CONDITIONS FOR ENTANGLEMENT IN SPIN SYSTEMS AND FOR MULTIPARTITE ENTANGLEMENT

BY

HONGJUN ZHENG

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, the City University of New York.

2012

This manuscript has been read and accepted for the

Graduate Faculty in Physics in satisfaction of the

Dissertation requirement for the degree of Doctor of Philosophy

Professor Mark Hillery

______________________ ______________________________

Date Chair of Examining Committee

Professor Igor. L. Kuskovsky

______________________ ______________________________

Date Executive Officer

The City University of New York

Abstract

CONDITIONS FOR ENTANGLEMENT IN SPIN SYSTEMS

AND FORMULTIPARTITE ENTANGLEMENT

by

HONGJUN ZHENG

This dissertation reports a series of studies of conditions for entanglement in spin systems and multipartite entanglement.

There have been numerous studies of entanglement in spin systems. These have usually focused on examining the entanglement between individual spins or determining whether the state of the system is completely separable. Here we present conditions that allow us to determine whether blocks of spins are entangled. We show that sometime these conditions can detect entanglement better than conditions involving individual spins. We apply these conditions to study entanglement in spin wave states, both when there are only a few magnons present and also at finite temperature.

We introduce two entanglement conditions that take the form of inequalities involving expectation values of operators. These conditions are sufficient conditions for entanglement that is if they are satisfied the state is entangled, but if they are not, one can say nothing about the entanglement of the state. These conditions are quite flexible, because the operators in them are not specified, and they are particularly useful in detecting multipartite entanglement.

We explore the range of utility of these conditions by considering a number

of examples of entangled states, and seeing under what conditions entanglement in them can be detected by the inequalities presented here. We explore the possibility of using quantum walks on graphs to find extra edges on a graph. We focus our attention on star graph, whose edges are like spokes coming out of a central hub. If there are N spokes, we show that a quantum walk can find an extra edge connecting two of the spokes or a spoke with a loop on it in $O(\sqrt{N})$ steps.

ACKNOWLEDGEMENTS

Though only my name appears on the cover of this dissertation, a great many people have contributed to its production. I owe my gratitude to all those people who have made this dissertation possible and because of whom my graduate experience has been one that I will cherish forever.

My deepest gratitude is to my advisor, Prof. Mark Hillery. I have been amazingly fortunate to have such a great advisor who gave me the freedom to explore on my own and at the same time the guidance to recover when my steps faltered. He taught me how to question thoughts and express ideas. His patience and support helped me overcome many crisis situations and finish this dissertation. I hope that one day I would become as good an advisor to my students as Mark has been to me.

I am grateful to Dr. Ho Trung Dung. As a good friend, he was always willing to help and give his best suggestions. I am also thankful to him for his brainstorming with me when I failed coming up with ideas. It had become a lonely lab after he went back to Vietnam.

I would like to thank my committee members Prof. Janos A. Bergou, Prof. Yuhang Ren, Prof. Neepa Maitra and Prof. Gerry Christopher for their guidance over the years. Thanks also go out to all the faculty and members in the Department of Astronomy and Physics at Hunter College for their support and help in the past five years.

Many friends have helped me stay sane through these difficult years. Their support and care helped me overcome setbacks and stay focused on my graduate study. I greatly value their friendship and I deeply appreciate their belief in me.

Most importantly, none of this would have been possible without the love and patience of my parents. My parents, to whom this dissertation is dedicated, have been a constant source of love, concern, support and strength all these years. Their unwavering faith and confidence in my abilities and in me is what has shaped me to be the person I am today.

Finally, my special thanks go to my girlfriend, Juan. She was always there cheering me up and stood by me through the good times and bad.

List of Figures

Table of Contents

Chapter 1

Introduction

1.1 Entanglement

1.1.1 Concept of entanglement

Quantum entanglement is a property of the quantum mechanical state of a system containing two or more objects, where the objects that make up the system are linked in such a way that the quantum state of any of them can not be adequately described without full mention of the others, even if the individual objects are spatially separated. Research into quantum entanglement was initiated by the EPR paradox paper of Albert Einstein, Boris Podolsky and Nathan Rosen [1] in 1935, and a letter by Erwin Schrödinger [2] in which the word "entanglement" was used to refer to quantum particles that had been in interaction with each other. Although these first studies focused on the counterintuitive properties of entanglement, gradually entanglement became recognized as a fundamental feature of quantum mechanics, and the focus of the research changed to its utilization as a resource.

Entanglement has proven to be a valuable resource in quantum information processing, for example, as we shall see, in superdense coding and teleportation. However, determining whether or not a state is entangled is often far from simple. Methods such as Bell's inequality [8], the Peres-Horodechi positive partial transpose condition [3], Duan's [4, 5] criteria exist, but are not always straightforward to apply. Recently, Mark Hillery [6] provided a class of inequalities to detect entanglement in two-mode states.

In this dissertation, we first present conditions that allow us to determine whether blocks of spins are entangled. We show that sometimes these conditions can detect entanglement better than conditions involving individual spins. We apply these conditions to study entanglement in spin wave states, both when there are only a few magnons and also at finite temperature. We will then introduce two entanglement conditions that are in the form of inequalities for multipartite systems. We explore the range of utility of these conditions by considering a number of examples of entangled states.

1.1.2 Definition of entanglement

Let us start from the definition of entanglement. Consider a quantum state [7] in a tensor product Hilbert space, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. A pure state is separable if it is of product form

$$|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\phi\rangle_B, \tag{1.1}$$

otherwise, it is entangled. A density matrix ρ_{AB} is separable it it is a mixture of product states, i.e if it is of the form

$$\rho_{AB} = \sum_i p_i \rho_{A_i} \otimes \rho_{B_i}, \tag{1.2}$$

where $0 \leqslant p_i \leqslant 1$, and $\sum_i p_i = 1$. If ρ_{AB} is not separable, it is entangled. For a pure state that is not entangled , measurements on systems A and B are not correlated. For a separable density matrix there are only classical correlations between measurements conducted on the two systems. As we share see, entangled states can lead to much stronger correlations than are possible classically.

1.2 Bell's inequality

The basic setup for Bell's inequalities consists of two observers, Alice and Bob, and a source that produces two-particle states. One particle is sent to Alice and the other to Bob. Alice can measure one of two observables for her particle, a_1 and a_2. These observables can each be either -1 or 1. Similarly Bob can measure either b_1 or b_2, and these can also be either -1 or 1. The idea is to run this experiment many times and compute the quantities $\langle a_i b_j \rangle$.

Let us first see how a hidden-variable theory would work. The source produces, along with the particles, instruction sets that go with them. We do not know which instruction set the source will produce, and so this, the instruction set, is our hidden variable. The adjective local is applied to this kind of a hidden-variable theory, because the instructions to Alice's particle do not depend on what Bob decides to measure. That is, the instruction set does not say something like, if Alice measures a_1, then she gets 1 if Bob measures b_1 and -1 if Bob measures b_2. We shall consider only local theories. We assume that each instruction set occurs with some probability. This is equivalent to assuming that we have a joint probability distribution for the variables, a_1, a_2,b_1, and b_2, which we shall denote as $P(a_1, a_2, b_1, b_2)$.We would then compute the expectation value $\langle a_1 b_1 \rangle$, as

$$\langle a_1 b_1 \rangle = \sum_{a_1=-1}^{1} ... \sum_{b_2=-1}^{1} a_1 b_1 P(a_1, a_2, b_1, b_2). \tag{1.3}$$

We now want to consider the quantity

$$\begin{aligned} S &= \langle a_1 b_1 \rangle + \langle a_1 b_2 \rangle + \langle a_2 b_1 \rangle - \langle a_2 b_2 \rangle \\ &= \sum_{a_1=-1}^{1} ... \sum_{b_2=-1}^{1} [a_1(b_1 + b_2) + a_2(b_1 - b_2)] P(a_1, a_2, b_1, b_2). \end{aligned} \tag{1.4}$$

Call the term in brackets multiplying the probability distribution X. We see that $X = a_1(b_1 + b_2)$ if $b_1 = b_2$, and $X = a_2(b_1 - b_2)$ if $b_1 = -b_2$. In both cases, $|X| = 2$, so that

$$|S| \leq 2 \sum_{a_1=-1}^{1} ... \sum_{b_2=-1}^{1} P(a_1, a_2, b_1, b_2) = 2. \tag{1.5}$$

This is the Bell's inequality. Now let us describe the same experiment using quantum mechanics, and assume that we are measuring the spins of two spin 1/2 particles. Assume

$$\begin{aligned} a_1 &= \sigma_{xa}, a_2 = \sigma_{ya}, \\ b_1 &= \sigma_{xb}, b_2 = \sigma_{yb}, \end{aligned} \tag{1.6}$$

and that the source puts out particles in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\pi/4}|11\rangle), \tag{1.7}$$

Note that $|\Psi\rangle$ is an entangled state. We have that $\langle a_1 b_1 \rangle$, $\langle a_1 b_2 \rangle$, and $\langle a_2 b_1 \rangle$ are all equal to $\sqrt{2}/2$, and $\langle a_2 b_2 \rangle$ is equal to $-\sqrt{2}/2$. This gives us $S = 2\sqrt{2}$, which violates Bell's inequality.

From this we can conclude: quantum mechanics cannot be described by a local hidden-variable theory, in the hidden-variable theory, the correlations came from a classical joint distribution function. Therefore, quantum mechanics can produce stronger correlation than classical systems can.

But quantum mechanics gives $|S| \leq 2\sqrt{2}$, and $|S| \geq 2$ requires entanglement.

1.3 Applications of entanglement

Now that we know what entanglement is, let us look at some interesting applications of entanglement that reveal its power as a resource for quantum information related tasks.

1.3.1 Superdense coding

Suppose that Alice want to send information to Bob using qubits, instead of classical bits. Alice would encode the classical information in a qubit and send it to Bob. After receiving the qubit, Bob recovers the classical information via measurement.

We assume that Alice and Bob share an entangled pair of qubits in the state $|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$. Out of a pair of qubits in this state, Alice has one of the qubits, labeled A, in her possession while the other, labeled B, is in Bob's possession. Bob then performs one of four operations on his qubit and send it back to Alice. Bob operation can be I, σ_x, σ_y or σ_z, and the corresponding Alice's state are $|\Phi_-\rangle$, $|\Psi_-\rangle$, $-i|\Psi_+\rangle$ and $-|\Phi_+\rangle$. The point is that Alice now has one of four orthogonal states and she can distinguish them perfectly. After performing a measurement in the Bell basis, Alice will know with certainty which of the four operations Bob performed. Bob sent only one particle, a single qubit, to Alice, but Alice can perfectly distinguish among four classical alternatives, i.e. one entangled qubit carried two classical bits of information.

1.3.2 Teleportation

Quantum teleportation is a process by which a qubit can be transmitted exactly (in principle) from one location to another, without the qubit being transmitted through the intervening space. Alice has a qubit, say A_1 in some quantum state $|\psi\rangle$, in her possession. She want to transfer the quantum state of her qubit A_1 onto Bob's qubit B. Alice may not even know what $|\psi\rangle$is. Measuring $|\psi\rangle$ and transmitting the classical information that is the result will not work; it is not enough information to reconstruct the state.

In the teleportation procedure Alice and Bob share an entangled pair A_2, B in the state $|\psi\rangle_{A_2,B} = \frac{1}{\sqrt{2}}(|01\rangle_{A_2B} - |10\rangle_{A_2B})$. The total state of the three qubits is then

$$\begin{aligned}
|\psi\rangle_{A_1}|\psi\rangle_{A_2B} &= \frac{1}{\sqrt{2}}(\alpha|0\rangle_{A_1} + \beta|1\rangle_{A_1})(|01\rangle_{A_2B} - |10\rangle_{A_2B}) \\
&= \frac{1}{\sqrt{2}}\{|\Phi_+\rangle_{A_1A_2}(-\sigma_z|\psi\rangle_B) + |\Phi_-\rangle_{A_1A_2}(-|\psi\rangle_B) \\
&+|\Psi_+\rangle_{A_1A_2}(-\sigma_x\sigma_z|\psi\rangle_B) + |\Psi_-\rangle_{A_1A_2}(\sigma_x|\psi\rangle_B)\}, \quad (1.8)
\end{aligned}$$

The key is in the last line. When the total three-qubit state is decomposed in terms of the four Bell basis states of the two qubits of Alice, the state of Bob's qubit associated with each of these terms is related in a simple way to the state to be teleported. When Alice measures her state in the Bell basis, she tells Bob over a classical channel what she got, and then Bob can apply the appropriate operator to his qubit to recover Alice's state.

All information about $|\psi\rangle$ is transferred to Bob, none is left with Alice. After teleportation Alice is left in possession of a Bell state. If someone else prepared the state of the original A_1 qubit for Alice, she will never learn its state in the process. Nevertheless, the state will be faithfully teleported to Bob's qubit B.

1.4 Conditions of separability

How can we tell if a given density matrix is separable? Necessary and sufficient conditions are known to exist for the simplest cases only. In general, there are no known necessary and sufficient conditions to determine whether the state is separable or entangled. There are , however, some sufficient conditions. One of them is Bell's inequality, but this is not a particularly strong criterion as there is a large class of entangled states that satisfies Bell inequalities.

1.4.1 Partial positive transpose condition

A stronger and more general test was found by Peres [3], which is known as the partial positive transpose (PPT) criterion. Consider a density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$ of arbitrary dimensions. We have the density matrix elements in some product basis $\rho_{m\mu;n\nu} = \langle m| \otimes \langle\mu|\rho|n\rangle \otimes |\nu\rangle$.

The partial transposition of ρ is the density matrix with the matrix elements

$$\rho^{T_B}_{m\mu;n\nu} = \rho_{m\nu;n\mu}. \tag{1.9}$$

The operator ρ^{T_B} depends on the basis in which the transpose is defined, but its eigenvalues do not. We say a state is PPT if $\rho^{T_B} \geq 0$. A separable state is always PPT. This is because if ρ_{AB} is separable then $\rho^{T_B}_{AB} = \sum_i p_i \rho_{A_i} \otimes \rho^{T}_{B_i}$, and if $\rho_{B_i} \geq 0$, then $\rho^{T}_{B_i} \geq 0$. Therefore, if a partial transpose is not positive the state is entangled. Thus, the PPT condition is sufficient. For $2 \otimes 2$ (two-qubits) and $2 \otimes 3$ (qubit-qutrit) systems the converse is also true: if a state is entangled the partial transpose is not positive. Thus, for these systems, the PPT condition is also necessary.

As an example, consider the two-qubit state

$$\rho_{AB} = p|\Phi_-\rangle\langle\Phi_-| + (1-p)|00\rangle\langle 00|. \tag{1.10}$$

It can be shown that if $p \leq 1/\sqrt{2}$ all Bell inequalities will be satisfied by this state.

Now, let us apply the PPT condition to the same state. In the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the above density matrix can be written as

$$\rho = \begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & \frac{p}{2} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{p}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.11}$$

Its partial transpose with respect to B is

$$\rho = \begin{pmatrix} 1-p & 0 & 0 & -\frac{p}{2} \\ 0 & \frac{p}{2} & 0 & 0 \\ 0 & 0 & \frac{p}{2} & 0 \\ -\frac{p}{2} & 0 & 0 & 0 \end{pmatrix}. \tag{1.12}$$

The eigenvalues can be determined from the secular equation $det(\rho^{T_B} - \lambda I) = 0$, which yields

$$(\frac{p}{2} - \lambda)^2(\lambda^2 - (1-p)\lambda - \frac{p^2}{2}) = 0, \tag{1.13}$$

so that the eigenvalues are $\lambda_{1,2} = \frac{p}{2}$ and $\lambda_{3,4} = \frac{1}{2}[(1-p) \pm (1-2p+2p^2)^{1/2}]$. Three of them are obviously positive. The fourth one, $\lambda_4 = \frac{1}{2}[(1-p) - (1-2p+2p^2)^{1/2}] < 0$ for $p > 0$. Therefore, for $p > 0$, the partial transpose is not positive and the state is entangled. Note that the Bell inequalities are not violated for $p \leq \frac{1}{\sqrt{2}}$, so the PPT condition is stronger than the condition of violating the Bell inequality.

1.4.2 Duan's condition

Another commonly used criterion for field modes is the one proved by Simon and by Duan,et al [4, 5]. Consider a two-mode system in which the annihilation operators for the modes are a and b. If a state satisfies the condition

$$[\Delta(x_a + x_b)]^2 + [\Delta(p_a - p_b)]^2 < 2, \tag{1.14}$$

where $x_a = (a^+ + a)/\sqrt{2}$, $p_a = i(a^+ - a)/\sqrt{2}$ and similar for x_b and p_b, then it is entangled.

Chapter 2

Entanglement Conditions for Spin Systems

2.1 Entanglement Condition for Two Mode Systems

A class of inequalities for detecting entanglement has been provided by Professor Mark Hillery [6]. These inequalities arise from examining uncertainty relations. Consider two modes of the electromagnetic field, where a and a^{+} are the annihilation and creation operators of the first mode and b and b^{+} are the annihilation and creation operators of the second mode. we define $L_1 = ab^{+} + a^{+}b$, $L_2 = i(ab^{+} - a^{+}b)$,and $L_3 = a^{+}a + b^{+}b$. $J_i = L_i/2$ satisfy the commutation relation $[J_k, J_m] = i\varepsilon_{kmn}J_n$. Entanglement conditions expressed in terms of angular momentum operators have been derived by a number of authors . On calculating the uncertainties of these variables and adding them, it follows that

$$(\Delta L_1)^2 + (\Delta L_2)^2 = 2(\langle (N_a+1)N_b\rangle + \langle N_a(N_b+1)\rangle - 2|\langle ab^+\rangle|^2). \quad (2.1)$$

Where $N_a = a^+a$ and $N_b = b^+b$. If the state is a product of a state in a mode and another state in the b mode. We then have that

$$(\Delta L_1)^2 + (\Delta L_2)^2 = 2(\langle (N_a+1)\rangle\langle N_b\rangle + \langle N_a\rangle\langle (N_b+1)\rangle - 2|\langle a\rangle\langle b^+\rangle|^2). \quad (2.2)$$

Schwarz inequality implies that $|\langle a\rangle|^2 \leq \langle N_a\rangle$ and $|\langle b\rangle|^2 \leq \langle N_b\rangle$,so we find that for a product state

$$(\Delta L_1)^2 + (\Delta L_2)^2 \geq 2(\langle N_a\rangle + \langle N_b\rangle). \quad (2.3)$$

This inequality can be extended to any separable state by using a result of Hofmann and Takeuchi [9]. For a density matrix $\rho = \sum_m p_m \rho_m$ and a variable S, we have that

$$(\Delta S)^2 \geq \sum_m p_m (\Delta S_m)^2, \quad (2.4)$$

where $(\Delta S)^2$ is the uncertainty of S calculated in the state ρ_m. If the original state ρ is separable, then all of the states ρ_m can be taken to be product states for which the inequality in Eq.(2.3) holds. Then Eq.(2.3) holds for any separable state. It can be easily shown that Eq.(2.3) is violated for the Bell state $|\psi_{01}\rangle = (|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b)/\sqrt{2}$.

An examination of the condition in Eq.(2.3) shows us that the state is entangled if

$$\langle N_a N_b \rangle < |\langle ab^+ \rangle|^2. \tag{2.5}$$

In stead of considering the operator ab^+, we consider $a^m(b^+)^n$. For a pure product state we have that

$$|\langle a^m(b^+)^n \rangle|^2 = |\langle a^m \rangle|^2 |\langle (b^+)^n \rangle|^2 \leq \langle (a^+)^m a^m \rangle \langle (b^+)^n b^n \rangle, \tag{2.6}$$

for a product state, it is also true that

$$|\langle a^m(b^+)^n \rangle|^2 \leq \langle (a^+)^m a^m (b^+)^n b^n \rangle. \tag{2.7}$$

Consider the density matrix for a general separable state given by $\rho = \sum_k p_k \rho_k$, where ρ is a density matrix corresponding to a pure product state, and p_k is the probability of ρ_k. Defining $A = a^m$ and $B = b^n$,

$$\begin{aligned} |\langle AB^\dagger \rangle| \quad & \leq \sum_k p_k |Tr(\rho_k AB^\dagger)| \\ & \leq \sum_k p_k (\langle A^\dagger A B^\dagger B \rangle_k)^{1/2}, \end{aligned} \tag{2.8}$$

where $\langle A^\dagger A B^\dagger B \rangle_k = Tr(\rho A^\dagger A B^\dagger B)$. We can now apply the Schwarz inequality to obtain

$$\begin{aligned} |\langle AB^\dagger \rangle| &\leq (\sum_k p_k)^{1/2} (\sum_k p_k \langle A^\dagger A B^\dagger B \rangle_k)^{1/2} \\ &\leq (\langle A^\dagger A B^\dagger B \rangle)^{1/2}. \end{aligned} \tag{2.9}$$

Therefore, we can conclude that the state is entangled if

$$|\langle AB^\dagger \rangle|^2 > \langle A^\dagger A B^\dagger B \rangle, \tag{2.10}$$

similarly, in case of product states we have that

$$|\langle ab \rangle| = |\langle a \rangle \langle b \rangle| \leq [\langle N_a \rangle \langle N_b \rangle]^{1/2}, \tag{2.11}$$

and now we want to show that this inequality is obeyed by all separable states. Therefore, a violation of this inequality implies that the state is entangled.

As before, consider the density matrix of a general separable state $\rho = \sum_k p_k \rho_k$, where ρ_k is a density matrix corresponding to a pure product state, and p_k is the probability of ρ_k. Again, setting $A = a^m$ and $B = b^n$, we have that

$$\begin{aligned} |\langle AB \rangle|^2 &\leq \sum_{k,l} p_k p_l |Tr(\rho AB)||Tr(\rho_l B^\dagger A^\dagger)| \\ &\leq \sum_{k,l} p_k p_l (\langle A^\dagger A \rangle_k \langle B^\dagger B \rangle_k \langle A^\dagger A \rangle_l \langle B^\dagger B \rangle_l)^{1/2}. \end{aligned} \tag{2.12}$$

In terms of the quantities $\langle A^\dagger A \rangle_k = Tr(A^\dagger A \rho_k) = x_k$ and $\langle B^\dagger B \rangle_k = Tr(B^\dagger B \rho_k) = y_k$, this inequality can be rewritten as

$$|\langle AB \rangle|^2 \leq \sum_k p_k^2 x_k y_k + \sum_{k>l} p_k p_l (x_k y_k x_l y_l)^{1/2}. \tag{2.13}$$

Next we consider $\langle A^\dagger A \rangle \langle B^\dagger B \rangle = \sum_k p_k^2 x_k y_k + \sum_{k>l} p_k p_l (x_k y_l + x_l y_k)$. As $(x_k y_l + x_l y_k) \geq (x_k y_k x_l y_l)^{1/2}$, we have the following inequality for all separable state

$$|\langle a^m b^n \rangle| \leq [\langle (a^\dagger)^m a^m \rangle \langle (b^\dagger)^n b^n]^{1/2}. \tag{2.14}$$

and we now have another entanglement condition

$$|\langle AB \rangle|^2 > \langle A^+ A \rangle \langle B^+ B \rangle. \tag{2.15}$$

2.2 Spin Systems

Now we want to apply our conditions to the spin systems. Spin systems have received a great deal of attention, and this has led to the formulation of several conditions for determining whether the state of a spin system is entangled. For example, in an N qubit system, with

$$J_l = \frac{1}{2}\sum_{k=1}^{N} \sigma_k^{(l)}, \tag{2.16}$$

with $l = 1, 2, 3$ and $\sigma_k^{(l)}$ being the Pauli matrices for the k^{th} qubit, the state is entangled if [10]

$$\frac{(\Delta J_3)^2}{\langle J_1\rangle^2 + \langle J_2\rangle^2} < \frac{1}{N}. \tag{2.17}$$

If this inequality is satisfied, the N-qubit state cannot be expressed as

$$\rho = \sum_j p_j \rho_j^{(1)} \otimes \rho_j^{(2)} \ldots \otimes \rho_j^{(N)}, \tag{2.18}$$

where the $1 \geq p_j > 0$ sum to one, and $\rho_j^{(k)}$ is a density matrix for the k^{th} qubit. A state that can be expressed in this form is known as completely separable.

Note that the condition (2.10) holds if the left-hand side is replaced by $|\langle A^\dagger B\rangle|^2$. In the following we shall use the form (2.10) or $|\langle A^\dagger B\rangle|^2 > \langle A^\dagger A B^\dagger B\rangle$ interchangeably.

These are sufficient conditions for entanglement; if they are not satisfied we cannot say whether the state is entangled or not. In spin systems, we have a collections of spins, and our subsystems are two non-overlapping subsets of the total set. We describe each subset by a collective spin, $\mathbf{J}_a$ for set a

and $\mathbf{J}_b$ for set b. Let J_{a-} be the angular momentum lowering operator for set a and J_{b-} be the angular momentum lowering operator for set b. The corresponding raising operators are J_{a+} and J_{b+}, respectively. Our entanglement conditions become

$$\begin{aligned} |\langle J_{a-}J_{b+}\rangle|^2 &> \langle J_{a+}J_{a-}J_{b+}J_{b-}\rangle, \\ |\langle J_{a-}J_{b-}\rangle|^2 &> \langle J_{a+}J_{a-}\rangle\langle J_{b+}J_{b-}\rangle. \end{aligned} \tag{2.19}$$

These inequalities differ from the ones discussed in the previous paragraph in that they detect entanglement between two blocks of spins, and not whether the state is completely separable or not. For example, if one is studying entanglement in a spin chain, one may simply be interested in whether the state is entangled or not, in which case Eq. (2.17) could be of use. However, one might instead wish to find out whether two blocks of spins are entangled, in which case the above equations, with J_a and J_b being collective spin operators for the respective blocks, could be useful.

2.3 Examples of states

One of the advantages of the entanglement conditions discussed above is that they allow us to look at the entanglement between blocks of spins rather than between individual spins. A standard approach when studying the entanglement in spin-1/2 systems is to choose two spins and calculate their concurrence. It is, however, quite possible that there is no entanglement between individual spins, but there is between blocks of spins. In that case, the method based on concurrence will fail. This can be illustrated by an example.

Let us consider four qubits, i.e. spin-1/2 particles, with qubits 1 and 2 in block a, and qubits 3 and 4 in block b. Each qubit has an orthonormal basis $\{|0\rangle, |1\rangle\}$, and a raising operator $\sigma^{(+)}$ and a lowering operator $\sigma^{(-)}$, where $\sigma^{(+)}|0\rangle = |1\rangle$, $\sigma^{(+)}|1\rangle = 0$, and $\sigma^{(-)} = (\sigma^{(+)})^{\dagger}$. Let us now consider the four-qubit state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle_{12}|00\rangle_{34} + \frac{1}{2}(|01\rangle_{12}|10\rangle_{34} + |10\rangle_{12}|01\rangle_{34}). \tag{2.20}$$

Tracing out qubits 2 and 4 we find the reduced density matrix for qubits 1 and 3

$$\rho_{13} = \frac{1}{2}|00\rangle_{13}\langle 00| + \frac{1}{4}(|01\rangle_{13}\langle 01| + |10\rangle_{13}\langle 10|), \tag{2.21}$$

which is separable. So, if we just look at qubits 1 and 3, i.e. one qubit in each block, we do not see any entanglement. However, setting

$$J_{a-} = \sigma_1^{(-)} + \sigma_2^{(-)}, \qquad J_{b-} = \sigma_3^{(-)} + \sigma_4^{(-)}, \tag{2.22}$$

we find that

$$\langle J_{a-}J_{b-}\rangle = \frac{1}{\sqrt{2}}, \qquad \langle J_{a+}J_{a-}\rangle\langle J_{b+}J_{b-}\rangle = \frac{1}{4}, \tag{2.23}$$

so that the second entanglement condition in Eq. (2.19) is satisfied. Therefore, by looking at entanglement between blocks, we see that the state is, in fact, entangled.

We first will proceed to examine two more complicated types of entangled states in order to see whether our entanglement conditions can show that these states are indeed entangled. In the first case, we will apply the conditions to both individual and collective spins in order to see which method yields a more sensitive test of entanglement.

2.3.1 Correlated sets of qubits

Suppose we have $2n$ qubits. We will divide the qubits into two blocks of n qubits each, and within each block, we will consider only those states of total spin $j = n/2$. In particular, we want to examine states of the form

$$|\Psi\rangle = \sum_{m=-j}^{j} c_m |j, m\rangle_a \otimes |j, m\rangle_b \tag{2.24}$$

with $j = n/2$, and the state with subscript a referring to the first block and the state with subscript b referring to the second. This is clearly an entangled state, and we want to see whether the entanglement conditions we have proposed will detect the entanglement. We will do this in two different ways. First, we will apply the second entanglement condition in Eq. (2.19) to the collective spin of each block. Next, we will choose one qubit from each block and apply the same condition to those two qubits.

The calculations for the condition using the collective spins is straightforward, and we find

$$\begin{aligned}\langle\Psi|J_{a-}J_{b-}|\Psi\rangle &= \sum_{m=-j+1}^{j}(j+m)(j-m+1)c_{m-1}^{*}c_m,\\ \langle\Psi|J_{a+}J_{a-}|\Psi\rangle &= \langle\Psi|J_{b+}J_{b-}|\Psi\rangle = \sum_{m=-j}^{j}|c_m|^2(j+m)(j-m+1)\end{aligned} \tag{2.25}$$

Therefore, the second entanglement condition in Eq. (2.19) becomes

$$\left|\sum_{m=-j+1}^{j}(j+m)(j-m+1)c_{m-1}^{*}c_m\right| > \sum_{m=-j}^{j}|c_m|^2(j+m)(j-m+1). \tag{2.26}$$

One possible choice of c_m is to set $c_m = \eta x^{j+m}$, for some $x > 0$ and η an appropriate normalization constant. This gives us

$$\left|\sum_{m=-j+1}^{j}(j+m)(j-m+1)x^{2(j+m)-1}\right| > \sum_{m=-j+1}^{j}(j+m)(j-m+1)x^{2(j+m)}. \tag{2.27}$$

This condition is clearly satisfied when $x < 1$, but not satisfied for $x > 1$.

Now let us see what happens if we just look at one qubit in each block. Let the qubit in the first block be qubit 1 and the one in the second block be qubit 2, and we will assume that each of these qubits is the first one in its respective block. Let us call the spin-down state of an individual qubit $|0\rangle$ and the spin-up state $|1\rangle$. The basis states for each block are n-fold tensor products of spin-up and spin-down states for each qubit in the block. The state $|j, m\rangle$ of n qubits with $j = n/2$ is the symmetric linear combination of all basis states in which there are $j+m$ ones and $j-m$ zeroes. There are $\begin{pmatrix} 2j \\ j+m \end{pmatrix}$ such states. The operator $\sigma_1^{(+)}\sigma_1^{(-)}$, where $\sigma_1^{(+)}$ and $\sigma_1^{(-)}$ are the raising and lowering operators for qubit 1, is just the projection onto states in which the state of the first qubit is $|1\rangle$. There are $\begin{pmatrix} 2j-1 \\ j+m-1 \end{pmatrix}$ states, and this implies that

$$\langle j,m|\sigma_1^{(+)}\sigma_1^{(-)}|j,m\rangle = \frac{\begin{pmatrix} 2j-1 \\ j+m-1 \end{pmatrix}}{\begin{pmatrix} 2j \\ j+m \end{pmatrix}} = \frac{j+m}{2j}. \tag{2.28}$$

This implies that

$$\langle\Psi|\sigma_1^{(+)}\sigma_1^{(-)}|\Psi\rangle = \sum_{m=-j}^{j} |c_m|^2 \frac{j+m}{2j}. \tag{2.29}$$

The expression for $\langle\Psi|\sigma_2^{(+)}\sigma_2^{(-)}|\Psi\rangle$ is identical.

We now want to compute $({}_a\langle j,m-1| \otimes {}_b\langle j,m-1|)\sigma_1^{(-)}\sigma_2^{(-)}(|j,m\rangle_a \otimes |j,m\rangle_b)$. The operator $\sigma_1^{(-)}\sigma_2^{(-)}$ will pick out the basis states with ones in the first slot of each block. By reasoning similar to that above, we have that

$$\begin{aligned}
&({}_a\langle j, m-1| \otimes {}_b\langle j, m-1|)\sigma_1^{(-)}\sigma_2^{(-)}(|j,m\rangle_a \otimes |j,m\rangle_b) \\
&= \frac{\begin{pmatrix} 2j-1 \\ j+m-1 \end{pmatrix}^2}{\begin{pmatrix} 2j \\ j+m-1 \end{pmatrix}\begin{pmatrix} 2j \\ j+m \end{pmatrix}} \\
&= \frac{(j+m)(j-m+1)}{(2j)^2}.
\end{aligned} \tag{2.30}$$

This gives us that

$$\langle\Psi|\sigma_1^{(-)}\sigma_2^{(-)}|\Psi\rangle = \sum_{m=-j+1}^{j} c_{m-1}^* c_m \frac{(j+m)(j-m+1)}{(2j)^2}. \tag{2.31}$$

Finally, the second entanglement condition in Eq. (2.19) with $A = \sigma_1^{(-)}$ and $B = \sigma_2^{(+)}$ becomes

$$\left|\sum_{m=-j+1}^{j} (j+m)(j-m+1)c_{m-1}^* c_m\right| > (2j)\sum_{m=-j}^{j} |c_m|^2 (j+m). \tag{2.32}$$

Comparing Eqs. (2.26) and (2.32) we note that $2j \geq j-m+1$ if $m \geq -j+1$, which is the entire range of the sum. This implies that at least for states of the type in Eq. (2.24), the collective spin condition is stronger, that is, it will be satisfied by more states, than the condition for individual spins.

2.3.2 Angular momentum intelligent states

We first want to find some spin states that satisfy our entanglement conditions. One possibility is to find states in which the spins of the two subsystems are highly correlated, and states in which the uncertainty of the sum (or difference) of the two spins is small will satisfy this condition.

Let us begin by looking at the uncertainty relation for the total spin,

$$\Delta(J_{a1} + J_{b1})\Delta(J_{a2} + J_{b2}) \geq \frac{1}{2}|\langle J_{a3} + J_{b3}\rangle|, \tag{2.33}$$

where J_{a1}, J_{a2}, and J_{a3} are the components of $\mathbf{J}_a$, and J_{b1}, J_{b2}, and J_{b3} are the components of $\mathbf{J}_b$. We would like to find the states which satisfy this relation as an equality. These states were first found in [11], and here we will follow the treatment given in [12]. These satisfy the eigenvalue equation

$$[(J_{1a} + J_{1b}) + i\lambda(J_{2a} + J_{2b})]|\Psi\rangle = \beta|\Psi\rangle, \tag{2.34}$$

where λ is real. This equation implies that

$$\langle\Psi|(J_{a1} + J_{b1})|\Psi\rangle = \mathrm{Re}(\beta), \quad \langle\Psi|(J_{a2} + J_{b2})|\Psi\rangle = (1/\lambda)\mathrm{Im}(\beta), \tag{2.35}$$

and

$$[\Delta(J_{a1} + J_{b1})]^2 = \frac{\lambda}{2}\langle J_{a3} + J_{b3}\rangle \quad [\Delta(J_{a2} + J_{b2})]^2 = \frac{1}{2\lambda}\langle J_{a3} + J_{b3}\rangle. \tag{2.36}$$

From these equations, we see that when λ is small, J_{a1} and $-J_{b1}$ are highly correlated, and when it is large, J_{a2} and $-J_{b2}$ are highly correlated. These states are spin analogs of two-mode squeezed state for light.

In order to solve Eq. (2.34), we first define a state

$$|\Psi'\rangle = e^{i\theta(J_{a1}+J_{b1})}|\Psi\rangle, \tag{2.37}$$

and insert the resulting expression for $|\Psi\rangle$ into Eq. (2.34) to give an equation for $|\Psi'\rangle$

$$\{(J_{a1} + J_{b1}) + i\lambda[(J_{a2} + J_{b2})\cos\theta - (J_{a3} + J_{b3})\sin\theta]\}|\Psi'\rangle = \beta|\Psi'\rangle. \tag{2.38}$$

Now choose $\lambda = -1/\cos\theta$, and θ to be in the range $\pi \geq \theta \geq \pi/2$, which implies that $\lambda > 1$, and

$$[(J_{a-} + J_{b-}) - i\sqrt{\lambda^2 - 1}(J_{a3} + J_{b3})]|\Psi'\rangle = \beta|\Psi'\rangle. \tag{2.39}$$

We now expand $|\Psi'\rangle$ as

$$|\Psi'\rangle = \sum_{n,m=-j}^{j} C_{nm}|n, m\rangle, \tag{2.40}$$

where we have set $|n, m\rangle = |j, n\rangle \otimes |j, m\rangle$. If we assume, for simplicity, that $C_{n,m} = 0$, unless $n = m$, our equation for $|\Psi'\rangle$ reduces to the recurrence relation

$$C_{m+1,m+1} = \frac{\beta + 2mi\sqrt{\lambda^2 - 1}}{(j+m+1)(j-m)} C_{m,m}, \quad m < j, \tag{2.41}$$

$$[\beta + 2ij\sqrt{\lambda^2 - 1}]C_{j,j} = 0, \quad m = j. \tag{2.42}$$

From the second equation, we see that either $\beta = -2ij\sqrt{\lambda^2 - 1}$, or $C_{j,j} = 0$. If $C_{j,j} = 0$, then it must be the case that $\beta = -2m_0 i\sqrt{\lambda^2 - 1}$ for some m_0. So,

$$C_{m,m} = (-2i\sqrt{\lambda^2 - 1})^{j+m} \frac{(m_0 + j)!(j-m)!}{(m_0 - m)!(j+m)!(2j)!} C_{-j,-j}, \tag{2.43}$$

for $m \leq m_0$ and $C_{mm} = 0$ for $m > m_0$. After grouping the m-independent constants $\frac{(m_0+j)!}{(2j)!} C_{-j,-j}$ into $C_{j,m_0}(\lambda)$, we have

$$\begin{aligned} |\Psi(j, m_0, \lambda)\rangle &= C_{j,m_0}(\lambda) e^{-i\theta(J_{1a}+J_{1b})} \\ & \sum_{m=-j}^{m_0} (-2j\sqrt{\lambda^2 - 1})^{j+m} \frac{(j-m)!}{(m_0 - m)!(j+m)!} |m, m\rangle \end{aligned} \tag{2.44}$$

We want to see if there is some range of parameters for which $|\langle J_{a-} J_{b-}\rangle|^2 > \langle J_{a+} J_{a-}\rangle\langle J_{b+} J_{b-}\rangle$, but for these states $\langle J_{a+} J_{a-}\rangle = \langle J_{b+} J_{b-}\rangle$, so we just need to show that $|\langle J_{a-} J_{b-}\rangle| > \langle J_{a+} J_{a-}\rangle$. We find that

$$\begin{aligned} \langle J_{a+} J_{a-}\rangle &= |C_{j,m_0}(\lambda)|^2 \sum_{m=-j}^{m_0} [4(\lambda^2 - 1)]^{j+m} [\frac{(j-m)!}{(m_0 - m)!(j+m)!}]^2 \\ & \times \left\{ \frac{\lambda^2 + 1}{2\lambda^2} [j(j+1) - m^2] - \frac{m}{\lambda} \right\}, \end{aligned} \tag{2.45}$$

and

$$\begin{aligned}\langle J_{a-}J_{b-}\rangle &= |C_{j,m_0}(\lambda)|^2 \sum_{m=-j}^{m_0} [4(\lambda^2-1)]^{j+m} [\frac{(j-m)!}{(m_0-m)!(j+m)!}]^2 \\ &\times \left[\frac{2i\sqrt{\lambda^2+1}}{\lambda}(m_0-m) - \frac{(\lambda^2-1)m^2}{\lambda^2} \right]. \qquad (2.46)\end{aligned}$$

Consider the simple case in which $j = 1$,$m_0 = -1$,and $m = -1$, so that the sum has only one term. The entanglement condition becomes

$$\left| \frac{\lambda^2-1}{\lambda^2} \right| > \frac{(\lambda+1)^2}{2\lambda^2}, \qquad (2.47)$$

and the state is entangled if $\lambda > 3$.

2.4 Local rotational invariance

Entanglement is not affected by local unitary transformations, and so, ideally, we would like our entanglement conditions to be invariant under local unitaries as well. It is not always possible to accomplish this, but we can sometimes obtain invariance under a subgroup of the group of local unitary transformations. In Ref. [13] in which entanglement between field modes was considered, it was possible to find entanglement conditions that are invariant under local Gaussian transformations of the field modes. These new conditions were stronger than the original ones, that is they detect entanglement in a larger set of states. Thus, making the conditions invariant under a subset of local unitary transformations strengthens them.

For the entanglement conditions we are considering in this paper, the obvious group of local unitaries consists of local rotations. Under the action of the rotation $R(\alpha,\beta,\gamma) = e^{-i\alpha J_1} e^{-i\beta J_2} e^{-i\gamma J_3}$, we have that

$$\begin{aligned} R^{-1} J_+ R &= [\frac{1}{2}(\cos\alpha + \cos\beta) + \frac{i}{2}\sin\alpha\sin\beta] e^{i\gamma} J_+ \\ &\quad +[\frac{1}{2}(\cos\beta - \cos\alpha) + \frac{i}{2}\sin\alpha\sin\beta] e^{-i\gamma} J_- \\ &\quad +[1 - i\sin\alpha\sin\beta] e^{i\gamma} J_3. \end{aligned} \tag{2.48}$$

Now suppose we start with the entanglement condition Eq. (2.10), and we want to find from it a condition that is invariant under local rotations of the a system (finding a condition that is invariant under local rotations of both a and b subsystems is possible, but it results in a 9×9 matrix, which is rather unwieldy). We note that what the local rotation on subsystem a does is to send both J_{a+} and J_{a-} into linear combinations of J_{a+}, J_{a-}, and J_{a3}. This suggests that we set $A = c_1^* J_{a-} + c_2^* J_{a+} + c_3^* J_{a3}$ and $B = J_{b-}$ in Eq. (2.10). The entanglement condition can then be written in the form

$$\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} M \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} > 0, \tag{2.49}$$

where M is a 3×3 matrix, whose elements are linear combinations of expectation values of products of angular momentum operators. In particular,

$$\begin{aligned}
M_{11} &= |\langle J_{a-}J_{b+}\rangle|^2 - \langle J_{a+}J_{a-}J_{b+}J_{b-}\rangle, \\
M_{12} &= \langle J_{a-}J_{b+}\rangle^*\langle J_{a+}J_{b+}\rangle - \langle J_{a+}^2 J_{b+}J_{b-}\rangle, \\
M_{13} &= \langle J_{a-}J_{b+}\rangle^*\langle J_{a3}J_{b+}\rangle - \langle J_{a+}J_{a3}J_{b+}J_{b-}\rangle, \\
M_{21} &= \langle J_{a+}J_{b+}\rangle^*\langle J_{a-}J_{b+}\rangle - \langle J_{a-}^2 J_{b+}J_{b-}\rangle, \\
M_{22} &= |\langle J_{a+}J_{b+}\rangle|^2 - \langle J_{a-}J_{a+}J_{b+}J_{b-}\rangle, \\
M_{23} &= \langle J_{a+}J_{b+}\rangle^*\langle J_{a3}J_{b+}\rangle - \langle J_{a-}J_{a3}J_{b+}J_{b-}\rangle, \\
M_{31} &= \langle J_{a3}J_{b+}\rangle^*\langle J_{a-}J_{b+}\rangle - \langle J_{a3}J_{a-}J_{b+}J_{b-}\rangle, \\
M_{32} &= \langle J_{a3}J_{b+}\rangle^*\langle J_{a+}J_{b+}\rangle - \langle J_{a3}J_{a+}J_{b+}J_{b-}\rangle, \\
M_{33} &= |\langle J_{a3}J_{b+}\rangle|^2 - \langle J_{a3}^2 J_{b+}J_{b-}\rangle.
\end{aligned} \tag{2.50}$$

If we change the state by a local rotation of system a, the effect on Eq. (2.49) is only to change the values of c_1, c_2 and c_3. This follows from the fact that when A is conjugated by the rotation R_a, the form of the operator stays the same, that is, it is a linear combination of J_{a+}, J_{a-}, and J_{a3}, but the coefficients multiplying the operators change. If the matrix M has a positive eigenvalue, then we can find values of c_1, c_2 and c_3 so that the above condition is satisfied, simply by choosing them to be the components of the vector corresponding to the positive eigenvalue. Therefore, our new entanglement condition becomes that M has a positive eigenvalue, and this condition is invariant under local rotations on system a.

Let us show that this new condition is stronger than our original condition. If the state we are considering is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|-j,-j+1\rangle + |-j+1,-j\rangle), \tag{2.51}$$

then

$$M = \begin{pmatrix} j^2 & 0 & 0 \\ 0 & -2j^2 & 0 \\ 0 & 0 & -j^3 \end{pmatrix}. \tag{2.52}$$

Noting that j^2 is positive, we see that the state is entangled. Because this condition is invariant under rotations of system a, it would also show that the state $R_a \otimes I_b|\Psi\rangle$ is entangled.

Now, let us see what happens if we apply our original condition to the state $R_a \otimes I_b|\Psi\rangle$. We begin by finding

$$\left|\langle\Psi|R_a^{-1}J_{a+}R_aJ_{b-}|\Psi\rangle\right|^2 = \frac{j^2}{4}[(\cos\alpha+\cos\beta)^2+\sin^2\alpha\sin^2\beta], \tag{2.53}$$

and

$$\begin{aligned}\langle\Psi|R_a^{-1}J_{a+}J_{a-}R_aJ_{b+}J_{b-}|\Psi\rangle &= \frac{j^2}{2}[(\cos\beta-\cos\alpha)^2+\sin^2\alpha\sin^2\beta] \\ &\quad +j^3(1+\sin^2\alpha\sin^2\beta). \end{aligned} \tag{2.54}$$

Therefore, the state is entangled according to the old condition if

$$\cos\alpha\cos\beta > j(1+\sin^2\alpha\sin^2\beta) + \frac{1}{4}(\cos\alpha\cos\beta - 1)^2. \tag{2.55}$$

This condition can be satisfied for only a limited range of α and β if j is small, and it cannot be satisfied at all if $j \geq 1$, which actually allows only for $j = 1/2$. Therefore, our new condition, which is invariant under rotations of system a, is considerably more powerful in that it detects entanglement in a much larger set of states.

2.5 Spin waves

The low-lying energy states of a system of spins coupled by exchange interactions are wavelike, as shown originally by Bloch for ferromagnets. The waves are called spin waves, and they correspond to excitations of definite energy called magnons. We will study the entanglement between spins, and blocks of spins for magnon states in a ferromagnet. We will first examine entanglement in states containing a small number of magnons, and then go on to study the case of a ferromagnet at low, but finite, temperature.

2.5.1 Small number of magnons

The Hamiltonian describing spins on a lattice interacting via a nearest-neighbor exchange interaction and an externally applied magnetic field is

[14,15]

$$H = -J \sum_{\mathbf{j},\delta} \mathbf{S_j} \cdot \mathbf{S_{j+\delta}} - 2\mu_0 H_0 \sum_{\mathbf{j}} S_{\mathbf{j}z}, \tag{2.56}$$

where the vectors δ connect the spin at site $\mathbf{j}$ with its nearest neighbors on a bravais lattice, J is the exchange integral, which is assumed to be positive, $\mu_0 = (g/2)\mu_B$ is the magnetic moment of the atoms, and $\mathbf{S_j}$ is the spin angular momentum operator of the atom at $\mathbf{j}$. H_0 is the intensity of a static magnetic field directed along the z axis, and we will take the limit as $H_0 \to 0^+$ to make the magnetic moments line up along the positive z axis when the system is in the ground state $|\Omega\rangle$. The z component of the total spin, $\mathcal{S}_z = \sum_j S_{jz}$ is a constant of the motion, and the ground state of the system simply has all of the spins pointing in the $+z$ direction.

For the case of a small number of spin waves, let us consider a line of N spins with periodic boundary conditions (the spin at $N+1$ is the same as the spin at 1). If the atoms have a spin of 1/2, then the state containing a single magnon is a linear combination of states with one spin flipped

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ikja} \sigma_j^{(-)} |\Omega\rangle, \tag{2.57}$$

where a is the spacing between spins, and $k = 2\pi n/(Na)$, where n is an integer in the range $-(N/2) < n \leq (N/2)$. The operator $\sigma_j^{(-)}$ is the spin lowering operator for the spin at site j, that is it maps the spin up state at site j to the spin down state at the same site.

Now let us examine the entanglement of this state using Eq. (2.10). Let

$$\begin{aligned} A &= S_{1+} = \sum_{j=1}^{m} \sigma_j^{(+)}, \\ B &= S_{2+} = \sum_{j=L+1}^{L+m} \sigma_j^{(+)}, \end{aligned} \tag{2.58}$$

where m is a number such that $2m < N$. This will allow us to see if there is entanglement between two blocks of spins each of size m and distanced from each other by $(L - m)$ spins. Our state is entangled if

$$|\langle S_{1+} S_{2-} \rangle|^2 > \langle S_{1-} S_{1+} S_{2-} S_{2+} \rangle. \tag{2.59}$$

For the single magnon state above, the right-hand side is zero, so as long as the left-hand side is non-zero, we can say that the blocks of spins are entangled. In fact, we find that

$$\langle S_{1+} S_{2-} \rangle = \frac{1}{N} \sum_{j_1=1}^{m} \sum_{j_2=L+1}^{L+m} e^{ika(j_2 - j_1)}. \tag{2.60}$$

If the size of the blocks is small compared to the wavelength of the spin wave, the term in the sum will all have approximately the same phase, and will add coherently. This would show that the blocks of spins are entangled for this state.

If we want to look at more than one magnon, more sophisticated techniques are required. We will make use of the Holstein-Primakoff transformation, which expresses the spin operators in terms of boson creation and annihilation operators, and allows us to approximately diagonalize the Hamiltonian. The Holstein-Primakoff transformation of the spin operator $\mathbf{S_j}$ to boson creation and annihilation operators $a_{\mathbf{j}}^{\dagger}$, $a_{\mathbf{j}}$ is given by

$$\begin{aligned} S_{\mathbf{j}+} &= S_{\mathbf{j}x} + iS_{\mathbf{j}y} = (2S - a_{\mathbf{j}}^{\dagger}a_{\mathbf{j}})^{1/2}a_{\mathbf{j}}, \\ S_{\mathbf{j}-} &= S_{\mathbf{j}x} - iS_{\mathbf{j}y} = a_{\mathbf{j}}^{\dagger}(2S - a_{\mathbf{j}}^{\dagger}a_{\mathbf{j}})^{1/2}, \\ S_{\mathbf{j}z} &= S - a_{\mathbf{j}}^{\dagger}a_{\mathbf{j}}, \end{aligned} \tag{2.61}$$

where

$$[a_{\mathbf{j}}, a_{\mathbf{l}}^{\dagger}] = \delta_{\mathbf{j},\mathbf{l}}. \tag{2.62}$$

If we consider only situations in which the number of flipped spins is small compared to the total number of spins, we can expand the square roots and keep only the first terms in the expansion. In addition we make a transformation from the spin operators, $a_{\mathbf{j}}^{\dagger}$ and $a_{\mathbf{j}}$, to the magnon variables, $b_{\mathbf{k}}^{\dagger}$and $b_{\mathbf{k}}$, defined by

$$b_{\mathbf{k}} = N^{-1/2} \sum_{\mathbf{j}} e^{i\mathbf{k}\cdot\mathbf{r_j}} a_{\mathbf{j}}, \tag{2.63}$$

where $\mathbf{r_j}$ is the position of spin $\mathbf{j}$. The magnon operators satisfy boson commutation relation:

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k},\mathbf{k}'}, \quad [b_{\mathbf{k}}, b_{\mathbf{k}'}] = 0. \tag{2.64}$$

When the number of flipped spins is much less than N, the Hamiltonian is diagonal in the magnon operators,

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \tag{2.65}$$

where

$$\omega_{\mathbf{k}} = 2JzS(1 - \gamma_{\mathbf{k}}) + 2\mu_0 H_0, \tag{2.66}$$

and

$$\gamma_{\mathbf{k}} = \frac{1}{z} \sum_{\delta} e^{i\mathbf{k}\cdot\delta}. \tag{2.67}$$

As was mentioned before, we will work in the limit $H_0 \to 0^+$, so that in the ground state the spins are lined up along the z axis. In these equations a center of symmetry is assumed so that $\gamma_{\mathbf{k}} = \gamma_{-\mathbf{k}}$, and z is the number of nearest neighbors each spin has.

Now we are in a position to consider the two-magnon state. We shall again consider the one-dimensional case, that is N spins in a line. We want to study the entanglement of the state

$$|\Psi\rangle = b_{k_1}^{\dagger} b_{k_2}^{\dagger} |0\rangle = \frac{1}{N} \sum_{u,v} e^{-ik_1 x_u} e^{-ik_2 x_v} a_u^{\dagger} a_v^{\dagger} |0\rangle \tag{2.68}$$

for $k_1 \neq k_2$. We shall examine the entanglement between two blocks consisting of m spins each, one beginning at spin 1 and the other beginning at spin

L, so that the blocks are separated by $L-m$ spins. Therefore, we choose

$$A=\sqrt{2S}\sum_{j=1}^{m} a_j, \qquad B=\sqrt{2S}\sum_{j=L+1}^{L+m} a_j, \tag{2.69}$$

in Eq. (2.10). We find

$$\langle A^\dagger A B^\dagger B\rangle = \frac{4S^2}{N^2}\{2xy+2xy\cos\left[La(k_1-k_2)\right]\}, \tag{2.70}$$

where $x=[\cos(k_1 ma)+1]/[\cos k_1 a+1]$, and $y=[\cos(k_2 ma)+1]/[\cos k_2 a+1]$, and

$$|\langle A^\dagger B\rangle|^2 = \frac{4S^2}{N^2}\{x^2+2xy\cos\left[La(k_1-k_2)\right]+y^2\}. \tag{2.71}$$

Therefore, the state is entangled if

$$(x-y)^2>0, \tag{2.72}$$

which is true as long as $x\neq y$. One situation where $x=y=1$ is when the block size is one $m=1$, implying that the condition (2.10) does not detect entanglement between individual spins in the two-magnon state. Recall that $k_1 a=\pi 2n_1/N$ and $k_2 a=\pi 2n_2/N$. If the block size m is such that $m2n_1/N=2l_1+1$ and $m2n_2/N=2l_2+1$ where l_1 and l_2 are integers, hence $x=y=0$ and no entanglement is found according to the inequality (2.72). The condition (2.72) indicates that in the ideal zero-temperature two-magnon state, entanglement is found regardless of how far the two blocks are separated. This no longer occurs in the more realistic non-zero temperature state we are going to investigate below.

2.5.2 Finite temperature

Now that we have seen that the entanglement condition, Eq. (2.10), is useful in detecting entanglement in states consisting of a few magnons, let us see whether it can also detect entanglement in a system of ferromagnetically interacting spins at a finite temperature, T. The density matrix for the system is now given by

$$\rho = \frac{1}{Z} e^{-\beta H}, \tag{2.73}$$

where $\beta = 1/(k_B T)$ and k_B is Boltzmann's constant. The partition function of the system, Z is given by

$$Z = \mathrm{Tr}(e^{-\beta H}) = \prod_{\mathbf{k}} \sum_{n_{\mathbf{k}}} e^{-\beta \omega_k n_{\mathbf{k}}} = \prod_{\mathbf{k}} \frac{1}{(1 - e^{-\beta \omega_k})}, \tag{2.74}$$

and $n_{\mathbf{k}}$ is the number of magnons with wave vector $\mathbf{k}$.

We first look for entanglement between two individual spins having radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ by employing the inequality in Eq. (2.10) with

$$A = S_{j_1+} = \sqrt{2S} a_1, \qquad B = S_{j_2+} = \sqrt{2S} a_2. \tag{2.75}$$

We then have that

$$\langle AB^{\dagger}\rangle = \langle S_{j_1+}S_{j_2-}\rangle$$
$$= \frac{2S}{N}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2} e^{-i\mathbf{k}_1\cdot\mathbf{r}_1+i\mathbf{k}_2\cdot\mathbf{r}_2}\mathrm{Tr}(b_{\mathbf{k}_1}b^{\dagger}_{\mathbf{k}_2}\rho). \tag{2.76}$$

Using the relationship $\sum_{n=0}^{\infty}(n+1)x^n = 1/(1-x)^2$, one obtains

$$\mathrm{Tr}(b_{\mathbf{k}_1}b^{\dagger}_{\mathbf{k}_2}\rho) = \frac{\delta_{\mathbf{k}_1,\mathbf{k}_2}}{1-e^{-\beta\omega_{k_1}}}, \tag{2.77}$$

so that

$$\langle AB^{\dagger}\rangle = \frac{2S}{N}\sum_{\mathbf{k}_1}\frac{e^{-i\mathbf{k}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)}}{1-e^{-\beta\omega_{k_1}}}. \tag{2.78}$$

This gives us the left-hand side of our inequality, and we now need to find the right-hand side. Using

$$\langle n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3}, n_{\mathbf{k}_4}, \ldots | b^{\dagger}_{\mathbf{k}_1} b_{\mathbf{k}_2} b^{\dagger}_{\mathbf{k}_3} b_{\mathbf{k}_4} | n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3}, n_{\mathbf{k}_4}, \ldots\rangle =$$
$$\delta_{\mathbf{k}_1,\mathbf{k}_2}\delta_{\mathbf{k}_3,\mathbf{k}_4} n_{\mathbf{k}_1} n_{\mathbf{k}_3} + \delta_{\mathbf{k}_1,\mathbf{k}_4}\delta_{\mathbf{k}_2,\mathbf{k}_3} n_{\mathbf{k}_1}(n_{\mathbf{k}_2}+1), \tag{2.79}$$

we obtain for the right hand side of Eq. (2.10)

$$\langle A^{\dagger}AB^{\dagger}B\rangle = \langle S_{j_1-}S_{j_1+}S_{j_2-}S_{j_2+}\rangle$$
$$= \left(\frac{2S}{N}\right)^2 \sum_{\mathbf{k}_1,\mathbf{k}_2}\left[\frac{e^{-\beta(\omega_{k_1}+\omega_{k_2})}}{(1-e^{-\beta\omega_{k_1}})(1-e^{-\beta\omega_{k_2}})} + \frac{e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot(\mathbf{r}_1-\mathbf{r}_2)}e^{-\beta\omega_{k_1}}}{(1-e^{-\beta\omega_{k_1}})(1-e^{-\beta\omega_{k_2}})}\right] \tag{2.80}$$

Equation (2.80) shows that $\langle A^\dagger A B^\dagger B\rangle$ can be separated into two parts: the first one represents the self correlation of the particles and is distance independent, while the second represents interparticle correlations and depends on the distance between the particles.

Let us now specialize to a cubic lattice with lattice constant a and $z = 6$. If $|\mathbf{k}a\,| \ll 1$, then

$$1 - \gamma_{\mathbf{k}} = 1 - \frac{1}{3}(\cos k_x a + \cos k_y a + \cos k_z a) \simeq \frac{1}{6}k^2 a^2, \tag{2.81}$$

and the magnon energy can be expressed as $\omega_k = Dk^2$, where $D = 2JSa^2$. To tackle the sums over $\mathbf{k}_j$ we note that

$$-\frac{\pi}{a} < k_j \leq \frac{\pi}{a} \tag{2.82}$$

and approximate the cube by a sphere, so that $k_j \leq \sqrt{3}\pi/a$. We replace the sums by integrals in a spherical coordinate system, and our entanglement inequality, Eq. (2.10) becomes, upon using Eqs. (2.78) and (2.80) and carrying out the angular integrations,

$$Q = I_1^2 - (I_2^2 + I_1 I_3) > 0, \tag{2.83}$$

where $|\langle AB^\dagger\rangle|^2 = I_1^2$, $\langle A^\dagger AB^\dagger B\rangle = I_2^2 + I_1 I_3$,

$$I_1 = \int_0^{y_0} dy f\left(\frac{y}{y_0}\frac{\Delta r}{a}\pi\sqrt{3}\right)\frac{y^2}{1-e^{-y^2}}, \tag{2.84}$$

$$I_2 = \int_0^{y_0} dy \frac{y^2 e^{-y^2}}{1-e^{-y^2}}, \tag{2.85}$$

$$I_3 = \int_0^{y_0} dy f\left(\frac{y}{y_0}\frac{\Delta r}{a}\pi\sqrt{3}\right)\frac{y^2 e^{-y^2}}{1-e^{-y^2}}, \tag{2.86}$$

and $f(x)$ is the familiar function

$$f(x) = \frac{\sin x}{x}. \tag{2.87}$$

Here $\Delta r = |\mathbf{r}_1 - \mathbf{r}_2|$ is the interatomic distance, $y_0 = \sqrt{\beta D}\sqrt{3}\pi/a$ and the dimensionless integration variable y is related to the wave vector component k by $y = \sqrt{\beta D}k$. Due to the presence of the exponentially decaying factor e^{-y^2} in the numerators of the integrands in I_2 and I_3, small values of y, $y \lesssim 1$, contribute most to these integrals. In the case of I_3, the fact that $y^2 e^{-y^2}/(1-e^{-y^2})$ is a decreasing function, causes that integral to be positive.

As T increases, the upper limit of the integrals, y_0, which is proportional to $\frac{1}{\sqrt{T}}$, tends to zero, and, as a result, $e^{-y^2} \to 1$ and $I_3 \to I_1$. Hence the inequality (2.83) becomes $I_1^2 - (I_2^2 + I_1^2) > 0$, which cannot be fulfilled. This means that our condition does not show the existence of entanglement in the high temperature limit, which is consistent with what we expect on physical grounds, i.e. that there is no entanglement at high temperature. As the temperature decreases, the upper integral limit y_0 increases. The integrand in I_1 is an oscillating function of y with a varying sign and an increasing magnitude, and the sign and value of I_1 are determined mostly by the contribution near y_0. For short distances and low temperatures, the absolute value of I_2 is typically much larger than those of I_1 and I_3, which makes it the leading factor in deciding the sign of Q.

In Fig. 2.1 we give a representative example of the distance dependence of Q. Positive values of Q indicate entanglement. It can be seen from Eq. (2.84) that I_1 is an oscillating function of the interatomic distance Δr, with a damping envelop. This shows up in the behavior of Q: If we allow for continuous values of $\Delta r/a$ we will see the damped oscillations more clearly. For short interatomic distances, entanglement is clearly observed. Q turns negative for the first time at $\Delta r/a = 13$. However, it can again become positive, meaning the reappearance of detectable entanglement at much larger distances before becoming permanently negative. In Fig. 2.1 the temperature is fixed. For lower temperatures, the shortest distance at which Q is found to be negative and the overall range over which Q is found to be positive increases.

The temperature dependence of Q is illustrated in Fig. 2.2 for different values of the inter-particle distance. It can be seen that as the temperature increases, Q monotonically decreases, and the shorter the interparticle distance, the later Q crosses into the negative range. In other words, as we would expect on physical grounds, lower temperatures and shorter inter-particle distances are more favorable for entanglement generation, and this parameter

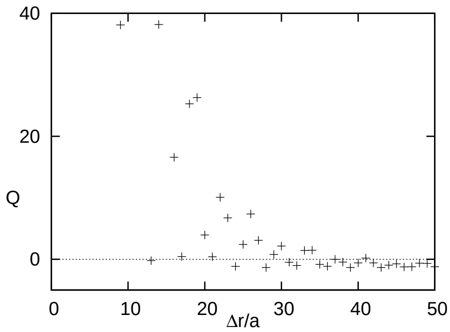

Figure 2.1: The quantity Q as a function of the interatomic distance, scaled by the lattice constant. A positive Q indicates entanglement. Some large (positive) values of Q are beyond the scope of the figure. The temperature is fixed at $\sqrt{2JS/(k_{\rm B}T)} = 7$.

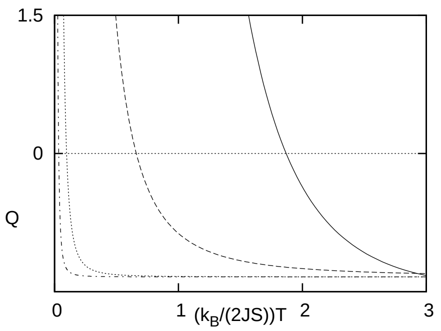

Figure 2.2: The quantity Q as a function of the dimensionless temperature for different inter-particle distances $\Delta r/a =$ (a) 1 (solid line), (b) 3 (dashed line), (c) 10 (dotted line), (d) 20 (dot-dashed line).

region is where our condition shows the presence of entanglement. If we assume some typical parameters for ferromagnets [14] $D \sim 0.5 \times 10^{-28}$ erg cm^2 and $a \sim 4\mathring{A}$, using $k_{\rm B} = 1.38 \times 10^{-16}$ erg K^{-1}, the temperatures at which Q turns negative, which is where entanglement is no longer detected, are 420K, 150K, 20K, and 8K for $\Delta r/a = 1$ (solid line), 3 (dashed line), 10 (dotted line), and 20 (dot-dashed line), respectively. Thus when the atoms are closer located, entanglement can be detected at higher temperatures. We are, of course, assuming that these temperatures are still considerably below the critical temperature, so that the spin-wave description remains valid.

Let us now proceed to use the condition (2.10) to investigate entanglement between blocks of m spins each, one beginning at spin 1 and the other beginning at spin L. With A and B chosen as in Eqs. (2.69), calculations similar to the derivation of Eqs. (2.83)-(2.87) show that the entanglement condition now takes on the form

$$Q = \left(\sum_{i=1}^{m}\sum_{j=L+1}^{L+m} I_{1ij}\right)^2 - \left\{\left[mI_2 + 2\sum_{i=1}^{m}\sum_{i'=i+1}^{m} I_{3ii'}\right]^2 + \left(\sum_{i=1}^{m}\sum_{j=L+1}^{L+m} I_{1ij}\right)\left(\sum_{i=1}^{m}\sum_{j=L+1}^{L+m} I_{3ij}\right)\right\} > 0, \qquad (2.88)$$

where I_{1ij} and I_{3ij} are given by the respective Eqs. (2.84) and (2.86) with $\Delta r = |\mathbf{r}_1 - \mathbf{r}_2|$ being replaced by $\Delta r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. Again $\langle A^\dagger A B^\dagger B\rangle$ (the term in the curly brackets) consists of two parts, the first representing the correlations between spins within a block and the second representing the

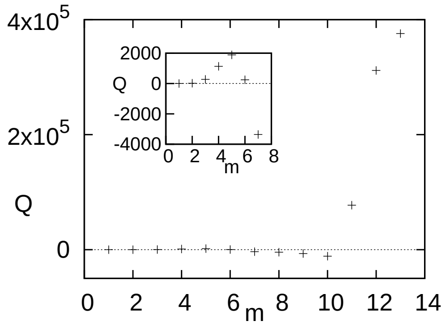

Figure 2.3: The quantity Q as a function of the block size m – the number of spins in each block for a fixed dimensionless temperature $\sqrt{2JS/(k_\mathrm{B}T)} = 7$ and for $L = 13$. The inset zooms in the part of the plot for small m.

correlations between blocks. Equation (2.88) is general in that the atoms can be arranged in an arbitrary manner in space. The only assumption used is that the two blocks do not overlap. As in the case of individual spins, in the high temperature limit $y_0 \to 0$, $I_{3ij} \to I_{1ij}$, indicating explicitly that the inequality (2.88) cannot be satisfied. At low temperatures, whether Q is positive or negative depends on the details of the terms in the sums over I_{1ij}.

In Fig. 2.3 we plotted Q as a function of the block size m, m being the number of spins contained in each block. It is assumed that each block consists of neighboring spins located along a straight line, one beginning at spin 1 and the other beginning at spin L so that the blocks are separated by $L-m$ spins. For the parameters used in Fig. 2.3, the case of individual spins $m = 1$ exhibits no entanglement (cf. Fig. 2.1). As the size of the blocks m increases (Fig. 2.3, inset), Q acquires positive values indicating a presence of inter-block entanglement. The change is not monotonic, however. As m increases further, the entanglement detected by our condition can disappear and reappear, being particularly strong for $m = 11, 12, 13$. The sign of Q obviously depends on whether the I_{1ij} add constructively or destructively. An examination of inter-block entanglement may thus offer much richer physics than simply a study of the entanglement between individual spins.

Chapter 3

Entanglement Condition for Multipartite

3.1 Introduction

The structure of entanglement in multipartite systems is much richer than that in the case of bipartite systems. Despite it that much effort has been spent on characterizing multipartite entanglement, detection, classification, and quantification of entanglement for arbitrary states of multipartite systems remains a formidable task [16, 17]. In this chapter we focus on the problem of multipartite entanglement detection using inequalities. One possible strategy in this approach is to use pairwise inequalities to check for entanglement in every possible bipartite cut in the system. This way on may gain detailed information on which subsystems are entangled [18, 19]. However, the amount of work required to perform the task may grow enormously with an increasing number of subsystems. It is desirable to have multipartite inequalities that would allow one to check for overall entanglement in multipartite systems in a straightforward and transparent manner.

Starting from the separability condition we derive the two conditions for multipartite entanglement

$$\Big|\Big\langle\prod_{k=1}^{n} A_k\Big\rangle\Big| > \prod_{k=1}^{n}\langle(A_k^{\dagger}A_k)^{n/2}\rangle^{1/n}, \tag{3.1}$$

$$\Big|\Big\langle\prod_{k=1}^{n} A_k\Big\rangle\Big| > \Big\langle\Big(\frac{1}{n}\sum_{k=1}^{n} A_k^{\dagger}A_k\Big)^{n/2}\Big\rangle, \tag{3.2}$$

where A_k is an operator that acts on the subsystem k. These are applicable to systems of continuous-variable type, discrete type, or a mixture between the two.

3.2 Separability Conditions

Consider a system consisting of n subsystems with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_n$. If the system is in a pure state, it is fully separable if and only if the state is a product of pure states describing n elementary subsystems. If the state is mixed, it is fully separable if ρ is a statistical mixture of product states

$$\rho = \sum_j p_j\rho_j = \sum_j p_j\rho_j^{(1)} \otimes \rho_j^{(2)} \cdots \otimes \rho_j^{(n)}. \tag{3.3}$$

Let A_k operate on $\mathcal{H}_k$, then we have that

$$\begin{aligned}\Big|\Big\langle\prod_{k=1}^{n} A_k\Big\rangle\Big| &= \Big|\sum_j p_j \prod_{k=1}^{n}\langle A_k\rangle_j\Big| \\ &\le \sum_j p_j \Big|\prod_{k=1}^{n}\langle A_k\rangle_j\Big| \\ &\le \sum_j p_j \prod_{k=1}^{n}\langle |A_k|^2\rangle_j^{1/2} \ . \end{aligned} \tag{3.4}$$

where $\langle A_k\rangle_j = \mathrm{Tr}\big(A_k\rho_j\big)$, $|A_k|$ denotes $\sqrt{A_k^\dagger A_k}$. In the first line we used the full separability of the state and in going from the second line to the third, we used the fact that any operator has a non-negative variance

$$|\langle A_k\rangle_j| \le \langle |A_k|^2\rangle_j^{1/2}.$$

We prove now a lemma.

Lemma: For any positive operator B we have that $\langle B\rangle^p \le \langle B^p\rangle,\ p > 1$.

Proof: First we write $\langle B\rangle$ in the form

$$\langle B\rangle = \sum_{l=1}^{m} \lambda_l \langle P_l\rangle, \tag{3.5}$$

where P_l is the projector corresponding to λ_l and $\langle P_l\rangle = \mathrm{Tr}(\rho P_l)$. We shall make use of the Hölder inequality [20], which reads as

$$\sum_{l=1}^{m} |x_l y_l| \le \Big(\sum_{l=1}^{m} |x_l|^p\Big)^{1/p}\Big(\sum_{l=1}^{m} |y_l|^q\Big)^{1/q}, \tag{3.6}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad p > 1,\ q > 1, \tag{3.7}$$

and the equality holds iff $|x_1|^{p-1}/|y_1| = |x_2|^{p-1}/|y_2| = \cdots = |x_m|^{p-1}/|y_m|$. For $p = q = 2$ it reduces to the Cauchy-Schwarz inequality. Substitution of

$$x_l = \lambda_l \langle P_l \rangle^{1/p}, \qquad y_l = \langle P_l \rangle^{1/q}, \tag{3.8}$$

where p and q satisfy Eq. (3.7), in the Hölder inequality, it follows that

$$\begin{aligned}
\sum_{l=1}^{m} |\lambda_l \langle P_l \rangle| &= \sum_{l=1}^{m} \lambda_l \langle P_l \rangle^{1/p} \langle P_l \rangle^{1/q} \\
&\leq \Big(\sum_{l=1}^{m} \lambda_l^p \langle P_l \rangle \Big)^{1/p} \Big(\sum_{l=1}^{m} \langle P_l \rangle \Big)^{1/q} \\
&= \Big(\sum_{l=1}^{m} \lambda_l^p \langle P_l \rangle \Big)^{1/p},
\end{aligned} \tag{3.9}$$

hence $\langle B \rangle \leq \langle B^p \rangle^{1/p}$. ■

We shall employ the generalized Hölder inequality [20], which reads

$$\Big(\sum_j p_j a_j^r b_j^r \ldots l_j^r \Big)^{\frac{1}{r}} \leq \Big(\sum_j p_j a_j^{r/\alpha} \Big)^{\alpha/r} \Big(\sum_j p_j b_j^{r/\beta} \Big)^{\beta/r} \ldots \Big(\sum_j p_j l_j^{r/\gamma} \Big)^{\gamma/r} \tag{3.10}$$

where

$$\sum_j p_j = 1, \qquad \alpha + \beta + \ldots + \gamma = 1. \tag{3.11}$$

Setting $r = 1$, $\alpha = \beta = \ldots = \gamma = \frac{1}{n}$, and $a_j = \langle |A_1|^2 \rangle_j^{1/2}$, $b_j = \langle |A_2|^2 \rangle_j^{1/2}, \ldots,$

$l_j = \langle |A_n|^2\rangle_j^{1/2}$, the inequality (3.10) yields readily

$$\begin{aligned}\sum_j p_j \prod_{k=1}^{n} \langle |A_k|^2\rangle_j^{1/2} &\leq \prod_{k=1}^{n}\Big(\sum_j p_j \langle |A_k|^2\rangle_j^{n/2}\Big)^{1/n} \\ &\leq \prod_{k=1}^{n}\Big(\sum_j p_j \langle |A_k|^n\rangle_j\Big)^{1/n} \\ &= \prod_{k=1}^{n} \langle |A_k|^n\rangle^{1/n}, \end{aligned} \tag{3.12}$$

where in the second step we made use of the lemma. This and Eq. (3.4) lead to

$$\Big|\Big\langle \prod_{k=1}^{n} A_k \Big\rangle\Big| \leq \prod_{k=1}^{n} \langle (A_k^\dagger A_k)^{n/2}\rangle^{1/n}. \tag{3.13}$$

Since all separable states must satisfy (3.13), a state that violates it is an entangled state and we obtain the multipartite entanglement condition (3.1).

To derive (3.2) we make use of the fact that the geometric means is smaller than or equal to the arithmetic means

$$\prod_{k=1}^{n} a_k^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n} a_k, \qquad a_k \geq 0, \tag{3.14}$$

where the equality holding iff $a_1 = a_2 = \cdots = a_n$. With $a_k = \langle |A_k|^2\rangle_j^{1/2}$, the inequality (3.14) yields

$$\begin{aligned}\prod_{k=1}^{n}\langle|A_k|^2\rangle_j^{1/2} &\leq \frac{1}{n^n}\left(\sum_{k=1}^{n}\langle|A_k|^2\rangle_j^{1/2}\right)^n \\ &\leq \frac{1}{n^n}n^{n/2}\left(\sum_{k=1}^{n}\langle|A_k|^2\rangle_j\right)^{n/2} \\ &= \frac{1}{n^{n/2}}\left\langle\sum_{k=1}^{n}|A_k|^2\right\rangle_j^{n/2} \\ &\leq \frac{1}{n^{n/2}}\left\langle\left(\sum_{k=1}^{n}|A_k|^2\right)^{n/2}\right\rangle_j , \end{aligned} \tag{3.15}$$

where in going from the first line to the second we applied the Cauchy-Schwarz inequality and in going from the third line to the fourth, we used the result of the lemma with $B = \sum_{k=1}^{n}|A_k|^2$ and $p = n/2$. The inequality (3.15) leads to

$$\sum_j p_j \prod_{k=1}^{n}\langle|A_k|^2\rangle_j^{1/2} \leq \frac{1}{n^{n/2}}\sum_j p_j\left\langle\left(\sum_{k=1}^{n}|A_k|^2\right)^{n/2}\right\rangle_j = \frac{1}{n^{n/2}}\left\langle\left(\sum_{k=1}^{n}|A_k|^2\right)^{n/2}\right\rangle . \tag{3.16}$$

Substituting this in Eq. (3.4), we arrive at the separability condition

$$\left|\left\langle\prod_{k=1}^{n}A_k\right\rangle\right| \leq \frac{1}{n^{n/2}}\left\langle\left(\sum_{k=1}^{n}A_k^\dagger A_k\right)^{n/2}\right\rangle. \tag{3.17}$$

Its violation yields the multipartite entanglement condition (3.2).

An inspection of the conditions (3.1) and (3.2) reveals that for states such that $A_k^\dagger A_k|\psi\rangle = A_{k'}^\dagger A_{k'}|\psi\rangle\ \forall k, k'$, these conditions are the same.

In the case of bipartite systems $n = 2$, the inequality (3.1) reduces to

$$|\langle AB\rangle|^2 > \langle A^\dagger A\rangle\langle B^\dagger B\rangle, \tag{3.18}$$

while the second inequality, Eq. (3.2), becomes $|\langle AB\rangle| > \frac{1}{2}(\langle A^\dagger A\rangle + \langle B^\dagger B\rangle)$ or

$$|\langle AB\rangle|^2 > \langle A^\dagger A\rangle\langle B^\dagger B\rangle + \frac{1}{4}(\langle A^\dagger A\rangle - \langle B^\dagger B\rangle)^2. \tag{3.19}$$

The bipartite entanglement condition (3.18) is exactly one of those previously derived in Ref. [6], while the condition (3.19) is weaker than the condition (3.18) because the second term in the right-hand side is positive. However, without specifying the state of the system, there seems to be no easy way to compare the two conditions for $n > 2$. In fact we find quite a few situations where the second condition, Eq. (3.2), can detect entanglement, while the first condition, Eq. (3.1) cannot.

3.3 Examples

3.3.1 GHZ-type states

Generalized GHZ state

Consider a system consisting of n subsystem of one spin $\frac{1}{2}$ each, being in the state

$$|\psi\rangle = \cos\theta|0\rangle^{\otimes n} + \sin\theta|1\rangle^{\otimes n}\ . \tag{3.20}$$

For the choice of A_k as

$$A_k = |0\rangle_k\langle 1|, \quad A_k^\dagger A_k = |1\rangle_k\langle 1|, \tag{3.21}$$

$A_k^\dagger A_k|\psi\rangle$ is independent of k and therefore the two conditions (3.1) and (3.2) coincide. Using Eqs. (3.20) and (3.21), we derive

$$\langle\psi|(\prod_{k=1}^{n} |0\rangle_k\langle 1|)|\psi\rangle = \cos\theta\sin\theta, \tag{3.22}$$

$$\Big(\prod_{k=1}^{n}\langle\psi|\big(|1\rangle_k\langle 1|\big)^{n/2}|\psi\rangle\Big)^{1/n} = \sin^2\theta. \tag{3.23}$$

If $|\cos\theta| > |\sin\theta|$, it can be seen that both entanglement conditions are satisfied, indicating the presence of entanglement. Alternatively, one can choose $A_k = |1\rangle_k\langle 0|$, $A_k^\dagger A_k = |0\rangle_k\langle 0|$. Then the left-hand sides of Eqs. (3.1) and (3.2) remain the same as in Eq. (3.22) while the right-hand sides are changed to $\cos^2\theta$, i.e., entanglement is detected if $|\sin\theta| > |\cos\theta|$. This choice of A_k thus complements the one given in Eqs. (3.21). The two choices detect entanglement in the state (3.20) for all values of θ, except for the case of $\cos\theta = \sin\theta$.

Since for $\sin 2\theta \leq 1/\sqrt{2^{n-1}}$ and n odd, the state (3.20) does not violate any n-party Bell inequalities for correlation functions which involve two dichotomic observables per local measurement station [21], including Mermin-Klyshko inequalities [22–24], for this state the conditions (3.1) and (3.2) are stronger than the mentioned criteria for entanglement detection.

If the state has one spin flipped with respect to the rest

$$|\psi\rangle = \cos\theta|1\rangle \otimes |0\rangle^{\otimes(n-1)} + \sin\theta|0\rangle \otimes |1\rangle^{\otimes(n-1)} \ , \tag{3.24}$$

then by choosing

$$A_1 = |1\rangle\langle 0|, \quad A_k = |0\rangle_k\langle 1|, \quad k > 1, \tag{3.25}$$

one can readily find that entanglement is detected for $|\cos\theta| > |\sin\theta|$. Generalization to cases where more spins are flipped is straightforward.

The conditions (3.1) and (3.2) are also robust against noise. It is not difficult to verify that for the state

$$\rho = p|\psi\rangle\langle\psi| + (1-p)|0\rangle^{\otimes n}\langle 0|, \qquad 0 < p < 1, \tag{3.26}$$

they work the same as discussed above. Interestingly, this holds no matter how large the amount of noise is, that is how close p is to zero, because p appears on both sides of the inequality and cancel each other out. One can assume a more general type of noise

$$\rho = p|\psi\rangle\langle\psi| + (1-p)\frac{I}{2^n}, \qquad 0 < p < 1, \tag{3.27}$$

where I is the unity operator. With the choice of A_k as in Eq. (3.21), the condtion (3.1) [or (3.2)] yields

$$|\cos\theta\sin\theta| > \sin^2\theta + \frac{1-p}{p}\frac{n}{2}\frac{1}{2^n}. \tag{3.28}$$

For fixed θ and p the second term in the right-hand side tends to zero as the number of parties n increases, i.e., the conditions work in the same way as in a noiseless system. For fixed θ and n, the conditions become harder to satisfy when the amount of noise increases $p \to 0$.

A partially separable state

The state (3.20) is a genuinely multipartite entangled state. We give now some examples on how the two conditions (3.1) and (3.2) work with a partially separable state. Consider again an ensemble of n spin $\frac{1}{2}$ particles, split into two groups of l and $(n-l)$ spins, each being in a generalized GHZ state

$$|\psi\rangle = \left[\cos\theta_1|0\rangle^{\otimes l} + \sin\theta_1|1\rangle^{\otimes l}\right] \otimes \left[\cos\theta_2|0\rangle^{\otimes(n-l)} + \sin\theta_2|1\rangle^{\otimes(n-l)}\right]. \tag{3.29}$$

Choosing A_k as in Eq. (3.21), the inequalities in Eqs. (3.1) and (3.2) become

$$|\cos\theta_1\sin\theta_1\cos\theta_2\sin\theta_2| > [(\sin\theta_1)^{2l}(\sin\theta_2)^{2(n-l)}]^{1/n}, \tag{3.30}$$

$$\begin{aligned} |\cos\theta_1\sin\theta_1\cos\theta_2\sin\theta_2| &> \left(\frac{n-l}{n}\right)^{n/2}\cos^2\theta_1\sin^2\theta_2 \\ &+ \left(\frac{l}{n}\right)^{n/2}\cos^2\theta_2\sin^2\theta_1 + \sin^2\theta_1\sin^2\theta_2. \end{aligned} \tag{3.31}$$

respectively. Note that the two conditions now behave differently. It is apparent from the above equations that entanglement cannot be detected if $\sin\theta_1 = 0$ or $\cos\theta_1 = 0$, which we exclude from further consideration. Since the inequalities (3.30) and (3.31) are rather involved, it is instructive to examine some special cases.

For $l = 1$ and $n = 3$, and $\sin\theta_1 = \cos\theta_1 = \frac{1}{\sqrt{2}}$, they become

$$|\cos\theta_2| > (4|\sin\theta_2|)^{1/3}, \tag{3.32}$$

$$|\cos\theta_2\sin\theta_2| > 1.09|\cos\theta_2\sin\theta_2| + (1.24|\sin\theta_2| - 0.44|\cos\theta_2|)^2 \tag{3.33}$$

Obviously, when θ_2 is close enough to 0 or π, the first inequality, Eq. (3.32), is satisfied meaning it can detect entanglement in the state, while there exists no θ_2 for which the second inequality, Eq. (3.33), is satisfied.

For $l = 2$ and $n = 4$, the two inequalities (3.30) and (3.31) read as

$$|\cos\theta_1\sin\theta_1\cos\theta_2\sin\theta_2| > |\sin\theta_1\sin\theta_2|, \tag{3.34}$$

$$|\cos\theta_1\sin\theta_1\cos\theta_2\sin\theta_2| > \frac{1}{2}(\cos\theta_1\sin\theta_2 - \cos\theta_2\sin\theta_1)^2 + 2\sin^2\theta_1\sin^2\theta_2, \tag{3.35}$$

respectively. It can be seen that the inequality (3.34) cannot be fufilled for any values of θ_1 and θ_2. Regarding the second inequality, we set for simplicity $\cos\theta_1\sin\theta_2 = \cos\theta_2\sin\theta_1$, to make it become $\cos^2\theta_2 > 2\sin^2\theta_2$, which clearly can be fufilled with θ_2 in the neighborhood of 0 and π. Thus the situation is opposite to that occuring in the case of $l = 1$ and $n = 3$

discussed above in that the second condition, not the first one, does better at detecting entanglement.

In the limit of large n, Eqs. (3.30) and (3.31) can be brought approximately to comparable form

$$|\cos\theta_2| > |\sin\theta_2|\frac{1}{|\cos\theta_1\sin\theta_1|}, \qquad (3.36)$$

$$|\cos\theta_2| > |\sin\theta_2|\frac{1}{|\cos\theta_1\sin\theta_1|}\Big[e^{-l/2} + \Big(1 - e^{-l/2}\Big)\sin^2\theta_1\Big], \qquad (3.37)$$

where in going from Eq. (3.30) to Eq. (3.36), we made the replacement $(\frac{\sin\theta_2}{\sin\theta_1})^{\frac{n-2l}{n}} \to \frac{\sin\theta_2}{\sin\theta_1}$, while and in going from Eq. (3.31) to Eq. (3.37), we used the relation $\left(\frac{n-l}{n}\right)^{n/2} = e^{-l/2} + O(\frac{1}{n})$ and dropped the second term in the right-hand side of Eq. (3.31). Two comments can be made regarding the inequalities (3.36) and (3.37). First, there exist θ_1 and θ_2 for which both or one of them are satisfied, meaning entanglement is detected. Second, since $\sin^2\theta_1 < 1$, the extra factor in Eq. (3.37) is less than unity resulting in that the condition (3.37) is more sensitive to entanglement compared to the other condition in the sense that there exist ranges of the parametters θ_1 and θ_2 for which the condition (3.37) can detect entanglement while the condition (3.36) cannot.

Some estimation on how little entanglement in a multipartite system is detectable by the condition (3.1) can be gained by studying the state

$$|\psi\rangle = \prod_{i=1}^{l\otimes}\big[\cos\theta_i|0\rangle_i + \sin\theta_i|1\rangle_i\big] \otimes \big[\cos\theta|0\rangle^{\otimes(n-l)} + \sin\theta|1\rangle^{\otimes(n-l)}\big]\ , \qquad (3.38)$$

where l parties are separated from the rest, which is in a GHZ state. For this state, the condition (3.1) reads as

$$|\cos\theta| > \frac{1}{\prod_{i=1}^{l} |\cos\theta_i(\sin\theta_i)^{1-2/n}|} |\sin\theta|^{(n-2l)/n}. \tag{3.39}$$

Since the denominator is less than one, the inequality can be satisfied when $l < n/2$. Though the number of separated parties has to be smaller than half the total number of parties in for the entanglement to be detected, in the case of large n, at least theoretically, using the condition (3.1) to look for entanglement is clearly less labour-intensive than using a bipartite condition to check every possible pairwise separations.

A mixed state

Consider the n-party state

$$\rho = \frac{1}{n} \sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i|, \tag{3.40}$$

where

$$|\psi_i\rangle = \left[\cos\theta_i|0\rangle + \sin\theta_i|1\rangle\right] \otimes \left[\cos\theta|0\rangle_{\bar{i}}^{\otimes(n-1)} + \sin\theta|1\rangle_{\bar{i}}^{\otimes(n-1)}\right], \tag{3.41}$$

$|0\rangle_{\bar{i}}^{\otimes(n-1)}$ and $|1\rangle_{\bar{i}}^{\otimes(n-1)}$ being states where the spin i is excluded. $|\psi_i\rangle$ represents a state where the spin i is separated, while from the remaining spins are in a generalized GHZ state. Though ρ is a statistical mixture of bipartite separable states, there is no overall bipartite splitting with respect to which the state is separable. Choosing A_k as in Eq. (3.21), the inequalities (3.1) and (3.2) read as

$$
\begin{aligned}
|\cos\theta\sin\theta\sum_{i=1}^{n}\cos\theta_i\sin\theta_i| &> \Big[\prod_{i=1}^{n}[\sin^2\theta_i+(n-1)\sin^2\theta]\Big]^{1/n}, \quad (3.42)\\
|\cos\theta\sin\theta\sum_{i=1}^{n}\cos\theta_i\sin\theta_i| &> \left(\frac{n-1}{n}\right)^{n/2}\sin^2\theta\sum_{i=1}^{n}\cos^2\theta_i\\
&+\left(\frac{1}{n}\right)^{n/2}\cos^2\theta\sum_{i=1}^{n}\sin^2\theta_i\\
&+\sin^2\theta\sum_{i=1}^{n}\sin^2\theta_i. \quad (3.43)
\end{aligned}
$$

It is instructive to consider the case of very large n, $\sin\theta_1=\cos\theta_1=\frac{1}{\sqrt{2}}$, $\sin\theta_i=0$ for $i\geq 2$, for which these are simplified greatly to become

$$|\cos\theta| > 2(n-1)|\sin\theta|, \quad (3.44)$$

$$|\cos\theta| > \Big[\frac{2}{\sqrt{e}}(n+\frac{1}{2})+1\Big]|\sin\theta|, \quad (3.45)$$

respectively. It can be seen that both inequalities can be satisfied if $|\cos\theta|$ is sufficiently close to one and both would perform worse as entanglement detection condition as the number of parties n increases, the first more so than the second.

3.3.2 Continuous-variable systems

Consider an n-mode squeezed vacuum field state

$$|\psi\rangle = \sqrt{1-x^2}\sum_{m=0}^{\infty} x^m |m\rangle^{\otimes n}, \tag{3.46}$$

where $0 < x < 1$. For the choice of A_k

$$A_k = a_k, \quad A_k^\dagger A_k = a_k^\dagger a_k, \tag{3.47}$$

the two conditions (3.1) and (3.2) are identical. One finds that

$$\langle\psi|(\prod_{k=1}^{n} a_k)|\psi\rangle = \frac{1}{x}(1-x^2)\sum_{m=0}^{\infty} x^{2m} m^{n/2}, \tag{3.48}$$

$$\Big(\prod_{k=1}^{n}\langle\psi|\big(a_k^\dagger a_k\big)^{n/2}|\psi\rangle\Big)^{1/n} = (1-x^2)\sum_{m=0}^{\infty} x^{2m} m^{n/2}. \tag{3.49}$$

A comparison of Eqs. (3.48) and (3.49) shows that the inequalities (3.1) and (3.2) are satisfied for any value of x in the range $0 < x < 1$. That is to say, these conditions can always detect entanglement in the multimode squeezed vacuum state.

We consider now an example of continuous variable systems where the two conditions (3.1) and (3.2) work differently, namely a modified four-mode squeezed vacuum state

$$|\psi\rangle = \sqrt{1-x^2}\sum_{m=0}^{\infty} x^m |m\rangle_1 |m\rangle_2 |m+1\rangle_3 |m+1\rangle_4, \tag{3.50}$$

where $0 < x < 1$. Choosing A_k as in Eq. (3.47), it can be found that

$$\langle\psi|(\prod_{k=1}^{4} a_k)|\psi\rangle = \frac{2x}{(1-x^2)^2} , \tag{3.51}$$

$$\Big(\prod_{k=1}^{4}\langle\psi|\big(a_k^\dagger a_k\big)^2|\psi\rangle\Big)^{1/4} = \frac{x(1+x^2)}{(1-x^2)^2} , \tag{3.52}$$

$$\frac{1}{16}\langle\psi|\Big(\sum_{k=1}^{4} a_k^\dagger a_k\Big)^2|\psi\rangle = \frac{1}{(1-x^2)^2}\frac{1}{4}(x^4+6x^2+1) , \tag{3.53}$$

where we have made use of the relations $\sum_{m=0}^{\infty} x^{2m} = \frac{1}{1-x^2}$, $\sum_{m=0}^{\infty} x^{2m} m = \frac{x^2}{(1-x^2)^2}$, and $\sum_{m=0}^{\infty} x^{2m} m^2 = \frac{x^2(1+x^2)}{(1-x^2)^3}$. From Eqs. (3.51) and (3.52) it can be inferred that the inequality (3.1) is satisfied for all x, meaning it always detects entanglement. A comparison of Eq. (3.51) and (3.53) however shows that the inequality (3.17) can be used for entanglement detection only for $x \gtrsim 0.1397$.

Chapter 4

Quantum Walk in star graph with extra edges

A quantum walk is a quantum version of a random walk [26]. Both types of walks occur on a graph, which is a set vertices connected by edges. A particle making a quantum walk behaves differently from one making a classical random walk, because the mathematical objects that govern its motion are amplitudes rather than probabilities, and this means that interference effects will play a role. There are two basic types of quantum walks, one in which time progresses in discrete steps [27] and the other in which time is continuous [28]. Here we shall be concerned with a particular version of the discrete-time walk known as the scattering quantum walk [29]. In this type of quantum walk the particle resides on the edges and scatters at the vertices at each time step.

We will consider what we call a star graph. This graph has a high degree of symmetry, which meas that analyzing walks on it becomes relatively simple, because the Hilbert space in which the walk occurs is of relatively small dimension [30,31].This graph has a central vertex, which we shall label 0,and N additional vertices, which we shall label 1 through N. The central vertex is connected to each of the other vertices by an edge, and, for now, the vertices $1, 2, ...N$ are not connected to each other by edges. In order to construct a quantum walk on this graph we first need a Hilbert space for the particle making the walk. We specify this by means of an orthonormal basis consisting of the states$|0, j\rangle, |j, 0\rangle, j = 1, 2, ...N$. The state $|0, j\rangle$ corresponds to the particle being on the edge between 0 and j and going from 0 to j, and the state$|j, 0\rangle$ corresponds to the particle again being on the edge between 0 and j, but now going from j to 0. Next we need a unitary operator that advances the walk one time step. That is provided by the collective action of unitaries at each vertex that tell how the particle scatters as it passes through that vertex. If U is the unitary that advances the walk one step, it acts on a particle entering the vertex 0 as

$$U|j, 0\rangle = -r|0, j\rangle + t \sum_{k=1, k\neq k}^{N} |0, k\rangle, \tag{4.1}$$

where $r = (N-2)/N$ and $t = 2/N$. That is, the particle has an amplitude of $-r$ of being reflected and an amplitude t of being transmitted to one of the other edges. We now need to choose what happens at the vertices 1 through

N. If we make the choice $U|0,j\rangle = |j,0\rangle$ for $j > 1$ and $U|0,1\rangle = -|1,0\rangle$, we obtain an implementation of the Grover search algorithm. Starting with an equal superposition of all of the basis states, after $O(\sqrt{N})$ steps the particle will be located on the edge connecting the vertices 0 and 1.

Here we wish to do something different. First, let's add an edge between vertices 1 and 2. The unitary operator will act as $U|0,j\rangle = |j,0\rangle$ for $j > 2$, and

$$\begin{aligned} U|0,1\rangle &= |1,2\rangle \\ U|0,2\rangle &= |2,1\rangle \\ U|1,2\rangle &= |2,0\rangle \\ U|2,1\rangle &= |1,0\rangle. \end{aligned} \tag{4.2}$$

Note that we have assumed that vertices 1 and 2 transmit the particle, and there is no reflection. One can put in an amplitude for reflection, but if it is not too large, this does not change our results appreciably. the walk resulting from this choice of U can be analyzed easily, because it stays within a fivedimensional subspace of the entire Hilbert space. Define the states

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{\sqrt{2}}|0,1\rangle + |0,2\rangle \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}|1,0\rangle + |2,0\rangle \\
|\psi_3\rangle &= \frac{1}{\sqrt{N-2}}\sum_{j=3}^{N}|0,j\rangle \\
|\psi_4\rangle &= \frac{1}{\sqrt{N-2}}\sum_{j=3}^{N}|j,0\rangle \\
|\psi_5\rangle &= \frac{1}{\sqrt{2}}|1,2\rangle + |2,1\rangle. \qquad (4.3)
\end{aligned}
$$

These states span a five-dimentional space we call S.The unitary transformation, U, that advances the walk one step acts on these states as follows:

$$
\begin{aligned}
U|\psi_1\rangle &= |\psi_5\rangle \\
U|\psi_2\rangle &= -(r-t)|\psi_1\rangle + 2\sqrt{rt}|\psi_3\rangle \\
U|\psi_3\rangle &= |\psi_4\rangle \\
U|\psi_4\rangle &= (r-t)|\psi_3\rangle + 2\sqrt{rt}|\psi_3\rangle \\
U|\psi_5\rangle &= |\psi_2\rangle. \qquad (4.4)
\end{aligned}
$$

The matrix that describes the action of U on S is given by

$$M = \begin{pmatrix} 0 & -(r-t) & 0 & 2\sqrt{rt} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2\sqrt{rt} & 0 & (r-t) & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.5}$$

The characteristic equation for this matrix is

$$\lambda^5 - (r-t)\lambda^3 + (r-t)\lambda^2 - 1 = 0. \tag{4.6}$$

One root of this equation is $\lambda = 1$, and if we factor out $(\lambda - 1)$ from the above equation, we are left with

$$\lambda^4 + \lambda^3 + 2t\lambda^2 + \lambda + 1 = 0. \tag{4.7}$$

We will use a perturbation expansion to find the roots of this equation with the transmission amplitude, t, as the small parameter. The zeroth order solutions are found by setting $t = 0$, and this gives us

$$\lambda^4 + \lambda^3 + \lambda + 1 = (\lambda + 1)(\lambda^3 + 1) = 0, \tag{4.8}$$

so the zeroth order roots are -1 twice, $e^{i\pi/3}$, and $e^{-i\pi/3}$. Setting $\lambda = -1 + \delta\lambda$, substituting into the above equation and keeping terms up to $(\delta\lambda)^2$ gives

$$3(\delta\lambda)^2 - 4t\delta\lambda + 2t = 0, \tag{4.9}$$

whose solution, keeping lowest order terms is

$$\delta\lambda = \pm i\sqrt{\frac{2t}{3}} = O(N^{-1/2}). \tag{4.10}$$

If we set $\lambda = e^{\pm i\pi/3} + \delta\lambda$, we find that $\delta\lambda = O(N^{-1})$, so these roots and their corresponding eigenvalues are not of interest, because they will not yield a quadratic speedup.

We now need to find the eigenvectors. If the components of the eigenvectors are denoted by x_j, where $j = 1, 2, ...5$, the eigenvector equations are

$$\begin{aligned}
-(r-t)x_2 + 2\sqrt{rt}x_4 = &\quad (-1 \pm i\Delta)x_1 \\
x_5 = &\quad (-1 \pm i\Delta)x_2 \\
2\sqrt{rt}x_2 + (r-t)x_4 = &\quad (-1 \pm i\Delta)x_3 \\
x_3 = &\quad (-1 \pm i\Delta)x_4 \\
x_1 = &\quad (-1 \pm i\Delta)x_5,
\end{aligned} \tag{4.11}$$

where $\Delta = (2t/3)^{1/2}$. To lowest order (the terms that were dropped are of order $1/\sqrt{N}$ or lower) the eigenvector corresponding to the eigenvalue

$-1 + i\Delta$ is

$$|v_1\rangle = \sqrt{\frac{1}{6}} \begin{pmatrix} 1 \\ 1 \\ -i\sqrt{\frac{3}{2}} \\ i\sqrt{\frac{3}{2}} \\ -1 \end{pmatrix}, \tag{4.12}$$

and the eigenvector corresponding to the eigenvalue $-1 - i\Delta$ is

$$|v_2\rangle = \sqrt{\frac{1}{6}} \begin{pmatrix} 1 \\ 1 \\ i\sqrt{\frac{3}{2}} \\ -i\sqrt{\frac{3}{2}} \\ -1 \end{pmatrix}. \tag{4.13}$$

For our initial state we choose

$$\begin{aligned} |\psi_{init}\rangle &= \frac{1}{2N}\sum_{j=1}^{N}(|0,j\rangle - |j,0\rangle) \\ &= \frac{1}{N}(|\psi_1\rangle - |\psi_2\rangle) + \sqrt{\frac{N-2}{2N}}(|\psi_3\rangle - |\psi_4\rangle), \end{aligned} \tag{4.14}$$

Which is in S. We find that

$$|\psi_{init}\rangle = \frac{i}{\sqrt{2}}(|v_1\rangle - |v_2\rangle). \tag{4.15}$$

Since the initial state is in S, and S is an invariant subspace of U , the entire walk will remain in S, and this reduces the complexity of our problem considerably. We should mention that the minus sign in the first expression for initial state is essential; if it is replaced by a plus sign, the search will fail.

In order to find the evolution of the quantum state for the walk, we find the eigenvalues and eigenstates of U restricted to S. This gives us the spectral representation of U and makes finding U^n, the operator that will advance the walk n steps, straightforward. We then find that, to good approximation assuming that N is large,

$$U^n|\psi_{init}\rangle = \frac{(-1)^n}{\sqrt{3}} \begin{pmatrix} \sin(n\Delta) \\ \sin(n\Delta \\ \sqrt{\frac{3}{2}}\cos(n\Delta) \\ -\sqrt{\frac{3}{2}}\cos(n\Delta) \\ -\sin(n\Delta) \end{pmatrix}, \tag{4.16}$$

where the first entry is the coefficient of $|\psi_1\rangle$, the second is the coefficient of $|\psi_2\rangle$, etc., and $\Delta = (2t/3)^{1/2}$. This is the state of the walk after n steps. From this equation and the definitions of $|\psi_1\rangle$ through $|\psi_5\rangle$, we see that when $n\Delta = \pi/2$, the particle is located on one of the edges leading to the extra edge or on the extra edge itself. This will happen when $n = O(\sqrt{N})$.

We now need to discuss how to interpret this result. It is reasonable to assume that if we are given a graph with an extra edge in an unknown location, we only have access to the edges connecting the central vertex to the outer ones, and not to the extra edge itself(if we had access to the extra edge, then we would have to know where it is). That is, in making a measurement, we can only determine which of the edges connecting central vertex to the outer ones the particle is on. If it is on the extra edge, we will not detect it. So, after n steps, where $n\Delta = \pi/2$, we measure the edges to which we have access to find out where the particle is. With probability 2/3 it will be on an edge connected to the extra edge, and with probability 1/3 it will be on the extra edge itself, in which case we won't detect it.

In comparing this procedure to a classical search for the extra edge, we shall assume that classically the graph is specified by an adjacency list, which is an efficient specification for sparse graphs. For each vertex of the graph, one lists the vertices not connected to the extra edge are connected only to the central vertex, and two of the outer vertices are connected to the central vertex and to each other. Searching this list classically would require $O(N)$ steps to find the extra edge, wile the quantum procedure will succeed in $O(\sqrt{N})$.

Chapter 5

Discussion

We have presented two entanglement conditions for spin systems that allow us to study the entanglement between blocks of spins. Most tests for entanglement in spin systems test for either complete separability or for entanglement between individual spins, and the results in this paper complement those. We have shown that in some cases the conditions involving blocks of spins can detect entanglement when tests of individual spins cannot. It was shown that our entanglement conditions can detect entanglement in intelligent spin states and in states of a spin chain containing a small number of spin waves. This latter result was then extended to show that entanglement in spin waves at finite temperature can also be detected.

We have presented two sufficient conditions for determining when multipartite states are entangled. These conditions are quite flexible, because the operators appearing in them can be chosen to best match the systems being considered. The conditions can be used to test for entanglement in discrete systems, continuous-variable systems, or mixture of the two. We have already seen that similar conditions for testing bipartite entanglement have proven useful in detecting entanglement in a variety of systems, including interacting spin systems and a collection of atoms interacting with the electromagnetic field.

We have shown that quantum walks can find extra edges in star graphs. The time complexity is $O(\sqrt{N})$ while the classical approach requires N steps.

Appendix

List of Publications

(1) H. Zheng, H. T. Dung, and M. Hillery, Phys. Rev. A **81**, 062311 (2010)

(2) M. Hillery, H. T. Dung, and H. Zheng, Phys. Rev. A **81**, 062322 (2010)

(3) E. Feldman, M. Hillery, H. Lee, D. Reitzner, H. Zheng, and V. Bužek, Phys. Rev. A **82**, 040301(R) (2010)

Bibliography

[1] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **47**(10): 777-780.

[2] E. Schrödinger, M. Born, Mathematical Proceedings of the Cambridge Philosophical Society **31**(04):555-563.

[3] A. Peres, Phys. Rev. Lett. 77, 1413C1415 (1996).

[4] R. Simon, Phys. Rev. Lett. 84, 2726 (2000).

[5] L. -M, Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000).

[6] M. Hillery and M. S. Zubairy, Phys. Rev. Lett. **96**, 050503 (2006).

[7] J. A. Bergou and M. Hillery, *A Short Course on Quantum Information*, page 18.

[8] J. S. Bell, On the Einstein Podolsky Rosen Paradox, Physics 1, 195-200 (1964).

[9] H. F. Hofmann and S. Takeeuchi, Phys. Rev. A **68**, 032103 (2003)

[10] A. Sørensen, L. -M. Duan J. I. Cirac, and P. Zoller, Nature **409**, 63 (2001).

[11] C. Aragone, G. Guerri, S. Salamo, and J. L. Tani, J. Phys. A **7**, L149 (1974).

[12] M. Hillery and L. Mlodinow, Phys. Rev. A **48**, 1548 (1993).

[13] M. Hillery, H. T. Dung, and J. Niset, Phys. Rev. A **80**, 052335 (2009).

[14] C. Kittel, *The Quantum Theory of Solids* (Wiley, New York, 1963), page 49.

[15] R. Feynman, *Statistical Mechanics* (Benjamin, Reading, 1974), page 198.

[16] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).

[17] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[18] M. Hillery and M. S. Zubairy, Phys. Rev. A **74**, 032333 (2006).

[19] E. Shchukin and W. Vogel, Phys. Rev. A **74**, 030302(R) (2006).

[20] G. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge Univ. Press, London, 1934).

[21] M. Zukowski, C. Brukner, W. Laskowski, and M. Wiesniak, Phys. Rev. Lett. **88**, 210402 (2002).

[22] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).

[23] M. Ardehali, Phys. Rev. A **46**, 5375 (1992)

[24] D. N. Klyshko, Phys. Lett. **A172**, 399 (1993).

[25] S. W, Ji, J. Kim, H. -W. Lee, M. S. Zubairy, and H. Nha, Phys. Rev. Lett. **105**, 170404 (2010).

[26] Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**(2), 1687 (1993)

[27] Y. Aharonov, A. Ambainis, J. Kempe, and U. Vazirani, In *Proc. of the 33rd ACM STOC*, 50-59 (2001).

[28] E. Farhi, and S. Gutmann, Phys. Rev. A **58**(2), 915 (1998).

[29] M. Hillery, J. Bergou, and E. Feldman, Phys. Rev. A **68**(3), 032314 (2003).

[30] D. Reitzner, M. Hillery, E. Feldman, and V. Bužek, Phys. Rev. A **79**(1), 012323, (2009).

[31] H. Krovi, and T. Brun, Phys. Rev. A **75**, 0622332 (2007)