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ANALYTICAL INVESTIGATION OF WAVE PROPAGATION AND REFLECTIONS IN TIMOSHENKO BEAMS

by

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ABSTRACT

The dynamic response of Timoshenko beams under transverse impact is studied theoretically with particular emphasis on discontinuities and their propagation.

The standard eigenfunction solution to the finite simply supported span subjected to a uniform velocity input is modified by isolating the leading discontinuities into a progressing wave expansion, resulting in improved convergence properties. Methods are developed for obtaining the leading reflected discontinuities for a general mixed boundary value problem involving the Timoshenko equations.

Asymptotic results for large slenderness ratios and times are obtained in the vicinity of the wave fronts by stationary phase and singular perturbation analyses.

Qualitative effects of Maxwell and Kelvin damping of the spring supports and in the material are shown.

The applicability of the elementary Euler Bernoulli theory to the analysis of the above problems is discussed.

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NOMENCLATURE

A	cross-sectional area; also characteristic matrix
A_s	cross-sectional area contributing to resistance to shear
A^x	coefficient matrix
A^t	coefficient matrix
B	coefficient matrix
B'	coefficient matrix
c_1	velocity of propagation of dilatational wave
c_2	velocity of propagation of shear wave
D	coefficient matrix
$\underline{d}$	constant vector
E	modulus of elasticity; also a constant
G	modulus of rigidity
G^ν	matrix of coefficients of progressing wave expansion
$\underline{g}$	vector whose elements are regular functions
H^ν	matrix of regular functions
$\underline{h}$	vector whose elements are regular functions
I	cross-sectional moment of inertia
J	diagonal matrix
K	spring parameter = sl^3/EI
k	index representing characteristic curve
L	linear differential operator --
l	span length
$\underline{l}_k$	left null vector corresponding to k'th characteristic

$\bar{M}$ bending moment

M non-dimensionalized bending moment $= \bar{M}/EAl$

$\tilde{M}$ Fourier transform of the bending moment

N number of terms in the progressing wave expansion

P coefficient matrix

p transformed space variable

Q μxn matrix representing the imposed boundary conditions

$\bar{q}$ lateral load

q non-dimensionalized lateral load; also transformed variable

$\underline{R}$ remainder of progressing wave expansion

r radius of gyration of cross-section; also a constant

$\underline{r}_K$ right null vector corresponding to k'th characteristic

S generalized function

s spring constant

T linear differential operator; also a diagonal matrix

T stretch variable $= \tau/\epsilon$

$\bar{t}$ time

t non-dimensionalized time $= c_2\bar{t}/l$

$\underline{U}$ vector representing dependent variables in canonical form

$\underline{U}'$ finite progressing wave series

$\underline{u}_i$ eigenfunctions

$\underline{u}$ vector representing dependent variables

$\bar{V}$ shear force

V non-dimensionalized shear force $= \bar{V}/A_sG$

$\tilde{V}$	Fourier transform of the shear
$\bar{v}$	vertical velocity
v	non-dimensionalized vertical velocity = $\bar{v}/c_2$
v_o	initial uniform velocity input
$\underline{W}$	vector representing dependent variables
$\bar{x}$	abscissa along beam axis
x	non-dimensionalized abscissa along beam axis = $\bar{x}/l$
$\underline{Y}$	vector representing dependent variables
$\underline{Y}_n$	eigenfunctions for Euler Bernoulli analysis
$\bar{y}$	vertical deflection
y	non-dimensionalized vertical deflection = $\bar{y}/l$
$\tilde{y}$	Fourier transform of vertical deflection
$\underline{Z}$	vector of dependent variables in canonical form
α	spring parameter = sl/A_sG
β_i	wave number
β	torsional spring parameter
γ	ratio of dilatational to shear wave speeds = c_1/c_2
Δ	reciprocal of stress relaxation time for Maxwell damping
δ	Kelvin damping parameter
ϵ	$(l/r)^{-1}$
$\bar{\eta}$	Maxwell damping parameter
η	Kelvin damping parameter
λ	real eigenvalue
λ'	complex eigenvalue
μ	number of required boundary conditions

ν index representing terms in progressing wave expansion

ξ coordinate axis along shear front

ρ density of beam

σ scalar multiplier

τ coordinate along dilatational front

φ characteristic curve

$\bar{\Psi}$ rotation of cross-section about neutarl axis

Ψ non-dimensionalized rotation of cross-section about neutral axis $=\bar{\Psi}(l/r)^{-2}$

$\tilde{\Psi}$ Fourier transform of rotation

$\hat{\Psi}$ Laplace transform of rotation

$\bar{\omega}$ angular velocity of cross-section

ω non-dimensionalized angular velocity of cross-section $= \bar{\omega}r^2/lc_2$

CHAPTER 1

INTRODUCTION

1.a Statement of Problem

This dissertation is a theoretical investigation of the dynamic response of beams under transverse impact with particular emphasis on discontinuities and their propagation. The problem of finite and semi infinite spans resting on spring and rigid supports is studied in detail. The transverse impact, suggested by applications in sonic boom and packaging problems, is imposed in the form of an initial transverse uniform velocity.

1.b Euler Bernoulli vs Timoshenko Analysis

In the analysis of a finite beam subjected to a transverse impact, a strict treatment using the theory of two or three dimensional elasticity is complicated by the stress wave reflections at the boundaries. Approximate models for describing beam behavior are limited to the elementary Euler Bernoulli theory and the Timoshenko bending mechanism.

For mixed boundary value problems such as will be considered in this work, discontinuities arise in the solution when the initial and boundary data do not satisfy certain consistency conditions. In the case of a rigidly supported span subjected to a uniform initial velocity, this lack of consistency at the boundary produces a discontinuity in the solution for the velocity itself. For spring supports, this inconsistency is not as acute and will be shown to produce a continuous solution with a discontinuous first derivative.

The Euler Bernoulli equation not being hyperbolic, is inadequate in analyzing these impact and near impact conditions as it yields the non physical result that waves of infinitesimal wave length and therefore discontinuities propagate with infinite velocity. Prescott[1] showed that this elementary theory yields satisfactory results only for the lower modes, and therefore cannot accurately describe the response of problems which are greatly influenced by induced discontinuities and their propagation.

The Timoshenko bending mechanism while retaining the one dimensional nature of the elementary theory, includes the effects of deformation due to transverse shear and rotary inertia producing a hyperbolic system of partial differential equations.

The Timoshenko equations will therefore be used as the equations governing the dynamic response of the impact problems considered in the present work.

1.c Timoshenko Equations

The response of a non-uniform beam subjected to a dynamic lateral loading $\bar{q}$, according to the Timoshenko bending mechanism is governed by the following four simultaneous partial differential equations

$$\begin{aligned} &\bar{M} + EI\,\bar{\psi}_{\bar{x}} = 0 \\ &\bar{V} - A_s G(\bar{y}_{\bar{x}} - \bar{\psi}) = 0 \\ &\bar{M}_{\bar{x}} - \bar{V} + \rho I\,\bar{\psi}_{\bar{t}\bar{t}} = 0 \\ &\bar{V}_{\bar{x}} - \rho A\,\bar{y}_{\bar{t}\bar{t}} + \bar{q} = 0 \end{aligned} \qquad (1\text{-}1)$$

where $\bar{x}$ is the space coordinate along the beam axis; $\bar{t}$ is time; $\bar{y}$ and $\bar{\psi}$ represent the deflection and rotation of a cross section, positive in the directions shown in figure 1; ρ is the density; A is the cross sectional area; A_s is the effective area in shear; E and G are the moduli of elasticity and rigidity; and I represents the moment of inertia.

The first two equations of (1-1) constitute the relationships between deformations and internal loadings assuming plane sections remain plane while the second two describe rotational and translational equilibrium of a beam element.

Equations (1-1) may be put in a more convenient form for analyzing uniform spans by differentiating the first two equations with respect to time, producing the following system of first order partial differential equations

$$\begin{aligned} &\bar{M}_{\bar{t}} + EI\,\bar{\omega}_{\bar{x}} = 0 \\ &\bar{\omega}_{\bar{t}} + \frac{1}{\rho I}(\bar{M}_{\bar{x}} - \bar{V}) = 0 \\ &\bar{V}_{\bar{t}} - A_s G(\bar{v}_{\bar{x}} + \bar{\omega}) = 0 \\ &\bar{v}_{\bar{t}} - \frac{1}{\rho A}\bar{V}_{\bar{x}} = \bar{q}/\rho A \end{aligned} \qquad (1\text{-}2)$$

where $\bar{v}$ and $\bar{\omega}$ are the lateral and angular velocities respectively.

Placing equations (1-2) in non dimensional form yields

$$\begin{aligned} &M_t + \omega_x = 0 \\ &\omega_t + \gamma^2 M_x - V = 0 \\ &V_t - v_x + (\ell/r)^2 \omega = 0 \\ &v_t - V_x + q = 0 \end{aligned} \qquad (1\text{-}3)$$

where

$$\begin{aligned} x &= \bar{x}/\ell \\ t &= c_2 \bar{t}/\ell \\ M &= \bar{M}/EA\ell \\ \omega &= \bar{\omega} r^2/\ell c_2 \\ V &= \bar{V}/A_s G \\ v &= \bar{v}/c_2 \\ q &= \bar{q}\ell/A_s G \end{aligned}$$

and

$$\begin{aligned} c_2 &= \sqrt{\frac{A_s G}{\rho A}} \\ \gamma &= \sqrt{\frac{EA}{A_s G}} \\ r^2 &= I/A \end{aligned}$$

The parameter γ is the ratio of dilatational to shear wave speeds and ℓ/r represents the ratio of length to radius of gyration of the cross section (slenderness ratio).

All subsequent analyses will be performed on these non dimensionalized equations.

Equations (1-3) may be written in the following first order matrix form

$$\underline{W}_t + A^x \underline{W}_x + B'\underline{W} = \underline{C} \tag{1-4}$$

where A^x and B' are 4x4 matrices with constant coefficients and $\underline{C}$ and $\underline{W}$ are vectors where $\underline{W}$ represents the four independent variables M, ω, V and v.

The hyperbolic structure of equations (1-4) as determined by the solutions to the characteristic equation

$$|A^x - c_k I| = 0 \tag{1-5}$$

reveals the equations to be totally hyperbolic with two modes of wave transmission, having velocities $c_1 = \pm\gamma$ and $c_2 = \pm 1$. c_1 and c_2 represent the velocities of propagation of dilatational and shear waves respectively.

For a uniform beam, the Timoshenko equations may also be expressed as two second order simultaneous partial differential equations with constant coefficients in y and ψ. In non dimensional form they are

$$\begin{aligned} y_{tt} - y_{xx} + \psi_x &= 0 \\ \psi_{tt} - \gamma^2 \psi_{xx} - (\ell/r)^2 (y_x - \psi) &= 0 \end{aligned} \tag{1-6}$$

with

$$\begin{aligned} M &= -\psi_x / (\ell/r)^2 \\ V &= y_x - \psi \end{aligned} \tag{1-7}$$

The initial conditions representing a uniform velocity are expressed by the following

For system (1-6):
$$y(x,0)=0$$
$$\Psi(x,0)=0$$
$$y_t(x,0)=V_0 \tag{1-8}$$
$$\Psi_t(x,0)=0$$

For system (1-4):
$$M(x,0)=0$$
$$\omega(x,0)=0$$
$$V(x,0)=0 \tag{1-9}$$
$$v(x,0)=V_0$$

The boundary conditions for a finite beam impinging against rigid supports are

For system (1-6):

At x=0 and x=1:
$$y=0$$
$$\Psi_x=0 \tag{1-10}$$

For system (1-4):

At x=0 and x=1:
$$M=0$$
$$v=0 \tag{1-11}$$

The case of spring supports for system (1-6) is given by

At x=0:
$$\Psi_x=0$$
$$y_x-\Psi-\alpha y=0$$

At x=1:
$$\Psi_x=0 \tag{1-12}$$
$$y_x-\Psi+\alpha y=0$$

where $\alpha = \frac{s\ell}{A_s G}$ or $\alpha = K\gamma^2/(\ell/r)^2$ and K is related to the spring constant s by the formula $K = s\ell^3/EI$.
The equivalent expressions for system (1-4) require the following modification: In order to express the condition of spring supports as a linear combination of the dependent variables, the auxiliary equation

$$y_t - v = 0 \tag{1-13}$$

is added to system (1-4), resulting in a system of five simultaneous equations. The boundary conditions are now given by

$$\begin{array}{lll} \text{At } x=0: & M=0 & \\ & V-\alpha y=0 & \\ \text{At } x=1: & M=0 & \\ & V+\alpha y=0 & \end{array} \tag{1-14}$$

1.d <u>Past Literature</u>

The following summarizes the various approaches that have been used in analyzing the Timoshenko equations together with a cross section of the problems solved.

Dengler and Goland[2] used Laplace and Fourier transforms to obtain the solution to an infinite beam subjected to an impulsive concentrated load. The solution required the numerical evaluation of the inverse Laplace transform except for the non physical case of coincident velocities of wave propagation where a closed form solution was obtained.

Similar numerical inversions were used by Boley and Chao[3] in studying the response of semi infinite spans to a variety of loading conditions such as step functions of moment and shear.

Zajac[4] used various expansions of the Laplace transform of the solution to a semi infinite beam to obtain term by term inverses with their corresponding regions of validity.

Leonard and Budiansky[5] considered the response of infinite and semi infinite beams to various loadings. Eigenfunction expansions were obtained and shown to smooth over the response in the vicinity of the wave fronts. Numerical integration along the characteristics and in some cases closed form solutions were carried out for the case of equal dilatational and shear wave speeds.

Plass[6] developed a numerical integration technique which accounted for distinct wave speeds and applied it to the solution of a semi infinite beam.

Chuo and Mortimer[7] developed general finite difference schemes which isolated the propagation of discontinuities along characteristics. The accuracy of the procedure decreases with time and becomes laborious when reflections are involved.

Normal mode solutions were developed by Anderson[8] for finite spans with various boundary conditions applied to a simply supported beam subjected to a concentrated loading at mid span.

The method of stationary phaze was used by Jones[9] in obtaining the response of semi infinite spans subjected to point loadings for large times. Asymptotic expressions were obtained for various regions of the space-time plane.

To complete this review, it should be mentioned that the effect of viscous damping mechanisms on the response of the Timoshenko beam has been studied. Ebner and Billington[10] introduced structural damping into the Timoshenko equations and obtained a steady state solution by means of numerical integration. Eigenfunction expansions were developed by Pan[11] for finite spans subjected to Kelvin or Maxwell damping.

A limited amount of experimental work has also been performed. Odaka and Nakahara[12] approximated an impact condition by dropping an elastic bar on a simply supported beam. Measurements of strain propagation yielded results that were in good agreement with the Timoshenko theory and showed the validity of the elementary theory away from the impacted area.

In summary, among the computational methods used, transform techniques generally require numerical inverses, numerical integration schemes become complicated by boundaries and yield little insight into the response, and normal mode solutions do not yield results near wave fronts.

1.e Difficulties in Numerical Calculations

The lack of reasonable convergence of standard normal mode solutions of hyperbolic systems in the vicinity of wave fronts can be directly related to the introduction of high discontinuities, which in the cases studied in this dissertation are due to inconsistent initial and boundary data. Their existence is easily seen from the dependence of the convergence of the standard eigenfunction solution for spring supports developed in section 2.a on the magnitude of the spring constant.

The effect of large values of the slenderness ratio and time proved equally troublesome as they were shown to yield solutions near the fronts with components of high oscillations having significant magnitudes. The coupling of the slenderness ratio and time can be explained through the interpretation of large l/r as a stretching of the span, in effect producing a large time delay between a disturbance and a fixed point.

1.f Methods of Analysis

A primary consideration in determining the methods of analysis used in this dissertation was not only to obtain the actual numerical response of the system but to yield insight into the nature of this response. While these methods are applied only to particular mixed boundary value problems involving the Timoshenko beam equations, they may be used in analyzing other boundary and initial conditions and in most cases may be generalized to linear one dimensional hyperbolic systems.

In Chapter 2 a standard eigenfunction solution is obtained for the finite spring supported span yielding results for moderate values of the spring parameter, slenderness ratio and time. The effect of Maxwell damping is also shown.

In Chapter 3 a progressing wave form of solution for first order hyperbolic initial value problems is discussed and shown to be equivalent to the term by term inversion of the Laurent expansion of the Fourier transform of the solution. The finite simply supported span is reduced to an initial value problem and its eigenfunction solution modified by isolating its leading discontinuities into a progressing wave series. This modification results in a marked improvement in convergence. The effect of Maxwell damping in the material is shown through the attenuation of the leading terms in the progressing wave expansion.

Methods for obtaining the reduction from a mixed boundary value problem to an initial value problem for the finite spring parameter are discussed in Chapter 4. One of these methods is then used to obtain the leading reflected discontinuities for various boundary conditions.

In Chapter 5 the leading terms of the progressing wave series for the simply supported span are modified to exhibit better behavior for large values of time. This increases the range of reasonable convergence of the solution to larger values of time.

Asymptotic results for large slenderness ratios are obtained for a semi infinite simply supported span in Chapter 6. The methods of stationary phase and singular perturbation are used to obtain results in the vicinity of the wave fronts. The effect of Maxwell damping on the shear response is also shown in these regions.

Chapter 7 discusses the applicability of the elementary Euler Bernoulli theory as a function of the slenderness ratio and spring parameter.

A discussion of the results and conclusions are given in Chapter 8.

CHAPTER 2

SOLUTION BY EIGENFUNCTION EXPANSION

2.a Linear Spring Supports

The response of a finite spring supported beam subjected to a uniform velocity input is not expected to exhibit high intensity discontinuities for a moderate spring parameter α and is therefore obtained by means of normal modes.

Expressing the Timoshenko equations in their second order form in matrix notation and assuming q=0 yields

$$\underline{u}_{tt} + T\underline{u} = 0 \tag{2-1}$$

where $\underline{u} = \begin{pmatrix} y \\ \psi \end{pmatrix}$ and $T\underline{u} = \begin{bmatrix} -y_{xx} + \psi_x \\ -r^2\psi_{xx} - (\mu/r)^2(y_x - \psi) \end{bmatrix}$

The above equations when applied to a finite spring supported span represent a conservative hyperbolic system and may be analyzed by means of a dual eigenfunction expansion. The following eigenvalue problem may be extracted

$$T\underline{u} = \lambda \underline{u} \tag{2-2}$$

with the homogeneous boundary conditions

At x=0:-

$$\psi_x = 0$$
$$y_x - \psi - \alpha y = 0 \tag{2-3}$$

At x=1: $\psi_x = 0$

$$y_x - \psi + \alpha y = 0$$

The operator T is self adjoint with respect to the positive definite inner product

$$(\underline{u}_1, \underline{u}_2) = \int_0^1 y_1 y_2 \, dx + (\ell/r)^{-2} \int_0^1 \psi_1 \psi_2 \, dx \tag{2-4}$$

which insures an eigenstructure consisting of real and positive eigenvalues and orthogonal eigenfunctions.

Proposing a solution in the form

$$u = \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} e^{i\beta x} \tag{2-5}$$

and substituting into equation (2-2) yields

$$\begin{bmatrix} \lambda - \beta^2 & i\beta \\ (\ell/r)^2 i\beta & \lambda - (\ell/r)^2 - \gamma^2\beta^2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0 \tag{2-6}$$

Thus λ and β are related by

$$(\lambda - \beta^2)(\lambda - \gamma^2\beta^2) - (\ell/r)^2 \lambda = 0 \tag{2-7}$$

Considering solutions representing symmetric elastic lines, equation (2-5) yields

$$\begin{aligned} y &= D_1 \cos \beta_1 (x - 1/2) + D_2 \cos \beta_2 (x - 1/2) \\ \psi &= \frac{\lambda - \beta_1^2}{\beta_1} D_1 \sin \beta_1 (x - 1/2) + \frac{\lambda - \beta_2^2}{\beta_2} D_2 \sin \beta_2 (x - 1/2) \end{aligned} \tag{2-8}$$

where $\pm\beta_1$ and $\pm\beta_2$ represent the roots of equation (2-7). Substitituting expressions (2-8) into (2-3) yields

$$\begin{bmatrix} \alpha \cos \beta_1/2 - \frac{\lambda}{\beta_1} \sin \beta_1/2 & \alpha \cos \beta_2/2 - \frac{\lambda}{\beta_2} \sin \beta_2/2 \\ (\lambda - \beta_1^2) \cos \beta_1/2 & (\lambda - \beta_2^2) \cos \beta_2/2 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = 0 \tag{2-9}$$

which results in the relation

$$\frac{\alpha - \frac{\lambda}{\beta_2} \tan \beta_2/2}{\lambda - \beta_2^2} - \frac{\alpha - \frac{\lambda}{\beta_1} \tan \beta_1/2}{\lambda - \beta_1^2} = 0 \tag{2-10}$$

and

$$\frac{D_1}{D_2} = - \frac{\cos \beta_2/2}{\cos \beta_1/2} \; \frac{\lambda - \beta_2^2}{\lambda - \beta_1^2} \tag{2-11}$$

This yields the eigenfunctions

$$\underline{U}_i(x) = \begin{pmatrix} \cos \beta_{1i}(x - 1/2) - \frac{\lambda_i - \beta_{1i}^2}{\lambda_i - \beta_{2i}^2} \, \frac{\cos \beta_{1i}/2}{\cos \beta_{2i}/2} \cos \beta_{2i}(x - 1/2) \\ \frac{\lambda_i - \beta_{1i}^2}{\beta_{1i}} \sin \beta_{1i}(x - 1/2) - \frac{\lambda_i - \beta_{1i}^2}{\beta_{2i}} \, \frac{\cos \beta_{1i}/2}{\cos \beta_{2i}/2} \sin \beta_{2i}(x - 1/2) \end{pmatrix} \tag{2-12}$$

where the eigenvalues λ_i are obtained from the simultaneous solution of equations (2-7) and (2-10).

The eigenvalues, being real and positive as previously mentioned, are numerically determined by isolating values of β_{1i} within an interval of π, obtaining λ_i and β_{2i} from equation (2-7) and using the method of interval halving on equation (2-10) to refine the solution.

Given the above orthogonal eigenfunctions, equation (2-1) reduces to solving

$$\ddot{f}_i(t) + \lambda_i f_i(t) = 0 \tag{2-13}$$

with initial conditions

$$f_i(0) = 0$$
$$\dot{f}_i(0) = \frac{(\underline{u}_t(x,0), \underline{u}_i(x))}{(\underline{u}_i(x), \underline{u}_i(x))} = A_i \qquad (2\text{-}14)$$

where A_i represents the i'th coefficient of the eigen-function expansion of the initial velocity

$$\underline{u}_t(x,0) = \begin{pmatrix} v_0 \\ 0 \end{pmatrix} \qquad (2\text{-}15)$$

and may be determined from equations (2-4) and (2-8). This results in the following expressions

$$(\underline{u}_t(x,0), \underline{u}_i(x)) = 2v_0\left[\frac{1}{\beta_{1i}} \sin\beta_{1i}/2 + G_i \frac{1}{\beta_{2i}} \sin\beta_{2i}/2\right] \qquad (2\text{-}16)$$

and

$$\begin{aligned}(\underline{u}_i(x), \underline{u}_i(x)) &= \frac{1}{2}\left(1 + \frac{1}{\beta_{1i}} \sin\beta_{1i}\right) + \frac{G_i^2}{2}\left(1 + \frac{1}{\beta_{2i}} \sin\beta_{2i}\right) \\ &+ \frac{4G_i}{\beta_{2i}^2 - \beta_{1i}^2}\left(\beta_{2i} \sin\beta_{2i}/2 \cos\beta_{1i}/2 - \beta_{1i} \cos\beta_{2i}/2 \sin\beta_{1i}/2\right) \\ &+ (\mathcal{I}/r)^{-1}\left\{\frac{E_i^2}{2}\left(1 - \frac{1}{\beta_{1i}} \sin\beta_{1i}\right) + \frac{F_i^2}{2}\left(1 - \frac{1}{\beta_{2i}} \sin\beta_{2i}\right)\right. \\ &+ \left.\frac{4E_iF_i}{\beta_2^2 - \beta_1^2}\left(\beta_{1i} \sin\beta_{2i}/2 \cos\beta_{1i}/2 - \beta_{2i} \cos\beta_{2i}/2 \sin\beta_{1i}/2\right)\right\}\end{aligned} \qquad (2\text{-}17)$$

where

$$E_i = \frac{\lambda_i - \beta_{1i}^2}{\beta_{1i}}$$
$$F_i = -\frac{\lambda_i - \beta_{1i}^2}{\beta_{2i}} \frac{\cos\beta_{1i}/2}{\cos\beta_{2i}/2}$$
$$G_i = -\frac{\lambda_i - \beta_{1i}^2}{\lambda_i - \beta_{2i}^2} \frac{\cos\beta_{1i}/2}{\cos\beta_{2i}/2}$$

Solving equation (2-13) yields

$$f_i(t) = \frac{1}{\sqrt{\lambda_i}} A_i \sin\sqrt{\lambda_i}\, t \qquad (2\text{-}18)$$

and the total solution to equation (2-1) is therefore

$$\underline{u}(x,t) = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\lambda_i}} A_i \, \underline{u}_i(x) \sin \sqrt{\lambda_i}\, t \tag{2-19}$$

or in component form

$$y = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\lambda_i}} A_i \left(\cos \beta_{1i}(x-1/2) + G_i \cos \beta_{2i}(x-1/2) \right) \sin \sqrt{\lambda_i}\, t \tag{2-20}$$

$$\psi = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\lambda_i}} A_i \left(E_i \sin \beta_{1i}(x-1/2) + F_i \sin \beta_{2i}(x-1/2) \right) \sin \sqrt{\lambda_i}\, t \tag{2-21}$$

Using the relationships

$$V = y_x - \psi$$

$$M = -\psi_x / (l/r)^2$$

the corresponding expressions for the shear and moment are

$$V = -\sum_{i=1}^{\infty} \sqrt{\lambda_i} A_i \left(\frac{1}{\beta_{1i}} \sin \beta_{1i}(x-1/2) - \frac{1}{\beta_{2i}} G_i \sin \beta_{2i}(x-1/2) \right) \sin \sqrt{\lambda_i}\, t \tag{2-22}$$

and

$$M = -(l/r)^{-2} \sum_{i=1}^{\infty} \frac{1}{\sqrt{\lambda_i}} A_i \left((\lambda_i - \beta_{1i}^2) \cos \beta_{1i}(x-1/2) + \beta_{2i} F_i \cos \beta_{2i}(x-1/2) \right) \sin \sqrt{\lambda_i}\, t \tag{2-23}$$

As shown by equation (2-19) the circular frequencies of the beam are the square roots of the eigenvalues.

Figures 2 through 5 show results for the moment at mid span and the shear at the support for various values of l/r and K. The results have been plotted for non dimensionalized times up to 5, that is, the time required for a shear wave to travel five span lengths. As expected, the physical quantity exhibiting the poorest convergence is the shear. Its convergence decreased with increasing l/r't and K and generally worsened in the vicinity of the shear wave fronts. These phenomena are discussed in Chapters 3 and 6.

All calculations were performed for a value of $\gamma=1.47$, which corresponds to a Poisson's ratio of .3 and a shear coefficient $(A_s/A)^{-1}$ equal to 1.2.

For the infinite spring parameter, i.e. rigid supports, the eigenstructure simplifies, as the values of β reduce to $n\pi$ where n is odd; the eigenvalues are given explicitly by equation (2-7) and the corresponding eigenfunctions reduce to

$$\underline{u}_n(x) = \begin{pmatrix} \sin n\pi(x-1/2) \\ \dfrac{(n\pi)^2-\lambda_n}{n\pi} \cos n\pi(x-1/2) \end{pmatrix} \tag{2-24}$$

It should be noted at this point that equation (2-7) yields two eigenvalues for each value of $n\pi$.

The analogous eigenseries solutions for this case exhibit unsatisfactory convergence and are of dubious value for values of x and t representing the vicinity of the wave fronts. This phenomenon is dealt with in Chapter 3.

2.b Viscoelastic Supports

The addition of a dissipative force on the linear spring supports reduces the problem to a non conservative system. The operator T in this case is not self adjoint, introducing the practical difficulties of complex eigenvalues and non orthogonal eigenfunctions.

Assuming a viscoelastic effect on the springs produced by Maxwell damping, i.e. $\dot{y}=\frac{1}{\alpha}\dot{V}+\frac{1}{\eta}V$, the boundary conditions become

$$\begin{aligned}
\text{At } x=0: \quad & \psi_x = 0 \\
& \dot{y} - \frac{1}{\alpha}(\dot{y}_x - \dot{\psi}) + \frac{1}{\eta}(y_x - \psi) = 0 \\
\text{At } x=1: \quad & \psi_x = 0 \\
& \dot{y} + \frac{1}{\alpha}(\dot{y}_x - \dot{\psi}) - \frac{1}{\eta}(y_x - \psi) = 0
\end{aligned} \qquad (2\text{-}25)$$

where $\eta^{-1}=2\sqrt{2}\,\theta/\sqrt{\alpha}$ and θ is the fraction of the critical value of an equivalent rigid span.

These boundary conditions produce eigenfunctions of similar form to (2-12) where now the eigenvalues must simultaneously satisfy equation (2-7) and the modified equation

$$\frac{\alpha + \frac{1}{i\sqrt{\lambda}\,\eta\,\beta_2}\left(\alpha(\lambda-2\beta_2^2)\right) - \frac{\lambda}{\beta_2}\tan\frac{\beta_2}{2}}{\lambda-\beta_2^2} - \frac{\alpha + \frac{1}{i\sqrt{\lambda}\,\eta\,\beta_1}\left(\alpha(\lambda-2\beta_1^2)\right) - \frac{\lambda}{\beta_1}\tan\frac{\beta_1}{2}}{\lambda-\beta_1^2} = 0 \qquad (2\text{-}26)$$

The complex eigenvalues λ_i' are obtained from a numerical double scanning procedure that determines the complex zeros of equation (2-26) subject to equation (2-7).

The initial data may be expanded in terms of the non orthogonal eigenfunctions $\underline{u}_i(x)$ by means of a system of simultaneous equations. Proposing an expansion of the initial velocity in the form

$$\dot{\underline{u}}(x,0) = \begin{Bmatrix} v_0 \\ 0 \end{Bmatrix} = \sum_{i=1}^{N} b_i \underline{u}'_i(x) \qquad (2\text{-}27)$$

the complex coefficients b_i are determined by solving the following set of N simultaneous equations

$$\left(\underline{u}'_j(x), \dot{\underline{u}}(x,0)\right) = \sum_{i=1}^{N} b_i \left(\underline{u}'_i(x), \underline{u}'_j(x)\right) \qquad (j=1,2\ldots N) \qquad (2\text{-}28)$$

The equivalent time function g(t) satisfying

$$\ddot{q}_i(t) + \lambda'_i \, q_i(t) = 0 \qquad (2\text{-}29)$$

with

$$q_i(0) = 0$$
$$\dot{q}_i(0) = b_i \qquad (2\text{-}30)$$

is

$$q_i(t) = \frac{1}{\sqrt{\lambda'_i}} b_i \sin \sqrt{\lambda'_i}\, t \qquad (2\text{-}31)$$

where $\sqrt{\lambda'_i}$ refers to the positive square root.

The N term eigenfunction solution for viscoelastic spring supports is therefore

$$\underline{u}'(x,t) = \mathrm{Re}\left\{ \sum_{i=1}^{N} \frac{1}{\sqrt{\lambda'_i}} b_i \, \underline{u}'_i(x) \sin \sqrt{\lambda'_i}\, t \right\} \qquad (2\text{-}32)$$

Figures 6 and 7 show the effect of this damping on the moment and shear responses. Calculations were performed for a value of θ=.10.

CHAPTER 3

PROGRESSING WAVE EXPANSION

The dependence of the convergence of the previously developed eigenfunction series on the spring parameter α may be attributed to the degree of consistency between the initial and boundary data.

Consider the Timoshenko equations in their first order form: In the case of an infinite spring parameter α, a discrepancy between the imposed boundary condition on v and its prescribed initial values evaluated at the boundaries produces a 'first order inconsistency', i.e. a discontinuity, in the solution for the velocity. For finite α, this inconsistency occurs in the first derivative of the prescribed function $(V-\alpha y)$ and will be shown to produce a solution which contains discontinuous first derivatives in the velocity and shear with magnitudes proportional to α. It is this lack of smoothness whose intensity is proportional to α, which produces the reduced rates of convergence as the spring parameter increases.

Courant[13] has shown for the case of first order hyperbolic systems that the leading discontinuities may be isolated in a progressing wave series of generalized functions leaving a remainder with a milder discontinuity. In this chapter the above method, which employs the fact that discontinuities in hyperbolic systems are restricted to propagate along characteristics, is discussed in general and used to modify the previous analysis.

3.a General Progressing Wave Expansion

Let the vector $\underline{U}(x,t)$ represent n dependent variables subject to the initial value problem

$$L[\underline{U}] = A^t \underline{U}_t + A^x \underline{U}_x + B\underline{U} = 0 \tag{3-1}$$

with

$$\underline{U}(x,0) = \sum_{\nu=0}^{N} \underline{g}^{\nu}(x) S_{\nu}(x) + \underline{R}(x) \tag{3-2}$$

where A^x, A^t and B are square nxn matrices; $\underline{g}^{\nu}(x)$ are n dimensional vectors of regular functions and $S_{\nu}(x)$ are generalized functions having discontinuities at x=0 and such that

$$\frac{d}{dx} S_{\nu}(x) = S_{\nu-1}(x) \tag{3-3}$$

which exhibits the fact that the generalized functions $S_{\nu}(x)$ are increasingly less singular as ν increases. $\underline{R}(x)$ is an n dimensional vector whose highest order singularity is at most the same order as that of $S_{N+1}(x)$.

In light of the form of the initial data which produces singularities on the characteristic curves $\varphi_k(x,t)=0$, Courant proposed a solution to equation (3-1) of the form

$$\underline{U}(x,t) = \sum_{\nu=0}^{N} \sum_{k=1}^{n} \underline{g}^{\nu,k}(x,t) S_{\nu}(\varphi_k) + \underline{R}(x,t) \tag{3-4}$$

where $\varphi_k(x,t)=0$ are the k characteristic curves passing through x=0. The generalized functions $S_{\nu}(\varphi_k)$ are determined by the nature of the singularities in the initial data leaving the coefficient vectors $\underline{g}^{\nu}(x,t)$ and the remainder term $\underline{R}(x,t)$ whose smoothness depends on N to be determined.

The series portion of equation (3-4) is called a 'progressing wave of degree N'. If this series can be extended to a convergent series as $N\to\infty$, it is called a 'complete progressing wave' and represents the total solution to the initial value problem. The existence of the solution in the above form has been proven by Ludwig[14].

In substituting equation (3-4) into equation (3-1) the generalized functions may be differentiated as if they were ordinary functions. Requiring the resulting coefficients of $S_\nu(\varphi_k)$ to vanish independently on each $\varphi_k(x,t)=$const. yields the following recursive relationships for each k

$$A\underline{g}^{0} = 0 \qquad (3\text{-}5)$$

$$L[\underline{g}^{\nu}] + A\underline{g}^{\nu+1} = 0 \qquad (\nu=0,1,\ldots N-1) \qquad (3\text{-}6)$$

$$S_N L[\underline{g}^{N}] + L[\underline{R}] = 0 \qquad (3\text{-}7)$$

where $A^k=c_k A^t+A^x$ is the characteristic matrix and is singular and of rank (n-1) due to the assumption of distinct characteristics. Define $\underline{l}^k$ and $\underline{r}^k$ to be the left and right null vectors of A^k respectively. That is, $\underline{l}A=0$ and $A\underline{r}=0$. The above assumptions imply the independence of the vectors $\underline{r}^k$.

Due to the singularity of A, the linear system of equations represented by equations (3-6) determines the vectors $g^{\nu+1}$ up to modulo $\underline{r}$. Therefore let

$$g^{\nu+1} = \sigma^{\nu+1}\underline{r} + \underline{h}^{\nu+1} \tag{3-8}$$

where the vector $\underline{h}^{\nu+1}$ may be obtained (modulo $\underline{r}$) from equation (3-6) and $\sigma^{\nu+1}$ is a scalar multiplier.

Multiplying equation (3-6) by the left null vector $\underline{l}$, and using expression (3-8) yields

$$\underline{l} L[\sigma^{\nu}\underline{r} + \underline{h}^{\nu}] = 0 \qquad (\nu = 1, 2 \ldots N-1) \tag{3-9}$$

but

$$\underline{l} L[\sigma^{\nu}\underline{r}] = \underline{l}\{A^{x}(\sigma\underline{r})_{\varphi}\varphi_{x} + A^{t}(\sigma\underline{r})_{\varphi}\varphi_{t} + B(\sigma\underline{r})\} \tag{3-10}$$

or

$$\underline{l} L[\sigma^{\nu} r] = \sigma^{\nu}\underline{l} L[r] + \underline{l}[A^{x}\varphi_{x} + A^{t}\varphi_{t}]\underline{r}\,\sigma_{\varphi} \tag{3-11}$$

Using the following lemma on differentiation along characteristic curves

$$\dot{\xi}_i = \underline{l} A^{\xi_i}\underline{r} \tag{3-12}$$

where ξ_i represent the independent variables and the dot denotes differentiation along the characteristic curve with respect to a suitable curve parameter, equation (3-11) becomes

$$\underline{l} L[\sigma^{\nu}\underline{r}] = \sigma\underline{l} L[\underline{r}] + \dot{\sigma} \tag{3-13}$$

and therefore equation (3-9) results in the ordinary differential 'transport' equation governing the determination of σ^{ν}

$$\dot{\sigma}^{\nu} + \sigma^{\nu}\underline{l} L[\underline{r}] + \underline{l} L[\underline{h}^{\nu}] = 0 \tag{3-14}$$

The initial data required for the solution of equation (3-14) for each k, is obtained by resolving the given initial data along the n characteristic curves passing through $\varphi(x,0)$ as follows

$$\sum_{k=1}^{n} \underline{g}^{\nu,k}(x,0) = \underline{g}^{\nu}(x) \tag{3-15}$$

or

$$\sum_{k=1}^{n} \left(\sigma^{\nu,k}(x,0)\,\underline{r}^{k} + \underline{h}^{\nu,k}(x,0)\right) = \underline{g}^{\nu}(x) \tag{3-16}$$

Equations (3-16) represent a system of linear equations which owing to the independence of the right null vectors $\underline{r}^{k}$ uniquely determine the initial values $\sigma^{\nu,k}(x,0)$.

Therefore the recursive procedure for determining the vectors $\underline{g}^{\nu,k}(x,t)$ is as follows: $\sigma^{0,k}(x,t)$ is directly determined from the transport equation (3-14) and the resolution (3-16) since from equation (3-5), $h^{0,k}=0$. These values are then used in equation (3-6) to determine $\underline{h}^{1,k}$ which determine the initial data and non homogeneous term of the transport equation whose solution yields $\sigma^{1,k}$, etc.

For N sufficiently large, the resulting problem

$$L[\underline{R}] = -S_N L[\underline{g}^N] \tag{3-17}$$

exhibits smooth initial data and a smooth non homogeneity.

3.b Application to Rigid Supports

To analyze the finite Timoshenko beam subjected to a uniform velocity input by means of the progressing wave expansion requires the reduction of this mixed boundary value problem to an initial value problem.

For an infinite spring parameter, i.e. rigid supports, the consideration of an infinite span with initial velocity $v(x,0)$ specified as $S_o'(x)$ where $S_o'(x)=v_o \operatorname{sgn}(x)$ produces a solution in which y and ψ are antisymmetrical and symmetrical respectively about x=0. This results in zero moment and deflection at x=0 and therefore represents a Cauchy initial value problem equivalent to the semi infinite span with a simple support at x=0. Using the principle of superposition, the initial velocity equivalent to the finite span of length 1 is expressed by $v_o S_o(x)$ where $S_o(x)$ is specified as

$$S_o(x) = \begin{cases} 1 & 0 \leq x < 1 \\ -1 & -1 < x < 0 \end{cases} \tag{3-18}$$

with a periodic extension of period 2.

Having reduced the finite problem to a Cauchy initial value problem it may now be analyzed by the method of progressing waves discussed in Section 3.a. The procedure will be used to isolate the leading discontinuities in M, ω, V and v in a finite progressing wave series of the form

$$\underline{U}'(x,t) = \sum_{\nu=1}^{N} \sum_{\kappa=1}^{4} q^{\nu,\kappa}(x,t)\, S_{\nu,\kappa}(\varphi_\kappa) \tag{3-19}$$

The elements of $\underline{U}'(x,t)$ representing v and ω will then be integrated to obtain the corresponding discontinuities in y and ψ which are denoted by the vector $\underline{u}'$. Finally, the remainder $\underline{R}$, which must satisfy the equation

$$\underline{R}_{tt} + T\underline{R} = -\underline{w} \tag{3-20}$$

where
$$\underline{w} = [\underline{u}'_{tt} + T\underline{u}'] \tag{3-21}$$

will be obtained by means of the previously developed eigenstructure.

For convenience, The Timoshenko equations are reduced to their canonical form. Expressing the Timoshenko equations in matrix form with q=0

$$\underline{W}_t + A^x \underline{W}_x + B'\underline{W} = 0 \tag{3-22}$$

where
$$\underline{W} = \begin{bmatrix} M \\ \omega \\ V \\ v \end{bmatrix} \qquad A^x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad B' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & (\ell/r)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving the characteristic equation

$$| A^x - c_k I | = 0 \tag{3-23}$$

yields the four characteristic curves $\varphi_i = x - c_i t$

$$\begin{aligned} \varphi_1(x,t) &= x - \gamma t \\ \varphi_2(x,t) &= x + \gamma t \\ \varphi_3(x,t) &= x - t \\ \varphi_4(x,t) &= x + t \end{aligned} \tag{3-24}$$

Calculating the left null vectors of

$$C^k = C^k - c^k I \tag{3-25}$$

yields

$$\underline{m}^1 = [\gamma \quad 1 \quad 0 \quad 0]$$
$$\underline{m}^2 = [\gamma \quad -1 \quad 0 \quad 0]$$
$$\underline{m}^3 = [0 \quad 0 \quad -1 \quad 1] \tag{3-26}$$
$$\underline{m}^4 = [0 \quad 0 \quad 1 \quad 1]$$

The canonical form is obtained by the change of variables

$$\underline{U}_{(k)} = \underline{m}^k \underline{W} \tag{3-27}$$

which yields the system of equations

$$I \underline{U}_t + T \underline{U}_x + B \underline{U} = 0 \tag{3-28}$$

where

$$U = \begin{bmatrix} \gamma M + \omega \\ \gamma M - \omega \\ \upsilon - V \\ \upsilon + V \end{bmatrix} \qquad B = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -(\ell/L)^2 & (\ell/L)^2 & 0 & 0 \\ (\ell/L)^2 & -(\ell/L)^2 & 0 & 0 \end{bmatrix} \tag{3-29}$$

and $$T = \mathrm{diag}\{\gamma, -\gamma, 1, -1\}$$

This normalization has the effect of diagonalizing the characteristic matrices

$$A^k = I - c_k T \tag{3-30}$$

to form

$$A^1 = \mathrm{diag}\{0, -2\gamma, -(\gamma+1), -(\gamma+1)\}$$
$$A^2 = \mathrm{diag}\{2\gamma, 0, (\gamma+1), (\gamma-1)\} \tag{3-31}$$
$$A^3 = \mathrm{diag}\{(\gamma-1), -(\gamma+1), 0, -2\}$$
$$A^4 = \mathrm{diag}\{(\gamma+1), -(\gamma-1), 2, 0\}$$

The corresponding left and right null vectors are

$$\underline{l}^k = [\underline{r}^k]^T = [\delta_{1k}, \delta_{2k}, \delta_{3k}, \delta_{4k}] \tag{3-32}$$

The initial data is now expressed by

$$\underline{U}(x,0) = \begin{bmatrix} 0 \\ 0 \\ v_0 \\ v_0 \end{bmatrix} S_0(x) \tag{3-33}$$

which together with equation (3-3) define the complete set of generalized functions $S^\nu(\varphi)$, the leading terms of which are described in Figure 8.

To obtain the first term of the progressing wave expansion, the initial data for the transport equation (3-14) is obtained through the resolution

$$\sum_{k=1}^{4} \sigma^{0,k}(x,0)\underline{r}^k = \begin{bmatrix} 0 \\ 0 \\ v_0 \\ v_0 \end{bmatrix} \tag{3-34}$$

which yields

$$\sigma^{0,1}(x,0) = \{0, 0, v_0, v_0\} \tag{3-35}$$

Noting that $\underline{l}^k L[\underline{r}^k] = 0$ for all k and the lack of dependence on x reduces the characteristic curve parameter to time, the transport equation yields

$$\sigma_t^{0,k}(x,t) = 0 \tag{3-36}$$

and therefore

$$\sigma^{0,1}(x,t) = \{0, 0, v_0, v_0\} \tag{3-37}$$

Writing the four column vectors $\underline{g}^{0,k}(x,t)$ as a single matrix G^0, the first term in the progressing wave expansion has coefficients

$$G^0 = [\underline{g}^{0,1}, \underline{g}^{0,2}, \underline{g}^{0,3}, \underline{g}^{0,4}] = \underline{\sigma}_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (3\text{-}38)$$

$\underline{h}^{1,k}$ may now be obtained from equations (3-6) and yield

$$H^1(x,t) = [\underline{h}^{1,1}, \underline{h}^{1,2}, \underline{h}^{1,3}, \underline{h}^{1,4}] = \frac{\underline{\sigma}_0}{2} \begin{pmatrix} 0 & 0 & -a & b \\ 0 & 0 & -b & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (3\text{-}39)$$

where $a = (\gamma-1)^{-1}$ and $b = (\gamma+1)^{-1}$.

Using expressions (3-16) to obtain the initial data $\sigma^{1,k}(x,0)$ results in

$$\sigma^{1,k}(x,0) = \sigma_0 \{ ab(\ell/r)^2, -ab(\ell/r)^2, 0, 0 \} \qquad (3\text{-}40)$$

Noting that $\underline{h}^{1,k}(x,t)$ are constants, the transport equations for $\nu=1$ reduce to

$$\sigma_t^{1,k}(x,t) = -\underline{\ell}^k B \underline{h}^{1,k}(x,t) \qquad (3\text{-}41)$$

and therefore the values of $\sigma^{1,k}(x,t)$ are expressed by

$$\sigma^{1,k}(x,t) = \sigma_0 \{ -ab(\ell/r)^2, -ab(\ell/r)^2, -ab(\ell/r)^2 t/2, ab(\ell/r)^2 t/2 \} \qquad (3\text{-}42)$$

Using equations (3-39) and (3-42), the vectors $\underline{g}^{1,k}(x,t)$ are

$$G^1 = \sigma_0 \begin{bmatrix} -ab(\ell/r)^2 & 0 & -a/2 & b/2 \\ 0 & -ab(\ell/r)^2 & -b/2 & a/2 \\ 0 & 0 & -ab(\ell/r)^2 t/2 & 0 \\ 0 & 0 & 0 & ab(\ell/r)^2 t/2 \end{bmatrix} \qquad (3\text{-}43)$$

In the same manner the vectors $\underline{g}^{2,k}(x,t)$ are determined to be

$$G^2 = \frac{\sigma_0}{2}(\ell/r)^2 \begin{pmatrix} ab(a+b)t & 0 & a^2bt/2 & ab^2t/2 \\ 0 & -ab(a+b)t & ab^2t/2 & a^2bt/2 \\ -a^2b & ab^2 & [ab(\ell/r)^2](t^2-1+4ab)/4 & -ab/2 \\ ab^2 & a^2b & -ab/2 & [ab(\ell/r)^2]^2(t^2-1-4ab)/4 \end{pmatrix} \qquad (3\text{-}44)$$

Terminating the progressing wave expansion at this point results in the smooth part $\underline{R}(x,t)$ having continuous second derivatives and therefore $L\{\underline{R}\}$ having continuous first derivatives. In calculating $S_N L\{\underline{g}^N\}$ which is required for the determination of the smooth part of the solution, it was found that considerable simplification results if the time dependent terms along the diagonal of G^2 were omitted. This has the effect of not satisfying the transport equation for $\nu=2$, which reduces the order of smoothness of $\underline{R}$ while the smoothness of $L\{\underline{R}\}$ remains unchanged.

In terms of the physical variables, the resulting expressions for the leading terms of the progressing wave expansion of $\underline{U}(x,t)$ are

$$M' = -\frac{1}{2\gamma(\gamma^2-1)}\left[S_1(\varphi_2)-S_1(\varphi_1)\right] + \frac{1}{2(\gamma^2-1)}\left[S_1(\varphi_4)-S_1(\varphi_3)\right]$$
$$+\frac{(\ell/r)^2 t}{4(\gamma^2-1)^2}\left[S_2(\varphi_4)+S_2(\varphi_3)\right] - \frac{(\ell/r)^2}{4(\gamma^2-1)^2}\left[S_3(\varphi_4)-S_3(\varphi_3)\right] \tag{3-45}$$

$$\omega' = \frac{(\ell/r)^2}{2(\gamma^2-1)}\left[S_1(\varphi_2)+S_1(\varphi_1)\right] - \frac{(\ell/r)^2}{2(\gamma^2-1)}\left[S_1(\varphi_4)+S_1(\varphi_3)\right]$$
$$-\frac{(\ell/r)^4 t}{4(\gamma^2-1)^2}\left[S_2(\varphi_4)-S_2(\varphi_3)\right] + \frac{(\ell/r)^4}{4(\gamma^2-1)^2}\left[S_3(\varphi_4)+S_3(\varphi_3)\right] \tag{3-46}$$

$$V' = \frac{1}{2}\left[S_0(\varphi_4)-S_0(\varphi_3)\right] + \frac{(\ell/r)^2 t}{4(\gamma^2-1)}\left[S_1(\varphi_4)+S_1(\varphi_3)\right]$$
$$-\frac{(\ell/r)^2\gamma}{2(\gamma^2-1)}\left[S_2(\varphi_2)-S_2(\varphi_1)\right] + \frac{(\ell/r)^2(\gamma^2+1)}{4(\gamma^2-1)^2}\left[S_2(\varphi_4)-S_2(\varphi_3)\right] \tag{3-47}$$
$$+\frac{(\ell/r)^2 t}{4(\gamma^2-1)^2}\left[S_3(\varphi_4)+S_3(\varphi_3)\right] - \frac{(\ell/r)^2}{4(\gamma^2-1)^2}\left[S_4(\varphi_4)-S_4(\varphi_3)\right]$$

$$v' = \frac{1}{2}\left[S_0(\varphi_4)+S_0(\varphi_3)\right] + \frac{(\ell/r)^2 t}{4(\gamma^2-1)}\left[S_1(\varphi_4)-S_1(\varphi_3)\right]$$
$$+\frac{(\ell/r)^2}{2(\gamma^2-1)^2}\left[S_2(\varphi_4)+S_2(\varphi_3)\right] - \frac{(\ell/r)^2}{2(\gamma^2-1)^2}\left[S_2(\varphi_2)+S_2(\varphi_1)\right] \tag{3-48}$$

and therefore integrating equations (3-46) and (3-48)

$$y' = \frac{1}{2}\left[S_1(\varphi_4)-S_1(\varphi_3)\right] + \frac{(\ell/r)^2 t}{4(\gamma^2-1)}\left[S_2(\varphi_4)+S_2(\varphi_3)\right]$$
$$+\frac{(3-\gamma^2)(\ell/r)^2}{4(\gamma^2-1)^2}\left[S_3(\varphi_4)-S_3(\varphi_3)\right] - \frac{(\ell/r)^2}{2\gamma(\gamma^2-1)^2}\left[S_3(\varphi_2)-S_3(\varphi_1)\right] \tag{3-49}$$

$$\psi' = \frac{(\ell/r)^2}{2\gamma(\gamma^2-1)}\left[S_2(\varphi_2)-S_2(\varphi_1)\right] - \frac{(\ell/r)^2}{2(\gamma^2-1)}\left[S_2(\varphi_4)-S_2(\varphi_3)\right]$$
$$-\frac{(\ell/r)^4 t}{4(\gamma^2-1)^2}\left[S_3(\varphi_4)+S_3(\varphi_3)\right] + \frac{(\ell/r)^4}{4(\gamma^2-1)^2}\left[S_4(\varphi_4)-S_4(\varphi_3)\right] \tag{3-50}$$

Equations (3-49) and (3-50) determine the vector $\underline{w}$ which when expanded in terms of the eigenfunctions (2-24) yields the solution to equation (3-20) in the form

$$\underline{R}(x,t) = \sum_{n=odd}^{\infty} f_n(t)\,\underline{u}_n(x) \tag{3-51}$$

where
$$f_n(t) = \frac{1}{\sqrt{\lambda_n}} \int_0^t Z_n(\tau) \sin\sqrt{\lambda_n}\,(t-\tau)\,d\tau \tag{3-52}$$

and
$$Z_n(t) = \frac{(\underline{w}, \underline{u}_n)}{(\underline{u}_n, \underline{u}_n)} \tag{3-53}$$

The total solution to the finite simply supported span is therefore the sum of $\underline{U}'$ and $\underline{R}$.

Results for shear and moment are plotted in Figures 9 through 16 for times up to $t=5$.

3.c Attenuation of Progressing Waves for Viscous Damping in the Material

To analyze the effect of viscous damping in the material on the progressing wave form of solution, the Maxwell model is studied. For the Kelvin model, the equations are no longer hyperbolic.

To obtain the equations for the viscoelastic Timoshenko beam, the elastic moduli E and G are replaced by their corresponding viscoelastic operators. For the Maxwell model this yields the relationships

$$\begin{aligned} E &\leftrightarrow 3Q/2P \\ G &\leftrightarrow Q/2P \end{aligned} \tag{3-54}$$

where

$$P = \frac{\partial}{\partial t} + \Delta$$
$$Q = 2G\frac{\partial}{\partial t} \tag{3-55}$$

and Δ is the reciprocal of the stress relaxation time.

Using these relationships in equations (1-1) the viscoelastic equations in non dimensional form become

$$\begin{aligned} M_t + \Delta M + \omega_x &= 0 \\ \omega_t + \gamma^2 M_x - V &= 0 \\ V_t + \Delta V - \sigma_x + (\ell/r)^2\omega &= 0 \\ \sigma_t - V_x &= 0 \end{aligned} \tag{3-56}$$

Note that the principal part of the equations and therefore the structure of the characteristics remains unchanged. Transforming to canonical form yields

$$\underline{U}_t + T\underline{U}_x + \bar{B}\underline{U} = 0 \tag{3-57}$$

where U and T are as previously defined and

$$\bar{B} = \frac{1}{2}\begin{bmatrix} \Delta & \Delta & 1 & -1 \\ \Delta & \Delta & -1 & 1 \\ -(\ell/r)^2 & (\ell/r)^2 & \Delta & -\Delta \\ (\ell/r)^2 & -(\ell/r)^2 & -\Delta & \Delta \end{bmatrix}$$

therefore

$$\underline{\ell}^k \bar{B}\, \underline{r}^k = \frac{1}{2}\Delta \qquad \text{for all } k \tag{3-58}$$

and the transport equation (3-14) yields

$$\sigma_t^\nu + \frac{1}{2}\Delta\sigma^\nu = -\underline{\ell}\, L[\underline{h}^\nu] \tag{3-59}$$

since there is no dependence on x.

Solving equation (3-59) results in

$$\sigma^{\nu}(x,t) = e^{-\frac{1}{2}\Delta t}\left\{\sigma^{\nu}(x,0) + \int_0^t e^{\frac{1}{2}\Delta\tau}\left(-\underline{\ell}L[\underline{h}^{\nu}]\right)d\tau\right\} \quad (3\text{-}60)$$

Equation (3-60) shows the attenuation of the progressing waves due to Maxwell damping. The limit as $\Delta \rightarrow 0$ is easily seen to reduce to the elastic case while the other extreme, i.e. $\Delta \rightarrow \infty$, has no relavant physical meaning.

For example, for $\nu=0$, equation (3-60) yields the coefficients analogous to (3-38) to be

$$G^{0} = \mathrm{diag}\left\{0,\ 0,\ e^{-\frac{1}{2}\Delta t},\ e^{-\frac{1}{2}\Delta t}\right\} \quad (3\text{-}61)$$

which exhibits the exponential decay associated with viscoelastic behavior.

3.d Relation of Progressing Wave Expansion to Fourier Transform Solution

The progressing wave form of solution may also be obtained by means of a Fourier transform analysis. For simplicity, this will be shown for the initial value problem $v(x,0)=v_0\operatorname{sgn}(x)$ which is equivalent to the semi infinite span with a rigid support subjected to an initial velocity $v(x,0)=v_0$. The method of superposition may then be used to extend the results to the finite case.

Taking the Fourier transform of the Timoshenko equations in their second order form

$$y_{tt} - y_{xx} + \psi_x = 0$$
$$\psi_{tt} - \gamma^2 \psi_{xx} - (\ell/r)^2 (y_x - \psi) = 0 \tag{3-62}$$

yields two coupled second order ordinary differential equations

$$\tilde{y}_{tt} + p^2 \tilde{y} - ip\tilde{\psi} = 0$$
$$\tilde{\psi}_{tt} + \gamma^2 p^2 \tilde{\psi} - (\ell/r)^2 (-ip\tilde{y} - \tilde{\psi}) = 0 \tag{3-63}$$

with transformed initial conditions

$$\tilde{y}(p,0) = \tilde{\psi}(p,0) = \tilde{\psi}_t(p,0) = 0$$
$$\tilde{y}_t(p,0) = -2/ip \tag{3-64}$$

where

$$\tilde{y}(p,t) = \int_{-\infty}^{\infty} y(x,t)\, e^{ipx}\, dx$$
$$\tilde{\psi}(p,t) = \int_{-\infty}^{\infty} \psi(x,t)\, e^{ipx}\, dx \tag{3-65}$$

Proposing a solution to equations 3-63 which identically satisfies the homogeneous initial conditions on $\tilde{y}$ and $\tilde{\psi}$

$$\tilde{y} = B \sin \omega t$$
$$\tilde{\psi} = -ipA \sin \omega t \tag{3-66}$$

Substituting equations (3-66) into equations (3-63) yields

$$\begin{pmatrix} -p^2 & -\omega^2 + p^2 \\ -\omega^2 + \gamma^2 p^2 + (\ell/r)^2 & -(\ell/r)^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{3-67}$$

Hence

$$A = \frac{p^2 - \omega^2}{p^2} B \tag{3-68}$$

and ω must satisfy

$$(p^2 - \omega^2)(\gamma^2 p^2 - \omega^2) - (\ell/r)^2 \omega^2 = 0 \tag{3-69}$$

The solution is therefore

$$\tilde{y} = B_1 \sin \omega_1 t + B_2 \sin \omega_2 t$$
$$\tilde{\psi} = -ip\left\{ \frac{p^2-\omega_1^2}{p^2} B_1 \sin\omega_1 t + \frac{p^2-\omega_2^2}{p^2} B_2 \sin\omega_2 t \right\} \tag{3-70}$$

where ω_1^2 and ω_2^2 (with $\omega_1^2 > \omega_2^2$) are the two roots of equation (3-69).

Applying the initial conditions on $\tilde{\psi}_t$ and $\tilde{y}_t$ yields

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \frac{p^2-\omega_1^2}{\omega_1} & \frac{p^2-\omega_2^2}{\omega_2} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{ip} v_0 \\ 0 \end{pmatrix} \tag{3-71}$$

whose solution is

$$B_1 = -2iv_0 \frac{p^2-\omega_2^2}{p\omega_1(\omega_1^2-\omega_2^2)}$$
$$B_2 = -2iv_0 \frac{p^2-\omega_1^2}{p\omega_2(\omega_2^2-\omega_1^2)} \tag{3-72}$$

Therefore the transformed solutions for $\tilde{y}$ and $\tilde{\psi}$ are

$$\tilde{y} = -2iv_0 \left\{ \frac{p^2-\omega_2^2}{p\omega_1(\omega_1^2-\omega_2^2)} \sin\omega_1 t + \frac{p^2-\omega_1^2}{p\omega_2(\omega_2^2-\omega_1^2)} \sin\omega_2 t \right\}$$
$$\tilde{\psi} = -2v_0 \left\{ \frac{(p^2-\omega_1^2)(p^2-\omega_2^2)}{p^2\omega_1(\omega_1^2-\omega_2^2)} \sin\omega_1 t + \frac{(p^2-\omega_2^2)(p^2-\omega_1^2)}{p^2\omega_2(\omega_2^2-\omega_1^2)} \sin\omega_2 t \right\} \tag{3-73}$$

Using the transform of equations (1-7)

$$\tilde{V} = -ip\tilde{y} - \tilde{\psi}$$
$$\tilde{M} = ip\tilde{\psi}/(l/r)^2 \tag{3-74}$$

results in the following expression for the Fourier transform of the shear and the moment

$$\tilde{V} = 2v_0 \left\{ \frac{\omega_1 (p^2-\omega_2^2)}{p^2(\omega_1^2-\omega_2^2)} \sin\omega_1 t + \frac{\omega_2(p^2-\omega_1^2)}{p^2(\omega_2^2-\omega_1^2)} \sin\omega_2 t \right\} \tag{3-75}$$

$$\tilde{M} = i v_0 (l/r)^{-2} \left\{ \frac{(p^2-\omega_1^2)(p^2-\omega_2^2)}{p\,\omega_1(\omega_1^2-\omega_2^2)} \sin \omega_1 t + \frac{(p^2-\omega_2^2)(p^2-\omega_1^2)}{p\,\omega_2(\omega_2^2-\omega_1^2)} \sin \omega_2 t \right\}$$

The above transformed solutions are all of the form

$$F_1(p) \sin \omega_1 t + F_2(p) \sin \omega_2 t \tag{3-76}$$

or equivalently

$$-\frac{v_0}{2} i \left\{ F_1(p) \left(e^{i\omega_1 t} - e^{-i\omega_1 t} \right) + F_2(p) \left(e^{i\omega_2 t} - e^{-i\omega_2 t} \right) \right\} \tag{3-77}$$

which may be rewritten as

$$\begin{aligned} -\frac{v_0}{2} i \Big\{ & F_1(p) \left[e^{i(\omega_1 - \gamma p)t} e^{ip\gamma t} - e^{-i(\omega_1 - \gamma p)t} e^{-ip\gamma t} \right] \\ & + F_2(p) \left[e^{i(\omega_2 - p)t} e^{ipt} - e^{-i(\omega_2 - p)t} e^{-ipt} \right] \Big\} \end{aligned} \tag{3-78}$$

Equation (3-69) defines $\pm\omega_1, \pm\omega_2$ as branches of a 4 valued algebraic function. The following asymptotic expressions, valid in the complex p plane, can be derived either from the explicit solution of equation (3-69) or by inspection of the equation after division by p^4.

$$\begin{aligned} \omega_1 &= \gamma p + \frac{\gamma (l/r)^2}{2(\gamma^2-1)} \frac{1}{p} + O\left(\frac{1}{p^3}\right) \\ \omega_2 &= p - \frac{(l/r)^2}{2(\gamma^2-1)} \frac{1}{p} + O\left(\frac{1}{p^3}\right) \end{aligned} \tag{3-79}$$

These expressions show that in the cut required to make the algebraic function single valued[15], the edge to ∞ is unessential and therefore ω_1 and ω_2 are analytic outside a circle C_0 containing the finite critical points. Hence each of the functions $F_1(p)e^{\pm i(\omega_1-\gamma p)t}$, $F_2(p)e^{\pm i(\omega_2-p)t}$ is also analytic in p for a fixed t outside C_0 and can be represented by a Laurent series. Therefore expression (3-78) may be expressed in the form

$$\sum_{n=1}^{N+1} \frac{A_n^1(t)}{(ip)^n} e^{ip\gamma t} + \frac{A_n^2(t)}{(ip)^n} e^{-ip\gamma t} + \frac{A_n^3(t)}{(ip)^n} e^{ipt} + \frac{A_n^4(t)}{(ip)^n} e^{-ipt} + \tilde{R}'(p,t) \qquad (3\text{-}80)$$

where the multipliers $e^{\pm ip\gamma t}$ and $e^{\pm ipt}$ simply represent a shift in the (x,t) space. Equation (3-80) may be rewritten as

$$\sum_{\nu=1}^{N+1} \sum_{k=1}^{4} \frac{A^{\nu,k}(t)}{(ip)^\nu} e^{ipc_k t} + \tilde{R}'(p,t) \qquad (3\text{-}81)$$

where c_k represent the characteristic velocities.

Inverting term by term yields

$$\sum_{\nu=0}^{N} \sum_{k=1}^{4} A^{\nu,k}(t)\, S^{\nu,k}(x-c_k t) + R'(x,t) \qquad (3\text{-}82)$$

which may be shown to be identical to the progressing wave expansion developed in Section 3.b as follows: The method of progressing waves yields a solution to the Timoshenko equations in the form

$$\underline{U}(x,t) = \sum_{\nu=0}^{N} \sum_{k=1}^{4} \underline{g}^{\nu,k}(t)\, S_\nu(\varphi^k(x,t)) + \underline{R}_N(x,t) \qquad (3\text{-}83)$$

where $S_0(\varphi^k(x,t))$ is the highest order singularity appearing in the solution. The functions $\underline{U}(x,t)$ being of compact support for a given t, and having discontinuities of the type previously described are Fourier transformable. The terms in the progressing wave expansion have Fourier transforms[16] and therefore so has the remainder $\underline{R}_N(x,t)$. This transform is expressed by

$$\hat{\underline{U}}(p,t) = \sum_{\nu=0}^{N} \sum_{k=1}^{4} \underline{g}^{\nu,k}(t)\, e^{ipc_k t} \frac{1}{(ip)^{\nu+1}} + \hat{\underline{R}}_N(p,t) \qquad (3\text{-}84)$$

while the direct Fourier transform approach yielded

$$\tilde{\underline{U}}(p,t) = \sum_{\nu=0}^{N} \sum_{k=1}^{4} \underline{A}^{\nu,k} e^{ipc_k t} \frac{1}{(ip)^{\nu+1}} + \tilde{\underline{R}}'_N(p,t) \qquad (3\text{-}85)$$

To show the identity of expressions (3-84) and (3-85) consider their respective equivalent forms

$$\hat{\underline{U}}(p,t) = \sum_{\nu=0}^{N-1} \sum_{k=1}^{4} \underline{g}^{\nu,k}(t)\, e^{ipc_k t} \frac{1}{(ip)^{\nu+1}} + \sum_{k=1}^{4} \underline{g}^{N,k}(t)\, e^{ipc_k t} \frac{1}{(ip)^{N+1}} \,. \qquad (3\text{-}86)$$

and

$$\tilde{\underline{U}}(p,t) = \sum_{\nu=0}^{N-1} \sum_{k=1}^{4} \underline{A}^{\nu,k} e^{ipc_k t} \frac{1}{(ip)^{\nu+1}} + \sum_{k=1}^{4} \underline{A}^{N,k} e^{ipc_k t} \frac{1}{(ip)^{N+1}} \qquad (3\text{-}87)$$

Proceding by induction and assuming the terms of equations (3-86) and (3-87) are identical up to and including $\nu = N-1$, one obtains the relationship

$$\sum_{k=1}^{4} \frac{e^{ipc_k t}}{(ip)^{N+1}} \left(\underline{g}^{N,k} - \underline{A}^{N,k}\right) = \tilde{\underline{R}}' - \hat{\underline{R}} \qquad (3\text{-}88)$$

which can only be satisfied if

$$\begin{aligned} \underline{g}^{N,k} &\equiv \underline{A}^{N,k} \\ \tilde{\underline{R}}' &\equiv \hat{\underline{R}} \end{aligned} \qquad (3\text{-}89)$$

since $\hat{\underline{R}}$ may be shown to be $o((ip)^{-(N+1)})$[16] and $\tilde{\underline{R}}'$ is by definition of $o((ip)^{-(N+1)})$.

Hence the identity of the progressing wave series obtained from the method of Fourier transforms and the series determined by the direct progressing wave expansion.

This result is quite remarkable in view of the fact that a term by term Fourier transformation of a Laurent series containing only terms of negative exponents, while always convergent, may not converge to the transform of the function represented by the Laurent series. The following example illustrates this:

Consider the function

$$\frac{1}{p^2+a^2} \tag{3-90}$$

in the Fourier transformed space p, where a is a real constant. Expression (3-90) is analytic outside a circle of radius a and may be rewritten as

$$\frac{1}{2ai}\left\{\frac{1}{p-ai}-\frac{1}{p+ai}\right\} \tag{3-91}$$

which yields an inverse transform given by

$$\frac{\pi}{a} H(ay)\, e^{-2\pi y|a|} \tag{3-92}$$

However, the term by term inverse of the Laurent expansion of (3-90) about p=0 which is

$$\frac{1}{p^2}-\frac{a^2}{p^4}+\frac{a^4}{p^6}-\ldots \tag{3-93}$$

converges to

$$\frac{\pi}{a}\, \mathrm{sgn}\, y\, e^{-2\pi a y} \tag{3-94}$$

which shows that the term by term inverse of the Laurent expansion of a function, convergent outside a circle of finite radius, will not in general be identical to the inverse directly obtained.

CHAPTER 4

PROGRESSING WAVE EXPANSIONS FOR GENERAL MIXED BOUNDARY VALUE PROBLEMS

For the finite span on rigid supports the extension of the initial data required to convert the mixed boundary value problem is immediately apparent. In the case of a finite spring constant or other more general boundary conditions, however, these extensions are not readily available. Possible methods for obtaining these functions which represent reflections of the given initial data are developed in this Chapter for the semi infinite span which together with the method of superposition may be used in the finite case.

4.a Reduction of Mixed Boundary Value Problem to Initial value Problem

Consider the solution to the semi infinite span with initial data in the form

$$\underline{u}(x,t) = \sum_{\nu=0}^{N} \underline{g}^{\nu}(x)\, S^{\nu}(x) \tag{4-1}$$

and subject to μ linear homogeneous boundary conditions, say for example at x=0, to be composed of two parts. The first part is determined as the solution to an infinite span subjected to the given initial data. This represents a Cauchy initial value problem and will yield a progressing wave form of solution

$$\underline{u}'(x,t) = \sum_{\nu=0}^{\infty} \sum_{k=1}^{4} \underline{g}^{k,\nu}(x,t)\, S_\nu(\varphi_k) \tag{4-2}$$

Evaluating equation (4-2) at the proposed boundary x=0 (Figure 17.a) and summing over k yields a generalized function series of the form

$$\underline{u}'(0,t) = \sum_{\nu=0}^{\infty} \underline{g}^{\nu}(t)\, S_\nu(t) \tag{4-3}$$

The second part of the solution is to be taken in the form

$$\underline{u}''(x,t) = \sum_{\nu=0}^{\infty} \sum_{k=1}^{\mu} \underline{f}^{k,\nu}(x,t)\, S_\nu(\varphi^k(x,t)) \tag{4-4}$$

where μ is the number of characteristics whose directions lie in the first quadrant of the (x,t) plane. The coefficients $\underline{f}^{k,\nu}$ will be determined in the manner similar to that for the Cauchy problem (Section 3.a) except that, instead of using initial conditions for t=0, the initial values are to be used for x=0 (Figure 17.b). Instead of the resolution of the initial data along characteristics expressed by (3-17), the values of $\sigma^{\nu,k}(0,t)$ will be determined from the boundary conditions

$$Q\left[\underline{u}'(0,t) + \underline{u}''(0,t)\right] = 0 \tag{4-5}$$

where Q is a μxn operator.

The desired reflection may now be obtained by extending $\underline{u}''(x,t)$ back to the negative x axis, that is, evaluating $\underline{u}''(x,t)$ at t=0.

The proof of existence of such an extension would require a proof of convergence of $u''(x,t)$ which is not attempted in this work.

A second and somewhat more direct method might be used. Consider an infinite span subjected to initial data composed of two parts

$$\underline{u}(x,0) = \sum_{\nu=0}^{m} \underline{q}^{\nu}(x)\, S_{\nu}(x) + \sum_{\nu=0}^{m} \bar{\underline{q}}^{\nu}(x)\, \bar{S}_{\nu}(x) \tag{4-6}$$

where $\bar{S}_{\nu}(-x) = S_{\nu}(x)(-1)^{\nu}$ (4-7)

and $S_{\nu}(x) = 0 \qquad x<0$ (4-8)

The first series is the actual data existing on a semi infinite span while the barred series represents the unknown extension. Assume the actual initial data to consist of a step input, that is, $S_0(x)=H(x)$, where $H(x)$ is the Heaviside function. The calculations for the progressing wave expansion are carried out in terms of the unknown vectors $\bar{\underline{q}}^{\nu}$ and evaluated at the proposed boundary to yield

$$\underline{u}(0,t) = \sum_{\nu=0}^{m}\sum_{k=1}^{4} \underline{q}^{\nu,k}(t)\, S^{\nu,k}(\varphi_k(0,t)) + \sum_{\nu=0}^{m}\sum_{k=1}^{4} \bar{\underline{q}}^{\nu,k}(t)\, \bar{S}^{\nu,k}(\varphi_k(0,t)) \tag{4-9}$$

Noting that, for the Timoshenko equations subjected to initial data which is independent of x, the transport equation (3-14) allows only polynomial functions of t for $g^{\nu,k}(x,t)$ and therefore

$$t\, S^{\nu,k}(\varphi_k(0,t)) = (\nu+1)\, S^{\nu+1,k}(\varphi_k(0,t)) \tag{4-10}$$

the known (unbarred) series in equation (4-9) may be rearranged to form a progressing wave expansion with constant coefficients. The required extension may be assumed to be of similar form and therefore equation (4-9) may be expressed by

$$\underline{u}(0,t) = \sum_{\nu=0}^{m} \sum_{k=1}^{4} \left\{ \underline{d}^{\nu,k} S^{\nu,k}(\varphi_k(0,t)) + \bar{\underline{d}}^{\nu,k} S^{\nu,k}(\varphi_k(0,t)) \right\} \tag{4-11}$$

The μ linear homogeneous boundary conditions are again represented by

$$Q[\underline{u}(0,t)] = 0 \tag{4-12}$$

where Q is a μxn matrix. Substituting equation (4-7) into equation (4-11) and summing over k yields two polynomials in t. Substituting this result into equation (4-12) and equating like coefficients of t (each ν) yields sets of linear simultaneous equations in the n unknowns $\bar{\underline{d}}^{\nu}$ which permit (n-μ) degrees of freedom. These values of $\bar{\underline{d}}^{\nu}$ may now be used to obtain the leading terms in the Taylor expansion of the required reflection, and therefore in the limit as $\nu \to \infty$, the function itself. The leading reflected discontinuities for various boundary conditions are obtained using this procedure in the following section.

4.b Reflection of Progressing Waves

The discontinuities in the waves reflected from an imposed boundary condition are characterized by the leading terms of the progressing wave expansion. To obtain the order and magnitude of these reflected discontinuities for various boundary conditions the method of matching solutions at the boundary developed in Section 4.a will be used. Results are obtained for the semi infinite span and their relation to the finite beam is discussed at the end of the chapter.

As previously discussed, obtaining boundary conditions in linear homogeneous forms for conditions other than a simple support may require the addition of auxiliary equations. For the physical boundary conditions considered in this chapter, these auxiliary equations take the form

$$\begin{aligned} y_t - v &= 0 \\ \psi_t - \omega &= 0 \end{aligned} \tag{4-13}$$

and when included in the basic equations (1-3) yield

$$\underline{Y}_t + P\underline{Y}_x + F\underline{Y} = 0 \tag{4-14}$$

where $$Y = \begin{pmatrix} M \\ \omega \\ V \\ v \\ y \\ \psi \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & (\ell/r)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -(\ell/r)^2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

These auxiliary equations introduce two additional null characteristics into the previous family of characteristic curves, that is,

$$\begin{aligned} \varphi_5 &= x - c_5 t \\ \varphi_6 &= x - c_6 t \end{aligned} \tag{4-15}$$

with $c_5=c_6=0$.

The modified canonical form is

$$\underline{Z}_t + J\underline{Z}_x + D\underline{Z} = 0 \tag{4-16}$$

where

$$\underline{Z} = \begin{pmatrix} \gamma M+\omega \\ \gamma M-\omega \\ v-V \\ v+V \\ y \\ \psi \end{pmatrix} \qquad D = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -(\ell/r)^2 & (\ell/r)^2 & 0 & 0 & 0 & 0 \\ (\ell/r)^2 & -(\ell/r)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ -(\ell/r)^2 & (\ell/r)^2 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4-17}$$

and

$$J = \mathrm{diag}\{\gamma, -\gamma, 1, -1, 0, 0\} \tag{4-18}$$

with the added characteristic matrices

$$A^5 = \mathrm{diag}\{\gamma, -\gamma, 1\ {-1}\ 0\ 0\}$$
$$A^6 = \mathrm{diag}\{\gamma, -\gamma, 1, -1, 0\ 0\} \tag{4-19}$$

and their associated null vectors

$$\underline{\ell}^5 = (\underline{r}^5)^T = [0\ 0\ 0\ 0\ 1\ 0]$$
$$\underline{\ell}^6 = (\underline{r}^6)^T = [0\ 0\ 0\ 0\ 0\ 1] \tag{4-20}$$

To illustrate the procedure the following two boundary conditions will be considered:

Vertical translation restrained by a linear spring exhibiting Kelvin damping represented by

$$V = \alpha y + \eta v \tag{4-21}$$

which in terms of the elements of the vector $\underline{Z}$ becomes

$$(\eta-1) Z^{(4)} + (\eta+1) Z^{(3)} + 2\alpha Z^{(5)} = 0 \tag{4-22}$$

and angular rotation restrained by a similar torsional viscoelastic spring represented by

$$M = -(\beta\psi + \delta\omega) \tag{4-23}$$

which becomes

$$-(\gamma\delta+1)Z^{(1)} - (\gamma\delta-1)Z^{(2)} - 2\gamma\beta Z^{(6)} = 0 \tag{4-24}$$

where (α,β) and (η,δ) are the spring and damping coefficients respectively.

In terms of the six dimensional vector $\underline{Z}$, the uniform velocity input for the semi infinite span is represented by

$$\underline{Z}(x,0) = \begin{bmatrix} 0 \\ 0 \\ v_0 \\ v_0 \\ 0 \\ 0 \end{bmatrix} S_a^{o}(x) \tag{4-25}$$

where $\quad S_a^o(x)=0 \qquad$ for $x<0$ (4-26)

while the extended initial data may be given by

$$\underline{Z}(x,0) = \underline{Z}_a(x)\, S_a^o(x) + \sum_{\nu=0}^{m} \underline{Z}_r^{\nu}(x)\, S_r^o(x) \tag{4-27}$$

where $\quad \underline{Z}_a(x) = \begin{bmatrix} 0 \\ 0 \\ v_0 \\ v_0 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{Z}_r^{\nu}(x) = \begin{bmatrix} Z_r^{\nu(1)}(x) \\ Z_r^{\nu(2)}(x) \\ Z_r^{\nu(3)}(x) \\ Z_r^{\nu(4)}(x) \\ Z_r^{\nu(5)}(x) \\ Z_r^{\nu(6)}(x) \end{bmatrix}$ (4-28)

$$S_r^{\nu}(x) = (-1)^{\nu} S_a^{\nu}(x) \tag{4-29}$$

and the subscripts a and r denote the actual and reflected initial data respectively.

As can be seen from the previous results for the four dimensional vector $\underset{\sim}{U}$, the first term in the progressing wave expansion of $\underset{\sim}{Z}$ becomes

$$\underset{\sim}{Z}^{o}(x,t) = \begin{pmatrix} 0 \\ 0 \\ \sigma_o S_a^o(\varphi_3) \\ \sigma_o S_a^o(\varphi_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} Z_r^{o(1)}(x)\, S_r^o(\varphi_1) \\ Z_r^{o(2)}(x)\, S_r^o(\varphi_2) \\ Z_r^{o(3)}(x)\, S_r^o(\varphi_3) \\ Z_r^{o(4)}(x)\, S_r^o(\varphi_4) \\ Z_r^{o(5)}(x)\, S_r^o(\varphi_5) \\ Z_r^{o(6)}(x)\, S_r^o(\varphi_6) \end{pmatrix} \tag{4-30}$$

Evaluating equation (4-30) at the proposed boundary x=0, and using relationships (4-26) and (4-29) yields

$$\underset{\sim}{Z}^{o}(0,t) = \begin{pmatrix} Z_r^{o(1)}(0)\, S_a^o(\gamma t) \\ 0 \\ Z_r^{o(3)}(0)\, S_a^o(t) \\ \sigma_o S_a^o(t) \\ 0 \\ 0 \end{pmatrix} \tag{4-31}$$

Applying the boundary conditions described by equations (4-22) and (4-24) results in

$$\begin{aligned} &[(\eta-1)\sigma_o + (\eta+1) Z_r^{o(3)}(0)]\, S_a^o(t) = 0 \\ &-(\gamma\delta+1)\, Z_r^{o(1)}(0)\, S_a^o(\gamma t) = 0 \end{aligned} \tag{4-32}$$

which yields

$$\begin{aligned} Z_r^{o(3)}(0) &= -\frac{(\eta-1)}{\eta+1}\,\sigma_o \\ Z_r^{o(1)}(0) &= 0 \end{aligned} \tag{4-33}$$

and where the value of the other elements remain arbitrary, for simplicity say

$$Z_r^{o(4)}(0) = Z_r^{o(3)}(0) \qquad (4\text{-}34)$$
$$Z_r^{o(2)}(0) = Z_r^{o(5)}(0) = Z_r^{o(6)}(0) = 0$$

Using calculations analogous to Section 3.b, the second term of the progressing wave expansion resulting from initial data (4-28) is obtained as

$$\underline{Z}^1(x,t) = v_0 \begin{bmatrix} (\ell/r)^2 ab\, S_a'(\varphi_1) - \frac{1}{2}(\ell/r)^2 a\, S_a'(\varphi_3) + \frac{1}{2}(\ell/r)^2 b\, S_a'(\varphi_4) \\ -(\ell/r)^2 ab\, S_a'(\varphi_2) - \frac{1}{2}(\ell/r)^2 b\, S_a'(\varphi_3) + \frac{1}{2}(\ell/r)^2 a\, S_a'(\varphi_4) \\ -\frac{1}{2}(\ell/r)^2 abt\, S_a'(\varphi_3) \\ \frac{1}{2}(\ell/r)^2 abt\, S_a'(\varphi_4) \\ 0 \\ 0 \end{bmatrix} +$$

$$\begin{bmatrix} \sigma_r^{1,1}(x,t) S_r'(\varphi_1) - \frac{1}{2}(\ell/r)^2 a Z_r^{o(3)}(x) S_r'(\varphi_3) + \frac{1}{2}(\ell/r)^2 b Z_r^{o(4)}(x) S_r'(\varphi_4) \\ \sigma_r^{1,2}(x,t) S_r'(\varphi_2) - \frac{1}{2}(\ell/r)^2 b Z_r^{o(3)}(x) S_r'(\varphi_3) + \frac{1}{2}(\ell/r)^2 a Z_r^{o(4)}(x) S_r'(\varphi_4) \\ -\frac{1}{2} a Z_r^{o(1)}(x) S_r'(\varphi_1) - \frac{1}{2} b Z_r^{o(2)}(x) S_r'(\varphi_2) + \sigma_r^{1,3}(x,t) S_r'(\varphi_3) \\ \frac{1}{2} b Z_r^{o(1)}(x) S_r'(\varphi_1) + \frac{1}{2} a Z_r^{o(2)}(x) S_r'(\varphi_2) + \sigma_r^{1,4}(x,t) S_r'(\varphi_4) \\ -\frac{1}{2}(\ell/r)^2 Z_r^{o(3)}(x) S_r'(\varphi_3) + \frac{1}{2}(\ell/r)^2 Z_r^{o(4)}(x) S_r'(\varphi_4) + \sigma_r^{1,5}(x,t) S_r'(\varphi_5) \\ \frac{ab}{a+b}(\ell/r)^2 Z_r^{o(1)}(x) S_r'(\varphi_1) - \frac{ab}{a+b}(\ell/r)^2 Z_r^{o(2)}(x) S_r'(\varphi_2) + \sigma_r^{1,6}(x,t) S_r'(\varphi_6) \end{bmatrix} \qquad (4\text{-}35)$$

where

$$\sigma_r^{1,k}(x,t) = \begin{pmatrix} \frac{1}{2}(\ell/r)^2 ab\left[\gamma Z_r^{o(1)}(x)t + b^{-1} Z_r^{o(3)}(x) - a^{-1} Z_r^{o(4)}(x)\right] + Z_r^{1(1)}(x) \\ \frac{1}{2}(\ell/r)^2 ab\left[\gamma Z_r^{o(2)}(x)t - a^{-1} Z_r^{o(3)}(x) + b^{-1} Z_r^{o(4)}(x)\right] + Z_r^{1(2)}(x) \\ -\frac{1}{2} ab\left[(\ell/r)^2 Z_r^{o(3)}(x)t - b^{-1} Z_r^{o(1)}(x) - a^{-1} Z_r^{o(2)}(x)\right] + Z_r^{1(3)}(x) \\ \frac{1}{2} ab\left[(\ell/r)^2 Z_r^{o(4)}(x)t - a^{-1} Z_r^{o(1)}(x) - b^{-1} Z_r^{o(2)}(x)\right] + Z_r^{1(4)}(x) \\ \frac{1}{2}(\ell/r)^2\left[Z_r^{o(3)}(x) - Z_r^{o(4)}(x)\right] + Z_r^{1(5)}(x) \\ \frac{ab}{a+b}(\ell/r)^2\left[Z_r^{o(1)}(x) + Z_r^{o(2)}(x)\right] + Z_r^{1(6)}(x) \end{pmatrix} \qquad (4\text{-}36)$$

Evaluating equation (4-35) at x=0 and using equations (4-26) and (4-29) together with the values ascribed to $\underline{Z}_r^{O}(0)$ yields

$$\underline{Z}_r^1(0,t) = \begin{pmatrix} \frac{v_0}{2}(\ell/r)^2\left(b - a\frac{\eta-1}{\eta+1}\right)S_a^1(t) + \left[v_0(\ell/r)^2 ab\frac{\eta-1}{\eta+1} - Z_r^{1(1)}(0)\right]S_a^1(\gamma t) \\ \frac{v_0}{2}(\ell/r)^2\left(a - b\frac{\eta-1}{\eta+1}\right)S_a^1(t) - v_0(\ell/r)^2 ab\, S_a^1(\gamma t) \\ -\frac{v_0}{2}ab(\ell/r)^2\frac{\eta-1}{\eta+1}S_a^1(t) - Z_r^{1(3)}(0)\, S_a^1(t) \\ \frac{v_0}{2}(\ell/r)^2 abt\, S_a^1(t) \\ -\frac{v_0}{2}(\ell/r)^2\frac{\eta-1}{\eta+1}S_a^1(t) \\ 0 \end{pmatrix} \tag{4-37}$$

Again applying boundary conditions (4-22) and (4-24) yields

$$\left[(\eta+1)Z_r^{1(3)}(0) + 2\alpha\left(v_0\frac{1}{2}(\ell/r)^2\frac{\eta-1}{\eta+1}\right)\right]S_a^1(t) = 0$$

$$\left[(1-\gamma\delta)\frac{v_0}{2}(\ell/r)^2\frac{2}{\gamma^2-1}\left(\frac{\eta-1}{\eta+1}\gamma - 1\right)\right]S_a^1(t) \tag{4-38}$$

$$+\left\{-(\gamma\delta+1)\left[(\ell/r)^2\frac{v_0}{\gamma^2-1}\frac{\eta-1}{\eta+1} - Z_r^{1(1)}(0) + (1-\gamma\delta)(\ell/r)^2\frac{v_0}{\gamma^2-1}\right\}S_a^1(\gamma t) = 0$$

Noting that

$$S_a^1(\gamma t) = \gamma\, S_a^1(t) \tag{4-39}$$

equations (4-38) result in

$$Z_r^{1(3)}(0) = -\alpha v_0(\ell/r)^2\frac{\eta-1}{(\eta+1)^2} \tag{4-40}$$

$$Z_r^{1(1)}(0) = v_0(\gamma\delta+1)^{-1}(\gamma^2-1)^{-1}(\ell/r)^2\, 2\eta\,(\gamma^2\delta+1-\gamma)(\eta+1)^{-1}$$

and the other elements of $Z_r^1(0)$ remain arbitrary. Again, for simplicity these are assigned the following values

$$\begin{aligned} Z_r^{1(4)}(0) &= Z_r^{1(3)}(0) \\ Z_r^{1(2)}(0) &= -Z_r^{1(1)}(0) \\ Z_r^{1(5)}(0) &= Z_r^{1(6)}(0) = 0 \end{aligned} \tag{4-41}$$

Physically these degrees of freedom represent the choices of reflecting the velocity or shear and the moment or angular velocity as initial data. The above selections represent reflections of vertical and angular velocities.

It should be noted at this point that the time dependent terms in equations (4-38) cancelled identically. As indicated in Section 4.a this will not be the case in general, requiring the transfer of these time dependent coefficients of order ν to the $(\nu+1)$ terms via the relationship

$$c_k t S^{\nu}(c_k t) = (\nu+1) S^{\nu+1}(c_k t) \tag{4-42}$$

thus producing constant coefficients for each order ν.

The determination of $\underline{Z}_r^o(o)$ and $\underline{Z}_r^1(0)$ may be summarized as follows: The first two terms of the progressing wave expansions of the required reflections involve both the vertical and angular velocities. The functions are sketched in Figure 18.a and their reduction in the case of zero damping in Figure 18.b. These figures reveal a discontinuous extension of the initial velocity which reduces to a continuous function for the case of zero damping of the vertical spring. The magnitude of the discontinuity in its first derivative is shown to be proportional to the spring parameter α and becomes infinite, corresponding to a discontinuous extension, in the limiting case of a rigid support.

The extension of the angular velocity is continuous and exhibits a continuous first derivative for a zero value of torsional spring damping. For finite δ this extension has a discontinuous first derivative which is independent of α, β and η.

Having determined the required reflection for a semi infinite span, the method of superposition may be used to obtain the equivalent reflection for a finite beam. The method however, becomes prohibitively tedious for large times as the procedures described above for the semi infinite span have to be performed for each integer multiple of the time required for a shear or dilatational wave to traverse the span, i.e. $t=m$; $t=\gamma m$, up to the desired response.

CHAPTER 5

MODIFIED PROGRESSING WAVES

As discussed in Section 3.d the expressions for the Fourier transforms of the physical quantities y,ψ,V and M in the semi infinite span may be expressed in the form

$$F_1(p)\sin\omega_1 t + F_2(p)\sin\omega_2 t \tag{5-1}$$

where ω_1 and ω_2 are the solutions to

$$(p^2-\omega^2)(\gamma^2 p^2-\omega^2) - (\ell/r)^2\omega^2 = 0 \tag{5-2}$$

A Laurent expansion of the above forms for large values of p revealed that the terms containing the leading discontinuities have the form

$$A^k(t)/(ip)^n \; e^{\pm ipc_k t}$$

and therefore yield inverses which do not diminish with time. This introduces numerical difficulties in isolating these leading discontinuities, as the solution is obtained as the difference of two large numbers representing the discontinuous and smooth components, for large times.

By introducing quantities of the form

$$B^k(t,p)/[(ip)^n + f(t)] \; e^{\pm ipc_k t}$$

these terms incorporating the discontinuities may be made to exhibit better behavior for large time, while the smoothness of the remainder remains unaffected since the $O(p^{-n})$ does not change.

5.a Modified Expression for Shear

Consider the expression representing the Fourier transform of the shear developed in Section 3.c

$$\tilde{V} = 2\left\{ \frac{\omega_1 (p^2 - \omega_2^2)}{p^2(\omega_1^2 - \omega_2^2)} \sin \omega_1 t + \frac{\omega_2 (p^2 - \omega_1^2)}{p^2(\omega_2^2 - \omega_1^2)} \sin \omega_2 t \right\} \tag{5-3}$$

Equation (5-3) may be rewritten in the form

$$\tilde{V} = -i\left\{ \frac{\omega_1 (p^2 - \omega_2^2)}{p^2(\omega_1^2 - \omega_2^2)} \left(e^{i(\omega_1 - \gamma p)t} e^{i\gamma p t} - e^{-i(\omega_1 - \gamma p)t} e^{-i\gamma p t} \right) + \frac{\omega_2 (p^2 - \omega_1^2)}{p^2(\omega_2^2 - \omega_1^2)} \left(e^{i(\omega_2 - p)t} e^{ipt} - e^{-i(\omega_2 - p)t} e^{-ipt} \right) \right\} \tag{5-4}$$

Using the asymptotic expansions of ω_1 and ω_2 for large values of p expressed by equations (3-79), the following approximations are obtained

$$\begin{aligned}
\frac{\omega_1 (p^2 - \omega_2^2)}{p^2(\omega_1^2 - \omega_2^2)} &= \frac{\gamma}{2(\gamma^2 - 1)^2} (l/r)^2 \frac{1}{p^3} + O\left(\frac{1}{p^5}\right) \\
\frac{\omega_2 (p^2 - \omega_1^2)}{p^2(\omega_2^2 - \omega_1^2)} &= \frac{1}{p} + \frac{\gamma^2 + 1}{2(\gamma^2 - 1)^2} (l/r)^2 \frac{1}{p^3} + O\left(\frac{1}{p^5}\right) \\
e^{i(\omega_1 - \gamma p)t} &= 1 + i\frac{1}{2} \frac{\gamma}{\gamma^2 - 1} (l/r)^2 \frac{t}{p} - \frac{1}{8} \frac{\gamma^2}{(\gamma^2 - 1)^2} (l/r)^4 \left(\frac{t}{p}\right)^2 + O\left(\frac{1}{p^3}\right) \\
e^{i(\omega_2 - p)t} &= 1 - i\frac{1}{2} \frac{1}{\gamma^2 - 1} (l/r)^2 \frac{t}{p} - \frac{1}{8} \frac{1}{(\gamma^2 - 1)^2} (l/r)^4 \left(\frac{t}{p}\right)^2 + O\left(\frac{1}{p^3}\right)
\end{aligned} \tag{5-5}$$

which reveal the terms of lowest order of (1/p), i.e. those that lead to the highest discontinuities, of equation (5-3) to be of the form

$$\left(-\frac{1}{ip} - \frac{Et}{p^2}\right) e^{ipt} + \left(\frac{1}{ip} - \frac{Et}{p^2}\right) e^{-ipt} \tag{5-6}$$

where

$$E = \frac{1}{2} \frac{(l/r)^2}{\gamma^2 - 1}$$

Adding and subtracting to equation (5-4) the expression

$$\frac{ip - Et}{p^2 + t^2} e^{ipt} + \frac{-ip - Et}{p^2 + t^2} e^{-ipt} \tag{5-7}$$

and inverting yields

$$V = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\Big[\frac{\omega_1(p^2-\omega_2^2)}{ip^2(\omega_1^2-\omega_2^2)}e^{i(\omega_2-p)t} + \frac{ip-Et}{p^2+t^2}\Big]e^{ip(t-x)}$$

$$+\Big[-\frac{\omega_1(p^2-\omega_2^2)}{ip^2(\omega_1^2-\omega_2^2)}e^{-i(\omega_2-p)t} + \frac{-ip-Et}{p^2+t^2}\Big]e^{-ip(t+x)} \tag{5-8}$$

$$+\frac{2\omega_2(p^2-\omega_1^2)}{p^2(\omega_2^2-\omega_1^2)}\sin\omega_1 t\Big\}dp - \frac{1}{2}\Big\{(-\operatorname{sgn}(x-t)+E)e^{-t|x-t|} + (\operatorname{sgn}(x+t)+E)e^{-t|x+t|}\Big\}$$

The two leading discontinuities have thus been isolated in the integrated term of equation (5-8) which exhibits exponential decay with time.

The magnitude of the next term $(O(1/p^3))$ is proportional to $(1/r)^4t^2$ which shows the dependence of the magnitude of the remainder of equation (5-4) on $1/r$ and t. Note therefore, that the basic difficulty of large values of $1/r$ and t remains.

A similar modification may be performed for the finite beam when considered as an infinite span subjected to the velocity input $v(x,0)=v_0S_0(x)$ as defined in Figure 8. The Poisson summation formula will then be used to obtain the smooth portion of the solution in infinite series form. The Fourier transform of the solution for the shear in the finite span is obtained from the method of superposition and equation (5-3):

$$\tilde{V} = 2\sum_{m=-\infty}^{\infty}(-1)^m\Big\{\frac{\omega_1(p^2-\omega_2^2)}{p^2(\omega_1^2-\omega_2^2)}\sin\omega_1 t + \frac{\omega_2(p^2-\omega_1^2)}{p^2(\omega_2^2-\omega_1^2)}\sin\omega_2 t\Big\}e^{ipm} \tag{5-9}$$

The Fourier transform of the progressing waves representing the two leading discontinuities in this case are expressed by

$$\sum_{m=-\infty}^{\infty}(-1)^m\left\{\left(-\frac{1}{ip}-\frac{Et}{p^2}\right)e^{ipt}+\left(\frac{1}{ip}-\frac{Et}{p^2}\right)e^{-ipt}\right\}e^{ipm} \qquad (5\text{-}10)$$

Adding and subtracting

$$\sum_{m=-\infty}^{\infty}(-1)^m\left\{\frac{ip-Et}{p^2+t^2}\,e^{ipt}+\frac{-ip-Et}{p^2+t^2}\,e^{-ipt}\right\}e^{ipm} \qquad (5\text{-}11)$$

to expression (5-10) yields

$$\sum_{m=-\infty}^{\infty}(-1)^m\left\{\frac{-ipt^2-Et^3}{p^2(p^2+t^2)}\,e^{ipt}+\frac{ipt^2-Et^3}{p^2(p^2+t^2)}\,e^{-ipt}\right\}e^{ipm} \qquad (5\text{-}12)$$

$$+\sum_{m=-\infty}^{\infty}(-1)^m\left\{\frac{ip-Et}{p^2+t^2}\,e^{ipt}+\frac{-ip-Et}{p^2+t^2}\,e^{-ipt}\right\}e^{ipm}$$

the second series of which, represents the isolation of the leading discontinuities in terms which decrease with time. This being established, expression (5-11) is added and subtracted to the integrand of (5-9) producing

$$\begin{aligned}\tilde{V}=\sum_{m=-\infty}^{\infty}(-1)^m\Bigg\{&\frac{\omega_1(p^2-\omega_2^2)}{p^2(\omega_1^2-\omega_2^2)}\sin\omega_1 t-\frac{2p}{p^2+t^2}\sin pt\\&+\frac{\omega_2(p^2-\omega_1^2)}{p^2(\omega_1^2-\omega_2^2)}\sin\omega_2 t-\frac{2Et}{p^2+t^2}\cos pt\Bigg\}e^{ipm}\\&+\sum_{m=-\infty}^{\infty}(-1)^m\left\{\frac{ip-Et}{p^2+t^2}\,e^{ipt}-\frac{-ip-Et}{p^2+t^2}\,e^{-ipt}\right\}e^{ipm}\end{aligned} \qquad (5\text{-}13)$$

The Poisson summation formula states that;

If $G(\xi)$ is the Fourier transform of $F(y)$,

then

$$\sum_{l=-\infty}^{\infty}F(l)=\sum_{j=-\infty}^{\infty}G(j) \qquad (5\text{-}14)$$

Applying this formula to the first series of equation (5-13) yields the following expression for the shear

$$\begin{aligned}V=4\sum_{n=1,3,5}^{\infty}\Bigg\{&\frac{\theta_{1n}(\pi^2n^2-\theta_{2n}^2)}{\pi^2n^2(\theta_{1n}^2-\theta_{2n}^2)}\sin\theta_{1n}t+\frac{\theta_{2n}(\pi^2n^2-\theta_{1n}^2)}{\pi^2n^2(\theta_{2n}^2-\theta_{1n}^2)}\sin\theta_{2n}t\\&+\frac{1}{2(\pi^2n^2+t^2)}\left[-\pi n\sin\pi nt-Et\cos\pi nt\right]\Bigg\}\cos\pi nx\end{aligned}$$

$$-\frac{1}{2}\sum_{\substack{n=-\infty\\(\text{odd})}}^{\infty}\left\{\left(-\operatorname{sgn}(n+x-t)+E\right)e^{-t|n+x-t|}+\left(\operatorname{sgn}(n+x+t)+E\right)e^{-t|n+x+t|}\right\} \quad (5\text{-}15)$$

where the even terms have been omitted as they represent modes symmetric about the mid span and the change of variables $n=p/\pi$ requires that the new variables θ_{1n} and θ_{2n} satisfy the equation

$$\left(\pi^2 n^2-\theta_n^2\right)\left(r^2\pi^2 n^2-\theta_n^2\right)-(l/r)^2\,\theta_n^2=0 \quad (5\text{-}16)$$

The first two terms in the first series of equation (5-15) represent an alternate form of the eigenfunction expansion for the shear developed in Chapter 2; The last term is the modification required by the isolation of the leading discontinuities into functions that diminish exponentially with time as given by the second series.

Equation (5-15) was used to extend the numerical results for the evaluation of the shear to larger values of l/r and t.

CHAPTER 6

ASYMPTOTIC ANALYSIS

In the previous methods of analysis reasonable convergence of the series solutions is limited to moderate values of the slenderness ratio and time. In particular, the convergence of the standard eigenfunction solution worsens in proportion to the product $1/r \cdot t$ while the magnitudes of a number of the leading terms in the progressing wave expansion are proportional to $(1/r)^2 t$.

The following asymptotic analyses are performed to obtain the nature of the response at large values of the slenderness ratio and time. While the methods employed are quite general, the following results are limited to the finite and semi infinite simply supported span.

6.a Method of Stationary Phase

The nature of the shear response in the semi infinite span in the vicinity of the wave fronts may be obtained by stationary phase analyses of the Fourier transform of its solution developed in Chapter 3.

The shear may be expressed in the following form

$$V = V_1 + V_2 \qquad (6\text{-}1)$$

where

$$V_1 = \frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty} G_1(p)\, e^{i(\omega_1 t - px)}\, dp - \int_{-\infty}^{\infty} G_1(p)\, e^{-i(\omega_1 t + px)}\, dp\right\} \qquad (6\text{-}2)$$

$$V_2 = \frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty} G_2(p)\, e^{i(\omega_2 t - px)}\, dp - \int_{-\infty}^{\infty} G_2(p)\, e^{-i(\omega_2 t + px)}\, dp\right\} \qquad (6\text{-}3)$$

$$G_1(p) = \frac{\omega_1 (p^2 - \omega_2^2)}{p^2(\omega_1^2 - \omega_2^2)} \tag{6-4}$$

$$G_2(p) = \frac{\omega_2 (p^2 - \omega_1^2)}{p^2(\omega_2^2 - \omega_1^2)} \tag{6-5}$$

and ω_1 and ω_2 satisfy the equation

$$(p^2 - \omega^2)(r^2 p^2 - \omega^2) - (\ell/r)^2 \omega^2 = 0 \tag{6-6}$$

The integrals (6-2) and (6-3) have a meaning as ordinary functions as can be seen from the following reasoning: The shear V can be identified with an ordinary function of x with bounded support $(-t \leq x \leq t)$ and piecewise continuous derivatives. It, therefore, satisfies sufficient conditions for the Fourier integral theorem and can therefore be obtained as the inverse Fourier transform of the right hand side of (3-75). Each of the four integrals appearing in V_1 and V_2 can be shown to be convergent, as long as $x^2 - t^2 \neq 0$, by use of equations (5-5) and, therefore, equation (6-1) is justified.

Consider the integral

$$\int_{-\infty}^{\infty} h(p)\, e^{itg(p)}\, dp \tag{6-7}$$

where t is large and positive and g(p) is a real function of p. According to the method of stationary phase, the major contribution to the integral arises from the values of p in the vicinity of the stationary points of g(p), i.e. where $\frac{d}{dp} g(p) = 0$. Applying this criterion to equation (6-1) for large times yields

$$\frac{d}{dp}\left(\omega \mp p\, x/t\right) = 0 \tag{6-8}$$

or

$$\frac{d\omega}{dp} = \pm\, x/t \tag{6-9}$$

In the vicinity of the shear wave front $x/t \approx 1$ and equation (6-9) reduces to finding the values of p for which

$$\frac{d\omega}{dp} = \pm 1 \tag{6-10}$$

Consider the integral V_2: ω_2 satisfies equation (6-10) for large values of $|p|$. The stationary points of the first integral of V_2 occurs for $p \to \infty$ while the stationary points of the second occur for $p \to -\infty$. Combining these two integrals and using the asymptotic expansions for $G_1(p)$ and $G_2(p)$ developed in Section 5.a the first approximation reduces V_2 to the integral[9]

$$V_2 = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1}{p} e^{ip(x-t)} e^{-\frac{iE}{p}t} dp \tag{6-11}$$

where $\quad E = [2(\gamma^2-1)]^{-1} (\ell/r)^2$

Integral (6-11) is tabulated and results in

$$V_2 \simeq - J_0 \left(2 \sqrt{\frac{t(x-t)}{2(\gamma^2-1)} (\ell/r)^2} \right) H(x-t) \tag{6-12}$$

For large values of $(l/r)^2 t$ equation (6-12) is expressed by

$$V_2 \simeq \left(\pi^2 t (x-t) E \right)^{-1/4} \sin \left(2\sqrt{t(x-t)E} - \pi/4 \right) \tag{6-13}$$

which reveals high oscilatory variations for large l/r and t with circular frequencies proportional to l/r.

The major contribution to V_1 occurs when ω_1 satisfies equation (6-10), which is attained when $|p| = o(l/r)$. This results in the ratio

$$V_1/V_2 = O(t^{-1/4}) \tag{6-14}$$

and therefore V_1 may be neglected for large time.

An analogous procedure may be followed for the region near the dilatational wave front. For this case (6-9) requires

$$\frac{d\omega}{dp} = \pm \gamma \tag{6-15}$$

which ω_1 satisfies for large values of $|p|$. Using the same asymptotic expansion for $G_1(p)$ leads to the following asymptotic expression for V_1

$$V_1 \simeq \frac{1}{\pi} \lim_{\beta \to 0} \int_{-\infty}^{\infty} \frac{E'}{(p+\beta)^2} e^{ip(\gamma t - x)} e^{\frac{i\gamma E t}{p+\beta}} dp \tag{6-16}$$

where $\quad E' = \frac{\gamma}{(\gamma^2-1)^2} (l/r)^2$

Evaluating equation (6-16)

$$V_1 \simeq \sqrt{2\gamma}\,(\gamma^2-1)^{-3/2}(l/r)\left(\frac{\gamma t - x}{t}\right)^{1/2} J_2\left(\sqrt{\frac{2\gamma (l/r)^2 t(\gamma t - x)}{\gamma^2-1}}\right) H(\gamma t - x) \tag{6-17}$$

which similarly contains high oscillatory variations for large values of l/r.

The integral V_2, having no stationary points in this region, may be shown to be $o((t(l/r))^{-2})$.

The solutions obtained represent the asymptotic behavior of the response in the vicinity of the wave fronts for large times and although developed for the semi infinite span, equation (6-12) is applicable to the finite case. This stems from the fact that, as indicated from previous results, for large l/r the shear response in the vicinity of the shear fronts is dominated by the reaction to the front itself. The resulting validity of equation (6-12) is shown in Figure 19.

Stationary phase approximations may also be used to determine the effect of damping in the material on the leading terms of the progressing wave expansion.

The response of a simply supported finite Timoshenko beam with Maxwell damping subjected to a uniform velocity input has been obtained by Pan[11] using an eigenfunction analysis and results in the following expression for the shear

$$V = \sum_{\substack{n=-\infty \\ (odd)}}^{\infty} \frac{1}{\sqrt{\Delta_n}} \frac{4n\pi\omega_n^2}{n^2\pi^2 + (\ell/r)^{-2}(n^2\pi^2-\omega_n^2)^2} e^{-\frac{t}{\eta}} \sin\sqrt{\Delta_n}t \cos n\pi x \tag{6-18}$$

where $$\Delta_n = \omega_n^2 - \eta^{-2} \tag{6-19}$$

and η represents the damping parameter previously defined and ω_n represent the solutions to

$$\omega_n^4 - [(\gamma^2+1) + (\ell/r)^2]\omega_n^2 + \gamma^2(n\pi)^4 = 0 \tag{6-20}$$

Using the Poisson summation formula to express equation (6-18) in integral form and considering only anti symmetric modes results in

$$V = \frac{1}{2\pi i} e^{-\frac{t}{\eta}} \sum_{\substack{k=-\infty \\ (odd)}}^{\infty} \int_{-\infty}^{\infty} \frac{4\omega^2}{\sqrt{\Delta}\,[p^2 + (\ell/r)^2(p^2-\omega^2)^2]} \left(e^{i\sqrt{\Delta}t} - e^{-i\sqrt{\Delta}t}\right)(-1)^k e^{ip(k-x)}\,dp \tag{6-21}$$

which may be rewritten as

$$V = e^{-\frac{t}{\eta}} (V_I + V_{II}) \tag{6-22}$$

where

$$\begin{aligned} V_I &= \frac{1}{2\pi i} \sum_{\substack{k=-\infty \\ (odd)}}^{\infty} \int_{-\infty}^{\infty} H_1(p)\left(e^{i(\sqrt{\Delta_I}t + p(k-x))} - e^{-i(\sqrt{\Delta_{II}}t - p(k-x))}\right)dp \\ V_{II} &= \frac{1}{2\pi i} \sum_{\substack{k=-\infty \\ (odd)}}^{\infty} \int_{-\infty}^{\infty} H_2(p)\left(e^{i(\sqrt{\Delta_{II}}t + p(k-x))} - e^{-i(\sqrt{\Delta_I}t - p(k-x))}\right)dp \\ H_1(p) &= \frac{4\omega_I^2}{\sqrt{\Delta_I}\,[p^2 + (\ell/r)^2(p^2-\omega_I^2)^2]} \\ H_2(p) &= \frac{4\omega_{II}^2}{\sqrt{\Delta_{II}}\,[p^2 + (\ell/r)^2(p^2-\omega_{II}^2)^2]} \end{aligned} \tag{6-23}$$

and ω_I, ω_{II} $(\omega_I > \omega_{II})$ represent the roots of equation (6-20) with $p = n\pi$.

The stationary points of V_I and V_{II} for large t, occur when

$$t \frac{\partial}{\partial p}\left(\sqrt{\Delta} \pm p\left(\frac{k-x}{t}\right)\right) = 0 \tag{6-24}$$

or equivalently when

$$\frac{1}{\sqrt{\Delta}} \frac{\partial \Delta}{\partial p} = \pm \frac{k-x}{t} \tag{6-25}$$

In the vicinity of a given shear front, for simplicity let k=0, equation (6-25) reduces to

$$\frac{1}{\sqrt{\Delta}} \frac{\partial \Delta}{\partial p} = \pm 1 \tag{6-26}$$

which Δ_{II} satisfies for large $|p|$.

In this region

$$\begin{aligned} \omega_{II} &= p - \frac{1}{2}\frac{1}{\gamma^2-1}(\ell/r)^2 \frac{1}{p} + O\left(\frac{1}{p^3}\right) \\ \Delta_{II}^{1/2} &= p - \left(\frac{1}{2}\frac{1}{\gamma^2-1}(\ell/r)^2 + \frac{1}{2\eta^2}\right)\frac{1}{p} + O\left(\frac{1}{p^3}\right) \end{aligned} \tag{6-27}$$

and the first approximation reduces V_{II} to the integral

$$V_{II} \simeq \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1}{p}\, e^{ip(x-t)}\, e^{\frac{irt}{p}}\, dp \tag{6-28}$$

where $r = [2(\gamma^2-1)]^{-1}(\ell/r)^2 + (2\eta^2)^{-1}$

Evaluating integral (6-28) yields

$$V_{II} \simeq -J_0\left(2\sqrt{t(x-t)r}\right) H(x-t) \tag{6-29}$$

V_I has no stationary points in this vicinity and the resulting contribution of V_I to the shear may be shown to be negligible for large times[17] and therefore the asymptotic expression for equation (6-18) is

$$V = e^{-\frac{t}{\eta}} J_0\left(2\sqrt{t(x-t)r}\right) H(x-t) \tag{6-30}$$

6.b Singular Perturbation

The asymptotic results obtained in Section 6.a which are valid in a loosely defined vicinity of the wave fronts are further restricted to large times. The method of singular perturbation is used in this section to develop asymptotic solutions of boundary layer type which are valid along the entire characteristic curves.

The following analysis is performed for the semi infinite span and uses the Timoshenko equations in their second order form

$$y_{tt} - y_{xx} + \psi_x = 0 \qquad (6\text{-}31)$$

$$\epsilon^2 (\psi_{tt} - \gamma^2 \psi_{xx}) - y_x + \psi = 0$$

where $\epsilon = (\ell/r)^{-1}$ (6-32)

The asymptotic behavior of equation (6-31) for small values of ϵ represents a problem in singular perturbation since the zero'th order problem is of lower order than the governing equation. Equation (6-31) yields a trivial solution for small ϵ, unless ψ exhibits large gradients which may be anticipated between the wave fronts. The following transformation to coordinates along the wave fronts is therefore considered (Figure 20).

$$\xi = \sqrt{\gamma^2+1}\,(\gamma-1)^{-1}(x-t) \qquad \tau = -\sqrt{2}\,(\gamma-1)^{-1}(x-\gamma t) \qquad (6\text{-}33)$$

Under this transformation ξ=const. and τ=const. represent the dilatational and shear wave fronts and equations (6-31) become

$$-2\sqrt{2}\sqrt{1+\gamma^2}\,\psi_{\xi\tau}+2(\gamma+1)\psi_{\tau\tau}+\sqrt{1+\gamma^2}\,\psi_{\xi}-\sqrt{2}\,\psi_{\tau}=0$$
$$\epsilon\left\{-(1+\gamma^2)(\gamma+1)\psi_{\xi\xi}+2\sqrt{2}\,\gamma\sqrt{1+\gamma^2}\,\psi_{\xi\tau}\right\}-\sqrt{1+\gamma^2}\,\psi_{\xi}+\sqrt{2}\,\psi_{\tau}+(\gamma-1)\psi=0 \qquad (6\text{-}34)$$

Consider the behavior of ψ behind the dilatational wave front: As previously described, the method of characteristics reveals that jumps in ψ and its derivatives are limited to occur across the wave fronts. It is therefore assumed that these large gradients of ψ occur in its derivatives exterior to the ξ coordinate axis and therefore a stretch variable of the form $T=\tau/f(\epsilon)$ is required. Utilizing the previously obtained results which exhibited variations with a circular frequency proportional to ϵ^{-1} and the observation that the number of terms required for convergence of the eigenfunction series solutions seems to be proportional to t/ϵ, f(t) is taken to be ϵ

Under the stretch $T=\tau/\epsilon$ equations (6-34) yield

$$-2\sqrt{2}\sqrt{1+\gamma^2}\,\epsilon\,\psi_{\xi T}-2(\gamma+1)\psi_{TT}+\sqrt{1+\gamma^2}\,\epsilon^2\psi_{\xi}-\sqrt{2}\,\epsilon\,\psi_{T}=0$$
$$-\epsilon^3(1+\gamma^2)(\gamma+1)\psi_{\xi\xi}+\epsilon^2 2\sqrt{2}\,\gamma\sqrt{1+\gamma^2}\,\psi_{\xi T}-\epsilon\sqrt{1+\gamma^2}\,\psi_{\xi}+\sqrt{2}\,\psi_{T}+(\gamma-1)\epsilon\psi=0 \qquad (6\text{-}35)$$

Neglecting higher order terms for derivatives of each order, the perturbed equations are

$$2(\gamma+1)\psi_{TT}-\sqrt{2}\,\epsilon\,\psi_{T}=0$$
$$\epsilon^2 2\sqrt{2}\,\gamma\sqrt{1+\gamma^2}\,\psi_{\xi T}+\sqrt{2}\,\psi_{T}+(\gamma-1)\epsilon\psi=0 \qquad (6\text{-}36)$$

The solution to equations (6-36) requires the following data along $\xi=0$ and $T=0$:

$$y(\xi,0)\,;\ y_T(\xi,0)\,;\ \psi(\xi,0)\,;\ \psi_\xi(\xi,0)\,;\ \psi(0,T) \tag{6-37}$$

Data along T=0 may be obtained from the given initial conditions

$$y(x,0)=\psi(x,0)=\psi_t(x,0)=0\quad;\ y_t(x,0)=v_0\,\mathrm{sgn}\,x$$

since the zone between the initial line and the ξ coordinate axis represents an undisturbed region. The resulting zero'th order data is:

$$y(\xi,0) = (1+\gamma^2)^{-1/2}\,\xi \tag{6-38}$$

$$\psi_T(\xi,0) = 0 \tag{6-39}$$

$$\psi(\xi,0) = 0 \tag{6-40}$$

$$\psi_\xi(\xi,0) = 0 \tag{6-41}$$

$$y_T(\xi,0) = 0 \tag{6-42}$$

To obtain initial values along $\xi=0$ the leading terms of the progressing wave expansion are used. The two leading terms of equation (3-50) yield

$$\psi(0,T) = \frac{1}{4\gamma}\,\frac{\gamma-1}{\gamma+1}\,T^2 \tag{6-43}$$

The region of validity of these terms and therefore of the initial data (6-43) are observed to be $O(\epsilon)$.

Equations (6-36) may now be solved together with the data specified by equations (6-38) thru (6-42). Integrating equation (6-36) and using conditions (6-40) and (6-42) yields

$$y_T = \frac{\sqrt{2}}{2(\gamma+1)}\,\epsilon\psi \tag{6-44}$$

Substituting this relationship into the second of equations (6-36) results in the following partial differential equation for Ψ.

$$\epsilon \Psi_{\xi\tau\tau} + a \Psi_{\tau} = 0 \tag{6-45}$$

where $a = \gamma \left(2\sqrt{2}\sqrt{1+\gamma^2}\,(\gamma+1)\right)^{-1}$

The Laplace transform of equation (6-45) from the T to the q space yields

$$\hat{\Psi}(\xi,q) = h(q)\, e^{-\frac{a}{\epsilon q}\xi} \tag{6-46}$$

Using the Laplace transform of equation (6-43) to solve for $h(q)$ results in the following expression for the Laplace transform of Ψ

$$\hat{\Psi}(\xi,q) = \frac{\gamma-1}{2\gamma(\gamma+1)}\,\frac{1}{q^3}\, e^{-\frac{a}{\epsilon q}\xi} \tag{6-47}$$

whose inverse is

$$\Psi(\xi,\tau) = \frac{\gamma-1}{2\gamma(\gamma+1)}\left(\frac{\tau\epsilon}{a\xi}\right) J_2(z) \tag{6-48}$$

where $z = 2\sqrt{\frac{a\xi\tau}{\epsilon}}$

Equation (6-44) may now be integrated and together with condition (6-38) yield the following expression for y.

$$y(\xi,\tau) = (1+\gamma^2)^{-1/2}\left\{\xi + \frac{\gamma-1}{8(\gamma+1)^3}\,\epsilon\left(\frac{\tau\epsilon}{a\xi}\right)^{3/2} J_3(z)\right\} \tag{6-49}$$

Using the relationships

$$V = y_x - \psi \quad ; \quad M = -\psi_x / (\ell/r)^2 \tag{6-50}$$

the resulting expressions for the moment and shear are

$$M(\xi,T) = \frac{-\epsilon^2}{2\gamma(\gamma+1)} \left\{ \sqrt{1+\gamma^2} \left(\frac{\epsilon T^3}{a\xi^3} \right)^{1/2} J_3(z) + \sqrt{2} \left(\frac{T}{a\epsilon\xi} \right)^{1/2} J_1(z) \right\} \tag{6-51}$$

$$V(\xi,T) = \frac{1}{\gamma-1}\xi - \left\{ \epsilon \frac{\sqrt{2}\sqrt{1+\gamma^2}}{4\gamma(\gamma+1)^2} \left(\frac{\epsilon T^2}{a\xi^2} \right) J_4(z) + \frac{\gamma}{2(\gamma+1)^2} \left(\frac{T\epsilon}{a\xi} \right) J_2(z) \right\} \tag{6-52}$$

Figures 21 and 22 show reasonable agreement between the asymptotic results for M and V expressed by equations (6-51) and (6-52) and the previously obtained solutions within the range of validity of the data. By increasing this range, the validity of the asymptotic solution may be similarly extended. For example, this is performed numerically for $\epsilon = 1/60$, by obtaining a polynomial approximation for $\psi(0,t)$ from the solution evaluated at $x=t$. This results in the following approximation for $\psi(0,T)$

$$\psi'(0,T) = (\epsilon T)^2 \left(a_3 T^2 + a_2 T + a_1 \right) \tag{6-53}$$

where $a_1 = 102.$ $\quad a_2 = -30 \quad a_3 = 74.$

and whose Laplace transform is expressed by

$$\bar{\psi}'(0,q) = \sum_{i=1}^{3} \frac{b_i}{q^{i+2}} \tag{6-54}$$

where $\quad b_i = \epsilon^2 a_i (i+1)!$

Using equation (6-53) as data for equation (6-46) instead of condition (6-43) yields

$$\hat{\Psi}'(\xi,q) = \sum_{i=1}^{3} \frac{b_i}{q^{i+2}} e^{-\frac{a}{\epsilon}q\xi} \tag{6-55}$$

and therefore

$$\Psi'(\xi,\tau) = \sum_{i=1}^{3} b_i \left(\frac{\tau\epsilon}{a\xi}\right)^{\frac{i+2}{2}} J_{i+2}(z) \tag{6-56}$$

and

$$y'(\xi,\tau) = \sum_{i=1}^{3} \frac{\sqrt{2}\,\epsilon}{2(\gamma+1)} b_i \left(\frac{\tau\epsilon}{a\xi}\right)^{\frac{i+3}{2}} J_{i+3}(z) \tag{6-57}$$

Substituting equations (6-56) and (6-57) into (6-50) yields the modified expressions for shear and moment

$$V' = \frac{-\sqrt{2}\,\epsilon}{2(\gamma+1)} \sum_{i=1}^{3} b_i \left(\frac{\tau\epsilon}{a\xi}\right)^{\frac{i+2}{2}} \left[\frac{\sqrt{1+\gamma^2}}{\gamma-1}\left(\frac{\tau}{\xi}\right) J_{i+4}(z) + \frac{\gamma^2}{\gamma-1} J_{i+2}(z)\right] \tag{6-58}$$

$$M' = -\epsilon^2 \sum_{i=1}^{3} \frac{1}{\gamma-1} b_i \left[\sqrt{1+\gamma^2}\left(\frac{\tau\epsilon}{a\xi}\right)^{\frac{i+2}{2}}\left(\frac{a\tau}{\xi\epsilon}\right) J_{i+3}(z) + \frac{\sqrt{2}}{\epsilon}\left(\frac{\tau\epsilon}{a\xi}\right)^{\frac{i+2}{2}} J_{i+1}(z)\right] \tag{6-59}$$

which are similarly plotted in Figures 21 and 22 revealing the extended range of validity of the solutions.

A similar analysis of the response in the vicinity of the shear wave front may be performed. In this case however, the resulting equations require data along the entire τ coordinate axis, which are not readily available.

CHAPTER 7

EULER BERNOULLI THEORY

Although the elementary Euler Bernoulli theory is unable to correctly describe the response of the finite simply supported span subjected to a uniform velocity input, its applicability to the case of spring supports is studied in this chapter.

The elementary Euler Bernoulli model of transverse bending results in the partial differential equation

$$y_{xxxx} + (\ell/r/\gamma)^2 y_{tt} = 0 \qquad (7\text{-}1)$$

while the initial and boundary values equivalent to the finite spring supported span subjected to a uniform velocity input are

For t=0:	$y=0$	(7-2)
	$y_t=v_0$	
For x=0:	$y_{xx}=0$	(7-3)
	$y_{xxx}-\alpha y=0$	
For x=1:	$y_{xx}=0$	(7-4)
	$y_{xxx}+\alpha y=0$	

The normal mode solution to the above mixed boundary value problem yields the solution[18]

$$y(x,t)= \sum_{n=1,3,5}^{\infty} \left(\frac{\ell/r}{\gamma}\right)^2 \frac{1}{\sqrt{\lambda_n}} Z_n \sin \frac{\sqrt{\lambda_n}\,\gamma}{\ell/r} t \; Y_n(x) \qquad (7\text{-}5)$$

where

$$Y_n(x) = \frac{\cosh \sqrt[4]{\lambda_n}(x-1/2)}{\cosh \sqrt[4]{\lambda_n}/2} + \frac{\cos \sqrt[4]{\lambda_n}\, x}{\cos \sqrt[4]{\lambda_n}/2} \quad (7\text{-}6)$$

$$Z_n = \frac{1}{2\sqrt{\lambda_n}\,(Y_n(x), Y_n(x))} \left(\tanh \sqrt[4]{\lambda_n}/2 + \tan \sqrt[4]{\lambda_n}/2 \right) \quad (7\text{-}7)$$

$$(Y_n(x), Y_n(x)) = \frac{1}{2}\left[\left(\cosh^2 \sqrt[4]{\lambda_n}/2\right)^{-1} + \left(\cos^2 \sqrt[4]{\lambda_n}/2\right)^{-1} + \frac{12}{\lambda_n}\frac{\alpha (l/r)^2}{\gamma^2} \right] \quad (7\text{-}8)$$

and the eigenvalues λ_n are the solutions to the equation

$$\tanh \sqrt[4]{\lambda_n}/2 + \tan \sqrt[4]{\lambda_n}/2 = \frac{2\alpha (l/r)^2}{\gamma^2} (\lambda_n)^{-3/4} \quad (7\text{-}9)$$

This results in the following expressions for the shear and moment respectively

$$V = \sum_{n=1,3,5}^{\infty} \frac{\sqrt[4]{\lambda_n}}{(l/r)\gamma} Z_n \sin \frac{\sqrt{\lambda_n}\,\gamma}{(l/r)} t \left\{ \frac{\sinh \sqrt[4]{\lambda_n}(x-1/2)}{\cosh \sqrt[4]{\lambda_n}/2} + \frac{\sin \sqrt[4]{\lambda_n}(x-1/2)}{\cos \sqrt[4]{\lambda_n}/2} \right\} \quad (7\text{-}10)$$

$$M = \sum_{n=1,3,5}^{\infty} \left[(l/r)\gamma\right]^{-1} Z_n \sin \frac{\sqrt{\lambda_n}\,\gamma}{(l/r)} t \left\{ \frac{\cosh \sqrt[4]{\lambda_n}(x-1/2)}{\cosh \sqrt[4]{\lambda_n}/2} - \frac{\cos \sqrt[4]{\lambda_n}(x-1/2)}{\cos \sqrt[4]{\lambda_n}/2} \right\} \quad (7\text{-}11)$$

As mentioned in the introduction, previous analyses have shown the elementary theory to adequately predict flexural stresses for problems whose high frequency components of solution produce minor contributions. The results obtained from equation (7-11) are consistent with these findings. Figure 23 reveals reasonable agreement between the Euler Bernoulli and the Timoshenko theories for the bending moment with a noticeable decrease in accuracy with increasing values of the spring parameter α. Similar results were obtained for all other values of l/r and α previously considered. Figure 24 shows the lack of agreement between the resulting shears which was evident for the full range of l/r and α.

CHAPTER 8

DISCUSSION AND CONCLUSIONS

8.a Discussion of Results

The mathematical difficulties encountered in this study of the Timoshenko beam subjected to transverse impact necessitated the use of various methods of analysis.

Assuming convergence when the last term represents a contribution of less than ten percent of the solution, the standard eigenfunction expansions for the finite spring supported span were found to exhibit reasonable convergence, e.g. a maximum of 160 terms, for moderate values of $1/r$, α and t. In general for fixed α, the convergence worsened in direct proportion to the product $(1/r)t$. Results for shear were obtained for values of $1/r$, α and t extending to 60, 10 and 5 respectively. This range was significantly larger for the bending moment and results were obtainable for values of $1/r$ extending to 200, or for times up to 15 for an $1/r$ ratio of 60.

Using the same convergence criterion, the eigenfunction solution for the simply supported span together with the finite progressing wave expansion yielded results up to a value of $(1/r)t$ approximately equal to 250 for the shear and 1200 for the bending moment. A significant adverse effect on the convergence of the shear was observed for values of x and t representing the vicinity of the shear wave fronts.

The modification of the progressing wave expansion permitting the isolation of the leading discontinuities of the shear in terms which diminish with time extended its range of convergence to larger values of l/r and t. In particular, for $l/r=60$ and t ranging from 3 to 5, the magnitudes of the isolated discontinuities were reduced from 10^4 to 10^2 yielding a reduction in the number of terms required to satisfy the convergence criterion from approximately 160 to 60.

The asymptotic results for large slenderness ratios, using the methods of stationary phase and singular perturbation revealed high oscillations in the vicinity of the wave fronts with frequencies proportional to l/r and showed good agreement with previously obtained solutions within their limited ranges of validity.

A comparison between the elementary Euler Bernoulli and Timoshenko theories revealed consistent results for the moment response but poor agreement for the shear for the full range of parameters considered.

As for the magnitudes of the response, in the case of rigid supports the moment response indicated an almost constant value of maximum flexural stress equal to $1.2(v_o/c_2)E$, which is attained at times less than one quarter of the fundamental period of the span, for values of l/r up to 60. The shear response is observed to be dominated by the shear wave fronts for large values of 1/r, that is the amplitudes are small away from the wave fronts with significant oscillations and magnitudes in the vicinity of these fronts. The introduction of a moderate amount of Maxwell damping in the spring supports, e.g. ten percent of the critical value of an equivalent rigid span, provided no significant reduction in the magnitudes of the response.

8.b Conclusions

Knowledge of the nature of the discontinuities present in the response of a Timoshenko beam were used to improve upon standard methods of analysis and in some cases generated new approaches.

To extend the range of applicability of the eigenfunction analysis of the Timoshenko equations to problems exhibiting high discontinuities, procedures were developed that theoretically allow the reduction of mixed boundary value problems representing finite spans to Cauchy initial value problems allowing the absorption of the resulting leading discontinuities in a separate progressing wave expansion. This reduction represents the determination of the reflections of the initial data and in general, because of extensive matrix manipulations, only the leading terms of the Taylor expansion of these required reflections are obtainable.

The progressing wave analysis was shown to yield identical results to the term by term inversion of the Laurent expansion of the Fourier transform of the solution. However, practically speaking no computational advantage is obtained using this alternate approach.

Asymptotic techniques provided accurate results for large slenderness ratios The method of stationary phase yielded results in a loosely defined region of large time while a singular perturbation analysis produces boundary layer type solutions in the vicinity of the wave fronts which are hampered by a need for initial data along the characteristics.

Results obtained by means of the Euler Bernoulli theory were consistent with previous investigations in showing the applicability of the elementary theory to predict flexural stresses and its inability to predict the shear stresses. The smoothing over of the response in the vicinity of the shear wave fronts was also in evidence.

In summary, the study of the discontinuities and their reflections produced by inconsistent mixed boundary value problems involving the Timoshenko equations did not by itself eliminate all computational problems However it did produce considerable information as to the nature of these results and when used in conjunction with eigenfunction and asymptotic analyses yielded solutions to particular boundary value problems.

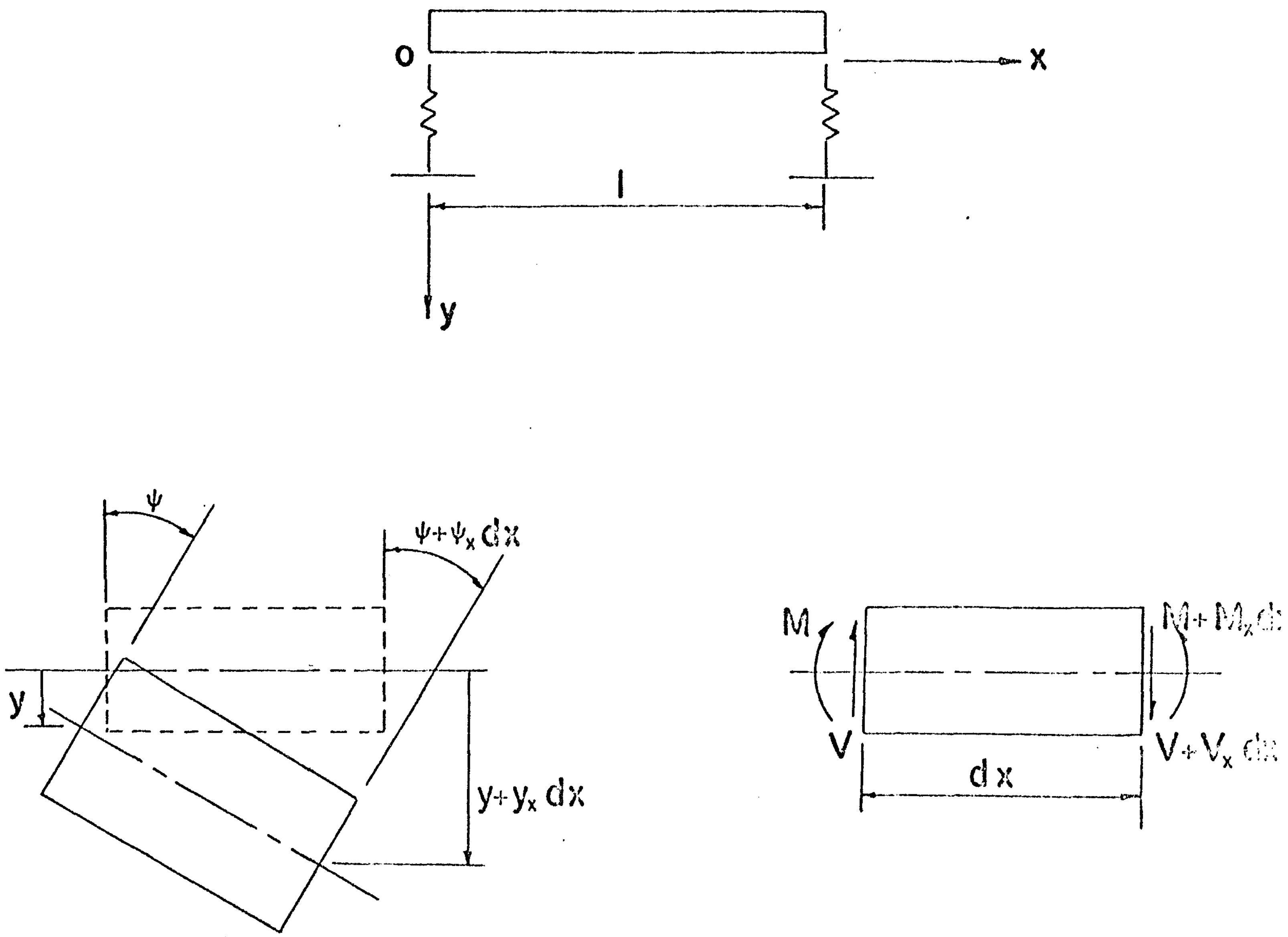

FIGURE I. NOTATION FOR A TIMOSHENKO BEAM

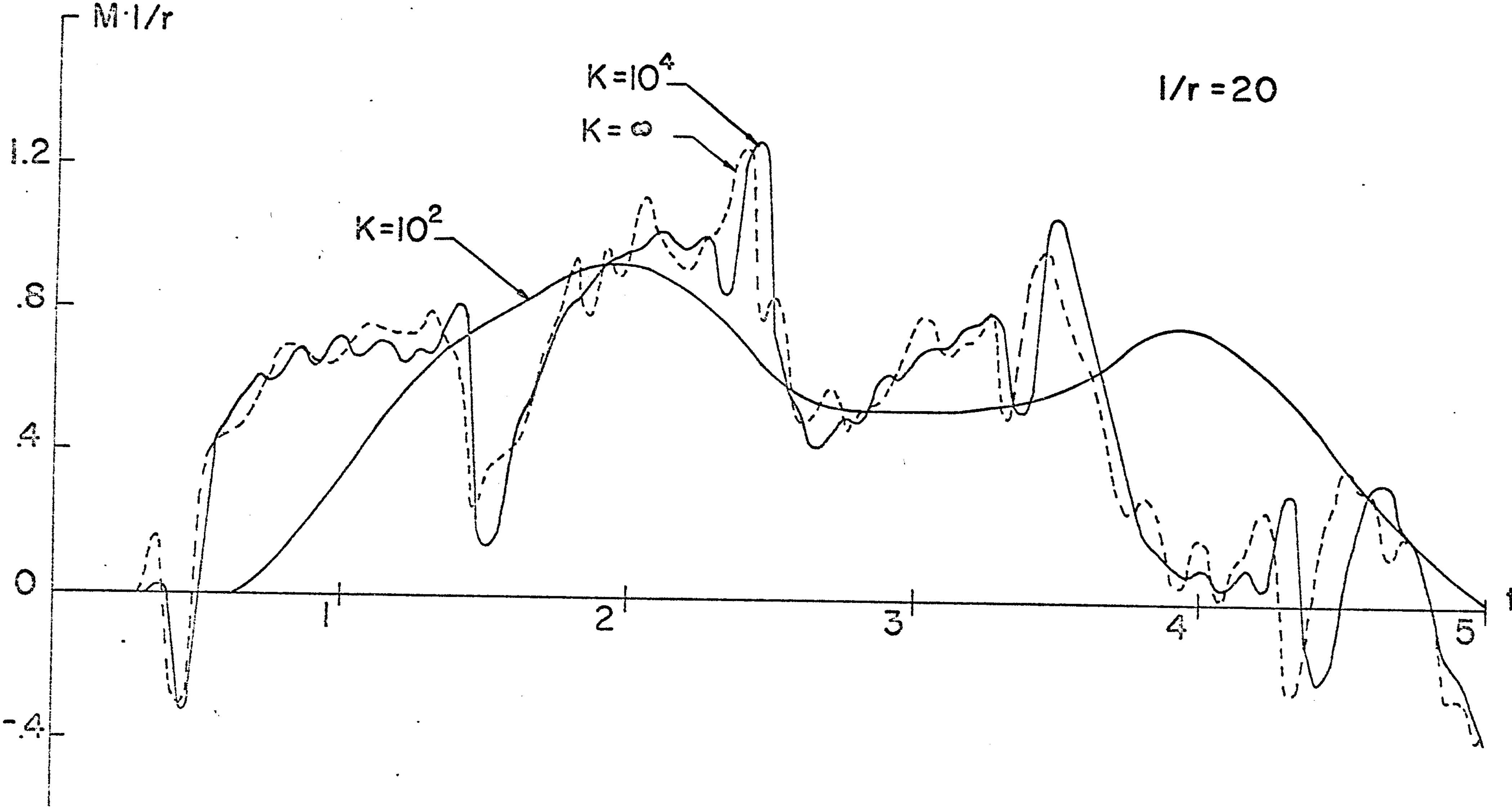

FIGURE 2. VARIATION OF MOMENT AT MID-SPAN; VARIOUS SPRING CONSTANT

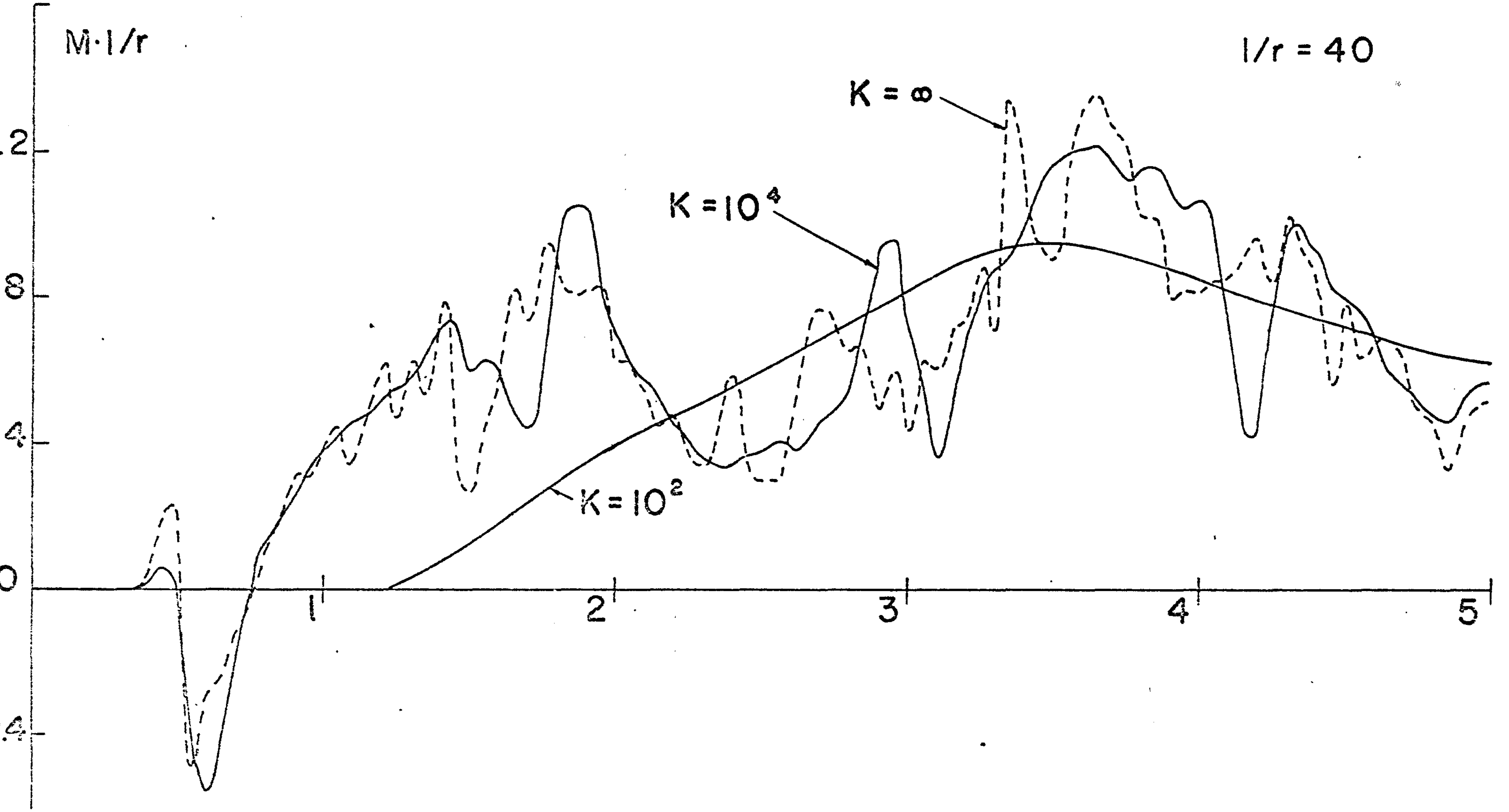

FIGURE 3. VARIATION OF MOMENT AT MID-SPAN; VARIOUS SPRING CONSTANTS

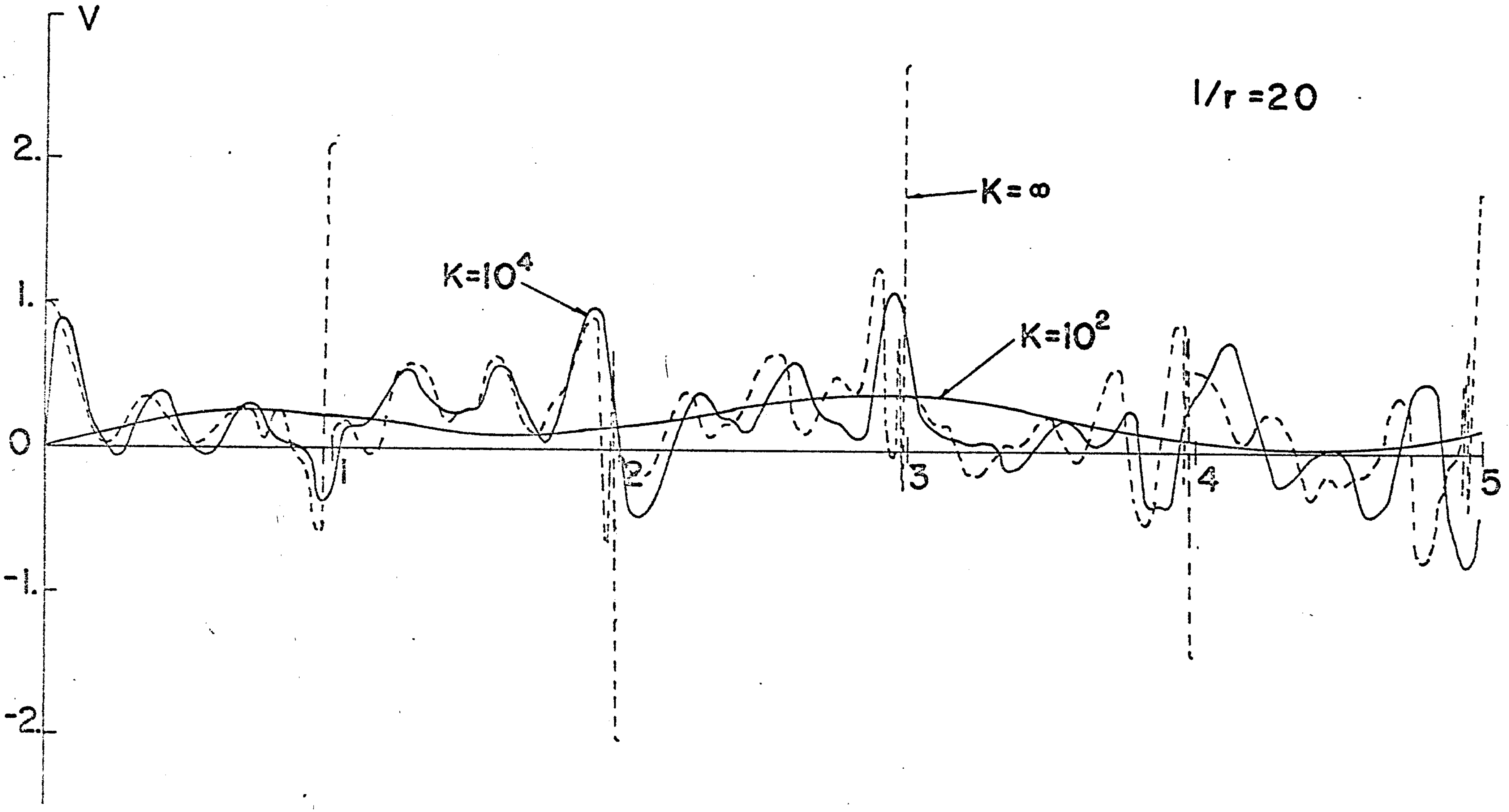

FIGURE 4. VARIATION OF SHEAR AT END-SPAN; VARIOUS SPRING CONSTANT

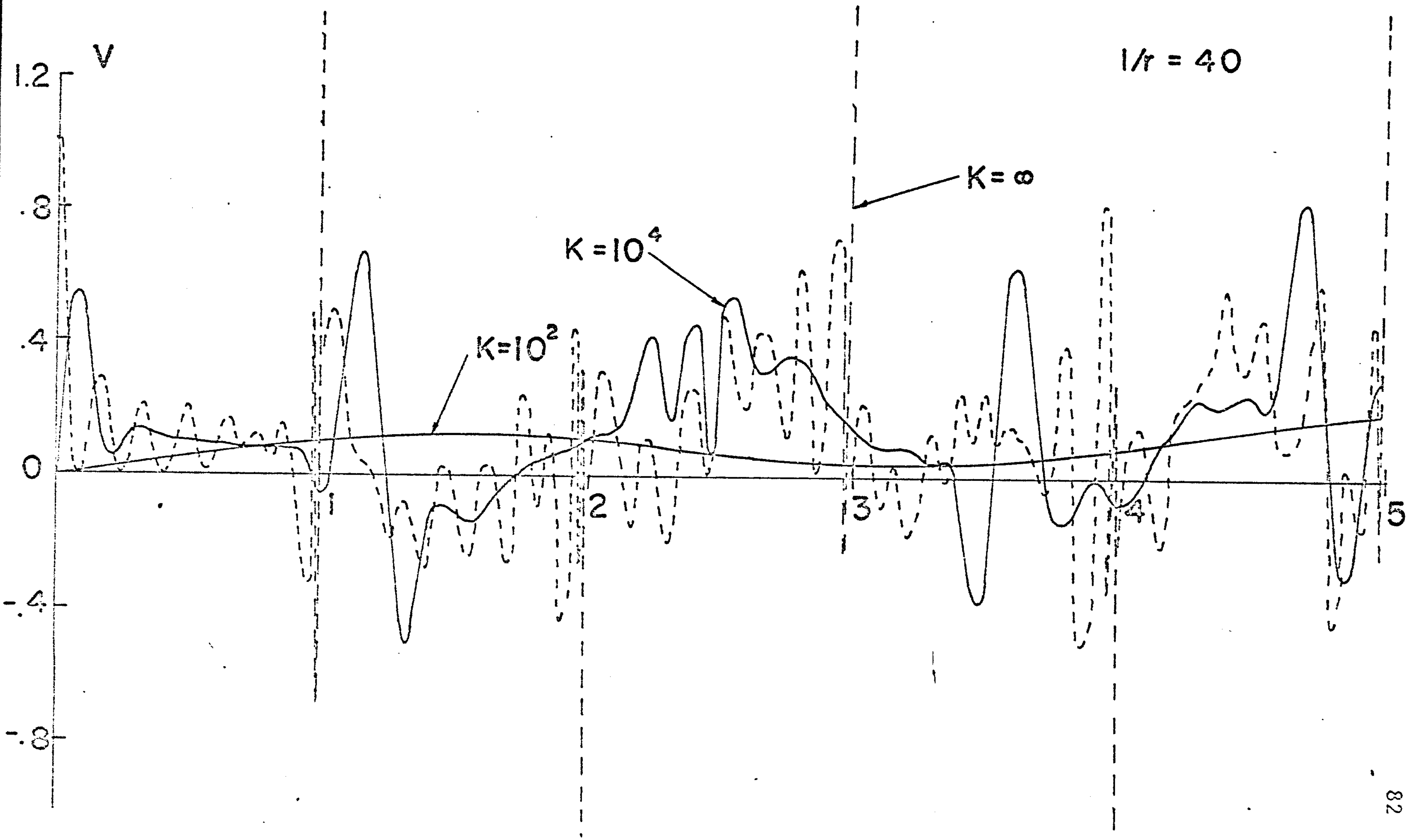

FIGURE 5. VARIATION OF SHEAR AT END-SPAN ; VARIOUS SPRING CONSTANT

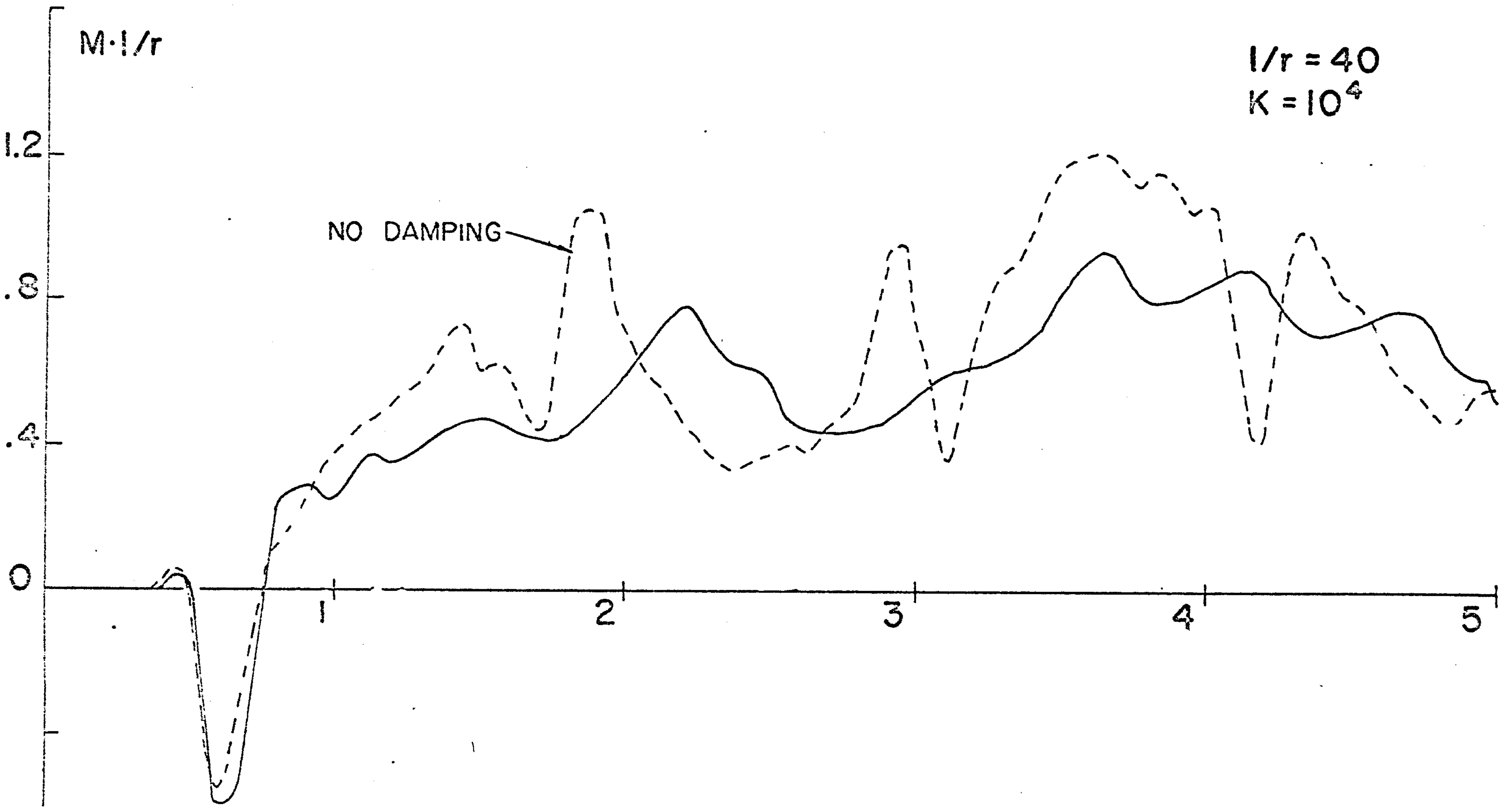

FIGURE 6. VARIATION OF MOMENT; EFFECT OF SPRING DAMPING

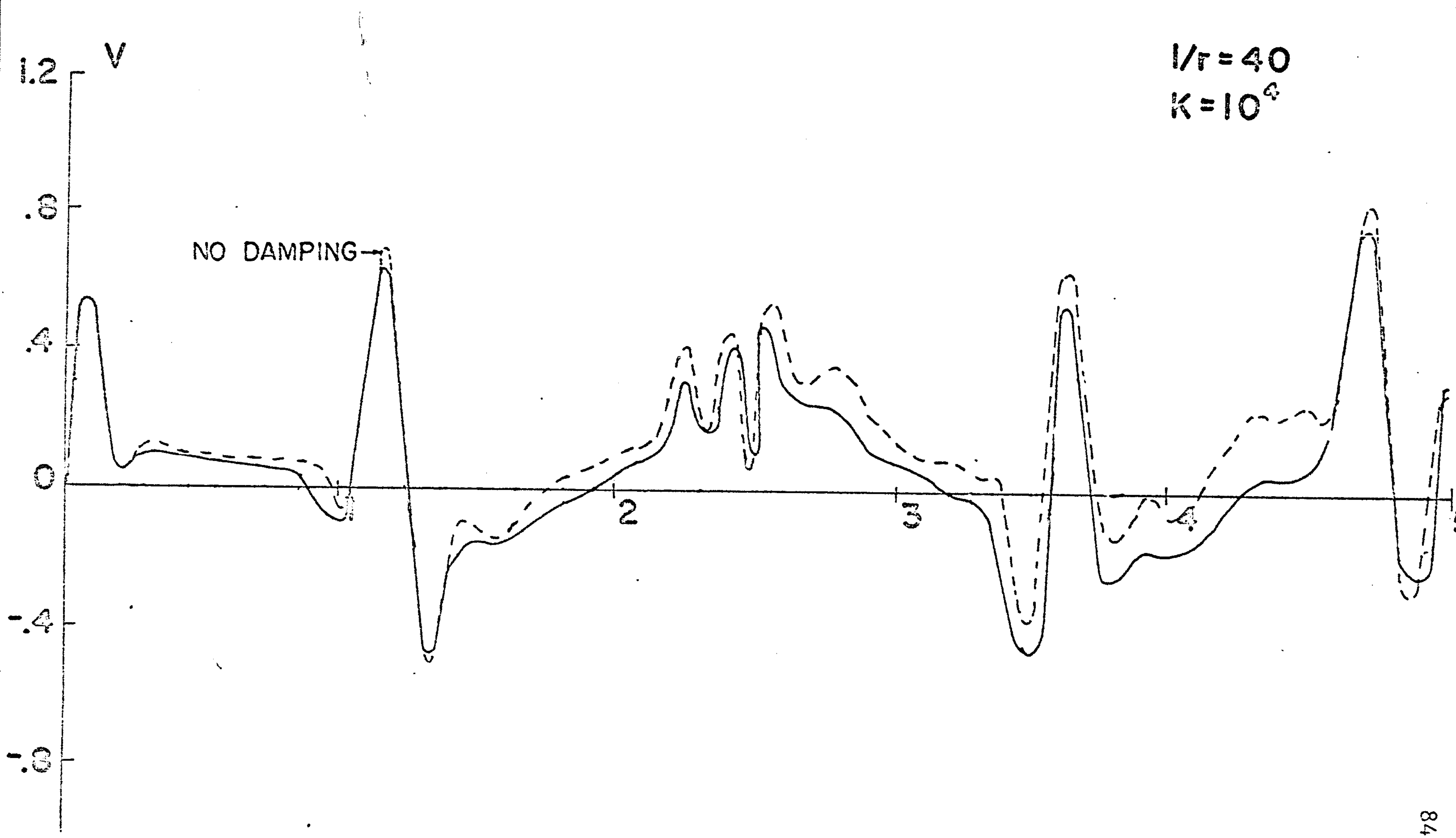

FIGURE 7. VARIATION OF SHEAR; EFFECT OF SPRING DAMPING

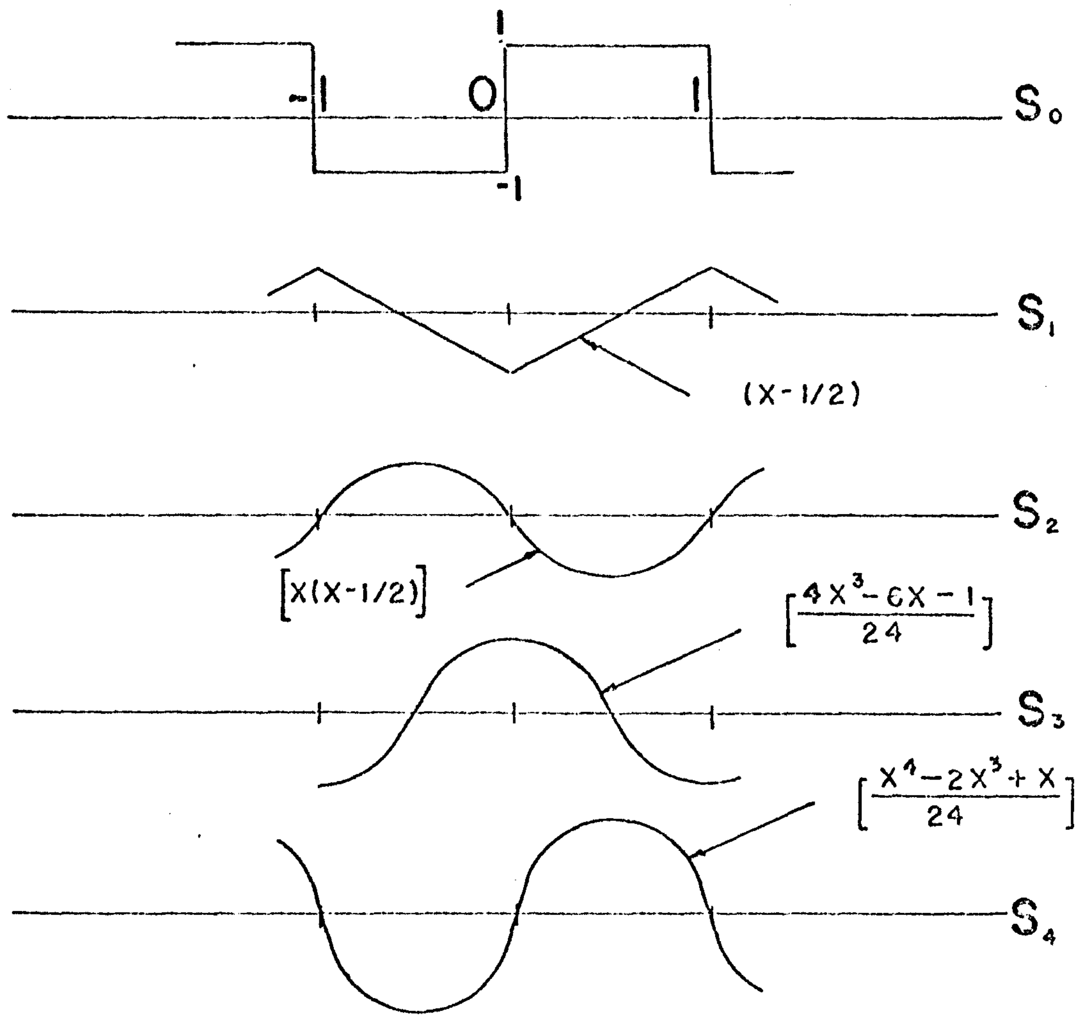

FIGURE 8. DEFINITION OF FUNCTIONS $S_i(x)$

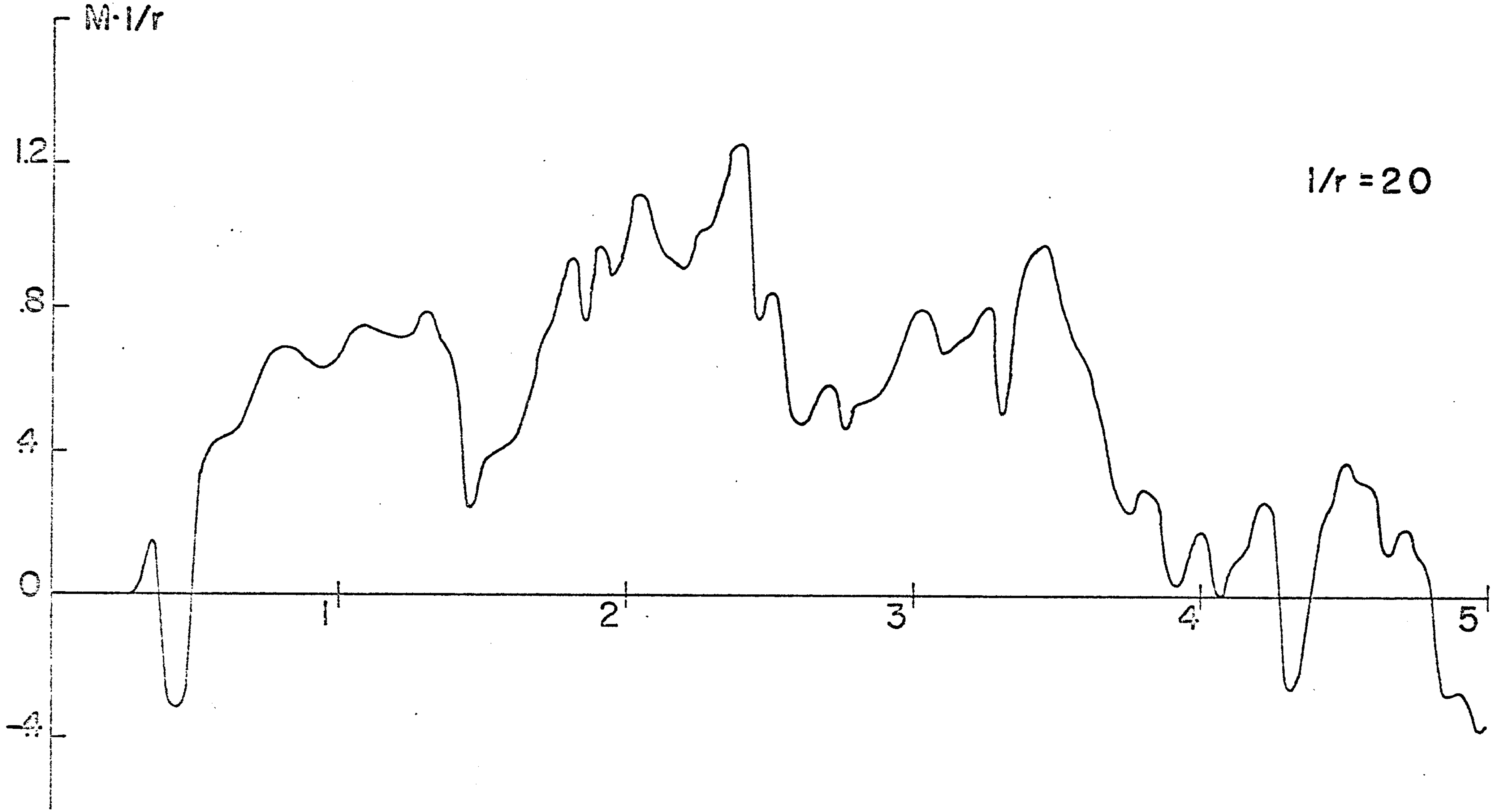

FIGURE 9. VARIATION OF MOMENT AT MID-SPAN ; RIGID SUPPORTS

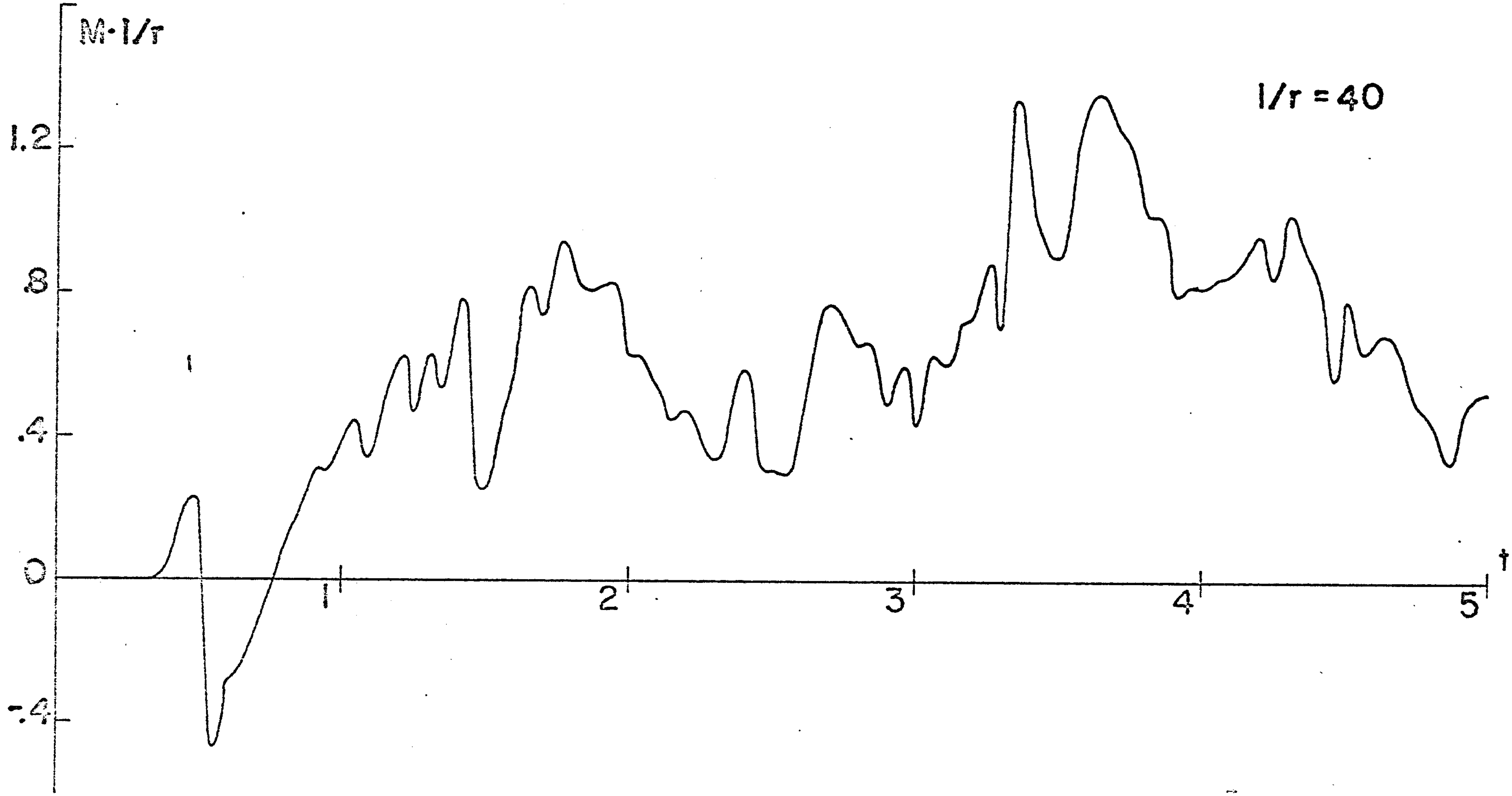

FIGURE 10. VARIATION OF MOMENT AT MID-SPAN ; RIGID SUPPORTS

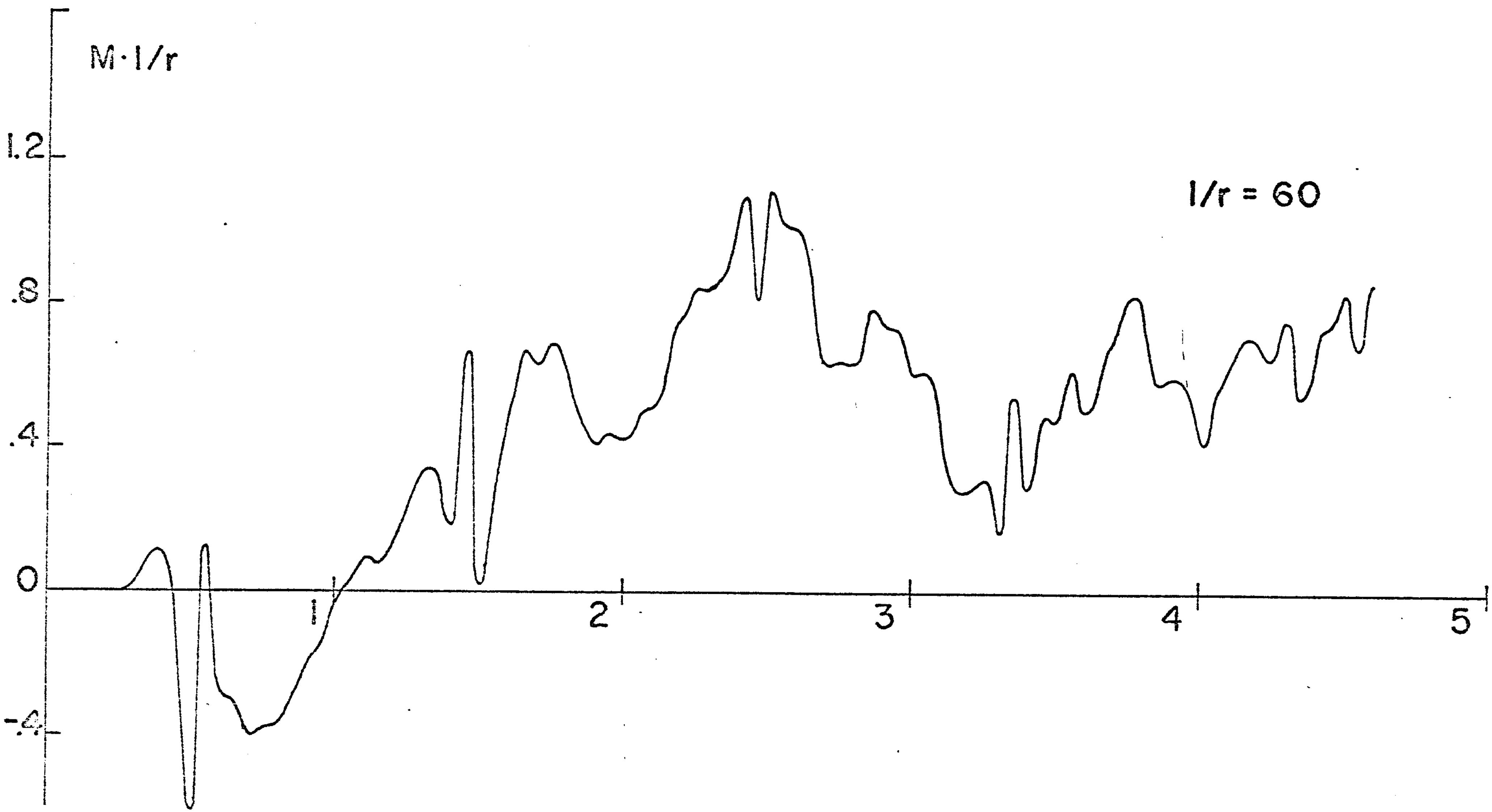

FIGURE 11. VARIATION OF MOMENT AT MID-SPAN ; RIGID SUPPORTS

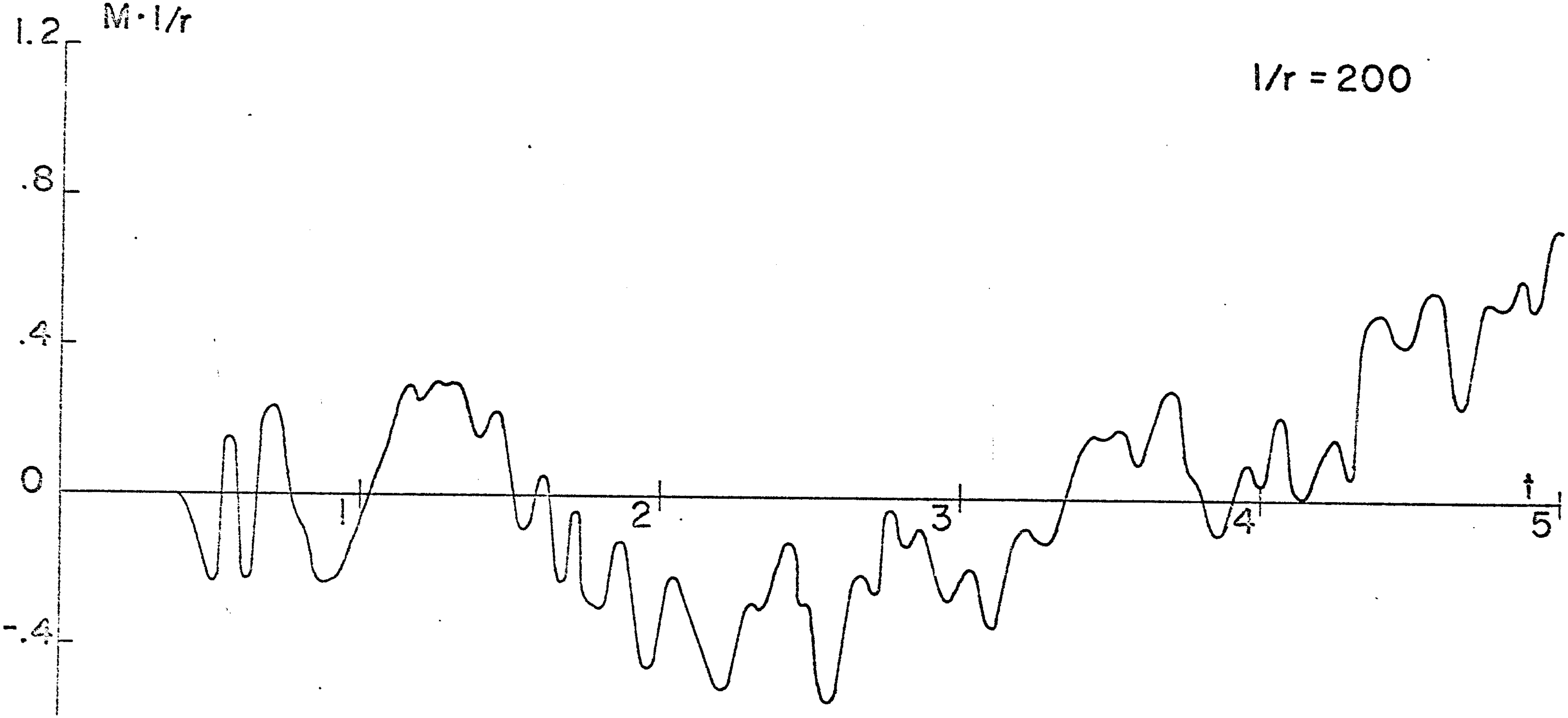

FIGURE 12. VARIATION OF MOMENT AT MID-SPAN; RIGID SUPPORTS

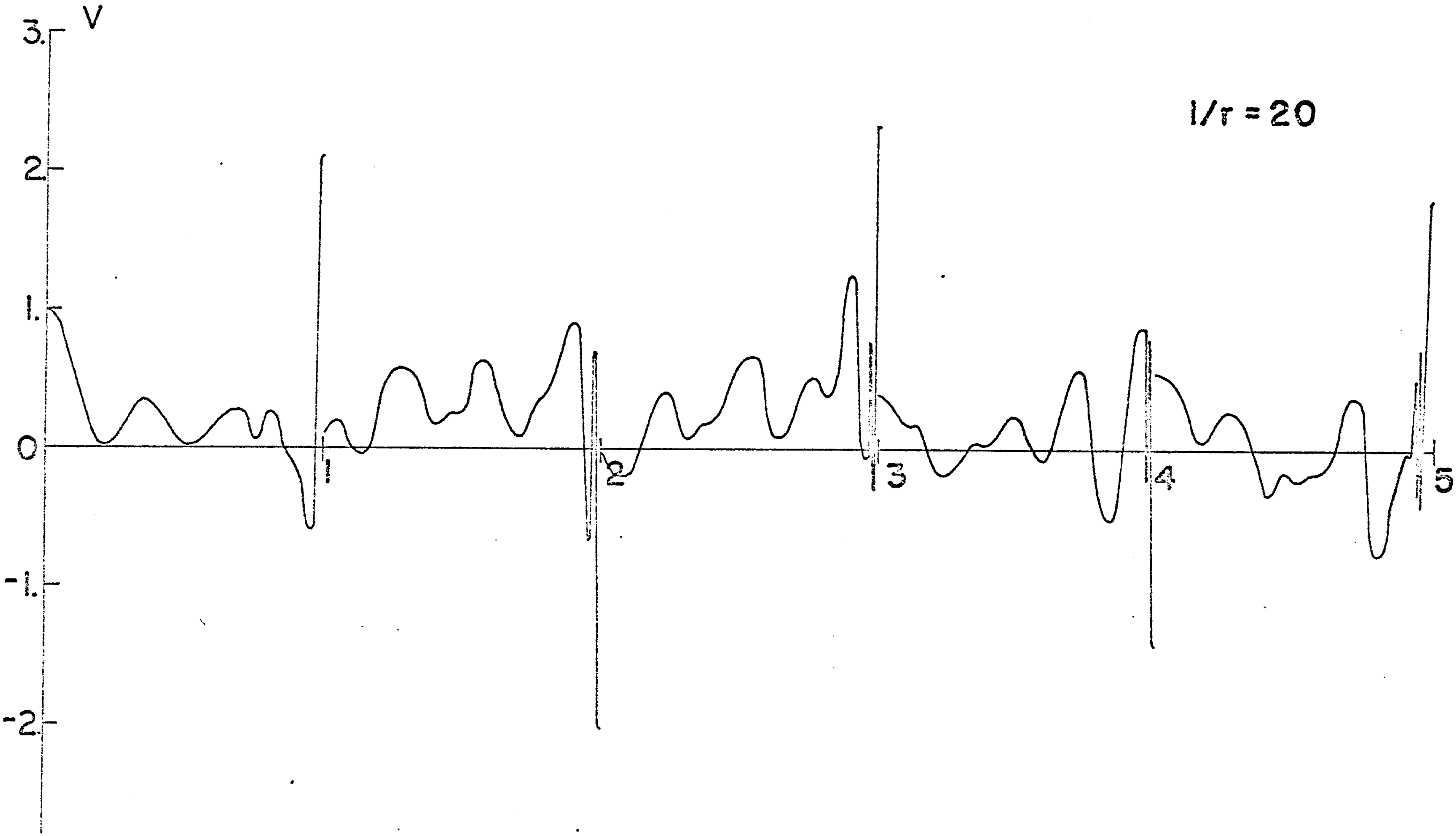

FIGURE 13. VARIATION OF SHEAR AT END-SPAN; RIGID SUPPORTS

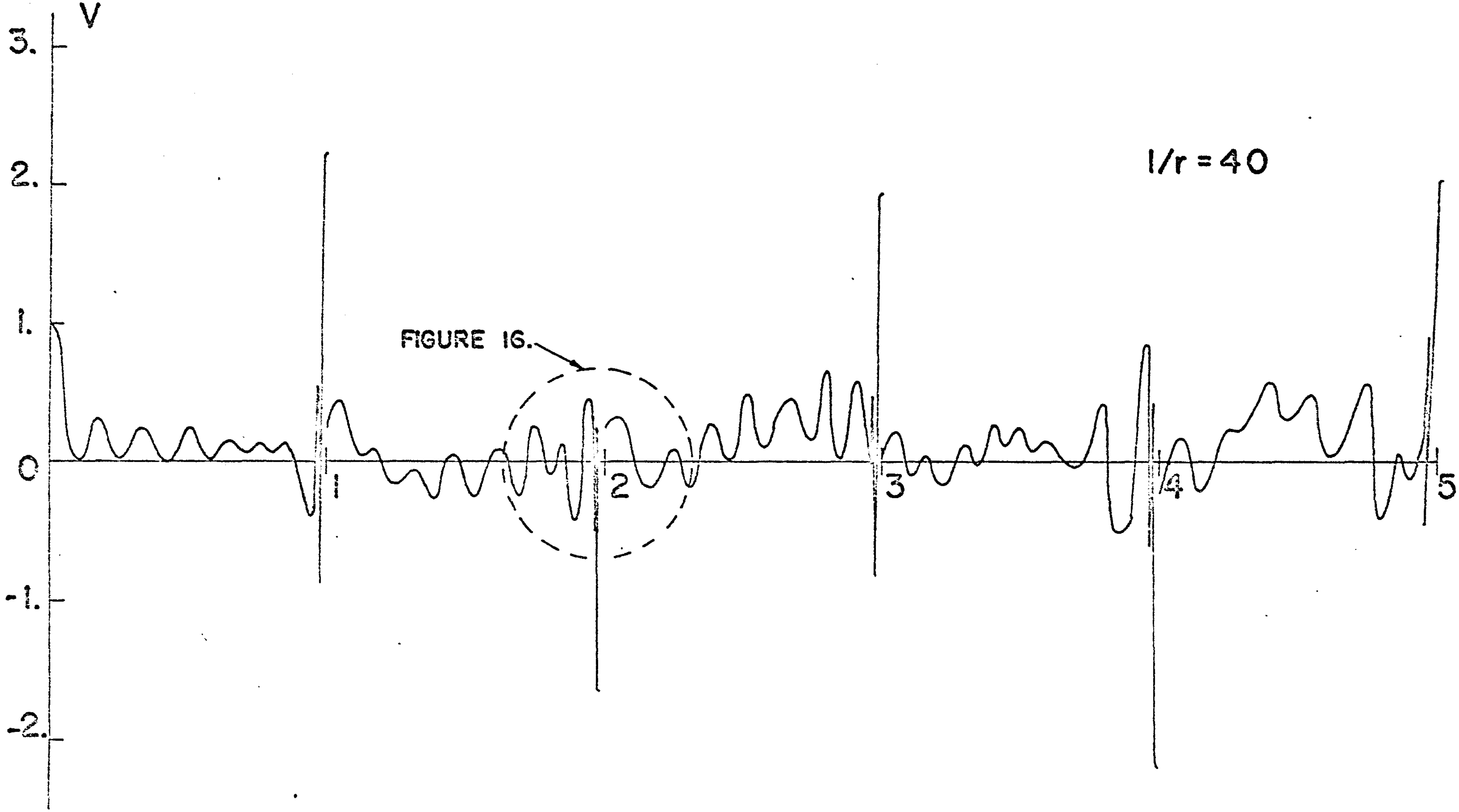

FIGURE 14. VARIATION OF SHEAR AT END-SPAN; RIGID SUPPORTS

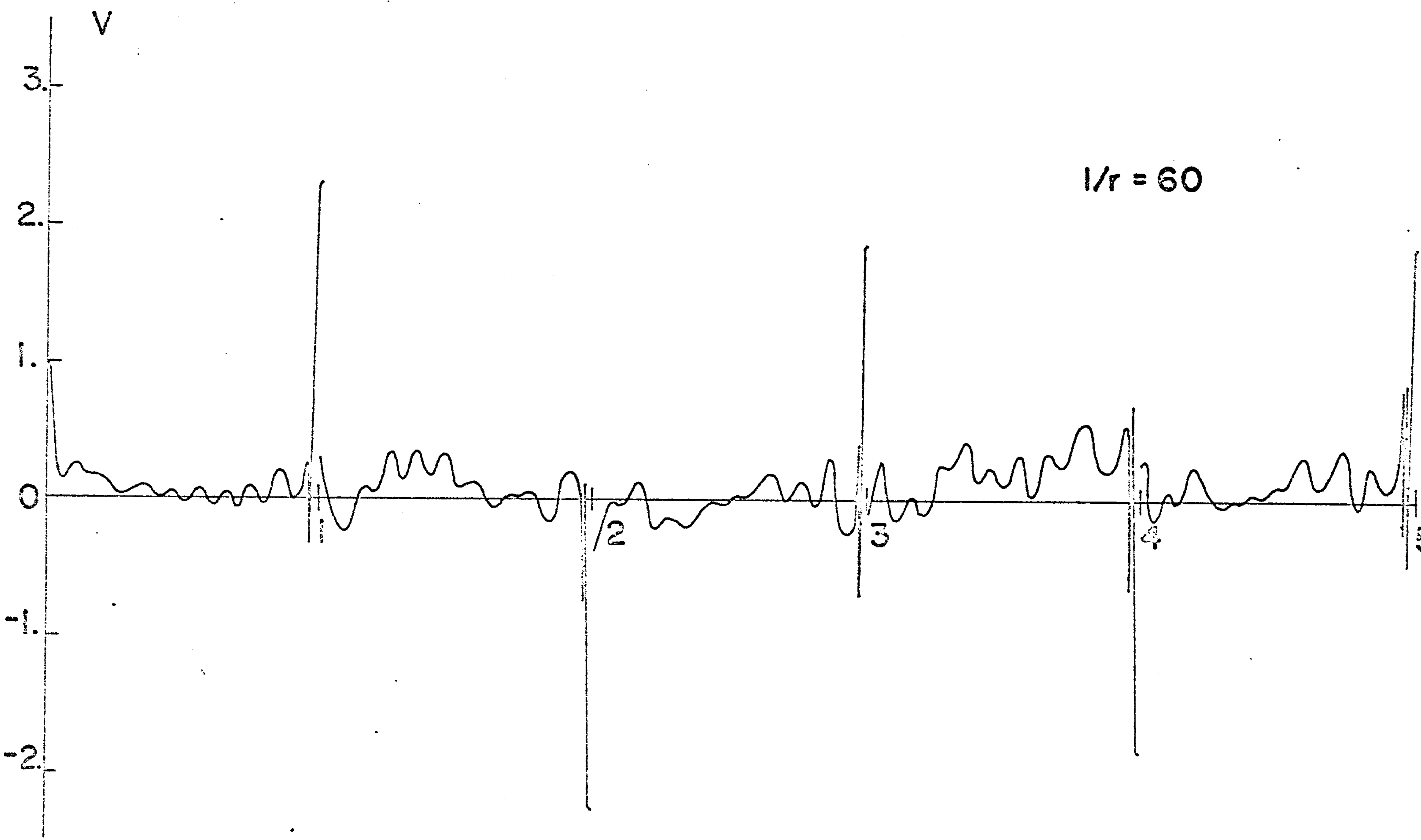

FIGURE 15. VARIATION OF SHEAR AT END-SPAN; RIGID SUPPORTS

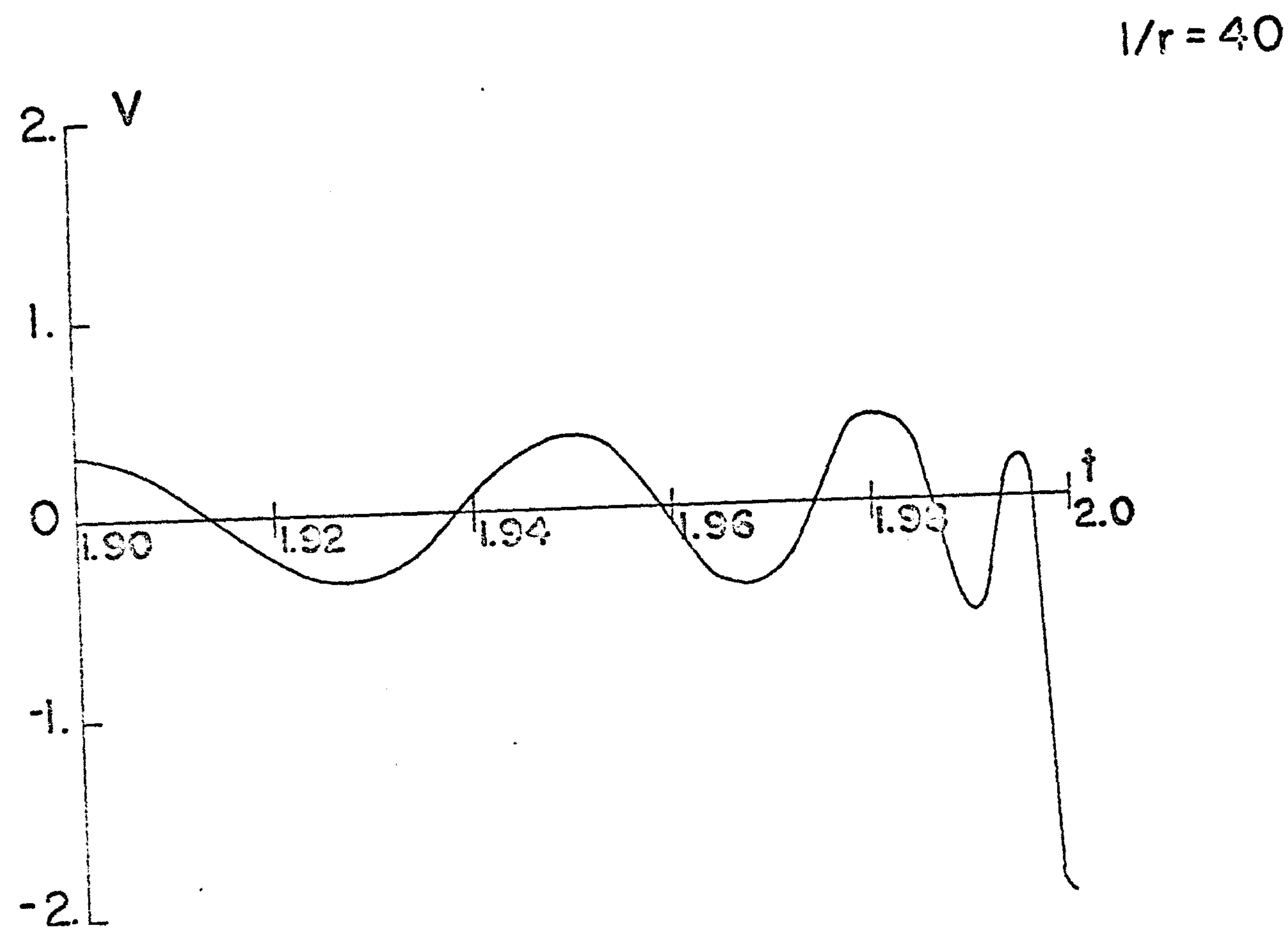

FIGURE 16. VARIATION OF SHEAR PRECEDING THE PASSAGE OF TYPICAL SHEAR WAVE FRONT; RIGID SUPPORTS (X=0)

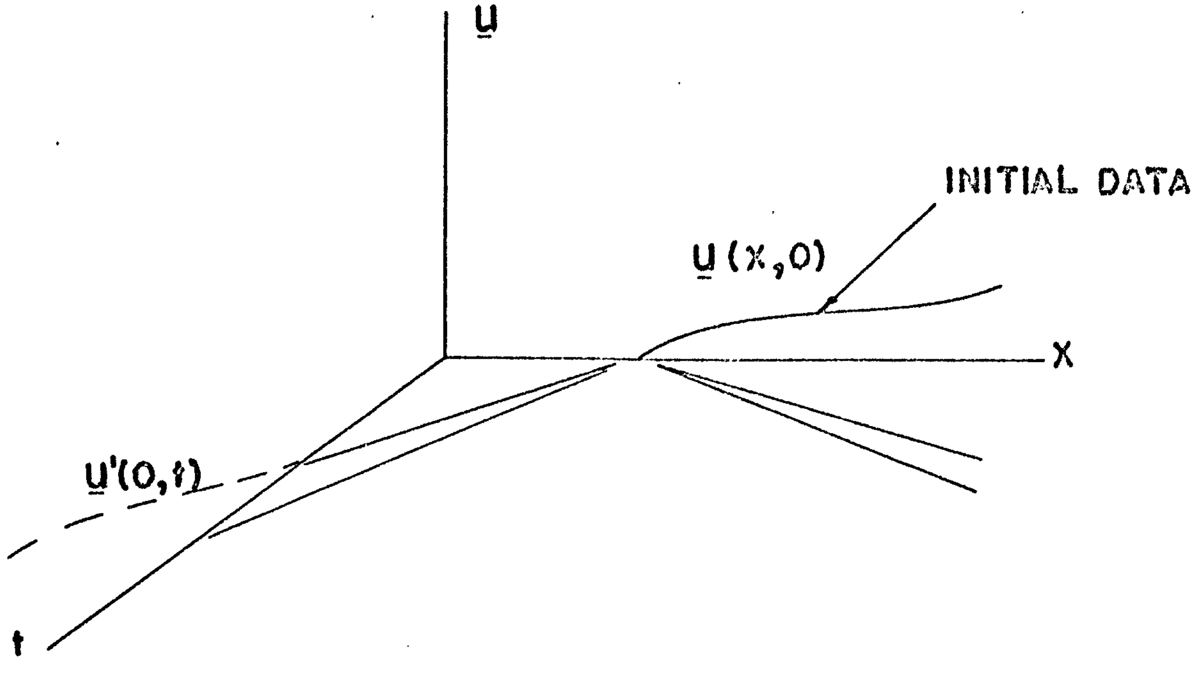

FIGURE 17a. INITIAL VALUE PROBLEM FOR U'(x,t)

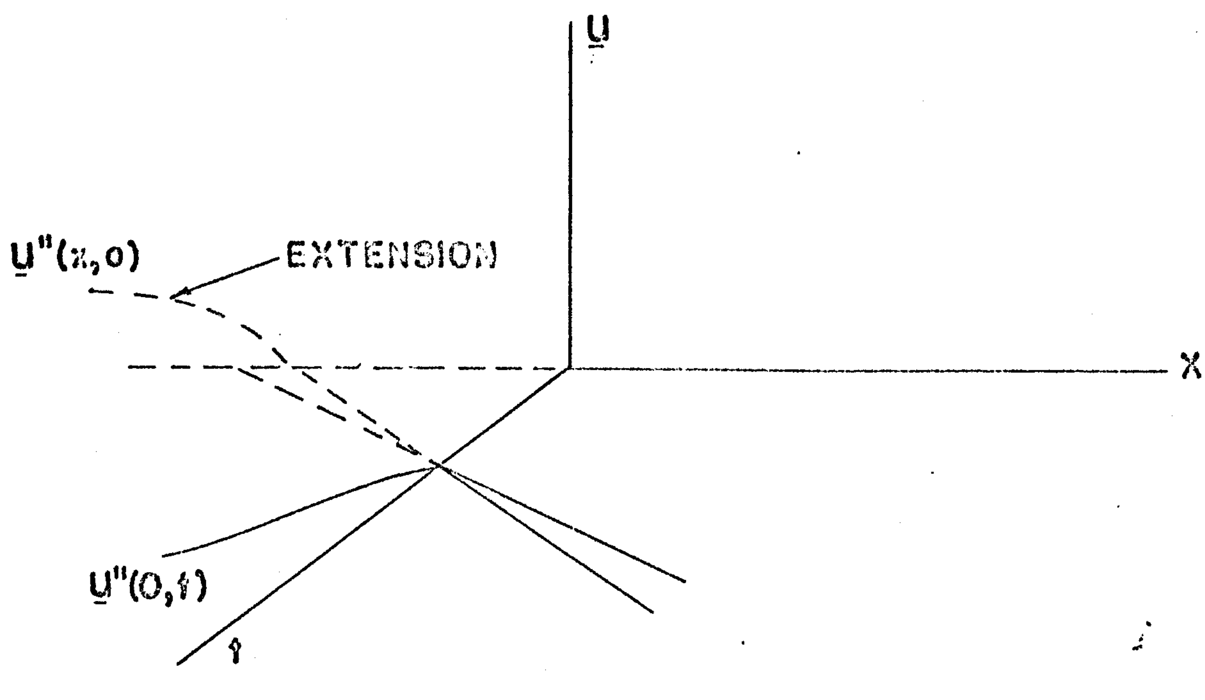

FIGURE 17b. INITIAL VALUE PROBLEM FOR U"(x,t)

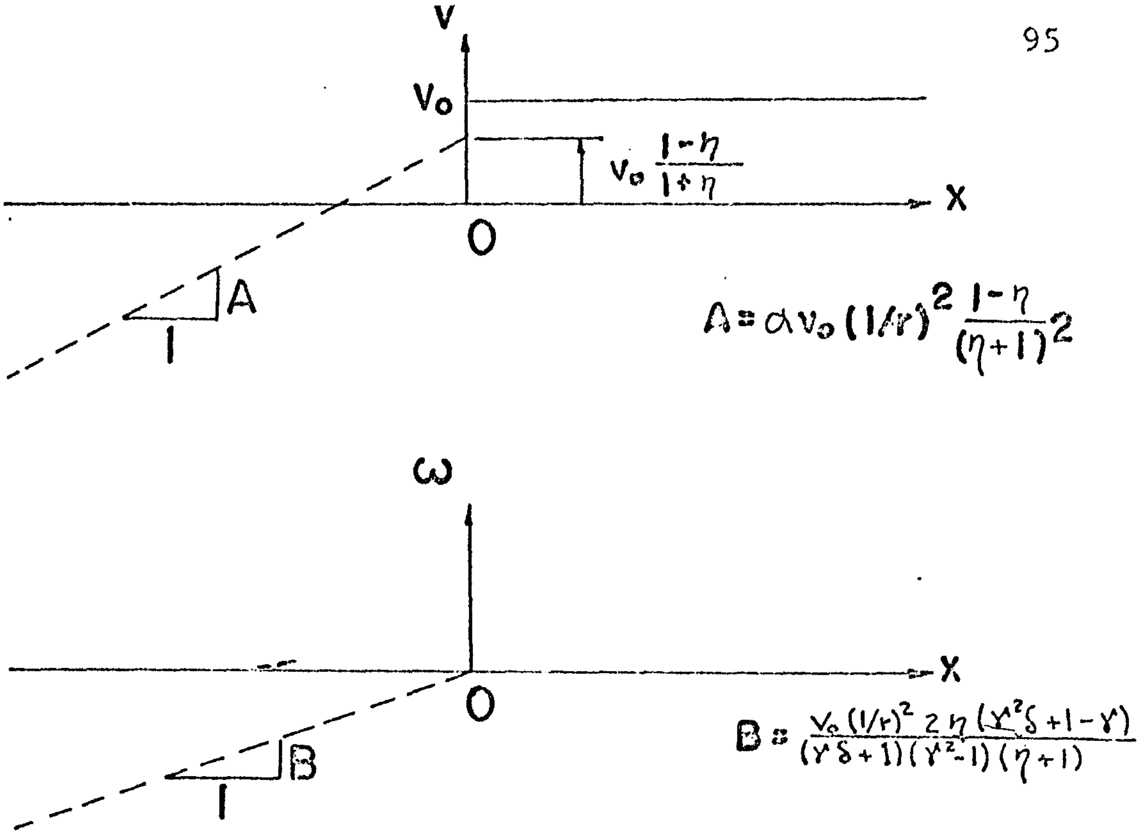

FIGURE 18a. LEADING REFLECTED DISCONTINUITIES: DAMPING

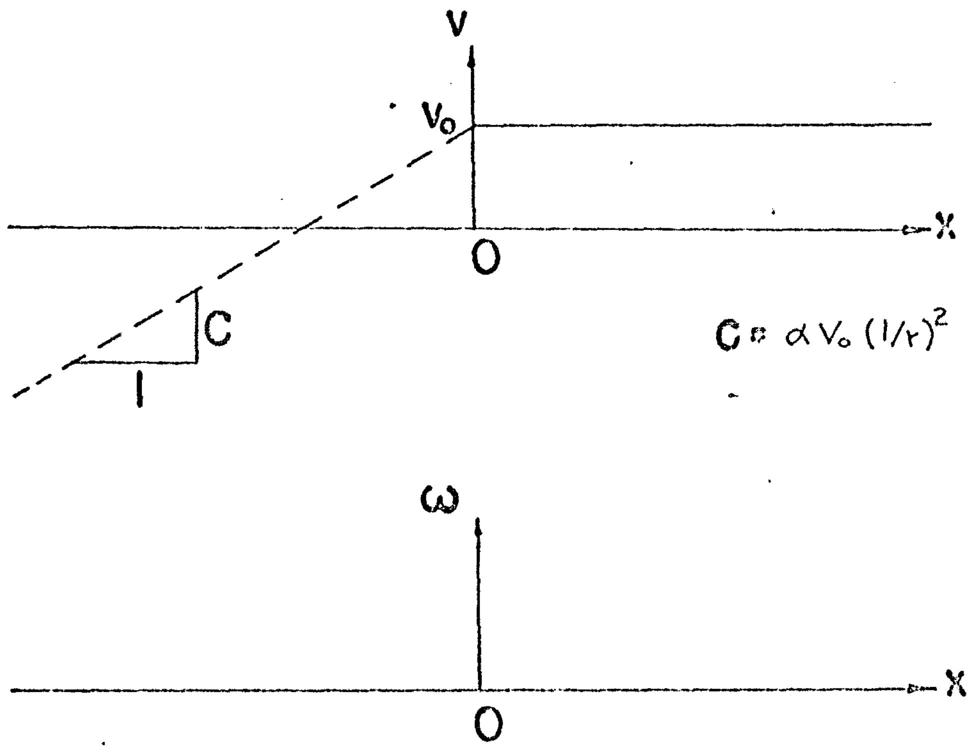

FIGURE 18b. LEADING REFLECTED DISCONTINUITIES: NO DAMPING

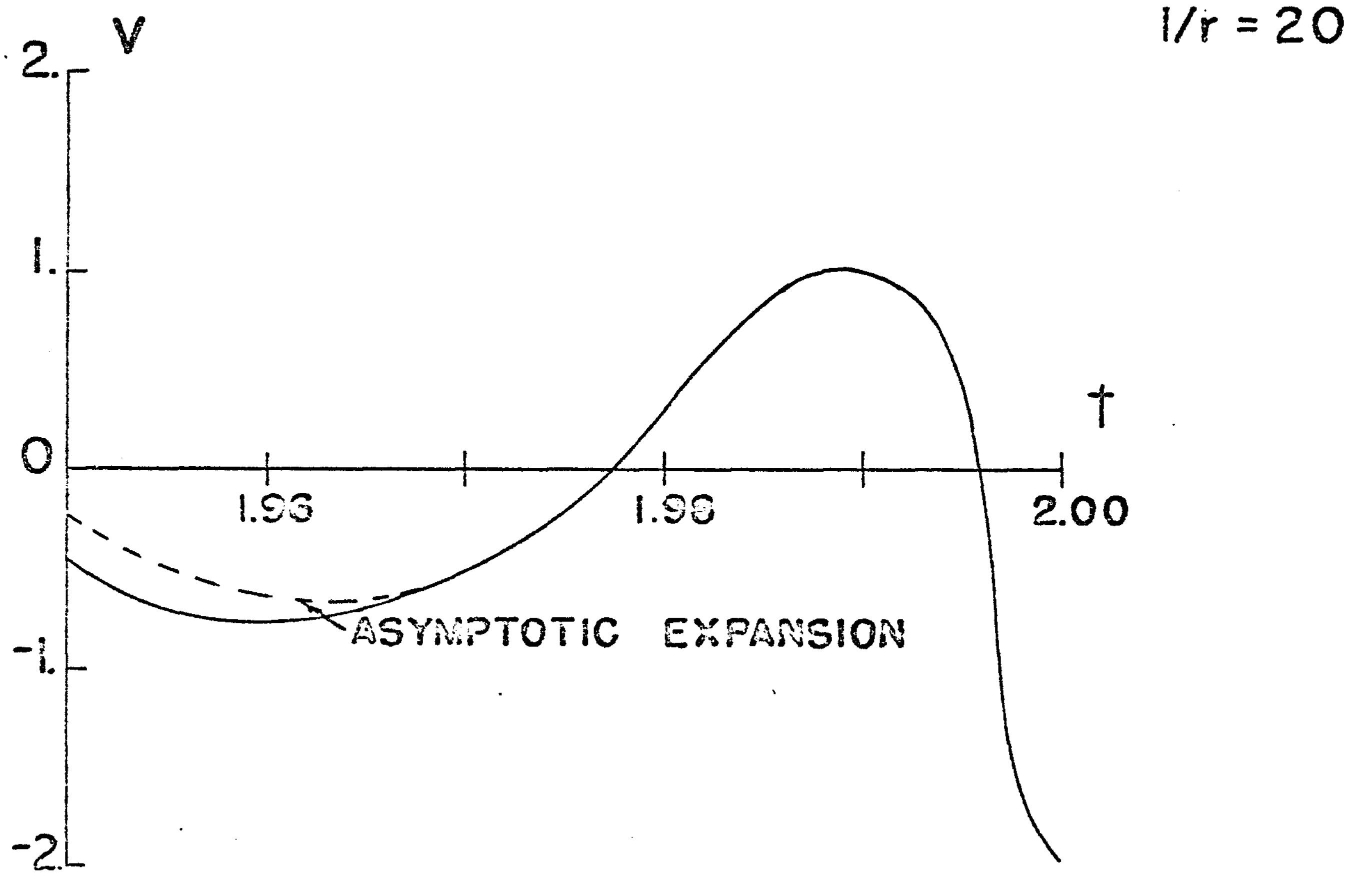

FIGURE 19. VARIATION OF SHEAR; ASYMPTOTIC EXPANSION USING STATIONARY PHASE ($x = 0$)

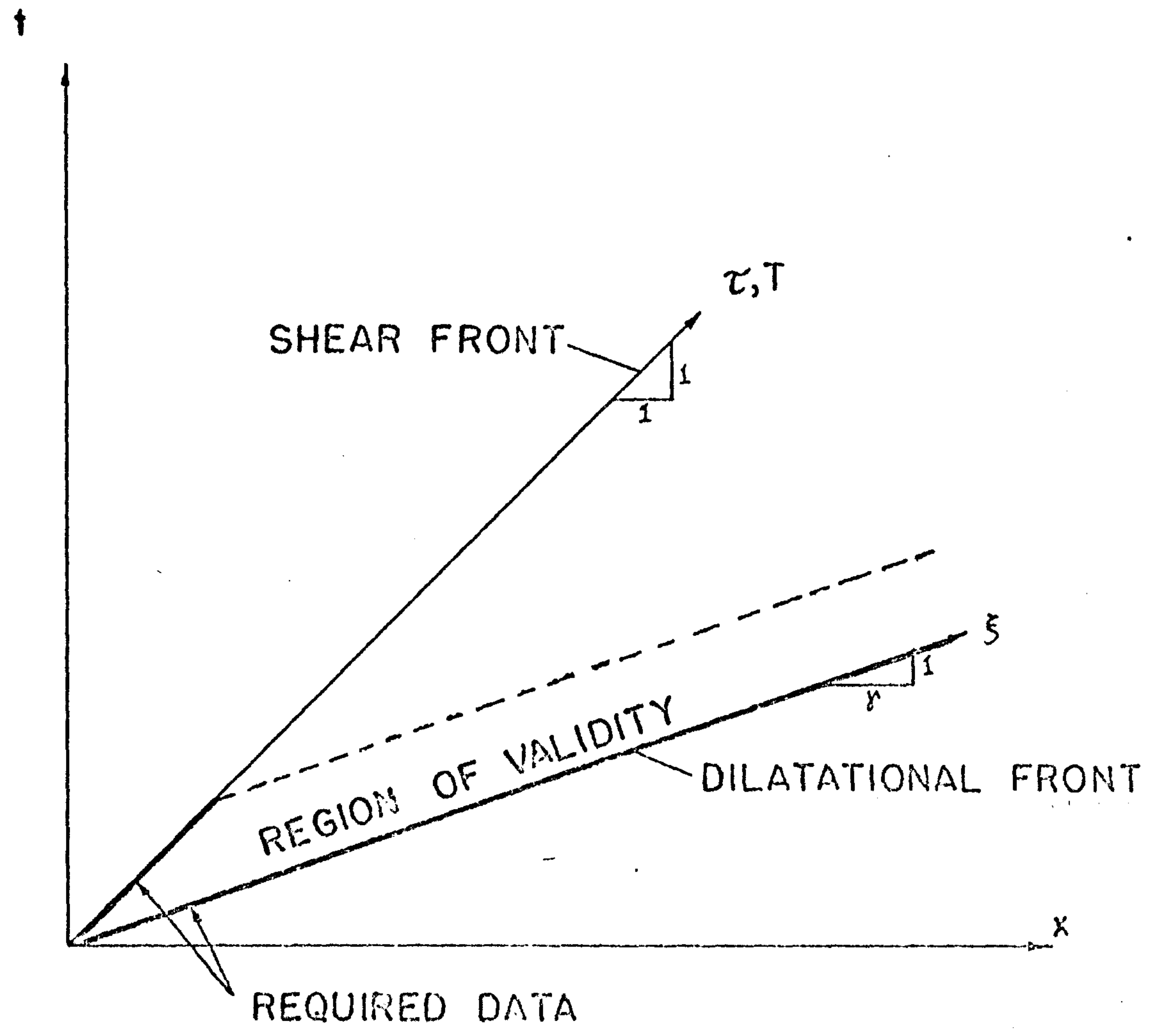

FIGURE 20. REGION OF VALIDITY OF SINGULAR PERTURBATION

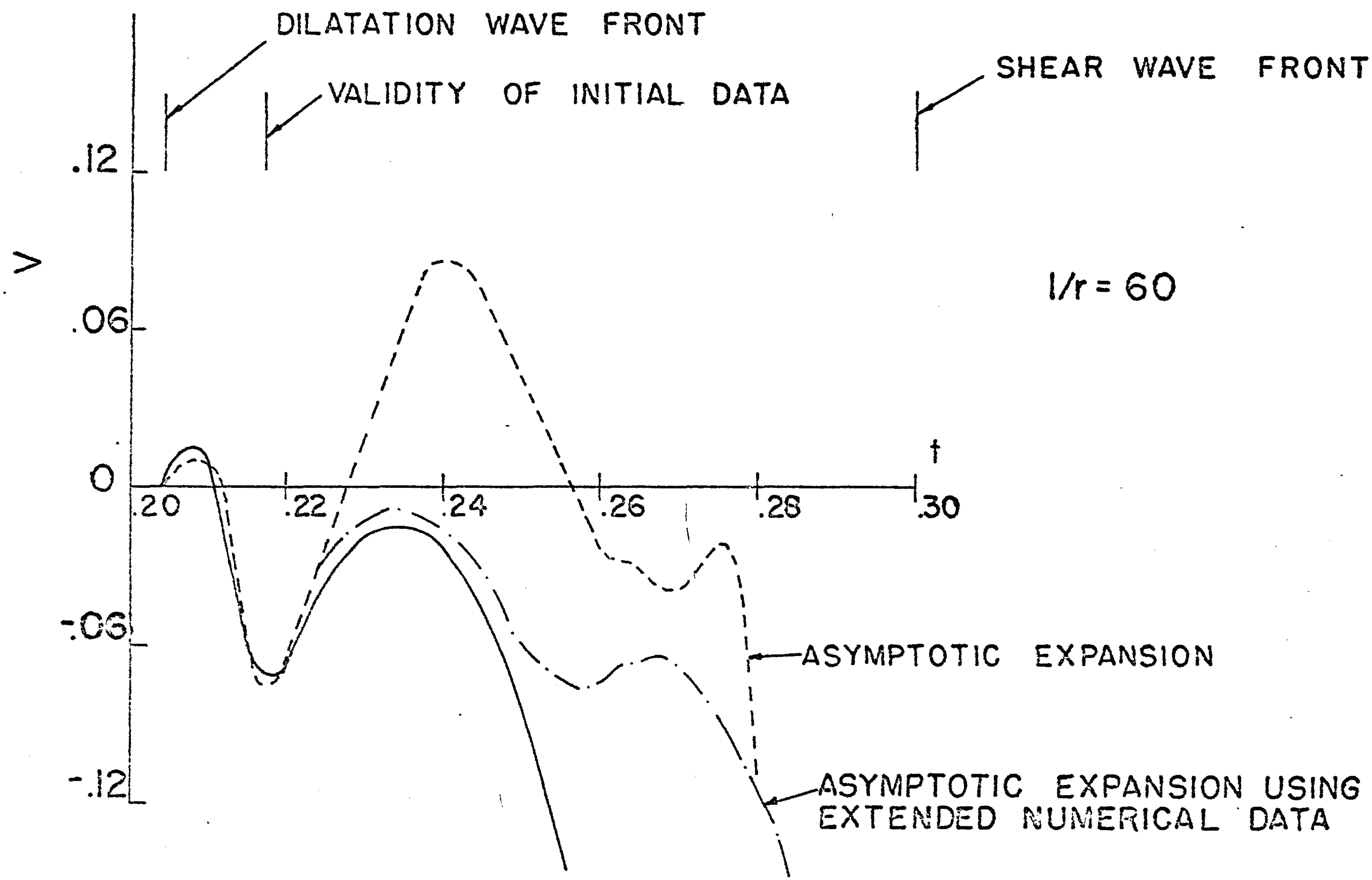

FIGURE 21. VARIATION OF MOMENT ; ASYMPTOTIC EXPANSION USING SINGULAR PERTURBATION (X = .3)

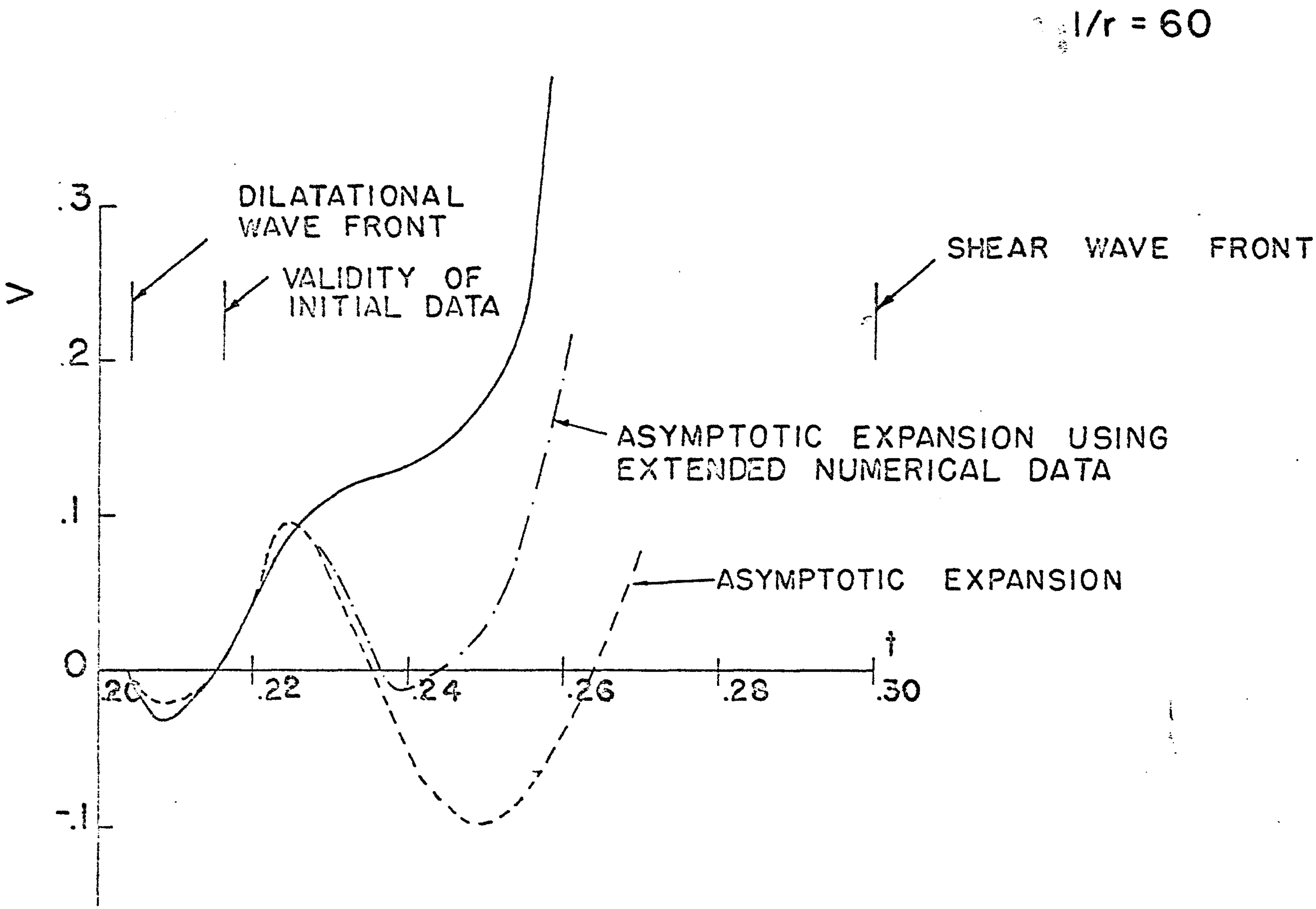

FIGURE 22. VARIATION OF SHEAR; ASYMPTOTIC EXPANSION USING SINGULAR PERTURBATION (X = .3)

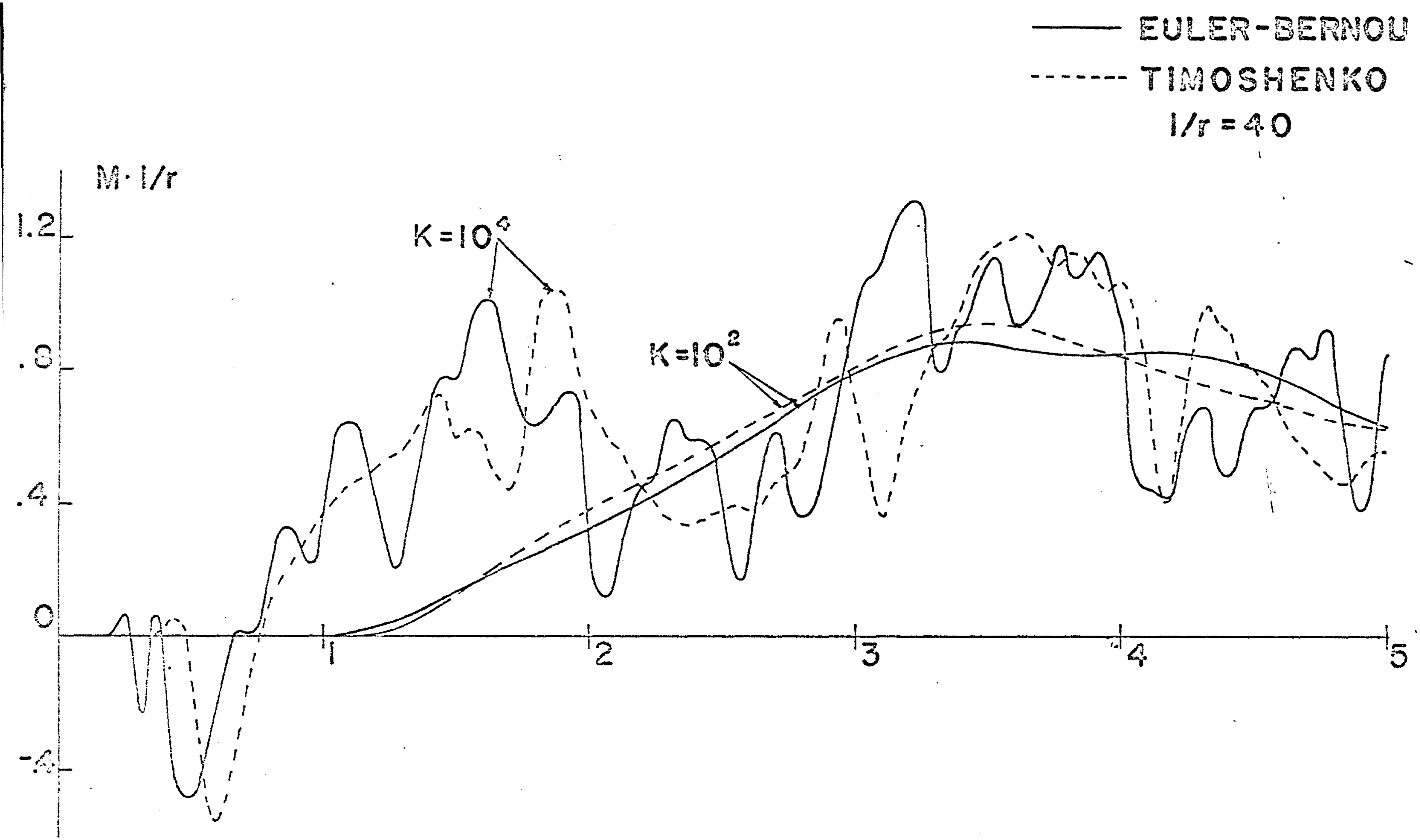

FIGURE 23. VARIATION OF MOMENT AT MID-SPAN ; VARIOUS SPRING CONSTAN
COMPARISON OF EULER-BERNOULLI AND TIMOSHENKO THEORIES

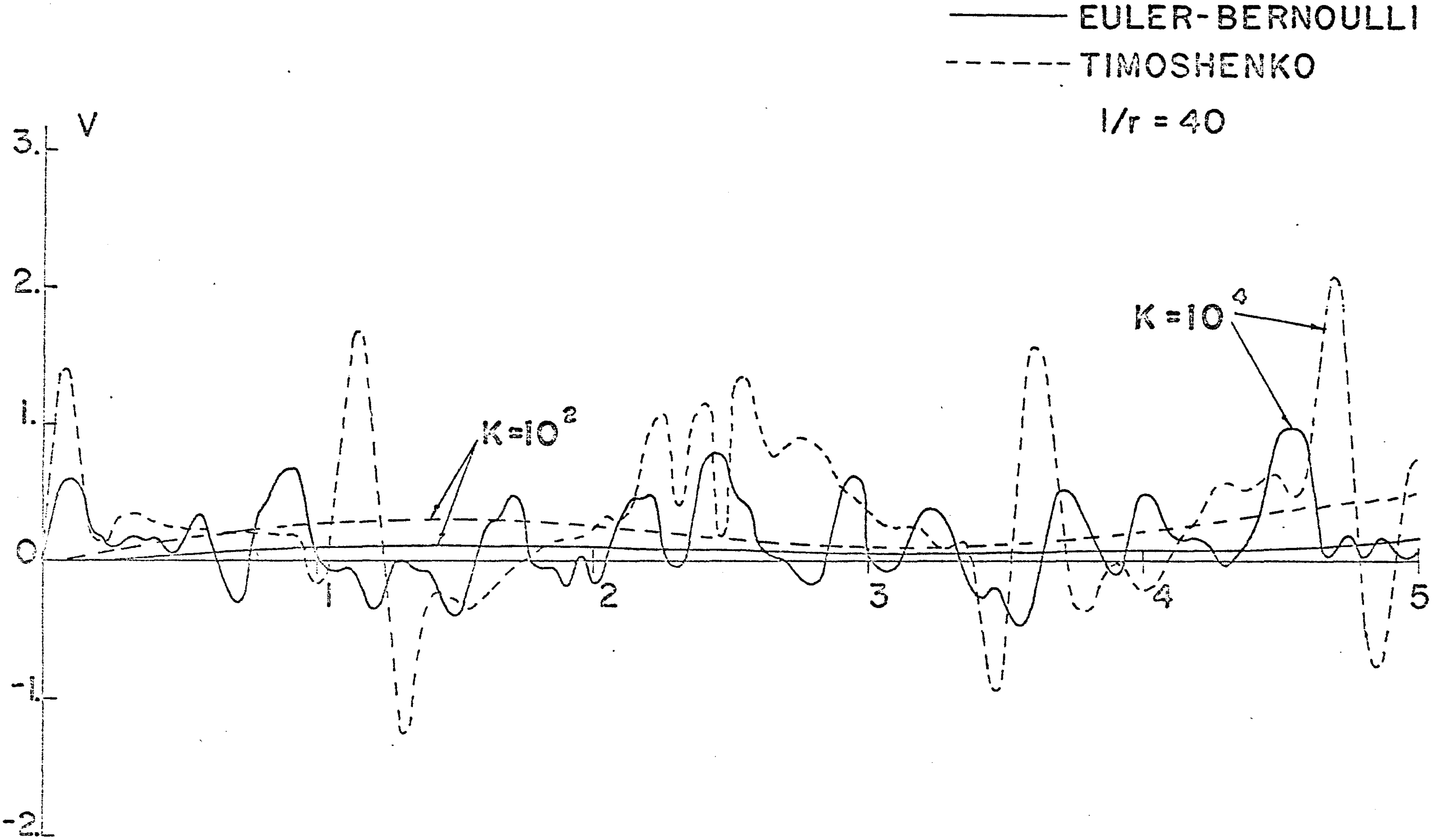

FIGURE 24. VARIATION OF SHEAR AT END-SPAN; VARIOUS SPRING CONSTANT
COMPARISON OF EULER-BERNOULLI AND TIMOSHENKO THEORIES

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