

69-7603

LOWY, Elie, 1923-

EFFECT OF VIBRONIC COUPLING ON RATE OF
SPREADING OF EXCITON WAVE PACKET--
HIGHER ORDER EFFECTS.

The City University of New York, Ph.D., 1969
Physics, molecular

University Microfilms, Inc., Ann Arbor, Michigan

EFFECT OF VIBRONIC COUPLING ON RATE OF
SPREADING OF EXCITON WAVE PACKET .

HIGHER ORDER EFFECTS

by

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A dissertation submitted to the
Graduate Faculty in Physics in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy,
The City University of New York.

1968

This manuscript has been read and accepted for the University Committee in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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ACKNOWLEDGEMENTS

It is my pleasant duty to acknowledge the help I received from many people in accomplishing this work. I am particularly grateful to Professor A. Bierman for suggesting this problem and for the illuminating guidance he provided. I would also like to thank the members of the Thesis Committee, Professors J. Birman, A. Abrahamson, A. Damask and N. Tzoar, for their sympathetic understanding. Above all I want to thank my wife, without whose patience, love and forbearance I would not have been able to pursue my studies and finish this work.

This research was supported by a contract from the United States Atomic Energy Commission.

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ABSTRACT

In a recent paper in the Journal of Chemical Physics, A. Bierman calculated a first correction to the rate of spreading of an exciton wave packet in an infinite one-dimensional crystal, paying attention to both the initial conditions and the effects of phonon-exciton coupling. The calculation was made assuming strong, nearest neighbor coupling and initial conditions in agreement with the Franck-Condon principle. The solution, carried through the second order perturbation (all odd number orders being zero) showed a time dependent decrease of the rate of spreading. For large t , this decrease goes as $t^{-1/2}$ or a constant, depending on whether $\hbar\omega < 2W$, $\hbar\omega = 2W$ or $\hbar\omega > 2W$. Here W is the intermonomer coupling energy and $\hbar\omega$ is the vibrational quantum of energy. In this paper we pursue this calculation to fourth order, showing that the rate of spreading is now increased by a term which goes as t^2 or $t^2/\ln t$ depending on whether $\hbar\omega \gg 2W$ or $\hbar\omega < 2W$.

I. INTRODUCTION

It is now well established that energy can be transferred in organic crystals over distances of the order of $100 \text{ \AA}$ ¹ via excitons.

One of the first experiments which proved this was performed by Bowen² who used an anthracene crystal containing traces of naphthacene. Pure anthracene fluoresces a violet color ($4030 - 4450 \text{ \AA}$) when excited by blue light while naphthacene fluoresces green under the same conditions. When shining light of $\lambda = 3650 \text{ \AA}$ on a solid solution of one naphthacene molecule for every 10^4 molecules of anthracene, we get a green fluorescence, due to the naphthacene whilst the normal violet emission of anthracene is entirely suppressed. Bowen showed very convincingly that the only possible explanation was resonance excitation and concluded that the excitation travels in anthracene until trapped in naphthacene and reemitted there.

Another experiment dating back to the same period

¹D. Curie, Luminescence in Crystals (Methuen 1963), 230
originally published in French by Dunod of Paris, 1960

²E. J. Bowen, J. Chem. Phys. 13, 306 (1945)

was reported by Bucher and Kaspers³. They used the complex myoglobin and $C O_2$ and irradiated it with light of various frequencies. When light of the wavelength of $5460 \text{ \AA}$ was used, the quantum yield, corresponding to separation of $C O_2$ from myoglobin, was unity. This wavelength would correspond to an excitation band of the heme part of the hemoglobin. By varying the wavelength of the exciting light, several energies were found to produce the same quantum yield. In particular, one such frequency, $\lambda = 2800 \text{ \AA}$ corresponds to an excitation band of the albumen part of the myoglobin molecule. Since the separation of $C O_2$ must occur through the heme molecule, it was concluded that the excitation energy was transferred from one part of the complex to another. A third type of experiment was performed by Bannister⁴ who found the same quantum yield of fluorescence of phycocyanin when the exciting light was of $\lambda = 4050 \text{ \AA}$ which is absorbed by the chromophore and with $\lambda = 2750 \text{ \AA}$ which is absorbed by the protein moiety.

We will be interested principally in energy

³T. Bucher; E. J. Kaspers, Biochem. et. Biophys. Acta. 1, 21 (1947)

⁴T. Bannister, Arch. Biochem. Biophys. 49, 222 (1954)

transfers in molecular crystals, defined as being those crystals in which the energies binding the individual units composing the crystal (atoms or molecules, henceforth called monomers) are significantly smaller than the energies of interaction between the composite parts (electrons or atoms) of these monomers. Hence when forming a solid body, these monomers preserve their individuality in the first approximation. Usually the forces between the monomers of such crystals are thought to be van der Waals forces of attraction^{5, 6}. In the zeroth approximation the ground state crystal wave function is then a product of the wave functions of the N individual monomers. If one of the monomers is in an excited state, we can characterize the crystal by a product of wave functions, $N-1$ of which are ground state functions of the separate monomers and one wave function corresponds to an excited state of a monomer. Since the excitation can be localized on any monomer, we would have N degenerate states. When we go to the first approximation, i.e., if we take into account the intermonomer attractive potential, the new wave function for the crystal - the

⁵ A. S. Davydov, Theory of Molecular Excitons (McGraw Hill 1962), 1

⁶ T. Förster, Comparative Effects of Radiation (J. Wiley & Sons, 1960), Ed. M. Burton, J. S. Kirby-Smith and J. L. Magee; 304 - 306.

exciton wave - will now be a sum of these wave functions with appropriate coefficients which depend on the intermonomer potentials. In effect these potentials thus remove the original degeneracy. The coefficients will then give the probability of finding the excitation on the corresponding monomers. Hence the energy is delocalized and shared by the whole crystal. When we consider the actual states of the crystal after a photon has been absorbed, we can admit two possibilities: either the crystal is in a stationary state or a wave packet is formed. If we have a stationary excited state so that the energy is shared by the whole crystal, it is possible that an impurity will trap the energy. We could then talk of energy transfer in the sense that the energy initially delocalized ends up by being concentrated at the trapping site. The difficulty lies in explaining how the initial stationary state arises. The energy arrives as a photon, a wave packet, whose dimensions are much smaller than those of the absorbing crystal. In addition, the photon impinges on the surface of the crystal and absorption could take place during the time that the photon penetrates into the crystal. The absorption of the photon is therefore not by the whole crystal, but only by a part of it - thus making the initial state necessarily unstationary. Hence it is

reasonable to use the wave packet model to analyze the energy transfer problem.

Physically one can imagine a number of mechanisms for disposing of the energy.

Reradiation is one possibility. The absorbed energy is either immediately reradiated or is transferred to another monomer which then reradiates all or part of this energy by luminescence or phosphorescence depending on whether the "acceptor" molecule is in a singlet or metastable state when excited. Fluorescence has a lifetime of the order of 10^{-8} sec. leading to a rate of transfer much slower than the rate we are considering² and thus unlikely to provide an explanation for the phenomena we are discussing.

A second mechanism for energy transfer is provided by the interaction with the phonons of the crystal. The electronic energy is then eventually degraded into heat.

The third possibility is that the energy is trapped in a lower lying state of an "impurity" particle. One has then in effect prepared one of the excited states of the defect. Hence implantation of known impurities is equivalent to using probes for investigating properties of the energy

transfer.

The characteristic of energy transfer one might want to investigate will depend on which one of the physical processes we are interested in, although some measurements are of interest in all three. In the case of trapping we would want to know the trapping probability, for instance. But in all cases of transfer of energy we want to know the distance traveled by the excitation and hence also the rates of transfer.

For some of these questions, as, for instance, for the distance traveled, a large amount of experimental evidence is available. On the other hand, little or no experimental facts are recorded about the rate of transfer of energy. We are then reduced at this point to rely on a purely theoretical analysis.

II. DISCUSSION OF THE PROBLEM

In the present work we investigate the rate of transfer of energy, taking the wave packet point of view. The particular model we choose is an infinite, one-dimensional chain of identical monomers. The rate of transfer of excitation will then be given by the motion of the center of mass of the wave packet and/or by the rate of spreading of the wave packet. Since the chain is infinite and composed of identical particles, the first moment of the packet will be zero by symmetry; consequently, there will be no migration of the center of mass. The rate of transfer of energy will then be identical to the rate of spreading of the wave packet. This rate of spreading is related to the second moment of the distribution function of the wave packet in the following manner

$$V = \frac{\sqrt{\Delta(t)}}{t}$$

where V is the rate of spreading and $\Delta(t)$ is the second moment defined as

$$\Delta(t) = \sum_r (r - \bar{r})^2 p_r$$

where p_r is the distribution function (probability of finding excitation at r) and $\bar{r}$ is the coordinate

of the center of mass. If $\bar{r} = 0$,

$$\Delta(t) = \sum_r r^2 P_r$$

Higher moments will then give details of the variation of this rate of spreading. In the present state of the experimental techniques it is difficult to check the rate of spreading, so that it is not interesting to calculate the detailed behavior of the rate of spreading and we omit consideration of the higher moments.

We choose $\frac{\sqrt{\Delta(t)}}{t}$ as the appropriate velocity of spreading rather than $\frac{d}{dt} \sqrt{\Delta(t)}$ for the following reason. The time derivative of $\sqrt{\Delta(t)}$ will give us the rate of spreading at the specific time " t " while $\frac{\sqrt{\Delta(t)}}{t}$ gives the average spreading rate over time " t ". Now experimentally one can at best obtain the time between absorption of the wave packet and the reemission of the excitation. Hence we will be able to obtain only the average rate of spreading.

We make now the following basic assumptions:

1. Since in organic crystals the ground state is

almost always a singlet state we will consider only transitions to excited singlet states, transitions to triplet states being very weak.⁷

2. There is little intermolecular overlap of electron orbitals and we may omit anti-symmetrization.

3. We shall assume only nearest neighbor coupling. We are considering only molecular crystals with van der Waals forces binding the molecules together; more specifically we assume dipole-dipole interaction. The interaction between monomers is then proportional to the inverse cube of the intermonomer distance R . The coupling between non-nearest neighbors becomes therefore sufficiently small, as compared to nearest neighbor coupling, so that it can be neglected.

Since transfer of excitation will occur only if there is interaction between monomers, it is clear that the

7

R. K. Clayton, Molecular Physics in Photosynthesis, Blaisdell Publishing Co. 1965, 81.

rate of transfer must depend on the magnitude of the intermonomer interaction energy. This energy must be compared with the energies which characterize the ground and excited states of the monomer as illustrated in Figures 1:

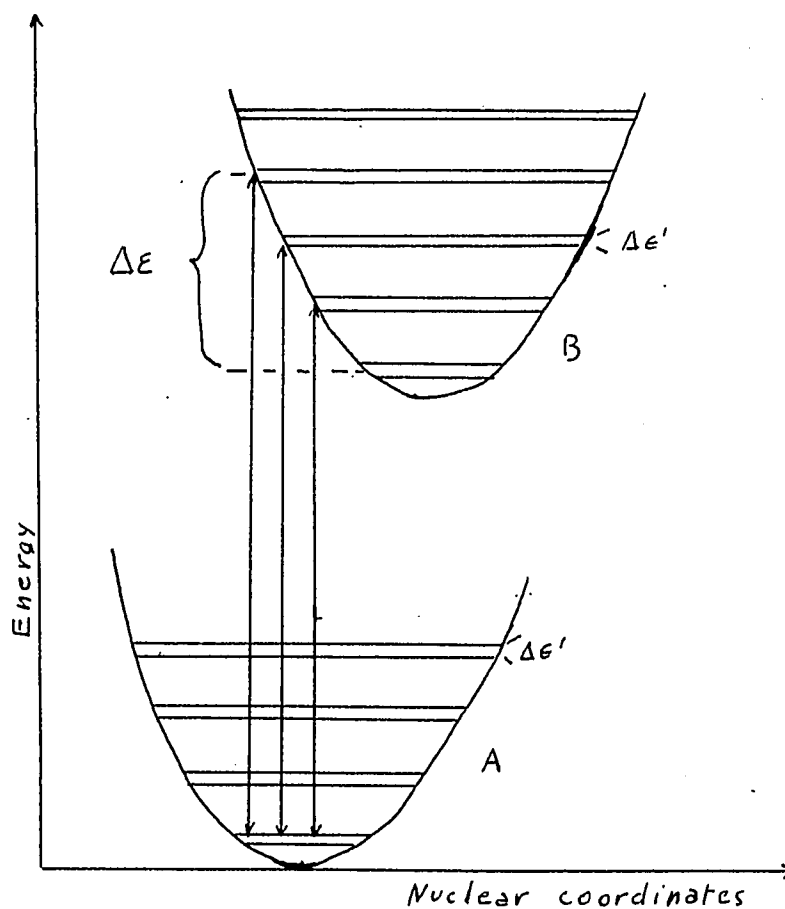


FIGURE 1

Curves A and B represent the electronic potential energy of the monomer ground state and an excited state with the corresponding vibrational levels. The energies are given as functions of the nuclear coordinates and the zero of energy is arbitrarily chosen at the equilibrium nuclear position of the ground state. As is well known, the equilibrium nuclear position is generally shifted for the excited state. Now, upon excitation, the nuclei will not immediately take on the new equilibrium position, but will be subject to the Franck-Condon principle.⁸

If the interaction energy W is much larger than the energy associated with the shift of nuclear equilibrium position ($\Delta\varepsilon$ in fig. 1), then the nuclei will not have had time to rearrange themselves according to the new equilibrium position, during the time that the excitation resides in that particular monomer. It would then be logical to expect that in this case ($W > \Delta\varepsilon$), the vibrational spectrum of the individual monomers would be replaced by the vibrational spectrum of the crystal.

Other quantities, whose relation with W , the intermonomer energy, will be important, are the vibrational quantum $\hbar\omega$ and the width of the vibrational band $\Delta\varepsilon'$.

⁸D. H. Dennison and K. T. Hecht, Quantum Theory II Aggregate of Particles (Academic Press) Ed. D. R. Bates page 264.

These will play a role in the interaction between the exciton wave and the phonons.

The importance of these relationships was already recognized by Franck and Teller⁹ in 1938, Davydov¹⁰ and Simpson and Peterson¹¹.

Förster¹² proposed the following classification based on the experimental facts:

1. Major Alteration in Absorption Spectra

"The vibronic envelopes of the spectra are completely different from those of the component spectra, even if the absorption region is approximately the same. This is the case in some dimers and crystals like anthracene at $\lambda = 2500 \text{ \AA}$ (absorption). Such profound alterations are obviously the result of fairly strong interaction between the components and it is natural to speak here of a strong coupling case."

2. Less Profound Alterations in Absorption Spectra

"There are examples where less profound alterations exist in the absorption spectra. The vibronic envelope is then retained, but the individual vibronic levels of the components are split

⁹J. Franck and E. Teller, J. Chem. Phys. 6, 861-872 (1938)

¹⁰A. S. Davydov, Zhur, Eksptl. i. Teoret. Fiz. 18, 210 (1948)

¹¹W. T. Simpson and D. L. Peterson, J. Chem. Phys. 26, 588 (1956)

¹²T. Förster in Modern Quantum Chemistry Part III (Academic Press, 1965), M. Burton, J. S. Kirby-Smith and J. L. Magee; 93-94.

in a characteristic way. Obviously these effects are also due to some kind of interaction between the components but of lesser magnitude. It has become customary to speak here of a weak coupling case."

3. Little or no Alterations in Absorption Spectra.

Many systems show very little alteration in their absorption spectra when compared with that of their components but have different luminescence and photochemical properties. Förster calls this the very weak coupling case.

In another publication¹³ Förster defines the limits of these three cases as follows:

1. Strong coupling $W \gg \Delta \epsilon \gg \Delta \epsilon'$
2. Weak coupling $\Delta \epsilon \gg W \gg \Delta \epsilon'$
3. Very weak coupling $\Delta \epsilon \gg \Delta \epsilon' \gg W$

Some controversy centers on the distinction between cases 2 and 3 as proposed by Förster. It was very tempting to make the distinction because case 3 would then be an example of localized excitation but experimentally it is very difficult to distinguish between them.

¹³

T. Förster, see ref. 6, page 307

Kasha¹⁴ then proposed to distinguish the cases differently by defining some limits for the ratio $\frac{W}{\Delta\epsilon}$. If $\frac{W}{\Delta\epsilon} \gg 1$ we have the very strong coupling case and when $\frac{W}{\Delta\epsilon} \ll 1$ we have the very weak coupling. All other values will then be called intermediate coupling. We can also look at these cases in a different light. Strong coupling can be interpreted as meaning that the time of residence of the excitation in any one monomer is very short. More precisely, consider the characteristic times involved. The period of intramonomer oscillation is of the order of $3 \times 10^{-14} \text{ sec}$. If the transfer time τ_t of electronic excitation is smaller than this number, we have the strong coupling case. If τ_t is between $3 \times 10^{-14} \text{ sec}$. (corresponding to a spacing of 1000 cm^{-1} between vibrational levels) and $3 \times 10^{-12} \text{ sec}$. (the period of collisional or lattice vibrations $\sim 10 \text{ cm}^{-1}$) we have weak binding. Otherwise we would be in the so-called very weak binding domain. Hence in the strong coupling case one expects that the rate of spreading of the initially localized wave packet should be independent of the vibrational spectrum; i.e., no phonon-exciton interaction. As the binding

¹⁴ Kasha, Radiation Research, 20, 55 (1963)

energy gets smaller, one should expect more exciton-phonon interaction and we will see that this is indeed the case.

As can be judged from all these comments, it appears that the classification according to comparative coupling strengths is rather ambiguous and incomplete.

In a note by McRae and Siebrand¹⁵, these authors clarify the ideas and criteria about strong and weak coupling in the following manner. Weak coupling is characterized by

$W \ll \Delta\epsilon$ and $W \ll 1$. Both W and $\Delta\epsilon$ are given in units of $\hbar\omega$, the energy quantum of a representative vibration. Strong coupling is then given by $\Delta\epsilon \ll W$ and $\Delta\epsilon \ll 1$.

Before any further discussion of the concept of strong and weak coupling, it will be useful to review briefly some of the work done by Magee¹⁶ and others to obtain the rate of energy transfer in a chain of molecules.

He assumes a linear system of N molecules with total wave function Ψ_j satisfying

$$H \Psi_j = E_j \Psi_j$$

and

¹⁵E. G. McRae and W. Siebrand, J. Chem. Phys. 41, 905-906 (1964)

¹⁶J. L. Magee, Comparative Effects of Radiation (J. Wiley 1960) 133-135

$$\psi_j = \sum_{k=1}^N \alpha_{jk} \eta_k$$

where η_k is the wave function of the system with the excitation located on the k^{th} monomer. Assuming that at the initial instant the excitation is located at a particular molecule (the p^{th} for instance), it follows after some calculation that the probability that the excitation will be located on the q^{th} molecule at time t is

$$P(q, p, t) = \left[J_{p-q} \left(\frac{2Wt}{\hbar} \right) \right]^2$$

where J_{p-q} is the Bessel function of order $p-q$ and W is $\int \eta_p^* H \eta_{p+1} dt$. Thus only nearest neighbor coupling is considered.

The spread of the excitation from the initial location is then obtained from

$$\overline{(p-q)^2} = \sum_{q=1}^N (q-p)^2 P(q, p, t)$$

For long chains this becomes

$$\overline{x^2} = 2 \sum_{n=0}^{\infty} n^2 J_n^2(z) = \frac{z^2}{2}$$

$$\text{where } z = \frac{2Wt}{\hbar}$$

The rate of spreading is then

$$V = \frac{\sqrt{2} W}{\hbar}$$

This result would presumably apply for all coupling strengths. Förster¹⁷ in a different derivation arrives at a similar result restricted to strong coupling and claims, without analytic proof, that for weak coupling the rate of transfer is

$$\frac{\sqrt{2} W |\mathcal{S}|^2}{\hbar}$$

where $|\mathcal{S}|^2$ is the overlap integral between excited and non-excited vibronic eigenfunctions.

A number of authors, Merrifield¹⁸, Magee and Funabashi¹⁹, arrive at the same conclusions. All the cited authors assume that initially one monomer is in an excited state of the isolated monomer. It is implicitly assumed that upon electronic excitation the nuclei of the excited monomer will also assume immediately the equilibrium position corresponding to the electronic excited state. But this violates the Franck-Condon principle. Bierman, in his paper "Motion of a Vibrationally Coupled Exciton Wave Packet

¹⁷T. Förster, see ref. 12, 105

¹⁸R. E. Merrifield, J. Chem. Phys. 28, 647 (1958)

¹⁹J. L. Magee and K. Funabashi, J. Chem. Phys. 34, 1715 (1961)

in an Infinite One-Dimensional Crystal"²⁰ starts off by assuming initial conditions in agreement with the Franck-Condon principle, that is to say that upon electronic excitation, the nuclei remain in the equilibrium position corresponding to the electronic ground state. In the very strong coupling limit he then again obtains Magee's result. But for less strong coupling one must use perturbation theory. The calculation of the first non-zero correction term is then the main topic of Bierman's paper. The main results are as follows. First he gets a two-parameter dependence; these parameters are the ratio of the intermonomer energy W to $\Delta\varepsilon$ and the ratio of the vibrational quantum $\hbar\omega$ to W . Secondly the rate of spreading of the wave packet is decreased from its value in the strong coupling limit by terms of the order of t , $\ln t$ or a constant depending on the value of the second parameter.

In the same paper it is also shown that if one assumes that, upon electronic excitation, the nuclei immediately assume the new equilibrium position, then one obtains for the velocity of energy transfer

$$(1) \quad v = \frac{\sqrt{2} W |S|^2}{\hbar}$$

valid for all coupling strengths and

²⁰A. Bierman, J. Chem. Phys., 46, 1484-1492 (1967). To be called henceforth Bierman's paper

which for strong coupling ($|S|^2=1$) again gives Magee's result. Thus the dependence on the parameter $\sigma = \frac{\hbar\omega}{2W}$ is due exclusively to the use of the Franck-Condon principle. In another paper²¹ Bierman shows that, for weak coupling, the rate of energy transfer is again time independent and is given by

$$(2) \quad V = \frac{\sqrt{2} W}{\hbar} \left(\sum_m |S_{m0}|^6 \right)^{1/2}$$

The difference between this expression for V and the one just above is due to the use of the Franck-Condon principle in fixing the initial conditions in (2) while this is not done in the derivation of (1).

As noted above the first order correction is always negative. It would be of interest to find out what the sign of the second order correction would be. Obtaining a higher order correction is equivalent to extending the domain of coupling strength, for which we would have a valid expression for the speed of energy transfer.

In addition a comparison of the terms of the first and second correction will define the domains of λ (and σ)

²¹ A. Bierman, J. Chem. Phys., 46, 2124-2126 (1967)

for which the truncation of the perturbation series, after the first or second correction, is valid.

It is then the purpose of this work to obtain the next order correction (actually the fourth order in $\lambda = \frac{\Delta\epsilon}{W}$) for the rate of spreading of the exciton wave packet.

III. REVIEW OF PAPER BY A. BIERMAN

Consider a one-dimensional array of N monomers labeled by K . Let the nuclear coordinates be labeled $\{Q_K\}$ and the electronic coordinates of the monomer by $\{q_K\}$. For fixed nuclear positions the electronic Hamiltonian of the K^{th} monomer is $\mathcal{H}_e(q_K, Q_K)$. We assume that the eigenvalue eq. for the isolated monomer is known:

$$(3) \quad \left\{ \mathcal{H}_e(q_K, Q_K) - \eta_\alpha(Q_K) \right\} \Lambda_\alpha = 0 \quad \alpha = 1, 2, \dots$$

The crystal Hamiltonian is then

$$(4) \quad \mathcal{H}(q, Q) = \sum_{K=1}^N \mathcal{H}_e(q_K, Q_K) + T_N + W$$

where T_N is the kinetic energy operator for the nuclei and

W represents the intermonomer electronic energy responsible for the crystal cohesion.

To construct eigenstates for $\mathcal{H}$ for the first excited electronic band, define an N -body wave function by

$$(5) \quad \phi_r = \Lambda_1(q_r, Q_r) \prod_{K \neq r}^N \Lambda_0(q_K, Q_K)$$

representing electronic excitation energy localized on the

r^{th} monomer in a given nuclear configuration of N non-interacting monomers.

Now suppose that at time $t=0$ energy is absorbed at one monomer. Since such an initial state does not correspond to an eigenstate of the Hamiltonian we have a wave packet. Let this wave packet be represented by the function $\Psi(q, Q, t)$ obeying the Schrödinger eq.

$$(6) \quad \mathcal{H}(q, Q) \Psi = \hbar i \frac{\partial \Psi}{\partial t}$$

$\Psi(t)$ can be represented by

$$(7) \quad \Psi(t) = \sum_{r=1}^N \xi_r(Q, t) \phi_r(q, Q)$$

and the probability of localizing the electronic energy at time t on the k^{th} monomer is given by

$$(8) \quad P_k(t) = \langle \xi_k(t) | \xi_k(t) \rangle$$

where the integration is over all nuclear coordinates. The equation for $\xi_k(Q, t)$ is obtained by replacing (7) in (6) and using (3). We get

$$(9) \quad \sum_{r=1}^N \left[\eta_1(Q_r) + \sum_{m \neq r} \eta_0(Q_m) + T_N + W \right] \xi_r \phi_r = \hbar i \sum_r \phi_r \frac{\partial \xi_r}{\partial t}$$

Now we assume that the Born-Oppenheimer approximation is valid for the isolated monomer. This can be expressed by

setting $[T_N, \phi_r] = 0$. Hence eq. (9) becomes, after multiplying by ϕ_s and integrating

$$(10) \quad \left[\sum_{m=1}^N \eta_0(Q_m) + \eta_1(Q_s) - \eta_0(Q_s) + T_N \right] \xi_s + \sum_{r=1}^N \langle \phi_s | W | \phi_r \rangle \xi_r = \hbar i \frac{\partial \xi_s}{\partial t}$$

$$s = -\frac{N}{2}, \dots, -1, 0, 1, \dots, \frac{N}{2}$$

We now assume that the matrix elements $\langle \phi_s | W | \phi_r \rangle$ are independent of Q . This is a reasonable assumption since their dependence on the Q 's appears only in the factor R^{-3} (R is the intermonomer distance) which will hardly change for motion of the nuclei (the change will be of much lower order of magnitude than the value of the matrix element itself).

Of course we also assume only nearest neighbor interaction.

Define now.

$$(11) \quad A(Q) = \sum_{m=1}^N \eta_0(Q_m) + \langle \phi_s | W | \phi_s \rangle$$

and

$$(12) \quad W = \langle \phi_s | W | \phi_{s \pm 1} \rangle \quad (\text{nearest neighbors})$$

and eq. (10) becomes

$$(13) \quad [A(Q) + \eta_1 - \eta_0 + T_N] \xi_s + W (\xi_{s+1} + \xi_{s-1}) = \hbar i \frac{\partial \xi_s}{\partial t}$$

These N coupled eqs. define the behavior of the vibronic wave packet.

Now one must discuss the specific initial state of the crystal. Bierman shows next that if one assumes first that the crystal is raised, upon excitation to an eigenstate of the Hamiltonian representing N non-interacting monomers, with one monomer, say the zeroth, being excited, all the others remaining in the ground state, then one eventually gets for

$$(14) \quad \xi_s(Q, t) = \chi_{os}(Q) (-i)^{|s|} \exp\left[-\frac{i}{\hbar} E'_0 t\right] \int_{|s|} \left(\frac{2W|s|^2}{\hbar} t\right)$$

where E'_0 is the vibrational ground state corresponding to the excited electronic level; and χ_{os} is the solution of

$$(15) \quad \left[A(Q) + \eta_i(s) - \eta_o(s) + T_N - E'_0 \right] \chi_{os}(Q) = 0$$

Then, for all coupling strengths, the wave packet spreads with constant rate

$$V = \frac{\sqrt{2} W}{\hbar} |S|^2$$

In the strong coupling limit where $|S|^2 = 1$ we recover Magee's result, namely that

$$V = \frac{\sqrt{2} W}{\hbar}$$

Since such an initial state would be in contradiction to the Franck-Condon principle, Bierman assumes a

different initial state.

Let $\Psi(o)$ represent a state of electronic excitation localized on one monomer with the nuclei still distributed in Q - space according to a probability function determined by the electronic ground state potential.

This means that

$$\Psi(o) = \chi_o(Q) \phi_o(q, Q)$$

where

$$(16) \quad [A(Q) + T_N - E_o] \chi_o(Q) = 0$$

and

$$\xi_s(Q, o) = \int_{s, o} \chi_o(Q)$$

E_o being the ground state vibronic state of the electronic ground state. In the extreme strong coupling limit ($\eta_1 - \eta_o = \text{constant}$) this condition is identical to the one usually used. They diverge most strongly in the weak coupling case. One investigates the implications of these new initial conditions by using $\eta_1 - \eta_o$ as a perturbation.

Assume now that

$$(17) \quad \begin{aligned} \eta_1(Q_r) &= \eta^* + \frac{k}{2} (Q_r - Q_o)^2 \\ \eta_o(Q_r) &= \frac{k}{2} Q_r^2 \end{aligned}$$

The strong coupling assumption is

$$(18) \quad \lambda \equiv \frac{k Q_0^2}{2 W} \ll 1 \quad \left(\frac{k}{2} Q_0^2 \text{ is } \Delta \varepsilon \right)$$

and one uses λ as the coefficient of expansion

Defining

$$(19) \quad \Omega^0 = \eta^* + \frac{k}{2} Q_0^2 + A(Q) + T_N$$

$$(20) \quad \lambda V_s = -\lambda \left(\frac{2 Q_s W}{Q_0} \right)$$

eq. (13) now becomes

$$(21) \quad \left(\Omega^0 + \lambda V_s \right) \xi_s + W (\xi_{s+1} + \xi_{s-1}) = \hbar i \frac{\partial \xi_s}{\partial t}$$

Now assume that

$$\xi_s = \xi_s^{(0)} + \lambda \xi_s^{(1)} + \lambda^2 \xi_s^{(2)} + \dots$$

and define E and F operators by

$$E \xi_s = \xi_{s+1}, \quad F \xi_s = \xi_{s-1}.$$

Then expansion of eq. (21) gives

$$(22a) \quad \left[\Omega^0 + W(E+F) \right] \xi_s^{(0)} = \hbar i \frac{\partial \xi_s^{(0)}}{\partial t}$$

$$(22b) \quad \left[\Omega^0 + W(E+F) \right] \xi_s^{(K)} + V_s \xi_s^{(K-1)} = \hbar i \frac{\partial \xi_s^{(K)}}{\partial t}$$

$K = 1, 2, \dots$

If the excitation was initially localized on the zeroth monomer and occurred in the ground vibrational state, then the solution of (22a) is

$$(23) \quad \xi_s^{(0)} = \chi_0(Q) \exp \left[-\frac{i}{\hbar} E_0 t \right] (-i)^{|s|} J_{|s|} \left(\frac{2W}{\hbar} t \right)$$

where

$$\left[\Omega^0(Q) - E_0 \right] \chi_0(Q) = 0$$

and

$$\chi_0 = \prod_{k=1}^N h_0(Q_k)$$

where $h_0(Q)$ represents the harmonic oscillator ground state.

The formal solution of (22b) is then

$$(24) \quad \xi_s^{(K)}(t) = (\hbar i)^{-1} \int_0^t d\tau \exp \left\{ -\left(\frac{i}{\hbar} \right) \left[\Omega^0 + W(E+F) \right] (t-\tau) \right\} V_s \xi_s^{(K-1)}(\tau)$$

The second moment of the vibronic wave packet is

$$\Delta(t) = \sum_{s=-\frac{N}{2}}^{s=\frac{N}{2}} s^2 \langle \xi_s(t) | \xi_s(t) \rangle$$

Expansion in powers of λ yields

$$(25) \quad \Delta(t) = \sum_{n=0}^{\infty} \lambda^n \Delta_n(t)$$

$$\Delta_0(t) = \sum_s s^2 \langle \xi_s^{(0)} | \xi_s^{(0)} \rangle = \frac{2W^2}{\hbar^2} t^2$$

$$\Delta_1(t) = 2 \operatorname{Re} \sum_s s^2 \langle \xi_s^{(1)} | \xi_s^{(0)} \rangle$$

$$\Delta_2(t) = \sum_s s^2 \langle \xi_s^{(1)} | \xi_s^{(1)} \rangle + 2 \operatorname{Re} \sum_s s^2 \langle \xi_s^{(0)} | \xi_s^{(2)} \rangle$$

$$\Delta_3(t) = 2 \operatorname{Re} \sum_s s^2 \langle \xi_s^{(1)} | \xi_s^{(2)} \rangle + 2 \operatorname{Re} \sum_s s^2 \langle \xi_s^{(0)} | \xi_s^{(3)} \rangle$$

$$\Delta_4(t) = \sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle + 2 \operatorname{Re} \sum_s s^2 \langle \xi_s^{(0)} | \xi_s^{(4)} \rangle + 2 \operatorname{Re} \sum_s s^2 \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle$$

It is shown in Appendix A that all odd-numbered $\Delta_n(t)$ are zero.

The remainder of Bierman's paper is concerned with the calculation of $\Delta_2(t)$ and the results for the case of

$$z \equiv \frac{2W}{\hbar} t \gg 1 \quad \text{are}$$

$$(26') \quad \lambda^2 \Delta_2(t) = -\frac{2}{3\pi} z^3 \left(\frac{\kappa Q_0^2}{2W} \right) \left(\frac{\hbar \omega}{2W} \right) \left[E \left(1 - \frac{\sigma^2}{4} \right) - \frac{\sigma^2}{4} K \left(1 - \frac{\sigma^2}{4} \right) \right] \quad \text{for } \sigma < 2$$

$$(26'') \quad \lambda^2 \Delta_2(t) = -\frac{1}{\pi} \left(\frac{\kappa Q_0^2}{2W} \right) z^2 \ln z \quad \text{for } \sigma = 2$$

$$(26''') \quad \lambda^2 \Delta_2(t) = -\frac{4}{\pi} \left(\frac{\kappa Q_0^2}{2W} \right) \left(\frac{2W}{\hbar \omega} \right) z^2 \left[\frac{1}{\pi} K \left(\frac{4}{\sigma^2} \right) - \frac{1}{2} \right] \quad \text{for } \sigma > 2$$

where

$$\sigma = \frac{\hbar \omega}{2W}$$

and E, K are complete elliptic integrals of first and second kind respectively.

IV. EVALUATION OF $\Delta_4(t)$

In the remainder of this paper we will be concerned with the evaluation of $\Delta_4(t)$. Due to the large amount of algebraic manipulations, the details are given in appendices.

As can be seen from eq. (25), $\Delta_4(t)$ consists of a sum of three terms. For each of these we must first replace $\xi_s^{(i)}$ by the expressions (24) and (23). By expanding the exponentials occurring in (24) we obtain products of infinite sums. Thus the first task is to reduce these sums. This will not always be possible and we will remain with three terms, each of which still contains one infinite sum. These terms will be discussed later. The other terms, nine in number, are four-dimensional integrals.

Next we perform two integrations on these nine terms. Up to this point we still have exact expressions, but it does not appear possible to continue without making some simplifying assumptions.

The second part of the work is started off by assuming that the constant upper limit in the integrals, " p ", becomes very large. As will be shown in the development,

this is a reasonable assumption. It is then possible to find the highest order term in ρ and evaluate its coefficient. In the last section we discuss the three terms in which the summation could not be effected and show that these terms are of lower order in ρ than the terms already evaluated.

1. Exact Calculations

A. Expansion of the three terms in $\Delta_4(t)$

(1) Expansion of $\sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle$

From eq. (25)

$$\begin{aligned} \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle &= \frac{1}{\hbar^4} \int_0^t d\tau \int_0^\tau d\tau' \int_0^\tau d\tau'' \int_0^{\tau''} d\tau''' \langle e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](t-\tau)} V_s \\ &\quad \times e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](\tau-\tau')} V_s \xi_s^0(\tau') \mid e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](t-\tau'')} V_s \\ &\quad \times e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](\tau''-\tau''')} V_s \xi_s^0(\tau''') \rangle \end{aligned}$$

We first expand all the exponentials of the form

$$e^{-\frac{i}{\hbar} W(E+F)(t-\tau')}$$

Then we note that $e^{-\frac{i}{\hbar} \Omega^0(t-\tau)}$ is the unitary operator which takes one from the Schrödinger to the Heisenberg picture; i.e.,

$$| e^{-\frac{i}{\hbar} \Omega^0(t-\tau)} V_{s+i-j} e^{-\frac{i}{\hbar} \Omega^0(\tau-t)} \chi_0 \rangle = | V_{s+i-j}(\tau-t) \chi_{0H} \rangle$$

Next we express $V_{s+i-j}(\tau-t)$ in terms of raising and lowering operators, i.e.,

$$V_{s+i-j}(\tau-t) = - \frac{2W}{Q_0} \left[a_{s+i-j}^+ e^{i\omega(t-\tau)} + a_{s+i-j} e^{-i\omega(t-\tau)} \right] \left(\frac{\hbar}{2M\omega} \right)^{1/2}$$

Hence the factor involving the raising and lowering operators becomes

$$\begin{aligned} & \frac{\hbar^2}{4M^2\omega^2} \left(\frac{2W}{Q_0} \right)^4 \left\{ \langle a_{s+i-j} a_{s+i-j+k-l}^+ \chi_0 | a_{s+m-n} a_{s+m-n+u-v}^+ \chi_0 \rangle \right. \\ & \quad \times e^{-i\omega(\tau-\tau'-\tau''+\tau''')} + \langle a_{s+i-j}^+ a_{s+i-j+k-l} \chi_0 | a_{s+m-n}^+ a_{s+m-n+u-v} \chi_0 \rangle \\ & \quad \left. \times e^{i\omega(\tau+\tau'-\tau''-\tau''')} \right\} \end{aligned}$$

From this expression we deduce that we have really three cases

- a) $\delta_{k,l} \delta_{u,v}$
- b) $\delta_{i-j, m-n} \delta_{k-l, u-v}$
- c) $\delta_{i-j, m-n+u-v} \delta_{i-j+k-l, m-n}$

For each of these cases the complete expression for the matrix element involves infinite sums over the indices $i, j, k, l,$

m, n, u, v subject to the conditions expressed by the Kronecker

de/tas.

By successive use of formulas derived from Graf's²²

formula

$$J_0(\omega) = \sum_{l=-\infty}^{\infty} J_l(u) J_l(v) \cos l\alpha$$

where

$$\omega = (u^2 + v^2 - 2uv \cos \alpha)^{1/2}$$

we can sum the series for cases "a" and "b" and reduce them to only one sum for case "c".

Putting now

$$\sigma = \frac{\hbar \omega}{2W}$$

$$p = \frac{W}{\hbar} t$$

$$q = \frac{W}{\hbar} \tau$$

$$q' = \frac{W'}{\hbar} \tau'$$

$$q'' = \frac{W}{\hbar} \tau''$$

$$q''' = \frac{W}{\hbar} \tau'''$$

the final results are

$$\sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle = I_{1a} + I_{1b} + I_{1c}$$

$$I_{1a} = 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \frac{(P-q+q')(P-q''+q''')}{q-q'-q''+q'''} \\ \times J_0(2q-2q') J_0(2q''-2q''') J_1(2q-2q'-2q''+2q''') \cos \sigma(2q-2q'-2q''+2q''')$$

$$\begin{aligned}
I_{1b} &= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \\
&\times \left\{ \frac{q' q'''}{q' - q'''} J_1(2q' - 2q''') J_0(2q - 2q' - 2q'' + 2q''') J_0(2q - 2q'') \right. \\
&+ \frac{(q - q')(q'' - q''')}{q - q' - q'' + q'''} J_0(2q' - 2q''') J_0(2q - 2q'') J_1(2q - 2q' - 2q'' + 2q''') \\
&+ \left. \frac{(p - q)(p - q'')}{q - q''} J_0(2q' - 2q''') J_0(2q - 2q' - 2q'' + 2q''') J_1(2q - 2q'') \right\} \\
&\times \cos \sigma(2q + 2q' - 2q'' - 2q''') \\
&= I_{1b,1} + I_{1b,2} + I_{1b,3}
\end{aligned}$$

$$\begin{aligned}
I_{1c} &= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} J_r(2q'' - 2q''') J_r(2q - 2q') \\
&\times \left\{ q' J_r(2q'' - 2q) J_{r+1}(2q''' - 2q') + q'^2 J_r(2q'' - 2q) [J_r(2q''' - 2q') + J_{r+2}(2q''' - 2q')] \right. \\
&+ (p - q'') J_r(2q''' - 2q') J_{r+1}(2q'' - 2q) + (p - q'')^2 J_r(2q''' - 2q') [J_r(2q'' - 2q) + J_{r+2}(2q'' - 2q)] \\
&+ q'(p - q'') [J_{r+1}(2q''' - 2q') J_{r+1}(2q'' - 2q) + J_{r+1}(2q''' - 2q') J_{r-1}(2q'' - 2q) \\
&+ J_{r-1}(2q''' - 2q') J_{r+1}(2q'' - 2q) + J_{r-1}(2q''' - 2q') J_{r-1}(2q'' - 2q)] \left. \right\} \\
&\times \cos \sigma(2q + 2q' - 2q'' - 2q''')
\end{aligned}$$

The detailed derivation of these expressions is given in Appendix B.

2. Expansion of $\sum_s s^2 < \xi_s^{(1)} | \xi_s^{(3)} >$

By following the same procedure as in section 1, one gets a sum of three terms:

$$\sum_s s^2 < \xi_s^{(1)} | \xi_s^{(3)} > = I_{2a} + I_{2b} + I_{2c}$$

$$\begin{aligned} I_{2a} = & -2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \int_0^{(2q''-2q''')} \\ & \times \left\{ \frac{q(q'-q''+q''')}{q-q'+q''-q'''} \int_0^{(2q-2q')} J_1(2q-2q'+2q''-2q''') \right. \\ & \left. + \frac{(p-q)(p-q')}{q-q'} \int_0^{(2q-2q'+2q''-2q''')} J_1(2q-2q') \right\} \cos \sigma(2q-2q'+2q''-2q''') \end{aligned}$$

$$I_{2a} = I_{2a,1} + I_{2a,2}$$

$$\begin{aligned} I_{2b} = & -2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \int_0^{(2q'-2q''')} \\ & \times \left\{ \frac{qq'''}{q-q'''} \int_0^{(2q-2q'+2q''-2q''')} J_1(2q-2q''') \right. \\ & \left. + \frac{(p-q)(p-q'+q''-q''')}{q-q'+q''-q'''} \int_0^{(2q-2q''')} J_1(2q-2q'+2q''-2q''') \right\} \\ & \times \cos \sigma(2q+2q'-2q''-2q''') \end{aligned}$$

$$\begin{aligned}
I_{2c} = & -2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_r J_r(2q' - 2q'') J_r(2q'' - 2q''') \\
& \times \left\{ q J_r(2q - 2q') J_{r+1}(2q''' - 2q) + q^2 J_r(2q - 2q') [J_r(2q''' - 2q) + J_{r+2}(2q''' - 2q)] \right. \\
& + (p-q) J_r(2q''' - 2q) J_{r+1}(2q - 2q') + (p-q)^2 J_r(2q''' - 2q) [J_r(2q - 2q') + J_{r+2}(2q - 2q')] \\
& + q(p-q) [J_{r+1}(2q''' - 2q) J_{r+1}(2q - 2q') + J_{r+1}(2q''' - 2q) J_{r-1}(2q - 2q') \\
& \quad \left. + J_{r-1}(2q''' - 2q) J_{r+1}(2q - 2q') + J_{r-1}(2q''' - 2q) J_{r-1}(2q - 2q') \right\} \\
& \times \cos \sigma(2q + 2q' - 2q'' - 2q''')
\end{aligned}$$

For details see Appendix C.

3. Expansion of $\sum_s s^2 < \xi_s^{(0)} | \xi_s^{(4)} >$

By the same procedure we get

$$\sum_s s^2 < \xi_s^{(0)} | \xi_s^{(4)} > = I_{3a} + I_{3b} + I_{3c}$$

$$\begin{aligned}
I_{3a} = & 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' J_0(2q - 2q') J_0(2q'' - 2q''') \\
& \times \frac{q(p-q+q'-q''+q''')}{p-2q+q'-q''+q'''} J_1(2p-4q+2q'-2q''+2q''') \cos \sigma(2q-2q'+2q''-2q''')
\end{aligned}$$

$$\begin{aligned}
I_{3b} &= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' J_0(2q' - 2q'') J_0(2q - 2q' + 2q'' - 2q''') \\
&\quad \times \frac{q(P - q + q''')}{P - 2q + q'''} J_1(2P - 4q + 2q''') \cos \sigma(2q + 2q' - 2q'' - 2q''') \\
I_{3c} &= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} J_r(2q - 2q') J_r(2q' - 2q'') \\
&\quad \times J_r(2q'' - 2q''') \left\{ q J_{r+1}(2P - 4q + 2q''') + q^2 [J_{r+2}(2P - 4q + 2q''') \right. \\
&\quad \left. + J_r(2P - 4q + 2q''')] \right\} \cos \sigma(2q + 2q' - 2q'' - 2q''')
\end{aligned}$$

The details are given in Appendix D.

B. Two Integrations of $I_{1a}, I_{1b}, I_{2a}, I_{2b}, I_{3a}, I_{3b}$

By the appropriate coordinate transformation it is possible to effect two integrations in integrals

$$I_{ia}, I_{ib} \quad ; \quad i = 1, 2, 3$$

The details are given in Appendix E and the results are letting

$$2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 = B, \quad Z = 2P$$

$$I_{1a} = \frac{B}{16} \int_0^z d\lambda \int_0^z d\xi \left[z^4 - z^3(2\lambda + 2\xi) + z^2(\xi^2 + 4\xi\lambda + \lambda^2) - z(2\lambda\xi^2 + 2\lambda^2\xi) \right. \\ \left. + \xi^2\lambda^2 \right] \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{2(\lambda - \xi)} \cos \sigma(\lambda - \xi)$$

$$I_{1b,1} = \frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 - 2z^3(2\lambda - \xi) + 6z^2(\lambda^2 - \lambda\xi) - 2z(2\lambda^3 - 3\lambda^2\xi) + \lambda^4 - 2\lambda^3\xi \right. \right. \\ \left. + \int_\lambda^z d\xi \left[z^4 - 2z^3\xi + 2z\xi^3 + \lambda^4 - \xi^4 \right] \right. \\ \left. + \int_{\lambda-z}^0 d\xi \left[z^4 - 2z(2\lambda - \xi) + 6z^2(\lambda^2 - \lambda\xi) - 2z(2\lambda^3 - 3\lambda^2\xi + \xi^3) + \lambda^4 \right. \right. \\ \left. \left. - 2\lambda^3\xi + 2\xi^2\lambda^2 - \xi^4 \right] \right\} \\ \times \frac{J_1(\xi) J_0(\lambda) J_0(\lambda - \xi)}{\xi} \cos \sigma(\lambda + \xi)$$

$$I_{1b,2} = \frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 + 6z^2\lambda\xi - z(6\lambda^2\xi - 2\lambda^3) - \lambda^4 + 2\lambda^3\xi \right] \right. \\ \left. + \int_\lambda^z d\xi \left[z^4 - 2z^3\lambda + 6z^2\lambda\xi - z(6\lambda\xi^2 - 2\xi^3 + 2\lambda\xi^3 - \xi^4) \right] \right. \\ \left. + \int_{\lambda-z}^0 d\xi \left[z^4 + 2z^3\xi - 2z(3\lambda^2\xi - \lambda^3 + \xi^3 - 3\lambda\xi^2) - \lambda^4 + 4\lambda^3\xi \right. \right. \\ \left. \left. - 6\lambda^2\xi^2 + 4\lambda\xi^3 - \xi^4 \right] \right\} \\ \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda + \xi)$$

$$\begin{aligned}
I_{1b,3} = & \frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 + 2z\lambda^3 - \lambda^4 \right] \right. \\
& + \int_\lambda^z d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6z^2(\xi^2 - \xi\lambda) + z(6\lambda\xi^2 - 4\xi^3) - 2\lambda\xi^3 + \xi^4 \right] \\
& + \int_{\lambda-z}^0 d\xi \left[z^4 + 4z^3\xi + 6z^2(\xi^2 - \xi\lambda) + z(4\xi^3 - 6\lambda\xi^2) - \lambda^4 + 2\lambda^3\xi - 2\lambda\xi^3 + \xi^4 \right] \\
& \times \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda + \xi)
\end{aligned}$$

$$\begin{aligned}
I_{2a,1} = & -\frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[3z^4 - 4z^3(\lambda + 2\xi) + 6z^2(\xi^2 + 2\lambda\xi) - 12z\lambda\xi^2 + \lambda^4 \right. \right. \\
& \left. \left. - 4\lambda^3\xi + 6\lambda^2\xi^2 \right] \right. \\
& + \int_\lambda^z d\xi \left[3z^4 - 4z^3(\lambda + 2\xi) + 6z^2(\xi^2 + 2\lambda\xi) - 12z\lambda\xi^2 - \xi^4 - 4\lambda\xi^3 \right] \\
& + \int_{\lambda-z}^0 d\xi \left[3z^4 - 8z^3(\lambda - \xi) + 6z^2(\lambda - \xi)^2 - (\lambda - \xi)^4 \right] \left. \right\} \\
& \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{2(\lambda - \xi)} \cos(\lambda - \xi)
\end{aligned}$$

$$\begin{aligned}
I_{2a,2} = & -\frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 + z^3(2\lambda - 4\xi) - 6z^2(\lambda^2 - \lambda\xi) + 2z\lambda^3 - 2\xi\lambda^3 + \lambda^4 \right] \right. \\
& + \int_\lambda^z d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6z^2(\xi^2 - \lambda\xi) + 2z(3\lambda\xi^2 - 2\xi^3 - \lambda^3) - \lambda^4 - 2\lambda\xi^3 + \xi^4 \right] \\
& + \int_{\lambda-z}^0 d\xi \left[z^4 - 2z^3(\lambda - 2\xi) - 6z^2(\lambda\xi - \xi^2) + 2z(\lambda^3 - 3\lambda\xi^2 + 2\xi^3) - \lambda^4 + 2\lambda^3\xi \right]
\end{aligned}$$

$$-2\lambda\xi^3 + \xi^4 \Big\}$$

$$\times \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{2\lambda} \cos \sigma(\lambda - \xi)$$

$$\begin{aligned} I_{2b,1} = & -\frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6z^2(\lambda\xi - \lambda^2) + 2z\lambda^3 + \lambda^4 - 2\lambda^3\xi \right] \right. \\ & + \int_\lambda^z d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6z^2(\xi^2 - \lambda\xi) + 2z(3\lambda\xi^2 - 2\xi^3) - 2\lambda\xi^3 + \xi^4 \right] \\ & + \int_{\lambda-z}^0 d\xi \left[z^4 + 2z^3(2\xi - \lambda) + 6z^2(\xi^2 - \lambda\xi) + 2z(\lambda^3 + 2\xi^3 - 3\lambda\xi^2) - \lambda^4 \right. \\ & \left. \left. + 2\lambda^3\xi - 2\lambda\xi^3 + \xi^4 \right] \right\} \end{aligned}$$

$$\times \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{2\lambda} \cos(\lambda + \xi)$$

$$\begin{aligned} I_{2b,2} = & -\frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[3z^4 - 4z^3(\lambda + 2\xi) + 6z^2(2\lambda\xi + \xi^2) - 12z\lambda\xi^2 + \lambda^4 \right. \right. \\ & \left. \left. - 4\lambda^3\xi + 6\lambda^2\xi^2 \right] \right. \\ & + \int_\lambda^z d\xi \left[3z^4 - 4z^3(\lambda + 2\xi) + 6z^2(2\lambda\xi + \xi^2) - 12z\lambda\xi^2 + 4\lambda\xi^3 - \xi^4 \right. \\ & \left. + \int_{\lambda-z}^0 d\xi \left[3z^4 - 8z^3(\lambda - \xi) + 6z^2(\lambda - \xi)^2 - (\lambda - \xi)^4 \right] \right\} \\ & \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{2(\lambda - \xi)} \cos \sigma(\lambda + \xi) \end{aligned}$$

$$\begin{aligned}
I_{3a} = & \frac{B}{32} \int_0^z d\lambda \int_0^{z-\lambda} d\xi (z-\lambda-\xi) J_0(\lambda) J_0(\xi) \\
& \times \left\{ -2(\lambda+\xi) J_0(\lambda+\xi) - (z-3\lambda-3\xi) J_0(z-2\lambda-2\xi) \right. \\
& + (z-\lambda-\xi)(z-2\lambda-2\xi) \left[J_1(\lambda+\xi) + J_1(z-2\lambda-2\xi) \right] \\
& - \left[1 + (z-\lambda-\xi)(z-2\lambda-2\xi) \right] \left[(\lambda+\xi) J_0(\lambda+\xi) + (z-2\lambda-2\xi) J_0(z-2\lambda-2\xi) \right] \\
& + \frac{\pi(\lambda+\xi)}{2} \left(H_0(\lambda+\xi) J_1(\lambda+\xi) - H_1(\lambda+\xi) J_0(\lambda+\xi) \right) \\
& \left. + \frac{\pi(z-2\lambda-2\xi)}{2} \left(H_0(z-2\lambda-2\xi) J_1(z-2\lambda-2\xi) - H_1(z-2\lambda-2\xi) J_0(z-2\lambda-2\xi) \right) \right\} \cos \sigma(\lambda+\xi)
\end{aligned}$$

$$\begin{aligned}
I_{3b} = & \frac{B}{32} \int_0^z d\lambda \int_0^\lambda d\xi (\lambda-\xi)(z-\lambda) J_0(\xi) J_0(\lambda-\xi) \\
& \times \left\{ J_0(z-2\lambda) - J_0(\lambda) + (z-\lambda) \left[-J_1(\lambda) - J_1(z-2\lambda) + (z-2\lambda) J_0(z-2\lambda) \right] \right. \\
& + \lambda J_0(\lambda) + \frac{\pi(z-2\lambda)}{2} \left(H_0(z-2\lambda) - H_1(z-2\lambda) J_0(z-2\lambda) \right) \\
& \left. + \frac{\pi\lambda}{2} \left(H_0(\lambda) J_1(\lambda) - H_1(\lambda) J_0(\lambda) \right) \right\} \cos \sigma(\lambda+\xi)
\end{aligned}$$

The details are given in Appendix E.

Before we proceed, it will be useful to show that all the integrals obtained up to now, the sum of which constitutes the term we want to obtain, go to zero when

$z \rightarrow 0$. This is evident by inspection of the integrals

$$I_{i \begin{smallmatrix} a, j \\ b \\ c \end{smallmatrix}}$$

. All the integrands are

finite when the arguments of the Bessel functions are zero. For the integrals I_{i_c} the sums reduce to a single term $J_0(x)$ which again is finite for $x=0$. Hence it is clear that $\Delta_4(t)$ is zero when $z=0$.

2. Evaluation of the Integrals for $z \rightarrow \infty$

Up to this point all the expressions are still exact. To proceed, it does not seem possible to get any further results without making the following assumption. As mentioned by Bierman, the case of $z \gg 1$ is physically quite reasonable, since $z = \frac{2W}{\hbar} t$ is about $10^{14}t$ (t in seconds), where we have assumed that $W \approx 0.05 \text{ eV}$. We will then let z become very large, in effect let $z \rightarrow \infty$. It is then possible to evaluate the integrals, or at least get the highest order term in z .

In the remainder of this paper we will then show that

(A) The terms which are the product of z^4 and a double integral are the highest order terms in z . Two cases will be possible

a) $\sigma \gg 2$ where the order is z^4

b) $\sigma < 2$ where the order is $z^4 \ln z$

(B) Only one term, namely $I_{1a} + 2I_{2a,1}$ contributes to these highest order terms in z .

(C) Finally we will justify our leaving aside the three non-reduced sums I_{1c} , I_{2c} and I_{3c} by showing that these integrals contribute at most terms of order z^3 .

A. Evaluation of the integrals (highest order terms in z only)

In this section we will obtain those terms which are of the highest order in z . It will be convenient to group together those integrals which have the same products of Bessel functions in the integrand. For easier reading only the general plans of the calculation are given, leaving the details in the appendices.

1. $I_{1b,2} + 2I_{2b,2}$

The combination $I_{1b,2} + 2I_{2b,2}$ is

$$= \frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[-2z^4 + 4z^3(\lambda + 2\xi) - 6z^2(\lambda\xi + \xi^2) + 2z\lambda(6\xi^2 - \right.$$

$$\begin{aligned}
& -3\lambda\xi + \lambda^2) - 2\lambda^4 + 6\lambda^3\xi - 6\lambda^2\xi^2] \\
& + \int_{\lambda}^z d\xi \left[-2z^4 + 2z^3(\lambda + 4\xi) - 6z^2(\lambda\xi + \xi^2) + 2z(3\lambda\xi^2 + \xi^3) - 2\lambda\xi^3 \right] \\
& + \int_{\lambda-z}^0 d\xi \left[-2z^4 + 2z^3(4\lambda - 3\xi) - 6z^2(\lambda - \xi)^2 + 2z(\lambda - \xi)^3 \right] \Bigg\} \\
& \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda + \xi) \\
& = I_1 + I_2 + I_3
\end{aligned}$$

The main stumbling block in the evaluation of these integrals are the Bessel functions. For z very large it is clear that there are domains of integration where it will be permitted to replace some, if not all the Bessel functions by their asymptotic expansion. Let V be the value of the argument of a Bessel function such that whenever the argument is larger than V the Bessel function may be replaced by its asymptotic expansion. By subdividing the domain of integration, say of I_1 , $0 < \lambda < z$, $0 < \xi < \lambda$ into three parts it can be shown that the z^4 term is the highest order term in z .

The three domains are

- 1) $0 < \lambda < V$, $0 < \xi < \lambda$
- 2) $V < \lambda < z$, $0 < \xi < V$
- 3) $V < \lambda < z$, $V < \xi < \lambda$

In domain (1) the integrals converge and it is clear that the z^4 term is the highest order term.

In domain (2) $J_0(\lambda)$ and $J_1(\lambda - \xi)$ can be replaced by their asymptotic expansion. Inverting the order of integration, one can effect the " λ " integration and the " ξ " integral remaining is clearly convergent. One remains with one term in z^4 as the highest order term, multiplied by a convergent integral similar to the one obtained in domain (1) and which can be incorporated in that integral by extending its upper limit.

In domain (3) both λ and ξ are large and we can use the asymptotic expansion of $J_0(\lambda)$ and $J_0(\xi)$. This accomplished, change of variables from λ and ξ to

$$\mu = \lambda + \xi \quad \nu = \lambda - \xi$$

allows one to do one integration (with respect to μ). A detailed analysis of the remaining integral then shows that no term will be of order in z higher than $z^{3/2}$.

The detailed analysis is to be found in Appendix F.

The same procedure is followed for I_2 and I_3 .

In this last integral it is necessary to subdivide the area of integration into 6 domains.

The details are given in Appendix F.

$$2. \quad \underline{I_{1a} + 2 I_{2a,1}}$$

$$\begin{aligned} I_{1a} + 2 I_{2a,1} &= \frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[3z^4 - 4z^3(2\lambda + \xi) + 6z^2(\lambda^2 + 2\lambda\xi) - 12z\lambda^2\xi - \lambda^4 + 4\lambda^3\xi \right] \right. \\ &\quad + \int_\lambda^z d\xi \left[3z^4 - 4z^3(2\lambda + \xi) + 6z^2(2\lambda\xi + \lambda^2) - 12z\lambda^2\xi + 6\lambda^2\xi^2 \right. \\ &\quad \left. \left. + 4\lambda\xi^3 + \xi^4 \right] \right. \\ &\quad \left. - \int_{\lambda-z}^0 d\xi \left[3z^4 - 8z^3(\lambda - \xi) + 6z^2(\lambda - \xi)^2 - (\lambda - \xi)^4 \right] \right\} \\ &\quad \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda - \xi) \\ &= \frac{B}{192} \left\{ {}^1I_2 + {}^2I_2 + {}^3I_2 \right\} \end{aligned}$$

The procedure for ${}^1I_2$ and ${}^2I_2$ is the same as before. For ${}^3I_2$ we get an important variation. The highest order terms in z will be different, depending on the value of σ .

a) For $\sigma \geq 2$ the highest order term in z is z^4

b) For $\sigma < 2$ the highest order term in z is $z^4 \ln z$

The details are given in Appendix G.

$$3. \quad \underline{I_{1b,3} + 2 I_{2b,1}}$$

$$I_{1b,3} + 2 I_{2b,1} =$$

$$\begin{aligned} & \frac{B}{192} \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[2z^3(2\xi - \lambda) + 6z^2(\lambda^2 - \lambda\xi) + 2\lambda^3\xi - 2\lambda^4 \right] \right. \\ & \quad \left. + \int_{\lambda-z}^0 d\xi \left[2z^3\lambda - 2z\lambda^3 \right] \right\} \\ & \quad \times \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda + \xi) \end{aligned}$$

This integral does not contribute a term of order z^4 ; all terms are of lower order as shown in Appendix H.

4. The calculations for the other integrals are similar except for I_{3a} and I_{3b} which can be handled without domain subdivisions by appropriate change of variables as shown in Appendix J and Appendix K . I_{3a} and I_{3b} are of order lower than z^4 and thus need not be considered any further.

The details of the evaluation of $I_{2a,2}$ and $I_{1b,1}$ are given in Appendix I.

B. Evaluation of the Coefficients of Remaining Terms

We must consider two cases

$$1. \quad \sigma < 2$$

$$2. \quad \sigma \gg 2$$

1. The case $\sigma < 2$ has already been solved completely when evaluating ${}^3I_2$ the details of which appeared in Appendix G.

Hence for $\sigma < 2$ the only remaining term is

$$\frac{\hbar}{4 Q_0^2 M \omega} \left[z^4 \ln z \right] \cos \left\{ \sin^{-1}(\sigma - 1) \right\}$$

2. For $\sigma \gg 2$ we tabulate first the results of section A.

The integrals which must be evaluated will not be greatly affected if the upper limit z is changed to infinity. We thus start from the following:

$$\begin{aligned} \frac{B}{192} z^4 \left\{ -2 \int_0^\infty d\lambda \int_0^\infty d\xi J_0(\lambda) J_0(\xi) \frac{J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda + \xi) \right. \\ \left. -2 \int_0^\infty d\lambda \int_0^\infty d\xi J_0(\lambda) J_0(\xi) \frac{J_1(\lambda + \xi)}{\lambda + \xi} \cos \sigma(\lambda - \xi) \right. \\ \left. +3 \int_0^\infty d\lambda \int_0^\infty d\xi J_0(\lambda) J_0(\xi) \frac{J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda - \xi) \right. \\ \left. -3 \int_0^\infty d\lambda \int_0^\infty d\xi J_0(\lambda) J_0(\xi) \frac{J_1(\lambda + \xi)}{\lambda + \xi} \cos \sigma(\lambda + \xi) \right\} \end{aligned}$$

$$\begin{aligned}
& - \int_0^\infty d\lambda \int_0^\infty d\xi \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda - \xi) \\
& - \int_0^\infty d\lambda \int_0^\infty d\xi \frac{J_0(\xi) J_0(\lambda + \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda + \xi) \\
& + \int_0^\infty d\lambda \int_0^\infty d\xi \frac{J_0(\lambda) J_0(\lambda - \xi) J_1(\xi)}{\xi} \cos \sigma(\lambda + \xi) \\
& + \int_0^\infty d\lambda \int_0^\infty d\xi \frac{J_0(\lambda) J_0(\lambda + \xi) J_1(\xi)}{\xi} \cos \sigma(\lambda - \xi) \Big\}
\end{aligned}$$

$$= \sum_i I_i \quad i = 1, 2, \dots, 8.$$

The same procedure is followed in all cases. Replace $J_{0,1}(\lambda \pm \xi)$ by an integral representation

$$J_0(\lambda \pm \xi) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos[(\lambda \pm \xi) \cos \theta] d\theta = \frac{2}{\pi} \int_0^1 \frac{\cos(\lambda \pm \xi)x}{\sqrt{1-x^2}} dx$$

Then invert the order of integration. It is then possible to integrate with respect to λ and ξ . As shown in Appendix L $I_1 + I_2$, $I_5 + I_6$, $I_7 + I_8$ are all zero.

$I_3 - I_4$ can be evaluated exactly and the final result is that the leading term for $\sigma \gg 2$ is

$$\frac{\hbar}{8 Q_0^2 M \omega} z^4 \left[2 + \sqrt{\sigma(\sigma+2)} - \sqrt{\sigma(\sigma-2)} \right]$$

For details see Appendix L.

The results are then

For $\sigma < 2$

$$(27) \quad \lambda^4 \Delta_4(t) = \frac{\lambda^3 \sigma}{4} \sqrt{2\sigma - \sigma^2} \, z^4 \ln 2z$$

For $\sigma \geq 2$

$$(28) \quad \lambda^4 \Delta_4(t) = \frac{\lambda^3 \sigma}{8} z^4 \left[2 + \sqrt{\sigma(\sigma+2)} - \sqrt{\sigma(\sigma-2)} \right]$$

V. DISCUSSION OF I_{1c} , I_{2c} , I_{3c}

We now consider those integrals which were ^{not} discussed in detail up to now, namely I_{1c} , I_{2c} and I_{3c} .

We will prove that they all are of order in z lower than 4 and hence do not contribute to the final result in the approximation which we are discussing, namely $z \rightarrow \infty$. For the discussion that follows it will be convenient to relabel and subdivide the three expressions

I_{1c} , I_{2c} and I_{3c} as follows:

$$R_1 = \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} q' J_r(2q''-2q''') J_r(2q-2q') J_r(2q''-2q) J_{r+1}(2q'''-2q') \\ \times \cos \sigma(2q+2q'-2q''-2q''')$$

$$R_2 = \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} q'^2 J_r(2q''-2q''') J_r(2q-2q') J_r(2q''-2q) [J_r(2q'''-2q') \\ + J_{r+2}(2q'''-2q')] \cos \sigma(2q+2q'-2q''-2q''')$$

$$R_3 = \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} (p-q'') J_r(2q''-2q''') J_r(2q-2q') J_r(2q'''-2q') \\ J_{r+1}(2q''-2q) \cos \sigma(2q+2q'-2q''-2q''')$$

$$R_4 = \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} (p-q'')^2 J_r(2q''-2q''') J_r(2q-2q') J_r(2q'''-2q') [J_r(2q''-2q) \\ + J_{r+2}(2q''-2q)] \cos \sigma(2q+2q'-2q''-2q''')$$

$$R_5 = - \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} q'(p-q'') J_r(2q''-2q''') J_r(2q-2q') [J_{r+1}(2q'''-2q') J_{r+1}(2q''-2q) \\ + J_{r+1}(2q'''-2q') J_{r-1}(2q''-2q) + J_{r-1}(2q'''-2q') J_{r+1}(2q''-2q) + J_{r-1}(2q'''-2q') J_{r-1}(2q''-2q)] \\ \cos \sigma(2q+2q'-2q''-2q''')$$

$$S_1 = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} q J_r(2q''-2q''') J_r(2q'-2q'') J_r(2q-2q') J_{r+1}(2q'''-2q) \\ \times \cos \sigma(2q+2q'-2q''-2q''')$$

$$S_2 = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} q^2 J_r(2q''-2q''') J_r(2q'-2q'') J_r(2q-2q') [J_r(2q'''-2q) \\ + J_{r+2}(2q'''-2q)] \cos \sigma(2q+2q'-2q''-2q''')$$

$$S_3 = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} (p-q) J_r(2q''-2q''') J_r(2q'-2q'') J_r(2q'''-2q) J_{r+1}(2q-2q') \\ \times \cos \sigma(2q+2q'-2q''-2q''')$$

$$S_4 = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} (p-q)^2 J_r(2q''-2q''') J_r(2q'-2q'') J_r(2q'''-2q) [J_r(2q-2q') \\ + J_{r+2}(2q-2q')] \cos \sigma(2q+2q'-2q''-2q''')$$

$$S_5 = + \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} q(p-q) J_r(2q''-2q''') J_r(2q'-2q'') [J_{r+1}(2q'''-2q) J_{r+1}(2q-2q') \\ + J_{r+1}(2q'''-2q) J_{r-1}(2q-2q') + J_{r-1}(2q'''-2q) J_{r+1}(2q-2q') + J_{r-1}(2q'''-2q) J_{r-1}(2q-2q') \\ \times \cos \sigma(2q+2q'-2q''-2q''')$$

$$T_1 = \int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} q J_r(2q-2q') J_r(2q'-2q'') J_r(2q''-2q''') J_{r+1}(2p-4q+2q' \\ \times \cos \sigma(2q+2q'-2q''-2q'''))$$

$$T_2 = \int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} q^2 J_r(2q-2q') J_r(2q'-2q'') J_r(2q''-2q''') \\ \times [J_r(2p-4q+2q''') + J_{r+2}(2p-4q+2q''')] \cos \sigma(2q+2q'-2q''-2q''')$$

The proof that these integrals are of lower order in z than z^4 now proceeds in four steps:

(1) Combining R_i and S'_i and changing variables, will enable us to perform one integration leaving a sum of 5 triple integrals (the integrands are still infinite sums).

(2) We prove that one of these terms is of order z^2 .

(3) We show that the remaining 4 terms must be of the same order in z , namely z^2 .

(4) The proof that the other terms R_i, S_i, T_i are of order in z lower than 4 is accomplished by comparison with $R_i + S_i$.

1. One Integral of $R_i + S'_i$

(a) By relabeling q and q' in S'_i we make the integrands of S'_i and R_i identical to each other.

The sum is then

$$R_i + S'_i = \int_0^P dq \int_0^P dq' \int_q^P dq'' \int_0^{q''} dq''' F - \int_0^P dq \int_q^P dq' \int_0^P dq'' \int_0^{q''} dq''' F$$

$$= A - B$$

where

$$F = \sum_r q' J_r(2q'' - 2q''') J_r(2q'' - 2q) J_r(2q - 2q') J_{r+1}(2q''' - 2q') \cos \sigma(2q + 2q' - 2q'' - 2)$$

(b) In integral A transform to new variables

c, d, b, μ defined by

$$c = 2q''' - 2q' \quad ; \quad d = 2q'' - 2q \quad ; \quad b = 2q - 2q' \quad ; \quad \mu = 2q'$$

For details see Appendix M.

(c) Proceed in the same fashion in integral B .

Relabel $c, -c$; $d, -d$; $b, -b$ and regroup terms.

(d) After some simple regroupings and cancellations A-B is much simplified. Now we perform the integration w.r. to μ . Each term contains a coefficient, the highest term of which is clearly p^2 . Thus the proof reduces to showing the "Z" order of the expression

$$(29) \quad p^2 \int_0^{2p} dd \left\{ \int_{d-2p}^0 dc \int_0^{2p+c-d} db + \int_0^d dc \left[- \int_0^c db + \int_0^{2p-d} db \right] \right. \\ \left. + \int_0^{2p} dc \int_0^c db \right\} \sum_{r=-\infty}^{\infty} J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \cos \pi(c+d)$$

2. Proof of Convergence of Integral $\int_0^{2p} dd \int_0^d dc \int_0^{c-d} db F$

Because of the particular limits of the integral it is possible to effect integration with respect to b (convolution integral) albeit in terms of an infinite sum. Next one proceeds as in the evaluation of the integrals in

part II by subdividing the domain of integration into appropriate subdomains.

In addition to making use of the asymptotic expansion of Bessel functions for large arguments we must use the asymptotic expansion for large order.

The details are given in Appendix N.

The result is that the integral

$$I_1 = \int_0^{2p} dd \int_0^d dc \int_0^{c-d} db F$$

is convergent

$$F = \sum_r J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \cos \sigma(c+d)$$

and that part of $R_1 + S_1$ is of order z^2 .

3. Evaluation of Other Terms in $R_1 + S_1$

Label the integrals appearing in 29, I_2, I_3, I_4, I_5 and I_1 is the integral evaluated above.

Use the integral representation

$$J_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-ir\theta + ix \sin \theta}$$

By replacing all the Bessel functions in I_1 by this representation, collecting terms in r and reversing the order of integration, we are able to integrate w.r. to "b" and remain with a quadruple integral w.r.

to $\theta_1, \theta_2, \theta_3, \theta_4$ or after relabeling,
 integrals w.r. to x, y, z, w .

Thus

$$I_1 = \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \int_{-\pi}^{\pi} d\theta_3 \int_{-\pi}^{\pi} d\theta_4 \left[\sum_r e^{-i\theta_2} e^{ir\varphi} \right] \\
\times \frac{1}{i(\gamma+z)} \left\{ \frac{e^{2ip(w+x+2\sigma)} - 1}{(x+y+\sigma)(w+x+2\sigma)} + \frac{e^{2ip(w+x-2\sigma)} - 1}{(x+y+\sigma)(w+x-2\sigma)} - \frac{e^{2ip(w-y+\sigma)} - 1}{(x+y+\sigma)(w-y+\sigma)} \right\}$$

+ 5 more terms of the same type

where

$$x = \sin \theta_1 ; \quad y = \sin \theta_2 , \text{ etc. } ; \quad \varphi = \theta_1 + \theta_2 + \theta_3 + \theta_4$$

The same is done with I_2, I_3, I_4, I_5 .

All these integrals have the same factor $\frac{\sum_r e^{-i\theta_2} e^{ir\varphi}}{i(\gamma+z)}$

and all the other terms appear in the same form. Hence
 all integrals are convergent.

The details are given in Appendix O.

4. Discussion of Other Terms R_i, S'_i, T_i

(a) For the other R_i and S'_i terms we group
 them as follows $R_2 + S'_2, R_3 + S'_3, R_4 + S'_4, R_5 + S'_5$. A
 comparison by inspection with $R_1 + S'_1$ makes it obvious that
 at most these terms are of order z^3 . The details appear
 in Appendix P.

(b) A separate analysis for T_1 and T_2 is necessary. It is very similar to the one performed on $R_1 + S_1$. The detailed steps are given in Appendix Q.

As a result T_1 is shown to be of order z^2

T_2 is shown to be of order z^3 .

VI. NUMERICAL CALCULATIONS

We have shown that for $z \gg 1$ ($z = \frac{2W}{\hbar} t$)

$$\lambda^4 \Delta_4(t) = \frac{\lambda^3 \sigma}{8} z^4 \left[2 + \sqrt{\sigma(\sigma+2)} - \sqrt{\sigma(\sigma-2)} \right] \quad \text{for } \sigma > 2$$

$$\lambda^4 \Delta_4(t) = 1.205 \lambda^3 z^4 \quad \text{for } \sigma = 2$$

$$\lambda^4 \Delta_4(t) = \frac{\lambda^3 \sigma}{4} \left[z^4 / \ln 2 z \right] \sqrt{2\sigma - \sigma^2} \quad \text{for } \sigma < 2$$

This means that the rate of spreading of the wave packet

$$v = \frac{(\Delta(t))^{1/2}}{t}$$

now becomes, after expanding the radical (assuming $\lambda^2 \Delta_2(t) + \lambda^4 \Delta_4(t) \ll \Delta_0$)

$$(30) \quad v = \frac{\sqrt{2} W}{\hbar} \left\{ 1 - A_1 + A_2 z^2 \right\} \quad \sigma > 2$$

$$(31) \quad v = \frac{\sqrt{2} W}{\hbar} \left\{ 1 - A_1' \ln z + A_2 z^2 \right\} \quad \sigma = 2$$

$$(32) \quad v = \frac{\sqrt{2} W}{\hbar} \left\{ 1 - A_1'' z + A_2'' z^2 / \ln 2 z \right\} \quad \sigma < 2$$

where v is given in units of intermonomer distance per second, and

$$A_1 = \frac{4}{\pi} \frac{\lambda}{\sigma} \left[\frac{1}{\pi} K\left(\frac{4}{\sigma^2}\right) - \frac{1}{2} \right]$$

$$A_2 = (1.25) \lambda^3 \sigma \left[2 + \sqrt{\sigma(\sigma+2)} - \sqrt{\sigma(\sigma-2)} \right]$$

$$A_2 (\sigma=2) = 1.205 \lambda^3$$

$$A_1' = \frac{\lambda}{\pi} \quad ; \quad A_1'' = \frac{2}{3\pi} \lambda \sigma \left[E\left(1 - \frac{\sigma^2}{4}\right) - \frac{\sigma^2}{4} K\left(1 - \frac{\sigma^2}{4}\right) \right]$$

$$A_2'' = \frac{\lambda^3 \sigma}{4} \sqrt{2\sigma - \sigma^2}$$

We will now evaluate $\Delta(t)$ and then V , the rate of spreading, for a number of values of σ and z . As a first step we will obtain the limits of validity of the formulas (30-31), i.e., find the limiting values of z for which these expressions for $\Delta(t)$ are valid. Next we will discuss plausible values of λ and z and finally calculate V for various values of σ and discuss the results.

1. Limits of z

As was shown, the final formulas (30 - 32) are valid only for large values of z . In accord with the derivation and proofs used to evaluate the integrals leading to eqs. (30 - 32), we can establish a lower limit on z :

z must be larger than such value of z which allows us to replace the zeroth and first order Bessel functions of z by their asymptotic expansion. Hence z must be larger than $15 - 20^{23}$. A reasonable value for z is then at least 30.

Next we must define the values of z for which

23

G. N. Watson; op. cit., page 359

it is plausible that the exact (unknown) series $\Delta(t) = \sum_{n=0}^{\infty} \lambda^n \Delta_n(t)$ is reasonably well approximated by the known truncated series $\Delta_0 + \lambda^2 \Delta_2(t) + \lambda^4 \Delta_4(t)$, where $\lambda^2 \Delta_2(t)$ is the expression $2\delta'$. This obviously requires that $|\lambda^2 \Delta_2| \ll \Delta_0$, $\lambda^4 \Delta_4 \ll |\lambda^2 \Delta_2|$. These values will in turn depend critically on σ . If we therefore denote the permissible values of z for $\Delta_2(t)$ by z_B (for Bierman) and the permissible values of z for $\Delta_4(t)$ by z_L (for Lowy), we obtain:

$$\underline{\sigma < 2}$$

$$z_B \ll \frac{3\pi}{4} \frac{1}{\lambda \sigma \left[E\left(1 - \frac{\sigma^2}{4}\right) - \frac{\sigma^2}{4} K\left(1 - \frac{\sigma^2}{4}\right) \right]}$$

$$z_L / \ln z_L \ll \frac{(0.85) \left[E\left(1 - \frac{\sigma^2}{4}\right) - \frac{\sigma^2}{4} K\left(1 - \frac{\sigma^2}{4}\right) \right]}{\lambda^2 \sqrt{2\sigma - \sigma^2}}$$

$$\underline{\sigma = 2}$$

$$\ln z_B \ll \frac{\pi}{2\lambda}$$

$$\frac{z_L^2}{\ln z_L} \ll \frac{0.264}{\lambda^2}$$

$$\sigma > 2$$

z_β can have any value.

$$(33) \quad z_L^2 \ll \frac{32}{\pi} \frac{\left[\frac{1}{\pi} K\left(\frac{4}{\sigma^2}\right) - \frac{1}{2} \right]}{\lambda^2 \sigma^2 [2 + \sqrt{\sigma(\sigma+2)} - \sqrt{\sigma(\sigma-2)}]}$$

As a result we obtain the following values:

σ	z_β smaller than			z_L smaller than		
	for $\lambda = .01$	for $\lambda = .05$	for $\lambda = .1$	for $\lambda = .01$	for $\lambda = .05$	for $\lambda = .1$
.25	957	191	95	1550	95	30
.50	490	99	49	1200	74	24
.75	342	68	34	1050	65	21
1	282	56	28	940	59	19
1.25	259	52	26	850	54	17
1.50	279	56	28	750	48	16
1.75	406	81	40	600	39	13
2	e^{157}	e^{31}	e^{16}	395	17	7
2.1	∞	∞	∞	35	1	1
2.5	∞	∞	∞	16	1	1

Table 1 : Permissible values of z

Thus it is immediately clear that for $\sigma > 2$ the final formulas (30 - 32) derived in this work cannot be applied at all, at least not for $\lambda \gg 0.01$. Nor will the results be entirely trustworthy ($\sigma > 1.75$) if λ is of the order of 5×10^{-2} even for $\sigma < 2$.

This means that, for these values of λ and σ , we cannot truncate the series after the third term.

2. Discussion of Plausible Values of z and λ

a) z

If we take into account only the zero order term of $\Delta(t)$, then V becomes

$$V = \frac{\sqrt{\Delta(t)}}{\hbar} = \frac{\sqrt{z}}{\hbar} W$$

where the units are "intermonomer distances per second".

Let this intermonomer distance be ℓ cm, then

$$V = \frac{\sqrt{z}}{\hbar} W \ell \frac{\text{cm}}{\text{sec.}}$$

For the DNA complex, the average distance between nucleotides is $3.4 \text{ \AA}$ ²⁴. Assume then that $\ell = 4 \times 10^{-8} \text{ cm}$.

Setting $W = 0.05 \text{ eV}$, then

$$V \approx 4 \times 10^6 \frac{\text{cm}}{\text{sec.}}$$

If we now assume that the excitation will spread over 10 monomers before being trapped and reemitted, then

²⁴

P. O. Löwdin; Advances in Quantum Chemistry, Vol. II (Academic Press, 1965), 234

$$v t = 10 \ell \quad \text{and} \quad t = \frac{10 \hbar}{\sqrt{2} W} \quad \text{so that}$$

$$z = \frac{2W}{\hbar} \frac{10 \hbar}{\sqrt{2} W} = 10\sqrt{2}$$

This value of z is too small if we wish to use it in the expressions (30 - 32).

On the other hand, if $z \approx 30$, a value for which the results (30 - 32) are valid, then it follows that we are dealing with a case where the excitation spreads over a total of 21 monomers, corresponding to a distance of $84 \text{ \AA}$. This is well within the range of the values found experimentally.²⁵

b) λ

An approximate evaluation of λ is rather difficult to make. From the discussion of part I of this section it follows that $\lambda = 0.01$ should give valid results when $z = 30$ and $\sigma \leq 2$.

²⁵

See Ref. 1.

It is possible to show that $\lambda=0.01$ is a plausible value in the following manner:

$$\lambda = \frac{k Q_0^2}{2 W} \quad ; \quad \sigma = \frac{\hbar \omega}{2 W} \quad ; \quad K = M \omega^2$$

Now $\hbar \omega$ is of the order of 1000 cm^{-1} . If we consider, as before, $W = 0.05 \text{ ev.}$, this becomes

$$W = 0.05 \text{ ev.} = 403 \text{ cm}^{-1}$$

Then

$$\sigma = 1.24$$

$$\lambda = \frac{M \sigma \omega Q_0^2}{\hbar}$$

let $M \sim 10^{-23} \text{ gram}$ ($M \text{ proton} = 1.67 \times 10^{-24} \text{ g}$)

and let

Q_0 be of ^avalue intermediate between the diameter of a nucleus and an atom, or

$$Q_0 \sim 10^{-10} \text{ cm}$$

Then $\lambda \sim 2.5 \times 10^{-2}$

which is of the same order of magnitude as 0.01 .

3. Values of $\Delta(t)$

For $\sigma > 2$ we have already established the

fact that the result (30) is not valid.

For $\sigma \leq 2$ we get the following results

($z = 30$) :

σ	$V (\lambda=0.01) \text{ in cm/sec}$	$V (\lambda=0.05) \text{ in cm/sec}$
.25	$4 \times 10^6 [1 - .0157 + .0152 \times 10^{-2}]$	$4 \times 10^6 [1 - .0783 + .019]$
.50	$4 \times 10^6 [1 - .0304 + .04 \times 10^{-2}]$	$4 \times 10^6 [1 - .152 + .0499]$
.75	$4 \times 10^6 [1 - .0439 + .067 \times 10^{-2}]$	$4 \times 10^6 [1 - .219 + .0836]$
1	$4 \times 10^6 [1 - .0533 + .092 \times 10^{-2}]$	$4 \times 10^6 [1 - .266 + .115]$
1.25	$4 \times 10^6 [1 - .0578 + .111 \times 10^{-2}]$	$4 \times 10^6 [1 - .289 + .139]$
1.50	$4 \times 10^6 [1 - .0536 + .12 \times 10^{-2}]$	$4 \times 10^6 [1 - .268 + .150]$
1.75	$4 \times 10^6 [1 - .0369 + .107 \times 10^{-2}]$	$4 \times 10^6 [1 - .184 + .133]$
2	$4 \times 10^6 [1 - .0108 + .108 \times 10^{-2}]$	

Table II : Values of V for $\lambda=0.01$ and $\lambda=0.05$

By differentiating A_2'' with respect to σ and equating to zero we get that $\Delta_4(t)$ has a maximum for $\sigma = 1.5$.

VII. DISCUSSION

We will now examine the physical meaning of the results. In particular we will discuss the behavior of v as a function of λ , z and σ and also suggest some possible experimental verification of the results of this work.

1. In the existing literature²⁶, the velocity of spreading of the wave packet is assumed to be

$$v = \frac{\sqrt{2}}{\hbar} W |S|^2$$

where $|S|^2$ is the overlap integral between excited and non-excited vibrational functions. As shown earlier, this result is arrived at by assuming initial conditions at variance with the Franck-Condon principle. It is also assumed in the literature²⁶ that $|S|^2$ and hence v decreases monotonically as one proceeds from strong to weak coupling. It is therefore of interest to examine in light of this revised theory, whether in fact the rate of spreading does decrease monotonically with increasing λ . Beginning with the truncated series $\Delta = \Delta_0 + \lambda^2 \Delta_2 + \lambda^4 \Delta_4$, it is trivially shown that while $\frac{\partial \Delta}{\partial \lambda}$ is initially negative, it becomes positive for

²⁶Förster, see ref. 6, 310
-65-

$$\lambda^2 > \frac{8}{9\pi} \frac{A}{B} \frac{1}{z \ln 2z}$$

where

$$A = E\left(1 - \frac{\sigma^2}{4}\right) - \frac{\sigma^2}{4} K\left(1 - \frac{\sigma^2}{4}\right)$$

$$B = \sqrt{2\sigma - \sigma^2}$$

On the other hand, eq. 32 is only valid if $z \ln 2z < \frac{8}{3\pi} \frac{A}{B\lambda^2}$.

There exists therefore a domain in λ , namely

$$\frac{8}{9\pi} \frac{A}{B} \frac{1}{z \ln 2z} < \lambda^2 < \frac{8}{3\pi} \frac{A}{B} \frac{1}{z \ln 2z}$$

for which the truncated series appears valid and yet $\frac{\partial \Delta}{\partial \lambda}$ is positive, i.e. the rate of spreading increases with increasing

λ . This "positive slope" domain is given in Table III

σ	
.25	.0585 < λ < .1015
.50	.0504 < λ < .0872
.75	.0468 < λ < .0810
1	.0439 < λ < .0760
1.25	.0416 < λ < .0720
1.50	.0386 < λ < .0669
1.75	.0340 < λ < .0588
2	.0182 < λ < .0316

Table III : "Positive slope domain".

It is possible that the result is spurious, being due to an inappropriately truncated series, and it would therefore be of interest to pursue this question to higher order terms in λ .

2. If we examine $\Delta(t)$ as a power series in λ we note the significant behavior of the coefficients of succeeding powers of λ .

$$\Delta(t) = f_0(z, \sigma) - \lambda f_1(z, \sigma) + \lambda^2 f_2(z, \sigma)$$

$f_2(z, \sigma)$ is a higher power of z than $f_1(z, \sigma)$ and $f_1(z, \sigma)$ is a higher power than $f_0(z, \sigma)$. Indeed $\frac{f_2}{f_1}$ is of order $z/n_1 z$ and $\frac{f_1}{f_0}$ is of order z . This means that even for small λ , V will change substantially if z gets large enough, or which is the same, if we wait long enough.

A possible explanation of this phenomenon is as follows. The coefficients of the different powers of λ reflect actually the interaction of the exciton wave with phonons, in f_1 one phonon, in f_2 two phonons. Thus the longer one waits the more phonons will be interacting with the exciton wave and presumably change the spreading of the wave.

3. We next examine the behavior of V as a

function of σ . From Table II, it is easy to obtain the following values of V .

σ	$V (\lambda = 0.01)$ in units of $4 \times 10^6 \frac{cm}{sec}$	$V (\lambda = 0.05)$ in units of $4 \times 10^6 \frac{cm}{sec}$
.25	.98445	.9407
.50	.96005	.8979
.75	.95677	.8646
1	.94622	.849
1.25	.94331	.85
1.50	.9476	.882
1.75	.99027	.949
2	.99029	

Table IV : values of V for $\lambda = .01$ and $\lambda = .05$

It is clear that V as a function of σ has a minimum, for σ smaller than 2 . $(|\Delta_2(t)|$ has a maximum at $\sigma = 1.25$ and $\Delta_4(t)$ has a maximum at $\sigma = 1.5$) We may then inquire whether evidence for the existence of such a minimum might be obtained experimentally. Of course it is important that any variation of σ should leave the other parameters λ and z unchanged. Now $\sigma = \frac{\hbar\omega}{2W}$ and ω itself is $\sqrt{\frac{k}{M}}$. In order to vary σ we may try to change M . This can be achieved by replacing an atom of the monomer by another one producing an "isotopic" monomer.

As an example we can cite an article by Davydov²⁷ who quotes from a paper of Broude, Sheka and Rashba²⁸ where they report on the study of the spectra of solid solutions of naphthalene in deuterionaphthalene. But difficulties will arise from the fact that one changes not only ω but also k and consequently λ . Even if it were possible to find one "isotopic" molecule without changing λ , it is highly improbable that one could find several different "isotopic" molecules (all without change in λ). These isotopic molecules will be needed to obtain a number of points in order to establish at least the probability of the existence of a minimum of the energy transfer velocity. An added difficulty is that the effect we want to investigate is rather small; 5% maximum variation in V for $\lambda = 0.01$ and 10% for $\lambda = 0.05$. We may conclude that at this stage it is not likely that this effect can be checked experimentally.

4. Finally, we may ask if other results of this work are susceptible to experimental verification. Although it might be difficult to check the exact form (or numerical values) of eq. 32, it is possible to suggest a way of checking the time dependence of V . One could for instance

²⁷A. S. Davydov, Soviet Physics USPEKHI, Sept. - Oct. 1964, 163

²⁸Broude, Sheka and Rashba, DAN SSSR 139, 1085 (1961); Soviet Phys. Doklady 6, 718 (1962)

form very pure crystals of anthracene of varying thickness.

Single crystals or monolayers of, say, naphthacene can be juxtaposed to these anthracene crystals by sublimation²⁹.

One has thus constructed a series of crystals with impurities (naphthacene) imbedded at various distances. By measuring the time between absorption by the anthracene and emission by the naphthacene and the known path length, it should be possible to check if the velocity of transfer of energy is constant or whether it is time dependent. Of course we have used naphthacene and anthracene only as an example. One would have to choose the crystal in such a way that λ is small enough (0.01 to 0.05) and $\sigma < 2$.

²⁹ A. S. Davydov, see Ref. 27, 158

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APPENDIX A

Proof that all $\Delta_{2n+1}(t)$ are Zero

We want to prove that all $\Delta_{2n+1}(t)$ are zero. Considering eqs. B2, B4 and B5 it is clear that all $\Delta_n(t)$ will be of the form (i.e. will contain factors of the form)

$$A1 \quad \langle a_m a_n^+ \dots \dots \chi_0 \mid a_{m'} a_{n'}^+ \dots \dots \chi_0 \rangle$$

where the total number of a 's and a^+ 's is equal to the order of "n" of $\Delta_n(t)$. If n is odd, then the "bra" and "ket" of (A1) must differ in the number of a 's and/or a^+ 's by an odd number.

For instance, in the case of $\Delta_3(t)$ we would get terms like

$$\langle a_m \chi_0 \mid a_{m'} a_{n'}^+ \chi_0 \rangle$$

But it is clear that then the matrix element must be zero because of the orthogonality of the $h_n(Q_k)$, the factor members of χ_0 .

APPENDIX B

Calculation of $\sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle$

From eq. 24

$$\begin{aligned} \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle &= \frac{1}{\hbar^4} \int_0^t d\tau \int_0^\tau d\tau' \int_0^\tau d\tau'' \int_0^{\tau''} d\tau''' \langle e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](t-\tau)} V_s \\ &\quad \times e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](\tau-\tau')} V_s \xi_s^{(0)}(\tau') | e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](t-\tau'')} V_s \\ &\quad \times e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](\tau''-\tau''')} V_s \xi_s^{(0)}(\tau''') \rangle \end{aligned}$$

B1

Now expand the exponentials in power series and use eq. 23.

We then find:

$$\begin{aligned} &| e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](t-\tau'')} V_s e^{-\frac{i}{\hbar} [\Omega^0 + W(E+F)](\tau''-\tau''')} V_s \xi_s^{(0)}(\tau''') \rangle = \\ &= e^{-\frac{i}{\hbar} E_0 t} \sum_{\substack{m,n \\ u,v}}^{\infty} \frac{(-\frac{i}{\hbar})^{m+n+u+v}}{m! n! u! v!} (t-\tau'')^{m+n} (\tau''-\tau''')^{u+v} (-i)^{15+m+n+u+v} \\ &\quad \times \int_{15+m-n+u-v}^{\infty} \left(\frac{2W}{\hbar} \tau''' \right) | e^{-\frac{i}{\hbar} \Omega^0(t-\tau'')} V_{s+m-n} e^{-\frac{i}{\hbar} \Omega^0(\tau''-\tau''')} V_{s+m-n+u-v} e^{-\frac{i}{\hbar} E_0(\tau''-t)} \chi_0; \end{aligned}$$

Thus $\langle \xi_s^{(2)} | \xi_s^{(2)} \rangle$ becomes

$$\begin{aligned} &\frac{1}{\hbar^4} \sum_{\substack{i,j,k,\ell \\ m,n,u,v}} \frac{(\frac{i}{\hbar} W)^{i+j+k+\ell}}{i! j! k! \ell! m! n! u! v!} \int_0^t d\tau \int_0^\tau d\tau' \int_0^\tau d\tau'' \int_0^{\tau''} d\tau''' (t-\tau)^{i+j} (\tau-\tau')^{k+\ell} \\ &\quad \times (t-\tau'')^{m+n} (\tau''-\tau''')^{u+v} \langle e^{-\frac{i}{\hbar} \Omega^0(t-\tau'')} V_{s+i-j} e^{-\frac{i}{\hbar} \Omega^0(\tau-\tau')} V_{s+i-j+k-\ell} \end{aligned}$$

B2

$$\begin{aligned} &\times e^{-\frac{i}{\hbar} E_0 \tau'} \chi_0 (-i)^{15+i-j+k-\ell} \int_{15+i-j+k-\ell}^{\infty} (2q') | e^{-\frac{i}{\hbar} \Omega^0(t-\tau'')} V_{s+m-n} e^{-\frac{i}{\hbar} (\tau''-\tau''')} \\ &\quad \times V_{s+m-n+u-v} e^{-\frac{i}{\hbar} E_0 \tau'''} \chi_0 (-i)^{15+m-n+u-v} \int_{15+m-n+u-v}^{\infty} (2q''') \rangle \end{aligned}$$

where

$$\frac{W}{\hbar} t = p \quad ; \quad \frac{W}{\hbar} \tau = q \quad ; \quad \frac{W}{\hbar} \tau' = q' \quad ; \quad \frac{W}{\hbar} \tau'' = q'' \quad ; \quad \frac{W}{\hbar} \tau''' = q''' .$$

First we group together all terms $(i)^\alpha (-i)^\beta$ and the J'_S .

It is shown in Appendix S that one always gets the following product

$$(-1)^{i+k+m+n} J_{s+i-j+k-l}^{(2q')} J_{s+m-n+u-v}^{(2q''')}$$

Next we observe that $e^{-i/\hbar \Omega^0(t-\tau)}$ is the unitary operator which takes one from the Schrödinger to the Heisenberg picture; i.e.

$$| e^{-i/\hbar \Omega^0(t-\tau)} V_{s+i-j} e^{-i/\hbar \Omega^0(\tau-t)} \chi_0 \rangle = | V_{s+i-j}(\tau-t) \chi_{0H} \rangle$$

Hence

$$\begin{aligned} & \langle e^{-i/\hbar \Omega^0(t-\tau)} V_{s+i-j} e^{-i/\hbar \Omega^0(\tau-\tau')} V_{s+i-j+k-l} e^{-i/\hbar E_0 \tau'} \chi_0 | e \\ & e^{-i/\hbar \Omega^0(t-\tau'')} V_{s+m-n} e^{-i/\hbar \Omega^0(\tau''-\tau''')} V_{s+m-n+u-v} e^{-i/\hbar E_0 \tau'''} \chi_0 \rangle \end{aligned}$$

becomes

$$B3 \quad e^{i/\hbar E_0(t-t)} \langle V_{s+i-j}(\tau-t) V_{s+i-j+k-l}(\tau'-t) \chi_0 | V_{s+m-n}(\tau''-t) V_{s+m-n+u-v}(\tau'''-t) \chi_0 \rangle$$

We now note that $V_s = \frac{-2W}{Q_0} Q_s$ which we can express in terms of raising and lowering operators as

$$B4 \quad V_{s+i-j}(\tau-t) = -\frac{2W}{Q_0} \left[a_{s+i-j}^+ e^{i\omega(t-\tau)} + a_{s+i-j} e^{-i\omega(t-\tau)} \right] \sqrt{\frac{\hbar}{2M\omega}}$$

Note that

$$a \chi_0 = 0$$

In terms of these operators eq. B3 now becomes

$$\begin{aligned} \text{B5} \quad & \frac{\hbar^2}{4M^2\omega^2} \left(\frac{2W}{Q_0} \right)^4 \left\{ \langle a_{s+i-j} a_{s+i-j+k-l}^\dagger \chi_0 | a_{s+m-n} a_{s+m-n+u-v}^\dagger \chi_0 \rangle e^{-i\omega(\tau-\tau'-\tau''+\tau''')} \right. \\ & \left. + \langle a_{s+i-j}^\dagger a_{s+i-j+k-l}^\dagger \chi_0 | a_{s+m-n}^\dagger a_{s+m-n+u-v}^\dagger \chi_0 \rangle e^{i\omega(\tau+\tau'-\tau''-\tau''')} \right\} \\ & = \frac{\hbar^2}{4M^2\omega^2} \left(\frac{2W}{Q_0} \right)^2 \{ A + B \} \end{aligned}$$

We must now analyze eq. B5 for possible cases according to the values of s, i, j , etc.

(1) Part A

$$I_1 = \langle a_{s+i-j} a_{s+i-j+k-l}^\dagger \chi_0 | a_{s+m-n} a_{s+m-n+u-v}^\dagger \chi_0 \rangle \cos \omega(\tau-\tau'+\tau''-\tau''')$$

In order that I_1 be different from zero, we must

have

$$k=l \quad ; \quad u=v \quad \text{or} \quad \delta_{k,l} \delta_{u,v}$$

(2) Part B

$$I_2 = \langle a_{s+i-j}^\dagger a_{s+i-j+k-l}^\dagger \chi_0 | a_{s+m-n}^\dagger a_{s+m-n+u-v}^\dagger \chi_0 \rangle \cos \omega(\tau+\tau'-\tau''-\tau''')$$

Here we must distinguish 2 cases:

$$(a) \quad i-j = m-n \quad \text{and} \quad i-j+k-l = m-n+u-v$$

or

$$\delta_{i-j, m-n} \quad \delta_{k-l, u-v}$$

$$(b) \quad i-j = m-n+u-v \quad \text{and} \quad i-j+k-l = m-n$$

or

$$\delta_{i-j, m-n+u-v} \quad \delta_{i-j+k-l, m-n}$$

For convenience we will label the integrals corresponding to the three cases B6, B7, B8 as I_{1a}, I_{1b}, I_{1c} .

In addition we note that all the integrals will have a factor

$$\frac{1}{\hbar^4} \frac{\hbar^2}{4 M^2 \omega^2} \left(\frac{2W}{Q_0} \right)^2 = \left(\frac{2W^2}{Q_0^2 M \hbar \omega} \right)^2 = A^2$$

We can also convert the variables $t, \tau, \tau', \tau'', \tau''', \tau'$ to

p, q, q', q'', q''' in the integrand by using the appropriate factors

$$\left(\frac{iW}{\hbar} \right)^\alpha \quad \text{and} \quad \left(\frac{-iW}{\hbar} \right)^\beta.$$

(1) The integral I_{1a}

Using the condition B6 we get for eq. B2 [$k=\ell$, $u=v$]

$$\langle \xi_s^{(2)} | \xi_s^{(2)} \rangle =$$

$$\begin{aligned} B9 \quad & A^2 \int_0^t d\tau \int_0^\tau d\tau' \int_0^t d\tau'' \int_0^{\tau''} d\tau''' \sum_{\substack{i,j \\ k,l \\ m,n}} \frac{(-1)^{i+k+m+u}}{i!j!k!l!m!n!u!v!} (p-q)^{i+j} (q-q')^{2k} \\ & \times (p-q'')^{m+n} (q''-q''')^{2u} J_{s+i-j}(2q') J_{s+m-n}(2q''') \cos \omega(\tau-\tau'-\tau''+\tau'''), \end{aligned}$$

By repeated use of the formula

$$B10 \quad \sum_u \frac{(-1)^u (p-q)^{2u+m-n}}{u!(u+m-n)!} = J_{m-n}(2p-2q); \quad \sum_{i,j} \frac{(-1)^i (p-q)^{i+j}}{i!j!} J_{s+i-j}(2q') = J_s(2p-2q+2)$$

eq. B9 becomes

$$B11 \quad A^2 \int_0^t d\tau \int_0^\tau d\tau' \int_0^t d\tau'' \int_0^{\tau''} d\tau''' J_s(2p-2q+2q') J_s(2p-2q''+2q''') J_0(2q-2q') J_0(2q''-2q''') \cos \omega(\tau-$$

From Graf's formula

$$J_0(\omega) = \sum_{m=-\infty}^{\infty} J_m(u) J_m(v) \cos m\alpha$$

where $\omega = (u^2 + v^2 - 2uv \cos \alpha)^{1/2}$, we get

$$B12 \quad \sum_{\ell=-\infty}^{\infty} \ell^2 J_\ell(u) J_\ell(v) = \frac{uv}{u-v} J_1(u-v)$$

Hence eq. B11 leads to

$$\begin{aligned}
 \sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle &= A^2 \int_0^t d\tau \int_0^{\tilde{\tau}} d\tau' \int_0^t d\tau'' \int_0^{\tilde{\tau}''} d\tau''' \sum_s s^2 J_s(2p - 2q + 2q') \\
 &\quad \times J_s(2p - 2q'' + 2q''') J_0(2q - 2q') J_0(2q'' - 2q''') \cos \omega(\tau - \tau' - \tau'' + \tau''') \\
 &= 2 A^2 \int_0^t d\tau \int_0^{\tilde{\tau}} d\tau' \int_0^t d\tau'' \int_0^{\tilde{\tau}''} d\tau''' \frac{(p - q + q')(p - q'' + q''')}{q - q' - q'' + q'''} J_0(2q - 2q') \\
 &\quad \times J_0(2q'' - 2q''') J_1(2q - 2q' - 2q'' + 2q''') \cos \omega(\tau - \tau' - \tau'' + \tau''')
 \end{aligned}$$

Let now $\sigma = \frac{\omega \hbar}{2W}$; we transform completely to

p, q, q', q'', q''' and obtain

$$\begin{aligned}
 I_{1\alpha} &= \sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle \\
 &= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^p dq \int_0^q dq' \int_0^p dq'' \int_0^{q''} dq''' \frac{(p - q + q')(p - q'' + q''')}{q - q' - q'' + q'''} \\
 &\quad \times J_0(2q - 2q') J_0(2q'' - 2q''') J_1(2q - 2q' - 2q'' + 2q''') \cos \sigma(2q - 2q' - 2q'' + 2q''')
 \end{aligned}$$

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(2) Integral I_{1b} .

Using the condition B7 ($\delta_{i-j, m-n} \delta_{k-l, u-v}$) we get
for eq. B2

$$\begin{aligned}
 < \xi_s^{(2)} | \xi_s^{(2)} >_{1b} = \\
 &= A^2 \int_0^t d\tau \int_0^{\tau} d\tau' \int_0^t d\tau'' \int_0^{\tau''} d\tau''' \sum_{\substack{i,j,k,l \\ m,u}} \frac{(-1)^{i+k+m+u}}{i! j! k! l! m! (m-i+j)! u! (u-k+l)!} \\
 &\quad \times (\rho-q)^{i+j} (q-q')^{k+l} (\rho-q'')^{2m+i+j} (q''-q''')^{2u+k+l} J_{s+i-j+k-l}(2q') \\
 &\quad \times J_{s+i-j+k-l}(2q''') \cos \omega(\tau+\tau'-\tau''-\tau''')
 \end{aligned}$$

By applying eq. B10 to summation over m and u ,
then using the following formula

$$\begin{aligned}
 &\sum_{k,l} \frac{(-1)^k}{k! l!} (q-q')^{k+l} J_{-k+l}(2q''-2q''') J_{s+i-j+k-l}(2q') J_{s+i-j+k-l}(2q''') \\
 B14 \quad &= \sum_{r=-\infty}^{\infty} J_{s+i-j+r}(2q') J_{s+i-j+r}(2q''') J_r(2q''-2q''') J_r(2q-2q')
 \end{aligned}$$

twice to the summation over i, j, k, l we get, summing over s ,

$$\begin{aligned}
 \sum_s s^2 < \xi_s^{(2)} | \xi_s^{(2)} >_{1b} = \\
 &= A^2 \int_0^t d\tau \int_0^{\tau} d\tau' \int_0^t d\tau'' \int_0^{\tau''} d\tau''' \sum_{r,t,s} s^2 J_{s+r+t}(2q') J_{s+r+t}(2q''') J_r(2q-2q') \\
 &\quad \times J_r(2q''-2q''') J_t(2\rho-2q'') J_t(2\rho-2q) \cos \omega(\tau+\tau'-\tau''-\tau''')
 \end{aligned}$$

Using first the following formula to sum over s

$$\begin{aligned}\sum_s s^2 J_{s+r+t}(2q') J_{s+r+t}(2q''') &= \sum_\ell [\ell^2 + (r+t)^2 - 2\ell(r+t)] J_\ell(2q') J_\ell(2q''') \\ &= \frac{2q'q'''}{q'-q'''} J_1(2q'-2q''') + (r+t)^2 J_0(2q'-2q''') ,\end{aligned}$$

then using this expression and eq. B12 to sum over r

and t we get

$$\begin{aligned}I_{1b} &= \sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle_{1b} \\ &= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \\ &\quad \times \left\{ \frac{q'q'''}{q'-q'''} J_1(2q'-2q''') J_0(2q-2q'-2q''+2q''') J_0(2q-2q'') \right. \\ &\quad + \frac{(q-q')(q''-q''')}{q-q'-q''+q'''} J_0(2q'-2q''') J_0(2q-2q'') J_1(2q-2q'-2q''+2q''') \\ &\quad \left. + \frac{(p-q)(p-q'')}{q-q''} J_0(2q'-2q''') J_0(2q-2q'-2q''+2q''') J_1(2q-2q'') \right\} \\ &\quad \times \cos \pi(2q+2q'-2q''-2q''') \\ &= I_{1b,1} + I_{1b,2} + I_{1b,3}\end{aligned}$$

(3) Integral I_{1c}

Using the conditions B8 ($\delta_{i-j, m-n+u-v} \delta_{i-j+k-l, m-n}$)

we get for eq. B2

$$\begin{aligned}
 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle_{1c} &= \\
 &= A^2 \int_0^t d\tau \int_0^\tau d\tau' \int_0^t d\tau'' \int_0^{\tau''} d\tau''' \sum_{\substack{i, m \\ n, k \\ u, v}} \left\{ \frac{(-1)^{i+k+m+u}}{i!(i-m+n-u+v)! m! n! k! (k+u-v)! u! v!} \right. \\
 &\quad \times (p-q)^{2i-m+n-u+v} (q-q')^{2k+u-v} (p-q'')^{m+n} (q''-q''')^{u+v} \\
 &\quad \times \left. J_{s+m-n}(2q') J_{s+m-n+u-v}(2q''') \right\} \cos \omega(\tau+\tau'-\tau''-\tau''')
 \end{aligned}$$

We first use eq. B10 twice to sum over i and over k

then use eq. B14 to sum over m, n, u, v we get

$$\begin{aligned}
 \sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle_{1c} &= \\
 &= A^2 \int_0^t d\tau \int_0^\tau d\tau' \int_0^t d\tau'' \int_0^{\tau''} d\tau''' \sum_{s, t, r=-\infty}^{\infty} \left\{ s^2 J_r(2q''-2q''') J_r(2q-2q') J_t(2p-2q'') \right. \\
 &\quad \times \left. J_{t+r}(2p-2q) J_{s+t+r}(2q''') J_{s+t}(2q') \right\} \cos \omega(\tau+\tau'-\tau''-\tau''')
 \end{aligned}$$

Let now $s+t=\ell$; $s=\ell-t$ and we get

$$\begin{aligned}
 \sum_s s^2 \langle \xi_s^{(2)} | \xi_s^{(2)} \rangle_{1c} &= A^2 \int_0^t d\tau \int_0^\tau d\tau' \int_0^t d\tau'' \int_0^{\tau''} d\tau''' \sum_{\ell, t, r} \left\{ (\ell-t)^2 J_r(2q''-2q''') \right. \\
 &\quad \times \left. J_r(2q-2q') J_{t+r}(2p-2q) J_{\ell+r}(2q''') J_\ell(2q') \right\} \cos \omega(\tau+\tau'-\tau''-\tau''')
 \end{aligned}$$

In order to further simplify this expression we must make use of the following formulas, derived in Appendix R

$$B17 \quad \sum_{m=-\infty}^{\infty} J_{n+m}(u) J_m(v) = J_n(u-v)$$

$$B18 \quad \sum_{m=-\infty}^{\infty} m J_{n+m}(u) J_m(v) = \frac{nv}{u-v} J_n(u-v)$$

$$B19 \quad \sum_{m=-\infty}^{\infty} m^2 J_{n+m}(u) J_m(v) = \frac{1}{2(u-v)} \left\{ (uv + nv^2) J_{n+1}(u-v) + (-uv + nv^2) J_{n-1}(u-v) \right\}$$

We first use these three formulas to effect the sum over ℓ in eq. B16, then repeat the procedure in summing over t and get eventually

$$\begin{aligned} \sum_s s^2 < \xi_s^{(2)} | \xi_s^{(2)} >_{1c} = \\ &= A^2 \int_0^t d\tau \int_0^{\tau} d\tau' \int_0^{\tau} d\tau'' \int_0^{\tau''} d\tau''' \sum_{r=-\infty}^{\infty} J_r(2q''-2q''') J_r(2q-2q') \\ &\times \left\{ \frac{J_r(2q''-2q)}{q'''-q'} \left[(q'q''' + r q'^2) J_{r+1}(2q'''-2q') + (-q'q''' + r q'^2) J_{r-1}(2q'''-2q') \right] \right. \\ &+ \frac{J_r(2q'''-2q')}{q''-q} \left[\left[(p-q)(p-q'') + (p-q'')^2 \right] J_{r+1}(2q''-2q) + \left[-(p-q)(p-q'') + r(p-q'') \right] J_{r-1}(2q''-2q) \right] \\ &\left. - \frac{2r^2(p-q'')q'}{(q'''-q')(q''-q)} J_r(2q'''-2q') J_r(2q''-2q) \right\} \cos \omega(\tau + \tau' - \tau'' - \tau''') \end{aligned}$$

We now note that

$$\sum_{r=-\infty}^{\infty} J_r J_r J_r [(u+r)J_{r+1} - (-u+r)J_{r-1}] = 0$$

then

$$\begin{aligned} & \sum_{r=-\infty}^{\infty} J_r J_r J_r [(u+r)J_{r+1} + (-u+r)J_{r-1}] \\ B_{20} \quad &= \sum_{r=-\infty}^{\infty} J_r J_r J_r [(u+r)J_{r+1} + (-u+r)J_{r-1} + (u+r)J_{r+1} - (-u+r)J_{r-1}] \\ &= 2 \sum_{r=-\infty}^{\infty} J_r J_r J_r (u+r)J_{r+1} \end{aligned}$$

Hence,

$$\begin{aligned} \sum_s s^2 < \xi_s^{(2)} | \xi_s^{(2)} >_{\mathcal{L}} &= 2 A^2 \int_0^t d\tau \int_0^{\tau} d\tau' \int_0^t d\tau'' \int_0^{\tau''} d\tau''' \sum_r J_r (2q'' - 2q''') \\ &\times J_r (2q - 2q') \left\{ \frac{q'q''' + rq'^2}{q''' - q'} J_{r+1} (2q''' - 2q') J_r (2q'' - 2q) \right. \\ B_{21} \quad &+ \frac{(p-q)(p-q'') + r(p-q'')^2}{q'' - q} J_r (2q''' - 2q') J_{r+1} (2q'' - 2q) \\ &\left. - \frac{r^2 q'(p-q'')}{(q''' - q')(q'' - q)} J_r (2q''' - 2q') J_r (2q'' - 2q) \right\} \cos \omega(\tau + \tau' - \tau') \end{aligned}$$

Now noting that:

1. $\frac{2\gamma}{z} J_\gamma(z) = J_{\gamma+1}(z) + J_{\gamma-1}(z)$
2. $\frac{q'q'''}{q''' - q'} = q' + \frac{q'^2}{q''' - q'}$
3. $\frac{(p-q)(p-q'')}{q'' - q} = p - q'' + \frac{(p-q'')^2}{q'' - q}$

Eq. B21 becomes

$$\begin{aligned}
& \sum_s s^2 < \xi_s^{(2)} | \xi_s^{(2)} > = \\
& = 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^p dq \int_0^q dq' \int_0^p dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} J_r(2q''-2q''') J_r(2q-2q') \\
& \quad \times \left\{ q' J_r(2q''-2q) J_{r+1}(2q'''-2q') + q'^2 J_r(2q''-2q) \left[J_{r+2}(2q'''-2q') \right. \right. \\
& \quad \quad \quad \left. \left. + J_r(2q'''-2q') \right] \right. \\
& \quad + (p-q'') J_r(2q'''-2q') J_{r+1}(2q''-2q) + (p-q'')^2 J_r(2q'''-2q') \left[J_{r+2}(2q''-2q) \right. \\
& \quad \quad \quad \left. + J_r(2q''-2q) \right] \\
& \quad - q'(p-q'') \left[J_{r+1}(2q'''-2q') J_{r+1}(2q''-2q) + J_{r+1}(2q'''-2q') J_{r-1}(2q''-2q) \right. \\
& \quad \quad \quad \left. + J_{r-1}(2q'''-2q') J_{r+1}(2q''-2q) + J_{r-1}(2q'''-2q') J_{r-1}(2q''-2q) \right] \left. \right\} \\
& \quad \times \cos \sigma(2q+2q'-2q''-2q''')
\end{aligned}$$

APPENDIX C

Calculation of $\sum_s s^2 \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle$

We again expand the exponentials of eq. B1 in a power series and apply the E and F operators. We then get

$$\begin{aligned}
 \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle &= -\frac{1}{\hbar^4} \int_0^t d\tau \int_0^t d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \times \\
 &\times \sum_{\substack{i,j \\ \kappa,\ell \\ m,n \\ u,v}} \frac{(i)^{i+j} (-i)^{\kappa+\ell+m+n+u+v}}{i! j! \kappa! \ell! m! n! u! v!} (i)^{|s+i-j|} (-i)^{|s+\kappa-\ell+m-n+u-v|} \\
 &\times (p-q)^{i+j} (p-q')^{\kappa+\ell} (q'-q'')^{m+n} (q''-q''')^{u+v} J_{|s+i-j|}^{(2q)} J_{|s+\kappa-\ell+m-n+u-v|}^{(2q''')} \\
 &\times \langle e^{-i/\hbar \Omega^0(t-\tau)} V_{s+i-j} e^{i/\hbar \Omega^0(t-\tau)} \chi_0 | e^{-i/\hbar \Omega^0(t-\tau')} V_{s+\kappa-\ell} e^{i/\hbar \Omega^0(t-\tau')} \\
 &\times e^{-i/\hbar \Omega^0(t-\tau'')} V_{s+\kappa-\ell+m-n} e^{i/\hbar \Omega^0(t-\tau'')} e^{-i/\hbar \Omega^0(t-\tau''')} V_{s+\kappa-\ell+m-n+u-} \\
 &\times e^{i/\hbar \Omega^0(t-\tau''')} \chi_0 \rangle
 \end{aligned}$$

As before it is easy to show that the terms $i^\alpha (-i)^\beta$ and the J 's always give

$$(-1)^{i+\kappa+m+u} J_{s+i-j}^{(2q)} J_{s+\kappa-\ell+m-n+u-v}^{(2q''')}$$

The proof is given in Appendix T.

We now proceed as in Appendix B and eventually obtain an expression, corresponding to eq. B5 which in terms of the raising and lowering operators is:

$$\frac{\hbar^2}{4M^2\omega^2} \left(\frac{2W}{Q_0}\right)^4 \langle a_{s+i-j}^+ e^{i\omega(t-\tau)} \chi_0 | (a_{s+k-l}^+ a_{s+k-l+m-n}^+ a_{s+k-l+m-n+u-v}^+ \\ \times e^{i\omega(3t-\tau'-\tau''-\tau''')} + a_{s+k-l}^+ a_{s+k-l+m-n}^+ a_{s+k-l+m-n+u-v}^+ e^{i\omega(t-\tau+\tau'+\tau''-\tau''')} \\ + a_{s+k-l}^+ a_{s+k-l+m-n}^+ a_{s+k-l+m-n+u-v}^+ e^{i\omega(t+\tau'-\tau''-\tau''')}) \chi_0 \rangle$$

We thus have a sum of three terms to consider:

$$C2 \quad \langle a_{s+i-j}^+ \chi_0 | a_{s+k-l}^+ a_{s+k-l+m-n}^+ a_{s+k-l+m-n+u-v}^+ \chi_0 \rangle e^{i\omega(2t+\tau-\tau'-\tau''-\tau''')}$$

$$C3 \quad \langle a_{s+i-j}^+ \chi_0 | a_{s+k-l}^+ a_{s+k-l+m-n}^+ a_{s+k-l+m-n+u-v}^+ \chi_0 \rangle e^{i\omega(\tau-\tau'+\tau''-\tau''')}$$

$$C4 \quad \langle a_{s+i-j}^+ \chi_0 | a_{s+k-l}^+ a_{s+k-l+m-n}^+ a_{s+k-l+m-n+u-v}^+ \chi_0 \rangle e^{i\omega(\tau+\tau'-\tau''-\tau''')}$$

It is immediately obvious that eq. C2 is always zero.

Expression C3 will be non-zero only if

$$u = v \quad \text{and} \quad i-j = k-l$$

Hence

$$C5 \quad \text{Case 2a is : } \delta_{u,v} \delta_{i-j, k-l}$$

For expression C4 we have two possibilities:

$$(1) \quad m = n \quad \text{and} \quad i-j = k-l+u-v$$

thus

$$C6 \quad \text{Case 2b is } \int_{m,n} \int_{i-j, k-l+u-v}$$

$$(2) \quad m-n+u-v=0 \quad \text{and} \quad i-j = k-l+m-n \quad \text{or}$$

thus

$$C7 \quad \text{Case 2c is } \int_{m-n, v-u} \int_{i-j, k-l+m-n}$$

For convenience we will label the integrals corresponding to the three cases I_{2a} , I_{2b} and I_{2c} respectively.

(1) The integral I_{2a}

Using condition C5 ($\int_{u,v} \int_{i-j, k-l}$) eq. C1

becomes

$$\begin{aligned} \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle_{2a} &= \\ &= -A^2 \int_0^t d\tau \int_0^t d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \sum_{\substack{i,j \\ k,l \\ m,n \\ u,v}} \frac{(-1)^{i+k+m+u}}{i! j! k! (k-i+j)! m! n! u! v!} (\rho-q)^{i+j} (\rho-q')^{2k} \\ &\quad \times (q'-q'')^{m+n} (q''-q''')^{2u} J_{s+i-j}(2q) J_{s+i-j+m-n}(2q''') \cos \omega(\tau-\tau'+\tau''-\tau'''). \end{aligned}$$

Using now formulas B10 for summation over u and k ,

then formula B10 for summation over m and n one remains

only with the summation over i, j and eventually s .

Next one uses formula B14 to sum over j , leaving one with

summation over a new index r and s . These summations are effected

by using eq. B12 and we finally obtain:

$$\begin{aligned} \sum_s s^2 \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle_{2a} &= -2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' J_0(2q'' - 2q''') \\ &\quad \times \left\{ \frac{q(q' - q'' + q''')}{q - q' + q'' - q'''} J_0(2q - 2q') J_1(2q - 2q' + 2q'' - 2q''') \right. \\ &\quad \left. + \frac{(\rho - q)(\rho - q')}{q - q'} J_0(2q - 2q' + 2q'' - 2q''') J_1(2q - 2q') \right\} \cos \sigma(2q - 2q' + 2q'' - 2q''') \\ &= I_{2a,1} + I_{2a,2} \end{aligned}$$

(2) Integral I_{2b}

Using condition C 6 ($\delta_{m,n} \delta_{i-j, k-\ell+u-v}$) eq. C1

becomes

$$\begin{aligned} \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle_{2b} = & \\ = -A^2 \int_0^t d\tau \int_0^t d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' & \sum_{\substack{i,j \\ k,\ell \\ m,u}} \frac{(-1)^{i+k+m+u}}{i! j! k! \ell! m! m! u! (u+k-\ell-i+j)!} (\rho-q)^{i+j} \\ & \times (\rho-q')^{k+\ell} (q'-q'')^{2m} (q''-q''')^{2u+k-\ell-i+j} \int_{s+i-j} (2q) J_{s+i-j} (2q''') \cos \omega(T+\tau'-\tau''-\tau) \end{aligned}$$

Using now formula B10 for summation over u and m , then formula B12 for summation over k and ℓ one remains only with the sums over i, j and eventually s . Next one uses eq. B14 to sum over j , leaving one with summations over a new index r and s . These sums are performed by using eq. B12 twice and we finally obtain (after transforming to p, q, q', q'' and q''' variables),

$$\begin{aligned} \sum_s s^2 \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle_{2b} = & \\ -2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^p dq \int_0^p dq' \int_0^{q'} dq'' \int_0^{q''} dq''' & J_0(2q'-2q'') \\ \times \left\{ \frac{qq'''}{q-q'''} J_0(2q-2q'+2q''-2q''') J_1(2q-2q''') \right. & \\ + \frac{(\rho-q)(\rho-q'+q''-q''')}{q-q'+q''-q'''} J_0(2q-2q''') J_1(2q-2q'+2q''-2q''') & \left. \right\} \\ \times \cos \omega(2q+2q'-2q''-2q''') & \end{aligned}$$

(3) Integral I_{2c}

Using condition C7, $(\int_{m-n, v-u} \int_{i-j, k-l+m-n})$, eq. C1

becomes

$$\begin{aligned}
 & \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle_{2c} = \\
 & = - A^2 \int_0^t d\tau \int_0^t d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \sum_{\substack{i,j \\ k,l \\ m,u}} \frac{(-1)^{i+k+m+u}}{i! j! k! l! m! (m+k-l-i+j)! u! (u-k+l+i-j)} \\
 & \quad \times (p-q)^{i+j} (p-q')^{k+l} (q'-q'')^{2m+k-l-i+j} (q''-q''')^{2u-k+l+i-j} \\
 & \quad \times J_{s+i-j}(2q) J_{s+k-l}(2q''') \cos \omega(\tau+\tau'-\tau''-\tau''')
 \end{aligned}$$

By the same methods as used for I_{1c} , i.e. use first eq. B10, then eq. B14 and finally eqs. B17, 18, 19 and B20, one gets

$$\begin{aligned}
 & \sum s^2 \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle_{2c} = \\
 & = 2A^2 \int_0^t d\tau \int_0^t d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \sum_r J_r(2q'-2q'') J_r(2q''-2q''') \\
 & \quad \times \left\{ \frac{qq''' + r q^2}{q''' - q} J_r(2q-2q') J_{r+1}(2q'''-2q) \right. \\
 & \quad + \frac{(p-q')(p-q) + r(p-q)^2}{q-q'} J_r(2q'''-2q) J_{r+1}(2q-2q') \\
 & \quad \left. - \frac{r^2 q(p-q)}{(q'''-q)(q-q')} J_r(2q'''-2q) J_r(2q-2q') \right\} \cos \omega(\tau+\tau'-\tau''-\tau''')
 \end{aligned}$$

Noting now that

$$\begin{aligned}
 1. \quad \frac{q q'''}{q''' - q} &= q + \frac{q^2}{q''' - q} \\
 2. \quad \frac{(p - q')(p - q)}{q - q'} &= -q + p + \frac{(p - q)^2}{q - q'} \\
 3. \quad \frac{2}{z} J_\nu(z) &= J_{\nu+1}(z) + J_{\nu-1}(z)
 \end{aligned}$$

and transforming to variables p, q, q', q'', q''' we finally get:

$$\begin{aligned}
 I_{2c} &= \sum_s s^2 \langle \xi_s^{(1)} | \xi_s^{(3)} \rangle_{2c} \\
 &= -2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^p dq \int_0^p dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_r J_r(2q' - 2q'') \bar{J}_r(2q'' - 2q''') \\
 &\quad \times \left\{ q J_r(2q - 2q') J_{r+1}(2q''' - 2q) + q^2 J_r(2q - 2q') \left[J_{r+2}(2q''' - 2q) + J_r(2q''' - 2q) \right] \right. \\
 &\quad + (p - q) J_r(2q''' - 2q) J_{r+1}(2q - 2q') + (p - q)^2 J_r(2q''' - 2q) \left[J_{r+2}(2q - 2q') \right. \\
 &\quad \left. \left. + J_r(2q - 2q') \right] \right\} \\
 &\quad + q(p - q) \left[J_{r+1}(2q''' - 2q) J_{r+1}(2q - 2q') + J_{r+1}(2q''' - 2q) J_{r-1}(2q - 2q') + J_{r-1}(2q''' - 2q) \right. \\
 &\quad \left. \times J_{r+1}(2q - 2q') + J_{r-1}(2q''' - 2q) J_{r-1}(2q - 2q') \right] \left. \right\} \cos \sigma(2q + 2q' - 2q'' - 2q''')
 \end{aligned}$$

C 10

APPENDIX D

Calculation of $\sum_s s^2 \langle \xi_s^{(0)} | \xi_s^{(4)} \rangle$

As in Appendices B and C, we again expand the exponentials in a power series and apply the E and F operators. We then get

$$\begin{aligned} \langle \xi_s^{(0)} | \xi_s^{(4)} \rangle &= \frac{1}{\hbar^4} \int_0^t d\tau \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \\ &\times \sum_{\substack{i,j,k \\ l,m,n \\ u,v}} \frac{i^{1s} (-i)^{1i+j+k+l+m+n+u+v} (-i)^{1s+i-j+k-l+m-n+u-v}}{i! j! k! l! m! n! u! v!} \\ &\times (p-q)^{i+j} (q-q')^{k+l} (q'-q'')^{m+n} (q''-q''')^{u+v} J_{1s}(2q) \\ &\times J_{1s+i-j+k-l+m-n+u-v}(2q''') \langle \chi_0 | V_{s+i-j}(\tau-t) V_{s+i-j+k-l}(\tau'-t) \\ &\times V_{s+i-j+k-l+m-n}(\tau''-t) V_{s+i-j+k-l+m-n+u-v}(\tau'''-t) \chi_0 \rangle \end{aligned}$$

where the V_α operators have already been expressed in the Heisenberg form.

As in the preceding sections it is easy to show that the terms $i^a, (-i)^b$ and the J 's always are

$$(-1)^{i+k+m+u} J_s(2q) J_{s+i-j+k-l+m-n+u-v}(2q''')$$

Proof is given in Appendix U.

We now rewrite the matrix elements in terms of raising and lowering operators, eventually obtaining

$$\frac{\hbar^2}{4H^2\omega^2} \left(\frac{2W}{Q_0}\right)^4 < \chi_0 \left| \left(a_{s+i-j} a_{s+i-j+k-l}^+ a_{s+i-j+k-l+m-n} \right. \right.$$

$$\times a_{s+i-j+k-l+m-n+u-v}^+ e^{i\omega(\tau-\tau'+\tau''-\tau''')}$$

$$\left. + a_{s+i-j} a_{s+i-j+k-l} a_{s+i-j+k-l+m-n}^+ a_{s+i-j+k-l+m-n+u-v}^+ e^{i\omega(\tau+\tau'-\tau''-\tau''')} \right) \chi_0$$

$$= \frac{\hbar^2}{4H^2\omega^2} \left(\frac{2W}{Q_0}\right)^4 \{A + B\}$$

(1) Conditions on A.

For non-zero values we must have that

D 1

$$3a \text{ is } \delta_{u,v} \delta_{k,l}$$

(2) Conditions on B.

Here we have two possibilities

$$(1) 3b; m=n \text{ and } k-l+m-n+u-v=0 \text{ or}$$

D 2

$$3b \text{ is } \delta_{m,n} \delta_{k-l,v-u}$$

$$(2) m-n+u-v=0 \text{ and } k-l+m-n=0 \text{ or}$$

D 3

$$3c \text{ is } \delta_{m-n,v-u} \delta_{m-n,l-k}$$

Again for convenience, the corresponding integrals will be labeled I_{3a} , I_{3b} and I_{3c} respectively.

By the same procedures as used in Appendices B and C we get:

$$I_{3a} = \sum_s s^2 \langle \xi_s^{(0)} | \xi_s^{(4)} \rangle_{3a}$$

$$= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' J_0(2q-2q') J_0(2q''-2q''') \\ \times \frac{q(p-q+q'-q''+q''')}{p-2q+q'-q''+q'''} J_1(2p-4q+2q'-2q''+2q''') \cos \sigma(2q-2q'+2q''-2q''')$$

D 4

$$I_{3b} = \sum_s s^2 \langle \xi_s^{(0)} | \xi_s^{(4)} \rangle_{3b}$$

$$= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' J_0(2q'-2q'') J_0(2q-2q'+2q''-2q''') \\ \times \frac{q(p-q+q''')}{p-2q+q'''} J_1(2p-4q+2q''') J \cos \sigma(2q+2q'-2q''-2q''')$$

D 5

$$I_{3c} = \sum_s s^2 \langle \xi_s^{(0)} | \xi_s^{(4)} \rangle_{3c}$$

$$= 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2 \int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_{r=-\infty}^{\infty} J_r(2q-2q') J_r(2q'-2q'') \\ \times J_r(2q''-2q''') \left\{ q J_{r+1}(2p-4q+2q''') + q^2 [J_{r+2}(2p-4q+2q''') \right. \\ \left. + J_r(2p-4q+2q''') \right] \Big\}$$

D 6

$$\times \cos \sigma(2q+2q'-2q''-2q''')$$

APPENDIX E

Two Integrations in I_{1a} , I_{1b} , I_{2a} , I_{2b} , I_{3a} , I_{3b} .

(1) Expression I_{1a}

Omitting the coefficient,

$$I_{1a}' = \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \frac{(P-q+q')(P-q''+q''')}{q-q'-q''+q'''} J_0(2q-2q') J_0(2q''-2q') \\ \times J_1(2q-2q'-2q''+2q''') \cos \sigma(2q-2q'-2q''+2q''')$$

(a) Let $2q''-2q''' = \xi$; $2q'' = \mu$; $2P = z$, then the integrals w. r. to q'' and q''' become

$$\frac{1}{4} \int_0^{2P} d\xi \int_{\xi}^z \frac{z-\xi}{2q-2q'-\xi} J_0(\xi) J_1(2q-2q'-\xi) \cos \sigma(2q-2q'-\xi) d\mu \\ = \frac{1}{4} \int_0^z d\xi \frac{(z-\xi)^2}{2q-2q'-\xi} J_0(\xi) J_1(2q-2q'-\xi) \cos \sigma(2q-2q'-\xi).$$

(b) Let $2q-2q' = \lambda$; $2q = \mu$, and get

$$I_{1a}' = \frac{1}{16} \int_0^z d\lambda \int_0^z d\xi \int_{\lambda}^z d\mu \frac{(z-\xi)^2(z-\lambda)}{2(\lambda-\xi)} J_0(\lambda) J_0(\xi) J_1(\lambda-\xi) \cos \sigma(\lambda-\xi)$$

and finally

$$I_{1a} = \frac{B}{16} \int_0^z d\lambda \int_0^z d\xi \left[z^4 - 2z^3(\lambda+\xi) + z^2(\xi^2 + 4\xi\lambda + \lambda^2) - 2z\lambda\xi(\xi+\lambda) + \xi^2\lambda^2 \right] \\ \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda-\xi)}{2(\lambda-\xi)} \cos \sigma(\lambda-\xi)$$

when

where $B = 2 \left(\frac{2\hbar}{Q_0^2 M \omega} \right)^2$

(2) Expression $I_{1b,1}$

Again we omit the coefficient

$$I_{1b,1}' = \int_0^P dq \int_0^P dq'' \int_0^q dq' \int_0^{q''} dq''' \frac{q' q'''}{q' - q'''} J_0(2q - 2q'') J_1(2q' - 2q''') \\ \times J_0(2q - 2q' - 2q'' + 2q''') \cos \sigma(2q + 2q' - 2q'' - 2q''')$$

(a) Considering first the last two integrals (w.r. to q' and q''') let

$$2q' - 2q''' = \xi \quad 2q' = \mu$$

We get, after integration w.r. to μ

$$\frac{1}{4} \int_0^P dq \int_0^P dq'' \left\{ \int_{-2q''}^0 d\xi \frac{J_1(\xi) J_0(2q - 2q'' - \xi)}{2\xi} \left(-\frac{\xi^3}{6} + \frac{8q''^3}{3} + 2\xi q''^2 \right) \right. \\ + \int_0^{2q-2q''} d\xi \frac{J_1(\xi) J_0(2q - 2q'' - \xi)}{2\xi} \left(\frac{8q''^3}{3} + 2\xi q''^2 \right) \\ \left. + \int_{2q-2q''}^{2q} d\xi \frac{J_1(\xi) J_0(2q - 2q'' - \xi)}{2\xi} \left(\frac{8q''^3}{3} - 2q^2 \xi + \frac{\xi^3}{6} \right) \right\} \\ \times J_0(2q - 2q'') \cos \sigma(2q - 2q'' + \xi) \\ = I_{1b,11} + I_{1b,12} + I_{1b,13}$$

(b) Let in all three terms

$$2q - 2q'' = \lambda \quad ; \quad 2q'' = \mu - \lambda \quad ; \quad 2q = \mu$$

After some rather lengthy but straightforward algebra we get
(including integration w.r. to μ)

$$I_{16,1} =$$

$$\begin{aligned} & \frac{B}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 - 2z^3(2\lambda - \xi) + 6z^2(\lambda^2 - \lambda\xi) - 2z\lambda^2(2\lambda - 3\xi) + \lambda^4 - 2\lambda^3\xi \right] \right. \right. \\ & \quad + \int_\lambda^z d\xi \left[z^4 - 2z^3\xi + 2z\xi^3 + \lambda^4 - \xi^4 \right] \\ & \quad + \int_{\lambda-z}^0 d\xi \left[z^4 - 2z(2\lambda - \xi) + 6z^2\lambda(\lambda - \xi) - 2z(2\lambda^3 - 3\lambda^2\xi + \xi^3) + \lambda^4 - 2\lambda^3\xi + 2\xi^2\lambda^2 - \xi^3 \right] \\ & \quad \times \left. \frac{J_1(\xi) J_0(\lambda) J_0(\lambda - \xi)}{\xi} \cos \sigma(\lambda + \xi) \right\} \end{aligned}$$

where

$$z = 2\rho$$

(3) Expression $I_{1b,2}$

$$I'_{1b,2} = \int_0^P dq \int_0^P dq'' \int_0^q dq' \int_0^{q''} dq''' \frac{(q-q')(q''-q''')}{q-q'-q''+q'''} J_0(2q-2q'') J_0(2q'-2q''') \\ \times J_1(2q-2q''-2q'+2q''') \cos \sigma(2q-2q''+2q'-2q''')$$

(a) Considering first the last two integrals (w.r.to q' and q''') let

$$2q'-2q''' = \xi \quad ; \quad 2q' = \mu$$

We get, after integration w.r. to μ ,

$$\frac{1}{4} \int_0^P dq \int_0^P dq'' \left\{ \int_{-2q''}^0 d\xi \left[q(\xi+2q'')^2 - \frac{(\xi+2q'')^3}{6} \right] J_0(2q-2q'') J_0(\xi) J_1(2q-2q''-\xi) \cos \sigma(2q-2q''+\xi) \right. \\ + \int_0^{2q-2q''} d\xi \left[4qq''^2 - 2\xi q''^2 - \frac{4q''^3}{3} \right] J_0(2q-2q'') J_0(\xi) J_1(2q-2q''-\xi) \cos \sigma(2q-2q''+\xi) \\ \left. + \int_{2q-2q''}^{2q} d\xi \left[\frac{(\xi-2q)^3}{6} + q''(\xi-2q)^2 \right] J_0(2q-2q'') J_0(\xi) J_1(2q-2q''-\xi) \cos \sigma(2q-2q''+\xi) \right\} \\ = I_{1b,21} + I_{1b,22} + I_{1b,23}$$

(b) Let in all three terms

$$2q''-2q = \lambda \quad ; \quad 2q'' = \mu$$

Having effected this transformation we will get a sum of

integrals, all of them of the form $\int d\lambda \int d\xi \int d\mu F$ with appropriate limits and integrands.

In all cases the μ integration is again trivial since it will not involve the Bessel function nor the cosine terms.

After some lengthy but straightforward algebra we get

$$I_{16,2} = \frac{B}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 + 6z^2\lambda\xi - 2z\lambda^2(3\xi - \lambda) - \lambda^4 + 2\lambda^3\xi \right] \right. \right. \\ + \int_\lambda^z d\xi \left[z^4 - 2z^3\lambda + 6z^2\lambda\xi - 2z\lambda(3\lambda\xi^2 - \xi^3) + 2\lambda\xi^3 - \xi^4 \right] \\ + \int_{-z+\lambda}^0 d\xi \left[z^4 + 2z^3\xi - z(6\lambda^2\xi - 2\lambda^3 + 2\xi^3 - 6\lambda\xi^2) - \lambda^4 + 4\lambda^3\xi \right. \\ \left. \left. - 6\lambda^2\xi^2 + 4\lambda\xi^3 - \xi^4 \right] \right\} \\ \times \frac{J_0(\lambda)J_0(\xi)J_1(\lambda-\xi)}{\lambda-\xi} \cos \sigma(\lambda+\xi) \left. \right\}$$

where

$$z = 2\rho$$

(4) Expression $I_{1b,3}$

$$I_{1b,3}' = \int_0^P dq \int_0^P dq'' \int_0^q dq' \int_0^{q''} dq''' \frac{(z-2q)(z-2q'')}{2(2q-2q'')} J_0(2q'-2q''') J_0(2q-2q'-2q''+2q') \\ \times J_1(2q-2q'') \cos \sigma(2q+2q'-2q''-2q''')$$

where $z = 2\rho$

As before,

(a) considering only the last two integrals (w.r. to q' and q''') let

$$2q'-2q''' = \xi \quad ; \quad 2q' = \mu$$

We get after transforming and integrating w.r. to μ

$$\frac{1}{4} \int_0^P dq \int_0^P dq'' \left\{ \int_{-2q''}^0 d\xi \frac{(z-2q)(z-2q'')(\xi+2q'')}{2(2q-2q'')} J_1(2q-2q'') J_0(2q-2q''-\xi) J_0(\xi) \right. \\ \times \cos \sigma(2q-2q''+\xi) \\ + \int_0^{2q-2q''} d\xi \frac{2q''(z-2q)(z-2q'')}{2(2q-2q'')} J_1(2q-2q'') J_0(2q-2q''-\xi) J_0(\xi) \cos \sigma(2q-2q''+\xi) \\ \left. + \int_{2q-2q''}^{2q} d\xi \frac{(2q-\xi)(z-2q)(z-2q'')}{2(2q-2q'')} J_1(2q-2q'') J_0(2q-2q''-\xi) J_0(\xi) \cos(2q-2q''+\xi) \right\} \\ = I_{1b,31} + I_{1b,32} + I_{1b,33}$$

(b) As before (i.e. in $I_{1b,1}$ and $I_{1b,2}$) let

$$2q''-2q = \lambda \quad ; \quad 2q'' = \mu$$

Having accomplished this transformation we will get a sum of

integrals, all of them of the form $\int d\lambda \int d\xi \int d\mu F$
with appropriate limits and integrands.

In every one of these terms the " μ " integration is trivial since μ will appear only in the form μ^k and since neither the Bessel function nor the cosine will be functions of μ .

Again after some lengthy but straightforward algebraic manipulations we get

$$\begin{aligned}
 I_{1b,3} = & \frac{B}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 + 2z\lambda^3 - \lambda^4 \right] \right. \right. \\
 & + \int_\lambda^z d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6z^2(\xi^2 - \xi\lambda) + 2z\lambda\xi^2(3\lambda - 2\xi) \right. \\
 & \left. \left. - 2\lambda\xi^3 + \xi^4 \right] \right. \\
 & + \int_{-z+\lambda}^0 d\xi \left[z^4 + 4z^3\xi + 6z^2(\xi^2 - \xi\lambda) + 2z\xi^2(2\xi - 3\lambda) \right. \\
 & \left. \left. - \lambda^4 + 2\lambda^3\xi - 2\lambda\xi^3 + \xi^4 \right] \right\} \\
 & \times \frac{J_0(\xi)J_0(\lambda-\xi)J_1(\lambda)}{\lambda} \cos \sigma(\lambda+\xi) \left. \right\}
 \end{aligned}$$

(5) Expression $I_{2a,1}$

$$I_{2a,1}' = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \frac{q(q' - q'' + q''')}{q - q' + q'' - q'''} J_0(2q - 2q') J_0(2q'' - 2q''') \\ \times J_1(2q - 2q' + 2q'' - 2q''') \cos \sigma(2q - 2q' + 2q'' - 2q''')$$

(a) Considering first the last two integrals (w.r. to q'' and q''') let

$$2q'' - 2q''' = \xi \quad ; \quad 2q'' = \mu$$

We get, after transformation to the new coordinates and integration w.r. to μ

$$\frac{1}{4} \int_0^P dq \int_0^P dq' \int_0^{2q'} d\xi J_0(2q - 2q') J_0(\xi) J_1(2q - 2q' + \xi) \frac{q(2q' - \xi)^2}{2q - 2q' + \xi} \cos \sigma(2q - 2q' + \xi)$$

(b) Let now

$$2q - 2q' = \lambda \quad ; \quad 2q = \mu$$

Having effected this transformation we will get a sum of integrals of the form $\int d\lambda \int d\xi \int d\mu$ with appropriate limits and integrands. Again, as in the other integrals, the " μ " integration is trivial since it does not involve the Bessel function or the cosines. Thus after some calculation we get

$$I_{2\alpha_1} = -\frac{\beta}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[3z^4 - 4z^3(\lambda + 2\xi) + 6z^2(\xi^2 + 2\lambda\xi) \right. \right. \right.$$

$$\left. \left. - 12z\lambda\xi^2 + \lambda^4 - 4\lambda^3\xi + 6\lambda^2\xi^2 \right] \right.$$

$$\left. + \int_\lambda^z d\xi \left[3z^4 - 4z^3(\lambda + 2\xi) + 6z^2(\xi^2 + 2\lambda\xi) \right. \right.$$

$$\left. \left. - 12z\lambda\xi^2 - 4\lambda\xi^3 - \xi^4 \right] \right.$$

$$\left. + \int_{-z+\lambda}^0 d\xi \left[3z^4 - 8z^3(\lambda - \xi) + 6z^2(\lambda - \xi)^2 - (\lambda - \xi)^4 \right] \right\}$$

$$\times \left\{ \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{2(\lambda - \xi)} \cos \sigma(\lambda - \xi) \right\}$$

E 5

(6) Expression $I_{2a,2}$.

$$I_{2a,2}' = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \int_0^{(2q''-2q''')} J_0(2q''-2q''') J_1(2q-2q') J_0(2q-2q'+2q''-2q''') \\ \times \frac{(z-2q)(z-2q')}{2(2q-2q')} \cos \sigma(2q-2q'+2q''-2q''')$$

(a) Considering first the last two integrals (w. r. to q'' and q''') let

$$2q'' - 2q''' = \xi \quad ; \quad 2q'' = \mu$$

We get after transformation to the new variables and integration with respect to μ

$$\frac{1}{4} \int_0^P dq \int_0^P dq' \int_0^{2q'} d\xi \frac{(2q'-\xi)(z-2q)(z-2q')}{2(2q-2q')} J_0(\xi) J_0(2q-2q'+\xi) J_1(2q-2q') \cos \sigma(2q-2q')$$

(b) Let now

$$2q - 2q' = \lambda \quad ; \quad 2q = \mu$$

As in $I_{2a,1}$, it is possible to effect one more integration after the transformation to λ, ξ and μ coordinates, so that one finally obtains:

$$I_{2a,2} = - \frac{B}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6\lambda z^2(\lambda - \xi) + 2z\lambda^3 \right. \right. \right.$$

$$\left. \left. - 2\xi\lambda^3 + \lambda^4 \right] \right.$$

$$\left. + \int_\lambda^z d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6\lambda z^2(\xi - \lambda) \right. \right.$$

$$\left. \left. - 2z(\xi - \lambda)^2(2\xi + \lambda) - \lambda^4 - 2\lambda\xi^3 + \xi^4 \right] \right.$$

$$\left. + \int_{-z+\lambda}^0 d\xi \left[z^4 - 2z^3(\lambda - 2\xi) - 6z^2\xi(\lambda - \xi) + 2z(\xi - \lambda)^2(2\xi + \lambda) \right. \right.$$

$$\left. \left. - \lambda^4 + 2\lambda^3\xi - 2\lambda\xi^3 + \xi^4 \right] \right\}$$

$$\times \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{2\lambda} \cos(\lambda - \xi) \left. \right\}$$

E 6

(7) Expression $I_{2b,1}$

$$I_{2b,1}' = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \frac{qq'''}{q-q'''} J_0(2q'-2q'') J_0(2q-2q'+2q''-2q''') \\ \times J_1(2q-2q''') \cos \sigma(2q+2q'-2q''-2q''')$$

(a) To be able to integrate we must first change the order of integration as follows

$$\int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' F = \int_0^P dq \int_0^P dq' \int_0^{q'} dq''' \int_{q'''}^{q'} dq'' F \\ = \int_0^P dq \int_0^P dq''' \int_{q'''}^P dq' \int_{q'''}^{q'} dq'' F$$

(b) Considering only the last two integrals (with respect to q' and q'') let

$$2q' - 2q'' = \xi \quad ; \quad 2q' = \mu$$

We get, after transformation to the new variables and integration w.r. to μ ,

$$\frac{1}{4} \int_0^P dq \int_0^P dq''' \int_0^{2P-2q'''} d\xi \frac{qq'''}{q-q'''} (2P-2q'''-\xi) J_0(\xi) J_0(2q-2q'''-\xi) \\ \times J_1(2q-2q''') \cos \sigma(2q-2q''' + \xi)$$

(c) Let now

$$2q - 2q''' = \lambda \quad ; \quad 2q = \mu$$

As before we can now effect one more integration (w.r. to μ)
 after transformation to λ, ξ and μ variables so that we
 finally get:

$$\begin{aligned}
 I_{2b,1} = & -\frac{B}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6z^2\lambda(\xi - \lambda) \right. \right. \right. \\
 & \left. \left. \left. + 2z\lambda^3 - 2\lambda^3\xi + \lambda^4 \right] \right. \right. \\
 & + \int_\lambda^z d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6z^2\xi(\xi - \lambda) \right. \\
 & \left. \left. + 2z\xi^2(3\lambda - 2\xi) - 2\lambda\xi^3 + \xi^4 \right] \right. \\
 & + \int_{-z+\lambda}^0 d\xi \left[z^4 + 2z^3(2\xi - \lambda) + 6z^2\xi(\xi - \lambda) + 2z(\xi - \lambda)^2(2\xi + \lambda) \right. \\
 & \left. \left. \left. - \lambda^4 + 2\lambda^3\xi - 2\lambda\xi^3 + \xi^4 \right] \right\} \right. \\
 & \times \left. \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{2\lambda} \cos \sigma(\lambda + \xi) \right\}
 \end{aligned}$$

(8) Expression $I_{2b,2}$

Using the same interchange of the order of integration as in $I_{2b,1}$ we have

$$I_{2b,2}' = - \int_0^P dq \int_0^P dq''' \int_{q'''}^P dq' \int_{q'''}^{q'} dq'' \frac{(p-q)(p-q'+q''-q''')}{q-q'+q''-q'''} \int_0^{2q'-2q''} \\ \times \int_0^{2q-2q'''} J_1(2q-2q'+2q''-2q''') \cos \sigma(2q+2q'-2q''-2q''')$$

(a) Considering the last two integrals only (w.r. to q' and q'') let

$$2q'-2q'' = \xi \quad ; \quad 2q' = \mu$$

We get, after transformation to the new variables and integration with respect to μ ,

$$\frac{1}{4} \int_0^P dq \int_0^P dq''' \int_0^{2-2q'''} d\xi \frac{(z-2q)(z-\xi-2q''')}{2(2q-2q'''-\xi)} \int_0^\xi J_0(\xi) \int_0^{2q-2q'''} \\ \times J_1(2q-2q'''-\xi) \cos \sigma(2q-2q''' + \xi)$$

(b) Let now

$$2q-2q''' = \lambda \quad ; \quad \xi = \xi \quad ; \quad 2q = \mu$$

As before, we can integrate once (w.r. to μ) after transformation to λ, ξ and μ variables so that we finally get:

$$I_{2b,2} = -\frac{B}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[3z^4 - 4z^3(\lambda + 2\xi) + 6z^2(2\lambda\xi + \xi^2) \right. \right. \right.$$

$$\left. \left. -12z\lambda\xi^2 + \lambda^4 - 4\lambda^3\xi + 6\lambda^2\xi^2 \right] \right.$$

$$\left. + \int_\lambda^z d\xi \left[3z^4 - 4z^3(\lambda + 2\xi) + 6z^2(2\lambda\xi + \xi^2) \right. \right.$$

E 8

$$\left. -12z\lambda\xi^2 + 4\lambda\xi^3 - \xi^4 \right]$$

$$+ \int_{-z+\lambda}^0 d\xi \left[3z^4 - 8z^3(\lambda - \xi) + 6z^2(\lambda - \xi)^2 \right.$$

$$\left. - (\lambda - \xi)^4 \right] \left. \right\}$$

$$\times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{2(\lambda - \xi)} \cos \sigma(\lambda + \xi)$$

(9) Expression I_{3a}

$$I'_{3a} = \int_0^p dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \frac{q(p-q+q'-q''+q''')}{p-2q+q-q''+q'''} J_0(2q-2q') \\ \times J_0(2q''-2q''') J_1(2p-4q+2q'-2q''+2q''') \cos \sigma(2q-2q'+2q''-2q''')$$

(a) Considering only the last two integrals (w.r.)
to q'' and q''') let

$$2q'' - 2q''' = \xi \quad ; \quad 2q'' = \mu$$

We get, after transformation to the new variables and integration with respect to μ ,

$$\frac{1}{4} \int_0^p dq \int_0^q dq' \int_0^{2q'} d\xi \frac{2q(2p-2q+2q'-\xi)(2q'-\xi)}{2(2p-4q+2q'-\xi)} J_0(2q-2q') J_0(\xi) \\ \times J_1(2p-4q+2q'-\xi) \cos \sigma(2q-2q'+\xi)$$

(b) Now let

$$2q - 2q' = \lambda \quad ; \quad 2q = \mu$$

and we get

$$\frac{1}{16} \int_0^z d\lambda \int_0^{z-\lambda} d\xi \int_{\xi+\lambda}^z d\mu \frac{\mu(z-\lambda-\xi)(\mu-\lambda-\xi)}{2(z-\mu-\lambda-\xi)} J_0(\lambda) J_0(\xi) \\ \times J_1(z-\mu-\lambda-\xi) \cos \sigma(\lambda+\xi) d$$

(c) The factor

$$\frac{\mu(\mu-\lambda-\xi)}{z-\mu-\lambda-\xi} = -\mu + \frac{\mu(z-2\lambda-2\xi)}{z-\mu-\lambda-\xi}$$

We then have two types of integrals, which despite the fact that the integrand contains Bessel functions, can be integrated exactly

$$(1) \int \mu J_1(z-\lambda-\xi-\mu) d\mu = I'$$

Let $z-\mu-\lambda-\xi = x$; $d\mu = -dx$, then

E 9

$$I' = \int x J_1(x) dx = -x J_0(x) + \int J_0(x) dx$$

and from Gradshteyn & Ryzhik, page 666, #6.511-6

E 10

$$\int_0^a J_0(x) dx = a J_0(a) + \frac{\pi}{a} [J_1(a) H_0(a) - J_0(a) H_1(a)] \quad a > 0$$

where H_0, H_1 are Struve functions.

E 11

$$(2) \int \frac{\mu(z-2\lambda-2\xi)}{z-\mu-\lambda-\xi} J_1(z-\mu-\lambda-\xi) d\mu \\ = -(z-2\lambda-2\xi) \int J_1(z-\mu-\lambda-\xi) d\mu + (z-\lambda-\xi)(z-2\lambda-2\xi) \int \frac{J_1(z-\mu-\lambda-\xi)}{z-\mu-\lambda-\xi}$$

This can be integrated since

E 12

$$\int J_1(x) dx = -J_0(x)$$

E 13

$$\int \frac{J_1(x)}{x} dx = \int [J_0(x) - J_1'(x)] dx = \int J_0(x) dx + J_1(x)$$

By using eqs. E9 - E13 we finally get

$$\begin{aligned}
 I_{3a} &= \frac{B}{32} \int_0^z d\lambda \int_0^{z-\lambda} d\xi (z-\lambda-\xi) J_0(\lambda) J_0(\xi) \\
 &\times \left\{ -2(\lambda+\xi) J_0(\lambda+\xi) - (z-3\lambda-3\xi) J_0(z-2\lambda-2\xi) \right. \\
 &\quad \left. + (z-\lambda-\xi)(z-2\lambda-2\xi) \left[J_1(\lambda+\xi) + J_1(z-2\lambda-2\xi) \right] \right. \\
 &\quad \left. - \left[1 + (z-\lambda-\xi)(z-2\lambda-2\xi) \right] \left[(\lambda+\xi) J_0(\lambda+\xi) \right. \right. \\
 &\quad \left. \left. + (z-2\lambda-2\xi) J_0(z-2\lambda-2\xi) \right] \right. \\
 &\quad \left. + \frac{\pi(\lambda+\xi)}{2} \left(H_0(\lambda+\xi) J_1(\lambda+\xi) - H_1(\lambda+\xi) J_0(\lambda+\xi) \right) \right. \\
 &\quad \left. + \frac{(z-2\lambda-2\xi)\pi}{2} \left(H_0(z-2\lambda-2\xi) J_1(z-2\lambda-2\xi) - H_1(z-2\lambda-2\xi) J_0(z-2\lambda-2\xi) \right) \right\} \\
 &\times \cos \sigma(\lambda+\xi)
 \end{aligned}$$

E 14

(10) Expression I_{3b}

We first note that

$$\int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' F = \int_0^P dq \int_0^q dq' \int_0^{q'} dq''' \int_{q'''}^{q'} dq'' F = \int_0^P dq \int_0^q dq''' \int_{q'''}^q dq' \int_{q'''}^{q'} dq'' F$$

so that eq. D5 becomes

$$I'_{3b} = \int_0^P dq \int_0^q dq''' \int_{q'''}^q dq' \int_{q'''}^{q'} dq'' \frac{q(z-2q+2q''')}{z-4q+2q'''} J_0(2q'-2q'') J_0(2q-2q'+2q''-2q''') \\ \times J_1(z-4q+2q''') \cos \sigma(2q+2q'-2q''-2q''')$$

(a) Considering only the last two integrals (w . r. to q' and q'') let

$$2q'-2q'' = \xi \quad ; \quad 2q' = \mu$$

We then get after transformation of variables and integration with respect to μ

$$\frac{1}{4} \int_0^P dq \int_0^q dq''' \int_0^{2q-2q'''} d\xi \frac{q(2q-2q'''-\xi)(z-2q+2q''')}{z-4q+2q'''} J_0(\xi) J_0(2q-2q'''-\xi) \\ \times J_1(z-4q+2q''') \cos \sigma(2q-2q''' + \xi)$$

(b) Now let

$$2q-2q''' = \lambda \quad 2q = \mu$$

and we get

$$\frac{1}{16} \int_0^z d\lambda \int_0^\lambda d\xi \int_\lambda^z d\mu \frac{\mu(\lambda-\xi)(z-\lambda)}{2(z-\mu-\lambda)} J_0(\xi) J_0(\lambda-\xi) J_1(z-\mu-\lambda) \cos \sigma(\lambda+\xi)$$

(c) Since

$$\frac{\mu}{z-\mu-\lambda} = -1 + \frac{z-\lambda}{z-\mu-\lambda}$$

using again eqs. E9 - E13 we finally obtain

$$\begin{aligned} I_{3b} &= \frac{\beta}{32} \int_0^z d\lambda \int_0^\lambda d\xi (\lambda-\xi)(z-\lambda) J_0(\xi) J_0(\lambda-\xi) \\ &\times \left\{ J_0(z-2\lambda) - J_0(\lambda) + (z-\lambda) \left[-J_1(\lambda) - J_1(z-2\lambda) + (z-2\lambda) J_0(z-2\lambda) \right. \right. \\ &\quad \left. \left. + \lambda J_0(\lambda) + \frac{\pi(z-2\lambda)}{2} (H_0(z-2\lambda) J_1(z-2\lambda) - H_1(z-2\lambda) J_0(z-2\lambda)) \right. \right. \\ &\quad \left. \left. + \frac{\pi\lambda}{2} (H_0(\lambda) J_1(\lambda) - H_1(\lambda) J_0(\lambda)) \right] \right\} \cos \sigma(\lambda+\xi) \end{aligned}$$

E 15

APPENDIX F

Evaluation of $I_{1b,2} + 2 I_{2b,2}$

We combine $I_{1b,2}$ and $2 I_{2b,2}$ because the product of Bessel function and the cosine in the integrand are the same.

Then

$$\begin{aligned}
 I_{1b,2} + 2 I_{2b,2} &= \\
 &= \frac{B}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[-2z^4 + 4z^3(\lambda + 2\xi) - 6z^2\xi(\lambda + \xi) + 2z\lambda(6\xi^2 - 3\lambda\xi + \lambda^2) - 2\lambda^4 \right. \right. \right. \\
 &\quad \left. \left. \left. + 6\lambda^3\xi - 6\lambda^2\xi^2 \right] \right. \right. \\
 &\quad \left. + \int_\lambda^z d\xi \left[-2z^4 + 2z^3(\lambda + 4\xi) - 6z^2\xi(\lambda + \xi) + 2z(3\lambda\xi^2 + \xi^3) - 2\lambda\xi^3 \right] \right. \\
 &\quad \left. + \int_{-z+\lambda}^0 d\xi \left[-2z^4 + 2z^3(4\lambda - 3\xi) - 6z^2(\lambda - \xi)^2 + 2z(\lambda - \xi)^3 \right] \right\} \\
 &\quad \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda + \xi) \\
 &= {}^1I_1 + {}^2I_1 + {}^3I_1
 \end{aligned}$$

We will discuss the three integrals separately although the procedure is the same for all three.

(a) ${}^1I_1$

The area of integration can be subdivided as shown in Figure F1:

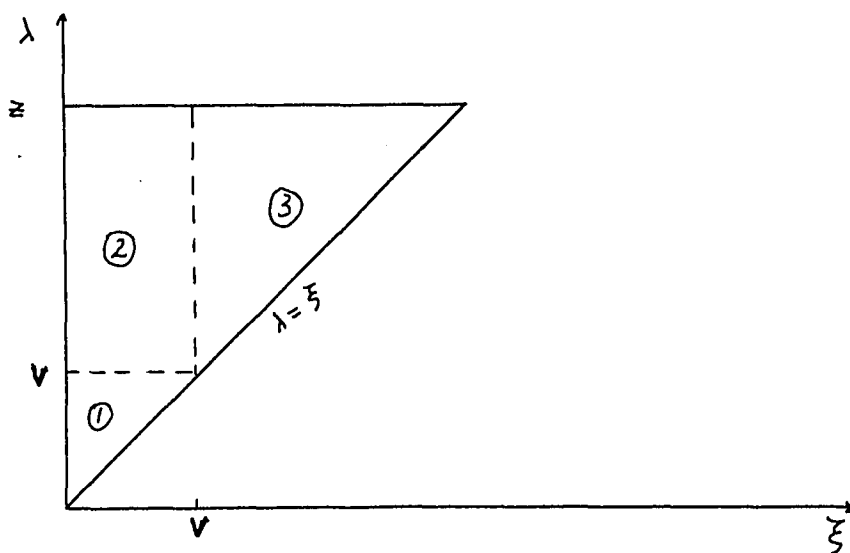


Fig. F1

v is such that when $x > v$ we can replace $J_0(x)$ by its asymptotic expansion, i.e.

$$F2 \quad J_0(x)_{x>v} \sim \frac{1}{\sqrt{\pi}} \frac{\cos x + \sin x}{\sqrt{x}}$$

We now examine the integration over the three regions separately, keeping in mind the fact that z is very large.

(1) Region (1)

The integral is now ${}^1I_{1,1}$ (the second subscript indicating the region of integration)

$${}^1I_{1,1} = \int_0^v d\lambda \int_0^\lambda d\xi F_1 f_1$$

where

$$F3 \quad F = J_0(\lambda) J_0(\xi) \frac{J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda + \xi)$$

and

$$F4 \quad f_i = -2z^4 + 4z^3(\lambda + 2\xi) - 6z^2\xi(\xi + \lambda) + 2z\lambda(6\xi^2 - 3\lambda\xi + \lambda^2) - 2\lambda^4 + 6\lambda^3\xi - 6\lambda^2\xi^2$$

Define now

$$K_j^i = \int_0^V d\lambda \int_0^\lambda d\xi \lambda^j \xi^i J_0(\lambda) J_0(\xi) [J_0(\lambda - \xi) + J_2(\lambda - \xi)] \cos \sigma(\lambda + \xi)$$

where

$$i = 0, 1, 2 \quad ; \quad j = 0, 2, 3, 4$$

It is obvious that the K_j^i thus defined are all finite and we get

$$F5 \quad {}^1I_{1,1} = -z^4 K_0^0 + 4z^3(K_1^0 + 2K_0^1) - 6z^2(K_0^2 + K_1^1) + 2z(6K_1^2 - 3K_2^1 + K_3^0) - 2K_4^0 + 6K_3^1 - 6K_2^2$$

Provided K_0^0 is different from zero, we see that $-z^4 K_0^0$ is the highest order term in z .

(2) Region (2)

The integrals are now ${}^1I_{1,2}$

$${}^1I_{1,2} = \int_V^z d\lambda \int_0^V d\xi F f_i$$

This is equal to

$$F6 \quad \int_V^{2V} d\lambda \int_0^V d\xi F f_i + \int_{2V}^z d\lambda \int_0^V d\xi F f_i$$

The first part is of the same type as ${}^1I_{1,1}$, converges (if $f_i = 1$) and will give the same result, namely that the highest order

term in z is $-z^4 \int_v^{2v} d\lambda \int_0^v d\xi F$. It can be joined to K_0^0 by redefining the upper limit of the " λ " integral in K_0^0 . We can invert the order of integration without any further difficulty and get a sum of terms of the form

$$z^{4-i-j} \int_0^v \xi^i J_0(\xi) d\xi \int_{2v}^z \lambda^{j-2} \left\{ \frac{\sin}{\cos} \right\} \alpha \xi \left\{ \frac{\sin}{\cos} \right\} \beta \lambda d\lambda$$

The ξ integration always gives a constant (dependent on v). For the λ integration, it is clear that the z dependence will be at most of order z^2 (for $j=4$) but in that case $i=0$ and z^2 must be the order of the integral. All other values of j will, when combined with the z coefficient, produce terms of order z^3 or lower. The only exception is the lower limit in the case $i=0$, $j=0$, where we would get (for example)

$$\begin{aligned} z^4 \int_0^v d\xi \cos \alpha \xi J_0(\xi) \int_{2v}^z d\lambda \lambda^{-2} \cos \beta \lambda d\lambda \\ = z^4 \int_0^v d\xi J_0(\xi) \cos \alpha \xi \left[-\frac{\sin \beta z}{\beta z^2} + \frac{\sin 2\beta v}{4\beta v^2} \right] \end{aligned}$$

Although the last term is of order z^4 , it can be taken care of by extending the limit on K_0^0 and is then contained in that same term. Note also that it would be small compared to K_0^0 due to v^2 in the denominator. In other words, the integral alone (without the z^4 factor) is convergent, in fact is small.

We conclude that in region (2) the only contribution to

a z^4 term is effectively taken care of by extending the limits of k_0^0 without affecting the convergence of this term.

(3) Region (3)

The integral to be evaluated is now

$$^1 I_{1,3} = \int_V^z d\lambda \int_V^\lambda d\xi J_0(\lambda) J_0(\xi) [J_0(\lambda-\xi) + J_2(\lambda-\xi)] \cos \sigma(\lambda+\xi) f_1(\lambda, \xi, z)$$

where use is made of the formula

$$\frac{J_1(z)}{z} = J_0(z) + J_2(z)$$

Since both λ and ξ are large in the domain of integration, we may use the asymptotic expansion for $J_0(\xi)$ and $J_0(\lambda)$, i.e.

$$J_0(\lambda) J_0(\xi) \sim \frac{1}{\pi \sqrt{\lambda \xi}} [\cos(\lambda-\xi) + \sin(\lambda+\xi)]$$

$\lambda, \xi \gg 1$

$$F7 \quad ^1 I_{1,3} = \int_V^z d\lambda \int_V^\lambda d\xi \frac{[\cos(\lambda-\xi) + \sin(\lambda+\xi)]}{\sqrt{\lambda \xi}} [J_0(\lambda-\xi) + J_2(\lambda-\xi)] \cos \sigma(\lambda+\xi) f_1(\lambda, \xi, z)$$

Now let

$$\lambda + \xi = \mu \quad ; \quad \lambda - \xi = \nu \quad ; \quad \lambda \xi = \frac{\mu^2 - \nu^2}{4}$$

The integral then becomes

$$^1 I_{1,3} \sim \frac{1}{\sqrt{\pi}} \int_0^{z-\nu} d\nu \int_{2\nu+\nu}^{2z-\nu} d\mu \frac{1}{\sqrt{\mu^2 - \nu^2}} (\cos \nu + \cos \mu) [J_0(\nu) + J_2(\nu)] \cos \sigma \mu f_1$$

Before we can evaluate this integral, we must transform $f_1(\lambda, \xi, z)$ to $f_1(\nu, \mu, z)$. $f_1(\lambda, \xi, z)$ and, incidentally, also $f_2(\lambda, \xi, z)$ and $f_3(\lambda, \xi, z)$ (parts of the integrands of $^2 I_1$ and $^3 I_1$) are composed of the sum of terms of the form

$$A \lambda^{j-\frac{1}{2}} \xi^{i-\frac{1}{2}} z^{4-j-i}$$

Since we want to prove that each such term multiplied by λ is convergent, it will be more convenient to first tabulate all possible terms and their equivalents in terms of μ and ν , then show that the individual terms of the sum, in terms of μ and ν , converge.

The possible terms are listed in Table F1:

Table F1

i	j	λ, ξ dependence	μ, ν dependence
0	0	$(\lambda, \xi)^{-1/2}$	$\frac{1}{\mu} + \frac{\nu^2}{2\mu^3}$
0	1	$(\lambda/\xi)^{1/2}$	$1 + \frac{\nu}{\mu}$
0	2	$\lambda^{3/2} \xi^{-1/2}$	$\frac{\mu}{2} + \nu + \frac{\nu^2}{2\mu}$
0	3	$\lambda^{5/2} \xi^{-1/2}$	$\frac{1}{4} [\mu^2 + 3\mu\nu + \frac{7}{2}\nu^2 + \frac{5}{2}\frac{\nu^3}{\mu}]$
0	4	$\lambda^{7/2} \xi^{-1/2}$	$\frac{1}{8} [\mu^3 + 4\mu^2\nu + \frac{13}{2}\mu\nu^2 + 8\nu^3 + 4\frac{\nu^4}{\mu}]$
1	0	$\lambda^{-1/2} \xi^{1/2}$	$1 - \frac{\nu}{\mu} + \frac{\nu^2}{2\mu^2}$
1	1	$(\lambda \xi)^{1/2}$	$\frac{\mu}{2} - \frac{\nu^2}{4\mu}$
1	2	$\lambda^{3/2} \xi^{1/2}$	$\frac{\mu^2}{4} + \frac{\mu\nu}{4} - \frac{\nu^2}{8} - \frac{\nu^3}{4\mu}$
1	3	$\lambda^{5/2} \xi^{1/2}$	$\frac{1}{8} [\mu^3 + 2\mu^2\nu + \frac{3}{2}\nu^2\mu + \nu^3 + \frac{\nu^4}{2\mu}]$
2	0	$\lambda^{-1/2} \xi^{3/2}$	$\frac{1}{2} [\mu - 2\nu + \frac{3\nu^2}{2\mu} - \frac{\nu^3}{2\mu^2}]$
2	1	$\lambda^{1/2} \xi^{3/2}$	$\frac{\mu^2}{4} - \frac{\mu\nu}{4} - \frac{\nu^2}{8} + \frac{\nu^3}{8\mu}$
2	2	$(\lambda \xi)^{3/2}$	$\frac{\mu^3}{8} - \frac{3}{16}\mu\nu^2 + \frac{\nu^4}{16\mu}$
3	0	$\lambda^{-1/2} \xi^{5/2}$	$\frac{1}{4} [\mu^2 - 3\mu\nu + \frac{7}{2}\nu^2 - \frac{5}{2}\frac{\nu^3}{\mu}]$
3	1	$\lambda^{1/2} \xi^{5/2}$	$\frac{1}{8} [\mu^3 - 2\mu^2\nu + \frac{\mu\nu^2}{2} + \nu^3 - \frac{\nu^4}{2\mu}]$
4	0	$\lambda^{-1/2} \xi^{7/2}$	$\frac{1}{8} [\mu^3 - 4\mu^2\nu + \frac{13}{2}\mu\nu^2 - \frac{1}{2}\nu^3]$

From Table F1 and the form of $f_1(\lambda, \xi, z)$ it is now clear that $I_{1,3}$ will be composed of a sum of terms of the form

$$\int_{2v+v}^{z-v} dv \int_{2v+v}^{2z-v} d\mu \mu^k v^l (\cos v + \sin \mu) \cos \sigma \mu [J_0(v) + J_2(v)]$$

where k can have the values $-1, 0, 1, 2, 3$

and l can have the values $0, 1, 2, 3, 4$

although not all combinations k, l occur. Only those terms which arise in I_1 are discussed in this section.

The next step is to integrate with respect to μ .

The integrals will be of the form

$$I_k = \int_{2v+v}^{2z-v} d\mu \mu^k (\cos v + \sin \mu) \cos \sigma \mu$$

where k has values $-1, 0, 1, 2, 3$.

In the integration use will be made of the fact that the limits

$2z+v$ and $2v-v$ are both large (since $0 < v < -z+v$).

In particular *sine* and *cosine* integrals can be evaluated in this case by using asymptotic expansions, hence

*

$$Ci(x) \sim \frac{\sin x}{x} - \frac{\cos x}{x^2}$$

$$Si(x) \sim -\frac{\cos x}{x} - \frac{\sin x}{x^2}$$

$$(Ci(x) = \int_0^x \frac{\cos x}{x} dx)$$

We get then

F9

$$I^{-1} = \frac{\cos \nu \sin \sigma (2z - \nu)}{\sigma (2z - \nu)} - \frac{1}{2} \frac{\cos[(\sigma+1)(2z - \nu)]}{(\sigma+1)(2z - \nu)} - \frac{1}{2} \frac{\cos[(1-\sigma)(2z - \nu)]}{(1-\sigma)(2z - \nu)} \\ - \frac{\cos \nu \sin \sigma (2v + \nu)}{\sigma (2v + \nu)} + \frac{1}{2} \frac{\cos[(\sigma+1)(2v + \nu)]}{(\sigma+1)(2v + \nu)} + \frac{1}{2} \frac{\cos[(1-\sigma)(2v + \nu)]}{(1-\sigma)(2v + \nu)}$$

For the other values of k only the form of the solution will be given.

F10

I^0 will then be of the form

$$I^0 \sim \cos \nu \left[\frac{\sin \alpha (2z - \nu)}{\alpha} - \frac{\sin \alpha (2v + \nu)}{\alpha} \right] - \frac{1}{2} \left[\frac{\cos \beta (2z - \nu)}{\beta} - \frac{\cos \beta (2v + \nu)}{\beta} \right]$$

I^1 will be of the form

F11

$$I^1 \sim \cos \nu \left[\frac{\cos \alpha (2z - \nu)}{\alpha^2} + \frac{(2z - \nu) \sin \alpha (2z - \nu)}{\alpha} - \frac{\cos \alpha (2v + \nu)}{\alpha^2} - \frac{(2v + \nu) \sin \alpha (2v + \nu)}{\alpha} \right. \\ \left. + \frac{1}{2} \left[\frac{\sin \beta (2z - \nu)}{\beta^2} - \frac{(2z - \nu) \cos \beta (2z - \nu)}{\beta} - \frac{\sin \beta (2v + \nu)}{\beta^2} + \frac{(2v + \nu) \cos \beta (2v + \nu)}{\beta} \right] \right]$$

F12

$$I^2 \sim \cos \nu \left[\frac{2(2z - \nu) \cos \alpha (2z - \nu)}{\alpha^2} + \left(\frac{(2z - \nu)^2}{\alpha} - \frac{2}{\alpha^3} \right) \sin \alpha (2z - \nu) - \frac{2(2v + \nu) \cos \alpha (2v + \nu)}{\alpha^2} \right. \\ \left. - \left(\frac{(2v + \nu)^2}{\alpha} - \frac{2}{\alpha^3} \right) \sin \alpha (2v + \nu) \right] + \frac{1}{2} \left[\frac{2(2z - \nu) \sin \beta (2z - \nu)}{\beta^2} - \left(\frac{(2z - \nu)^2}{\beta} - \frac{2}{\beta^3} \right) \right. \\ \left. \times \cos \beta (2z - \nu) - \frac{2(2v + \nu) \sin \beta (2v + \nu)}{\beta^2} + \left(\frac{(2v + \nu)^2}{\beta} - \frac{2}{\beta^3} \right) \cos \beta (2v + \nu) \right]$$

F13

$$I^3 \sim \cos \nu \left[\left(\frac{3(2z - \nu)^2}{\alpha^2} - \frac{6}{\alpha^4} \right) \cos \alpha (2z - \nu) + \left(\frac{(2z - \nu)^3}{\alpha} - \frac{6(2z - \nu)}{\alpha^3} \right) \sin \alpha (2z - \nu) \right. \\ \left. - \text{same with } (2v + \nu) \text{ instead of } (2z - \nu) \right] \\ + \frac{1}{2} \left[\left(\frac{3(2z - \nu)^2}{\beta^2} - \frac{6}{\beta^4} \right) \sin \beta (2z - \nu) - \left(\frac{(2z - \nu)^3}{\beta} - \frac{6(2z - \nu)}{\beta^3} \right) \cos \beta (2z - \nu) \right. \\ \left. - \text{same with } (2v + \nu) \text{ instead of } (2z - \nu) \right]$$

Next we proceed to the last integration, w.r. to ν . We then have integrals of the form

$$F14 \quad \int_0^{z-\nu} I^k \nu^\ell [J_0(\nu) + J_2(\nu)] d\nu$$

The $k\ell$ combinations possible here are (with corresponding powers of z)

	$k = -1$	$\ell = 0, 1, 2, 3, 4$	$\rightarrow$	z^4, z^3, z^2, z, c
	$k = 0$	$\ell = 0, 1, 2, 3$	$\rightarrow$	z^3, z^2, z, c
F15	$k = 1$	$\ell = 0, 1, 2$	$\rightarrow$	z^2, z, c
	$k = 2$	$\ell = 0, 1$	$\rightarrow$	z, c
	$k = 3$	$\ell = 0$	$\rightarrow$	c $c = \text{constant}$

For all integrals of form F14 we follow the procedure:

$$\begin{aligned} \int_0^{z-\nu} I^k \nu^\ell [J_0(\nu) + J_2(\nu)] d\nu &= \int_0^\nu I^k \nu^\ell [J_0(\nu) + J_2(\nu)] d\nu \\ &+ \int_\nu^{z-\nu} I^k \nu^\ell [J_0(\nu) + J_2(\nu)] d\nu = A + B \end{aligned}$$

It is clear that A is finite for $k=0, 1$. For $k=1, 2$ the integral behaves as z and z^2 respectively so that according to F15 the expression is at most of order z^3 . For $k=0$ we again get at most a z^3 order of magnitude. But for $k=-1$, at least part of I^{-1} behaves as $\frac{1}{\nu}$ and together with F15 and the fact that $\int_0^\nu \nu^\ell [J_0(\nu) + J_2(\nu)] d\nu$ is convergent we have here a term of order z^4 . But since there is ν in the denominator it can be included in K_0^0 by extending the limits without affecting the

convergence of K_0° as extended. (We would add effectively

$$\int_0^V I^{-1} [J_0(\nu) + J_2(\nu)] d\nu \text{ to } K_0^\circ).$$

In part B we need a lengthier discussion.

$$F16 \quad B = \int_V^{z-V} I^K (-1)^\ell \nu^\ell [J_0(\nu) + J_2(\nu)]$$

Clearly ν is always large and $J_0(\nu) + J_2(\nu)$ can be replaced by its asymptotic expansion:

$$J_0(\nu) + J_2(\nu) \sim \frac{2}{\sqrt{\pi}} \nu^{-3/2} (\sin \nu - \cos \nu)$$

Now denote the possible terms as

$$F17 \quad I_m^{K, \ell} = z^m \int_V^{z-V} I^K (-1)^\ell \nu^{\ell-3/2} (\sin \nu - \cos \nu) d\nu$$

where m indicates the power of z associated with the values of K, ℓ used as shown in the third column of F15. By using the following expressions it is clear by inspection that all the $I_m^{K, \ell}$ are of order $z^{3/2}$.

The formulas used are

$$\frac{1}{2z - \nu} \sim \frac{1}{2z} \left(1 + \frac{\nu}{2z} + \dots\right) \sim \frac{1}{2z}$$

It is known that:

$$\int \sqrt{\lambda} \sin \beta \lambda d\lambda = \frac{1}{2} (i\beta)^{-3/2} \gamma(\frac{3}{2}, i\beta\lambda) - \frac{i}{2} (-i\beta)^{-3/2} \gamma(\frac{3}{2}, -i\beta\lambda) *$$

where $\gamma(a, b)$ is the incomplete gamma function.

For λ large, i.e. if the limits are large as is the case here, this becomes

$$\int \sqrt{\lambda} \sin \beta \lambda d\lambda \sim - \frac{\lambda^{3/2} \cos \beta \lambda}{\beta}$$

and similarly

$$\int \sqrt{\lambda} \cos \beta \lambda d\lambda \sim \frac{\lambda^{3/2} \sin \beta \lambda}{\beta}$$

$$\int \frac{\cos \beta \lambda}{\sqrt{\lambda}} d\lambda \sim \frac{\sin \beta \lambda}{\beta \sqrt{\lambda}}$$

$$\int \frac{\sin \beta \lambda}{\sqrt{\lambda}} d\lambda \sim \frac{\cos \beta \lambda}{\beta \sqrt{\lambda}}$$

$$F 18 \quad \int \lambda^{\frac{2n+1}{2}} \sin \beta \lambda d\lambda \sim - \lambda^{\frac{2n+1}{2}} \frac{\cos \beta \lambda}{\beta}$$

$$F 19 \quad \int \lambda^{\frac{2n+1}{2}} \cos \beta \lambda d\lambda \sim \lambda^{\frac{2n+1}{2}} \frac{\sin \beta \lambda}{\beta}$$

Using these formulas and the expressions for I'' (F9 - 13) it is easily seen that all terms are of order in z lower than 4 (in fact $z^{3/2}$). In conclusion, we have shown that all the integrals which are multiplied by z^4 converge and the highest order term in z is then

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 183 # 2.632, 1 - 4

$$z^4 \left\{ \int_0^V d\lambda \int_0^\lambda d\xi + \int_V^{2V} d\lambda \int_0^V d\xi + \int_{2V}^Z d\lambda \int_0^V d\xi + \int_V^Z d\lambda \int_V^\lambda d\xi \right\} J_0(\lambda) J_0(\xi) \frac{J_1(\lambda-\xi)}{\lambda-\xi} \times \cos \sigma(\lambda+\xi)$$

F 20

$$= z^4 \int_0^Z d\lambda \int_0^\lambda d\xi J_0(\lambda) J_0(\xi) \frac{J_1(\lambda-\xi)}{\lambda-\xi} \cos \sigma(\lambda+\xi)$$

(b) ${}^2I_1$

$${}^2I_1 = \int_0^Z d\lambda \int_\lambda^Z d\xi \left[-2z^4 + 2z^3(\lambda + 4\xi) - 6z^2(\lambda\xi + \xi^2) + 2z\xi^2(3\lambda + \xi) - 2\lambda\xi^3 \right]$$

$$\times J_0(\lambda) J_0(\xi) \frac{J_1(\lambda-\xi)}{\lambda-\xi} \cos \sigma(\lambda+\xi)$$

But we can simply change the order of integration to get

$$\int_0^Z d\xi \int_0^\xi d\lambda \left[-2z^4 + \dots \right] \times \dots$$

Now we have exactly the same expression as in ${}^1I_1$, except that λ and ξ have been interchanged. All the conclusions arrived at in ${}^1I_1$ must also be valid here. We thus remain with

$$z^4 \int_0^Z d\lambda \int_\lambda^Z d\xi J_0(\lambda) J_0(\xi) \frac{J_1(\lambda-\xi)}{\lambda-\xi} \cos \sigma(\lambda+\xi)$$

$$(c) {}^3I_1$$

$$F21 \quad {}^3I_1 = \int_0^z d\lambda \int_{-z+\lambda}^0 d\xi \left[-2z^4 + 2z^3(4\lambda - 3\xi) - 6z^2(\lambda - \xi)^2 + 2z(\lambda - \xi)^3 \right] \\ \times \frac{J_0(\lambda) J_0(\xi)}{\lambda - \xi} \frac{J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda + \xi)$$

Here the situation is somewhat more involved than in ${}^2I_1$.

The area of integration may be subdivided as shown in Fig. F2.

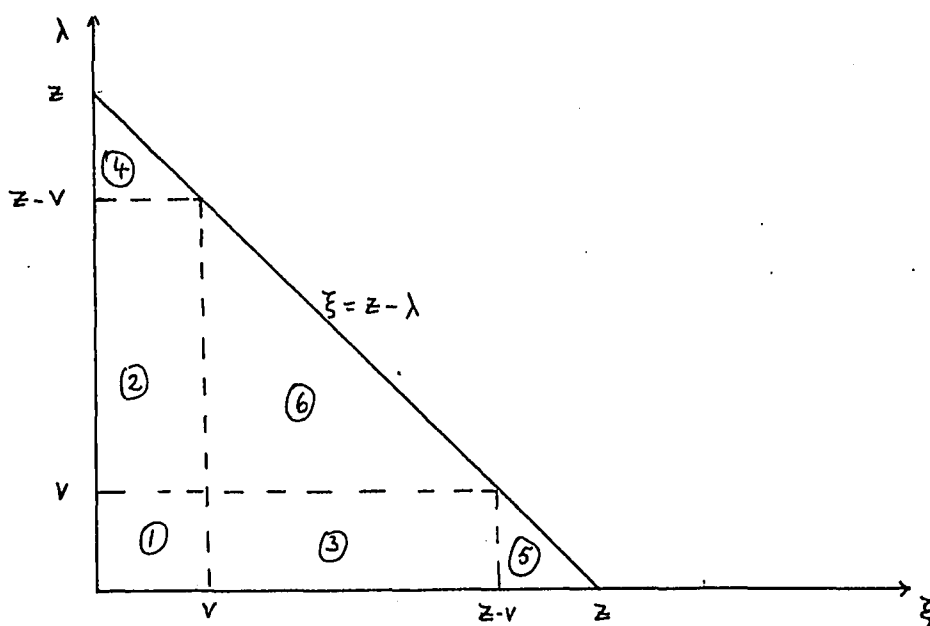


Fig. F2

The integral has been transformed by letting $\xi \rightarrow -\xi$. Thus

$${}^3I_1 = \int_0^z d\lambda \int_0^{z-\lambda} d\xi J_0(\lambda) J_0(\xi) [J_0(\lambda + \xi) + J_2(\lambda + \xi)] \cos \sigma(\lambda - \xi) f_3(-\xi) \\ = \int_0^z d\lambda \int_0^{z-\lambda} d\xi F_3(\lambda, \xi) f_3(-\xi)$$

We will now evaluate the integral in each of the 6 regions and will show that the highest order term in z is

$$F22 \quad -z^4 \int_0^z d\lambda \int_0^{z-\lambda} d\xi J_0(\lambda) J_0(\xi) \frac{J_1(\lambda + \xi)}{\lambda + \xi} \cos \sigma(\lambda - \xi)$$

(1) Region (1)

As for I_1 , $\int_0^V d\lambda \int_0^V d\xi F_3$ is clearly finite. Hence the highest order term in z for region (1) is

F 23

$$- z^4 \int_0^V d\lambda \int_0^V d\xi F_3$$

(2) Region 2

Here λ and $\lambda + \xi$ are both large so that $J_0(\lambda)$ and $\frac{J_1(\lambda + \xi)}{\lambda + \xi}$ may be replaced by the asymptotic expansion. We then get an expression entirely similar to the one obtained in the case of I_1 (region (2)) except for the fact that here we have the factor $\cos \sigma(\lambda - \xi)$ instead of $\cos \sigma(\lambda + \xi)$ and the upper limit for λ is $z - V$ instead of z ; but this clearly does not affect the results obtained for $I_1^{(2)}$ which will also be valid here. Hence the highest order term in z^4 (and the only one) is

F 24

$$- z^4 \int_0^{z-V} d\lambda \int_0^V d\xi F_3$$

(3) Region (3)

Interchange of ξ and λ makes the integral in this region identical to the one over region (2). Thus giving the same result, namely that the highest order term in z is

F 25

$$-z^4 \int_0^V d\lambda \int_V^{z-v} d\xi F_3$$

(4) Region (4)

Here λ is always large and so is $\lambda + \xi$. We can then use the asymptotic expansion for $J_0(\lambda)$ and $\frac{J_1(\lambda + \xi)}{\lambda + \xi}$. In addition $\lambda \gg \xi$. Using these approximations and interchanging the order of integration we get a sum of terms as follows:

F 26

$$i_j^3 I_1^{(4)} \sim \frac{2}{\pi} \int_0^V J_0(\xi) d\xi \int_{z-v}^{z-\xi} d\lambda \lambda^{j-2} \xi^i z^{4-i-j} \left\{ [\sin(\lambda + \xi) - \cos(\lambda + \xi)] \right. \\ \left. \times (\sin \lambda + \cos \lambda) \cos \sigma(\lambda - \xi) \right\}$$

The terms in curly brackets become

$$A = \frac{1}{2} \left\{ \cos(\sigma-1)\xi [\sin \sigma \lambda - \cos(2+\sigma)\lambda] - \sin(\sigma-1)\xi [\cos \sigma \lambda + \sin(2+\sigma)\lambda] \right. \\ \left. - \cos(\sigma-1)\xi [\sin \sigma \lambda + \cos(2-\sigma)\lambda] + \sin(\sigma+1)\xi [\cos \sigma \lambda + \sin(2-\sigma)\lambda] \right\}$$

It will be sufficient for our purposes to analyze a general term of the form

F 27

$$z^{4-i-j} \int_0^V J_0(\xi) \xi^i d\xi \int_{z-v}^{z-\xi} d\lambda \lambda^{j-2} \cos \alpha \lambda = i_j I'$$

where α is a constant

(a) when $j = 0$

$$\int_{z-v}^{z-\xi} d\lambda \frac{\cos \alpha \lambda}{\lambda^2} \approx \alpha \left[\frac{\cos \alpha(z-v)}{\alpha^2(z-v)^2} - \frac{\cos \alpha(z-\xi)}{\alpha^2(z-\xi)^2} + \frac{\cos \alpha(z-\xi)}{\alpha(z-\xi)} - \frac{\cos \alpha(z-v)}{\alpha(z-v)} \right]$$

where the fact that $z \gg 1$, $z-v \gg 1$, $z-\xi \gg 1$ has been taken into account.

It is clear that integration with respect to ξ will not change the z dependence. From eq. F27 we see then that we will get at most a z^3 term.

$$(b) \quad j = 1, 2, 3, 4$$

It is now easy to see that all the terms will be of order lower than in z^4 . The λ integration will give terms of order $1/z$, a constant, z , z^2 for $j = 1, 2, 3, 4$ respectively. Again the ξ integration will not change this z dependence. Hence we finally get at most a term of order z^2 (for $j = 1, 2, 3, 4; i=0$). Then region (4) does not contribute at all to z^4 terms.

(5) Region 5

If we interchange the "names" of λ and ξ , the integral is identical to ${}^3I_1^{(4)}$ (eq. F26) and we get the same results.

(6) Region 6

The integral is

$${}^3I_1^{(6)} = \int_v^{z-v} d\lambda \int_v^{z-\lambda} d\xi \quad F_3 \quad f_3(-\xi)$$

Now λ , ξ and $\lambda + \xi$ are all large. Thus we can

replace all the Bessel functions by their asymptotic expansion. When this is done, the integrand is a function of $\lambda + \xi$ and $\lambda - \xi$. We then let

$$\lambda + \xi = \mu \quad ; \quad \lambda - \xi = \nu \quad ; \quad \lambda = \frac{\mu + \nu}{2}$$

and get

F28

$${}^3I_1^{(0)} \approx \frac{\sqrt{2}}{2\pi^{3/2}} \left\{ \int_{-z+2\nu}^0 d\nu \int_{2\nu-\nu}^z d\mu + \int_0^{z-2\nu} d\nu \int_{2\nu+\nu}^z d\mu \right\} \\ \times \frac{(\sin \mu - \cos \mu) \left(\sin \frac{\mu+\nu}{2} - \cos \frac{\mu+\nu}{2} \right) \left(\sin \frac{\mu-\nu}{2} + \cos \frac{\mu-\nu}{2} \right) \cos \sigma \nu}{\mu^{5/2}} \\ \times [-z^4 + 2z^3(\mu+3\nu) - 6z^2\nu^2 - 4z\nu^3 - \nu^4]$$

where use has been made of the fact that $\mu \gg \nu$ so that

$$(\mu^2 - \nu^2)^{-1/2} \approx \frac{1}{\mu} \quad \left(1 + \frac{\nu^2}{2\mu^2}\right) \sim \frac{1}{\mu}$$

By combining the trigonometric functions we finally get for the integrand of eq. F28

F29

$$\frac{A}{\mu^{5/2}} (1 - \cos 2\mu + \sin 2\mu + \sin \mu \cos \nu - \cos \mu \cos \nu) \cos \sigma \nu \\ [-z^4 + 2z^3(\mu+3\nu) - 6z^2\nu^2 - 4z\nu^3 - \nu^4]$$

It is evident by inspection that the highest order term in z must come from the "1" in the curved bracket. Let us then analyze this term. We have then a sum of integrals of the form

$$F30 \quad \left\{ \int_{-z+2v}^0 dv \int_{2v-v}^z d\mu + \int_0^{z-2v} dv \int_{2v+v}^z d\mu \right\} \frac{\cos \sigma v}{\mu^{5/2}} z^{4-j} v^j$$

where $j = 0, 1, 2, 3, 4$

The only term omitted is the term containing $\frac{z^3 \cos \sigma v}{\mu^{3/2}}$ which we will treat separately.

Integrating F30 w.r. to μ we get

$$F31 \quad \left\{ \int_{-z+2v}^0 dv \left[\frac{1}{z^{3/2}} - \frac{1}{(2v-v)^{3/2}} \right] + \int_0^{z-2v} dv \left[\frac{1}{z^{3/2}} - \frac{1}{(2v+v)^{3/2}} \right] \right\} \cos \sigma v z^{4-j} v^j$$

Considering now first only the $z^{-3/2}$ terms in the above expression it is easy to see that we get at most terms of the form

$$z^{5/2} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (z \pm 2v)$$

For the other two terms a change of variable (from v to $\lambda = 2v \pm v$) produces terms in z^4 but multiplied by $\frac{\cos 2\sigma v}{(2v)^{3/2}}$ which are clearly very small. Indeed as in 'I', this can be taken into account by extending the limits in eq. F23.

As for the term

$$\left\{ \int_{-z+2v}^0 dv \int_{2v-v}^z d\mu + \int_0^{z-2v} dv \int_{2v+v}^z d\mu \right\} z^3 \frac{\cos \sigma v}{\mu^{3/2}}$$

we get after the " μ " integration

$$\begin{aligned}
& A z^3 \left\{ \int_{-z+2v}^0 dv \left[\frac{1}{z^{1/2}} - \frac{1}{(2v-z)^{1/2}} \right] + \int_0^{z-2v} dv \left[\frac{1}{z^{1/2}} - \frac{1}{(2v+v)^{1/2}} \right] \right\} \cos \sigma v \\
& = A z^3 \left\{ \frac{1}{\sqrt{z}} [\sin(-z+2v) - \sin(z-2v)] - \text{terms of order } z^{-1/2} \cos 2\sigma v \dots \right\} \\
& \sim O(z^{5/2})
\end{aligned}$$

We have thus shown that we remain with only z^4 times the integral F23 with extended limits. We may indeed let v in that integral $\rightarrow \infty$ without affecting the convergence of the integral nor changing its value appreciably.

APPENDIX G

Evaluation of $I_{1a} + 2 I_{2a,1}$

As can be seen from eqs. E1 and E5 it is natural to combine these two expressions since they have the same products of Bessel functions and cosine functions in their integrands. If we combine these expressions we get:

$$\begin{aligned}
 I_2 &= I_{1a} + 2 I_{2a,1} \\
 &= \frac{\beta}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi \left[3z^4 - 4z^3(2\lambda + \xi) + 6z^2\lambda(2\xi + \lambda) - 12z\lambda^2\xi - \lambda^4 + 4\lambda^3\xi \right] \right. \right. \\
 &\quad \left. + \int_\lambda^z d\xi \left[3z^4 - 4z^3(2\lambda + \xi) + 6z^2\lambda(2\xi + \lambda) - 12z\lambda^2\xi + 6\lambda^2\xi^2 + 4\lambda\xi^3 + 3 \right. \right. \\
 G1 \quad &\quad \left. \left. - \int_{-z+\lambda}^0 d\xi \left[3z^4 - 8z^3(\lambda - \xi) + 6z^2(\lambda - \xi)^2 - (\lambda - \xi)^4 \right] \right\} \right. \\
 &\quad \left. \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda - \xi) \right\} \\
 &= \frac{\beta}{192} \left\{ {}^1I_2 + {}^2I_2 + {}^3I_2 \right\}
 \end{aligned}$$

If we now compare I_2 to I_1 we note that the only important difference between the two is that in I_2 we have $\cos \sigma(\lambda - \xi)$ and in I_1 we had $\cos \sigma(\lambda + \xi)$. This does not affect the proof in regions (1) and (2). In region (3) it

changes the picture. Consider first $I_2^{(3)}$, where the right superscript (3) indicates the region of integration.

As in $I_1^{(3)}$, we can expand $J_0(\lambda)$ and $J_0(\xi)$, then change variables from λ and ξ to $\mu = \lambda + \xi$ and $\nu = \lambda - \xi$, and get a sum of terms of the form (compare eq. E3).

$$G-2 \quad I_2^{(3)} = A z^{4-i-j} \int_0^{z-\nu} d\nu \int_{\nu+2\nu}^{2z-\nu} d\mu \frac{(\mu+\nu)^j (\mu-\nu)^i}{z^{i+j} \sqrt{\mu^2 - \nu^2}} \frac{J_1(\nu)}{\nu} (\cos \nu + \sin \mu) \cos \sigma \nu$$

The term containing $\sin \mu \cos \sigma \nu$ will lead to the same results as obtained for $I_1^{(3)}$, namely a convergent integral, highest order term of order z^4 .

The other term containing $\cos \nu \cos \sigma \nu$ requires more careful scrutiny.

Now we are interested in the term which has z^4 as coefficient since it will be the highest order term in z . This means that $i = j = 0$. Note also that $\mu \gg \nu$ so that

$$(\mu^2 - \nu^2)^{-1/2} \sim \frac{1}{\mu} \left(1 + \frac{\nu^2}{2\mu^2}\right) \sim \frac{1}{\mu}$$

The integral is then, after integration w.r. to μ ,

$$A z^4 \int_0^{z-\nu} d\nu \frac{J_1(\nu)}{\nu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu] [\ln(2z-\nu) - \ln(\nu+2\nu)]$$

(a) Consider first only the term containing $\ln(2z-\nu)$

we can expand $\ln(2z - \nu)$ since $2z \gg \nu$

$$\ln(2z - \nu) = \ln 2z + \ln\left(1 - \frac{\nu}{2z}\right) \sim \ln 2z - \frac{\nu}{2z}$$

We now have two integrals

$$\Lambda_1 = A z^4 / \ln 2z \int_0^{z-\nu} d\nu \frac{J_1(\nu)}{\nu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu]$$

$$\Lambda_2 = - \frac{A z^4}{2z} \int_0^{z-\nu} d\nu J_1(\nu) [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu]$$

For Λ_1 , we must consider two cases based on the following expression.*

$$\begin{aligned} \int_0^{\beta} J_\nu(\alpha x) \cos \beta x \frac{dx}{x} &= \frac{1}{\nu} \cos\left(\nu \sin^{-1} \frac{\beta}{\alpha}\right) & \beta \leq \alpha \\ &= \frac{\alpha^\nu \cos \frac{\nu \pi}{2}}{\nu (\beta + \sqrt{\beta^2 - \alpha^2})^\nu} & \beta \geq \alpha \end{aligned}$$

First consider $\sigma \gg 2$. Then the integral in Λ_1 , is

$$\int_0^{z-\nu} d\nu \frac{J_1(\nu)}{\nu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu] = \int_0^\infty d\nu - \int_{z-\nu}^\infty d\nu ;$$

for $\sigma \gg 2$ the first integral is clearly zero. The second integral is such that we can replace $J_1(\nu)$ by its asymptotic

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 743 #6.693, 2

expansion. We get:

$$- \frac{2}{\sqrt{\pi}} \int_{z-v}^{\infty} dv \frac{(\sin v - \cos v)}{v^{3/2}} [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$$\text{which by eqs. F18 and F19} \quad \simeq \quad \frac{\left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\} \alpha (z-v)}{(z-v)^{3/2}}$$

$$\text{and clearly} \quad \frac{1/h z \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\} \alpha z}{(z-v)^{3/2}} \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty$$

Thus Λ_1 , behaves as z^α where $\alpha \leq 4$ for $\sigma \geq 2$

For $\sigma < 2$ we get

$$\Lambda_1 = [A z^4 / h z] \cos(\sin^{-1}(\sigma-1))$$

which is of higher order in z than z^4

As for Λ_2 it is clear that it will be at most of order z^3 .

(b) The term containing $1/h(\nu+2\nu)$

We have the integral

$$\begin{aligned} \int_0^{z-v} dv \frac{J_1(\nu)}{\nu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu] 1/h(\nu+2\nu) \\ = \int_0^{z-v} dv \varphi(\nu) = \int_0^v \varphi d\nu + \int_v^{z-v} \varphi d\nu \end{aligned}$$

The first integral is certainly finite. For the second, we expand $\frac{J_1(\nu)}{\nu}$ asymptotically and get terms of the form

$$\int_V^{z-V} \frac{1/h(\nu+2\nu)}{\nu^{3/2}} \cos \alpha \nu (\sin \nu - \cos \nu) d\nu$$

but $\int_0^\infty \frac{1/h \nu}{\nu} \sin \alpha \nu d\nu = -\frac{\pi}{2} (C + \ln a) \quad *$ $C = \text{Euler's const.}$

and the integral we are considering must be smaller and is indeed finite.

We notice that for $\sigma > 2$ we have now a term in z^4 . But since the integral is convergent we can include it in the term in z^4 coming from $I_2^{(0)}$ by extending its limits as usual.

We have considered only the case $i=0, j=0$. It is easily shown that the other values of i and j contribute terms of order lower than z^4 . This is done in Appendix V.

We conclude that for $\sigma \gg 2$ we have z^4 term

$$G3 \quad 3 z^4 \int_0^z d\lambda \int_0^\lambda d\xi J_0(\xi) J_0(\lambda) \frac{J_1(\lambda-\xi)}{\lambda-\xi} \cos \sigma(\lambda-\xi)$$

For $\sigma < 2$ we have a term of order $z^4 / h_2 z$

$$G4 \quad [3 z^4 / h_2 z] \cos [\sin^{-1}(\sigma-1)]$$

For $I_2^{(3)}$ we can proceed in a more direct manner as follows:

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 598 #4.421, 1

First change from ξ to $-\xi$, hence

$$G_5 \quad {}^3I_2 = -\frac{B}{192} \int_0^z d\lambda \int_0^{z-\lambda} d\xi \left[3z^4 - 8z^3(\lambda+\xi) + 6z^2(\lambda+\xi)^2 - (\lambda+\xi)^4 \right] \\ \times \frac{J_0(\lambda) J_0(\xi) J_1(\lambda+\xi)}{\lambda+\xi} \cos \sigma(\lambda+\xi)$$

Let now

$$\lambda + \xi = \mu \quad ; \quad \lambda = \lambda \quad ; \quad \xi = \mu - \lambda$$

Then

$${}^3I_2 = -\frac{B}{192} \int_0^z d\mu \int_0^\mu d\lambda \left[3z^4 - 12z^3\mu + 6z^2\mu^2 - \mu^4 \right] \\ \times \frac{J_0(\lambda) J_0(\mu-\lambda) J_1(\mu)}{\mu} \cos \sigma\mu$$

Using Gradshteyn & Ryzhik, page 668 #6.513, 7

$$G_6 \quad \int_0^\mu d\lambda J_0(\lambda) J_0(\mu-\lambda) = \sin \mu$$

we get

$${}^3I_2 = -\frac{B}{2(192)} \int_0^z d\mu \left[3z^4 - 12z^3\mu + 6z^2\mu^2 - \mu^4 \right] \\ \times \left[\sin(1+\sigma)\mu + \sin(1-\sigma)\mu \right] \frac{J_1(\mu)}{\mu}$$

But

$$\begin{aligned} * \int_0^\infty J_\nu(ax) \sin bx \, dx &= \frac{\sin(\nu \sin^{-1} \frac{b}{a})}{\sqrt{a^2 - b^2}} & b < a \\ &= \infty \text{ or } 0 & b = a \\ &= \frac{a^\nu \cos \frac{\nu\pi}{2}}{\sqrt{b^2 - a^2} (b + \sqrt{a^2 - b^2})^\nu} & b > a \end{aligned}$$

$$\begin{aligned}
 ** \int_0^\infty J_\nu(ax) \sin bx \frac{dx}{x} &= \frac{1}{\nu} \sin(\nu \sin^{-1} \frac{b}{a}) & b \leq a \\
 &= \frac{a^\nu \sin \frac{\nu \pi}{2}}{\nu (b + \sqrt{b^2 - a^2})^\nu} & b \geq a
 \end{aligned}$$

Hence it is clear that only the z^4 term need be retained, all the others being of lower order in z . Indeed for $z \rightarrow \infty$ we can integrate exactly.

Concluding, we have shown that the only terms to be taken into account in I_2 are

$$\begin{aligned}
 I_2 \sim \frac{3\beta z^4}{192} &\left\{ \int_0^z d\lambda \int_0^z d\xi \frac{J_0(\lambda) J_0(\xi) J_1(\lambda-\xi)}{\lambda-\xi} \cos \sigma(\lambda-\xi) \right. \\
 &\left. - \int_0^z d\lambda \int_0^{z-\lambda} d\xi \frac{J_0(\lambda) J_0(\xi) J_1(\lambda+\xi)}{\lambda+\xi} \cos \sigma(\lambda+\xi) \right\}
 \end{aligned}$$

where we get different results according to whether $\sigma \geq 2$ or $\sigma < 2$ as shown previously.

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 730 #6.671, 1

**Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 743 #6.693, 1

APPENDIX H

Evaluation of $I_{1b,3} + 2I_{2b,1}$

From eqs. E4 and E7 we get

$$\begin{aligned}
 I_3 &= I_{1b,3} + 2I_{2b,1} \\
 &= \frac{B}{192} \left\{ \int_0^z d\lambda \left\{ \int_0^\lambda d\xi [2z^3(2\xi - \lambda) + 6z^2(\lambda^2 - \lambda\xi) + 2\lambda^3\xi - 2\lambda^4] \right. \right. \\
 &\quad \left. \left. + \int_{-z+\lambda}^0 d\xi [2z^3\lambda - 2z\lambda^3] \right\} \right. \\
 &\quad \left. \times \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda + \xi) \right\}
 \end{aligned}$$

H1

Comparing this expression to ${}^0I_{2a,2}$ and ${}^3I_{2a,2}$ it is clear that all the results arrived at there are valid also here.

The fact that now we have $\cos \sigma(\lambda + \xi)$ instead of $\cos \sigma(\lambda - \xi)$ as in $I_{2a,2}$ makes it even more certain that no term in H1 is of order higher than z^3 . This is illustrated by eq. I3 which would in the case here always contain a factor $\cos \sigma\mu$ instead of $\cos \sigma\gamma$ which reduces the value of the integrand for the integration.

Hence we conclude that we can neglect I_3 contributions for the case of large z .

APPENDIX I

Evaluation of $I_{2a,2}$ and $I_{1b,1}$

(1) $I_{2a,2}$

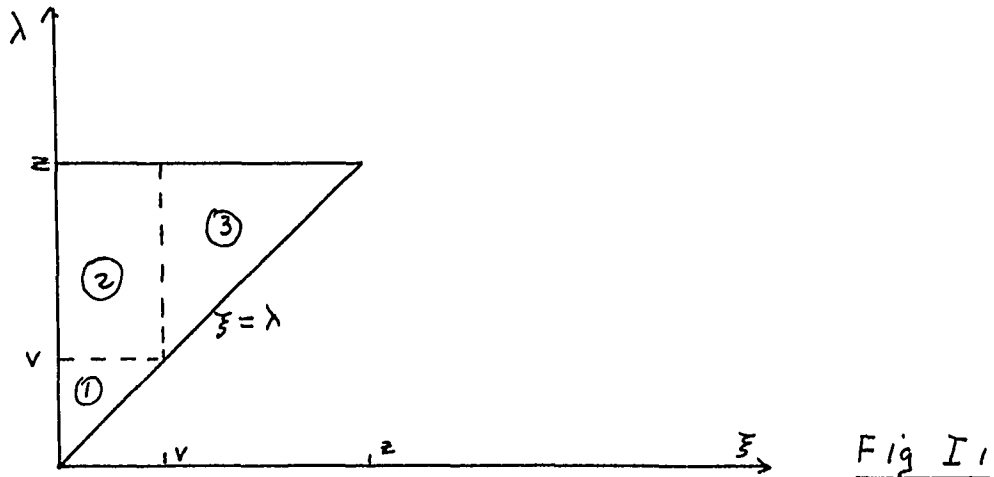
Following the same pattern as in I_1 we will show that the z^4 terms are the highest order terms in z .

(a) $I_{2a,2}$

From eq. E6

$$I_1 \quad I_{2a,2} = - \int_0^z d\lambda \int_0^\lambda d\xi \left[z^4 + z^3(2\lambda - 4\xi) + 6z^2\lambda(\lambda - \xi) + 2z\lambda^3 - 2\xi\lambda^3 + \lambda^4 \right] \times \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda - \xi)$$

The area of integration can again be subdivided as shown



It is evident that in area (1), the highest order in z is

$$I_2 \quad z^4 \int_0^v d\lambda \int_0^\lambda d\xi \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda - \xi) \text{ which is clearly convergent.}$$

In area (2) we again subdivide into two domains:

$$\int_V^{2V} d\lambda \int_0^V d\xi + \int_{2V}^{\infty} d\lambda \int_0^V d\xi$$

The first integral is finite and can be incorporated into eq. 12 by extending the limits.

In the second integral we know that both λ and $\lambda - \xi$ are large, so that $J_0(\lambda - \xi)$ and $J_1(\lambda)$ can be replaced by their asymptotic expansion.

When we do this we get eventually a sum of terms of the form (inverting also the order of integration and letting $\sqrt{\lambda - \xi} \sim \lambda^{1/2} (1 - \frac{\xi}{\lambda})$)

$$z^{4-i-j} \int_0^V J_0(\xi) d\xi \xi^i \int_{2V}^{\infty} \lambda^{j-2} \left\{ \frac{\cos}{\sin} \right\} \alpha \xi \left\{ \frac{\cos}{\sin} \right\} \beta \lambda d\lambda$$

where $i = 0, 1, 2$; $j = 0, 1, 2, 3, 4$

The integration with respect to λ will produce terms of the form

$$z^{4-i-j} \left\{ \frac{\sin}{\cos} \right\} \alpha \xi \left[\frac{\left\{ \frac{\sin}{\cos} \right\} \beta z}{\beta z^{2-j}} - \frac{\left\{ \frac{\sin}{\cos} \right\} (\beta 2V)}{\beta (2V)^{2-j}} \right]$$

The integration with respect to ξ will give only a $\left\{ \frac{\sin}{\cos} \right\} \alpha z$. Thus it is clear by inspection that all the terms will be of order in z lower than 4 with the possible exception of

a term for $i = j = 0$, using the ν dependent part only. But this term, in z^4 , can be included in eq. I2 by extending the limits. It is a small term anyhow because of the $(2\nu)^{-2}$ factor.

In area (3) ξ and λ are both large and we can use the asymptotic expansion for $J_0(\xi)$ and $J_1(\lambda)$. When this is done we change variables, letting

$$\lambda + \xi = \mu \quad ; \quad \lambda - \xi = \nu$$

and we eventually obtain a sum of terms of the form

$$I_3 \quad z^K \int_0^{z-\nu} d\nu \int_{\nu+2\nu}^{2z-\nu} d\mu J_0(\nu) (\sin \nu - \cos \mu) \cos \sigma \nu \nu^\ell \mu^m d\mu$$

where ℓ ranges from 0 to 3 and m from -2 to +2

while K is less well defined, since eq. I3 does not represent exactly any of the terms of eq. I1. We need a table like table F1 of I_1 to show the term correspondence:

Table I1

i	j	K	λ, μ
0	0	4	$\mu^{-2} (1 - \frac{\nu}{\mu})$
1	0	3	$\sim \frac{1}{2\mu}$
0	1	3	$\sim \frac{1}{2\mu} (1 + \frac{\nu^2}{2\mu^2})$
1	1	2	$\sim \frac{1}{4} (1 - \frac{\nu}{\mu})$
0	2	2	$\sim \frac{1}{4} (1 + \frac{\nu}{\mu} + \frac{\nu^2}{2\mu^2} + \dots)$
0	3	1	$\sim \frac{1}{8} (\mu + 2\nu + \frac{\nu^2}{\mu} + \dots)$
1	3	0	$\sim \frac{1}{16} (\mu^2 + \nu\mu - \frac{\nu^2}{2} + \dots)$

We now examine eq. I3 keeping in mind the value of K , the exponent of z that goes with the values of i, j or ℓ and m .

(1) $K=4$

Clearly the highest order term in I3 is the $\sin \nu \cos \sigma \nu$ term. The " μ " integration will produce a difference of two terms, one of the order of z^{-1} , the other ν^{-1} . The next integration, w.r. to ν , will not change this fact. We end up then with a term of order z^3 , which we can neglect, and a term of order $z^4(\nu)^{-1}$ which can again be included in eq. I2 since the integral itself must be convergent and small due to $\frac{1}{\nu}$ factor.

(2) $K=3$

Here we get, from Table I/ , " $\frac{1}{\mu}$ " factors. Upon integration these give $1/n z$. The " ν " integration does not change the z dependence and we get at most a term of order $z^3/n z$ which we neglect since it is of lower order than z^4 .

(3) $K=2$

Upon integration w.r. to μ we get terms of order z . We end up with terms at most of order z^3 .

(4) $K=1, K=0$

The " μ " integration produces respectively z^2 and z^3

dependence and after ν integration we again have terms of at most order z^3 .

We can conclude that only $i=0, j=0$ contribute to the z^4 term.

$$(b) {}^2I_{2a,2}$$

from eq. E6

$${}^2I_{2a,2} = \int_0^z d\lambda \int_\lambda^z d\xi \left[z^4 + 2z^3(\lambda - 2\xi) + 6z^2\xi(\xi - \lambda) + 2z(-2\xi^3 + 3\lambda\xi^2 - \lambda^3) - \lambda^4 - 2\lambda\xi^3 + \xi^4 \right] \frac{J_0(\xi)J_0(\lambda - \xi)J_1(\lambda)}{\lambda} \cos \sigma(\lambda - \xi)$$

The area of integration is now

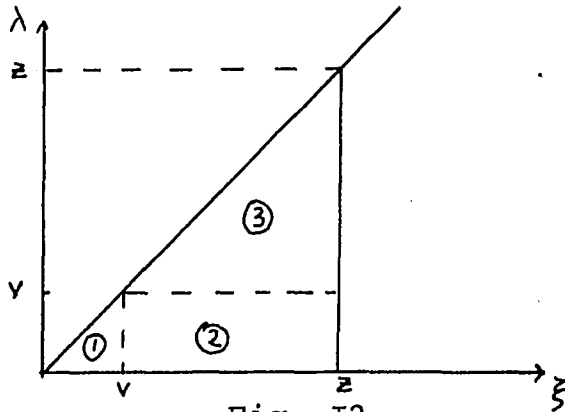


Fig. I2

The proof is similar to the one for $'I_{2a,2}$

In region (1) we get exactly the same results as for $'I_{2a,2}$. In region (2) we simply interchange the "names" of λ and ξ and get the same results as for $'I_{2a,2}^{(2)}$.

In region (3), since both λ and ξ are large, $J_0(\xi)$ and $J_1(\lambda)$ are replaced by their asymptotic expansion. Then

the change of variables

$$\lambda + \xi = \mu \quad ; \quad \lambda - \xi = \nu \quad ; \quad (\mu \gg \nu)$$

gives a sum of terms of the form

$$\int_{-z+\nu}^0 d\nu \int_{2\nu-\nu}^{2z+\nu} d\mu (\sin \nu - \cos \mu) J_0(\nu) \nu^\ell \mu^m \cos \sigma \nu$$

where again

$$m = -2, -1, 0, 1, 2 \quad ; \quad \ell = 0, 1, 2, 3$$

and we have, except for sign, the eq. I3 again, which leads

us to the same results as for $I_{2\alpha,2}^{\circ}$, namely that the highest order term is

$$I_4 \quad z^4 \int_0^z d\lambda \int_\lambda^z d\xi \frac{J_0(\xi) J_0(\lambda - \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda - \xi)$$

$$(c) \quad {}^3 I_{2\alpha,2}$$

From eq. E6 (omitting the constant coefficient) and

letting $\xi \rightarrow -\xi$

$${}^3 I_{2\alpha,2} = \int_0^z d\lambda \int_0^{z-\lambda} d\xi [z^4 - 2z^3(\lambda + 2\xi) + 6z^2\xi(\xi + \lambda) + 2z(+2\xi^3 + 3\lambda\xi^2 - \lambda^3) - \lambda^4 + 2\lambda\xi^3 - 2\lambda^3\xi + \xi^4] \frac{J_0(\xi) J_0(\lambda + \xi) J_1(\lambda)}{\lambda} \cos \sigma(\lambda + \xi)$$

Here it is possible to effect one integration exactly.

Let

$$\lambda + \xi = \mu \quad ; \quad \lambda = \lambda$$

Then we get

$$^3I_{2a,1} = \int_0^z d\mu \int_0^\mu d\lambda \frac{J_0(\mu-\lambda) J_0(\mu) J_1(\lambda)}{\lambda} \cos \sigma\mu [-2\lambda(\mu-z)^3 + (\mu-z)^4]$$

but

$$\int_0^\mu d\lambda \frac{J_1(\lambda) J_0(\mu-\lambda)}{\lambda} = J_1(\mu) \quad *$$

$$\int_0^\mu d\lambda J_0(\mu-\lambda) J_1(\lambda) = J_0(\mu) - \cos \mu \quad **$$

Then

$$I5 \quad ^3I_{2a,2} = \int_0^z d\mu J_0(\mu) \cos \sigma\mu [-2(\mu-z)^3 (J_0(\mu) - \cos \mu) - (\mu-z)^4 J_1(\mu)]$$

For the integration w.r. to μ , we proceed as follows:

$$I6 \quad ^3I_{2a,2} = \int_0^v d\mu + \int_v^z d\mu$$

where again v is such that for $\mu > v$, $J_0(\mu)$ and $J_1(\mu)$ may be replaced by their asymptotic expansion.

For the first integral we have the following types of integrals:

$$\left. \begin{aligned} (1) \quad & \int_0^v d\mu J_0^2(\mu) \mu^i z^{3-i} \cos \sigma\mu \\ (2) \quad & \int_0^v d\mu J_0(\mu) \cos \sigma\mu \cos \mu \mu^i z^{3-i} \\ (3) \quad & \int_0^v d\mu J_0(\mu) J_1(\mu) \mu^i z^{4-i} \cos \sigma\mu \end{aligned} \right\} \quad i = 0, 1, 2, 3$$

$$i = 0, 1, 2, 3, 4$$

All the integrals, the z term apart, are convergent; the highest

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 678 #6.533, 1

**Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 668 #6.513, 8

order term in z , namely z^4 , arises from integral 3 with $i=0$. It is also the only one of that order in z .

For the second integral in I6, i.e. with limits v to z , we can use the asymptotic expansion of the Bessel function. We have then three terms:

$$(1) \frac{z}{\pi} \int_v^z \frac{(\sin \mu + \cos \mu)^2}{\mu} (\mu^3 - 3\mu^2 z + 3\mu z^2 - z^3) \cos \sigma \mu d\mu$$

$$(1) \frac{z}{\pi} \int_v^z \frac{(\sin \mu + \cos \mu)^2}{\mu} (\mu^3 - 3\mu^2 z + 3\mu z^2 - z^3) \cos \sigma \mu d\mu$$

which will give no term of order in z higher than z^3 as is evident by inspection.

$$(2) \frac{z}{\pi} \int_v^z \frac{(\sin \mu + \cos \mu)}{\sqrt{\mu}} [\cos(\sigma+1)\mu - \cos(\sigma-1)\mu] (\mu^3 - 3\mu^2 z + 3\mu z^2 - z^3) d\mu$$

here again the highest order term in z is of order z^3

(using expression F18, F19)

$$(3) \frac{z}{\pi} \int_v^z \frac{\cos 2\mu \cos \sigma \mu}{\mu} (\mu - z)^4 d\mu$$

Here we have one term of order z^4 due to the z^4 coefficient multiplying the integral (lower limit). But this is clearly of the type that we can include in integral (3) of the first type above by extending the limit.

Thus for ${}^3I_{2\alpha,2}$ it has been shown that we must keep only

I 7

$$z^4 \int_0^z d\lambda \int_0^{z-\lambda} J_0(\xi) J_0(\lambda+\xi) \frac{J_1(\lambda)}{\lambda} \cos \sigma(\lambda+\xi) d\xi$$

the integral itself being convergent.

$$(2) I_{1b,1}$$

As shown in eq. E2 $I_{1b,1}$ is of the form,

$$\int_0^z d\lambda \left\{ \int_0^\lambda d\xi f_1 + \int_\lambda^z d\xi f_2 + \int_{-z+\lambda}^0 d\xi f_3 \right\} \frac{J_1(\xi) J_0(\lambda) J_0(\lambda-\xi)}{\xi} \cos \sigma(\lambda+\xi)$$

By interchanging the order of integration this becomes

$$\begin{aligned} \int_0^z d\xi \left\{ \left[\int_\xi^z d\lambda f_1 + \int_0^\xi d\lambda f_2 \right] \frac{J_1(\xi) J_0(\lambda) J_0(\lambda-\xi)}{\xi} \cos \sigma(\lambda+\xi) \right. \\ \left. + \int_0^{z-\xi} d\lambda \frac{J_1(\xi) J_0(\lambda) J_0(\lambda+\xi)}{\xi} f_3 \cos \sigma(\lambda-\xi) \right\} \end{aligned}$$

In this form all we need to do is to relabel λ, ξ and ξ, λ and $I_{1b,1}$ is nearly the same as $I_{2a,2}$ except for the fact that here we have $\cos \sigma(\lambda+\xi), \cos \sigma(\lambda-\xi)$ instead of $\cos \sigma(\lambda-\xi), \cos \sigma(\lambda+\xi)$ in

$I_{2a,2}$. But this fact clearly does not affect the proof for the first two terms (those involving f_1 and f_2). The third term is similar to ${}^3I$, and the analysis would be nearly identical to the one used for ${}^3I$. Hence we will get that the highest order terms in $I_{1b,1}$ and the only ones to be retained are

$$\begin{aligned} \frac{B}{192} z^4 \left\{ \int_0^z d\lambda \int_0^{z-\lambda} d\xi \frac{J_1(\xi) J_0(\lambda) J_0(\lambda-\xi)}{\xi} \cos \sigma(\lambda+\xi) \right. \\ \left. + \int_0^z d\lambda \int_0^{z-\lambda} d\xi \frac{J_1(\xi) J_0(\lambda) J_0(\lambda+\xi)}{\xi} \cos \sigma(\lambda-\xi) \right\} \end{aligned}$$

I 8

APPENDIX J

Evaluation of I_{3a}

In eq. E14 label the successive terms of the sum of integrals as

$$I_{3a}^1, I_{3a}^2, \dots \text{ etc.}$$

We will now evaluate each of these terms separately, omitting the constant coefficient.

$$(1) I_{3a}^1$$

J1

$$I_{3a}^1 = \int_0^z d\lambda \int_0^{z-\lambda} \frac{1}{2} (\lambda + \xi)(z - \lambda - \xi) J_0(\lambda) J_0(\xi) J_0(\lambda + \xi) \cos \sigma(\lambda + \xi) d\xi$$

Let

$$\lambda + \xi = \mu \quad ; \quad \lambda = \lambda \quad ; \quad \xi = \mu - \lambda$$

Then

$$\begin{aligned} I_{3a}^1 &= 2 \int_0^z d\mu \int_0^\mu d\lambda \mu(z - \mu) J_0(\mu) J_0(\lambda) J_0(\mu - \lambda) \cos \sigma \mu \\ &= \int_0^z d\mu \mu(z - \mu) J_0(\mu) [\sin(1 + \sigma)\mu + \sin(1 - \sigma)\mu] \end{aligned}$$

by (G6)

We thus have two types of terms:

$$(a) \quad z \int_0^z \mu J_0(\mu) \sin \alpha \mu d\mu = z \int_0^v d\mu + z \int_v^z d\mu$$

The first term with integral from σ to v is clearly

convergent. In the second term we may replace $J_0(\mu)$ by

$$\frac{\sin \mu + \cos \mu}{\sqrt{\mu}} \quad \text{and the integral is then of order } \sqrt{z} \left\{ \frac{\sin}{\cos} \right\} \beta z.$$

In both cases the order is much lower than z^4 .

$$(b) \int_0^z \mu^2 J_0(\mu) \sin \alpha \mu d\mu = \int_0^V d\mu + \int_V^z d\mu$$

The first integral is clearly finite. In the second integral we can use the asymptotic expansion for $J_0(\mu)$ and we get after integration 4 terms of type $z^{\frac{3}{2}} \left\{ \frac{\sin}{\cos} \right\} \beta z$ (B a constant)

Hence I_{3a}' will not contribute

$$J2 \quad (2) \quad I_{3a}^2 = \int_0^z d\lambda \int_0^{z-\lambda} d\xi (z-3\lambda-3\xi)(z-\lambda-\xi) J_0(\lambda) J_0(\xi) J_0(z-2\lambda-2\xi) \\ \times \cos \sigma(\lambda+\xi)$$

$$\text{Again } \mu = \lambda + \xi, \quad \lambda = \lambda$$

$$I_{3a}^2 = \frac{1}{2} \int_0^z d\mu (z-3\mu)(z-\mu) J_0(z-2\mu) [\sin(1+\sigma)\mu + \sin(1-\sigma)\mu]$$

Clearly the highest order term will be the one containing z^2 as a factor. This term is

$$z^2 \int_0^z d\mu J_0(z-2\mu) \sin \alpha \mu = z^2 \int_{-z}^z d\xi J_0(\xi) \left[\sin \frac{\alpha}{2} z \cos \alpha \frac{\xi}{2} - \cos \frac{\alpha}{2} z \sin \alpha \frac{\xi}{2} \right]$$

The second term, involving $\sin \alpha \frac{\xi}{2}$, being odd gives zero. We remain with

$$z^2 \sin \frac{\alpha}{2} z \int_{-z}^z J_0(\xi) \cos \frac{\alpha}{2} \xi d\xi$$

For $z \rightarrow \infty$ the integral is finite for all α except $\frac{\alpha}{2} = 1$

for $\frac{\alpha}{2} = 1$ or $|1 \pm \sigma| = 2$, i.e. $\sigma = 1, 3$. But the integral behaves as $z^{1/2}$ and with its coefficients we get at most $z^{5/2}$ dependence.

(3) Evaluation of I_{3a}^3

$$I_{3a}^3 = \int_0^z d\lambda \int_0^{z-\lambda} d\xi (z-\lambda-\xi)^2 (z-2\lambda-2\xi) [J_1(\lambda+\xi) + J_1(z-2\lambda-2\xi)] \\ \times J_1(\xi) J_0(\lambda) \cos \sigma(\lambda+\xi)$$

Let $\lambda + \xi = \mu$; $\lambda = \lambda$ and after integration w.r. to λ

$$I_{3a}^3 = \frac{1}{2} \int_0^z d\mu (z-\mu)^2 (z-2\mu) [J_1(\mu) + J_1(z-2\mu)] [\sin(1+\sigma)\mu + \sin(1-\sigma)\mu]$$

It is clear that the z^3 term will be the highest order term in z . Separate now the sum of two Bessel functions.

$$(a) \quad z^3 \int_0^z J_1(\mu) [\sin(1+\sigma)\mu + \sin(1-\sigma)\mu] d\mu$$

Since $\sigma \neq 0$, $\sigma \pm 1 \neq 1$ and the integral is a constant*

$$\int_0^\infty J_1(\alpha x) \sin \beta x dx = 0 \quad \text{if } \beta < \alpha$$

$$\infty \quad \text{if } \beta = \alpha$$

$$\frac{\alpha}{\sqrt{\beta^2 - \alpha^2} (\beta + \sqrt{\beta^2 - \alpha^2})} \quad \text{if } \beta > \alpha$$

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 730. #6.671 (1)

$$(b) \quad z^3 \int_0^z J_1(z-2\mu) d\mu [\sin(1+\sigma)\mu + \sin(1-\sigma)\mu]$$

Let $\xi = z - 2\mu$ then we get

$$z^3 \int_{-z}^z d\xi J_1(\xi) \left[\sin \frac{1+\sigma}{2} z \cos \frac{1+\sigma}{2} \xi - \cos \frac{1+\sigma}{2} z \sin \frac{1+\sigma}{2} \xi \right. \\ \left. + \sin \frac{1-\sigma}{2} z \cos \frac{1-\sigma}{2} \xi - \cos \frac{1-\sigma}{2} z \sin \frac{1-\sigma}{2} \xi \right]$$

We have to take into account only the even integrands so that we get

$$z^3 \left\{ \cos \frac{1+\sigma}{2} z \int_{-z}^z d\xi J_1(\xi) \sin \frac{1+\sigma}{2} \xi \right. \\ \left. + \cos \frac{1-\sigma}{2} z \int_{-z}^z d\xi J_1(\xi) \sin \frac{1-\sigma}{2} \xi \right\}$$

and we again get $A z^3$ of order smaller than z^4 .

$$(4) \quad I_{3a}^4$$

$$I_{3a}^4 = \int_0^z d\mu \int_0^\mu d\lambda (z-\mu)^2 (z-2\mu) \mu J_0(\lambda) J_0(\mu-\lambda) J_0(\mu) \cos \sigma\mu \\ = \frac{1}{2} \int_0^z d\mu (z-\mu)^2 (z-2\mu) \mu J_0(\mu) [\sin(1+\sigma)\mu + \sin(1-\sigma)\mu] d\mu \\ = \int_0^V d\mu + \int_V^z d\mu$$

In the first integral it is evident that the highest order term in z must be z^3 .

In the second integral we can use the asymptotic expansion for $J_0(\mu)$ and get

$$\int_V^z (z^3 - 4\mu z^2 + 5\mu^2 z - 2\mu^3) \sqrt{\mu} (\sin \mu - \cos \mu) [\sin(1+\sigma)\mu + \sin(1-\sigma)\mu] d\mu$$

By E14 and E15 it is immediately clear that the highest order term is of order $z^{7/2} \left\{ \frac{\sin}{\cos} \right\} \beta z$ which is of lower order than z^4 .

$$(5) I_{3a}^5$$

$$I_{3a}^5 = \int_0^z d\lambda \int_0^{z-\lambda} d\xi (z-\lambda-\xi)^2 (z-2\lambda-2\xi)^2 J_0(z-2\lambda-2\xi) J_0(\lambda) J_0(\xi) \cos \sigma(\lambda+\xi)$$

$$\text{Let } \mu = \lambda + \xi \quad ; \quad \lambda = \lambda$$

and integrate with respect to λ to obtain

$$\frac{1}{2} \int_0^z d\mu (z-\mu)^2 (z-2\mu)^2 J_0(z-2\mu) [\sin(1+\sigma)\mu + \sin(1-\sigma)\mu]$$

$$\text{Now let } \xi = z - 2\mu$$

Then the integrals become a sum of two terms each of the form

$$\int_{-z}^z d\xi \xi^2 \frac{(z+\xi)^2}{4} J_0(\xi) \left[\sin \frac{\alpha}{2} z \cos \frac{\alpha}{2} \xi + \cos \frac{\alpha}{2} z \sin \frac{\alpha}{2} \xi \right]$$

The term in z^2 (which will be of highest order) is then

$$\begin{aligned} & \frac{2z^2}{4} \int_0^z d\xi \xi^2 J_0(\xi) \sin \frac{\alpha}{2} z \cos \frac{\alpha}{2} \xi \\ &= z^2 \left[\int_0^V d\xi + \int_V^z d\xi \right] \end{aligned}$$

The first integral is clearly finite. In the second we can use the asymptotic expansion for $J_0(\xi)$ and end up with a term of the form

which is again of order in z lower than 4 .

(6) Evaluation of I_{3a}^6

$$I_{3a}^6 = \int_0^z d\lambda \int_0^{z-\lambda} d\xi (z-\lambda-\xi)(\lambda+\xi) \left[H_0(\lambda+\xi) J_1(\lambda+\xi) - H_1(\lambda+\xi) J_0(\lambda+\xi) \right] \\ \times J_0(\lambda) J_0(\xi) \cos \sigma(\lambda+\xi)$$

Again let $\lambda + \xi = \mu$; $\lambda = \lambda$ and λ will occur only in the last two Bessel functions so that we can integrate w.r. to λ and remain with

$$\frac{1}{2} \int_0^z d\mu \mu (z-\mu) \left[H_0(\mu) J_1(\mu) - H_1(\mu) J_0(\mu) \right] \left[\sin(1+\sigma)\mu + \sin(1-\sigma)\mu \right] \\ = \int_0^v d\mu + \int_v^z d\mu$$

In the first integral we get at most a term of order z . In the second integral we may use the asymptotic expansion. Noting that, for large μ

$$J_3 \quad H_0(\mu)_{\mu \gg 1} \sim \sqrt{\frac{2}{\pi\mu}} \sin\left(\mu - \frac{\pi}{4}\right) + \frac{2}{\pi} \left(\frac{1}{\mu} \dots\right) \dots$$

$$J_4 \quad H_1(\mu)_{\mu \gg 1} \sim -\sqrt{\frac{2}{\pi\mu}} \cos\left(\mu - \frac{\pi}{4}\right) - \frac{2}{\pi} \dots$$

From which it follows that

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 933 #8.554

$$H_0(\mu) J_1(\mu) - H_1(\mu) J_0(\mu) \sim \frac{2}{\pi\mu}$$

Then the second integral above becomes

$$\int_V^Z d\mu (Z\mu - \mu^2) \frac{2}{\pi\mu} [\sin(1+\sigma)\mu + \sin(1-\sigma)\mu]$$

$$\sim Z \left\{ \frac{\sin}{\cos} \right\} \beta Z$$

(7) Evaluation of I_{3a}^7

$$I_{3a}^7 = \int_0^Z d\lambda \int_0^{Z-\lambda} d\xi (Z-\lambda-\xi)(Z-2\lambda-2\xi) [H_0(Z-2\lambda-2\xi) J_1(Z-2\lambda-2\xi) \\ - H_1(Z-2\lambda-2\xi) J_0(Z-2\lambda-2\xi)] J_0(\lambda) J_0(\xi) \cos \sigma(\lambda+\xi)$$

As before let $\lambda + \xi = \mu$; $\lambda = \lambda$, integrate w.r. to λ , then let $\xi = Z - 2\mu$; we get

$$\frac{1}{2} \int_{-Z}^Z d\xi \frac{(Z+\xi)}{2} \xi [H_0(\xi) J_1(\xi) - H_1(\xi) J_0(\xi)] [\sin \alpha(Z-\xi) + \sin \beta(Z-\xi)]$$

The $Z\xi$ term gives integrals of the form

$$Z \cos \alpha Z \int_0^Z \xi d\xi [H_0(\xi) J_1(\xi) - H_1(\xi) J_0(\xi)] \sin \frac{\alpha}{2} \xi \\ \sim Z \cos \alpha Z \left[\int_0^V F d\xi + \frac{2}{\pi} \int_V^Z \sin \alpha \xi d\xi \right]$$

(F is the integrand as shown just above)

which is at most of order $Z \left\{ \frac{\sin}{\cos} \right\} \beta Z$

(8) Evaluation of I_{3a}^8

$$I_{3a}^8 = \frac{\pi}{2} \int_0^z d\lambda \int_0^{z-\lambda} d\xi (z-\lambda-\xi)^2 (\lambda+\xi) (z-2\lambda-2\xi) \left[H_0(\lambda+\xi) J_1(\lambda+\xi) \right. \\ \left. - H_1(\lambda+\xi) J_0(\lambda+\xi) \right] J_0^*(\lambda) J_0(\xi) \cos \sigma(\lambda+\xi)$$

Again let $\lambda + \xi = \mu$; $\lambda = \lambda$ integrate w.r. to λ
and get

$$\frac{\pi}{4} \int_0^z d\mu \left[z^3 \mu - 4z^2 \mu^2 + 5z \mu^3 - 2\mu^4 \right] \left[H_0(\mu) J_1(\mu) - H_1(\mu) J_0(\mu) \right] \\ \times \left[\sin(1+\sigma)\mu + \sin(1-\sigma)\mu \right]$$

Considering only term in $z^3 \mu$, the integral is the same as for I_{3a}^6 , except of course that here we have z^3 instead of z . Thus we get as highest power of z , at most terms like $z^3 \left\{ \frac{\sin}{\cos} \right\} \beta z$.

(9) Evaluation of I_{3a}^9

$$I_{3a}^9 = \int_0^z d\lambda \int_0^{z-\lambda} d\xi (z-\lambda-\xi)^2 (z-2\lambda-2\xi)^2 J_0^*(\lambda) J_0(\xi) \cos \sigma(\lambda+\xi) \\ \times \left[H_0(z-2\lambda-2\xi) J_1(z-2\lambda-2\xi) - H_1(z-2\lambda-2\xi) J_0(z-2\lambda-2\xi) \right]$$

Again let $\lambda + \xi = \mu$; $\lambda = \lambda$ integrate w.r. to λ
then let $\xi = z - 2\mu$ and get

$$\sim \int_{-z}^z d\xi \xi^2 (z+\xi)^2 \left[H_0(\xi) J_1(\xi) - H_1(\xi) J_0(\xi) \right] \left[\sin \alpha (z-\xi) + \sin \beta (z-\xi) \right]$$

The highest order term is $z^2 \xi^2$.

We consider then integral of the form

$$\begin{aligned}
& z^2 \int_0^z \xi^2 \left[H_0(\xi) J_1(\xi) - H_1(\xi) J_0(\xi) \right] \sin \alpha z \cos \alpha \xi \, d\xi \\
& = z^2 \left[\int_0^V F \, d\xi + \int_V^z \xi \sin \alpha z \cos \alpha \xi \, d\xi \right]
\end{aligned}$$

where F is the integrand above.

The first integral must be of order z^2 , the second of order $z^3 \left\{ \frac{\sin}{\cos} \right\} z$.

In conclusion we have shown that I_{3a} does not contain any term of order 4 in z or higher. We can then neglect this part.

APPENDIX K

Evaluation of I_{3b}

It will now be useful to separate I_{3b} into two parts as follows: From eq. E15

$$\begin{aligned}
 I_{3b} = & \frac{1}{32} \left\{ \int_0^z d\lambda \int_0^\lambda d\xi (\lambda - \xi) (z - \lambda) J_0(\xi) J_0(\lambda - \xi) \right. \\
 & \left[-J_0(\lambda) - (z - \lambda) J_1(\lambda) - (z - \lambda)\lambda J_0(\lambda) + \frac{\pi}{2}(z - \lambda)\lambda [H_0(\lambda) J_1(\lambda) \right. \\
 & \left. \left. - H_1(\lambda) J_0(\lambda)] \right] \cos \sigma(\lambda + \xi) \right. \\
 & + \int_0^z d\lambda \int_0^\lambda d\xi (\lambda - \xi) (z - \lambda) J_0(\xi) J_0(\lambda - \xi) \\
 & \left[J_0(z - 2\lambda) - (z - \lambda) J_1(z - 2\lambda) - (z - \lambda)(z - 2\lambda) J_0(z - 2\lambda) \right. \\
 & \left. - (z - \lambda)(z - 2\lambda) \frac{\pi}{2} [H_0(z - 2\lambda) J_1(z - 2\lambda) - H_1(z - 2\lambda) J_0(z - 2\lambda)] \right] \cos \sigma(\lambda + \xi) \Big\} \\
 = & I_{3b}^1 + I_{3b}^2 + I_{3b}^3 + I_{3b}^4 + I_{3b}^5 + I_{3b}^6 + I_{3b}^7 + I_{3b}^8
 \end{aligned}$$

It is nearly intuitively clear that the first four terms are of order in z lower than z^4 , especially if one compares these terms to the corresponding terms of I_{3a} . For completeness sake the proof of this assertion is given in Appendix W.

For the last four terms the procedure has to be varied

somewhat, due to presence of $z-2\lambda$ as an argument of the Bessel functions.

$$I_{3b}^5 = \int_0^z d\lambda \int_0^\lambda d\xi (\lambda-\xi) (z-\lambda) J_0(\xi) J_0(\lambda-\xi) J_0(z-2\lambda) \cos \sigma(\lambda+\xi) d\xi$$

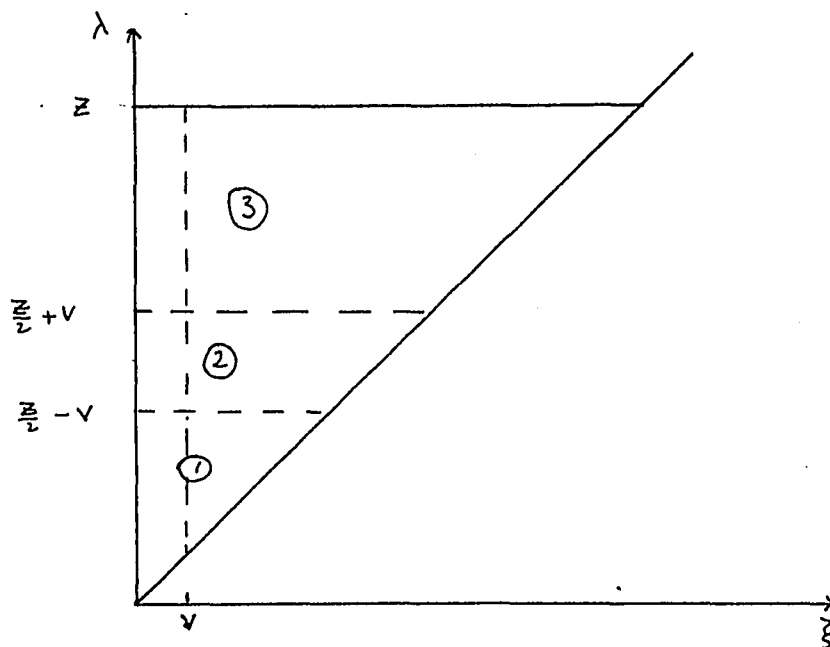


Fig. K1

We now subdivide the domain of integration as shown in the figure K1 and consider only the $z\lambda$ term (the highest order in z)

(1) First consider the domain (2). We can again subdivide it into two domains such that

$$z \int_{z/2-v}^{z/2+v} d\lambda \lambda J_0(z-2\lambda) \left[\int_0^v d\xi + \int_v^\lambda d\xi \right] J_0(\xi) J_0(\lambda-\xi) \cos \sigma(\lambda+\xi) d\xi$$

(a) Consider first the domain where $0 \leq \xi \leq v$. Here λ is

large and much greater than ξ so that we can expand $J_0(\lambda - \xi)$.

Thus the ξ integral becomes a sum of terms of the form

$$\int_0^V J_0(\xi) \left\{ \frac{\sin}{\cos} \right\} (\sigma \pm 1) \lambda \left\{ \frac{\sin}{\cos} \right\} (\sigma \pm 1) \xi d\xi = K \left\{ \frac{\sin}{\cos} \right\} (\sigma \pm 1) \lambda$$

where K is a constant

The λ integral for this part is

$$z \int_{\frac{z}{2} - V}^{\frac{z}{2} + V} \lambda J_0(z - 2\lambda) \left\{ \frac{\sin}{\cos} \right\} (\sigma \pm 1) \lambda d\lambda$$

let $z - 2\lambda = \mu$, then the integral becomes

$$\frac{z}{2} \int_{-2V}^{2V} (z - \mu) J_0(\mu) \left\{ \frac{\sin}{\cos} \right\} \beta z \left\{ \frac{\sin}{\cos} \right\} \beta \mu d\mu$$

which is at most of order $z^2 \left\{ \frac{\sin}{\cos} \right\} \beta z$ where $\beta = \sigma \pm 1$.

(b) Now consider the domain $v \leq \xi \leq \lambda$

Now ξ is large, but $\lambda - \xi$ can have any value. Here

$J_0(\xi)$ can be replaced by $\frac{2}{\sqrt{\pi \xi}} (\sin \xi + \cos \xi)$. The integral w.r.

to ξ is then

$$\left\{ \frac{\sin}{\cos} \right\} \beta \lambda \int_v^\lambda d\xi J_0(\lambda - \xi) \left\{ \frac{\cos}{\sin} \right\} \alpha \xi d\xi$$

where α and β are constants

dependent on σ .

Let now $\lambda - \xi = \mu$; $\lambda = \lambda$; $\xi = \lambda - \mu$ and we

get

$$\left\{ \frac{\sin}{\cos} \right\} \beta \lambda \int_0^{\lambda - v} d\xi J_0(\mu) \left\{ \frac{\cos}{\sin} \right\} \alpha \mu \left\{ \frac{\sin}{\cos} \right\} \alpha \lambda d\xi$$

which like similar integrals encountered earlier is equal to

$$K \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \gamma \lambda \quad \text{where } K \text{ and } \gamma \text{ are constants.}$$

The integration with respect to λ is now similar to the one encountered in part (a) so that we again get a term of the form $z^2 \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \beta z$.

(2) In domains (1) and (3), $z - 2\lambda$ is always larger than V so that $J_0(z - 2\lambda)$ can be replaced by

$$\frac{2}{\sqrt{\pi} \sqrt{z - 2\lambda}} [\sin(z - 2\lambda) + \cos(z - 2\lambda)]$$

By the same reasoning as in domain (2) we then arrive at a z dependence of at most $z^{3/2} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \beta z$.

$$\underline{I_{3b}^6, I_{3b}^7, I_{3b}^8}$$

For I_{3b}^6 we have one more z factor, thus giving at most a term of order $z^3 \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \beta z$.

For I_{3b}^7 we notice that although we now have as highest order factor of the Bessel functions, the term $z^2 \lambda(z - 2\lambda)$, this becomes (in domain (2), we integrate w.r. to λ)

$z^2 \mu(z - \mu)$ which gives eventually, at most, a term of order

$$z^3 \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \beta z$$

The same will hold for I_{3b}^8 when use is made of J_3 and J_4 in the appropriate places.

We conclude by stating that $\bar{I}_{3b}$ will not contribute to the term we are investigating.

APPENDIX L

Evaluation of Coefficients of Remaining Terms

In all the integrals we begin in the same way, by replacing $\int_{0,1} (\lambda \pm \xi)$ by an integral representation, namely:

$$L1 \quad J_0(\lambda \pm \xi) = \frac{2}{\pi} \int_0^{\pi/2} \cos [(\lambda \pm \xi) \cos \theta] d\theta = \frac{2}{\pi} \int_0^1 \frac{\cos(\lambda \pm \xi)x}{\sqrt{1-x^2}} dx^*$$

$$L2 \quad J_1(\lambda \pm \xi) = \frac{2}{\pi} (\lambda \pm \xi) \int_0^{\pi/2} \sin^2 \theta \cos [(\lambda \pm \xi) \cos \theta] d\theta \\ = \frac{2}{\pi} (\lambda \pm \xi) \int_0^1 \sqrt{1-x^2} \cos(\lambda \pm \xi)x dx$$

Start now with I_1 ,

$$I_1 = \int_0^\infty d\lambda \int_0^\infty d\xi J_0(\lambda) J_0(\xi) \frac{J_1(\lambda - \xi)}{\lambda - \xi} \cos \sigma(\lambda + \xi) \\ = \frac{2}{\pi} \int_0^1 dx \sqrt{1-x^2} \int_0^\infty J_0(\lambda) d\lambda \int_0^\infty J_0(\xi) \cos \sigma(\lambda + \xi) \cos x(\lambda - \xi) d\xi \\ = \frac{1}{\pi} \int_0^1 dx \sqrt{1-x^2} \int_0^\infty d\lambda J_0(\lambda) \int_0^\infty d\xi J_0(\xi) \left\{ \cos(\sigma-x)\lambda \cos(\sigma+x)\xi - \sin(\sigma-x)\lambda \sin(\sigma+x)\xi \right. \\ L3 \quad \left. + \cos(\sigma+x)\lambda \cos(\sigma-x)\xi - \sin(\sigma+x)\lambda \sin(\sigma-x)\xi \right\}$$

We now make use of the formulas

$$L4 \quad \int_0^\infty J_0(ax) \sin(bx) dx = \begin{matrix} 0 & \text{if} & 0 < b < a \\ \infty & \text{if} & a = b \\ \frac{1}{\sqrt{b^2 - a^2}} & \text{if} & 0 < a < b \end{matrix}$$

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 952 #8.411 (4)

L 5

$$\begin{aligned}
 \int_0^\infty J_0(ax) \cos bx \, dx &= \frac{1}{\sqrt{a^2 - b^2}} \quad \text{if } 0 < b < a \quad * \\
 &= \infty \quad \text{if } a = b \\
 &= 0 \quad \text{if } 0 < a < b
 \end{aligned}$$

Now apply these to expression L3. We have then several cases possible, $\sigma > 2$, $1 < \sigma < 2$, $\sigma < 1$

Consider only $\sigma > 2$

By L5 integration of $\cos$ terms gives zero and we remain with only the sine terms.

If we now consider I_2 , we first note that for $z \rightarrow \infty$, the limits are identical to those of I_1 . Using the same integral representation for $J_1(\lambda + \xi)$, the expression corresponding to L3 is

$$\begin{aligned}
 I_2 = \frac{1}{\pi} \int_0^1 \sqrt{1-x^2} \, dx \int_0^\infty J_0(\lambda) \, d\lambda \int_0^\infty d\xi J_0(\xi) \{ &\cos(\sigma-x)\lambda \cos(\sigma+x)\xi \\
 &+ \sin(\sigma-x)\lambda \sin(\sigma+x)\xi + \cos(\sigma+x)\lambda \cos(\sigma-x)\xi + \sin(\sigma+x)\lambda \sin(\sigma-x)\xi \}
 \end{aligned}$$

When $\sigma > 2$, the terms containing cosines are zero and the other terms, the sine terms, cancel out with I_1 . Thus it is clear that, for $\sigma > 2$, $I_1 + I_2 = 0$. Rather than examine the cases $\sigma < 2$, we will now examine first all the I_i for $\sigma > 2$.

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 731 #6.671 (7, 8)

By the same procedure as used for I_1 , we get

L7

$$I_{3,4} = \frac{1}{\pi} \int_0^1 \sqrt{1-x^2} dx \int_0^\infty J_0(\lambda) d\lambda \int_0^\infty J_0(\xi) \left\{ \cos(\sigma-x)\lambda \cos(\sigma-x)\xi \right. \\ \left. \pm \sin(\sigma-x)\lambda \sin(\sigma-x)\xi + \cos(\sigma+x)\lambda \cos(\sigma+x)\xi \pm \sin(\sigma+x)\lambda \sin(\sigma+x)\xi \right\}$$

where the upper sign corresponds to I_3 and lower sign to

I_4 . Again the cosine terms are zero and we get for $I_3 - I_4$ using L4.

$$I_3 - I_4 = \frac{2}{\pi} \int_0^1 \sqrt{1-x^2} dx \int_0^\infty J_0(\lambda) d\lambda \left[\frac{\sin(\sigma-x)\lambda}{\sqrt{(\sigma-x)^2 - 1}} + \frac{\sin(\sigma+x)\lambda}{\sqrt{(\sigma+x)^2 - 1}} \right] \\ = \frac{2}{\pi} \int_0^1 \sqrt{1-x^2} dx \left\{ \frac{1}{(\sigma-x)^2 - 1} + \frac{1}{(\sigma+x)^2 - 1} \right\} \\ = \frac{2}{\pi} \int_0^1 \sqrt{1-x^2} dx \left\{ \frac{x^2 + \sigma^2 - 1}{(x^2 + \sigma^2 - 1)^2 - 4\sigma^2 x^2} \right\} \\ = \frac{2}{\pi} \int_0^1 \frac{\sqrt{(1-z)z}}{z} \left\{ \frac{z + \sigma^2 - 1}{[z - (\sigma+1)^2][z - (\sigma-1)^2]} \right\} dz$$

We have thus a sum of two integrals

$$A = \int_0^1 dz \frac{\sqrt{z(1-z)}}{(z-\alpha)(z-\beta)} \quad B = \int_0^1 \frac{(\sigma^2-1)\sqrt{z(1-z)}}{z(z-\alpha)(z-\beta)} dz$$

From G. & R. 83

$$\int_0^1 \frac{z dz}{\sqrt{(1-z)z}} = -\sqrt{z(1-z)} \Big|_0^1 + \frac{1}{2} \int_0^1 \frac{dz}{\sqrt{z(1-z)}}$$

$$\int_0^1 \frac{dz}{\sqrt{z(1-z)}} = -\sin^{-1}(-2z+1) \Big|_0^1 = +\pi$$

$$\begin{aligned} \int_0^1 \frac{dz}{(z-\alpha)\sqrt{z(1-z)}} &= - \int_{-\frac{1}{\alpha}}^{-\frac{1}{\alpha-1}} \frac{dt}{\sqrt{-1+(1-2\alpha)t+\alpha(1-\alpha)t^2}} \\ &= - \frac{\pi}{\sqrt{\alpha(\alpha-1)}} \end{aligned}$$

Then

$$A = \frac{1}{\alpha-\beta} \left\{ -\left(\frac{1}{2}-\alpha\right)\pi + \pi\sqrt{\alpha(\alpha-1)} \right\} + \frac{\pi}{\beta-\alpha} \left\{ \beta - \frac{1}{2} + \sqrt{\beta(\beta-1)} \right\}$$

2) B

$$(\sigma^2-1) \int_0^1 \frac{\sqrt{z(1-z)}}{z(z-\alpha)(z-\beta)} dz = (\sigma^2-1) \left\{ \int_0^1 \left[\frac{\sqrt{z(1-z)}}{z\alpha\beta} + \frac{\sqrt{z(1-z)}}{\alpha(\alpha-\beta)(z-\alpha)} - \frac{\sqrt{z(1-z)}}{\beta(\alpha-\beta)(z-\beta)} \right] dz \right\}$$

We need now *

$$\int_0^1 \frac{\sqrt{z(1-z)}}{z} dz = \sqrt{z(1-z)} \Big|_0^1 + \frac{1}{2} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} = -\frac{\pi}{2}$$

$$B = (\sigma^2-1)\pi \left\{ -\frac{1}{2\alpha\beta} + \frac{1}{\alpha(\alpha-\beta)} \left(\alpha - \frac{1}{2} + \sqrt{\alpha(\alpha-1)} \right) - \frac{1}{\beta(\alpha-\beta)} \left(\beta - \frac{1}{2} + \sqrt{\beta(\beta-1)} \right) \right\}$$

When we now replace α and β by their value

$(\sigma+1)^2$ and $(\sigma-1)^2$ we get after some simple

algebra

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 84 #2.267 (1)

L 8

$$I_3 - I_4 = 2 + \sqrt{\sigma(\sigma+2)} - \sqrt{\sigma(\sigma-2)}$$

For I_5, I_6 we proceed as for the others, replacing $\int_0 (\lambda \mp \xi)$ by the representation L2 to get

L 9

$$I_{5,6} = \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^\infty d\lambda \frac{J_1(\lambda)}{\lambda} \int_0^\infty d\xi J_0(\xi) \left\{ \cos(\sigma-x)\lambda \cos(\sigma-x)\xi \right. \\ \left. \pm \sin(\sigma-x)\lambda \sin(\sigma-x)\xi + \cos(\sigma+x)\lambda \cos(\sigma+x)\xi \pm \sin(\sigma+x)\lambda \sin(\sigma+x)\xi \right\}$$

where the upper signs correspond to I_5 , the lower ones to I_6 . As for I_1 and I_2 the cosine terms are zero for $\sigma > 0$ and the other terms will just cancel out. Thus $I_5 + I_6 = 0$ for $\sigma > 0$.

For $I_{7,8}$ we proceed in the same manner and get

L 10

$$I_{7,8} = \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^\infty d\xi \frac{J_1(\xi)}{\xi} \int_0^\infty d\lambda J_0(\lambda) \left\{ \cos(\sigma-x)\lambda \cos(\sigma+x)\xi \right. \\ \left. \mp \sin(\sigma-x)\lambda \sin(\sigma+x)\xi + \cos(\sigma+x)\lambda \cos(\sigma-x)\xi \mp \sin(\sigma+x)\lambda \sin(\sigma-x)\xi \right\}$$

where again the upper signs correspond to I_7 , the lower ones to I_8 . For $\sigma > 2$ we again get zero.

We conclude that for $\sigma > 2$, the leading term is of the form

L 11

$$\frac{3B}{192} z^4 [2 + \sqrt{\sigma(\sigma+2)} - \sqrt{\sigma(\sigma-2)}] = \frac{1}{8} \frac{\hbar}{Q_0^2 M \omega} z^4 [2 + \sqrt{\sigma(\sigma+2)} - \sqrt{\sigma(\sigma-2)}]$$

APPENDIX M

A. Transformation of $R_1 + S_1$ from q, q', q'', q''' to c, b, d, u

(1) In S_1 let $q \leftrightarrow q'$

$$M1 \quad S_1 = - \int_0^P dq' \int_0^P dq \int_0^q dq'' \int_0^{q'''} dq''' \sum_r q' J_r(2q'' - 2q''') J_r(2q'' - 2q) J_r(2q - 2q') J_{r+1}(2q''' - 2q) \times \cos \sigma(2q + 2q' - 2q'' - 2q''')$$

The integrand of S_1 is now identical to the integrand of R_1 .

Next

$$\begin{aligned} \int_0^P dq' \int_0^P dq \int_0^q dq'' \int_0^{q'''} dq''' &= \int_0^P dq \left[\int_0^q dq' + \int_q^P dq' \right] \left[\int_0^P dq'' - \int_q^P dq'' \right] \int_0^{q'''} dq''' \\ S_1 &= - \left\{ \int_0^P dq' \int_0^q dq \int_0^P dq'' - \int_0^P dq \int_0^q dq' \int_q^P dq'' + \int_0^P dq \int_q^P dq' \int_0^P dq'' \right. \\ &\quad \left. - \int_0^P dq \int_q^P dq' \int_q^P dq'' \right\} \int_0^{q'''} dq''' F \end{aligned}$$

where F is the integrand of M1.

Combining now $R_1 + S_1$, we get

$$\begin{aligned} R_1 + S_1 &= \int_0^P dq \int_0^P dq' \int_q^P dq'' \int_0^{q'''} dq''' F - \int_0^P dq \int_q^P dq' \int_0^P dq'' \int_0^{q'''} dq''' F \\ &= A - B \end{aligned}$$

(2) Transformation of A

We will express A in terms of the following variables.

$$c = 2q''' - 2q'$$

$$d = 2q'' - 2q$$

$$b = 2q - 2q'$$

$$\mu = 2q'$$

with an "intermediary" variable $\lambda = 2q$ which will eventually disappear.

Thus the integrand will become

$$F = \frac{\mu}{2} \sum_r J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \cos \sigma(c+d)$$

(a) Starting with

$$A = \int_0^P dq \int_q^P dq'' \int_0^P dq' \int_0^{q'} dq''' F$$

let

$$2q''' - 2q' = c$$

$$\text{and } \mu = 2q'$$

$$A = \int_0^P dq \int_q^P dq'' \left\{ \int_{-2p}^{2q''-2p} dc \int_{-c}^{2p} d\mu F + \int_{2q''-2p}^0 dc \int_{-c}^{2q''-c} d\mu F + \int_0^{2q''} dc \int_0^{2q''-c} d\mu F \right\}$$

(b) Considering only $\int_0^P dq \int_q^P dq''$ let

$$2q'' - 2q = d \quad ; \quad 2q = \lambda$$

Then

$$A = \frac{1}{16} \int_0^{2p} dd \int_0^{2p-d} d\lambda \left[\int_{-2p}^{-2p+d+\lambda} dc \int_{-c}^{2p} d\mu F + \int_{-2p+d+\lambda}^0 dc \int_{-c}^{d+\lambda-c} d\mu F \right. \\ \left. + \int_0^{d+\lambda} dc \int_0^{d+\lambda-c} d\mu F \right]$$

(c) Now interchange the order of λ and c ,

then let

$$b = \lambda - \mu \quad ; \quad \mu = \mu$$

To get finally

$$A = \frac{1}{16} \int_0^{2p} F dd \left\{ \int_{-2p+d}^0 dc \left[\int_{2p-d+2c}^{2p-d+c} db \int_{-c}^{2p-d-b} d\mu + \int_{-d}^{2p-d+2c} db \int_{2p-d+c-b}^{2p-d+b} d\mu \right. \right. \\ \left. + \int_{c-d}^{-d} db \int_{2p-d+c-b}^{2p} d\mu \right] + \int_{-2p}^{-2p+d} dc \left[\int_{-d}^{2p-d+c} db \int_{-c}^{2p-d-b} d\mu \right. \\ \left. + \int_0^{-d} db \int_{-c}^{2p} d\mu + \int_{-2p}^c db \int_{-b}^{2p} d\mu \right] + \int_{-2p+d}^0 dc \left[\int_c^{2p+c-d} db \int_{-c}^{2p+c-d-b} d\mu \right. \\ \left. + \int_{c-d}^c db \int_{-b}^{2p+c-d-b} d\mu \right] + \int_0^d dc \left[\int_0^{2p-d} db \int_0^{2p-d-b} d\mu \right. \\ \left. + \int_{c-d}^0 db \int_{-b}^{2p-d-b} d\mu \right] + \int_d^{2p} dc \int_{c-d}^c db \int_{-b}^{2p+c-d-b} d\mu \left. \right\}$$

M2

(3) Transformation of B

(a) We use the same sequence of transformations, i.e.,
first let

$$2q''' - 2q' = c \quad ; \quad 2q' = \mu$$

then let

$$2q'' - 2q = d \quad ; \quad 2q = \lambda$$

and follow by interchanging the order of λ and c to
finally let

$$\lambda - \mu = b \quad ; \quad \mu = \mu$$

and obtain

Thus

$$\begin{aligned}
B = \int_0^{2p} dd \left\{ \int_{-2p+d}^0 dc \left[\int_{2p-d+2c}^{2p-d+c} db \int_{-c}^{2p-d-b} d\mu + \int_{-d}^{2p-d+c} db \int_{2p-d+c-b}^{2p-d-b} d\mu + \int_{c-d}^{-d} db \int_{2p-d+c-b}^{2p} d\mu \right] \right. \\
+ \int_{-2p}^{-2p+d} dc \left[\int_{-d}^{2p-d+c} db \int_{-c}^{2p-d-b} d\mu + \int_c^{-d} db \int_{-c}^{2p} d\mu + \int_{-2p}^c db \int_{-b}^{2p} d\mu \right] \\
+ \int_{-p+\frac{d}{2}}^0 dc \left[\int_c^0 db \int_{-c}^{-c-b} d\mu + \int_{c-d}^c db \int_{-b}^{-c-b} d\mu \right] + \int_{-2p+d}^{-p+\frac{d}{2}} dc \left[\int_c^{2p+2c-d} db \int_{-c}^{2p+c-d} d\mu \right. \\
+ \left. \int_{c-d}^c db \int_{-b}^{2p+c-d-b} d\mu \right] + \int_0^{-p+\frac{d}{2}} dc \left[\int_{2p+2c-d}^{2p+c-d} db \int_{-c}^{2p-d-b} d\mu + \int_{c-d}^{2p+2c-d} db \int_{2p+c-d-b}^{2p-d-b} d\mu \right. \\
+ \left. \int_{-p+\frac{d}{2}}^{-2p+d} dc \left[\int_0^{2p+c-d} db \int_{-c}^{2p-d-b} d\mu + \int_{c-d}^0 db \int_{-c-b}^{2p-d-b} d\mu \right] + \int_0^d dc \int_{c-d}^0 db \int_{-b}^{2p-d-b} d\mu \right. \\
+ \left. \left. \int_{-2p+d}^0 dc \int_{c-d}^0 db \int_{-c-b}^{2p-d-b} d\mu \right] \right\} \\
+ \int_{-2p}^0 dd \left\{ \int_{-2p}^d dc \left[\int_{2p+2c-d}^{2p+c} db \int_{-c}^{2p-b} d\mu + \int_0^{2p+2c-d} db \int_{2p+c-d-b}^{2p-b} d\mu + \int_{c-d}^0 db \int_{2p+c-d-b}^{2p} d\mu \right] \right. \\
+ \int_{-p+\frac{d}{2}}^d dc \int_{c-d}^0 db \int_{-c}^{-c-b} d\mu + \int_{-2p}^{-p+\frac{d}{2}} dc \int_{c-d}^{2p+2c-d} db \int_{-c}^{2p+c-d-b} d\mu + \int_{-p+\frac{d}{2}}^d dc \left[\int_{2p+2c-d}^{2p+c} db \int_{-c}^{2p-b} d\mu \right. \\
+ \left. \int_{c-d}^{2p+2c-d} db \int_{2p+c-d-b}^{2p-b} d\mu \right] + \int_{-p+\frac{d}{2}}^{-2p} dc \left[\int_0^{2p+c} db \int_{-c}^{2p-b} d\mu + \int_{c-d}^0 db \int_{-c-b}^{2p-b} d\mu \right] + \int_{-2p}^d dc \int_{c-d}^0 db \int_{-c-b}^{2p-b} d\mu \left. \right\}
\end{aligned}$$

M3

(b) In the part of B where the " d " integration is from $-2p$ to 0 , let

$$c \rightarrow -c$$

$$d \rightarrow -d$$

$$b \rightarrow -b$$

so that this part becomes

$$-\int_0^{2p} dd \left\{ \int_d^{2p} dc \left[\int_{-2p+c}^{-2p+2c-d} db \int_c^{2p+b} d\mu + \int_{-2p+2c-d}^0 db \int_{2p-c+d+b}^{2p+b} d\mu + \int_0^{c-d} db \int_{2p-c+d-b}^{2p} d\mu \right] \right.$$

$$+ \int_d^{\frac{d}{2}-p} dc \int_0^{c-d} db \int_c^{c+b} d\mu + \int_{p+\frac{d}{2}}^{2p} dc \int_{-2p+2c-d}^{c-d} db \int_c^{2p-c+d+b} d\mu$$

$$+ \int_{p+\frac{d}{2}}^d dc \left[\int_{-2p+c}^{-2p+2c-d} db \int_c^{2p+b} d\mu + \int_{-2p+2c-d}^{c-d} db \int_{2p+b-c+d}^{2p+b} d\mu \right]$$

$$+ \int_{2p}^{p+\frac{d}{2}} dc \left[\int_{-2p+c}^0 db \int_c^{2p+b} d\mu + \int_0^{c-d} db \int_{c+b}^{2p+b} d\mu \right] + \int_0^{2p} dc \int_0^{c-d} db \int_{c+b}^{2p+b} d\mu \left. \right\}$$

$$\times \frac{\mu}{2} J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \cos \sigma(c+d)$$

(c) Next we group all those terms containing $\frac{d}{2}$ in the limits for the " c " integration in the following way

$$1) \text{ if } \int_0^{p+\frac{d}{2}} dc \quad \dots \quad \text{leave it that way}$$

$$\text{if } \int_{-2p+d}^{-p+\frac{d}{2}} dc \rightarrow \int_{-2p+d}^0 dc - \int_{-p+\frac{d}{2}}^0 dc$$

The terms containing $\frac{d}{2}$ in the limits will cancel out.

(4) Now we evaluate $A - B$.

After some obvious cancellations and some additional simple manipulations we get

$$\begin{aligned}
 A - B = \int_0^{2p} dd \left\{ \int_{d-2p}^0 dc \int_0^{2p+c-d} db \int_{-c}^{2p-d-b} d\mu \right. \\
 + \int_0^d dc \left[- \int_0^{c-d} db \int_c^{2p} d\mu + \int_0^{c-d} db \int_{-b}^{2p+c-d-b} d\mu - \int_0^c db \int_{-b}^{2p+c-d} d\mu \right. \\
 \left. + \int_0^{2p-d} db \int_0^{2p-d-b} d\mu \right] + \int_0^{2p} dc \left[\int_0^{c-d} db \int_c^{2p} d\mu - \int_0^{c-d} db \int_{-b}^{2p+c-d-1} d\mu \right. \\
 \left. + \int_0^c db \int_{-b}^{2p+c-d-b} d\mu \right] \left. \right\} \sum_r \frac{\mu}{2} J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \cos \sigma(c+d)
 \end{aligned}$$

Integration w. r. to μ now yields in each of these integrals a " p^2 " term which, for $p \rightarrow \infty$, must be the highest order term in p . It will then be sufficient to show that the triple integrals multiplying p^2 are convergent.

Thus we will now show that

$$\begin{aligned}
 p^2 \int_0^{2p} dd \left\{ \int_{d-2p}^0 dc \int_0^{2p+c-d} db + \int_0^d dc \left[\int_0^{2p-d} db - \int_0^c db \right] + \int_0^{2p} dc \int_0^c db \right\} \\
 \times \sum_{r=-\infty}^{\infty} J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \cos \sigma(c+d)
 \end{aligned}$$

is of order p^2 .

APPENDIX N

B. Proof that $R_1 + S_1$ is of order p^2

The proof that M5 is of order p^2 will be accomplished in two steps as stated earlier.

Consider the integral

$$N1 \quad \int_0^{2p} dd \int_0^d dc \int_0^{c-d} db \sum_r J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \cos \sigma(c+d)$$

First we note that

$$\begin{aligned} \sum_r J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) &= \sum_{r=0}^{\infty} J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \\ &\quad - \sum_{r=1}^{\infty} J_r(b-c+d) J_r(b) J_r(d) J_{r-1}(c) \end{aligned}$$

$$= A - B$$

Now let

$$N2 \quad \int_0^{2p} dd \int_0^d dc \int_0^{c-d} db A \cos \sigma(c+d) = I_A \quad ; \quad \int_0^{2p} dd \int_0^d dc \int_0^{c-d} db B \cos \sigma(c+d) = I_B$$

But*

$$\int_0^z J_\mu(x) J_\nu(z-x) dx = 2 \sum_{s=0}^{\infty} (-1)^s J_{\mu+\nu+2s+1}(z) \quad \mu, \nu > -1$$

$$\text{Hence } \int_0^{c-d} db J_r(b-c+d) J_r(b) = (-1)^{r+1} \int_0^{d-c} J_r(b) J_r(d-c-b) db$$

$$= 2 (-1)^{r+1} \sum_{s=0}^{\infty} (-1)^s J_{2r+2s+1}(d-c) \quad r > 0 \quad \text{Thus}$$

$$I_A = 2 \int_0^{2p} dd \int_0^d dc \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{r+s+1} J_r(d) J_{r+1}(c) J_{2r+2s+1}(d-c) \cos \sigma(c+d)$$

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 668 #6.513 (6)

$$I_B = 2 \int_0^{2p} dd \int_0^d dc \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} (-1)^{r+s+1} J_r(cd) J_{r-1}(c) J_{2r+2s+1}(d-c) \cos \sigma(c+d)$$

Next we look at

$$N3 \quad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{r+s+1} \int_0^{2p} J_r(d) \int_0^d J_{r+1}(c) J_{2r+2s+1}(d-c) \cos \sigma(c+d) dc$$

The domain of integration is

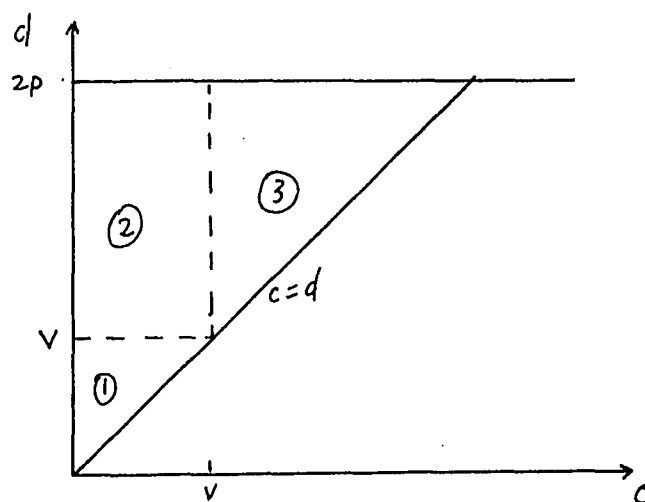


Fig. N1

Consider now $J_n(x)$

(1) For n not too large, there is a v such that for all x larger than v , $J_n(x)$ can be replaced by

$$\sqrt{\frac{2}{\pi n}} \cos \left(x + n \frac{\pi}{2} - \frac{\pi}{4} \right)$$

(2) For n large, and $n \gg x$, we can replace

$J_n(x)$ by

$$\frac{e^{n(th\alpha - \alpha)}}{\sqrt{2\pi n th\alpha}} \quad \text{where} \quad ch\alpha = \frac{n}{x}$$

Now V as shown in Figure $N/$ is such as to satisfy requirement (1). Now let us analyze the three regions (1), (2) and (3).

(1) Region (1)

Here the integral is

$$\int_0^V dd \int_0^d J_r(d) J_{r+1}(c) J_{2r+2s+1}(d-c) \cos \sigma(c+d) dc$$

(a) Clearly this integral is finite. If we consider only those values of r , and $2r+2s+1$ for which (1) applies we deal with a finite number of such terms and their sum is finite.

(b) When r is large, then from requirement (2)

$$\begin{aligned} ch\alpha &= \frac{r}{x} \sim sh\alpha & ; & \quad th\alpha \sim 1 \\ ch\alpha &= \frac{e^\alpha}{2} & ; & \quad e^\alpha = 2ch\alpha \\ \alpha &= \ln(2ch\alpha) = \ln\left(2\frac{r}{x}\right) \end{aligned}$$

then

$$\begin{aligned} \frac{e^{r(1-\alpha)}}{\sqrt{2\pi r}} &\sim \frac{e^r e^{-r \ln(2\frac{r}{x})}}{\sqrt{2\pi r}} = \frac{e^r (e^{\ln(2\frac{r}{x})})^{-r}}{\sqrt{2\pi r}} \\ &= \frac{e^r (\frac{2r}{x})^{-r}}{\sqrt{2\pi r}} = \frac{e^r x^r}{\sqrt{2\pi r} r^r} \end{aligned}$$

and the integral becomes (since $d-c < V$)

$$\int_0^V dd \int_0^d dc \frac{e^{4r+2s+1} d^r c^{r+1} (d-c)^{2r+2s+1} (\cos \sigma d \cos \sigma c - \sin \sigma d \sin \sigma c)}{(2\pi)^{3/2} [r(r+1)(2r+2s+1)]^{1/2} r^r r^{r+1} (2r+2s+1)^{2r+2s+1}}$$

for each value of r and s this integral $\rightarrow 0$ and the sum is finite.

Since there also will be only a finite number of values of r where $r \sim c$ we can conclude that over region (1) the sum of the integrals I_A is finite. The same reasoning would apply also to I_B .

(2) Region (2)

The integral is now

$$\int_v^{2p} J_r(d) dd \int_0^v J_{r+1}(c) J_{2r+2s+1}(d-c) \cos \sigma(c+d) dc$$

First we subdivide this integral into

$$\int_v^{2v} dd \int_0^v dc + \int_{2v}^{2p} dd \int_0^v dc$$

The first integral is very much like the integral over region (1) and all the conclusions arrived at there apply to too. Now for the second integral we have

$$\int_{2v}^{2p} dd \int_0^v dc J_r(d) J_{r+1}(c) J_{2r+2s+1}(d-c) \cos \sigma(c+d)$$

(a) r and s not too large

Now $d-c > v$ so that for r not too large (really for $2r+2s+1 < d-c$) we have

$$\begin{aligned} J_{2r+2s+1}(d-c) &\sim \sqrt{\frac{2}{\pi(d-c)}} \cos \left[d-c - \frac{\pi}{2}(2r+2s+1) - \frac{\pi}{4} \right] \\ &= \sqrt{\frac{1}{\pi(d-c)}} \left\{ \cos \left[d-c - (2r+2s+1) \frac{\pi}{2} \right] + \sin \left[d-c - (2r+2s+1) \frac{\pi}{2} \right] \right\} \\ &= \sqrt{\frac{1}{\pi(d-c)}} \left\{ (-1)^{r+s} \sin(d-c) - (-1)^{r+s} \cos(d-c) \right\} \end{aligned}$$

Recalling now that we originally had

$$\sum_r \sum_s (-1)^{r+s} \dots$$

the factor $(-1)^{r+s}$ disappears.

We also have, since d is always large

$$\begin{aligned} J_r(d) &\sim \sqrt{\frac{1}{\pi d}} \left[\cos\left(d - r\frac{\pi}{2}\right) + \sin\left(d - r\frac{\pi}{2}\right) \right] \\ &= \sqrt{\frac{1}{\pi d}} (-1)^{r/2} (\cos d + \sin d) && \text{if } r \text{ even} \\ &= \sqrt{\frac{1}{\pi d}} (-1)^{r-1/2} (\sin d - \cos d) && \text{if } r \text{ odd} \end{aligned}$$

Thus for a finite number of values of r and s we have

$$\frac{1}{\pi} \int_0^V J_{r+1}(c) dc \int_{2V}^{2P} \frac{[\sin(d-c) - \cos(d-c)] [\sin d \pm \cos d] \cos \sigma(c+d)}{\sqrt{d(d-c)}} dd$$

which is clearly finite.

Summing it over r and s (both not too large) gives us a finite result.

(b) For r large, we can expand $J_{r+1}(c)$ to get

$$J_{r+1}(c) \sim \frac{e^{r+1} c^{r+1}}{\sqrt{2\pi(r+1)} (r+1)^{r+1}}$$

The integral then becomes

$$\int_{2V}^{2P} J_r(d) dd \int_0^V dc J_{2r+2s+1}(d-c) \frac{e^{r+1} c^{r+1}}{\sqrt{2\pi(r+1)} (r+1)^{r+1}} \cos \sigma(d+c)$$

The denominator becomes dominant very rapidly so that

each such integral $\rightarrow 0$ and summing over r and s gives a finite result.

Thus we have shown that in region (2) the integral is finite and we again have p^2 dependence.

(3) Region (3)

The integral is now

$$\int_v^{2p} dd \int_v^d J_r(d) J_{r+1}(c) J_{2r+2s+1}(d-c) \cos \sigma(c+d)$$

(a) r is not too large, i.e. such that asymptotic expansion is valid for all values of c and d for $J_r(d), J_{r+1}(c)$ for $c, d > v$. Note that we now have a finite number of values of r .

Since both c and d are large we get,

$$\begin{aligned} J_r(d) J_{r+1}(c) &\sim \frac{2}{\pi} \frac{1}{\sqrt{cd}} \cos\left(d - r\frac{\pi}{2} - \frac{\pi}{4}\right) \cos\left(c - (r+1)\frac{\pi}{2} - \frac{\pi}{4}\right) \\ &= \frac{1}{\pi \sqrt{cd}} \left\{ \cos\left(d-c + \frac{\pi}{2}\right) + \cos\left(d+c - r\pi - \frac{\pi}{2}\right) \right\} \\ &= - \frac{1}{\pi \sqrt{cd}} \left\{ \sin(d-c) - (-1)^r \cos(d+c) \right\} \end{aligned}$$

The integral is then

$$- \frac{1}{\pi} \int_v^{2p} dd \int_v^d dc \frac{J_{2r+2s+1}(d-c)}{\sqrt{cd}} \left\{ \sin(d-c) - (-1)^r \cos(d+c) \right\} \cos \sigma(c+d)$$

Let

$$d - c = v \quad ; \quad d = \frac{\mu + v}{2}$$

$$d + c = \mu \quad ; \quad c = \frac{\mu - v}{2}$$

The limiting lines are

$$c = v \quad \rightarrow \quad \mu - v = 2v \quad ; \quad d = 2p \rightarrow \mu + v = 2p$$

$$c = d \quad \rightarrow \quad v = 0$$

$$d = v \quad \rightarrow \quad \mu + v = 2v$$

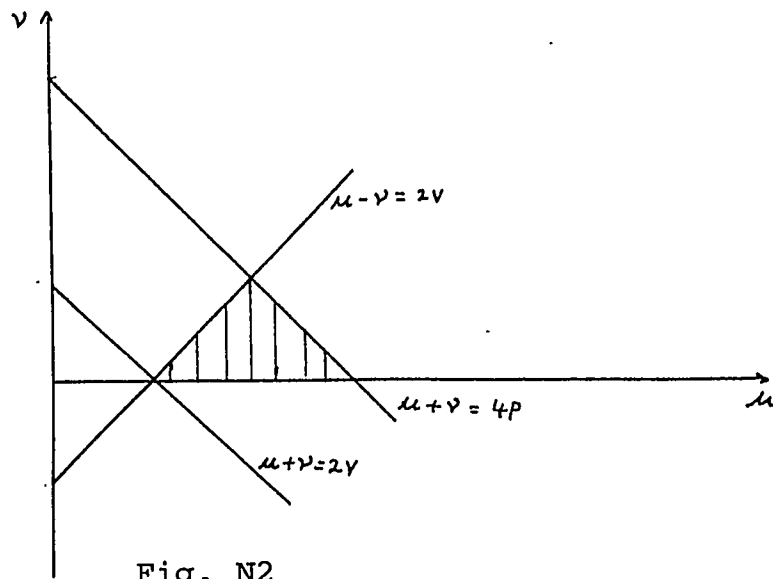


Fig. N2

The integral now becomes

$$\int_0^{2p-v} d\nu \int_{2v+\nu}^{4p-\nu} d\mu \frac{J_{2r+2s+1}(\nu)}{\sqrt{\mu^2 - \nu^2}} \left\{ \sin \nu - (-1)^r \cos \mu \right\} \cos \sigma \mu$$

Now $\mu \gg \nu$ so that

$$(\mu^2 - \nu^2)^{-1/2} \sim \frac{1}{\mu} \left(1 + \frac{\nu^2}{2\mu^2}\right) \sim \frac{1}{\mu} \quad \text{and}$$

$$\int_{2v+\nu}^{4p-\nu} d\mu \left\{ \sin \nu - (-1)^r \cos \mu \right\} \frac{\cos \sigma \mu}{\mu}$$

$$= \sin \nu \left\{ \frac{\sin \sigma(4p-\nu)}{\sigma(4p-\nu)} - \frac{\sin \sigma(2v+\nu)}{\sigma(2v+\nu)} \right\} - \frac{(-1)^r}{2} \left\{ \frac{\sin[(1-\sigma)(4p-\nu)]}{(1-\sigma)(4p-\nu)} \right. \\ \left. - \frac{\sin[(1-\sigma)(2v+\nu)]}{(1-\sigma)(2v+\nu)} + \frac{\sin[(1+\sigma)(4p-\nu)]}{(1+\sigma)(4p-\nu)} - \frac{\sin[(1+\sigma)(2v+\nu)]}{(1+\sigma)(2v+\nu)} \right\}$$

Next we get as a typical term

$$\int_0^{2p-v} d\nu J_{2r+2s+1}(\nu) \left\{ \frac{\sin \nu \sin \sigma(4p-\nu)}{\sigma(4p-\nu)} - \frac{\sin \nu \sin \sigma(2v+\nu)}{\sigma(2v+\nu)} \right\}$$

The first term obviously goes to zero due to the $4\rho-\nu$ in the denominator.

The second term can be disposed of as follows (remembering that we eventually sum over a finite number of values of r but an infinite number of s)

Let the integral be divided into two parts

$$\int_0^V d\nu + \int_V^{2\rho-\nu} d\nu$$

We now repeat the procedure used in region (1) for the first integral. For each value of s , $\int_0^V d\nu$ is finite. For a limited number of values of s (s relatively small) the sum over s is clearly finite. For $s > V$ we replace $J_{2r+2s+1}^{(\nu)}$ by

$$\frac{e^{2r+2s+1} \nu^{2r+2s+1}}{\sqrt{2\pi}(2r+2s+1) (2r+2s+1)^{2r+2s+1}}$$

when the integral $\rightarrow 0$ so that the sum over s is finite.

For the second integral, we can use the asymptotic expansion of $J_{2r+2s+1}^{(\nu)}$ for s relatively small and again get a finite result. For large $2r+2s+1$ we can use either the expansion for large order or note that when $2r+2s+1 \sim \nu$ (both large) the $\frac{1}{\nu}$ term dominates and we again get the integral $\rightarrow 0$ and the sum over s is finite.

We have thus shown that for r not too large we have a finite result in region (3).

(b) For r large we use the fact that in region (3) either $d-c$ is small or otherwise c must be $\ll d$.

In the first case

$$J_{2r+2s+1}(d-c) \sim \frac{e^{2r+2s+1}(d-c)^{2r+2s+1}}{(2r+2s+1)^{2r+2s+1}}$$

and the integral $\rightarrow 0$. In the second case

$$J_{r+1}(c) \sim \frac{e^{r+1} c^{r+1}}{(r+1)^{r+1}} \quad \text{and the integral} \rightarrow 0.$$

In both cases the sum over s and r will be finite.

We have thus the result that eq. N3 is finite and

$$I_A \sim p^2.$$

Quite evidently we get the same conclusions for I_B .

We now note that the same conclusions would be arrived at if instead of $J_{r+1}(c)$ we had $J_r(c)$ which means that R_2 (with the limits as given here) would be of order p^3 . Similar results will be obtained for $R_4, R_5, S_2, S_3, S_4, S_5$. Thus all R_i and S_i would produce the same results provided we consider only those integrals in d, c, b which have as limits 0 and $2p$, 0 and d , 0 and $d-c$ respectively.

APPENDIX O

Proof that other Terms are of Order

We must now tackle the second point of the proof, namely to show that the remaining terms in $M5$ are of the same type as the integral shown to be convergent in App N

The integrals are then

$$I_1 = \int_0^{2\rho} dd \int_0^d dc \int_0^{c-d} db \sum_r J_r(b-c+d) J_r(b) J_r(d) J_{r+1}(c) \cos \sigma(c+d)$$

$$I_2 = \int_0^{2\rho} dd \int_{-2\rho+d}^0 dc \int_0^{2\rho+c-d} db F$$

$$I_3 = \int_0^{2\rho} dd \int_0^d dc \int_0^c db F$$

$$I_4 = \int_0^{2\rho} dd \int_0^d dc \int_0^{2\rho-d} db F$$

$$I_5 = \int_0^{2\rho} dd \int_0^{2\rho} dc \int_0^c db F$$

We will now use an integral representation of the Bessel functions.

$$J_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-ir\theta + ix \sin \theta} \quad *$$

Replacing now the Bessel functions in I_1 by the expression 01 and interchanging the order of integration, we

get

$$I_1 = \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \int_{-\pi}^{\pi} d\theta_3 \int_{-\pi}^{\pi} d\theta_4 \int_0^{2\rho} dd \int_0^d dc \int_0^{c-d} db \sum_r e^{-ir\theta_1 + id \sin \theta_1 - i(r+1)\theta_2 + i r \sin \theta_2} \\ \times e^{-ir\theta_3 + ib \sin \theta_3 - ir\theta_4 + i(b-c+d) \sin \theta_4} \left[\frac{e^{i\sigma(c+d)} + e^{-i\sigma(c+d)}}{2} \right]$$

*Gradshteyn & Ryzhik: Table of Integrals, Series & Products, Academic Press (Second Ed. 1966) page 952 #8.411 (1)

Collecting now terms in r , letting

$$\sin \theta_1 = w \quad ; \quad \sin \theta_2 = x \quad ; \quad \sin \theta_3 = y \quad ; \quad \sin \theta_4 = z$$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = \varphi$$

and integrating w.r. to b, c and d successively one obtains

$$I_1 = \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \int_{-\pi}^{\pi} d\theta_3 \int_{-\pi}^{\pi} d\theta_4 \left[\sum_r e^{-i\theta_2} e^{ir\varphi} \right] I_1'$$

where

$$\begin{aligned} I_1' = \frac{1}{i(\gamma+z)} & \left\{ \frac{e^{2ip(w+x+2\sigma)} - 1}{(x+y+\sigma)(w+x+2\sigma)} + \frac{e^{2ip(w+x-2\sigma)} - 1}{(x+y+\sigma)(w+x-2\sigma)} - \frac{e^{2ip(w-y+\sigma)} - 1}{(x+y+\sigma)(w-y+\sigma)} \right. \\ & - \frac{e^{2ip(w-y-\sigma)} - 1}{(x+y-\sigma)(w-y-\sigma)} - \frac{e^{2ip(w+x+2\sigma)} - 1}{(x-z+\sigma)(w+x+2\sigma)} - \frac{e^{2ip(w+x-2\sigma)} - 1}{(x-z-\sigma)(w+x-2\sigma)} \\ & \left. + \frac{e^{2ip(w+z+\sigma)} - 1}{(x-z+\sigma)(w+z+\sigma)} + \frac{e^{2ip(w+z-\sigma)} - 1}{(x-z-\sigma)(w+z-\sigma)} \right\} \end{aligned}$$

Proceeding in the same way for the other I_i' 's we eventually get

$$I_2 = \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \int_{-\pi}^{\pi} d\theta_3 \int_{-\pi}^{\pi} d\theta_4 \left[\sum_r e^{-i\theta_2} e^{ir\varphi} \right] I_2'$$

where

$$\begin{aligned} I_2' = -\frac{1}{i(\gamma+z)} & \left\{ \frac{e^{2pi(\gamma+z)} (e^{2pi(w-y+\sigma)} - 1)}{(x+y+\sigma)(w-y+\sigma)} + \frac{e^{2pi(\gamma+z)} (e^{2pi(w-y-\sigma)} - 1)}{(x+y-\sigma)(w-y-\sigma)} \right. \\ & - \frac{e^{2pi(\gamma+z)} e^{-2pi(x+y+\sigma)} (e^{2pi(w+x+2\sigma)} - 1)}{(x+y+\sigma)(w+x+2\sigma)} - \frac{e^{2pi(\gamma+z)} e^{-2pi(x+y-\sigma)} (e^{2pi(w+x-2\sigma)} - 1)}{(x+y-\sigma)(w+x-2\sigma)} \\ & - \frac{e^{2pi(w+z+\sigma)} - 1}{(x-z+\sigma)(w+z+\sigma)} - \frac{e^{2pi(w+z-\sigma)} - 1}{(x-z-\sigma)(w+z-\sigma)} + \frac{e^{-2pi(x-z+\sigma)} (e^{2pi(w+x+2\sigma)} - 1)}{(x-z+\sigma)(w+x+2\sigma)} + \frac{e^{-2pi(x-z-\sigma)} (e^{2pi(w+x-2\sigma)} - 1)}{(x-z-\sigma)(w+x-2\sigma)} \end{aligned}$$

$$I_3 = \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \int_{-\pi}^{\pi} d\theta_3 \int_{-\pi}^{\pi} d\theta_4 \left[\sum_r e^{-i\theta_2} e^{-i\varphi r} \right] I_3'$$

where

$$I_3' = -\frac{1}{i(\gamma+z)} \left\{ \frac{e^{2\pi i(\gamma+w+x+y+z+2\sigma)} - 1}{(x+y+\sigma)(w+x+y+z+2\sigma)} + \frac{e^{2\pi i(\gamma+w+x+y+z-2\sigma)} - 1}{(x+y-\sigma)(w+x+y+z-2\sigma)} \right. \\ - \frac{e^{2\pi i(\gamma+w+z+\sigma)} - 1}{(x+y+\sigma)(w+z+\sigma)} - \frac{e^{2\pi i(\gamma+w+z-\sigma)} - 1}{(x+y-\sigma)(w+z-\sigma)} - \frac{e^{2\pi i(\gamma+w+x+2\sigma)} - 1}{(w+x+2\sigma)(x-z+\sigma)} \\ \left. - \frac{e^{2\pi i(\gamma+w+x-2\sigma)} - 1}{(w+x-2\sigma)(x-z-\sigma)} + \frac{e^{2\pi i(\gamma+w+z+\sigma)} - 1}{(x-z+\sigma)(w+z+\sigma)} + \frac{e^{2\pi i(\gamma+w+z-\sigma)} - 1}{(x-z-\sigma)(w+z-\sigma)} \right\}$$

$$I_4 = \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \int_{-\pi}^{\pi} d\theta_3 \int_{-\pi}^{\pi} d\theta_4 \left[\sum_r e^{-i\theta_2} e^{-i\varphi r} \right] I_4'$$

where

$$I_4' = -\frac{1}{i(\gamma+z)} \left\{ \frac{e^{2\pi i(\gamma+z)} (e^{2\pi i(\gamma+w+x-y-z+2\sigma)} - 1)}{(x-z+\sigma)(w+x-y-z+2\sigma)} + \frac{e^{2\pi i(\gamma+z)} (e^{2\pi i(\gamma+w+x-y-z-2\sigma)} - 1)}{(x-z-\sigma)(w+x-y-z-2\sigma)} \right. \\ - \frac{e^{2\pi i(\gamma+z)} (e^{2\pi i(\gamma+w-y+\sigma)} - 1)}{(x-z+\sigma)(w-y+\sigma)} - \frac{e^{2\pi i(\gamma+z)} (e^{2\pi i(\gamma+w-y-\sigma)} - 1)}{(x-z-\sigma)(w-y-\sigma)} \\ - \frac{e^{2\pi i(\gamma+w+x+2\sigma)} - 1}{(x-z+\sigma)(w+x+2\sigma)} - \frac{e^{2\pi i(\gamma+w+x-2\sigma)} - 1}{(x-z-\sigma)(w+x-2\sigma)} + \frac{e^{2\pi i(\gamma+w+z+\sigma)} - 1}{(x-z+\sigma)(w+z+\sigma)} \\ \left. + \frac{e^{2\pi i(\gamma+w+z-\sigma)} - 1}{(x-z-\sigma)(w+z-\sigma)} \right\}$$

$$I_5 = \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \int_{-\pi}^{\pi} d\theta_3 \int_{-\pi}^{\pi} d\theta_4 \left[\sum_r e^{-i\theta_2} e^{-irp} \right] I_5'$$

where

$$I_5' = - \frac{1}{i(\gamma+z)} \left\{ \begin{aligned} & \frac{e^{2\pi i(\chi+\gamma+\sigma)} (e^{2\pi i(w+z+\sigma)} - 1)}{(w+z+\sigma)(\chi+\gamma+\sigma)} + \frac{e^{2\pi i(\chi+\gamma-\sigma)} (e^{2\pi i(w+z-\sigma)} - 1)}{(w+z-\sigma)(\chi+\gamma-\sigma)} \\ & - \frac{e^{2\pi i(w+z+\sigma)} - 1}{(\chi+\gamma+\sigma)(w+z+\sigma)} - \frac{e^{2\pi i(w+z-\sigma)} - 1}{(\chi+\gamma-\sigma)(w+z-\sigma)} \\ & - \frac{e^{2\pi i(\chi-z+\sigma)} (e^{2\pi i(w+z+\sigma)} - 1)}{(\chi-z+\sigma)(w+z+\sigma)} - \frac{e^{2\pi i(\chi-z-\sigma)} (e^{2\pi i(w+z-\sigma)} - 1)}{(\chi-z-\sigma)(w+z-\sigma)} \\ & + \frac{e^{2\pi i(w+z+\sigma)} - 1}{(\chi-z+\sigma)(w+z+\sigma)} - \frac{e^{2\pi i(w+z-\sigma)} - 1}{(\chi-z-\sigma)(w+z-\sigma)} \end{aligned} \right\}$$

Now consider the expressions $I_1', I_2', I_3', I_4', I_5'$.

First we note that for $\sigma > 2$ the only possible singularity occurs when $y = -z$ but that occurs in the same way in all five expressions. So that if I_1 is convergent, so are all the others.

For $\sigma < 2$, the possible zeros of the denominators will not give any trouble because:

(1) Wherever a term containing w occurs in the denominator the same term appears in the numerator as the exponent of e with an accompanying (-1) term and precisely in the form necessary for cancelling out

E.g.

$$\frac{e^{2ip(w+x+2\sigma)} - 1}{w+x+2\sigma}$$

(2) The other terms will cancel in pairs at the singularity. Here it is necessary to show this in detail.

(a)

$$(1) \quad \frac{e^{2ip(w+x+2\sigma)} - 1}{(x+y+\sigma)(w+x+2\sigma)} - \frac{e^{2ip(w-y+\sigma)} - 1}{(x+y+\sigma)(w-y+\sigma)}$$

for $y = -x - \sigma$ this becomes (let $x+y+\sigma = \epsilon$)

$$\frac{1}{\epsilon} \left\{ \frac{e^{2ip(w+x+2\sigma)} - 1}{w+x+2\sigma} - \frac{e^{2ip(w+x+2\sigma)} - 1}{w+x+2\sigma} \right\} \rightarrow K$$

$$(2) \quad \frac{e^{2ip(w+x-2\sigma)} - 1}{(w+x-2\sigma)(x+y-\sigma)} - \frac{e^{2ip(w-y-\sigma)} - 1}{(x+y-\sigma)(w-y-\sigma)}$$

for $x = -y + \sigma$ this becomes (let $x+y-\sigma = \epsilon$)

$$\frac{1}{\epsilon} \left\{ \frac{e^{2ip(w-y-\sigma)} - 1}{(w-y-\sigma)} - \frac{e^{2ip(w-y-\sigma)} - 1}{(w-y-\sigma)} \right\} \rightarrow K$$

$$(3) - \frac{e^{2ip(w+x+2\sigma)} - 1}{(x-z+\sigma)(w+x+2\sigma)} + \frac{e^{2ip(w+z+\sigma)} - 1}{(x-z+\sigma)(w+z+\sigma)}$$

for $x = z - \sigma$ this becomes $(x-z+\sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ - \frac{e^{2ip(w+z+\sigma)} - 1}{w+z+\sigma} + \frac{e^{2ip(w+z+\sigma)} - 1}{w+z+\sigma} \right\} \rightarrow K$$

$$(4) \frac{e^{2ip(w+z-\sigma)} - 1}{(x-z-\sigma)(w+z-\sigma)} - \frac{e^{2ip(w+x-2\sigma)} - 1}{(x-z-\sigma)(w+x-2\sigma)}$$

for $x = z + \sigma$ $(x-z-\sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ \frac{e^{2ip(w+z-\sigma)} - 1}{w+z-\sigma} - \frac{e^{2ip(w+z-\sigma)} - 1}{w+z-\sigma} \right\} \rightarrow K$$

(b) I_2'

$$(1) e^{2ip(\gamma+z)} \left\{ \frac{e^{2pi(w-\gamma+\sigma)} - 1}{(x+\gamma+\sigma)(w-\gamma+\sigma)} - \frac{e^{-2pi(x+\gamma+\sigma)} (e^{2pi(w+x+2\sigma)} - 1)}{(x+\gamma+\sigma)(w+x+2\sigma)} \right\}$$

let $\gamma = -x - \sigma$ then $(x+\gamma+\sigma = \epsilon)$

$$\frac{e^{2pi(-x+z-\sigma)}}{\epsilon} \left\{ \frac{e^{2pi(w+x+2\sigma)} - 1}{w+x+\sigma} - \frac{e^{2pi(w+x+2\sigma)} - 1}{w+x+\sigma} \right\} \rightarrow K$$

$$(2) - \frac{e^{2pi(w+z+\sigma)} - 1}{(x-z+\sigma)(w+z+\sigma)} + \frac{e^{-2pi(x-z+\sigma)} (e^{2pi(w+x+2\sigma)} - 1)}{(x-z+\sigma)(w+x+2\sigma)}$$

let $x = z - \sigma$ then $(x-z+\sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ - \frac{e^{2pi(w+z+\sigma)} - 1}{w+z+\sigma} + \frac{e^{2pi(w+z+\sigma)} - 1}{w+z+\sigma} \right\} \rightarrow K$$

$$(3) \frac{e^{2pi(\gamma+z)}}{x+\gamma-\sigma} \left\{ \frac{e^{2pi(w-\gamma-\sigma)} - 1}{w-\gamma-\sigma} - \frac{e^{-2pi(x+\gamma-\sigma)} (e^{2pi(w+x-2\sigma)} - 1)}{w+x-2\sigma} \right\}$$

let $\gamma = -x + \sigma$ then $(x+\gamma-\sigma = \epsilon)$

$$\frac{e^{2pi(\sigma-x+z)}}{\epsilon} \left\{ \frac{e^{2pi(w+x-2\sigma)} - 1}{w+x-2\sigma} - \frac{e^{2pi(w+x-2\sigma)} - 1}{w+x-2\sigma} \right\} \rightarrow K$$

$$(4) - \frac{e^{2\pi i(w+z-\sigma)} - 1}{(x-z-\sigma)(w+z-\sigma)} + \frac{e^{-2\pi i(x-z-\sigma)} (e^{2\pi i(w+x-2\sigma)} - 1)}{(x-z-\sigma)(w+x-2\sigma)}$$

for $x = z + \sigma$ we get $(x-z-\sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ - \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z-\sigma} + \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z-\sigma} \right\} \rightarrow K$$

(c) I_3'

$$(1) \frac{1}{x+y+\sigma} \left\{ \frac{e^{2\pi i(w+x+y+z+2\sigma)} - 1}{w+x+y+z+2\sigma} - \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} \right\}$$

for $x = -y - \sigma$ $(x+y+\sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} - \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} \right\} \rightarrow K$$

$$(2) \frac{1}{x-z+\sigma} \left\{ \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} - \frac{e^{2\pi i(w+x+2\sigma)} - 1}{w+x+2\sigma} \right\}$$

for $x = z - \sigma$ $(x-z+\sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} - \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} \right\}$$

$$(3) \frac{1}{x+y-\sigma} \left\{ \frac{e^{2\pi i(w+x+y+z-2\sigma)} - 1}{w+x+y+z-2\sigma} - \frac{e^{2\pi i(w+z-\sigma)} - 1}{w+z-\sigma} \right\}$$

for $x = -y + \sigma$ we get $(x+y-\sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ \frac{e^{2\pi i(w+z-\sigma)} - 1}{w+z-\sigma} - \frac{e^{2\pi i(w+z-\sigma)} - 1}{w+z-\sigma} \right\} \rightarrow K$$

$$(4) \frac{1}{x-z-\sigma} \left\{ \frac{e^{2\pi i(w+z-\sigma)} - 1}{w+z-\sigma} - \frac{e^{2\pi i(w+x-2\sigma)} - 1}{w+x-2\sigma} \right\}$$

for $x = z + \sigma$ we get $(x-z-\sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ \frac{e^{2\pi i(w+z-\sigma)} - 1}{w+z-\sigma} - \frac{e^{2\pi i(w+z-\sigma)} - 1}{w+z-\sigma} \right\} \rightarrow K$$

(d) I_4'

$$(1) \frac{1}{x-z+\sigma} \left\{ \frac{e^{2\pi i(\gamma+z)} (e^{2\pi i(w+x-y-z+2\sigma)} - 1)}{(w+x-y-z+2\sigma)} - \frac{e^{2\pi i(\gamma+z)} (e^{2\pi i(w-y+\sigma)} - 1)}{w-y+\sigma} \right\}$$

for $x = z - \sigma$ we get $(x - z + \sigma = \epsilon)$

$$\frac{e^{2\pi i(\gamma+z)}}{\epsilon} \left\{ \frac{e^{2\pi i(w-y+\sigma)} - 1}{w-y+\sigma} - \frac{e^{2\pi i(w-y+\sigma)} - 1}{w-y+\sigma} \right\} \rightarrow K$$

$$(2) \frac{1}{x-z+\sigma} \left\{ \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} - \frac{e^{2\pi i(w+x+2\sigma)} - 1}{w+x+2\sigma} \right\}$$

for $x = z - \sigma$ we get $(x - z + \sigma = \epsilon)$

$$\frac{1}{\epsilon} \left\{ \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} - \frac{e^{2\pi i(w+z+\sigma)} - 1}{w+z+\sigma} \right\} \rightarrow K$$

$$(3) \frac{e^{2\pi i(\gamma+z)}}{w-z-\sigma} \left\{ \frac{e^{2\pi i(w+x-y-z-2\sigma)} - 1}{w+x-y-z-2\sigma} - \frac{e^{2\pi i(w-y-\sigma)} - 1}{w-y-\sigma} \right\}$$

for $x = z + \sigma$ we get $(x - z - \sigma = \epsilon)$

$$\frac{e^{2\pi i(\gamma+z)}}{\epsilon} \left\{ \frac{e^{2\pi i(w-y-\sigma)} - 1}{w-y-\sigma} - \frac{e^{2\pi i(w-y-\sigma)} - 1}{w-y-\sigma} \right\} \rightarrow K$$

$$(4) \frac{1}{x-z-\sigma} \left\{ \frac{e^{2\pi i(w+z-\sigma)} - 1}{w+z-\sigma} - \frac{e^{2\pi i(w+x-2\sigma)} - 1}{w+x-2\sigma} \right\}$$

for $z = x - \sigma$ we get $(z - x + \sigma = \epsilon)$

$$(e) I_5' \frac{1}{\epsilon} \left\{ \frac{e^{2\pi i(w+x-2\sigma)} - 1}{w+x-2\sigma} - \frac{e^{2\pi i(w+x-2\sigma)} - 1}{w+x-2\sigma} \right\} \rightarrow K$$

$$(1) \frac{e^{2\pi i(w+y+\sigma)} [e^{2\pi i(w+z+\sigma)} - 1] - [e^{2\pi i(w+z+\sigma)} - 1]}{(x+y+\sigma)(w+z+\sigma)}$$

for $x+y+\sigma = \epsilon$ we get

$$\frac{1}{\epsilon} \frac{e^{2\pi i(w+z+\sigma)} - 1 - (e^{2\pi i(w+z+\sigma)} - 1)}{w+z+\sigma} \rightarrow K$$

$$(2) \quad \frac{e^{2\pi i(x-z+\sigma)} [e^{2\pi i(w+z+\sigma)} - 1] - [e^{2\pi i(w+z+\sigma)} - 1]}{(x-z+\sigma)(w+z+\sigma)}$$

for $x-z+\sigma = \epsilon$ we get

$$\frac{1}{\epsilon} \frac{e^{2\pi i(w+z+\sigma)} - 1 - (e^{2\pi i(w+z+\sigma)} - 1)}{w+z+\sigma} \rightarrow K$$

$$(3) \quad \frac{e^{2\pi i(x+y-\sigma)} [e^{2\pi i(w+z-\sigma)} - 1] - [e^{2\pi i(w+z-\sigma)} - 1]}{(x+y-\sigma)(w+z-\sigma)}$$

for $x+y-\sigma = \epsilon$ we get

$$\frac{1}{\epsilon} \frac{e^{2\pi i(w+z-\sigma)} - 1 - (e^{2\pi i(w+z-\sigma)} - 1)}{w+z-\sigma} \rightarrow K$$

$$(4) \quad \frac{e^{2\pi i(x-z-\sigma)} [e^{2\pi i(w+z-\sigma)} - 1] - [e^{2\pi i(w+z-\sigma)} - 1]}{(x-z-\sigma)(w+z-\sigma)}$$

for $x-z-\sigma = \epsilon$ we get

$$\frac{1}{\epsilon} \frac{e^{2\pi i(w+z-\sigma)} - 1 - (e^{2\pi i(w+z-\sigma)} - 1)}{w+z-\sigma}$$

Thus divergence might occur only due to those parts of the five integrals which are the same in all. But I have shown that one of them, namely I_1 is convergent. Hence all must be convergent.

APPENDIX P

Analysis of all R_i and S_i

Having shown that $R_1 + S_1$ is of order p^2 , it is now a simple matter to show that all the other R_i, S_i are of the same type and will be, at most, of order p^3 in p .

1) $R_2 + S_2$

$$R_2 = \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_r q'^2 J_r(2q'' - 2q''') J_r(2q - 2q') J_r(2q'' - 2q) \\ \times [J_r(2q''' - 2q) + J_{r+2}(2q''' - 2q)] \cos \sigma(2q + 2q' - 2q'' - 2q''')$$

$$S_2 = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_r q^2 J_r(2q'' - 2q''') J_r(2q' - 2q'') J_r(2q - 2q') \\ \times [J_r(2q''' - 2q) + J_{r+2}(2q''' - 2q)] \cos \sigma(2q + 2q' - 2q'' - 2q''')$$

In S_2 let $q \leftrightarrow q'$ and we get

$$S_2 = - \int_0^P dq' \int_0^P dq \int_0^q dq'' \int_0^{q''} dq''' \sum_r q'^2 J_r(2q'' - 2q''') J_r(2q'' - 2q) J_r(2q - 2q') \\ \times [J_r(2q''' - 2q') + J_{r+2}(2q''' - 2q')] \cos \sigma(2q + 2q' - 2q'' - 2q''')$$

The integrands are now identical. We then get, by a procedure identical to the one used for $R_1 + S_1$, that

$$R_2 + S_2 = \left\{ \int_0^P dq \int_0^P dq' \int_0^q dq'' \int_0^{q''} dq''' - \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \right\} \\ \times \left\{ \sum_r q'^2 J_r(2q'' - 2q''') J_r(2q'' - 2q) J_r(2q - 2q') [J_r(2q''' - 2q) + J_{r+2}(2q''' - 2q')] \right. \\ \left. \times \cos \sigma(2q + 2q' - 2q'' - 2q''') \right\}$$

This last expression differs from the corresponding expression for $R_1 + S_1$ in two points. First we have here J_r and J_{r+2} instead of $J_{r+1} (2q''' - 2q')$. But clearly this does not affect the results: substituting r or $r+2$ for $r+1$ in $R_1 + S_1$ would have changed neither the proof nor the result. The second difference is in the factor " q'^2 " instead of " p^2 ". This would lead to a " p^3 " factor after the μ integration, so that we obtain a p^3 rather than p^2 dependence.

Hence we are led to the conclusion that $R_2 + S_2$ behaves as p^3 .

2) $R_3 + S_3$

$$R_3 = \int_0^p dq \int_0^q dq' \int_0^p dq'' \int_0^{q''} dq''' \sum_r (\rho - q'') J_r(2q'' - 2q''') J_r(2q - 2q') J_r(2q''' - 2q') J_{r+1}(2q'' - 2q) \\ \times \cos \sigma(2q + 2q' - 2q'' - 2q''')$$

$$S_3 = \int_0^p dq \int_0^p dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_r (\rho - q) J_r(2q'' - 2q''') J_r(2q' - 2q'') J_r(2q''' - 2q) J_{r+1}(2q - 2q') \\ \times \cos \sigma(2q + 2q' - 2q'' - 2q''')$$

We first note that for large ρ it is the ρ term in $(\rho - q'')$ and in $(\rho - q)$ factors which will give the dominant dependence. We can then omit q'' and q factor and consider only the ρ term.

In that case the same substitution in S_3 of

$q \leftrightarrow q'$ will lead to

$$R_3 + S_3 = \int_0^p dq \left\{ \int_0^p dq' \int_0^p dq'' \int_0^{q''} dq''' - \int_0^p dq' \int_0^p dq'' \int_0^{q''} dq''' \right\} \\ \times \sum_r \rho J_r(2q'' - 2q''') J_r(2q - 2q') J_r(2q''' - 2q') J_{r+1}(2q'' - 2q) \cos \sigma(2q + 2q' - 2q'' - 2q''')$$

Again this expression differs from the corresponding

$R_1 + S_1$ expression only in the term $J_r(2q''' - 2q') J_{r+1}(2q'' - 2q)$ instead of $J_{r+1}(2q''' - 2q') J_r(2q'' - 2q)$ and in replacing the q' factor by ρ .

The $r+1, r$ interchange certainly does not affect the results obtained in $R_1 + S_1$. In the " μ " integration we again obtain ρ^2 as in $R_1 + S_1$ so that we again have that

$$R_3 + S_3 \sim \rho^2$$

3) $R_4 + S_4$

$$R_4 = \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_r (p-q'')^2 J_r(2q''-2q''') J_r(2q-2q') J_r(2q'''-2q') \\ \times [J_r(2q''-2q) + J_{r+2}(2q''-2q)] \cos \sigma(2q+2q'-2q''-2q''')$$

$$S_4 = \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_r (p-q)^2 J_r(2q''-2q''') J_r(2q'-2q'') J_r(2q'''-2q) \\ [J_r(2q-2q') + J_{r+2}(2q-2q')] \cos \sigma(2q+2q'-2q''-2q''')$$

We again note that the p^2 term in $(p-q'')^2$ and $(p-q)^2$ will lead to highest p dependence. By a process identical to the one used for $R_3 + S_3$ and $R_4 + S_4$ we will obtain

$$R_4 + S_4 = \int_0^P dq \left\{ \int_0^P dq' \int_q^P dq'' - \int_q^P dq' \int_0^P dq'' \right\} \int_0^{q''} dq''' \sum_r p^2 J_r(2q''-2q''') \\ \times J_r(2q-2q') J_r(2q'''-2q') [J_r(2q''-2q) + J_{r+2}(2q''-2q)] \cos \sigma(2q+2q'-2q''-2q''')$$

which is clearly now of order p^3 .

Note: We do not really have to consider only the p^2 term: everything would go through in identical fashion, if we considered separately q''^2 , $-2pq''$ and q^2 , $-2pq$ terms respectively.

4) $R_5 + S_5$

$$R_5 = \int_0^P dq \int_0^q dq' \int_0^P dq'' \int_0^{q''} dq''' \sum_r q'(p-q'') J_r(2q''-2q''') J_r(2q-2q') \\ \left[J_{r+1}(2q'''-2q') J_{r+1}(2q''-2q) + J_{r+1}(2q'''-2q') J_{r-1}(2q''-2q) \right. \\ \left. + J_{r-1}(2q'''-2q') J_{r+1}(2q''-2q) + J_{r-1}(2q'''-2q') J_{r-1}(2q''-2q) \right] \cos \sigma(2q+2q'-2q''-2q''')$$

$$S_5 = - \int_0^P dq \int_0^P dq' \int_0^{q'} dq'' \int_0^{q''} dq''' \sum_r q(p-q) J_r(2q''-2q''') J_r(2q'-2q'') \\ \left[J_{r+1}(2q'''-2q) J_{r+1}(2q-2q') + J_{r+1}(2q'''-2q) J_{r-1}(2q-2q') \right. \\ \left. + J_{r-1}(2q'''-2q) J_{r+1}(2q-2q') + J_{r-1}(2q'''-2q) J_{r-1}(2q-2q') \right] \cos \sigma(2q+2q'-2q''-2q''')$$

Again note that in both R_5 and S_5 the dominant terms in the factors $q'(p-q'')$ and $q(p-q)$ in R_5 and S_5 respectively are $q'p$ and qp respectively. Omitting the terms $q'q''$ (R_5) and q^2 (S_5) we have the same case as before. Eventually we find that

$$R_5 + S_5 \sim p^3$$

APPENDIX Q

Analysis of T_1 and T_2

A separate analysis is required for T_1 and T_2

First we change the order of integration

$$\int_0^P dq \int_0^q dq' \int_0^{q'} dq'' \int_0^{q''} dq''' F = \int_0^P dq' \int_0^{q'} dq'' \int_{q'}^P dq \int_0^{q''} dq''' F$$

where
$$F = q \sum_r J_r(2q-2q') J_r(2q'-2q'') J_r(2q''-2q''') J_{r+1}(2p-4q+2q''') \\ \times \cos \sigma(2q+2q'-2q''-2q''')$$

Next we make a series of changes of variables

- (1) let $2q'''-2q = a$; $2q = \mu$
- (2) let $2q''-2q' = b$; $2q' = \lambda$
- (3) interchange the order of λ and a
- (4) let $c = \mu - \lambda$; $\mu = \mu$
- (5) let $\mu - 2p = \lambda$

And we finally get

$$T_1 = \frac{1}{16} \int_{-2p}^0 db \left\{ \int_{-2p}^0 da \left[\int_c^{b-a} dc \int_{-b+c-2p}^{a-b-c} d\lambda - \int_0^{b-a} dc \int_{b-a}^{c+a-b} d\lambda - \int_{b-a}^{2p+b} dc \int_{c-b+2p}^0 d\lambda \right. \right. \\ \left. \left. + \int_0^{2p+q} dc \int_{c-a-2p}^0 d\lambda + \int_0^{b-a} dc \int_{-2p-a}^{-2p+c-a} d\lambda + \int_{b-a}^{2p+a} dc \int_{-2p+c-b}^{-2p+c-a} d\lambda \right] \right. \\ \left. + \int_{2p+a}^{2p+b} dc \int_{-2p+c-b}^0 d\lambda \right\}$$

$$+ \int_b^0 da \left[- \int_0^{b-a} dc \int_{-2p-b+c}^{a-b+c} d\lambda + \int_0^{2p+a} dc \int_{-2p+c-b}^{-2p-a+c} d\lambda + \int_{2p+a}^{2p+b} dc \int_{-2p+c-b}^0 d\lambda \right. \\ \left. - \int_0^{b-a} dc \int_{a+c-b}^0 d\lambda - \int_{b-a}^{2p+a} dc \int_{-2p-b+c}^{-2p-a+c} d\lambda - \int_{2p+a}^{2p+b} dc \int_{-2p-b+c}^0 d\lambda \right]$$

Q1

$$+ \int_{-p+\frac{b}{2}}^0 da \left[- \int_0^{2p+a} dc \int_{-2p+c-b}^{-2p+c-a} d\lambda - \int_{2p+a}^{2p+b} dc \int_{-2p-b+c}^0 d\lambda + \int_0^{b-a} dc \int_{-2p-b+c}^{a-b+c} d\lambda \right. \\ \left. + \int_{b-a}^{2p+b} dc \int_{-2p+c-b}^0 d\lambda + \int_0^{b-a} dc \int_{a-b+c}^0 d\lambda - \int_0^{2p+a} dc \int_{-2p-a+c}^0 d\lambda \right] \Bigg\}$$

$$\times \left(\frac{\lambda}{2} + p\right) \sum_{r=-\infty}^{\infty} J_r(c) J_r(b) J_r(a+c-b) J_{r+1}(a-\lambda) \cos \sigma(a+b)$$

By appropriate grouping of the terms as shown in

Appendix X we finally get

Q2

$$T_1 = A \int_{-2p}^0 db \left\{ \int_{-2p}^0 da \int_0^{b-c} dc \int_{-2p-a}^0 d\lambda - \int_b^0 da \int_0^{b-a} dc \int_{-2p+c-a}^0 d\lambda \right\} \\ \times \sum_{r=-\infty}^{\infty} p J_r(b) J_r(c+a-b) J_r(c) J_{r+1}(a-\lambda) \cos \sigma(a+b)$$

Now consider the integral

$$\int_{-2p-a}^0 d\lambda J_{r+1}(a-\lambda) \quad \text{Let } \mu = a - \lambda, \text{ then}$$

$$= \int_a^{2p+a} d\mu J_{r+1}(\mu)$$

By the mean value theorem this is

$$(2p+a) J_{r+1}(\xi) \quad \text{where } \xi \text{ is some definite value}$$

between a and $2p+2a$.

Then we have, for the first integral in Q_2

$$\int_{-2p}^0 db \int_{-2p}^0 da \int_0^{b-a} dc \sum_r p(2p+2a) J_r(b) J_r(c+a-b) J_r(c) J_{r+1}(\xi) \cos \sigma(a+b)$$

But

$$\int_0^{b-a} dc J_r(c) J_r(c+a-b) = (-1)^r \int_0^{b-a} J_r(c) J_r(b-a-c) = 2(-1)^r \sum_{s=0}^{\infty} (-1)^s J_{2r+2s+1}(b-a),$$

(for $r > 0$)

Thus we get (to highest order in p)

$$\sim \int_{-2p}^0 db \int_{-2p}^0 da \left[p^2 \sum_{r=0}^{\infty} J_r(b) J_{r+1}(\xi) \sum_{s=0}^{\infty} (-1)^{r+s} J_{2r+2s+1}(b-a) + p^2 \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} (-1)^{r+s} J_r(b) J_{r-1}(\xi) J_{2r+2s+1}(b-a) \right] \times \cos \sigma(a+b)$$

Now ξ must be between a and $2p+2a$, let it be αa where α is some number (+ or -). We have then essentially the same integral as was discussed in $R_1 + S_1$, and which was shown to be of order of p^2 .

The same applies to the second integral

$$\int_{-2p}^0 db \int_b^0 da \int_0^{b-a} dc \int_{-2p+c-a}^0 d\lambda \sum_r p J_r(b) J_r(c+a-b) J_r(c) J_{r+1}(a-\lambda) \\ \times \cos \sigma(a+b)$$

Thus it has been shown that T_1 is of order p^2 .
 By the same token T_2 is of order p^3 since everything goes through in the same fashion except that we have μ^2 instead of μ ; this becomes $(\lambda-p)^2$ so that we start the evaluation with a factor p^2 instead of p . The other difference is that instead of $J_{r+1}(a-\lambda)$ we now have $J_r(a-\lambda)$ and $J_{r+2}(a-\lambda)$ but this does not affect the result and we obtain finally that T_2 is of order p^3 .

APPENDIX R

Derivation of Formulas B17, B18, B19

One starts from Graf's formula*

$$R_1 \quad J_\nu(\tilde{\omega}) \left\{ \frac{u - v e^{-i\phi}}{u - v e^{i\phi}} \right\}^{1/2} = \sum_{m=-\infty}^{\infty} J_{\nu+m}(u) J_m(v) e^{im\phi}$$

where

$$\tilde{\omega} = (u^2 + v^2 - 2uv \cos \phi)^{1/2}$$

To get B18 take derivatives on both sides with respect to ϕ then evaluate at $\phi = 0$. This produces

$$\frac{i v v}{u - v} J_\nu(u - v) = \sum_m J_{\nu+m}(u) J_m(v) i m$$

which is B18

To get B19 differentiate R_1 twice with respect to ϕ then evaluate for $\phi = 0$.

Eq. B17 is obtained directly from R_2 by setting $\phi = 0$.

*Watson, Theory of Bessel Functions, 359

APPENDIX S

Initial Coefficient of $\sum s^2 < \xi_s^{(2)} | \xi_s^{(2)} >$

Grouping together all the "i" ($\sqrt{-1}$) and the

J 's of expression B2 we then get

$$(i)^{i+j+k+l} (-i)^{m+n+u+v} (i)^{1s+i-j+k-l} (-i)^{1s+m-n+u-v} J_{1s+i-j+k-l}^{(2q')} \\ \times J_{1s+m-n+u-v}^{(2q''')}$$

$$(1) \text{ Let } s+i-j+k-l > 0 \quad \text{and} \quad s+m-n+u-v > 0$$

we then get

$$(i)^{i+j+k+l+s+i-j+k-l} (-i)^{m+n+u+v+s+m-n+u-v} J_{s+i-j+k-l}^{(2q')} J_{s+m-n+u-v}^{(2q''')} \\ = (-1)^{i+k+m+u} J_{s+i-j+k-l}^{(2q')} J_{s+m-n+u-v}^{(2q''')}$$

$$(2) \quad s+i-j+k-l > 0 \quad ; \quad s+m-n+u-v < 0$$

then we get

$$(i)^{s+2i+2k} (-i)^{-s+2n+2v} J_{s+i-j+k-l}^{(2q')} (-1)^{s+m-n+u-v} J_{s+m-n+u-v}^{(2q''')} \\ = (-1)^{i+k+m+u} J_{s+i-j+k-l}^{(2q')} J_{s+m-n+u-v}^{(2q''')}$$

$$(3) \quad s+i-j+k-l < 0 \quad ; \quad s+m-n+u-v > 0$$

then we get

$$(i)^{-s+2j+2l} (-i)^{s+2m+2u} J_{s+i-j+k-l}^{(2q')} (-1)^{s+i-j+k-l} J_{s+m-n+u-v}^{(2q''')} \\ = (-1)^{j+l+i-j+k-l+m+u} J_{s+i-j+k-l}^{(2q')} J_{s+m-n+u-v}^{(2q''')} \\ = (-1)^{i+k+m+u} J_{s+i-j+k-l}^{(2q')} J_{s+m-n+u-v}^{(2q''')}$$

$$(4) \quad s+i-j+k-\ell < 0 \quad ; \quad s+m-n+u-v < 0$$

then we get

$$\begin{aligned} & (i)^{-s+2j+2\ell} (-i)^{-s+2m+2v} J_{-s-i+j-k+\ell}^{(2q')} J_{-s-m+n-u+v}^{(2q''')} \\ &= (-1)^{j+\ell+n+v+s+i-j+k-\ell+s+m-n+u-v} J_{s+i-j+k-\ell}^{(2q')} J_{s+m-n+u-v}^{(2q''')} \\ &= (-1)^{i+k+m+u} J_{s+i-j+k-\ell}^{(2q')} J_{s+m-n+u-v}^{(2q''')} \end{aligned}$$

Thus the same result is always obtained, namely

$$(-1)^{i+k+m+u} J_{s+i-j+k-\ell}^{(2q')} J_{s+m-n+u-v}^{(2q''')}$$

APPENDIX T

Initial Coefficient of $\sum s^i \zeta \xi_s^{(1)} | \xi_s^{(3)} \rangle$

As in Appendix C group together all the " i " ($\sqrt{-1}$) terms to get

$$(i)^{i+j} (-i)^{k+l+m+n+u+v} (i)^{|s+i-j|} (-i)^{|s+k-l+m-n+u-v|} J_{|s+i-j|}^{(2q)} J_{|s+k-l+m-n+u-v|}^{(2q')}$$

(1) If $s+i-j > 0$ and $s+k-l+m-n+u-v > 0$ we get

$$\begin{aligned} & (i)^{s+2i} (-i)^{s+2k+2m+2u} J_{s+i-j}^{(2q)} J_{s+k-l+m-n+u-v}^{(2q''')} \\ & = (-1)^{i+k+m+u} J_{s+i-j}^{(2q)} J_{s+k-l+m-n+u-v}^{(2q''')} \end{aligned}$$

(2) If $s+i-j < 0$ and $s+k-l+m-n+u-v < 0$ then we

get

$$\begin{aligned} & (i)^{-s+2j} (-i)^{-s+2l+2n+2v} (-1)^{s+i-j} J_{s+i-j}^{(2q)} (-1)^{s+k-l+m-n+u-v} J_{s+k-l+m-n+u-v}^{(2q''')} \\ & = (-1)^{i+k+m+u} J_{s+i-j}^{(2q)} J_{s+k-l+m-n+u-v}^{(2q''')} \end{aligned}$$

(3) If $s+i-j < 0$ and $s+k-l+m-n+u-v > 0$ we get

$$\begin{aligned} & (i)^{-s+2j} (-i)^{s+k+m+u+v} J_{s+i-j}^{(2q)} (-1)^{s+i-j} J_{s+k-l+m-n+u-v}^{(2q''')} \\ & = (-1)^{i+k+m+u} J_{s+i-j}^{(2q)} J_{s+k-l+m-n+u-v}^{(2q''')} \end{aligned}$$

(4) If $s+i-j > 0$ and $s+k-l+m-n+u-v < 0$ we get

$$\begin{aligned} & (i)^{s+2i} (-i)^{-s+2l+2n+2v} J_{s+i-j}^{(2q)} (-1)^{s+k-l+m-n+u-v} J_{s+k-l+m-n+u-v}^{(2q''')} \\ & = (-1)^{i+k+m+u} J_{s+i-j}^{(2q)} J_{s+k-l+m-n+u-v}^{(2q''')} \end{aligned}$$

APPENDIX U

Initial Coefficient of $\sum s^2 < \xi_s^{(0)} | \xi_s^{(4)} >$

As in Appendices C and D group together all the " i " ($\sqrt{-1}$) terms to get

$$(i)^{|s|} (-i)^{|s+i-j+k-\ell+m-n+u-v|+i+j+k+\ell+m+n+u+v} J_{s|}^{(2q)} J_{s+i-j+k-\ell+m-n+u-v}^{(2q''')}$$

(1) If $s > 0$ and $s+i-j+k-\ell+m-n+u-v > 0$ we get

$$(-1)^{i+k+m+u} J_s^{(2q)} J_{s+i-j+k-\ell+m-n+u-v}^{(2q''')}$$

(2) If $s < 0$ and $s+i-j+k-\ell+m-n+u-v < 0$ we get

$$(-1)^{i+k+m+u} J_s^{(2q)} J_{s+i-j+k-\ell+m-n+u-v}^{(2q''')}$$

(3) If $s < 0$ and $s+i-j+k-\ell+m-n+u-v > 0$ we get

$$\begin{aligned} & (-i)^s (i)^{-s} (-i)^{i-j+k-\ell+m-n+u-v+i+j+k+\ell+m+n+u+v} (-1)^s J_s^{(2q)} J_{s+i-j+k-\ell+m-n+u-v}^{(2q''')} \\ & = (-1)^{i+k+m+u} J_s^{(2q)} J_{s+i-j+k-\ell+m-n+u-v}^{(2q''')} \end{aligned}$$

(4) If $s > 0$ and $s+i-j+k-\ell+m-n+u-v < 0$ we get

$$\begin{aligned} & i^s (-i)^{-s} (-i)^{-i+j-k+\ell-m+n+u+v+i+j+k+\ell+m+n+u+v} J_s^{(2q)} J_{s+i-j+k-\ell+m-n+u-v}^{(2q''')} \\ & \quad \times (-1)^{s+i-j+k-\ell+m-n+u-v} \\ & = (-1)^{i+k+m+u} J_s^{(2q)} J_{s+i-j+k-\ell+m-n+u-v}^{(2q''')} \end{aligned}$$

APPENDIX V

Details of Appendix G

Here we investigate those cases of eq. G 2 for $i, j \neq 0$, considering only the term $\cos \nu \cos \sigma \nu$.

Hence the general expression is

$$i, j \text{ I} = A z^{4-i-j} \int_0^{z-\nu} d\nu \int_{\nu+2\nu}^{2z-\nu} d\mu \frac{(\mu+\nu)^j (\mu-\nu)^i}{2^{i+j+1} \sqrt{\mu^2 - \nu^2}} \frac{J_i(\nu)}{\nu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu]$$

(1) $i=1, j=0$ or $i=0, j=1$

We get

$$i, j \text{ I} = A z^3 \int_0^{z-\nu} d\nu \int_{\nu+2\nu}^{2z-\nu} d\mu \frac{\mu \pm \nu}{\mu} \frac{J_i(\nu)}{\nu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu]$$

The integral can then be split in two terms

$$\Delta^1 = \int_0^{z-\nu} d\nu \int_{\nu+2\nu}^{2z-\nu} d\mu \frac{J_i(\nu)}{\nu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu]$$

$$\Delta^2 = \int_0^{z-\nu} d\nu \int_{\nu+2\nu}^{2z-\nu} d\mu \frac{J_i(\nu)}{\mu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu]$$

(1) $\Delta^1 = \int_0^{z-\nu} d\nu (2z - 2\nu - 2\nu) \frac{J_i(\nu)}{\nu} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu]$

$$= 2z \underset{\sigma \geq 2}{[0]} - 2 \int_0^{z-\nu} d\nu J_i(\nu) [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu] - 2\nu \underset{\sigma \geq 2}{[0]}$$

The second integral is finite. This can be seen by splitting it into

$$\int_0^{\nu} d\nu + \int_{\nu}^{z-\nu} d\nu$$

The first is obviously convergent and in the second we get

$$\int_{\nu}^{z-\nu} \frac{\sin \nu - \cos \nu}{\nu^{1/2}} [\cos(\sigma+1)\nu + \cos(\sigma-1)\nu] \sim \frac{1}{(z-\nu)^{1/2}}$$

In all cases (provided $\sigma \gg 1$) we get terms of lower order than z^4 .

For $\sigma < 1$ we get at most terms of order z^4 which is lower than z^4/hz .

$$(2) \Delta^2 = \int_0^{z-v} dv J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v] [1/h(2z-v) - 1/h(v+2v)]$$

$$\text{using again } 1/h(2z-v) = 1/h2z(1 - \frac{v}{2z}) \approx 1/h2z - \frac{v}{2z^2}$$

We now have three types:

$$(a) 1/h2z \int_0^{z-v} dv J_1(v) \cos \alpha v$$

$$\text{When } z \rightarrow \infty \int_0^{\infty} dv J_1(v) \cos \alpha v$$

$$= \frac{1}{\sqrt{\alpha^2-1} (\alpha + \sqrt{\alpha^2-1})} \quad \text{if } \alpha > 1$$

$$= \frac{\cos \sin^{-1} \alpha}{\sqrt{\alpha^2-1}} \quad \text{if } \alpha < 1$$

$$\sim z^{1/2} \quad \text{if } \alpha = 1$$

but even in the case $\alpha = 1$, we have

as z dependence $(z^3/hz) z^{1/2} < z^4$.

(b) The term containing $\frac{v}{2z}$ is

$$\frac{1}{2z} \int_0^{z-v} v J_1(v) \cos \alpha v dv = \frac{1}{2z} \left[\int_0^v dv + \int_v^{z-v} dv \right]$$

$$\sim \frac{1}{2z} \int_v^{z-v} \frac{v(\sin v - \cos v) \cos \alpha v}{v^{1/2}} dv \sim \frac{\left\{ \frac{1}{\cos} \right\} \beta(z-v)}{2z^{1/2}}$$

$$(c) \int_0^{z-v} J_1(v) \cos \alpha v \ln(v+2v) = \int_0^v dv + \int_v^{z-v} dv$$

The first integral converges.

The second integral is now

$$\int_v^{z-v} \frac{\sin v - \cos v}{\sqrt{v}} \cos \alpha v \ln(v+2v) dv$$

Thus we investigate an integral of the form

$$\int_v^{z-v} \frac{\left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\} \beta v}{v^{1/2}} \ln(v+2v) dv$$

By parts

$$x = \ln(v+2v)$$

$$dy = \frac{\left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\} \beta v}{v^{3/2}}$$

$$dx = \frac{dv}{v+2v}$$

$$y \sim \frac{\left\{ \begin{smallmatrix} -\cos \\ \sin \end{smallmatrix} \right\} \beta v}{\beta \sqrt{v}} \quad \text{for large } v$$

We then get

$$\frac{\left\{ \begin{smallmatrix} -\cos \\ \sin \end{smallmatrix} \right\} \beta(z-v)}{\beta \sqrt{z-v}} \ln(z+2v) - \int_v^{z-v} \frac{\left\{ \begin{smallmatrix} -\cos \\ \sin \end{smallmatrix} \right\} \beta v}{\beta \sqrt{v} (v+2v)} dv$$

↓
~ z^{-3/2}

In any case we finally obtain a term of order in z lower than z^4 , after multiplication by z^3 .

Thus we have shown that for $i=0, j=1$ and $i=1, j=0$ we obtain only terms of order in z lower than 4.

$$(2) \quad i=2, j=0 \quad ; \quad i=1, j=1$$

$${}_{2,0}I = z^2 \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu \frac{(\mu^2 + v^2 + 2\mu v)}{\mu} \frac{J_1(v)}{v} [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

We now have three types

$${}_2\Delta_1 = \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu \mu \frac{J_1(v)}{v} [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$${}_2\Delta_2 = \int_0^{z-v} dv \int_{v+2v}^{2z-v} \frac{v}{\mu} d\mu J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$${}_2\Delta_3 = \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$$(a) \quad {}_2\Delta_1 = \frac{1}{2} \int_0^{z-v} dv \frac{J_1(v)}{v} [\cos(\sigma+1)v + \cos(\sigma-1)v] [4z^2 - 4zv - 4v^2 - 4vv]$$

By previous calculations we have

$$z^2 \left[0 \right]_{\sigma \geq 2} - 4z \left[\frac{1}{(z-v)^{1/2}} \right]_{\sigma \geq 2} - 4v^2 \left[0 \right]_{\sigma \geq 2} - 4v \left[\frac{1}{(z-v)^{1/2}} \right]$$

All of which are of order lower than z^4 even when multiplied by z^2 .

$$(b) \quad {}_2\Delta_2 = \int_0^{z-v} dv v J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v] [\ln(2z-v) - \ln(v+2v)]$$

$$\text{but } \ln(2z-v) \sim \ln 2z - \frac{v}{2z}$$

$$\begin{aligned} (1) \quad & \int_0^{z-v} dv v \ln 2z J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v] \\ &= \ln 2z \left[\int_0^v dv + \int_v^{z-v} v J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v] \right] \end{aligned}$$

The first integral gives a constant. The second becomes

$$\int_v^{z-v} v^{1/2} (\sin v - \cos v) [\cos(\sigma+1)v + \cos(\sigma-1)v] dv$$

$$\sim (z-v)^{1/2} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \beta(z-v)$$

Thus for this part we get finally as z dependence

$$\sim z^{5/2} / n z$$

(2) All other terms are clearly of lower order in z .

$$(c) {}_2\Delta_3 = \int_0^{z-v} J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v] (2z - 2v - 2v)$$

From $i=1, j=0$ parts a and b this will produce

$$\text{at most } z^{7/2} < z^4.$$

$$(3) \quad i=2, j=1 \quad ; \quad i=0, j=3$$

$${}_{0,3}^{2,1}I = z \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu \frac{[\mu^3 + \alpha\mu^2v + \alpha\mu v^2 + v^3]}{\mu} \frac{J_1(v)}{v} [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

We now get the following integrals:

$${}_3\Delta_1 = \int_0^{z-v} dv (8z^3 - 12z^2v + 6zv^2 - 2v^3 - 6v^2v - 12vv^2 - 8v^3) \frac{J_1(v)}{v} (\cos \alpha v + \cos \beta v)$$

$${}_3\Delta_2 = \int_0^{z-v} dv (4z^2 - 4zv - 4v^2 - 4vv) J_1(v) (\cos \alpha v + \cos \beta v)$$

$${}_3\Delta_3 = \int_0^{z-v} dv (2z - 2v - 2v) v J_1(v) (\cos \alpha v + \cos \beta v)$$

$${}_4\Delta_3 = \int_0^{z-v} dv v^2 J_1(v) [1/h(z-2v) - 1/h(v+2v)] (\cos \alpha v + \cos \beta v)$$

$$(a) \quad {}_3\Delta_1$$

The only new type of integral is

$$-\int_0^{z-v} dv v^2 J_1(v) \cos \alpha v = \int_0^v dv + \int_v^{z-v} dv$$

The first integral is a constant; the second becomes

$$\int_v^{z-v} dv v^{3/2} (\sin v - \cos v) \cos \alpha v \sim (z-v)^{3/2} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \beta(z-v)$$

Using previous calculations we now get for z dependence of ${}_3\Delta_1$

$$\sim z^4 [0] - z^3 \left[\frac{1}{(z-v)^{1/2}} \right] + z^2 [(z-v)^{1/2}] - z [(z-v)^{3/2}] < z^4$$

(b) ${}_3\Delta_2$

By previous calculations (namely (a) of ${}_2\Delta_1$)

we get for ${}_3\Delta_2$

$$z^3 [0] - z^2 \left[\frac{1}{(z-v)^{1/2}} \right]$$

(c) ${}_3\Delta_3$

By (a), ${}_2\Delta_1$ and by ${}_3\Delta_1$ we get for ${}_3\Delta_3$

$$z^2 [(z-v)^{1/2} \left\{ \frac{\sin}{\cos} \right\} \beta(z-v)] - z [(z-v)^{3/2} \left\{ \frac{\sin}{\cos} \right\} \beta(z-v)] < z^4$$

(d) ${}_3\Delta_4$

Using the same procedure as in ${}_2\Delta_2$ we get for

$$\ln(2z-v) \sim \ln 2z - \frac{v}{2z}$$

$$\text{then } \ln 2z \int_0^{z-v} v^2 J_1(v) \cos \alpha v \sim \ln 2z [(z-v)^{3/2} \left\{ \frac{\sin}{\cos} \right\} \beta(z-v)]$$

and ${}_3\Delta_4$ is at most of order

$$\ln z [z(z-v)^{3/2}] \left\{ \frac{\sin}{\cos} \right\} \beta(z-v) \sim z^{5/2} \ln z < z^4$$

$$(4) \quad i=0, j=4 \quad ; \quad i=3, j=1 \quad ; \quad i=2, j=2$$

$$\begin{aligned} {}_{0,4}^{3,1} \Delta_{2,2} I &= \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu (\mu^4 + a\mu^3v + b\mu^2v^2 + c\mu v^3 + dv^4) \frac{J_1(v)}{\mu v} \\ &\quad \times [\cos(\sigma+1)v + \cos(\sigma-1)v] \end{aligned}$$

$${}_4\Delta_1 = \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu \mu^3 \frac{J_1(v)}{v} [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$${}_4\Delta_2 = \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu \mu^2 J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$${}_4\Delta_3 = \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu \mu v J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$${}_4\Delta_4 = \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu v^2 J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$${}_4\Delta_5 = \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu \frac{v^3}{\mu} J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$$(a) \quad {}_4\Delta_1 =$$

$$\int_0^{z-v} dv [16z^4 - 32z^3v + 24z^2v^2 - 8zv^3 + v^4] \frac{J_1(v)}{v} [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

The only new type of integral is

$$\int_0^{z-v} dv v^3 J_1(v) \cos \alpha v = \int_0^v dv + \int_v^{z-v} dv$$

The first integral is a constant. The second becomes

$$\int_v^{z-v} v^{5/2} (\sin v - \cos v) \cos \alpha v dv \sim (z-v)^{5/2} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \beta(z-v)$$

By previous calculations, the z dependence of ${}_4\Delta_1$ is then ($\sigma \gg 1$)

$$z^4 [0] + z^3 [(z-v)^{-1/2}] + z^2 [(z-v)^{1/2}] + z [(z-v)^{3/2}] + [(z-v)^{5/2}]$$

$$< z^4$$

$$(b) {}_4\Delta_2 = \int_0^{z-v} dv [8z^3 - 12z^2v + 6zv^2 - v^3] J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

From previous calculations it is clear that for large z we have

$${}_4\Delta_2 \sim z^{5/2} \text{ at most.}$$

$$(c) {}_4\Delta_3 = \int_0^{z-v} dv [4z^2 - 4zv + v^2] v J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

Again from previous calculations we have

$${}_4\Delta_3 \sim z^{5/2} \text{ at most.}$$

$$(d) {}_4\Delta_4 = \int_0^{z-v} dv [2z - v - 2v] v^2 J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v]$$

$$\text{Then } {}_4\Delta_4 \sim z^{5/2}$$

$$(e) {}_4\Delta_5 = \int_0^{z-v} dv v^3 J_1(v) [\cos(\sigma+1)v + \cos(\sigma-1)v] [\ln 2z - \frac{v}{2z} - \ln(v+2v)]$$

$$\text{from } {}_4\Delta_1 \quad (a) \quad \text{we get } {}_4\Delta_5 \sim (\ln 2z)(z-v)^{5/2} < z^4$$

APPENDIX W

Evaluation of I_{3b}

Evaluation of I'_{3b}

$$I'_{3b} = \int_0^z d\lambda \int_0^\lambda d\xi (\lambda - \xi)(z - \lambda) J_0(\xi) J_0(\lambda - \xi) J_0(\lambda) \cos \sigma(\lambda + \xi)$$

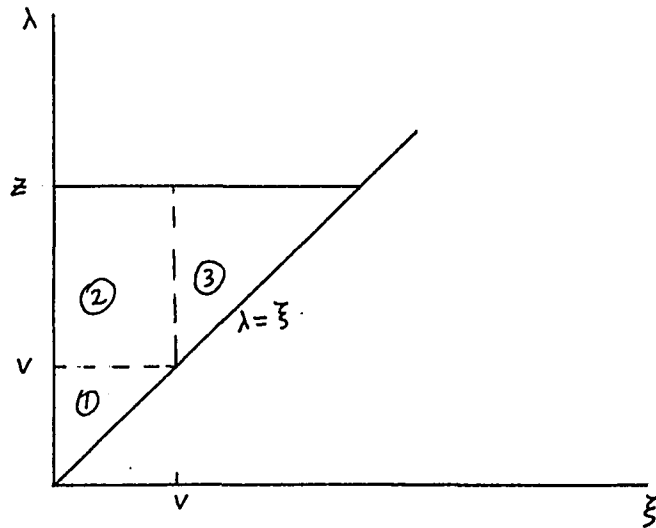


Fig. W1

Divide into three regions:

$$I'_{3b} = \underbrace{\int_0^v d\lambda \int_0^\lambda d\xi}_{(1)} + \underbrace{\int_0^v d\xi \int_v^z d\lambda}_{(2)} + \underbrace{\int_v^z d\lambda \int_v^\lambda d\xi}_{(3)}$$

The first integral gives at most a constant times z thus of much lower order in z than z^4 .

The second integral can be subdivided into two

$$\int_0^v d\xi \int_v^{2v} d\lambda + \int_0^v d\xi \int_v^\xi d\lambda$$

The first of these integrals yields the same result as the integral over ① .

In the second integral we may expand $J_0(\lambda - \xi) J_0(\lambda)$ thus

$$J_0(\lambda - \xi) J_0(\lambda) \sim \frac{2}{\pi} \frac{[\sin(\lambda - \xi) + \cos(\lambda - \xi)]}{\sqrt{\lambda} \sqrt{\lambda - \xi}} (\sin \lambda + \cos \lambda)$$

but $\lambda \gg \xi$, then $(\lambda - \xi)^{1/2} \sim \frac{1}{\sqrt{\lambda}} (1 + \frac{\xi}{2\lambda})$

and the integral is

$$\int_0^V J_0(\xi) d\xi \int_{2V}^z d\lambda \frac{(z\lambda - z\xi + \lambda\xi)}{\lambda} [\sin(\lambda - \xi) + \cos(\lambda - \xi)] (\sin \lambda + \cos \lambda) \times \cos \sigma(\lambda + \xi)$$

The " λ " integration will yield at most term of order $z \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\} \beta z$ whilst the " ξ " integration yields a constant.

Thus we again get only terms of order lower than 4 in z .

For integration over domain (3) we can expand only

$J_0(\xi) J_0(\lambda)$ to get

$$J_0(\xi) J_0(\lambda) \sim \frac{2}{\pi} \frac{1}{\sqrt{\lambda\xi}} [\cos(\lambda - \xi) + \sin(\lambda + \xi)]$$

$$\int_V^z d\lambda \int_V^\lambda d\xi \frac{(z\lambda - z\xi - \lambda^2 + \lambda\xi)}{\sqrt{\lambda\xi}} J_0(\lambda - \xi) \cos \sigma(\lambda + \xi) [\cos(\lambda - \xi) + \sin(\lambda + \xi)]$$

Let now $\lambda + \xi = \mu$, $\lambda - \xi = \nu$ and get

$$\frac{1}{2} \int_0^{z-V} d\nu \int_{\nu+2V}^{2z-\nu} d\mu \frac{\nu(2z-\mu-\nu)}{\sqrt{\mu^2-\nu^2}} J_0(\nu) \cos \sigma \mu (\cos \nu + \sin \mu)$$

Comparison with $'I_{1,3}$ where the same integral occurs shows immediately that the highest order in z is of lower order than z^4 .

Evaluation of I_{3b}^2 .

$$I_{3b}^2 = \int_0^z d\lambda \int_0^\lambda d\xi J_0(\xi) J_0(\lambda - \xi) J_1(\lambda) (z - \lambda)^2 (\lambda - \xi) \cos \sigma(\lambda + \xi)$$

Since the expansion for large λ of $J_1(\lambda)$ is

$$\frac{\sin \lambda - \cos \lambda}{\sqrt{\lambda}}$$

we see that by following the same procedure as in I_{3b}' we get at most a term in z^2 which is again negligible when compared to z^4 .

Evaluation of I_{3b}^3

$$I_{3b}^3 = \int_0^z d\lambda \int_0^\lambda d\xi \lambda (z-\lambda)^2 (\lambda-\xi) J_0(\lambda) J_0(\xi) J_0(\lambda-\xi) \cos \sigma(\lambda+\xi)$$

Using again the same three domains already displayed in I_{3b}^1 , we see that the highest order term is $z^2 \lambda^2$. In domain (1) we get only a z^2 term. In domain (2) we get

$$z^2 \int_0^v d\xi J_0(\xi) \int_{2v}^z \frac{\lambda^2}{\lambda} [\sin(\lambda-\xi) + \cos(\lambda-\xi)] (\sin \lambda + \cos \lambda) \cos \sigma(\lambda+\xi)$$

The " λ " integration yields terms of the form $z^3 \left\{ \frac{\sin}{\cos} \right\} \beta z$ which become multiplied by a constant when the " ξ " integration is performed.

In domain (3) we get

$$z^2 \int_0^{z-v} dv \int_{v+2v}^{2z-v} d\mu \frac{(\mu+v)^2}{\mu} J_0(v) \cos \sigma \mu [\cos v + \sin \mu]$$

and again we get, by consulting $z, I \dots$, at most terms like

$$z^{5/2} \left\{ \frac{\sin}{\cos} \right\} \beta z \ll z^4.$$

Evaluation of I_{3b}^4

$$I_{3b}^4 = \int_0^z d\lambda \int_0^\lambda d\xi (\lambda - \xi) (z - \lambda)^2 \lambda [H_0(\lambda) J_1(\lambda) - H_1(\lambda) J_0(\lambda)] J_0(\xi) J_0(\lambda - \xi) \cos \sigma(\lambda)$$

As before we subdivide the area of integration into three domains

$$\int_0^V d\lambda \int_0^\lambda d\xi \quad (1) \quad + \int_V^z d\lambda \int_0^V d\xi \quad (2) \quad + \int_V^z d\lambda \int_V^\lambda d\xi \quad (3)$$

The first integral gives a constant, thus we have at most a z^2 term.

The second integral becomes first

$$\int_V^z d\lambda \int_0^V d\xi = \int_0^V d\xi \int_V^{2V} d\lambda + \int_0^V d\xi \int_{2V}^z d\lambda$$

The first integral on the right is similar to the integral over domain (1).

$$\text{We remain thus } \frac{2}{\pi} \int_0^V d\xi \int_{2V}^z d\lambda (\lambda - \xi) \lambda (z^2 - 2z\lambda + \lambda^2) \frac{[\sin(\lambda - \xi) + \cos(\lambda - \xi)]}{\lambda^{3/2}}$$

$$\int_0^V d\xi \int_{2V}^z d\lambda (\lambda - \xi) (z^2 - 2z\lambda + \lambda^2) \lambda [H_0(\lambda) J_1(\lambda) - H_1(\lambda) J_0(\lambda)] J_0(\lambda - \xi) J_0(\xi) \cos \sigma(\lambda) \\ \int_0^V d\xi \int_{2V}^z d\lambda (\lambda - \xi) (z^2 - 2z\lambda + \lambda^2) \lambda [H_0(\lambda) J_1(\lambda) - H_1(\lambda) J_0(\lambda)] J_0(\lambda - \xi) J_0(\xi) \cos \sigma(\lambda)$$

Since $\lambda \gg 1, \lambda \gg \xi$ we get, using J_3 and J_4

$$\frac{2}{\pi} \int_0^V d\xi \int_{2V}^z d\lambda (\lambda - \xi) \lambda (z^2 - 2z\lambda + \lambda^2) \frac{[\sin(\lambda - \xi) + \cos(\lambda - \xi)]}{\lambda^{3/2}} \left(1 + \frac{\xi}{2\lambda}\right) J_0(\xi) \cos \sigma(\lambda)$$

In the " λ " integration we get at most (in z) terms

of the type

$$\int_0^v d\xi z^{5/2} \left\{ \frac{\sin}{\cos} \right\} \beta z J_0(\xi) \left\{ \frac{\sin}{\cos} \right\} \sigma \xi \sim z^{5/2} \ll z^4$$

In domain (3) we can expand $J_0(\xi)$ and $-H_1(\lambda)J_0(\lambda) + H_0(\lambda)J_1(\lambda)$

and get

$$\int_v^z d\lambda \int_v^\lambda d\xi J_0(\lambda-\xi) \frac{(\sin \xi + \cos \xi)}{\lambda \sqrt{\xi}} \cos \sigma(\lambda+\xi) (\lambda^2 - \lambda\xi)(z^2 - 2z\lambda + \lambda^2)$$

Let $\lambda + \xi = \mu$; $\lambda - \xi = \nu$ and get

$$\int_0^{z-v} d\nu \int_{\nu+2\nu}^{2z-\nu} d\mu \frac{\nu(\mu+\nu)(2z-\mu-\nu)^2}{4(\mu+\nu)\sqrt{\mu-\nu}} J_0(\nu) \left[\sin \frac{\mu-\nu}{2} + \cos \frac{\mu-\nu}{2} \right] \cos \sigma \mu$$

which becomes a sum of eight terms when we rearrange the

trigonometric terms. Each of these terms will be of the form

$$\int_0^{z-v} d\nu \int_{\nu+2\nu}^{2z-\nu} d\mu \frac{\nu(2z-\mu-\nu)^2}{\mu} J_0(\nu) \left\{ \frac{\sin}{\cos} \right\} \alpha \nu \left\{ \frac{\sin}{\cos} \right\} \beta \mu$$

The integration with respect to μ gives at most terms

of the type

$$\int_0^{z-v} d\nu \frac{\nu z^2}{2z-\nu} \left\{ \frac{\sin}{\cos} \right\} \nu (2z-\nu) J_0(\nu) = \int_0^v d\nu + \int_v^{z-v} d\nu$$

The first integral will be of order z . The second,

where we can use expansion for $J_0(\nu)$ gives

$$\int_0^{z-v} d\nu \frac{z^2 \sqrt{\nu}}{2z-\nu} \left\{ \frac{\sin}{\cos} \right\} \nu (2z-\nu) (\sin \nu + \cos \nu) \sim z^{3/2} \left\{ \frac{\sin}{\cos} \right\} \beta z \ll z^4$$

APPENDIX X

Reduction of T_1 (Q1)

In eq. Q1 number the terms 1, 2, 3, ... 19 .

The sequence is as follows:

$$3 + 7$$

$$3 + 7 + 6$$

$$3 + 7 + 6 + 4$$

$$3 + 7 + 6 + 4 + 5 \rightarrow \int_0^{b-c} dc \int_{-2p-a}^0 d\lambda$$

which are the first two integrations followed by $\int_{-2p}^0 db \int_{-2p}^0 da$

Next

$$9 + 12$$

$$8 + 11$$

$$\text{and } 10 + 13 = 0$$

$$\text{to get } - \int_0^{b-a} dc \int_{-2p+c-a}^0 d\lambda$$

Finally

$$15 + 17 = 0$$

$$16 + 18 = 0$$

$$14 + 19 = 0$$