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Massless fractionally charged states and Gepner compactification

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MASSLESS FRACTIONALLY CHARGED STATES AND GEPNER COMPACTIFICATION

by

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Table of Contents

List of Tables

1. Introduction

Four-dimensional string theory often predicts the existence of exotic particles carrying a fractional electric charge $\frac{k}{n}e$ where k, n are integers and e is the charge of the electron. These states appear as the result of a symmetry breaking process – the so-called gauge embedding – involved in the construction of a realistic gauge group. Although, these states are massive ($\sim M_{Planck}$) for generic string vacua, some of them appear to be massless (at the Planck scale) for a particular class of vacua, namely, the orbifold[1] compactifications. We propose to study the occurrence of massless fractionally charged states (FCS) in the spectrum of another class of vacua referred to as Gepner compactifications. In the real world, these states might appear as light (100 Gev) exotic particles. One can imagine they were produced abundantly in the hot plasma of the early universe and that a few of them survived the cooling down due to the expansion. As the result of their exotic charges they would constitute stable relics potentially accessible to detection. Assuming a mass of order 100 Gev or so, standard cosmology predicts concentration higher than the experimental bounds by at least ten orders of magnitude. The interest in these particles reside in the following. As explained later, string theory is plagued with a huge number of four-dimensional classical vacua, around which we can develop a perturbative expansion. Taking the cosmological argument seriously may provide us with an important constraint for model building. On the other hand, it may turn out that these specific vacua are of some relevance once non-perturbative physics is taken into account. For example, they possess enhanced symmetry groups which suggest them as candidates for non-perturbative vacua. In such a case, we might close our eyes to the cosmological constraints, envision the actual existence of such particles and consider searching for them in large machines like the SSC.

While string theory provides us with a few ten-dimensional classical vacua, it possesses a wealth of four-dimensional solutions each of which, from a microscopic point of view, seems as good as any other. Initially, the latter were thought of as the result of the compactification of six space dimensions on a compact manifold such as a Calabi-Yau manifold[2] or an orbifold.[1] Calabi-Yau compactification forms a large class of solutions preserving N=1 spacetime supersymmetry. The structure of a Calabi-Yau manifold is determined by a set of parameters called moduli; varying them amounts to deforming continuously the manifold. To each modulus corresponds a field with a flat potential such that, giving a vacuum expectation value (vev) to this field will slightly distort the manifold at no cost of energy. One modulus, of particular importance to us, is the radial mode R corresponding to the variation of the size of the manifold. An orbifold is a (six-dimensional) torus whose points have been identified by some elements of its discrete symmetry group. It is a limiting case of a Calabi-Yau manifold where all the curvature has been pushed to some singular points. It has the advantage of being completely solvable; this is to be opposed to a generic Calabi-Yau manifold which cannot be exactly solved even in the limit of large radius ($R \gg M_{Planck}^{-1}$). An orbifold possesses moduli associated to its singularities; giving them a vev amounts to restoring the smoothness of the manifold. A standard Calabi-Yau manifold or orbifold, without Wilson lines, leads to the gauge group $E_6 \times \tilde{E}_8$; its massless spectrum contains in addition to a gravity multiplet, the matter fields, coming in multiplets of the $\mathbf{27}$ and $\overline{\mathbf{27}}$ representations

$$\mathbf{27} = \mathbf{10}^{1} + \mathbf{16}^{-1/2} + \mathbf{1}^{-2}$$

$$\overline{\mathbf{27}} = \mathbf{10}^{-1} + \overline{\mathbf{16}}^{1/2} + \mathbf{1}^{2}$$

and the gauge bosons decomposing as

$$\mathbf{78} = \mathbf{45}^0 + \mathbf{16}^{-3/2} + \overline{\mathbf{16}}^{1/2} + \mathbf{1}^0$$

The $\mathbf{16}^{-1/2}$ contains one full generation of quarks and leptons and the $\overline{\mathbf{16}}^{1/2}$ one full anti-generation (opposite chirality). Hence, $\#\mathbf{27} - \#\overline{\mathbf{27}}$ gives us the net number of generations of the theory. It turns out that the latter is half the Euler number of the internal manifold – a purely topological quantity.

Calabi-Yau and orbifold compactifications form only a subset of possible constructions. A general four-dimensional string vacuum is the result of gluing together a four-dimensional spacetime with a so-called internal sector. The presence of the latter is required to cancel-out an anomaly occuring while quantizing the theory – the superconformal anomaly.* As we have just seen, the internal sector may be suitably described by the propagation of the string in some hidden compactified dimensions. However,this doesn't have to be the case and the internal sector might only be describing some additional degrees of freedom carried by the string. More abstractly, the internal sectors can be thought of as superconformal theories having the correct central charge $c = 9$. An additional set of completely solvable four-dimensional string vacua was provided by Gepner.[3] He constructed, out of interacting fields, superstring and heterotic string vacua, which are tachyon free and modular invariant and have spacetime supersymmetry with the correct spin-statistics. The internal sector of these models is a tensor product of N=2

⋆ String theory can be suitably described by a set of fields propagating on a two-dimensional surface (worldsheet) whose dynamics respects a local symmetry called superconformal symmetry. A short description of the construction of a string theory is given in the appendix.

superconformal minimal models $k_1, ..., k_r$ with central charge

$$c = \sum_{i=1}^{r} \frac{3k_i}{k_i + 2} = 9 \ .$$

N=2 minimal models are among the simplest exactly solvable conformal field theories with two worldsheet supersymmetries and unitary representations.[3] A well-studied example is the 3^5 theory,[3] corresponding to five internal minimal models with $k_i = 3$. It contains 101 generations(**27**), 1 antigeneration($\overline{\mathbf{27}}$) and 330 scalars; all with their supersymmetric partners and their antiparticles. Even though the geometrical interpretation of such constructions is a priori unclear, Gepner further showed that these constructions describe Calabi-Yau manifolds at certain special points of their moduli spaces. At such points, the gauge group is supplemented with some extra U(1) currents and the massless spectrum with an equal number of modes which act as Higgses for these symmetries as we move away from these special points by giving expectation values to some of the moduli.[4,5]

Constructions such as Calabi-Yau compactifications often lead to models with a large number of generations. The number of generations is usually reduced to a more reasonable number by dividing out the internal sector with one of its discrete symmetries, most simply taken to be a Z_n symmetry. In Gepner's construction, each minimal model possesses a Z_{k_i+2} discrete symmetry; in addition, there may be some permutation symmetry S, arising when the tensor product contains two or more identical minimal models, leading to a discrete group$G = Z_{k_1+2} \times ... \times Z_{k_r+2} \times S$. The action of the modding out on the Hilbert space is to select the string excitations which respect that symmetry with the action of a projection operator. Also, because the string is an extended object, it is necessary to include states describing excitations along strings closing up to a Z_n symmetry – they constitute the so-called twisted sectors. For example, in the 3^5 theory, we can mod out with $x = (0,1,2,3,4) \in Z_5 \times Z_5 \times Z_5 \times Z_5 \times Z_5$ leading to 21 generations, 1 anti-generation and 210 scalar singlets, 70 of which coming from the untwisted sector. Modding out further by another $Z_5 \subset S : i \to i+1$, yields a four generation model[3] which has been shown to correspond to the maximally symmetric quintic hypersurface in the CP^4 Calabi-Yau manifold.[3,9]

It was suggested in ref.2 that one could use the Hosotani mechanism[7] to break the gauge group of the heterotic string; namely, one could let the Z_n group, mentioned above, act in the gauge group as well. In a geometrical language, this amounts to letting the strings propagate in a topologically non-trivial background gauge configuration in the internal manifold.* The latter is made possible by the Z_n modding out which makes the internal manifold multiply-connected. Only the gauge transformations which leave the background configuration invariant survive this mechanism. Such a Wilson line symmetry breaking would lead to a more reasonable gauge group such as, for instance, $SU(3)_c \times SU(2)_w \times SU(2)_r \times U(1)^n$ times another E_8 which in turn could be broken. This was first done in the framework of Calabi-Yau compactification.[8] Wen and Witten[10] showed that this procedure could lead to the existence of states carrying a fractional electric charge $\frac{k}{n}.e$, where k is an integer and e, the charge of the electron. These states arise in the twisted sectors of the theory – i.e. the sectors obtained from strings closed up to a Z_n phase – and have a mass of order R, because they must wind around the multiply

* This background ought to be pure gauge in order not to spoil the equations of motion defining the superstring vacuum and cannot depend on the spacetime coordinates in order to preserve Lorentz invariance.

connected compact manifold of radius R. To understand the origin of the fractional charge notice that an electron is described by an untwisted string. This is enough to normalize the electric charge current. Consider a string twisted by a Z_n–symmetry, there is no longer any freedom to fix its quantum of electric charge. To see that, glue n such strings together to form a single string which lies now in the untwisted sector† and has its charge multiple of e or $\frac{e}{3}$. As a consequence, the original string carries an electric charge multiple of $\frac{e}{n}$ or $\frac{e}{3n}$. Athanasiu et al.[14] showed that the same procedure applied to orbifold compactifications leads to the production of *massless* fractionally charged states as well.‡

For a state in the twisted sector to become part of the massless physical spectrum, two conditions are to be fulfilled:

1. The state has to survive the projection required to ensure the invariance of the spectrum under the symmetries we modded out with.
2. The mass receives two contributions: the energy of excitations and the energy of the twist, carried by the ground state of the corresponding sector. These two terms must cancel-out exactly.

These conditions constitute a highly non-trivial constraint, which suggests that not any theory can successfully fulfil them. If they correspond to a Calabi-Yau manifold, these theories are necessarily at special points in the moduli space, since, as mentioned above, the twisted states are very massive in the large radius limit. An orbifold is indeed such a theory. In this thesis we search for massless fraction-

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$\dagger$ Because $g \in Z \Rightarrow g^n = 1$.

$\ddagger$ They provided an example of a Z_5 orbifold which, unfortunately, doesn't exist because the cristallographic group of a six-dimensional lattice cannot possesse a Z_5 symmetry.[1] However, one can show that similar phenomena occur with a Z_7 orbifold.

ally charged states in Gepner constructions. This is of interest, first, to examine how generic this phenomenon is, second, because Calabi-Yau compactification has proven to be a more promising candidate for constructing realistic models than orbifold compactification. This work consists of three parts.

1. Development of a formalism which implements the Wilson line symmetry breaking on a general Gepner construction, allowing us to write down explicitly the two conditions described above. Although such a formalism was written in a quite general context by de Alwis,[15] once transposed to Gepner construction, it restricts us to a Wilson line completely within the $SO(10)$ subgroup of E_6. This is because de Alwis'approach does not apply to the case where the gauge group contains a generator which generates the $U(1)$ current of the N=2 superconformal algebra of the internal sector.

2. Explicit analysis of some models to verify that, indeed, for generic Wilson lines, massless particles occur in the twisted sectors. Because of the relatively large number of quantum numbers to be handled, this requires the use of a computer.

3. Verification that these states are massless as a consequence of being at a special point in the moduli space. This is done by computing couplings of moduli to these states.

We find massless color singlets and color triplets as well. We confirm that they arise, as suggested in ref. 14, for generic Wilson lines. Looking at an explicit model, we show that all the massless fractionally charged states we found pair–up and gain mass when the modulus R is given an expectation value. This reproduces the result of Wen and Witten in the large radius limit. We consider only Z_n Wilson lines with n odd. The reason is that, in such a case, modular invariance is a much more powerful constraint acting on the partition function, the latter being our main tool to determine the massless spectrum of the theory. Essentially, when n is odd, modular transformations mix most of the conformal families of the theory leaving only a few undetermined phases which, in turn, can be inferred by requiring the consistency of the projectors found. In contrast, when n is even, numerous phases are undetermined by modular invariance and more requirements are to be imposed on the construction to extract the physical spectrum. For definiteness, we consider the 3^5 theory — i.e. the tensor product of five minimal models of level $k = 3$ — modded out with Z_5 Wilson lines. It is a low generation model and has the advantage of being well studied in both the context of Calabi–Yau[2] compactification and conformal field theory.[3]

This work is organized as follow. Chapter 2 gives an account on the experimental status of the detection of fractionally electric charged particles and a brief description of their would-be cosmological abundance. Chapter 3 introduces the reader to modular invariance, the main tool used in this work, while chapter 4 shows how to use this tool to extract information on the spectrum of a string theory. In chapter 5, we analyze Gepner's compactification with the most general Z_n Wilson line. Finally, we present our results and conclusions.

2. Experimental limits and cosmological argument

2.1 Experimental limits.

A large number of experiments searching for particles with exotic charge have already been conducted. Their initial purpose was the detection of the existence of free quarks originating either as byproducts from high energy processes or as remnants from an early epoch of the universe when quarks were believed to be unconfined. Later, other motivations came along such as the hypothetical existence of magnetic monopoles[17] and the prediction of new particles such as color singlets with a charge $\frac{n}{3}e$ predicted in some GUT models.* The search has been made in particle accelerators, cosmic rays and stable matter and, in a variety of ways. A full analysis of these experiments can be found in ref. 18 and their results in ref. 19. Here, we will content ourselves with a brief description and a few comments.

Accelerator searches consist of heavy ion collisions, $p\bar{p}$ reactions, deep inelastic scattering and e^+e^- annihilations. These experiments usually make assumptions about properties of the particles. Most of them have been directed to the detection of free quarks with a charge $\frac{n}{3}e$ in a range of mass below 25 Gev. So far, no detectable production has been observed which translates into an upper bound for the rate of production.[19]

Searchs in cosmic rays have also been mostly directed toward the detection of particles carrying a charge $\frac{n}{3}e$. They have been unsuccessful so far. The latest results come from Kamiokande II[20] which sets an upper limit of $\sim 10^{-15} cm^{-2} s^{-1} sr^{-1}$ to the flux of particles with charges $e/3$ and $2e/3$.

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* For instance GUT models based on SU(7).[16]

There are two types of searches in stable matter:

1. Searches for particles with nonintegral charge in ion beams, covering a limited range of mass ≤ 100 Gev.
2. Measurements of the residual electric charge on neutralized samples isolated by levitation. This method is largely insensitive to the nature of the material analyzed.

Many materials have been examined including extra-terrestrial ones. Except for a false alarm[22] none have reported a positive result. The typical bound is an abundance less than 10^{-21} per nucleon. This bound has more information than the previous ones. For it not only limits the amount of FCS from cosmic rays that could have accumulated in the earth since its formation but it also gives a stringent limit on the abundance of primordial relics from the early universe. As we will see in detail in the next section, in the framework of standard cosmology, such a constraint is in disagreement with the very existence of these particles at all. Before leaving this section, it is worth mentionnig some criticisms raised by Lackner & Zweig[21] against these experiments.† A free FCS with a positive charge is expected to acquire an electron to form a hydrogen-like atom with charge $-\frac{2}{3}e$ or $-\frac{1}{3}e$, if negatively charged it could be part of the negative cloud of an atom and, if interacting strongly, it could be bound in a nucleus. Such atoms would have different chemical properties from the ordinary ones. As a consequence,

1. the concentration of FCS is expected to be different in different substances.
2. such atoms could form neutral molecules that would elude their detection.

† Besides the fact that most of these experiments — including the latter type — confine themselves to the detection of an electric charge of $\frac{n}{3}e$ only.

3. these atoms could be removed from a given substance by either geochemical or by technological purification processes.

Based on this argument, Lackner & Zweig suggested a list of substances more likely to be enriched in fractionally charged particles. These substances haven't been tested so far.

2.2 PREDICTED ABUNDANCES FROM COSMOLOGY.

The origin of the cosmological bounds is easy to understand. The main assumption is that these particles were at an epoch in thermal equilibrium with the hot plasma filling the early universe. Let us consider a species i and its antiparticle $\bar{i}$. At high enough temperature particles i and $\bar{i}$ are relativistic and as abundant as the photons.‡ Their number density is

$$n_i^{eq} = n_{\bar{i}}^{eq} = \frac{g_i}{2} n_\gamma \qquad (2.1)$$

with

$$n_\gamma = 2\frac{\zeta(3)}{\pi^2}T^3 \qquad \zeta(3) = 1.202 \quad , \qquad (2.2)$$

and g_i is the effective number of degrees of freedom of particle i. As the universe expands, the temperature decreases (as $\frac{1}{R}$). When $T < m_i$ the distribution becomes non-relativistic and starts dropping exponentially as

‡ To obtain the minimum abundance which survives the early universe, we assume that i and $\bar{i}$ are equally abundant.

$$n_i^{eq} \simeq g_i(\frac{x}{2\pi})^{3/2} e^{-x} n_\gamma; \quad x = \frac{m_i}{T} \quad . \tag{2.3}$$

It would become negligibly small if the equilibrium could be maintained long enough. However, the latter holds only when the typical rate Γ of the reaction $i + \bar{i} \leftrightarrow all$ is fast enough to neglect the expansion rate H, namely,

$$\Gamma \gg H \quad .$$

When this condition is violated, the participating species cannot maintain equilibrium. Consequently, particles i and $\bar{i}$ cease to be produced and, quickly, cease to annihilate; their total number has been frozen-out. The time it occurs depends on the reactivity of the particles: the stronger their interactions with the plasma, the later they decouple and the smaller is their population.

Now, we reproduce this argument on a more quantitative footing.[§] The freeze-out temperature T^* is defined by the condition

$$\frac{\Gamma}{H}\Big|_{T^*} \sim 1 \tag{2.4}$$

H, in a radiation-dominated universe, is given by

$$H = \frac{T^2}{M_{Pl}} \sqrt{\frac{4\pi^3 g(T)}{45}} \tag{2.5}$$

with M_{Pl} the Planck mass 10^{19} Gev and $g(T)$, the total effective number of degrees of freedom of the relativistic particles in the plasma; for $T > 100$ Gev, $g(T)$ is of

§ There are many levels of analysis: from a numerical integration of the rate equation to a crude dimensional analysis. They are usually in reasonable agreement.

the order of 100. $\Gamma = n_i^{eq} < v\sigma >$ is the typical annihilation rate for the reaction $i\bar{i} \leftrightarrow all$. Using equations (2.3) and (2.5) equation (2.4) becomes

$$x_*^{\frac{1}{2}} e^{-x_*} \simeq \sqrt{\frac{4\pi^3 g(T^*)}{45}} \frac{\pi^2}{2\zeta(3)} \frac{(2\pi)^{\frac{3}{2}}}{g_i} \frac{1}{M_{Pl}} \frac{1}{m_i < v\sigma >^*}$$

taking the logarithm of this expression and neglecting logarithmic corrections gives

$$x_* \simeq ln(m_i^{Gev} < v\sigma >^*_{Gev^2}) + 40. \tag{2.6}$$

On the other hand, equation (2.4) can conveniently be written

$$\begin{aligned} \frac{n_i^*}{n_\gamma^*} &\sim \frac{H}{n_\gamma < v\sigma >}|_{T*} \\ &\sim \frac{x^*}{m_i^{Gev} < v\sigma >^*_{Gev^2}} 10^{-17} \end{aligned} \tag{2.7}$$

which gives the number of remnant particles i (or $\bar{i}$) per photon at the freeze-out temperature T^*. This is not yet the final answer. As T drops below T^*, many other species go out of equilibrium and their subsequent annihilations increase the number of photons. From conservation of entropy S in a comoving volume, we have

$$S \approx g(T^*) n_\gamma^* = g(T^{now}) n_\gamma^{now}$$

or

$$\begin{aligned} n_\gamma^{now} &\simeq n_\gamma^* \frac{g(T^*)}{4} \\ &\simeq -25\ n_\gamma^* \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{n_i}{n_\gamma}|_{now} &\sim \frac{x_*}{25} \frac{1}{m_i < v\sigma >^*} 10^{-17} \\ &\sim \frac{10^{-17}}{m_i < v\sigma >^*} \end{aligned},$$

we have used the empiric fact that $x^*/25$ is a number of order one (cf. equ. (2.6)). Finally, using $\frac{n_{baryon}}{n_\gamma^{now}} \sim 10^{-11}$ we find for the actual abundance

$$N_i \sim N_{\bar{i}} \sim \frac{10^{-6}}{m_i^{Gev} < v\sigma >^*_{Gev^2}} \quad \text{per nucleon} \quad . \tag{2.8}$$

As a first application, consider a charged heavy lepton L of mass ~ 100 Gev. $L\bar{L}$ annihilate through γ, Z and $W^\pm$ exchanges. $< v\sigma >^*$ is approximately $10\pi\frac{\alpha^2}{M_L^2}$. From (2.8), we obtain for the residual concentration

$$\begin{aligned} N_L \sim N_{\bar{L}} &\sim \frac{10^{-6}}{10\pi\alpha^2} M_L^{Gev} \\ &\sim 10^{-4} M_L^{Gev} \\ &\sim 10^{-2} \quad \text{per nucleon} \quad . \end{aligned}$$

A more accurate calculation[25] gives 10^{-4} per nucleon.

A second example is a heavy free quark Q for which $< v\sigma >^* \sim \frac{\alpha_s^2}{M_Q^2} \sim \frac{10^{-2}}{M_Q^2}$. This yields

$$N_Q \sim N_{\bar{Q}} \sim \frac{10^{-6}}{10^{-2}} M_Q^{Gev} \sim 10^{-4} M_Q^{Gev} \sim 10^{-2} \quad \text{per nucleon} \quad .$$

while (2.6) gives for the freeze-out temperature $T^* \sim \frac{M_q}{30} \sim 30$ Gev. T^* is typically one or two orders of magnitude above the critical temperature of the quark–hadron phase transition $T_c \sim \Lambda_{QCD} \sim 300$ Mev. During the latter, some of the Q and $\bar{Q}$ are expected to combine with the much more abundant ordinary quarks to form color singlets. These composite objects interact with a typical hadronic cross–section $< v\sigma >^c \sim 1$ and, as a result, annihilate further. The freeze–out abundance for these processes is approximately given by (2.4) $\frac{\Gamma}{H}\big|_{T_c} \sim 1$, that is, using equation (2.8)

$$N_Q \sim N_{\bar{Q}} \sim \frac{10^{-6}}{M_Q^{Gev}} \sim 10^{-8} \quad \text{per nucleon} \quad .$$

Ref. 23 finds $\sim 10^{-11}$ per nucleon.* The discrepancy comes from a more careful calculation in ref. 23 of the subsequent annihilation of particles Q and $\bar{Q}$ at T_c.

These estimates are in drastic conflict with the experimental bound of 10^{-21} per nucleon.

⋆ These authors argue that subsequent annihilations in stars may decrease the abundance by ten orders of magnitude.

3. Modular invariance

In this chapter, we introduce the concept of modular invariance and show how to use it to extract the physical spectrum of the theory. At first sight, the Hilbert space $\mathcal{H}_{phys}$ of a string theory would appear to be the tensor product† of the left–handed modes with the right–handed modes

$$| >_L \otimes | >_R .$$

Such a space turns out to be far too large; it contains spacetime bosons and fermions with the wrong statistics — which may not even fit into supersymmetric multiplets if one wants to enforce spacetime supersymmetry — and, furthermore, leads to an inconsistency with reparametrization invariance itself. To eliminate these inconsistencies, one has to truncate the Hilbert space $\mathcal{H}$ to the physical space $\mathcal{H}_{phys}$with the use of a *projection operator* $\mathcal{P}$ whose action is defined as

$$\begin{aligned} &\mathcal{P}|physical >= |physical > \\ &\mathcal{P}|unphysical >= 0 \end{aligned} \qquad . \tag{3.1}$$

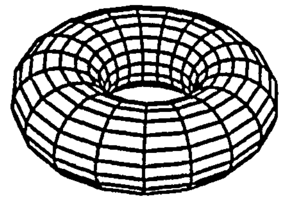
3.1 The partition function.

A place to look for this operator is the *partition function* of the theory i.e. the

† I refer the reader to the appendix.

one –loop amplitude with no external leg:

$$\mathcal{Z} =$$



$$= \int \mathcal{D}[\phi, \Psi] e^{-S[\phi,\Psi]}$$
$$= Tr_{\mathcal{H}_{phys}} e^{-\beta H} \qquad (3.2)$$

where the path integral is taken over fields with suitable boundary conditions. Indeed, (3.2) can be written

$$\begin{aligned} \mathcal{Z} &= Tr_{\mathcal{H}_{phys}} e^{-\beta H} \\ &= Tr_{\mathcal{H}} \mathcal{P} e^{-\beta H} \mathcal{P} \quad , \\ &= Tr_{\mathcal{H}} \mathcal{P} e^{-\beta H} \end{aligned} \qquad (3.3)$$

where now the trace is taken over the untroncated spectrum $\mathcal{H}$. The operator $\mathcal{P}$ is defined by a series of physical requirements. For instance, spin–statistics requires that $\mathcal{P}$ be such that

$$\begin{aligned} \mathcal{Z} &= Tr_{\mathcal{H}} \mathcal{P} e^{-\beta H} \\ &= Tr_{bosons} e^{-\beta H} - Tr_{fermions} e^{-\beta H} \end{aligned} ,$$

and spacetime supersymmetry requires that this quantity be rewritten

$$\begin{aligned} \mathcal{Z} &= Tr_{\mathcal{H}} \mathcal{P} e^{-\beta H} \\ &= Tr_{susy} e^{-\beta H} \end{aligned} .$$

and vanishes.

These two conditions restrict the form of the projector $\mathcal{P}$ but usually don't give us enough information to find its structure. What does, is the enforcement of the invariance of $\mathcal{Z}$ under a particular subclass of reparametrisations

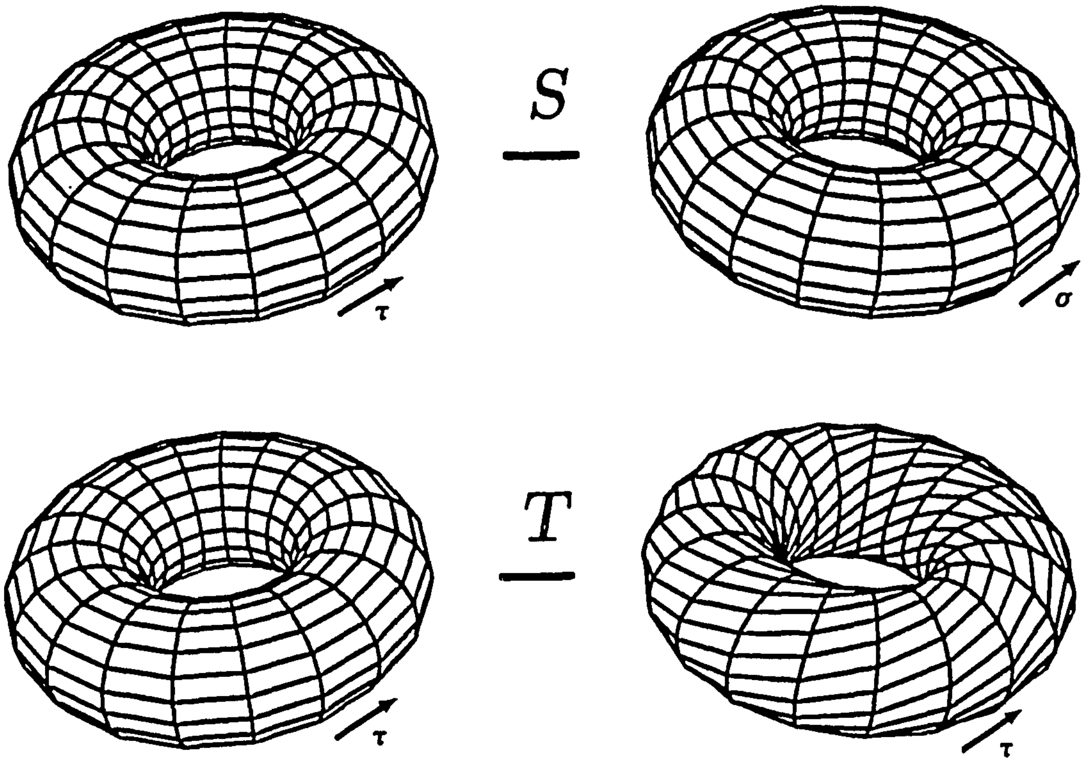
$$\mathcal{Z} \longrightarrow \mathcal{Z}' = \mathcal{Z}$$

$$(\sigma, \tau) \longrightarrow (\sigma', \tau')$$

called modular transformations, described below.

3.2 Modular invariance.

Reparametrisations continuously connected to the identity are taken care of by the standard Fadeev-Popov gauge fixing procedure leaving us with conformal invariance (appendix). However, on a torus, one can perform global* reparametrisations such as



Their existence reflects the fact that a torus has two independent non–contractible loops. Such transformations are called *modular transformations.* They form a group, the *modular group*, $SL(2,\mathbf{Z})$. To analyse these transformations, we project the torus on the complex w–plane according to

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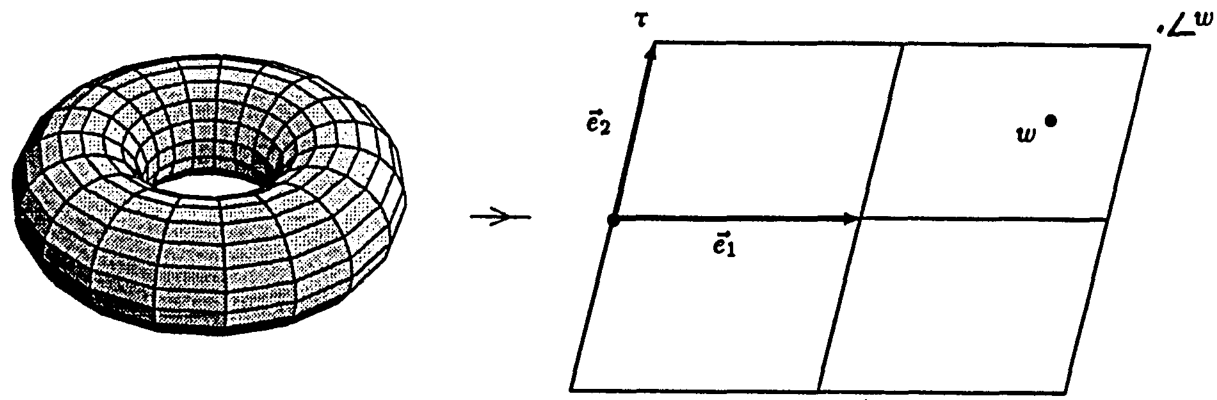
⋆ Transformations which are no connected to the identity.

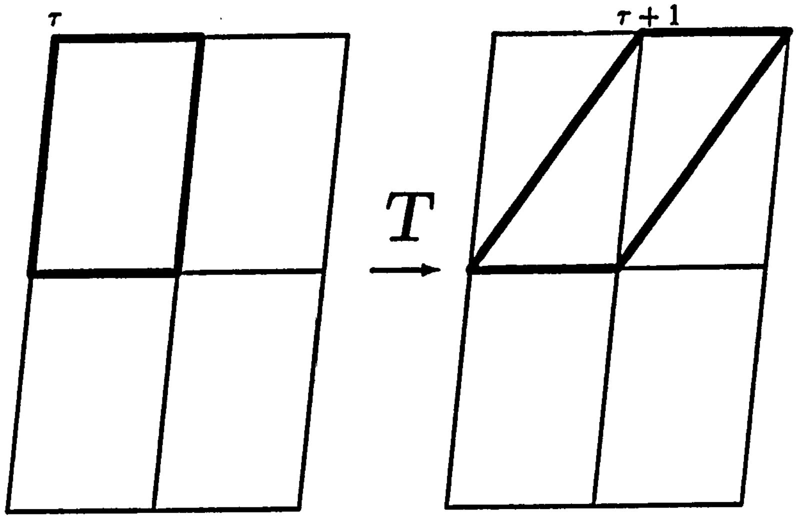


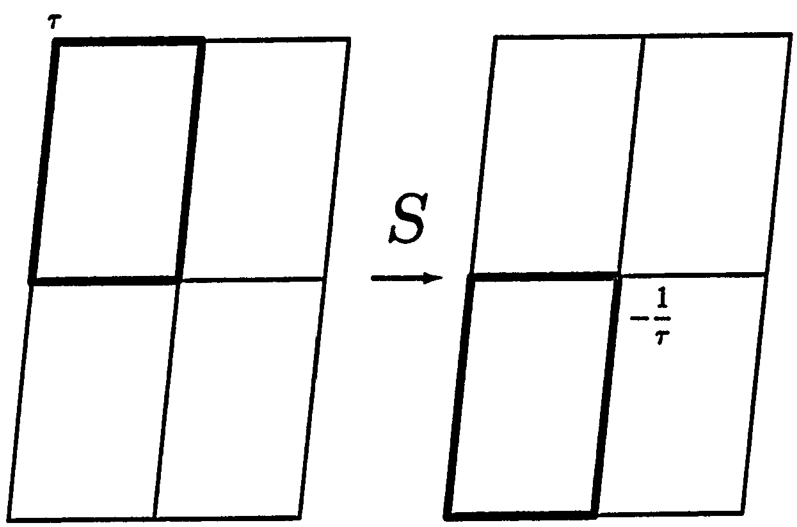
This is a two-dimensional lattice. We can make use of conformal symmetry to orient $\vec{e}_1$ on the real axis and rescale it to one; $\vec{e}_2$ becomes an arbitrary complex number; $\tau = \tau_1 + i\tau_2$ is called the *modular parameter*.† An arbitrary point on the torus can be written as $w = \sigma_1 + \sigma_2\tau$. It is now clear that the partition function $\mathcal{Z}$ has to be summed over all possible *inequivalent* torii

$$\begin{aligned}\mathcal{Z} &= \sum_{\tau\in\mathcal{D}_\tau} \int e^{-S} \\ &= \sum_{\tau\in\mathcal{D}_\tau} \mathcal{Z}(\tau,\bar{\tau})\end{aligned}$$

where $\mathcal{D}_\tau$ is the *fundamental domain* of inequivalent torii under modular transformations. It is not difficult to show that the modular transformations T and S act on the parameter τ in the following manner (up to rescalings)

† To avoid any confusion with the modular parameter τ, we rename the coordinates (σ,τ) to (σ_1,σ_2).





Combining T and S successively generates the most general transformation, under which,

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}\,; \quad ad - bc = 1; \quad a, b, c, d \in \mathbf{Z}.$$

The sum over the fundamental domain $\sum_{\tau\in\mathcal{D}_\tau}$ can be shown to be $\int_{\tau\in\mathcal{D}_\tau}\frac{d\tau_1 d\tau_2}{\tau_2^2}$ and is — as expected — a modular invariant quantity. The condition of invariance of the partition function can now be written

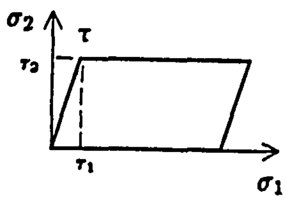
$$\mathcal{Z}(\tau,\bar{\tau}) = \mathcal{Z}(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}) \quad .$$

In practice, it is enough to verify that $\mathcal{Z}$ is invariant under S and T, only, namely

$$\begin{aligned} T \equiv \quad & \mathcal{Z}(\tau,\bar\tau) = \mathcal{Z}(\tau+1,\bar\tau+1) \\ S \equiv \quad & \mathcal{Z}(\tau,\bar\tau) = \mathcal{Z}(-\frac{1}{\tau},-\frac{1}{\bar\tau}) \end{aligned} \quad . \tag{3.4}$$

Let us now write an explicit form for the quantity $\mathcal{Z}(\tau,\bar\tau)$

$$\mathcal{Z}(\tau,\bar\tau) =$$



(3.5)

$$= Tr_{\mathcal{H}_{phys}} e^{-2\pi\tau_2 H} e^{2i\pi\tau_1 P}$$

H is the hamiltonian and P the worldsheet momentum on the w–complex plane. They are related to the corresponding quantities defined on the z–plane (cf. appendix), $L_0+\bar L_0$ and $i(L_0-\bar L_0)$, by a conformal transformation $w(z)$, which has the effect of shifting the conformal dimensions by the amount $(\frac{c}{24},\frac{\bar c}{24})$.* Substituting in (3.5), we obtain

$$\mathcal{Z}(\tau,\bar\tau) = Tr_{\mathcal{H}_{phys}} q^{L_0-\frac{c}{24}} \bar q^{\bar L_0-\frac{\bar c}{24}}.$$

or equivalently,

$$\mathcal{Z}(\tau,\bar\tau) = Tr_{\mathcal{H}} \mathcal{P} q^{L_0-\frac{c}{24}} \bar q^{\bar L_0-\frac{\bar c}{24}}. \tag{3.6}$$

The analysis of the structure of (3.6) is the subject of the next chapter.

* Due to the non tensorial nature of the stress–energy tensor.

4. The partition function

A generic four–dimensional string vacuum is constructed gluing together many differents CFT's (cf. appendix): the spacetime sector composed of free bosons and free fermions, the ghost sector, the internal sector and, for the heterotic string, the gauge sector. This structure should reflect itself in the Hilbert space $\mathcal{H}$ and the partition function $\mathcal{Z}(\tau, \bar{\tau})$ (3.6). In this chapter, we intend to factorize the partition function (3.6) into as many pieces as possible with definite modular transformation properties. In the process, we will illustrate the use of modular invariance to extract the physical spectrum of certain type of theories. We will encounter three types of projection: the projection of states with negative norm whose existence is related to the indefinite spacetime metrics, the GSO projection enforcing spacetime supersymmetry and the projection of states "non–covariant" under internal discrete symmetries. We will end this chapter with a discussion of gauge symmetry breaking with Wilson–line along which the generation of fractional electric charges.

4.1 Ghost sector.

The first important simplification is the cancellation of the ghosts (b, c) and superghosts (β, γ) contributions with the timelike and longitudinal spacetime fields $X^{0,1}(\tau, \bar{\tau})$ and $\Psi^{0,1}(\tau, \bar{\tau})$. This cancellation can be viewed as the realisation of unitarity in the theory with the projection of negative norm states.[26] This allows us to ignore the ghosts and work only with tranverse spacetime fields X^i and Ψ^i transforming in the group SO(2). This is the first illustration of the action of the projector $\mathcal{P}$.

4.2 THE BOSONIC PARTITION FUNCTION.

The second simplification is the factorization of the worldsheet bosons contribution

$$\mathcal{Z}(\tau,\bar\tau) = \mathcal{Z}_{X^i_{SO(2)}}(\tau,\bar\tau) \times \mathcal{Z}_{\Psi^i_{SO(2)},internal,(gauge)}(\tau,\bar\tau).$$

It occurs because the modular group acts trivially on it. Let us show this explicitly by computing the bosonic contribution. The boson action (see the appendix) in coordinates $z, \bar z$ is

$$S \sim \int dz d\bar z \partial_z X^1 \partial_{\bar z} X^1 + \int dz d\bar z \partial_z X^2 \partial_{\bar z} X^2 \quad .$$

The equation of motion is readily found to be

$$\partial_z \partial_{\bar z} X^i = 0 \qquad i = 1,2 \quad .$$

Its solution, $X^i(z,\bar z) = X^i_L(z) + X^i_R(\bar z)$, has the "Fourier" expansion

$$\begin{aligned} X^i_L(z) &= x^i_L - i\frac{p^i}{2} ln(z) + i\sum_{n\neq 0} \frac{1}{n}\alpha^i_n(\bar z)^{-n} \\ X^i_R(z) &= x^i_R - i\frac{p^i}{2} ln(\bar z) + i\sum_{n\neq 0} \frac{1}{n}\bar\alpha^i_n \bar z^{-n} \end{aligned} ,$$

dictated by the boundary conditions*

$$X^i(z,\bar z) = X^i(e^{2i\pi} z, e^{-2i\pi}\bar z) \quad ,$$

$x^i = x^i_L + x^i_R$ are the center of mass coordinates of the string and p^i their momentum

* Remember the definition of $z = e^{\tau + i\sigma}$.

conjugates. Quantization leads to the following commutation relations

$$\begin{cases} [x^i, p^i] = i\delta_{i,j} \\ [\alpha_n^i, \alpha_m^j] = n\alpha_n^i \delta_{n+m,0}\delta_{i,j} \\ [\alpha_n^i, \bar{\alpha}_m^j] = n\bar{\alpha}_n^i \delta_{n+m,0}\delta_{i,j} \\ [\alpha_n^i, \bar{\alpha}_m^j] = 0, \qquad n, m \in \mathbf{Z}_{\{0\}} \end{cases} .$$

A generic state of $\mathcal{H}$ is obtained acting on the vacuum $|0>$ with the raising operators α_{-n}, $\bar{\alpha}_{-n}$ and the spacetime translation operator $e^{ik\cdot x}$:

$$|state >= (\alpha_{-n_1}^{i_1})^{p_1}(\alpha_{-n_2}^{i_2})^{p_2}\dots(\alpha_{-n_l}^{i_l})^{p_l} \otimes (\bar{\alpha}_{-n_1}^{i_1})^{p_1}(\bar{\alpha}_{-n_2}^{i_2})^{p_2}\dots(\bar{\alpha}_{-n_l}^{i_l})^{p_l}\ e^{ik\cdot x}|0> \quad .$$

It is an eigenvector of L_o^{bosons} (similarly for $\bar{L}_o^{bosons}$)

$$\begin{aligned} &L_o^{bosons}|state >= (\frac{k\cdot k}{2} + N)|state > \\ &N = p_1 n_1 + \dots + p_l n_l \end{aligned} .$$

With this information it is a simple exercise to compute $\mathcal{Z}_{bosons}(\tau, \bar{\tau})$. We have successively,[†]

$$\begin{aligned} \mathcal{Z}_{bosons}(\tau,\bar{\tau}) &= Tr_{\mathcal{H}} q^{L_o^{bosons} - \frac{2}{24}} \bar{q}^{\bar{L}_o^{bosons} - \frac{2}{24}} \\ &= (q\bar{q})^{-\frac{1}{12}} \times \int_{-\infty}^{+\infty} d^2k\ e^{-2\pi\tau_2 k^2} \times [\prod_{n=1}^{\infty}(1-q^n)]^2 \times [\prod_{n=1}^{\infty}(1-\bar{q}^n)]^2 \\ &= (2\tau_2)^{-1}|\eta(\tau)|^{-4} \end{aligned} \tag{4.1}$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \quad , \tag{4.2}$$

† Each boson field $X^i(\sigma,\tau)$ contributes on unit to the central charge c (cf. appendix).

is the *Dedekind eta function* which transforms under modular transformations as

$$
\begin{aligned}
T \equiv \qquad & \eta(\tau+1) = e^{i\frac{\pi}{12}}\eta(\tau) \\
S \equiv \qquad & \eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}}\eta(\tau)
\end{aligned} \quad . \tag{4.3}
$$

With these transformations and $Im(-\frac{1}{\tau}) = (\tau\bar{\tau})^{-1}Im(\tau)$, it is easy to verify that (4.1) is a modular invariant quantity and can, consequently, be factored–out.

4.3 Character of a representation.

In general, such a factorization doesn't occur for the other factors in $\mathcal{Z}$. Without making any assumption we can decompose the partition function in a simple and general form:

$$
\mathcal{Z}(\tau,\bar{\tau}) = (2\tau_2)^{-1}|\eta(\tau)|^{-4}\sum_{\alpha,\beta}\mathcal{P}_{\alpha,\beta}\mathcal{B}_{\alpha}^{SO(2)}(\tau)\mathcal{B}_{\beta}^{SO(2)}(\bar{\tau})\mathcal{Z}_{internal}^{\alpha,\beta}(\tau,\bar{\tau})
$$

with

$$
\begin{aligned}
\mathcal{B}_{\alpha}^{SO(2)}(\tau) &= Tr_{\mathcal{R}_{\alpha}^{left}} q^{L_0^f - \frac{c^f}{24}} \\
\mathcal{B}_{\beta}^{SO(2)}(\bar{\tau}) &= Tr_{\mathcal{R}_{\beta}^{right}} \bar{q}^{L_0^f - \frac{c^f}{24}} \\
\mathcal{Z}_{internal}^{\alpha,\beta}(\tau,\bar{\tau}) &= Tr_{\mathcal{H}_{internal}^{\alpha,\beta}} q^{L_0^{int} - \frac{c^{int}}{24}} \bar{q}^{L_0^{int} - \frac{c^{int}}{24}}
\end{aligned} \quad , \tag{4.4}
$$

which follows from the decomposition of the Hilbert space as

$$
\mathcal{H} = \mathcal{H}_{bosons} \otimes \sum_{\alpha,\beta} \mathcal{R}_{SO(2)}^{\alpha} \otimes \mathcal{H}_{internal}^{\alpha,\beta} \otimes \mathcal{R}_{SO(2)}^{\beta}.
$$

$\mathcal{B}_\alpha$ is the *character* of the representation $\mathcal{R}_\alpha$ of the affine SO(2) algebra describing the transverse fields Ψ_L^i. $\mathcal{R}_\alpha$ contains one irreducible multiplet of highest weight states and their descendants. $\mathcal{Z}^{\alpha,\beta}$ is the contribution from the internal sector; its structure is for now left unspecified but it is clear that it is also a sum of products of left– and right–moving characters of irreducible representations of the internal algebra. The projector $\mathcal{P}$ is now understood as a set of integers $\mathcal{P}_{\alpha,\beta}$ selecting the combinations of characters which are allowed by modular invariance, unitarity and any other requirement we are to implement in the theory. Our next task is to evaluate the characters $\mathcal{B}_\alpha^{SO(2)}$.

4.4 The transverse fermions.

Let Ψ^i, $\quad i = 1, 2$ be two free Weyl spinors of the same chirality. They evolve according to the fermion action (see the appendix) which, in $z, \bar{z}$ coordinates, is

$$S \sim \int \Psi^1 \partial_{\bar{z}} \Psi^1 + \Psi^2 \partial_{\bar{z}} \Psi^2 \quad . \tag{4.5}$$

The equation of motion $\partial_{\bar{z}} \Psi^i = 0$ shows that Ψ^i are holomorphic fields. They are primary with $h = 1/2$ and admit the Fourier expansion

$$\Psi^i(z) = \sum_n \Psi^i_n z^{-n-\frac{1}{2}} \tag{4.6}$$

with

$$\{\Psi^i_n, \Psi^j_m\} = \delta^{ij} \delta_{n+m,0} \quad .$$

It is convenient to define $\Psi = \frac{1}{2}(\Psi^1 + i\Psi^2)$ and $\bar{\Psi} = \frac{1}{2}(\Psi^1 - i\Psi^2)$; their action is

$$S \sim \int \bar{\Psi} \partial_{\bar{z}} \Psi \quad . \tag{4.7}$$

they have the expansions

$$\Psi(z)=\sum_{n\geq 0}\Psi_n z^{-n-\frac{1}{2}}+\sum_{n\geq 1}\bar{\Psi}_{-n}z^{n-\frac{1}{2}}$$
$$\bar{\Psi}(z)=\sum_{n\geq 1}\bar{\Psi}_n z^{-n-\frac{1}{2}}+\sum_{n\geq 0}\Psi_{-n}z^{n-\frac{1}{2}}$$

with their modes satisfying the commutation relations

$$\begin{aligned}&\{\Psi_n,\Psi_m\}=\{\bar{\Psi}_n,\bar{\Psi}_m\}=\delta_{n+m,0}\\&\{\Psi_n,\bar{\Psi}_m\}=0\end{aligned}\quad . \tag{4.8}$$

The modding n is specified by the boundary conditions

$$\Psi(ze^{2i\pi})=\pm\Psi(z)\qquad\begin{cases}P\\A\end{cases}.$$

or, in the w-plane*

$$\Psi(w+1)=\mp\Psi(w)\qquad\begin{cases}A\\P\end{cases}$$

corresponding to the Neveu–Schwartz(NS) and Ramond(R) sectors with modding $n+\frac{1}{2}\in\mathbf{Z}$ and $n\in\mathbf{Z}$, respectively. Consider first the NS sector. The spectrum is

$\|0>$	$\Psi_{-\frac{1}{2}}\|0>,\bar{\Psi}_{-\frac{1}{2}}\|0>$
$\Psi_{-\frac{1}{2}}\bar{\Psi}_{-\frac{1}{2}}\|0>$	$\Psi_{-\frac{3}{2}},\bar{\Psi}_{-\frac{3}{2}}\|0>$
$\Psi_{-\frac{3}{2}}\bar{\Psi}_{-\frac{1}{2}}\|0>,\Psi_{-\frac{1}{2}}\bar{\Psi}_{-\frac{3}{2}}\|0>$	$\Psi_{-\frac{5}{2}},\bar{\Psi}_{-\frac{5}{2}}\|0>,\dots$
$\Psi_{-\frac{5}{2}}\bar{\Psi}_{-\frac{1}{2}}\|0>,\Psi_{-\frac{3}{2}},\bar{\Psi}_{-\frac{3}{2}}\|0>,\dots$	$\Psi_{-\frac{7}{2}},\bar{\Psi}_{-\frac{7}{2}}\|0>,\dots$
...	...

* $\Psi(w)=(\frac{\partial z}{\partial w})^{\frac{1}{2}}\Psi(z)$ and $z=e^{2i\pi w}$ explain the change in sign in the boundary conditions.

They arrange themselves into two conformal families corresponding to the two irreducible representations of primary fields $|0>$ with $h=0$, and $(\Psi_{-\frac{1}{2}}|0>, \bar{\Psi}_{-\frac{1}{2}}|0>)$, with $h=\frac{1}{2}$. These representations are called *Adjoint*(A) and *Vector*(V), respectively. Observe that the descendants obtained from $|0>$ with an even (odd) number of creation operators are in the A (V) representation. We can make use of this fact to compute the corresponding characters. Define the operator $(-1)^F$such that $\{(-1)^F, \Psi^i\}=0$ and $(-1)^F|0>=0$. It has the obvious property

$$\begin{cases} (-1)^F|(A)> &= |(A)> \\ (-1)^F|(V)> &=-|(V)> \end{cases} .$$

We can write

$$\begin{aligned} \mathcal{B}^{SO(2)}_{\substack{(A)\\(V)}}(\tau) &= Tr^{left}_{\substack{(A)\\(V)}} q^{L_0^f-\frac{c^f}{24}} \\ &= Tr_{NS}\frac{1\pm(-1)^F}{2}q^{L_0^f-\frac{c^f}{24}} \\ &= \frac{1}{2}Tr_{NS}q^{L_0^f-\frac{c^f}{24}} \pm \frac{1}{2}Tr_{NS}(-1)^F q^{L_0^f-\frac{c^f}{24}} \end{aligned} \qquad (4.9)$$

Repeting the analysis we made for the bosons (4.1), we obtain

$$\begin{aligned} Tr_{NS}q^{L_0^f-\frac{c^f}{24}} &= q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}})(1-q^{n+\frac{1}{2}}) \\ &= \frac{\Theta\left[{0 \atop -\frac{1}{2}}\right](0,\tau)}{\eta(\tau)} \\ Tr_{NS}(-1)^F q^{L_0^f-\frac{c^f}{24}} &= q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}})(1+q^{n+\frac{1}{2}}) \\ &= \frac{\Theta\left[{0 \atop 0}\right](0,\tau)}{\eta(\tau)} \end{aligned} , \qquad (4.10)$$

where $\eta(\tau)$ is the eta Dedekind function introduced earlier and

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = e^{2i\pi a(b+z)}q^{\frac{a^2}{2}}\prod_{n=0}^{\infty}(1+q^{n+a+\frac{1}{2}}e^{2i\pi(z+b)})(1+q^{n-a+\frac{1}{2}}e^{-2i\pi(z+b)})(1-q^{n+1})$$

is a *classical theta function* with characteristics a and b.[27] (4.10) has a simple interpretation in the path–integral formalism. Denote ${}^{P}\underset{A}{\square}$ and (${}^{A}\underset{A}{\square}$) the functional integral $\int D\Psi e^{-S}$ over the torus with A boundary conditions in the σ–direction and $P(A)$ boundary conditions in the τ–direction. When transposed in the operator language they reproduce (4.10)

$$
\begin{aligned}
Tr_{NS} q^{L_0^f - \frac{c^f}{24}} &= {}^{A}\underset{A}{\square} \\
Tr_{NS} (-1)^F q^{L_0^f - \frac{c^f}{24}} &= {}^{P}\underset{A}{\square}
\end{aligned}
\quad . \tag{4.11}
$$

This correspondence makes clear that the existence of different conformal families is linked to the different boundary conditions of the fermions on the torus. We can make use of that connection to infer the existence of two conformal families in the R sector corresponding to the *Spinor (S)* and *Anti-spinor ($\bar{S}$)* representations whose characters are

$$
B^{SO(2)}_{\substack{(S) \\ (\bar{S})}}(\tau) = Tr_R \frac{1 \pm (-1)^F}{2} q^{L_0^f - \frac{c^f}{24}} \quad . \tag{4.12}
$$

Analysis of the spectrum[27] shows that they correspond to primary fields with $h = \frac{1}{8}$. An explicit calculation yields

$$
\begin{aligned}
Tr_R(-1)^F q^{L_0^f - \frac{c^f}{24}} &= {}^{\mathsf{P}}\underset{\mathsf{P}}{\square} \\
&= \frac{\Theta\left[{-\frac{1}{2} \atop 0}\right](0,\tau)}{\eta(\tau)} \\
Tr_R q^{L_0^f - \frac{c^f}{24}} &= {}^{\mathsf{A}}\underset{\mathsf{P}}{\square} \\
&= \frac{\Theta\left[{-\frac{1}{2} \atop -\frac{1}{2}}\right](0,\tau)}{\eta(\tau)}
\end{aligned}
\quad . \tag{4.13}
$$

Characters (4.12) can be expressed in a more compact form:

a) make use of the Jacobi triple product identity[27]

$$
\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{n\in Z} q^{\frac{(n+a)^2}{2}} e^{2i\pi(n+a)(z+b)} \quad . \tag{4.14}
$$

b) define an index s modulo 4 labelling the four representations

s	representation
0	(A)
1	$(\bar{S})$
2	(V)
3	(S)

c) verify that

$$
\mathcal{B}_s = \frac{1}{\eta(\tau)} \sum_{n=even+\frac{s}{2}} q^{\frac{n^2}{2}} \quad . \tag{4.15}
$$

Let us show it explicitly for $s = 2$ (V); the other cases are treated similarly.

From (4.10) and (4.12)

$$
\begin{aligned}
\mathcal{B}_s &= \frac{1}{2}(\Theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau) - \Theta\begin{bmatrix}0\\-\frac{1}{2}\end{bmatrix}(0,\tau)) \\
&= \frac{1}{2}\sum_{n\in Z} q^{\frac{n^2}{2}} - \frac{1}{2}\sum_{n\in Z} q^{\frac{n^2}{2}} e^{2i\pi n/2} \\
&= \sum_{n\in Z} q^{\frac{n^2}{2}} \frac{(1-(-1)^n)}{2} \\
&= \sum_{n=even+\frac{s}{2}} q^{\frac{n^2}{2}} \qquad , \; s=2
\end{aligned}
$$

In summary, we have found a closed form for the characters of the spacetime transverse fermions; there is one of them for each of the four irreducible representations reflecting the four different *spin structures* on the torus: the (A) and (V) representations in the NS–sector and the (S) and $(\bar{S})$ representations in the R sector. This result can be generalized to $2n$ fermions forming an affine $SO(2n)$ algebra. The characters are, then

$$
\mathcal{B}_s = \frac{1}{\eta(\tau)^n} \sum_{\sum n_i = even+\frac{s}{2}} q^{\frac{\vec{n}\cdot\vec{n}}{2}} \quad . \tag{4.16}
$$

These characters transform in a well–defined way under modular transformations; they form an *irreducible representation of the modular group*[3]

$$
\begin{aligned}
T &\equiv \quad \mathcal{B}_s^{SO(2n)}(\tau+1) = \mathcal{T}_{ss'}\mathcal{B}_{s'}(\tau) \\
S &\equiv \quad \mathcal{B}_s^{SO(2n)}(-\frac{1}{\tau}) \; = \mathcal{S}_{ss'}\mathcal{B}_{s'}(\tau)
\end{aligned} \tag{4.17}
$$

with $\mathcal{T}_{ss'}$ and $\mathcal{S}_{ss'}$ unitary matrices given by

$$\mathcal{T}_{ss'} = diag\,(1 \quad e^{\frac{i\pi n}{4}} \quad -1 \quad e^{\frac{i\pi n}{4}})\,e^{-in\frac{\pi}{12}}$$

$$\mathcal{S}_{ss'} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i^{-n} & -1 & -i^{-n} \\ 1 & -1 & 1 & -1 \\ 1 & -i^{-n} & -1 & i^{-n} \end{pmatrix} \quad . \tag{4.18}$$

As an illustration, let us compute an example.

4.5 TEN–DIMENSIONAL SUPERSTRING—AN EXAMPLE.

The structure of the ten–dimensional superstring is summarized in the appendix. The internal sector extends the spacetime dimension d+2 from four to 10. The transverse fields are $X^i_L, X^i_R, \Psi^i_L, \Psi^i_R$, $i = 1, ...d$. The bosonic fields contribute as before as an overall factor $Im(\tau)^{-\frac{d}{2}}|\eta(\tau)|^d$. The partition function can be written

$$\mathcal{Z}(\tau,\bar{\tau}) = Im(\tau)^{-\frac{d}{2}}|\eta(\tau)|^d \sum_{s,s'} \mathcal{P}_{s,s'}\mathcal{B}_s^{SO(d)}(\tau)\mathcal{B}_{s'}^{SO(d)}(\bar{\tau}).$$

According to (4.18), modular transformations act on the $\mathcal{B}$'s through the matrices $\mathcal{T}$ and $\mathcal{S}$*

$$\mathcal{T}_{ss'} = diag\,(1 \quad -1 \quad -1 \quad -1)\,e^{-i\frac{\pi}{3}}$$

$$\mathcal{S}_{ss'} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad . \tag{4.19}$$

---

* $\bar{\mathcal{B}}$ being a function of $\bar{\tau}$ transform under $\mathcal{T}^\dagger$ and $\mathcal{S}^\dagger$.

An obvious choice for $\mathcal{P}_{s,s'}$ is $\delta_{s,s'}$ leading to the modular invariant combination $\mathcal{B}_{(A)}\bar{\mathcal{B}}_{(A)} + \mathcal{B}_{(V)}\bar{\mathcal{B}}_{(V)} + \mathcal{B}_{(S)}\bar{\mathcal{B}}_{(S)} + \mathcal{B}_{(\bar{S})}\bar{\mathcal{B}}_{(\bar{S})}$. This is an unfortunate choice for it is positive definite and consequently cannot describe properly spacetime fermions with the correct spin–statistics. We need a minus sign! The structure of the matrix $\mathcal{S}_{ss'}$ suggests that a minus sign always accompanies the (V) representation. Under S the latter transforms as

$$S \equiv \qquad (V) \longrightarrow \frac{1}{2}[(A) - (\bar{S}) + (V) - (S)] \quad ,$$

the others as

$$S \equiv \qquad \begin{aligned} (A) &\longrightarrow \frac{1}{2}[(A) + (\bar{S}) + (V) + (S)] \\ (\bar{S}) &\longrightarrow \frac{1}{2}[(A) + (\bar{S}) - (V) - (S)] \\ (S) &\longrightarrow \frac{1}{2}[(A) - (\bar{S}) - (V) + (S)] \end{aligned} \quad .$$

Only the conbinations $(V) + (A)$, $(V) - (S)$ and $(V) - (\bar{S})$ are invariant under S, while under T

$$T \equiv \qquad \begin{aligned} (V) + (A) &\longrightarrow [-(V) + (A)]e^{-i\frac{\pi}{3}} \\ (V) - (S) &\longrightarrow [-(V) - e^{i\pi}(S)]e^{-i\frac{\pi}{3}} \\ (V) - (\bar{S}) &\longrightarrow [-(V) - e^{i\pi}(\bar{S})]e^{-i\frac{\pi}{3}} \end{aligned} \quad .$$

Hence, $(V) + (A)$ is ruled–out and modular invariance doesn't distinguish between (S) and $(\bar{S})$. Our final modular invariant partition function becomes

$$\begin{aligned} \mathcal{Z}(\tau,\bar{\tau}) &= Im(\tau)^{-\frac{d}{2}}|\eta(\tau)|^{d}(\mathcal{B}_{(V)} - \mathcal{B}_{(S)})(\bar{\mathcal{B}}_{(V)} - \bar{\mathcal{B}}_{(S)}) \\ &= Im(\tau)^{-\frac{d}{2}}|\eta(\tau)|^{d}[\mathcal{B}_{(V)}\bar{\mathcal{B}}_{(V)} + \mathcal{B}_{(S)}\bar{\mathcal{B}}_{(S)}] - [\mathcal{B}_{(V)}\bar{\mathcal{B}}_{(S)} + \mathcal{B}_{(S)}\bar{\mathcal{B}}_{(V)}] \end{aligned} \tag{4.20}$$

Sectors $(V)(V)$ and $(S)(S)$ describe consistently the spacetime bosons of the theory, while sectors $(V)(S)$ and $(S)(V)$ their spacetime supersymmetric partners;

furthermore, $\mathcal{Z}(\tau,\bar{\tau})$ has the correct spin–statistics and can be shown to vanish.

We have projected–out the (A) representation in the NS sector and the $(\bar{S})$ representation in the R sector. This projection is called the *GSO projection.* It amounts to keeping in the spectrum the states with an odd number of creation operators Ψ_{-n}, $\bar{\Psi}_{-n}$ acting on the ground state, namely,

$$(-1)^F|state>= -|state> \quad .$$

$\frac{1-(-1)^F}{2}$ is the GSO projector for the superstring and is the second projector $\mathcal{P}$ that we encounter. To conclude this section, notice that the construction (4.20) is modular invariant only because $e^{\frac{i\pi(d/2)}{4}}$ in (4.19) is 1 which requires that d be a multiple of 8; this is indeed the case for a ten–dimensional spacetime.

4.6 Gauge sector and the heterotic string.

Gepner proposed a general and systematic way of constructing a four–dimensional string theory from a ten–dimensional superstring. The idea (appendix) is to replace the left-moving fermions $\Psi^{SO(2n)}$ with an affine SO(2n+X) algebra at level 1 in a way which doesn't alter the modular invariance properties of the partition function. This is done if one can find a unitary matrix $M_{ss'}$ satisfying

$$\mathcal{B}_s^{SO(2n+X)} = M_{ss'}\mathcal{B}_{s'}^{SO(2n)} \quad \text{with} \quad \begin{cases} \mathcal{S}^{SO(2n)} & = M\mathcal{S}^{SO(2n+X)}M^{-1} \\ \mathcal{T}^{SO(2n)} & = M\mathcal{T}^{SO(2n+X)}M^{-1} \end{cases} \quad . \quad (4.21)$$

A solution exists only for $X = 13$ in which case M produces the following mapping

$$(\mathcal{B}_{(A)},\mathcal{B}_{(S)},\mathcal{B}_{(V)},\mathcal{B}_{(S)})^{SO(2n)} \longrightarrow (\mathcal{B}_{(V)},-\mathcal{B}_{(\bar{S})},\mathcal{B}_{(A)},-\mathcal{B}_{(S)})^{SO(2n+13)} \quad . \quad (4.22)$$

Once considering more general gauge group, another solution exists. It is an affine

$SO(2n+6) \times E_8$ at level 1 with the mapping

$$(\mathcal{B}_{(A)}, \mathcal{B}_{(\bar{S})}, \mathcal{B}_{(V)}, \mathcal{B}_{(S)})^{SO(2n)} \longrightarrow \mathcal{R}^{\tilde{E}_8}_{(A)} \otimes (\mathcal{B}_{(V)}, -\mathcal{B}_{(\bar{S})}, \mathcal{B}_{(A)}, -\mathcal{B}_{(S)})^{SO(2n+13)} \quad . \tag{4.23}$$

In (4.23), $\mathcal{R}^{\tilde{E}_8}_{(A)}$ is the character of the adjoint of E_8 and can be shown to be equivalent to $\tilde{\mathcal{B}}^{SO(16)}_{(A)} + \tilde{\mathcal{B}}^{SO(16)}_{(S)}$ and, from (4.19), can be seen to transform under T and S as

$$\begin{cases} T \equiv & \mathcal{R} \longrightarrow e^{-i\pi 2/3}\, \mathcal{R} \\ S \equiv & \mathcal{R} \longrightarrow \mathcal{R} \end{cases} \quad .$$

The reader is invited to verify that with such a replacement modular invariance is unaffected.

As an example, let us do the substitution in the ten-dimensional superstring. From (4.20), we immediately obtain

$$\begin{aligned} \mathcal{Z}(\tau,\bar{\tau}) &= Im(\tau)^{-\frac{d}{2}} |\eta(\tau)|^d (\mathcal{R}^{\tilde{E}_8}_{(A)})(\mathcal{B}_{(A)} + \mathcal{B}_{(S)})(\bar{\mathcal{B}}_{(V)} - \bar{\mathcal{B}}_{(S)}) \\ &= Im(\tau)^{-\frac{d}{2}} |\eta(\tau)|^d (\tilde{\mathcal{B}}_{(A)} + \tilde{\mathcal{B}}_{(S)})(\mathcal{B}_{(A)} + \mathcal{B}_{(S)})(\bar{\mathcal{B}}_{(V)} - \bar{\mathcal{B}}_{(S)}) \quad . \\ &= Im(\tau)^{-\frac{d}{2}} |\eta(\tau)|^d (\mathcal{R}^{\tilde{E}_8}_{(A)})(\mathcal{R}^{E_8}_{(A)})(\bar{\mathcal{B}}_{(V)} - \bar{\mathcal{B}}_{(S)}) \end{aligned} \tag{4.24}$$

The reader can easily check modular invariance of (4.24) using (4.19). The GSO projection has been reversed in the left sector. The gauge left-movers are now in the adjoint of $\tilde{E}_8 \times E_8$ which can be shown to contain 256 massless gauge bosons — and their superpartners — realizing the gauge group $\tilde{E}_8 \times E_8$. The same construction applied to a four-dimensional heterotic string leads to the gauge group $\tilde{E}_8 \times E_6$, studied in the next chapter.

4.7 Lattice formulation of an affine SO(2N) algebra at level 1.

There is a nice and useful interpretation of (4.16). Consider the R^n lattice Λ generated by $\{\pm e_i \pm e_j\}_{i \neq j}$ with $e_i = (0, \ldots, 1^{ith}, \ldots, 0)$. Then (4.16) can be written

$$
\begin{aligned}
\mathcal{B}_s &= \frac{1}{\eta(\tau)^n} \sum_{\sum n_i = even + \frac{s}{2}} q^{\frac{\vec{n}\cdot\vec{n}}{2}} \\
&= \frac{1}{\eta(\tau)^n} \sum_{w = w_s + \Lambda} q^{\frac{w \cdot w}{2}}
\end{aligned}
\qquad (4.25)
$$

with the correspondence

s	weight w_s	representation
0	$(0,\ldots,0)$	(A)
1	$(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$	$(\bar{S})$
2	$(0,\ldots,0,1)$	(V)
3	$(\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2})$	(S)

There is a one–to–one correspondence between the *weights* w and the states of the four conformal families. Assembling the $2n$ fermions in n complex fermions $\Psi^i, \bar{\Psi}^i, \quad i = 1,\ldots,n$ as in section 4.4, we have for example, $|0> \equiv (0,\ldots,0)$; $\bar{\Psi}^i_{-\frac{1}{2}}\Psi^j_{-\frac{1}{2}}|0> \equiv (0,\ldots,-1^{i^{th}},\ldots,1^{j^{th}},\ldots,0)$ in the (A) representation, $\Psi^i_{-\frac{1}{2}}|0> \equiv (0,\ldots,1^{i^{th}},\ldots,0)$ in the (V) representation and the ground state in the R sector is in correspondence with $(\pm\frac{1}{2},\ldots,\pm\frac{1}{2},\pm\frac{1}{2})$.One can convince oneself that the conformal dimension of the state $|w>$ is simply $h = \frac{w\cdot w}{2}$. Modular transformations (4.17) take the closed form

$$
\begin{aligned}
&\mathcal{B}_s^{SO(2n)}(\tau+1) = e^{2i\pi(\frac{w_s^2}{2}-\frac{c}{24})}\ \mathcal{B}_s^{SO(2n)}(\tau) \\
&\mathcal{B}_s^{SO(2n)}(\frac{-1}{\tau}) = \frac{1}{2}\sum_{s'} e^{-2i\pi w_s\cdot w'_s}\mathcal{B}_{s'}^{SO(2n)}(\tau)
\end{aligned}
\qquad (4.26)
$$

This interpretation reflects the existence of another description[27] of the affine SO(2n) algebra in terms of free bosons living on the lattice Λ and brings us also very close to the structure of the underlying Lie algebra of $SO(2n)$; in particular, Λ is nothing but the lattice of the roots of $SO(2n)$ which, in the gauge sector, are the weights of the massless gauge bosons. This description is particularly convenient to compute the gauge quantum numbers of a given state. A given current is described by a certain direction $\vec{Q}$ in the weight lattice and the corresponding charge of a state of weight w is simply the projection along this axis

$$\hat{Q}|w>= \vec{Q}\cdot w\ |w> \quad . \tag{4.27}$$

This is used in the next chapter to compute the electric charge of the massless spectrum in Gepner's models.

4.8 More general boundary conditions.

More general boundary conditions can be imposed on the worldsheet fermions which are compatible with the action (4.7), namely,

$$\Psi^i(ze^{2i\pi}) = \pm e^{2i\pi a_i}\Psi^i(z) \quad \left\{ \begin{array}{l} \text{NS} \\ \text{R} \end{array} \right. \qquad a_i \in \mathbf{Z} \quad . \tag{4.28}$$

It is clear that taking a_i not all identical doesn't respect any initial $SO(2n)$ symmetry. For instance, $a_1 = \ldots = a_{n-1} \neq a_n$ breaks the $SO(2n)$ algebra to an $SO(2n-2)\times SO(2)$ algebra. Indeed, this procedure will be used to break the gauge symmetry as explained shortly. Consider fermions with such *twisted* boundary conditions $\{a_i\}$ on the torus,

$$\Psi^i(w+1) = \mp e^{2i\pi a_i}\Psi^i(w) \quad \left\{ \begin{array}{l} \text{NS} \\ \text{R} \end{array} \right. \qquad a_i \in \mathbf{Z} \quad . \tag{4.29}$$

and denote their character $Tr_{\mathcal{H}^{a_i}} q^{L_0^f - \frac{c^f}{24}}$ symbolically by $\underset{a}{\Box}$. It is clear that the modular transformation S — which exchanges σ_1 with σ_2 — forces us to consider the sector ${}^{a}\Box$ as well, with non trivial boundary conditions in the σ_2-direction.

Fields of the latter satisfy

$$\begin{cases} \Psi^i(w+\tau) & = \mp\, e^{2i\pi a_i}\Psi^i(w) \\ \Psi^i(w+1) & = \mp\, \Psi^i(w) \end{cases} \qquad . \tag{4.30}$$

On another hand, a T transformation maps $\square_{a}$ to ${}^{a}\square_{a}$. More generally,

$$\begin{aligned} T \equiv & \qquad {}^{b}\square_{a} \longrightarrow {}^{a+b}\square_{a} \\ & \qquad\qquad\qquad\qquad . \\ S \equiv & \qquad {}^{b}\square_{a} \longrightarrow {}^{a}\square_{-b} \end{aligned} \tag{4.31}$$

To compute ${}^{b}\square_{a}$, one can write it in the operator language

$${}^{b}\square_{a} = Tr_{\mathcal{H}^a}\hat{g}^b q^{L_0^f - \frac{c^f}{24}}$$

with $\hat{g}$ an operator whose action is defined as

$$\begin{aligned} \hat{g}\Psi^i_{-n}\hat{g}^{-i} &= e^{2i\pi a_i}\Psi^i_{-n} \\ \hat{g}|0> &= |0> \end{aligned}$$

Then one proceeds as in the untwisted case with the modding n such that $\Psi^i(z) = \sum_n \Psi^i_n z^{-n-\frac{1}{2}}$ with $n + b^i + \frac{1}{2} \in \mathcal{Z}$ in the NS sector and $n + b^i \in \mathcal{Z}$ in the R sector. In the case in hand, this yields[15]

$$\begin{aligned} B_s^{SO(2n)}\begin{bmatrix} a \\ b \end{bmatrix} &= {}^{b}\square_{a} \\ &= \frac{\Theta_s\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}{\eta(\tau)^n} \end{aligned} \tag{4.32}$$

with

$$\begin{aligned}\Theta_s\begin{bmatrix}a\\b\end{bmatrix} &= \sum_{\sum n_i = even+\frac{s}{2}} q^{\frac{(\vec{n}+\vec{a})^2}{2}} e^{2i\pi(\vec{n}+\vec{a})\cdot\vec{b}} \\ &= \frac{1}{\eta(\tau)^n}\sum_{\Lambda} q^{\frac{(w_s+\Lambda+a)^2}{2}} e^{2i\pi(w_s+\Lambda+a)\cdot b}\end{aligned} \quad . \tag{4.33}$$

Modular transformations of (4.32) are easily worked out using (4.26)

$$\begin{aligned}\mathcal{B}_s\begin{bmatrix}a\\b\end{bmatrix}(\tau+1) &= e^{-i\pi a^2}\ e^{2i\pi(\frac{w_s^2}{2}-\frac{c}{24})}\ \mathcal{B}_s\begin{bmatrix}a\\a+b\end{bmatrix}(\tau) \\ \mathcal{B}_s\begin{bmatrix}a\\b\end{bmatrix}(\frac{-1}{\tau}) &= \frac{1}{2}e^{2i\pi a\cdot b}\ \sum_{s'} e^{-2i\pi w_s\cdot w'_s}\mathcal{B}_{w'_s}\begin{bmatrix}-b\\a\end{bmatrix}(\tau)\end{aligned} \quad . \tag{4.34}$$

Now, consider the partition function of a theory containing twisted boundary conditions. Formally, it has the form

$$\mathcal{Z} = \sum_{u,t=0}^{n-1}\frac{1}{n}\mathcal{Z}\begin{bmatrix}ux\\tx\end{bmatrix}$$

with $x \in \mathbf{Z}$ and

$$\mathcal{Z}\begin{bmatrix}ux\\tx\end{bmatrix} = Tr_{\mathcal{H}^u}\hat{g}_u^t q^{L_o-\frac{c}{24}}$$

where $\hat{g}_u^t$ implements the twist t times in the σ_2–direction in the sector twisted u times and is so chosen that $\mathcal{Z}$ is modular invariant. We have

$$\begin{aligned}\mathcal{Z} &= \sum_{u=0}^{n-1}\ Tr_{\mathcal{H}^u}\ (\frac{1}{n}\sum_{t=0}^{n-1}\hat{g}_u^t)\ q^{L_o-\frac{c}{24}} \\ &= \sum_u Tr_{\mathcal{H}^u}\ \mathcal{P}_u\ q^{L_o-\frac{c}{24}}\end{aligned} \quad .$$

$\mathcal{P}_u$ is required to satisfy $\mathcal{P}_u^2 = \mathcal{P}_u$. Hence,

$$\begin{aligned}\mathcal{Z} &= \sum_{u=0}^{n-1} Tr_{\mathcal{H}^u}\ \mathcal{P}_u\ q^{L_o-\frac{c}{24}}\ \mathcal{P}_u \\ &= \sum_u Tr_{\mathcal{H}^u_{phys}}\ q^{L_o-\frac{c}{24}}\end{aligned} \tag{4.35}$$

with $\mathcal{H}^u_{phys}$ the truncated Hilbert space in the sector twisted u times. The role of $\mathcal{P}_u$ is to render the spectrum of the theory fully consistent with the Z_n–symmetry we modded with. This is the third and last projector $\mathcal{P}$ we encounter.

In the lattice formulation (4.33), twisted boundary conditions have a simple interpretation: they amount to shifting the root lattice Λ by the twist $\{a_i\}$. As a result, the weight of a twisted state $|w>_a$ becomes

$$|w>_a = |w + a> \quad .$$

An important implication is that the charge $\hat{Q}_i$ of any state twisted by a is shifted by an amount $\hat{Q}_i \cdot a$. This is elaborated in the next section.

4.9 Orbifold construction and gauge symmetry breaking.

Twisted boundary conditions are usually introduced for two major reasons.

Orbifold — Twisted boundary conditions can lead to the construction of new internal sectors from already known ones. The typical example is the orbifold. A Z_n–orbifold is a compactification on a six–dimensional torus whose points are identified through a Z_n–symmetry $z_i \equiv e^{2i\pi b_i} z_i$ with z_i, $i = 1, 2, 3$ a set of complex coordinates. To preserve worldsheet supersymmetry the complex fermions Ψ^i, forming an $SO(6)$ affine algebra are to be identified accordingly. From (4.30), we know that this identification amounts to introducing the sector ${}^{\mathrm{b}}\Box$ which, as a result of modular invariance, leads to the setting described above. Of more direct concern, in Gepner's models the internal sector possesses a Z_n symmetry which can be used to mod out the theory

$$\Phi_i^{int} \longrightarrow g^k \Phi_i^{int} , \qquad g^n = 1 \quad .$$

The motivation of this procedure is that it leads to a new internal sector with a smaller number of generations of chiral fermions.[3]

Symmetry breaking with Wilson lines — As alluded earlier, twisted boundary conditions imposed on the fermions describing the gauge sector lead to the process of gauge symmetry breaking. Modular invariance requires that they accompany the "orbifoldization" of the internal sector, namely

$$\begin{cases} \Phi_i^{int} \longrightarrow g^k \, \Phi_i^{int} , \qquad g^n = 1 & (4.36) \\ \Psi^i \longrightarrow e^{2i\pi k b_i} \Psi^i , \qquad n b^i \in \mathbf{Z} & (4.37) \end{cases} .$$

This is the abstract realization in the CFT language of the implementation of a Wilson line–like construction[7,9] in a geometrical language: the modding out (4.36) defines an internal space with non–contractible loops while (4.37) implies that when going around such a loop, the charged fields Ψ^i pick–up a phase. This can be shown[9] to be equivalent to giving an expectation value to some gauge fields $< \mathbf{A} >=< \delta_\alpha A^\alpha > \neq 0$ with $\{A^\alpha\}$ commuting with each other and $n\delta_\alpha \in \mathbf{Z}$. The unbroken gauge group is generated by the generators commuting with $\mathbf{A}$.

A twisted sector ${}^{a}\Box$ defines strings wrapped around non–trivial topological "structures" in the internal manifold. Their excitations correspond to weights w shifted by a. As we have learned in the previous section, their charges Q are shifted by an amount $Q \cdot a$ with respect to the untwisted strings whose excitations describe in particular quarks and leptons. This is the origin of the fractional electric charge already described in a more pictorial way in the introduction. Of course, the shift is non–trivial only if the background gauge field **A** is not orthogonal to the electric charge axis, namely, iff $Q \cdot a \notin \mathbf{Z}$.

We have introduced all the tools necessary to carry–on our main task: to analyse the charge content of the massless spectrum of Gepner's compactifications with realistic gauge group broken with Wilson lines. This is the subject of the next chapter.

5. Wilson lines in Gepner compactification

In chapter 3, we introduced modular invariance of the one–loop amplitude with no external leg (the partition function) as a tool to extract the physical spectrum of a closed string vacuum. Chapter 4 was devoted to the analysis of the structure of the partition function of a four–dimensional string vacuum. We have shown how the partition function factorizes in a sum of products of characters of irreducible representations of different affine algebras. We have computed explicitly the characters of the gauge and spacetime sectors. We have also illustrated the power of modular invariance in extracting the GSO projectors for the ten–dimensional superstring and heterotic string. Furthermore, we have sketched how to construct the projector of an "orbifoldized theory" and described in words how the latter, once accompanied with a Wilson line acting in the gauge group, can lead to the appearance of states with fractional electric charge. In this chapter we go through a similar but much more detailed analysis of the massless spectrum of four–dimensional string vacua constructed by Gepner[3] with general Wilson lines. We begin by describing the structure of the internal sector.

5.1 Review of Gepner models.

Gepner succeeded in constructing, out of interacting fields, superstring and heterotic string vacua, which are tachyon free and modular invariant and have spacetime supersymmetry with the correct spin-statistics.[3] The internal sector of these models is a tensor product of N=2 superconformal minimal models $k_1, ..., k_r$

with central charge

$$c = \sum_{i=1}^{r} \frac{3k_i}{k_i + 2} = 9 \quad . \tag{5.1}$$

Each primary field splits into holomorphic and anti-holomorphic parts, labelled by a set of quantum numbers $\Pi_i(l_i, q_i, s_i)$. Its conformal dimension is

$$h = \sum_i \frac{l_i(l_i + 2) - q_i^2}{4(k_i + 2)} + \frac{s_i^2}{8} \quad . \tag{5.2}$$

Each N=2 algebra contains a $U(1)$ current; the corresponding r $U(1)$ charges are given by

$$Q_i = \frac{-q_i}{k_i + 2} + \frac{s_i}{2} \tag{5.3}$$

$|q_i - s_i| \leq l_i$, $0 \leq l_i \leq k_i$, $l_i + q_i + s_i =$ even ; $s_i = 0, 2$ modulo(4) correspond to the NS sector and $s_i = 1, -1$ modulo(4) to the R sector.

The character of the corresponding conformal family is given by $\prod \chi_{q_i}^{l_i(s_i)}(\tau, 0, 0)$ where

$$\chi_q^{l(s)}(\tau, z, u) = e^{-2i\pi u} Tr_{\mathcal{H}_q^{l(s)}} e^{2i\pi z J_o} q^{L_o - \frac{c}{24}} \tag{5.4}$$

$\mathcal{H}_q^{l(s)}$ contains the primary field (l, q, s) and its descendants obtained by acting with an even number of superconformal generators $G^{\pm}$, which change the conformal dimension by an integer and the $U(1)$ charge by an even integer. These characters have been constructed by Gepner and Qiu[28] and shown to have the following modular transformation properties

$$\begin{aligned} \chi_q^{l(s)}(\tau + 1, z, u) &= \mathcal{T}_{qq}^{ll(ss)} \, \chi_q^{l(s)}(\tau, z, u) \\ \chi_q^{l(s)}(\frac{-1}{\tau}, \frac{z}{\tau}, u + \frac{cz^2}{6\tau}) &= \mathcal{S}_{qq'}^{ll'(ss')} \, \chi_{q'}^{l'(s')}(\tau, z, u) \end{aligned} \tag{5.5}$$

where the matrices $\mathcal{T}$ and $\mathcal{S}$ are independent of z and u:

$$
\begin{aligned}
\mathcal{T}^{ll(ss)}_{qq} &= e^{2i\pi[\frac{l(l+2)-q^2}{4(k+2)}+\frac{s^2}{8}]} \\
\mathcal{S}^{ll'(ss')}_{qq'} &= \sin(\frac{\pi(l+1)(l'+1)}{k+2})\; e^{i\pi\frac{qq'}{k+2}}\; e^{-i\pi\frac{ss'}{2}} \qquad && l'+q'+s'=\text{even} \\
&= 0 && l'+q'+s'=\text{odd}
\end{aligned}
$$

In the superstring case, the spacetime sector is constructed out of d free transverse bosons and d free transverse fermions ($d=2$). The character of the fermionic part is (4.25)

$$
\mathcal{B}^{SO(2)}_{s_o}(\tau) = \frac{1}{\eta(\tau)} \sum_{n=even+\frac{s}{2}} q^{\frac{n^2}{2}} \tag{5.6}
$$

where $\eta(\tau)$ is the Dedekind function and the numerator an $SO(2)$ theta function at level 1; s_o takes the values $0,1,2,3$ labelling respectively the four representations; adjoint(A), antispinor($\bar{S}$),vector(V) and spinor(S).

The total conformal dimension is $c_T = 9 + \frac{3}{2}(d+2) = 15$. This is enough to construct a superstring theory.[3] To obtain an heterotic string theory, following Gepner and, as explained in detail in the previous chapter, we replace the d left spacetime fermions by a Kac-Moody algebra at level one corresponding to $\tilde{E}_8 \times SO(8+d)$. The corresponding character is

$$
\mathcal{B}_{s_o,\tilde{s}_o}(\tau) = \mathcal{B}^{SO(16)}_{\tilde{s}_o,1}\; \mathcal{B}^{SO(8+d)}_{s_o+V}(\tau) \tag{5.7}
$$

with $\tilde{s}_o$ the values corresponding to the (A) and (S) representations of $SO(16)$ (making-up altogether the adjoint of $\tilde{E}_8$) and s_o+V the representation corresponding to w_{s_o}, but shifted by the element $(0,\ldots,0,1)$ of (V). The above choice is uniquely selected by the requirement that the substitution doesn't affect the modular invariance properties. The central charge is now $c = 9 + d + 2 + \frac{24+d}{2} = 26$, as required for an heterotic string theory.

To write down the partition function we need further conventions. Define

$$\bar{l} = (\bar{l}_1, ..., \bar{l}_r) \qquad \bar{\mu} = (\bar{q}_1, ..., \bar{q}_r; \bar{s}_1, ..., \bar{s}_r; \bar{s}) = (\bar{\mu}'; \bar{s}) \qquad \text{on the right,}$$

$$l = (l_1, ..., l_r) \qquad \mu = (q_1, ..., q_r; s_1, ...s_r; w, \tilde{w}) = (\mu'; w, \tilde{w}) \quad \text{on the left,}$$

$$\bar{\beta} = (1, 1, ...; 1, 1...; 1) \qquad \bar{\beta}_i = (0, 0, ...; 0, .., 2^{i^{th}}, .., 0; 2)$$

$$\beta = (1, 1, ...; 1, 1...; w_{\tilde{S}}, w_{\tilde{A}}) \quad \beta_i = (0, 0, ...; 0, .., 2^{i^{th}}, ..0; w_V, w_{\tilde{A}})$$

$$\bar{\varrho} = (0\mathrm{mod}(2(k_1 + 2), ...; 0\mathrm{mod}(4), ...; 0)$$

$$\varrho = (0\mathrm{mod}(2(k_1 + 2), ...; 0\mathrm{mod}(4), ...; \Lambda, \tilde{\Lambda})$$

$$\bar{Q} = \bar{n}\bar{\beta} + \bar{m}_i\bar{\beta}_i + \bar{p}\bar{\varrho} \qquad Q = n\beta + m_i\beta_i + p\varrho \qquad n, \bar{n}, m_i, \bar{m}_i, p, \bar{p} \in \mathbf{Z}$$

Also define the products:

$$\bar{\mu}'^1 \cdot \bar{\mu}'^2 = \sum_i \frac{-\bar{q}_i^1\bar{q}_i^2}{2(k_i + 2)} + \frac{\bar{s}_i^1\bar{s}_i^2}{4} ,$$

$$\bar{\mu}^1 \cdot \bar{\mu}^2 = \bar{\mu}'^1 \cdot \bar{\mu}'^2 + \frac{\bar{s}^1\bar{s}^2}{4} \qquad \mu^1 \cdot \mu^2 = \mu'^1 \cdot \mu'^2 + w^1 \cdot w^2 + \tilde{w}^1 \cdot \tilde{w}^2 .$$

With these definitions, the conformal dimensions and the total internal $U(1)$ charges of the primary field $|\bar{\mu}; \mu>$ are

$$\bar{h} = \sum_i \frac{\bar{l}_i(\bar{l}_i + 2)}{4(k_i + 2)} + \frac{\bar{\mu} \cdot \bar{\mu}}{2} \qquad h = \sum_i \frac{l_i(l_i + 2)}{4(k_i + 2)} + \frac{\mu \cdot \mu}{2}$$
$$\bar{Q}_{int} = 2\bar{\mu}' \cdot \bar{\beta} \qquad Q_{int} = 2\mu' \cdot \beta.$$

while the total $U(1)$ charge (on the right) is $\bar{Q}_T = 2\bar{\mu} \cdot \bar{\beta}$. Useful relations are

$\frac{\varrho\cdot\varrho}{2} \in \mathbf{Z}$, $\mu\cdot\varrho \in \mathbf{Z}$ and $\bar{\beta}'^2 =$ odd.

The partition function constructed by Gepner is[3]

$$\mathcal{Z}(\tau,\bar{\tau}) = Im(\tau)^{\frac{-d}{2}}|\eta(\tau)|^{2d}\sum_{l,\bar{l}} N_{l,\bar{l}} \sum_{\substack{\bar{\mu}_o\in\mathcal{G}\\ \mu_o=\bar{\mu}_o}} \mathcal{Z}^{*\bar{l}}_{\bar{\mu}_o}(\bar{\tau})\mathcal{Z}^{l}_{\mu_o}(\tau) \tag{5.8}$$

where

$$\begin{aligned}\mathcal{Z}^{\bar{l}}_{\bar{\mu}_o} &= \sum_{\bar{\mu}=\bar{\mu}_o+\bar{Q}} (-1)^{\bar{n}}\mathcal{B}^{SO(2)}_{\bar{s}}\prod_i \chi^{\bar{l}_i(\bar{s}_i)}_{\bar{q}_i}(\bar{\tau})\\ \mathcal{Z}^{l}_{\mu_o}(\tau) &= \sum_{\mu_o=\mu_o+Q} \mathcal{B}^{\tilde{E}_8\times SO(10)}_{s_o+V,\bar{s}}\prod_i \chi^{l_i(s_i)}_{q_i}(\tau)\end{aligned} \tag{5.9}$$

$N_{l\bar{l}}$ is any SU(2) affine invariant[28] (that, for simplicity, we take diagonal). $\mu_o \in \mathcal{G}$ are the requirements: $\bar{\mu}_o\cdot\bar{Q} + \frac{\bar{Q}^2}{2} \in \mathbf{Z}$, or equivalently, $\mu\cdot Q + \frac{Q^2}{2} \in \mathbf{Z}$. When $\bar{Q} = \bar{\beta}_i$, it means that we take all the internal sectors simultaneously in NS or R (independently on the left and on the right); when $\bar{Q} = \bar{\varrho}$, this is trivially satisfied and $\bar{Q} = \bar{\beta}$ means that we select states with odd total $U(1)$ charge Q_T – including the spacetime contribution. On the right, the latter condition enforces spacetime supersymmetry; on the left, it guarantees that the $U(1)$ of the internal sector extends the gauge group from $SO(10)$ to E_6. This is nothing but the usual GSO projection. The reader can expand (5.8) in a form similar to (4.24). Spacetime supersymmetry is realized by a shift of the quantum numbers by $\pm\bar{\beta}$:

$$|\bar{\mu},\mu> \longleftrightarrow |\bar{\mu} \pm \bar{\beta},\mu> \quad , \tag{5.10}$$

with the sign depending on the superconformal charge $\bar{Q}_T = 2\bar{\mu}\cdot\bar{\beta}$.

The spectrum obtained from (5.8) contains the matter fields coming in multiplets of the **27** and $\overline{\mathbf{27}}$ representations :

$$\begin{aligned} \mathbf{27} &= \mathbf{10}^{1} + \mathbf{16}^{-1/2} + \mathbf{1}^{-2} \\ \overline{\mathbf{27}} &= \mathbf{10}^{-1} + \overline{\mathbf{16}}^{1/2} + \mathbf{1}^{2} \end{aligned}$$

and the gauge bosons decompose as

$$\mathbf{78} = \mathbf{45}^{0} + \mathbf{16}^{-3/2} + \overline{\mathbf{16}}^{3/2} + \mathbf{1}^{0}$$

The well known example is the 3^5 theory, corresponding to 5 internal minimal models with $k_i = 3$. It contains 101 generations(**27**), 1 antigeneration($\overline{\mathbf{27}}$) and 330 scalars; all, with their supersymmetric partners and their antiparticles.

5.2 ORBIFOLD CONSTRUCTION.

The number of generations is usually very large,[29] which suggests modding-out the theory by one of the discrete symmetries of the internal sector. Each minimal model possesses a Z_{k_i+2} discrete symmetry

$$\Phi_{q_i}^{l_i(s_i)} \longrightarrow e^{2i\pi\frac{q_i}{k_i+2}}\Phi_{q_i}^{l_i(s_i)} \quad .$$

In addition, there may be some permutation symmetry S leading to a discrete group $G = Z_{k_1+2} \times ... \times Z_{k_r+2} \times S$.

The construction has been worked out by Gepner and Qiu.[28] Define $y =$

$(2x_1,...,2x_r;0...0;0), x_i \in \mathbf{Z}$, such that

$$ny_i = 0 \bmod 2(k_i+2) \tag{5.11}$$

or

$$ny = \varrho \quad .$$

(5.11) ensures that y generates a Z_n subgroup of G. Then consider the expression

$$\mathcal{Z}^l_{\mu_o}\begin{bmatrix} S \\ R \end{bmatrix}(\tau) = \sum_Q e^{2i\pi(\mu+S)\cdot R}\, \mathcal{B}_{s_o+V,\bar{s}_o} \prod_i \chi^{l_i(s_i)}_{q_i+S_i} \tag{5.12}$$

as the building blocks (in the path-integral language) containing the fields twisted by $R = ty$ in the time direction in the sector twisted by $S = uy$ in the space direction; $0 \leq t, u < n$. It is left the exercise to the reader to show that they transform under S and T as

$$\begin{aligned} \mathcal{Z}^l_{\mu_o}\begin{bmatrix} S \\ R \end{bmatrix}(\tau+1) &= e^{-i\pi S^2}\mathcal{T}_{\mu_o\mu_o}\, \mathcal{Z}^l_{\mu_o}\begin{bmatrix} S \\ R+S \end{bmatrix}(\tau) \\ \mathcal{Z}_{\mu_o}{}^l\begin{bmatrix} S \\ R \end{bmatrix}(\frac{-1}{\tau}) &= e^{2i\pi R\cdot S}\mathcal{S}_{\mu_o\nu_o}\, \mathcal{Z}^l_{\nu_o}\begin{bmatrix} -R \\ S \end{bmatrix}(\tau), \end{aligned} \tag{5.13}$$

providing $\mu_o, \nu_o \in \mathcal{G}$. Substituting this character for the corresponding one in (5.8) and summing over all the values of t and u yields the new partition function*

$$\mathcal{Z}(\tau,\bar{\tau}) = \frac{1}{n}\sum_{t,u=0}^{n-1} e^{-i\pi R\cdot S} \sum_{\substack{\mu_o\in\mathcal{G} \\ \mu_o=\bar{\mu}_o}} \mathcal{Z}^{*\bar{l}}_{\bar{\mu}_o}(\bar{\tau})\mathcal{Z}^l_{\mu_o}\begin{bmatrix} S \\ R \end{bmatrix}(\tau). \tag{5.14}$$

The additional phase is required for modular invariance, which is readily checked using the formula (5.13) above. The projector is obtained summing the phases

* From now on, we drop the bosonic contribution and some unimportant coefficients.

over all the values of t. We have successively,

$$\begin{aligned}
\mathcal{Z}(\tau,\bar\tau) = & \frac{1}{n}\sum_{u=0}^{n-1} \mathcal{Z}^{*\bar{l}}_{\bar\mu_o}(\bar\tau)\sum_{t=0}^{n-1} e^{-i\pi R\cdot S} \sum_{\substack{\mu_o\in\mathcal{G}\\ \mu_o=\bar\mu_o}} e^{2i\pi(\mu+S)\cdot R}\mathcal{B}_{s_o+V,\bar s_o}\prod_i \chi^{l_i(s_i)}_{q_i+S_i} \\
= & \sum_{u=0}^{n-1} \mathcal{Z}^{*\bar{l}}_{\bar\mu_o}(\bar\tau) \sum_{\substack{\mu_o\in\mathcal{G}\\ \mu_o=\bar\mu_o}} \left\{\sum_{t=0}^{n-1}\frac{1}{n}e^{2i\pi[\frac{R\cdot S}{2}+R\cdot\mu]}\right\}\mathcal{B}_{s_o+V,\bar s_o}\prod_i \chi^{l_i(s_i)}_{q_i+S_i} \qquad . \\
= & \sum_{u=0}^{n-1} \mathcal{Z}^{*\bar{l}}_{\bar\mu_o}(\bar\tau) \sum_{\substack{\mu_o\in\mathcal{G}\\ \mu_o=\bar\mu_o}} \mathcal{P}_s(\mu)\, \mathcal{B}_{s_o+V,\bar s_o}\prod_i \chi^{l_i(s_i)}_{q_i+S_i}
\end{aligned}$$

(5.11) implies that $\mathcal{P}_s(\mu) = \frac{1}{n}\sum_{t=0}^{n-1} e^{2i\pi[u\frac{y\cdot y}{2}+\mu\cdot y]t}$ satisfies $\mathcal{P}_s^2 = \mathcal{P}_s$. As a consequence,

$$\mathcal{P}_s(\mu) = \begin{cases} 1 & \text{iff} \quad u\frac{y\cdot y}{2}+\mu\cdot y\in\mathbf{Z} \\ \\ 0 & \text{iff} \quad u\frac{y\cdot y}{2}+\mu\cdot y\notin\mathbf{Z} \end{cases} \qquad . \tag{5.15}$$

To preserve spacetime supersymmetry we require further that $|\bar\mu;...>$ and $|\bar\mu+\bar\beta;...>$ be simultaneously projected out by (5.15), which gives the condition

$$y\cdot\beta\in\mathbf{Z} \quad . \tag{5.16}$$

The final form for the twisted partition function is

$$\mathcal{Z}(\tau,\bar\tau) = \sum_{u=0}^{n-1} \sum_{\substack{\mu_o\in\mathcal{G}\\ \mu_o=\bar\mu_o\\ u\frac{y\cdot y}{2}+\mu_o\cdot y\in\mathbf{Z}}} \mathcal{Z}^{*\bar{l}}_{\bar\mu_o}(\bar\tau)\mathcal{Z}^{l}_{\mu_o+uy}(\tau)$$

where the trace is now taken only over the states $|\bar\mu;\mu+uy>$ satisfying ($\mu=\bar\mu+Q$), $u\frac{y\cdot y}{2}+\mu\cdot y\in\mathbf{Z}$ and the GSO projection in the sector twisted u times. (5.16) ensures also that the E_6 gauge group stays unbroken. For example, in the 3^5 theory, we can mod out with $y = 2(0,1,2,3,4)$ leading to 21 generations, 1 anti-generation and 210 scalar singlets, 70 of which coming from the untwisted sector. Modding out further by another $Z_5 \subset S : i \to i+1$, yields a four generation model[3] which has been shown to correspond to the maximally symmetric quintic hypersurface in CP^4 Calabi-Yau manifold.[3,9]

In order to break the gauge symmetry, we wish to embed the action of the discrete group into the gauge group.[8] This leads to a Wilson line-like construction. Before going to the general case, let's first study the simpler case of the Wilson line embedded in $SO(10)$.

5.3 Wilson line in SO(10).

The realization of a Wilson line like construction in a CFT was explained in the previous chapter. We wish to embed the action of the Z_n group into the $SO(10)$ subgroup of E_6. The corresponding partition function was worked out by de Alwis[15] in a quite general context. Let us apply his procedure in our specific case.

In section 4.8 we learned that the action of a Z_n transformation is implemented by a shift α in the root lattice $\Lambda_{SO(10)}$. This lead to the following modification of the characters of $SO(10)$ (5.7),

$$\mathcal{B}_s^{SO(10)}\begin{bmatrix} a \\ b \end{bmatrix} = C \sum_{\Lambda\in\Lambda_{SO(10)}} q^{\frac{(w_s+\Lambda+a)^2}{2}} \; e^{2i\pi(w_s+\Lambda+a)\cdot b} \quad , \tag{5.17}$$

where the dot product is defined with a suitable metric (that we can choose to be diagonal) and C is a constant. Recall their transformations properties (4.34) under

S and T

$$\mathcal{B}_s\begin{bmatrix}a\\b\end{bmatrix}(\tau+1)=e^{-i\pi a^2}\;e^{2i\pi(\frac{w^2}{2}-\frac{c}{24})}\;\mathcal{B}_s\begin{bmatrix}a\\a+b\end{bmatrix}(\tau)$$
$$\mathcal{B}_s\begin{bmatrix}a\\b\end{bmatrix}(\frac{-1}{\tau})=\frac{1}{2}e^{2i\pi a\cdot b}\;\sum_{s'}e^{-2i\pi w_s\cdot w'_s}\mathcal{B}_{s'}\begin{bmatrix}-b\\a\end{bmatrix}(\tau)\,. \tag{5.18}$$

To incorporate this into our partition function we define

$$\mathcal{Z}^l_{\mu_o}\begin{bmatrix}S & a\\R & b\end{bmatrix}=\sum_Q e^{2i\pi(\mu+S)\cdot R}\;\mathcal{B}_{s_o+V,\bar{s}_o}\begin{bmatrix}a\\b\end{bmatrix}\prod_i\chi^{l_i(s_i)}_{q_i+S_i} \tag{5.19}$$

as the new building blocks of the twisted theory. We readily obtain their modular transformation properties

$$\mathcal{Z}^l_{\mu_o}\begin{bmatrix}S & a\\R & b\end{bmatrix}(\tau+1)=e^{-i\pi a^2}e^{-i\pi S^2}\mathcal{T}_{\mu_o\mu_o}\;\mathcal{Z}^l_{\mu_o}\begin{bmatrix}S & a\\R+S & b+a\end{bmatrix}(\tau)$$
$$\mathcal{Z}^l_{\mu_o}\begin{bmatrix}S & a\\R & b\end{bmatrix}(\frac{-1}{\tau})=e^{2i\pi a\cdot b}\;e^{2i\pi R\cdot S}\;\mathcal{S}_{\mu_o\nu_o}\;\mathcal{Z}^l_{\nu_o}\begin{bmatrix}-R & -b\\S & a\end{bmatrix}(\tau)\,. \tag{5.20}$$

Substituting (5.13) with (5.19) into (5.14), we obtain the new partition function:

$$\mathcal{Z}(\bar{\tau},\tau)=\sum_{t,u=1}^{n-1}e^{-i\pi a\cdot b}\;e^{-i\pi R\cdot S}\sum_{\substack{\mu_o\in\mathcal{G}\\ \mu_o=\bar{\mu}_o}}\mathcal{Z}^{*\bar{l}}_{\bar{\mu}_o}(\bar{\tau})\mathcal{Z}^l_{\mu_o}\begin{bmatrix}S & a\\R & b\end{bmatrix}(\tau)\,. \tag{5.21}$$

Proceeding as in (5.14), we extract the projector

$$\mu\cdot T\;+\;\frac{u}{2}T^2\in\mathbf{Z},\quad\text{with}\quad T=(y;\alpha)$$

or more explicitly,

$$\mu'\cdot y\;+\;w\cdot\alpha\;+\;\frac{u}{2}(y^2\;+\;\alpha^2)\in\mathbf{Z} \tag{5.22)1$$

selecting the state $|\bar{\mu};\mu+uT>$ in the sector twisted u times.

The projector is well-defined* iff $\mu' \cdot ny + w \cdot n\alpha \in \mathbf{Z}$ and $\frac{n}{2}(y^2 + \alpha^2) \in \mathbf{Z}$. Remembering that $\mu \in \mathcal{G}$ (5.14), the first condition is satisfied iff $nT = Q$. Finally, both will be satisfied simultaneously iff

$$\begin{aligned} &ny = \varrho, \\ &n\alpha \in \Lambda_{SO(10)}, \\ &\frac{n}{2}\alpha^2 \in \mathbf{Z}, \end{aligned} \tag{5.23}$$

a result consistent with the ones obtained in ref. 15. We could as well include a simultaneous twist $\tilde{\alpha}$ in $\tilde{E}_8$. This would lead to

$$\begin{aligned} &ny = \varrho, \\ &n\alpha \in \Lambda_{SO(10)} \ ; \ n\tilde{\alpha} \in \Lambda_{\tilde{E}_8}, \\ &\frac{n}{2}(y^2 + \alpha^2 + \tilde{\alpha}^2) \in \mathbf{Z}. \end{aligned} \tag{5.24}$$

This case was extensively studied by Font et al.[30] As noted by these authors, the twisted sectors can generate other gauge bosons than the ones projected out from E_6 extending the unbroken group to an "unexpected" one. This occurs when modding out the 3^5 theory with $y = 2(0,0,0,-1,1)$ as seen explicitly in section 5.2.

5.4 General Wilson line in E_6.

To break E_6 more generally, we need to have the Wilson line acting on the $U(1)$ of $E_6 \supset SO(10) \times U(1)$ as well; the latter being nothing else but the diagonal current of the r internal theories.

⋆ By which we mean $\mathbf{P}^2 = \mathbf{P}$, for the projector $\mathbf{P}$.

From the structure of the character (5.4) we immediately see how to proceed: a twist in the $U(1)$ in the time direction (in the path integral language) will be implemented in giving a non-zero value to the parameter z in $\chi(\tau, z, u)$. However, the modular transformation S forces us to define twisted sectors, describing strings that close up to a $U(1)$ gauge transformation: $e^{2i\pi\eta Q}$. Let's pause here and review a few features of the N=2 superconformal algebra:[31,5]

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
[L_m, G_r^{\pm}] &= (\frac{m}{2} - r)G_{m+r}^{\pm} \\
[L_m, J_n] &= -nJ_{m+n} \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0} \\
[J_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm}
\end{aligned}
\quad . \tag{5.25}
$$

This algebra admits a continuous range of realizations depending on the boundary conditions imposed on $G^{\pm}(z)$: $G^{\pm}(e^{2i\pi z}) = e^{2i\pi\eta}G^{\pm}(z)$, $\eta \in [0,1]$. $\eta = 0, 1/2$ correspond, respectively, to R and NS sectors. In general, $G^{\pm}(z) = \sum_n G_n^{\pm} z^{-n-3/2}$, $n \in \mathbf{Z} + 1/2 \pm \eta$.

The different realizations are isomorphic to each other. To a field $f(z)$ in a given realization with charge Q and conformal dimension h, corresponds a field $f^{\eta}(z)$ in the realization "shifted" by η with charge and conformal dimension

$$
\begin{aligned}
Q^{\eta} &= Q \ + \ \eta\frac{c}{3} \\
h^{\eta} &= h \ + \ \eta Q \ + \ \eta^2\frac{c}{6} \, .
\end{aligned} \tag{5.26}
$$

The corresponding character is

$$
\chi_p^{\eta}(\tau, 0, 0) = Tr_{\mathcal{H}_p^{\eta}} e^{2i\pi\tau(L_o - \frac{c}{24})}
$$

$\mathcal{H}_p^\eta$, the representations obtained from $\mathcal{H}$ by the shifted η, are the required twisted sectors. As shown in ref. 32, $\chi_p^\eta(\tau,z,u) = \chi_p(\tau, z+\eta\tau, u-\frac{1}{6}\eta^2\tau c - \frac{1}{3}\eta z c)$. We will define $\chi_p^\eta(\tau,\xi,0)$ as the building blocks corresponding to the fields twisted by $\xi = t\epsilon$ in the time direction in the sector twisted by $\eta = u\epsilon$. Their modular transformation properties are (5.5)

$$\begin{aligned} \chi_p^\eta(\tau+1,\xi,0) &= e^{-i\pi\eta^2\frac{c}{3}}\ \mathcal{T}_{pp}\ \chi_p^{-\xi}(\tau,\eta+\xi,0) \\ \chi_p^\eta(\frac{-1}{\tau},\xi,0) &= e^{2i\pi\xi\eta\frac{c}{3}}\ \mathcal{S}_{pp'}\ \chi_{p'}^{-\xi}(\tau,\eta,0)\ . \end{aligned} \tag{5.27}$$

For generality, let's apply this construction to each internal theory. The procedure is, by now, familiar. Define

$$\mathcal{Z}^l_{\mu_o}\begin{bmatrix} S & a & \boldsymbol{\eta} \\ R & b & \boldsymbol{\xi}\end{bmatrix} = (\ldots)\sum_Q e^{2i\pi R\cdot(\mu+S)}\mathcal{B}_{s_o+V,\bar{s}_o}\begin{bmatrix} a \\ b\end{bmatrix}\prod_i \chi^{\eta_i l_i(s_i)}_{q_i+S_i}(\tau,\xi_i,0)$$

as the piece of the partition function corresponding to a twist $t(y;\alpha;\epsilon_i)$ in the time direction in the sector twisted by $u(y;\alpha;\epsilon_i)$. For simplicity, we arrange the ϵ_i into a vector: $\boldsymbol{\varepsilon} = (2\epsilon_1,\ldots,2\epsilon_r;2\epsilon_1,\ldots,2\epsilon_r;0)$, with which the modular transformation properties take the form:

$$\begin{aligned} \mathcal{Z}^l_{\mu_o}\begin{bmatrix} S & a & \boldsymbol{\eta} \\ R & b & \boldsymbol{\xi}\end{bmatrix}(\tau+1) &= \mathcal{T}_{\mu_o\mu_o}\ e^{-i\pi(S^2+a^2+\boldsymbol{\eta}^2)}\ \mathcal{Z}^l_{\mu_o}\begin{bmatrix} S & a & \boldsymbol{\eta} \\ R+S & a+b & \boldsymbol{\eta}+\boldsymbol{\xi}\end{bmatrix}(\tau) \\ \mathcal{Z}^l_{\mu_o}\begin{bmatrix} S & a & \boldsymbol{\eta} \\ R & b & \boldsymbol{\xi}\end{bmatrix}(\frac{(-1)}{\tau}) &= e^{2i\pi(R\cdot S+a\cdot b+\boldsymbol{\eta}\cdot\boldsymbol{\xi})}\ \mathcal{S}_{\mu_o\nu_o}\ \mathcal{Z}^l_{\nu_o}\begin{bmatrix} -R & -b & -\boldsymbol{\xi} \\ S & a & \boldsymbol{\eta}\end{bmatrix}(\tau) \end{aligned} \tag{5.28}$$

where $\mu_o,\nu_o \in \mathcal{G}$ and we have used

$$\frac{c_i}{3}\eta_i\xi_i = \frac{\eta_i\xi_i}{2(k_i+2)} + \frac{\eta_i\xi_i}{2} = \boldsymbol{\eta}\cdot\boldsymbol{\xi} \quad \text{and} \quad \frac{c_i}{6}\eta_i^2 = \frac{\boldsymbol{\eta}\cdot\boldsymbol{\eta}}{2}\ .$$

The complete partition function describing a theory with such a twist is

$$\mathcal{Z}(\bar{\tau},\tau) = (\ldots)\sum_{t,u} e^{-i\pi(R\cdot S+a\cdot b+\xi\cdot\eta)} \sum_{\substack{\mu_o\in\mathcal{G}\\ \mu_o=\bar{\mu}_o}} \mathcal{Z}^l_{\bar{\mu}_o}(\bar{\tau})\mathcal{Z}^l_{\mu_o}\begin{bmatrix} S & a & \eta \\ R & b & \xi \end{bmatrix}(\tau) \quad . \tag{5.29}$$

This is the most general partition function we can describe in this context. The projector is

$$\mu\cdot T \;+\; \frac{u}{2}T^2 \in \mathbf{Z}, \quad \text{with} \quad T = (y+\varepsilon;\alpha,\tilde{\alpha})$$

or, explicitly

$$y\cdot\mu' \;+\; \epsilon_i Q_i \;+\; \alpha\cdot w \;+\; \tilde{\alpha}\cdot\tilde{w} \;+\; \frac{u}{2}(y^2+\frac{c_i}{3}\epsilon_i^2+\alpha^2+\tilde{\alpha}^2) \in \mathbf{Z} \tag{5.30}$$

projecting out the primary states $|\bar{\mu};(\mu'+uy)^{u\varepsilon};w+u\alpha,\tilde{w}+u\tilde{\alpha}>$. The consistency conditions are, as before:

$$nT\cdot\mu \in \mathbf{Z}$$
$$\frac{n}{2}T\cdot T \in \mathbf{Z}\ .$$

They are solved by

$$\begin{aligned} &ny = \varrho, \qquad y\ .\ \beta \in \mathbf{Z} \\ &n(\varepsilon;\alpha,\tilde{\alpha}) = Q \\ &\frac{n}{2}(\varepsilon^2+\alpha^2+\tilde{\alpha}^2) \in \mathbf{Z} \end{aligned} \tag{5.31}$$

where we have also included a twist in $\tilde{E}_8$.

What we have shown is how to embed a Wilson line into $\tilde{E}_8\times E_6\times U(1)^{r-1}$. For our purpose, we consider a twist in $\tilde{E}_8\times E_6$ only. The shift that breaks E_6 into a group H containing the standard model can be written in general[33,34]

$\delta = (-c, c, a, b, c, 0)^{dual}$; $na, nb, nc \in \mathbf{Z}$. We wish to express it in the basis that occurs naturally in our construction, namely, the one where every weight of E_6 is written as a weight of $SO(10)$ supplemented with the charge Q of the internal $U(1)$. Let $e_i = (0, ..., 1^{ith}, ..., 0)$, $i = 1, ..., 5$, be the vector basis of $SO(10)$. Any weight of E_6 can be written $g_i e_i + Q\sqrt{\frac{3}{c}}K$, with K a unit vector corresponding to the $U(1)$ axis.* For example,

$$
\begin{aligned}
45^0 &\equiv \pm e_i \pm e_j \\
16^{-3/2} &\equiv \frac{1}{2}(\pm e_1 \pm ... \pm e_5) - \frac{\sqrt{3}}{2}K \quad \text{odd \# of } + , \\
\overline{16}^{3/2} &\equiv \frac{1}{2}(\mp e_1 \mp ... \mp e_5) + \frac{\sqrt{3}}{2}K \quad \text{even \# of } + .
\end{aligned}
$$

In this basis, $\delta = \alpha_i e_i + \sqrt{3}\epsilon K$; with $\alpha_1 = -\alpha_2 = -\alpha_5 = \frac{b+c}{2}$, $\alpha_3 = -\alpha_4 = a - \frac{b+c}{2}$ and $\epsilon = \frac{b-3c}{2}$; ϵ being the common value of the ϵ_i 's we considered above. With such a setting, conditions (5.31) are satisfied when:

$$
\begin{aligned}
&ny = \varrho \qquad y.\beta \in \mathbf{Z}, \\
&na, nb, nc \in \mathbf{Z} \quad n\tilde{\alpha} \in \Lambda_{\tilde{E}_8}, \\
&\frac{n}{2}(\delta \cdot \delta \ + \ \tilde{\alpha}^2) \in \mathbf{Z}
\end{aligned}
\tag{5.32}
$$

where $\delta \cdot \delta = 2(4c^2 + b^2 + a^2 - cb - ba - ac)$.

The projector is given by

$$
y \cdot \mu \ + \ \alpha \cdot w \ + \ \tilde{\alpha} \cdot \tilde{w} \ + \ \epsilon Q \ + \ \frac{u}{2}(y^2 \ + \ \tilde{\alpha}^2 \ + \ \delta^2) \in \mathbf{Z} \tag{5.33}
$$

* As before, c is the central charge of the internal theory (= 9).

selecting the primary state $|\bar{\mu};(\mu'+uy)^{u\epsilon};w+u\alpha,\tilde{w}+u\tilde{\alpha}>$ with ΠΙmass satisfying

$$\frac{M^2}{8}=h(\tilde{\mu}')+\bar{N}-1/2$$
$$\frac{M^2}{8}=h(\mu'+uy)+\frac{(w+u\alpha)^2}{2}+\frac{(\tilde{w}+u\tilde{\alpha})^2}{2}+u\epsilon Q+\frac{3}{2}(u\epsilon)^2+N-1 \tag{5.34}$$

in the sector twisted u times.

6. Results and conclusion.

As explained in section 5.4, each set of parameters (a, b, c) characterizes a Wilson line which breaks E_6 into a gauge group containing the standard group $SU(3)_c \times SU(2)_w \times U(1)_Y$. The latter is embedded into E_6 in a general but well defined way, which means that the quantum numbers of each state are uniquely assigned.[33] In particular the electric charge is represented by the axis $\frac{1}{3}(3, -2, 3, -3, 2, -2)^{dynkin}$ in the Dynkin basis and the electric charge of the state with weight $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)^{dual}$ is $Q_{em} = \frac{1}{3}(3\alpha - 2\beta + 3\gamma - 3\delta + 2\epsilon - 2\zeta)$ times e the charge of the electron. A simple way of understanding why a twisted sector contains fractionally charged states[14,5] is to realize that the corresponding ground state is created out of the ground state of the untwisted sector, acting with a twist field whose weight is precisely given by $(-c, c, a, b, c, 0)^{dual}$; consequently it carries an electric charge $Q_{em} = (a - b - c)e$ and every state in the sector has its electric charge shifted by the same amount. As an example, consider the Z_7 orbifold. This orbifold is obtained by dividing out the six-dimensional torus with the Z_7 group generated by $z_i \rightarrow e^{2i\pi\theta_i} z_i$, where z_i, $i = 1, 2, 3$, are complex coordinates on the torus and $\theta_i = (\frac{1}{7}, \frac{2}{7}, -\frac{3}{7})$. The embedding into the gauge group is made using $(a, b, c) = (\frac{2}{7}, \frac{3}{7}, \frac{1}{7})$ and $\tilde{\alpha} = (\frac{1}{7}, \frac{1}{7}, 0, ..., 0)$ breaking E_6 into $SU(3)_c \times SU(2)_w \times SU(2)_r \times U(1)^2$. Using formulas similar to (5.33) and (5.34), we find, as an example, the massless representations $(\mathbf{1_c}, \mathbf{1_w}, \mathbf{1_r})$, $(\mathbf{1_c}, \mathbf{2_w}, \mathbf{1_r})$ and $(\mathbf{3_c}, \mathbf{2_w}, \mathbf{1_r})$ in the singly twisted sector and the twisted ground state carries the electric charge $-\frac{2}{7}e$.

Even though the results of section 5.4 (equations (5.33) and (5.34)) are quite simple, the construction of the massless spectrum is a tedious task. We designed a computer program that does it for us. The inputs are the parameters of the minimal models and the Z_n twist T:$(y; a, b, c; \tilde{\alpha})$, the output is the spectrum of the theory. We consider the 3^5 theory described earlier, modded out by the elements $y_1 = 2(0,1,2,3,4)$ and $y_2 = 2(0,0,0,1,-1)$ coupled to different gauge embeddings leading to theories $\frac{3^5}{Z_5}$ with, respectively, four and forty-four generations and a variety of unbroken gauge groups. We also consider the former case further in modding out with the Z_5 element $i \to i+1,\ i = 1, \ldots, 5$, corresponding to a cyclic permutation of the five internal minimal models. This leads to a four generation model.

6.1 RESULTS.

The results are presented in tables 6.1 and 6.2. A large number of states occur in the twisted sectors. Though we haven't scanned all the possible Wilson lines, we haven't found one which doesn't lead to massless fractionally charged states.

In these tables, we also mention when new gauge bosons appear in the twisted sectors, leading to a larger gauge group than the one expected from simply breaking $E_6 \times \tilde{E}_8$. This extended gauge symmetry is another confirmation that one is at a special point in the moduli space and is expected to be broken while moving away from this point.

Now we describe in some detail the twisted spectrum of two four generation models T_1 and T_2, both obtained from the $\frac{3^5}{Z_5 \times Z_5}$ theory acted on with the Wilson lines

$$W_1 \equiv (a,b,c)_1 = \frac{1}{5}(-2,-3,-1) \ ; \ \tilde{\alpha}_1 = \frac{1}{5}(2,2,0,0,0,0,0,0)$$
$$W_2 \equiv (a,b,c)_2 = \frac{1}{5}(1,0,-1) \qquad ; \ \tilde{\alpha}_2 = \frac{1}{5}(2,2,0,0,0,0,0,0)$$

for which

$$\alpha_1 = \frac{1}{5}(-2,2,0,0,2) \qquad ; \ \epsilon = 0$$
$$\alpha_2 = \frac{1}{10}(-1,1,3,-3,1) \ ; \ \epsilon = \frac{3}{10}.$$

The gauge group is in both cases $SU(3)_c \times SU(2)_w \times SU(2)_r \times U(1) \times U(1)' \times \tilde{U}(1) \times \tilde{E}_7$.* The two theories differ only in the way the hypercharge is distributed among I_r^3 and $U(1)$. The purpose is to illustrate a case where the Wilson line is completely within the $SO(10)$ subgroup of E_6 (W_1) and a case where it is not (W_2), in order to check the consistency of the formalism of section 5.4. For completeness, we give the directions of I_r^3 and of the $U(1)$'s in the dual basis:†

$$W_1 \equiv I_r^3 = (0,0,\frac{1}{2},0,0,0)^{dual}$$
$$U = (-1,1,2,3,1,0,)^{dual}$$
$$U' = (1,-1,0,1,-1,0)^{dual} = 2\sqrt{3}K$$
$$W_2 \equiv I_r^3 = (0,0,0,\frac{1}{2},0,0)^{dual}$$
$$U = (-1,1,-1,0,1,0)^{dual}$$
$$U' = \frac{-1}{2}(-1,1,3,2,1,0)^{dual}$$

⋆ By convention $U(1)'$ is taken orthogonal to the hypercharge.

†

- The assignments for $SU(3)_c \times SU(2)_w \times U(1)_Y$ are the standard ones given in ref.33.
- K was introduced in section 3.

- In the $\frac{3^5}{Z_5}$ we have six $U(1)$'s, five coming from the five currents of the minimal models. However, here, only the diagonal one survives the projection by the second Z_5.

The spectra are shown in tables 6.3 and 6.4.‡ Some further comments are in order.

1. The labelling of the states is a little different from before; the correspondence is $|\bar{\mu};(\mu+uy)^{u\epsilon};w+u\alpha;\tilde{w}+u\tilde{\alpha}>\equiv\prod l_{i q_i}^{\bar{q}_i \bar{s}_i}{}_{s_i}(w_1,...,w_5)$; $\tilde{w}$ turns out to be $(0,...,0)$.

2. Before modding out with $i\rightarrow i+1$, states appear in groups of five, obtained from one another by the cyclic permutation $i\rightarrow i+1$. This is so, because the latter is the only permutation symmetry surviving the modding out by y_1. As far as the $\frac{3^5}{Z_5}$ theory is concerned they are different states but, in the $\frac{3^5}{Z_5\times Z_5}$ theory, only the diagonal superposition of all five states survive the projection. In tables 6.3 and 6.4, we write one of these states but it is understood that we are in the four generation model.

3. Finally, only one helicity is represented, the other being the CPT conjugate.

As seen from tables 6.3 and 6.4, states appear in generation-anti-generation pairs. This is expected for, as mentioned earlier the discrete symmetry we are modding out with acts freely and consequently doesn't affect the Euler characteristic of the underlying manifold. Notice that the right quantum numbers are the same for the generation and the anti-generation, except that they are in opposite order. The intimate connection between the two theories appears in an elegant manner. A one-to-one correspondence exists between the states of a given representation; the right quantum numbers are identical and the left ones, if not identical, are obtained from one another by shifting with one or two β (and with the opposite sign for the anti-generations). This is not a surprise because we know

‡ The scalars only.

that β ($= (\beta', \tilde{S})$) on the left corresponds to a gauge boson in the $\mathbf{16}^{-3/2}$ and is what we need to transform W_1, which is completely in $SO(10)$, into W_2, which is not.

$5 \times (a,b,c)$	$\tilde{\alpha}.\tilde{\alpha}$	unbroken gauge group	*#of states* $\frac{3^5}{Z_5}$	*#of states* $\frac{3^5}{Z_5 \times Z_5}$
$(2,4,1)$	0	$SU(5) \times SU(2)_w \times U(1)$	160	32
$(1,-2,-1)$	0	$SU(5) \times SU(2)' \times U(1)$	120	24
$(-1,3,0)$	4/25	$SU(3)_c \times SU(3)_w \times U(1)^2$	90	18
$(1,1,1)$	4/25	$SU(3)_c \times SU(2)_w \times SU(3) \times U(1)$	90	18
$(-1,1,0)$	4/25	$SU(3)_c \times SU(3)_w \times SU(2) \times U(1)$	90	18
$(2,3,1)$	8/25	$SU(3)_c \times SU(2)_w \times SU(2)_r \times U(1)^2$	50	10
$(1,0,-1)$	8/25	$SU(3)_c \times SU(2)_w \times SU(2)_r \times U(1)^2$	50	10
$(-2,0,-1)$	8/25	$SU(3)_c \times SU(2)_w \times SU(2)_r \times U(1)^2$	50	10

Table 6.1. 3^5 theory modded out with y_1. Number of massless states the singly twisted sector.

$5 \times (a,b,c)$	$\tilde{\alpha}.\tilde{\alpha}$	unbroken gauge group	*#of states* $\frac{3^5}{Z_5}$	*new gauge bosons*
$(2,0,-1)$	0	$SU(5) \times SU(2)_w \times U(1)$	578	5
$(-1,3,0)$	4/25	$SU(3)_c \times SU(3)_w \times U(1)^2$	> 300	0
$(1,1,1)$	4/25	$SU(3)_c \times SU(2)_w \times SU(3) \times U(1)$	372	0
$(-1,1,0)$	4/25	$SU(3)_c \times SU(3)_w \times SU(2) \times U(1)$	360	0
$(2,3,1)$	8/25	$SU(3)_c \times SU(2)_w \times SU(2)_r \times U(1)^2$	240	1

Table 6.2. 3^5 theory modded out with y_2. Number of massless states the singly twisted sector.

u	$\prod l_{i\,q_i s_i}^{\bar q_i \bar s_i}(w_1,...,w_5)$	$(c,w,r)^{U,U'}$	Y	Q_{em}
1	$0^{00}_{-1-1}0^{00}_{11}1^{-10}_{21}2^{-20}_{31}2^{-20}_{51}\left\{\begin{smallmatrix}(+--+-)\\(+-+--)\end{smallmatrix}\right.$	$(1,1,2)^{.6,-.5}$	$\left\{\begin{smallmatrix}-1.2\\.8\end{smallmatrix}\right.$	$\left\{\begin{smallmatrix}-.6\\.4\end{smallmatrix}\right.$
-1	$2^{-20}_{-5-1}2^{-20}_{31}1^{-10}_{21}0^{00}_{11}0^{00}_{-1-1}\left\{\begin{smallmatrix}(-+-++)\\(-++-+)\end{smallmatrix}\right.$	$(1,1,2)^{-.6,.5}$	$\left\{\begin{smallmatrix}-.8\\1.2\end{smallmatrix}\right.$	$\left\{\begin{smallmatrix}-.4\\.6\end{smallmatrix}\right.$
1	$0^{00}_{11}1^{-10}_{21}2^{-20}_{31}2^{-20}_{-5-1}0^{00}_{-1-1}\left\{\begin{smallmatrix}(+----)\\(+-++-)\end{smallmatrix}\right.$	$(1,2,1)^{.6,.5}$	$\left\{\begin{smallmatrix}-.2\\-.2\end{smallmatrix}\right.$	$\left\{\begin{smallmatrix}-.6\\.4\end{smallmatrix}\right.$
-1	$2^{-20}_{31}1^{-10}_{21}0^{00}_{11}0^{00}_{-1-1}2^{-20}_{51}\left\{\begin{smallmatrix}(-+--+)\\(-++++)\end{smallmatrix}\right.$	$(1,2,1)^{-.6,-.5}$	$\left\{\begin{smallmatrix}.2\\.2\end{smallmatrix}\right.$	$\left\{\begin{smallmatrix}-.4\\.6\end{smallmatrix}\right.$
1	$0^{00}_{00}1^{-10}_{10}1^{-10}_{32}2^{-20}_{42}1^{-10}_{-3-2}(00000)$	$(1,1,1)^{-2.4,0}$	.8	.4
-1	$2^{-20}_{42}1^{-10}_{32}1^{-10}_{10}0^{00}_{00}1^{-10}_{-3-2}(00000)$	$(1,1,1)^{2.4,0}$	−.8	−.4
2	$1^{-10}_{10}2^{-20}_{42}0^{00}_{00}1^{-10}_{32}1^{-10}_{-3-2}\left\{\begin{smallmatrix}(0-100-1)\\(1-1000)\\(1000-1)\end{smallmatrix}\right.$	$(3,1,1)^{-.8,0}$	$\frac{.8}{3}$	$\frac{.4}{3}$
-2	$1^{-10}_{32}0^{00}_{00}2^{-20}_{42}1^{-10}_{10}1^{-10}_{-3-2}\left\{\begin{smallmatrix}(-10001)\\(-11000)\\(01001)\end{smallmatrix}\right.$	$(3,1,1)^{.8,0}$	$-\frac{.8}{3}$	$-\frac{.4}{3}$
2	$0^{00}_{11}2^{-20}_{31}0^{00}_{-1-1}1^{-10}_{21}2^{-20}_{51}\left\{\begin{smallmatrix}(+--+-)\\(+-+--)\end{smallmatrix}\right.$	$(1,1,2)^{-1.8,-.5}$	$\left\{\begin{smallmatrix}-.4\\1.6\end{smallmatrix}\right.$	$\left\{\begin{smallmatrix}-.2\\.8\end{smallmatrix}\right.$
-2	$2^{-20}_{31}0^{00}_{11}2^{-20}_{-5-1}1^{-10}_{21}0^{00}_{-1-1}\left\{\begin{smallmatrix}(-+-++)\\(-++-+)\end{smallmatrix}\right.$	$(1,1,2)^{1.8,.5}$	$\left\{\begin{smallmatrix}-1.6\\.4\end{smallmatrix}\right.$	$\left\{\begin{smallmatrix}-.8\\.2\end{smallmatrix}\right.$
2	$1^{-10}_{21}2^{-20}_{-5-1}0^{00}_{11}2^{-20}_{31}0^{00}_{-1-1}\left\{\begin{smallmatrix}(+----)\\(+-++-)\end{smallmatrix}\right.$	$(1,2,1)^{-1.8,.5}$	$\left\{\begin{smallmatrix}.6\\.6\end{smallmatrix}\right.$	$\left\{\begin{smallmatrix}-.2\\.8\end{smallmatrix}\right.$
-2	$1^{-10}_{21}0^{00}_{-1-1}2^{-20}_{31}0^{00}_{11}2^{-20}_{51}\left\{\begin{smallmatrix}(-+--+)\\(-++++)\end{smallmatrix}\right.$	$(1,2,1)^{1.8,-.5}$	$\left\{\begin{smallmatrix}-.6\\-.6\end{smallmatrix}\right.$	$\left\{\begin{smallmatrix}-.8\\.2\end{smallmatrix}\right.$
2	$0^{00}_{22}2^{-20}_{42}1^{-10}_{-10}1^{-10}_{32}1^{-10}_{-3-2}(1-100-1)$	$(1,1,1)^{1.2,1}$	−.4	−.2
-2	$2^{-20}_{20}0^{00}_{00}1^{-10}_{52}1^{-10}_{10}1^{-10}_{-3-2}(-11001)$	$(1,1,1)^{-1.2,-1}$	.4	.2
2	$1^{-10}_{10}1^{-10}_{52}0^{00}_{00}2^{-20}_{20}1^{-10}_{-3-2}(1-100-1)$	$(1,1,1)^{1.2,-1}$	−.4	−.2
-2	$1^{-10}_{32}1^{-10}_{-10}2^{-20}_{42}0^{00}_{22}1^{-10}_{-3-2}(-11001)$	$(1,1,1)^{-1.2,1}$	.4	.2
2	$0^{00}_{00}1^{-10}_{32}3^{-30}_{52}0^{00}_{22}1^{-10}_{52}(1-100-1)$	$(1,1,1)^{1.2,1}$	−.4	−.2
-2	$1^{-10}_{32}0^{00}_{00}3^{-30}_{30}1^{-10}_{10}0^{00}_{-2-2}(-11001)$	$(1,1,1)^{-1.2,-1}$	.4	.2
2	$0^{00}_{00}1^{-10}_{32}0^{00}_{-2-2}1^{-10}_{10}3^{-30}_{30}(1-100-1)$	$(1,1,1)^{1.2,-1}$	−.4	−.2
-2	$1^{-10}_{32}0^{00}_{00}1^{-10}_{52}0^{00}_{22}3^{-30}_{52}(-11001)$	$(1,1,1)^{-1.2,1}$	.4	.2

Table 6.3. Massless spectrum in the twisted sectors of theory T_1.

u	$\prod l_{i\,q_i s_i}^{\,\bar{q}_i \bar{s}_i}(w_1,...,w_5)$	$(c,w,r)^{U,U'}$	Y	Q_{em}
1	$\left\{\begin{array}{c} 0^{00}_{-2-2}0^{00}_{00}1^{-10}_{10}2^{-20}_{20}2^{-20}_{40}(00000) \\ 0^{00}_{-1-1}0^{00}_{11}1^{-10}_{21}2^{-20}_{31}2^{-20}_{51}(+--+-) \end{array}\right.$	$(1,1,2)^{.6,-.5}$	$\left\{\begin{array}{c} .8 \\ -1.2 \end{array}\right.$	$\left\{\begin{array}{c} .4 \\ -.6 \end{array}\right.$
-1	$\left\{\begin{array}{c} 2^{-20}_{-5-1}2^{-20}_{31}1^{-10}_{21}0^{00}_{11}0^{00}_{-1-1}(-++-+) \\ 2^{-20}_{-40}2^{-20}_{42}1^{-10}_{32}0^{00}_{22}0^{00}_{00}(00000) \end{array}\right.$	$(1,1,2)^{-.6,.5}$	$\left\{\begin{array}{c} 1.2 \\ -.8 \end{array}\right.$	$\left\{\begin{array}{c} .6 \\ -.4 \end{array}\right.$
1	$0^{00}_{00}1^{-10}_{10}2^{-20}_{20}2^{-20}_{42}0^{00}_{-2-2}\left\{\begin{array}{c} (00-10) \\ (00010) \end{array}\right.$	$(1,2,1)^{.6,.5}$	$\left\{\begin{array}{c} -.2 \\ -.2 \end{array}\right.$	$\left\{\begin{array}{c} -.6 \\ .4 \end{array}\right.$
-1	$2^{-20}_{42}1^{-10}_{32}0^{00}_{22}0^{00}_{00}2^{-20}_{-4-2}\left\{\begin{array}{c} (000-10) \\ (00100) \end{array}\right.$	$(1,2,1)^{-.6,-.5}$	$\left\{\begin{array}{c} .2 \\ .2 \end{array}\right.$	$\left\{\begin{array}{c} -.4 \\ .6 \end{array}\right.$
1	$0^{00}_{00}1^{-10}_{10}1^{-10}_{32}2^{-20}_{42}1^{-10}_{-3-2}(00000)$	$(1,1,1)^{-2.4,0}$	.8	.4
-1	$2^{-20}_{42}1^{-10}_{32}1^{-10}_{10}0^{00}_{00}1^{-10}_{-3-2}(00000)$	$(1,1,1)^{2.4,0}$	−.8	−.4
2	$1^{-10}_{0-1}2^{-20}_{31}0^{00}_{-1-1}1^{-10}_{21}1^{-10}_{-4-3}\left\{\begin{array}{c} (---+-) \\ (+--++) \\ (++-+-) \end{array}\right.$	$(3,1,1)^{-.8,0}$	$\frac{.8}{3}$	$\frac{.4}{3}$
-2	$1^{-10}_{43}0^{00}_{11}2^{-20}_{-5-1}1^{-10}_{21}1^{-10}_{-2-1}\left\{\begin{array}{c} (--+-+) \\ (-++--) \\ (+++-+) \end{array}\right.$	$(3,1,1)^{.8,0}$	$-\frac{.8}{3}$	$-\frac{.4}{3}$
2	$\left\{\begin{array}{c} 0^{00}_{00}2^{-20}_{20}0^{00}_{-2-2}1^{-10}_{10}2^{-20}_{40}(00000) \\ 0^{00}_{11}2^{-20}_{31}0^{00}_{-1-1}1^{-10}_{21}2^{-20}_{51}(+--+-) \end{array}\right.$	$(1,1,2)^{-1.8,-.5}$	$\left\{\begin{array}{c} 1.6 \\ -.4 \end{array}\right.$	$\left\{\begin{array}{c} .8 \\ -.2 \end{array}\right.$
-2	$\left\{\begin{array}{c} 2^{-20}_{31}0^{00}_{11}2^{-20}_{-5-1}1^{-10}_{21}0^{00}_{-1-1}(-++-+) \\ 2^{-20}_{42}0^{00}_{22}2^{-20}_{-40}1^{-10}_{32}0^{00}_{00}(00000) \end{array}\right.$	$(1,1,2)^{1.8,.5}$	$\left\{\begin{array}{c} .4 \\ -1.6 \end{array}\right.$	$\left\{\begin{array}{c} .2 \\ -.8 \end{array}\right.$
2	$1^{-10}_{10}2^{-20}_{42}0^{00}_{00}2^{-20}_{20}0^{00}_{-2-2}\left\{\begin{array}{c} (00-100) \\ (00010) \end{array}\right.$	$(1,2,1)^{-1.8,.5}$	$\left\{\begin{array}{c} .6 \\ .6 \end{array}\right.$	$\left\{\begin{array}{c} -.2 \\ .8 \end{array}\right.$
-2	$1^{-10}_{32}0^{00}_{00}2^{-20}_{42}0^{00}_{22}2^{-20}_{-4-2}\left\{\begin{array}{c} (000-10) \\ (00100) \end{array}\right.$	$(1,2,1)^{1.8,-.5}$	$\left\{\begin{array}{c} -.6 \\ -.6 \end{array}\right.$	$\left\{\begin{array}{c} -.8 \\ .2 \end{array}\right.$
2	$0^{00}_{00}2^{-20}_{20}1^{-10}_{-3-2}1^{-10}_{10}1^{-10}_{50}(00-110)$	$(1,1,1)^{1.2,1}$	−.4	−.2
-2	$2^{-20}_{42}0^{00}_{22}1^{-10}_{-30}1^{-10}_{32}1^{-10}_{-10}(001-10)$	$(1,1,1)^{-1.2,-1}$	.4	.2
2	$1^{-10}_{0-1}1^{-10}_{41}0^{00}_{-1-1}2^{-20}_{1-1}1^{-10}_{-4-3}(+--+-)$	$(1,1,1)^{1.2,-1}$	−.4	−.2
-2	$1^{-10}_{43}1^{-10}_{01}2^{-20}_{-5-1}0^{00}_{33}1^{-10}_{-2-1}(-++-+)$	$(1,1,1)^{-1.2,1}$	.4	.2
2	$0^{00}_{-2-2}1^{-10}_{10}3^{-30}_{30}0^{00}_{00}1^{-10}_{30}(00-110)$	$(1,1,1)^{1.2,1}$	−.4	−.2
-2	$1^{-10}_{-50}0^{00}_{22}3^{-30}_{52}1^{-10}_{32}0^{00}_{00}(001-10)$	$(1,1,1)^{-1.2,-1}$	.4	.2
2	$0^{00}_{-1-1}1^{-10}_{21}0^{00}_{-3-3}1^{-10}_{0-1}3^{-30}_{2-1}(+--+-)$	$(1,1,1)^{1.2,-1}$	−.4	−.2
-2	$1^{-10}_{43}0^{00}_{11}1^{-10}_{-4-1}0^{00}_{33}3^{-30}_{-4-1}(-++-+)$	$(1,1,1)^{-1.2,1}$	.4	.2

Table 6.4. Massless spectrum in the twisted sectors of theory T_2.

6.2 CONCLUSION.

To conclude we wish to show how the generation-anti-generation pairs gain a mass while moving away from Gepner's point in the moduli space of the corresponding Calabi-Yau manifold. This is expected since, as explained earlier,[10] the states are massive at infinite radius. We consider, in particular, giving an expectation value to the gauge singlet field R, whose internal quantum numbers are[4]

$$X\ X\ X\ X\ Z\ +\ \textit{cyclic permut.} \qquad ;\ X\ =\ 1_{10}^{-10} \quad Z\ =\ 1_{12}^{-10}.$$

We are looking for a non zero coupling of the form $< \mathcal{G}\overline{\mathcal{G}}R^n >$ where $\mathcal{G}, \overline{\mathcal{G}}$ is the pair of states considered. It suffices to find those couplings for the theory T_1 because of the mapping between T_1 and T_2 described in the last section.

We found that all the pairs gain mass through $< \mathcal{G}\overline{\mathcal{G}}R >$ except the last two, which gain mass through $< \mathcal{G}\overline{\mathcal{G}}R^2 >$. To allow the reader to easily check this result, we have written in tables three and four the five sets of internal quantum numbers of each state of a pair in a permutation symmetry configuration such that the couplings are $< \mathcal{G}\overline{\mathcal{G}}(X\ X\ X\ X\ Z) >$ and $< \mathcal{G}\overline{\mathcal{G}}(X\ X\ X\ X\ Z)(X\ X\ Z\ X\ X) >$ respectively.

As an example, let's consider the first and last pairs of table 6.3.

$$
\begin{array}{lcccccccl}
 & \mathcal{G} & 0^{00}_{-1-1} & 0^{00}_{11} & 1^{-10}_{21} & 2^{-20}_{31} & 2^{-20}_{51} & scalar \\
pair\ \#\ 1 & \overline{\mathcal{G}} & 2^{-20}_{-5-1} & 2^{-20}_{31} & 1^{-10}_{21} & 0^{00}_{11} & 0^{00}_{-1-1} & scalar \\
 & R & 1^{-3-2}_{10} & 1^{-3-2}_{10} & 1^{-3-2}_{10} & 1^{-3-2}_{10} & 1^{-3-2}_{12} & F-term
\end{array}
$$

$$
\begin{array}{lccccccll}
 & \mathcal{G} & 0^{00}_{00} & 1^{-10}_{32} & 0^{00}_{-2-2} & 1^{-10}_{10} & 3^{-30}_{30} & scalar & -1\ picture \\
 & \overline{\mathcal{G}} & 1^{-10}_{32} & 0^{00}_{00} & 1^{-10}_{52} & 0^{00}_{22} & 3^{-3-2}_{52} & scalar & 0\ picture \\
pair\ \#\ 2 & R & 1^{-2-1}_{10} & 1^{-2-1}_{10} & 1^{-2-1}_{10} & 1^{-2-1}_{10} & 1^{-2-1}_{12} & fermion & -\frac{1}{2}\ picture \\
 & R & 1^{-2-1}_{10} & 1^{-2-1}_{10} & 1^{-2-1}_{12} & 1^{-2-1}_{10} & 1^{-2-1}_{10} & fermion & -\frac{1}{2}\ picture
\end{array}
$$

We went from the scalar to the fermion to the F-term by successive spacetime supersymmetry transformations,[4] which amount in the internal sector to a shift by $\bar{\beta}' = (0^{-1-1}_{00})^5$. The 0 picture is obtained from the -1 picture by applying $G^+_{-\frac{1}{2}}$ to the internal part of the scalar field:[36]

$$G^+_{-\frac{1}{2}} \prod_i l_{i\,q_i s_i}^{\bar{q}_i \bar{s}_i} = \sum_{j=1}^{r} l_{1\,q_1 s_1}^{\bar{q}_1 \bar{s}_1} \ldots l_{j\,q_j s_j}^{\bar{q}_j \bar{s}_j - 2} \ldots l_{5\,q_5 s_5}^{\bar{q}_5 \bar{s}_5} \;;$$

only one element of the sum contributes non trivially to the coupling, it is the one written above. These couplings can be computed exactly,[4] although, for our purpose, it is not necessary to do so. The reasons are the strong constraints imposed on the couplings by the large internal symmetry group of the minimal models. We content ourself to verify the existence of the required couplings in checking that the following selection rules are satisfied:[4,36]

– Conservation of the 5 internal $U(1)$ charges Q_i and $\bar{Q}_i$:

$$\begin{cases} \sum_j \bar{Q}_i^j = 0 \\ \sum_j Q_i^j = 0 \end{cases} .$$

– $SU(2)$ selection rules:*

$$\begin{cases} \sum_j (\bar{q}_i^j - \bar{s}_i^j) = 0 \bmod(k_i) \\ \sum_j (q_i^j - s_i^j) = 0 \bmod(k_i) \end{cases}.$$

This enforces the idea that Gepner compactifications are exceptional points in the moduli space with enhanced symmetries which conspire to produce exotic massless states. Because of these enhanced symmetries, it is plausible that the true ground states of string theory lie at such points. Of course, if we take the cosmological constraints seriously, this must not be the case and avoiding such points can be viewed as a constraint on model building.† As explained earlier, the fractional electric charge of those states is a consequence of the one of the corresponding twisted ground state given in general by $Q_{em} = (a - b - c)e$. An apparent way out is to impose the condition[14,5] $a = b + c \bmod(1)$ which would prevent the occurrence of any fractional electric charge in the theory. This condition is non trivial. It greatly restricts the number of Wilson lines. It has, however, a drawback:[39] it leads to an unbroken $SU(5)$ gauge group containing $SU(3)_c \times SU(2)_w$,[33] which implies leptoquark gauge bosons, leading to fast decay of the proton. It was suggested in ref. 38 that exotic states could be confined under an additional color force provided by the breaking of E_6. A simple counting shows that the rank of the responsible gauge group is at most two, which suggests an $SU(2)$ or an $SU(3)$. Thus this mechanism is able to hide from detection FCS which carry an electric charge multiple of $\frac{e}{2}$ or

⋆ They originate from the correspondence between N=2 superconformal minimal model k_i and an SU(2) algebra at level k_i.[35]

† Similar issues are addressed in ref. 37.

$\frac{e}{3}$. Finally, an alternative is to keep our mind open to the existence of particles with exotic electric charge, in spite of the cosmological constraints.

APPENDIX

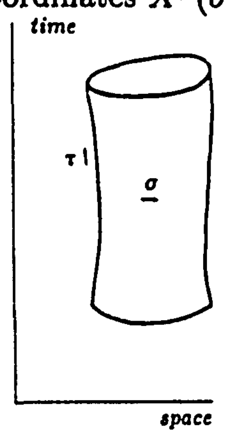
String theory. An overview

A string can be *open* or *closed.* In the present work we are concerned with closed string theories. They appear in three major types: ***bosonic string***, ***superstring*** and ***heterotic string***. A closed string is a closed loop of (presumably) planckian size $\sim 10^{-33}$ cm propagating and waving in spacetime. Its excitations can be split into two independent sets of modes: the ***left–movers*** and the ***right–movers***. After quantization, particles are described as superpositions of such modes with a mass given in proper units by $mass^2 \simeq N_{left} + N_{right} + a$; where N counts the number of excitations and a is a number. A ***string vacuum*** is nothing but a classical solution for the propagation of a string in spacetime.

In this appendix, we go over some selected aspects of the construction of a string vacuum aimed to familiarize the unacquainted reader with the tools and the terminology used in the text. I refer the reader to excellent reviews on the subject, in particular refs. 26, 40,27.

7.1 String theory as a 2-D conformal field theory.

While moving in the spacetime, a closed string sweeps a *worldsheet* on which its coordinates $X^\mu(\sigma, \tau)$ are parametrized by two fictitious worldsheet coordinates



σ and τ. It is clear from such a formulation that a string theory can be described by a $1+1$-dimensional field theory. A suitable action is

$$S[g_{\alpha\beta}, X^\mu] \sim \int d\sigma d\tau \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \quad . \qquad \text{(A.1)}$$

This action is rich in local symmetries:

— Reparametrization invariance :

$$
\begin{aligned}
&\sigma^\alpha \rightarrow \tilde{\sigma}^\beta(\sigma^\alpha) && \sigma^\alpha = (\sigma, \tau) \\
&X^\mu(\sigma^\alpha) = X^\mu(\tilde{\sigma}^\beta) && . \\
&g^{\alpha\beta} \rightarrow \tilde{g}^{\gamma\delta} = \frac{\partial\tilde{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\tilde{\sigma}^\delta}{\partial\sigma^\beta} g^{\alpha\beta}
\end{aligned}
\tag{A.2}
$$

— Weyl invariance :

$$
\begin{aligned}
&\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha = \sigma^\alpha \\
&X^\mu(\sigma^\alpha) = X^\mu(\tilde{\sigma}^\beta) \quad . \\
&g^{\alpha\beta} \rightarrow \tilde{g}^{\alpha\beta} = \Gamma(\sigma) g^{\alpha\beta}
\end{aligned}
$$

This local symmetry reflects the arbitrariness of the parameters σ, τ and some arbitrariness in the choice of the two–dimensional metric $g_{\alpha\beta}$. A gauge fixing procedure can be implemented by choosing $g_{\alpha\beta}$ to be $h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which leaves us with a restricted class of local symmetries:

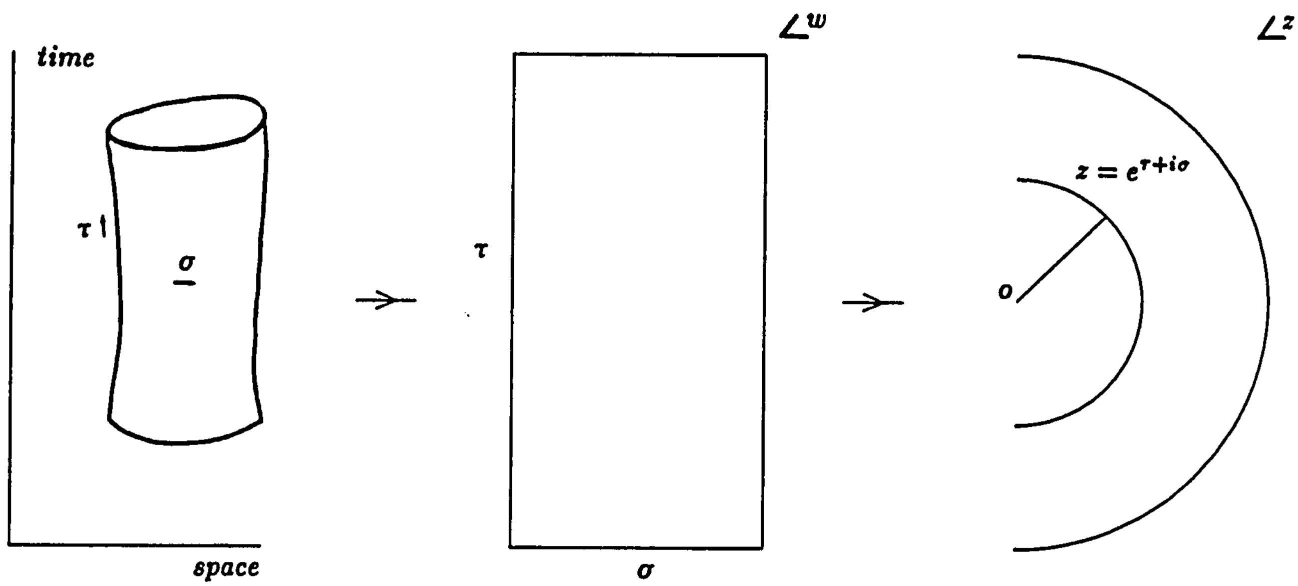
$$
\left\{ \begin{array}{c} \sigma^\alpha \rightarrow \tilde{\sigma}^\beta(\sigma^\alpha) \\ g^{\alpha\beta} \rightarrow \tilde{g}^{\alpha\beta} = \Gamma(\sigma) g^{\alpha\beta} \end{array} \right. \times \left\{ \begin{array}{c} \tilde{\sigma}^\beta \rightarrow \tilde{\sigma}^\beta \\ \tilde{g}^{\alpha\beta} \rightarrow \Gamma^{-1}(\sigma) \tilde{g}^{\alpha\beta} = g^{\alpha\beta} \end{array} \right. \quad . \tag{A.3}
$$

This subclass of reparametrizations are called *conformal transformations* they rescale the metric by a local factor,* which is then removed by a Weyl rescaling. A field theory with such a property is called a *conformal field theory* (CFT). Perturbations around a string vacuum are described by such a theory.

⋆ The angle between two vectors is left unchanged.

7.2 A SHORT TRIP TO CFT.

A CFT in two dimensions is generally formulated on the complex z-plane. This can be done by the following successive transformations:



A CFT is a strongly restricted theory. It is described by an ∞-dimensional algebra, the Virasoro algebra:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}(n^3 - n)\delta_{n+m,0} \, . \\
[L_n, \bar{L}_m] &= 0
\end{aligned} \tag{A.4}$$

$c(\bar{c})$ is a c-number called the central charge; it takes the value $c(\bar{c})$=1 for one massless scalar field such as $X^\mu(\sigma, \tau)$. The generators are the "Fourier modes"

$L_n, \bar{L}_n$, $n \in \mathbf{Z}$, of the holomorphic and anti-holomorphic components of the stress-energy tensor $T(z)$, $\bar{T}(\bar{z})$:

$$T(z) = \sum_{n=-\infty}^{n=+\infty} L_n z^{-n-2} \quad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{n=+\infty} \bar{L}_n \bar{z}^{-n-2}.$$

L_n generates the infinitesimal transformation $z \longrightarrow z + \epsilon z^{n+1}$; similarly for $\bar{L}_n$. In particular,

$$L_0 + \bar{L}_0 \equiv \quad \begin{aligned} z &\longrightarrow (1+\epsilon)z \\ \bar{z} &\longrightarrow (1+\epsilon)\bar{z} \end{aligned} \tag{A.5}$$

and

$$i(L_0 - \bar{L}_0) \equiv \quad \begin{aligned} z &\longrightarrow (1+i\epsilon)z \\ \bar{z} &\longrightarrow (1-i\epsilon)\bar{z} \end{aligned} \tag{A.6}$$

corresponding respectively to translation in time $\tau \longrightarrow \tau + \epsilon$ and translation in "space" $\sigma \longrightarrow \sigma + \epsilon$ respectively*.The first operator is the *hamiltonian* H the second is the *worldsheet momentum* P and describes the spin of the string.

The quantization of the theory on the z-plane is particularly easy. Time-ordering is replaced by radial-ordering. This is called *radial quantization.* The algebra has a natural decomposition in holomorphic and anti-holomorphic parts. This reflects the decomposition of the excitations along the string in left-movers and right-movers. As a consequence, a general field splits itself into two components $\phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$, which can be separately expanded in Fourier modes:

$$\begin{aligned} \phi(z) &= \sum_{n+h \in \mathbf{Z}} \phi_n z^{n-h} \\ \bar{\phi}(\bar{z}) &= \sum_{n+h \in \mathbf{Z}} \bar{\phi}_n \bar{z}^{n-h} \end{aligned}.$$

$(h, \bar{h})$ are the *conformal dimensions* of the fields. $\phi(z,\bar{z})$ acting on the vacuum

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* As can be seen straightforwardly from the relations $\tau = ln z\bar{z}$ and $2i\sigma = ln\frac{z}{\bar{z}}$.

$|0>$ at $z = \bar{z} = 0$ creates a state $|\phi>$ in the far past ($\tau = -\infty$) with energy $h + \bar{h}$ and spin $h - \bar{h}$. Not every field behaves nicely under conformal transformations. The ones that do so are called *primary fields*; they are characterized by

$$\phi(w, \bar{w}) = \left(\frac{\partial z}{\partial w}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\bar{h}} \phi(z, \bar{z}) \quad .$$

They create *highest weight states* satisfying

$$\begin{aligned} L_n|\phi> &= 0 \quad , n > 0 \\ L_0|\phi> &= h\ |\phi> \end{aligned} \quad .$$

The other states are called *descendants*; they are obtained in acting on the highest weight states with the raising operators L_{-n} which increase the conformal dimension h by n. Each primary field and its infinite tower of descendants define a *conformal family*. Primary fields play a crucial role in the theory for they can be shown to be the only ones describing physical states.† In a pure bosonic theory they ought to have conformal dimension (1,1).‡ For example, $\phi_a \bar{\phi}_b e^{ik^\mu X_\mu}|0>$ has conformal dimensions

$$\begin{aligned} h &= \frac{k^\mu k_\mu}{2} + h_a \\ \bar{h} &= \frac{k^\mu k_\mu}{2} + \bar{h}_b \end{aligned} \quad ,$$

h_a, $\bar{h}_b$ being the conformal dimensions of $\phi_a(z)$ and $\bar{\phi}_b(\bar{z})$ respectively. The condition $(h, \bar{h}) = (1, 1)$ with $k^\mu k_\mu = -m^2$ implies

$$\begin{aligned} \frac{m^2}{2} &= h_a - 1 \\ &= \bar{h}_b - 1 \end{aligned}$$

† This results from a BRST invariance analysis.[26]

‡ In particular, they have spin 0.

or

$$m^2 = h_a + \bar{h}_b - 2, \tag{A.7}$$

a formula advocated in the introduction of this appendix. One also has to impose the condition $L_n|\phi_a >= \bar{L}_n|\bar{\phi}_b >= 0, \quad n > 0$ which provides additional transversality conditions.

7.3 THE BOSONIC STRING.

For the case initially considered we have four free scalar fields $X^\mu(\sigma^\alpha) = X^\mu_{left}(z) + X^\mu_{right}(\bar{z})$ with a total central charge $c = \bar{c} = 4$. However the implementation of gauge–fixing with a Fadeev–Popov procedure introduces additional ghost fields (b,c) which contribute with a central charge $c_{gh} = 26$, altogether yielding a total central charge $c = -22$. This is not satisfying because $c < 0$ doesn't lead to a unitary theory. Furthermore, in the quantum theory, conformal symmetry is anomalous with an anomaly proportional to c. To make the theory anomaly–free, one can add 22 additional free scalar fields $X^i(\sigma^\alpha)$ and interpret them as additional space coordinates extending the dimensionality of the spacetime from four to 26. More generally, one can add to the theory an internal sector described by a general CFT with $c = 22$. The advantage is that spacetime remains four–dimensional; the disadvantage is that there are additional degrees of freedom propagating along the string. However, after some thoughts, this is welcome for we have to account for the full structure of the real world, in particular, for chiral fermions and the gauge symmetry of the Standard Model.

7.4 THE SUPERSTRING.

The introduction of spacetime fermions can be implemented by modifying the original action (A.1) to

$$S[g_{\alpha\beta}, X^\mu] \sim \int d\sigma^\alpha \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\Psi}^\mu \gamma^\alpha \partial_\alpha \Psi_\mu + (...) \quad (A.8)$$

where Ψ^μ is a set of free left and right moving Weyl fermions — each of which contributing $c = \frac{1}{2}$ — and γ^α, a set of two–dimensional gamma matrices, (...) are additional terms necessary to implement worldsheet supersymmetry. The conformal algebra is now extended to a superconformal algebra. To fix this extended gauge symmetry, a pair of additional ghost (β, γ) are to be introduced. They increase the central charge of the theory by an amount $c_{super} = +11$. The total central charge is now

$$\begin{aligned} c =& 4(1 + \frac{1}{2}) - 26 + 11 \\ =& -9 \end{aligned} \quad .$$

Quantization, as before, leads to an anomaly which can be cancelled in introducing an internal sector with central charge $c_{int} = 9$.* In four dimensions, such a theory is known to lead to spacetime supersymmetry if and only if the internal sector has an N=2 superconformal symmetry. Because Gepner's models have such a property we will assume it implicitely, from now on. Schematically, we have the following

⋆ Interpretation of the sector as a set of additional pairs of fields (X^μ, Ψ^μ) leads to a spacetime with ten dimensions.

structure

N=2 superconformal field theory $\bar{c}_{int} = c_{int} = 9$	
spacetime sector X^μ_L, Ψ^μ_L c=6	spacetime sector X^μ_R, Ψ^μ_R $\bar{c} = 6$

7.5 THE HETEROTIC STRING.

Such a superstring construction can be shown to be too restrictive to accomodate the gauge group of the Standard Model simultaneously with its fermion content.[41] This motivates for an hybrid version of a bosonic and superstring theories. The construction is shown below.

gauge sector Kac–Moody algebra $c = 13$	
N=2 superconformal field theory $\bar{c}_{int} = c_{int} = 9$	
spacetime sector X^μ_L $c = 4$	spacetime sector X^μ_R, Ψ^μ_R $\bar{c} = 6$

It amounts to replacing the left–moving world sheet fermions $\Psi^\mu_{left}(z)$ with a set of holomorphic fields $j^a(z) = \sum_{n+h \in Z} j^a_n z^{-n-1}$ forming a chiral current algebra called a *Kac-Moody algebra*:

$$[j^a_n, j^b_m] = i f_{abc} j^c_{n+m} + k n \delta^{ab} \delta_{m+n,0} \quad , \qquad \text{(A.9)}$$

k is a c–number called the level of the algebra.

A Kac–Moody algebra is a Lie algebra —generated by j_0^a — supplemented with an infinite number of generators j_n^a, $n \neq 0$ forming an ∞–dimensional affine algebra. It implies the Virasoro algebra (A.4) and is the simplest way of introducing explicitely a gauge structure in the theory. The algebras considered so far have level $k = 1$. There are two candidates making the substitution consistent. They correspond to the Lie Algebras $SO(10) \times \tilde{E}_8$ and $SO(26)$; both contribute to the central charge by $c_{gauge} = 13$. The group $SO(10) \times \tilde{E}_8$ is particularly interesting for the following reason. The internal N=2 superconformal algebra of the internal sector contains a $U(1)$ current which is shown to extend the gauge group from $SO(10) \times \tilde{E}_8$ to $E_6 \times \tilde{E}_8$. Instead of $SO(10)$ and $SO(26)$, E_6 contains complex representations which are necessary to allow spacetime fermions with opposite chirality to be in different gauge representations. In particular, the massless spectrum, in addition to a gravity multiplet, contains the matter fields coming generically in multiplets of the **27** and $\overline{\mathbf{27}}$ representations :

$$\begin{aligned}
\mathbf{27} &= \mathbf{10}^{1} + \mathbf{16}^{-1/2} + \mathbf{1}^{-2} \\
\overline{\mathbf{27}} &= \mathbf{10}^{-1} + \overline{\mathbf{16}}^{1/2} + \mathbf{1}^{2}
\end{aligned}$$

The $\mathbf{16}^{-1/2}$ contains one full generation of quarks and leptons and the $\overline{\mathbf{16}}^{1/2}$ one full anti-generation (opposite chirality). Hence, $\#\mathbf{27} - \#\overline{\mathbf{27}}$ gives us the net number of generations of the theory.

7.6 THE INTERNAL SECTOR.

In addition to have a central charge $c = 9$, the internal sector is submitted to certain constraints. We will mention only two of them.

Unitarity—A basic requirement in a quantum field theory is unitarity of the S–matrix which, in particular, implies that states have positive norm. As a consequence, the internal sector has to be unitary. This is a strong constraint on a CFT with $c < 1$. There is only a discrete series of them — the so–called *minimal series* — which satisfy this requirement. They have central charge

$$c = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}\ ; \quad 1 \leq p \leq m-1,\ 1 \leq q \leq p. \tag{A.10}$$

and each of them contains a finite number of primary fields with conformal dimensions

$$h_{p,q}(m) = 1 - \frac{6}{m(m+1)}, \quad m = 2, 3, 4, \ldots. \tag{A.11}$$

A similar series exists for superconformal field theories. These theories are exactly solvable and are to be combined together to form a CFT with $c = 9$. This is how Gepner proceeded.

Modular invariance—The second constraint is modular invariance. Modular invariance is a powerful constraint on a string theory. Because it is the main tool used in this work, chapter 3 is devoted to it.

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