housekeeping
▪ Listening to Teacher
▪ Listening to Students
▪ Discussing
▪ Investigating
▪ Writing/Reading/Reflecting
▪ Using Technology
▪ Practicing
▪ Active Participation (listening to students, discussing, investigating, writing, technology)</td></tr>
<tr><td>Textual Adaptation</td><td>Unadapted</td><td>Adapted</td></tr>
<tr><td rowspan="3">Role</td><td>Metaphor</td><td>Incidental</td></tr>
<tr><td>Metaphor</td><td>Central</td></tr>
<tr><td>Incidental</td><td>Central</td></tr>
<tr><td>Modeling</td><td>Given</td><td>Developed</td></tr>
<tr><td>Elaboration</td><td>Elaborated</td><td>Not elaborated</td></tr>
</table>

These t-tests provide a quantitative analysis of differences between lessons with different features concerning the treatment of real-world contexts that complements and expands upon the qualitative analysis and descriptions developed in the framework for categorizing lessons. Focusing on the variables that result in statistically significant differences can in turn pinpoint the features of lessons that could potentially have the greatest impact on the instructional experiences provided to students in classroom lessons.

Limitations of the Study

Although this study incorporates a total large number of classroom observations and a large number of observations per teacher, the representativeness of this sample poses a few limitations. First, because the time span of this project spans two separate school years, a variety of courses, as indicated in Table 2, is actually represented, and this may influence the frequency with which real-world contexts are included. Second, although the observations were scheduled to capture a broad swatch of lessons in consecutive clusters across the school year, this sampling does not guarantee that the lessons are representative of everyday practice in those classes, or practices specific to particular mathematical topics. Third, these observations were conducted in connection with the CTMUY project, which did not have the inclusion of real-world contexts as its sole or primary focus. Further, because the schedule of classroom observations is arranged in advance and in coordination with teachers, it is possible that they may have tried to present lessons which reflect greater than normal quality or effort.

In terms of the dependent variables in this study, the different subject areas, courses, and grade level at which different teachers taught across different years makes it impossible to use a single measure of student achievement, or to control for students' prior achievement in different areas of mathematics, beyond measures of eighth grade mathematics achievement which are lagged differently depending on students' current grade levels. The three dependent variables in this study are thus directed at the processes that unfolded within classroom lessons but cannot necessarily be compared with other studies where the focus has been on student achievement drawing on standardized assessments.

In terms of the sample selection, the teachers in this study, although committed to the project in the general, may not be generally representative of classroom practice, and there are no baseline measures with which to compare their practices before the professional development intervention and after. Further, differences in practice may be closely associated with differences in teacher beliefs, knowledge, or practices, but the chosen emphasis of this study is to focus on the teaching practices, rather than the characteristics of teachers (Hiebert & Grouws, 2007; Hiebert & Morris, 2012). Despite this qualification or caveat, it may be that different practices are differentially accessible or possible for teachers to consistently implement. That is, there may still be other variables which mediate the delivery of the target practices of this study which could pose additional challenges or obstacles to implementation or professional development.

Chapter 4

Results

This chapter is organized into six sections. Each of the first five sections is divided into two parts, corresponding to the two research questions. To answer the first research question, I report relative frequencies of the different groups of lessons. Then, I present a detailed interpretative analysis of emerging themes and subcategories. The relative frequencies answer the question of *to what extent* or *if* certain features were present. The descriptive analysis provides a preliminary account of *how or in what ways* those features unfolded in the classroom and what functions they served within the lesson.   The second part of each of the first five sections answers the second research question and is a quantitative comparison of three kinds of measures: a) cognitive demand; b) instructional environment; c) the distribution of student participation categories. For each section I outline implications for classroom practice and in-service teacher professional development.

The first five sections of this chapter are as follows. First, I analyze whether or not mathematical lessons included real-world contexts, with attention to the mathematical topics. Second, I analyze the extent to which the real-world contexts were presented in an adapted form compared to how they were given by texts or textbooks. Third, I consider the role that the real-world context plays within the mathematical lesson (metaphoric, incidental or central), relative to the main mathematical task. Fourth, among the real-world contexts which served a central role, I describe how mathematical models were either developed by students or given to them to apply. Fifth, I present how often real-world contexts were elaborated and how this elaboration was actually carried out. Throughout, I analyze how new categories are related to previous ones, such as how the degree of adaptation is related to the three different roles identified for real-world contexts. Finally, in a sixth section, I cross-tabulate each of these findings by teacher to look for patterns of correlation which might suggest that teacher characteristics or practices might significantly affect the other dependent variables.

Section 1: Inclusion of Real-World Contexts

Of the 71 lessons observed in total (18 lessons for each of 4 teachers), 37 (52%) of them included real-world contexts. This percentage is in line the 57% Jackson et al. (2011) reported, but somewhat larger than the 22% found by the TIMSS video study (Hiebert et al., 2003). One possible caveat is that Jackson et al. explicitly solicited teachers to include lessons with "problem solving situations" in which the role of the real-world context was central to the lesson. The Hiebert et al. analysis was also focused on the individual problems covered in eighth grade classes, while by contrast this study is focused at the lesson level. On the other hand, this relative frequency would still indicate that in nearly half of all mathematics lessons there are no real-world contexts included at all.

Table 5 presents a tabulation of the mathematical topics associated with the thirty-seven lessons in which there were real-world contexts. These topics reflect the distribution of courses among the teachers in the study. Algebra is the common subject, and although quadratic functions are the most common topic within algebra, the other three topics of linear functions, equations, and systems of linear equations are all closely connected and part of the Integrated Algebra curriculum.

Other lessons covering these topics did not include real-world contexts. On the other hand, there were some topics for which the lessons never included real-world contexts: graphing of functions, both linear and quadratic; algebraic manipulations, including square roots and rational expressions; and geometric proofs.   Graphing of functions was treated largely in the

Table 5. Real-Word Contexts by Mathematical Subject and Topic

Subject	Topic	Frequency
Algebra	Quadratic functions	7
	Linear functions	6
	Writing and solving equations	4
	Systems of linear equations	2
Geometry	Area and volume	5
	Triangles	2
Probability & Combinatorics	Probability	4
	Combinatorics	2
Statistics	Histograms	4
Set theory	Union, intersection, and complement	1

context of describing changes or relationships among different graphs as related to parameters in the algebraic equation, using the language of the representation but with no connections to the real world. Similarly, algebraic manipulations were treated as procedures. Geometric proofs, a genre specific to school mathematics, were also unconnected to the real world.

One way to understand the presence or absence of real-world contexts within mathematics lessons is to analyze pairs of consecutive lessons on the same topic taught by the same teacher. The purpose of these pairings is to illustrate that in connected lessons the differences between lessons were not due to the presence or absence of real-world contexts. In each of these three pairs, there is both a topic connection as well as similarities in how student participation is distributed. The level of cognitive demand is also similar for these lessons.

For example, teacher A taught a pair of lessons on finding the area of composite figures which emphasized that students could use a variety of strategies in these problems. In the first lesson, students worked at first reproducing given figures using Geoboards. A Geoboard is a tool consisting of a plastic board on which there are short plastic rods jutting up vertically, arranged in a regular grid pattern. Students stretch rubber bands around these rods to create geometrical shapes, mostly polygons. The teacher then selected students to present their strategies and then they presented different strategies to the whole class (COI, 9/30/10). Students' strategies included both additive methods, where shapes were decomposed into smaller and simpler shapes, and subtractive methods, where shapes were formed by cutting pieces off of larger shapes.

In the lesson that followed, the teacher provided the class with an outline map of various neighborhoods in Brooklyn and had different groups of students compute the area of different regions of Brooklyn in comparison with the community of Harwood (COI, 10/1/10). In addition to computing the area of various polygons drawn out on grid paper, students needed to also scale those quantities to what they would be in the real-world setting. Teacher A then facilitated a whole-class discussion in which they made different scalar comparisons of regions compared to the home neighborhood of Harwood and compared the class estimate with the actual figure for the total area of Brooklyn, reflecting as a class on possible sources of error. This pair of lessons illustrates one way that mathematical procedures and concepts can be applied to analyze a local and relevant real-world setting.

A pair of lessons from teacher C's class illustrates how real-world contexts can partially support student understanding. In the first lesson of the pair, the main topic was to write the equation of a line in slope-intercept form given two points (COI, 3/2/11). These problems were completed in a presentation led by the teacher in which he referred to and applied the algebraic formula for slope and the slope-intercept form of the equation of a line, but provided little interpretation of what slope might mean in a real-world context. In the second lesson, students were given tables about related variables and they were able to fill in missing values, citing how the numbers in one column "goes by" a quantity equal to the slope (COI, 3/3/11). These two lessons illustrate how posing a problem in terms of a real-world context and using a representation that includes a display of data can provide students with a means to interpret a mathematical concept in terms of that context.

In the direction of decontextualization, there was a pair of consecutive lessons in teacher D's class in which the first lesson was set in a real world context, but the second did not refer extensively to that context (COI, 11/15/10 and 11/16/10). The first lesson was about a rancher attempting to maximize the area enclosed by a rectangular fence with a fixed perimeter, and students engaged in discussion with their teacher about how different fences would look and whether hardware supply stores would actually sell sections of fence in half-foot lengths (COI, 11/15/10). Although the second lesson was a continuation of the same mathematical topic, the problem was largely treated as one about related geometric and numerical quantities rather than a real-world scenario (COI, 11/16/10). Instead of focusing on interpreting the real-world context, the second lesson instead was focused on developing algebraic and graphical representations through algebraic manipulation.

The different measures of cognitive demand, instructional environment, and student participation distributions for these three pairs of lessons are presented in table 6. These pairs of lessons illustrate on a small, qualitative scale that within the same instructional sequence there were not large differences in cognitive demand, instructional environment, or student participation distribution that could be directly attributed to or associated with the inclusion or omission of real-world contexts. In particular, in terms of the ratings of mathematical discourse, there are examples where a lesson including real-world context had a higher rating (teacher C, 11/15/10 versus 11/16/10) but also where there was a lower rating in a lesson including a real-world context (teacher A, 10/1/10 versus 9/30/10).

Table 6. Pairs of Consecutive Lessons Including Real-World Contexts or Not

	Teacher					
	A		C		D	
Topic	Area of Polygons		Equations of lines		Maximizing area	
Date	09/30/10	10/01/10	03/02/11	03/03/11	11/15/10	11/16/10
Real-World Context	No	Yes	No	Yes	Yes	No
Cognitive Demand	High	High	Low	Low	Low	Low
Instructional Environment						
Discourse	4	3	3	3	3	2
Intellectual Support	4	4	2	3	3	3
Depth of Knowledge	4	3	2	2	2	2
Engagement	4	4	3	3	3	3
Participation Categories						
Housekeeping	0%	0%	13%	15%	0%	6%
Listening to teacher	0%	0%	4%	4%	0%	0%
Listening to students	35%	0%	0%	0%	0%	0%
Discussing	4%	38%	17%	17%	50%	42%
Investigating	0%	31%	0%	0%	50%	52%
Writing	10%	0%	0%	0%	0%	0%
Technology	50%	0%	0%	0%	0%	0%
Practicing	0%	31%	67%	65%	0%	0%
Active	100%	69%	17%	17%	100%	94%

Quantitative comparison: One difference in student participation. Overall, there was only one statistically significant difference in terms of the cognitive demand, ratings of instructional environment, and student participation distributions based upon whether or not real-world contexts were included in the 71 observed lessons. Lessons which did not include real-world contexts included more time spent listening to students. These results overall suggest that the mere inclusion or omission of real-world contexts is not, by itself, an influential factor in level of cognitive demand, instructional environment or student participation. This is discussed in the sections that follow.

Measure A: No differences in cognitive demand. Overall, the inclusion of the real-world contexts was not associated with the level of cognitive demand. Of the 71 lessons which were observed in total, 18, or 25%, of lessons were rated as having high cognitive demand. Lessons of high cognitive demand were equally represented among lessons which did include real-world contexts (11 out of 41, or 27%) and those which did not include real-world contexts (7 out of 30, or 23%). A complete cross-tabulation of cognitive demand by presence of real-world contexts is presented in table 7. There was no association between the inclusion of real-world contexts and the level of cognitive demand, with Fisher's exact test returning p=0.79.

Table 7. Cognitive Demand by Presence of Context

	Cognitive Demand		
Real-World Context	High	Low	Total
Present	10	27	37
Absent	8	26	34
Total	18	53	71

Measure B: No differences in instructional environment. Table 8 displays the mean and standard deviation of instructional environment ratings for lessons with real-world contexts compared to those without. Across all four of the ratings of instructional environment, lessons *without* real-world contexts rated slightly higher, but these differences were not statistically significant.

Table 8. Instructional Environment by Presence of Real World Context

Rating	Real-World Contexts Yes (n=37)	Real-World Contexts No (n=34)	t	df
Mathematical discourse	2.30	2.62	-1.449	69
	(0.88)	(0.99)		
Intellectual Support	2.81	3.18	-1.797	69
	(0.67)	(0.80)		
Depth of Knowledge	2.03	2.18	-0.849	69
	(0.80)	(0.67)		
Engagement	2.54	2.79	-1.245	69
	(0.87)	(0.85)		

Note: for all tables reporting means, figures in parentheses represent the corresponding standard deviations.

Measure C: More time spent listening to students in lessons without real-world contexts. Table 9 displays the relative frequencies of participation categories compared across whether or not those lessons contained real-world contexts. Each of these percentages represents the average share, expressed as a percentage, of class time that was spent with student engaging in a participation category.

There was only one difference in the distribution of student participation categories when lessons which included real-world contexts were compared to those that did not. In lessons including real-world contexts, 2% of class time was spent listening to students, while in lessons without real-world contexts, 6% of class time was spent listening to students. This difference was statistically significant, $t(42)=-2.049$, $p=0.047$. In part, the prevalence of listening to students as a participation structure in lessons without real-world contexts may be due to the practices described above in the pair of lessons from teacher A's class concerning the area of figures and displayed in table 6. In the first lesson, over one third of the class period was spent listening to student presentations about their strategies for finding the areas of different shapes

Table 9. Participation Structures by Presence of Real-World Contexts

	Real-World Contexts			
Participation Category	Yes (n=37)	No (n=34)	t	df
Housekeeping	0.04 (0.06)	0.04 (0.06)	0.464	69
Listening to Teacher	0.21 (0.21)	0.15 (0.16)	1.422	69
Listening to Students	0.02 (0.04)	0.06 (0.11)	-2.049*	42
Discussing	0.17 (0.18)	0.22 (0.28)	-0.245	43
Investigating	0.22 (0.24)	0.22 (0.28)	-0.102	69
Writing/Reading/Reflecting	0.02 (0.07)	0.03 (0.09)	-0.559	69
Using Technology	0.01 (0.02)	0.02 (0.09)	-0.915	69
Practicing	0.31 (0.28)	0.29 (0.31)	0.217	69
Active Participation	0.42 (0.36)	0.45 (0.36)	0.318	69

Note: *=p<0.05

(COI, 9/30/10). By contrast, the second lesson had more discussion but no time spent on extended student presentations (COI, 10/1/10). The student presentations in the first lesson focused on mathematical strategies and procedures rather than real-world contexts or the connections between mathematical procedures and real-world contexts. Unpacking these connections with the real-world was more typically handled as discussing.

A more detailed analysis of the distribution of the participation category of *listening to students* shows that this category occurred (had a nonzero number of minutes during the lesson) in five out of thirty-seven lessons (14%) including real-world contexts but in eleven out of thirtyfour lessons (32%) that did not include real-world contexts. The participation category of "listening to students" was associated with lessons *not* including real-world contexts in two ways. First, more lessons is both on the level of the number of lessons in which that category is present. Second, the average percentage share of class-time spent listening to students was higher in lessons without real-world contexts.

Discussion and summary. The three quantitative results of 1) no difference in cognitive demand 2) no difference in instructional environment, and 3) no difference in student participation between lessons with and without real-world context (with the exception of more student presentations and listening to students in lessons without real-world contexts) suggest that the inclusion or omission of real-world contexts by itself does not relate to cognitive demand, ratings of instructional environment, or student participation. High cognitive demand in lessons without real-world contexts could be achieved by making connections between different mathematical representations such as graphs, tables, and equations. Similarly, aspects of instructional environment which include discourse, intellectual support, depth of student knowledge and student engagement could be achieved regardless of whether lessons included real-world contexts. The only difference in participation categories was in listening to students, as a number of lessons without real-world contexts had extensive student presentations of solution methods to mathematical problems.

One further implication of these findings is that factors related to actual treatment of the real-world contexts within the unfolding of the mathematics lesson needs to be more closely analyzed. Additional factors influential in shaping difference in cognitive demand include the nature of adaptation, role, mathematical modeling, and elaboration detailed in the following four sections. The remainder of this chapter only includes the 37 lessons with real-world contexts.

Section 2: Level of adaptation

This section begins with a qualitative analysis of features of the low and high levels of adaptation that were observed in how real-world contexts were presented to students. This analysis focuses largely on the textual form with which students were presented real-world contexts and any additional framing that supplemented given information about the real-world situations and associated mathematical problems. The second part of this section details differences based upon the level of adaptation on the three kinds of measures related to the second research question: cognitive demand, instructional environment, and student participation.

Low level of adaptation. Of the 37 lessons which included real-world contexts, 17 lessons (46%) had the real-world context presented to students with a low level of textual adaptation. Previous studies of teachers' use of curriculum have focused on teachers' enactment or engagement with Standards-based reform curricula, such as the Connected Mathematics Project (Choppin, 2011), or the Investigations curriculum (Remillard & Bryans, 2004). By contrast, although New York City previously had a standardized text, Prentice Hall Mathematics A, many of the lessons observed in this study were taken from the internet. This finding may also be contrasted with secondary teachers' self-reported behavior as reported in Gainsburg (2009). On that survey, a large majority of secondary teachers (46 out of 55) claimed that they supplied or invented the real-world contexts and connections that were present in their lessons.

There were two forms in which real-world contexts were presented to students with low levels of teacher adaptation. In the first form, entire lesson plans, handouts, or worksheets were copied wholesale from internet resources. In the second form, problems were cut and pasted from textbook sources and photocopies of these compilations were produced and distributed to students. Three themes emerged from closer reading of these seventeen lessons in which teacher adaptation of the material was low.

1) Low quality planning
2) Ignoring contexts seeming to be of little relevance to students
3) Predetermining how students approach the problem.

Low quality planning. Use of real-world contexts presented with low levels of adaptation in the form of photocopies was correlated with lack of adequate planning for lessons. For example, in one lesson, teacher C distributed a worksheet taken directly from an internet source during a lesson on linear functions (COI, 3/3/11). Upon reading one of the data tables which students were supposed to model using a linear function, teacher C exclaimed, "I've been bamboozled again!" as he told students to skip that exercise because the equation would need to be quadratic.

In other lessons, the use of the instructional materials reflected teachers' lack of understanding of the mathematics presented by the question and the implicit instructional plans. In a lesson in teacher B's classroom, students were given a photocopied worksheet problem about the graph of the height of an arrow as related to a camera that was taking photos of it every half second, with the positions at those moments represented by points on the graph (COI, 2/8/10). The worksheet also included a sequence of photocopied questions. This line of questioning began with asking the height of the arrow when the camera takes its first picture. The intention of the sequence of questions was to have students realize that the arrow returns to that initial height at 4.2 seconds, even though that is not a plotted point on the graph, because the plotted points represent different moments that the camera took a snapshot of the arrow. As implemented in the classroom, this question was skipped in the whole-class discussion. The teacher also interpreted the first question about the initial height to be about reading the value of the graph at 0.5 seconds, incorrectly telling students that the value of the height represented by that point is 32 m, when in fact it is 34 m. Teacher B then led the class discussion toward settling upon the maximum height occurring at 2 seconds and a height of 46 meters, even though the graph and symmetry would indicate that the maximum occurs at 2.1 seconds.

Taken together, these examples suggest that in many cases, photocopied lessons were not read carefully with clear instructional goals in mind, or that the intended purpose of the curriculum materials was changed by the teacher in the course of the lesson due to a mismatch in understanding. In this sense, the implementation of the materials was a form of adaptation, but the textual form in which students received the materials was not altered in advance to better coordinate with the teacher's actual goals or implementation.

Ignoring irrelevant contexts. A second pattern that emerged with these low-adaptation real-world contexts was their low relevance to students. One example is from a worksheet distributed during a lesson in teacher B's class which was taken directly from a USA Today lesson plan about the in-state tuition rates in 2002-3 in various states (COI, 9/29/10). This data was not only out of date but would not be relevant to students living in the state of New York. A similar example was in teacher C's class, in which he had taken data from a curriculum resource available on-line and changed the names of the locations to be local to the state of New York and the community of Harwood (COI, 9/27/10). Other examples which arose in problems involving linear regression or models using quadratic functions included textbook problems involving the production of gold in Ghana, the shape of the Water Arc monument, eruptions of Old Faithful, and household income in the 1980s. Of particular note in these cases is the potential of these sources to serve as real-world data and opportunities to develop statistics or informal inferential reasoning. Instead, these problems tended to be treated in the classroom as if they were purely numerical problems without either interpreting the context to reach a preliminary mathematical model or interpreting the meaning of a constructed mathematical model back into the real-world context.

Predetermining student approaches. A third potential effect of using curriculum resources with little adaptation was to predetermine how students would approach the problem. In the problem about college cost data described above in teacher B's class, the handout dictated that students would find the mean cost per state as well as providing the intervals for the histograms in advance (COI, 9/29/10). These printed instructions determined how students would approach the problem and removed the potential complexities, such as adjusting or weighting the mean based upon based upon attendance within that state.

Similarly, in a lesson about the motion of a quadratic projectile in teacher D's class, there was a pre-printed set of questions taken from the textbook that outlined a given approach to analyzing the situation through a fixed sequence of representations (COI, 10/7/10). This sequencing of questions was similar in its reliance on a textbook as a source to how teacher B treated this topic, as described in the problem about the flight of an arrow (COI, 2/8/10).

One last example from teacher B's class was a handout describing the "Game of Beano" which was taken directly from an internet source. In this lesson, students were to play a game in which they first placed a given number of beans on a game board labeled with numbers from 2 to 12. As students took turns rolling pairs of dice, they could remove one bean from the number corresponding to the sum of the values on the dice. The goal of the game was to be the first player to remove all beans. The handout distributed to students contained a fixed sequence of representations, including a two-dimensional addition table which students were instructed to fill out (COI, 5/5/11). According to the instructions on the worksheet, students were then to fill out a bar chart indicating the theoretical probability distribution of the sums. In this case, the worksheet predetermined how students would approach the problem as they worked in pairs. In particular, the worksheet omitted the sum of 1, which is not possible to get as the sum of the values rolled on two dice. Students were thus not given as much of an opportunity to explore or express the misconception that that sum is a possible outcome. Although one student actually mentioned this possibility in this discussion, the worksheet was set up in order to avoid and not address this potential complication.

High levels of adaptation. The previous lesson can be contrasted with a lesson in teacher A's class which was based upon the same game, but which introduced a different operation, multiplication, to combine the values of the dice and allowed for students to devise their own representations for tabulating the different outcomes (COI, 4/22/10). This task was presented to students through a sequence of teacher-produced overhead slides. In this case, the teacher specified that students would work in pairs but produce their own product, the basic tools they would be given, and a detailed, general outline for the product that they needed to produce. While students had a common outline that the teacher had designated for the work, they had freedom in their representational strategies. Three practices characterized the higher level of adaptation: the lowering of demand or simplification of tasks, the posing of problems phrased by the teacher, and the addition of contextual information not necessarily essential to the mathematics of the task. I comment on each of these below.

Lowering demand. In a lesson in teacher B's class, the original task, photocopied from a textbook, gave two quantities for vertical motion of a firework: the initial height and the initial velocity. Teacher B distributed a handout to the class on which she had written by hand a quadratic equation that incorporated those two quantities (COI, 3/10/11). The difficulty of that task was thus reduced as students only had to substitute specific values or solve equations for a function that was already given to them.

Sometimes lowering the demand of a task could be seen to support students in making other connections. In one lesson in teacher C's class, as he orally told students about a ticket-price problem, teacher C wrote hybrid verbal equations on the board:

Mom:	1 adult movie ticket + 2 child tickets = $ 23
Dad:	1 adult movie tickets + 1 child ticket= $ 17

Teacher C created these numbers on the spot, as he changed the larger dollar amount a couple of times so that the final answers would work out nicely. The form of the problem thus was already compatible with the equations that they would need to solve in the remainder of that lesson, when they were assigned to practice on purely symbolic problems on the worksheet (COI, 5/2/11).

Posing problems. Other problems were posed or given by the teacher. For example in a lesson on multiple representations of word problems, teacher B gave students a problem about hourly wages to represent in terms of tables, equations, and graphs (COI, 11/11/10):

> Example #1: Jamel worked h hours at the Boys and Girls club. He earns $6 an hour. **(Underline the important information in above).**

In another class on operations on sets such as union, intersection, and complement, teacher B posed to students a question about the jersey numbers of the players on a basketball team as an illustration of the operation of complement (COI, 9/17/09).

Broader problems were posed in teacher A's classes. For example, in two separate lessons in different school years, teacher A posed the task of using a single letter-sized (8 ½ x 11) sheet of cardstock to create a container which would have the greatest volume as measured by the total amount of rice that it could contain. There were, however, variations across the two instances of this lesson in how this task was given to students. In the first version of the lesson, students were told to work collaboratively in groups in order to plan and then create a container (COI, 12/15/09). The task that was presented to students as a PowerPoint slide is below:

Task: Using ONLY an 8.5x11 sheet of paper, tape, and scissors, plan and then build a model of a container that will hold the most ______. Group members please note your container must have a lid or top. It can't have an "open" side.

In the second version of the lesson, each group received a type of solid including a small plastic model (such as rectangular prism, cone, or cylinder) and they had to individually create a container that was also that shape, with the goal of being the student in the group with the great volume (COI, 3/4/11).

The basics: Using just one sheet of 8.5x11 inch paper, build a container. The restrictions: * Your container must have a top and a bottom. * If you're sitting at a table with a particular 3-dimensional object "theme" your container must be similar to that one. The challenge: Within your table, construct the container that holds the most rice.

Both of these cases reflect the teacher's planned modifications to this task as well as explicit written instructions about how the work was to be conducted.

Adding information. In another lesson taught by teacher D, the main mathematical topic was permutations, which was introduced through the context of finishers in the Belmont Stakes horse race (COI, 5/6/11). Although this race is in some sense local to New York City, students did not say anything during the whole-class discussion or small group work that suggested that they were familiar with the context of horse racing. Nonetheless, teacher D provided the class, through a PowerPoint slide, the complete list of the top 12 finishers in the latest race. The context of horse racing does provide motivation for permutations because the order in which they finish matters. The additional information supplied by the teacher, however, did not appear to deepen students' mathematical knowledge or the level of the mathematical discourse. Another lesson in teacher D's class, which actually showed a low level of adaptation, also had this approach to supplying additional information, but the material came directly from a textbook resource (COI, 10/8/10)

Quantitative comparison: Lower cognitive demands, lower ratings of instructional environment, and more practicing in lessons including real-world contexts with low adaptation. The foregoing qualitative analysis of the level of textual adaptation is consistent with the of results of quantitative comparisons of lessons with and without textual adaptation in real-world contexts according to the three kinds of measures in Research Question 2 —a) cognitive demand, b) instructional environment, and c) student participation. As we will see, lessons including real-world contexts with low levels of adaptation had lower cognitive demand and lower ratings of instructional environment. In terms of student participation categories, zero time was spent in lessons with low levels of adaptation listening to students, and more time was spent on practicing.

Measure A: Lower cognitive demand in lessons with low levels of adaptation. Among lessons including real-world contexts, 10 out of 37, or 27%, were rated as being at a high level of cognitive demand. By contrast, among lessons with low levels of adaptation, only one out of seventeen, or 6%, was rated at a high level of cognitive demand. Nearly half of lessons, 9 out of 20, or 45%, with a high level of adaptation were rated at a high level of cognitive demand. Table 10 is a complete cross tabulation of cognitive demand by adaptation. The association of low adaptation with low cognitive demand is statistically significant, by Fisher's exact test, p=0.01.

Table 10. Cognitive Demand by Adaptation

	Cognitive Demand		
Adaptation	High	Low	Total
High	9	11	20
Low	1	16	17
Total	10	27	37

The lower cognitive demands of lessons with low adaptation reflect the extent to which the emphasis was on completing mathematical procedures without making connections to the real-world contexts. These real-world contexts were presented as not having possibly problematic features, or were treated without further explicit supports provided by the teacher.

Measure B: Lower instructional environment ratings in lessons with low levels of adaptation. Table 11 presents the comparisons of lessons with high or low level of adaptation by the teacher of an identifiable source text across the four ratings of instructional environment –– discourse, intellectual support, depth of knowledge, and engagement.

The differences in all four categories of instruction environment are compatible with what was observed in the qualitative analysis. Lessons with low levels of adaptation had a lower mean rating of mathematical discourse, t(35)=2.914, p=0.006. This difference is consistent with the predetermined flow of questions as well as the tendency for these lessons to focus on short, closed responses from students to teacher questions. The lower levels of intellectual support in lessons including real-world contexts with low levels of adaptation, t(35)=2.212, p=0.034 is also

Table 11. Instructional Environment by Adaptation

Rating	Adaptation High (n=20)	Adaptation Low (n=17)	T	df
Mathematical discourse	2.65 (0.88)	1.88 (0.70)	2.914**	35
Intellectual Support	3.10 (1.02)	2.47 (0.62)	2.212*	35
Depth of Knowledge	2.35 (0.81)	1.65 (0.61)	2.936**	35
Engagement	2.85 (0.99)	2.18 (0.53)	2.518*	30

Note: **=p<0.01, *=p<0.05

consistent with fewer and narrower opportunities for students to take intellectual risks given the less open nature of the mathematical tasks. As a result, students attained lower ratings for depth of knowledge, t(35)=2.936, p=0.006. Finally, lower ratings of student engagement are evidenced in classes which showed low levels of adaptation by the teacher, t(30)=2.518, p=0.017. This may be due in part to the selection of real-world contexts that were neither relevant to the students' experiences nor bridged to their experiences . In the case of engagement, Levene's test for the equality of variances rejected the null hypothesis that the variances were equal, F(2,35)=11.431, p=0.002, indicating that lessons including real-world contexts and low levels of adaptation had less variation in their overall lower ratings of engagement. That is, lessons including real-world contexts with low levels of adaptation were more uniformly unengaging.

Measure C: Less listening to students but more practicing in lessons with real-world contexts with low levels of adaptation. Table 12 summarizes the mean relative frequencies and standard deviations of the eight student participation categories as well as the aggregated "active" student participation category.

Table 12. Participation Structures by Adaptation

	Adaptation			
	High (n=20)	Low (n=17)	t	df
Housekeeping	0.04 (0.06)	0.05 (0.06)	-0.461	35
Listening to Teacher	0.24 (0.20)	0.18 (0.21)	0.942	35
Listening to Students	0.03 (0.06)	0.00 (0.00)	2.189*	19
Discussing	0.18 (0.18)	0.17 (0.20)	0.126	35
Investigating	0.29 (0.23)	0.14 (0.24)	1.968	35
Writing/Reading/Reflecting	0.05 (0.08)	0.01 (0.02)	2.018	22
Using Technology	0.01 (0.02)	0.01 (0.03)	-0.115	35
Practicing	0.18 (0.20)	0.46 (0.30)	-3.426*	35
Active Participation	0.54 (0.36)	0.32 (0.35)	1.928	35

Note: *=p<0.05, **=p<0.01

On average, more time was spent *listening to students* in lessons with adapted real world contexts (3% of total class time) than lessons where the real-world contexts were presented to students with low levels of adaptation, t(19)=2.189, p=0.041. Indeed, within the 17 lessons in which real-world contexts were presented with low adaptation, not a single lesson had any time spent *listening to students*. The absence of *listening to students* among low adaptation lessons accounts for the statistically significant difference in the relative frequency of *listening to students* between lessons including real-world contexts and those without. Further, students in those lessons with unadapted real-world contexts spent on average nearly half (46%) of the class time practicing, compared to 18% on average in lessons with adapted real-world contexts,

$t(35)=-3.426$, $p=0.016$. The nature of this practicing was on exercises of a similar form but occasionally varying real-world contexts, but which were generally not conducive to extended student presentations.

Discussion and summary. Lessons in which real-world contexts were taken wholesale from another textual source with low levels of teacher adaptation were generally focused on the execution and repetitive practice of procedures without making connections. Since the rating of cognitive demand allows for connections between procedures, concepts, representations, and real-world situations, this association suggests that teachers bear a large degree of responsibility for actually providing supports or other framing to assist students in fully making connections. The impact of this lack of support and planning is reflected on all four measures of instructional environment. Opportunities provided to students to participate in the classroom were also constrained by providing them fewer opportunities to present to the whole class their work, with nearly half of the class time on average spent practicing in lessons with low levels of adaptation.

At the same time, this issue of the level of adaptation of curriculum sources and the means by which teachers create instructional resources in print for use in their lessons extends beyond the inclusion of real-world contexts. Other lessons also showed low levels of adaptation from textbooks or internet test preparation resources. This pattern indicates that focusing on the need for high quality instructional planning in professional development may have farther reaching impact than on the teaching of real-world contexts. At the same time, there may be further knowledge, specific to the teaching of real-world contexts, which is also required in the planning of lesson and the preparation of textual materials for students.

Section 3: Instructional Roles Played by Real-World Contexts

In this section, I analyze and describe three roles which real-world contexts played in mathematics lessons. I also subdivide each of these roles into subcategories and provide descriptive examples of each. Overall, three roles were identified for real-world contexts. First, some real-world contexts were used as the basis for motivating metaphors or explanations which were not necessarily the main focus of mathematical analysis. Second, in other instances real-world contexts played an incidental role in the lesson, where they were either mentioned tangentially and then ignored or mentioned among many other real-world contexts without further depth. Third, sometimes a real-world context was integrated into the central mathematical task of the lesson.

Motivating metaphors. One role that real-world contexts played in lessons was as part of the mathematical motivation or metaphor for the main subject of the class. These cases accounted for seven lessons out of the 37 lessons including real-world contexts, or 19%. These attempts were not always successful. For example, as an introductory "Do Now" exercise in teacher C's class, students were asked which weighs more, a pound of feathers or pound of coal (COI, 9/17/09). In their discussions with each other and the teacher, students had a variety of responses which included all three options: that the feathers were heavier, that the coal was heavier, and that they weigh the same. The whole-class discussion was, however, resolved by the teacher's authority and the rest of the lesson required students to perform the same operation to both sides of an equation to maintain "balance". There was no clear connection made between students' varied responses and the content of the actual lesson.

An example of a more successful form of motivation was in teacher C's lesson on solving linear systems. In this case, teacher C led an extended discussion of a problem involving buying combinations of hamburgers and hot dogs, which he wrote on the board in sentence form. This problem, which drew upon students' reasoning about real-world transactions, served as motivation for the basic operations of adding and scaling equations that students were then asked to perform on systems of linear equations in purely symbolic form (COI, 5/3/11). Other examples of successful motivation are detailed below in the section on elaboration.

Incidental roles associated with practicing on worksheets. Incidental roles were twice as common as motivating metaphors, accounting for 14 out 37, or 38%, of all lessons including real-world contexts. A real-world context included in a lesson was considered incidental if it was part of a problem posed or framed in a real-world context but this context was subsequently ignored and not referred to again as students did the problem. One example of ignored context was in teacher B's class, when students were given a worksheet problem about drawing reasonable lines of fit for scatterplots of the North latitude and April mean temperature of various world cities (COI, 12/10/09). Other than a clarification by the teacher that "mean" refers to average, the teacher did not frame the problem with any further reference to what the problem was about. Students worked on the problem without thinking about what the variables meant and what insight those meanings might provide into the mathematical relationships, the choice of linear model, and mathematical representation. In this case, students could have reasoned that locations with higher North latitude would be closer to the North Pole and therefore tend to be colder with lower mean temperatures in April. This interpretation would have been a real-world prediction of negative correlation. Examples of these subsequently ignored incidental contexts included worksheets where there were multiple contexts that were not further discussed or interpreted, with the different exercises treated purely numerically, as if independent of the stated real-world context. For example, in the line of best fit example given above, the same mathematical procedure (create a scatter plot, draw a line of fit, find the equation of that line, use that line to find some other points for specified values) was applied regardless of the real-world context in which the problem was posed.

Indeed, incidental roles were associated with real-world contexts that had low levels of adaptation. While overall slightly less than half of lessons had real-world contexts presented with low levels of adaptation, among the fourteen lessons in an incidental role, 10 (71%) had a low level of adaptation. Less than half of all real-world contexts in a central role were presented with low levels of adaptation, while none of those in which the real-world context served as a motivating metaphor did so. A complete cross-tabulation is shown in table 13. This association between incidental roles and low adaptation was statistically significant, $\chi^2(N=37, 2)=9.642$, $p=0.008$.

Table 13. Cross-tabulation of Role by Adaptation

	Role			
Adaptation	Metaphor	Incidental	Central	Total
High	7	4	9	20
Low	0	10	7	17
Total	7	14	16	37

The relationship between low adaptation and incidental roles also suggests that in these lessons teachers did not plan the lessons around the real-world contexts, but rather may have inserted them as an additional exercise in the application of a given mathematical procedure without further attention to the particular details of the real-world context.

Central roles. Real-world contexts were in a central role in 16 out of 37 lessons (43%), and were only slightly more common than incidental roles. A more detailed discussion of the role of mathematical modeling within central contexts is the substance of the next section and provides most of the qualitative analysis of central roles. In some cases an entire class period was devoted to the problem in which a real-world context played a central role, while in other cases investigation of that central real-world context extended across multiple lessons (e.g., in teacher D's class the two lessons about projectile motion, COI 10/7/10 and 10/8/10). Approximately half (7 out of 16, or 44%) of real-world contexts in a central role were presented with a low level of adaptation.

Quantitative comparisons: Lower cognitive demands, lower discourse and knowledge ratings, and less active participation in lessons including real-world contexts in incidental roles. Lessons which put real-world contexts in incidental roles had lower cognitive demands, lower discourse and knowledge ratings, and less active forms of participation when compared with lessons which used real-world contexts centrally or as metaphors. Other differences were not consistent for non-incidental roles, with more investigating observed in lessons where the real-world contexts were central compared to when contexts were incidental, and more listening to students when real-world contexts served as a motivating metaphor compared to in incidental roles.

Measure A: Low cognitive demands for lessons with real-world contexts in an incidental role. Overall, slightly more than one quarter (27%, or 10 out of 37) lessons including real-world contexts were rated at a high level of cognitive demand. The percentage of lessons with a real-world context serving the role of a motivating metaphor rated as high cognitive demand was greater, at 56%, or 4 out 7. Lessons with real-world contexts in an incidental role were never rated at a high level of cognitive demand. By contrast, 6 out of 16, or 38%, of lessons with real-world contexts in a central role were rated at a high level of cognitive demand. These results put together indicate that low cognitive demand is associated with incidental roles for real-world contexts. This association was statistically significant, $\chi^2(N=37, 2)=9.294$, p=0.01. Table 14 displays the complete cross-tabulation of cognitive demand by role.

Table 14. Cognitive Demand by Role

	Cognitive Demand		
Role	High	Low	Total
Metaphor	4	3	7
Incidental	0	14	14
Central	6	10	16
Total	10	27	37

The lower cognitive demands of lessons in which real-world contexts played an incidental role reflects the extent to which these lessons did not make extensive connections between the contexts and the mathematics being studied by students.   In addition, the emphasis on repetitive practice which was common to lessons in which real-world contexts were treated as incidental meant fewer connections between mathematical concepts and procedures as well.

Measure B: Lower mathematical discourse and depth of knowledge ratings for lessons with real-world contexts in an incidental role. There were two dimensions of instructional environment — mathematical discourse and depth of student knowledge — for which there were statistically significant differences in mean ratings across different roles. The mean instructional environment ratings and standard deviations across the three roles are displayed below in table 15.

Table 15. Instructional Environment by Role

Rating	Role			Metaphor-Incidental		Metaphor-Central		Incidental-Central	
	Metaphor (n=7)	Incidental (n=14)	Central (n=16)	T	df	t	df	t	df
Mathematical discourse	2.71 (1.11)	1.86 (0.66)	2.50 (0.82)	2.226*	19	0.519	21	-2.345*	28
Intellectual Support	3.00 (0.82)	2.43 (0.65)	3.06 (1.06)	1.752	19	-0.138	21	-1.938	28
Depth of Knowledge	2.57 (0.79)	1.57 (0.65)	2.19 (0.75)	3.114**	19	1.114	21	-2.392*	28
Engagement	2.86 (0.90)	2.21 (0.58)	2.69 (1.01)	1.994	19	0.381	21	-1.538	28

Note: *=p<0.05; **=p<0.01

Lessons in which a real-world context played an incidental role had lower ratings of mathematical discourse than those lessons in which the real-world context served as a metaphor, t(19)=2.226, p=0.040, and those in which they were part of the central mathematical task, t(28)=-2.348, p=0.026. This lower rating reflects the lack of discussion to communicate and negotiate mathematical meanings in lessons where the real-world context was not deeply addressed, with the real-world contexts being only incidental to the main mathematical discussion. Similarly, the depth of knowledge attained in lessons where the real-world context was in an incidental role was lower than in lessons in which the context served as a motivating metaphor, t(19)=3.114, p=0.004, and lessons in which real-world contexts were central to the mathematics lesson, t(28)=-2.392, p=0.024. This lower mean rating reflects how students did not make connections between the real-world context and the mathematical topic.

Measure C: More practicing and less listening to students, investigating, and active participation in lessons with incidental real-world contexts. Table 16 displays the mean relative frequencies and standard deviations of the eight student participation categories and the aggregated category of "active" student participation. Overall there were five statistically significant differences in four different participation categories, and in each case these differences involved lessons with real-world contexts in incidental roles compared to those which served central roles or as motivating metaphors.

Table 16. Participation Structures by Role

	Role			Metaphor-Incidental		Metaphor-Central		Incidental-Central
Rating	Metaphor (n=7)	Incidental (n=14)	Central (n=16)	t	df	t	df	t
Housekeeping	0.06	6.00	0.02	0.134	19	1.265	7	1.900
	(0.08)	(0.07)	(0.03)					
Listening to Teacher	0.25	0.21	0.19	0.405	19	0.739	21	0.291
	(0.14)	(0.24)	(0.21)					
Listening to Students	0.05	0.00	0.01	2.497*	10	1.579	21	-0.933
	(0.05)	(0.00)	(0.05)					
Discussing	0.15	0.11	0.24	0.545	19	-1.052	21	-1.927
	(0.14)	(0.13)	(0.22)					
Investigating	0.21	0.07	0.35	1.726	9	-1.334	21	-3.908**
	(0.20)	(0.14)	(0.26)					
Writing/Reading/Reflecting	0.07	0.01	0.03	1.443	7	1.004	21	-1.066
	(0.10)	(0.03)	(0.07)					
Using Technology	0.01	0.01	0.00	0.503	19	1.000	6	1.000
	(0.04)	(0.03)	(0.00)					
Practicing	0.19	0.54	0.17	-3.380**	19	0.474	21	4.715**
	(0.21)	(0.22)	(0.23)					
Active	0.48	0.19	0.63	1.762	8	-0.968	21	-4.192**
	(0.40)	(0.24)	(0.32)					

Note: *=p<0.05, **=p<0.01

In terms of the participation category of practicing, lessons in which real-world contexts functioned in an incidental role had more practicing (54% of total class time, on average) when compared to 19% of class time in lessons with real-world contexts as a motivating metaphor, $t(19)=3.380$, $p=0.003$, and 17% of class time in lessons with contexts in a central role, $t(28)=4.715$, $p=0.005$. These differences are in line with the tendency of lessons using real-world contexts incidentally to focus on practice on largely worksheet problems which were treated as unconnected to each other.

When compared to lessons in which real-world contexts were central, lessons which used contexts incidentally contained less investigating in incidental (7%) compared to 35% of class time in lessons with real-world contexts in central roles, $t(23)=-3.908$, $p=0.001$. This difference reflects how central real-world contexts allowed for more investigating of numerical patterns, connections among representations, or connections with the real-world context. Although there also was more investigating (21%) in lessons where real world contexts served as a motivating metaphor, this difference was not statistically significant. More time was spent listening to students (5%) in those lessons with real-world contexts as a motivating metaphor. By contrast, listening to students was not at all present in lessons with incidental real-world contexts, and this difference was statistically significant when compared to lessons in which real-world contexts were part of a motivating metaphor, $t(6)=2.497$, $p=0.047$.

Discussion and summary. Analysis of the roles that real-world contexts play within mathematics lessons leads to two different conclusions. First, incidental roles were associated with both low levels of adaptation as well as lower cognitive demand and instructional environment ratings. These associations can be explained by considering how much of the class time was spent on the repetitive practice of predetermined mathematical procedures on a variety of problems without a need to consider the real-world context in which those problems were set. Because these problems were selected precisely because a targeted mathematical procedure would work, there was very little intellectual work for students to actually do. Further, when compared to lessons in which real-world contexts served in a central role, lessons in which real-world contexts were incidental had less investigating.

A second conclusion is that motivating metaphors have potential as a meaningful means to include real-world contexts, comparable to that of directly analyzing or mathematizing those real-world contexts in a central role. This approach to incorporating real-world contexts offers an alternative to a problem-based approach in which students solve word problems set in real-world contexts. This role for real-world contexts provides instances of what Mosvold (2008) has characterized as a typical element of Japanese lessons in which real-world connections are made in a non-problem setting, but the subsequent mathematical focus of the lesson is more abstract. Motivating metaphors did not differ significantly from lessons which put real-world contexts in a central role, but did have higher levels of cognitive demand and ratings of mathematical discourse and depth of student knowledge than lessons which used real-world contexts incidentally.

Taken together, both of these conclusions suggest that if real-world contexts are to be included in mathematics lessons, they need to be addressed directly and clearly by the teacher and the class. There is, however, more than one general approach. By also having real-world contexts serve as motivating metaphors, teachers can expand their repertoire of lesson types beyond the problem-focused model which is a central design principle in much of mathematics education (e.g. Jackson et al., 2011). Further analysis of how real-world contexts in central roles were treated is given in the following section, with a focus on the nature of mathematical modeling.

Section 4: Developing and Using Given Models

Sixteen out of 37 lessons (or 43%) were classified as having real-world contexts in a central role, meaning that the context was the main focus or setting of the lesson's mathematical activity. In each case, students' approaches to modeling were classified either as developing those models with the support of the teacher, or as being given those models by the teacher or the instructional materials. The first part of this section provides a qualitative analysis of features of students' work developing models and features of the models that were given to students. The second part of this section makes quantitative comparisons across developed and given models on three kinds of measures: cognitive demand, ratings of instructional environment, and student participation distributions.

Ways to develop models. Eight lessons out of the sixteen lessons (50%) including real-world contexts had students develop mathematical models in order to solve a problem. Three general themes emerged from analyzing how those models were developed and how they functioned:

1) Devising general strategies
2) Using physical models
3) Reasoning with quantities

Devising general strategies. In several cases, the modeling process involved coming up with general strategies for solving certain kinds of problems. One example was in a lesson in teacher A's class about finding the areas of different regions of the borough of Brooklyn (COI, 10/1/10). Students had the freedom to choose their own approaches, including different ways to dissect the polygon into simpler shapes or subtractive approaches comparing the area of the polygon in question with that of a larger but simpler shape.

In other cases involving counting and systematic lists, students had freedom to choose their own representations and strategies for systematic listing. These included a lesson in teacher D's class about counting the number of possible handshakes (COI, 9/14/09) as well as a lesson in teacher A's class about listing all of the possible products of the values of two dice (COI, 4/22/10). In the case of the dice-products of problems, students worked with systematic lists, tree diagrams, and tabular representations.

Physical models. Two lessons previously described illustrated the use of physical models which students had to develop in order to complete a mathematical task. These lessons were observed in teacher A's class and both involved constructing a container with the largest possible volume given a fixed letter-sized piece of cardstock (COI, 12/15/09 & 3/3/11). In these cases, because these lessons were students' first introductions to the topic of volume, students could devise their own construction techniques as they created physical models, and their work had to take in consideration not only the total area but the given dimensions and constraints of using adhesive tape. The effectiveness of these physical constructions was further checked physically by using rice and a measuring cup to compare and measure the volumes actually contained by the students' creations.

Reasoning with quantities. A third way in which students worked to develop mathematical models was by reasoning with quantities. Often done with numerical tables, this kind of reasoning was frequently facilitated by the teacher. For example, in the second lesson of a pair on modeling the different possible areas enclosed by a rectangular fence of fixed perimeter in teacher D's class, teacher D led a whole class discussion about the quantities involved, listing different possible pairs of lengths and widths (COI, 11/16/10). Although students had difficulty with connecting the observed numerical relationships with the algebraic derivation from the perimeter formula, the quadratic model that they ultimately developed with extensive assistance from the teacher was grounded very differently compared to other quadratic functions which were given in advance to students.

In teacher C's class, there was an example of a story problem taken from a textbook resource with low adaptation which nonetheless engaged students in developing numerical and graphical models. In this case, the problem was about the rates of water consumption of two families. Rather than being given the equations, students had to reason through the different parameters including the daily rate of change and the initial amount and how they related to different parts of the equation as well as the graph of the linear function (COI, 2/9/10).

Challenges of given models. Two of the examples of mathematical models that were given to students to apply have already been discussed extensively. In one example, which was given as an example of real-world data given with a low level of adaptation, students were given in-state college cost data told how to construct a histogram from that data, with bins already specified (COI, 9/29/10). In another lesson about probability distributions, students played the game of "Beano" in order to investigate it, but the representation of an addition table gave away much of mathematics that was implicit in the task (COI, 5/5/11).

The other six examples of given models all involved the real-world context of the projectile motion or vertical motion under the influence of gravity. With two examples from teacher D's class and four from teacher B's class, quadratic functions modeling projectile motion were given to students, usually with height as a function of time. Although the textual treatment in teacher D's class, which was taken from a textbook with a low level of adaptation, had more exposition about the meaning of the different parameters in the problem (COI, 10/8/10), this issue was not the focus of the lessons. Although many of these lessons included actual physical demonstrations of either parabolic arcs or the actual dropping of objects which kept the real-world context at the forefront, all six lessons shared a disconnect between the features of the mathematical model and given information in the real-world context. In two similar lessons in teacher B's class in two different school years, the formula for the vertex was further given to students as a formula rather than developed algebraic or from graphical or numerical properties of the model (COI, 2/9/10 and 3/10/11).

Quantitative comparisons: Developing models associated with higher cognitive demands, higher ratings of instructional environment, and less listening to the teacher compared to given models. As suggested by the qualitative analysis above, lessons with central real-world contexts in which students developed models or solutions to the problems had higher cognitive demands, instructional environment ratings, and less listening to the teacher than lessons where they were given models to apply.

Measure A: Developing models associated with higher cognitive demands. Among lessons including real-world contexts in a central role, 6 out of 16 or 38%, were rated as high cognitive demand. By comparison, 6 out of 8, or 75% of lessons in which mathematical models were developed were rated as high cognitive demand, but none of the eight lessons in which students were given models were rated as high cognitive demand. By Fisher's exact test, this association was statistically significant, p=0.007. The complete cross-tabulation is shown in table 17.

Table 17. Cognitive Demand by Modeling

	Cognitive Demand		
Modeling	High	Low	Total
Given	0	8	8
Developed	6	2	8
Total	6	10	16

This result suggests that when students were given mathematical models to apply, the cognitive demand was lowered as students did not need to make connections with the procedures that they were carrying out.

Measure B: Developing models associated with higher instructional environment ratings. All four ratings of instructional environment were lower for given mathematical models compared to developed ones. These mean instructional ratings are displayed in table 18.

Table 18. Instructional Environment by Modeling

	Models			
Rating	Developed (n=8)	Given (n=8)	t	Df
Mathematical discourse	3.13 (0.35)	1.88 (0.64)	4.830***	14
Intellectual Support	3.88 (0.84)	2.25 (0.46)	4.816***	14
Depth of Knowledge	2.75 (0.46)	1.63 (0.52)	4.583***	14
Engagement	3.38 (1.06)	2.00 (0.00)	3.667**	7

Note: ***=p<0.001, **=p<0.01

These differences are consistent with the mathematical processes involved in developing models of real-world contexts. Mathematical discourse is necessary as features of the real-world context are described, analyzed, and quantified, while strategies are devised, implemented, and evaluated, $t(14)=4.830$, $p<0.001$. Meanwhile, to successfully develop models, students need to take intellectual risks and be supported in those attempts to offer new ideas, in contrast with lessons where they are given models where there are few risks in following the given instructions, $t(14)=4.816$, $p<0.001$. By actually developing models and exploring connections between representations and the real-world context, students achieve a greater depth of knowledge, t(14)=4.583, p<0.001. Because there is more intellectual work for students to do and they are supported as they do the difficult work, lessons which had students developing models had higher levels of student engagement, t(7)=3.667, p=0.008. Indeed, each of the eight lessons in which students were given models were rated uniformly as a "2" on the five-point scale for student engagement, correspond to passive disengagement.

Measure C: Developing models associated with less listening to the teacher. Table 19 displays the mean relative frequencies for eight student participation categories as well as the aggregated "active" participation category compared across whether models were developed or given.

Table 19. Participation Structures by Modeling

	Models			
	Developed (n=8)	Given (n=8)	T	Df
Housekeeping	0.03 (0.04)	0.01 (0.02)	1.610	11
Listening to Teacher	0.07 (0.09)	0.30 (0.23)	-2.643*	9
Listening to Students	0.03 (0.07)	0.00 (0.00)	1.000	7
Discussing	0.29 (0.17)	0.20 (0.27)	0.839	14
Investigating	0.38 (0.27)	0.33 (0.26)	0.434	14
Writing/Reading/Reflecting	0.04 (0.09)	0.02 (0.03)	0.850	14
Using Technology	0.00 (0.00)	0.00 (0.00)	0.000	14
Practicing	0.15 (0.25)	0.15 (0.23)	0.000	14
Active Participation	0.74 (0.23)	0.54 (0.39)	1.134	14

Note: *=p<0.05

There was only one statistically significant difference between developed or given models and type of participation structure. In lessons where models were developed, students spent overall an average of 7% of class time listening to the teacher, less than the average of 30% within those lessons in which models were given to them to apply, $t(9)=-2.643$, $p=0.044$. This difference suggests that one of the primary modes of transmitting information, including the mathematical models to use, was listening to the teacher. Although there was slightly more investing when models were being developed (38% versus 33%), this difference was not statistically significant. The difference was more in the substance of the investigation, with students investigating features of a given model, such as one for quadratic functions which was already designated at the beginning, or investigating along a predetermined course as outlined by a handout or the given mathematical model.

Discussion and summary. The quantitative analysis provides evidence that developing mathematical models for solving problems set in real-world contexts is a more desirable practice than giving those models to students to apply. The qualitative analysis moreover suggests that there are a variety of ways in which students can be supported in order to develop mathematical models. Indeed, the lesson with the highest rating of cognitive demand among all 71 lessons, the sole lesson rated at the level of "doing mathematics", was the lesson in teacher A's class where students developed methods for representing and computing different probabilities of the products of the values rolled on two dice (4/22/10).

Conversely, given models are less desirable, and typically involve a great deal more of the passive participation category of listening to the teacher. At the same time, a single topic—vertical or projectile motion modeled by quadratics—accounted for three quarters of the lessons in which models were given. This concentration suggests that perhaps teachers need more pedagogical content knowledge in how to support students as they develop quadratic models in connection with this real-world context. Teacher D was able to work with students in modeling areas enclosed by rectangles with a fixed perimeter with a quadratic function (COI, 10/8/10), so the difficulty in modeling projectile motion may be specific to these problems with uniform acceleration (i.e. $h(t)=- ½ gt^2+v_ot+h_o$). Focusing on this topic in teacher professional development could have a large impact on how well teachers assist students in genuinely developing these quadratic models.

At the same time, the actual instances of developing models described in the eight lessons in many cases still fall short of what is meant by mathematical modeling in the mathematics education literature. For instance, the two lessons in which students needed to develop strategies for maximizing the volume contained for a given surface area did not involve further algebraic models or representations because of its location within the larger curricular unit. Similarly in other lessons such as those about solving systems of linear equations (teacher C, 2/9/10 and 5/2/11) a maximizing area enclosed by a rectangular fence (teacher D, COI, 11/15/10), while modeling was present as a process, the actual real-world contexts were rendered less problematic by the information that was given and the assumptions that were made. That is, in many cases it would be difficult to imagine an alternative approach to the one that was actually developed within the lesson. These problems would correspond to what has been termed “standard applications” rather than full-fledged modeling problems (Niss, et al., 2007).

Section 5: Elaboration

Overall, 15 out of 37, or 41% of lessons including real-world contexts elaborated upon those real-world contexts. In the first part of this section, I provide a qualitative analysis of that elaboration, focusing on the nature of student participation as well as the kinds of support and explanation provided by the teacher. The second section compares lessons in which the real-world context was elaborated with those which were not elaborated based upon three kinds of measures: cognitive demand, instruction environment, and student participation distributions.

Varieties and functions of elaboration. Three ways to think about elaboration emerged:

1) breadth and depth in terms of student participation
2) connection with other categories, in particular with role
3) mode and phase.

Breadth and depth of student participation. Elaboration varied along two dimensions: breadth and depth. *Broad elaboration* involves a large number of students, while *deep elaboration* involves sustained sharing ideas with an individual student. As an example of broad elaboration which involved multiple students, at the beginning of a lesson about the centroid of a triangle, teacher A had students write down everything that they knew about the word "middle." She had supplied them with prompts such as, "middle child", "middle of the week", and "middle of the road". Teacher A then asked students to share with the whole class what they had found. Students enthusiastically raised their hands to participate within whole-group discussion, sharing different opinions, perspectives, and experiences.

> Ms. A then points out a student in the back of the room who is twirling a protractor on the point of his pencil on the eyepiece and asks him where he is twirling. The teacher and student discuss how the eyepiece is in the middle... Another student then shares about how he, as the middle child, is closer to his older brother and his younger brother, but

> how his older and younger brothers aren't as close to each other. There is then a loud and animated discussion about when the middle of the week is. Students are disagreeing loudly about whether the middle is Wednesday and Thursday, depending on whether or not they are counting the school week or the whole week (starting on Monday). Ms. A is calling on students to offer arguments about why Wednesday is the middle of the school week, and they offer different ideas such as how there are two days on either side, or by counting off the days with pauses (Monday-Tuesday, Wednesday, Thursday-Friday). Ms. A then brings up what the middle of the road means, and one student interprets this to mean cutting the road in half (across the road itself as if crossing it), and says that it can't be done, which leads to an aside by a student who says if he had a "big ass" protractor and compass that he could indeed cut a line in half. (COI, 2/11/10)

In this case, the cumulative effect of elaboration was to give students multiple ways to understand what "middle" might mean as well as what it might mean to be in the middle.

These multiple perspectives served as a bridge to understanding what the "middle" of a triangle—its centroid—would be. In this example, multiple students contributed multiple examples of a single central concept. These included adopting the perspective of being in the middle and having the same amount of something (days of the week, brothers) on either side. The prompts provided by the teacher were structured and specific forms of the "contexting" questions described by Evans (1999). Answering these questions and discussing the answers in a whole group allowed students to connect the mathematical concepts to personal experiences. At the same time, this example of broad elaboration is distinct from the description of the "contextual relationships" described in Jackson et al. (2011). In this case, the real-world contexts of "middle" were not the central focus of the lesson, but did serve as a *motivating metaphor* to the notion of centroid (center of gravity) which students would then construct and investigate in the case of triangles.

In the lesson following this example of broad elaboration, there was an instance of deeper elaboration, in which a single student explained at length an analogy after teacher A showed the class an image of a classmate who was break-dancing and balanced upside-down on one hand. The student said:

> on the three sides… equal balance, just sitting on the tip of my pen. His body got equal balance, the top part is equal balanced to the bottom part.. not too heavy on the hand and so… on the triangle we did the slice and all that, equal weights, sitting on the tip of our pen. (COI, 2/12/10)

Teacher A then further elaborated by drawing upon what other students had said as well as engaging another student's question and making explicit connections between the real-world example and the mathematical concepts:

> Teacher A asks what those slices were and then names two students Edwin and Katheryn, crediting them as having described it: "cut it at an angle, cut it in half". She says that Katheryn had said it was the midpoint, and the name of the line itself is the median. Martin asks what the dot is, and teacher A says the hand is the centroid and the arms and legs are the medians. Teacher A calls them cross-sections of balance, gesturing across her own hips and talking about the balance and weight coming down. (COI, 2/12/10)

Here, as in the previous lesson, the role of the real-world context was as a motivating metaphor to provide a common, grounding experience in which to anchor the mathematical ideas. The rest of the lesson was focused on problems involving medians and midsegments of triangles, problems that were by themselves decontextualized. Yet, the deep nature of the elaboration in the introduction of this concept also allowed for students to explicitly connect the previous lesson to the new topic.

In another lesson about negative space and area of composite figures, teacher A showed students a sequence of three images involving negative space (see Appendix II). She then asked students to tell the class what they saw in them, as an instance of broad not very deep elaboration:

> For the classic faces/vase optical illusion, students explain both how they see faces in profile as well as a "candle holder" or, after Ms. A calls it a "chalice", a "pimp cup". Another student picks up a trophy sitting on the teacher's desk in the front of the room

> and shows it to those around him. As Ms. A shows other images, such as that of a dog eating a cat eating a rat, students clamor to contribute, as others express their disbelief or make references to “therapy”, or Rorsharch images. Students then talk to each other and try to help each other with a negative space image in which “EVIL” is written inside “GOOD”. Ms. A summarizes these examples by telling the class about negative space as a concept and technique in art, mentioning the name of the students’ art teacher. (COI, 12/15/09).

Here, the cumulative effect of these three examples was to provide a range of instances of negative space, which provided parallel support to the variety of composite figures which students then worked on in groups. Similar to the lesson involving medians and centroids, this motivating metaphor connected students’ experiences or shared perceptions of new images that were not necessarily familiar into solving mathematical problems that did not have the same level of contextualization.

This lesson can be contrasted with a lesson in teacher C’s class on the same mathematical topic. Although the students were asked to create cut-outs of quarter-circle sectors from different colored sheets of paper and rearrange these pieces, the problem of finding the area of composite figures was addressed without further elaboration to what the real-world context might resemble or seem like (COI, 12/10/09). Further, there was only one sequence of steps which students were expected to complete, rather than a variety of problems that they would then have a reason to share with their classmates.

These two dimensions of breadth and depth are useful tools for describing the practice because they reflect the different levels at which elaboration can take place, whether at the level of the whole-class discussion or teachers’ individual tutorial work with students. For instance, in teacher C’s classroom, which was not typically marked by much elaboration, in one session which otherwise was typical of the test-preparation routines of that class, teacher C sat for an extended period of time and contextualized a problem about volume and cubes by showing a textbook as an example of a rectangular prism. For a different problem, the teacher took out pocket change and used it as a way of explaining powers of 2. These instances, however, were confined to the students to which they directly and individually addressed and did not appear to have helped other students in the class (COI, 4/16/10). This example is of a teacher-directed, deep, but not broad form of elaboration.

As two dimensions for considering participation in general, breadth and depth could help to develop teachers' self-reflection as well as planning for future lessons. Because of the limited time available for any class lesson, any instructional sequence necessarily must make trade-offs in terms of breadth, depth, and other considerations of instructional priorities. The examples in which elaboration were particularly effective and well-integrated into the lesson had breadth and depth well-suited to the target mathematical understanding, even when the role of the context was not central to the lesson, but rather serving as carefully planned, motivating metaphor that provided multiple entry points for students.

Versatility across roles. Elaboration can be compared to the other three variables articulated so far, including adaptation and role. Although lessons with high adaptation were more likely to be elaborated (10 out of 20, or 50%) than lessons with low elaboration (5 out of 17, or 29%), this association was not statistically significant, by Fisher's exact test, $p=0.37$. A complete cross-tabulation is presented in table 20.

Table 20. Cross-tabulation of Adaptation by Elaboration

	Adaptation		
Elaboration	High	Low	Total
Yes	10	5	15
No	10	12	22
Total	20	17	37

Compared to the overall 41% of real-world contexts which were elaborated, real-world contexts in incidental roles were much less likely to be elaborated (2 out 14, or 14%), while those in central roles (8 out of 16, or 50%) or serving as a motivating metaphor (5 out of 7, or 71%) were more likely to be elaborated. Indeed, the association of elaboration with motivating metaphors and unelaborated contexts with incidental roles was statistically significant, $\chi^2(N=37, 2)=7.368$, $p=0.025$. Table 21 displays a complete cross-tabulation of elaboration by the different roles that a real-world context could take in a lesson.

Table 21. Cross-tabulation of Role by Elaboration

	Role			
Elaboration	Metaphor	Incidental	Central	Total
Yes	5	2	8	15
No	2	12	8	22
Total	7	14	16	37

This association suggests that elaboration is well-suited to providing students broadened opportunities to participate in the development of motivating metaphors which are grounded in real-world contexts. Indeed, many of the detailed examples which were provided in the previous section on breadth and depth were examples of elaboration which helped to develop a motivating metaphor for the lesson.

Finally, among the sixteen lessons in which real-world contexts played central roles, there was no association between elaboration and whether models were developed by students or given to them to apply. The complete cross-tabulation is in table 22.

Table 22. Cross-tabulation of Modeling by Elaboration

	Models		
Elaboration	Developed	Given	Total
Yes	5	3	8
No	3	5	8
Total	8	8	16

Mode and phase. As conceptualized by Chapman (2006) and operationalized into a fine-grained coding schema by Depaepe et al. (2010), teachers' approaches for explaining word problems with their classes can be divided into paradigmatic and narrative modes. Elaboration as conceptualized by this study is more closely aligned with the narrative mode. This narrative mode is focused on the humanistic interpretations and connections' to students' personal experiences, with what Chapman calls "resonances".

Depaepe et al. (2010) placed emphasis on a further dimension for analysis, that of *entry* and *exit* into the mathematical understanding of the problem. This distinction is between what allows students to enter from their experiences or understandings of the real-world context into the problem, and then what allows them to exit from the mathematical operations or procedures back into an understanding or interpretation of the world. It is thus possible to further characterize the substance of the elaboration to complement the previous analysis of the participatory characteristics of the elaboration in terms of the particular narrative mode and the entry and exit conditions. The classification scheme that they developed for individual moves that teachers make in whole-class discussion does, however, tacitly assume that the real-world context is associated with a problem that is for the instructional moment in a central role. That is, it does not apply as well to examples where in the coding developed for this study the real-world context plays the role of a motivating metaphor.

Nonetheless, the notion of "entry" thought of more broadly is still useful for considering how in cases where the role of a real-world context is to serve as a motivating metaphor, the purpose of elaboration is to bridge from students' experiences, perceptions, perspectives, or opinions into a common prompt and from there into some mathematical concepts or procedures. Examples of this invitation of students' personal and lived experiences, perceptions, or opinions toward a more formal mathematical context are characteristic of the examples given above, as well as a discussion in teacher A's class about the fairness of casinos (COI, 4/26/11). In this lesson, students drew upon personal experiences with casinos or stories that they had heard about them to speculate about different ways a casino might not be fair to players, and specific strategies that casinos may employ to ensure a profit. The main mathematical task was then about a specific game.

There were three examples of elaboration in the exit function for central real-world contexts. Two of these lessons, in separate years, were about maximizing the volume of a closed container made from a fixed letter sized sheet of cardstock (COI, 12/15/09 & 3/4/11). In one implementation of that lesson, the different groups created a single container of their own design, and winner was what another student called "a five pound sugar bag". The teacher and students then talked about the different dimensions of cylinders, making references to Pixie Sticks (a long, narrow tube of colored sugar) and straws in general (COI, 12/15/09). Similarly, at the end of a different lesson where each small group had a designated kind of solid to build their container in, different groups elaborated on how their models connect back with real-world contexts:

> Ms. A calls on each of the groups to talk about what they found. These reports involve students holding up their shapes and reporting the volumes, sometimes calling them by other names, such as a flat pyramidal solid with a narrow base a "fishing boat". Another one is called a "purse". Students are making comparisons of the different solids in their groups by saying some are wider, or larger. Students are making these comparisons and saying things like a shape is "wider, it's more volume to it" or that one solid "has more base and height". One student explains the comparison of a narrow cone with a wider one by saying that the wider cone will have more volume (COI, 3/4/11).

The closing whole-class discussion is focused on what would happen if a different unit of measure, such as grapes, were used and how different objects could be used for filling. Students suggest units of measures such as Twizzlers (a long, red sweet licorice), pencils, and pickles as being well-suited to the cylindrical container. These examples are instances of elaboration in the exit phase and also of the relatively rare intervention of taking realistic considerations into account.

In another example, students worked to compute an approximation of the area of Brooklyn based upon a scale diagram (COI, 10/1/10). Pairs of students had worked independently to compute area on a scale diagram and then scale that area up to the area of the corresponding region in the real-world. The teacher then led a discussion comparing the average of the students' computations (83.6 square miles) with the actual area of Brooklyn (75 square miles). The class then made other comparisons with Brooklyn's population and its population density. These are further examples of elaboration in the exit phase of a problem set in a real-world context.

Quantitative comparisons: Elaboration associated with higher cognitive demands, higher instructional environmental ratings, and less listening to the teacher. When lessons with elaborated real-world contexts were compared to lessons where the real-world contexts were left unelaborated, there were statistically significant differences in the cognitive demand and all four of the instructional environment ratings. Among the participation categories, however, the only category in which there was a statistically significant difference was that students spent less time listening to the teacher in lessons where the real-world context was elaborated,.

Measure A: Elaboration associated with higher cognitive demands. Overall, 10 out of 37, or 27% of all lessons including real-world contexts were rated at a high level of cognitive demand. By comparison, among contexts which were elaborated, 8 out of 15, or 53% were rated as of high cognitive demand, while just 2 out of 22, or 9% of unelaborated contexts were rated as having high cognitive demand. The association between elaboration and high cognitive demand was statistically significant by Fisher's exact test, p=0.01. Table 23 displays the complete cross-tabulation of cognitive demand by elaboration.

Table 23. Cognitive Demand by Elaboration

	Cognitive Demand		
Elaboration	High	Low	Total
Elaborated	8	7	15
Not Elaborated	2	20	22
Total	10	27	37

This association reflects two possible ways elaboration supports higher cognitive demands. Elaboration provides students the opportunity to make more connections between mathematical procedures and real-world contexts. Alternatively, elaboration could also broaden participation and deepen student engagement in the lesson to enable making more connections.

Measure B: Elaborated real-world contexts occurred in lessons with higher instructional environment ratings. There were statistically significant differences in all four mean ratings of instructional environment when lessons with elaborated real-world contexts were compared to those with unelaborated contexts. Table 24 displays a complete cross-tabulation of instructional environment by elaboration.

Table 24. Instructional Environment by Elaboration

Rating	Elaboration Yes (n=15)	Elaboration No (n=22)	T	df
Mathematical discourse	2.87 (0.74)	1.91 (0.75)	3.826**	35
Intellectual Support	3.33 (0.82)	2.45 (0.80)	3.253**	35
Depth of Knowledge	2.47 (0.74)	1.73 (0.70)	3.071**	35
Engagement	3.07 (0.96)	2.18 (0.59)	3.182**	21

Note: **=p<0.01

Because elaboration is primarily conducted through oral communication, opportunities to further explore features of the real-world context supported higher ratings in mathematical discourse as well, t(35)=3.826, p=0.001. Students were given the opportunity through their discussion of these features and the intellectual support of the lesson to take more intellectual risks both in terms of the mathematics as well as in thinking about or with the real-world situation, t(35)=3.2353, p=0.003. By doing so, students were able to develop deeper mathematical knowledge, t(35)=3.071, p=0.003. Finally, as a result of these diverse ways to participate as well as the meaning that was being co-constructed through elaboration, students in those lessons were more engaged, t(21)=3.182, p=0.003.

Measure C: Lessons with elaborated real-world contexts had more discussing and active participation but less practicing. Consistent with higher ratings of mathematical discourse, lessons in which real-world contexts were elaborated had more discussing (25%) than lessons where contexts were not elaborated (12%), t(35)=2.300, p=0.038. Conversely, lessons where the real-world context were elaborated had less practicing (18%) compared to lessons where the real-world contexts were unelaborated (39%), t(35)=-2.395, p=0.022. Overall, there was also more active participation in lessons with elaborated real-world contexts (65%) compared to those where real-world contexts were not elaborated (30%), t(35)=3.285, p=0.002. Table 25 presents a complete tabulation of the relative frequencies of the eight individual participation categories and the aggregate "active" participation category compared across whether or not the lesson incorporated elaboration.

Table 25. Participation Structures by Elaboration

	Elaboration			
Participation Category	Yes (n=15)	No (n=22)	t	df
Housekeeping	0.03 (0.06)	0.05 (0.06)	-0.669	35
Listening to Teacher	0.13 (0.14)	0.26 (0.23)	-2.160	35
Listening to Students	0.03 (0.06)	0.01 (0.03)	1.426	18
Discussing	0.25 (0.21)	0.12 (0.15)	2.300*	35
Investigating	0.31 (0.23)	0.16 (0.24)	1.906	35
Writing/Reading/Reflecting	0.06 (0.09)	0.01 (0.03)	2.037	16
Using Technology	0.01 (0.03)	0.00 (0.02)	0.273	35
Practicing	0.18 (0.26)	0.39 (0.28)	-2.395*	35
Active	0.65 (0.35)	0.3 (0.30)	3.285**	35

Note: *=p<0.05, **=p<0.01

While there were other differences compatible with the qualitative analysis of the nature of elaboration in these lessons, such as more investigating, more writing, and less listening to the teacher, these differences were not statistically significant.

Discussion and summary. Elaboration is a high-impact practice which can support student learning and therefore should be encouraged in all lessons. Further, students are empowered to offer their own connections with real-world contexts in the course of mathematics lessons when prompts are chosen that are closely related to the target mathematics being studied. This practice of inviting student contributions of real-world contexts contrasts with the approach of using only those real-world contexts which come from textbook or internet sources with low levels of adaptation. One negative recommendation then is against inserting real-world contexts from texts directly into lessons without more opportunities for elaboration and contextualization.

Elaboration has been defined as various ways for broadening and deepening student participation in the discussion of ideas, and in the explanation, interpretation, and qualification of the real-world contexts that are brought up in connection with mathematics lessons. While in some cases this coincides with what Jackson et al. (2011) describe as the discussion of "contextual relationships," elaboration does not need to occur only in connection with a real-world context that has a central role, a condition that is assumed in many other studies. Rather, elaboration of a real-world context can serve as the motivating metaphor for a lesson, which then moves into more abstract mathematical representations and procedures.

Elaboration can further be analyzed in terms of the breadth and depth of student participation, with other effective practices including writing in response to a prompt before having a whole-class discussion. Elaboration is further compatible with and mutually supportive of a variety of classroom participation categories which maximize active student participation. Lessons in which real-world contexts were elaborated had more discussing and less practicing.

Effective elaboration for motivating metaphors may actually occur in several phases.

First, an appealing prompt *creates openings* for multiple students to respond with their opinions, perceptions, or perspectives. These may include connections to their personal stories or experiences. Second, a whole-class discussion *shares multiple students' ideas*. Prior to this, other instructional moves which allow students more time to think, such as reflecting first in writing and taking turns when sharing in whole-class discussion, can provide more level opportunities for all students to participate. Third, the teacher *makes more explicit connections* to the mathematical topic of the lesson. This basic outline for elaboration of motivating metaphors once again reflects the need for careful planning on the part of the teacher. This planning includes anticipating students' responses, which can further be integrated into the original prompt.

Section 6: Cross-tabulation by Teacher

This study differs substantially from other studies because it intensively observed a small number of individual teachers multiple times over the course of two years. By contrast, other studies have sought to maximize the total number of teachers observed, but with a far smaller number of observations per teacher. For example, Jackson et al. (2011) observed more than 100 teachers, but each for no more than three class periods or lessons. While the sampling design for this study reflects greater depth into teachers' practices, the smaller sample size of four teachers is less representative of the total population of mathematics teachers. That is, with a small sample, differences associated with teacher characteristics might be conflated with differences in classroom practices. This section examines this possibility and explores alternative hypotheses and interpretations for the foregoing results.

Cross-tabulation of the five focal variables of this study (presence of contexts, adaptation, role, modeling, and elaboration) can demonstrate the extent to these practices were associated with particular teachers. Table 26 displays this complete cross-tabulation. The remainder of this chapter considers each of these five variables in turn. The analysis in each section focuses on describing "profiles" of teacher approaches based upon the overall distribution of the five focal variables. After considering each of these variables in turn, I provide general sketches of each of the four teachers, and then consider the extent to which the main conclusions of this study stand relative to these differences associated with teachers.

Table 26. Cross-tabulation of Variables by Teacher

		Teacher				
		A	B	C	D	Total
Total observations		17	18	18	18	71
Real-world contexts	Absent	9	5	7	13	34
	Present	8	13	11	5	37
Adaptation	High	8	6	3	3	20
	Low	0	7	8	2	17
Role	Metaphor	4	1	2	0	7
	Incidental	0	6	7	1	14
	Central	4	6	2	4	16
Models	Given	0	6	0	2	8
	Developed	4	0	2	2	8
Elaboration		7	2	2	4	15

Inclusion of real-world contexts. There were three profiles in terms of the inclusion of real-world contexts. Teachers B and C included real world contexts in a majority of the lessons in their classrooms (72% and 61%, respectively). Among teacher A's lessons, slightly less than half included a real-world context (8 out of 17, or 47%). Teacher D had the lowest frequency of real-world contexts in his lessons (5 out of 18, or 28%).

These differences could be attributed to the teacher, but another possible explanation is subject area. Compared to Geometry, Algebra is more closely associated with story problems that involve real-world contexts. Indeed, teacher D taught Geometry in the first school year of this study, but Algebra in the second. The percentages of lessons incorporating real-world contexts reflect this difference in subject area. In the Geometry year, only one out of 10 (10%) lessons contained real-world contexts, while in the Algebra year, four out of eight (50%) lessons contained real-world contexts. Given this discrepancy, the nature of the subject and existing instructional materials may be more influential than teacher-level variables.

Adaptation. The use of existing instructional materials with low levels of adaptation was associated with two of four teachers in this sample. Teacher C presented students with instructional materials containing real-world contexts with low levels of adaptation, as observed in 8 out of the 11 (72%) lessons containing real-world contexts. In multiple cases, these included worksheets or test-review problems which were reproduced with no edits to the core content. This practice of providing students with worksheets or photocopied instructional materials without much adaptation was also observed in a majority of teacher B's lessons (7 out of 13, or 54%). Teacher B more frequently combined, via cut-and-paste, real-world contexts from multiple textbooks or sources. Teacher D also took some language and exposition directly from a textbook source in a pair of lessons, but retyped this information on a handout and PowerPoint slides.

By contrast, all eight of teacher A's lessons including real-world contexts reflected some higher degree of adaptation in terms of the information or prompt that she presented to students. This difference suggests that the teachers in this sample may have significantly dissimilar orientations toward the curriculum materials that they are provided, and ways of approaching the creation of new curriculum materials. It is, however, difficult to draw more inferences about teacher practices orteachers' perspectives on their planning and adaptation processes. This question would be worth investigating within the context of the larger CTMUY research project and would need to incorporate more input and reflective interviews with teachers.

Instructional roles. Compared across all four teachers, there were four different profiles in terms of how real-world contexts played instructional roles within the lesson. Differences hinged on the use of real-world contexts in incidental roles, in which they were neither a central problem context nor a motivating metaphor for the main mathematical topic of the lesson. This practice was associated with the lessons of teachers B and C. Teacher C in particular most frequently treated real-world contexts incidentally. In 7 out of 11, or 63% of lessons observed, although real-world contexts were mentioned they were not explored in greater depth. While teacher B also had real-world contexts in incidental roles, an equal number (6 out of 13, or 46%) of lessons had real-world contexts as the central setting of a problem for the lesson.

A third pattern was demonstrated by teacher D, who focused on real-world contexts as the central setting of a problem in four out of five lessons (80%). Finally, teacher A had an approach to real-world contexts in her lessons which equally balanced real-world contexts as motivating metaphors and as central problem contexts. The roles played by real-world contexts may also be associated with teacher practices around adaptation, as teacher A tended to pose more cognitively demanding and realistically grounded problems to her students than teachers B and C, who did not substantially adapt real-world contexts in routine problems derived from existing curriculum sources.

Modeling. Given that modeling was analyzed only for real-world contexts which served as the setting for a central mathematical problem in a lesson, the extent to which developing models was associated with teachers partially reflects their use of real-world contexts in different instructional roles. Students in teacher B's lessons were asked to apply models which they were given, in lessons set in the context of projectile motion modeled by quadratic functions. As noted earlier, half of teacher D's lessons were also on projectile motion modeled with given quadratic models.

Teacher C's lessons provided an interesting contrast. Although overall only two of teacher C's lessons placed real-world context in a central role, in both of these lessons, students developed rather than applied given models. In this respect, teacher C resembled teacher A in how all lessons in which real-world contexts were in a central role had models developed with or by students, rather than being given to them in order to apply.

Elaboration. Elaboration was distributed across all four teachers insofar as for each teacher, at least two lessons including real-world contexts reflected teachers or students elaborating features of the real-world context. When analyzed as a percentage of lessons, however, there were two groups of teachers. Teachers A and D engaged their students in elaboration in a large majority of lessons (88% and 80%, respectively), while teachers B and C engaged in elaboration less frequently (15% and 18%, respectively). The lower frequency and depth of elaboration is consistent with differences in terms of how frequently teachers B and C used curriculum resources with a low level of adaptation, and in incidental roles within the lesson.

Discussion and summary. Based upon the above analysis of differences across the four teachers in the study, it is possible to develop profiles of types of teachers, what other researchers have even referred to as caricatures (Lambdin & Preston, 1995). Teachers B and C might be characterized as "by the book," but drawing upon multiple existing sources for real-world contexts. In contrast with other studies on the implementations of Standards-based curriculum, the sources for these classroom exercises can vary from lesson to lesson (e.g., from different internet resources), or even within a single handout or worksheet within a single lesson. In these cases the real-world contexts are treated as incidental, as the main focus of the lesson is the repetitive application of learned procedures to exercises selected for that purpose.

While Teacher D shares some of these characteristics in terms of drawing upon some curriculum materials with low levels of adaptation, his use of curriculum materials tended to be more centered on a single textbook in each of the two years he taught first Geometry and then

Algebra. The modeling processes in his lessons asked students to develop rather than to apply models which they were given. Teacher A, on the other hand, might be caricatured as an innovator who designs, refines, and repackages real-world contexts in a way that is more highly adapted to her students' interests and experiences, thereby opening up spaces for students to elaborate.

To further continue this line of reasoning, one interpretation of these distinctive profiles would be to attribute differences in the three measures of classroom practice (cognitive demand, instructional environment ratings, and participation categories) to teacher characteristics. This line of reasoning, however, is not fruitful because it does not recognize the agency of teachers to reflect upon, modify, or refine their practices. Alternatively, the focus could be on teaching practices or instructional routines within the classroom (Hiebert & Morris, 2012).

While to develop conclusions to inform educational policy or professional development design would require addressing individual teachers based upon their existing levels of knowledge, beliefs, and practice, a focus on what teachers actually do in the classroom may provide them with more concrete suggestions for change. Further, this approach contrasts sharply with the current emphasis in the policy environment on comparing teachers to each other, as linked to evaluation, continuation of service, or distribution of financial incentives. While the results for this section show that there is differential implementation of the different variables in this study across teachers, each of these variables is still within the locus of control of teachers—that is, they are variables about teach*ing* not teachers. While this study may have also identified effective individuals who are implementing these practices in their teaching, it is only through a focus on productive practices that are generalizable across individuals that the complex work of school and instructional improvement can be achieved.

Chapter 5

Conclusions and Recommendations

This chapter extends the findings of this study to make recommendations in two broad areas of practice and research. I conceive of practice as including classroom teaching and learning as well as teacher professional development. Both practice and research are further framed by changes in mathematics education policy, and I situate this chapter in the current policy environment.

First, I outline two of the major policy changes currently underway in mathematics education: the implementation of the new Common Core State Standards in Mathematics and the incorporation of classroom observation data into new teacher evaluation systems as part of Race to the Top. Second, I reinterpret the findings as they inform, reflect, or extend the notion of culturally relevant mathematics pedagogy. Third, I consider implications for practice, both for teaching and learning in the classroom and for the design and implementation of professional development for teachers. Finally, I outline a research agenda and suggest new ways of collaborating with teachers toward nurturing teacher change and growth in more full and robust implementation of CureMap in their regular classroom practice.

Policy Currents and Openings

Common Core State Standards. One aspect of mathematics education policy is the body of standards that define what school mathematics is and the testing regime that determines how student achievement will be measured. The Common Core State Standards, adopted by 46 states and the District of Columbia, is one such effort to move toward standards that are more consistent across states. Although they have stated that they are meant to be aligned, coherent, and focused (Cobb & Jackson, 2011), at the high school level, the Common Core State Standards in Mathematics (CCSSM) have primarily re-sequenced the coverage of content area topics, retaining a heavy emphasis on symbolic manipulation such as polynomial identities. At the same time, there is the potential to leverage the change that the new standards present by focusing on the Common Core Standards for Mathematical Practice.

One of these Mathematical Practices is to “model with mathematics”. Here, “model” is a verb rather than a noun, which is directly in line with this study’s findings about developing models in order to analyze a central real-world context. At the same time, Modeling is one of the six content standards within the CCSSM at the high school level, but modeling as a process and Mathematical Practice may need to be an explicit and separate focus of professional development around the standards (Tam, 2011). Further, teachers’ understanding of mathematical modeling is often not far less robust than what is described in the theoretical research literature (Niss et al., 2007).

At the same time, the implementation of these new Standards provides an opening for changes on three levels beyond the realm of classroom practice: instructional materials, standardized assessments, and professional development. New instructional materials in the form of textbooks, model units, formative assessment tasks, are being generated by the for-profit sector and states and school districts. As the degree of implementation increases, new assessments are being designed which will be aligned to the new Standards and the basis for measuring students’ progress toward the universal proficiency required by NCLB. The design of these assessments has not been completed and needs to fully incorporate modeling (Leong, 2011). Finally, and perhaps the area with the biggest potential for effecting change in teachers’ practices is the large investments that states and districts need to make in professional development for teachers.

Teacher evaluation systems. As a component of the federally sponsored Race to the Top competitions, states could compete for funding based upon proposals that incorporated a variety of features named by the United Stated Department of Education. Common to many of these systems are revised systems of teacher evaluation. While many of these include substantial portions devoted to gains in student achievement on standardized tests, most states have incorporated some form of revised rating system for teachers. This trend has also been paralleled by the Measures of Effective Teaching project which has coordinated and combined multiple classroom observation protocols with student surveys and achievement data. In both cases, incentive payments are tied to teacher performance based upon these observation protocols.

While the most prevalent observation protocols, including the CLASS and the Danielson Framework for Teaching, as well as the "lite" version of the Mathematics Quality of Instruction instrument do not perfectly align with the scales used in this study, there are openings for change. Because of the relatively high personal, professional, and financial stakes tied to teacher evaluation, there is an opportunity to catalyze teacher growth if directed in specific and concrete ways. Providing teachers with tools such as elaboration, or alternatives such as using real-world contexts as motivating metaphors, or perspectives on doing mathematics such as mathematical modeling, can be tied to specific kinds of growth on these observation protocols.

Such an approach toward teaching practices would contrast sharply with the tendency to focus on teachers, and their characteristics in terms of educational background, content knowledge, pedagogical content knowledge, or other such variables. A focus instead on practices that directly influence student experiences in the classroom might be more productive than the current emphasis on accountability which has been trickling down from the school-level more and more to rest on individual teachers measured by student achievement scores.

Implications for Culturally Relevant Mathematics Pedagogy

Taken as a whole, the findings of this study directly inform the framework of culturally relevant mathematics pedagogy (CureMap). CureMap consists of specifications to mathematics of the three core outcomes of culturally relevant pedagogy: academic achievement, cultural competence, and critical consciousness. Three dimensions make up CureMap's framework: teaching mathematics for understanding, centering instruction on students, and developing students' abilities to be critical about and with mathematics. I analyze how the findings presented in each of the five sections in this chapter inform or reflect these three components of CureMap.

Teaching mathematics for understanding. One way to measure the first component of CureMap, teaching for mathematics for understanding, is through examining the cognitive demands of tasks and the different ratings of instructional environment. In this case, when it comes to real-world contexts in mathematics lessons, it was not a matter of *if* but *how*—inclusion of real-world contexts by itself was not associated with statistically significant differences in cognitive demand or instructional environment. More influential was the nature of the planning and design that went into the lesson. Lessons where there were low levels of adaptation of real-world contexts and lessons where real-world contexts played an incidental role had lower ratings of cognitive demand as well as instructional environment in terms of both mathematical discourse and depth of student understanding. Lessons with low levels of adaptation also had lower ratings of intellectual support and student engagement. These results suggest that to fully support the goal of teaching mathematics for understanding, real-world contexts need to be integrated more fully into the planning and delivery of lessons, and placed in more central or motivating roles.

Indeed, one major finding of this study is the articulation of a new target role for real-world contexts: as a motivating metaphor. In these instances, even though the bulk of the mathematical work of the period did not focus on the real-world context, in the successful instances of developing motivating metaphors, multiple connections and entry points into the main conceptual topic of the lesson were developed in the first part of the lesson. This pattern of bringing in real-world contexts is not typical of instruction in the United States, but is more commonly found in Japan (Mosvold, 2008). As such, placing real-world contexts in the role of motivating metaphors provides an alternative for mathematical topics which do not necessarily have elegant real-world applications, or for which a more appropriate focus would be more abstract relationships between mathematical representations, structures, or concepts.

A third feature analyzed by this study is the nature of the mathematical modeling in which students are involved as they investigate central real-world contexts within mathematics lessons. Lessons were categorized by whether or not students were supported in developing mathematical models. While half of lessons had students applying models that they were given, the majority of these lessons were focused on a single topic, that of vertical or projectile motion modeled by quadratics. Because developing models was associated with higher cognitive demands as well as higher ratings in all four aspects of instructional environment, targeting this particular topic for curriculum and professional development could substantially improve students' experiences with mathematical modeling.

Centering instruction on students' experiences and participation. Explicit focus on real-world contexts that were either "local" or "relevant" in a central role, as originally anticipated by the CureMap framework, was infrequent. Aside from one lesson on the area of various neighborhoods in Brooklyn, a large majority of the real-world contexts that were the main subject of mathematical tasks were not what CureMap would consider "local". This pattern suggests that teachers may need a good deal more time and preparation in order to incorporate students' lived experiences into central problem contexts.

On the other hand, elaboration as a classroom practice provides an alternative approach, especially when real-world contexts are used as part of motivating metaphors for mathematical concepts or procedures. By promoting multiple perspectives and facilitating student engagement, lessons with elaboration of real-world contexts were associated with higher cognitive demands and ratings of instructional environment. Examples of elaborated motivating metaphors provide robust examples of cases where the main mathematical tasks of the lesson are more abstract or noncontextualized than the original entry into them provided by the motivating prompt and the sharing of student experiences. One possible structure for elaboration for this purpose was outlined above.

Another way to approach the goal of "centering" on students is to examine patterns in the distribution of student participation categories. The category of "listening to students" was overall not very common, but it was less common in lessons including real-world contexts. Further, it was not at all present in lessons including real-world contexts with low levels of adaptation or with contexts in incidental roles. This absence indicates that changes in planning practices and the positioning of real-world contexts are necessary to facilitate this mode of student participation which puts their voices at the center of the mathematical discussion.

From the perspective of other key active participation structures such as discussing and investigating, the findings suggest that certain features of lessons are more supportive of and conducive to productive uses of these participation categories. For example, putting real-world contexts at the center of the lesson was associated with more time spent investigating compared to incidental roles for those contexts. Elaboration was associated with significantly more time spent discussing.

Being critical about and with mathematics. The final component of CureMap has not yet been fully addressed by the lessons which were observed. For students to be critical with mathematics, teachers would need to produce and bring in data and examples that are relevant to students' experiences and engage in developing mathematical models with them that would surface and address inequities. No lessons observed within the sample took this approach, but there are promising uses of real-world data and elaboration in the exit phase which could serve as components of this more critical approach.

Nor did observed lessons provide students with opportunities to be critical about mathematics. Again the exit phase plays an essential role, as perhaps students could work on modeling or application problems where mathematics fails to make sense, and then tie this back into the assumptions within mathematical models again. Modeling as an iterative process thus has potential for developing these more sophisticated forms of critical consciousness.

Developing the analogical reasoning embodied by metaphors has the potential for further support students becoming more critical *about* mathematics. If students frequently explore and develop analogies, they can analyze disanalogies to criticize limitations of mathematical explanations and algorithms. For example, how does the common analogy from balanced mass scales to solving equations extend to inequalities? In the case of inequalities, multiplying by negative numbers requires students to revisit, revise, rephrase, and refine the analogy with reference to the RWC. These doubts can lead either to criticizing the commonplace analogy, or to extending the analogy to negative weights, using structures such as balloons or counterweights on pulleys.

Implications for Practice

In this section, I consider two domains of practice: teaching and learning in the classroom and teacher professional development and then synthesize the connections between these two.

Teaching and learning in the classroom. One way to understand the findings of this study is in terms of two negative recommendations: against planning that does not incorporate high levels of adaptation of curriculum materials and against using real-world contexts in incidental roles that do not fully explore the meanings embedded in real-world situations. Alternatively, the focus could be placed on the positive structures that teachers should strive to incorporate into their regular classroom practices. Taken this way, there are three recommendations for teacher practice:

1) Consider the potential of real-world contexts as a motivating metaphor
2) Facilitate and support mathematical modeling as a process
3) Incorporate structures for elaboration of real-world contexts

It may be that teachers are particularly prone to thinking of real-world contexts in incidental or central roles given that is their default positioning in the school curriculum. The examples articulated here provide robust existence proofs that motivating metaphors are feasible in this school setting. These lesson plans could be developed into curriculum resources for other teachers to draw upon as a lower stakes way to enter into the practice. Other motivating metaphors, examples, and prompts could also be designed and collected in this way to expand the repertoire to which teachers can refer.

Modeling as a Mathematical Practice is also one of the six content domains at the high school level. As such, it could serve as a recurring theme throughout courses such as Integrated Algebra. It may be helpful to then develop regular scaffolding practices within the classroom which support the process of modeling by providing students with general heuristics within the process. Indeed, some of the interventions for elaboration around entry and exit could provide guidance for the kinds of cognitive moves teachers and students make while engaging with a problem requiring mathematical modeling. Drawing upon existing libraries of mathematical tasks that better exemplify modeling problems could also facilitate this process, as the cognitive demand achieved during the lesson is limited by the nature of the task.

Elaboration is actually closely tied to both of these goals, as it provides more points of entry into motivating metaphors. Elaboration also provides different ways for students to participate as they enter and exit into the mathematics involved in modeling real-world situations. There is also need for further development of classroom participation norms such as accountable talk or other productive and reciprocal ways for students to interact with each other and with the teacher in both small-group and whole-class discussions.

Teacher professional development. The above three recommendations can be directly incorporated into the goals for ongoing teacher professional development. Three structural modifications might facilitate future professional development on the role of real-world contexts within a framework of culturally relevant pedagogy. First, developing in-school norms with stronger accountability structures and alignment in terms of classroom practices and shared beliefs about students and content would facilitate the quality of lessons within a group of teachers. These shared structures could include agreements about lesson planning elements and appropriate instructional resources, and the use and adaptation of text-based resources. With this coding framework in place, teachers could be asked to bring in classroom artifacts and lesson plans in order to reflect upon how their process of instructional design incorporates real-world contexts and with what kinds of sources, roles, and elaboration invited.

Second, the findings on elaboration should be further operationalized into discrete moves that teachers can make in order to broaden and deepen student participation. Elaboration could take on the form of different routine activities which students can work on with varying kinds of mathematical content, such as the observed examples of writing individually before sharing as a whole group. Making elaboration a more explicit goal of professional development activities can thus serve as another tool in a teacher's repertoire for facilitating student participation. These classroom routines and structures would complement the kinds of orchestrating moves that teachers make in whole-class discussions on a more informal and formative level (Stein et al., 2008).

Finally, other ways of breaking down norms of privacy within a group of teachers at the same school, such as video case studies or regular inter-visitations, might be able to refocus professional conversation on the work that teachers do and the professional responsibilities that they share. Looking at student work on the common assessments that will emerge from the implementation of the CCSSM may also provide this common experience that can begin more productive and reflective conversations about practice.

Implications for Research

In this section, I first consider other studies that could be done with the corpus of data accumulated for this project. I then consider how to design and frame new research projects with teachers as collaborators in classroom and professional development settings.

Research associated with this project. Because the nature of teachers' approaches to planning lessons emerged as an important factor in terms of adaptation, more research needs to be done in terms of how teachers approach curriculum design. Further, because teachers were engaged in both regular professional development meeting at the school site as well as other institutes, case studies could examine the extent to which teachers' beliefs about students, mathematics, and teaching further influence their classroom practices. Given the two year scope of the project, teacher change, growth, or stability are also topics with important implications for the design and delivery of future professional development.

Mathematics education research at large. This study suggests that there are two ways to think about future research in mathematics education. One perspective is on the topics or phenomena that would be interesting to research further. Another perspective would be on the design of future research models and the integration of those models with ongoing professional development.

Key topics which arose in this project and require future investigation can be summarized under the following headings and with the following possible research questions.

1) Instructional planning
    i. How do teachers plan for incorporating real-world contexts into mathematics lessons?
    ii. What collaborative structures can change teachers' practices around planning and integrating real-world contexts?
    iii. What tools or conceptual resources or professional development activities can change teachers' planning practices?
2) Modeling
    i. How do teachers understand the new Common Core modeling standards?
    ii. How do teachers implement modeling standards in their lessons?
    iii. What professional development experiences can change teachers' beliefs about and practices with modeling?

iv. What are the impacts of modeling experiences on student engagement and attitudes toward mathematics?

3) Elaboration

i. What are teacher moves or prompts support elaboration?

ii. What professional development experiences support teachers' incorporation of elaboration into lesson?

iii. How can prompts that maximize elaboration be designed?

In terms of research design, given the baseline for teachers' practices that now exists through this study, a more action-research-oriented model might facilitate teachers' growth over time. By creating design experiments in collaboration with teachers, they might see modeled in their own classrooms existence proofs of the kinds of participatory practices such as elaboration that are the goals of professional development. Other research designs, such as dual design research, may also be effective for more directly investigating both teacher and student learning through an iterative process that is more directly collaborative with researchers and connected directly to professional development (Smit & van Erde, 2011).

At the same time, the issue of teacher evaluation through classroom observation has become more and more prominent as the federal Race to the Top program has come to require both test scores and classroom observations as part of revised evaluation systems. The instruments used in this study offer one means to assess the quality of the instructional environment, but a more detailed, leveled rubric for elaboration could complement efforts to measure certain kinds of productive academic interactions in the classroom. In practice, a ratings rubric for elaboration could share features with the rubric for "contextual relationships" described by Jackson et al. (2011), but would generalize to discussions of real-world contexts that are not necessarily the central problem-solving situation.

Returning to the conceptual framework of culturally relevant mathematics pedagogy, this dissertation in its focus on real-world contexts has articulated one component that overlaps with multiple aspects of CureMap. For the quantity and quality of mathematics lessons containing real-world contexts to sustain growth, curriculum policy, teacher education and professional development, and research will all need to align. With renewed efforts around this focus, teachers may be able to develop multiple approaches and tools. This dissertation has put forth three general practices: elaboration, motivating metaphors, and mathematical modeling. While these exist at different levels of accessibility for teachers, together they present promise for deepening the understanding and engagement of students in urban high schools.

Appendix I: Classroom Observation Instrument

Date:	
School:	

Teacher (pseudonym)	
Mathematics course:	

Period:		Scheduled time:	____________ to __________

Time instruction began:		Time instruction ended:	

Number of students in class during lesson		
Genders of students:	female	male

Note any print resources used in this lesson and attach copies to this report.	
Briefly describe the classroom (e.g., walls, charts, maintenance, etc.)	

I. Description of Lesson

Describe what happened in this lesson, including enough rich detail that readers have a sense of having been there. Include:

a. where this lesson fit with the overall unit
b. the focus of this lesson
c. instructional materials used, if any
d. A synopsis of the structure/flow of the lesson
e. Nature and quality of lesson activities, including lecture, class discussion, problem-solving/investigation, seatwork
f. Roles of the teacher and students in the intellectual work of the lesson (e.g. providing problems or questions, proposing conjectures or hypotheses, developing/applying strategies of procedures; drawing, challenging or verifying conclusions)
g. Roles of any other adults in the classroom

This description should stand on its own. Do not be concerned if you repeat information you have already provided elsewhere.

II. CUREMAP

a. How, if at all, did the content of the lesson correspond to mathematizing local or relevant contexts?
b. How, if at all, did the content of the lesson utilize mathematics to describe or analyze societal inequities?
c. How, and under what conditions, did the teacher encourage links between students' informal reasoning and more formal, canonical or sophisticated mathematical thinking
d. How, if at all, did teacher make connections between students' out of school knowledge and mathematical analysis of the problem context?

- **Mathematical Tasks**
  a. Divide the lesson into the classroom activities with which students were engaged. Briefly identify and list these activities in chronological order. Include a descriptive label, identify how students were grouped, and indicate the number of minutes devoted to each activity.
  b. Select the task that occupied the most amount of class time. What appeared to be the goal of the task, from the teacher's perspective? How do you know that?
  c. What appeared to be the mathematical goals of the task, from the teacher's perspective? How do you know that?
  d. As the task was enacted in the classroom, rate its level of cognitive demand (doing mathematics, procedures with connections, procedures without connections, memorization, other). Justify your choice.
  e. What types of participation were made accessible by the task? (What is involved in the "doing" of this task?)
  f. What types of participation were made accessible by the teacher? (In other words, how was "doing" of the task organized by the teacher?)

IV. Participation Structures

1. How many minutes during the lesson were spent:		
	a. on instructional activities ?	
	b. on housekeeping unrelated to the lesson/interruptions/ other non-instructional activities?	
2. Mark time spent on any of the following activities:		
a. Listened to a presentation:	1. By teacher (includes: demonstrations, lectures, media presentations, extensive procedural instructions)	
	2. By student (would include informal, as well as formal, presentations of their work)	
	3. By guest speaker/"expert" serving as a resource	
b. Engaged in discussion/seminar:	1. Whole group	
	2. Small groups/pairs	
c. Engaged in	1. Worked with manipulatives	

problem solving or investigation:	2. Played a game to build or review knowledge/skills	
	3. Followed specific instructions in an investigation	
	4. Had some latitude in designing an investigation	
	5. Recorded, represented and/or analyzed data	
	6 Recognized patterns, cycles or trends	
	7. Evaluated the validity of arguments or claims	
	8. Provided an informal justification or formal proof	
d. Engaged in reading/reflection/ writing about mathematics or science:	1. Read about mathematics/science	
	2. Answered textbook/worksheet questions	
	3. Reflected on readings, activities, problems individually or in groups	
	4. Prepared a written report	
	5. Wrote a description of plan, procedure, or problem-solving process	
	6. Wrote reflections in a notebook or journal	
e. Used technology/audio-visual resource:	1. To develop conceptual understanding	
	2. To learn or practice a skill	
	3. To collect data (e.g., probeware)	
	4. As an analytic tool (e.g., spreadsheets or data analysis)	
	5. As a presentation tool	
	6. For word processing or communications tool (e.g., e-mail, Internet)	
f. Practiced skills	1. Students compete worksheets or answer textbook exercises	
	2. Review/practice to prepare students for Regents test	

V. Instructional materials

Which best describes the source of the instructional materials upon which this lesson was based? (Check one)

	Materials designated for this course from a commercially published textbook
	Materials designated for this course developed by district, school or other non-commercial source
	Materials selected or adapted by the teacher, from a commercially published textbook
	Materials selected or adapted by the teacher, from a non-commercial source
	Materials developed by the teacher

Describe the instructional materials, including publisher, title, date, pages or URL, if applicable.	

List any connections to summer	

math content activities.	
Describe any connections to processes and structures from summer institute	

- **Design of lesson**

	1	2	3	4	5
The design of the lesson reflected careful planning and organization					
The instructional strategies and activities used in this lesson reflected attention to students' experience, preparedness, prior knowledge, and/or learning styles					
The instructional strategies and activities reflected attention to uses of access, equity, and diversity for students (eg cooperative learning, language appropriate strategies/,materials)					
The design of the lesson encouraged a collaborative approach to learning among the students					
The focus and direction of the lesson was often determined by ideas originating with the students					

VII. Implementation

	1	2	3	4	5
The teacher was able to "read" the students' level of understanding and adjusted instruction accordingly					
The teacher had a solid grasp of the mathematics content inherent in this lesson					
Elements of abstraction were encouraged when it was important to do so					
Connections with other content disciplines and or real world phenomena were explored and valued					
Teacher used a variety of representations					
Students used a variety of representations					

- **Discourse and communication**

a. What, if anything, did the teacher do to uncover student thinking? Describe the manner in which the teacher provided opportunities for students to make their thinking public.

b. What, if anything, did the teacher do to draw on students' language or other cultural resources?

c. What questions (or kinds of questions) did students ask during the lesson?

d. How do the students and teacher demonstrate cultural competence through the use of students' vernaculars or "street" language?

IX. Mathematical discourse and communication

1	2	3	4	5
Virtually no features of mathematical discourse and communication occur, or what occurs is of a fill in the blank nature.	Sharing and the development of collective understanding among a few students (or between a single student and the teacher) occur briefly.	There is at least one sustained episode of sharing and developing collective understanding about mathematics that involves (a) a small group of students or b) a small group of students and the teacher. Or, brief episodes of sharing and developing collective understandings occur sporadically throughout the lesson.	There are many sustained episodes of sharing and developing collective understandings about mathematics in which many students (20-50%) participate.	The creation and maintenance of collective understandings permeates the entire lesson. This could include the use of a common terminology and the careful negotiation of meanings. Most students (50-90%) participate
			Rating:	
Detailed explanation				

X. Distribution of Mathematical Authority

a. What did the teacher tell or teach to the students and under what circumstances?
b. What did the teacher let students discover on their own and under what circumstances?
c. How and by whom was the correctness of a mathematical answer or approach determined?
d. (If lesson deals with real-world context) How, if at all, did teacher make connections between students' out of school knowledge and mathematical analysis of the problem context?

XI. Intellectual Support

1	2	3	4	5
Intellectual	Intellectual	Intellectual	Intellectual	Intellectual

support is negative; action/comments by teacher or students result in put-downs of students' academic efforts; students interfere with one another's efforts to learn; and classroom atmosphere for learning is negative.	support is mixed. Both negative and positive behaviors or comments by teacher or students concerning students' academic efforts are observed. The teacher fails to call on students who want to participate repeatedly.	support is neutral or mildly positive. Evidence may be mainly in the form of verbal approval for student effort and work. However, such support tends to be given to students who are already taking initiative in the class, and it tends not to be given to those who are reluctant participants or less articulate or skilled in the subject.	support from the teacher is clearly positive, and there is some evidence of intellectual support among students for their peers. Evidence of special efforts by the teacher take the form of expressions that convey high academic expectations for all, mutual respect, and a need to try hard and risk initial failure.	support is strong; the class is characterized by high academic expectations, challenging work, strong effort, mutual respect, and assistance in achievement for all students. Both teacher and students demonstrate number of these attitudes by soliciting and welcoming contributions from all students who are expected to put forth their best efforts. Broad participation may be an indication that low-achieving students receive intellectual support for learning.
			Rating:	
Detailed explanation:				

XII. Depth of Knowledge and Student Understanding

1	2	3	4	5
Knowledge is very thin because concepts are treated trivially	Knowledge remains superficial and fragmented.	Knowledge is treated unevenly during instruction; deep	Knowledge is relatively deep because the students provide	Knowledge is very deep because the teacher

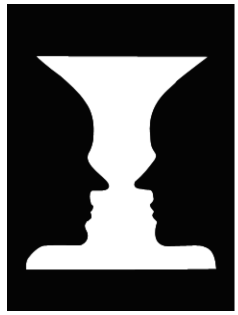
or presented as nonproblematic; students are involved in the coverage of information they are to remember.	Underlying concepts and ideas might be mentioned or covered, but only a superficial acquaintance or trivialized understanding of these ideas is evident.	understanding of some mathematical concepts is countered by superficial understanding of some other ideas. At least one idea may be presented in-depth and its significance grasped by some (10-20%) students, but in general the focus is not sustained.	information, arguments, or reasoning that demonstrates the complexity of one or more ideas. The teacher structures the lesson so that many students (20-50%) do at least one of the following: sustain a focus on a significant topic for a period of time; or demonstrate their understanding of the problematic nature of information and/or ideas; or demonstrate understanding by arriving at a reasoned, supported conclusions or explain how they solved a relatively complex problem.	successfully structures the lesson so that most students (50-90%) do at least one of the following: sustain a focus on a significant topic for a period of time; or demonstrate their understanding of the problematic nature of information and/or ideas; or demonstrate understanding by arriving at a reasoned, supported conclusions or explain how they solved a relatively complex problem. I general, students' reasoning, explanations, and arguments demonstrate fullness and complexity of understanding.
			Rating:	
Detailed explanation:				

XIII. Engagement

1	2	3	4	5
Disruptive disengagement. Students are frequently off-	Passive disengagement. Students appear lethargic and are	Sporadic or episodic engagement. Most students	Engagement is widespread. Most students (50-90%), most	Serious engagement. Almost all of the students (90% or

task, as evidenced by gross inattention or serious disruptions by many students (20-50%); this is the central characteristics during much of the class.	only occasionally on-task carrying out assigned activities; for substantial portions of the time, nay students (20-50% are either clearly off-task or nominally on-task but not trying very hard	(50%-90%) , some of the time (20-50%) are engaged in class activities, but this engagement is uneven, mildly enthusiastic, or depending on frequent prodding from the teacher.	of the time (50-90%), are on-task pursuing the substance of the lesson; most students seem to be taking the work seriously and seem to be trying hard.	more) are deeply involved, almost all of the time (90% or more), in pursuing the substance of the lesson.
			Rating:	
Detailed explanation:				

Appendix II
Images from Teacher A's lesson, 12/9/09



References

Baranes, R., Perry, M., & Stigler, J. (1989). Activation of real-world knowledge in the solution of word problems. *Cognition and Instruction*, 6(4), 287-318.

Barton, B. (1996). Making sense of ethnomathematics: Ethnomathematics is making sense. *Mathematics Teaching and Learning*, 31 (1/2), 201-233.

Bishop, A. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics*, 19(2), 179-191.

Boaler, J. (1993a). The role of contexts in the mathematics classroom: Do they make mathematics more "real"? *For the Learning of Mathematics*, 13(2), 12-17.

Boaler, J. (1993b). Encouraging the transfer of "school" mathematics to the "real world" through the integration of process, content, and culture. *Educational Studies in Mathematics*, 25(4), 341-373.

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education*, 29(1), 41-62.

Bonner, E., & Adams, T. (2012). Culturally resposnsive teaching in the context of mathematics: A grounded theory case study, *Journal of Mathematics Teacher Education*, doi: 10.1007/s10857-011-9198-4

Boyd, D., Grossman, P., Hammerness, K., Lankford, H., Loeb, S., Ronfeldt, M., & Wyckoff, J. (in press). Recruiting effective math teachers: Evidence from New York City. *American Educational Research Journal.* doi:10.3102/0002831211434579

Carraher, D., & Schliemann, A. D. (2002). The transfer dilemma. *Journal of the Learning Sciences*, 11(1), 1-24.

Carraher, T. N., Carraher, D., & Schliemann, A. D. (1987). *Journal for Research in Mathematics Education*, 18(2), 83-97.

Chapman, O. (2006). Classroom practices for context of mathematics word problems. *Educational Studies in Mathematics*, 62(2), 211-230.

Civil, M. (2002). Everyday mathematics, mathematicians' mathematics, and school mathematics: Can we bring them together? *Journal for Research in Mathematics Education Monograph*, 11, 40-62.

Cobb, P., & Jackson, K. (2011). Assessing the Common Core Standards: Opportunities for improving measures of instruction. *Educational Researcher*, 40(4), 186-8.

Cooper, B., & Harries, T. (2009). Realistic contexts, mathematics assessment, and social class. In L. Verschaffel, B. Greer, W. Van Dooren, & S. Mukhopadhyay (Eds.) *Words and worlds: Modelling verbal descriptions of situations* (pp. 93-110). Rotterdam: Sense.

Depaepe, F., De Corte, E., & Verschaffel, L. (2010). Teachers' approaches towards word problem solving: Elaborating or restricting the problem context. *Teaching and Teacher Education*, 26, 152-160.

Dewolf, T., Van Dooren, W., & Verschaffel, L. (2011). Upper elementary school children's understanding and solution of a quantitative problem inside and outside the mathematics class. *Learning and Instruction* , 21, 770-780.

Dierdorp, A., Bakker, A., Eijkelhof, H., & van Maanen, J. (2011). Authentic practices as contexts for learning to draw inferences beyond correlated data. *Mathematical Thinking and Learning*, 13, 132-151.

Doerr, H. M. (2006). Teachers' ways of listening and responding to students' emerging mathematical models. *ZDM*, 38(3), 255-268.

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical

reasoning about data. *Journal for Research in Mathematics Education,* 34(2), 130-136.
Domínguez, H. (2011). Using what matters to students in bilingual mathematics problems. *Educational Studies in Mathematics,* 76, 305-328.
Evans, J. (1999). Building bridges: Reflections on the problem of transfer in mathematics. *Educational Studies in Mathematics,* 39(1/3), 23-44.
Foote, M., Smith, B., & Gellert, M. (2011). Evolution of (urban) mathematics teachers' identity, *Journal of Urban Mathematics Education,* 4(2), 67-95.
Gainsburg, J. (2009). How and why second mathematics teachers make (or don't make) real-world connections in teaching. In L. Verschaffel, B. Greer, W. Van Dooren, & S. Mukhopadhyay (Eds.) *Words and worlds: Modelling verbal descriptions of situations. (pp. 265-281).* Rotterdam: Sense.
Gay, G. (2002). Preparing for culturally responsive teaching. *Journal of Teacher Education,* 53(2), 106-116.
González, L. (2009). Teaching mathematics for social justice: Reflections on a community of practice for high school mathematics teachers. *Journal of Urban Mathematics Education,* 2(1), 21-55.
Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1(2), 155-177.
Gravemeijer, K. (2002). Preamble: From models to modeling. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.) *Symbolizing, modeling and tool use in mathematics education.* (pp. 7-22). Dordecht: Kluwer.
Greer, B. (1997). Modelling reality in mathematics classrooms: The case of word problems. Learning and Instruction, 7(4), 293-307.
Greer, B., Mukhopadhyay, S., Powell, A. B., & Nelson-Barber, S. (2009). *Culturally responsive mathematics pedagogy.* New York: Routledge.
Grossman, P., & Stodolsky, S. (1994). Considerations of content and the circumstances of secondary school teaching. *Review of Research in Education,* 20, 179-221.
Guberman, S. (2004). A comparative study of children's out of school activities and arithmetical achievements. *Journal for Research in Mathematics Education,* 35(2), 117-150.
Gutiérrez, K., & Rogoff, B. (2003). Cultural ways of learning: Individual traits or repertoires of practice. *Educational Researcher,* 32(5), 19-25.
Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education*, *34*, 37-73.
Gutstein, E., Lipman, P., Hernández, P., and de los Reyes, R. (1997). Culturally relevant mathematics teaching in a Mexican American context. *Journal for Research in Mathematics Education*, 38(6), 708-37.
Heck, D. J., Weiss, I. R., & Pasley, J. D. (2011). *A priority research agenda for understanding the influence of Common Core State Standards for Mathematics.* Chapel Hill, NC: Horizon Research.
Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28, 524-549.
Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., et al. (2003). *Teaching mathematics in seven countries. Results from the TIMSS 1999 video study.* Washington, DC: National Center for Educational Statistics.
Hiebert, J. & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grouws

(Ed.), *Handbook of Research on Mathematics Teaching and Learning (*pp. 65-97). Reston, VA : NCTM.
Hiebert, J., & Grouws, D. (2007). The effects of classroom mathematics teaching on students' learning. In F. Lester (Ed.) *Second Handbook of Research on Mathematics Teasching and Learning* (pp. 371-404). Reston, VA: National Council of Teachers of Mathematics.
Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal,* 42, 371-406.
Hoyles, C., Noss, R. and Pozzi, S. (2001). Proportional reasoning in nursing practice. *Journal for Research in Mathematics Education*, 32(1), 4-27.
Ingersoll, R., & Perda, D. (2010). Is the supply of mathematics and science teachers sufficient? *American Educational Research Journal*, 47(3), 563-594.
Inoue, H. (2008). Minimalism as a guiding principle: Linking mathematical learning to everyday knowledge. *Mathematical Thinking and Learning*, 10, 36-67.
Izsák, A. (2003). "We want a statement that is always true": Criteria for good algebraic representations and the development of modeling knowledge. *Journal for Research in Mathematics Education*, 34(3), 191-227.
Jablonka, E. & Gellert, U. (2010). Equity concerns about mathematical modeling. In B. Atweh et al. (Eds.) *Mapping equity and quality in mathematics education*. (pp. 223-236), New York: Springer.
Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2011, April). Investigating how setting up cognitively demanding tasks is related to opportunities to learn in middle-grades mathematics classrooms. Paper presented at the Research Pre-session of the Annual meeting of the National Council of Teachers of Mathematics.
Kaiser, G., & Maass, K. (2007). Modelling in lower secondary mathematics classroom—problems and opportunities. In P. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 99-108), New York: Springer.
Keijzer, R., & Terwel, J. (2003). Learning for mathematical insight: A longitudinal comparative study on modeling. *Learning and Instruction,* 13, 285-304.
Kitchen, R., DePree, J., Celedón-Pattichis, S., & Brinkerhoff, J. (2007). *Mathematics education at highly effective schools that serve the poor: Strategies for change*. Mahwah, NJ: Erlbaum.
Koedinger, K., & Nathan, M. (2004). The real story behind story problems: Effects of representation on quantitative reasoning. *Journal of the Learning Sciences,* 13(2), 129-164.
Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal*, 32(3), 465-491.
Ladson-Billings, G. (1997). It doesn't add up: African American Students' mathematics achievement. *Journal for Research in Mathematics Education,* 28(6), 697-708.
Lambdin, D., & Preston, R. (1995). Caricatures in innovation: Teacher adaptation to an investigation-oriented middle school mathematics curriculum. *Journal of Teacher Education*, 46, 130-140.
Langrall, C., Nisbet, S., Mooney, E., & Jansem, S. (2011). The role of context expertise when comparing data. *Mathematical Thinking and Learning*, 13, 47-67.
Larson, C., Harel, G., Oehrtman, M., Zandieh, M., Rasmussen, C., Speiser, R., et al. (2010). In R. Lesh, et al. (Eds.), *Modeling students' mathematical modeling competencies* (pp. 61-71). New York: Springer.
Lee, J. (in press). Prospective elementary teachers' perceptions of real-life connections reflected

in posing and evaluating story problems. *Journal of Mathematics Teacher Education.*
Legé, J. (2007). "To model, or to let them model?" That is the question. In W. Blum, P. Galbraith, H. Nenn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 425-433). New York: Springer.
Leonard, J. (2008). *Culturally specific pedagogy in the mathematics classroom: Strategies for teachers and students.* New York: Routledge.
Leonard, J., Napp, C., & Adeleke, S. (2009). The complexities of culturally relevant pedagogy: A case study of two secondary mathematics teachers and their ESOL students. *The High School Journal*, 93(1), 3-22.
Leonard, J., Brooks, W. Barnes-Johnson, J., Berry, R. (2010). The nuances and complexities of teaching mathematics for cultural relevance and social justice. *Journal of Teacher Education*, 61(3), 261-270.
Leong, R. K. (2011). Assessment of mathematical modeling. *Journal of Mathematics Education at Teachers College*, 2(1), 62-65.
Lesh, R., & Caylor, B. (2009). Differeing conceptions of problem solving in mathematics education, science education, and professional schools. In L. Verschaffel, B. Greer, W. Van Dooren, & S. Mukhopadhyay (Eds.) *Words and worlds: Modelling verbal descriptions of situations* (pp. 333-350). Rotterdam: Sense.
Lipman, P. (2011). *The new political economy of urban education: Neoliberalism, race, and the right to the city*. New York: Routledge.
Masingila, J. O., Davidenko, S., & Prus-Wisnowska, E. (1996). Mathematics learning and practice in and out of school: A framework for connecting these experiences. *Educational Studies in Mathematics*, 31(1/2), 175-200.
McCulloch, A., & Marshall, P. (2011). K-2 teachers' attempts to connect out-of-school experiences to in-school mathematics learning. *Journal of Urban Mathematics Education*, 4(2), 44-66.
Meagher, M., & Brantlinger, A. (2011). When am I going to learn to be a mathematics teacher? A case study of a novice New York City Teaching Fellow, *Journal of Urban Mathematics Education*, 4(2), 96-130.
Molyneux-Hodgson, S., Rojano, T., Sutherland, R., & Ursini, S. (1999). Mathematical modeling: The interaction of culture and practice. *Educational Studies in Mathematics*, 39(1/3), 167-183.
Morrison, K. A., Robbins, H. H., & Rose, D. G. (2008). Operationalizing culturally relevant pedagogy: A synthesis of classroom-based research. *Equity & Excellence in Education*, 41(4), 433-452.
Moses, R. P. & Cobb, C. E. (2001). *Radical equations: Math literacy and civil rights*. Boston, Massachussetts: Beacon Press.
Mosvold, R. (2008). Real-life connections in Japan and the Netherlands: National teaching patterns and cultural beliefs. *International Journal for Mathematics Teaching and Learning.*
Nasir, N. S. (2000). "Points ain't everything": Emergent goals and average and percent understandings in the play of basketball among African American students. *Anthropology & Education Quarterly*, 31(3), 283-205.
Nasir, N., S, & Hand, V. (2008). From the court to the classroom: Opportunities for engagement, learning, and identity in basketball and classroom mathematics. *Journal of the Learning Sciences*, 17(2), 143-179.

Nasir, N. S., Hand, V., & Taylor, E. V. (2008). Culture and mathematics in school: Boundaries between "cultural" and "domain" knowledge in mathematics classrooms and beyond. *Review of Research in Education,* 32, 187-239.

Pfannkuch, M. (2011). The role of context in developing informal statistical inferential reasoning. *Mathematical Thinking and Learning,* 13, 27-46.

Pollak, H. O. (1968). On some of the problems of teaching applications of mathematics. *Educational Studies in Mathematics*, 1 (1/2), 24-30.

Presmeg, N. (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior,* 17(1), 25-32.

Remillard, J. & Bryans, M. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education*, 35(5), 352-288.

Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, 75(2), 211-246.

Reusser, K., & Stebler, R. (1997). Every word problem has a solution—the social rationality of mathematical modeling in schools. *Learning and Instruction,* 7(4), 309-327.

Richland, L. E., Holyoak, K. J., & Stigler, J. (2004). Analogy use in eighth-grade mathematics classrooms. *Cognition and Instruction,* 22(1), 37-60.

Richland, L. E. (2007). Cognitive supports for analogies in mathematics classrooms. *Science,* 315, 1128-1129.

Rubel, L. H. (2012). Centering the teaching of mathematics on urban youth: learning together about our students and their communities. In J. Bay-Williams and R. Speer (Eds). *Professional Collaborations in Mathematics Teaching and Learning: Seeking Success for All* (NCTM 2012 Yearbook). Reston, VA: NCTM.

Rubel, L. H. (2010). Centering the teaching of mathematics on urban youth: equity pedagogy in action (pp. 25-39). In M. Q. Foote (Ed.) *Mathematics Teaching and Learning in K-12: Equity and Professional Development*. New York: Palgrave.

Rubel, L. H., & Chu, H. (2012). Reinscribing urban: High school mathematics teaching in low-income, urban communities of color. *Journal of Mathematics Teacher Education,* 15(1), 39-52. doi: 10.1007/s10857-011-9200-1.

Rubel, L. H., Chu, H., & Shookhoff, L. (2011). Learning to map and mapping to learn our students' worlds. *Mathematics Teacher,* 104 (8), 586-591.

Rubel, L. H., & Monroe, S. L. (2012, April). *Culturally relevant mathematics pedagogy and student participation.* Paper presented at the annual Research Pre-Session of the National Council of Teachers of Mathematics.

Schwartz, D. L, & Martin, T. (2004). Inventing to prepare for future learning: The hidden efficiency of encouraging original student production in statistics instruction. *Cognition and Instruction*, 22(2), 129-184.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review,* 57, 1-22.

Sierpinska, A. (1995): Mathematics: "in context", "pure" or "with applications"? A contribution to the question of transfer in the learning of mathematics. *For the Learning of Mathematics*, 15(1), 2-15.

Smit, J., & van Eerde, H. A. A. (2011). A teacher's learning process in dual design research: learning to scaffold language in a multilingual mathematics classroom. ZDM, 43, 889-900.

Stein, M. K., Engle, R., Smith, M., & Hughes, E. (2008). Orchestrating productive mathematics discussions: Five practices for helping teachers to move beyond show and tell. Mathematical Thinking and Learning, 10(4), 313-340.
Stein, M. K., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing Standards-based mathematics instruction: A case for professional development.* New York: Teachers College Press.
Sullivan, P., Zevenbergen, R., & Mousley, J. (2003). The contexts of mathematics tasks and the context of the classroom: Are we including all students? *Mathematics Education Research Journal*, 15(2), 107-121.
Tam, K. C. (2011). Modeling in Common Core State Standards. *Journal of Mathematics Education at Teachers College*, 28-33.
Taylor, E. V. (2009). The purchasing practice of low-income students: The relationship to mathematical development. *Journal of the Learning Sciences*, 18, 370-415.
Taylor, E. V. (in press). Supporting children's mathematical understandings: Professional development focused on out-of-school practices. *Journal of Mathematics Teacher Education.* doi: 10.1007/s10857-011-9187-7.
Terwel, J., van Oers, B., van Dijk, I., & van den Eeden, P. (2009). Are representations to be provided or generated in primary mathematics education? Effects on transfer. *Educational Research and Evaluation*, 15(1), 25-44.
Turner, E., Drake, C., McDuffie, A. R., Aguirre, J., Bartell, T. G., & Foote, M. (2012). Promoting equity in mathematics teacher preparation: a framework for advancing teacher learning of children's multiple mathematics knowledge bases. *Journal of Mathematics Teacher Education*, 15(1), 67-82. doi: 10.1007/s10857-011-9196-6
Turner, E., Varley, M., Simic-Muller, K., & Diez-Palomar, J. (2009). "Everything is math in the whole world": Integrating critical and community knowledge in authentic mathematical investigations with elementary Latina/o students. *Mathematical Thinking and Learning,* 11, 136-157.
United States Department of Education. (2012). *Teacher shortage areas nationwide listing 1990-1991 through 2012-2013*. Washington, DC: Author.
Van den Heuvel-Panhuizen, M., & Wijers, M. (2005) Mathematics standards and curricula in the Netherlands, *ZDM*, 37(4), 287-307.
Van den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics,* 25(2), 2-9, 23.
Van Dijk, I., van Oers, B., & Terwel, J. (2003). Providing or designing? Constructing models in primary maths education. *Learning and Instruction*, 13, 53-72.
Verschaffel, L, De Corte, E., & Lasure, S. (1994). Realistic considerations in mathematical modeling of school arithmetic word problems. *Learning and Instruction*, 4, 273-294.
Verschaffel, L, De Corte, E., & Borghard, I. (1997). Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modeling of school word problems. *Learning and Instruction*, 7(4), 339-359.
Verschaffel, L., Van Dooren, W., Chen, L., & Stessens, K. (2009). The relationship between posing and solving division-with-remainder problems among Flemish upper elementary school children. In L. Verschaffel, B. Greer, W. Van Dooren, & S. Mukhopadhyay (Eds.) *Words and worlds: Modelling verbal descriptions of situations* (pp. 143-160). Rotterdam: Sense.
Vomvoridi-Ivanovic (2012). Using culture as a resource in mathematics: The case of four

Mexican-American prospective teachers in a bilingual after-school program. *Journal of Mathematics Teacher Education*, 15(1), 53-66.
Villegas, A., & Lucas, T. (2002). *Educating culturally responsive teachers.* Albany, NY: State University of New York Press.
Wager, A. (2012). Incorporating out-of-school mathematics: from cultural context to embedded practice, *Journal of Mathematics Teacher Education*, 15(1), 9-23. doi: 10.1007/s10857-011-9199-3.
Warwick, J. Some reflections on the teaching of mathematical modeling. *The Mathematics Educator,* 17(1), 32-41.