The derivative of intersection local time in the plane

by

Gregory Markowsky

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Jay Rosen

Date

Chair of Examining Committee

Jozef Dodziuk

Date

Executive Officer

Supervisory committee

Jay Rosen
Michael Marcus
Antonia Foldes

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Advisor: Professor Jay Rosen

We begin by introducing Brownian motion, intersection local time, and symmetric stable processes. We then state some basic results concerning intersection local time and its derivative. The remainder of the thesis is devoted to the proofs of the main theorems, which pertain to the asymptotics of a family of integrals related to the intersection local times of Brownian motion and symmetric stable processes in $\mathbf{R}^2$. These integrals depend on a parameter ε, and we show that, upon dividing by a suitable function of ε, these integrals converge in law to a one dimensional Brownian motion as ε converges to 0.

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Contents

1 Introduction

Brownian motion is a random process which is fundamental to probability theory. It is named for British botanist Robert Brown, who in 1827 observed particles within liquid undergoing a seemingly random, jerky motion. At the time, the hypothesis that matter was made of atoms was not as universally accepted as now, and debate continued on this subject throughout the 1800's and early 1900's. In 1905, Albert Einstein published a paper concerning this motion, in which he argued that such motion was in fact predicted by theory. Heat in a substance is the jiggling and movement of the molecules composing the substance, and a particle in a liquid will be constantly struck from all directions by the molecules of the liquid. The effect on the particle of each collision between the particle and a liquid molecule is very small. However, the sum of many such collisions can account for Brownian motion. This explanation is now generally accepted.

The mathematical theory of Brownian motion was first rigorized by the American mathematician Norbert Wiener in 1921 (it is for this reason that Brownian motion is sometimes referred to as the *Wiener process*). Wiener showed that there is a probability measure, now known as *Wiener measure*, on the space of continuous paths which give a satisfactory realization of

Brownian motion. This theory has grown extensively since that time, and has developed applications in fields far removed from physics, perhaps most notably in mathematical finance.

Brownian motion in one dimension is defined to be a random process $B_t = B_t(\omega)$ with the following properties:

1) If $t > s$ then $B_t - B_s \sim N(0, t-s)$, where $N(0, t-s)$ is a normally distributed random variable with mean 0 and variance $t - s$.

2) B_t has independent increments. That is, if $t_1 < ... < t_n$ then $B_{t_1}, (B_{t_2} - B_{t_1}), ..., (B_{t_n} - B_{t_{n-1}})$ is a set of independent random variables.

3) On a set of probability 1, the function $t \longrightarrow B_t$ is continuous for all t.

Brownian motion in n dimensions is simply the process obtained by assigning an independent Brownian motion to each coordinate. It is not trivial to show that Brownian motion exists, but by now there are many constructions of the process. Perhaps the most intuitive way is to obtain Brownian motion as the limiting process of random walks properly scaled. In fact, some theorems about Brownian motion can be proved by proving the corresponding statement for a random walk and then passing to the limit. The reverse technique also can be used.

Countless properties of the Brownian paths have been developed over the years. One that is relevant to this thesis is the fact that, for any y, the set

$\{t : B_t = y\}$ has Lebesque measure 0 a.s. This is a simple consequence of Fubini's theorem, as

$$E[\int_0^\infty 1_{\{B_t=y\}}dt] = \int_0^\infty E[1_{\{B_t=y\}}]dt = \int_0^\infty 0dt = 0 \tag{1.1}$$

This opens the possibility of defining a process L_t^y, called the *local time at 0*, as

$$L_t^a = \lim_{\varepsilon \longrightarrow 0} \int_0^t f_\varepsilon(B_s - a)ds \tag{1.2}$$

provided that the limit exists, where $f_\varepsilon(x) = \frac{1}{\varepsilon}f(\frac{x}{\varepsilon})$, and f is a measurable function with $\int f = 1$. We can also write

$$L_t^a = \int_0^t \delta(B_s - a)ds \tag{1.3}$$

where δ denotes the Dirac delta function. (1.3) is a formal equation which is equivalent to (1.2). In fact, for Brownian motion in one dimension the limit in (1.2) does exist for all t on a set of probability 1, and defines a very useful process. The most commonly used application of L_t is the following *Occupation time formula*, valid for any positive Borel function f:

$$\int_0^t f(X_s)ds = \int_{\mathbf{R}} f(a)L_t^a da \tag{1.4}$$

In two dimensions it is not difficult to show that for any nonzero $x \in \mathbf{R}^2$ we have $P(B_t = x \text{ for any } t) = 0$, and this eliminates the possibility of the same type of local time in $\mathbf{R}^2$. However, instead of placing a measure on the set $\{t : B_t = y\}$ in $\mathbf{R}^2$ we can do an analogous thing with the set $\{(t,s) : B_t - B_s = 0\}$

The intersection local time (abbreviated henceforth as ILT) is defined in any dimension d as

$$\lim_{\varepsilon \longrightarrow 0} \int_0^t \int_0^s f_\varepsilon(B_s - B_u) du ds \tag{1.5}$$

provided that such an integral exists. Again, we can formally write

$$\int_0^t \int_0^s \delta(B_s - B_u) du ds \tag{1.6}$$

with the understanding that (1.5) and (1.6) are equivalent. Just as the local time puts a random measure on the set $\{s : B_s = 0\}$ in $\mathbf{R}_+$, the intersection local time puts a measure on the set $\{u < s : B_u = B_s\}$ in $\mathbf{R}_+^2$. We remark here in passing that the reason for stipulating that $u < s$ is that we will later examine (1.5) with δ replaced with δ'. As δ' is an odd function this integral would be zero if taken over the entire region $\{s, u < t\}$.

In one dimension the ILT exists, though is perhaps not especially interesting, as it can be expressed in terms of the local time via the occupation times formula (1.4). The formal calculation, using (1.6), is:

$$
\begin{aligned}
(1.7) \qquad & \int_0^t \int_0^t \delta(B_s - B_u) du ds \\
&= \int_0^t \int_{\mathbf{R}} \delta(B_s - y) L_t^y dy ds \\
&= \iint_{\mathbf{R}^2} \delta(x - y) L_t^y L_t^x dx dy \\
&= \int_{\mathbf{R}} (L_t^y)^2 dy
\end{aligned}
$$

In two dimensions things get more interesting. It turns out that the integral (1.5) is not defined, as it blows up near the set $\{s = u\}$. However, it was shown by Varadhan in [6] that, if we let f_ε be a family of functions converging weakly to δ and let

$$
(1.8) \qquad \alpha_\varepsilon(T) = \int_0^T \int_0^s f_\varepsilon(B_s - B_u) du ds
$$

then we have

Theorem 1 *$\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]$ converges pathwise as $\varepsilon \longrightarrow 0$ to a finite random variable, known as the renormalized self-intersection local time.*

Brownian paths do not self intersect in dimensions greater than 3, so the only other dimension in which Brownian ILT could exist would be 3.

In [7] Yor dealt with this question. Before stating his result, let us give some more terminology. $C(R^{+})$ denotes the space of continuous paths with the topology generated by the uniform norm. By *convergence in law* (also known as *convergence in distribution*) we mean weak convergence on $C(R^{+})$. Yor proved the following theorem (with $\alpha_\varepsilon(T)$ defined by the same equation (1.8) but in 3 dimensions instead of 2):

Theorem 2

$$\Big\{\frac{1}{\sqrt{\log(1/\varepsilon)}}\Big(\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]\Big), T \geq 0\Big\} \tag{1.9}$$

converges in law as $\varepsilon \longrightarrow 0$ *to the process* $\{\frac{1}{\sqrt{2\pi}}B_T, T \geq 0\}$, *where* B_T *is a one dimensional Brownian motion.*

A similar result was obtained by Rosen in [4] involving symmetric stable processes. The symmetric stable process of index $1 < \beta \leq 2$ is the process with independent increments and transition function

$$f_t(x) = \frac{1}{(2\pi)^2}\int e^{ipx - tp^\beta} d^2p \tag{1.10}$$

f depends on β, but we hide the β in the notation just to keep the expression simple. Rosen considered the process

$$\alpha_\varepsilon(T) = \int_0^T \int_0^t f_\varepsilon(X_t - X_s) ds dt \tag{1.11}$$

where now X is a symmetric stable process. He proved the following:

Theorem 3 *If* $4/3 < \beta \leq 2$, *then* $\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]$ *converges pathwise as* $\varepsilon \longrightarrow 0$ *to a finite random variable. If* $\beta = 4/3$, *then*

$$\Big\{\frac{1}{\sqrt{\log(1/\varepsilon)}}\Big(\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]\Big), T \geq 0\Big\} \tag{1.12}$$

converges in law as $\varepsilon \longrightarrow 0$ *to* $\{k(\beta)B_T, T \geq 0\}$ *where* $k(\beta)$ *is a constant which depends on* β. *Finally, if* $1 < \beta < 4/3$, *then*

$$\Big\{\varepsilon^{2/\beta - 3/2}\Big(\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]\Big), T \geq 0\Big\} \tag{1.13}$$

converges in law as $\varepsilon \longrightarrow 0$ *to* $\{k(\beta)B_T, T \geq 0\}$ *where* $k(\beta)$ *is a constant depending on* β.

If $\beta = 2$ then X is Brownian motion, and this gives a different proof of Varahdan's renormalization. The method employed by Rosen in this paper also gives an alternate proof of Theorem 2.

In this thesis we will consider a similar question to the one which Rosen solved. In [5] Rosen introduced the notion of the derivative of the intersection local time of Brownian motion in $\mathbf{R}^1$. It is defined formally as

$$\alpha'_t = \int_0^t \int_0^s \delta'(B_s - B_u)\,du\,ds \tag{1.14}$$

Rosen was able to show that this integral converges, and proved an occupation time formula, as well as some other facts about α'_t. A natural question is whether this will work in $\mathbf{R}^2$, or whether, perhaps, we could do something along the lines of Varadhan's renormalization. The main result in this paper, however, is that for the derivative of intersection local time in $\mathbf{R}^2$ we in fact obtain a theorem very much like Theorem 2.

Specifically, we begin, as in [4], by letting f_ε be the function on $\mathbf{R}^2$ defined by

$$f_\varepsilon(x) = \frac{1}{(2\pi)^2} \int e^{ipx - \varepsilon p^2} d^2p \tag{1.15}$$

where X_t is a two dimensional Brownian motion. Two abbreviations of notation occur in the above, namely p^2 denotes $|p|^2$ and ipx denotes $i(p \cdot x)$. These notations will be used throughout where convenient. The expression for f_ε is in fact equal to the Gaussian density function with mean 0 and variance ε, as it is the inverse Fourier Transform of $e^{-\varepsilon p^2}$. It will be easier to work with in this form. For ease of notation, let f'_ε denote the partial derivative of f with respect to x_1, where $x = (x_1, x_2)$. So

$$f'_\varepsilon(x) = \frac{i}{(2\pi)^2} \int p_1 e^{ipx - \varepsilon p^2} d^2p \tag{1.16}$$

where $p = (p_1, p_2)$. For any set $A \subseteq \mathbf{R}_+^2$ we let

$$\alpha'_\varepsilon(A) = \int\int_A f'_\varepsilon(X_t - X_s)dsdt \tag{1.17}$$

and we set $\alpha'_\varepsilon(T) = \alpha'_\varepsilon(D(T))$, where D_T denotes the triangle $\{(s_i, t_i)|0 \le s_i \le t_i \le T\}$. We will show that $\alpha'_\varepsilon(T)$ does not approach a random variable as $\varepsilon \longrightarrow 0$. In fact, we will prove

Theorem 4 *$\Big\{(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T), T \ge 0\Big\}$ converges in law to $\Big\{\frac{\sqrt{5}}{\pi 8\sqrt[4]{2}} W_T, T \ge 0\Big\}$ as $\varepsilon \longrightarrow 0$, where W_T is a one-dimensional Brownian motion.*

Remark: f'_ε is an odd function, so $E[\alpha'_\varepsilon(T)] = 0$, which is why we need not subtract the expectation to obtain convergence.

We also will prove an analogous theorem about symmetric stable processes. We let X_t be a symmetric stable process of index β with $1 < \beta < 2$, and let f_ε be defined as in (1.10). Then

$$f'_t(x) = \frac{i}{(2\pi)^2}\int p_1 e^{ipx - tp^\beta} d^2p \tag{1.18}$$

Again we will consider

$$\alpha'_\varepsilon(A) = \int\int_A f'_\varepsilon(X_t - X_s)dsdt \tag{1.19}$$

We will let $\alpha'_\varepsilon(T) = \alpha'_\varepsilon(D_T)$, and we will prove the following:

Theorem 5 *$\{\varepsilon^{3/\beta-3/2}\alpha'_\varepsilon(T), T \geq 0\}$ converges in law to $\{c(\beta)W_T, T \geq 0\}$ as $\varepsilon \longrightarrow 0$ where W_T is a one-dimensional Brownian motion and $c(\beta)$ is given by*

$$\frac{1}{2\sqrt{2}\pi^2}\Big(\int\int \frac{1}{p^\beta}\frac{1}{q^\beta}\frac{1}{(p+q)^\beta}e^{-(p^\beta+q^\beta)}p_1q_1dpdq+\int\int \frac{1}{p^{2\beta}}\frac{1}{(p+q)^\beta}e^{-(p^\beta+q^\beta)}p_1q_1dpdq\Big)$$
(1.20)

Included in the proof is the definition of the second integral in the definition of $c(\beta)$. This integral does not converge absolutely for $\beta \geq 3/2$, so we must clarify what is meant by it. The details will be covered in section 6.

2 Outline of the proof

The outline of the proof of Theorem 4 follows closely the proofs of Theorems 1, 2, and 3 given in [4], though the details, given in sections 3 and 4 below, are quite different. The reader may refer to that paper to see a slightly different presentation of the ideas of this section. We will show first that the moments of $\alpha'_\varepsilon(T)(\log(1/\varepsilon))^{-1}$ converge to the moments of a Brownian motion times $\frac{\sqrt{5}}{\pi\sqrt{128\sqrt{2}}}$. We have

$$E(\alpha'_\varepsilon(T)^n) = \frac{i^n}{(2\pi)^{2n}} \int_{(\mathbf{R}^2)^n} \int_{D_T^n} e^{-\varepsilon \sum_j p_j^2} \prod_{j=1}^n p_{j,1} E[\prod_{j=1}^n e^{ip_j(X_{t_j}-X_{s_j})}] \prod_{j=1}^n ds_j dt_j d^2 p_j \tag{2.1}$$

This is obtained by combining n copies of the integral in (1.17), using the definition (1.16) of f_ε. Now, if n is odd, then the integrand is an odd function of p, and the expectation is therefore 0. Since all odd moments of Brownian motion are 0, we need only show that the even moments converge to the right values. The $2n$-th moment of Brownian motion at time T is $\frac{(2n)!}{2^n n!}T^n$, so we will show that

$$E[\alpha'_\varepsilon(T)^{2n}] = \frac{(2n)!}{2^n n!}(\frac{\sqrt{5}}{\pi\sqrt{128\sqrt{2}}}(\log(1/\varepsilon)))^{2n} T^n + o(\log(1/\varepsilon))^{2n} \tag{2.2}$$

This expectation is given by (2.1) with n replaced by $2n$. The expectation on the right side of (2.1) will depend on the ordering of the s_j's and t_j's in D_T^{2n}. By independence, the expectation will factor into the product of several expectations, each corresponding to a component of the set $\bigcup_j [s_j, t_j]$. Each of these resulting expectations are then integrated separately. Following [4], we will say that a component consisting of m intervals $[s_j, t_j]$ is of order m. As in [4], it turns out that the dominant contribution comes from regions consisting of n components of order 2. Suppose, for the time being, that we are integrating over only such regions. We will hold fixed the initial points of the components $r_1 < ... < r_n$. In this case we will show(Section 3) that each of the n components contribute

$$(\frac{5}{\pi^2 128\sqrt{2}}(\log(1/\varepsilon)))^2 + o(\log(1/\varepsilon))^2 \tag{2.3}$$

The contribution of each configuration with n components of order 2 to (2.1) is therefore

$$\begin{aligned}\Big((\frac{5}{\pi^2 128\sqrt{2}}(\log(1/\varepsilon))^2 + o(\log(1/\varepsilon)^2)\Big)^n \int_{0\le r_1\le ... \le r_n \le T} dr_1 ... dr_n \\ = \frac{T^n}{2^n n!}(\frac{\sqrt{5}\sqrt{2}}{\pi\sqrt{128\sqrt{2}}}(\log(1/\varepsilon)))^{2n} + o(\log(1/\varepsilon))^{2n} \\ = \frac{T^n}{2^n n!}(\frac{\sqrt{5}}{\pi 8\sqrt[4]{2}}(\log(1/\varepsilon)))^{2n} + o(\log(1/\varepsilon))^{2n}\end{aligned} \tag{2.4}$$

where we have used the identity

$$\int_{0\le r_1\le\ldots\le r_n\le T} dr_1\ldots dr_n = T^n/n!$$

There are $(2n)!$ different ways to order the set $\{s_1,\ldots,s_{2n}\}$, and each contributes (2.4), so the total is

$$\frac{(2n)!}{2^n n!}\Big(\frac{\sqrt{5}}{\pi 8\sqrt[4]{2}}(\log(1/\varepsilon))\Big)^{2n} T^n + o(\log(1/\varepsilon))^{2n} \tag{2.5}$$

which is, in fact, what we were aiming for. We must therefore show that the contribution from all regions which have other than n components of order 2 is $o(\log(1/\varepsilon))^{2n}$. We will do this in Section 4 by showing that each component of order m with $m\ge 3$ contributes $o(\log(1/\varepsilon))^m$. Note that any component of order 1, or indeed of any odd order, will in fact contribute 0, since the integrand is an odd function. It will follow from all of this that the moments of $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T)$ converge to the right values. We will prove that the processes $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T)$ are tight in Section 5, so that there is a limiting process which has all of the same moments as $\frac{\sqrt{5}}{\pi\sqrt{128\sqrt{2}}}W_t$. In addition, this process has independent increments, as will also be shown in Section 5. These facts, taken together, will prove Theorem 4.

3 Components of order 2

Here we will deal with the aforementioned components of order 2. We'll begin by proving

Proposition 1

$$(3.1) \qquad E[\alpha'_\varepsilon(T)^2] = T\log(1/\varepsilon)^2\left(\frac{10}{128\sqrt{2}\pi^2} + o(1)\right)$$

To begin the proof, we write

$$(3.2)\ E[\alpha'_\varepsilon(T)^2] = -\frac{1}{(2\pi)^4}\int\int_{D_T^2} e^{-\varepsilon(p^2+q^2)}E[e^{ip(X_{t_1}-X_{s_1})+iq(X_{t_2}-X_{s_2})}] \\ p_1q_1d^2sd^2tdpdq$$

where D_T denotes the triangle $\{(s_i,t_i)|0 \leq s_i \leq t_i \leq T\}$. For simplicity we'll assume for the time being that $T = 1$. To handle the expectation in the integrand, we must consider different orderings of the s's and t's. By symmetry, we may assume $s_1 < s_2$. We'll suppress the $\frac{-1}{(2\pi)^4}$ in front of the integral for the time being.

Case 1: $s_1 < s_2 < t_2 < t_1$

We rewrite the exponent in the expectation as $ip(X_{t_1}-X_{t_2})+i(p+q)(X_{t_2}-X_{s_2}) + ip(X_{s_2} - X_{s_1})$, and then factor the expectation using independence. As a result, the expectation becomes

$$e^{-p^2(a+c)-(p+q)^2 b} \tag{3.3}$$

where $a = t_1 - t_2$, $b = t_2 - s_2$, and $c = s_2 - s_1$. Upon making this linear transformation, the integral in question becomes

$$\int_0^T \Big[\int\int\int_{a+b+c\le t_1} dadbdc \int\int e^{-p^2(a+c+\varepsilon)} e^{-(p+q)^2 b} e^{-q^2\varepsilon} p_1 q_1 dpdq\Big] dt_1 \tag{3.4}$$

Case 2: $s_1 < s_2 < t_1 < t_2$

We rewrite the exponent in the expectation as $iq(X_{t_2} - X_{t_1}) + i(p+q)(X_{t_1} - X_{s_2}) + ip(X_{s_2} - X_{s_1})$, and proceed as in Case 1. The integral in question here is then

$$\int_0^T \Big[\int\int\int_{a+b+c\le t_1} dadbdc \int\int e^{-p^2(c+\varepsilon)} e^{-(p+q)^2 b} e^{-q^2(a+\varepsilon)} p_1 q_1 dpdq\Big] dt_1 \tag{3.5}$$

where now $a = t_2 - t_1$, $b = t_1 - s_2$, and $c = s_2 - s_1$.

Case 3: $s_1 < t_1 < s_2 < t_2$

Here the expectation factors, and since the integrand of (3.2) is then an odd function of p, the contribution to (3.2) of this case is 0.

Remark: Before we begin, let us clear up a technical point that will be necessary later. To do the computations, we will in fact integrate da, db, and dc first. However, if we were to begin with dp and dq, so that the integrals in cases 1 and 2 were of the form

$$\frac{-1}{(2\pi)^4}\int\int\int\int h(a,b,c,t)dadbdcdt \tag{3.6}$$

then h would be everywhere negative. Intuitively, this is because when p_1 and q_1 are the same sign, $|p+q|$ is larger than when they are opposite signs. In order to rigorously prove this, just note that the map $\phi((p_1,p_2),((q_1,q_2)) = ((p_1,p_2),(-q_1,q_2))$ is a linear isometry which maps $U = \{p_1q_1 > 0\}$ bijectively onto $V = \{p_1q_1 < 0\}$, and the function $|p+q|$ is greater at $(p,q) \in U$ than at $\phi(p,q) \in V$. It will be pointed out later where we have used this fact.

We'll attack case 1 first. In order to compute the required integral, we'll first examine the following:

$$\int\int\int_{0<a,b,c\leq 1} dadbdc \int\int e^{-p^2(a+c+\varepsilon)}e^{-(p+q)^2b}e^{-q^2\varepsilon}p_1q_1dpdq \tag{3.7}$$

Upon integrating da,db, and dc, we obtain

$$\begin{aligned} &\int\int \frac{(1-e^{-p^2})^2}{p^4}\frac{(1-e^{-(p+q)^2})}{(p+q)^2}e^{-\varepsilon(p^2+q^2)}p_1q_1dpdq \\ &= \int\int \frac{(1-e^{-p^2/\varepsilon})^2}{p^4}\frac{(1-e^{-(p+q)^2/\varepsilon})}{(p+q)^2}e^{-(p^2+q^2)}p_1q_1dpdq \end{aligned} \tag{3.8}$$

Upon converting to polar coordinates, with $p = re^{i\theta}$, $q = se^{i\phi}$, we arrive at

$$\int\int\int \frac{(1-e^{-r^2/\varepsilon})^2}{r^2} s^2 cos(\phi) e^{-(r^2+s^2)} \\ \int_0^{2\pi} \frac{(1-e^{-|re^{i\theta}+se^{i\phi}|^2/\varepsilon})}{|re^{i\theta}+se^{i\phi}|^2} cos(\theta) d\theta d\phi dr ds \tag{3.9}$$

We'll isolate the $d\theta$ integral. We may replace θ with $\theta+\phi$. Note that

$$\frac{(1-e^{-|re^{i(\theta+\phi)}+se^{i\phi}|^2/\varepsilon})}{|re^{i(\theta+\phi)}+se^{i\phi}|^2} = \int_0^{1/\varepsilon} e^{-|re^{i(\theta+\phi)}+se^{i\phi}|^2 x} dx \tag{3.10}$$

so the $d\theta$ integral is

$$\int_0^{1/\varepsilon}\int_0^{2\pi} e^{-|re^{i(\theta+\phi)}+se^{i\phi}|^2 x} cos(\theta+\phi) d\theta dx \\ = \int_0^{1/\varepsilon}\int_0^{2\pi} e^{-|re^{i\theta}+s|^2 x}[cos(\theta)cos(\phi) - sin(\theta)sin(\phi)] d\theta dx \tag{3.11}$$

We expand this and pull the $cos(\phi)$ and $sin(\phi)$ terms outside of the $d\theta$ integral. Integrating the $sin(\phi)$ term with the $cos(\phi)$ term already present outside the $d\theta$ integral gives 0. Integrating the $cos(\phi)$ together with the other $cos(\phi)$ term already present gives $\pi/2$. We have therefore eliminated ϕ from (3.9). The contribution of (3.10) to (3.9) is therefore equal to

$$\frac{\pi}{2}\int_0^{1/\varepsilon} e^{-(r^2+s^2)x} \int_0^{2\pi} e^{-2rsxcos(\theta)} cos(\theta) d\theta dx \tag{3.12}$$

By [2] (p. 958, 8.431.5 with $\nu = 1$, replace θ with $\theta + \pi$), what remains of the $d\theta$ integral is equal to -$2\pi I_1(2rsx)$, where I_1 denotes the modified Bessel function of the first kind. Thus, (3.9) is equal to

$$-\pi^2 \int_0^\infty \int_0^\infty \int_0^{1/\varepsilon} \frac{(1-e^{-r^2/\varepsilon})^2}{r^2} s^2 e^{-(r^2+s^2)(x+1)} I_1(2rsx)\,dxdrds \tag{3.13}$$
$$= -\pi^2 \int_0^\infty \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} e^{-r^2 y}(1-e^{-r^2/\varepsilon}) s^2 e^{-(r^2+s^2)(x+1)} I_1(2rsx)\,dydxdrds$$

Expand this into two integrals, and rewrite the first one as

$$-\pi^2 \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} e^{-s^2(x+1)} s^2 \int_0^\infty e^{-r^2(y+x+1)} I_1(2rsx)\,drdxdyds \tag{3.14}$$

By [2] (p. 711,6.618.4) the dr integral is equal to

$$\frac{\sqrt{\pi}}{2\sqrt{x+y+1}} e^{\frac{(2sx)^2}{8(x+y+1)}} I_{1/2}\left(\frac{(2sx)^2}{8(x+y+1)}\right) \tag{3.15}$$

This is $\frac{(e^{\frac{s^2x^2}{x+y+1}}-1)}{\sqrt{2}sx}$, since $I_{1/2}(z) = \frac{1}{\sqrt{2\pi z}}(e^z - e^{-z})$ by [2] (p.967 8.467). Thus, (3.13) is

$$\frac{-\pi^2}{\sqrt{2}} \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} e^{-s^2(x+1)} s \frac{(e^{\frac{s^2x^2}{x+y+1}}-1)}{x}\,dxdyds \tag{3.16}$$
$$= \frac{-\pi^2}{\sqrt{2}} \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} \frac{1}{x} \int_0^\infty s\left(e^{-s^2(x+1-\frac{x^2}{x+y+1})} - e^{-s^2(x+1)}\right) dsdydx$$

Since

$$\int_0^\infty s e^{-bs^2} ds = (1/2)b^{-1} \tag{3.17}$$

we see that (3.14) is

$$\begin{aligned} \frac{-\pi^2}{2\sqrt{2}} \int_0^M \int_0^M \frac{1}{x}((x+1-\frac{x^2}{x+y+1})^{-1} - (x+1)^{-1})dxdy \\ = \frac{-\pi^2}{2\sqrt{2}} \int_0^M \int_0^M \frac{x}{(x+1)(xy+2x+y+1)} dxdy \end{aligned} \tag{3.18}$$

where we have substituted $M = 1/\varepsilon$ to simplify what follows. This last integral is explicitly computable, but it is easier to calculate the derivative and then apply L'Hospital's rule. To do so, we'll make use off the following lemma:

Lemma 1 *If $h(x, y, M)$ is bounded, continuous in x and y on $\{x, y \geq 0\}$, differentiable in M with bounded derivative, then*

$$\begin{aligned} \frac{d}{dM} \int_0^M \int_0^M h(x,y,M)dxdy = \int_0^M h(M,y,M)dy \\ + \int_0^M h(x,M,M)dx + \int_0^M \int_0^M \frac{d}{dM} h(x,y,M)dxdy \end{aligned} \tag{3.19}$$

In the case of (3.18) the integrand does not depend on M, so the last term is 0. We include the last term because it will be used later. By the lemma, the derivative with respect to M of (3.18) is

$$\begin{aligned}(3.20)\qquad &\frac{-\pi^2}{2\sqrt{2}}\Big[\frac{M}{M+1}\int_0^M \frac{dy}{(M+1)y+2M+1}\\ &+\int_0^M \frac{x}{(x+1)((M+2)x+M+1)}dx\Big]\\ &=\frac{-\pi^2}{2\sqrt{2}}\Big[\frac{M}{M+1}\int_0^M \frac{dy}{(M+1)y+2M+1}+\\ &\int_0^M \frac{1}{(x+1)}-\frac{M+1}{((M+2)x+M+1)}dx\Big]\end{aligned}$$

Performing the integration gives

$$\begin{aligned}(3.21)\qquad \frac{-\pi^2}{2\sqrt{2}}\Big[&\frac{M}{(M+1)^2}\log\Big(\frac{(M+1)M+2M+1}{2M+1}\Big)+log(M+1)\\ &-\frac{(M+1)}{(M+2)}\log\Big(\frac{(M+2)M+M+1}{M+1}\Big)\Big]\end{aligned}$$

The expression inside the brackets is asymptotic to $\frac{2\log M}{M}$. This is an immediate consequence of the following easily verified facts:

1) $\frac{M}{(M+1)^2}=\frac{1}{M}+O(\frac{1}{M^2})$

2) $\log\Big(\frac{(M+1)M+2M+1}{2M+1}\Big)=\log M+O(1)$

3) $log(M+1)=log(M)+O(\frac{1}{M})$

4) $\log\Big(\frac{(M+2)M+M+1}{M+1}\Big)=\log M+O(\frac{1}{M})$

5) $\frac{(M+1)}{(M+2)}=1-\frac{1}{(M+2)}=1-\frac{1}{M}+O(\frac{1}{M^2})$.

Thus, (3.20) is equal to $\frac{\log M}{M}(\frac{-\pi^2}{\sqrt{2}}+o(1))$, and it follows from l'Hospital's rule that (3.18) is equal to $(\log M)^2(\frac{-\pi^2}{2\sqrt{2}}+o(1))$. Recall that we split (3.13) into two integrals. We must now deal with the second, namely

$$(3.22)\quad -\pi^2\int_0^\infty\int_0^{1/\varepsilon}\int_0^{1/\varepsilon}e^{-s^2(x+1)}s^2\int_0^\infty e^{-r^2(y+x+1+1/\varepsilon)}I_1(2rsx)drdxdyds$$

We can follow steps (3.14)-(3.18) exactly, with the only difference being that we have $y+M$ in place of y. We get

$$\begin{aligned}(3.23)\quad &\frac{-\pi^2}{2\sqrt{2}}\int_0^M\int_0^M\frac{1}{x}((x+1-\frac{x^2}{x+y+1+M})^{-1}-(x+1)^{-1})dxdy\\ &=\frac{-\pi^2}{2\sqrt{2}}\int_0^M\int_0^M\frac{x}{(x+1)(xy+2x+y+1+M(x+1))}dxdy\end{aligned}$$

where, again, $M=1/\varepsilon$. We take the derivative as before, using Lemma 1, and this time the integrand depends on M:

$$\begin{aligned}(3.24)\qquad \frac{-\pi^2}{2\sqrt{2}}\Big[&\frac{M}{M+1}\int_0^M\frac{dy}{(M+1)y+2M+1+M(M+1)}\\ &+\int_0^M\frac{x}{(x+1)((2M+2)x+2M+1)}dx\\ &-\int_0^M\int_0^M\frac{x}{(xy+2x+y+1+M(x+1))^2}dxdy\Big]\end{aligned}$$

The first term

$$\frac{M}{M+1}\int_0^M \frac{dy}{(M+1)y+2M+1+M(M+1)} \tag{3.25}$$

is bounded above by

$$\int_0^M \frac{dy}{M^2} = \frac{1}{M} \tag{3.26}$$

We may ignore it, as it is $o(\log M/M)$. The second term

$$\begin{aligned} &\int_0^M \frac{x}{(x+1)((2M+2)x+2M+1)}dx \\ = &\int_0^M \frac{1}{(x+1)} - \frac{2M+1}{((2M+2)x+2M+1)}dx \end{aligned} \tag{3.27}$$

is the same as the second integral in (3.20), with $2M$ replacing M. We can follow steps (3.20) and (3.21), and use fact (3) above along with

6) $\log\left(\frac{(2M+2)M+2M+1}{M+1}\right) = \log M + O(\frac{1}{M})$

7) $\frac{(M+1)}{(M+2)} = 1 - \frac{1}{(2M+2)} = 1 - \frac{1}{2M} + O(\frac{1}{M^2})$

to see that this term is asymptotic to $(1/2)\log(M)/M$. The third term

$$\int_0^M \int_0^M \frac{x}{(xy+2x+y+1+M(x+1))^2}dxdy \tag{3.28}$$

is bounded above by

$$\int_0^M \int_0^M \frac{dxdy}{M^2(x+1)^2} = O(1/M) \tag{3.29}$$

and may also be ignored. Thus, (3.22) is $(\log(1/\varepsilon))^2(\frac{-\pi^2}{8\sqrt{2}}+o(1))$. We have found the asymptotics for (3.22) and (3.14). They are

$$(3.30) -\pi^2\int_0^\infty\int_0^{1/\varepsilon}\int_0^{1/\varepsilon} e^{-s^2(x+1)}s^2\int_0^\infty e^{-r^2(y+x+1)}I_1(2rsx)drdxdyds = (\log(1/\varepsilon))^2(\frac{-\pi^2}{2\sqrt{2}}+o(1))$$

and

$$(3.31) -\pi^2\int_0^\infty\int_0^{1/\varepsilon}\int_0^{1/\varepsilon} e^{-s^2(x+1)}s^2\int_0^\infty e^{-r^2(y+x+1+1/\varepsilon)}I_1(2rsx)drdxdyds = (\log(1/\varepsilon))^2(\frac{-\pi^2}{8\sqrt{2}}+o(1))$$

Combining these as in (3.13), we see that (3.7) is $\log(1/\varepsilon)^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$

Now let us consider

$$(3.32)\qquad \int\int\int_{0<a,b,c\leq t} dadbdc\int\int e^{-p^2(a+c+\varepsilon)}e^{-(p+q)^2b}e^{-q^2\varepsilon}p_1q_1dpdq$$

By a simple scaling we now show that this is equal to $(\log(t/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$. The scaling is as follows:

$$(3.33)\qquad \int\int\int_{0<a,b,c\leq t} dadbdc\int\int e^{-p^2(a+c+\varepsilon)}e^{-(p+q)^2b}e^{-q^2\varepsilon}p_1q_1dpdq$$

$$= \int\int\int_{0<a,b,c\leq t} dadbdc \int\int e^{-p^2 t(\frac{a+c+\varepsilon}{t})} e^{-(p+q)^2 t\frac{b}{t}} e^{-q^2 t(\frac{\varepsilon}{t})} p_1 q_1 dpdq$$

$$= t^2 \int\int\int_{0<a,b,c\leq 1} dadbdc \int\int e^{-p^2 t(a+c+\varepsilon/t)} e^{-(p+q)^2 tb} e^{-q^2 t(\varepsilon/t)} (\sqrt{t}p_1)(\sqrt{t}q_1) dpdq$$

Now replace (p,q) with $(p/\sqrt{t}, q/\sqrt{t})$. The t^2 in front of the integral is canceled, and we are left with

$$\begin{aligned}(3.34)\int\int\int_{0<a,b,c\leq 1} dadbdc \int\int e^{-p^2(a+c+\varepsilon/t)} e^{-(p+q)^2 b} e^{-q^2(\varepsilon/t)} p_1 q_1 dpdq \\ = (\log(t/\varepsilon))^2\left(\frac{-3\pi^2}{8\sqrt{2}} + o(1)\right)\end{aligned}$$

We must now examine the same integral, but over the region $\{a+b+c < t_1\}$ rather than $\{0 < a,b,c \leq t_1\}$. However, the remark following Case 3 shows that if $U \subseteq V$, then

$$(3.35) \qquad \left|\int_{a,b,c\in U}\right| \leq \left|\int_{a,b,c\in V}\right|$$

(Recall that both are negative) We also note that

$$(3.36) \qquad \frac{(\log(t/\varepsilon))^2}{(\log(1/\varepsilon))^2} \longrightarrow 1$$

We can then write

$$(3.37) \qquad \left|\int_{a,b,c<(t_1/3)}\right| \leq \left|\int_{a+b+c<t_1}\right| \leq \left|\int_{a,b,c<t_1}\right|$$

The first and last integrals are both $(log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$, and it follows that the middle one is as well. Thus, the integrand in the dt_1 integral (3.4) is $(\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$, with the $o(1)$ term uniformly bounded on $\{0<\delta<t_1<T\}$. We will split up (3.4) as:

$$(3.38)\int_0^T\int\int\int_{a+b+c\leq t_1} dadbdcdt_1\int\int e^{-p^2(a+c+\varepsilon)}e^{-(p+q)^2b}e^{-q^2\varepsilon}p_1q_1dpdq$$
$$=\int_0^\delta h(t_1,\varepsilon)dt_1+\int_\delta^T h(t_1,\varepsilon)dt_1$$

where h denotes the result after doing the integrals in the other variables. We know that, for the second integral, $h(t_1,\varepsilon)(\log(1/\varepsilon))^{-2}\longrightarrow\frac{-3\pi^2}{8\sqrt{2}}$ uniformly. Thus, the second integral is $(T-\delta)(\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$. The first integral is bounded above in absolute value by(assuming $\delta<1$)

$$(3.39)\Big|\int_0^\delta\int\int\int_{a+b+c\leq 1} dadbdcdt_1\int\int e^{-p^2(a+c+\varepsilon)}e^{-(p+q)^2b}e^{-q^2\varepsilon}p_1q_1dpdq\Big|$$
$$=\delta(\log(1/\varepsilon))^2(\frac{3\pi^2}{8\sqrt{2}}+o(1))$$

By letting $\delta\longrightarrow 0$, we may finally conclude that the contribution from Case 1 is

$$(3.40)\qquad T(\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}}+o(1))$$

Similar techniques will yield Case 2. We'll give only the outline here. Recall that we are evaluating

$$(3.41)\int_0^T \int\int\int_{a+b+c\leq t_1} dadbdcdt_1 \int\int e^{-p^2(c+\varepsilon)}e^{-(p+q)^2 b}e^{-q^2(a+\varepsilon)}p_1q_1dpdq$$

As before, we begin by changing the domain to $\{0 \leq a, b, c \leq 1\}$ and integrating da, db, and dc. We arrive at

$$(3.42) \qquad \int\int \frac{(1-e^{-p^2})}{p^2}\frac{(1-e^{-q^2})}{q^2}\frac{(1-e^{-(p+q)^2})}{(p+q)^2}e^{-\varepsilon(p^2+q^2)}p_1q_1dpdq$$
$$= \int\int \frac{(1-e^{-p^2/\varepsilon})}{p^2}\frac{(1-e^{-q^2/\varepsilon})}{q^2}\frac{(1-e^{-(p+q)^2/\varepsilon})}{(p+q)^2}e^{-(p^2+q^2)}p_1q_1dpdq$$

We convert to polar coordinates again:

$$(3.43) \qquad \int\int\int(1-e^{-r^2/\varepsilon})(1-e^{-s^2/\varepsilon})cos(\phi)e^{-(r^2+s^2)}$$
$$\int_0^{2\pi}\frac{(1-e^{-|re^{i\theta}+se^{i\phi}|^2/\varepsilon})}{|re^{i\theta}+se^{i\phi}|^2}cos(\theta)d\theta d\phi drds$$

We follow the same steps for the $d\theta$ integral as before(steps (3.9)-(3.13)) to arrive at

$$(3.44) \qquad -\pi^2\int_0^\infty\int_0^\infty\int_0^{1/\varepsilon}(1-e^{-r^2/\varepsilon})(1-e^{-s^2/\varepsilon})$$
$$e^{-(r^2+s^2)(x+1)}I_1(2rsx)dxdrds$$

We'll expand this into 4 integrals and do each separately. The first is

$$-\pi^2\int_0^\infty\int_0^{1/\varepsilon} e^{-s^2(x+1)}\Big(\int_0^\infty e^{-r^2(x+1)}I_1(2rsx)dr\Big)dxds \tag{3.45}$$

The dr integral, as in step (3.15), is equal to $\frac{(e^{\frac{s^2x^2}{x+1}}-1)}{\sqrt{2}sx}$, and we obtain:

$$\frac{-\pi^2}{\sqrt{2}}\int_0^\infty\int_0^{1/\varepsilon} e^{-s^2(x+1)}\frac{(e^{\frac{s^2x^2}{x+1}}-1)}{sx}dxds \tag{3.46}$$

We use the identity

$$s\int_0^{\frac{x^2}{x+1}} e^{s^2t}dt = \frac{(e^{\frac{s^2x^2}{x+1}}-1)}{s} \tag{3.47}$$

And the integral in question becomes

$$\frac{-\pi^2}{\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}\int_0^{\frac{x^2}{x+1}}\int_0^\infty e^{-s^2(x+1-t)}sdsdtdx \tag{3.48}$$

Note that $t \leq \frac{x^2}{x+1} < x+1$, which implies $(x+1-t) > 0$, so there's no problem with convergence. Tackling this integral again reduces to basic calculus. Begin by substituting $u = s^2(x+1-t)$ to get

$$\frac{-\pi^2}{2\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}\int_0^{\frac{x^2}{x+1}}\frac{1}{x+1-t}\int_0^\infty e^{-u}dudtdx \tag{3.49}$$

$$= \frac{-\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \int_0^{\frac{x^2}{x+1}} \frac{1}{x+1-t} dt dx$$
$$= \frac{-\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} log\Big(\frac{x+1}{x+1-x^2/(x+1)}\Big) dx$$

If $M = 1/\varepsilon$, then, by the fundamental theorem of calculus, $\frac{d}{dM}$ of the above is

$$\frac{-\pi^2}{2\sqrt{2}} \frac{\log\left(\frac{M+1}{M+1-M^2/(M+1)}\right)}{M} \tag{3.50}$$

Since

$$\log\Big(\frac{M+1}{M+1-M^2/(M+1)}\Big) = \log(M) + O(1) \tag{3.51}$$

we see that (3.50) is $\frac{\log M}{M}(\frac{-\pi^2}{2\sqrt{2}} + o(1))$, and thus our original integral (3.45) is $(\log M)^2(\frac{-\pi^2}{4\sqrt{2}} + o(1))$. Recall that (3.44) was divided into four integrals. The remaining three give a contribution of

$$-\pi^2 \int_0^\infty \int_0^\infty \int_0^{1/\varepsilon} (-e^{-r^2/\varepsilon} - e^{-s^2/\varepsilon} + e^{-r^2/\varepsilon} e^{-s^2/\varepsilon}) \\ e^{-(r^2+s^2)(x+1)} I_1(2rsx) dx dr ds \tag{3.52}$$

The integral corresponding to the first and second term will be identical, and so if we follow steps (3.45) through (3.49) we get

$$\text{(3.53)} \qquad \frac{-\pi^2}{2\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}log\Big(\frac{x+1+1/\varepsilon}{x+1+1/\varepsilon-x^2/(x+1+1/\varepsilon)}\Big)dx$$
$$+2\frac{\pi^2}{2\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}log\Big(\frac{x+1+1/\varepsilon}{x+1+1/\varepsilon-x^2/(x+1)}\Big)dx$$
$$=\frac{\pi^2}{2\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}log\Big(1-\frac{x^2}{(x+1+1/\varepsilon)^2}\Big)dx$$
$$-2\frac{\pi^2}{2\sqrt{2}}\int_0^{1/\varepsilon}\frac{1}{x}log\Big(1-\frac{x^2}{(x+1)(x+1+1/\varepsilon)}\Big)dx$$

Each of these integrals is $O(log(1/\varepsilon))$. To see this, note that the absolute values of the integrands are bounded above by $g(x) = \frac{-1}{x}\log(1-x^2)$ on $\{0 < x < \frac{1}{2}\}$. $g(x)$ is bounded on this interval, so the integral over $\{0 < x < \frac{1}{2}\}$ is bounded. For $\{x > \frac{1}{2}\}$ the integrand is bounded by $\frac{-1}{x}\log(\frac{1}{2})$, since $1/\varepsilon > x$, and it follows from this that the integral is $O(\log(1/\varepsilon)$. This proves that (3.43) is $\log(1/\varepsilon)^2(\frac{-\pi^2}{4\sqrt{2}} + o(1))$. The remainder of the proof that the integral we began with in case 2, namely (3.5), is $\log(1/\varepsilon)^2(\frac{-\pi^2}{4\sqrt{2}} + o(1))$ is identical to the steps (3.35) through (3.40).

Combining our work in cases 1 and 2, and reinserting the constant $\frac{-1}{(2\pi)^4}$ which was suppressed throughout, we see that

$$\text{(3.54)} \qquad E[\alpha'_\varepsilon(T)^2] = T\log(1/\varepsilon)^2\Big(\frac{5}{128\sqrt{2}\pi^2} + o(1)\Big)$$

We assumed at the outset that $s_1 < s_2$, so this must be multiplied by 2 to obtain the correct answer. □

Corollary 1. *The contribution to (2.1) of any component of order 2 is* $\log(1/\varepsilon)^2(\frac{5}{128\sqrt{2}\pi^2}+o(1))$. *That is, if* r_j *is the left endpoint of a component of order 2, and* r_{j+1} *is the maximal right endpoint(see (2.4)), then the integral over this region is* $\log(1/\varepsilon)^2(\frac{5}{128\sqrt{2}\pi^2}+o(1))(r_{j+1}-r_j)$.

Proof: Suppose that the component of order 2 is composed of $[s_k, t_k]$ and $[s_{k'}, t_{k'}]$. Then the integral in question is

$$\frac{-1}{(2\pi)^4}\iint_{s_k,t_k,s_{k'},t_{k'}\in[r_j,r_{j+1}]} e^{-\varepsilon(p^2+q^2)}E[e^{ip(X_{t_k}-X_{s_k})+iq(X_{t_{k'}}-X_{s_{k'}})}]p_1q_1d^2sd^2tdpdq \tag{3.55}$$

We can write $X_{t_k}-X_{s_k}$ as $(X_{t_k}-X_{r_j})-(X_{s_k}-X_{r_j})$, and likewise for $X_{t_{k'}}-X_{s_{k'}}$. Since $X_{r_j+t}-X_{r_j}$ is itself a Brownian motion we see that (3.55) is equal to (3.2) with D_T replaced by $D_{r_{j+1}-r_j}$. □

4 Components of order $n \geq 3$

We now turn to components of order n, where $n \geq 3$. We'll show

Proposition 2 *The contribution to (2.1) of any component of order* $n \geq 3$ *is* $o(\log(1/\varepsilon)^n)$.

This entire section is devoted to the proof of this proposition. Suppose that a component of order n is can be formed by a specific arrangement of intervals $[s_1, t_1], ..., [s_n, t_n]$ corresponding to the variables $p_1, ..., p_n$. Let $\{r_1, ..., r_{2n}\}$ be a relabeling of the s_i's and t_i's so that $0 := r_0 \leq r_1 \leq r_2 \leq ... \leq r_{2n}$. We split up the expectation in the integrand by independence and change coordinates as we did in the order 2 case. The contribution of this arrangement of intervals is then given by the integral

$$\int e^{-\varepsilon \sum p_i^2} \prod_i (p_i)_1 \Big(\int_{\sum c_j < T} \prod_j e^{-u_j^2 c_j} \prod_j dc_j \Big) \prod_i dp_i \tag{4.1}$$

where $c_j = r_j - r_{j-1}$ and, as before, $(p_i)_1$ denotes the x-coordinate of p_i. Each u_j is a linear combination of p_i's, and is ordered in the natural way. That is, $u_1 = p_1$, $u_2 = p_1 + p_2$, $u_3 = p_2$ or p_1 or $p_1 + p_2 + p_3$, etc. For each j, either $u_j - u_{j-1} = p_i$ or $u_j - u_{j-1} = -p_i$ for some i. In the first case we'll refer to u_j as increasing (abbreviated as $u_j \uparrow$) and in the second case we'll say u_j is decreasing (abbreviated as $u_j \downarrow$). To simplify some of the notation that follows, let $u_0 = u_{2n} = 0$. We will suppose first that our component contains no isolated intervals. That is, there does not exist an interval $[s_i, t_i]$ such that $t_k, s_k \notin (s_i, t_i)$ for all k. In this case, we may obtain a sufficient bound for (4.1) by replacing the integrand with the absolute value of the integrand. We may then also replace the region $\sum c_j < T$ with $0 < c_j < T$ for all j and use the simple fact that

$$\int_0^T e^{-u_j^2 c_j} dc_j \leq \frac{k}{(1+|u_j|)^2} \tag{4.2}$$

for a constant k to reduce our problem to bounding

$$\int \frac{e^{-\varepsilon \sum p_i^2} \prod |p_i|}{\prod_{j=1}^{2n-1}(1+|u_j|)^2} \prod dp_i \tag{4.3}$$

We will eventually need the following lemmas:

Lemma 2

$$\int_{R^2} \frac{e^{-\varepsilon u^2}}{(1+|u|)^2} d^2u = O(\log(1/\varepsilon)) \tag{4.4}$$

Proof: Replace u with $u/\sqrt{\varepsilon}$ and the integral becomes

$$\int_{R^2} \frac{e^{-u^2}}{(\sqrt{\varepsilon}+u)^2} d^2u \tag{4.5}$$

The integral over $|u| > 1$ is clearly $O(1)$, and for $|u| < 1$ it suffices to bound

$$\int_{|u|<1} \frac{1}{(\sqrt{\varepsilon}+u)^2} d^2u \tag{4.6}$$

Converting to polar coordinates this is

$$2\pi \int_0^1 \frac{r}{(\sqrt{\varepsilon}+r)^2} dr \leq 2\pi \int_0^1 \frac{1}{\sqrt{\varepsilon}+r} dr = O(\log(1/\varepsilon)) \tag{4.7}$$

□

Lemma 3 *There is a constant $c < \infty$ such that, independently of ε, a and $n \geq 1$, we have*

$$\int_{R^2} \frac{e^{-\varepsilon u^2}}{(1+|u|)^2(1+|a+u|)^n} d^2u < c \tag{4.8}$$

Proof: We can ignore the numerator. By Hölder's inequality the integral is bounded by

$$\Big(\int \frac{1}{(1+|u|)^3} d^2u\Big)^{2/3}\Big(\int \frac{1}{(1+|u+a|)^{3n}} d^2u\Big)^{1/3} < c < \infty \tag{4.9}$$

□

These lemmas motivate the intuition behind our approach, which we first describe informally. The lemmas essentially say that a square in the denominator gives a log, whereas a cube or higher gives a constant. We will write

(4.3) in such a way that we can cancel the $\prod |p_i|$ in the numerator with powers in the denominator. We will use the Cauchy-Schwarz inequality to cut down on the number of different terms in the numerator, and we will change variables by a linear transformation. After all of this we will obtain a product of a collection of integrals of the form in Lemma 2 with at least one integral in the form of Lemma 3. Each one of the type in Lemma 2 contributes a $\log(1/\varepsilon)$, whereas the Lemma 3 type doesn't. When we multiply everything out, the power of $\log(1/\varepsilon)$ will be less than n. (As a side note, Lemma 2 also indicates why we are initially not considering the case of the isolated intervals. In that case there is some p_i which is only present as a term in one u_j, so that if we were to put the absolute value inside the integral as we are doing here, we would have only a square of p_i in the denominator with a $|p_i|$ present in the numerator. This would essentially give us

$$\int \frac{e^{-\varepsilon p_i^2}}{1+|p_i|} d^2 p_i \tag{4.10}$$

And this is only $O(\sqrt{1/\varepsilon})$, which is not good enough.)

To make good on this approach, we will need a way to make sure that, after we cancel the terms in the numerator, we have enough terms left in the denominator to obtain adequate convergence. In terms of the sheer number of powers in the denominator there is no problem. Lemma 2 suggests we need more than $2n$ powers on the bottom, but there are n powers on top versus $4n-2$ on the bottom, for a total of $3n-2$ on the bottom. This is enough since $n \geq 3$. The tricky part is making sure that we have a proper assortment on the bottom. The details are rather involved, so we first will prove several technical lemmas. To state the first lemma, we need another bit of terminology: We will say that p_i is *t-free* if there is no t_k contained in (s_i, t_i). Note that if p_i is t-free and $[s_i, t_i]$ is not an isolated interval, then at least one s_k is contained in (s_i, t_i). We have the following lemma, which was proved in [5].

Lemma 4 *The span of the decreasing u_j's is equal to the span of the set of all p_i's which are not t-free. Furthermore, suppose that for each t-free p_i we chose $u(p_i)$ to be any one of the increasing u_j's which contains p_i as a term. Then, if we let $D = \{set\ of\ decreasing\ u_j\text{'}s\} \bigcup \{set\ of\ all\ u(p_i)\text{'}s\}$, D spans the entire set $\{p_1, ..., p_n\}$*

Proof: To begin with, suppose that p_i is t-free. Then p_i only appears as a term in increasing u_j's, and so is not in the span of the decreasing u_j's. Conversely, suppose we have a configuration of intervals such that the set of decreasing u_j's does not span the set of all p_i's which are not t-free. Let

p_{i_o} be the non-t-free p with largest s value which is not in the span of the decreasing u_j's. That is, if $s_i > s_{i_o}$ and p_i is not t-free, then p_i is in the span of decreasing u_j's. Now, let u_{j_o} be the u with largest j value which contains p_{i_o}, i.e. such that $u_{j_o} - u_{j_o+1} = p_{i_o}$. Then u_{j_o+1} is decreasing, and we will obtain a contradiction if we can show that u_{j_o} is in the span of decreasing u_j's. If u_{j_o} is decreasing there is nothing to prove, so suppose u_{j_o} is increasing. Let $v > 0$ be chosen as small as possible so that u_{j_o-v} is decreasing. The fact that p_{i_o} is not t-free implies that p_{i_o} appears as a term in u_{j_o-v}. Furthermore, we can write $p_{i_o} = u_{j_o-v} + (u_{j_o-v+1} - u_{j_o-v}) + ... + (u_{j_o} - u_{j_o-1}) - u_{j_o+1}$. Now, u_{j_o-v} and u_{j_o+1} are decreasing, and the terms $(u_{j_o-v+1} - u_{j_o-v}), ..., (u_{j_o} - u_{j_o-1})$ are all equal to p_i's which have the properties that (i) they are not t-free, because they appear as a term in u_{j_o+1}, and (ii) they have larger s values than p_{i_o}. We conclude that they are in the span of the decreasing u_j's, which means that p_{i_o} is as well. This is a contradiction, and establishes the first part of the lemma. To prove the second part, just note that $u(p_i)$ contains p_i as a term as well as several other p_k's which cannot be t-free. These p_k's are in the span of D then, and thus p_i is as well. $\square$

In order to state the next lemma, we must consider (4.3) again. Let $u_{j_1}, ..., u_{j_n}$ be the increasing u's in order. That is, $u_{j_i} - u_{j_i-1} = p_i$. We see that (4.3) is bounded by

$$\int \frac{e^{-\varepsilon \sum p_i^2} \prod(|u_{j_i}| + |u_{j_i - 1}|)}{\prod_{j=1}^{2n-1}(1 + |u_j|)^2} \prod dp_i \tag{4.11}$$

Expand the numerator completely, and break this integral into the sum of many integrals, each of which we do individually. Each of these integrals has a product of $|u|$'s in the numerator, but no u can appear more than twice. This allows us to cancel all of the u's in the numerator with u's in the denominator(Note: The word "canceling", in this context, means replacing $\frac{|u|}{1+|u|}$ with 1). We arrive at the following integral:

$$\int \frac{e^{-\varepsilon \sum p_i^2}}{\prod_{j=1}^{2n-1}(1 + |u_j|)^{m_j}} \prod dp_i \tag{4.12}$$

where $m_j = 0, 1$, or 2, depending on what power of u_j appeared in the numerator. The following lemma relates m_j with the properties of u_j in the configuration of intervals.

Lemma 5 *1. If $u_j \downarrow$ then $m_j, m_{j-1} \geq 1$.*

2. If $u_j \downarrow$ and $m_j = 1$, then $u_{j+1} \uparrow$ and $m_{j+1} \geq 1$.

3. If u_j, $u_{j+1} \downarrow$ then $m_j = 2$.

Proof: Each term in the numerator is of the form $(|u_j| + |u_{j-1}|)$ where $u_j \uparrow$. We see that we can only have $m_j = 0$ if u_j appears in two terms in the numerator, and this can only happen if both u_j and u_{j+1} are increasing. This proves (1). If $u_j \downarrow$ then u_j appears at most once in the numerator in the term $(|u_{j+1}| + |u_j|)$ where u_{j+1} must be increasing. Furthermore, u_{j+1} can appear in at most one other term, and so if $m_j = 1$ then $m_{j+1} \geq 1$. This proves (2). As for (3), if u_j, $u_{j+1} \downarrow$ then u_j does not appear in the numerator at all, so $m_j = 2$. □

We now turn our attention to (4.3). Suppose that we can form sets $A = \{a_1, ..., a_r\}$ and $B = \{b_1, ..., b_s\}$ with the following properties:

i) Each of $a_1, ..., a_r$ and $b_1, ..., b_s$ are equal to some u_j.

ii) A and B each span $\{p_1, ..., p_n\}$.

iii) If $a_i = u_j$ or $b_i = u_j$, then $m_j \geq 1$.

iv) If $a_i = b_k = u_j$, then $m_j = 2$.

Note that if we can find such sets we can, simply by deleting elements if necessary, find two sets A_o and B_o which satisfy the above properties and which each have n elements. So in the calculations which follow we'll assume that $n = s = r$, even though when we eventually construct A and B they may have more than n elements. Given two such sets, we could bound (4.12) by

$$\int \frac{e^{-\varepsilon \sum p_i^2}}{K(a_1, ..., a_n) \prod_{j=1}^n (1+|a_j|)(1+|b_j|)} \prod dp_i \tag{4.13}$$

where K is of the form $(1+|u_j|)$ for some j. Recall that we have $3n-2$ powers of u's in the denominator, so there will always be at least one term left over after choosing our sets A and B. This term will contain a linear combination of p_i's, but since A spans $\{p_1, ..., p_n\}$ we may write it as a linear combination of a_j's. It is irrelevant what the linear combination present in K actually is, except that it must be nonzero, of course. Now, we can apply the Cauchy-Schwarz inequality to bound (4.13) by

$$\begin{aligned}\Big(\int \frac{e^{-\varepsilon \sum p_i^2}}{K(a_1, ..., a_n)^2 \prod_{j=1}^n (1+|a_j|)^2} \prod dp_i\Big)^{1/2} \\ \times\Big(\int \frac{e^{-\varepsilon \sum p_i^2}}{\prod_{j=1}^n (1+|b_j|)^2} \prod dp_i\Big)^{1/2}\end{aligned} \tag{4.14}$$

There is a constant $c > 0$ so that $\sum p_i^2 > c \sum a_i^2$ and $\sum p_i^2 > c \sum b_i^2$; this is because the functions $\sum a_i^2$ and $\sum b_i^2$ are homogeneous of degree 2 in the p_i's and bounded on $\sum p_i^2 = 1$. Thus, (4.14) is bounded by

$$\begin{aligned}\Big(\int \frac{e^{-\varepsilon c \sum a_j^2}}{K(a_1, ..., a_n)^2 \prod_{j=1}^n (1+|a_j|)^2} \prod dp_i\Big)^{1/2} \\ \times\Big(\int \frac{e^{-\varepsilon c \sum b_j^2}}{\prod_{j=1}^n (1+|b_j|)^2} \prod dp_i\Big)^{1/2}\end{aligned} \tag{4.15}$$

Now, we apply a linear change of coordinates to these integrals so that we are integrating with respect to a_j and b_j instead of p_i. Relabel if necessary so that a_1 is one of the a's which appears as a term in $K(a_1, ..., a_n)$. We see that the first integral in (4.15) is bounded by a constant times

$$\int \Big(\int \frac{e^{-c\varepsilon a_1^2}}{K(a_1, ..., a_n)^2 (1+|a_1|)^2} da_1 \Big) \prod_{j=2}^{n} \frac{e^{-c\varepsilon a_j^2}}{(1+|a_j|)^2} da_j \tag{4.16}$$

By Lemma 3 the inner integral is $O(1)$ and by Lemma 2 the others are all $O(\log(1/\varepsilon))$. Lemma 2 also shows that the second integral in (4.15) is $O(\log^n(1/\varepsilon))$. We see that (4.15) is $O((\log(1/\varepsilon))^{n-(1/2)}$, and this shows that (4.1) is $o((\log(1/\varepsilon))^n)$, which is what we set out to prove.

All that remains, then, is to show that we can always find sets A and B of this form. For this, we'll use Lemmas 4 and 5. Lemma 4 gives us a good first initial candidate for A and B. We can let A be equal to the set of (distinct) decreasing u_j's together with elements $u(p_i)$ for each t-free p_i (Recall that all decreasing u_j's have $m_j \geq 1$). The possible problem with this is that every increasing u_i, and in particular each possibility for $u(p_i)$, appears at least once in the numerator of (4.11), so that we need to make sure that we really can appropriately choose the $u(p_i)$'s. Nevertheless, as will be shown below, this works for A. B cannot be chosen the same way, however. This is because if u_j is decreasing but u_{j+1} is increasing, then u_j appears exactly once in the numerator of (4.11) and we may have $m_j = 1$, so that u_j cannot be in both A and B. B will have to be formed in a different manner.

In order to eventually create the set B, we'll begin by creating an increasing collection of sets B_n by considering decreasing u_j's for increasing values of j. Start with the smallest j such that u_j is decreasing. If $m_j = 2$, then let $B_1 = \{u_j\}$. If $m_j = 1$, then we know from Lemma 5 that u_{j+1} is increasing. We can then choose $d > 0$ such that $u_j = u_{j+1} - (u_{j+d} - u_{j+d+1})$; d is simply chosen to be the largest value such that u_{j+d} contains $p_i = u_{j+1} - u_j$ as a term. let $B_1 = \{u_{j+1}, u_{j+d}, u_{j+d+1}\}$. We will essentially repeat this for each decreasing u_j. Suppose that the set B_n has already been formed. Let j be as small as possible so that $u_j \notin span\{B_n\}$ and u_j is decreasing. If $m_j = 2$, then let $B_{n+1} = B_n \bigcup \{u_j\}$. If $m_j = 1$, then let $B_{n+1} = B_n \bigcup \{u_{j+1}, u_{j+d}, u_{j+d+1}\} \backslash \{u_j\}$, where again d is such that $u_{j+d+1} - u_{j+d} = u_{j+1} - u_j$. The reason for subtracting the set $\{u_j\}$ is that it may already be in the set B_n, having having been of the form $u_{j'+d'}$ or $u_{j'+d'+1}$ for an earlier j'. Repeat this process through all of the decreasing u_j's. The final set obtained, say B_N, will span the set of decreasing u_j's. To see this, suppose to the contrary, and let u_{j_o} be the u_j with largest j value which is not in the span of B_N. Clearly then $m_{j_o} = 1$, which means that an earlier B_n must have contained $u_{j_o+1}, u_{j_o+d_o}$, and $u_{j_o+d_o+1}$. Any of these elements which are increasing must be present in B_N, and any decreasing ones have larger j values than u_{j_o}, which means they are in the span of B_N. Thus, $u_{j_o} = u_{j_o+1} - (u_{j_o+d_o} - u_{j_o+d_o+1})$ is also in the span of B_N, a contradiction. The set B_N also satisfies property (iii) above. This will be shown using the following lemma:

Lemma 6 *1. If u_i is in B_N then u_i is either decreasing or else neighbors on a decreasing interval(i.e. at least one of u_{i-1} and u_{i+1} is decreasing).*

2. If u_i, u_{i+1} are both increasing and $u_i \in B_N$ then $u_{i-1} \downarrow$, $m_{i-1} = 1$, and $m_i \geq 1$.

Proof: If u_i is increasing and in BN then u_i must be of the form u_{j+1}, u_{j+d}, or u_{j+d+1} for some j where j and d are as in the construction of the set B' above, i.e. u_j is decreasing, $m_j = 1$, and $u_j = u_{j+1} - (u_{j+d} - u_{j+d+1})$. It is always true that u_{j+d+1} is decreasing, so this cannot be u_i. (1) is proved by noting that u_{j+1}, u_{j+d} are neighbors to the decreasing intervals u_j, u_{j+d+1} respectively. If, in addition, the situation in (2) arises then u_i cannot be of the form u_{j+d} since in that case u_{j+d+1} would be decreasing. Thus, u_i is of the form u_{j+1}. In order for u_{j+1} to be included in B' it was necessary that $u_j \downarrow$ and $m_j = 1$. By part 2 in Lemma 5 $m_i \geq 1$. □

If $u_j \in B_N$ then either $u_j \downarrow$, in which case $m_j \geq 1$ by part 1 of Lemma 5, or $u_j \uparrow$ in which $m_j \geq 1$ by part 1 of Lemma 5 or part 2 of Lemma 6, depending on whether $u_{j+1} \downarrow$ or $\uparrow$. Thus, B_N satisfies (iii) as claimed. Let $B' = B_N$ and A' be the set of all decreasing u_j's. We know from the discussion above that B' satisfies (i) and (iii). A' clearly satisfies (i), and satisfies (iii) by part 1 of Lemma 5. A' and B' together satisfy (iv) because of the way that B' was constructed, and both span the set of all decreasing u_j's. We need now only extend them to sets A and B which span all of $\{p_1, ..., p_n\}$. A' and B' already span the set of all non-t-free p_i's, by the first part of Lemma 4. In light of the second part of Lemma 4, all that remains is to show that, for any t-free p_i, we can always choose $u_1(p_i), u_2(p_i)$ which contain p_i as a term, and which we may include in A and B respectively without violating rules (iii) and (iv).

Suppose p_i is t-free, and k is chosen as large as possible so that $s_i < s_{i+1} < ... < s_{i+k} < t_i$(Note: k here is not the same as d above; they differ by 1). Let $u_j - u_{j-1} = p_i$. The term $p_i p_{i+1} ... p_{i+k}$ in (4.1) becomes $(|u_j| + |u_{j-1}|)...(|u_{j+k}| + |u_{j+k-1}|)$ in (4.11), with $u_j, ..., u_{j+k}$ not appearing anywhere else in the numerator. If $k > 1$ we can just note that, upon expanding this expression, the sum of the powers of u_{j+k} and u_{j+k-1} in the numerator is at most two. This means that $m_{j+k} + m_{j+k-1}$ must be at least 2, and we can choose $u_1(p_i), u_2(p_i)$ as some combination of u_{j+k} and u_{j+k-1}. It is possible that u_{j+k} is already in B', and so we must make sure that if

$u_1(p_i) = u_{j+k} \neq u_2(p_i)$ that we interchange $u_1(p_i)$ and $u_2(p_i)$, so that u_{j+k} is not in both A and B, so as to not violate (iv). Note that if $k > 1$ then $u_{j+k-1} \notin B'$ by Lemma 6, since $u_{j+k-1} \uparrow$ and neighbors only on increasing intervals. In the case that $k = 1$ we still have $m_j + m_{j+1} \geq 2$, but now it is possible that both u_j and u_{j+1} are in B', since both neighbor upon intervals which may be decreasing. However, if this is the case then, since $u_j, u_{j+1} \uparrow$, we have by Lemma 6 $u_{j-1} \downarrow$ and $m_{j-1} = 1$. Recall that we have the term $(|u_j| + |u_{j-1}|)(|u_{j+1}| + |u_j|)$ in the numerator, with u_{j-1}, u_j, u_{j+1} appearing nowhere else in the numerator. The sum of the powers of u_{j-1}, u_j, and u_{j+1} is two, and thus $m_{j-1} + m_j + m_{j+1} = 4$. Since $m_{j-1} = 1$, one of m_j and m_{j+1} is 2. We can then let $u_1(p_i) = u_2(p_i) = u_j$ or u_{j+1}, depending on whether m_j or m_{j+1} is 2. This handles the case $k = 1$. (If $k = 0$ then we would have an isolated interval, and this argument doesn't work. This is the only place where we used the fact that we had no isolated intervals.) Doing this for each t-free p_i we create the sets A and B, which are guaranteed by Lemma 4 to satisfy the property (ii). A and B also satisfy properties (i), (ii), and (iv) by construction, so we have completed the proof in the case where no isolated intervals are present.

Now for the isolated intervals case. Recall that the integral which gives us the contribution from this configuration is

$$\int e^{-\varepsilon\sum p_i^2}\prod(p_i)_1\prod_{j=1}^{2n-1}\Big(\int_{\sum t_j<T}\prod_j e^{-u_j^2 t_j}\prod_j dt_j\Big)\prod dp_i \tag{4.17}$$

As mentioned before, here we cannot replace the integrand with its absolute value, for in that case each isolated interval would contribute a $\sqrt{1/\varepsilon}$ to the integral. Cancelation occurs in the integral, however, since the integrand is positive in some regions and negative in others. It turns out that it is enough to integrate each of the variables corresponding to isolated intervals first, and then to bring the absolute value inside the integral. After we have "removed" the initial set of isolated intervals in this fashion, we will have created a new configuration of intervals, which may again contain isolated intervals. We can remove these isolated intervals by a different method than was used for the first set. This brings us to a new configuration, which may again have isolated intervals, which we again remove, etc. After a finite number of steps we either have removed all intervals or we have arrived at an arrangement with no isolated intervals. In the second case we are reduced to the case we have already done, and the first is handled easily in a slightly different way.

Let us bring in some definitions in order to make this rigorous. Let our initial configuration of intervals be denoted K_0, and let K_m be the configuration of intervals obtained upon removing the isolated intervals from K_{m-1}. We will say $p_i \in K_m$ to mean that the interval (s_i, t_i) appears in the configuration K_m, and we will define the *order* of K_m to be the number of p_i's in K_m. Let $u_{(m,1)}, ..., u_{(m,n_m)}$ be the linear combinations of p_i's which appear in the configuration K_m, ordered from left to right. Let us define I_m to be the set of all j values corresponding to isolated intervals in K_m; that is, $I_m = \{j : u_{(m,j)} = p_i$ where (s_i, t_i) contains no s_k or t_k in $K_m\}$. A $\hat{p}$ will refer to the p associated to an isolated interval. That is, if $j \in I_m$ and p_i is the p which appears only in $u_{(m,j)}$, label p_i as $\hat{p}_{m,j}$. We can bound (4.17) by

$$\int \prod e^{-\varepsilon p_i^2} |p_i| \Big(\int \prod_{j \notin I_0} e^{-u_{(0,j)}^2 t_j} \Big| \prod_{j \in I_0} \int\int e^{-\varepsilon \hat{p}_{(0,j)}^2} (\hat{p}_{(0,j)})_1 e^{-u_{(0,j)}^2 t_j} dt_j d\hat{p}_{(0,j)} \Big| \prod_{j \notin I_0} dt_j \Big) \prod dp_i \tag{4.18}$$

The first and last products over all i such that $p_i \neq \hat{p}_{(0,j)}$ for all j. We will get a good bound on the $dt_j d\hat{p}_j$ integrals. Note that we have suppressed the region of integration in t_j, since it may be quite complicated. We do know that the upper limit of integration is bounded above by T, and this allows us to get a sufficient bound, as the following lemma shows.

Lemma 7 *For any a with $0 < a < T$ and any $k \in \mathbf{R}^2$, we have*

$$\Big| \int \int_0^a e^{-\varepsilon p^2} p_1 e^{-(p+k)^2 t} dt dp \Big| = |k| O(log(1/\varepsilon)) \tag{4.19}$$

independently of a.

Proof:

$$\int\int_0^a e^{-\varepsilon p^2} p_1 e^{-(p+k)^2 t} dt dp \tag{4.20}$$
$$= \int_0^a \Big(\int e^{-\varepsilon p_1^2} p_1 e^{-(p_1+k_1)^2 t} dp_1 \int e^{-\varepsilon p_2^2} e^{-(p_2+k_2)^2 t} dp_2 \Big) dt$$

Now, for $p, k \in R^2$, $(p+k)^2 = p^2 + k^2 + 2p \cdot k$, so this is

$$\begin{aligned} &\int_0^a e^{-k^2 t} \Big(\int e^{-\varepsilon p_1^2} p_1 e^{-(p_1^2+2p_1k_1)t} dp_1 \int e^{-\varepsilon p_1^2} e^{(-p_2^2+2p_2k_2)t} dp_2 \Big) dt \\ &= \int_0^a e^{-k^2 t} e^{\frac{k^2t^2}{\varepsilon+t}} \Big(\int e^{-(\varepsilon+t)(p_1+\frac{k_1 t}{\varepsilon+t})^2} p_1 dp_1 \int e^{-(\varepsilon+t)(p_2+\frac{k_2 t}{\varepsilon+t})^2} dp_2 \Big) dt \\ &= \int_0^a e^{-k^2 t} e^{\frac{k^2t^2}{\varepsilon+t}} \Big(\int e^{-(\varepsilon+t)p_1^2} (p_1 - \frac{k_1 t}{\varepsilon+t}) dp_1 \int e^{-(\varepsilon+t)p_2^2} dp_2 \Big) dt \end{aligned} \tag{4.21}$$

We now split the p_1 integral into two pieces, and we see that the first one,

$$\int e^{-(\varepsilon+t)p_1^2} p_1 dp_1 \tag{4.22}$$

is 0 by symmetry (this is what will give us the extra convergence). We use the fact that, for $d = 1, 2$ we have

$$\int e^{-(\varepsilon+t)p_d^2} dp_d = \frac{c}{\sqrt{\varepsilon+t}} \tag{4.23}$$

for some constant c. We'll also replace $\frac{t}{\varepsilon+t}$ and $e^{-k^2 t} e^{\frac{k^2t^2}{\varepsilon+t}}$ by the trivial bound of 1. This shows us that we can bound (4.21) by

$$c^2|k_1|\int_0^a \frac{1}{\varepsilon+t}dt \tag{4.24}$$

Since $a < T$, this is $|k|O(\log(1/\varepsilon))$, independently of a. □

We integrate the $\hat{p}_{0,j}$'s first, and by the previous lemma each one gives $|u_{0,j-1}|O(\log(1/\varepsilon))$ ($|u_{0,j-1}|$ is the u_j which appears immediately before and after the isolated interval corresponding to $\hat{p}_{0,j}$). (4.18) is thus

$$O(\log(1/\varepsilon))^{|I_0|}\int \prod_{p_i\neq\hat{p}_{(0,j)}\forall j} e^{-\varepsilon p_i^2}|p_i| \prod_{(j+1)\in I_0} |u_{(0,j)}| \Big(\int \prod_{j\notin I_0} e^{-u^2_{(0,j)}t_j} \prod_{j\notin I_0} dt_j\Big) \prod_{p_i\neq\hat{p}_{(0,j)}\forall j} dp_i \tag{4.25}$$

Since the integrand is now positive we can extend the region of integration for the t_i's to be $0 < t_i < T$ and use (4.2) to bound (4.25) by

$$O(\log(1/\varepsilon))^{|I_0|}\int \prod_{p_i\neq\hat{p}_{(0,j)}\forall j} e^{-\varepsilon p_i^2}|p_i| \prod_{(j+1)\in I_0} |u_{(0,j)}| \prod_{j\notin I_0}\frac{1}{1+u^2_{(0,j)}} \prod_{p_i\neq\hat{p}_{(0,j)}\forall j} dp_i \tag{4.26}$$

Suppose that $u_{m,j}$ is an isolated interval in K_m. Then $u_{m,j-1} = u_{m,j+1}$. We will say in this case that $u_{m,j-1}$ contains $u_{m,j}$. If $u_{m,j} = u_{m',j'}$, where $m > m'$, and $u_{m',j'+1}$ is isolated in $K_{m'}$, we will also say that $u_{m,j}$ contains $u_{m',j'+1}$. We will let $l_{m,j}$ denote the total number of isolated intervals which the interval $u_{m,j}$ contained in all $K_{m'}$'s, where $m' < m$. Each $u_{1,j}$ which contained one or more isolated intervals in K_0 will appear to a power $l_{1,j}$ in the numerator of (4.26) as a result of Lemma 7, but the term $(1+u_{1,j}^2)$ will also appear an extra $l_{1,j}$ times in the denominator. We see that (4.26) is

$$O(\log(1/\varepsilon))^{|I_0|}\int e^{-\varepsilon\sum_{p_i\in K_1}p_i^2}\prod_{p_i\in K_1}|p_i|\prod_{1\le j\le n_1}|u_{(1,j)}|^{l_{1,j}}\frac{1}{(1+u_{(1,j)}^{2+2l_{1,j}})}\prod_{p_i\in K_1}dp_i \tag{4.27}$$

We must have some idea how the integral (4.26) can be bounded as we remove successive stages of isolated intervals, and Lemma 9 below gives us that. The following lemma prepares us to prove Lemma 9.

Lemma 8

$$\int e^{-\varepsilon p^2}|p|\frac{1}{(1+|k+p|)^m}dp = (1+|k|)O(1) + O(\log(1/\varepsilon)) \tag{4.28}$$

if $m=3$, *and is* $(1+|k|)O(1)$ *if* $m>3$.

Proof: (4.28) is bounded by

$$\int e^{-\varepsilon(p-k)^2}(|p|+|k|)\frac{1}{(1+|p|)^m}dp \tag{4.29}$$

Divide this into two integrals. The one with $|k|$ in the numerator is bounded by

$$|k|\int \frac{1}{(1+|p|)^m}dp = |k|O(1) \tag{4.30}$$

The other is bounded by

$$c\int e^{-\varepsilon(p-k)^2}\frac{1}{1+|p|^{m-1}}dp \tag{4.31}$$

Again if $m > 3$ this is $O(1)$. If $m = 3$, divide the region into $\{|p| > 2|k|\}$ and $\{|p| < 2|k|\}$. On $\{|p| > 2|k|\}$ we can bound the integral by

$$\int e^{\varepsilon p^2/2}\frac{1}{1+p^2} = O(log(1/\varepsilon)) \tag{4.32}$$

by Lemma 2. On $\{|p| < 2|k|\}$ we can bound it by

$$\int_{|p|<2|k|}\frac{1}{1+p^2}dp \leq \log(|k|+1) \leq |k| \tag{4.33}$$

These bounds combine to prove the lemma. □

Lemma 9 *Suppose that K_m contains isolated intervals. Then (4.17) is*

$$O(\log(1/\varepsilon))^{|I_0|+\ldots+|I_m|}\int e^{-\varepsilon\sum_{p_i\in K_{m+1}}p_i^2}\prod_{p_i\in K_{m+1}}|p_i| \tag{4.34}$$

$$\prod_{1\leq j\leq n_{m+1}} (1+|u_{(m+1,j)}|)^{l_{m+1,j}} \frac{1}{(1+u_{(m+1,j)}^{2+2l_{m+1,j}})} \prod_{p_i\in K_{m+1}} dp_i$$

Proof: By induction. We know that it is true for $m = 0$(see(4.27)). Assume that it is true for $m-1$, so (4.17) is

$$O(\log(1/\varepsilon))^{|I_0|+\ldots+|I_{m-1}|} \int e^{-\varepsilon\sum_{p_i\in K_m} p_i^2} \prod_{p_i\in K_m} |p_i| \prod_{1\leq j\leq n_m} (1+|u_{(m,j)}|)^{l_{m,j}} \frac{1}{(1+u_{(m,j)}^{2+2l_{m,j}})} \prod_{p_i\in K_m} dp_i \tag{4.35}$$

We will integrate the variables in K_m corresponding to isolated intervals. We can rewrite the integral in (4.35) as

$$\int e^{-\varepsilon\sum_{p_i\in K_m, i\notin \hat{I}_m} p_i^2} \prod_{p_i\in K_m, i\notin \hat{I}_m} |p_i| \prod_{j\notin I_m} \frac{(1+|u_{(m,j)}|)^{l_{m,j}}}{(1+u_{(m,j)}^{2+2l_{m,j}})} \Big(\prod_{j\in I_m} \int |\hat{p}_{m,j}| e^{-\varepsilon \hat{p}_{m,j}} \frac{(1+|u_{(m,j)}|)^{l_{m,j}}}{(1+u_{(m,j)}^{2+2l_{m,j}})} d\hat{p}_{m,j}\Big) \prod_{p_i\in K_m i\notin \hat{I}_m} dp_i \tag{4.36}$$

It is simply to verify that

$$\frac{(1+|u_{(m,j)}|)^{l_{m,j}}}{(1+u_{(m,j)}^{2+2l_{m,j}})} \leq K \frac{1}{(1+u_{(m,j)}^{2+l_{m,j}})} \tag{4.37}$$

For some constant K depending on $l_{m+1,j}$. Each $\hat{p}_{m,j}$ integral is

$$(1+|u_{m,j}-\hat{p}_{m,j}|)O(\log(1/\varepsilon)) \tag{4.38}$$

by Lemma 8. Plugging this into (4.36) and relabeling the u's with index $m+1$ instead of m gives (4.34). $\square$

To complete the proof of Proposition 2, let us consider several cases. Recall that *order* refers to how many intervals $[s_i, t_i]$ make up a configuration.

Case 1: There is a K_m of order greater than or equal to 3 which contains no isolated intervals.

In this case our integral in (4.34) is almost the same as what would have been obtained if we had started with the configuration K_m. The only difference is the presence of the extra powers $l_{m,j}$, which in fact cause greater convergence. Thus, by what we did earlier in this section, the remaining integral is $o(\log(1/\varepsilon))^{|K_m|}$. Since

$$|I_0| + ... + |I_{m-1}| + |K_m| = n \tag{4.39}$$

we see that (4.17) is $o(\log(1/\varepsilon))^n$, which is what we set out to prove.

Case 2: There is a K_m of order 2 with no isolated intervals.

As before we get $O(\log(1/\varepsilon))^{|I_0|+...+|I_{m-1}|}$ times an integral nearly identical to what we would have had if starting with K_m. Again there will be extra factors which aid convergence. The integral in question can be bounded by one of the following integrals:

$$
\begin{aligned}
(4.40)\qquad & \int\int \frac{1}{(1+|p|)^2(1+|q|)^2(1+|p+q|)^3} e^{-\varepsilon(p^2+q^2)}|p||q|dpdq \\
& \int\int \frac{1}{(1+|p|)^2(1+|q|)^3(1+|p+q|)^2} e^{-\varepsilon(p^2+q^2)}|p||q|dpdq \\
& \int\int \frac{1}{(1+|p|)^3(1+|q|)^2(1+|p+q|)^2} e^{-\varepsilon(p^2+q^2)}|p||q|dpdq
\end{aligned}
$$

And therefore the following lemma completes the proof in this case.

Lemma 10 *Each of the integrals in (4.40) is* $o(\log(1/\varepsilon))^2$

Proof: This is fairly straightforward to calculate using Lemmas 2 and 8. For example, by the Cauchy-Schwarz inequality and symmetry we can bound the first integral by

$$
(4.41)\qquad k\int\int \frac{1}{(1+|p|)^4(1+|p+q|)^3} e^{-\varepsilon(p^2+q^2)}|p||q|dpdq
$$

The dq integral is $(1+|p|)O(1)+O(\log(1/\varepsilon))$ by Lemma 8, and thus 4.41 is

$$
(4.42) kO(1)\int\int \frac{1}{(1+|p|)^2} e^{-\varepsilon p^2} dp + O(\log(1/\varepsilon))\int\int \frac{1}{(1+|p|)^3} e^{-\varepsilon p^2} dp
$$

which is $O(\log(1/\varepsilon))$ by Lemma 2 and the fact that $\frac{1}{(1+|p|)^3} \in L^1$.

The second and third integrals are identical with p and q interchanged, so we need only do one, let us say the second one. This is bounded by

$$k\int\int\frac{1}{(1+|p|)(1+|q|)^2(1+|p+q|)^2}e^{-\varepsilon(p^2+q^2)}dpdq \tag{4.43}$$

By Cauchy-Schwarz, this is bounded by

$$\Big(\int\int\frac{1}{(1+|p|)^2(1+|q|)^2}e^{-\varepsilon(p^2+q^2)}dpdq\Big)^{1/2}\Big(\int\int\frac{1}{(1+|q|)^2(1+|p+q|)^4}e^{-\varepsilon(p^2+q^2)}dpdq\Big)^{1/2} \tag{4.44}$$

This first integral is $O(\log(1/\varepsilon))^2$ by Lemma 2, and the second one is $O(\log(1/\varepsilon))$, using Lemma 2 in conjunction with the fact that

$$\int\frac{1}{(1+|p+q|)^4}dp = O(1) \tag{4.45}$$

As a simple alternate proof, one can recall our proof for the case with no isolated intervals where we constructed the sets A and B. Here it is simple to verify in each case that we can form two sets with the same properties. The lemma is then proved by the reasoning in steps (4.13) through (4.16) above. □

Case 3: There is a K_m consisting of just one interval.

Here we must examine in closer detail the proof of Lemma 9. First of all, if there was ever an isolated interval in some $K_{m'}$ which contained two or more isolated intervals in $K_{m'-1}$, then the variable corresponding to that interval, say $u_{(m',j)}$, would have had $l_{m',j} \geq 2$. In that case, by Lemma 8, the contribution to (4.36) of the $\hat{p}_{m',j}$ integral is $O(1)(1 + |u_{(m',j)} - \hat{p}_{m',j})|$. We see that we can replace the term $O(\log(1/\varepsilon))^{|I_0|+...+|I_{m-1}|}$ in (4.34) with $o(\log(1/\varepsilon))^{|I_0|+...+|I_{m-1}|}$, which will finish the proof. Thus we need only consider the case where $s_1 < s_2 < ... < s_n < t_n < ... < t_2 < t_1$. In this case, consider what happens as we remove the first three intervals (recall that we are assuming that there are at least three intervals). After removing (s_n, t_n) and then (s_{n-1}, t_{n-1}) we have

$$O(\log(1/\varepsilon)) \int e^{-\varepsilon \sum_{1\leq i\leq n-2} p_i^2} \prod_{1\leq i\leq n-2} |p_i| \\ (1 + |u_{(2,n-2)}| + O(\log(1/\varepsilon))) \prod_{1\leq j\leq n_2} \frac{1}{(1 + u_{(2,j)}^2)} \prod_{1\leq i\leq n-2} dp_i \tag{4.46}$$

Note that $u_{2,n-2} = p_1 + ... + p_{n-2}$. We can expand this into two integrals, namely

$$O(\log(1/\varepsilon)) \int e^{-\varepsilon \sum_{1\leq i\leq n-2} p_i^2} \prod_{1\leq i\leq n-2} |p_i| \\ (1 + |u_{(2,n-2)}|) \prod_{1\leq j\leq n_2} \frac{1}{(1 + u_{(2,j)}^2)} \prod_{1\leq i\leq n-2} dp_i \tag{4.47}$$

and

$$O(\log(1/\varepsilon))^2 \int e^{-\varepsilon \sum_{1\le i\le n-2} p_i^2} \prod_{1\le i\le n-2} |p_i| \prod_{1\le j\le n_2} \frac{1}{(1+u_{(2,j)}^2)} \prod_{1\le i\le n-2} dp_i \tag{4.48}$$

The integral in (4.47) is $O(\log(1/\varepsilon))^{n-2}$ by the same technique as was used to prove Lemma 9. Thus, (4.47) is $O(\log(1/\varepsilon))^{n-1}$. As for (4.48), when we remove the next interval, (s_{n-2}, t_{n-2}), we have no powers of $|u_{2,n-2}|$ in the numerator, and by Lemma 8 we do not pick up an $O(\log(1/\varepsilon))$ term. Thus, (4.48) is $o(\log(1/\varepsilon))^n$ as well. This completes the proof of Proposition 2.

5 Completing the proof

All that remains is to prove that the processes $\alpha'_\varepsilon(T)$ are tight and that the limit process has independent increments. Both are essentially corollaries of the following lemma:

Lemma 11 *If $b \leq c$, then $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon([a,b]\times[c,d]) \longrightarrow 0$ in L^n, for any $n \geq 3$.*

Proof: To compute $E[\alpha'_\varepsilon([a,b]\times[c,d])]^n$, we multiply the integrals together as before (see (2.1)). Now, however, we have $s_i \leq b \leq c \leq t_i$ for all i, and it follows from this that the only configurations of intervals that can appear here are ones containing just one component of order n. We have shown that these components contribute $o(\log(1/\varepsilon))$ to the the expectation, and this is enough to prove the lemma $\square$

Now that we have this lemma, we can show that the processes $\alpha'_\varepsilon(T)(\log(1/\varepsilon))^{-1}$ are tight. We will show that

$$E[(\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon(T)-\alpha'_\varepsilon(S))]^{2n} \leq k(T-S)^n \tag{5.1}$$

where k depends on $n \geq 2$ but can be chosen independently of ε, S, and T, provided they are sufficiently small. This will prove tightness by, for example, Theorem 12.3 in [1]. We can rewrite the left side of (5.1) as

$$E[(\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon([0,S]\times[S,T])+\alpha'_\varepsilon(D_T\bigcap\{s,t\geq S\}))]^{2n} \tag{5.2}$$

We know by the lemma that $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon([0,S]\times[S,T])\longrightarrow 0$ in L^{2n}, so that (5.2) is bounded by

$$kE[(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(D_T\bigcap\{s,t\geq S\})]^{2n} \tag{5.3}$$

Suppressing the *log* for the time being, this is given by

$$\frac{(-1)^n}{(2\pi)^{4n}}\iint_{D_T^{2n}\bigcap\{s,t\geq S\}} e^{-\varepsilon\sum_j p_j^2}\prod_{j=1}^n p_{j,1}E[\prod_{j=1}^n e^{ip_j(X_{t_j}-X_{s_j})}]\prod_{j=1}^n ds_j dt_j d^2p_j \tag{5.4}$$

If we rewrite $X_{t_j}-X_{s_j}$ as $(X_{t_j}-X_S)-(X_{s_j}-X_S)$, and let $\beta_t=X_{S+t}-X_S$ be a new Brownian motion this is

$$\frac{(-1)^n}{(2\pi)^{4n}}\iint_{D(T-S)^{2n}} e^{-\varepsilon\sum_j p_j^2}\prod_{j=1}^n p_{j,1}E[\prod_{j=1}^n e^{ip_j(\beta_{t_j}-\beta_{s_j})}]\prod_{j=1}^n ds_j dt_j d^2p_j \tag{5.5}$$

which is equal to(reinserting the *log*)

$$kE[(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T-S)]^{2n} \tag{5.6}$$

And this is $O(1)|T-S|^n$, as we showed earlier. This establishes tightness. We can write

$$(5.7) \qquad (\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon(T)-\alpha'_\varepsilon(S))$$

$$= (\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon([0,S]\times[S,T])+\alpha'_\varepsilon(D_T \bigcap\{s,t\geq S\}))$$

Since $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon([0,S]\times[S,T]) \longrightarrow 0$ and $\alpha'_\varepsilon(D_T \bigcap\{s,t\geq S\}) \in \bigcup\{\sigma(X_t - X_s) : S\leq s,t\leq T\}$, we see that $(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T)$ has asymptotically independent increments. This shows that the limit process, W_T, has independent increments, and completes the proof of Theorem 1.

6 Symmetric stable processes

We will now prove Theorem 5. The proof of this theorem is, naturally, very similar to the proof in the Brownian motion case, so we will in many cases just refer to steps undertaken in the previous proof. In particular, the general outline (Sections 2 and 5) is identical in both cases; the only difference lies in some of the calculations.

The main difficulty is in showing that the integrals corresponding to components of order two converge. Proceeding as in section 3, the first integral is

$$\begin{aligned} &\int\!\!\int \frac{(1-e^{-p^\beta})}{p^\beta}\frac{(1-e^{-q^\beta})}{q^\beta}\frac{(1-e^{-(p+q)^\beta})}{(p+q)^\beta}e^{-\varepsilon(p^\beta+q^\beta)}p_1q_1dpdq \qquad (6.1)\\ &= \varepsilon^{3-6/\beta}\int\!\!\int \frac{(1-e^{-p^\beta/\varepsilon})}{p^\beta}\frac{(1-e^{-q^\beta/\varepsilon})}{q^\beta}\frac{(1-e^{-(p+q)^\beta/\varepsilon})}{(p+q)^\beta}e^{-(p^\beta+q^\beta)}p_1q_1dpdq \end{aligned}$$

In order to prove that this integral converges as $\varepsilon \longrightarrow 0$, it is enough to show that

$$\int\!\!\int \frac{1}{p^{\beta-1}}\frac{1}{q^{\beta-1}}\frac{1}{(p+q)^\beta}e^{-(p^\beta+q^\beta)}dpdq \qquad (6.2)$$

converges, and then to apply the dominated convergence theorem. We need only consider the integral over $\{|p|,|q| < 1\}$, for in order to evaluate the integral over, say, $A = \{|p| > 1\}$ we may divide A into the disjoint union of $B = \{|q| < 1/2\} \bigcap A$, $C = \{|p+q| < 1/2\} \bigcap A$, and $D = A - (B \bigcup C)$. The integrals over B and C are both bounded because $\beta < 2$ i.e. we have only integrable singularities. And the integral over D is bounded by a constant times

$$\int \int e^{-(p^\beta + q^\beta)} dpdq < \infty \tag{6.3}$$

So we must consider the integral

$$\int \int_{\{|p|,|q|<1\}} \frac{1}{p^{\beta-1}} \frac{1}{q^{\beta-1}} \frac{1}{(p+q)^\beta} dpdq \tag{6.4}$$

We manipulate the integral as follows:

$$\begin{aligned} &\int_{|p|<1} \frac{1}{p^{\beta-1}} \int_{\{|q|<1\}} \frac{1}{q^{\beta-1}} \frac{1}{(p+q)^\beta} dqdp \\ = &\int_{|p|<1} \frac{1}{p^{3\beta-2}} \int_{\{|q|<1\}} \frac{1}{(q/|p|)^{\beta-1}} \frac{1}{(p/|p| + q/|p|)^\beta} dqdp \end{aligned} \tag{6.5}$$

The argument of p (thought of as a complex number) is irrelevant, so we may replace $p/|p|$ by 1, and substitute $q' = q/|p|$ to get

$$\int_{|p|<1} \frac{1}{p^{3\beta-4}} \Big(\int_{\{|q|<1/|p|\}} \frac{1}{q^{\beta-1}} \frac{1}{(1+q)^\beta} dq \Big) dp \tag{6.6}$$

If $\beta > 3/2$ then the dq integral is bounded independently of $|p|$ (since then $\frac{1}{q^{\beta-1}}\frac{1}{(1+q)^{\beta}} \in L^1$), so that (6.4) is bounded by a constant times

$$\int_{\{|p|<1\}} \frac{1}{p^{3\beta-4}} dp \tag{6.7}$$

which is finite, as $\beta < 2$. If $\beta < 3/2$ (resp. $\beta = 3/2$), then the dq integral in (6.6) is $O(|p|)^{2\beta-3}$ (resp. $O(|\log|p||)$), so that (6.4) is bounded by a constant times

$$\int_{\{|p|<1\}} \frac{1}{p^{\beta-1}} dp \tag{6.8}$$

when $\beta < 3/2$ and

$$\int_{\{|p|<1\}} \frac{|\log(|p|)|}{p^{1/2}} dp \tag{6.9}$$

when $\beta = 3/2$. These integrals are both finite.

The second configuration of intervals gives rise to the following:

$$\int\int \frac{(1-e^{-p^{\beta}/\varepsilon})^2}{p^{2\beta}} \frac{(1-e^{-(p+q)^{\beta}/\varepsilon})}{(p+q)^{\beta}} e^{-(p^{\beta}+q^{\beta})} p_1 q_1 dp dq \tag{6.10}$$

This integral is more difficult as for some β the integrand is not in L^1 were we to remove the terms involving ε (there is a non-integrable singularity at $p = 0$

when $\beta \geq 3/2$). We will first show that (6.10) is bounded independently of ε. We isolate the dq integral:

$$\int \frac{(1-e^{-(p+q)^\beta/\varepsilon})}{(p+q)^\beta} e^{-q^\beta} q_1 dq \tag{6.11}$$

We will show that this is $|p|O(1)$ (the O here refers to ε). Because we will refer to this result later, we isolate it as a lemma (which we state in slightly greater generality).

Lemma 12 *For any a with $0 < a < T$ and any $p \in \mathbf{R}^2$, we have*

$$\Big|\int \frac{1-e^{-(p+q)^\beta a/\varepsilon}}{(p+q)^\beta} e^{q^\beta} q_1 dq\Big| = |p|O(1) \tag{6.12}$$

independently of a.

Proof: We can drop the $e^{-(p+q)^\beta a/\varepsilon}$ term. (6.12) is bounded by

$$\Big|\int \frac{1}{q^\beta} e^{-(q-p)^\beta}(q_1 - p_1) dq\Big| \tag{6.13}$$

Expand the $(q_1 - p_1)$ term. The second term is bounded by

$$|p| \int \frac{1}{q^\beta} e^{-(q-p)^\beta} dq \tag{6.14}$$

The integrand is bounded by the function

$$\frac{1}{q^\beta}1_{\{|q|<1\}} + e^{-(q-p)^\beta}1_{\{|q|\geq 1\}} \tag{6.15}$$

which is bounded in L^1 independently of p. Thus, (6.14) is $|p|O(1)$. To bound the first term we subtract

$$\int \frac{1}{q^\beta} e^{-q^\beta} q_1 dq \tag{6.16}$$

which is 0 by symmetry. This gives us

$$\begin{aligned} &\Big|\int \frac{1}{q^\beta}(e^{-(q-p)^\beta} - e^{-q^\beta})q_1 dq\Big| \\ &\leq \int \frac{1}{q^\beta}|e^{-(q-p)^\beta} - e^{-q^\beta}||q|dq \end{aligned} \tag{6.17}$$

We split this up into the integral over the region $\{|q| < 2|p|\}$ and $\{|q| \geq 2|p|\}$. The integral over the first region is bounded by

$$\begin{aligned} &k\int_{\{|q|<2|p|\}} |q|^{1-\beta} dq \\ &= k\int_0^{2|p|} r^{2-\beta} dr \\ &= k|p|^{3-\beta} \end{aligned} \tag{6.18}$$

Here k is a constant which may change from line to line. This is $(|p| + |p|^2)O(1)$. On the region $\{|q| \geq 2|p|\}$ suppose first that $e^{-(q-p)^\beta} \geq e^{-q^\beta}$.

Then

$$\begin{aligned}
(6.19) \qquad & |e^{-(q-p)^\beta} - e^{-q^\beta}| \\
& \leq e^{-(|q|-|p|)^\beta} - e^{-q^\beta} \\
& = \beta \int_{|q|-|p|}^{|q|} x^{\beta-1} e^{-x^\beta} dx \\
& \leq k|p||q|^{\beta-1} e^{-(|q|-|p|)^\beta}
\end{aligned}$$

The last inequality is the length of the interval being integrated over multiplied by a term which bounds the integrand. Plugging this into (6.17) gives a bound of

$$(6.20) \qquad k|p| \int_{|q| \geq 2|p|} e^{-(|q|-|p|)^\beta} dq \leq k|p| \int_{|q| \geq 2|p|} e^{-(|q|/2)^\beta} dq = O(|p|)$$

In the case $e^{-(q-p)^\beta} < e^{-q^\beta}$ we have

$$\begin{aligned}
(6.21) \qquad & |e^{-(q-p)^\beta} - e^{-q^\beta}| \\
& \leq e^{-|q|^\beta} - e^{-(|q|+|p|)^\beta} \\
& = \beta \int_{|q|}^{|q|+|p|} x^{\beta-1} e^{-x^\beta} dx \\
& \leq k|p|(|q|+|p|)^{\beta-1} e^{-|q|^\beta}
\end{aligned}$$

Since $|q| \geq 2|p|$ this is $k|p||q|^{\beta-1}e^{-|q|^\beta}$ Thus, the contribution to (6.17) of this region is bounded by

$$k|p|\int e^{-|q|^\beta}dq = |p|O(1) \tag{6.22}$$

This shows that

$$\int \frac{1}{q^\beta}e^{-(q-p)^\beta}q_1 dq = (|p|+|p|^2)O(1) \tag{6.23}$$

It is also $O(1)$, however, since the integrand is bounded by

$$\frac{1}{q^{\beta-1}}1_{\{|q|<1\}} + e^{-(q-p)^\beta} \tag{6.24}$$

which is bounded in L^1 independently of p. So (6.23) is $|p|O(1)$ for p small, and $O(1)$ for p large. We conclude that (6.23) is $|p|O(1)$ for all p. □

This lemma allows us to see that (6.10) is bounded by

$$k\int \frac{(1-e^{-p^\beta/\varepsilon})^2}{p^{2\beta}}e^{-p^\beta}|p|^2 dp \tag{6.25}$$

The extra powers of p in the numerator are enough to convert our singularity at 0 into an integrable one, and it follows that (6.25) is bounded by

$$k\int \frac{1}{p^{2\beta}}e^{-p^\beta}|p|^2 dp < \infty \tag{6.26}$$

We have showed that (6.10) is bounded independently of ε. This alone does not show that (6.10) converges. However, convergence is proved using the same ideas, as follows. Let the value of (6.10) be denoted by $A(\varepsilon)$. We will show that, for any $\delta > 0$, there is an $\varepsilon' > 0$ such that if $0 < \varepsilon_1, \varepsilon_2 < \varepsilon'$ then $|A(\varepsilon_1) - A(\varepsilon_2)| < \delta$. This will prove convergence. We will assume below that $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon'$. We have

$$\begin{aligned}(6.27) \qquad & A(\varepsilon_1) - A(\varepsilon_2) \\ = \int\int \Big(& \frac{(1-e^{-p^\beta/\varepsilon_1})^2}{p^{2\beta}} \frac{(1-e^{-(p+q)^\beta/\varepsilon_1})}{(p+q)^\beta} - \frac{(1-e^{-p^\beta/\varepsilon_2})^2}{p^{2\beta}} \frac{(1-e^{-(p+q)^\beta/\varepsilon_2})}{(p+q)^\beta}\Big) \\ & e^{-(p^\beta+q^\beta)} p_1 q_1 \, dp dq\end{aligned}$$

We will rewrite the difference

$$(6.28) \quad (1-e^{-p^\beta/\varepsilon_1})^2(1-e^{-(p+q)^\beta/\varepsilon_1}) - (1-e^{-p^\beta/\varepsilon_2})^2(1-e^{-(p+q)^\beta/\varepsilon_2})$$

as

$$\begin{aligned}(6.29) \qquad & (1-e^{-p^\beta/\varepsilon_1})^2[(1-e^{-(p+q)^\beta/\varepsilon_1}) - (1-e^{-(p+q)^\beta/\varepsilon_2})] \\ & +[(1-e^{-p^\beta/\varepsilon_1})^2 - (1-e^{-p^\beta/\varepsilon_2})^2](1-e^{-(p+q)^\beta/\varepsilon_2})\end{aligned}$$

and handle each term in this sum separately. The first one gives rise to the integral

$$\int\int \frac{(1-e^{-p^\beta/\varepsilon_1})^2}{p^{2\beta}} \frac{(e^{-(p+q)^\beta/\varepsilon_2} - e^{-(p+q)^\beta/\varepsilon_1})}{(p+q)^\beta} e^{-(p^\beta+q^\beta)} p_1 q_1 \, dp dq \tag{6.30}$$

As in step (6.13) the dq integral is

$$\int \frac{e^{-q^\beta/\varepsilon_2}(1-e^{-q^\beta/\varepsilon_3})}{q^\beta} e^{-(q-p)^\beta}(q_1-p_1)\,dq \tag{6.31}$$

where $\varepsilon_3 = \frac{\varepsilon_1\varepsilon_2}{\varepsilon_2-\varepsilon_1} > 0$ This is in turn bounded by

$$\int e^{-q^\beta/\varepsilon'} \frac{1-e^{-q^\beta/\varepsilon_3}}{q^\beta} e^{-(q-p)^\beta}(q_1-p_1)\,dq \tag{6.32}$$

We may now follow steps (6.13) through (6.26), and it is straightforward to verify in each case that the extra $e^{-q^\beta/\varepsilon'}$ term allows us to replace the $O(1)$ by $o(1)$ (the o now refers to ε'). This implies that (6.30) can be made arbitrarily small by choosing ε' sufficiently small. As for the second integral

$$\iint \frac{((1-e^{-p^\beta/\varepsilon_1})^2 - (1-e^{-p^\beta/\varepsilon_2})^2)}{p^{2\beta}} e^{-(p^\beta+q^\beta)} p_1 q_1 \, dp dq \tag{6.33}$$

We can rewrite $((1-e^{-p^\beta/\varepsilon_1})^2 - (1-e^{-p^\beta/\varepsilon_2})^2)$ as

$$((1-e^{-p^\beta/\varepsilon_1}) + (1-e^{-p^\beta/\varepsilon_2}))e^{-p^\beta/\varepsilon_2}(1-e^{-p^\beta/\varepsilon_3}) \tag{6.34}$$

and we see that we can bound (6.33) by

$$k\int \frac{e^{-p^\beta/\varepsilon'}|p|e^{-p^\beta}}{p^{2\beta}}\Big|\int \frac{(1-e^{-(p+q)^\beta/\varepsilon_2})}{(p+q)^\beta}e^{-q^\beta}q_1dq\Big|dp \tag{6.35}$$

We have shown above that the dq integral is $(|p|+|p|^2)O(1)$, and that this implies that the entire integral converges. Furthermore, as $\varepsilon' \longrightarrow 0$, the dominated convergence theorem implies that the value of the integral approaches zero. Again we see that if we choose ε' sufficiently small we can make (6.33) arbitrarily small. This shows that if ε_n is a sequence converging to zero then $A(\varepsilon_n)$ converges. Thus, $\lim_{\varepsilon\longrightarrow 0} A(\varepsilon)$ exists, and we define

$$\int\int \frac{1}{p^{2\beta}}\frac{1}{(p+q)^\beta}e^{-(p^\beta+q^\beta)}p_1q_1dpdq \tag{6.36}$$

to be this limit. This completes the calculation for components of order 2.

For a component of order $n \geq 3$ we have the following integral:

$$\int e^{-\varepsilon\sum p_i^\beta}\prod_i (p_i)_1\Big(\int_{\sum c_j<T}\prod_j e^{-u_j^\beta c_j}\prod_j dc_j\Big)\prod_i dp_i \tag{6.37}$$

We must show that this is $o(\varepsilon^{-(3n/\beta-3n/2)})$. This would be a bit of a chore were it not that we have done almost all of the work already in the Brownian motion case. For instance, suppose we have a configuration with no isolated intervals. Then (6.37) can be bounded by(see (4.3))

$$\int \frac{e^{-\varepsilon \sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1}(1+|u_j|)^\beta} \prod dp_i \tag{6.38}$$
$$= \varepsilon^{-(3n/\beta-2n-1)} \int \frac{e^{-\sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1}(\varepsilon+|u_j|)^\beta} \prod dp_i$$

We are done if we can bound this integral effectively. We know from earlier work that if β were replaced by 2 in this integral then it would be $O(\log(1/\varepsilon))^n$, which is certainly good enough. We can bound as follows using Holder's inequality:

$$\int \frac{e^{-\sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1}(\varepsilon+|u_j|)^\beta} \prod dp_i \tag{6.39}$$
$$\leq \Big(\int \frac{e^{-\sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1}(\varepsilon+|u_j|)^2} \prod dp_i\Big)^{\beta/2} \Big(\int e^{-\sum p_i^\beta} \prod |p_i| \prod dp_i\Big)^{(2-\beta)/2}$$

A quick examination of the proofs of Lemmas 2, 3, 8, and 10 will show that the conclusions of these lemmas remain valid if any e^{-p^2}'s in the hypotheses are replaced by e^{-p^β}. We can conclude that (6.39) is $O(\log(1/\varepsilon))^n$, and this component is therefore sufficiently bounded. We do the same thing in the isolated interval case, with Lemma 7 replaced by Lemma 12. This completes the proof of Theorem 5.

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