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A RELATIVISTIC INFINITE COMPONENT WAVE EQUATION WITH
THE INTERNAL STRUCTURE OF A THREE-DIMENSIONAL
HARMONIC OSCILLATOR

by

GERALD J. PORTNER

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Abstract

A RELATIVISTIC INFINITE COMPONENT WAVE EQUATION WITH THE INTERNAL STRUCTURE OF A THREE-DIMENSIONAL HARMONIC OSCILLATOR

by

Gerald J. Portner

Advisor: Professor Ngee Pong Chang

A representation of the group $SU(3,1)$ which contains a Lorentz four-vector is used as the basis of an infinite component wave equation. The wave equation has a linearly increasing harmonic oscillator $(\text{mass})^2$ spectrum, the rest frame wave functions being those of a non-relativistic three-dimensional harmonic oscillator. Spacelike solutions of the wave equation are expressible as a linear combination of four timelike solutions so that the Hilbert space of timelike solutions is complete. Properties of the wave equation, its solutions and the associated currents are discussed. A lowest order, ghost free, scattering amplitude is obtained, and some of its analytic properties are discussed. A possible extension of the model to half integral spin is described.

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1. Introduction

There is considerable evidence from Regge analysis of scattering data for linear trajectories of particle resonances. Many attempts have been made to explain this. The Dual Resonance model is one major attempt at providing a clue to the fundamental structure of space-time which would lead to linear trajectories. Another attempt, pioneered by Nambu^{1,2} and elaborated in many papers by Fronsda1, Barut, and others, assigns infinite families of resonances to infinite multiplets of an internal symmetry group. This idea dates back to Majorana³ who was apparently the first to utilize the infinite dimensional representations of the Lorentz group and to provide a mass spectrum for elementary particles $\left(m = \frac{k}{j+1/2} \right)$. The infinite multiplet approach uses group theory to determine properties of the states, internal quantum numbers, degeneracy structure of mass levels, and other observables. It also uses wave functions with an infinite number of components to describe the infinite tower of discrete states of an isolated system, e.g., an atom or a hadron. The hydrogen atom model^{1,2,4} has been actively explored in this way.

From the point of view of basic quark dynamics the infinite component idea is a very attractive one. The infinite number of states can be viewed as the excitations of the fundamental constituents.

In particular, the linear trajectories, i.e., equal spacing, are suggestive of an internal oscillator dynamics. There have been a number of non-relativistic oscillator models of quark dynamics which have had some success in describing particle data.⁵

The problem with relativistic oscillators is their timelike modes which introduce ghost states into the theory. Although this is true for dual models, an infinite number of fundamental frequencies are introduced here (thus the "string" model) which, as has recently been proved,⁶ can serve to cancel the timelike ghosts. Feynman, Kislinger and Ravndal,⁷ have discussed the relativistic oscillator states. They introduce subsidiary condition to eliminate the timelike modes; however, they point out that leaving out the timelike excitations in summing over states gives results which are too large, since the timelike contributions have not been subtracted away. To correct this they introduce an over-all quenching factor to adjust matrix elements.

In this thesis we would like to discuss a new model of relativistic oscillator states, without ghosts. The annihilation and creation operators for the timelike modes $(a_0, a_0^\dagger)$ are simply replaced by $\sqrt{C+C}$,

$$\mathcal{C} = \vec{a}^\dagger \cdot \vec{a} \quad . \quad \text{The group generated by the operators}$$

$$M_{\mu\nu} = -i(a_\mu a_\nu^\dagger - a_\nu a_\mu^\dagger), \quad S_{\mu\nu} = q_\mu a_\nu^\dagger + a_\nu a_\mu^\dagger \quad (\text{with the replacement above})$$

is $SU(3,1)$, the minimal relativistic extension of the $SU(3)$ symmetry

group of the three dimensional harmonic oscillator. Models based on this representation have been considered by Nambu and Fronsda1.

Since in the representation considered, there are only the tensors

$M_{\mu\nu}$, $S_{\mu\nu}$ and no four vectors, wave equations based on it have necessarily been of the form

$$\{ S_{\mu\nu} p_\mu p_\nu + f(p^2) \} \psi = 0 \quad (1.1)$$

In the model considered here, a four vector can be constructed; as a consequence, the wave equation above can be "linearized" to the form

$$\{ A \cdot p + p^2 \} \psi = 0 \quad (1.2)$$

with $\{ A_\mu, A_\nu \} \propto S_{\mu\nu}$.

This equation leads to a $(\text{mass})^2$ spectrum that has the quantum numbers

of a three dimensional oscillator, and there are no ghosts. The al-

gebra of the operators $M_{\mu\nu}$, $S_{\mu\nu}$, A_μ becomes an infinite dimension-

al one--unlike most infinite component wave equations studied in the

literature; for instance, the Majorana or $O(4,2)$ type. We believe

this to be a significant difference which may make this model more

viable than previous attempts.

Most of the infinite component wave equations which have been proposed suffer from various diseases, the most famous being the

existence of spacelike solutions. Usually this implies that a success-

ful quantization cannot be carried out since spacelike states are unphysical. In the model considered here, there are "spacelike" solutions to the wave equation but they are not independent of the timelike solutions. In fact, the "spacelike" solutions are simple linear combinations of timelike ones. The spacelike states are not new physical modes, and the timelike states alone span the complete Hilbert space.

An explicit quantization of the wave equation (1.2) has been carried out by Bacry and Chang⁸ using only timelike states to construct a causal field. This is not prevented by the Grodsky-Streater theorem which does not apply here. For, the central assumption of the Grodsky-Streater theorem is that the representation in which the field transforms is holomorphic in the extended tube, i.e., holomorphic with respect to the complex Lorentz group. This condition is not true for this representation, and the assumptions of the theorem of finite covariance are not satisfied. Hence the Grodsky-Streater theorem is not applicable. (See reference 7 and Section 5.) There is, as yet, no prescription for a quantum field theory of interactions based on the wave equation.

The aim of this thesis is to explore the properties of the model at the first quantized level. We consider a physical picture of the oscillator states, completeness relations, form factors, and have cal-

culated some lowest order scattering amplitudes. Another goal in this study has been to acquire techniques and experience in calculation at the first quantized level which, hopefully, can be applied to the second quantized field theory.

We have also investigated a generalization of the wave equation to include half integral spin.

2. The Relativistic Harmonic Oscillator

A representation of the group $SU(3,1)$ can be constructed using the usual annihilation and creation operators of the three-dimensional harmonic oscillator. This possibility has been noted in the literature^{2,9,10} and we will take these previous constructions as our starting point.

Consider first the $O(3,1)$ subgroup. The angular momentum operators are constructed in the usual way,

$$\vec{L} = i \vec{a} \times \vec{a}^\dagger, \quad \vec{a} = \frac{1}{\sqrt{2}} (\vec{\xi} + i \vec{\eta}) \quad (2.1)$$

where $\vec{\xi}$ and $\vec{\eta}$ are the internal coordinate and momentum of the oscillator. Then consider the operator $\sqrt{\mathcal{C} + \mathcal{C}}$, where $\mathcal{C} = \vec{a}^\dagger \cdot \vec{a}$ and c is an arbitrary constant. This operator has the following properties with respect to $\vec{a}$ and $\vec{a}^\dagger$:

$$\begin{aligned} \vec{a} \sqrt{\mathcal{C} + c} &= \sqrt{\mathcal{C} + c + 1} \vec{a} \\ \vec{a}^\dagger \sqrt{\mathcal{C} + c} &= \sqrt{\mathcal{C} + c - 1} \vec{a}^\dagger \end{aligned} \quad (2.2)$$

These relations follow from taking matrix elements of both sides of equation (2.2) in a basis in which $\mathcal{C}$ is diagonal. Then the boost operators can be constructed as follows:

$$\vec{K} = i \sqrt{\mathcal{C} + c + 1} \vec{a} - i \sqrt{\mathcal{C} + c} \vec{a}^\dagger \quad (2.3)$$

Using equation (2.2) the boost operators satisfy

$$[K_i, K_j] = -i \epsilon_{ijk} L_k \quad (2.4)$$

and since $\vec{L}$ commutes with $\mathcal{C}$

$$[L_i, K_j] = i\epsilon_{ijk} K_k \quad (2.5)$$

The operators $\vec{K}$ and $\vec{L}$ are therefore a representation of the homogeneous Lorentz group. The $SU(3,1)$ representation has the following properties with respect to the constant c .²

- i) $c \gg 0$, unitary infinite dimensional
- ii) c an integer < 0 , $\mathcal{C} < |c|$, finite non-unitary
- iii) c an integer ≤ 0 , $\mathcal{C} > |c|$, discrete unitary.

In this work we shall be interested in the infinite dimensional unitary representations. We also require that $\sqrt{\mathcal{C}+c} > 0$, and so restrict ourselves to $c > 0$.

Setting

$$M_{ij} = \epsilon_{ijk} L_k$$

$$M_{i4} = -i K_i$$

the $M_{\mu\nu}$ satisfy¹¹

$$[M_{\mu\nu}, M_{\lambda\rho}] = i\delta_{\mu\lambda} M_{\nu\rho} + i\delta_{\nu\rho} M_{\mu\lambda} - i\delta_{\nu\lambda} M_{\mu\rho} - i\delta_{\mu\rho} M_{\nu\lambda} \quad (2.6)$$

The symmetric generators of the $SU(3,1)$ group are constructed as follows:

$$S_{ij} = i(a_i a_j^\dagger + a_j a_i^\dagger)$$

$$S_{i4} = \sqrt{c+c+1} a_i + \sqrt{c+c} a_i^\dagger \quad (2.7)$$

$$S_{44} = -2i(c+c)$$

These operators satisfy

$$\begin{aligned} [S_{\mu\nu}, S_{\lambda\rho}] &= i\delta_{\mu\rho} M_{\lambda\nu} - i\delta_{\nu\lambda} M_{\mu\rho} + i\delta_{\nu\rho} M_{\lambda\mu} - i\delta_{\mu\lambda} M_{\nu\rho} \\ [M_{\mu\nu}, S_{\lambda\rho}] &= i\delta_{\mu\rho} S_{\nu\lambda} - i\delta_{\nu\lambda} S_{\mu\rho} - i\delta_{\nu\rho} S_{\mu\lambda} + i\delta_{\mu\lambda} S_{\nu\rho} \end{aligned} \quad (2.8)$$

Equations (2.6) and (2.8) form the Lie Algebra of the group generators of $SU(3,1)$. The harmonic oscillator states are a basis for the irreducible triangular representations of the $SU(3)$ subgroup ($n_1 + n_2 + n_3 = N$). The properties of this representation are well known; we shall, however, be interested in a generalization of it.

The motivation for doing this is that we would like to write a wave equation of the form

$$(\Gamma \cdot p + p^2) \psi(p) = 0$$

where Γ_μ transforms like a four-vector under the operators $M_{\mu\nu}$ defined above. The mass spectrum will then be given by the eigenvalues of Γ_0 in the frame $\vec{p} = 0$

$$\Gamma_0 \psi(\vec{p}=0) = m \psi(\vec{p}=0)$$

and we would like the mass spectrum to be that of the three-dimensional non-relativistic harmonic oscillator, as previously discussed. The difficulty in writing such an equation is that there are no four-vectors in this representation, only tensors $M_{\mu\nu}$, $S_{\mu\nu}$. This has

led a number of authors to consider wave equations of the form

$$(S_{\mu\nu} p_\mu p_\nu + \alpha p^4 + \beta p^2 + \delta) \psi(p) = 0$$

with a mass spectrum related to $S_{\mu\mu}$.^{2,12,13}

To achieve the generalization we want, we adjoin two representations of the previous type in the following manner:

$$\vec{L} = \begin{pmatrix} \vec{L} & 0 \\ 0 & \vec{L} \end{pmatrix} \quad (2.9)$$

$$\vec{K} = \begin{pmatrix} \vec{K}(c) & 0 \\ 0 & \vec{K}(c-1) \end{pmatrix}$$

where $\vec{K}(c)$ is the boost (equation (2.3)) and $\vec{K}(c-1)$ is obtained from $\vec{K}(c)$ by letting $c \rightarrow c-1$. These operators still satisfy the $O(3,1)$ algebra since the constant c is arbitrary.

This new representation now admits a four-vector

$$A_\mu \equiv \left\{ \begin{pmatrix} 0 & \vec{a} \\ \vec{a}^\dagger & 0 \end{pmatrix}, i \begin{pmatrix} 0 & -\sqrt{c+c} \\ -\sqrt{c+c} & 0 \end{pmatrix} \right\} \quad (2.10)$$

which satisfies

$$[M_{\mu\nu}, A_\rho] = i\delta_{\mu\rho} A_\nu - i\delta_{\nu\rho} A_\mu \quad (2.11)$$

and

$$A_\mu A_\mu = - \begin{pmatrix} c-3 & 0 \\ 0 & c \end{pmatrix} \quad (2.12)$$

The four-vector A_μ is related to the generators $M_{\mu\nu}$ and $S_{\mu\nu}$ in the following way¹⁴.

$$\begin{aligned} [A_\mu, A_\nu] &= -i\sigma_3 M_{\mu\nu} \\ \{A_\mu, A_\nu\} &= -i\sigma_3 S_{\mu\nu} + (1+\sigma_3)\delta_{\mu\nu} \\ \{\sigma_3, A_\mu\} &= 0 \end{aligned} \quad (2.13)$$

The generators $S_{\mu\nu}$ in this representation are

$$S_{ij} = i \begin{pmatrix} a_i a_j^\dagger + a_j a_i^\dagger - 2\delta_{ij} & 0 \\ 0 & -a_i^\dagger a_j - a_j^\dagger a_i \end{pmatrix}, \quad S_{44} = \begin{pmatrix} -2i(c+c+1) & 0 \\ 0 & 2i(c+c) \end{pmatrix} \quad (2.14)$$

$$S_{i4} = \begin{pmatrix} \sqrt{c+c+1} a_i + \sqrt{c+c} a_i^\dagger & 0 \\ 0 & -\sqrt{c+c} a_i - \sqrt{c+c-1} a_i^\dagger \end{pmatrix}$$

The presence of the operator σ_3 in the commutation relations (2.13) of the operators A_μ makes their commutator algebra infinite; i.e., it does not close. It is this property which makes this representation inequivalent to most other infinite component theories where the vector operators are chosen to satisfy $[A_\mu, A_\nu] = \alpha M_{\mu\nu}$.

The Casimir operators of this representation are

$$C_1 = \frac{1}{2} \{ M_{\mu\nu} M_{\mu\nu} - S_{\mu\nu} S_{\mu\nu} \} = 2 \begin{pmatrix} (c+1)(c-2) & 0 \\ 0 & c(c-3) \end{pmatrix}$$

$$C_2 = \sum_{\mu} S_{\mu\mu} = -2i\sigma_3 \begin{pmatrix} c+1 & 0 \\ 0 & c \end{pmatrix} \quad (2.15)$$

and those of the $O(3,1)$ subgroup are

$$C_3 = \frac{i}{8} \epsilon_{\mu\nu\lambda\rho} M_{\mu\nu} M_{\lambda\rho} = \vec{K} \cdot \vec{L} = 0$$

$$C_4 = \frac{1}{2} M_{\mu\nu} M_{\mu\nu} = L^2 - K^2$$

$$= \begin{pmatrix} [(c+c+1)(c+c+2)]^{\frac{1}{2}} \vec{a} \cdot \vec{a} + [(c+c)(c+c-1)]^{\frac{1}{2}} \vec{a}^+ \cdot \vec{a}^+ - \vec{a}^+ \cdot \vec{a}^+ \vec{a} \cdot \vec{a} - c(c+3+2c) - (c+1)(2c-1) & 0 \\ 0 & [(c+c)(c+c+1)]^{\frac{1}{2}} \vec{a} \cdot \vec{a} + [(c+c-1)(c+c-2)]^{\frac{1}{2}} \vec{a}^+ \cdot \vec{a}^+ - \vec{a}^+ \cdot \vec{a}^+ \vec{a} \cdot \vec{a} - c(c+1+2c) - c(2c-3) \end{pmatrix}$$

The relation (2.15) among the $S_{\mu\mu}$ has the significance that the upper and lower components of the solutions to the wave equation are characterized by the constants $c+1$ and c respectively. In the Bacry-Chang paper, which is outlined in section (3), these constants are the degree

of homogeneity of the monomials from which solutions to the wave equation are constructed.

The form of the Casimir operator for the homogeneous Lorentz group, $\frac{1}{2} M_{\mu\nu} M_{\mu\nu}$ indicates that the representation of the Lorentz group is reducible; in fact, it is infinitely reducible. This is demonstrated in Appendix A. The infinite reducibility of the representation allows this model to avoid the conditions of the Grodsky-Streater theorem. This will be discussed in a later section.

3. Wave Equation

With the four-vector A_μ we can write a wave equation

$$(\alpha A \cdot p + p^2) \psi(p) = 0 \quad (3.1)$$

In the frame $\vec{p} = 0$ this becomes

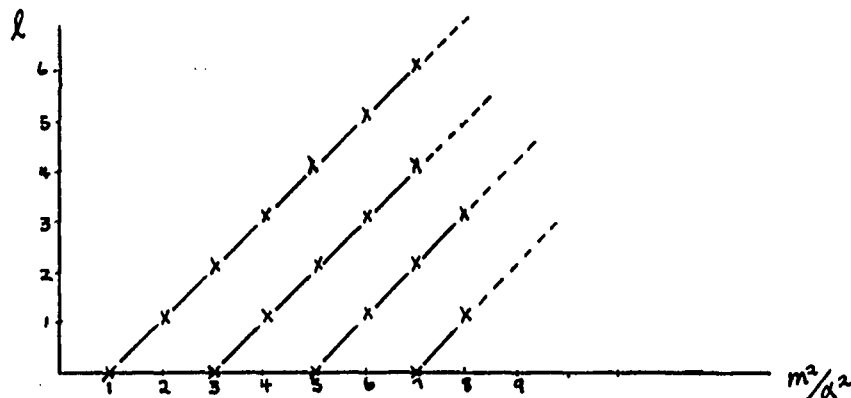
$$-\alpha m A_0 \psi(0) = m^2 \psi(0) \quad (3.2)$$

$$\alpha \begin{pmatrix} 0 & \sqrt{\tau+c} \\ \sqrt{\tau+c} & 0 \end{pmatrix} \psi(0) = m \psi(0)$$

The solutions of (3.2) are

$$\psi_{n,l,m}(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} |n,l,m\rangle \\ \pm |n,l,m\rangle \end{pmatrix} \quad (3.3)$$

with $m = \pm \alpha \sqrt{2n + l + c}$. The states $|n,l,m\rangle$ are the eigenfunctions of the non-relativistic three dimensional harmonic oscillator, which results in a linearly increasing $(\text{mass})^2$ spectrum.



Harmonic Oscillator Trajectories $m^2 = \alpha^2(2n + l + c)$, $(c=1)$

The two component nature of the representation allows for positive and negative energies, so that doubling the dimensionality of the original representation does not introduce any redundant components. In order to obtain a more physical picture of the solutions of the wave equation, we examine in detail the structure of the ground state wave function.

We first boost the ground state wave function at rest to momentum p using the boost operator defined in equation (2.9):

$$\begin{aligned} \psi_0(p) &= e^{-i\vec{k} \cdot \vec{\xi}(p)} \psi_0(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\vec{k}(c) \cdot \vec{\xi}(p)} |0\rangle \\ e^{-i\vec{k}(c-) \cdot \vec{\xi}(p)} |0\rangle \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left(\frac{(n+c)!}{c!} \right)^{1/2} \left(\frac{m}{p_0} \right)^{c+1} \left(\frac{\vec{p} \cdot \vec{a}^+}{p_0} \right)^n |0\rangle \\ \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left(\frac{(n+c-1)!}{(c-1)!} \right)^{1/2} \left(\frac{m}{p_0} \right)^c \left(\frac{\vec{p} \cdot \vec{a}^+}{p_0} \right)^n |0\rangle \end{pmatrix} \end{aligned} \quad (3.4)$$

with $\vec{\xi}(p) = \hat{p} \sinh^{-1} \frac{p}{m}$, $m = \alpha\sqrt{c}$

The method of calculating equation (3.4) and the general boost matrix elements is given in Appendix B. In terms of the internal coordinates $\vec{\xi}$ of the oscillator the ground state can be represented in this non-manifestly covariant representation by

$$\langle \vec{\xi} | 0 \rangle = \frac{e^{-\frac{\xi^2}{2}}}{(\sqrt{\pi})^{3/2}} \quad (3.5)$$

and $\sqrt{c+c} |0\rangle = \sqrt{c} |0\rangle$

The physical interpretation of the state (3.4) is not obvious, and so for our discussion we consider a manifestly covariant representation by introducing the internal time (ξ_0) of the oscillator through the relation

$$\langle \xi, \xi_0 | 0 \rangle_c = e^{-\frac{\xi^2}{2}} e^{-\frac{\xi_0^2}{2}} H_c(\xi_0) \quad (3.6)$$

where $H_c(\xi_0)$ is the Hermite polynomial. The constant c will be associated with

$$\mathcal{C}_0 = a_0^\dagger a_0, \quad [a_0, a_0^\dagger] = 1, \quad \mathcal{C}_0 |0\rangle_c = c |0\rangle_c$$

i.e.

$$\sqrt{c+c} |0\rangle \longrightarrow \sqrt{c+c_0} |0\rangle_c = \sqrt{c} |0\rangle_c$$

Then we note that the coefficients appearing in the boosted ground state (3.4) (restricting the motion to be along the z direction)

$$\begin{aligned} & (-)^n \left(\frac{(n+c)!}{n! c!} \right)^{1/2} \left(\frac{m}{p_0} \right)^{c+1} \left(\frac{p_z}{p_0} \right)^n |0, 0, n\rangle \\ &= (-)^n \left(\frac{(n+c)!}{n! c!} \right)^{1/2} \left(\frac{1}{\gamma} \right)^{c+1} v^n |0, 0, n\rangle \end{aligned} \quad \begin{aligned} v &= \text{velocity} \\ \gamma &= \sqrt{1-v^2} \end{aligned}$$

are precisely the coefficients which would be obtained by the action of

the operator $\frac{(-a_0^\dagger a_3^\dagger v)^n}{n!}$ on $|0\rangle_c$. Then using the ground state $|0\rangle_c$ the action of the boost is equivalent to

$$\left(\frac{1}{\gamma} \right)^{c+1} e^{-a_0^\dagger a_3^\dagger v} |0\rangle_c = e^{-\frac{\xi_1^2}{2}} e^{-\frac{\xi_2^2}{2}} e^{-\frac{\xi_0^2}{2}} \sum_{n=0}^{\infty} \left(\frac{1}{\gamma} \right)^{c+1} \frac{1}{n!} \frac{(-v)^n}{2^n} H_n(\xi_0) \frac{H_{n+c}(\xi_0)}{\sqrt{2^c c!}} \quad (3.6)$$

and $c \rightarrow c - 1$ for the lower component. The factors $\left(\frac{(n+c)!}{n!c!}\right)^{1/2}$ are absorbed by the normalization of $|0,0,n\rangle_c$. Replacing the Hermite polynomials by their integral representation

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x+iu)^n e^{-u^2} du$$

then interchanging the order of summation and integration in equation (3.6) it becomes

$$|0,v\rangle_c = \frac{e^{-\frac{\xi_1^2}{2}}}{\pi^2} e^{-\frac{\xi_2'^2}{2}} e^{-\frac{\xi_0^2}{2}} H_c(\xi_0') \quad (3.7)$$

where $\xi_3' = \gamma(\xi_3 + v\xi_0)$, $\xi_0' = \gamma(\xi_0 + v\xi_3)$

This is the wave function in a frame in which the oscillator is moving at velocity v expressed in terms of the internal coordinates of the oscillator.

Thus, the boost in fact performs a Lorentz transformation on the internal coordinates of the oscillator, if the ground state is re-interpreted as above. Although the internal time coordinate has not appeared in the non-covariant formulation of the representation, the analysis above establishes a connection between this model and the relativistic oscillator representation of Bacry and Chang. In their

covariant representation the vector A_μ is

$$A_\mu = \begin{pmatrix} 0 & \frac{\partial}{\partial z_\mu} \\ z_\mu & 0 \end{pmatrix} \quad (3.8)$$

A_μ are Hermitian operators acting in the Hilbert space of functions spanned by the monomials

$$\frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{z_0^{n+C}} \begin{pmatrix} \pm \frac{\sqrt{n+C}}{z_0} \\ 1 \end{pmatrix} \quad \begin{aligned} n_1, n_2, n_3 &\geq 0 \\ n &= n_1 + n_2 + n_3 \end{aligned} \quad (3.9)$$

The A_μ 's generate an infinite dimensional algebra obtainable from the relations

$$A_\mu A_\nu = \frac{1}{2} S_{\mu\nu} - \frac{i}{2} \sigma_3 \bar{Z}_{\mu\nu} + \frac{1}{2} \delta_{\mu\nu} (1 + \sigma_3) \quad (3.10)$$

$$\{A_\mu, \sigma_3\} = 0$$

and $S_{\mu\nu}$ and $\bar{Z}_{\mu\nu}$ generate a unitary representation of $SU(3,1)$.

The z 's are complex and satisfy the $SU(3,1)$ invariant restriction

$$|z_0|^2 > |z_1|^2 + |z_2|^2 + |z_3|^2 \quad . \quad \text{The monomials (3.6) are}$$

eigenfunctions of A_0 with eigenvalues $\pm \sqrt{n+C}$. The quantization

based on this representation will be discussed in Section 5. If

the z 's are identified with internal oscillator coordinates ξ_μ

in the following manner:

$$z_\mu = \frac{1}{\sqrt{2}} \left(\xi_\mu - \frac{\partial}{\partial \xi_\mu} \right), \quad \frac{\partial}{\partial z_\mu} = \frac{1}{\sqrt{2}} \left(\xi_\mu + \frac{\partial}{\partial \xi_\mu} \right)$$

$$\frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{z_0^{n+c}} \longrightarrow N e^{-\frac{\xi_1^2}{2}} e^{-\frac{\xi_2^2}{2}} H_{n_1}(\xi_1) H_{n_2}(\xi_2) H_{n_3}(\xi_3) H_{n+c}(\xi_0) \quad (3.11)$$

the properties of the $\tilde{z}$ representation are preserved. This identification makes a connection between the non-covariant representation and the $\tilde{z}$ representation. The internal time coordinate is on an equal footing with the space coordinate; that is, the timelike oscillators have positive metric. Thus the ghost problem in the usual formulation of the relativistic harmonic oscillator is eliminated.

The non-manifestly covariant oscillator described here is a "projection" of the Bacry-Chang formulation in the sense that time-like operators are replaced by the result of their action on the wave function, i.e., by numbers $\sqrt{n+c}$ etc. Therefore a time coordinate never appears in this model, and this explains its non-manifest covariance.

4. Spacelike Solutions and Completeness Relations

As is well known, one of the problems which has plagued infinite component theories has been that of spacelike solutions. These unphysical states had to be included in completeness relations and made an appearance in physical processes. In this model there are also "spacelike" states but they do not pose such a problem since, as will be demonstrated below, the "spacelike" states are not linearly independent of timelike states, namely, they are contained in the Hilbert space of timelike solutions.

Writing the wave equation (3.1) in a standard spacelike frame

$$p = (0, 0, p, 0)$$

$$\left\{ \alpha p A_3 + p^2 \right\} \psi(p) = 0 \quad (4.1)$$

$$\alpha p \begin{pmatrix} 0 & a_3 \\ a_3^+ & 0 \end{pmatrix} \psi(p) = -p^2 \psi(p)$$

This equation has "spacelike" solutions

$$\psi_K(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} |n_1 n_2 n_3\rangle \\ \mp |n_1 n_2 n_3+1\rangle \end{pmatrix} \quad (4.2)$$

$$\text{with } p = \pm \alpha \sqrt{n_3+1} \quad \text{or} \quad m_{sp} = ik = \pm i \alpha \sqrt{n_3+1}$$

The solutions (4.2) hold for any values of n_1 and n_2 , and therefore the spacelike states here are infinitely degenerate. There are also lightlike solutions to the wave equation but they are not normalizable.

To see the connection between the "spacelike" and timelike solutions we note that due to the completeness of the harmonic oscillator wave functions the timelike states at rest are complete. The completeness relation is

$$\sum_{\substack{n_1, n_2, n_3 \\ \text{pos. energy}}} \psi_{n_1, n_2, n_3}^{(+)}(0) \psi_{n_1, n_2, n_3}^{(+)\dagger}(0) + \sum_{\substack{n_1, n_2, n_3 \\ \text{neg. energy}}} \psi_{n_1, n_2, n_3}^{(-)}(0) \psi_{n_1, n_2, n_3}^{(-)\dagger}(0) \\ = \sum_{n_1, n_2, n_3} \begin{pmatrix} |n_1, n_2, n_3\rangle \langle n_1, n_2, n_3| & 0 \\ 0 & |n_1, n_2, n_3\rangle \langle n_1, n_2, n_3| \end{pmatrix} = 1 \quad (4.3)$$

Then the "spacelike" solution in its rest frame can be constructed out of linear combinations of positive and negative energy timelike states at rest. For example

$$\psi_{k=\alpha \sqrt{n_3+1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} |n_1, n_2, n_3\rangle \\ -|n_1, n_2, n_3+1\rangle \end{pmatrix} = \frac{(1+\sqrt{3})}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} |n_1, n_2, n_3\rangle \\ |n_1, n_2, n_3\rangle \end{pmatrix} + \\ + \frac{(1-\sqrt{3})}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} |n_1, n_2, n_3+1\rangle \\ -|n_1, n_2, n_3+1\rangle \end{pmatrix} = \psi_{n_1, n_2, n_3}^{(+)}(0) + \psi_{n_1, n_2, n_3}^{(-)}(0) + \psi_{n_1, n_2, n_3+1}^{(-)}(0) - \psi_{n_1, n_2, n_3+1}^{(+)}(0) \quad (4.4)$$

If we boost ψ_k with a given velocity, the boost does not take ψ_k out of the space of timelike states, so that a spacelike state will still be represented as a linear combination of timelike states, all with the same velocity but different momenta. From equation (4.3) the completeness relation for a given velocity is

$$\sum_{\substack{n_1, n_2, n_3 \\ p \text{ and neg. energy}}} \psi_{n_1, n_2, n_3}(\xi) \psi_{n_1, n_2, n_3}^\dagger(\xi) = 1 \quad (4.5)$$

There are some additional orthogonality and completeness relations which can be derived. Using the wave equation and its adjoint

$$\begin{aligned} \{ \alpha A \cdot p + p^2 \} \psi_N(p) &= 0 \\ \psi_{N'}^\dagger(p') \{ \alpha A \cdot p' + p'^2 \} &= 0 \end{aligned} \quad (4.6)$$

a conserved current can be constructed

$$\begin{aligned} J_\mu(p, p') &= \psi_{N'}^\dagger(p') j_\mu(p, p') \psi_N(p) \\ j_\mu(p, p') &= \alpha A_\mu + (p + p')_\mu \end{aligned} \quad (4.7)$$

satisfying

$$(p - p') \cdot J(p, p') = 0 \quad (4.8)$$

In particular for $\vec{p} = \vec{p}'$ equation (4.8) becomes

$$(p_0 - p'_0) J_0(p, p') = 0 \quad (4.9)$$

which results in an orthogonality relation for a given momentum $\vec{p}$

$$\psi_{n'_1 n'_2 n'_3}^\dagger(\vec{p}) j_0(p, p') \psi_{n_1 n_2 n_3}(\vec{p}) = p_0 \delta_{n_1 n'_1} \delta_{n_2 n'_2} \delta_{n_3 n'_3} \quad (4.10)$$

Following the same reasoning, there is also an orthogonality relation for "spacelike" states at the same momentum.

$$\psi_k(\vec{p}) j_0(p, p') \psi_k(\vec{p}) = p_0 \delta_{n_1 n'_1} \delta_{n_2 n'_2} \delta_{n_3 n'_3} \quad (4.11)$$

And a similar relation holds for "spacelike" and timelike states at the same momentum. Provided the value of the momentum is consistent with both particles being on their respective mass shells, we have from equation (4.9)

$$(p_0 - p'_0(k)) \psi_k(\vec{p}) j_0(p, p') \psi_N(p) = 0 \quad (4.12)$$

and since $p_0 \neq p'_0(k)$ for equal spatial components of momenta

$$\psi_k(\vec{p}) j_0(p, p') \psi_N(p) = 0 \quad (4.13)$$

These orthogonality relations can be used to write a completeness relation at a fixed momentum with the metric $j_0(p, p')$. But since $j_0(p, p')$ contains two different momenta, it is convenient to write the completeness relation in coordinate space. To do this we define the wave functions

$$\psi_N^{(\pm)}(x) = \int \frac{d^3 p}{p_0} e^{\pm i p \cdot x} \psi_N^{(\pm)}(p)$$

$p^2 = -\alpha^2 (N + c)$

$$\psi_k^{\pm}(x) = \int d^4 p e^{\pm i p \cdot x} \psi_k^{\pm}(p) \delta(p^2 - k^2)$$

(4.14)

In this representation

$$j_0(p, p') \longrightarrow \alpha A_0 + i \frac{\overleftrightarrow{\partial}}{\partial t}$$

Then the completeness relation can be written in the form

$$\sum_{(\pm) n_1 n_2 n_3} \int \frac{d^3 p}{p_0} \psi_{n_1 n_2 n_3}^{(\pm)}(p) \psi_{n_1 n_2 n_3}^{\dagger(\pm)}(p) e^{\pm i p \cdot (x-y)} \left\{ A_0 + i \frac{\overleftrightarrow{\partial}}{\partial y_0} \right\}$$

$$\sum_{(\pm) n_1 n_2 n_3} \int d^4 p \delta(p^2 - k^2) \psi_k^{(\pm)}(p) \psi_k^{\dagger(\pm)}(p) e^{\pm i p \cdot (x-y)} \left\{ A_0 + i \frac{\overleftrightarrow{\partial}}{\partial y_0} \right\} = \delta(\vec{x} - \vec{y})$$

(4.15)

with the instruction, set $x_0 = y_0$ after differentiation.

This completeness relation is a delta function only with respect to solutions of the wave equation; i.e., denoting (4.15) by $K(x,y)$, then for any solution of the wave equation $\phi_{N'}(y)$

$$\int d^3y K(x,y) \phi_{N'}(y) = \phi_{N'}(x)$$

The completeness (4.15) follows from the orthogonality relations derived from current conservation at equal spatial momenta. The derivative $\frac{\overleftrightarrow{\partial}}{\partial y_0}$ picks out the proper signs of $p_0 + p_0'$ in $j_0(p,p')$ giving the correct form for the fourth component of the conserved current.

The main point to be made about this completeness relation is that the presence of spacelike solutions is often quoted as 'proof' that spacelike solutions cause difficulties for any infinite component wave equation. But the purpose for putting quotation marks around "spacelike" in the preceding discussion has been to draw attention to the fact that they are not independent physical solutions of the wave equation. Equation (4.4) shows that a "spacelike" solution is indeed a linear superposition of timelike solutions at the same velocity, but, of course, in terms of momenta they are at different (but well defined) momentum.

To underscore this point we write the spacelike solution at momentum $\vec{p}$ in terms of timelike states of different momenta. If ψ_k

is boosted with velocity u to bring it to momentum p then a timelike

solution boosted with the same velocity will have momentum $p' = \frac{M_N}{K} (\vec{p}^2 - k^2)^{1/2}$

Then the "spacelike" state can be written as

$$\psi_{n_2 k}^{(+)}(\vec{p}) = \psi_{n_1 n_2 n_3}^{(+)}(\vec{p}') + \psi_{n_1 n_2 n_3}^{(-)}(\vec{p}') + \psi_{n_1 n_2 n_3+1}^{(-)}(\vec{p}'') - \psi_{n_1 n_2 n_3+1}^{(+)}(\vec{p}'')$$

$$K = \alpha \sqrt{n_3+1}$$

$$N = n_1 + n_2 + n_3 \quad p' = \frac{\sqrt{N+1} C}{\sqrt{n_3+1}} (\vec{p}^2 - k^2)^{1/2}, \quad p'' = \frac{\sqrt{N+1} C}{\sqrt{n_3+1}} (\vec{p}^2 - k^2)^{1/2}$$

With $\psi_K(p)$ written in this manner the completeness relation is expressed entirely in terms of timelike states.

The Majorana equation is the prototype of an equation for which the spacelike solutions are independent solutions of the equation; they are also independent of the timelike solutions. In this theory the completeness relation includes a continuum of spacelike solutions which are normalized in the delta-function sense with respect to the same metric Γ^0 as the timelike solutions;¹⁵

$$\sum_{s, \lambda} \psi_{\lambda, \vec{p}}^{M(s)} \psi_{\lambda, \vec{p}}^{\dagger M(s)} \Gamma^0 + \sum_{\epsilon = \pm \lambda} \int_{k/p}^{\infty} d\tau \psi_{\lambda, \vec{p}, \epsilon}^{L(\tau)} \psi_{\lambda, \vec{p}, \epsilon}^{\dagger L(\tau)} \Gamma^0 = 1$$

$s \equiv \text{spin}$
 $\lambda \equiv \text{helicity}$

It turns out that for the free field quantization of the oscillator, the relevant completeness relation is the one for timelike states at fixed velocity--equation (4.5). Presumably for purposes of current algebra it is the completeness relation at fixed p which would be

relevant. Since the question of whether a successful quantization can be carried out with only timelike states is a crucial one, we briefly outline the procedure in the next section.

5. Quantization of the Relativistic Oscillator

The work presented in this section has been carried out by Professors H. Bacry and N.P. Chang, and is contained in reference (8). It is presented here to indicate that a free quantized field can be constructed from timelike states alone, and to show the connection between the quantized and unquantized versions. The correspondence is quite close so that many features of the quantized version will be contained in the unquantized one.

The starting point is the wave equation

$$(i\cancel{A}\cdot\partial - \partial^2) \psi(x, \bar{z}) = 0 \quad (5.1)$$

where $\cancel{A}_\mu$ and the monomials $\frac{\bar{z}_{\mu_1}^{n_1} \bar{z}_{\mu_2}^{n_2} \bar{z}_{\mu_3}^{n_3}}{\bar{z}_0^{n+C}}$ have been defined in Section 3, p.19 .

The field is defined as $(u_0 = \sqrt{u^2 + 1}, \bar{z}_\mu^t = \bar{z}_\mu + (\bar{z} \cdot u) u_\mu)$

$$\begin{aligned} \psi(x, \bar{z}) &= \sum_{n=0}^{\infty} \int \frac{d^3 u}{u_0} \frac{(n+C-1)!}{n!(C-1)!} \bar{z}_{\mu_1}^t \bar{z}_{\mu_2}^t \cdots \bar{z}_{\mu_n}^t \left\{ \left(\frac{\sqrt{n+C}}{-\bar{z} \cdot u} \right) e^{i\sqrt{n+C} u \cdot x} a_{\mu_1 \dots \mu_n}(\vec{p}) \right. \\ &\quad \left. + \left(\frac{-\sqrt{n+C}}{-\bar{z} \cdot u} \right) e^{-i\sqrt{n+C} u \cdot x} a_{\mu_1 \dots \mu_n}^+(\vec{p}) \right\} \\ &= \sum_{n=0}^{\infty} \psi_n(x, \bar{z}) \end{aligned} \quad (5.2)$$

where $a_{\mu_1 \dots \mu_n}(\vec{p})$ are annihilation operators for physical particles of mass $\sqrt{n+C}$ and momentum $\vec{p} = \sqrt{n+C} \vec{u}$. $\psi(x, z)$ satisfies the wave equation (5.1) and transforms under Lorentz transformations as

$$U(1) \psi(x, z) U^\dagger(1) = \psi(1x, 1z) \quad (5.3)$$

The commutator is obtained from (5.2)

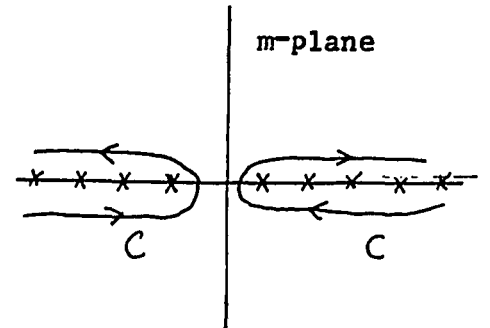
$$\begin{aligned} [\psi(x, z), \psi^\dagger(x', z')] &= \hat{\Delta}(x-x', z, z'^*) \\ &= \left\{ \hat{\Delta}_1 + i \not{A} \cdot \frac{\partial}{\partial x} \hat{\Delta}_2 \right\} I(z, z'^*) \end{aligned} \quad (5.4)$$

where
$$I(z, z'^*) = \left(\frac{1}{(-z, z'^*)^C} \begin{pmatrix} \frac{C}{-z, z'^*} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$\hat{\Delta}_1 \propto \int \frac{d^3 \vec{u}}{u_0} \int_C \frac{dm}{m^2} \frac{m e^{im u \cdot x}}{(\not{A} \cdot u)^2}$$

$$\hat{\Delta}_2 \propto \int \frac{d^3 \vec{u}}{u_0} \int_C \frac{dm}{m} \frac{e^{im u \cdot x}}{m^2 (\not{A} \cdot u)^2}$$



The proof that $\hat{\Delta}(x-x', z, z'^*)$ vanishes for spacelike separations is given in (BC). The individual fields $\psi_n(x, z)$ do not respect micro-causality; their commutators behave as a decreasing exponential for

space-like separations.

The Grodsky - Streater theorem¹⁶ does not apply to the commutator (5.4) "because this representation of the Lorentz group is reducible into an infinite number of unitary irreducible representations of $SL(2, \mathbb{C})$, and this representation is no longer holomorphic in the complex homogeneous Lorentz group." (BC) This quantization also respects the spin and statistics connection since, for the boson operators, anticommutation rules would not respect causality.

$$(CPT) \psi(x, z) (CPT)^{-1} = \sigma_3 \psi^*(-x, z)$$

The wave equation is invariant under this operation since $\{\sigma_3, A_\mu\} = 0$

As regards the O'Raifeartaigh theorem¹⁷, the space-time translation generators with respect to the Hilbert space of solutions of equation (5.1) are represented by

$$P_\mu = (A \cdot u) u_\mu$$

These do not commute with the $SU(3, 1)$ generators and so the representation space is not that of a finite dimensional group containing the Poincare group as a subgroup. The translation operators together with the $SU(3, 1)$ generators generate an infinite dimensional Lie algebra. (The above applies to the first quantized representation as well.)

This completes the discussion of the second quantization. There

has as yet been no satisfactory prescription for discussing interacting field theory, therefore we shall proceed at the first quantized level. The purpose as mentioned in the introduction is to follow the work of Nambu, Fronsda1 and others to determine if there are differences between their models and this one, and to learn more of the physics in this model.

6. Properties of the Oscillator States

Using the wave function (3.4) we can calculate the ground state current matrix elements

$$\begin{aligned} \mathcal{T}_\mu(p, p') &= \psi_0^\dagger(p') j_\mu(p, p') \psi_0(p) \\ &= \frac{1}{2} \frac{(p+p')_\mu}{\left(-\frac{p \cdot p'}{m^2}\right)^c} = \frac{(p+p')_\mu}{2} \frac{1}{\left(1 - t/2m^2\right)^c} \end{aligned} \quad (6.1)$$

The form factor $F(t) = \frac{1}{\left(1 - t/2m^2\right)^c}$ is of the type which appears in many infinite component theories and is due to the factors $(\cosh \xi)^{-1}$ which appear in the boost matrix elements. It is the same ground state form factor obtained in Reference 9. In general, however, our results are different from theirs, due to the two component nature of this representation which generally gives two separate contributions to form factors and scattering amplitudes.

Since the mass of the ground state is $M = \alpha\sqrt{C}$ the form factor can be written as

$$F(t) = \frac{1}{\left(1 - \frac{t}{2\alpha^2 c}\right)^c} \quad (6.2)$$

This form factor has a non-relativistic limit which can be obtained by

letting the speed of light go to infinity, or, equivalently, letting the ground state energy go to infinity. This limit is reached by letting $c \rightarrow \infty$ rather than $\alpha^2 \rightarrow \infty$, since in this limit the boost operator becomes

$$K_i \sinh^{-1} \frac{p}{m} = (i\sqrt{\kappa+c+1} a_i - i\sqrt{\kappa+c} a_i^\dagger) \sinh^{-1} \frac{p}{\alpha c}$$

$$\xrightarrow{c \rightarrow \infty} i(a_i - a_i^\dagger) \frac{p}{\alpha}$$

which is a Euclidean translation operator in the internal oscillator space. It is not quite a Galilean transformation for the Schroedinger equation because it does not involve the time variable. Taking the limit $c \rightarrow \infty$ in equation (6.2) with $t = -k^2$, $F(\vec{k}) = e^{-k^2/2\alpha^2}$ which is the ground state form factor for a non-relativistic particle of mass μ in an oscillator potential if we identify $2\alpha^2 = 4\mu\omega$.

Although the form factor has a reasonable non-relativistic limit, it is not clear that it should be regarded as the correct relativistic form factor. The formulation as presented thus far is for a free particle and contains no interactions, and the nature of the correct vertex operator is crucial to producing realistic results. For example, if the model is regarded as describing a composite system, then the interaction of the system with the electromagnetic field should occur at the position of the constituent charges and not at the over-

all CM position. Therefore the vertex would involve some function of the internal oscillator coordinates.

Another feature of this form factor is that its t dependence is tied to the value of the mass via the value of c . The constant α^2 is not completely free if the mass spectrum is to have a reasonable slope, so that we cannot expect a good experimental fit from $\left(1 - \frac{t}{2m^2}\right)^{-c}$ alone. For example, if the ground state is identified with the π meson, this requires c to be vanishingly small and α^2 to be ~ 1.33 in order that the π trajectory be described by $\alpha_\pi(t) = \frac{t}{\alpha^2} - c \approx .75t$. This would give the form factor a much too slow fall off in t , i.e.,

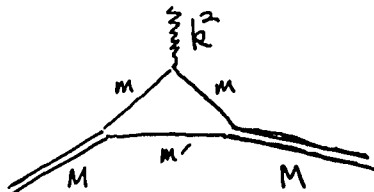
$\left(1 - \frac{t}{2m^2}\right)^{-0.15}$ The next highest energy state $M_1^2 = \alpha^2(c+1)$ has a form factor $\sim \left(1 - \frac{t}{2m^2}\right)^{-(c+2)}$ which also has $e^{-k^2/2\alpha^2}$ as its non-relativistic limit.

An analogy can be made with the current matrix element for the Dirac electron, i.e.

$$J_0(p,0) = \bar{u}_s(p) \gamma_0 u_s(0) = \cosh \frac{\xi(p)}{2} = \left(1 - \frac{t}{4m^2}\right)^{1/2}$$

The factor $\left(1 - \frac{t}{4m^2}\right)^{1/2}$ is not generally regarded as the form factor for the electron. It is perhaps in this sense that $\left(1 - \frac{t}{2m^2}\right)^{-c}$ should be thought of as a universal factor for all current matrix elements with another overall $F_i(t)$ coming from the specific dynamical structure of the vertex operator.

Finally, we note the anomalous threshold at $t = 2m^2$. An explanation for this anomalous threshold has been offered by H. Kleinert and C. Fronsdal. If one considers the infinite component equation as describing a composite system, then triangle graphs of the form



can produce anomalous singularities. Kleinert is able to show in his hydrogen atom model that the anomalous threshold appearing there is in fact due to this graph. However, in the hydrogen atom the binding energy is very small compared to the constituent masses, while in the oscillator model the total mass is always larger than the masses of its hypothetical constituents, and the binding energy is infinite. In any case, the condition for an anomalous threshold to occur is¹⁸

$$t > t_0 = 4m^2 - \frac{1}{m'^2} (M^2 - m^2 - m'^2)^2 \quad (6.3)$$

If we suppose $m = m'$, then for the anomalous threshold to occur at

$$t_0 = 2M^2, \text{ equation (6.3) implies } m = \frac{M}{\sqrt{2}} = \frac{\alpha\sqrt{C}}{2}. \text{ Now}$$

looking at the excited states of the oscillator, they will have form

$$\text{factors of the form } \sim \left(\frac{1}{1 - \frac{t}{2M^2}} \right)^{C+N'} \text{ which also have } e^{-\frac{k^2}{2\alpha^2}}$$

as a non-relativistic limit. It is not possible for the triangle singularities to account for all these anomalous thresholds because of the relation $M_N^2 = \alpha^2(c+N)$. For example, if the first excited state has an anomalous threshold at $t = 2M_1^2 = 2\alpha^2(c+1)$, substituting this in equation (6.3) we obtain

$$2\alpha^2(c+1) = 2\alpha^2c - \frac{2}{\alpha^2c} \left(\alpha^2(c+1) - \alpha^2c \right)^2$$

This is satisfied only if $c = -1$ and therefore $M_1^2 = 0$

This explanation does not therefore seem satisfactory to account for the threshold behavior. It is curious to note that the form factor leads to the same charge radius ($\frac{3}{2\alpha^2}$) as the non-relativistic Gaussian $e^{-k^2/2\alpha^2}$. We have no good explanation of this anomalous threshold beyond the view mentioned above that a full current matrix element will have, in general, an overall factor $F_1(t)$ coming from the specific dynamical structure of the vertex operator.

The magnetic moments of the states can be obtained from

$$\psi_{n,l,m}^\dagger(p) \vec{M} \psi_{n,l,m}(p) = \lim_{\vec{q} \rightarrow 0} \frac{e}{2m_N} \int d^3x \vec{x} \times \psi_{n,l,m}^\dagger(\vec{p} + \vec{q}) \vec{J}(\vec{p}, \vec{q}) \psi_{n,l,m}(p) e^{-i\vec{q} \cdot \vec{x}}$$

which yields the static moments

$$\langle \vec{M} \rangle = \frac{e}{2m_N} \langle \vec{L} \rangle$$

and $g = 1$ for all states, which is in agreement with the non-

relativistic oscillator.

The average radii of the states are just those of the non-relativistic oscillator.

$$\langle r^2 \rangle = N_{l,l}^2 \int_0^\infty dr (r^2)^{l+2} e^{-r^2} \left[L_l^{l+1/2}(r^2) \right]^2 = \frac{2l+3/2}{\mu\omega}$$

Making the identification as before $\mu\omega = 2\alpha^2$

$$\langle r^2 \rangle = \frac{M_N^2 - (M_0^2 - 3/2\alpha^2)}{(2\alpha^4)} \sim_{N \gg C} \frac{M_N^2}{2\alpha^4}$$

The expanding size of the resonances leads to an exponential decrease in ℓ of the partial decay widths for large ℓ ; i.e., for purely angular excitations $M^2 \propto \ell$. (See next section.) This is expected from the large centrifugal barrier for large angular momentum. The Veneziano amplitude yields widths roughly characteristic of a size $R \sim \ln n$, (n, ℓ both large). The oscillator gives a more rapid increase in size, $R \sim \sqrt{N}$.

We next consider the scattering of the infinite component system.

7. Scattering Amplitudes

In this section we report on some initial attempts at obtaining lowest order scattering amplitudes in this model. The quantum field theory that we have described is still at the free field stage, so that no perturbation theory "Feynman" diagrams can yet be drawn for this model. To do this would require a judicious choice of interaction vertices and propagator much like the construction in the dual resonance theory. As far as we know this is the only successful construction of this type available in the literature.

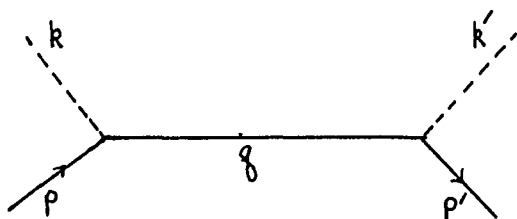
The aim here is the more modest one of investigating the properties of the simplest lowest order scattering amplitudes, without the proper interaction vertices and propagator. The usual means of constructing a propagator is to use the inverse wave operator. This is the method used in the dual resonance theory where the inverse wave operator does not have a spectrum of spacelike states (apart from an isolated spacelike point), and so there are no difficulties with this procedure. In our case, the inverse wave operator does contain "spacelike" solutions. Therefore a scattering amplitude calculated with this propagator will, when s is continued from timelike ($s > 0$) to spacelike ($s < 0$) values, exhibit spacelike poles. These features would indicate that the simple inverse wave operator should not be used for all s . However, lacking a complete analytic prescription for a propagator, we shall take the

view that in the $s > 0$ physical region we are correct in using the inverse wave operator, which in this region has contributions only from the time-like solutions. This amounts to using the Bohr "compound nucleus" resonance picture to describe the scattering amplitude.

The examples studied verify explicitly the absence of ghosts in the amplitude, and we have explored to some extent the analyticity properties of the amplitude. The results obtained here are in general agreement with the previous calculations of Fronsda¹³ and Nambu², the differences being due to the linearized wave equation of our formulation.

The amplitudes have poles at the energies corresponding to the time-like solutions of the wave equation, but since these are Born amplitudes there are no two body unitarity cuts. The poles at an energy have contributions from angular momenta $N, N-2, N-4, \dots$. These daughters arise from the degeneracy of the harmonic oscillator spectrum.

We consider first the elastic scattering of a scalar particle with the ground state of the infinite component wave equation. We take the lowest order scattering to be given by the diagram



where the solid line represents the infinite component system. The vertices are taken to be 1, and for the propagator we use the inverse

wave operator $\frac{1}{\alpha A \cdot q + q^2}$. The amplitude is then

$$\begin{aligned}
 A(p, p', q) &= \psi_0^\dagger(p') \frac{1}{\alpha A \cdot q + q^2} \psi_0(p) \\
 &= \frac{1}{2} \frac{1}{m^2 + q^2} \left(\frac{\bar{z}^c}{(-\frac{p \cdot p'}{m^2})} \right)^c \left\{ \frac{\bar{z}}{(-\frac{p \cdot p'}{m^2})} - m^2 \left(\frac{1}{p \cdot q} + \frac{1}{p' \cdot q} + \frac{1}{q^2} \right) \right\} {}_2F_1 \left(c+1, c + \frac{q^2}{\alpha^2}; c+1 + \frac{q^2}{\alpha^2}; 1 - z \right) \\
 &\quad + \frac{1}{2q^2} \left(\frac{1}{(-\frac{p \cdot p'}{m^2})} \right)^c \quad \quad \quad z = \frac{p \cdot p' q^2}{p \cdot q p' \cdot q}, \quad m^2 = \alpha^2 c \quad (7.1)
 \end{aligned}$$

This calculation is outlined in Appendix C. Briefly, the method is to do the calculation in the center of mass frame where q is purely timelike and the propagator is diagonal. Then Lorentz invariance is used to re-express the variables in an arbitrary frame. Therefore (7.1) holds for $q^2 < 0$. In order to study the amplitude in more detail we rewrite (7.1) in the CM frame where $q^2 = -s$.

$$\begin{aligned}
 A(s, t) &= \frac{1}{2} \frac{1}{m^2 - s} \left[\frac{4m^2 s}{(s + m^2 - \mu^2)^2} \right]^c \left[\frac{4m^2 (2s + m^2 - \mu^2)}{(s + m^2 - \mu^2)^2} + \frac{m^2}{s} \right] {}_2F_1 \left(c+1, c - \frac{s}{\alpha^2}; c+1 - \frac{s}{\alpha^2}; \frac{s(t-u) + (m^2 - \mu^2)^2}{(s + m^2 - \mu^2)^2} \right) \\
 &\quad - \frac{1}{2s} \left(\frac{1}{(1 - \frac{t}{2m^2})} \right)^c
 \end{aligned}$$

and for equal masses $k^2 = p^2 = -m^2 = -\mu^2$

$$\begin{aligned}
 A(s, t) &= \frac{1}{2} \frac{1}{m^2 - s} \left(\frac{4m^2}{s} \right)^c \frac{q m^2}{s} {}_2F_1 \left(c+1, c - \frac{s}{\alpha^2}; c+1 - \frac{s}{\alpha^2}; \frac{t-u}{s} \right) \\
 &\quad - \frac{1}{2s} \left(\frac{1}{(1 - \frac{t}{2m^2})} \right)^c \quad (7.2)
 \end{aligned}$$

It is necessary to distinguish between the two cases of equal and unequal masses since the behavior at $s \rightarrow 0$ is different in the two instances. For equal mass the behavior near $s = 0$ is $\sim \frac{1}{s}$ while for unequal masses it is $\sim \frac{1}{m^2 - \mu^2}$. In the following we shall, for simplicity, consider mainly the case with all particles having the same mass.

The parameters in the hypergeometric function give a simple pole whenever $\frac{s}{\alpha^2} = n + c + 1$ with residue proportional to $(\cos \theta)^n$.

The residues are easily obtained from the original series from which the amplitude was obtained. (See Appendix C.)

$$A(s,t) = \frac{1}{2\alpha^2} \left(\frac{4m^2}{s} \right)^c \left\{ \sum_{n=0}^{\infty} \left[\frac{g m^2}{s} \frac{(n+c)!}{n! c!} + \frac{(n+c-1)!}{n! (c-1)!} \right] \left(\frac{p^2}{p_0^2} \right)^n \frac{(\cos \theta)^n}{n+c-\frac{s}{\alpha^2}} \right\} \quad (7.3)$$

$(\cos \theta)^n$ can be written as a sum of Legendre polynomials with positive coefficients.²⁰

$$\begin{aligned} (\cos \theta)^{2n} &= \frac{P_0}{2n+1} + \sum_{k=1}^{\infty} \frac{(4k+1) 2n(2n-2) \cdots (2n-2k+2)}{(2n+1)(2n+3) \cdots (2n+2k+1)} P_{2k}(\cos \theta) \\ (\cos \theta)^{2n+1} &= \frac{3}{2n+3} P_1(\cos \theta) + \sum_{k=1}^{\infty} \frac{(4k+3) 2n(2n-2) \cdots (2n-2k+2)}{(2n+3)(2n+5) \cdots (2n+2k+3)} P_{2k+1}(\cos \theta) \end{aligned} \quad (7.4)$$

Therefore, there are no states with imaginary coupling constants (ghosts) which contribute to the scattering. The no ghost feature is a natural consequence of having no time component oscillator modes.

Also, only even or odd Legendre Polynomials appear in the residues.

This is to be expected from the degeneracy of the harmonic oscillator spectrum. For a fixed $N = 2n + L$, only even or odd L are permitted depending on whether N is even or odd. Using (7.4) the residues can be read off from (7.3).

$$S_N \equiv d^2(n+c) \quad , \quad n=0 \quad \text{Res.} = \frac{1}{2} q (4)^c$$

$$n=1 \quad \text{Res.} = \frac{1}{2} \left(\frac{4c}{c+1} \right)^c q c \frac{p^2}{p_0^2} \cos \theta$$

$$n \geq 2 \quad \text{Res.} = \frac{1}{2} \left\{ \left(\frac{4c}{n+c} \right)^c \frac{q}{\Gamma(c)} \frac{\Gamma(n+c)}{n!} \left(\frac{p^2}{p_0^2} \right)^n \right\} \left\{ \sum_{\substack{l=2 \\ l(\text{even})}}^n (2l+1) \frac{n(n-2)\dots(n-l+2)}{(n+1)(n+3)\dots(n+l+1)} P_l(\cos \theta) \right. \\ \left. + \sum_{\substack{l=3 \\ l(\text{odd})}}^n (2l+1) \frac{(n-1)(n-3)\dots(n-l+2)}{(n+2)(n+4)\dots(n+l+1)} P_l(\cos \theta) \right\}$$

(7.5)

The partial decay widths for the decay of the resonances to the incident particles can be determined from these residues.

$$\Gamma_{M_N(l) \rightarrow 2M_0} = \frac{p}{128\pi^2} \frac{1}{M_{N(l)}^2} \frac{1}{2l+1} \text{Res.}_{N(l)}(\theta=0) \Big|_{s=M_N^2}$$

$$= \frac{q}{256\pi^2} d \frac{\Gamma(n+c)}{\Gamma(c)} \left(\frac{4c}{n+c} \right)^c \left(\frac{n-3c}{n+c} \right)^{n+1/2} \frac{1}{(n-l)!! (n+l+1)!! (n+c)^{1/2}}$$

For the leading trajectory $n = l \rightarrow \infty$

$$\Gamma \sim \frac{e^{-n \ln 2}}{n^2} \sim \frac{e^{-\langle n^2 \rangle \ln 2}}{(\langle n^2 \rangle)^2}$$

while for a low value of l , say $l = 2$ and $n \rightarrow \infty$

$$\Gamma \sim \frac{1}{n^{5/2}}$$

This is in agreement with the partial widths calculated from the Veneziano amplitude which for large l decreases roughly exponentially.¹⁹

The above indicates that for large S the contributions to the scattering come mainly from the daughter trajectories.

To obtain the behavior of the amplitude for large s the limiting procedure must be done cautiously since the parameters of the hypergeometric function become large and the argument goes to one. However, no matter how the limit is taken, the leading behavior is $\sim \frac{1}{s} F(t)$ where $F(t)$ is sensitive to how the limit is taken but the s dependence is not. To obtain an explicit expression for the asymptotic behavior in s we take the limit in the following manner. Write the variable x as

$$x = \frac{\vec{p} \cdot \vec{p}'}{p_0 p'_0} = \frac{t-u}{s} = 1 + \frac{2t - 4m^2}{s}$$

Then, using the relations for the analytic continuation of the hypergeometric function which connect ${}_2F_1(\alpha, \beta, \gamma; z)$ with linear combinations of ${}_2F_1(\alpha', \beta', \gamma'; 1-z)$, we obtain

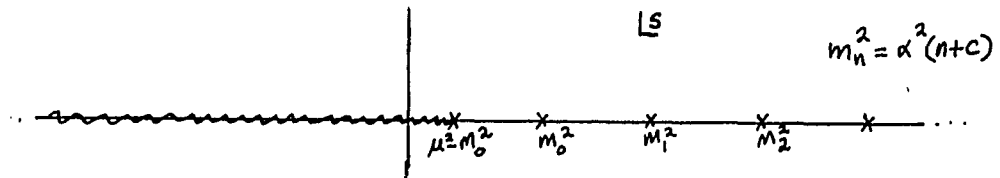
$$A(s, t) = \frac{1}{2} \frac{q_m^2}{s} \left\{ \left(\frac{4m^2}{s} \right)^c \frac{\Gamma(c - \frac{s}{\alpha^2}) \Gamma(-c)}{\alpha^2 \Gamma(-\frac{s}{\alpha^2})} \frac{1}{\left(1 - \frac{(4m^2 - 2t)}{s} \right)^{c-5/2}} + \frac{1}{m^2} \left(\frac{1 - \frac{t}{2m^2}}{1 - \frac{t}{2m^2}} \right)^c {}_2F_1 \left(1, -\frac{s}{\alpha^2}; 1-c; \frac{4m^2 - 2t}{s} \right) \right\} - \frac{1}{2s} \frac{1}{\left(1 - \frac{t}{2m^2} \right)^c} \quad (7.6)$$

Then, taking the limit as $|s| \rightarrow \infty$ (t fixed 0), and keeping $-s$ off the negative real axis to avoid the poles, the asymptotic form becomes

$$A(s, t) \xrightarrow{|s| \rightarrow \infty} \frac{1}{s} \left\{ e^{-\frac{(4m^2 - 2t)}{\alpha^2}} \frac{q_c \Gamma(-c) (-4c)^c}{2} + \frac{q}{2} \frac{1}{(1 - t/2m^2)} {}_1F_1(1, 1-c; \frac{2t - 4m^2}{\alpha^2}) - \frac{1}{2} \frac{1}{(1 - t/2m^2)^c} \right\} \quad (7.7)$$

Thus, there are no unreasonable features at high energy. The $1/s$ dependence is characteristic of a Born term and is also in agreement with the large s behaviour of Fronsda's amplitude.

In addition to the poles in s there is a left hand cut in the s plane due to the factor $\left[\frac{4m^2 s}{(s + m^2 - \mu^2)^2} \right]^c$. The cut runs from $\mu^2 - m^2$ or 0 (whichever is greater) to $-\infty$. The origin of this cut is the factor ρ_0^{-c} which should be thought of as kinematic in the same sense as the factor $(1 - \frac{t}{2m^2})^{-c}$. The normal threshold cut at $s = 4m^2$ is not present and is replaced here by the string of poles along the positive real axis. The analytic properties in the $s < 0$ region do not necessarily reflect those of the physical scattering amplitude due to the properties of the inverse wave operator in this region.



As a function of $\cos\theta$ the amplitude (7.2) is analytic within the Lehmann ellipse and, in fact, has a much larger region of analyticity in the $\cos\theta$ plane. The analytic properties of the amplitude are obtained from those of the hypergeometric function.

The hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ is a single valued

analytic function in the z plane, cut from $z = 1$ to ∞ , i.e.,

$|\arg(-z)| < \pi$ for $|z| \geq 1$.²¹ The variable in the amplitude (7.2) is

$$\frac{\vec{p} \cdot \vec{p}'}{p_0 p_0'} = \frac{p^2 \cos \theta}{p_0^2} \quad \text{which is always } < 1 \text{ in the physical region of the}$$

s channel. For s continued out of this region there is an s dependent

cut in $\cos \theta$ from $\text{Re} \frac{p^2 \cos \theta}{p_0^2} \geq 1$ to ∞ (equal mass). In the t plane

this results in a cut from $t = 2m^2 \rightarrow \infty$, the same cut which comes from

the form factor $(1 - \frac{t}{2m^2})^{-c}$ in the second term of the amplitude.

To do a partial wave analysis of the amplitude it is convenient to consider it in its original form as a sum of two hypergeometric functions.

$$\begin{aligned} A(s, t) &= \frac{1}{2} \frac{1}{m^2 - s} \left(\frac{4m^2}{s} \right)^c \left\{ \frac{8m^2}{s} {}_2F_1 \left(c+1, c-\frac{s}{2}; c+1-\frac{s}{2}; \frac{p^2 \cos \theta}{p_0^2} \right) + {}_2F_1 \left(c, c-\frac{s}{2}; c+1-\frac{s}{2}; \frac{p^2 \cos \theta}{p_0^2} \right) \right\} \\ &= \frac{1}{2} \frac{1}{m^2 - s} \left(\frac{4m^2}{s} \right)^c \left\{ \sum_{l=0}^{\infty} (2l+1) \left\{ \frac{8m^2}{s} A_l^1(s) + A_l^2(s) \right\} P_l(\cos \theta) \right\} \quad (7.8) \end{aligned}$$

where

$$A_l^1(s) = \left(\frac{s-4m^2}{2s} \right)^{\frac{l}{2}} \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\Gamma(2n+l+c+1)}{\Gamma(c) \Gamma(n+l+\frac{3}{2})} \frac{1}{n!} \left(\frac{s-4m^2}{2s} \right)^{2n} \frac{1}{(2n+l+c-\frac{s}{2})}$$

$$A_l^2(s) = \left(\frac{s-4m^2}{2s} \right)^{\frac{l}{2}} \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{\Gamma(2n+l+c)}{\Gamma(c-1) \Gamma(n+l+\frac{3}{2})} \frac{1}{n!} \left(\frac{s-4m^2}{2s} \right)^{2n} \frac{1}{(2n+l+c-\frac{s}{2})} \quad (7.9)$$

These partial wave amplitudes are identical for even and odd l due to the absence of a left hand cut in the $\cos\theta$ plane. Therefore the usual definitions²² of partial wave amplitudes of even and odd signature are identical. i.e.,

$$f_l^{\pm}(s) = \frac{1}{\pi} \int_{z_0}^{\infty} dx Q_l(x) [F_1(s, x) \pm F_2(s, -x)]$$

$$F_2(s, -x) = 0 \quad \text{if } F(s, t) \text{ real for } \cos\theta \text{ negative.}$$

In order to continue the partial wave amplitude to complex l it must satisfy Carlson's theorem to be the unique interpolation. Letting $|l| \rightarrow \infty$ in equation (7.9), and then summing the series, we obtain

$$|A_l^i(s)| \rightarrow \left| \left(\frac{s - 4m^2}{2s} \right)^l \frac{1}{l^{3/2} - c} e^{l \left(\frac{s - 4m^2}{2s} \right)^2} \right| < e^{\pi |l|}$$

so that $A_l^i(s)$ may be continued to complex l . If s is fixed $-1 < \frac{s}{\alpha^2} < 0$ then the three poles furthest to the right in the l plane are at $l = -c$, $\frac{s}{\alpha^2} - c$, $-1 - c$. Therefore, for large t the leading behavior is governed by the fixed pole at $-c$ which comes from the $(1 - \frac{t}{2m^2})^{-c}$ term. The second pole at $\frac{s}{\alpha^2} - c$ gives Regge-like behavior since the particles in this model are on trajectories $\alpha(s) = \frac{s}{\alpha^2} - c$. This large t behavior can be verified

directly from the amplitude. Transforming the hypergeometric function to continue it into the region $s < 0$, $t \geq 4m^2$ we obtain

$$A(s,t) = \frac{1}{2} \frac{1}{(1-\frac{t}{2m^2})^c} \left\{ \frac{q}{4} \frac{1}{(1-\frac{t}{2m^2})} \left[-\frac{1}{s+\alpha^2} {}_2F_1\left(c+1, 1; 2+\frac{s}{\alpha^2}; \frac{s}{4m^2-2t}\right) \right. \right. \\ \left. \left. + \frac{\Gamma(c-\frac{s}{\alpha^2})\Gamma(1+\frac{s}{\alpha^2})}{m^2\Gamma(c)} \left(\frac{4m^2-2t}{s}\right)^{1+\frac{s}{\alpha^2}} \left(1-\frac{s}{4m^2-2t}\right)^{\frac{s}{\alpha^2}-c} \right] - \frac{1}{s} \right\} \quad (7.10)$$

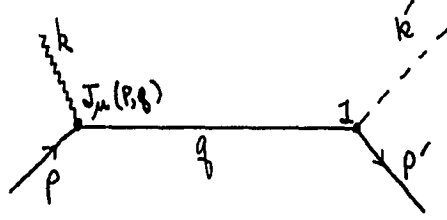
Then letting $|t|$ go to infinity with $s < 0$

$$A(s,t) \longrightarrow -\left(\frac{-t}{2m^2}\right)^{-c} \frac{1}{2s} + \left(\frac{-t}{2m^2}\right)^{\frac{s}{\alpha^2}-c} \frac{q}{8m^2} \left(\frac{4m^2}{s}\right)^{1+\frac{s}{\alpha^2}} \frac{\Gamma(c-\frac{s}{\alpha^2})\Gamma(1+\frac{s}{\alpha^2})}{\Gamma(c)} \\ - \left(\frac{-t}{2m^2}\right)^{-c-1} \frac{q}{8(s+\alpha^2)} \quad (7.11)$$

This is valid for $s \neq 0$; at $s = -\alpha^2$ there is only an apparent pole since the residues of the poles in the second and third terms cancel. Equation (7.11) agrees with the large t behavior obtained from the partial wave analysis

The crossed graph for the process we are considering has not been included in the amplitude since it contributes poles in u in the physical region of the s channel. These poles occur at the "spacelike" masses (See Appendix C.) In addition, the crossed amplitude is not the analytic continuation of $A(s,t)$, $s \rightarrow u < 0$, and there is the problem of the spacelike poles. Questions of crossing symmetry and elimination of the spurious spacelike poles must await the formulation of an interacting quantum field theory of this model. These questions are, of course, related to the completeness relations and the construction of the correct Green's function and interaction vertices. Our continuing investigation will focus on these matters.

We next consider an amplitude with a modification of one of the vertices; one scalar and one vector vertex.



for example, the interaction of a vector particle with the ground state of the infinite component system producing a scalar particle.

The method of calculation is outlined in Appendix C. The amplitude for this process is:

$$\begin{aligned}
 A_{\mu}(p, p', q) = & \frac{1}{2} \frac{m^2}{m^2 + q^2} \frac{1}{p \cdot q} \left[\frac{z}{(-p \cdot p')/m^2} \right]^c {}_2F_1 \left(c+1, c+\frac{q^2}{\alpha^2}; c+1+\frac{q^2}{\alpha^2}; 1-z \right) \left[p_{\mu} \left(\frac{q^2}{p' \cdot q} + 1 \right) - m^2 g_{\mu} \left(\frac{1}{p'} \right) \right. \\
 & + \frac{1}{2} \frac{m^2}{m_1^2 + q^2} \left[\frac{z}{(-p \cdot p')/m_1^2} \right]^c \left\{ -m_1^2 \left(\frac{1}{p' \cdot q} + \frac{1}{q^2} \right) {}_2F_1 \left(c+2, c+1+\frac{q^2}{\alpha^2}; c+2+\frac{q^2}{\alpha^2}; 1-z \right) + \frac{m_1^2 + q^2}{q^2} \frac{1}{z^{c+1}} \right\} \times \\
 & \times \left\{ \frac{p'_{\mu} q^2}{p \cdot q p' \cdot q} - \frac{g_{\mu} p \cdot p' q^2}{(p \cdot q)^2 p' \cdot q} \right\} + \frac{g_{\mu}}{2q^2} \frac{m^2}{p \cdot q} \frac{1}{(-p \cdot p')/m^2}^c + (p_{\mu} + q_{\mu}) A(p, p', q)
 \end{aligned}
 \tag{7.12}$$

where $z = \frac{p \cdot p' q^2}{p \cdot q p' \cdot q}$, $m^2 = \alpha^2 c$, $m_1^2 = \alpha^2 (c+1)$ and $A(p, p', q)$ is the previous amplitude equation (7.2). In the center of mass with

$$q = (0, 0, 0, i\sqrt{s}) \quad \text{and for equal mass particles this becomes:}$$

$$\begin{aligned}
A_{\mu}(s,t) = & \frac{3}{2} \frac{m^2}{m^2-s} \left[\frac{4m^2}{s} \right]^c {}_2F_1\left(c+1, c-\frac{s}{\alpha^2}; c+1-\frac{s}{\alpha^2}; \frac{t-u}{s}\right) \left[\frac{p_{\mu}}{s} + \frac{q_{\mu}}{s} \left(3-\frac{2m^2}{s}\right) \right] \\
& - \frac{6m^2 m_1^2}{m_1^2-s} \left[\frac{4m^2}{s} \right]^c {}_2F_1\left(c+2, c+1-\frac{s}{\alpha^2}; c+2-\frac{s}{\alpha^2}; \frac{t-u}{s}\right) \left[\frac{p'_{\mu}}{s^2} + 2q_{\mu} \frac{p \cdot p'}{s^2} \right] \\
& + \frac{p'_{\mu}}{2s} \frac{1}{\left(1-\frac{t}{2m^2}\right)^{c+1}} - \frac{(p_{\mu}+q_{\mu})}{2s} \frac{1}{\left(1-\frac{t}{2m^2}\right)^c}
\end{aligned}$$

(7.13)

The general structure of this amplitude is the same as that of the scalar amplitude. The leading behavior for large s is $\sim \frac{1}{\sqrt{s}}$ and for large t the fixed pole behavior $\sim t^{-c}$ is still present, as well as the Regge term $\sim t^{\frac{s}{\alpha^2}-c}$.

In summary, we have examined some of the properties of lowest order scattering amplitudes in the relativistic oscillator model. The amplitudes have the same general structure as those obtained by Fronsdal and Nambu in their $SU(3,1)$ models, i.e., hypergeometric functions with similar arguments and parameters. We would expect, however, that the two component nature of this representation and the linearized propagator would, in higher orders, yield different results. This may be particularly true when more complicated vertex functions are considered. The calculational techniques used here may prove useful once an interacting field theory of this model is developed.

8. Half Integral Spin Relativistic Oscillator

We next describe an attempt to extend the previous model to one with half-integral spins. In order to do this one could simply append a spinor to each component of the wave function. This would require increasing the number of components of the 2-component oscillator representation so that the spin and "orbital" operators commute. However, this procedure would not change the mass spectrum or alter any of the previous results in any significant way since the spin and internal structure of the oscillator would then be independent. Instead, we look for a model where the mass operator is spin dependent. One such model that we have considered here has an undesirable feature, namely, the mass spectrum has two branches, one of which is infinitely degenerate (the mass being independent of angular momentum). This difficulty is similar to one discovered by Abarbanel and Frishman²³ in a somewhat different context. They found that a field transforming as a finite dimensional $(\otimes)$ infinite irreducible representation $([\frac{1}{2}, 0] \otimes [0, C])$ with the conditions of covariance, locality, and a mass spectrum bounded from below leads to this infinite degeneracy. The model considered here transforms as $[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})] \otimes$ infinitely reducible representation. As will be indicated below, we can trace this infinite degeneracy to a large spin orbit coupling

which can either add to or cancel the angular momentum contribution to the mass, thus giving two series of masses one of which is spin independent. However, in spite of the infinite degeneracy of one sequence of masses, the form factors for the lowest non-degenerate states have a reasonable non-relativistic limit. This model also gives an indication of some of the problems involved in introducing half-integral spin in a non-trivial way. A description of this representation follows.

In order to combine a spin-half representation with our previous representation it is first necessary to enlarge it to an 8 x 8 matrix. The reason for this is that in order to retain the properties of the previous representation we require first that the angular momentum operator be expressible as $\vec{J} = \vec{L} + \frac{\vec{\sigma}}{2}$. Adjoining a two dimensional spin space to the two component oscillator requires enlarging the space to four components. To keep the representation unitary we require a metric operator η such that $\eta D_{spin}^{+}(1) \eta^{-1} = D_{spin}^{-}(1)$. Such a metric does not exist for the $[\frac{1}{2}, 0]$ or $[0, \frac{1}{2}]$ representation.²⁴ Therefore we must further enlarge the representation to eight components to incorporate a Dirac representation. Thus we define 8 x 8 gamma matrices analagous to the Dirac matrices:

$$\Gamma_\mu = \begin{pmatrix} 0 & 0 & -i\sigma_\mu & 0 \\ 0 & 0 & 0 & -i\sigma_\mu \\ -i\hat{\sigma}_\mu & 0 & 0 & 0 \\ 0 & -i\hat{\sigma}_\mu & 0 & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (8.1)$$

$$\sigma_\mu \equiv (\vec{\sigma}, i), \quad \hat{\sigma}_\mu \equiv (-\vec{\sigma}, i)$$

Using these matrices the homogeneous Lorentz group generators are:

$$\frac{\Sigma_{\mu\nu}}{2} = \frac{1}{4i} [\Gamma_\mu, \Gamma_\nu], \quad \Sigma_{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma_k & & 0 \\ & \sigma_k & \\ 0 & & \sigma_k \end{pmatrix}, \quad \Sigma_{i4} = \begin{pmatrix} -\sigma_i & & 0 \\ & -\sigma_i & \\ 0 & & \sigma_i \end{pmatrix} \quad (8.2)$$

Now extend the previous two component representation to eight components

$$\vec{K} = \begin{pmatrix} \vec{K}^{(c)} & 0 \\ 0 & \vec{K}^{(c-1)} \end{pmatrix} \longrightarrow \mathbb{K} = \begin{pmatrix} \vec{K}^{(c)} & & 0 \\ & \vec{K}^{(c-1)} & \\ 0 & & \vec{K}^{(c)} & \\ & & & \vec{K}^{(c-1)} \end{pmatrix} \quad (8.3)$$

where each $\vec{K}$ in $\mathbb{K}$ is two components, and the orbital angular momentum is completely diagonal.

$$\mathbb{L} = \begin{pmatrix} \vec{L} & & 0 \\ & \vec{L} & \\ 0 & & \vec{L} & \\ & & & \vec{L} \end{pmatrix}$$

Then

$$[\mathbb{K}, \Sigma_{\mu\nu}] = [\mathbb{L}, \Sigma_{\mu\nu}] = 0$$

and the new Lorentz group generators are

$$\vec{J} = \mathbb{L} + \frac{\mathbb{K}}{2}, \quad N_i = K_i + i \frac{\Sigma_{i4}}{2} \quad (8.4)$$

The vector $A_\mu = \begin{pmatrix} 0 & a_\mu \\ a_\mu^* & 0 \end{pmatrix}$ then becomes

$A_\mu = \begin{pmatrix} A_\mu & 0 \\ 0 & A_\mu \end{pmatrix}$ where each A_μ is four components
i.e.

$$A_\mu = \begin{pmatrix} 0 & 0 & a_\mu & 0 \\ 0 & 0 & 0 & a_\mu \\ a_\mu^* & 0 & 0 & 0 \\ 0 & a_\mu^* & 0 & 0 \end{pmatrix} \quad (8.5)$$

With these definitions the vector

$$V_\mu \equiv -i \Gamma_5 \frac{\Sigma_{\mu\nu}}{2} A_\nu$$

is a four vector under transformations generated by $\vec{J}$ and $\vec{N}$.

$$\vec{V} = \frac{i}{2} \begin{pmatrix} \vec{\sigma} \times \vec{A} & 0 \\ 0 & -\vec{\sigma} \times \vec{A} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \vec{\sigma} A_0 & 0 \\ 0 & \vec{\sigma} A_0 \end{pmatrix} \quad (8.6)$$

$$V_4 = -\frac{i}{2} \begin{pmatrix} \vec{\sigma} \cdot \vec{A} & 0 \\ 0 & \vec{\sigma} \cdot \vec{A} \end{pmatrix}$$

V_μ does not satisfy the algebra $[V_\mu, V_\nu] = -i \Gamma_5 M_{\mu\nu}$ as in the two dimensional case.

Using the four vector V_μ we can write a wave equation

$$(\lambda V \cdot p + p^2) \psi(p) = 0 \quad (8.7)$$

$$\bar{\psi}(p) (\lambda V \cdot p + p^2) = 0$$

$$\bar{\psi}(p) = \psi^\dagger(p) \Gamma_4$$

In the rest frame this becomes

$$m \frac{\lambda}{2} \begin{pmatrix} \vec{\sigma} \cdot \vec{A} & 0 \\ 0 & \vec{\sigma} \cdot \vec{A} \end{pmatrix} \psi(0) = m^2 \psi(0) \quad (8.8)$$

so that $\psi(0)$ has the form $\begin{pmatrix} \phi(0) \\ \phi(0) \end{pmatrix}$

The redundancy of the components wave function at rest is a consequence of requiring that spin and internal coordinate operators commute. In any other frame the components are distinguished by the spin part of the transformation.

The mass spectrum is then determined by the eigenvalue equation

$$\begin{aligned} \frac{\lambda}{2} \vec{\sigma} \cdot \vec{A} \phi(0) &= m \phi(0) \\ \frac{\lambda}{2} \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{a} \\ \vec{\sigma} \cdot \vec{a}^+ & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= m \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \end{aligned} \quad (8.9)$$

To solve this equation we express $\vec{a}$ and $\vec{a}^+$ in terms of internal coordinates so that

$$\begin{aligned} \vec{\sigma} \cdot \vec{a} &= \frac{\vec{\sigma} \cdot \hat{r}}{\sqrt{2}} \left\{ r + \frac{\partial}{\partial r} - \frac{\vec{\sigma} \cdot \vec{l}}{r} \right\} \\ \vec{\sigma} \cdot \vec{a}^+ &= \frac{\vec{\sigma} \cdot \hat{r}}{\sqrt{2}} \left\{ r - \frac{\partial}{\partial r} + \frac{\vec{\sigma} \cdot \vec{l}}{r} \right\} \end{aligned}$$

Introducing the operator

$$\mathcal{K} \equiv \vec{\sigma} \cdot \vec{l} + 1$$

$$\begin{aligned} \mathcal{K} \phi_k(\theta, \phi) &= -k \phi_k(\theta, \phi) & k=l & \quad j=l-1/2 \\ & & & = -l-1 \quad j=l+1/2 \end{aligned}$$

then equation (8.9) becomes

$$\begin{aligned}\vec{\sigma} \cdot \hat{r} \left\{ r + \frac{\partial}{\partial r} - \frac{(\lambda-1)}{r} \right\} \phi_2(\vec{r}) &= \sqrt{2} m \left(\frac{2}{\lambda} \right) \phi_1(\vec{r}) \\ \vec{\sigma} \cdot \hat{r} \left\{ r - \frac{\partial}{\partial r} + \frac{(\lambda-1)}{r} \right\} \phi_1(\vec{r}) &= \sqrt{2} m \left(\frac{2}{\lambda} \right) \phi_2(\vec{r})\end{aligned}\quad (8.10)$$

There are two sets of solutions to this equation:

$$\begin{aligned}\phi^a(\vec{r}) &= N e^{-\frac{r^2}{2}} \left(\begin{array}{l} r^l L_n^{l+1/2}(r^2) \phi_k(\theta, \phi) \\ \vec{\sigma} \cdot \hat{r} r^l \left\{ \begin{array}{l} -\frac{\sqrt{n+1}}{r} L_{n+1}^{l-1/2}(r^2) \\ \frac{r}{(n+l+3/2)^{1/2}} L_n^{l+3/2}(r^2) \end{array} \right\} \phi_k(\theta, \phi) \end{array} \right. \begin{array}{l} \text{for } k=l \\ \text{for } k=-l-1 \end{array} \left. \right) \\ m^2 &= \left(\frac{\lambda}{2} \right)^2 \begin{cases} 2(n+1) & k=l \\ 2(n+l+3/2) & k=-l-1 \end{cases} \\ \phi^b(\vec{r}) &= N e^{-\frac{r^2}{2}} \left(\begin{array}{l} \vec{\sigma} \cdot \hat{r} r^l \left\{ \begin{array}{l} \frac{(n+l+1/2)^{1/2}}{r} L_n^{l-1/2}(r^2) \\ -\frac{r}{\sqrt{n}} L_{n-1}^{l+3/2}(r^2) \end{array} \right\} \phi_k(\theta, \phi) \\ r^l L_n^{l+1/2}(r^2) \phi_k(\theta, \phi) \end{array} \right. \begin{array}{l} \text{for } k=l \\ \text{for } k=-l-1 \end{array} \left. \right) \\ m^2 &= \left(\frac{\lambda}{2} \right)^2 \begin{cases} 2(n+l+1/2) & k=l \\ 2n & k=-l-1 \end{cases}\end{aligned}\quad (8.11)$$

where $L_n^\alpha(r^2)$ are the Laguerre polynomials.

The origin of the infinite degeneracy is that the operator $\mathcal{K}$ has positive and negative eigenvalues (see equation (8.18) and the discussion following it), and there does not seem to be any grounds for rejecting the degenerate solutions. A spectrum similar to this with eigenvalues dependent on $|k|$ has been considered by N.V.V.J. Swamy.²⁵ His equation is a Dirac equation with an equivalent oscillator potential which can be written in the form²⁶

$$\left\{ \chi^2 \begin{pmatrix} 0 & -i(s_x \vec{r} \cdot \vec{r} + i \vec{r} \cdot \vec{p}) \\ i(\vec{r} \cdot \vec{r} s_x - i \vec{r} \cdot \vec{p}) & 0 \end{pmatrix} + \begin{pmatrix} m_0 & 0 \\ 0 & -m_0 \end{pmatrix} \right\} \psi(\vec{r}) = E \psi(\vec{r}) \quad (8.12)$$

with eigenvalues $E = [m_0^2 + 4\chi^2(n + |k| + 1/2)]^{1/2}$ where

$$s_x = \frac{\vec{r} \cdot \vec{l} + 1}{|\vec{r} \cdot \vec{l} + 1|} = \frac{\chi}{|\chi|}$$

Introducing a factor S_χ appropriately into equation (8.10) removes the infinite degeneracy but also has the effect of destroying the algebra of the two operators $\vec{K}, A_\mu$. Attempts along these lines have so far not been successful.

Returning to the solutions (8.11), the lowest mass states can be written in terms of $\vec{a}$ and $\vec{a}^\dagger$ as

$$\begin{aligned} \phi^\alpha &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} |0\rangle \chi_s \\ \frac{\vec{r} \cdot \vec{a}^\dagger}{\sqrt{3}} |0\rangle \chi_s \end{pmatrix} & \phi^\beta &\equiv \frac{1}{\sqrt{6}} \begin{pmatrix} \vec{r} \cdot \vec{a}^\dagger |0\rangle \chi_s \\ \frac{\vec{a} \cdot \vec{a}^\dagger}{\sqrt{2}} |0\rangle \chi_s \end{pmatrix} \\ m_\alpha &= \frac{\lambda}{2} \sqrt{3}, \quad j = \frac{1}{2} & m_\beta &= \frac{\lambda}{2} \sqrt{2}, \quad j = \frac{1}{2} \end{aligned} \quad (8.13)$$

It should be noted that the mass spectrum is independent of the constant c which appears in the boosts. Perhaps this is an indication that a different method of introducing spin half is required. Other properties of the particle, however, do depend on c . For example, the magnetic moments of the states can be calculated as before

$$\bar{\psi}(p,s) \vec{M} \psi(p,s) = \lim_{\vec{q} \rightarrow 0} \frac{e}{2m} \int d^3x \vec{x} \times \vec{J}(p+q) e^{-i\vec{q} \cdot \vec{x}} \quad (8.14)$$

where $J_\mu(p,p') = \bar{\psi}(p',s) \{ \lambda V_\mu + (p+p')_\mu \} \psi(p,s)$ is the conserved current derived from the wave equation. Carrying out this calculation for the states ϕ^α and ϕ^β (equation 8.13) we find for the static moments

$$\begin{aligned} \vec{M}^\alpha &= \frac{2}{3} c \frac{e}{2m_\alpha} \vec{\sigma} & g-2 &= \frac{2}{3} (2c-3) \\ \vec{M}^\beta &= - \frac{(4+2c)}{3} \frac{e}{2m_\beta} \vec{\sigma} & g-2 &= - \frac{2}{3} (7+2c) \end{aligned} \quad (8.15)$$

The form factor for the state ϕ^α is

$$\bar{\psi}^\alpha(p',s) J_0 \psi(p,s) = \frac{(1-\frac{t}{4m^2})^{3/2}}{(1-\frac{t}{2m^2})^{c+1}} \left\{ \frac{1}{3} \left[5 - \frac{2}{(1-\frac{t}{4m^2})} \right] + \frac{2}{3} c \frac{t}{4m^2} \right\} \quad (8.16)$$

To get the non-relativistic limit of this form factor we have to let m^2 and c go to infinity simultaneously; therefore we make the following identification: $m^2 \equiv \alpha^2 c$. Then letting $c \rightarrow \infty$ with $t = -k^2$ we obtain

$$\bar{\psi}(p) J_0 \psi(p) \rightarrow e^{-\frac{k^2}{2\alpha^2}} \left(1 - \frac{k^2}{6\alpha^2}\right) \quad (8.17)$$

Now we may ask what non-relativistic Schroedinger equation gives a form factor of this form. The answer is that a harmonic oscillator Hamiltonian with a spin orbit coupling will yield the form (8.17)

$$\left\{ \frac{\vec{a}^+ \cdot \vec{a} + \vec{a} \cdot \vec{a}^+}{2} \pm \vec{\sigma} \cdot \vec{l} + \beta \right\} \psi = E \psi \quad (8.18)$$

For the + sign (8.18) can be written as

$$\left\{ \vec{\sigma} \cdot \vec{a} \vec{\sigma} \cdot \vec{a}^+ + \left(\beta - \frac{3}{2}\right) \right\} \psi = E \psi \quad (8.19)$$

Equation (8.19) indicates the origin of the infinite degeneracy since

$\vec{\sigma} \cdot \vec{A}$ is essentially a factorization of equation (8.18). This factorization can be done only if the coefficients of $\vec{a}^+ \cdot \vec{a}$ and $\vec{\sigma} \cdot \vec{l}$ are the same—resulting in a spin orbit coupling of the same strength as the $\vec{a}^+ \cdot \vec{a}$ term.

A simple solution of (8.19) is $\psi^{(1)} = \frac{\vec{\sigma} \cdot \vec{a}^\dagger}{\sqrt{3}} |0\rangle$

$$\left\{ \vec{\sigma} \cdot \vec{a} \vec{\sigma} \cdot \vec{a}^\dagger + (\beta - \frac{3}{2}) \right\} \psi^{(1)} = (\beta + \frac{1}{2}) \psi^{(1)}$$

Then $\psi^{(1)\dagger} e^{i\vec{k} \cdot \vec{x}} \psi^{(1)} = \frac{1}{3} \langle 0 | \vec{\sigma} \cdot \vec{a} e^{i\vec{k} \cdot \frac{(\vec{a} + \vec{a}^\dagger)}{\sqrt{2}}} \vec{\sigma} \cdot \vec{a}^\dagger | 0 \rangle$ (8.20)

$$= e^{-\frac{k^2}{4}} \left(1 - \frac{k^2}{6} \right)$$

restoring the constants we have

$$F(k^2) = e^{-\frac{k^2}{4\mu\omega}} \left(1 - \frac{k^2}{6\mu\omega} \right) \quad (8.21)$$

letting $2\alpha^2 \rightarrow 4\mu\omega$ (8.17) becomes

$$F(k^2) = e^{-\frac{k^2}{4\mu\omega}} \left(1 - \frac{k^2}{12\mu\omega} \right)$$

which is almost identical to (8.21).

Similarly, for the state ϕ^β we have

$$\bar{\psi}_{(p,s)}^\beta \mathcal{D}_0 \psi_{(p,s)}^\beta = \frac{\left(1 - \frac{t}{4m^2}\right)^{1/2}}{\left(1 - \frac{t}{2m^2}\right)^{c+1}} \left\{ 1 - \frac{t}{12m^2} \left(3 - \frac{t}{2m^2}\right) + \frac{(c+2)t}{6m^2} \left(1 - \frac{t}{4m^2}\right) \times \right. \\ \left. \times \left[2 + \frac{1}{\left(1 - \frac{t}{2m^2}\right)} + \frac{(c+3)t}{2m^2} \frac{\left(1 - \frac{t}{4m^2}\right)}{\left(1 - \frac{t}{2m^2}\right)^2} \right] \right\}$$

which in the limit $c \rightarrow \infty$ reduces to

$$e^{-\frac{k^2}{2\alpha^2}} \left\{ 1 - \frac{k^2}{2\alpha^2} + \frac{k^4}{12\alpha^4} \right\}$$

Making the same identification as before this becomes

$$e^{-\frac{k^2}{4\mu\omega}} \left\{ 1 - \frac{k^2}{4\mu\omega} + \frac{k^4}{48\mu^2\omega^2} \right\}$$

while the solution $\psi^{(2)} = \frac{\vec{a} \cdot \vec{a}^\dagger}{\sqrt{6}} |0\rangle$ of equation (8.19)

has the form factor

$$e^{-\frac{k^2}{4\mu\omega}} \left\{ 1 - \frac{k^2}{3\mu\omega} + \frac{k^4}{24\mu^2\omega^2} \right\}$$

The agreement of the form factors in the non-relativistic limit is not exact because the non-relativistic equation (8.19) contains only one sign of $\vec{\sigma} \cdot \vec{L}$ while the relativistic wave function has both signs $(\pm)$.

A heuristic interpretation of the spin orbit term is that if the particle has spin there is an additional interaction energy due to the Thomas precession, $-\vec{S} \cdot \vec{\omega}_T$

where $\vec{\omega}_T \approx \frac{1}{2c^2} \vec{v} \times \dot{\vec{v}}$ for $v \ll c$

$$\dot{\vec{v}} = -\omega^2 \vec{r}, \quad \therefore \vec{\omega}_T = -\frac{\omega^2}{2mc^2} \vec{L}$$

$$-\vec{S} \cdot \vec{\omega}_T = \frac{\hbar\omega^2}{4mc^2} \vec{\sigma} \cdot \vec{L}$$

Now, roughly in this model $k\omega \sim mc^2$ so there is an additional interaction $\propto \vec{\sigma} \cdot \vec{L}$. So the addition of this term is reasonable.

If we square the operator $\vec{\sigma} \cdot \vec{A}$ we obtain

$$\begin{aligned}
 (\vec{\sigma} \cdot \vec{A})^2 &= \begin{pmatrix} \vec{\sigma} \cdot \vec{a} \vec{\sigma} \cdot \vec{a}^+ & 0 \\ 0 & \vec{\sigma} \cdot \vec{a}^+ \vec{\sigma} \cdot \vec{a} \end{pmatrix} \\
 &= \begin{pmatrix} \vec{a}^+ \cdot \vec{a} + \vec{\sigma} \cdot \vec{L} + 3 & 0 \\ 0 & \vec{a}^+ \cdot \vec{a} - \vec{\sigma} \cdot \vec{L} \end{pmatrix}
 \end{aligned}$$

which is equation (8.18) with both signs of the $\vec{\sigma} \cdot \vec{L}$ term. Thus it seems that the wave equation $(\lambda p \cdot V + p^2) \psi(p) = 0$ is in effect a relativistic version of equation (8.18), i.e., we have found a four vector whose square is essentially the non-relativistic oscillator equation with spin orbit coupling. But equation (8.18) already contains the infinite degeneracy due to the positive and negative eigenvalues of $\vec{\sigma} \cdot \vec{L}$.

In summary, this model is a relativistic generalization of a harmonic oscillator with spin orbit coupling. The infinite degeneracy will have to be removed by a different means of adjoining a finite dimensional spin half representation with the oscillator representation.

A description of baryons will require resolution of this difficulty, and is still being investigated.

9. Summary

We have studied some of the properties of the relativistic oscillator. The model has the attractive features of being ghost free, and of having spacelike solutions which are really only linear combinations of timelike solutions. Therefore, they do not present a barrier to quantization; moreover, a free field quantization can be carried out.

We have not yet made any serious confrontation with experiment since, at this stage, the model is not sufficiently complex to account for a wide range of data. The question of interactions is a crucial one--particularly the electromagnetic interaction. Accordingly, we have been experimenting with various vertex functions in an attempt to produce a more realistic form factor.

We have obtained ghost free scattering amplitudes which are reasonable and which one would expect from a model with a discrete mass spectrum. They are also in agreement with those obtained by others using the $SU(3,1)$ oscillator representation. Here again, a vertex function must be found which suitably excites the internal structure of the oscillator. The fact that we have a four vector A_μ is a new feature of the representation, and allows for additional freedom in constructing possible interactions. The infinite reducibility

of the representation and the infinite dimensional algebra generated by the A_μ 's, allow this model to circumvent the conditions of some of the no-go theorems pertaining to infinite component theories. Thus, one can proceed in elaborating the model without the burden of these theorems.

A half integral spin version of this model, which will retain its desirable features, is still being pursued.

Appendix A

Infinite Reducibility of the Representation

In a basis in which $\mathcal{C}$ is diagonal the angular momentum content for each value of $\mathcal{C}$ is $\mathcal{C}, \mathcal{C}-2, \dots \begin{cases} 0 & \mathcal{C} \text{ even} \\ 1 & \mathcal{C} \text{ odd} \end{cases}$. Now a unitary irreducible representation of the homogeneous Lorentz group can contain an irreducible representation of the rotation group at most once.* Therefore, in this harmonic oscillator representation every value of $\mathcal{C} = N$ contains angular momentum $N, N-2, \dots$ and so each value of $\mathcal{C}$ occurs an infinite number of times.

In terms of the boosted wave function this infinite reducibility is clear. For example

$$e^{ik_3^{(u)} \zeta(p_3)} |0\rangle = \sum_{n=0}^{\infty} \alpha_n \left(\frac{p_3 a_3^+}{p_0} \right)^n |0\rangle$$

each term $\left(\frac{p_3 a_3^+}{p_0} \right)^n |0\rangle$ contains values of ℓ from

$$N, N-2, \dots, \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Therefore a boosted wave function

contains each value of ℓ an infinite number of times.

* M.A. Naimark, Linear Representation of the Lorentz Group.
Pergamon Press, New York, 1964.

Appendix B

The Boosted Oscillator Wave Function and Boost Matrix Elements

In order to obtain an explicit expression for the boosted wave function we must determine the action of $e^{-i\vec{k}\cdot\vec{\xi}}$ on a wave function at rest. The boost operator in the form

$$K_i(c) = i\sqrt{c+c+1} a_i - i\sqrt{c+c} a_i^\dagger \quad (\text{B.1})$$

is very unwieldy for purposes of calculation; however, it can be written in the following form

$$e^{i k_i \xi} = e^{-i \alpha_i} e^{i \frac{\beta_i}{2}} e^{-(c+c_i) \xi} e^{-i \frac{\beta_i}{2}} e^{i \alpha_i} e^{-(1+c) \xi}$$

$$\alpha_i = i\sqrt{c+c+1} a_i, \quad \beta_i = -i\sqrt{c+c} a_i^\dagger \quad (\text{B.2})$$

where $\xi = \sinh^{-1} u$ and u is the four velocity. The identity (B.2)

can be verified using the relation

$$e^{iA} B e^{-iA} = B + i[A, B] + \frac{i^2}{2!} [A, [A, B]] + \dots$$

since the series terminates after a few terms when the operators α, β, c are used. With this identity the boosted wave function can be determined. In particular, consider the ground state ($m = \alpha\sqrt{c}$) boosted

along the z direction. Using (B.2) the boost matrix elements are

$$\langle n_1, n_2, n_3 | e^{-ik_z \xi} | 0, 0, 0 \rangle = \left(\frac{(n_3 + c)!}{n_3! c!} \right)^{1/2} \frac{(-\tanh \xi)^{n_3}}{(\cosh \xi)^{c+1}} \delta_{n_1, 0} \delta_{n_2, 0}$$

so that

$$e^{-ik_z \xi} | 0 \rangle = \sum_{n=0}^{\infty} \left(\frac{(n+c)!}{n! c!} \right)^{1/2} (-)^n \frac{(\tanh \xi)^n}{(\cosh \xi)^{c+1}} | 0, 0, n \rangle$$

(B.3)

setting $\tanh \xi = \frac{u}{u_0} = \frac{p}{p_0}$, $\cosh \xi = u_0 = p_0/m$

$$| 0, 0, n \rangle = \frac{a_3^{\dagger n}}{\sqrt{n!}} | 0 \rangle$$

equation (B.3) becomes

$$e^{-ik_z \xi} | 0 \rangle = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left(\frac{(n+c)!}{c!} \right)^{1/2} \left(\frac{m}{p_0} \right)^{c+1} \left(\frac{p_3 a_3^{\dagger}}{p_0} \right)^n | 0 \rangle$$

(B.4)

If we now rotate equation (B.4) so as to rotate p_3 into an arbitrary direction, we have a simple form for the ground state wave function with momentum $\vec{p}$.

$$\psi_0(\vec{p}) = \frac{1}{\sqrt{2}} \left(\sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left(\frac{(n+c)!}{c!} \right)^{1/2} \left(\frac{m}{p_0} \right)^{c+1} \left(\frac{\vec{p} \cdot \vec{a}^{\dagger}}{p_0} \right)^n | 0 \rangle \right)$$

$$\sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left(\frac{(n+c-1)!}{(c-1)!} \right)^{1/2} \left(\frac{m}{p_0} \right)^c \left(\frac{\vec{p} \cdot \vec{a}^{\dagger}}{p_0} \right)^n | 0 \rangle$$

(B.5)

Although this wave function is quite simple in form, the presence of the square roots of factorials makes many calculations difficult since one must do sums of the type $\sum_{n=0}^{\infty} \sqrt{n+c} a_n x^n$. For example, the calculation of $\langle x^2 \rangle$ for the particle moving in the z direction leads to this type of sum.

The general boost matrix element can also be derived from (B.2)

$$\begin{aligned} f_{N',N}(\xi) &\equiv \langle n'_1, n'_2, n'_3 | e^{ik_z \xi} | n_1, n_2, n_3 \rangle \\ &= \delta_{n_1, n'_1} \delta_{n_2, n'_2} \left(\frac{n_3! (n'_3 + n_1 + n_2 + c)!}{n'_3! (n_1 + n_2 + n_3 + c)!} \right)^{\frac{1}{2}} \frac{(\tanh \xi)^{n'_3}}{(\cosh \xi)^{n_1 + n_2 + c + 1}} \times \\ &\quad \times \left[(\cosh 2\xi - 1)(\cosh 2\xi + 1) \right]^{\frac{-n_3}{2}} 2^{n_3} P_{n_3}^{(n'_3 - n_3, c + n_1 + n_2)}(\cosh 2\xi) \end{aligned}$$

(B.6)

where $P_n^{(\alpha, \beta)}(x)$ is the Jacobi polynomial. This result holds for $n'_3 \geq n_3$; for $n'_3 < n_3$, $f_{N,N'}(-\xi)$ is used.

An identity analogous to (B.2) can be written for rotations:

(B.7)

$$e^{-iM_{ij}\theta} = e^{(a_i a_j^\dagger - a_j a_i^\dagger)\theta} = e^{i a_j a_i^\dagger} e^{\frac{i}{2} a_i a_j^\dagger} e^{-i(c_i - c_j)} e^{\frac{i}{2} a_i a_j^\dagger} e^{-i a_j a_i^\dagger}$$

This is useful for evaluating rotation matrix elements in the basis

$|n_1, n_2, n_3\rangle$, since this basis is most convenient for evaluating boosts.

Appendix CCalculation of Scattering Amplitude

$$A(s,t) = \text{diagram} = \psi_0^\dagger(p') \frac{1}{\alpha A_0(p+k) + (p+k)^2} \psi_0(p) \quad (C.1)$$

Evaluating this in the center of mass frame

$$\vec{p} + \vec{k} = 0, \quad s = -(p+k)^2 = (p_0 + k_0)^2$$

it becomes

$$A(s,t) = \psi_0^\dagger(p') \frac{1}{-\alpha A_0 \sqrt{s} - s} \psi_0(p) = \int_0^1 dx x^{-s-1} \psi_0^\dagger(p') e^{-\lambda A_0} \psi_0(p) \quad (C.2)$$

where $\lambda = \alpha \sqrt{s} \ln x$

now

$$e^{-\lambda A_0} \psi_0(p) = e^{\lambda \begin{pmatrix} 0 & \sqrt{p+c} \\ \sqrt{p+c} & 0 \end{pmatrix}} \begin{pmatrix} \sum_{n=0}^{\infty} a_n(p) |0\rangle \\ \sum_{n=0}^{\infty} b_n(p) |0\rangle \end{pmatrix} \quad (C.3)$$

where

$$a_n(p) = \frac{(-)^n}{\sqrt{2}} \left(\frac{(n+c)!}{c!} \right)^{\frac{1}{2}} \left(\frac{m}{p_0} \right)^{c+1} \frac{1}{n!} \left(\frac{\vec{p} \cdot \vec{a}^\dagger}{p_0} \right)^n$$

for $b_n(p)$ let $c \rightarrow c-1$ in $a_n(p)$

Then evaluating (C.3) using

$$\sqrt{n+c} \left(\frac{\vec{p} \cdot \vec{a}^+}{\rho_0} \right)^n |0\rangle = \sqrt{n+c} \left(\frac{\vec{p} \cdot \vec{a}^+}{\rho_0} \right)^n |0\rangle$$

we obtain

$$e^{-\lambda A_0} \psi_0(p) = \left(\sum_{n=0}^{\infty} \left\{ a_n(p) \cosh(\lambda \sqrt{n+c}) + b_n(p) \sinh(\lambda \sqrt{n+c}) \right\} |0\rangle \right) \quad (C.4)$$

$$\left(\sum_{n=0}^{\infty} \left\{ b_n(p) \cosh(\lambda \sqrt{n+c}) + a_n(p) \sinh(\lambda \sqrt{n+c}) \right\} |0\rangle \right)$$

Substituting (C.4) in (C.3)

$$A(s,t) = \frac{1}{2} \int_0^1 dx x^{-s-1} \sum_{n=0}^{\infty} \left\{ \frac{(n+c)!}{n! c!} \omega^n \left(\frac{m^2}{\rho_0 \rho'_0} \right)^{c+1} \cosh(\lambda \sqrt{n+c}) + \frac{(n+c-1)!}{n! (c-1)!} \omega^n \left(\frac{m^2}{\rho_0 \rho'_0} \right)^c \cosh(\lambda \sqrt{n+c}) \right. \\ \left. + \left(\frac{m^2}{\rho_0 \rho'_0} \right)^c \left(\frac{m}{\rho_0} + \frac{m}{\rho'_0} \right) \frac{\sqrt{n+c}}{\sqrt{c}} \frac{(n+c-1)!}{n! (c-1)!} \omega^n \sinh(\lambda \sqrt{n+c}) \right\} \quad (C.5)$$

$$\omega = \frac{\vec{p} \cdot \vec{p}'}{\rho_0 \rho'_0}$$

Doing the x integration ($\lambda = \alpha \sqrt{s} \ln x$)

$$\int_0^1 dx x^{-s-1} \cosh(\lambda \sqrt{n+c}) = \frac{1}{\alpha^2 (n+c - s/\alpha^2)} \quad (C.6)$$

$$\int_0^1 dx x^{-s-1} \sinh(\lambda \sqrt{n+c}) = \frac{\sqrt{n+c}}{\alpha \sqrt{s}} \frac{1}{(n+c - s/\alpha^2)}$$

Then

$$A(s,t) = \frac{1}{2\alpha^2} \left\{ \left(\frac{m^2}{\rho_0 \rho'_0} \right)^c \left(\frac{m^2}{\rho_0 \rho'_0} + \frac{m^2}{\sqrt{s} \rho_0} + \frac{m^2}{\sqrt{s} \rho'_0} \right) \sum_{n=0}^{\infty} \frac{(n+c)!}{c!} \frac{\omega^n}{n!} \frac{1}{(n+c - s/\alpha^2)} \right. \\ \left. + \left(\frac{m^2}{\rho_0 \rho'_0} \right)^c \sum_{n=0}^{\infty} \frac{(n+c-1)!}{(c-1)!} \frac{\omega^n}{n!} \frac{1}{(n+c - s/\alpha^2)} \right\} \quad (C.7)$$

We then note the following identity

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n+a-1)!}{(a-1)! n!} \frac{\omega^n}{n+b} &= \sum_{n=0}^{\infty} \frac{\Gamma(n+a)}{\Gamma(a)} \frac{\Gamma(n+b)}{\Gamma(b)} \frac{\Gamma(b+1)}{\Gamma(n+b+1)} \frac{1}{b} \frac{\omega^n}{n!} \\ &= \frac{1}{b} {}_2F_1(a, b; b+1; \omega) \end{aligned} \quad (C.8)$$

which is used to rewrite (C.7)

$$\begin{aligned} A(s, t) &= \frac{1}{2} \frac{1}{m^2 - s} \left(\frac{m^2}{p_0 p'_0} \right)^c \left\{ \left(\frac{m^2}{p_0 p'_0} + \frac{m^2}{\sqrt{s} p_0} + \frac{m^2}{\sqrt{s} p'_0} \right) {}_2F_1 \left(c+1, c-\frac{s}{\alpha^2}; c+1-\frac{s}{\alpha^2}; \frac{\vec{p} \cdot \vec{p}'}{p_0 p'_0} \right) \right. \\ &\quad \left. + {}_2F_1 \left(c, c-\frac{s}{\alpha^2}; c+1-\frac{s}{\alpha^2}; \frac{\vec{p} \cdot \vec{p}'}{p_0 p'_0} \right) \right\} \end{aligned} \quad (C.9)$$

Rewriting the second term of (C.9) using the relation

$$(b-a) {}_2F_1(a, b; c; z) + a {}_2F_1(a+1, b; c; z) = b {}_2F_1(a, b; c; z)$$

$$\begin{aligned} A(s, t) &= \frac{1}{2} \frac{1}{m^2 - s} \left(\frac{m^2}{p_0 p'_0} \right)^c \left(\frac{m^2}{p_0 p'_0} + \frac{m^2}{\sqrt{s} p_0} + \frac{m^2}{\sqrt{s} p'_0} + \frac{m^2}{s} \right) {}_2F_1 \left(c+1, c-\frac{s}{\alpha^2}; c+1-\frac{s}{\alpha^2}; \frac{\vec{p} \cdot \vec{p}'}{p_0 p'_0} \right) \\ &\quad - \frac{1}{2s} \left(\frac{1}{-p \cdot p'} \right)^c \end{aligned} \quad (C.10)$$

we then note that the invariant variable

$$1-z = 1 - \frac{p \cdot p' q^2}{p \cdot q p' \cdot q} \quad \text{reduces to} \quad \frac{\vec{p} \cdot \vec{p}'}{p_0 p'_0} \quad \text{in the}$$

frame

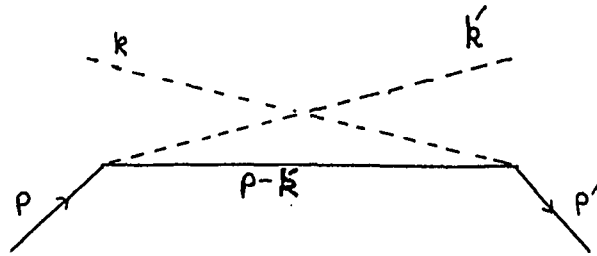
$$q = (0, 0, 0, i\sqrt{s})$$

so that (C.10) can be re-expressed

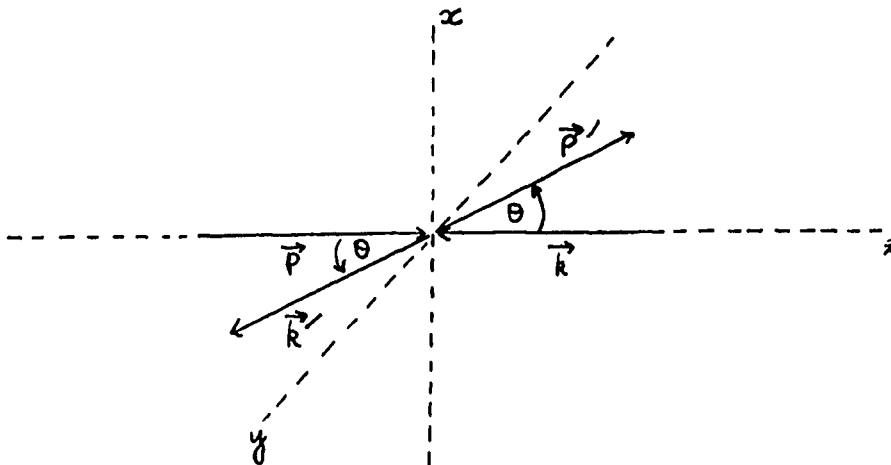
using $z, p \cdot p', p \cdot q, p' \cdot q, q^2$ to obtain

$$A(p, p', q) = \frac{1}{2} \frac{1}{m^2 + q^2} \left(\frac{\bar{z}}{(-p \cdot p')^c} \right)^c \left\{ \left(\frac{z}{(-p \cdot p')^c} \right)^{-m^2} \left(\frac{1}{p \cdot q} + \frac{1}{p' \cdot q} + \frac{1}{q^2} \right) \right\} {}_2F_1 \left(c+1, c+\frac{q^2}{\alpha^2}; c+1+\frac{q^2}{\alpha^2}; 1-z \right) \\ + \frac{1}{2q^2} \left(\frac{1}{(-p \cdot p')^c} \right)^c \quad z = \frac{p \cdot p' q^2}{p \cdot q p' \cdot q}, \quad m^2 = \alpha^2 c$$

As a matter of interest we calculate the crossed graph and, for comparison, the analytic continuation of the amplitude above is given. They are not identical, but both exhibit the spacelike poles. As discussed in Section 7, the crossed graph, calculated with the inverse wave operator as propagator, is not included in the amplitude. It is presented here only to indicate the calculational techniques and to exhibit its functional form explicitly.



This graph is evaluated for the special case of equal mass particles in the CM frame with the following configuration of momenta.



Then the propagator becomes ($\vec{p}$ = CM momentum)

$$\begin{aligned} \frac{1}{\alpha A \cdot (p-k) + (p-k)^2} &= \frac{1}{\alpha p (A_1 \sin \theta + A_3 (1 + \cos \theta)) - u} \\ &= e^{-iJ_2 \frac{\theta}{2}} \frac{1}{\alpha A_3 \sqrt{-u} - u} e^{iJ_2 \frac{\theta}{2}} \end{aligned} \quad (C.11)$$

and the calculation proceeds as before with the use of *

$$\langle n_1, n_2, n_3 | e^{iJ_2 \frac{\theta}{2}} | 0, 0, n \rangle = \delta_{n_1, 0} \delta_{n_3 + n_1, n} (-\sin \frac{\theta}{2})^{n_1} (\cos \frac{\theta}{2})^{n-n_1} \sqrt{\frac{n!}{n_1! (n-n_1)!}} \quad (C.12)$$

and $\psi_0^\dagger(p') = \psi_0^\dagger(0) e^{ik_3 \xi(p')} e^{iJ_2 \theta}$ to obtain

$$\begin{aligned} A^2(s, t, u) = & -\frac{1}{2} \left(\frac{4m^2}{s-t} \right)^c \left\{ \frac{8m^2}{s-t} \frac{1}{u+\alpha^2} {}_2F_1 \left(c+1, \frac{u}{\alpha^2}+1; \frac{u}{\alpha^2}+2; \frac{u}{t-s} \right) \right. \\ & \left. + \frac{1}{u} {}_2F_1 \left(c, \frac{u}{\alpha^2}; \frac{u}{\alpha^2}+1; \frac{u}{t-s} \right) \right\} \end{aligned} \quad (C.13)$$

The analytic continuation of (C.9) to $s < 0$, $u \geq 4m^2$ is obtained from the continuation of the hypergeometric functions which yields

$$\begin{aligned} A^1(s, t, u) = & -\frac{1}{2} \left(\frac{4m^2}{u-t} \right)^c \left\{ \frac{8m^2}{u-t} \frac{1}{s+\alpha^2} {}_2F_1 \left(c+1, \frac{s}{\alpha^2}+1; \frac{s}{\alpha^2}+2; \frac{s}{t-u} \right) \right. \\ & \left. + \frac{1}{s} {}_2F_1 \left(c, \frac{s}{\alpha^2}; \frac{s}{\alpha^2}+1; \frac{s}{t-u} \right) \right\} + \frac{q}{2s} \frac{\Gamma(c-\frac{s}{\alpha^2}) \Gamma(1+\frac{s}{\alpha^2})}{\Gamma(c)} \left(\frac{s}{u-t} \right)^{c-\frac{s}{\alpha^2}} \left(\frac{4m^2}{u-t} \right)^c \end{aligned} \quad (C.14)$$

*This is obtained by the use of the identity (B.7) in Appendix B.

The calculation of the amplitude with the vector vertex is done by making use of the following identity

$$\begin{aligned} \alpha A_\mu \psi_0(p) &= \alpha \begin{pmatrix} 0 & a_\mu \\ a_\mu^+ & 0 \end{pmatrix} \begin{pmatrix} \phi^1(p) \\ \phi^2(p) \end{pmatrix} \\ &= - \begin{pmatrix} p_\mu \phi^1(p) \\ \alpha^2 \frac{\partial}{\partial p_\mu} \phi^2(p) \end{pmatrix} \end{aligned} \quad (C.15)$$

Then, since p_μ and $\frac{\partial}{\partial p_\mu}$ commute with any functions of a_μ and a_μ^+ , the calculation can be done as before, the derivative being taken after the final result is calculated.

The identity (C.15) allows one, in principle, to calculate the action of any function of A_μ on the ground state wave function when it appears in some matrix element. For example,

$$\begin{aligned} \psi_0^+(p') e^{i \frac{k \cdot A}{\alpha}} \psi_0(p) &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-i)^{2n}}{(2n)!} \left\{ \left(k \cdot \frac{\partial}{\partial p} \frac{p \cdot k}{\alpha^2} \right)^n \left(\frac{-p \cdot p'}{m^2} \right)^{-(c+1)} + \left(\frac{p \cdot k}{\alpha^2} k \cdot \frac{\partial}{\partial p} \right)^n \left(\frac{-p \cdot p'}{m^2} \right)^{-c} \right\} \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-i)^{2n+1}}{(2n+1)!} \left\{ \frac{p \cdot k}{\alpha^2} \left(k \cdot \frac{\partial}{\partial p} \frac{p \cdot k}{\alpha^2} \right)^n \left(\frac{-p \cdot p'}{m^2} \right)^{-(c+1)} + k \cdot \frac{\partial}{\partial p} \left(\frac{p \cdot k}{\alpha^2} k \cdot \frac{\partial}{\partial p} \right)^n \left(\frac{-p \cdot p'}{m^2} \right)^{-c} \right\} \end{aligned} \quad (C.16)$$

becomes for the special case $k^2 = 0$

$$\begin{aligned} \psi_0^+(p') e^{i \frac{k \cdot A}{\alpha}} \psi_0(p) &= \frac{1}{2} \left(\frac{-p \cdot p'}{m^2} \right)^{-c} \left\{ \frac{1}{\left(\frac{-p \cdot p'}{m^2} \right)} {}_1F_1(c+1; \frac{1}{2}; z) + {}_1F_1(c; \frac{1}{2}; z) \right. \\ &\left. - \frac{i}{2\alpha^2} \frac{k \cdot (p+p')}{\left(\frac{-p \cdot p'}{m^2} \right)} {}_1F_1(c+1; \frac{3}{2}; z) \right\} \quad z = \frac{p \cdot k \cdot p' \cdot k}{4\alpha^2 p \cdot p'} \end{aligned} \quad (C.17)$$

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24. Recently L.C. Biedenharn, M.Y. Han and H. van Dam have proposed

the metric $(-)^{|\vec{r}, \vec{l}+1|} \frac{(\vec{r}, \vec{l}+1)}{|\vec{r}, \vec{l}+1|} \mathcal{P}^{\text{ext.}}$ in their "Two component alternative to Dirac's Equation", Phys. Rev. Lett. 27, 1171 (1971) and Phys. Rev. D6, 500 (1972) where $\mathcal{P}^{\text{ext.}}$ is the external space

parity operator. It may be possible to utilize this metric in this model; however, the interpretation of $\mathcal{P}^{\text{ext.}}$ is not the conventional one and there is still some controversy surrounding the interpretation of their equation.* This alternative is still being explored.

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