72-5076

LATCH, Dana May, 1943-
ON DERIVED FUNCTORS OF LIMIT.

The City University of New York, Ph.D.,
1971
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

# ON DERIVED FUNCTORS OF LIMIT

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1971

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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# ACKNOWLEDGMENTS

I am deeply indebted to my adviser, Professor Alex Heller, for his guidance and encouragement, as well as for his patience. Also I wish to thank my husband Daniel for his love and understanding.

Lastly, I wish to express my thanks to The City University of New York and to the National Science Foundation for financial support given me while writing this paper.

## INTRODUCTION

It is well known [11] that if $\mathcal{A}$ is a cocomplete abelian category and $\Omega$ is any small category, then the functor $\operatorname{colim}_{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$ is right exact; and that if $\mathcal{A}$ has enough projectives, the left derived functors $L_* \operatorname{colim}_{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$ exist. One is naturally led to study the vanishing of these left derived functors as a measure of the exactness of $\operatorname{colim}_{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$. Dually, if $\mathcal{A}$ is a complete abelian category with enough injectives, one is interested in the vanishing of the right derived functors of $\operatorname{Lim}_{\Omega^{op}} : \mathcal{A}^{\Omega^{op}} \to \mathcal{A}$ for small $\Omega$.

The problem was first studied by Roos [15] who considered conditions under which derived functors of colimit exist, and showed that these functors were related to the homology of the simplicial realization of $\Omega$. Using these results, Laudal [10] and Nobeling [13] exhibited conditions on $\Omega$ under which higher derived functors vanished.

This paper explores another method for determining when derived functors of high degree vanish for $\Omega$, a pre-ordered $\downarrow$-finite set. The problem is divided into two parts:

(i) an "infinite" part which depends on the cardinality of $\Omega$

(ii) a "finite" part which vanishes if $\Omega$ is directed.

As a first step, $\mathcal{A}$ is embedded in the AB5 category $D(\mathcal{A})$ of directed diagrams over $\mathcal{A}$. The embedding $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is shown to be exact, Ext-preserving, and projective-preserving; but most important, to satisfy a universal extension property.

$D(\mathcal{A})$ is more or less equivalent to Grothendieck's category of Pro-objects of $\mathcal{A}$ [6] and is similar to the cocontinuous extension of $\mathcal{A}$ studied by Hilton [8].

If $\mathcal{A}$ is cocomplete, we apply the universal extension property to $id_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ to get a coreflection $\Psi : D(\mathcal{A}) \to \mathcal{A}$ of $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ . Then these two functors together give rise to a factorization of

$$\underset{\Omega}{\text{colim}} : \mathcal{A}^{\Omega} \to \mathcal{A} \quad \text{into} \quad \mathcal{A}^{\Omega} \xrightarrow{\underset{\Omega}{\text{colim}}\, J^{\Omega}} D(\mathcal{A}) \xrightarrow{\Psi} \mathcal{A} .$$

When $\Omega$ is a pre-ordered $\downarrow$-finite set, we apply the Grothendieck Two Functor theorem [5] to the above factorization of $\underset{\Omega}{\text{colim}} : \mathcal{A}^{\Omega} \to \mathcal{A}$ . This results in the spectral sequence $E^2 = (L_*\Psi)(L_*\underset{\Omega}{\text{colim}})(J^{\Omega}(\bar{A}))$ which converges to $(L_*\underset{\Omega}{\text{colim}})(\bar{A})$ . It is here that the two parts of the problem become apparent:

(i) the derived functors of $\Psi : D(\mathcal{A}) \to \mathcal{A}$ give the "infinite" part

(ii) the derived functors of $\underset{\Omega}{\text{colim}}\, J^{\Omega} : \mathcal{A}^{\Omega} \to D(\mathcal{A})$, the "finite" part.

Finally using a generalization of a result of Osofsky [14], and the above spectral sequence, we determine when the right derived functors of $\underset{\Omega^{op}}{\text{Lim}} : \mathcal{A}b^{\Omega^{op}} \to \mathcal{A}b$ will vanish. If $\Omega$ is a $\downarrow$-finite ordered set such that the cardinality of $\Omega$ is less than $\aleph_n$ and if $\left(L_p \underset{\Omega}{\text{colim}}\right) : \mathcal{A}b^{\Omega} \to \mathcal{A}b$ vanishes for every $p > k$ , then $\left(R^q \underset{\Omega^{op}}{\text{Lim}}\right) : \mathcal{A}b^{\Omega^{op}} \to \mathcal{A}b$ will vanish for $q > n + 1 + k$ .

# CHAPTER I

## PRELIMINARIES

### Section 1: Projectives in $\mathcal{A}^{\Omega}$.

It is well known that if $\Omega$ is a small category and $\mathcal{A}$ is cocomplete abelian category, then $\mathcal{A}^{\Omega}$, the category of all diagrams of type $\Omega$, i.e., covariant functors from $\Omega$ to $\mathcal{A}$, is a cocomplete abelian category [11].

Most of the properties of $\mathcal{A}^{\Omega}$ appear "pointwise;" e.g., if $\eta : \bar{A}' \to \bar{A}$ is a natural transformation, then $\ker(\eta)$ is given by $(\ker(\eta))_{\omega} = \ker(\eta_{\omega} : A'_{\omega} \to A_{\omega})$ .

In addition, if $\mathcal{A}$ has enough projectives, then $\mathcal{A}^{\Omega}$ will also have enough projectives. But the definition of a projective $\bar{P}$ is not a "pointwise" definition. Because this paper is concerned with conditions under which functors from $\mathcal{A}^{\Omega}$ to particular abelian categories preserve projectives, it is useful to understand the construction of projectives in $\mathcal{A}^{\Omega}$ .

Suppose $\mathcal{A}$ is a cocomplete abelian category and $\Omega$ is a small category. Then for each object $\omega \in \Omega$, there are two associated functors between $\mathcal{A}$ and $\mathcal{A}^{\Omega}$.

The first is the canonical evaluation functor $ev_{\omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$ defined by $ev_{\omega}(\bar{A}) = A_{\omega}$ where $\bar{A} \in \mathcal{A}^{\Omega}$. It is exact since exactness in $\mathcal{A}^{\Omega}$ is "pointwise."

The second functor is $E_{\omega} : \mathcal{A} \to \mathcal{A}^{\Omega}$ which is constructed in the following way. For each object $X \in \mathcal{A}$, let $(E_{\omega}X)_{\mu} = \coprod_{\omega \xrightarrow{a} \mu} X$ ,

$\mu \in \text{ob}\Omega$ and let $(E_\omega X)(b) : (E_\omega X)\mu \to (E_\omega X)\mu'$ , $b : \mu \to \mu'$ in $\Omega$ , be the canonical morphism such that $(E_\omega X)(b)u_a = u_{ba}$ where $u_a : A \to \coprod_{\omega \overset{a}{\to} \mu} A$ is the natural inclusion into the coproduct. Similarly, for each morphism $f : X \to Y$ , there is a natural transformation $(E_\omega f) : E_\omega X \to E_\omega Y$ defined by $(E_\omega f)u_a = u_a \cdot f$ .

<u>Lemma 1.1</u>: $E_\omega : \mathcal{A} \to \mathcal{A}^\Omega$ is the coadjoint of $ev_\omega : \mathcal{A}^\Omega \to \mathcal{A}$ .

<u>Proof</u>: It suffices to show $\mathcal{A}^\Omega(E_\omega X, A)$ is isomorphic to $\mathcal{A}(X, A_\omega)$ for $X \in \mathcal{A}$ and $\bar{A} \in \mathcal{A}^\Omega$. Define $R : (E_\omega X, A) \to (X, A_\omega)$ by $R(\eta) = \eta_\omega \cdot u_{id_\omega}$ , $\eta : E_\omega X \to \bar{A}$ . Also define $S : (X, A_\omega) \to (E_\omega X, \bar{A})$ by $S(\theta)_\mu \cdot u_a = A(a) \cdot \theta$ , $\theta : X \to A_\omega$ . Then it is easily seen that $R \cdot S$ and $S \cdot R$ are both identities, and therefore $(E_\omega X, \bar{A}) \cong (X, A_\omega)$ .

In particular, when $\Omega$ is a pre-ordered set, there can be at most one morphism from $\omega$ to $\mu$ in $\Omega$ , and therefore

$$(E_\omega X)_\mu = \begin{cases} X & \text{if } \mu \geq \omega \\ 0 & \text{otherwise} \end{cases} \text{, for } X \in \mathcal{A} .$$

<u>Corollary 1.2</u>: $E_\omega : \mathcal{A} \to \mathcal{A}^\Omega$ preserves projectives.

<u>Proof</u>: This follows immediately from the fact that $E_\omega : \mathcal{A} \to \mathcal{A}^\Omega$ is coadjoint of the exact functor $ev_\omega : \mathcal{A}^\Omega \to \mathcal{A}$ .

<u>Proposition 1.3</u>: If $\mathcal{A}$ has enough projectives, then $\mathcal{A}^\Omega$ also has enough projectives.

<u>Proof</u>: Suppose $\bar{A} \in \mathcal{A}^\Omega$ . For each $\omega \in \Omega$ , there is a projective $P_\omega$ and an epimorphism $p_\omega : P_\omega \to A_\omega$ . Since $E_\omega$ is coadjoint to $ev_\omega$ , there is an unique natural transformation $E_\omega p_\omega : E_\omega P_\omega \to \bar{A}$ with $(E_\omega p_\omega)_\omega = p_\omega$ and $E_\omega P_\omega$ projective. Set $\bar{P} = \coprod_\omega E_\omega P_\omega$ and $\bar{p} = \coprod_\omega E_\omega p_\omega$ .

Then $\bar{p} : \bar{P} \to \bar{A}$ is an epimorphism with projective $\bar{P}$ , since coproducts preserve epimorphisms and projectives.

A projective $\bar{P}$ in $\mathcal{A}^{\Omega}$ is called a <u>canonical projective</u> if $\bar{P} = \coprod_{\omega} E_{\omega} P_{\omega}$ . The above proposition insures that $\mathcal{A}^{\Omega}$ has enough canonical projectives. Therefore, to show that a functor $\Phi : \mathcal{A}^{\Omega} \to \mathcal{C}$ preserves projectives, it is only necessary to show that it maps a canonical projective in $\mathcal{A}^{\Omega}$ to a projective in $\mathcal{C}$ .

## Section 2: A Construction in the Category of Directed Ordered Sets.

The following construction in the category of directed (upward) ordered sets and order preserving maps relates a directed ordered set $\Lambda$ to the directed ordered set $\mathfrak{F}(\Lambda)$ , the set of all finite subsets of $\Lambda$ ordered by inclusion. The construction actually yields a cofinal order preserving map $\theta : \mathfrak{F}(\Lambda) \to \Lambda$ .

Let $\Lambda$ be a directed (upward) ordered set. Then there is a "multiplication" $m : \Lambda \times \Lambda \to \Lambda$ such that $m(\lambda_0, \lambda_1) \geq \lambda_0, \lambda_1$ . Generally, there are many such m's ; we pick one. We define "inductively" set maps $m_n : \Lambda^{n+1} \to \Lambda$ , $n \geq 0$ by:

$$m_0 = id_{\Lambda} : \Lambda \to \Lambda$$

$$m_1 = m : \Lambda \times \Lambda \to \Lambda$$

$$m_n : \Lambda^{n+1} \to \Lambda , \; m\,(m_{n-1}(\{\lambda_1, \ldots, \lambda_n\}), \ldots, m_{n-1}(\{\lambda_0, \ldots, \hat{\lambda}_i, \ldots, \lambda_n\}), \ldots, m_{n-1}(\{\lambda_0, \ldots, \lambda_{n-1}\}) .$$

It is clear that for each $n \geq 0$ ,

$$m_n(\lambda_0, \ldots, \lambda_n) \geq \lambda_0, \ldots, \lambda_n .$$

Using the definition of coproduct in $S$ , the category of sets, there is a unique map $m_* : \coprod_{n \geq 0} \Lambda^{n+1} \to \Lambda$ . Even though the set of elements of $\mathfrak{F}(\Lambda)$ is isomorphic to $\bigcup_{n\geq 0} \Lambda^{n+1}$ , the map $m_* : \bigcup_{n\geq 0} \Lambda^{n+1} \to \Lambda$ is not generally order preserving.

Thus using $m_*$ , we define a new map $\theta : \mathfrak{F}(\Lambda) \to \Lambda$ which will be shown to preserve the ordering of $\mathfrak{F}(\Lambda)$ and be cofinal:

(i) $\theta_0 : \Lambda \to \Lambda$ is defined by $\theta_0(\{\lambda\}) = \lambda$ and

(ii) $\theta_n : \Lambda^{n+1} \to \Lambda$ is defined by

$$\theta_n(\{\lambda_0,\ldots,\lambda_n\}) = m_n(\theta_{n-1}(\{\lambda_1,\ldots,\lambda_n\}),\ldots,\theta_{n-1}(\{\lambda_0,\ldots,\hat{\lambda}_i,\ldots,\lambda_n\}),\ldots,\theta_{n-1}(\{\lambda_0,\ldots,\lambda_{n-1}\}) .$$

From the definition of $m_*$ , it is clear that

$$(*) \qquad \theta_n(\{\lambda_0,\ldots,\lambda_n\}) \geq \theta_{n-1}(\{\lambda_0,\ldots,\hat{\lambda}_i,\ldots,\lambda_n\}),\ 0 \leq i \leq n$$

Set $\theta = \bigcup_{n\geq 0} \theta_n : \mathfrak{F}(\Lambda) \to \Lambda$ .

Proposition 1.4: $\theta : \mathfrak{F}(\Lambda) \to \Lambda$ is a cofinal order preserving map of directed ordered sets.

Proof: Suppose $\{\lambda_0,\ldots,\lambda_m\} \subseteq \{\bar{\lambda}_0,\ldots,\bar{\lambda}_n\}$ for $m \leq n$ . Without loss of generality, we assume that $\{\lambda_0,\ldots,\lambda_m\} \subseteq \{\lambda_0,\ldots,\lambda_m,\lambda_{m+1},\ldots,\lambda_n\}$ . Using formula (*) recursively $n - k$ times, it follows that

$$\theta_n(\lambda_0,\ldots,\lambda_n) \geq \theta_{n-1}(\lambda_0,\ldots,\lambda_{n-1}) \geq \ldots \geq \theta_m(\lambda_0,\ldots,\lambda_m) ,$$

and therefore $\theta$ is order preserving. Obviously $\theta$ is cofinal and onto, since $\theta(\{\lambda\}) = \lambda \geq \lambda$ , for every $\lambda \in \Lambda$ .

## Section 3: A Factorization of $\operatorname{colim}_{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$

The factorization which is developed below, is used a number of times in Chapter III.

Suppose $\Omega$ is an ordered set. Then $\mathfrak{F}(\Omega)$ clearly satisfies a number of conditions:

(i) $\emptyset$ is the initial element of $\mathfrak{F}(\Omega)$ , i.e., $\emptyset \subseteq u$ for every $u \in \mathfrak{F}(\Omega)$

(ii) $\mathfrak{F}(\Omega)$ is $\downarrow$-finite, i.e. $In(u) = \{v \mid v \subseteq u\}$ is finite for every $u \in \mathfrak{F}(\Omega)$

(iii) $\mathfrak{F}(\Omega)$ is directed (upward)

(iv) if the cardinality of $\Omega$ is less than $\aleph_n$ , then the cardinality of $\mathfrak{F}(\Omega)$ is also less than $\aleph_n$ .

Let $W : \mathcal{A}^{\Omega} \to \mathcal{A}^{\mathfrak{F}(\Omega)}$ denote the functor which is given by $(W\bar{A})_u = \operatorname{colim}_u \bar{A}/u$ with $(W\bar{A})_u^v : (W\bar{A})_u \to (W\bar{A})_v$ the canonical map of colimits induced by the inclusion $u \subseteq v$ .

**Proposition 1.5:** If $\mathcal{A}$ is a cocomplete abelian category and $\Omega$ is an ordered set, then

$$\begin{array}{ccc} \mathcal{A}^{\Omega} & \xrightarrow{\quad W \quad} & \mathcal{A}^{\mathfrak{F}(\Omega)} \\ & {\scriptstyle \operatorname{colim}_{\Omega}} \searrow \quad \swarrow {\scriptstyle \operatorname{colim}_{\mathfrak{F}(\Omega)}} & \\ & \mathcal{A} & \end{array}$$

commutes.

**Proof:** Let $A_\omega \xrightarrow{p_\omega} \operatorname{colim}_\Omega \bar{A}$ , $A_\omega \xrightarrow{q^u_\omega} (W\bar{A})_u$ , and $(W\bar{A})_u \xrightarrow{r_u} \operatorname{colim}_{\mathfrak{F}(\Omega)}(W\bar{A})$ denote the canonical injections for each respective colimit. From the definition of colimit, it suffices to find two morphisms $f : \operatorname{colim}_\Omega \bar{A} \to \operatorname{colim}_{\mathfrak{F}(\Omega)} W\bar{A}$ and $g : \operatorname{colim}_{\mathfrak{F}(\Omega)} W\bar{A} \to \operatorname{colim}_\Omega \bar{A}$ such that $f \cdot g$ and $g \cdot f$ are both identities. To define $f : \operatorname{colim}_\Omega \bar{A} \to \operatorname{colim}_{\mathfrak{F}(\Omega)} W\bar{A}$ set $f_\omega = r_{\{\omega\}} \cdot q^{\{\omega\}}_\omega : A_\omega \to \operatorname{colim}_{\mathfrak{F}(\Omega)} W\bar{A}$ . Since $\omega \in u$ implies $(W\bar{A})^u_{\{\omega\}} \cdot q^{\{\omega\}}_\omega = q^u_\omega$ , it follows easily that $f_{\omega'} A^{\omega'}_\omega = f_\omega$ whenever $\omega \le \omega'$ . Therefore, there is a unique morphism $f : \operatorname{colim}_\Omega \bar{A} \to \operatorname{colim}_{\mathfrak{F}(\Omega)} W\bar{A}$ such that $f \cdot p_\omega = r_{\{\omega\}} \cdot g^{\{\omega\}}_\omega$ . $g : \operatorname{colim}_{\mathfrak{F}(\Omega)} W\bar{A} \to \operatorname{colim}_\Omega \bar{A}$ is the unique morphism defined by the property $gr_u : (W\bar{A})_u \to \operatorname{colim}_\Omega \bar{A}$ is the canonical morphism induced by $u \subseteq \Omega$ . By standard arguments $f \cdot g$ and $g \cdot f$ are both identities.

# CHAPTER II

## THE AB5 CATEGORY $D(\mathcal{A})$

For any category $\mathcal{C}$, we shall construct an associated category $D(\mathcal{C})$, of directed diagrams over $\mathcal{C}$, and an enbedding $J : \mathcal{C} \hookrightarrow D(\mathcal{C})$ In particular, if $\mathcal{A}$ is abelian with enough projectives, $D(\mathcal{A})$ is an AB5 category with enough projectives. Furthermore, the embedding $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is exact, Ext-preserving and projective-preserving. $D(\mathcal{A})$ also solves the universal problem of extending right exact functors from $\mathcal{A}$ to a cocomplete abelian category $\mathcal{B}$, to cocontinuous functors from $D(\mathcal{A})$ to $\mathcal{B}$.

Throughout this chapter, $\mathcal{C}$ will denote an arbitrary category with a zero object, $\mathcal{A}$ an abelian category.

### Section 1: The Category $D(\mathcal{C})$ and the Embedding $J : \mathcal{C} \hookrightarrow D(\mathcal{C})$.

We shall say that $\Omega$ is a ↓-finite ordered set if $In(\omega) = \{\omega' | \omega' \le \omega\}$ is finite for every $\omega \in \Omega$. Analgously, $\Omega$ is ↑-finite if $T(\omega) = \{\omega' | \omega' \ge \omega\}$ is finite for every $\omega \in \Omega$.

We shall construct $D(\mathcal{C})$ as a quotient of a category $PD(\mathcal{C})$. The objects of $PD(\mathcal{C})$ are pairs $(\Lambda,\bar{A})$ where $\Lambda$ is a ↓-finite directed ordered set with initial element $0$, and $\bar{A} : \Lambda \to \mathcal{C}$ is a diagram in $\mathcal{C}^{\Lambda}$ such that $\bar{A}_0 = 0$. The morphisms of $PD(\mathcal{C})$, called pre-morphisms, are pairs $(\eta,\bar{n}) : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ where $\eta : \Lambda \to \Gamma$ is an ordered preserving map such that $\eta(0) = 0$, and $\bar{n} : \bar{A} \to \bar{B}\cdot\eta$ is a map of diagrams in $\mathcal{C}^{\Lambda}$. Composition is given by the rule

$$(\xi,\bar{s})\cdot(\eta,\bar{n}) = (\xi\cdot\eta,\bar{s}_{\eta}\cdot\bar{n}), \quad (\bar{s}_{\eta}\cdot\bar{n})_{\lambda} = s_{\eta(\lambda)}\cdot n_{\lambda} ,$$

where

$$(\eta,\bar{n}) : (\Lambda,\bar{A}) \to (\Gamma,\bar{B}) \quad \text{and} \quad (\xi,\bar{s}) : (\Gamma,\bar{B}) \to (\Delta,\bar{C}) .$$

In particular, if $(\eta,\bar{n}) : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ is a pre-morphism and $\varphi : \Lambda \to \Gamma$ is an order preserving map such that $\varphi \geq \eta$ (i.e., $\varphi(\lambda) \geq \eta(\lambda)$ for every $\lambda \in \Lambda$) and $\varphi(0) = 0$, then $(\varphi,\bar{n}^{\varphi}) : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ is also a pre-morphism where $(\bar{n}^{\varphi})_{\lambda} = B_{\eta(\lambda)}^{\varphi(\lambda)} \cdot n_{\lambda}$ .

In order to construct $D(\mathcal{C})$, the category of directed diagrams over $\mathcal{C}$, we introduce a congruence in $PD(\mathcal{C})$. Suppose $(\eta,\bar{n})$ and $(\mu,\bar{m})$ are both pre-morphisms from $(\Lambda,\bar{A})$ to $(\Gamma,\bar{B})$. Then $(\mu,\bar{m}) \sim (\eta,\bar{n})$ if and only if $\bar{n}^{\varphi} = \bar{m}^{\varphi}$ for some $\varphi \geq \eta,\mu$. The following technical lemma is needed to show $\sim$ is actually a congruence in $PD(\mathcal{C})$.

<u>Lemma 2.1</u> If $\varphi : \Lambda \to \Gamma$ and $\psi : \Lambda \to \Gamma$ are two order and basepoint preserving mappings between $\downarrow$-finite directed ordered sets with initial element, then there is a basepoint and order preserving mapping $\epsilon : \Lambda \to \Gamma$ such that $\epsilon \geq \varphi,\Psi$ .

<u>Proof</u>: Since $\Lambda$ is a $\downarrow$-finite ordered set,

$$\Lambda = \bigcup_{r\geq 0} \Lambda_r \quad \text{where}$$

$$\Lambda_0 = \{0\}$$

$$\Lambda_1 = \{\lambda | \lambda' < \lambda \text{ implies } \lambda' = 0\}$$

$$\Lambda_r = \{\lambda | \lambda' < \lambda \text{ implies } \lambda' \in \bigcup_{i=0}^{r-1} \Lambda_i\} .$$

We define $\epsilon : \Lambda \to \Gamma$ inductively on the sets $\Lambda_r$. Let $\epsilon(0) = 0$. Suppose $\lambda_r \in \Lambda_r$, and set $\Gamma(\lambda_r)$ equal to $\{\Psi(\lambda_r), \varphi(\lambda_r), \epsilon(\lambda') | \lambda' < \lambda\}$ .

(Note that since $\lambda' < \lambda_r$ if and only if $\lambda' \in \Lambda_{r'}$, $r' < r$, $\epsilon(\lambda')$ has already been defined inductively.) $\Gamma(\lambda_r) \subseteq \Gamma$ is finite since $\Lambda$ is $\downarrow$-finite. Therefore there is an $\epsilon(\lambda_r) \in \Gamma$ such that $\epsilon(\lambda_r)$ is larger than any element of $\Gamma(\lambda_r)$. Let $\epsilon = \bigsqcup_{r\geq 0} \epsilon_r : \Lambda \to \Gamma$.

$\epsilon : \Lambda \to \Gamma$ is order preserving, for $\lambda' < \lambda$ insures that $\epsilon(\lambda) \geq \epsilon(\lambda')$ since $\epsilon(\lambda') \in \Gamma(\lambda)$. Also, from the definition of $\epsilon : \Lambda \to \Gamma$, $\epsilon \geq \varphi, \Psi$.

<u>Proposition 2.2</u> $\sim$ is a congruence in $PD(\mathcal{C})$.

<u>Proof</u>: From the definition of $\sim$, it is clear that $\sim$ is reflexive and symmetric. To show that $\sim$ is transitive, suppose $(\eta,\bar{n}) \sim (\mu,\bar{m})$ and $(\mu,\bar{m}) \sim (\rho,\bar{p})$ are premorphisms from $(\Lambda,\bar{A})$ to $(\Gamma,\bar{B})$. Then there are order and basepoint preserving maps $\varphi,\Psi : \Lambda \to \Gamma$ such that $\varphi \geq \eta,\mu$, $\Psi \geq \mu,\omega$, and $\bar{n}^{\varphi} = \bar{m}^{\varphi}$, $\bar{m}^{\Psi} = \bar{p}^{\Psi}$. The above lemma insures that there is an $\epsilon : \Lambda \to \Gamma$ such that $\epsilon \geq \Psi,\varphi$, and therefore $\epsilon \geq \eta,\rho$. But for every $\lambda \in \Lambda$,

$$n_\lambda^\epsilon = B_{\eta(\lambda)}^{\epsilon(\lambda)} n_\lambda = B_{\varphi(\lambda)}^{\epsilon(\lambda)} n_\lambda^\varphi = B_{\varphi(\lambda)}^{\epsilon(\lambda)} m_\lambda^\varphi = B_{\mu(\lambda)}^{\epsilon(\lambda)} m_\lambda$$

and

$$p_\lambda^\epsilon = B_{\rho(\lambda)}^{\epsilon(\lambda)} p_\lambda = B_{\Psi(\lambda)}^{\epsilon(\lambda)} p_\lambda^\Psi = B_{\Psi(\lambda)}^{\epsilon(\lambda)} m_\lambda^\Psi = B_{\mu(\lambda)}^{\epsilon(\lambda)} m_\lambda$$

together imply $\bar{p}^{\epsilon} = \bar{n}^{\epsilon}$. Thus, $(\eta,\bar{n}) \sim (\rho,\bar{p})$ and $\sim$ is transitive.

Suppose $(\eta,\bar{n}) \sim (\mu,\bar{m})$ with $\varphi \geq \eta,\mu$ and $(\xi,\bar{s}) : (\Gamma,\bar{B}) \to (\Delta,\bar{C})$. Then $\xi\varphi \geq \xi\eta, \xi\mu$. Also $(\bar{s}_\eta \cdot \bar{n})^{\xi\varphi} = (\bar{s}_\mu \cdot \bar{m})^{\xi\varphi}$ since for every $\lambda \in \Lambda$,

$$(\bar{s}_\eta \cdot \bar{n})_\lambda^{\xi\varphi} = C_{\xi\eta(\lambda)}^{\xi\varphi(\lambda)} s_{\eta(\lambda)} n_\lambda = s_{\varphi(\lambda)} B_{\eta(\lambda)}^{\varphi(\lambda)} n_\lambda$$

$$= s_{\varphi(\lambda)} n_\lambda^\varphi = s_{\varphi(\lambda)} m_\lambda^\varphi$$

$$= C_{\xi\mu(\lambda)}^{\xi\varphi(\lambda)} s_{\mu(\lambda)} m_\lambda = (\bar{s}_\mu \cdot \bar{m})_\lambda^{\xi\varphi}$$

Therefore $(\xi,\bar{s})\cdot(\eta,\bar{n}) \sim (\xi,\bar{s})\cdot(\mu,\bar{m})$ . A similar argument shows that composition on the right commutes with $\sim$ . Thus $\sim$ is a congruence in $PD(\mathcal{C})$ .

$D(\mathcal{C})$ , <u>the category of directed diagrams</u> over $\mathcal{C}$, is defined to be the quotient category $PD(\mathcal{C})/\sim$ .

<u>Remark:</u> Actually this category is equivalent to the category whose objects are $(\Lambda,\bar{A})$ where $\Lambda$ is only $\downarrow$-finite and directed, and whose morphisms do not necessarily preserve initial elements. Also the assumption that $\mathcal{C}$ has a zero object is not needed. However, by adding the basepoints, many of the constructions in the paper are simplified.

Suppose $\operatorname{colim}_\Lambda \bar{A}$ and $\operatorname{colim}_\Gamma \bar{B}$ exist in $\mathcal{C}$ . Then a pre-morphism $(\eta,\bar{n}) : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ defines a morphism <u>$\operatorname{colim}(\eta,\bar{n})$</u> : $\operatorname{colim}_\Lambda \bar{A} \to \operatorname{colim}_\Gamma \bar{B}$ in $\mathcal{C}$ by the rule $\operatorname{colim}(\eta,\bar{n})\cdot u_\lambda = u_{\eta(\lambda)} n_\lambda$ for $\lambda \in \Lambda$ , where $u_\lambda : A_\lambda \to \operatorname{colim}_\Lambda \bar{A}$ and $u_\gamma : B_\gamma \to \operatorname{colim}_\Gamma \bar{B}$ are canonical inclusions.

<u>Proposition 2.3:</u> If $\operatorname{colim}_\Lambda \bar{A}$ and $\operatorname{colim}_\Gamma \bar{B}$ exist in $\mathcal{C}$ and $(\eta,\bar{n})$ and $(\mu,\bar{m})$ are congruent pre-morphisms from $(\Lambda,\bar{A})$ to $(\Gamma,\bar{B})$ , then $\operatorname{colim}(\eta,\bar{n}) \cong \operatorname{colim}(\mu,\bar{m})$ .

<u>Proof:</u> Let $\varphi \geq \eta,\mu$ such that $\bar{n}^\varphi = \bar{m}^\varphi$ . Then for every $\lambda$ ,

$$\mathrm{colim}(\eta,\bar{n})u_\lambda = u_{\eta(\lambda)}n_\lambda = u_{\varphi(\lambda)}n_\lambda^\varphi \quad \text{and} \quad \mathrm{colim}(\mu,\bar{m})u_\lambda = u_{\mu(\lambda)} = u_{\varphi(\lambda)}m_\lambda^\varphi$$

together insure that $\mathrm{colim}(\eta,\bar{n}) \cong \mathrm{colim}(\mu,\bar{m})$ .

Furthermore, if $\mathcal{C}$ is cocomplete, then there is a functor $\mathrm{colim} : PD(\mathcal{C}) \to \mathcal{C}$ which is defined by setting $\mathrm{colim}\,(\Lambda,\bar{A})$ equal to $\mathrm{colim}_\Lambda \bar{A}$ and $(\eta,\bar{n})$ equal to the unique morphism from $\mathrm{colim}_\Lambda \bar{A}$ to $\mathrm{colim}_\Gamma \bar{B}$ defined above. Proposition 2.3 shows that this functor factors through $D(\mathcal{C})$.

Lastly, we construct a functor $J : \mathcal{C} \hookrightarrow D(\mathcal{C})$ in the following fashion. Let $\mathbb{1}$ denote the directed ordered set $\{0,1\}$ with ordering $0 < 1$ . For each object $A \in \mathcal{C}$ , set $J(A)$ equal to $(\mathbb{1},\hat{A})$ where $\hat{A}_0 = 0$ and $\hat{A}_1 = A$ . Similarly, for each morphism $f : A \to B$ in $\mathcal{C}$, $J(f)$ is defined to be $(id_{\mathbb{1}},\hat{f})$ , where $\hat{f}_0 = 0$ and $\hat{f}_1 = f$ . Clearly, $J : \mathcal{C} \hookrightarrow D(\mathcal{C})$ is an embedding.

Thus, we have constructed the category of directed diagrams over $\mathcal{C}$ , $D(\mathcal{C})$ , and embedded $\mathcal{C}$ into $D(\mathcal{C})$ . This category is equivalent to the dual of the category of Pro-objects of $\mathcal{C}^{op}$ studied by Grothendieck [6].

In the next sections, $D(\mathcal{C})$ is shown to be an abelian category whenever $\mathcal{C}$ is abelian. In fact, $D(\mathcal{C})$ is actually an AB5 category. Furthermore, the embedding $J : \mathcal{C} \hookrightarrow D(\mathcal{C})$ is exact, Ext-preserving, and projective-preserving as well as satisfying a universal extension property.

## Section 2: $\mathcal{A}$ abelian implies $D(\mathcal{A})$ is abelian and cocomplete.

Throughout the rest of this chapter $\mathcal{A}$ will denote an abelian category. The fact that $D(\mathcal{A})$ is abelian is demonstrated by the following series of lemmas.

Lemma 2.4 $D(\mathcal{O})$ is additive; i.e., $D(\mathcal{O})((\Lambda,\bar{A}),(\Gamma,\bar{B}))$ is an abelian group for every $(\Lambda,\bar{A}),(\Gamma,\bar{B}) \in D(\mathcal{O})$ .

Proof: Suppose $(\eta,\bar{n})$ and $(\mu,\bar{m})$ are pre-morphisms from $(\Lambda,\bar{A})$ to $(\Gamma,\bar{B})$ . Lemma 2.2 insures that there is a $\varphi : \Lambda \to \Gamma$ such that $\varphi \geq \eta,\mu$ . Define $[(\eta,\bar{n})] + [(\mu,\bar{m})] = [(\varphi,\bar{n}^{\varphi} + \bar{m}^{\varphi})]$ , where $[\ ]$ denotes the equivalence class of a pre-morphism in $D(\mathcal{O})$ .

To show that the definition of addition is independent of the choice of representatives and of $\varphi : \Lambda \to \Gamma$ , suppose $(\eta,\bar{n}) \sim (\eta',\bar{n}')$ and $(\mu,\bar{m}) \sim (\mu',\bar{m}')$ with $\theta \geq \eta,\eta'$ and $\omega \geq \mu,\mu'$ such that $\bar{n}^{\theta} = \bar{n}'^{\theta}$ and $\bar{m}^{\omega} = \bar{m}'^{\omega}$ . Choose $\Psi : \Lambda \to \Gamma$ so that $\Psi \geq \theta,\omega,\varphi,\varphi'$ . Then for every $\lambda \in \Lambda$ ,

$$\begin{aligned}
(\bar{n}'^{\varphi'} + \bar{m}'^{\varphi'})^{\Psi}_{\lambda} &= B^{\Psi(\lambda)}_{\varphi'(\lambda)}(n'^{\varphi'}_{\lambda} + m'^{\varphi'}_{\lambda}) \\
&= B^{\Psi(\lambda)}_{\eta'(\lambda)} n'_{\lambda} + B^{\Psi(\lambda)}_{\mu'(\lambda)} m'_{\lambda} \\
&= B^{\Psi(\lambda)}_{\theta(\lambda)} B^{\theta(\lambda)}_{\eta'(\lambda)} n'_{\lambda} + B^{\Psi(\lambda)}_{\omega(\lambda)} B^{\omega(\lambda)}_{\mu'(\lambda)} m'_{\lambda} \\
&= B^{\Psi(\lambda)}_{\theta(\lambda)} B^{\theta(\lambda)}_{\eta(\lambda)} n_{\lambda} + B^{\Psi(\lambda)}_{\omega(\lambda)} B^{\omega(\lambda)}_{\mu(\lambda)} m_{\lambda} \\
&= B^{\Psi(\lambda)}_{\varphi(\lambda)} n^{\varphi}_{\lambda} + B^{\Psi(\lambda)}_{\varphi(\lambda)} m^{\varphi}_{\lambda} \\
&= (\bar{n}^{\varphi} + \bar{m}^{\varphi})^{\Psi}_{\lambda}
\end{aligned}$$

insures that $(\varphi',\bar{n}'^{\varphi'} + \bar{m}'^{\varphi'}) \sim (\varphi,\bar{n}^{\varphi} + \bar{m}^{\varphi})$ . (We call such an argument a standard argument and similar arguments will be left to the reader for the remainder of the chapter.)

Clearly $[(\eta,\bar{0})]$ is a zero object, where $\bar{0} : \bar{A} \to \bar{B}\cdot\eta$ ; and $[(\eta,-\bar{n})]$ is the additive inverse of $[(\eta,\bar{n})]$ . Thus $D(\mathcal{O})$ is an additive category.

In the next lemma, a construction for arbitrary coproducts in $D(\mathcal{A})$ is given. A "proof by construction" is also needed later in the chapter in order to show that $D(\mathcal{A})$ is actually AB5 .

<u>Lemma 2.5</u>: $D(\mathcal{A})$ has arbitrary coproducts.

<u>Proof</u>: Suppose $\{(\Lambda^i,\bar{A}^i)_{i\in I}\}$ is a collection of objects of $D(\mathcal{A})$ . It suffices to construct $(\Lambda,\bar{A}) \in D(\mathcal{A})$ along with universal morphisms $[(\mu^i,\bar{u}^i)] : (\Lambda^i,\bar{A}^i) \to (\Lambda,\bar{A})$ .

As a first step, define $\Lambda$ to be $\{\lambda = (\lambda i) \in \prod_i \Lambda^i \,|\, \lambda i = 0$ except on a finite subset of $I\}$ having the natural ordering of $\prod_i \Lambda^i$ . For each $\lambda \in \Lambda$ , let $I_\lambda$ be the <u>support of $\lambda$</u> , i.e. the unique finite subset such that $\lambda_i = 0$ if and only if $i \in I\backslash I_\lambda$ . If $\lambda \le \lambda'$ , then $I_\lambda \le I_{\lambda'}$ .

Clearly, $(0) \in \Lambda$ is initial. $\Lambda$ is $\downarrow$-finite since for every $\lambda \in \Lambda$ , $In(\lambda) \cong \prod_{i\in I_\lambda} In(\lambda_i)$ is a finite set. Further $\Lambda$ is also directed because each $\Lambda^i$ is directed and the choice of non-zero $\lambda^i$'s can be restricted to a finite set.

Next, the diagram $\bar{A} : \Lambda \to \mathcal{A}$ is given by $A_\lambda = \coprod_{i\in I_\lambda} A^i_{\lambda_i}$ for each $\lambda \in \Lambda$ . If $\lambda \le \lambda'$ in $\Lambda$ , then $A_\lambda^{\lambda'} : A_\lambda \to A_{\lambda'}$ is the canonical map of coproducts induced by $I_\lambda \subseteq I_{\lambda'}$ , i.e. $A_\lambda^{\lambda'} u^i_{\lambda_i} = \begin{cases} u^i_{\lambda'_i} (A^i)^{\lambda'_i}_{\lambda_i} & \text{if } i \in I_\lambda \\ \text{otherwise } 0 \end{cases}$

where $u^i_{\lambda_i} : A^i_{\lambda_i} \to \coprod_{i\in I_\lambda} A^i_{\lambda_i}$ and $u^i_{\lambda'_i} : A^i_{\lambda'_i} \to \coprod_{i\in I_{\lambda'}} A^i_{\lambda'_i}$ are the canonical inclusions into the coproduct.

Lastly, $[(\mu^i,\bar{u}^i)] : (\Lambda^i,\bar{A}^i) \to (\Lambda,\bar{A})$ is defined by:

$$\text{(i)}\quad \mu^i : \Lambda^i \to \Lambda\ ,\ \mu^i(\lambda_i) = (\hat{\lambda}_i)_j = \begin{cases} 0, & i \neq j \\ \lambda_i, & i = j \end{cases}$$

$$\text{(ii)}\quad \bar{u}^{\,i} : A^i \to \bar{A}\ ,\ (\bar{u}^{\,i})_{\lambda_i} = u^i_{\lambda_i} : A^i_{\lambda_i} \to A_{\hat{\lambda}_i}\ .$$

Here $u^i_{\lambda_i}$ is the canonical inclusion into the coproduct which obviously gives a natural transformation from the definition of $\bar{A} : \Lambda \to \mathcal{Q}$ .

To show that $((\Lambda,\bar{A}),[(\mu^i,\bar{u}^{\,i})])$ is a coproduct, suppose that $(\Gamma,\bar{B}) \in D(\mathcal{Q})$ and $[(\eta^i,\bar{n}^{\,i})] : (\Lambda^i,\bar{A}^{\,i}) \to (\Gamma,\bar{B})$ . Then it suffices to construct $[(\eta,\bar{n})]:(\Lambda,\bar{A})\to(\Gamma,\bar{B})$ such that $[(\eta,\bar{n})\cdot(\mu^i,\bar{u}^{\,i})] = [(\eta^i,\bar{n}^{\,i})]$. To this end, we first define $\eta : \Lambda \to \Gamma$ inductively. Recall that $\Lambda = \bigcup_{r \geq 0} \Lambda_r$ where $\Lambda_0 = \{0\}$ , $\Lambda_1 = \{\lambda | \lambda' < \lambda$ implies $\lambda' = 0\}$ , and $\Lambda_r = \{\lambda | \lambda' < \lambda$ implies $\lambda \in \Lambda_{r-1}\}$ . Set $\eta(0) = 0$ . For each $\lambda \in \Lambda_r$ , let $\Gamma(\lambda)$ be the finite subset of $\Gamma, \{\eta^i(\lambda_i) | i \in I_\lambda\} \cup \{\eta(\lambda') | \lambda' < \lambda\}$ . Since $\Gamma$ is directed, choose $\eta(\lambda)$ larger than any of the elements of $\Gamma(\lambda)$ . Clearly, $\eta : \Lambda \to \Gamma$ is an order and basepoint preserving mapping. Next, we define $\bar{n} : \bar{A} \to \bar{B}\cdot\eta$ by the rule

$$n_\lambda u^i_{\lambda_i} = \begin{cases} B^{\eta(\lambda)}_{\eta^i(\lambda_i)} \cdot n^i_{\lambda_i} & \text{for } i \in I_\lambda \\ 0 & \text{otherwise} . \end{cases}$$

Then $B^{\eta(\lambda')}_{\eta(\lambda)} n_\lambda = n_{\lambda'} A^{\lambda'}_\lambda$ insures that $\bar{n} : \bar{A} \to \bar{B}\cdot\eta$ is a map of diagrams and $[(\eta,\bar{n})] : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ such that $[(\eta,\bar{n})\cdot(\mu^i,\bar{u}^{\,i})]$ equals $[(\eta^i,\bar{n}^{\,i})]$ in $D(\mathcal{Q})$ . Thus $D(\mathcal{Q})$ has arbitrary coproducts.

In particular, if I is itself a finite set, then

$$\coprod_I (\Lambda^i,\bar{A}^{\,i}) = (\prod_{i\in I} \Lambda^i, \bar{A}) \quad \text{where } \bar{A}_\lambda = \coprod_{i \in I} A^i_{\lambda_i}\ .$$

<u>Corollary 2.6</u>: $D(\mathcal{Q})$ has finite products.

<u>Proof</u>: The proof is immediate from the fact that any additive category

with finite coproducts also has finite products [11].

The next lemma is a key lemma in the sense that it is used to show not only that $D(\mathcal{A})$ has kernels and cokernels, and is normal and conormal; but also to demonstrate that short exact sequences in $D(\mathcal{A})$ are equivalent to short exact sequences in $\mathcal{A}^\Lambda$, $\Lambda$ some $\downarrow$-finite directed ordered set with initial element.

<u>Lemma 2.7</u> (<u>Lifting lemma</u>): Suppose $[(\eta,\bar{n})] : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ in $D(\mathcal{A})$. Then there is a $\downarrow$-finite directed ordered set $\Delta$ with initial element $\bar{A}',\bar{B}' \in \mathcal{A}^\Delta$, and a map of diagrams $\bar{n}' : \bar{A}' \to \bar{B}'$ such that $(\Lambda,\bar{A}) \cong (\Delta,\bar{A}')$, $(\Gamma,\bar{B}) \cong (\Delta,\bar{B}')$ in $D(\mathcal{A})$ and $[(id_\Delta,\bar{n}')]$ is equivalent to $[(\eta,\bar{n})]$ in $D(\mathcal{A})$ via these isomorphisms.

<u>Proof</u>: Suppose $[(\eta,\bar{n})] : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ in $D(\mathcal{A})$. Let $\Delta$ be the ordered subset $\Lambda \times \Gamma$ whose elements are $(\lambda,\gamma)$ such that $\eta(\lambda) \le \gamma$. Since $\eta(0) = 0 \le 0$, $(0,0) \in \Delta$, and $\Delta$ has an initial element. For any $(\lambda,\gamma),(\lambda',\gamma')$ in $\Delta$, choose $\lambda'' \ge \{\lambda,\lambda'\}$ and $\gamma'' \ge \{\gamma,\gamma',\eta(\lambda'')\}$. Then $(\lambda'',\gamma'')$ is larger than $(\lambda,\gamma),(\lambda',\gamma')$, and $\Delta$ is directed. Since $In(\lambda,\gamma) \subseteq (In(\lambda) \times In(\gamma))$, a finite set, for every $(\lambda,\gamma) \in \Delta$, $\Delta$ is $\downarrow$-finite.

Also, there are two <u>cofinal</u> order preserving mappings $\rho_\Lambda : \Delta \to \Lambda$, $\rho_\Lambda(\lambda,\gamma) = \lambda$ and $\rho_\Gamma : \Delta \to \Gamma$, $\rho_\Gamma(\lambda,\gamma) = \gamma$. Let $\bar{A}' = \rho_\Lambda^* \bar{A}$, $A'(\lambda,\gamma) = A_\lambda$, and $\bar{B} = \rho_\Gamma^* \bar{B}$, $B'_{(\lambda,\gamma)} = B_\gamma$. Further, define $\bar{n}' : \bar{A}' \to \bar{B}'$ by the rule $n'_{(\lambda,\gamma)} = B^\gamma_{\eta(\lambda)} n_\lambda$; then $[(id_\Delta,\bar{n}')] : (\Delta,\bar{A}') \to (\Delta,\bar{B}')$ in $D(\mathcal{A})$.

Next, if $< id_\Delta,\eta > : \Lambda \to \Delta$ is the unique order and basepoint preserving mapping given by $<id,\eta>(\lambda) = (\lambda,\eta(\lambda))$, and $< 0,id_\Gamma > : \Gamma \to \Delta$ is given by $< 0,id_\Gamma >(\gamma) = (0,\gamma)$, then there is a commutative diagram in $D(\mathcal{A})$

$$\begin{array}{ccc}
(\Lambda,\bar{A}) & \xrightarrow{(\eta,\bar{n})} & (\Gamma,\bar{B}) \\
\Big\downarrow [(\langle id_{\Lambda},\eta\rangle,\bar{a})] & & \Big\downarrow [(\langle 0,id_{\Gamma}\rangle,\bar{b})] \\
(\Delta,\bar{A}') & \xrightarrow{(id_{\Delta},\bar{n}')} & (\Delta,\bar{B}')
\end{array}$$

where $a_\lambda = id_{A_\lambda} : A_\lambda \to A'_{(\lambda,\eta(\lambda))}$ and

$b_\gamma = id_{B_\gamma} : B_\gamma \to B'_{(0,\gamma)}$ .

In addition, $[(\rho_\Lambda,\bar{a}')]$ , $a'_{(\lambda,\gamma)} = id_{A_\lambda}$ and $[(\rho_\Gamma,\bar{b}')]$ , $b'_{(\lambda,\gamma)} = id_{B_\gamma} : B'_{(\lambda,\gamma)} \to B_{\rho_\Gamma(\lambda,\gamma)}$ are shown, by a standard argument, to be the respective inverses of $[(\langle id_A,\eta\rangle,\bar{a})]$ and $[(\langle 0,id_\Gamma\rangle,\bar{b})]$ in $D(\mathcal{A})$ . Also, standard arguments show that all the choices made are independent of the representatives, and the lemma is proved.

Thus, any morphism in $D(\mathcal{A})$ can be "regarded" as a morphism in $\mathcal{A}^\Delta$ for some $\downarrow$-finite directed $\Delta$ with initial element.

<u>Proposition 2.8</u> $D(\mathcal{A})$ is a cocomplete abelian category.

Proof: From the above lemma, any morphism in $D(\mathcal{A})$ is equivalent to a map of diagrams in $\mathcal{A}^\Delta$ for some $\downarrow$-finite directed $\Delta$ with initial element. Therefore, in order to show that $D(\mathcal{A})$ has kernels, it suffices to define a kernel for $[(id_\Delta,\bar{n})] : (\Delta,\bar{A}) \to (\Delta,\bar{B})$ and show that this is indeed a kernel in $D(\mathcal{A})$ . Let

$0 \to (\Delta,\bar{K}) \xrightarrow{(\Delta,\bar{k})} (\Delta,\bar{A})$ be the pointwise kernel of

$(id_\Delta,\bar{n}):(\Delta,\bar{A})\to(\Delta,\bar{B})$ in $\mathcal{A}^\Delta$. Suppose $[(\xi,\bar{s})]:(\dot{\Omega},\bar{C})\to(\Delta,\bar{A})$ is such that $[(id_\Delta,\bar{n})\cdot(\xi,\bar{s})] = [(\xi',0)]$. Then there is a $\varphi \geq \xi,\xi'$ such that $(\bar{n}_\xi\cdot\bar{s})^\varphi = (\bar{0})^\varphi$, and for every $\omega \in \Omega$, $B^{\varphi(\omega)}_{\xi(\omega)} n_{\xi(\omega)} s_\omega = B^{\varphi(\omega)}_{\xi(\omega)}\cdot 0 = 0$. From the pointwise definition of kernel in $\mathcal{A}^\Delta$, there is a unique $c_\omega : C_\omega \to K_{\varphi(\omega)}$ such that $A^{\varphi(\omega)}_{\xi(\omega)} s_\omega = k_{\varphi(\omega)}\cdot c_\omega$. Clearly, $\bar{c} : C \to K\cdot\varphi$ is a map of diagrams such that $[(id_\Delta,\bar{k})\cdot\varphi,\bar{c})]$ equals $[(\xi,\bar{s})]$ in $D(\mathcal{A})$. By a standard argument, all the choices above are shown to be independent of the representatives, insuring that

$$\ker([(id_\Delta,\bar{n})]) = [0 \to (\Delta,\bar{K}) \xrightarrow{(id_\Delta,\bar{k})} (\Delta,\bar{A})] \text{ in } D(\mathcal{A}) .$$

The dual argument shows that cok $[(id_\Delta,\bar{n})]$ in $D(\mathcal{A})$ is isomorphic to $\text{cok}(\bar{n})$ in $\mathcal{A}^\Delta$. From the pointwise definition of kernel and cokernel, it is clear that $D(\mathcal{A})$ is both normal and conormal. Lastly, since $D(\mathcal{A})$ has arbitrary coproducts and cokernels, it is cocomplete.

<u>Corollary 2.9</u>. If

$$(*) \qquad 0 \to (\Lambda',\bar{A}') \xrightarrow{[(\eta',\bar{n}')]} (\Lambda,\bar{A}) \xrightarrow{[(\eta'',\bar{n}'')]} (\Lambda'',\bar{A}'') \to 0$$

is a short exact sequence in $D(\mathcal{A})$, then there is a $\downarrow$-finite directed ordered set $\Delta$, with initial element, $0 \to \bar{B}' \xrightarrow{\bar{m}''} B \xrightarrow{\bar{m}''} \bar{B}'' \to 0$ exact in $\mathcal{A}^\Delta$ such that

$$0 \longrightarrow (\Delta,\bar{B}') \xrightarrow{[(id_\Delta,\bar{m}')]} (\Delta,\bar{B}) \xrightarrow{[(id_\Delta,\bar{m}'')]} (\Delta,\bar{B}'') \longrightarrow 0$$

is equivalent to $(*)$ in $D(\mathcal{A})$.

<u>Proof</u>: This is immediate from the lifting lemma and the definition of kernels and cokernels in $D(\mathcal{A})$.

$D(\mathcal{A})$ has been shown to be a cocomplete abelian category. However, in order to show that $D(\mathcal{A})$ is AB5, it is necessary to actually construct the colimit for $\Omega$, a $\downarrow$-finite directed ordered set with initial element.

## Section 3: $D(\mathcal{A})$ is AB5 .

**Proposition 2.10:** Suppose $\tilde{A} = \{(\Lambda^{\omega},\bar{A}^{\omega}),[(\eta_{\omega}^{\omega'},\bar{n}_{\omega}^{\omega'})]\}$ is in $D(\mathcal{A})^{\Omega}$ , $\Omega$ ↓-finite directed ordered set with initial element. Let $\Delta$ be the ordered set

$$\{(\omega,\varphi)\,|\,\varphi \in \prod_{\gamma\in In(\omega)} \Lambda^{\gamma} \text{ and } \gamma < \gamma' \text{ in}$$

$$In(\omega) \text{ insures that } \eta_{\gamma}^{\gamma'}\varphi_{\gamma} \le \varphi_{\gamma'}\}$$

with ordering given by the rule:

$$(\omega,\varphi) \le (\omega',\varphi') \text{ if and only if } \omega \le \omega'$$

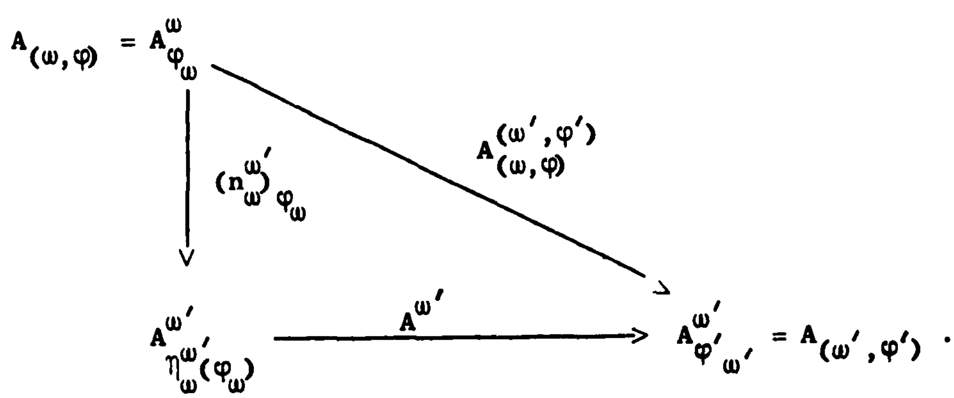
$$\text{and } \varphi'/In(\omega) \ge \varphi .$$

Define $\bar{A} : \Delta \to \mathcal{A}$ by:

(i) $A_{(\omega,\varphi)} = A^{\omega}_{\varphi_{\omega}}$ , and

(ii) $A^{(\omega',\varphi')}_{(\omega,\varphi)} = ((A^{\omega'})^{\varphi'_{\omega'}}_{\eta^{\omega'}_{\omega}(\varphi_{\omega})}) \cdot ((n^{\omega'}_{\omega})_{\varphi_{\omega}})$ where

$(\omega,\varphi) \le (\omega',\varphi')$ , i.e.



Then $(\Delta,\bar{A}) \in D(\mathcal{A})$ and $(\Delta,\bar{A})$ is isomorphic to $\operatorname{colim}_{\Omega}(A^{\omega},\bar{A}^{\omega})$ in $D(\mathcal{A})$ .

<u>Proof</u>: In order that $(\Delta,\bar{A}) \in D(\mathcal{A})$ , it suffices to show that $\Delta$ is $\downarrow$-finite, directed ordered set with initial element.

Suppose $(\omega,\varphi)$ and $(\omega',\varphi')$ are in $\Delta$ . Since $\Omega$ is directed, there is an $\omega'' \geq \omega,\omega'$ . Clearly both $\varphi$ and $\varphi'$ can be considered as elements of $\prod_{\gamma\in In(\omega'')} \Lambda^{\gamma}$ . Define $\varphi'' \in \prod_{\gamma\in In(\omega'')} \Lambda^{\gamma}$ inductively on the usual decomposition of $In(\omega'') = \bigcup_{i\geq 0} (\Omega_i \cap In(\omega''))$ . Choose $\varphi''_0 \geq \varphi_0,\varphi'_0$ in $\Lambda^{\bullet}$ . For $\gamma \in \Omega_i \cap In(\omega'')$ , let $\varphi''$ be an element of $\Lambda^{\gamma}$ larger than any element of the finite set $\{\varphi_{\gamma},\varphi'_{\gamma},\varphi''_{\gamma'},\eta_{\gamma'}(\varphi''_{\gamma'})$ where $\gamma' < \gamma\}$ . Then $(\omega'',\varphi'') \in \Delta$ and $(\omega'',\varphi'') \geq (\omega,\varphi),(\omega',\varphi')$ ; therefore $\Delta$ is directed.

Further, $(0,0_{\Lambda^{\circ}})$ , where $0$ is the initial element of $\Omega$ and $0_{\Lambda^{\circ}}$ is the initial element of $\Lambda^{\circ}$ , serves as the initial element of $\Delta$ . For every $(\omega,\varphi) \in \Delta$ , $In(\omega,\varphi)$ is contained in the finite set $In(\omega) \times In(\varphi)$ , and $\Delta$ is $\downarrow$-finite . Therefore, $(\Delta,\bar{A}) \in D(\mathcal{A})$ .

For each $\omega \in \Omega$ , let $[(\rho^{\omega},\bar{p}^{\omega})] : (\Lambda^{\omega},\bar{A}^{\omega}) \to (\Delta,\bar{A})$ be the morphism in $D(\mathcal{A})$ defined by:

(i) $\rho^{\omega}(\lambda_{\omega}) = (\omega,\hat{\lambda}_{\omega})$ , where $(\hat{\lambda}_{\omega})_{\gamma} = \begin{cases} 0, \gamma \neq \omega \\ \lambda_{\omega}, \gamma = \omega \end{cases}$

(ii) $\bar{p}^{\omega}_{\lambda_{\omega}} = id_{A^{\omega}_{\lambda_{\omega}}}$ , since $A(\omega,\hat{\lambda}_{\omega}) = A^{\omega}_{\lambda_{\omega}}$ .

Then $[(\rho^{\omega'},\bar{p}^{\omega'})]\cdot[(\eta^{\omega'}_{\omega},\bar{n}^{\omega'}_{\omega})] = [(\rho^{\omega},\bar{p}^{\omega})]$ , for $\omega < \omega'$ .

Suppose $\{[(\xi^{\omega},\bar{s}^{\omega})] : (\Lambda^{\omega},\bar{A}^{\omega}) \to (\Gamma,\bar{B})\}$ such that

$[(\xi^{\omega'},\bar{s}^{\omega'})]\cdot[(\eta_{\omega}^{\omega'},\bar{n}_{\omega}^{\omega'})] = [(\xi^{\omega},\bar{s}^{\omega})]$ for $\omega < \omega'$ . Let $\xi : \Delta \to \Gamma$ be the order and basepoint preserving mapping defined by $\xi(\omega,\varphi) = \xi^{\omega}(\varphi_{\omega})$ , and $\bar{s} : \bar{A} \to \bar{B}\cdot\xi$ the natural transformation $s_{(\omega,\varphi)} = s^{\omega}_{\varphi_{\omega}}$ . Then $[(\xi,\bar{s})][(\rho^{\omega},\bar{p}^{\omega})] = [(\xi^{\omega},\bar{s}^{\omega})]$ . Standard arguments show that $[(\xi,\bar{s})] : (\Delta,\bar{A}) \to (\Gamma,\bar{B})$ is unique and independent of the choice of representatives. Thus $\{(\Delta,\bar{A}),[(\rho^{\omega},\bar{p}^{\omega})]\} \cong \operatorname{colim}_{\Omega}(\Lambda^{\omega},\bar{A}^{\omega})$ in $D(\mathcal{A})$ .

For any small category $\Theta$ , let $J^{\Theta} : \mathcal{A}^{\Theta} \to (D(\mathcal{A}))^{\Theta}$ be the functor defined by

$$J^{\Theta}(\bar{A})_{\theta} = J(A_{\theta}) = (\mathbf{1},\hat{A}_{\theta}) ,$$

where $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is the embedding given in section 1.

Corollary 2.11: If $(\Lambda,\bar{A}) \in D(\mathcal{A})$ , then $(\Lambda,\bar{A})$ is isomorphic to $\operatorname{colim}_{\Lambda} J^{\Lambda}(\bar{A})$ in $D(\mathcal{A})$ .

Proof: By proposition 2.10, $\operatorname{colim} J^{\Lambda}(\bar{A}) \cong (\Delta,\bar{A}')$ where

(i) $\Delta = \{(\lambda,\varphi) \mid \varphi \in \prod_{In(\lambda)} \mathbb{1}$ , $\varphi_{\gamma}$ is either 0 or 1 , and $\varphi_{\lambda} = 0$ if and only if $\varphi_{\gamma} = 0$ for every $\gamma < \lambda\}$ , and

(ii) $$A'_{(\lambda,\varphi)} = \begin{cases} 0 & \text{if } \varphi_{\lambda} = 0 \\ A_{\lambda} & \text{if } \varphi_{\lambda} = 1 . \end{cases}$$

Then $(\xi,\bar{s}) : (\Lambda,\bar{A}) \to (\Delta,\bar{A}')$ , given by

(i) $\xi(\lambda) = (\lambda,(1))$ where $(1)_{\gamma} = 1$ for every $\gamma \leq \lambda$ and

(ii) $s_{\lambda} = id_{A_{(\lambda)}}$

and $(\eta,\bar{n}) : (\Delta,\bar{A}') \to (\Lambda,\bar{A})$ , given by

(i) $\eta(\lambda,\varphi) = \varphi_{\lambda}$ and

(ii) $n_{(\lambda,\varphi)} = id_{A_{\varphi_\lambda}}$ ,

are inverses of each other in $D(\mathcal{A})$ . Therefore, $(\Lambda,\bar{A}) \cong \underset{\Lambda}{\text{colim}} J^{\Lambda}(\bar{A})$ .

For any abelian category $\mathcal{B}$ , there is an associated category Seq $\mathcal{B}$ , <u>the category of short exact sequences in</u> $\mathcal{B}$ , whose

(i) objects are $0 \to B' \to B \to B'' \to 0$ exact in $\mathcal{B}$ , and

(ii) morphisms are commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & B_1' & \to & B_1 & \to & B_1'' & \to & 0 \\ & & \downarrow{f'} & & \downarrow{f} & & \downarrow{f''} & & \\ 0 & \to & B_2' & \to & B_2 & \to & B_2'' & \to & 0 \end{array}$$

with the obvious composition.

In general, this category is not abelian.

<u>Proposition 2.12</u> . $D(\text{Seq } \mathcal{A})$ and $\text{Seq } (D(\mathcal{A}))$ are equivalent categories.

<u>Proof</u>: Define $U : D(\text{Seq } \mathcal{A}) \to \text{Seq } (D(\mathcal{A}))$ by the rule:

(i) $U(\Lambda,\tilde{A}) = 0 \to (\Lambda,\bar{A}') \xrightarrow{[(id_\Lambda,\bar{n}')]} (\Lambda,\bar{A}) \xrightarrow{[(id_\Lambda,\bar{n}'')]} (\Lambda'',\bar{A}'') \to 0$

where $\tilde{A} = (0 \to \bar{A}' \xrightarrow{\bar{n}'} \bar{A} \xrightarrow{\bar{n}''} \bar{A}'' \to 0)$ in $\mathcal{A}^\Lambda$ , and

(ii) $U(\tilde{\eta},\tilde{f}) = ([(\eta,\bar{f}')],[(\eta,\bar{f})],[(\eta,\bar{f}'')])$ ,

where $(\tilde{\eta},\tilde{f}) : (\Lambda,\tilde{A}) \to (\Gamma,\tilde{B})$ is the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bar{A}' & \xrightarrow{\bar{n}'} & \bar{A} & \xrightarrow{\bar{n}''} & \bar{A}'' & \longrightarrow & 0 \\ & & \downarrow{f'} & & \downarrow{f} & & \downarrow{f''} & & \\ 0 & \to & \bar{B}'\cdot\eta & \xrightarrow{\bar{m}'} & \bar{B}\cdot\eta & \xrightarrow{\bar{m}''} & \bar{B}\cdot\eta & \to & 0 \end{array}$$ in $\mathcal{A}^\Lambda$ .

From corollary 2.9 and the definition of $D(\mathcal{A})$ , U is both representative and faithful.

To show that $U$ is full, suppose

$$(2.13)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & (\Lambda,\bar{A}') & \xrightarrow{[(id_\Lambda,\bar{n}')]} & (\Lambda,\bar{A}) & \xrightarrow{[(id_\Lambda,\bar{n}'')]} & (\Lambda,\bar{A}'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle [(\xi',\bar{s}')]} & & \downarrow{\scriptstyle [(\xi,\bar{s})]} & & \downarrow{\scriptstyle [(\xi'',\bar{s}'')]} & & \\ 0 & \longrightarrow & (\Gamma,\bar{B}') & \xrightarrow{[(id_\Gamma,\bar{m}')]} & (\Gamma,\bar{B}) & \xrightarrow{[(id_\Gamma,\bar{m}'')]} & (\Gamma,\bar{B}'') & \longrightarrow & 0 \end{array}$$

is a morphism in $\mathrm{Seq}\ (D(\mathcal{A}))$ between $U(\Lambda,\tilde{A})$ and $U(\Gamma,\tilde{B})$ ; i.e., there are $\varphi \geq \xi,\xi'$ and $\Psi \geq \xi,\xi''$ such that $(\bar{m}'\cdot\bar{s}')^{\varphi} = (\bar{s}\cdot\bar{n}')^{\varphi}$ and $(\bar{m}''\cdot\bar{s})^{\Psi} = (\bar{s}''\cdot\bar{n}'')^{\Psi}$ . Filter $\Lambda = \bigcup \Lambda_r$ as before, and define $\epsilon : \Lambda \to \Gamma$ inductively as follows. Set $\epsilon(0) = 0$ . Suppose $\epsilon$ is already defined on $\Lambda_{r-1}$ and $\lambda \in \Lambda_r$ . Let $\Gamma(\lambda)$ denote the finite subset $\{\Psi(\lambda),\varphi(\lambda),\epsilon(\lambda')\,|\,\lambda' < \lambda\}$ of $\Gamma$ . Choose $\epsilon(\lambda) \geq \Gamma(\lambda)$ . Clearly, $\epsilon : \Lambda \to \Gamma$ is order preserving and $\epsilon \geq \xi',\xi,\xi''$ . A standard argument shows that $([(\xi',\bar{s}')],[(\xi,\bar{s})],[(\xi'',\bar{s}'')])$ equals $([(\epsilon,\bar{s}'^{\epsilon})], [(\epsilon,\bar{s}^{\epsilon})],[(\epsilon,\bar{s}''^{\epsilon})])$ in $\mathrm{Seq}\ (D(\mathcal{A}))$ , and $U$ is full.

Therefore $U : D(\mathrm{seq}\ \mathcal{A}) \to \mathrm{seq}\ (D(\mathcal{A}))$ is an equivalence of categories.

<u>Theorem 2.14</u> $D(\mathcal{A})$ is AB5 .

<u>Proof:</u> Suppose $\Gamma$ is any directed ordered set. Then $(\mathfrak{J}(\Gamma),\tilde{A}\cdot\theta) \xrightarrow{(\theta,id)} (\Gamma,\tilde{A})$ induces an isomorphism of colimits in $D(\mathcal{A})$ where $\theta : \mathfrak{J}(\Gamma) \to \Gamma$ is the cofinal map defined in proposition 1.4 and

$\tilde{A} : \Gamma \to D(\mathcal{A})$ . Since $\mathfrak{F}(\Gamma)$ is a $\downarrow$-finite directed ordered set with initial element $\emptyset$ , it suffices to show that $\operatorname{colim}_{\Omega} : D(\mathcal{A})^{\Omega} \to D(\mathcal{A})$ is exact for $\Omega$ is a $\downarrow$-finite directed ordered set with initial element.

Suppose $0 \to \tilde{A}' \to \tilde{A} \to \tilde{A}'' \to 0$ is a short exact sequence in $D(\mathcal{A})^{\Omega}$ . By Proposition 2.12,

$(0 \to \tilde{A}' \to \tilde{A} \to \tilde{A}'' \to 0) \in (D(\text{seq } \mathcal{A}))^{\Omega}$ , i.e. for every $\omega \in \Omega$ ,

$$0 \to \tilde{A}'_{\omega} \to \tilde{A}_{\omega} \to \tilde{A}''_{\omega} \to 0 \text{ is isomorphic to}$$

$$0 \to (\Lambda^{\omega},\bar{A}'^{\omega}) \xrightarrow{[(id_{\Lambda^{\omega}},\bar{n}'^{\omega})]} (\Lambda^{\omega},\bar{A}^{\omega}) \xrightarrow{[(id_{\Lambda^{\omega}},\bar{n}''^{\omega})]} (\Lambda^{\omega},\bar{A}''^{\omega}) \to 0 ,$$

and $\omega \le \omega'$ gives a commutative diagram similar to (2.13).

Therefore, using the construction for colimit in Proposition 2.10, $\operatorname{colim}_{\Omega} (0 \to \tilde{A}' \to \tilde{A} \to \tilde{A}'' \to 0)$ is isomorphic to $0 \to (\Delta,\bar{A}') \xrightarrow{[(id_{\Delta},\bar{n}')]} (\Delta,\bar{A}) \xrightarrow{[(id_{\Delta},\bar{n}'')]} (\Delta,\bar{A}'') \to 0$ . Thus $D(\mathcal{A})$ is AB5 .

## Section 4: The Embedding $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ .

In this section, we show that $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is exact, Ext-preserving, and projective-preserving. Also $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ satisfies a universal extension property for right exact functors from $\mathcal{A}$ to cocomplete abelian categories. Furthermore, if $\mathcal{A}$ has enough projectives, then $D(\mathcal{A})$ will also have enough projectives, so that derived functors from $D(\mathcal{A})$ can be defined.

Proposition 2.15: If $\mathcal{A}$ is abelian, then $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is exact, projective-preserving and Ext-preserving (i.e. $\mathrm{Ext}^*(J(A),J(B)) \cong \mathrm{Ext}^*(A,B)$) .

<u>Proof:</u> $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is exact since $0 \to A' \xrightarrow{n'} A \xrightarrow{n''} A'' \to 0$ exact in $\mathcal{A}$ insures $0 \to (1,\hat{A}') \xrightarrow{[(id,\hat{n}')]} (1,\hat{A}) \xrightarrow{[(id,\hat{n}'')]} (1,\hat{A}'') \to 0$ is exact in $D(\mathcal{A})$ , by the definition of exactness there.

Suppose $P \in \mathcal{A}$ is projective. By Corollary 2.9, it suffices to consider

$$\begin{array}{ccc} & & J(P) \\ & & \Big\downarrow {\scriptstyle [(\xi,\bar{s})]} \\ (\Lambda,\bar{A}) & \xrightarrow{[(id_\Lambda,\bar{n})]} & (\Lambda,\bar{A}'') \longrightarrow 0 \text{ , in } D(\mathcal{A}) \text{ .} \end{array}$$

Since $P$ is projective in $\mathcal{A}$ , there is a $s_1' : P \to A_{\xi(1)}$ such that $n_{\xi(1)} s_1' = s_1$ . Therefore, by a standard argument $[(id_\Lambda,\bar{n})][(\xi,\bar{s}')] = [(\xi,\bar{s})]$ , and $J(P)$ is projective in $D(\mathcal{A})$ .

In order to show that $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is Ext-preserving, it suffices to extend any diagram

$$\begin{array}{ccc} & & J(Y) \\ & & \Big\downarrow {\scriptstyle [(\xi,\bar{s})]} \\ (\Lambda,\bar{A}) & \xrightarrow{[(id_\Lambda,\bar{n})]} & (\Lambda,\bar{A}'') \longrightarrow 0 \end{array}$$

in the following manner [12].

Let

$$\begin{array}{ccc} X & \xrightarrow{n_1'} & Y \\ \Big\downarrow {\scriptstyle s_1'} & & \Big\downarrow {\scriptstyle s_1} \\ A_{\xi(1)} & \xrightarrow{n_{\xi(1)}} & A''\xi(1) \to 0 \end{array}$$

be the pullback diagram in $\mathcal{A}$. Then $X \xrightarrow{n_1'} Y \longrightarrow 0$ is exact in $\mathcal{A}$. Clearly,

$$\begin{array}{ccccc} J(X) & \xrightarrow{J(n')} & J(Y) & \dashrightarrow & 0 \\ \downarrow{\scriptstyle [(\xi,\bar{s}')]} & & \downarrow{\scriptstyle [(\xi,\bar{s})]} & & \\ (\Lambda,\bar{A}) & \xrightarrow{(id_\Lambda,\bar{n})} & (\Lambda,\bar{A}'') & \longrightarrow & 0 \end{array}$$

commutes in $D(\mathcal{A})$, and has exact top row.

<u>Proposition 2.16</u>: If $\mathcal{A}$ has enough projectives, then $\{J(P) \mid P$ projective in $\mathcal{A}\}$ is a class of small projective generators in $D(\mathcal{A})$ .

<u>Proof</u>: From the above proposition, $J(P)$ is projective for $P$ projective in $\mathcal{A}$. To see that $J(P)$ is small in $D(\mathcal{A})$, suppose $[(\xi,\bar{s})] : J(P) \to \coprod_{i\in I} (\Lambda^i,\bar{A}^i)$. By Lemma 2.5, $\coprod_{i\in I} (\Lambda^i,\bar{A}^i) \cong (\Lambda,\bar{A})$

where

(i) $\Lambda = \{(\lambda_i) \in \prod_i \Lambda^i \mid \lambda_i = 0$ except on a finite subset of $I\}$

and

(ii) $\bar{A}_\lambda = \coprod_{i\in I_\lambda} A^i_{\lambda_i}$ .

Clearly

$$\begin{array}{ccc} J(P) & \longrightarrow & \coprod_{i\in I_{\xi(1)}} (\Lambda^i,\bar{A}^i) \\ & {\scriptstyle [(\xi,\bar{s})]} \searrow & \downarrow \\ & & \coprod_{i\in I} (\Lambda^i,\bar{A}^i) \end{array}$$

commutes, where $\xi'(1)$"="$\xi(1)$, $I_{\xi(1)}$ is the support of $\xi(1)$, and

$s_1' \text{"="} s_1$ .

Next, we show that $\{J(P)\}$ generate. Suppose $[(\eta,\bar{n})] : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ is not zero, i.e., there is a $\lambda_0 \in \Lambda$ such that $n_{\lambda_0} : A_{\lambda_0} \to B_{\eta(\lambda_0)}$ is not zero. Since $\mathcal{A}$ has enough projectives, there is a $P \in \mathcal{A}$ and epimorphism $P \xrightarrow{s} A_{\lambda_0} \longrightarrow 0$ . Obviously,

$J(P) \xrightarrow{[(\xi,\bar{s})]} (\Lambda,\bar{A}) \xrightarrow{[(\eta,\bar{n})]} (\Gamma,B)$ is not zero, where $\xi(1) = \lambda_0$ , $\xi(0) = 0$ , and $s_1 = s$ .

The above proof actually shows that $\{J(A) \mid A \in \mathcal{A}\}$ is a class of small generators in $D(\mathcal{A})$ .

<u>Corollary 2.17</u> If $\mathcal{A}$ has enough projectives, then so does $D(\mathcal{A})$ .

<u>Proof</u>: Suppose $(\Lambda,\bar{A}) \in D(\mathcal{A})$ , and for each $\lambda \in \Lambda$ , let $P_\lambda \xrightarrow{\epsilon_\lambda} A_\lambda \longrightarrow 0$ with $P_\lambda$ projective in $\mathcal{A}$ . From the definition of colimit and the fact $D(\mathcal{A})$ is AB5 ,

$$\coprod_{\lambda\in\Lambda} J(P_\lambda) \longrightarrow \coprod_{\lambda\in\Lambda} J(A_\lambda) \longrightarrow \operatorname*{colim}_{\Lambda} J(A_\lambda)$$

is an epimorphism in $D(\mathcal{A})$ . But $\coprod_{\lambda\in\Lambda} J(P_\lambda)$ is projective and corollary 2.11 together insure that $\coprod_{\lambda\in\Lambda} J(P_\lambda) \longrightarrow (\Lambda,\bar{A}) \longrightarrow 0$ .

Thus $D(\mathcal{A})$ has enough projectives.

<u>Theorem 2.17</u> (<u>Universal Extension Theorem</u>):

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor from $\mathcal{A}$ to a cocomplete abelian category $\mathcal{B}$ . Then there is a right exact functor $G : D(\mathcal{A}) \longrightarrow \mathcal{B}$ such that $G \cdot J = F$ , where $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is the canonical embedding. Furthermore, if $\mathcal{B}$ is AB5 and $F: \mathcal{A} \to \mathcal{B}$

is exact, then the extension $G : D(\mathcal{A}) \to \mathcal{B}$ is also exact.

Proof: Suppose $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor. Let $F^{\Lambda} : \mathcal{A}^{\Lambda} \to \mathcal{B}^{\Lambda}$ be the associated functor of diagram categories. Then we define $G : D(\mathcal{A}) \to \mathcal{B}$ in the following fashion:

(i) for each $(\Lambda,\bar{A}) \in D(\mathcal{A})$, set $G(\Lambda,\bar{A}) = \operatorname{colim}_{\Lambda} F^{\Lambda}(\bar{A})$

(ii) if $[(\eta,\bar{n})] : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ in $D(\mathcal{A})$, then set

$$G([(\eta,\bar{n})]) = \operatorname{colim}_{\Lambda} F^{\Lambda}(\bar{n}) \text{ , where } \operatorname{colim}_{\Lambda} F^{\Lambda}(\bar{n})\cdot p_{\lambda} = p_{\eta(\lambda)}\cdot F(n_{\lambda}) \text{ .}$$

A standard argument shows that the definition of $G : D(\mathcal{A}) \to \mathcal{B}$ is independent of the choice of representative. Clearly $G\cdot J = F$, since $\operatorname{colim} F^{\mathbb{1}}(\hat{A}) = F(\hat{A}_1) = F(A)$.

By Proposition 1.5, $\operatorname{colim}_{\Theta} \tilde{A} \cong \operatorname{colim}_{\mathfrak{J}(\Theta)} W\tilde{A}$ in $D(\mathcal{A})$, where $\Theta$ is a small category. Therefore, it suffices to show $G(\operatorname{colim}_{\Omega} \tilde{A}) \cong \operatorname{colim}_{\Omega} G(\bar{A}^{\omega})$, where $\Omega$ is a $\downarrow$-finite directed ordered set with initial element. In this case, $\operatorname{colim}_{\Omega} \tilde{A} = ((\Delta,\bar{A}),[(\rho^{\omega},\bar{p}^{\omega})])$ as defined in Lemma 2.10. Therefore, it suffices to show $\operatorname{colim}_{\Delta} F^{\Delta}(\bar{A}) \cong \operatorname{colim}_{\Omega}(\operatorname{colim}_{\Lambda^{\omega}} F^{\Lambda^{\omega}}(\bar{A}^{\omega}))$, where $q_{\lambda_{\omega}} : F(A^{\omega}_{\lambda_{\omega}}) \to \operatorname{colim}_{\Lambda^{\omega}} F^{\Lambda^{\omega}}(\bar{A}^{\omega})$ and $r_{\omega} : \operatorname{colim}_{\Lambda^{\omega}} F^{\Lambda^{\omega}}(\bar{A}^{\omega}) \to \operatorname{colim}_{\Omega}(\operatorname{colim}_{\Lambda^{\omega}} F^{\Lambda^{\omega}}(\bar{A}^{\omega}))$ are the canonical injections. Then $v : \operatorname{colim}_{\Omega}(\operatorname{colim}_{\Lambda^{\omega}} F^{\Lambda^{\omega}}(\bar{A}^{\omega})) \to \operatorname{colim}_{\Delta} F^{\Delta}(\bar{A})$ defined by the rule $v\cdot r_{\omega} = G([(\rho^{\omega},\bar{p}^{\omega})])$ and $t : \operatorname{colim}_{\Delta} F^{\Delta}(\bar{A}) \to \operatorname{colim}_{\Omega}(\operatorname{colim}_{\Lambda^{\omega}} F^{\Lambda^{\omega}}(\bar{A}^{\omega}))$ defined by the rule $t\cdot s_{(\omega,\varphi)} = r_{\omega}\cdot q_{\varphi_{\omega}}$ are inverses. Thus $G(\operatorname{colim}_{\Omega} \tilde{A}) \cong \operatorname{colim}_{\Omega} G(A^{\omega})$ in $\mathcal{B}$ and $G : D(\mathcal{A}) \to \mathcal{B}$ is cocontinuous.

Proposition 2.12, the right exactness of $F^{\Lambda} : \mathcal{A}^{\Lambda} \to \mathcal{A}$, corollary 2.9, and the right exactness of $\operatorname{colim}_{\Lambda} : \mathcal{A}^{\Lambda} \to \mathcal{A}$ together insure that $G : D(\mathcal{A}) \to \mathcal{A}$ is right exact.

The uniqueness of $G : D(\mathcal{A}) \to \mathcal{B}$ follows immediately from Corollary 2.11, and the facts that $G$ is cocontinuous and $G \cdot J = F$.

If $\mathcal{B}$ is AB5 and $F : \mathcal{A} \to \mathcal{B}$ is exact, then $F^{\Lambda} : \mathcal{A}^{\Lambda} \to \mathcal{B}^{\Lambda}$ is exact. Therefore Corollary 2.9 and $\mathcal{B}$ AB5 together imply that $G : D(\mathcal{A}) \to \mathcal{B}$ is exact.

It is this unique extension property which enables us, in the next chapter, to factorize $\operatorname{colim}_{\Theta} : \mathcal{A}^{\Theta} \to \mathcal{A}$ into $\operatorname{colim}_{\Theta} J^{\Theta} : \mathcal{A}^{\Theta} \to D(\mathcal{A})$ and $\Psi : D(\mathcal{A}) \to \mathcal{A}$ whenever $\mathcal{A}$ is cocomplete; and to develop a spectral sequence which is an important tool of this paper.

# CHAPTER III

## A SPECTRAL SEQUENCE FOR $\underset{\Omega}{\mathrm{Colim}} : \mathcal{A}^{\Omega} \to \mathcal{A}$

### Section 1: $\mathcal{A}$ cocomplete insures $\mathcal{A}$ coreflective subcategory of $D(\mathcal{A})$

Suppose $\mathcal{A}$ is a cocomplete abelian category.

**Lemma 3.1:** There is a cocontinuous (and consequently right exact functor) $\Psi : D(\mathcal{A}) \to \mathcal{A}$ defined by:

(i) if $(\Lambda,\bar{A}) \in D(\mathcal{A})$, $\Psi(\Lambda,\bar{A}) = \underset{\Lambda}{\mathrm{colim}}\ A_\lambda$

(ii) if $(\eta,\bar{n}) : (\Lambda,\bar{A}) \to (\Gamma,\bar{B})$ is a morphism in $D(\mathcal{A})$, then

$$\Psi(\eta,\bar{n}) = \underset{\Lambda}{\mathrm{colim}}\ (\bar{n}) \quad \text{where} \quad \underset{\Lambda}{\mathrm{colim}}\ (\bar{n})\cdot p_\lambda = p_{\eta(\lambda)} n_\lambda \ .$$

Furthermore, $\Psi \cdot J = \mathrm{id}_{\mathcal{A}}$ .

**Proof:** This is simply Theorem 2.17 where $F = \mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ .

**Proposition 3.2:** $\mathcal{A}$ is a coreflective subcategory of $D(\mathcal{A})$ , i.e. the functor $\Psi : D(\mathcal{A}) \to \mathcal{A}$ is the coadjoint of $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ .

**Proof:** It suffices to show

$$D(\mathcal{A})((\Lambda,\bar{A}),J(X)) \cong \mathcal{A}(\Psi(\Lambda,\bar{A}),X)$$

for $(\Lambda,\bar{A}) \in D(\mathcal{A})$ and $X \in \mathcal{A}$ . From Corollary 2.11, the definition of colimit, and the fact that $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is a full embedding, it follows that

$$D(\mathcal{A})((\Lambda,\bar{A}),J(X)) \cong D(\mathcal{A})(\underset{\Lambda}{\mathrm{colim}}\ J^{\Lambda}(\bar{A}),J(X))$$

$$\cong \underset{\Lambda^{op}}{\mathrm{Lim}}\ D(\mathcal{A})(J(A_\lambda),J(X))$$

$$\cong \lim_{\Lambda^{op}} \mathcal{A}(A_\lambda, X)$$

$$\cong \mathcal{A}\,(\operatorname{colim}_{\Lambda} A_\lambda, X)$$

$$= \mathcal{A}(\Psi(\Lambda,\bar{A}), X) \ .$$

Thus, $\mathcal{A}$ is a coreflective subcategory of $D(\mathcal{A})$ .

For $\Theta$ , a small category, let $\Psi^\Theta : D(\mathcal{A})^\Theta \to \mathcal{A}^\Theta$ denote the canonical functor induced by $\Psi : D(\mathcal{A}) \to \mathcal{A}$ .

<u>Corollary 3.3</u>: If $\Theta$ is any small category, then $\Psi(\operatorname{colim}_\Theta J^\Theta(\bar{A})) \cong \operatorname{colim}_\Theta (\bar{A})$ for $\bar{A} \in \mathcal{A}^\Theta$ .

<u>Proof</u>: Since $\Psi : D(\mathcal{A}) \to \mathcal{A}$ is cocontinuous and $\Psi\cdot J \cong \mathrm{id}_{\mathcal{A}}$ ,

$$\Psi(\operatorname{colim}_\Theta J^\Theta(\bar{A})) \cong \operatorname{colim}_\Theta \Psi\cdot J(A_\theta)$$

$$\cong \operatorname{colim}_\Theta (A_\theta)$$

$$\cong \operatorname{colim}_\Theta (\bar{A}) \ .$$

By Corollary 3.3, $\operatorname{colim}_\Theta : \mathcal{A}^\Theta \to \mathcal{A}$ is factored into $\operatorname{colim}_\Theta J^\Theta : \mathcal{A}^\Theta \to D(\mathcal{A})$ and $\Psi : D(\mathcal{A}) \to \mathcal{A}$ . It is this factorization for $\Theta = \Omega$ , a $\downarrow$-finite ordered set, which enables us to derive the spectral sequence which is the major tool of this thesis.

As a first step, we prove a series of lemmas to show that $\operatorname{colim}_\Omega J^\Omega : \mathcal{A}^\Omega \to D(\mathcal{A})$ preserves projectives. For the remainder of this chapter, we assume that $\mathcal{A}$ has enough projectives so that the left derived functors are well-defined, and that $\Omega$ is a $\downarrow$-finite ordered set not necessarily directed.

Section 2: $\Omega$ $\downarrow$-finite ordered set insures that $\underset{\Omega}{\operatorname{colim}} J^{\Omega} : \mathcal{A}^{\Omega} \to D(\mathcal{A})$ preserves projectives.

Recall that Proposition 1.3 guarantees that $\mathcal{A}^{\Omega}$ has enough canonic·l projectives; i.e., projectives of the form $\coprod_{\omega \in \Omega} E_\omega P_\omega$ where $E_\omega : \mathcal{A}$ · djoint to $ev_\omega : \mathcal{A}^{\Omega} \to \mathcal{A}$ , and $P_\omega$ is projective in $\mathcal{A}$ .

Lemma 3.4: For every $\omega \in \Omega$ ,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad J \quad} & D(\mathcal{A}) \\ \downarrow E_\omega & & \downarrow E_\omega \\ \mathcal{A}^{\Omega} & \xrightarrow{\quad J^{\Omega} \quad} & (D(\mathcal{A}))^{\Omega} \end{array}$$

commutes.

Proof: It suffices to show that

$$J^{\Omega}(E_\omega X)_\gamma = E_\omega(J(X))_\gamma$$

for $\gamma \in \Omega$ and $X \in \mathcal{A}$ . But this is true, since by definition,

$$J^{\Omega}(E_\omega X)_\gamma = J((E_\omega X)_\gamma) = \begin{cases} J(X) & \text{if } \gamma \geq \omega \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$E_\omega(J(X))_\gamma = \begin{cases} J(X) & \text{if } \gamma \geq \omega \\ 0 & \text{otherwise} . \end{cases}$$

Lemma 3.5: If $P \in \mathcal{A}$ is projective in $\mathcal{A}$ , then $J^{\Omega}(E_\omega P) = E_\omega(J(P))$ is projective in $D(\mathcal{A})^{\Omega}$ .

Proof: Since $E_\omega \dashv ev_\omega$ and $ev_\omega : D(\mathcal{A})^{\Omega} \to D(\mathcal{A})$ is exact, $E_\omega : D(\mathcal{A}) \to D(\mathcal{A})^{\Omega}$ preserves projectives. Therefore, $E_\omega(J(P))$ is projective in $D(\mathcal{A})^{\Omega}$ , because $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ preserves projectives.

Lemma 3.6: If $\{X_\omega\}_{\omega \in \Omega}$ is any collection of objects in $\mathcal{A}$ and $\Omega$ is a $\downarrow$-finite ordered set, then $J^{\Omega}(\coprod_\omega E_\omega X_\omega) \cong \coprod_\omega E_\omega(J(X_\omega))$ .

<u>Proof</u>: Suppose that $\{X_\omega\}_{\omega\in\Omega}$ is given and that $\gamma$ is an arbitrary element of $\Omega$. From the definitions of $J^\Omega : \mathcal{A}^\Omega \to D(\mathcal{A})^\Omega$ and $E_\omega : \mathcal{A} \to \mathcal{A}^\omega$, it follows that

$$J^\Omega(\coprod_{\omega\in\Omega} E_\omega X_\omega)_\gamma = J((\coprod_\omega E_\omega X_\omega)_\gamma)$$

$$\cong J(\coprod_{\omega\in In(\gamma)} X_\omega) .$$

On the other hand,

$$(\coprod_{\omega\in\Omega} J^\Omega(E_\omega X_\omega))_\gamma = \coprod_{\omega\in\Omega} J((E_\omega X_\omega)_\gamma)$$

$$\cong \coprod_{\omega\in In(\gamma)} J(X_\omega) .$$

But since $In(\gamma)$ is finite, it is the maximal element of $\mathfrak{F}(In(\gamma))$. Therefore, using Proposition 1.5,

$$\coprod_{\omega\in In(\gamma)} J(X_\omega) \cong \operatorname*{colim}_{u\in\mathfrak{F}(In(\gamma))} J(\coprod_{\omega\in u} X_\omega)$$

$$\cong J(\coprod_{\omega\in In(\gamma)} X_\omega) ,$$

and the lemma follows.

It is here that the utility (for this argument) of the $\Omega$'s being $\downarrow$-finite is apparent. The functor $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ is not necessarily cocontinuous, i.e., $J : \mathcal{A} \hookrightarrow D(\mathcal{A})$ does not necessarily commute with arbitrary coproducts.

<u>Corollary 3.7</u>: $J^\Omega : \mathcal{A}^\Omega \to D(\mathcal{A})^\Omega$ preserves canonical projectives, i.e. $J^\Omega(\coprod_{\omega\in\Omega} E_\omega P_\omega) \cong \coprod_{\omega\in\Omega} E_\omega(J(P_\omega))$ .

<u>Proof</u>: This is immediate from lemma 3.5, lemma 3.6, and the fact that the coproduct of projectives is projective.

Proposition 3.8: $\operatorname{colim}_{\Omega} J^{\Omega} : \mathcal{A}^{\Omega} \to D(\mathcal{A})$ preserves projectives, whenever $\Omega$ is a $\downarrow$-finite ordered set.

Proof: Both $\operatorname{colim}_{\Omega} : D(\mathcal{A})^{\Omega} \to D(\mathcal{A})$ and $J^{\Omega} : \mathcal{A}^{\Omega} \to D(\mathcal{A})^{\Omega}$, by lemma 3.7, preserve projectives, and therefore their composition, $\operatorname{colim}_{\Omega} J^{\Omega} : \mathcal{A}^{\Omega} \to D(\mathcal{A})$, also preserves projectives.

## Section 3: The Spectral Sequence for $\operatorname{colim}_{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$.

Theorem 3.9. If $\Omega$ is a $\downarrow$-finite ordered set, $\mathcal{A}$ is cocomplete abelian category, with projectives, and $\bar{A} \in \mathcal{A}^{\Omega}$, then there is a first quadrant spectral sequence with

$$E^2_{pq} = (L_p \Psi)(L_q \operatorname{colim}_{\Omega})(J^{\Omega}(\bar{A}))$$

converging to

$$(L_{p+q} \operatorname{colim}_{\Omega}) \ (\bar{A})$$

Proof: The hypotheses of the "Grothendieck Two Functor Theorem" [5] are satisfied since $\Psi \cdot \operatorname{colim}_{\Omega} J^{\Omega}(\bar{A}) \cong \operatorname{colim}_{\Omega} (\bar{A})$, $\bar{A} \in \mathcal{A}^{\Omega}$, $\Psi : D(\mathcal{A}) \to \mathcal{A}$ is right exact, and $\operatorname{colim}_{\Omega} J^{\Omega} : \mathcal{A}^{\Omega} \to D(\mathcal{A})$ preserves projectives.

Therefore, applying this theorem of Grothendieck and Corollary 3.7, there is a spectral sequence with

$$E^2_{pq} = (L_p \Psi)(L_q \operatorname{colim}_{\Omega})(J^{\Omega}(\bar{A}))$$

converging to

$$(L_{p+q} \operatorname{colim}_{\Omega})(\bar{A}) .$$

Since $D(\mathcal{A})$ is AB5 and $J^{\Omega} : \mathcal{A}^{\Omega} \to D(\mathcal{A})^{\Omega}$ is exact, we have the following corollary:

Corollary 3.10: If $\Lambda$ is a $\downarrow$-finite directed ordered set, $\mathcal{A}$ is a cocomplete abelian category with projectives, and $\bar{A} \in \mathcal{A}^{\Lambda}$, then $(L_p \Psi)(\operatorname{colim}_{\Omega} J^{\Omega}(\bar{A})) \cong (L_p \operatorname{colim}_{\Omega})(\bar{A})$ for every $p \geq 0$.

Proof: Since $\operatorname{colim}_{\Omega} J^{\Omega} : \mathcal{A}^{\Omega} \to D(\mathcal{A})$ is exact, $(L_q \operatorname{colim}_{\Omega} J^{\Omega})(\bar{A}) = 0$ for $q > 0$. From the definition of convergence of a spectral sequence, it follows that

$$(L_p \Psi)(\operatorname{colim}_{\Omega} J^{\Omega}(\bar{A})) \cong (L_p \operatorname{colim}_{\Omega})(\bar{A}) .$$

## Section 4: An Alternative Form for the Spectral Sequence.

Recall that $W : \mathcal{A}^{\Omega} \to \mathcal{A}^{\mathfrak{F}(\Omega)}$ is the functor defined by $(W\bar{A})_u = \operatorname{colim}_u \bar{A}/u$, where $\mathfrak{F}(\Omega)$ is the $\downarrow$-finite directed ordered set consisting of all finite subsets of $\Omega$. Proposition 1.5 shows that

(i) $\operatorname{colim}_{\mathfrak{F}(\Omega)} (W\bar{A}) \cong \operatorname{colim}_{\Omega} (\bar{A})$ in $\mathcal{A}$, and

(ii) $\operatorname{colim}_{\mathfrak{F}(\Omega)} J^{\mathfrak{F}(\Omega)}(W\bar{A}) \cong \operatorname{colim}_{\Omega} J^{\Omega}(\bar{A})$ in $D(\mathcal{A})$.

Using these facts, we are able to evaluate the derived functors of $\operatorname{colim}_{\Omega} J^{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$, and express the spectral sequence in an alternative form.

Lemma 3.11: If $\Omega$ is a $\downarrow$-finite ordered set, then

$$(L_q \operatorname{colim}_{\Omega} J^{\Omega})(\bar{A}) \cong \operatorname{colim}_{\mathfrak{F}(\Omega)} J^{\mathfrak{F}(\Omega)}((L_q W)(\bar{A}))$$

$$\cong \operatorname{colim}_{u \in \mathfrak{F}(\Omega)} J((L_q \operatorname{colim}_u)(\bar{A}/u))$$

for every $\bar{A} \in \mathcal{A}^{\Omega}$.

<u>Proof</u>: Suppose $(\tilde{X}, \epsilon) = (\ldots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} \bar{A} \to 0)$ is a projective resolution of $\bar{A}$ in $\mathcal{A}^{\Omega}$. Since $\operatorname{colim}_{\Omega} J^{\Omega}(\bar{A}) \cong \operatorname{colim}_{\mathfrak{I}(\Omega)} J^{\mathfrak{I}(\Omega)}(W\bar{A})$, $H_*(\operatorname{colim}_{\Omega} J^{\Omega}(\tilde{X})) \cong H_*(\operatorname{colim}_{\mathfrak{I}(\Omega)} J^{\mathfrak{I}(\Omega)}(W\tilde{X})$. But $\operatorname{colim}_{\mathfrak{I}(\Omega)} J^{\mathfrak{I}(\Omega)} : \mathcal{A}^{\mathfrak{I}(\Omega)} \to D(\mathcal{A})$ exact, insures that

$$H_*(\operatorname{colim}_{\Omega} J^{\Omega}(\tilde{X})) \cong \operatorname{colim}_{\mathfrak{I}(\Omega)} J^{\mathfrak{I}(\Omega)} H_*(W\tilde{X})$$

since homology commutes with exact functors. By definition,

$$H_*(W\tilde{X})_u = L_*(W\bar{A})_u = (L_* \operatorname{colim}_u)(\bar{A}/u) ,$$

and the result follows.

<u>Theorem 3.12</u>: If $\mathcal{A}$ is a cocomplete abelian category with projectives, $\Omega$ is a $\downarrow$-finite ordered set, and $\bar{A} \in \mathcal{A}^{\Omega}$, then there is a first quadrant spectral sequence with

$$E^2_{pq} = (L_p \operatorname{colim}_{\mathfrak{I}(\Omega)})(L_q \operatorname{colim}_u)(\bar{A}/u)$$

$$= (L_p \operatorname{colim}_{\mathfrak{I}(\Omega)})(L_q W)(\bar{A})$$

converging to $(L_{p+q} \operatorname{colim}_{\Omega})(\bar{A})$ .

<u>Proof</u>: By Theorem 3.9, there is a first quadrant spectral sequence with

$$E^2_{pq} = (L_p \Psi)(L_q \operatorname{colim}_{\Omega} J^{\Omega})(\bar{A})$$

converging to $(L_{p+q} \operatorname{colim}_{\Omega})(\bar{A})$ . Using Lemma 3.11, we substitute $\operatorname{colim}_{\mathfrak{I}(\Omega)} J^{\mathfrak{I}(\Omega)}(L_q W)(\bar{A})$ for $((L_q \operatorname{colim}_{\Omega} J^{\Omega})(\bar{A}))$ to get

$$E^2_{pq} \cong (L_p \Psi)(\operatorname{colim}_{\mathfrak{I}(\Omega)} J^{\mathfrak{I}(\Omega)}((L_q W)(\bar{A}))) .$$

Lastly, since $\mathfrak{J}(\Omega)$ is a $\downarrow$-finite directed ordered set, we are able to apply Corollary 3.10 to get

$$E^2_{pq} \cong (L_p \operatorname*{colim}_{\mathfrak{J}(\Omega)})(L_q W)(\bar{A}) \quad .$$

Thus, from the factorization

$$\begin{array}{ccc} & \mathcal{A}^{\Omega} & \\ {\scriptstyle \operatorname*{colim}_{\Omega} J^{\Omega}} \swarrow & & \searrow {\scriptstyle \operatorname*{colim}_{\Omega}} \\ D(\mathcal{A}) & \xrightarrow{\ \Psi\ } & \mathcal{A} \end{array} \quad ,$$

we get a spectral sequence which involves derived functors of colimit over a directed ordered set, namely $\mathfrak{J}(\Omega)$ . In the next chapter, we develop a method, based on the cardinality of the directed ordered set, of determining when higher derived functors of colimit vanish over directed ordered sets. Since the cardinality of $\mathfrak{J}(\Omega)$ is the same as that of $\Omega$ when $\Omega$ is infinite, the above spectral sequence, in various cases, will tell us when higher derived functors of $\operatorname*{colim}_{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$ vanish.

# CHAPTER IV

## CARDINALITY, DIRECTED ORDERED SETS AND DERIVED FUNCTORS OF LIMIT

In this chapter, it is shown that knowledge of the cardinality of directed ordered set $\Lambda$ is enough to determine when higher derived functors of $\underset{\Lambda}{\text{colim}} : \mathcal{A}^{\Lambda} \to \mathcal{A}$ vanish. We shall suppose that $\mathcal{A}$ is an AB4 category with injectives.

To show this, we first generalize a lemma of M. Auslander [2], dealing with dimension of modules, to homological dimension in an AB5 category with projectives. From this lemma, we are able to prove a generalization of a result of B.L. Osofsky [14] which insures that the homological dimension of $\underset{\Lambda}{\text{colim}}\, X_\lambda, (X_\lambda$ projective in $\mathcal{A}$ for every $\lambda \in \Lambda)$, is zero in high degrees.

Finally, using these results and the definition of limit in $\mathcal{A}b$, it follows that if $\mathcal{A}$ is AB4 category with enough injectives and $\Lambda$ is a directed ordered set with $\|\Lambda\| \le \aleph_n$, then $(L_p \underset{\Lambda}{\text{colim}})(\bar{A}) = 0$ for $p > n + 1$.

Because the proofs in this chapter depend on an understanding of homological dimension, we recall that the <u>homological dimension</u> of $A \in \mathcal{A}$, h.d.(A), is defined to be the least non-negative integer such that $0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to A \to 0$ is exact with $P_i$ projective in $\mathcal{A}$, $0 \le i \le n$; if no such integer exists, then h.d(A) is defined to be $\infty$. P is projective if and only if h.d.(P) = 0. Furthermore, h.d.(A) = n if and only if $\text{Ext}^n(\_,A) \neq 0$ and

$Ext^{n+k}(\_,A) = 0$ for every $k > 0$. Therefore, $0 \to A \to B \to C \to 0$ exact in $\mathcal{A}$ insures:

(i) if $h.d(B) = h.d(A)$, then $h.d(C) \leq 1 + h.d(A)$ and

(ii) if $h.d(B) < h.d(A)$, then $h.d(C) = 1 + h.d(A)$.

Section 1: M. Auslander's Lemma.

In this section, we assume that $\mathcal{A}$ is an AB5 category with projectives.

Lemma 4.1. (M. Auslander): Let $A \in \mathcal{A}$, I be a non-empty well-ordered set and $\{A_j\}_{j \in I}$ be a family of sub-objects of $A, A_j \xrightarrow{v_j} A$, such that if $i \leq j$ in I, $A_i \xrightarrow{v_i^j} A_j$. If $A = \bigcup_{j \in I} A_j$ and $h.d.(A_j / \bigcup_{i<j} A_i) \leq n$ for every $j \in I$, then $h.d.(A) \leq n$.

Proof: Let $B_j$ denote $A_j / \bigcup_{i<j} A_i$ for every $j \in I$. Since A is AB5, for every $j \in I$, there is a short exact sequence

(4.2) $$0 \to \bigcup_{i<j} A_i \to A_j \to B_j \to 0 \quad \text{in } \mathcal{A}.$$

The proof is by induction on the homological dimension of the $B_j$'s. First consider the case when $n = 0$, i.e. $h.d.(B_j) = 0$ and $B_j$ is projective for every $j \in I$. Then the short exact sequence (4.2) splits with retraction $w_j : B_j \to A_j$, and $A_j \cong B_j \oplus \bigcup_{i<j} A_j$, for every $j \in I$. Thus it suffices to show that $A = \bigcup_{j \in I} A_j$ is isomorphic to $\coprod_{j \in I} B_j$, because $\coprod_{j \in I} B_j$ is projective and $h.d.(\coprod_{j \in I} B_j) = 0 = h.d.(A)$.

Since $\mathcal{A}$ is AB5 and $B_j$ is a sub-object of $\bigcup_{j \in I} A_j$ via

$v_j w_j : B_j \hookrightarrow \bigcup_{i \in I} A_j$ for every $j \in I$, $\coprod_j B_j$ is a subobject of $A$, i.e., $\coprod_{j \in I} B_j \hookrightarrow \bigcup_{j \in I} A_j$. Next, suppose $\bigcup_{j \in I} A_j = A$ is not a subobject of $\coprod_{j \in I} B_j$. Let $j_0$ be the minimal element of $I$ such that $A_{j_0}$ is not included in $\coprod_{j \in I} B_j$. (One exists since $I$ is well-ordered.) Then for every $i < j_0, A_i \hookrightarrow \coprod_j B_j$. $\mathcal{C}$ AB5 again insures that $\bigcup_{i<j_0} A_i$ is a subobject of $\coprod_{j \in I} B_j$. But $A_{j_0} = B_{j_0} \oplus \bigcup_{i \in j_0} A_i$ is a subobject of $\coprod_{j \in I} B_j$, which yields a contradiction. Therefore, $A$ is a subobject of $\coprod_{j \in I} B_j$ and $A \cong \coprod_j B_j$.

We now assume that the lemma is true in the cases when $h.d.(B_j) \le n - 1$. For each $j \in I$, there is a commutative diagram

$$(4.3) \qquad \begin{array}{ccccccccc} & & & & & & A_j & & \\ & & & & & \overset{w_j}{\nearrow} & \downarrow{\scriptstyle p_j} & & \\ 0 & \longrightarrow & K'_j & \longrightarrow & X_j & \xrightarrow{\beta_j} & B_j & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

with exact bottom row and $X_j$ projective.

Using (4.3), we construct a commutative diagram.

(4.4)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigcup_{i<j} K_i & \longrightarrow & \coprod_{i<j} X_i & \xrightarrow{\alpha'_j} & \bigcup_{i<j} A_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_j & \longrightarrow & \coprod_{i\leq j} X_j & \xrightarrow{\alpha_j} & A_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow p_j & & \\
0 & \longrightarrow & K'_j & \longrightarrow & X_j & \xrightarrow{\beta_j} & B_j & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $K_j = \ker\left(\coprod_{i\leq j} X_j \xrightarrow{\alpha_j} A_j\right)$ . It is clear that all the columns and the bottom row are exact. In particular, $K'_j \cong (K_j / \bigcup_{i<j} K_i)$ . A transfinite induction shows that $\alpha_j : \coprod_{i\leq j} X_j \to A_j$ is an epimorphism, and therefore the middle row is exact. The "nine lemma" applied to (4.4), insures that the top row is exact, i.e.,

$$0 \longrightarrow \bigcup_{i>j} K_j \longrightarrow \coprod_{i<j} X_j \xrightarrow{\alpha'_j} \bigcup_{i<j} A_i \longrightarrow 0$$

is exact for every $j \in I$ .

Define $K = \ker(\coprod_j X_j \to A)$ , and consider $\{K_j\}_{j\in I}$ . If $i \leq j$ , then $K_i \hookrightarrow K_j$ by construction since $A_i \hookrightarrow A_j$ . But $\mathcal{A}$ AB5 implies that

(4.5) $$0 \to \bigcup_{j \in I} K_j \to \coprod_{j \in I} X_j \to \bigcup_{j \in I} A_j \to 0$$

is exact in $\mathcal{A}$ since

$$0 \longrightarrow K_j \longrightarrow \coprod_{i \leq j} X_j \xrightarrow{\alpha_j} A_j \longrightarrow 0$$

is exact for every $j \in I$. By definition, $K = \bigcup_{j \in I} K_j$. Furthermore,

since $h.d.(B_j) \leq n$ for every $j \in I$ and

$$0 \longrightarrow K'_j \longrightarrow X_j \xrightarrow{\beta_j} B_j \longrightarrow 0$$

is exact, with $X_j$ projective,

$$h.d.(K_j / \bigcup_{i<j} k_i) = h.d.(K'_j) \leq n - 1 .$$

Therefore, $\{K_j\}$ satisfies the inductive hypothesis and $h.d.(K) \leq n - 1$.

But $\coprod_{j \in I} X_j$ projective and (4.5) together insure that $h.d.(A) = h.d.(K) + 1 \leq n$, and the induction is complete.

The above lemma is actually used in a slightly different form.

Corollary 4.6: Suppose $I$ is well-ordered, $A \in \mathcal{A}$ and $\{A_j\}_{j \in I}$ is a family of subobjects of $A$ such that $A = \bigcup_{j \in I} A_j$, $i \leq j$ implies $A_i \hookrightarrow A_j$, and $j$ a limit ordinal implies $A_j = \bigcup_{i<j} A_i$. If $h.d.(A_j) \leq n$, for every $j \in I$, then $h.d.(A) \leq n + 1$.

Proof: By hypothesis,

$$\bigcup_{i<j} A_i = \begin{cases} A_{j'} & \text{if } j = j' + 1 \\ A_j & \text{if } j \text{ is a limit ordinal} . \end{cases}$$

Therefore,

$$B_j = A_j/(\bigcup_{i<j} A_j) = \begin{cases} A_{j'+1}/A_{j'} & \text{if } j = j' + 1 \\ 0 & \text{if } j \text{ is a limit ordinal.} \end{cases}$$

From the exact sequence (4.2), it follows that $h.d.(B_j) \leq n + 1$ for every $j \in I$. Therefore $\{A_j\}_{j\in I}$ satisfies the hypothesis of lemma 4.1, and $h.d.(A) \leq n + 1$.

Section 2: Osofsky's Proposition

Using Auslander's lemma for R-modules and a lemma of I. Berstein [3], M.B. Osofsky showed that if $\Lambda$ is a directed ordered set with $||\Lambda|| \leq \aleph_n$, and $\{M_\lambda\}$ is a directed system of groups such that each $M_\lambda$ is a $R_\lambda$-module, then $h.d._R(\operatorname{colim}_\Lambda M_\lambda) \leq n{+}1 + \sup h.d._{R_\lambda}(M_\lambda)$, where $R = \operatorname{colim}_\Lambda R_\lambda$ [14].

We prove a more general version of this proposition for AB5 categories with projectives. The proof is done by induction on the cardinality of the directed (upwards) ordered set $\Lambda$, $||\Lambda|| \leq \aleph_n$. As a first step, we generalize Berstein's lemma to get the case for $n = 0$, $||\Lambda|| \leq \aleph_0$.

Lemma 4.7: (Berstein) Suppose $||\Lambda|| \leq \aleph_0$ and $\bar{X} \in \mathcal{A}^\Lambda$ such that $X_\lambda$ is projective in $\mathcal{A}$ for every $\lambda \in \Lambda$. Then $h.d.(\operatorname{colim}_\Lambda X_\lambda) \leq 1$.

Proof: If $||\Lambda|| < \aleph_0$, then $\Lambda$ is a finite directed ordered set and $\Lambda$ has a maximal element $\lambda_0 \in \Lambda$. Therefore, $\operatorname{colim}_\Lambda X_\lambda = X_{\lambda_0}$ and

and $h.d.(\operatorname{colim}_{\Lambda} X_\lambda) = h.d.(X_{\lambda_0}) = 0$ .

Next, suppose $\|\Lambda\| = \aleph_0$ . Then there is a totally ordered cofinal subset $\Lambda' \subseteq \Lambda$ such that $\|\Lambda'\| = \aleph_0$ . Without loss of generality, we suppose that $\Lambda = \{0,1,2,\ldots\} = \mathbb{Z}^+$ .

For every integer $s$ ,

$$(4.8) \qquad 0 \longrightarrow \coprod_{r<s} X_n \xrightarrow{\sigma_s} \coprod_{r\leq s} X_r \longrightarrow X_s \longrightarrow 0$$

is exact in $\mathcal{A}$ where $\sigma_s \cdot u_r = u_r - u_{r+1} X_r^{r+1}$ , $u_r : X_r \to \coprod_{r\leq s} X_r$ the canonical injection. It is clear that the exact sequences (4.8) define an exact sequence of diagrams in $\mathcal{A}^\Lambda$ . Since $\mathcal{A}$ is AB5 and $\Lambda$ is directed, $0 \to \coprod_s X_s \xrightarrow{\sigma} \coprod_s X_s \to \operatorname{colim}_{\Lambda} X_s \to 0$ is exact in $\mathcal{A}$ . Therefore $h.d.(\operatorname{colim}_{\Lambda} X_s) \leq h.d.(\coprod_s X_s) + 1$ . But $\coprod_s X_s$ projective insures that $h.d.(\operatorname{colim}_{\Lambda} X_s) \leq 1$ .

A set theoretic lemma of Osofsky [14] is stated here without proof. It is used to complete the induction.

<u>Lemma 4.9</u>: Assume $n > 0$ and $\Lambda = \{\gamma | \gamma < \aleph_n\}$ . Then there exist directed subsets $\{\Lambda_\gamma | \gamma < \aleph_n\}$ of $\Lambda$ such that:

(i) $\Lambda = \bigcup_{\gamma<\aleph_n} \Lambda_\gamma$

(ii) $\Lambda_\gamma \subseteq \Lambda_{\gamma'}$ for $\gamma < \gamma'$

(iii) $\|\Lambda_\gamma\| = \aleph_{n-1}$ for all $\gamma < \aleph_n$ .

<u>Proposition 4.10 (Osofsky)</u>: Suppose $\Lambda$ is a directed ordered set with

cardinality less than or equal to $\aleph_n$ . If $\bar{X} \in \mathcal{A}^{\Lambda}$ with $X_\lambda$ projective for every $\lambda \in \Lambda$ , then $\text{h.d.}(\operatorname{colim}_{\Lambda} X_\lambda) \leq n + 1$ .

<u>Proof</u>: The first step of the induction was done in lemma 4.7. Suppose the proposition is true for the case $n - 1$ . Let $\Lambda = \{\gamma \mid \gamma < \aleph_n\}$ . By lemma 4.9, $\Lambda = \bigcup_{\gamma < \aleph_n} \Lambda_\gamma$ where each $\Lambda_\gamma$ is directed with $\|\Lambda_\gamma\| = \aleph_{n-1}$ .

For each $\gamma < \aleph_n$, define $A_\gamma = \operatorname{colim}_{\Gamma_\gamma} \bar{X}/\Gamma_\gamma$ , where $\Gamma_\gamma = \bigcup_{\beta<\gamma} \Lambda_\beta$ . Since $\|\Gamma_\gamma\| = \aleph_{n-1}$ , the inductive hypothesis applies to $\bar{X}/\Gamma_\gamma$ , and therefore $\text{h.d.}(A_\gamma) \leq n$ .

Next, set

$$B_\gamma = \bigcup_{\lambda<\lambda' \subseteq \Gamma_\gamma} \operatorname{Im}(u_\lambda - u_{\lambda'} X_\lambda^{\lambda'}) \quad \text{and}$$

$$B = \bigcup_{\lambda<\lambda' \subseteq \Lambda} \operatorname{Im}(u_\lambda - u_{\lambda'} X_\lambda^{\lambda'}) ,$$

where $u_\lambda : X_\lambda \to \coprod_{\lambda\in\Gamma_\gamma} X_\gamma$ and $u_\lambda : X_\gamma \to \coprod_{\lambda\in\Lambda} X_\lambda$ are the canonical inclusions into the coproduct.

For $\gamma < \gamma'$ , $B_\gamma \hookrightarrow B_{\gamma'}$ since $\Gamma_\gamma \subseteq \Gamma_{\gamma'}$ . $\mathcal{A}$ AB5 insures that

$$B = \operatorname{colim}_{\gamma<\aleph_n} B_\gamma = \bigcup_{\gamma<\aleph_n} B_\gamma .$$

If $\alpha$ is a limit ordinal less than $\aleph_n$ , then $\Gamma_\alpha = \bigcup_{\gamma<\alpha} \Lambda_\gamma = \bigcup_{\gamma<\alpha} \Gamma_\gamma$ , implies $B_\alpha = \bigcup_{\gamma<\alpha} B_\gamma$ . Furthermore, for each $\gamma \in \Lambda$ , $\text{h.d.}(B_\gamma) \leq n - 1$ since $0 \to B_\gamma \to \coprod_{\lambda\in\Gamma_\gamma} X_\lambda \to A_\gamma \to 0$ is exact with $\coprod_{\lambda\in\Gamma_\gamma} X_\lambda$ projective.

Therefore $\{B_\gamma\}_{\gamma<\aleph_n}$ with $B = \bigcup_{\gamma<\aleph_n} B_\gamma$ satisfy the hypothesis of Corollary 4.6 and h.d.(B) $\le n$ .

From the definition of colimit,

$$0 \to B \to \coprod_{\lambda\in\Lambda} X_\lambda \to \operatorname*{colim}_{\Lambda} X_\lambda \to 0$$

is exact with $\coprod_{\lambda\in\Lambda} X_\lambda$ projective. Therefore

$$\text{h.d.}(\operatorname*{colim}_{\Lambda} X_\lambda) = 1 + \text{h.d.}(B) \le 1 + n .$$

Section 3: The Result for $\mathcal{A}b$ .

In this section, we will use the Osofsky proposition to show that if $\Lambda$ is directed, $\|\Lambda\| \le \aleph_n$ , then $\left(R^q\operatorname*{Lim}_{\Lambda}\right) : \mathcal{A}b^{\Lambda} \to \mathcal{A}b$ , is zero for every $q > n + 1$ .

First we show that the right derived functors of $\operatorname*{Lim}_{\Omega} : \mathcal{A}b^{\Omega} \to \mathcal{A}b$ , $\Omega$ a small category, can be characterized in terms of Ext-functors.

Lemma 4.11: Suppose $\bar{A} \in \mathcal{A}b^{\Omega}$ and $\hat{\mathbb{Z}}$ is the constant diagram over $\Omega$ with value $\mathbb{Z}$ . Then

$$\left(R^q\operatorname*{Lim}_{\Omega}\right)(\bar{A}) \cong \operatorname{Ext}^q(\hat{\mathbb{Z}},\bar{A}) \quad \text{for every} \quad q \ge 0 .$$

Proof: Let $\wedge : \mathcal{A}b \to \mathcal{A}b^{\Omega}$ be the functor which assigns to each $G \in \mathcal{A}b$ the constant diagram $\hat{G} : \Omega \to \mathcal{A}b$ , By definition $\wedge : \mathcal{A}b \to \mathcal{A}b^{\Omega}$ is the coadjoint of $\operatorname*{Lim}_{\Omega} : \mathcal{A}b^{\Omega} \to \mathcal{A}b$ , i.e.,

$$\mathcal{A}b^{\Omega}(\hat{G},\bar{A}) \cong \mathcal{A}b(G,\operatorname*{Lim}_{\Omega} \bar{A})$$

Since $\mathcal{A}b(\mathbb{Z},H) \cong H$ for every $H \in \mathcal{A}b$ ,

$$\mathcal{A}b^{\Omega}(\hat{\mathbb{Z}},\bar{A}) \cong \mathcal{A}b(\mathbb{Z},\operatorname*{Lim}_{\Omega} \bar{A}) \cong \operatorname*{Lim}_{\Omega} \bar{A} .$$

Furthermore $\bar{Q}$ injective in $\mathcal{A}b^{\Omega}$ insures that $\mathrm{Ext}^q(\hat{\mathbb{Z}},\bar{Q}) = 0$ for all $q > 0$ . Therefore,

$$\mathrm{Ext}^q(\hat{\mathbb{Z}},\bar{A}) \cong \left(R^q \underset{\Omega}{\mathrm{Lim}}\right)(\bar{A}) \text{ , for every } q \geq 0 .$$

We use this fact to prove the following proposition.

Proposition 4.12: Suppose $\Lambda$ is a directed (downward) ordered set of cardinality $\aleph_n$ . Then $R^q \underset{\Lambda}{\mathrm{Lim}} : \mathcal{A}b^{\Lambda} \to \mathcal{A}b$ is zero for every $q > n + 1$ .

Proof: Recall that if $E_\lambda : \mathcal{A}b \to \mathcal{A}b^{\Lambda}$ is the coadjoint of $ev_\lambda : \mathcal{A}b^{\Lambda} \to \mathcal{A}b$ , evaluation at $\lambda$ , then $E_\lambda : \mathcal{A}b \to \mathcal{A}b^{\Lambda}$ preserves projectives. In particular, $E_\lambda \mathbb{Z} \in \mathcal{A}b^{\Lambda}$ is projective in $\mathcal{A}b^{\Lambda}$ for every $\lambda \in \Lambda$ , since $\mathbb{Z}$ is projective in $\mathcal{A}b$ .

Furthermore, it is clear that $\underset{\Lambda^{op}}{\mathrm{colim}}\, E_\lambda \mathbb{Z} \cong \hat{\mathbb{Z}}$ , and $\|\Lambda^{op}\| = \|\Lambda\| = \aleph_n$ . Therefore, since $\{E_\lambda \mathbb{Z}\}$ and $\Lambda^{op}$ satisfy the hypothesis of Osofsky's proposition,

$$\mathrm{h.d.}(\hat{\mathbb{Z}}) \cong \mathrm{h.d.}(\underset{\Lambda^{op}}{\mathrm{colim}}\, E_\lambda \mathbb{Z}) \leq n + 1 .$$

From the definition of homological dimension and lemma 4.11, it follows that $\left(R^q \underset{\Lambda}{\mathrm{Lim}}\right)(\bar{A}) = 0$ for $q > n + 1$ and $\bar{A} \in \mathcal{A}b^{\Lambda}$ .

## Section 4: Generalization to AB4 Category $\mathcal{A}$ with Injectives.

The theorem developed in this section shows that the vanishing of higher derived functors of $\underset{\Lambda}{\mathrm{colim}} : \mathcal{A}^{\Lambda} \to \mathcal{A}$ , $\Lambda$ directed, depends only on the cardinality of $\Lambda$ .

Suppose $\mathcal{A}$ is an AB4 category with injectives.

Lemma 4.13: If $B \in \mathcal{A}$ such that $\mathcal{A}(B,Q) = 0$ for every injective $Q \in \mathcal{A}$, then $B = 0$.

Proof: Since $\mathcal{A}$ has enough injectives, there is an injective $Q$ in $\mathcal{A}$ and a monomorphism $\beta : B \to Q$. But $\mathcal{A}(B,\_) : \mathcal{A} \to \mathcal{A}b$ left exact insures that $0 \longrightarrow \mathcal{A}(B,B) \xrightarrow{\mathcal{A}(B,\beta)} \mathcal{A}(B,Q)$ is exact in $\mathcal{A}b$. Since $\mathcal{A}(B,Q) = 0$, $0 = \mathcal{A}(B,\beta)\cdot id_B = \beta\cdot id_B$; and $\beta : B \to Q$ a monomorphism implies that $id_B = 0$ or $B \cong 0$.

Theorem 4.14: If $\Lambda$ is a directed (upwards) ordered set, $||\Lambda|| \le \aleph_n$, then

$$(L_q \operatorname*{colim}_{\Lambda})(\bar{A}) = 0$$

for every $q > n + 1$ and $\bar{A} \in \mathcal{A}^{\Lambda}$.

Proof: Since $\mathcal{A}$ is AB4, the results of J.E. Roos [15] apply, i.e., for every $\bar{A} \in \mathcal{A}^{\Lambda}$, there is a "coflabby" resolution of $\bar{A}$, $\coprod_* \bar{A} \to \bar{A} \to 0$ where $(\coprod_q \bar{A})_\lambda = \coprod_{\lambda \le \lambda_0 < \ldots < \lambda_q} A_{\lambda_q}$, is such that

$$H_*(\operatorname*{colim}_{\Lambda} \coprod_* \bar{A}) \cong (L_* \operatorname*{colim}_{\Lambda})(\bar{A}) .$$

Suppose $Q \in \mathcal{A}$ is injective. Since $\mathcal{A}(\_,Q) : \mathcal{A}^{op} \to \mathcal{A}b$ is exact,

$$\mathcal{A}((L_q \operatorname*{colim}_{\Lambda})(\bar{A}),Q) \cong H^q(\mathcal{A}(\operatorname*{colim}_{\Lambda} \coprod_* \bar{A}),Q)) .$$

A duality argument shows that

$$\mathcal{A}(\operatorname*{colim}_{\Lambda} \coprod_* \bar{A},Q) \cong \operatorname*{Lim}_{\Lambda^{op}} \prod^* (\bar{A},\hat{Q}) ,$$

where $\prod^*(\bar{A},\hat{Q})$ is the canonical "flabby" resolution of $(\bar{A},\hat{Q}) \in \mathcal{A}b^{\Lambda^{op}}$.

Therefore,

$$\mathcal{A}((L_q \operatorname*{colim}_{\Lambda})(\bar{A}),Q) \cong (R^q \operatorname*{Lim}_{\Lambda^{op}})(\bar{A},\hat{Q}) .$$

Since $\Lambda$ is a directed (upward) ordered set, $||\Lambda|| \leq \aleph_n$, $\Lambda^{op}$ is a directed (downward) ordered set with $||\Lambda^{op}|| \leq \aleph_n$. Thus applying proposition 4.12,

$$0 = R^q \underset{\Lambda^{op}}{\mathrm{Lim}}(\bar{A}, \hat{Q}) \cong \mathcal{A}((L_q \underset{\Lambda}{\mathrm{colim}})(\bar{A}), Q)$$

for every $q > n + 1$. Finally lemma 4.13 insures that $(L_q \underset{\Lambda}{\mathrm{colim}})(\bar{A}) = 0$ for $q > n + 1$.

The hypothesis that $\mathcal{A}$ has enough injectives is stronger than is needed. For instance, the following weaker hypothesis on $\mathcal{A}$ might be used: For each $\bar{A} \in \mathcal{A}^{\Lambda}$, there is a full exact small abelian subcategory $\mathcal{B}(\bar{A}) \in \mathcal{A}$ such that:

(i) $\bar{A}$ and colim $\coprod_* \bar{A}$ are contained in $\mathcal{B}(\bar{A})$.

(ii) there is a collection $\{Q_p \mid p \geq 0\}$ of objects in $\mathcal{B}(\mathcal{A})$ with

(a) $\mathcal{A}(\_, Q_p) : \mathcal{B}(\mathcal{A}) \to \mathcal{A}b$ exact for every $p \geq 0$,

(b) $0 \to (L_p \underset{\Lambda}{\mathrm{colim}})\bar{A} \xrightarrow{\alpha_p} Q_p$ existing in $\mathcal{B}(\mathcal{A})$ for every $p \geq 0$,

(c) $\mathcal{A}(L_p \underset{\Lambda}{\mathrm{colim}}\, \bar{A}, Q_p) = 0$ for every $p \geq 0$.

In this case, the necessary embeddings into $\mathcal{A}b$ would exist and the above theorem would follow similarly.

# CHAPTER V

## CONCLUSIONS

In this chapter, the results of the last chapter are applied to the spectral sequence developed in Chapter III to give a concrete way of determining when higher derived functors of limit and colimit vanish.

### Section 1: Finiteness Lemma.

The following lemma develops a way of determining when the higher derived functors of $\operatorname{colim}_{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$ vanish, based on when the higher derived functors of $\operatorname{colim}_{\Omega} : \mathcal{A}b^{\Omega} \to \mathcal{A}b$ vanish.

Suppose $\mathcal{A}$ is an AB4 category with a generating class of small projectives.

Lemma 5.1: Suppose $\Omega$ is any small category such that $L_p \operatorname{colim}_{\Omega} : \mathcal{A}b^{\Omega} \to \mathcal{A}b$ is zero for every $p > k$, then $\left(L_p \operatorname{colim}_{\Omega}\right): \mathcal{A}^{\Omega} \to \mathcal{A}$ is also zero for every $p > k$.

Proof: Suppose $\bar{A} \in \mathcal{A}^{\Omega}$ and $\coprod_{*} \bar{A} \to \bar{A} \to 0$ is the canonical coflabby resolution of $\bar{A}$, i.e.,

$$(\coprod_p \bar{A})_{\omega} = \coprod_{\omega \to \omega_0 \to \omega_1 \to \cdots \to \omega_p} A_{\omega_0 \to \omega_1 \to \cdots \to \omega_p}$$

where

$$A_{\omega_0 \to \cdots \to \omega_p} = A_{\omega_p} .$$

Let P be <u>any</u> small projective in the generating family. Then $(\hat{P}, \coprod_* \bar{A}) \to (\hat{P}, \bar{A}) \to 0$ is the canonical coflabby resolution of $(\hat{P}, \bar{A}) \in \mathcal{A}b^{\Omega}$ since $\mathcal{A}(P,\_) : \mathcal{A} \to \mathcal{A}b$ is exact and commutes with coproducts. Therefore, $(L_p \operatorname{colim}_{\Omega})(\hat{P}, \bar{A}) = H_p(\operatorname{colim}_{\Omega} \coprod_* (\hat{P}, \bar{A}))$ is isomorphic to $(P, H_p(\operatorname{colim}_{\Omega} \coprod_* \bar{A})) = \mathcal{A}(P, (L_p \operatorname{colim}_{\Omega})(\bar{A}))$, again using the fact that P is small and projective. By hypothesis, $(L_p \operatorname{colim}_{\Omega})(\hat{P}, \bar{A}) = 0$ for $p > k$. Therefore $\mathcal{A}(P, (L_p \operatorname{colim}_{\Omega})(\bar{A}))$ is also zero for every $p > k$ and for every small projective generator P in $\mathcal{A}$; and it follows that $(L_p \operatorname{colim}_{\Omega})(\bar{A}) = 0$ for every $p > k$.

<u>Section 2</u>: <u>The Vanishing of Higher Derived Functors of Limit and Colimit</u>.

<u>Theorem 5.2</u>:

Suppose $\mathcal{A}$ is an AB4 category with projectives and injectives. If $\Omega$ is any $\downarrow$-finite, ordered set with $\|\Omega\| \leq \aleph_n$ such that $L_p \operatorname{colim}_{\Omega} : \mathcal{A}b^{\Omega} \to \mathcal{A}b$ is zero for every $p > k$, then $L_r \operatorname{colim}_{\Omega} : \mathcal{A}^{\Omega} \to \mathcal{A}$ is zero for every $r > k + n + 1$.

<u>Proof</u>: By Theorem 3.9 and Theorem 3.12, there is a first quadrant spectral sequence with

$$E^2_{pq} = (L_p \Psi)(L_q \operatorname{colim}_{\Omega} J^{\Omega})(\bar{A})$$

$$\cong (L_p \operatorname{colim}_{\mathfrak{F}(\Omega)})(L_q W)(\bar{A})$$

converging to $(L_{p+q} \operatorname{colim})(\bar{A})$ for every $\bar{A} \in \mathcal{A}^{\Omega}$.

First hold p constant. Since $D(\mathcal{A})$ has a generating class of small projectives and is AB5 , lemma 5.1 applies and $(L_p\Psi)(L_q\operatorname{colim}_{\Omega})(J^{\Omega}(\bar{A}))$ is zero for every $q > k$ . (Note that this slightly different form for $E^2$ is possible since $J^{\Omega} : \mathcal{A}^{\Omega} \to (D(\mathcal{A}))^{\Omega}$ preserves projectives and is exact.)

Next, let q be held constant. $\|\Omega\| \le \aleph_n$ implies that $\|\mathfrak{J}(\Omega)\| \le \aleph_n$ . Since $\mathcal{A}$ is AB4 with injectives, $\mathfrak{J}(\Omega)$ is a directed (upward) ordered set and $(L_qW)(\bar{A}) \in \mathcal{A}^{\mathfrak{J}(\Omega)}$ , Theorem 4.14 applies; and $E^2_{pq} = (L_p\operatorname{colim}_{\mathfrak{J}(\Omega)})(L_qW)(\bar{A}) = 0$ for $p > n + 1$ .

Thus the spectral sequence yields $(L_r\operatorname{colim}_{\Omega})(\bar{A}) = 0$ for $r > k + n + 1$ .

For the convenience of the reader, the dual statement is included.

Theorem 5.3 (Dualization):

Suppose $\mathcal{A}$ is an AB4* category with projectives and injectives. If $\Omega$ is a $\uparrow$-finite ordered set, such that $\|\Omega\| \le \aleph_n$ and $(L_q\operatorname{colim}_{\Omega^{op}}) : \mathcal{A}b^{\Omega^{op}} \to \mathcal{A}b$ is zero for every $q > k$ , then $(R^r\operatorname{Lim}_{\Omega}) : \mathcal{A}^{\Omega} \to \mathcal{A}$ is also zero for every $r > k + n + 1$ .

In particular we can apply this Theorem to the case when $\mathcal{A} = \mathcal{A}b$ to get the following result promised in the Introduction.

Corollary 5.4: If $\Omega$ is any $\uparrow$-finite ordered set with $\|\Omega\| \le \aleph_n$ such that $L_q\operatorname{colim}_{\Omega^{op}} : \mathcal{A}b^{\Omega^{op}} \to \mathcal{A}b$ is zero for every $q > k$ , then $(R^r\operatorname{Lim}_{\Omega}) : \mathcal{A}b^{\Omega} \to \mathcal{A}b$ is also zero for every $r > k + n + 1$ .

## Section 3: The Homological Dimension of Constant Diagrams.

Suppose that $\Omega$ is a $\uparrow$-finite ordered set and $\mathcal{M}_R$ the AB5 category of modules over a ring $R$.

In this section, we show that the homological dimension of a constant diagram of R-modules, $\hat{M} : \Omega \to \mathcal{M}_R$, depends on the cardinality of $\Omega$. As in the case when $R = \mathbb{Z}$ and $\mathcal{M}_R = \mathcal{A}b$, we get the following lemma.

Lemma 5.5: Suppose $\bar{A} \in \mathcal{M}_R^{\Omega}$ and $\hat{R}$ is the constant diagram over $\Omega$ with value $R$. Then $(R^*\underset{\Omega}{\mathrm{Lim}})(\bar{A}) \cong \mathrm{Ext}^*(\hat{R},\bar{A})$.

For the remainder of this section, we assume that $\Omega$ represents a $\uparrow$-finite ordered set such that $\|\Omega\| \leq \aleph_n$ and $(L_q \underset{\Omega^{op}}{\mathrm{colim}}) : \mathcal{A}b^{\Omega^{op}} \to \mathcal{A}b$ is zero for every $q > k$.

Corollary 5.6: $h.d(\hat{R}) \leq k + n + 1$.

Proof: This is immediate from the definition of homological dimension, Theorem 5.3, and lemma 5.5.

Lemma 5.7: If $P$ is a projective R-module, then $h.d. (\hat{P}) \leq k + n + 1$.

Proof: If $F$ is a free R-module, then $F \cong \underset{I}{\oplus} R$. Therefore, from corollary 5.6,

$$h.d(\hat{F}) = h.d. (\underset{I}{\oplus} \hat{R}) = \sup_{I} h.d. (\hat{R}) \leq k + n + 1 .$$

If $P$ is a projective R-module, then $P$ is a direct summand of a free R-module, i.e., $F = P \oplus P'$. Since homological dimension of a direct sum is the maximum of the homological dimensions of the summands, and $\hat{F} = \hat{P} \oplus \hat{P}'$, $h.d.(\hat{P}) \leq h.d. (\hat{F}) \leq k + n + 1$.

We note that even though $P$ is a projective $R$-module, $\hat{P}$ is not necessarily projective in $\mathcal{M}_R^{\Omega}$.

<u>Theorem 5.8</u>:

Suppose $M$ is any $R$-module, then $\text{h.d.}(\hat{M}) \leq k + n + 1 + \text{h.d.}_R(M)$.

<u>Proof</u>: The proof is by induction on $\text{h.d.}_R(M)$. The case when $\text{h.d.}_R(M) = 0$, i.e., $M$ is projective, has been done in Lemma 5.7. Suppose that $\text{h.d.}(\hat{M}') \leq k + n + 1 + \text{h.d}_R(M')$ whenever $\text{h.d.}_R(M') \leq r - 1$.

Let $M$ be an arbitrary $R$-module with $\text{h.d}_R(M) \leq r$. Then there is a short exact sequence

$$0 \to M' \to P \to M \to 0$$

in $\mathcal{M}_R$ with $P$ a projective $R$-module. In particular, $\text{h.d}_R(M') < \text{h.d}_R(M)$, since $\text{h.d}_R(M) = 1 + \text{h.d}_R(M')$. Further, since $\wedge : \mathcal{M}_R \to \mathcal{M}_R^{\Omega}$ is exact,

$$0 \to \hat{M}' \to \hat{P} \to \hat{M} \to 0$$

is exact in $\mathcal{M}^{\Omega}$. By inductive hypothesis, $\text{h.d.}(\hat{M}') \leq k + n + 1 + \text{h.d.}_R(M')$ and therefore

$$\text{h.d.}(\hat{M}) \leq 1 + \text{h.d.}(\hat{M}') \leq k + n + 1 + \text{h.d.}_R(M) .$$

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## AUTOBIOGRAPHICAL STATEMENT

Dana May Latch was born in New York City on August 29, 1943. She was graduated from Walt Whitman High School in 1961, received a Bachelor of Arts degree (cum laude) from Harpur College, State University of New York in 1965, and a Master of Arts degree from Queens College, The City University of New York in 1967. In February 1967, she entered the Ph.D. Program in Mathematics at the Graduate Center, The City University of New York.

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