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Multicolor Ramsey numbers for disjoint unions of graphs

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MULTICOLOR RAMSEY NUMBERS FOR DISJOINT UNIONS OF GRAPHS

by

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ABSTRACT

MULTICOLOR RAMSEY NUMBERS FOR DISJOINT UNIONS OF GRAPHS

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Saoping Loo

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Let $G_1, G_2, \ldots, G_c$ be simple graphs, i.e., graphs without loops or multiple edges. The *Ramsey number* $r(G_1, G_2, \ldots, G_c)$ is the smallest integer n such that if the edges of the complete graph K_n are colored arbitrarily with c colors, then for some i, the subgraph in color i contains a copy of G_i. Let mG denote m disjoint copies of some graph G. In this thesis we study the 3–color Ramsey numbers for large disjoint unions of graphs. Results are given which, in principle, permit the Ramsey numbers $r(n_1G_1, G_2, G_3)$, $r(n_1G_1, n_2G_2, G_3)$, and $r(n_1G_1, n_2G_2, n_3G_3)$ to be exactly evaluated when G_i are connected non–bipartite graphs, provided that the n_i are sufficiently large. Such evaluations are often possible in practice, as shown by several examples. For instance, when n_1, n_2, n_3 are sufficiently large,

$$r(n_1K_3, n_2K_3, n_3K_3) = 3(n_1 + n_2 + n_3).$$

Also in this thesis, when n is sufficiently large, results are given which, in principle, permit the Ramsey numbers $r(nF, nG, nH)$ to be evaluated exactly for a large class of connected graphs F, G, and H, where, some or all of these graphs may be bipartite.

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1. INTRODUCTION

Let G and H be simple graphs. The Ramsey number $r(G, H)$ is defined to be the smallest integer n such that, if the edges of the complete graph K_n are colored red and blue, either the red subgraph contains a copy of G or the blue subgraph contains a copy of H. An important case is that in which G and H are disjoint unions of graphs chosen from some fixed finite set $\mathcal{G}$ of connected graphs. Various special cases of this have been considered, but those general results that are known are contained primarily in [1], [2] and [5]. The most general results are contained in [2]. In [2], when G is a disjoint union of m graphs and H is a disjoint union of n graphs chosen from $\mathcal{G}$, then $r(G, H)$ is exactly determined when m and n are large. We will state the main results after introducing a few notation and definitions. In general, we will use the notation of Harary[11]. In particular, p and β represent the order and the vertex independence number, respectively. We use $V(G)$ and $E(G)$ to denote the sets of vertices and edges of a graph G. For any subset S of $V(G)$, the symbol $\langle S \rangle$ represents the subgraph of G induced by the vertex set S. For any two subsets S_1, S_2 of $V(G)$, S_1S_2 denotes the set of edges of G joining S_1 to S_2. We use xy to denote the edge connecting vertices x and y, and use xS denote the set of edges of G joining x to S.

In this thesis, when we say a *c–coloring* of some graph we always mean a coloring of the edges of that graph with c colors. We often just will say coloring instead of c–coloring.

A c–coloring of K_n is called $(G_1, G_2, \ldots, G_c)$*–good* if for all i, $1 \leq i \leq c$, the monochromatic subgraph in color i contains no copy of G_i.

A $(G_1, G_2, \ldots, G_c)$*–tie* is a c–colored graph M such that for each i, $1 \leq i \leq c$, the monochromatic subgraph in color i contains a copy of G_i.

A $(G_1, G_2, \ldots, G_c)$*–good coloring* of K_n is called $(G_1, G_2, \ldots, G_c)$*–critical* if $n = r(G_1, G_2, \ldots, G_c) - 1$.

Define a *c–canonical coloring* of K_n to be a coloring for which the vertices may be partitioned into c sets $V_1, V_2, \ldots, V_c$, where all the edges of $\langle V_i \rangle$ are the same color, and all edges joining V_i to V_j are the same color. We call the edges joining V_i to V_j the connecting edges of V_i and V_j. We note that in such a canonical coloring, one or more V_i may be empty.

Define a ***nearly–canonical coloring*** of K_n with ***exceptional*** set X to be a c–coloring such that $\langle V(K_n) - X \rangle$ is canonically colored, with all the edges joining any single vertex of X to each V_i being in one color different from color i. We will be concerned with cases in which the size of X is small. We will often designate a nearly canonical coloring by the tuple $(C_1, C_2, \ldots, C_c, X)$. This is slight abuse in language, since the coloring of $\langle X \rangle$ is often unspecified, but no confusion should arise.

Here are three results from 2–color case:

Theorem 1.1 ([1]). Given a finite set $\mathcal{G}$ of graphs, there is a constant c_0 such that if G and H are disjoint unions of graphs from $\mathcal{G}$, then

$$p(G)+p(H)-\min(\beta(G),\beta(H))-1 \le r(G,H) \le p(G)+p(H)-\min(\beta(G),\beta(H))+c_0. \blacksquare$$

Theorem 1.2 ([3]). Given $\mathcal{G}$, there is a constant c_1 for which the following holds: Let G and H have m and n components, respectively, the components all being chosen from $\mathcal{G}$. If m and n are sufficiently large, then there is a (G, H)–critical coloring which is nearly canonical, with the exceptional set X satisfying $|X| \le c_1$. ■

This theorem is so powerful that from it some exact Ramsey numbers can be determined easily. For example, a consequence of this theorem is the following:

Theorem 1.3 ([3]). If k and l are fixed, then if m and n are large, with $m \le n$,

$$r(mK_k, nK_l) = (k-1)m + ln + r(K_{k-1}, K_{l-1}) - 2. \qquad \blacksquare$$

Proofs of the above theorems can be all found in [2].

For multicolor cases, the early works can be found in [12], [13], and [14]. The main results of these papers are stated as follows:

Theorem 1.4 ([13]). If $n_1, \ldots, n_c$ are positive integers and $n_1 = \max\{n_1, \ldots, n_c\}$ then

$$r(n_1P_2, \ldots, n_cP_2) = n_1 + 1 + \sum_{i=1}^{c}(n_i - 1). \qquad \blacksquare$$

Theorem 1.5 ([12]). If $d < p,\ p > 2$, and $n_1, \ldots, n_d$ are positive integers, then

$$r(K_p, n_1P_2, \ldots, n_dP_2) = p + \sum_{i=1}^{d} 2(n_i - 1). \qquad \blacksquare$$

Theorem 1.6 ([12]). If $d \geq p > 2$ and $n_1 \geq n_2 \geq \cdots \geq n_d > 0$ then

$$r(K_p, n_1P_2, \ldots, n_dP_2) = p + \sum_{i=1}^{p-1}(n_i - 1) + \sum_{i=1}^{d}(n_i - 1). \qquad \blacksquare$$

Theorem 1.7 ([14]). If $c,\ p_1, \ldots, p_c$ are positive integers, all greater than 2, then there are constants λ_1 and λ_2 (depending only on $c,\ p_1, \ldots, p_c$) such that

$$\sum_{i=1}^{c} p_i m_i + \lambda_1 \leq r(m_1K_{p_1}, \ldots, m_cK_{p_c}) \leq \sum_{i=1}^{c} p_i m_i + \lambda_2$$

for all positive integers $m_1, \ldots, m_c$. $\blacksquare$

In this thesis we concentrate on 3–color case though our results of Section 2 through Section 4 can be easily generalized to c–color cases for $c \geq 3$. Our main attention is on the Ramsey number $r(n_1F, n_2G, n_3H)$ for large n_1, n_2, n_3. We will see that when F, G, H are connected non–bipartite graphs, without too much difficulty, similar results of [1], [2], and [5] are obtained mostly by using the techniques developed in these papers. But it becomes complicated if one or more of F, G, H is bipartite. In such cases, the problem of determing the Ramsey number $r(n_1F, n_2G, n_3H)$ is not completely solved at this stage, and it is not clear what can be done in general. However, similar results are obtained for a fairly large class of connected graphs F, G, and H, even when some are bipartite.

In Section 2 we discuss the Ramsey number $r(nF, G, H)$ for large n and connected non-bipartite graphs G, H. We have:

Theorem 2.1. Let F be a connected graph, $n \geq 1$ be a positive integer, $p = p(F)$, and let G, H be connected non-bipartite graphs. Then there is a constant c_2 such that

$$np \leq r(nF, G, H) \leq np + c_2. \qquad \blacksquare$$

The proof of this theorem is easy and will be omitted. Actually, we will give the proof of the similar Theorem 3.1, from which one can easily draw the proof of this theorem.

Theorem 2.2. Let F, G, H be as in Theorem 2.1, and let $r = r(nF, G, H)$. Then there is a (nF, G, H)–critical coloring, $(A, \emptyset, \emptyset, X)$, of K_{r-1} which is nearly canonical, with the exceptional set X having the property that no red F contains a vertex of X. ■

As a special case of this theorem, we have:

Theorem 2.3. If n is sufficiently large, then

$$r(nK_3, K_3, K_3) = 3n + 10. \qquad \blacksquare$$

This theorem can be generalized.

Theorem 2.4. If n is sufficiently large, then

$$r(nK_p, K_3, K_3) = 3n + 2(r(K_p, K_3) - 1). \qquad \blacksquare$$

In Section 3 we discuss the Ramsey numbers $r(n_1F, n_2G, H)$ for connected non-bipartite graphs F, G, H and large n_i, $i = 1, 2$. We get:

Theorem 3.1. Let F, G, H be connected non-bipartite graphs, with $p(F) = p_1$, $p(G) = p_2$, $p(H) = p_3$. Then for $n_i \geq 1$, $1 \leq i \leq 3$, there is a constant c_3 (depending only on

F, G, H) such that

$$\sum_{i=1}^{3} n_i p_i \leq r(n_1F, n_2G, n_3H) \leq \sum_{i=1}^{3} n_i p_i + c_3. \qquad \blacksquare$$

Theorem 3.2. Let F, G, H, p_1, p_2 be as in Theorem 3.1, and let $r = r(n_1F, n_2G, H)$, then there is a (n_1F, n_2G, H)–critical coloring, (A, B, C, X), of K_{r-1} which is nearly canonical, with the exceptional set X having the property that no red F or blue G in the K_{r-1} contains a vertex of X. ■

As a very special case of Theorem 3.2, we have:

Theorem 3.3 If n_1, n_2 are sufficiently large, then

$$r(n_1K_3, n_2K_3, K_3) = 3(n_1 + n_2) + 5. \qquad \blacksquare$$

In Section 4 we discuss the Ramsey numbers $r(n_1F, n_2G, n_3H)$ for large n_i and connected non–bipartite graphs F, G, H. We get:

Theorem 4.1. Suppose F, G, H are connected non-bipartite graphs.
Let $r = r(n_1F, n_2G, n_3H)$. Then if n_1, n_2, n_3 are sufficiently large, there is a (n_1F, n_2G, n_3H)–critical coloring, (A, B, C, X), of K_{r-1} which is nearly canonical, with the exceptional set X having the property that no red F, blue G, or green H contains a vertex of X. ■

This is one of our main theorems; the significance of this theorem is that it permits us to get exact evaluations in some particular cases. As an illustration of what can be done, we have the following very special case:

Theorem 4.2. If n_1, n_2, n_3 are sufficiently large, then

$$r(n_1K_3, n_2K_3, n_3K_3) = 3(n_1 + n_2 + n_3). \qquad \blacksquare$$

In Section 5 we still discuss the 3–color Ramsey numbers but where we do not require F, G, H to be non–bipartite. In other words, one or more of them may be connected bipartite graphs. In this section, more general results are given, but to state those results, many new definitions need to be introduced. This takes a lot space, so we don't state them here and leave it to the section.

In Section 6 are some closing remarks.

2. THE RAMSEY NUMBER $r(nF, G, H)$

Lemma 2.1. Suppose $Q(A, X)$ is a c–colored complete bipartite graph with parts A and X. Then there is a subset A' of A, where $|A'| \geq \frac{|A|}{c^{|X|}}$, such that the edges joining any given vertex of X to the vertices of A' are monochromatic.

Proof. Let $|X| = l$. Label the vertices of X by x_i, $1 \leq i \leq l$. Partition A into c subsets $A_{1,1}, A_{2,1}, \ldots, A_{c,1}$ such that the edges joining x_1 to each of these sets are monochromatic. Let A_1 be the largest set (if the largest sets are not unique, arbitrarily choose one of them). Then the size of A_1 is at least $\frac{|A|}{c}$. Partition A_1 into c subsets $A_{1,2}, A_{2,2}, \ldots, A_{c,2}$ such that the edges joining x_2 to each of these sets are monochromatic. Let A_2 be the largest set. Then the size of A_2 is at least $\frac{|A|}{c^2}$. Repeat the above procedure. This, finally, reaches a set A_l such that the edges joining x_i to A_l are monochromatic, $i = 1, \ldots, l$, and $|A_l| \geq \frac{|A|}{c^l}$. Then $A' = A_l$ is the desired subset of A. ∎

Theorem 2.1. Let F be a connected graph of order p, and let G, H be connected non–bipartite graphs. Then there is a constant c_2 such that for $n \geq 1$

$$np \leq r(nF, G, H) \leq np + c_2.$$

The proof of this theorem is easy and is omitted.

Theorem 2.2. Let F, G, H be as in Theorem 2.1, and let $r = r(nF, G, H)$. Then if n is sufficiently large, there is an (nF, G, H)–critical coloring of K_{r-1} which is nearly canonical, with the exceptional set X having the property that no red F contains a vertex of X.

Proof. Suppose χ is a (nF, G, H)–critical coloring of K_{r-1}. Let $V = V(K_{r-1})$. From Theorem 2.1, we know $r \geq np_1$, where $p_1 = p(F)$. If n is sufficiently large, the K_{r-1} must contain a red complete graph K_m on a set of vertices A_1, where m will be chosen (implicitly) later. In fact, we need only that $r \geq r(K_m, G, H)$.

Find as many disjoint red F as possible in the graph $\langle V - A_1 \rangle$, denoting the vertices of these F by T_1. Let $X_1 = V - A_1 - T_1$. Clearly, $|X_1| \leq r(F, G, H)$, since $\langle X_1 \rangle$ cannot contain a red F, and by the hypothesis on χ, the entire K_{r-1} does not contain a blue G or green H. Suppose now that some vertex of X_1 is joined by red edges to at least $\delta(F)$ vertices of A_1. Then this vertex and some $p_1 - 1$ vertices of A_1 span a red F. Transfer these vertices to T_1, and continue this process as long as possible. This yields three sets A_2, T_2, X_2 such that no vertex of X_2 has as many as $\delta(F)$ red edges going to A_2, $\langle T_2 \rangle$ can be partitioned into disjoint red F, and $\langle A_2 \rangle$ is a red complete graph. if n was chosen large enough, A_2 is still large compared to X_2, since no more than $r(F, G, H)(p_1 - 1)$ vertices have been removed from A_1.

Move those vertices of A_2 that have red edges going to X_2 to T_2; if necessary, move additional vertices so that the total number of vertices moved is a multiple of p_1. This yields three new sets A_3, T_3, X_3 such that there are no red edges between A_3 and X_3, $\langle T_3 \rangle$ can be partitioned into disjoint red F, and $\langle A_3 \rangle$ is a red complete graph. if n was chosen large enough, A_3 is still large compared to X_3, since fewer than $r(F, G, H)(\delta(F) - 1) + p_1$ vertices have been moved from A_2.

Let $l = |X_3|$. Label the vertices of X_3 by x_i, $i = 1, \ldots, l$. By Lemma 2.1, if A_3 is large enough, there is a subset A_4 of A_3, where $|A_4| \geq \frac{|A_3|}{2^l}$, such that the edges joining any given vertex of X_4 to vertices of A_3 are in one color(either blue or green). Transfer all the vertices of A_3 that are not in A_4 to T, along with sufficient vertices of A_4 so that the number of vertices moved is a multiple of p_1. This yields two sets A^*, T^* such that all the edges joining any single vertex of X_3 to the vertice of A^* are in the same color, $\langle T^* \rangle$ can be partitioned into disjoint red F, and $\langle A^* \rangle$ is a red complete graph. This graph is still large if n was chosen large enough. X_3 is not affected, but set $X = X_3$ for consistency.

Because χ is critical, the total number of red F in $\langle A^* \cup T^* \rangle$ is less than n. Let $A = A^* \cup T^*$. We now give a new coloring χ' to $\langle V \rangle = \langle A \cup X \rangle$. Color $\langle A \rangle$ red. For any $x \in X$, color all the edges joining x to the vertices of A by whatever color (x, A^*) was in $\langle A^* \cup X \rangle$ under χ. Also, we leave the coloring of $\langle X \rangle$ unchanged. Then under the new coloring χ' the K_{r-1} contains no red nF, since all the red F are in $\langle A \rangle$. In order to prove that χ' is (nF, G, H)–good, we need to show that K_{r-1} contains no blue G or green H. Suppose that K_{r-1} has a blue G under χ'. Then this blue G cannot lie entirely in $\langle X \cup A^* \rangle$, for otherwise K_{r-1} would have the same blue G under χ already. So this blue G must use some vertices of A which are not in A^*. Let $S = A^* - V(G)$. If A^* is large enough, S is still large. For any vertex v of $V(G)$ that is not in $A^* \cup X$, we can replace it by a vertex in S, and get a new blue G, because under the new coloring, for any vertex $y \in V$ and any two vertices u_1, u_2 of A, $u_1 \neq y$, $u_2 \neq y$, the edge $u_1 y$ has the same color as $u_2 y$. If we had made A^*, and therefore S, large enough, then we can continue the above replacing process until we get a blue G that has no vertex outside $A^* \cup X$. That is, we will have a blue G which lies entirely in $\langle X \cup A^* \rangle$, a contradiction. This shows that under χ' the K_{r-1} contains no blue G. The same argument proves that the K_{r-1} contains no green H. Hence χ' is the desired coloring. ∎

Applying Theorem 2.2, we can now prove Theorem 2.3.

Theorem 2.3. If n is sufficiently large, then

$$r(nK_3, K_3, K_3) = 3n + 10.$$

Proof. Let $s = 3n + 10$, $r = r(nK_3, K_3, K_3)$. We first prove $r \geq s$ by exhibiting an (nK_3, K_3, K_3)–good coloring of K_{s-1}. Partition $V = V(K_{s-1})$ into three sets: $V = A \cup B \cup C$ where $|A| = 3n - 1$, $|B| = |C| = 5$. Color $\langle A \rangle$ red, AB green and AC blue. Since $r(K_3, K_3) = 6$, we can find a 2–coloring χ of $\langle B \rangle$ by red and blue that contains no monochromatic triangles. Label the vertices of B and C by b_i, c_i, $1 \leq i \leq 5$, respectively. Color $\langle C \rangle$ as follows:

$c_i c_j$ is red $\iff$ $\chi(b_i b_j)$ is red, $c_i c_j$ is green $\iff$ $\chi(b_i b_j)$ is blue. Because $\langle B \rangle$ contains no monochromatic triangles, $\langle C \rangle$ contains no monochromatic triangles. Now color BC as follows: Color $b_i c_j$ by the same color that $b_i b_j$ has, $i \neq j$, $1 \leq i, j \leq 5$. Color $b_i c_i$ green.

We claim that the graph $Q = \langle B \cup C \rangle$ colored above contains no monochromatic triangles. It is obvious that Q has no green triangles. Since $b_i c_j$ has the same color as $b_i b_j$, Q contains no monochromatic triangles in the form $b_i b_j c_k$, for otherwise $b_i b_j b_k$ would be a monochromatic triangle in $\langle B \rangle$. Since there is no blue edge in $\langle C \rangle$ and Q has no green triangles, if Q contains a monochromatic triangle in the form $b_i c_j c_k$, it must be a red one. But this is impossible because if $c_j c_k$, $b_i c_j$, $b_i c_k$ are red, so are $b_j b_k$, $b_i b_j$ $b_i b_k$. Summarizing the above, we have proved that Q contains no monochromatic triangles. Since AB is green, and $\langle B \rangle$ has no green edges, then the K_{s-1} contains no monochromatic triangle in the form $ab_i b_j$, where $a \in A$, $b_i, b_j \in B$. The K_{s-1} contains no monochromatic triangle in the form $ac_i c_j$ where $a \in A$, $c_i, c_j \in C$. It's obvious that K_{r-1} contains no monochromatic triangle in the form $ab_i c_j$ with $a \in A$ $b_i \in B$ $c_j \in C$. Hence K_{s-1} colored above contains $n-1$ red triangles but no blue or green triangles. Thus we have proved $r \geq s$.

Next we prove $r \leq s$. By Theorem 2.2, we may suppose (A, X) is a (nK_3, K_3, K_3)– nearly canonical critical coloring χ of K_{r-1}, where $\langle A \rangle$ is red, there is no red edge between A and X, and $\langle X \rangle$ contains no monochromatic triangles. Then all the red triangles are in $\langle A \rangle$, hence $|A| \leq n-1$. Let v be a vertex of A. Partition X into two sets Y, Z such that all the edges joining v to the vertices of Y are blue and those joining v to the vertices of Z are green. Since under χ K_{r-1} contains no blue or green triangles, there are no blue edges in $\langle Y \rangle$ and no green edges in $\langle Z \rangle$. Thus $\langle Y \rangle$ and $\langle Z \rangle$ are 2–colored subgraphs. Since $\langle X \rangle$, and therefore $\langle Y \rangle$ and $\langle Z \rangle$, contain no monochromatic triangles, $|Y| \leq 5$, $|Z| \leq 5$, hence $|X| \leq 10$. Then

$$r - 1 \leq 3n - 1 + |X| \leq 3n - 1 + 10, \quad \text{so} \quad r \leq 3n + 10. \qquad \blacksquare$$

Using the same method one can easily generalize this result:

Theorem 2.4. If n is sufficiently large, then

$$r(nK_p, K_3, K_3) = np + 2r(K_p, K_3) - 2.$$

3. THE RAMSEY NUMBER $r(n_1F, n_2G, H)$

In this and the next section, we suppose that F, as well as G and H, are connected non–bipartite graphs, and we set $p(F) = p_1,\ p(G) = p_2,\ p(H) = p_3$.

We state Theorem 3.1 in the form suitable in this section as well as in the next section.

Theorem 3.1. Suppose F, G, H are connected non-bipartite graphs, with $p(F) = p_1,\ p(G) = p_2,\ p(H) = p_3$ then for $n_i \geq 1, \quad 1 \leq i \leq 3$, there is a constant c_3 (depending only on F, G, H) such that

$$n_1p_1 + n_2p_2 + n_3p_3 \leq r(n_1F, n_2G, n_3H) \leq n_1p_1 + n_2p_2 + n_3p_3 + c_3. \qquad (*)$$

Proof. Let $n = n_1p_1 + n_2p_2 + n_3p_3 - 1$. To prove the left-hand side of $(*)$, we need to give a (n_1F, n_2G, n_3H)–good coloring of K_n.
We mention that if n were replaced by $n - 2$, the coloring in question would be almost trivial(it would be 3–canonical), but we will work a bit harder to get a better lower bound.
First, consider K_5, the complete graph on 5 vertices. Label the vertices as a, b, c, x, y. Color the edges of K_5 as follows: ab green, bc red, ac blue, xy red, ax blue, ay green, bx green, by red, cx red, and cy blue. Then under this coloring, K_5 has the following properties:
(1). It contains no red F, no blue G or green H.
(2). vertices a, b, c are incident to no red, blue, green edges respectively. Replace vertex a by a complete red graph with $n_1p_1 - 1$ vertices. Join every other vertex of the K_5 to each vertex of the red graph by an edge of whatever color that vertex was originally joined to a. The new graph Q contains $n_1 - 1$ red F, but no blue G or green H. In Q replace b

by a complete blue graph with n_2p_2-1 vertices. Join every vertex of Q to each vertex of the blue complete graph by an edge of whatever color that vertex was originally joined to b. The new graph Q_1 contains n_1-1 red F, n_2-1 blue G but no green H. In Q_1 replace vertex c by a complete green graph with n_3p_3-1 vertices. Join every other vertex of Q_1 to each vertex of the green graph by an edge of whatever color that vertex was originally joined to c. The new graph Q_2 contains n_1-1 disjoint red F, n_2-1 disjoint blue G, and n_3-1 disjoint green H, but no more. The number of vertices of this graph is

$$2+(n_1p_1-1)+(n_2p_2-1)+(n_3p_3-1)=n.$$

Hence we get a (n_1F, n_2G, n_3H)–good coloring of K_n.

For the right-hand side of $(*)$, use the obvious fact

$$r(n_1G_1,\dots,n_iG_i,\dots,n_kG_k)\le r(n_1G_1,\dots,(n_i-1)G_i,\dots,n_kG_k)+|G_i|,$$

where $n_i\ge 2$. Employ induction one by one on integers n_1, n_2 and n_3, starting from the trivial fact $r(F,G,H)\le r(F,G,H)$. This leads to

$$r(n_1F,n_2G,n_3H)\le (n_1-1)p_1+(n_2-1)p_2+(n_3-1)p_3+r(F,G,H).$$

In the above inequality, let

$$c_3=r(F,G,H)-(p_1+p_2+p_3).$$

This gives us the right-hand side of $(*)$. ∎

Next we present three lemmas that are needed to prove Theorem 3.2 and Theorem 4.1.

Lemma 3.1. Let Q be a 3–colored graph that contains a red m_1F, a blue m_2G, and a green m_3H, where the order of Q is less than $m_1p_1+m_2p_2+m_3p_3-c_3$, and let $r=r(n_1F,n_2G,n_3H)$. Then any (n_1F,n_2G,n_3H)– critical coloring of K_{r-1} cannot contain a subgraph isomorphic to Q.

Proof. Let χ be a critical coloring of K_{r-1}. Suppose to the contrary that under χ the K_{r-1} contains a subgraph isomorphic to Q. Delete the vertices of this subgraph from the K_{r-1}. From Theorem 3.1 we know that the remaining graph S has at least

$$(n_1 - m_1)p_1 + (n_2 - m_2)p_2 + (n_3 - m_3)p_3 + c_3$$

vertices. Again by Theorm 4.1, S must contain a red $(n_1 - m_1)F$, a blue $(n_2 - m_2)G$, or a green $(n_3 - m_3)H$. If S contains a red $(n_1 - m_1)F$, these red F together with the m_1 red F of Q yields a red n_1F in the original K_{r-1}, a contradiction. A blue $(n_2 - m_2)G$ or a green $(n_3 - m_3)H$ in S give similar contradictions. ∎

Lemma 3.2. Let $k \geq 1$ be given, and let $r = r(n_1F, n_2G, n_3H)$, then when n_i are sufficiently large, $1 \leq i \leq 3$, any (n_1F, n_2G, n_3H)–critical coloring of K_{r-1} contains a canonically colored subgraph with red, blue and green set each of order k. Furthermore, we can require that all the edges connecting the red and blue sets are green, those connecting the blue and green sets are red, and those connecting the red and green sets are blue.

Proof. For large n_1, n_2, n_3, let y be the largest integer such that

$$r - 1 \geq \max\{r(K_y, n_2G, n_3H), r(n_1F, K_y, n_3H), r(n_1F, n_2G, K_y)\}. \qquad (**)$$

By Theorem 3.1, we know that $r \geq n_1p_1 + n_2p_2 + n_3p_3$, and if n_1, n_2, n_3 are large enough, y can be made as large as we wish. Let χ be a (n_1F, n_2G, n_3H)–critical coloring of K_{r-1}. Then the K_{r-1} contains no red n_1F, blue n_2G or green n_3H.

Because of (**), K_{r-1} must contain red, blue, and green complete subgraphs each of order y. Let A, B, C denote the vertex sets of the red, blue, and green complete graphs, respectively. Any two of these three sets may have at most one vertex in commom, if so, delete those commom vertices from these sets, this reduces each set by at most 2. So when y is big enough, we may assume that A, B, and C are disjoint. From a well known theorem [8], the 3–colored complete bipartite graph $Q(A, B)$ joining A to B must contain a monochromatic complete bipartite subgraph $Q(A_1, B_1)$, with each part of order at least $\log_3 y$. If y is large enough, $\log_3 y$ is still large compared to any given constant.

Next we want to show that $Q(A_1, B_1)$ must be green. Suppose to the contrary that $Q(A, B)$ contains a red complete bipartite subgraph $K_{m,m}$ with m large. Let $\beta = \beta(F)$ be the independence number of F. Let

$$\alpha = \min\{\lfloor\frac{m}{p_1 - \beta}\rfloor, \lfloor\frac{m}{\beta}\rfloor, \lfloor\frac{m}{p_2}\rfloor\}.$$

Then this red $K_{m,m}$ together with the red and blue K_m on the two parts of it must contain a $(\alpha F, \alpha G)$–tie M that has

$$\alpha(p_1 + p_2 - \beta) = \alpha p_1 + \alpha p_2 - \alpha\beta$$

vertices. This means that the critically colored K_{r-1} contains a subgraph M which contains a red αF and a blue αG, and the order of M is $\alpha p_1 + \alpha p_2 - \alpha\beta$. When m is large enough that $\alpha\beta > c_3$, we get a result that contradicts Lemma 3.1. Hence when y is sufficiently large $Q(A, B)$ cannot contain a large red complete bipartite subgraph. This means that when y is large, $Q(A_1, B_1)$ cannot be red. The same arguement shows that it cannot be blue either. Hence it must be green.

Arbitrarily choose a subset C_1 of C such that $|C_1| = \min\{|A_1|, |B_1|\}$. Then the 3–colored graph $Q(A_1, C_1)$ contains a complete monochromatic bipartite subgraph $Q(A_2, C_2)$ with each part of size at least $\log_3 |C_1|$. If $\log_3 |C_1|$ is large, we can use the above method to show that the complete bipartite graph $Q(A_2, C_2)$ must be blue.

Arbitrarily choose a subset B_2 of B_1 such that $|B_2| = |C_2|$. Then the 3–colored complete bipartite graph $Q(B_2, C_2)$ contains a monochromatic complete bipartite subgraph $Q(B_3, C_3)$ with each part of size at least $\log_3 |C_2|$. When $\log_3 |C_2|$ is large, $Q(B_3, C_3)$ must be red.

Finally, arbitrarily choose a subset A_3 of A_2 such that $|A_3| = \min\{|B_3|, |C_3|\}$. The subgraph $\langle A_3 \cup B_3 \cup C_3 \rangle$ has the following properties:

(1). $\min\{|A_3|, |B_3|, |C_3|\} \geq \log_3 \log_3 \log_3 y$.

(2). $\langle A_3 \rangle$ red, $\langle B_3 \rangle$ blue, and $\langle C_3 \rangle$ green.

(3). A_3B_3 green, B_3C_3 red, and C_3A_3 blue.

Let $k = \log_3 \log_3 \log_3 y$. Then graph $\langle A_3 \cup B_3 \cup C_3 \rangle$ contains a subgraph that satisfies the requirement. ∎

By using the same method as in the proof of Lemma 3.2 we immediately get:

Lemma 3.3. Let $k \geq 1$ be given, and let $r = r(n_1F, n_2G, H)$, then for n_i sufficiently large, $1 \leq i \leq 2$, any (n_1F, n_2G, H)–critical coloring of K_{r-1} contains a canonically colored subgraph $\langle A \cup B \rangle$ with $\langle A \rangle$ red, $\langle B \rangle$ blue, all the edges joining A to B green, and $|A| = |B| = k$. ∎

Theorem 3.2. Let F, G, H be connected non–bipartite graphs with $p_1 = p(F)$, $p_2 = p(G)$, and let $r = r(n_1F, n_2G, H)$, then if n_1, n_2 are sufficiently large, there is a (n_1F, n_2G, H)–critical coloring χ of K_{r-1} which is nearly canonical, with the exceptional set X having the property that no red F or blue G in the K_{r-1} contains a vertex of X.

Proof. Let χ be a (n_1F, n_2G, H)–critical coloring of K_{r-1}, and let $V = V(K_{r-1})$. Our plan is to partition V into sets A, B, T, X, where T is spanned by disjoint red F and blue G, and where $Q_0 = \langle A \cup B \cup X \rangle$ has a nearly–canonical coloring that no red F or blue G in Q_0 contains a vertex of X. We will then show that this nearly- canonical coloring can be extended to a complete graph on $r - 1$ vertices, yielding the desired critical coloring.

By Lemma 3.3, the 3–colored K_{r-1} must contain two large sets A_1 and B_1 of vertices such that $\langle A_1 \cup B_1 \rangle$ is canonically colored, with red A_1, blue B_1, A_1B_1 green, and $|A_1| = |B_1| = k$. This k can be made as large as we wish by providing sufficiently large n_i, $i = 1, 2$. In the remainig graph $\langle V - A_1 - B_1 \rangle$ find as many disjoint red F and blue G as possible and denote the vertices of these red F and blue G by T_1. Let $X_1 = V - A_1 - B_1 - T_1$, Then $|X_1| < r(F, G, H)$, since $\langle X_1 \rangle$ contains no red F, blue G, or green H.

We will now reduce X_1, A_1, and B_1 in several steps, adding red F and blue G to T_1, ultimately arriving at the A, B, T, X described at the begining of the proof. As a first step, suppose that X_1 contains a vertex with at least $\delta(F)$ red edges joining it to A_1, then it is easy to see that this vertex, together with some vertices of A_1, form a red F. Remove such red F in turn, moving them to T_1. This yields new sets of vertices A_2, T_2, X_2. If we have chosen n_1, n_2, and therefore A_1, B_1, large enough, then A_2 and $B_2 = B_1$ are still large compared to X_2.

Similarly, if some vertex of X_2 has at least $\delta(G)$ blue edges going to B_2, then this vertex, together with some vertices of B_2, form a blue G. Remove such blue G in turn, moving them to T_2. This yields new sets of vertices B_3, T_3, X_3. If n_1, n_2 are large, then $A_3 = A_2$ and B_3 are still large compared to X_3.

At this point, fewer than $\delta(F)|X_3|$ red edges join X_3 to A_3, and fewer than $\delta(G)|X_3|$ blue edges join X_3 to B_3. Hence, for all but at most $\delta(F)|X_3|$ special vertices of A_3, only blue and green edges join a vertex of A_3 to X_3; and for all but at most $\delta(G)|X_3|$ special vertices of B_3, only red and green edges join a vertex of B_3 to X_3. Each of these special vertices of A_3, and indeed any vertices of A_3, are part of a red F that lies in $\langle A_3 \rangle$. Likewise, each of these special vertices of B_3 are part of a blue G that lies in $\langle B_3 \rangle$. Transfer all these special vertices of A_3 and B_3 to T_3, along with sufficient other vertices of A_3 and B_3, in the form of disjoint red F and blue G. This yields new sets A_4, B_4, T_4; X_3 is not affected, but set $X_4 = X_3$ for consistency.

If n_1, n_2 are large enough, then A_4 and B_4 are still large compared to X_4, and by construction, all edges between X_4 and A_4 are either blue or green; all edges between X_4 and B_4 are either red or green; and all edges between A_4 and B_4 are green.

If there is a red F using some vertices of X and some vertices of B_4, move the vertices of this red F to T_4. Repeat this procedure until there is no such a red F. If there is a blue G using some vertices of A_4 and some vertices remaining in X_4, move the vertices of this blue G to T_4. Repeat this procedure until there is no such a blue G left. This yields new sets A_5, B_5, T_5, X_5.

If n_1, n_2 are large enough, then A_5 and B_5 are still large compared to X_5, and by the construction, all edges between A_5 and B_5 are green, all edges between A_5 and X_5 are either blue or green, and all edges between B_5 and X_5 are either red or green, and there is no red F or blue G in $\langle A_5 \cup B_5 \cup X_5 \rangle$ contains a vertex in X_5. Furthermore, T_5 is spanned by disjoint red F and blue G.

Starting from sets A_5, B_5, T_5, X_5, using a method similar to that used in the proof of Theorem 2.2, we get a new partition of V : $V = A \cup B \cup T \cup X$, where A, B, X are subsets of the sets A_5, B_5, X_5 respectively, and T is obtained from T_5 by adding the vertices of some disjoint red F and blue G to T_5. The new sets A, B, T, X have the same properties as A_5, B_5, T_5, X_5. and in addition, all the edges joining any single vertex of X (if any) to the vertices of A are in the same color, and all the edges joining any single vertex of X to the vertices of B are in the same color.

Hence, $\langle A \cup B \cup X \rangle$ has a nearly-canonical coloring with exceptional set X, and with no red F or blue G in $\langle A \cup B \cup X \rangle$ having a vertex in X. Furthermore, T is spanned by some disjoint red F and blue G. If n_1, n_2 were chosen large enough, A and B will be large relative to X.

We will now give a nearly-canonical coloring of $\langle V \rangle = K_{r-1}$, essentially by expanding A and B while eliminating T. Let $\langle T \rangle$ contains j disjoint red F and l disjoint blue G. Move all the j red F to A and move all the l blue G to B.

This yields new sets A_6, B_6; X is still not affected. No vertex remains in T. Now we have a new partition of V : $V = A_6 \cup B_6 \cup X$. Give K_{r-1} a nearly canonical coloring, χ', with A_6, B_6 and X being the red, blue, and the exceptional sets respectively, with green edges between A_6 and B_6. The color of the edges from any vertex of X to the vertices of A_6 is the same as the color of the edges from this vertex to the vertices of A under coloring χ; the color of the edges joining any vertex of X to the vertices of B_6 is the same as the color of the edges joining this vertex to the vertices of B under χ. Also, we leave the coloring of $\langle X \rangle$ unchanged. It remains to show that this coloring is (n_1F, n_2G, H)–good.

First it is easy to see that the number of of disjoint red F in $\langle A_6 \rangle$ is less than n_1, for otherwise K_{r-1} would contain a red n_1F under χ. For the same reason, the number of disjoint blue G is less than n_2. Because there are no red edges between X and A_6, there is no red F that uses vertices of both A_6 and X. So any additional red F other than those in $\langle A_6 \rangle$ must be in $\langle B_6 \cup X \rangle$. Let $B_6 = B \cup C$. Suppose under χ' there is a red F in $\langle B_6 \cup X \rangle$. This red F cannot lie entirely in $\langle B \cup X \rangle$, since otherwise this would also be a red F under χ that lies in $\langle B \cup X \rangle$. So this red F must use some vertices of C. Let D denote these vertices. In this case, under the new coloring χ', all the vertices of B_6 are in the same position; that is, the color of the edge joining a vertex y of B_6 to some other vertex v of V is exactly the same as the color of the edge joining any other vertex z of B_6 to v. Hence, as long as $B - V(F)$ contains enough vertices, we can replace D by the same number of vertices of $B - V(F)$ and get a red F that lies entirely in $\langle B \cup X \rangle$, a contradiction. This shows that under the new coloring, the K_{r-1} contains no red F outside $\langle A_6 \rangle$, hence it contains no red n_1F, and there is no red F that contains a vertex of X. A similar argument will prove that under χ' the K_{r-1} contains no blue G outside $\langle B_6 \rangle$, and therefore it contains no blue n_2G, and there is no blue G that has a vertex in X. By the hypothesis, the K_{r-1} contains no green H under χ. By the construction and the new coloring of $\langle X \rangle$, it contains no green H. The similar argument will show that the K_{r-1} cannot have a green H that uses vertices of X, A_6 or B_6. Since H is not a bipartite graph, there is no green H in $\langle A_6 \cup B_6 \rangle$. Thus under χ' K_{r-1} contains no no green H at all. Then χ' is (n_1F, n_2G, H)–good. ∎

Theorem 3.3 If n_1, n_2 are sufficiently large, then

$$r(n_1K_3, n_2K_3, K_3) = 3(n_1 + n_2) + 5.$$

Proof. Let $r = r(n_1K_3, n_2K_3, K_3)$, and let $s = 3(n_1 + n_2) + 5$. We first prove $r \geq s$ by exhibiting a (n_1K_3, n_2K_3, K_3)–good coloring of the K_{s-1}. Let $V = V(K_{s-1})$. Partition V into sets A, B, X, with $|A| = 3n_1 - 1, |B| = 3n_2 - 1$, and $|X| = 6$. Label the vertices of X by x_i, y_i, z_i, $i = 1, 2$. Color all the edges of $\langle A \rangle$ red, all the edges of $\langle B \rangle$ blue, AB green, all the edges joining x_i, y_i to the vertices of A blue, all the edges joining z_i to the vertices of A green, all the edges joining y_i, z_i to the vertices of B red, and all the edges joining x_i to the vertices of B green. Also color x_1x_2 red, y_1y_2 green, z_1z_2 blue, x_1y_1 and x_2y_2 red, x_1y_2 and x_2y_1 green, y_1z_1 and y_2z_2 blue, y_1z_2 and y_2z_1 green, x_1z_1 and x_2z_2 blue, and x_1z_2 and x_2z_1 red. It is easy to check that $\langle X\rangle$ contains no monochromatic triangles under this coloring. Because there is no red edge between A and the rest of the vertices, and $\langle A\rangle$ is red, there is no monochromatic triangle that uses two vertices of A and one vertex outside A. The same is true for B. There is no monochromatic triangle auv with $a \in A, u, v \in X$, since there is no blue edge in $\langle x_1, x_2, y_1, y_2\rangle$ and z_1z_2 is blue. There is no monochromatic triangle of the form buv with $b \in B, u, v \in X$, since there is no red edge in $\langle y_1, y_2, z_1, z_2\rangle$ and x_1x_2 is red. There is no monochromatic triangle abu with $a \in A$, $b \in B$ and $u \in X$, since all the edges between A and B are green and there is no vertex $u \in X$ such that both ua and ub are green for some $a \in A$, $b \in B$. Hence all the nomochromatic triangles are in $\langle A\rangle$ and $\langle B\rangle$. But $\langle A\rangle$ contains only $n_1 - 1$ disjoint red triangles but no blue or green ones, $\langle B\rangle$ contains only $n_2 - 1$ disjoint blue triangles but no red or green ones. Hence the above coloring is (n_1K_3, n_2K_3, K_3)–good.

Next we use Theorem 3.2 to prove $r \leq s$. By Theorem 3.2 we know there is a nearly–canonical coloring, (A, B, X), of the K_{r-1} which is (n_1K_3, n_2K_3, K_3)–good, with $\langle A\rangle$ red, $\langle B\rangle$ blue, all the edges between A and B green, and there are no red edges between X and A, no blue edges between X and B, and there is no red or blue triangle that contains a vertex of X. Let v be a vertex of A. Partition X into two sets Y and Z so that the edges joining v to the vertices (if any) of Y are blue, and the edges joining v to the vertices (if any) of Z are green. Since all the edges between A and B are green, and all the edges between v and Z are green, no edge between B and Z can be green, for otherwise the K_{r-1} would contain a green triangle, which contradicts the fact that the given coloring is (n_1K_3, n_2K_3, K_3)–good. Hence all the edges between B and Z are red. Because there is no red or blue triangle that contains a vertex of X, and because all the edges between Z and A are green and those between Z and B are red, there are no red or green edges in $\langle Z\rangle$. Thus $\langle Z\rangle$ is monochromatic. But there is no monochromatic triangle in $\langle X\rangle$, so $|Z| \leq 2$. Let b be a vertex of B. Partition Y into two sets Y_1, Y_2 such that all the edges joining b and the vertices of Y_1 are red, and those joining b and the vertices of Y_2 are green. Now since all the edges between v and Y_1 are blue, and all the edges between b and Y_1 are red, there is neither a blue nor a red edge in $\langle Y_1 \rangle$, for otherwise the K_{r-1} would contain a monochromatic triangle that uses vertices of X which contradicts our hypothesis. Then $\langle Y_1 \rangle$ is monochromatic. Since $\langle Y_1 \rangle$ cannot contain a monochromatin triangle, $|Y_1| \leq 2$. The same argument proves that $|Y_2| \leq 2$. Then

$$|X| = |Y_1 \cup Y_2 \cup Z| = |Y_1| + |Y_2| + |Z| \leq 6.$$

Thus

$$r(n_1K_3, n_2K_3, K_3) - 1 = |A \cup B \cup X| = |A| + |B| + |X| \leq (3n_1 - 1) + (3n_2 - 1) + 6.$$

That is,

$$r(n_1K_3, n_2K_3, K_3) \leq 3(n_1 + n_2) + 5. \quad \blacksquare$$

This theorem can be generalized in several directions. The proof of such generalizations is complicated, we do not introduce them here.

4. THE RAMSEY NUMBER $r(n_1F, n_2G, n_3H)$

Theorem 4.1. Suppose F, G, H are connected non-bipartite graphs.
Let $r = r(n_1F, n_2G, n_3H)$, then if n_1, n_2, n_3 are sufficiently large, there is a (n_1F, n_2G, n_3H)–critical coloring of K_{r-1} which is nearly canonical, with the exceptional set X having the property that no red F, blue G, or green H contains a vertex of X.

Proof. Consider any (n_1F, n_2G, n_3H)–critical coloring, χ, of K_{r-1} by red, blue, and green. denote the vertices of K_{r-1} by V. Oor plan is to partition V into five sets A, B, C, T, X, where T is the vertex set of some disjoint red F, blue G and green H, and under this coloring $Q = \langle A \cup B \cup C \cup X \rangle$ is nearly canonically colored with exceptional set X, such that any red F, blue G, or green H in Q contains no vertex of X, and the edges joining any vertex of X to vertices of each of A, B, and C are in one color. We will then show that this nearly canonical coloring of Q can be extended to a complete graph on $|V|$ vertices, yielding the desired critical coloring.

By Lemma 3.2, the K_{r-1} contains three large vertex sets A, B, C each of size k such that $\langle A \cup B \cup C \rangle$ is canonically colored, with $\langle A \rangle$ red, $\langle B \rangle$ blue, $\langle C \rangle$ and AB green, BC red and AC blue.

In $W = V - (A \cup B \cup C)$, find as many disjoint red F, blue G, and green H as possible. Use T to denote the vertex set of all these red F, blue G, and green H. Let $X = W - T$. Then by the definition of T and X, there is no red F, blue G, or green H in $\langle X \rangle$. Hence we have $|X| < r(F, G, H)$. Since k can be made as large as we wish if n_1, n_2, n_3 are large enough, we can assume that $|X|$ is very small compared to k.

In X, if any vertex has at least $\delta(F)$ red edges to A, move this vertex to T along with enough vertices of A to form a red F. Repeat this procedure as long as possible. This yields new sets A_1, X_1, T_1, corresponding to A, X, T, respectively. Now in X_1, if some vertex has at least $\delta(G)$ blue edges to B, move this vertex to T_1 along with enough vertices of B to form a blue G. Repeat this procedure as long as possible. This yields new sets B_1, X_2, T_2. Now in X_2, if some vertex has at least $\delta(H)$ green edges to C, move this vertex to T, along with enough vertices of C to form a green H. Repeat this procedure as long as possible. This yields new sets C_1, X_3, T_3. If we have chosen n_i large enough, A_1, B_1, C_1 are still large compared to X_3.

Now we have a new partition: $V = A_1 \cup B_1 \cup C_1 \cup X_3 \cup T_3$, where T_3 is spanned by some disjoint red F, blue G, and green H, and $\langle X_3 \rangle$ contains no red F, blue G, or green H. Furthermore, there is no vertex of X_3 that has $\delta(F)$ red edges going to A_1, $\delta(G)$ blue edges going to B_1, or $\delta(H)$ green edges going to C_1.

Let $Q_1 = \langle A_1 \cup B_1 \cup C_1 \cup X_3 \rangle$. If Q_1 contains some red F, blue G, and green H that use some vertices of X_3, move the vertices of a maximal set of disjoint such red F, blue G, and green H to T_3. This yields new sets A_2, B_2, C_2, X_4, T_4. If n_i are large enough,

A_2, B_2, C_2 are still large compared to X_4 since the total number of vertices moved from Q_1 cannot exceed $|X_3|\max\{p_1, p_2, p_3\}$, and $|X_3| \leq |X| \leq r(F, G, H)$ is a constant. We have a new partition: $V = A_2 \cup B_2 \cup C_2 \cup X_2 \cup T_4$, where $\langle A_2 \rangle$ is red, $\langle B_2 \rangle$ is blue, $\langle C_2 \rangle$ is green, A_2B_2 is green, B_2C_2 is red, and C_2A_2 is blue; T_4 is spanned by disjoint red F, blue G, and green H.

Let $Q_2 = \langle A_2 \cup B_2 \cup C_2 \cup X_4 \rangle$. Then Q_2 contains no red F, blue G, or green H that uses a vertex of X_4. Since A_2B_2 is green, A_2C_2 is blue, and F is not bipartite, Q_2 contains no red F that uses a vertex outside A_2; i.e., any red F of Q_2 must be entirely in $\langle A_2 \rangle$. Similarly, any blue G of Q_2 must be in $\langle B_2 \rangle$, and any green H of Q_2 must be in $\langle C_2 \rangle$. Furthermore, there is no vertex of X_4 that has $\delta(F)$ red edges going to A_2, or $\delta(G)$ blue edges going to B_2, or $\delta(H)$ green edges going to C_2. So the total number of red edges between A_2 and X_4 is no more than $|X_4|(\delta(F) - 1)$, the total number of blue edges between B_2 and X_4 is no more than $|X_4|(\delta(G) - 1)$, and the number of green edges between C_2 and X_4 is no more than $|X_4|(\delta(H) - 1)$. Move the vertices of A_2 to T_4 that have red edges joining to X_4, if necessary, move additional vertices of A_2 to T_4 so that the total number of vertices moved is a multiple of p_1. Perform similar operation to sets B_2 and C_2. This procedure yields new sets A_3, B_3, C_3, T_5; X_4 is not affected, but we set $X_5 = X_4$ for consistency.

Let $Q_3 = \langle A_3 \cup B_3 \cup C_3 \cup X_5 \rangle$. In Q_3, there is no red edge between X_5 and A_3, no blue edge between X_5 and B_3, and no green edge between X_5 and C_3. If n_i are large enough, A_3, B_3, C_3 are still large compared to X_5. Let $|X_5| = l$. By Lemma 2.1, there are subsets A_4, B_4, C_4 of sets A_3, B_3, C_3 respectively, such that $|A_4| \geq \frac{|A_3|}{2^l}$, $|B_4| \geq \frac{|B_3|}{2^l}$, $|C_4| \geq \frac{|C_3|}{2^l}$, and the edges joining any given vertex of X_5 to vertices of each of the sets A_4, B_4, and C_4 are monochromatic.

Move all the vertices of A_3 that are not in A_4 to T_5, along with some vertices of A_4 so that the total number of vertices moved from A_3 is a multiple of p_1. We use A_5 denote the remaining part of A_4. Then $|A_5| \geq \frac{|A_3|}{2^l} - p_1 + 1$. We can make $|A_3|$ sufficiently large so that A_5 is still large compared to X_5.

Applying the above steps to B_3 and C_3, reducing B_4 by at most $p_2 - 1$, and reducing C_4 by at most $p_3 - 1$, we get new sets B_5 and C_5. If n_1, n_2, n_3 are large enough, B_5 and C_5 are still large compared to X_5. This process produces a new set T_5 corresponding to T_4. T_5 is also spanned by a set of disjoint red F, blue G, and green H.

Let $Q_5 = \langle A_5 \cup B_5 \cup C_5 \cup X_5 \rangle$. Then from the construction of Q_5, we know that Q_5 is nearly canonically colored with red set A_5, blue set B_5, green set C_5, and exceptional set X_5. A_5, B_5 ,C_5 are still large compared to X_5. furthermore, A_5B_5 is green, B_5C_5 is red, and A_5C_5 is blue.

Now we have a new partition:

$$V = A_5 \cup B_5 \cup C_5 \cup X_5 \cup T_5,$$

where T_5 is spanned by disjoint red F, blue G, and green H. Suppose there are m_1 disjoint red F, m_2 disjoint blue G, and m_3 disjoint green H in $\langle T_5 \rangle$. Since χ is a critical coloring of K_{r-1}, the number of disjoint red F in K_{r-1} is less than n_1, the number of disjoint blue G is less than n_2, and the number of disjoint green H is less than n_3.

Now add all the vertices of the m_1 disjoint red F of $\langle T_5 \rangle$ to A_5 and denote the set by V_1, add all the vertices of the m_2 disjoint blue G to B_5 and denote the this new set by V_2, and add all the vertices of the m_3 disjoint green H to C_5 and denote this new set by V_3. By the construction of T_5, we know that after the above procedure, there is no vertex left. Hence we get a new partition of V :

$$V = V_1 \cup V_2 \cup V_3 \cup X_5.$$

Now we Give a new coloring of K_{r-1} as follows:

Color $\langle V_1 \rangle$ red, $\langle V_2 \rangle$ blue, $\langle V_3 \rangle$ green, V_1V_2 green, V_2V_3 red, and V_3V_1 blue. Also we leave the coloring of $\langle X_5 \rangle$ unchanged. The edge xv_1 has the same color as xA_5 had under χ, where $x \in X_5$, $v_1 \in V_1$.

The edge xv_2 has the same color as xB_5 had under χ, where $v_2 \in V_2$.

The edge xv_3 has the same color as xC_5 had under χ, where $v_3 \in V_3$.

We claim that the above coloring χ' of K_{r-1} is (n_1F, n_2G, n_3H)–critical coloring which satisfies our requirement.
First, by the construction we can see that $\langle V_1 \rangle$ contains fewer than n_1 disjoint red F, $\langle V_2 \rangle$ contains fewer than n_2 blue G, and $\langle V_3 \rangle$ contains fewer than n_3 green H.

Next, there is no red edge between X_5 and V_1 since there was no red edge between X_5 and A_5 under χ. For the same reason, there is no blue edge between X_5 and V_2, no green edge between X_5 and V_3. Then there is no red edge joining V_1 to the rest of the K_{r-1}, no blue edge joining V_2 to the rest of the K_{r-1}, and no green edge joining V_3 to the rest of the K_{r-1}. Since $\langle X_5 \rangle$ contains no red F, blue G, or green H, if under χ' the K_{r-1} contains a red F that is not entirely in $\langle V_1 \rangle$, it must use some vertices of X_5, some vertices of V_2, and some vertices of V_3. Suppose the K_{r-1} contains a red F that is not in $\langle V_1 \rangle$. Let $V(F)$ denote the vertex set of this red F, and set $V(F) \cap X_5 = S_1$, $V(F) \cap V_2 = S_2$, and $V(F) \cap V_3 = S_3$, where S_2 and S_3 are not both empty. S_1 cannot be empty because F is not a bipartite graph. Let

$$S_2 = (S_2 \cap B_5) \cup (S_2 \cap (V_2 - B_5)), \qquad S_3 = (S_3 \cap C_5) \cup (S_3 \cap (V_3 - C_5)).$$

Then because B_5, C_5 are large sets, we can replace $S_2 \cap (V_2 - B_5)$ by the same number of vertices from $B_5 - S_2$ and replace $S_3 \cap (V_3 - C_5)$ by the same number of vertices from $C_5 - S_3$ and get another red F. But this red F uses vertices only in X_5, B_5, C_5, this is impossible because under χ Q_5 contains no red F outside $\langle A_5 \rangle$. This proves that under χ' K_{r-1} contains no red F outside $\langle V_1 \rangle$. The same argument shows that the K_{r-1} contains no blue G outside $\langle V_2 \rangle$, and no green H outside $\langle V_3 \rangle$. Hence under χ' the K_{r-1} contains no red n_1F, no blue n_2G, and no green n_3H. Furthermore, any red F, blue G, or green H in K_{r-1} contains no vertex of X_5. Hence χ' is a nearly canonical coloring of K_{r-1} that satisties our requirement. ∎

Theorem 4.2. If n_1, n_2, n_3 are sufficiently large,then

$$r(n_1K_3, n_2K_3, n_3K_3) = 3(n_1 + n_2 + n_3).$$

Proof. By Theorem 3.1, we have $r(n_1K_3, n_2K_3, n_3K_3) \geq 3(n_1 + n_2 + n_3)$. Let $r =$

$r(n_1K_3, n_2K_3, n_3K_3)$. By Theorem 4.1, there is a (n_1K_3, n_2K_3, n_3K_3)–critical coloring χ of K_{r-1} which is nearly canonical with the exceptional set X having the property that no monochromatic triangle in the K_{r-1} uses a vertex of X. Suppose χ is such a coloring based on the partition $V = A \cup B \cup C \cup X$. Here A, B, C, and X are the red, blue, green, and exceptional sets respectively, and AB is green, BC is red, and AC is blue. Besides, under χ, no monochromatic triangle in K_{r-1} contains a vertex of X, and there is no red edge between X and A, no blue edge between X and B, and no green edge between X and C. From this we can see that all the red triangles are in $\langle A \rangle$, all the blue triangles are in $\langle B \rangle$, and all the green triangles are in $\langle C \rangle$. Since *chi* is a critical coloring of the K_{r-1}, we must have $|A| \leq 3n_1 - 1$, $|B| \leq 3n_2 - 1$, $|C| \leq 3n_3 - 1$. From the above facts we have:

$$r(n_1K_3, n_2K_3, n_3K_3) \leq 3(n_1 + n_2 + n_3) + |X| - 2.$$

Since $r(n_1K_3, n_2K_3, n_3K_3) \geq 3(n_1 + n_2 + n_3)$, thus $|X| \geq 2$.
Next we want to show that $|X| = 2$, so that $r(n_1K_3, n_2K_3, n_3K_3) = 3(n_1 + n_2 + n_3)$.

Suppose to the contrary that $|X| \geq 3$. Let x, y, z be three vertices in X. From of Theorem 4.1, we can further require that xA, yA, zA are each monochromatic. The same is true for the blue set B and green set C. Let $Y = \{x, y, z\}$. We first show that AY, BY, or CY cannot be monochromatic. Without loss of generality, suppose to the contrary that AY is monochromatic. Then AY is either blue or green. If it is blue, since AC is also blue, can be no blue edge between C and Y, for otherwise we would have a blue triangle which uses some vertex of Y. So CY must be red. Then since BC is red, can be no red edge between B and Y, so BY is green. But in this case whatever color the edge (x, y) has we will get a monochromatic triangle that contains x and y, a contradiction. The same argument proves that AY cannot be green.
Next we show that among the following nine sets of edges

$$xA,\ yA,\ zA,\ xB,\ yB,\ zB,\ xC,\ yC,\ zC$$

no four of them can be in the same color. Without loss of generality, suppose that xA, yA are both blue. Since AC is blue, xC, yC must be both red. Since CB is red, xB, yB

must be both green. Because xA, yA, zA cannot be all in one color, zA must be green. For the same reason, zB is red and zC is blue. Hence we know for the above nine edge sets, no four of them can be in the same color. Thus if xA, yA are blue, we must have

xC, yC, zB are all red,

xA, yA, zC are all blue,

xB, yB, zA are all green.

But in this case whatever color the edge xy has, we will have a monochromatic triangle which uses x and y, a contradiction. Summarizing all the above, we have proved that $|X| = 2$. Therefore $r(n_1K_3, n_2K_3, n_3K_3) = 3(n_1 + n_2 + n_3)$. ∎

5. THE GENERAL CASE.

In this section, we suppose that F, G, H are connected graphs, $p(F) = p_1$, $p(G) = p_2$, $p(H) = p_3$, $\beta(F) = \beta_1$, $\beta(G) = \beta_2$, $\beta(H) = \beta_3$, and $p_1 \geq p_2 \geq p_3$. Some or all of these graphs may be bipartite.

Define the (F, G, H)*–balance number* to be the largest integer t such that the following hold:

(1). There is a 3–canonical coloring, χ, of the complete graph K_t that contains a red F, a blue G, and a green H; and

(2). Under coloring χ, the removal of any vertex of K_t leaves a graph no longer having all of the three: a red F, a blue G, and a green H.

Such a canonical coloring χ is called an (F, G, H)*–balanced coloring.* We use (A, B, C) to denote the canonical coloring of a complete graph K_n on n vertices, where under this coloring $\langle A \rangle$ is red, $\langle B \rangle$ is blue, $\langle C \rangle$ is green, and the edges between any two of the three sets are monochromatic.

If a 3–canonically colored complete graph K_s has the above two properties for some given graphs F, G and H, we say the K_s has property MM ("maximum–minimum") over (F, G, H), or simply that K_s has property MM when it's clear. It is obvious that the (F, G, H)–balance number t exists for any given connected graphs F, G, H.

For example, as in Section 4, suppose F, G and H are connected non-bipartite graphs, then the (F,G,H)-balance number is $p(F)+p(G)+p(H)$, there is a unique (F,G,H)-balanced coloring given by the partition (A,B,C), where $|A|=p(F)$, $|B|=p(G)$, $|C|=p(H)$, with AB, BC, and AC being green, red, and blue respectively.

Let χ be a canonical coloring of a complete graph K_s with red, blue, and green sets A, B, and C. By increasing the size of the sets A, B, and C, we get a canonical coloring χ' of some complete graph K_q, where $q \geq s$. We call χ' an extension of χ to K_q.

Let χ be a (F,G,H)-balanced coloring of the K_t given by the partition $V = A \cup B \cup C$, where t is the (F,G,H)-balance number. For any positive integer m, let χ_m be the extension of χ to the complete graph K_{mt}, with the red, blue and green sets A_m, B_m and C_m, where $|A_m| = m|A|$, $|B_m| = m|B|$, $|C_m| = m|C|$. If under χ_m the K_{mt} contains exactly m disjoint red F, m disjoint blue G, and m disjoint green H, and the removal of any vertex of the K_{mt} would destroy the above property, then we call χ_m the balanced extension of χ to degree m. If for every positive integer m the extension χ_m of χ is balanced, then we call the coloring χ of K_t is expandable. Otherwise χ is not expandable. It's not difficult to find an example of an (F,G,H)-balanced coloring for some connected graphs F,G,H that is not expandable. The following is such an example: Let $F=G=K_3$, and let $H=K_{4,4}$, the complete bipartite graph with each part having 4 vertices. Partition the vertex set of K_{14} into three sets A,B,C such that $|A|=|B|=3$, $|C|=8$. Let (A,B,C) be a canonical coloring, χ, of the K_{14} with $\langle A\rangle$ red, $\langle B\rangle$ blue, $\langle C\rangle$ green, AB green, BC red, and AC blue. It's trivial to verify that χ is a (F,G,H)-balanced coloring, but it is not expandable.

By the special use of this section, we modify the definition of a tie graph. Let t be the (F,G,H)-balance number, an (F,G,H)-tie is a 3-colored graph Q that contains a red F, a blue G, and a green H, with $|Q| \leq t$. The graph Q is not required to be a complete graph.

Under the above definitions, we have:

Lemma 5.1. Suppose χ is a expandable (F, G, H)–balanced coloring of K_t with partition $V(K_t) = A \cup B \cup C$. Then any red F in the K_t must use all the vertices of A, any blue G must use all the vertices of B, and any green H must use all the vertices of C.

Proof. Without loss of generality, suppose to the contrary that there is a red F in the K_t that uses only $k < |A|$ vertices of A. Because χ is an expandable (F, G, H)–balanced coloring of the K_t, then the complete graph K_{mt} under coloring χ_m, the balanced extension of χ on K_t, contains exactly m disjoint red F, m disjoint blue G, and m disjoint green H, but no more. On the other hand, in the same K_{mt} we can find m disjoint red F which uses mk vertices of A_m, the extension of the red A. Remove these m red F from K_{mt}; then there are $m(|A| - k) \geq m$ vertices in the remaining set of the red A_m. If $m \geq p_1$, we can find at least one more red F in the remaining graph. This means that under χ_m K_{mt} contains at least $m + 1$ disjoint red F, a contradiction. This shows that under χ any red F in K_t must use all vertices of A. By symmetry, under χ any blue G in K_t must use all of B, and any green H must use all of C. ■

Lemma 5.2. Let $k \geq 1$ be given. Suppose t is the (F, G, H)–balance number, where $t > p_1 + p_2$. Then if n is sufficiently large, any 3–coloring of K_v, where $v = nt - 3$, must contain a red nF, a blue nG, a green nH, or a canonically colored subgraph with red, blue and green sets each of order k.

Proof. Let $v = nt - 3$, and consider any 3–coloring of K_v. Remove a maximal set of disjoint blue G, and then remove a maximal set of disjoint green H. If this yields n blue G or green H, we are done. Suppose not. We use M to denote the remaining graph. Then since $t > p_1 + p_2 \geq p_2 + p_3$, at least n vertices remain that do not induce a blue G or a green H. Let l be the largest integer for which $r(K_l, G, H) \leq n$; then M contains a red K_l. If n is large enough, then l can be made as large as we please (although perhaps much smaller than n). The same argument will show that the graph contains a blue K_l and a green K_l. It may happen that the red K_l shares a common vertex with each of the blue and green K_l. And the blue K_l may also share a vertex with the green K_l. If so, delete these common vertices, reducing l by at most 2.

The same method used in the proof of Lemma 3.2 will show that the 3–colored $K_{l,l,l}$ that joins the red, blue and green K_l contains a subgraph $K_{j,j,j}$ with $j \geq \log_3 \log_3 \log_3 l$ such that the edges joining any two parts of this complete 3–partite graph are in the same color. That is, the red, blue and green K_l each contains a K_j such that the complete bipartite graph that joins any two of the red, blue and green K_j is monochromatic. Therefore, if n, and hence l, is large enough, we can make $j \geq k$. Then the $K_{j,j,j}$ is the desired subgraph. ■

Using the above two lemmas we now prove:

Theorem 5.1. Suppose t is the (F,G,H)–balance number, where $t > p_1 + p_2$, and suppose that χ is an expandable balanced coloring of K_t given by partition $V(K_t) = A \cup B \cup C$. Then

$$nt - 2 \leq r(nF, nG, nH) \leq nt + c_4,$$

where c_4 is a constant depending only on F, G and H.

Proof. For the lower bound, consider the balanced extension χ_n of χ on the complete graph K_{nt} with partiton $V(K_{nt}) = A_n \cup B_n \cup C_n$. Then by the definition of χ_n, this K_{nt} contains exactly n disjoint red F, n disjoint blue G, and n disjoint green H, but no more. By Lemma 5.1, any red F must use at least $|A|$ vertices of A_n, so any n disjoint red F must use at least $n|A| = |A_n|$ vertices of A_n. That is, any n disjoint red F must use all the vertices of A_n. Thus if we delete one vertex from A_n, the remaining graph no longer contains n disjoint red F. The same results are true for the blue G and green H. Now remove one vertex from each of A_n, B_n, C_n; then the remaining graph contains no red nF, blue nG, or green nH, and it has $nt-3$ vertices. Thus $nt-2 \leq r(nF, nG, nH)$.

We now prove the upper bound. Given a positive integer k, let n_0 be the minimum integer corresponding to this k in Lemma 5.2, so that if a 3–colored complete graph on $n_0t - 3$ vertices contains no red n_0F, blue n_0G, or green n_0H, then it must contain a canonically colored subgraph with each part of order k. It is obvious that n_0 is a constant which depends only on F, G, H and k. Find c_4 so that the inequality holds for all $n \leq n_0$, and now apply induction. Consider any 3–coloring K_v, where $v = (n+1)t + c_4$; we need to show the existence of a red $(n+1)F$, a blue $(n+1)G$, or a green $(n+1)H$. Here $n \geq n_0$, and we know that $r(nF, nG, nH) \leq nt + c_4$.

By Lemma 5.2, this 3–colored complete graph must contain one of the following: a red $(n+1)F$, a blue $(n+1)G$, a green $(n+1)H$, or a canonically colored subgraph with each part of order k, and if n is large enough, this k can be made as large as we wish. If any but the last case happens, we are done. So suppose that the 3–colored complete graph contains a canonically-colored subgraph Q with each part of order k. Then if $k \geq p_1$, this Q contains a subgraph that contains exactly one disjoint red F, one disjoint blue G, and one disjoint green H. Let Q_1 be a minimal subgraph of Q that has the above property, i.e., the removal of any vertex from Q_1 will make the remaining graph either contain no red F, no blue G, or no green H. Then by the definition of t, this Q_1 has no more than t vertices. Delete Q_1 from the K_v. Then the remaining graph has at least $nt + c_4$ vertices. By the induction hypothesis, this graph must contain a red nF, a blue nG, or a green nH. In each of these three cases, the corresponding case holds in the original K_v. By the induction principle, we know that $r(nF, nG, nH) \leq nt + c_4$ is true for all n. ∎

To get the main theorem of this section, we need three more lemmas.

Lemma 5.3. Let t be the (F, G, H)–balance number, where $t > p_1 + p_2$, and let $r = r(nF, nG, nH)$. Suppose there exists an expandable balanced coloring of the K_t. Let c_4 be the constant occurring in Theorem 5.1. Let Q be a 3–colored graph such that, for some m, Q contains a red mF, a blue mG and a green mH, but $p(Q) \leq mt - c_4 - 3$. Then if n is sufficiently large, no (nF, nG, nH)–critical coloring of K_{r-1} can contain a copy of Q.

Proof. We will show that the desired result holds for all $n \geq m$. Suppose the contrary, so that some (nF, nG, nH)–critical coloring of K_{r-1}, contains a copy of Q. By Theorem 5.1, $r - 1 \geq nt - 3$. Delete the vertices of the Q that occurs in this K_{r-1}, yielding a K_s, where $s \geq (n-m)t + c_4$. This K_s cannot contain a red $(n-m)F$, a blue $(n-m)G$, or a green $(n-m)H$, for such a graph, together with graphs in Q, would yield a red nF, or a blue nG, or a green nH in our K_{r-1}. But this contradicts Theorem 5.1, since s is too large. ∎

Lemma 5.4. Let t be the (F, G, H)–balance number, where $t > p_1 + p_2$. Suppose there is an expandable balanced coloring χ. Then there is a y, depending only on F, G, H, such that the following holds: For any $n \geq y$, any 3–coloring of K_v, where $v = nt - 3$, contains a red nF, or a blue nG, or a green nH, or $n - y$ disjoint (F, G, H)–ties.

Proof. Let $v = nt - 3$. Set $k = p_1$, take n_0 to be the minimum n so that the Lemma 5.2 holds for this k. That is, any 3–colored K_{n_0t-3} contains a red n_0F, or a blue n_0G, or a green n_0H, or a canonically colored subgraph with each part of order k. Set $y = n_0 + 3$. We proceed by induction on n. The case $n = y$ is trivil, so assume the theorem to have been proved for $n - 1$. Consider a 3–colored K_v. For $n \geq n_0$, by Lemma 5.2, this K_v must contain a red nF, a blue nG, or a green nH (if any of these happens, we are done,) or a canonically colored subgraph with each part of order k. But it is staightforward that such a canonically colored subgraph contains a (F, G, H)–tie which has at most t vertices. Remove this tie, leaving at least $(n - 1)t - 3$ vertices. By the induction hypothesis, the remaining graph contains either a red $(n - 1)F$, a blue $(n - 1)G$, a green $(n - 1)H$, or $n - y - 1$ disjoint (F, G, H)–ties. In each of these four cases, the corresponding case holds in the original K_v. ∎

Lemma 5.5. Let t be the (F, G, H)–balance number, with $t > p_1 + p_2$, and let χ be an expandable–balanced coloring of K_t given by partition (A, B, C), then any red F cannot use all the vertices of B or C, any blue G cannot use all the vertices of A or C, and any green H cannot use all the vertices of A or B.

Proof. Let χ be an expandable (F, G, H)–balanced coloring of K_t given by partition (A, B, C), where t is the (F, G, H)–balance number. We only prove that any red F in this K_t cannot use all the vertices of B or C; the other cases follow in the same way.

First, we prove that any red F in the K_t cannot use all the vertices of B. Suppose to the contrary that there is a red F which uses all the vertices of B. By Lemma 5.1 we know that this red F must also use all the vertices of A. Then $p_1 \geq |A|+|B|$. But $|C| \leq p_3$, so $t = |A|+|B|+|C| \leq p_1 + p_3 \leq p_1 + p_2$, a contradiction. The same argument shows that any red F cannot contain all the vertices of C. ∎

Now we are ready for the main theorem of this section.

Theorem 5.2. Let t be the (F, G, H)–balance number, where $t > p_1 + p_2$, and let $r = r(nF, nG, nH)$. Suppose every (F, G, H)–balaced coloring of K_t is expandable. Then if n is sufficiently large, there is a (nF, nG, nH)–critical coloring of K_{r-1} which is nearly canonical, with red set A, blue set B, green set C, and the exceptional set X having the properties that no (F, G, H)–tie contains a vertex of X.

Proof. Let n be large, and consider any (nF, nG, nH)–critical coloring of K_{r-1}. By Theorem 5.1, we know that $r \geq nt-2$. Denote the vertex set of this K_{r-1} by V. Our plan is to partition V into sets A', B', C', T', X', where T' is spanned by a set of (F, G, H)–ties, and where $M = \langle X' \cup A' \cup B' \cup C' \rangle$ has a nearly canonical coloring, with no (F, G, H)–tie in M having a vertex in the exceptional set X'. We will then show that this nearly canonical coloring can be extended to a complete graph on $|V| = r-1$ vertices, yielding the desired coloring.

By Lemma 5.2, we have three sets A_1, B_1, C_1 of vertices each of order k such that $Q = \langle A_1 \cup B_1 \cup C_1 \rangle$ is canonically colored, with red set A_1, blue set B_1, and green set C_1. k can be made arbitrarily large if n is large enough, although it may be much smaller than n. First we claim that the coloring on Q is an extension of some (F, G, H)–balanced coloring χ on K_t. Suppose to the contrary that the coloring on Q is an extension of a canonical coloring χ' on a complete graph K_l which contains a red F, a blue G and a green H, and χ' is not a (F, G, H)–balanced coloring. We may require l to be the minimum with the above properties. That is, the removal of any vertex from this K_l leaves a graph no longer containing all the following three: a red F, a blue G, and a green H. Then by the definition of the (F, G, H)–balance number, t, we have $l \leq t-1$. Hence the Q contains a subgraph Q_1 which contains a red mF, a blue mG, and a green mH for some integer m, and $|Q_1| \leq ml \leq m(t-1) = mt - m$. If we have chosen n, and therefore k, large enough, m is larger than $c_4 - 2$. This means that the critically colored K_{r-1} contains a subgraph Q_1 which has at most $mt - c_4 - 2$ vertices and which contains a red mF, a blue mG, and a green mH. This contradicts Lemma 5.3. This shows that the coloring on Q is an extension of an (F, G, H)–balance coloring χ on K_t. Let the balanced coloring χ of K_t be given with red set Y_1, blue set Y_2, and green set Y_3. Set $|Y_1| = a$, $|Y_2| = b$, $|Y_3| = c$, then $a + b + c = t$. We now reduce the size of A_1, B_1, and C_1 so that so that $|A_1| = ma$, $|B_1| = mb$, $|C_1| = mc$, where m is as large as possible.

Next we apply Lemma 5.4 to $\langle V - A_1 - B_1 - C_1 \rangle$. From the the size of A_1, B_1, C_1, we see that $\langle V - A_1 - B_1 - C_1 \rangle$ has at least $(n-m)t - 3$ vertices, and it cannot contain a red $(n-m)F$, a blue $(n-m)G$, or a green $(n-m)H$. Therefore, it must contain at least $n - m - y$ (F, G, H)–ties, where y does not depend on m. Let T_1 denote the vertices of a maximal set of these ties, and set $X_1 = V - A_1 - B_1 - C_1 - T_1$. Since $|V| \leq nt + c_4$, where c_4 is as in Theorem 5.1, we have $|X_1| \leq yt + c_4$, a constant. So $|X_1|$ is much smaller than m if m was made large enough. It is possible that X_1 is empty.

We now reduce A_1, B_1, C_1 and X_1 in several steps, adding ties to T_1, ultimately arriving at the A', B', C', T', X' described at the beginning of the proof. As a first step, suppose that X_1 contains a vertex with at least $\Delta(H)$ green edges joining it to C_1. If so, then from the condition $t > p_1 + p_2$ and Lemma 5.5, it is easy to see that this vertex, together with some vertices of A_1, B_1, and C_1, span an (F, G, H)–tie, and indeed a join. Remove such ties in turn, moving them to T_1. Call the altered sets of vertices A_2, B_2, C_2, T_2, X_2. If we have chosen n, and therefore A_1, B_1, and C_1, large enough, A_2, B_2, and C_2 are still large compared to X_2.

At this point, fewer than $\Delta(H)|X_2|$ green edges join X_2 to C_2. Hence, for all but at most $\Delta(H)|X_2|$ special vertices of C_2, only red and blue edges join a vertex of C_2 to X_2. Each of these special vertices of C_2, and indeed any of the vertices of C_2, are part of a tie, using other vertices of A_2, B_2, and C_2. Transfer all of the exceptional vertices to T_2, along with sufficient other vertices of A_2, B_2, and C_2, in the form of ties. This yields new sets A_3, B_3, C_3, T_3; X_2 is not affected, but we set $X_3 = X_2$ for consistency.

If n is large enough, then A_3, B_3, and C_3 are still large compared to X_3, and by the construction, all the edges between X_3 and C_3 are either red or blue. It is easy to see that any vertex of X_3 with at least $\Delta(G)$ blue edges joining it to B_3 is part of a tie which uses only this vertex and some vertices of A_3, B_3, and C_3. So we proceed similarly to the last two steps. We first remove any vertex of X_3 with $\Delta(G)$ blue edges joining it to B_3 by incorporating it in a tie and moving the tie to T_3. This process yields new sets A_4, B_4, C_4, T_4, X_4. Then, remove any vertex of B_4 joined to any vertex of X_4 by a blue edge in a similar manner, yielding new sets A_5, B_5, C_5, T_5, X_5.

If n is large enough, then A_5, B_5, and C_5 are still large compared to X_5, and by construction, all the edges between X_5 and C_5 are either red or blue, all the edges between X_5 and B_5 are either red or green. At this stage, we also know that any vertex of X_5 with at least $\Delta(F)$ red edges joining it to A_5 is part of a tie which uses only this vertex and some vertices of A_5, B_5, and C_5. So we can proceed similarly to the above steps. We first remove any vertex of X_5 with $\Delta(F)$ red edges joining it to A_5 by incorporating it in a tie and moving the tie to T_5. This process yields new sets A_6, B_6, C_6, T_6, X_6. Finally, remove any vertex of A_6 joined to any vertex of X_6 by a red edges in a similar manner, yielding new sets A_7, B_7, C_7, T_7, X_7.

T_7 is spanned by a set of ties. If n is large enough, then A_7, B_7, and C_7 are still large compared to X_7. Let $l = |X_7|$. By Lemma 2.1, there are subsets A_8, B_8, and C_8 of the sets A_7, B_7, and C_7 such that

$$|A_8| \geq \frac{|A_7|}{2^l}, \quad |B_8| \geq \frac{|B_7|}{2^l}, \quad |C_8| \geq \frac{|C_7|}{2^l},$$

and the edges joining any given vertex of X_7 to vertices of each of the sets A_8, B_8 and C_8 are monochromatic. Transfer all the vertices of A_7 that are not in A_8, all the vertices of B_7 that are not in B_8, and all the vertices of C_7 that are not in C_8, to T_7, along with sufficient vertices of A_8, B_8 and C_8, in the form of (F, G, H)–ties. This yields new sets

$A_9, B_9, C_9, T_9; X_7$ is not affected, but we set $X_9 = X_7$ for consistency.

If n is large enough, then A_9, B_9, and C_9 are still large compared to X_9, and by construction, we see that $P = \langle A_9 \cup B_9 \cup C_9 \cup X_9 \rangle$ has a nearly canonical coloring with exceptional set X_9. T_9 is spanned by a set of ties.

Finally, if P contains (F, G, H)–ties that use some vertex of X_9, move all the vertices of such ties to T_9, this yields new sets $A_{10}, B_{10}, C_{10}, T_{10}$ and X_{10}. If n is large enough, A_{10}, B_{10} and C_{10} are still large compared to X_{10}. The graph $P_1 = \langle A_{10} \cup B_{10} \cup C_{10} \cup X_{10} \rangle$ has all the properties as P, but in addition, there is no (F, G, H)–tie in P_1 that contains a vertex of X_{10}. T_{10} is spanned by disjoint (F, G, H)–ties. Therefore we may set

$$A' = A_{10},\ B' = B_{10},\ C' = C_{10},\ T' = T_{10},\ X' = X_{10},$$

since (A', B', C', T', X') represents the partition of V we sought at the beginning of the proof.

We now give a nearly-canonical coloring of $\langle V \rangle = K_{r-1}$, essentially by expanding A', B' and C' while eliminating T'. Let T' contain α disjoint (F, G, H)–ties, each tie has at most $t = a + b + c$ vertices, hence T' has at most αt vertices. Extend the nearly canonical coloring of the graph $S = \langle A' \cup B' \cup C' \cup X' \rangle$ to the complete graph W on $q = |A'| + |B'| + |C'| + |X'| + \alpha t$ vertices simply by increasing the red set by $a\alpha$ vertices, the blue set by $b\alpha$ vertices and the green set by $c\alpha$ vertices. Denote the new red, blue, and green set by A, B, and C respectively, X' is not affected, set $X = X'$ for consistency.

It is easy to see that $q \geq |V|$, and it is clear that no (F, G, H)–tie in W contains a vertex of X. We need to show that this coloring is (nF, nG, nH)–good.

We claim that the number of disjoint red F in W is no more than that in the original graph $\langle V \rangle = K_{r-1}$. Suppose S contains x disjoint red F. Of course K_{r-1} contains at least $x + \alpha$ disjoint red F. If we can show that the number of disjoint red F in W is no more than $x + \alpha$, then our claim is proved. To the contrary, suppose that W contains at least $x + \alpha + 1$ disjoint red F. Choose such a set of red F, and let $\mathcal{R}$ denote those members of the set which have a vertex in X. Let Y denote the set of vertices of the members of $\mathcal{R}$, together with any remaining vertices in X. Let $\mathcal{R}$ have z members. Then $\langle V(W) - Y \rangle$ is canonically colored, with red, blue, and green sets A^*, B^*, and C^*, respectively (say). This graph contains $x + \alpha + 1 - z$ disjoint red F. Since $|Y| \leq p_1|X|$ is a constant, if we have chosen n, and therefore A', B', and C', large enough, we have

$$|A^*| \geq a\alpha, \ |B^*| \geq b\alpha, \ |C^*| \geq c\alpha.$$

Thus we may delete $a\alpha, b\alpha$ and $c\alpha$ vertices from A^*, B^*, and C^* respectively, without exhausting them. Call these reduced sets A_1^*, B_1^* and C_1^*. The graph $\langle A_1^* \cup B_1^* \cup C_1^* \rangle$ is still canonically colored, and it contains $(x + \alpha + 1 - z) - \alpha = x + 1 - z$ disjoint red F. Now, $L = \langle Y \cup A_1^* \cup B_1^* \cup C_1^* \rangle$ is colored isomorphically to $S = \langle A' \cup B' \cup C' \cup X' \rangle$. But L, and hence S, contains $x + 1$ disjoint red F, a contradiction.

Hence we have proved that W has no more disjoint red F than $\langle V \rangle$. Similar argument shows that W contains no more disjoint blue G or green H than $\langle V \rangle$. Since the coloring of $\langle V \rangle$ is (nF, nG, nH)–critical the coloring of W is (nF, nG, nH)–good. Furthermore, since $|W| \geq |V|$, the coloring of W must also be critical (and in fact $|W| = |V|$). ∎

It seems that this theorem is a direct generalization of Theorem 4.1, but if we read the statements of both theorems carefully, we see that the result of Theorem 5.2 is slightly weaker than Theorem 4.1. In fact, in Theorem 4.1 we say that there is no red F, blue G, or green H that contains a vertex of the exceptional set. But in Theorem 5.2 we cannot guarantee this. Instead, we only can say that there is no (F, G, H)-tie that contains a vertex of the exceptional set.

In practice, the balance number for some special connected graphs F, G, and H is often not hard to determine. In most cases, it is also not difficult to check that every (F, G, H)–balanced coloring is expandable since the number of non-isomorphic balanced colorings is small, often not exceeding 4. We can apply Theorem 5.2 to such special triples. In fact, in determining the Ramsey number $r(nF, nG, nH)$ for specifically-given graphs F, G, H, the most difficult work is often to determine the size and coloring of the critical exceptional set X. As an example, we prove the following:

Theorem 5.3. If n is sufficiently large, then

$$r(nK_3, nK_3, nK_{1,2}) = 8n - 2.$$

Proof. Let $r = r(nK_3, nK_3, nK_{1,2})$, $q = 8n-2$, where n is large. We first prove $r \geq q$ by exhbiting a $(nK_3, nK_3, nK_{1,2})$–good coloring of K_{q-1}. We claim that the $(K_3, K_3, K_{1,2})$–balance number $t = 8$. We will give a $(K_3, K_3, K_{1,2})$–balanced coloring and show it is expandable, and therefore by applying Theorem 5.1 we have $r \geq q$. Let $V = V(K_8)$. Partition V into three sets A, B, and C, where $|A| = |C| = 3$, $|B| = 2$. Now color K_8 as follows: $\langle A \rangle$, $\langle B \rangle$, and $\langle C \rangle$ being colored red, blue, and green respectively, AB and AC are blue, and BC is red. It is easy to see that the K_8 colored this way contains exactly one red triangle, and one disjoint green 2–star, $K_{1,2}$. Since $|B| = 2$, AB is blue, but BC is red, this K_8 contains one disjoint blue triangle. If we delete one vertex from A, the remaining graph contains no red triangle; if delete one from B, then the remaining graph contains no blue triangle; if delete a vertex from C, then the remaining graph contains no green 2–star. It is trivial to verify that any canonical coloring of K_9 does not have property MM. Thus we have $t = 8$.

It is also easy to check show that the above coloring χ of K_8 is expandable. There are only two non-isomorphic $(K_3, K_3, K_{1,2})$–balanced colorings of K_8. One is χ. The other one, χ', is given by the coloring (B, A, C), with $\langle B \rangle$ red, $\langle A \rangle$ blue, $\langle C \rangle$ green, BA, AC red, and BC blue. *chi'* is also expandable. Thus we have proved that $t = 8$ and every $(K_3, K_3, K_{1,2})$–balance coloring is expandable. So by Theorem 5.1, $r \geq 8n - 2$.

Now we apply Theorem 5.2 to prove that $r \leq 8n - 2$. Here, we have $F = G = K_3$, $H = K_{1,2}$, and $p_1 = p_2 = p_3 = 3$. Hence the condition $t > p_1 + p_2$ is satisfied, and by the above results, we know that every $(K_3, K_3, K_{1,2})$–balanced coloring is expandable. Then by Theorem 5.2, there is a $(nK_3, nK_3, nK_{1,2})$– critical coloring, τ, of K_{r-1} which is nearly canonical with red, blue, green, and exceptional sets A', B' ,C', and X', and with no $(K_3, K_3, K_{1,2})$–tie having a vertex in X'. Furthermore, the canonical coloring of $Q = \langle A' \cup B' \cup C' \rangle$ is an extension of some $(K_3, K_3, K_{1,2})$–balanced coloring. Without loss of generality, suppose the coloring of Q is an extension of χ. Then $A'B'$ and $A'C'$ are blue, and $B'C'$ is red. Now let x be a vertex of X' (if any). xA' cannot be green, for otherwise, under coloring τ, K_{r-1} contains a $(K_3, K_3, K_{1,2})$–tie that uses x, a contrdiction. So xA' must be blue. Since $A'B'$, $A'C'$, and xA' are blue, xB' or xC' cannot be blue, for otherwise K_{r-1} contains a blue triangle that uses x, then we get a $(K_3, K_3, K_{1,2})$–tie that uses x, contradiction. But xB' or xC' cannot be green either. Hence xB' and xC' must be red. This is also impossible, because $B'C'$ is red, and if xB' and xC' are red, then K_{r-1} contains a red triangle, and therefore a $(K_3, K_3, K_{1,2})$–tie that uses x. Summarizing all the above, we know that X' must be empty. Since τ is $(nK_3, nK_3, nK_{1,2})$-critical, $|A'| \leq 3n - 1$, $|C'| \leq 3n - 1$, $|B'| \leq 2n - 1$. Then

$$r - 1 = |A'| + |B'| + |C'| \leq 8n - 3, \qquad \text{so} \qquad r \leq 8n - 2. \qquad \blacksquare$$

This is our first example in which the critical exceptional set is empty. Most of the time this doesn't happen. This example also shows that the lower bound given by Theorem 5.1 is best possible.

It is often possible to get the Ramsey number directly without using Theorem 5.2, as shown by the next theorem. The interesting part of the next theorem is that the result given by Theorem 5.2 even holds for all $n \geq 1$. The reader can refer [5] for a similar result and proof of the next theorem.

Theorem 5.4. For $n \geq 1$, $r(nK_3, nK_{1,2}, nK_{1,2}) = 7n - 2$.

Proof. Let $r = r(nK_3, nK_{1,2}, nK_{1,2})$, $s = 7n - 2$. We first prove $r \geq s$ by exhibiting an $(nK_3, nK_{1,2}, nK_{1,2})$–good coloring of K_{s-1}. Let $V = V(K_{s-1})$. Partition V into three sets A, B, and C, where $|A| = n - 1$, $|B| = |C| = 3n - 1$. Give a 3–canonical coloring (A, B, C) of K_{s-1} as follows: $\langle A \rangle$ red, $\langle B \rangle$ blue, $\langle C \rangle$ green, and AB, BC, and AC being all red. It is easy to verify that the above coloring of K_{s-1} is $(nK_3, nK_{1,2}, nK_{1,2})$–good. Thus $r \geq 7n - 2$.

Next we prove $r \leq 7n - 2$ by using induction on n. When $n = 1$, $7n - 2 = 5$. Consider any 3–coloring of K_5. If it contains no blue or green 2–star, then it has at most four blue and green edges, then it has at least six red edges, hence it must contain a red triangle. So when $n = 1$, the theorem is true.

Suppose the theorem is true for all $1 \leq n \leq N$.
Now for $n+1$, we need to show that any 3–coloring of the complete graph on $7(n+1)-2$ vertices must contain a red $(n+1)K_3$, or a blue or a green $(n+1)K_{1,2}$. Let $s' = 7(n+1)-2$, and consider any 3–coloring of $K_{s'}$. Our plan is to find a subgraph Q of this $K_{s'}$ that contains each of a red triangle, a blue and a green 2–star, and that has no more than 7 vertices. Then delete this Q from $K_{s'}$, and apply the induction hypothesis to the remaining graph.

If this $K_{s'}$ contains a red $(n+1)K_3$, or a blue or green $(n+1)K_{1,2}$, we are done, so suppose not. Since $n \geq 1, 7(n+1) - 2 \geq 12 > 5$, $K_{s'}$ must contain a red triangle, a blue or a green 2–star. Without loss of generality, suppose it has a green 2–star. Delete a maximal set of disjoint green 2–stars from the $K_{s'}$, denote the remaining graph by M. M has has at least $7(n+1) - 2 - 3n + 1 = 4n + 6 \geq 10$ vertices. Hence W must contain a red triangle or a blue 2–star. Without loss of generality, suppose it contains a blue 2–star. Delete a maximal set of disjoint blue 2–stars from M, and denote the remaining graph by M_1. This graph has at least $4n + 6 - 3n = n + 6$ vertices. Hence it must contain a red triangle.

We have proved that $K_{s'}$ contains a red triangle, a blue and a green 2–star, and they are all disjoint. Let a_1, a_2, a_3 be the vertices of a red triangle X, b_1, b_2, b_3 the vertices of a blue 2–star Y, and c_1, c_2, c_3 the vertices of a green 2–star Z, where all these vertices are distinct. Consider the graph $P = \langle\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}\rangle$. We claim that P contains a subgraph with at most 7 vertices that contains a red triangle, a blue and a green 2–star.

Consider the edges between X and Y. If there are at least 4 blue edges among them, then some vertex of Y must have 2 blue edges going to X. In this case, P contains a subgraph with 4 vertices that contains a red triangle and a blue 2–star, and this subgraph uses one vertex of Y and all three vertices of X. This subgraph together with the green 2–star form a subgraph of P with at most 7 vertices that contains a red triangle, and blue and green 2–stars. In this case our claim is true.

If there is at least one blue edge between X and Y, say a_1b_1, then P has a subgraph with 5 vertices which contains a red triangle and a blue 2–star, and this subgraph uses all the vertices of X and two vertices of Y. In this case, consider the edges between X and Z. If there is at least one green edge between X and Z, it is easy to see that P has a subgraph which uses all three vertices of X and two vertices of each of Y and Z, and which contains each of a red triangle, a blue and a green 2–star, again, our claim is true. So suppose there is no green edges between X and Z. In this case, if there are at least three blue edges between X and Z, then there is either a blue 2–star between X and Z, or a blue edge incident to a_1. In the first case P has a subgraph that only uses all the vertices of X and all vertices of Z, and, which contains a red triangle, and blue and green 2–stars; in the second case, P contains the desired subgraph that uses all the vertices of X and Z, and vertex b_1 of Y. So in both cases, our claim is true. Now suppose there are at most two blue edges but no green edge between X and Z. Then there are at least seven red edges between X and Z. In this case, at least one vertex of Z, say c_1, is joined by red edges to all vertices of X. Then P contains the desired subgraph that uses all vertices of Z, two vertices of X including a_1, and two vertices of Y including b_1.

Summarizing above we see that if there is a blue edge beween X and Y, P contains the desired subgraph. Similarly, if there is a green edge between X and Z, P contains the desired subgraph.

Now suppose that there is no blue edge between X and Y, and no green edge between X and Z. If there are four green edges between X and Y, or four blue edges between X and Z, as proved above P contains the desired subgraph.

So there are at most three disjoint green edges between X and Y. Then there are at least six red edges between X and Y, so P contains a subgraph that contains a red triangle and a blue 2–star, this subgraph uses all the vertices of Y and two vertices of X. Let a_1, a_2, x, b_1, b_2 be the five vertices of this subgraph, where $T = \langle a_1, a_2, x \rangle$ is a red triangle, and where $\langle x, b_1, b_2 \rangle$ contains a blue 2–star. Consider the edges between T and Z. If there are at least three blue edges between T and Z, then either there is a blue 2–star, or a blue edge incident to x; as proved before, in each of these two cases, P contains the desired subgraph. If there is at least one green edge between T and Z, P also contains the desired subgraph. Hence suppose there are at most two blue edges, and no green edge between T and Z. Then there are at least seven red edges between T and Z. In this case, at least one vertex of Z, say c_1, is joined to each vertex of T by red edge, then there is a red triangle that uses c_1, x, and one of the vertices a_1, a_2. Then it is easy to see that in this case P contains the desired subgraph which uses all the vertices of Z, and all vertices x, b_1, b_2 and one of a_1, a_2. Our claim is still true.

Summarizing all the above we see that our claim is true; that is P, and therefore $K_{s'}$, contains a subgraph Q of order at most 7, and Q contains a red triangle, and blue and green 2–stars.

Delete Q from $K_{s'}$. The remaining graph M' has at least $s' - 7 = 7n - 2$ vertices. By the induction hypothesis, M' must contain a red nK_3, a blue or a green $nK_{1,2}$, in each of these three cases, the corresponding case holds in the original $K_{s'}$. Hence the theorem is true for $n+1$. By the induction principle, the theorem is true for all $n \geq 1$. ∎

Using the methods of Theorem 5.4, one can easily prove:

Theorem 5.5. For all $n \geq 1$, $r(nK_3, nK_{1,2}, K_{1,2}) = 4n + 1$. ∎

Note that in this theorem the third graph is $K_{1,2}$, instead of $nK_{1,2}$.

6. CLOSING REMARKS

The results given in Section 2 through Section 4 can be generalized to c–color cases for any $c \geq 3$. The techniques to generalize these results are all contained in this thesis. It would be possible to give further exact values in practice, but we will not do so here. It might be worthwhile to do so elsewhere, but even more valuable would be to extend the theory, improving Theorem 5.1 and 5.2. It would be very desirable to solve the problem of determining the Ramsey number , $r(nF, nG, nH)$, for large n and for all connected graphs F, G and H, completely.

Another significant question is that of how large n_1, n_2, n_3 and n must be for Theorem 2.2, 3.2, 4.2 and 5.2 to hold. It is not hard to show that it suffices to take n_1, n_2, n_3 and n no larger than a triple exponential, respectively, involving the order of F, G and H. On the other hand, as in Theorem 5.4, the long-run behavior can begin at very small n for some special cases, usually 1 or 2.

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