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The mental representation of number in young children: Pictures, action, and language

Stevens, Patricia Joy, Ph.D.

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THE MENTAL REPRESENTATION OF NUMBER IN YOUNG CHILDREN: PICTURES, ACTION, AND LANGUAGE

by

Patricia J. Stevens

A dissertation submitted to the Graduate Faculty in Psychology in partial fulfillment of the requirements for the degree of Doctor of Philosophy, the City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Psychology in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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CHAPTER I

Introduction

There seems to be broad agreement among cognitive, developmental, and educational psychologists on a constructivist assumption about how mathematics is learned (Resnick, 1989). Recently a great deal of attention has been paid to children's early representations of number and number knowledge. While it may be generally agreed that the knowledge children bring with them to a learning situation will influence what they learn, there is less clarity and agreement on the nature and origin of children's mathematical representations. This presents a problem for educators who want to fine tune their instruction to the needs and abilities of individual children.

Representation: A Special Kind of Memory

Mandler's (1983) distinction between two usages of the term representation is well known; it can refer to symbols, or it can refer to knowledge and the way it is organized. This distinction resembles Piaget's (1977) distinction between memory in the strict sense and memory in the broad sense.

Bruner (1964) has provided another definition of representation which will be used in this study.

> "For the most important thing about memory is not storage of past experience, but rather the retrieval of what is relevant in some usable form. This depends upon how past experience is coded and processed so that it may indeed be relevant and usable in the present when needed. The end product of such a system of coding and processing is what we may speak of as a representation."

This study will treat mathematics as a representational system, and investigate aspects of it's development in early elementary school children. But first, seven research examples will be presented to illustrate ways in which different researchers have looked at children's representations of number.

Culturally Provided Symbols as Shapers of Representations

Miura, Kim, Chang, and Okamoto (1988), and Miura and Okamoto (1989), have used the construction of quantities with base-ten blocks by elementary school children as evidence of the internal representation of number. Miura et al. point out that the spoken numerals of Oriental languages describe precisely what is represented by the base 10 numeration system. For example, 46 would be four-ten(s)-six. Twelve is read as ten-two. They argue that this language difference may result in fundamental differences between Asian and American students in the cognitive representation of number. Speakers of Asian languages learn place value as they learn the number names, so this information is built into their representations of number. Since place value understanding is useful in addition and subtraction with regrouping, as well as multiplication and division, Miura et al. believe that better cognitive representations which include information on place value may importantly contribute to the mathematics achievement of Asian students.

In their studies, Miura and her colleagues compared the way American children and speakers of Asian languages constructed numbers with tens and unit blocks. In the second study cited above, they also compared American and Japanese students' understanding of place value. They found that (1) the block constructions of the Asian students in the first of two trials used more ten blocks than the constructions of the American students, who, in the first of two trials, used more unit blocks. Over two trials, more Asian children than American children were able to construct numbers in two ways, using different combinations of ten and unit blocks. (2) Japanese children scored significantly higher than American children on a five item test of place value understanding. The block construction task was too difficult for American Kindergarten students, who therefore could not be included in the study. Since they were not able to clearly and consistently count up to 50, they could not do the task.

Miura et al. stated that their results suggest that language differences may positively affect the Asian children's understanding of place value, and their subsequent mathematics achievement. Moreover, since more Asian than American children were able to construct numbers in two ways, they concluded that the Asian children had greater flexibility of number manipulation. Two assumptions appear to underlie this research. First, the construction of numbers with blocks will provide evidence about the nature of number representation. Second, the meaning content of number words, whether Asian or English, is present in the mental representations of numbers in children's minds in such a way as to hinder or facilitate the use of procedures to operate on the numbers.

Representations as Tools for Thinking About a Domain

Miller and Stigler (1987) looked at the developmental course of counting in English and Chinese. For these researchers, culturally provided conventional representations of numbers serve as tools for thinking about a domain. Moreover, while the culture provides the symbols, the understanding of the symbols is based on learning to count. Their study looked at patterns of counting errors in American and Chinese preschoolers in order to distinguish features of counting due to language from more intrinsic aspects of cognitive development. The children were asked to count as high as they could in order to test their number generation process; and they were given an object counting task to test their ability to determine the numerosity of sets. These researchers found that (1) while American children made more errors overall than Chinese children, patterns of errors were similar for both groups of students. Counting includes coordinating the production of number names and distinguishing items which are already counted from those which are not. Differences in number naming errors were not associated with the ability to tag objects properly when counting. This, say Miller and Stigler, suggests that processes for generating numbers and strategies for tagging objects develop independently. (2) For both American and Chinese children, errors were most likely to occur at the decades (i.e., at 20, 30, etc.). There was no significant difference between groups for these errors, even though it is easier to generate a new decade number name in Chinese. Miller and Stigler stated that coordinating the incrementing of two series, decades and units, is a general problem which is independent of language. Miller and Stigler concluded that while counting forms the basis for children's early understanding of numerical symbols, counting based representations of numbers are elaborated during elementary school as children learn other features of numbers. Therefore, the effects of early differences in the acquisition of counting on later mathematical achievement have yet to be determined.

Mental Representations of Numeric Principles: Implicit and Explicit Knowledge

Gelman and Gallistel (1978) and Gelman and Baillargeon (1983) postulate implicit knowledge of mathematical principles in young children which structure and support the development of mathematical understanding. For example, a stable order principle and one-to-one correspondence are two of five implicit principles involved in the acquisition of counting. Gelman and Baillargeon (1983) offer the following evidence that preschoolers possess these principles. First, Gelman and Gallistel (1978) found systematic, if sometimes incorrect, counting procedures in children as young as 2 1/2 years, for example, counting 2,6,10. These procedures revealed a stable order principle, although the children had not yet acquired the necessary list of verbal tags. These procedures suggest, say the authors, "a scheme in search of a list". Second, Gelman and Gallistel (1978) reported spontaneous counting by young children, as well as spontaneous self correction of errors. This suggests the use of a scheme as a point of reference, and also as a motivator of practice. Gelman and Baillargeon (1983) state that there is evidence for structural change in the development of number concepts which involves the Piagetian concepts of assimilation and accommodation. A major development in the understanding of number is a shift of attention from number per se to reasoning about relationships between numbers. This includes conscious access to numeric principles, which preschoolers do not possess. There is something about structural change in the organization of number knowledge which allows it to become explicit.

Representations: "Protoquantitative Reasoning Schemas"

For Resnick (1989), infants and preschoolers have nonlinguistic "protoquantitative reasoning schemas", such as a part-whole schema, an increase/decrease schema, and implicit knowledge of the additive property of quantities. Groen and Resnick (1977) found that 4 1/2 year old children spontaneously invented addition algorithms such as adding on. For example, in order to add two plus five, a child might use what the authors called the minimum addend strategy, incrementing five two times, as in "five, six, seven." Resnick and Omanson (1987) report a series of studies with elementary school children in the second and third grades, of errors in subtraction. They found that (1) elementary school children not only learn more numbers, but they learn codified directly taught procedures, such as subtraction with regrouping, for operating on the numbers. They extensively practice these procedures so that they become automatic and embedded within the developing representational system. Resnick and Omanson (1987) identified four sets of principles which are necessary to do subtraction correctly: (1) additive composition of quantities, (2) conventions of place value notation, (3) calculation through partitioning (column by column), and (4) recomposition and conservation of the minuend. The first of these is described by Resnick as a non-linguistic protoquantitative schema. The others are taught in school, and may also be learned informally in the preschool years. When children automatically do subtraction algorithms, they do not appear to consciously access the principles, and this can result in errors. For Resnick and Omanson, the understanding which is necessary for correct subtraction involves conscious access both to principles and to the quantitative meaning of numeric symbols. These findings invite speculation about ways in which developing mathematical representation is interpenetrated and supported by other representational systems such as language or perhaps event representations. Indeed Resnick and Omanson describe subtraction which results in systematic errors as syntactic. To them, this means that it reflects rules for symbol manipulation, but does not embody knowledge of quantity principles or the quantitative meaning of the numeric symbols. They suggest that semantic knowledge needs to be "mapped on" to syntactic knowledge.

Piaget:Number Knowledge Depends On General Cognitive Structures

Gelman et. al.'s implicit numeric principles, and Resnick et. al.'s protoquantitative reasoning schemas are understood to be domain-specific mental structures. For Piaget (1965), the child's understanding of number results from the development of a system of general operational structures. Early reflexive schemas form the basis for later more complex structures which result from the child's actions on objects. For Piaget (1965), a notion of quantity is the same as the conservation of quantity; "The child discovers true quantification only when he is capable of constructing wholes that can be preserved." So, a child does not understand what a number means unless she also understands that it represents a set, and has reached a stage of development in which seriation and classification are coordinated. For Piaget that cannot occur until the stage of concrete operations, which is distinguished by the reversibility of relations. Only then can the child be said to have an understanding of number, which is at the same time a hierarchical class and a series.

For Gelman et. al., one-to-one correspondence is one of five counting principles; and counting forms the basis of the child's understanding of number. For Piaget, one-to-one correspondence is a "tool used by the mind in comparing two sets". Correspondence is not true correspondence until it becomes quantitative instead of perceptual; and until it becomes operational and reversible. In a Piagetian view of the development of mathematical knowledge, there are three major disagreements with the studies described above. First, the child's understanding of number, in Piagetian theory, develops as part of a system of general operational structures. The nature of the organization of knowledge is general rather than domain specific. Therefore one-to-one correspondence, which Gelman et. al. describe as an implicit counting principle; and part/whole relationships, which Resnick et. al. describe as protoquantitative schemas, serve more general logical purposes which go beyond the understanding of number. Second, the reorganization of mental structures results in qualitatively distinct stages of development. For all of the implicit mathematical principles and protoquantitative schemas Gelman et al. and Resnick et al. have found in their subjects, Piaget would probably have said, "____ becomes true ____ only when it becomes operational and is linked with____". This is in contrast to Gelman et.

al.'s structural change in implicit counting principles. Third, action on objects results in the processes of assimilation and accommodation by means of which the child actively constructs knowledge. This is more important than the initial form or guidance of innate schemas.

Representation of Number: Part of a System of Shared Meanings Derived From Events

For Nelson (1985), children develop a system of shared meanings. Their representations are not merely internalizations of what they see and hear. Instead, representations are based on experience of relationships in the world, and are mediated by spoken language. Nelson uses general event representations to model natural knowledge systems which result from children's interactions with the social world. These event representations or scripts, are "schematic, relating diverse elements in a holistic structure by means of different types of relationships". Fulani (1984) has used this model to describe the development of children's early understanding of number symbols. In a study with preschoolers, she found that numbers first appear embedded in scripts. The context specifies what to do with the numbers and how to talk about them, so the child doesn't need to understand underlying mathematical relationships. Children's informal number symbol use and knowledge is context or script-bound. Preschoolers must learn to distinguish between numbers as labels (such as the M16 bus) and numbers as quantities. Fulani applied Vygotsky's theory of spontaneous and scientific concepts to the development of number knowledge, although, as she herself points out, Vygotsky's definition of spontaneous concepts is not easily applied to numerical concepts. Spontaneous concepts, she says, or informal number knowledge derived from event representations, provide a groundwork for scientific concepts, or formally organized mathematics lessons in school, which then raise the spontaneous concepts to a new level of awareness. This view of the development of mathematical knowledge would benefit from a richer characterization of spontaneous numeric concepts which would include numeric relationships as well as the use of numbers as labels. However, Fulani's research draws attention to event representations as sources of implicit numerical knowledge, including Resnick's protoquantitative schemas as well as Gelman's counting principles.

<u>Numerical Understandings: Goal Directed Adaptations</u>

For Saxe (1985), and Saxe, Guberman and Gearhart (1987), general cognitive processes operate within social interactions, which provide a context for novel cognitive development. For Saxe et al.(1987), children's numerical understandings are goal directed adaptations to their numerical environment. At the same time, the child's numerical understandings actually can help to generate environments. Children engage in numerical activities with adults or more capable peers, such as setting the table and getting the correct number of forks and spoons, or playing counting games with their mothers, or comparing numbers of candies with friends.

Saxe et al.'s description of these activities differs from event representations in several ways. They do not appear to be temporally or spatially organized, nor is a schematic organization indicated. A mother-child dyad, with the mother teaching the child how to count, is an example of an activity in a dynamic environment which can be effected by the child. The organization of the activity is provided by the task which the dyad is trying to achieve. Saxe et al. state that the goal structure of the activity emerges through reciprocal adjustments by mother and child. In other words, task organization emerges through the interaction of the participants. The form this organization would take once it gets inside the child's mind is not clear. Whereas information and relationships within an event are structured schematically within an event representation, this does not appear to be the case for information and relationships contained within the goal structures of activities. In a study of mother-child dyads, in which (1) a battery of assessment tasks was given to two- and four-year-old children, (2) mothers were interviewed about their children's numerical activities, and (3) mothers taught their children counting and number reproduction activities, Saxe et al. (1987) found that (1) children actively participate in socially organized experiences, in GENERATED environments; which are both a product of the child's numerical understandings, and a source of new understandings. (2) Mothers adjusted the goal structures of the tasks to their children's abilities. (3) Children generate goals in order to perform tasks in these environments, and the goals that are generated at different ages reflect different levels of complexity.

While Vygotsky or Leontiev would look at the structure of the task, no doubt pointing out that the structure of the task exists both inside and outside of the child's mind, Saxe et al.'s study shows that what a child brings to the task can affect its structure. Whatever the structure of the internalized information, the result is different levels of "numerical understandings". Saxe et al. (1987) propose a four stage developmental sequence of numerical functions. The stages differ with respect to the complexity of the correspondence operations involved. For example, the first stage includes counting with number words, and involves one-to-one correspondence between words and objects. We have seen this before, included in Gelman et al.'s (1978) counting principles. In contrast, the fourth stage includes elementary arithmetic operations with sets. Sets are described as "summations of correspondences". Therefore, elementary arithmetic involves "relating and manipulating summations of correspondences". Saxe does not specify how information extracted from socially organized activities would alter the child's representation of mathematical knowledge so as to allow more complex functions to emerge. For Saxe et al., numerical experiences are not contained either in the activities or in the minds of the participants; they emerge in interactions during efforts to accomplish a task.

Summary of Research

The seven approaches to researching children's mathematical understanding which have been presented offer a diverse set of possible contributors to the development of a representational system dealing with number. Perhaps the simplest approach, that of Miura et al., assumes that the meaning content of culturally provided number words actually shapes the representation of number. For these researchers, children seem to get the meaning as they learn the words. This gives speakers of Asian languages, which contain information about the base ten system in the number words, an initial advantage over English-speaking children in mathematical achievement.

Miller and Stigler say that the culture provides the symbols, but counting gives them meaning. Gelman et al. and Resnick et al. discuss domain specific structures such as implicit mathematical principles or protoquantitative reasoning schemas. These structures get developed to the point where the child has conscious access to them and can reason about numeric relationships. Another way of thinking about having conscious access to principles is to say that the principles have meaning for the child. What is it about the mental representation of number which would make it easier to access implicit knowledge and make it explicit? Could it be a matter of well established linkages within and between systems of information?

Miller and Stigler, Gelman et al., and Resnick et al. discuss mathematical understanding as a specific domain; in contrast to Piaget, for whom it is part of more general purpose cognitive structures.

In Nelson's research, children develop a system of shared meanings derived from events. For Saxe et al., numerical understandings are goal directed adaptations to a social environment. In all of these approaches, language and meaning play a role, albeit in different ways or with different emphases.

A preliminary study was designed to investigate two related themes suggested by the above studies.

The Contribution Of The Meaning Of Number Words To Numerical Understanding

Miura et. al. suggest that Asian number words, which include information about the base ten system, actually shape the mental representation of number and facilitate the flexible use of base ten knowledge. If English-speaking children could construct numbers with blocks in ways which show flexible knowledge of the base-10 system this would suggest the importance of other contributors to numerical understanding beyond the number words.

The Contribution of Different Modes Of Representation To Numerical Understanding

Bruner's (1973) characterizations of representation as enactive, iconic, or symbolic provide additional ways of thinking about children's mathematical knowledge. According to Bruner, it is through processing systems of action, imagery, and language that humans construct models of the world which allow them to function. Through representation, one preserves one's encounters with events, and the representation is in some medium: the actions required by an event, pictures, or symbols. For Bruner, growth would involve successive mastery of these three forms of representation along with the ability to translate one into the other.

Will children reveal the same level of quantitative knowledge in all three modes of representation? Piaget (1977) accounted for decalage between different types of conservation by saying that knowledge which is mastered at the level of action must be mastered again at the level of thought, and this is done unevenly. If there is unevenness of development within the domain of mathematical knowledge along the lines of Bruner's three modes of representation, then this suggests alternative ways of structuring and restructuring thought.

It may be that the ability to translate knowledge from one modality to another is associated with the conscious access to numeric principles which Gelman et al. and Resnick et al. say comes with development. If so, it would be useful to know whether information develops evenly in these modalities.

CHAPTER II

Preliminary Study

A preliminary study was conducted in order to address two questions. (1) Would English-speaking children be able to construct numbers with blocks in ways which show flexibly used knowledge of the base ten system? That is, would they use a variety of block constructions with different numeric combinations? (2) Would children demonstrate different levels of number knowledge in different representational modes?

Method

Subjects

Forty children consisting of ten children (five boys and five girls) each from grades Kindergarten through three were selected from a New York City public school located in Queens. The school was not Chapter 1 eligible; that is, it did not qualify for the federal funding for compensatory education which is provided to schools in severely disadvantaged neighborhoods. Thus the research problems associated with extreme poverty were avoided, although the school's population was mixed both with regard to ethnicity and socio-economic status (SES).

The classroom teachers were asked to select children distributed over a range of mathematics achievement, ranging from low to medium to high. These teacher assessments were validated for second and third grade students by collecting their scores on the Metropolitan Achievement Test (MAT). However, students in Kindergarten and first grade are not given standardized tests in this school.

Procedure

Children were individually taken from their classrooms to another room in their school to do a task which included an introduction, reading eight two-digit numbers aloud from a card, drawing pictures of what the numbers meant, locating these numbers on a number line, constructing the numbers with tens and units blocks, and verbally contrasting pairs of numbers. The task was designed to measure quantitative knowledge in Bruner's enactive (block constructions), iconic (drawings), and symbolic modes of representation (reading numbers from a card, and verbal contrasts).

Introduction. The introduction included an explanation of the task, a demonstration of the values of the blocks, and an opportunity to practice using the number 24. The following sentences were used in the introduction.

1. Before we begin, I'll show you what we'll be doing. First, I'll ask you to read a number from a card, like this. Go ahead and read the number.
2. Next, I'd like you to draw me that number. Make me a drawing that shows what 24 means. If you had to explain what 24 meant to a younger child, what kind of drawing would help you explain it? (Children who could not think of drawings were asked what kind of pictures their teacher might put on the board to explain it.)
3. Now I'd like you to show me where that number is on this number line.
4. Now I'd like you to show me that number using these blocks. The yellow blocks stand for ones, and the red blocks stand for tens. So, ten yellow blocks equals one red block (demonstration).

Numbers Used. The following eight two digit numbers were used in random order: 43, 14, 41, 32, 24, 23, 34, 13. Six of them (43/34, 14/41, 32/23) were numbers whose digits were reversed, and were used for the verbal contrast of number pairs at the end of the procedure.

Materials. Numbers were written on large index cards. Paper and pens were provided for drawings. A number line was drawn which showed the numbers from one to 50, with longer vertical lines coming down from the decade numbers. This study was not planned as a replication of Miura et al.'s, so base-ten blocks were not used. Red and yellow TYCO blocks, which are the same as LEGOS, were used instead. The basic blocks were uniform in size (1.5 by .5 inches) and shape, only differing in color. Only 39 yellow unit blocks were provided in order to see what children would do when they had to construct numbers in the 40s.

Block Constructions. For each of the eight numbers, unlike the introduction, once a child constructed a number with blocks he/she was asked, "Can you think of another way to show me that number?" This question was asked as often as necessary in order to get the child to produce a maximum number of constructions. Two constructions were possible for numbers in the teens (i.e., for the number 13, either one red and three yellows; or 13 yellows could be used.), three constructions for numbers in the twenties, four constructions for the thirties, and five possible constructions for numbers in the forties.

If a child answered "No", and other constructions were possible, additional prompts such as "Are you sure?", or "I bet you can think of another way to show me that number" were used.

Verbal Contrasts of Number Pairs. After reading numbers from a card, drawing the numbers, locating them on a number line, and constructing them with blocks, children were asked to verbally contrast up to five (time permitting) number pairs, which were combinations of the numbers they had just worked with. They were asked, for example, "How is 32 different from 23? Write the numbers down and tell me how they're different." In addition to 32 and 23, the other contrasts were 14/13, 41/14, 43/34, and finally 24/34.

Changes in Procedure

After piloting with the first 11 students, it became clear that some changes had to be made in the procedure.

Order of Administration. Since some children in grades Kindergarten and one had difficulty with the larger numbers, the order of administration, instead of random, became: 14, 23, 13, 32, 24, 34, 43, 41. In this way, at least partial data could be gathered from the younger children.

Number of Drawings. Since the amount of time for which a child could be taken from a classroom was limited, and since some children seemed to enjoy drawing and took a long time with this part of the task, drawings were requested only for the first four numbers. If children took a very long time, they were only requested for the first two numbers.

Awareness of the Number of Yellow Blocks. At the end of each block construction, children were asked "Now what do we have here? __ reds and how many yellows?" This provided information on whether a child knew the number of yellow blocks, or had to count them.

Questions Asked During Verbal Contrasts. In addition to asking how one number was different from another, three more questions were asked, depending on the child's initial response. (1) Which number is bigger? (2) How do you know that number is bigger? (3) How much bigger is __ from __?

Results

Reading Numbers Aloud From A Card

Four out of ten Kindergarten students and two first graders had difficulty reading the two digit numbers. For example, they read 24 as "two..four". One Kindergarten student read 14 as "one..four". The other 34 children were able to read the numbers accurately from the cards.

Locating Numbers on a Number Line

Two Kindergarten students were unable to do this without help. Five other Kindergarten students and two first graders moved their fingers back and forth along the number line until they found the number they were looking for. Three Kindergarten students, eight first graders, and all of the second and third graders were able to easily locate the numbers on the number line.

Drawings

Table 1 shows the responses of students to a request for a drawing of what a number meant. About a fourth of the students made from zero to eight drawings, and the rest made from zero to four drawings. Students who produced two or more drawings, with a few exceptions, did not change their style. They consistently remained within one of the following categories.

Unable To Produce a Drawing. Seven students in Kindergarten, and one at each of the other grade levels, were unable to produce a drawing. Since the classrooms all had numerous drawings on the walls, it is difficult to say that the students were unable to draw. It is possible that they were unable to make sense of the experimental request, or perhaps were unable to connect drawings with numbers.

Numbers Used As Labels. One of the three Kindergarten students who produced a drawing, three students each in grades one and two, and one student in the third grade used numbers as labels. Typical

TABLE 1

Categories of Drawings For Students in Grades K Through Three

Category of Drawing	Grade K	1	2	3
Unable to produce a drawing	7	1	1	1
Numbers used as labels	1	3	3	1
Drew actual number of objects	2	3	3	3
Drew one example of actual number of objects	0	2	0	1
Wrote numbers expressing number composition; i.e., 10,20,30,40,1 for 41	0	1	0	0
Wrote math problems whose answer was the number	0	0	2	1
Used talley marks or arrays of lines, blocks, or numbers to show base ten information	0	0	1	3

examples were the Kindergarten student who said, "I'll draw a person who's 24 years old.", and drew a cartoon figure with 24 on the forehead. A first grader said, "I'll draw a birthday cake for a 24th birthday." Then she actually drew six candles and one big candle with 24 written on it. A second grader drew a road sign with 43 on it to represent a speed limit of 43 miles per hour.

Twenty percent of all the students, or 24 percent of the students who produced pictures, fell into this category.

Drew Actual Number, or Examples of an Actual Number of Objects. The other two Kindergarten students who produced pictures, and three students at each of the other grade levels, drew the actual number of objects. One Kindergarten student said, "I'll draw trees, the number of 24." A first grader said, "You mean like 24 things?", and proceeded to draw 24 different things, explaining each one as he went along.

In addition to these 11 students, two first graders and one third grader drew one example of the actual number of objects. One first grader said "24 books", drew one book, pointed to it and said, "That's one." Another child said "24 pictures", drew a picture in a picture frame, and said, "I'll just draw one."

When these two categories are added together, the total of 14 students represents 35 percent of all the students, and 42 percent of the students who produced drawings. If these drawings are taken as evidence of knowledge of one-to-one correspondence, then this knowledge was demonstrated by more than a third of the students, and evenly spread across grades.

Other Kinds of Drawings. None of the Kindergarten students, one first grader, three second graders, and four third graders produced responses in which the influence of school math could be discerned. Some of these included talley sticks or arrays of blocks. Others were not actually drawings, but because of their representational content, they are included in these categories. A first grader (the same child who drew 24 "things") in order to "draw" 43, wrote 10,20, 30, 40, 41, 42, 43. Two second graders and one third grader composed mathematics problems whose answers were the desired numbers. For example, 10+10+4=24 was produced instead of a drawing. Others drew either tally sticks or arrays of blocks or Xs and Os which showed the composition of numbers and base ten information.

Judging from the drawings these students produced, we might expect that only students in the last categories would construct numbers using both tens and units blocks. We might expect the students who demonstrated one-to-one correspondence in their drawings to only use unit blocks. However, that was not the case.

Block Constructions

Table 2 shows the types of block constructions students produced, and their distribution by grade.

Kindergarten. Two Kindergarten students were unable to do the task. After reading the numbers, one digit at a time, they attempted to arrange blocks into configural patterns which looked like the numbers on the card. One of these students produced a drawing with the number used as a label, and the other did not produce a drawing. Two numbers into the task, the procedure was discontinued for these students.

Five Kindergarten students only used yellow unit blocks in their constructions. Four of them had difficulty counting, and could not construct the higher numbers. They were able to construct numbers in the teens and twenties, and three of them constructed numbers in the thirties. None of these students produced drawings.

Three Kindergarten students produced a variety of block constructions, using not only units blocks, but

TABLE 2

Types of Block Constructions For Students in Grades K Through Three

Type of Construction	Grade K	Grade 1	Grade 2	Grade 3
Unable to construct number, or used configural patterns	2	1	0	0
Only used yellow unit blocks	5	1	0	0
Only used number of red blocks equal to highest number of tens in the number (i.e. four reds for 43) plus yellows for the units. (Maximum reds)	0	0	0	1
Only used red blocks plus groups of yellows for tens, plus yellows for units (< maximum reds)	0	0	0	0
Used both units and maximum reds constructions	0	1	1	0
Used maximum and < maximum reds	0	0	0	1
Used units, maximum reds, and < maximum reds	3	7	9	8

also using different combinations of tens and units blocks. They had no difficulty in counting. One did not produce a drawing, but two drew the actual number of objects. Their drawings did not reflect the knowledge of the base ten system which was revealed in their block constructions.

Eighty percent of the Kindergarten students were able to do at least part of the task, and forty percent did the complete task. Three out of 10 showed flexibly used base ten knowledge in their block constructions, which is not taught to Kindergarten students in school. These three also used units constructions, demonstrating knowledge of one-to-one correspondence as well.

First Grade. One first grader was unable to do the task. He tried to use unit blocks, but did not know how to count well enough to stop at the desired number. He had attempted to draw "24 people", but actually kept drawing and went past 24. Three numbers into the task, the procedure was discontinued.

One student only used yellow unit blocks, and did not produce a drawing. Another student began with only unit constructions, but on the fourth number, switched to using red blocks for the maximum number of tens in the number. This student produced a drawing with a number used as a label.

Seven out of ten first graders produced block constructions with tens and units blocks. Six of these also used units constructions. These students demonstrated both knowledge of one-to-one correspondence, and also flexibly used base ten knowledge in their block constructions. Five of these students drew the actual number of objects, so their drawings only revealed knowledge of one-to-one correspondence.

Second and Third Grade Students. All twenty of the second and third grade students demonstrated base ten knowledge in their block constructions, and for 19 of them, this knowledge was flexibly used in a variety of constructions. One student only produced constructions which used red blocks for the maximum amount of tens in the number. For 13 of these 20 students, the knowledge of a base ten system revealed in their block constructions was not revealed in their drawings.

Verbal Data

Awareness of the Number of Yellow Blocks In Base Ten Block Constructions. Out of all the children across grades who used base ten block constructions, the only time there was an awareness of the number of yellow blocks was when the maximum number of red blocks had been used. For example, when four red blocks and three yellow blocks were used to construct 43, it was easy for the children to see that there were three yellow blocks.

However, when other combinations of tens and unit blocks were used, all of the children counted up with yellow blocks. For example, in order to construct 32, one child put down two red blocks, then started adding yellows by twos, saying, "20,22,24,,26,28,30, and 32". He had to count the yellow blocks to find out that he had 12. The counting up appeared to be a procedure, and the children had access to the end result, but not to the component parts.

Verbal Contrasts of Number Pairs. Three of the children in Kindergarten, and most of the children in grades one through three who did base ten block constructions, gave at least one answer similar to the following example: "32 has a three before the 2, and 23 has a two before the three." This surface description of the numeric symbols does not reveal the base ten knowledge included in the students' block constructions. All of the students who were asked however, knew which number was bigger than another. Because of time constraints, it is unfortunate that follow up questions could not be asked systematically of all the subjects in the preliminary study.

The children in this study demonstrated flexibly used base ten knowledge in their block constructions, but did not demonstrate the same level of quantitative knowledge in their drawings and verbal contrasts of number pairs. Moreover, the attention paid in the verbal contrasts to the surface appearance of the numeric symbols did not seem to be linked to the meaning of the symbols.

Twenty five percent of the subjects did not produce drawings, however this cannot be taken as evidence that these children were not iconically representing quantitative information.

Therefore another study was designed to further investigate these phenomena, which would include time for systematic follow up questions about number contrasts. Since the number line, and having an insufficient number of unit blocks for the higher numbers did not substantially increase the data about quantitative knowledge in three modes of representation, these features of the procedure were eliminated.

CHAPTER III

Method

Subjects

Fifty four English-speaking children were selected from grades Kindergarten, one, and two, in the same school which was used for the pilot study. This school was not Chapter 1-eligible; that is, it did not qualify for Federal funding for compensatory education for children in educationally disadvantaged neighborhoods. Chapter 1 eligibility is linked to the number of children in the school who qualify for free lunches, and also to the number of children in the school who are in the lowest quartile on standardized achievement tests in reading and math. The school was in Queens, and the student population was ethnically diverse.

While the pilot study included students in grades Kindergarten through three, the results for second and third graders were similar enough that this study only included grades Kindergarten through two.

Subject Selection

Parental consent letters were distributed to all children in six classes, since the school had two classes at each grade level. Of the students whose parents returned consent letters, the following groups of students were not included in this study: Asian students, Limited-English-Proficient students, and students who had been included in the pilot study the previous year.

This resulted in a sample of 19 Kindergarten students (seven boys and 12 girls), 17 first grade students (eight boys and nine girls), and 18 second grade students (nine boys and nine girls). Their ages ranged from five years, four months to six years (in Kindergarten), from six years, two months to seven years, two months (in first grade), and from seven years, eight months to eight years, five months (in second grade).

Teacher Ratings

When parental consent letters were sent out, teachers were asked to assign either a low, medium, or high math ability rating to each of the English-speaking, non-Asian students on their class rosters. Each teacher was asked to think about math activities done in her classroom, and to base her ratings on each child's performance on those activities. The above sample of students had the following ratings: (1) the 19 Kindergarten students included three rated as low, seven rated as medium, and nine rated as high; (2) the

17 first grade students included three rated as low, eight rated as medium, and six rated as high; (3) the 18 second grade students included four rated as low, nine rated as medium, and five rated as high.

When the consent letters were distributed, one of the teachers commented that the parents of children rated as low ability would probably not consent to their children's participation in the study. This prediction was bourne out, since only 10 students, or 19 percent of the total sample, had a teacher rating of low ability.

Standardized Test Scores

In the school used in this study, standardized tests are not administered to children in Kindergarten and first grade. The math score on the Metropolitan Achievement Test (M.A.T.) was collected in N.C.E.s (normal curve equivalents) for second grade students. These scores ranged from 52 N.C.E.s to 99 N.C.E.s. Since an N.C.E. of 50 is considered grade level performance, all of the second grade students were at or above grade level in math achievement.

Procedure

The procedure used in this study was similar to that used in the pilot study, with modifications suggested by the pilot data. The entire procedure was audiotaped and transcribed.

Materials

Numbers were written on large index cards, and paper and pens were provided for drawings. The same red and yellow tyco blocks as in the preliminary study were used.

The use of tyco blocks presents a contrast to the base-ten blocks used by Miura et al.(1988). Base-ten blocks consist of small unit blocks plus larger ten blocks, which are equal in size to ten of the unit blocks, and which have unit-sized segments marked off on them. These blocks are considered an age-appropriate method for instructing young children in the base-ten features of the number system. Since the purpose of this study was to discover what knowledge of the base-ten system children would reveal through action on objects, the construction of numbers with blocks which did not present a concrete clue of differential size was more appropriate.

Introduction

The introduction was the same as it was in the preliminary study. First, students were asked to read a two digit number, 24, from a card, in order to ascertain their knowledge of two digit numeric symbols.

Next, they were asked to draw a picture of what that number means. In other words, they were told, suppose a younger child were to ask "What does 24 mean?" What kind of a picture could they draw which would help explain to that child what 24 means? If they could not think of anything, they were asked what kind of a picture their teacher might put on the blackboard to show what 24 means. If they still could not think of anything, the experimenter moved on to the next part of the introduction.

Next, they were asked to "show me 24 using these blocks." The experimenter had a pile of red and a pile of yellow tyco blocks. The basic blocks were used, which only differed in color. The children were told, "Each yellow block equals one, and each red block equals 10. In other words, one red block (the experimenter pulled out a red block) equals 10 yellow blocks" (the experimenter pulled out 10 yellow blocks). If the child looked puzzled, this information and the demonstration were repeated. So, the information which the child was given, both verbally and by demonstration, about the blocks was: (1) yellow blocks equal one, and red blocks equal 10, and (2) one red block equals 10 yellow blocks.

After the child produced 24 in some way with the blocks, she was told that we would be doing this with five numbers, and asked if she had any questions.

Choice of Numbers

The following five numbers were used: 12, 16, 21, 26, and 36. The selection of these numbers was based on the following considerations. First, these numbers provide a range of two (12 and 16) to three (21 and 26) to four (36) possible block constructions. The maximum number of block constructions for each number is the sum of the maximum number of tens (i.e. three 10s in 36) plus one (for constructions using unit blocks). So for 36, a child could produce 36 yellow blocks, three reds and six yellows, two reds and 16 yellows, and one red and 26 yellows; for a total of four possible constructions. The maximum number of constructions a child could produce in the course of the task was 14.

Second, pilot data showed that some children in Kindergarten and first grade had difficulty with numbers in the 30s and 40s. The numbers were selected in order to maximize data collected at all three grade levels.

Lastly, four numbers (12 and 21; and 16 and 26) were selected with verbal contrasts of number pairs (see below) in mind.

Drawings

Since pilot data showed that the type of drawing produced tended to remain the same from number to number, children were only asked to produce drawings for the first two numbers, 12 and 16.

Picture Choice

Since pilot data showed that some children did not produce drawings, a picture choice task was added in this study. For the next two numbers (21 and 26), the children were told, "Now instead of asking you to draw me a picture, I have some pictures for you to look at. Here are five pictures. Look at all five, and tell me which one best explains what (21) (26) means." These pictures (see Appendix A) were based on the drawings produced by children in the pilot study.

Two of them used numbers as labels. A numbered candle on a birthday cake or a doll with a number on its forehead or clothing were drawings the pilot subjects used as examples of someone's age. Two other pictures showed the actual number of objects; i.e.

either 21 balloons or dots, or 26 cats or triangles. In the pilot study these kinds of drawings were taken as evidence of one-to-one correspondence.

A fifth picture showed either an array of red tens and yellow units blocks equal to 21 or an array of Xs and Os standing for 26.

Block Constructions

For each number, after a child had produced an initial construction, he/she was asked, "Can you think of any other way to show me that number?" Children who said no were encouraged by the experimenter; "Oh I bet you can think of another way." Children who still could not think of another construction went on to the next number.

The first four numbers, 12, 16, 21, and 26, were preceded by either drawings or pictures. The fifth number, 36, was not.

Verbal Contrasts of Number Pairs

In order to investigate the children's verbal expression of number knowledge, they were asked to contrast two pairs of numbers: 12 and 21, and 16 and 26. Pilot data suggested that contrasts of numbers whose digits were reversed, as well as numbers which were 10 away from each other, would produce some interesting responses. For each number pair, the children were shown the numbers written next to each other, and were asked, "Suppose you had these two numbers, 12 (or 16) and 21 (or 26). How is 12 (or 16) different from 21 (or 26)?"

If the child made no reference to the size of the numbers in his/her response, he/she was asked, "Which number is bigger?" All of the children were asked "How do you know that (21) (26) is bigger than (12) (16)?" They were also asked how much bigger one number was from the other.

Sequence of Tasks

After the introduction, the following sequence was followed: (1) Children were asked to read numbers from a card, produce drawings of what the numbers meant, and produce block constructions, for the numbers 12 and 16. (2) Children were asked to read numbers from a card, choose pictures which best explained what they meant, and produce block constructions, for the numbers 21 and 26. (3) Children were asked to read 36 from a card and produce block constructions. (4) Children were asked to verbally contrast two number pairs: 12 and 21, and 16 and 26. Based on their replies, they were asked which number was bigger. (5) All children were asked how they knew one number was bigger than another.

Finally, they were asked how much bigger one number was from another.

Ratings on Tasks

Students' drawings, picture choices, block constructions, verbal contrasts of number pairs, and explanations of how they knew one number was bigger than another, were given ratings from one to three, according to the amount of quantitative knowledge they revealed.

Drawings

If a student did not produce a drawing, or if a drawing did not directly include quantitative information, a rating of one was given. For example, a cartoon which used a number as a label would be given a rating of one. These drawings provided no evidence of a child's knowledge of number composition. If a student drew the actual number of objects, such as 12 lollipops for the number 12, a rating of two was given. These drawings provide evidence of one-to-one correspondence. If a student's drawing included other quantitative information, such as tally marks, or arrays standing for tens and units, a rating of three was given, for quantitative knowledge besides one-to-one correspondence. Appendix B includes examples of drawings at each rating level.

Picture Choices

The ratings for picture choices exactly parallel the ratings for drawings. A rating of one indicates either no choice, or the choice of a picture in which a number was used as a label. A rating of two indicates a choice of a picture with the actual number of objects. A rating of three indicates the choice of a picture of an array of tens and units blocks.

Block Constructions

If a student did not produce a block construction, or if blocks were merely placed in configural patterns mimicking the numeric symbol, a rating of one was given. If a student used only units blocks to construct a number, a rating of two was given, indicating the use of one-to-one correspondence in block constructions. If block constructions, in addition to one-to-one correspondence, revealed base-ten knowledge with combinations of tens and unit blocks, a rating of three was given.

A rating of three, besides indicating base-ten knowledge, also indicates flexible knowledge of number composition. Students demonstrated that numbers could be constructed in at least two ways.

This rating scheme does not, however, differentiate between levels of flexibility in the use of tens and units blocks. For example, a student who, for the number 26, produced 26 yellow blocks, and then produced two reds and six yellows, got a rating of three. However, another student who produced a third construction of one red block and 16 yellow blocks, also got a rating of three.

Verbal Contrasts of Number Pairs

If students could not verbalize any difference between two numbers; or if they merely contrasted the surface appearance of the numbers, they were given a rating of one. For example, in a digit-location contrast of 12 and 21, the student merely notes that in 12, the one comes before the two; and in 21, the two comes before the one.

If students volunteered a size contrast, saying for example that 21 is more than 12, a rating of two was given. Any other quantitative contrast, such as noting that 21 is in the twenties, would receive a rating of three.

A rating of two on verbal contrasts refers to a number size comparison; in contrast to ratings of two on drawings, picture choices, and block constructions; which indicate one-to-one correspondence.

Bigness Explanations

Students who could not explain how they knew that one number was bigger than another were given a rating of one. If a number sequence explanation was given, in which the size of a number was linked to its place in a number sequence, a rating of two was given. Other quantitative explanations such as base-ten explanations (21 is in the twenties), or math facts explanations (26 is 10 more than 16) were given a rating of three.

Number-By Number and Overall Ratings

Students were rated on each of the five numbers included in the task. (1) For the numbers 12 and 16, they were rated on drawings and blocks. (2) For the numbers 21 and 26, they were rated on picture choices and blocks. (3) For the number 36, they were rated on block constructions. (4) For both number pairs, they were rated on verbal contrasts and bigness explanations.

Based on these number by number ratings, each student received an overall rating on drawings, picture choice, blocks, verbal contrasts, and bigness explanations. This overall rating consisted of the highest rating achieved with any number. Thus the overall rating reflects demonstrated knowledge, whether or not it was demonstrated consistently.

To illustrate the kinds of verbal responses students produced, Appendix C includes all of the students'bigness explanations. Table 3 summarizes the meaning of the ratings for each part of the task.

TABLE 3

Interpretation of Ratings of Quantitative Knowledge

	Rating		
Task	1	2	3
Drawings	No Evidence	One-to-One	Base-10 or other
Pictures	No Evidence	One-to-One	Base-10 or other
Blocks	No Evidence	One-to-One	One-to-one plus base-10
Verbal Contrasts	No Evidence	Knowledge of number size	Base-10 or math facts
Bigness Explanations	No Evidence	Knowledge of number sequence	Base-10 or math facts

NOTE: A rating of three on blocks does not distinguish 1between levels of flexible base-10 knowledge.

CHAPTER IV

Results

Performance By Grade

Six out of 19 Kindergarten students were unable to do the task owing to unfamiliarity with two-digit numbers and/or inability to count, and had to be discontinued after the introduction or the first or second number. All 17 first graders, and all 18 second graders were able to complete the task. The six Kindergarten students are included in the following analyses, having been given a rating of one (no evidence) on all of the performance ratings.

Drawings

Table 4 shows what students in each grade did when they were asked to draw a picture of what a number means, along with the ratings given for each response.

Kindergarten. Thirteen of the Kindergarten students (including the six who were discontinued) received a rating of one on drawings. One of these students produced a cartoon which used a number as a label; the others did not produce a drawing. Six Kindergarten students achieved a rating of two by drawing the actual number of objects; and none of the Kindergarten students achieved a rating of three.

TABLE 4

Number of Students By Grade in Drawing Response Categories[a]

Response	Rating	Grade K	Grade 1	Grade 2
No drawing	1	12	0	3
Drew cartoon using number as label	1	1	3	1
Drew actual number of objects	2	6	14	10
Drew array showing number composition or math problem	3	0	0	4

[a]These data were gathered at the end of the school year.

First Grade. All of the first grade students produced drawings. Three drew cartoon pictures which used numbers as labels, and 14 drew the actual number of objects.

Second Grade. Three second graders did not produce drawings, one drew a cartoon picture with a number as label, and 10 drew the actual number of objects. In addition, one second grader drew talley marks to show what 12 meant, and two drew arrays of blocks which included bcth tens and units. Another drew math problems; for example, eight circles, followed by a plus sign, followed by eight circles, equals 16.

As grade goes up, the number of students who receive higher ratings on drawings increases (Kendall's tau =.44, $p<.001$), and grade is a useful predictor of ratings on drawings (Lambda[a]=.32).

Picture Choice

Table 5 shows the choices students made when they were asked to pick the picture which best explained what a number meant. Three Kindergarten students, five first graders, and one second grader chose pictures

[a]Lambda is a proportional reduction of error measure (Norusis, 1982) which is the percentage by which the prediction error on one variable is reduced when the values of a second variable are known.

TABLE 5

Number of Students By Grade in Picture Choice Categories[a]

Picture Choice	Rating	Grade K	Grade 1	Grade 2
None[b]	1	6	0	0
Number Used as Label	1	3	5	1
Actual Number of Objects	2	10	11	16
Array of Tens and Units	3	0	1	1

[a]These data were gathered at the end of the school year.

[b]These students were discontinued.

which used numbers as labels. (The six discontinued Kindergarten students did not choose pictures.) Ten Kindergarten students, 11 first graders, and 16 second graders chose pictures of the actual number of objects. One first grader and one second grader chose the picture of tens and units blocks.

As grade goes up, the rating on picture choice goes up (Kendalls Tau=.36, p=.002), and grade is a useful predictor of the rating on picture choice (Lambda=.28).

Reasons For Choosing Pictures With Numbers Used As Labels. The nine students who chose pictures with numbers used as labels offered the following kinds of explanations for their choices. One Kindergarten student said, "Because you're gonna be 21 years old." The other two Kindergarten students either just pointed to the number, or said "Because it has 21 on it."

Of the five first graders, three pointed to the numbers and said things like, "Cause it has a 21 on the forehead", or, "Because it has the number on the birthday candle 21 on top." Two referred to age in explanations like "Cause someone's turning 21", or "Cause if she (the doll with 26 on the skirt) walks around, people will look at her skirt and know she's 26."

The second grader, who chose pictures of birthday cakes, said, "Cause um if it's someone's birthday, and if they're 21, that explains it." However, this child (who produced base ten block constructions) also said, when he chose a birthday cake with 26 on it, "Cause um you can just put a candle there instead that has 26 on it instead of drawing 26 candles"; and he pointed to the picture of 26 cats.

Reasons For Choosing Exact Number of Objects. Seven of the ten Kindergarten students who chose pictures of the exact number of objects chose the pictures without apparently counting, and three did visible counting (one object at a time) before making their choices. The children who did not appear to count offered the following kinds of reasons for their choices: "Cause you can draw dots what makes the number", or "I didn't count them cause the whole page is covered", or "There could be 26 triangles there cause it looks like 26", or "Because there's more".

Two of the eleven first graders who picked the actual number of objects also chose without apparently counting first. The reasons they gave were, "Cause it has a lot", and "I'd get balloons to count." Three first graders also did not appear to count, however their explanations indicated that they did. For example, one child who picked 21 dots said, "Because it has two tens (two columns of 10) and a one..it has twenty um twenty and one more is 21." This child only did unit block constructions. Another picked 21 dots "Because it has things (points to rows of two) so you can count them (two by two) um you know it's 21." This child did base ten block constructions. Another child said, "I counted in my mind." The rest of the first graders in this category counted objects in the pictures one at a time.

Of the 16 second graders who referred to an actual number of objects in their picture choices, eight did not physically count the objects. One child picked both triangles and cats for the number 26 because "Both of them look like 26." Two others simply said, "Cause it has 21", or "Cause there are 21", after picking either balloons or dots.

Five children gave explanations which showed that they had done some counting. One said "Cause it has more than any..yeah than the other one..um that one (the birthday cake) only shows 10 (candles). Cause I looked at it (the cats) and it seemed like 26 so I started counting and there were." Another who picked 26 cats said, "If you don't know like 20 like 10 plus 16 (points to rows of cats) you could count them." Another who picked 26 cats said "This you can count by fives and then you put one." Yet another said, "Because if you like cats you can look at them and you could see like a line (points at row of cats) yeah like 5, 10, 15, 20, 6." A child who picked 21 dots said, "Because it's two groups of 10 and one one."

The other eight second graders in this category visibly counted objects in the pictures. They counted objects one at a time, and also by twos. Some of them went directly to the pictures of cats, triangles, balloons, or dots, and started counting. Others counted birthday candles, stripes on the doll's clothing, or blocks; ruled out those pictures, and then started counting cats, triangles, balloons or dots. For example, one child counted the red and yellow blocks, "Two, four, six, eight..no", and went on to choose 26 cats and triangles.

Two of these children actually chose pictures of birthday cakes after counting the candles. They are included in this category, because after finding only 8 or 10 candles in the pictures, they said, "It has 21 candles..yeah cause they're on the back of it when it goes around", or "Cause it has 21 candles..the rest of the candles are in the back." These children showed imagination as well as knowledge of one-to-one correspondence.

Finally, one second grader, after counting 26 triangles one by one, chose the picture of tens and units blocks "Because it's more easier to describe."

Block Constructions

Table 6 shows the block constructions produced by students at each grade level. This table includes the six Kindergarten students who were discontinued.

Kindergarten. Of the 13 Kindergarten students who were able to complete the task, two placed blocks in the configural patterns of the numeric symbols, nine used either red or yellow blocks as units, and two students produced both units and base ten constructions. One of these actually produced all possible combinations of block constructions.

One of the kindergarten students who produced both unit and base ten constructions succeeded with numbers in the teens and twenties. For example, "Two reds equals 20 so if I add one that's 21." However when she got to the number 36 she had a problem. First she put out six yellow blocks, and said, "First, 1,2,3,4,5,6..now how do you do this 30 deal?" She couldn't figure out what to do, and didn't produce a construction for 36.

Four Kindergarten students who only produced unit constructions, and none of the first or second graders, used both yellow and red blocks as units. They did not seem to make sense of the information that red blocks equalled ten and yellow blocks equalled one. One student who was asked "Red blocks equal what?", replied "Ten." Then she was asked "Yellow blocks equal what?", and replied, "One." Nevertheless she produced "Twelve different colored blocks."

First Grade. One first grader placed blocks in configural patterns, and four only produced units constructions. Twelve out of 17 first graders showed base ten knowledge in their block constructions, and 10 of these demonstrated varying degrees of flexibility in their use of base ten knowledge. Three first graders used all possible block combinations.

Second Grade. None of the second graders used configural patterns, and none relied solely on units constructions. All eighteen second graders showed base ten knowledge in their blocks constructions, and 17 of them showed varying degrees of flexibility. Six second graders used all possible block constructions.

As grade goes up, the rating on blocks goes up, (Kendall's tau =.68, $p<.001$), and grade is a useful

TABLE 6

Number of Students By Grade in Block Construction Categories[a]

Block Construction	Rating	Grade K	1	2
Unable to do task[b]	1	6	0	0
Configural pattern	1	2	1	0
Only used reds or yellows as units	2	4	1	0
Only used yellows as units	2	5	3	0
Only used maximum reds constructions	3	0	2	1
Used yellows as units plus maximum reds constructions	3	1	1	3
Used yellows as units, maximum and <maximum reds	3	0	6	8
Used all combinations	3	1	3	6

[a]These data were gathered at the end of the school year.

[b]These students were discontinued.

predictor of ratings on block construction (Lambda=.42).

Awareness of the Number of Yellow Blocks In Base Ten Block Constructions. The majority of students who produced base ten block constructions counted up with yellow blocks. For example, to produce 16, they would take a red block and say "ten", and then add yellows, counting up to 16. Or, for 26, they might take two red blocks, and say "21,22,23,24,25,26." Both of these are examples of maximum reds constructions, since the number of red blocks matches the number of tens in the number. Alternatively, for a number in the twenties or thirties, a child could take one red block and count up. This could result in 11 yellow blocks for 21, 16 for 26, and either 16 or 26 yellow blocks in 36.

When asked how many yellow blocks they had, the majority of students who produced maximum reds constructions readily said, without visible counting, that there were six yellow blocks. However when there were 11 or more yellow blocks, most of the students had to count them.

One out of two student Kindergarten students, seven out of 12 first graders, and 15 out of 18 second graders, in maximum reds constructions, said there were six yellow blocks without pause. One second grader said

"Three plus three is six." Five second graders did not apparently have to count higher numbers. Two of them, referring to constructions of 21, said that there were 11 yellow blocks with no pause. Two others, referring to constructions of 26 with 16 yellow blocks, said, "Ten..and then there's six", or "ten yellows then six." Another, referring to a construction of 36 with 26 yellow blocks, said " two tens (a red block and a group of 10 yellows), and another 10 and a six". These answers reflected the sequence in which they had built up the number of blocks.

All of the other students had counted up with yellow blocks, and when asked how many there were, had to count them. One Kindergarten student, five first graders, and three second graders had to count six. Two Kindergarten students, 10 first graders, and 12 second graders, or 96 percent of the total of 27 students who produced block constructions with less than the maximum number of red blocks, had to count in order to find out that there were 11, 16, or 26 yellow blocks.

Verbal Contrasts of Number Pairs

Table 7 shows students' responses when they were asked to verbally contrast pairs of numbers. All 13 Kindergarten students who did this task contrasted the

TABLE 7

Verbal Contrasts of Number Pairs[a]

Type of Contrast	Rating	Grade K	Grade 1	Grade 2
No contrast	1	6	0	0
Digit location contrast	1	13	14	7
Number size comparison	2	0	3	10
Base-10 or math facts	3	0	0	1

[a]These data were gathered at the end of the school year.

location of digits in 12 and 21, and in 16 and 26. For example, one child said, "Ah..ah cause the 21 has a two in the front, and the 12 has the two in the back."

None of the Kindergarten students volunteered any quantitative information in their verbal contrasts. All 13, however, when asked which number was bigger, answered correctly.

Fourteen first graders also relied on digit location in their verbal contrasts, although when asked, knew which numbers were bigger. Three of the first graders volunteered size comparisons when asked to contrast the numbers. For example one child said, "Twenty one is higher than twelve." Another said, "Um somebody growing up could be 26, and some kid could be 16, and you could tell the difference, how taller or shorter."

Seven of the second graders used digit location to contrast the numbers (but identified the larger numbers when asked), and ten of them volunteered size comparisons. The second graders who volunteered size comparisons sometimes used digit location to support their comparisons, for example, "The one is over here and the two is over here..one (points to 21) is a higher one and one (points to 12) is a lower one." Another second grader said "One's an odd number and one's an even number..and um one's a higher number and one's a lower number."

One of the second graders volunteered a math facts verbal contrast, which also served as a bigness explanation. He said, "well, they're 10 away", when comparing 16 to 26.

As grade went up, the rating on verbal contrasts went up (Kendall's Tau =.50, p=.000), and grade is useful as a predictor of ratings on verbal contrasts (Lambda=.25).

Bigness Explanations

Table 8 shows students' explanations of how they knew one number was bigger than another. Three of the 13 Kindergarten students who did the task could not come up with an explanation. Six produced a number sequence explanation. For example, one student said, "Because, 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17, 18,19,20,21..21 comes last!" Another student said, "Twelve comes first and 21 comes second." Another said, "Sixteen is by the 12 and 26 is by the 24".

Four Kindergarten students produced initial digit explanations which suggested knowledge of place value, such as, "Cause it's a two (points to two in 26), and the one is starting in this (points to one in 16), so it's (points to 26) higher."

One first grader could not produce an explanation. Six first graders produced number sequence explanations, such as "Because when you count up to 10, it's 11,12. When you count up to 19, it's 20,21." Ten first graders produced other sorts of explanations. Four of these used initial digit explanations such as "Cause it's..the one is in front there (12), and the two is in front there (21), which suggested knowledge of place value. Four first graders explicitly referred to place value in their explanations. For example, one child said, "Cause it's in the teens (16), and 26 is in the twenties." Two second graders gave more concretely quantitative explanations such as "Because if you have um..twen..um 16 um boxes, and you need more for your other toys at home um and you have um 26 toys you know uh you'd need 26 boxes and that's more."

All of the second graders were able to explain how they knew that one number was bigger than another. Five of them produced number sequence explanations, such as "Cause when you count you pass 12 before you pass 21." Seven used initial digit explanations which suggested place value knowledge such as "The first number (in 21) is bigger." Three children explicitly referred to place value, for example, "Let's see..there's a two in the

TABLE 8

Bigness Explanations[a]

Explanation	Rating	Grade K	Grade 1	Grade 2
No explanation	1	9[b]	1	0
Number sequence	2	6	6	5
Initial digit (place value)	3	4	4	7
Explicit place value	3	0	4	3
Quantity of objects	3	0	2	0
Math facts	3	0	0	3

[a]These data were gathered at the end of the school year.

[b]Six of these nine students were discontinued.

tens place (21) and only a one in the tens place (12)". Three children gave math facts explanations such as "Twenty six take away 10 is 16", or "Well because they're 10 away". None of the second graders referred to quantities of objects.

As grade went up, the ratings on bigness explanations went up (Kendalls Tau =.47, p<.001), and grade is a useful predictor of these ratings (Lambda=.24).

The Difference Between Two Numbers

None of the Kindergarten students, three first graders, and 12 second graders knew or could figure out how much bigger one number was than another. The students who figured it out either counted on their fingers or used a subtraction rule such as "six take away six..two take away one." Some used math facts such as "It's ten away." In contrast, one of the Kindergarten students didn't try to figure it out. When she was asked how much bigger 26 was than 16, she replied, "A lot much bigger."

Performance By Teacher Ratings

Table 9 shows the Kendall's tau values and their probability levels by grade when teacher ratings are

TABLE 9

Relationship Between Teacher Ratings Of Students' Math Ability And Student Ratings

	Grade		
Student Ratings	K	1	2
Blocks			
Kendall's tau	.37	.51	--
Probability	.04	.02	--
Drawings			
Kendall's tau	.47	.15	.21
Probability	.02	.27	.16
Picture Choice			
Kendall's tau	.42	.20	-.45
Probability	.03	.19	.02
Verbal Contrasts			
Kendall's tau	--	.32	.22
Probability	--	.10	.17
Bigness Explanations			
Kendall's tau	.06	-.06	.22
Probability	.39	.40	.17

associated with ratings on blocks, drawings, picture choices, verbal contrasts, and bigness explanations.

Teacher ratings accurately predict student ratings on block construction in Kindergarten and the first grade, but not in the second grade. All of the second graders got a rating of three on the blocks task, including those rated at low or medium math ability levels, indicating a ceiling effect for second graders on this task.

Teacher ratings also predict ratings on drawings and picture choices for Kindergarten students; but not for first graders. There is a significant negative relationship between teacher ratings and ratings on picture choice for the second grade. These findings may be explained by the fact that the majority of first and second grade students drew or chose pictures which illustrated one-to- one correspondence, with a rating of two, including those students rated as high ability.

All of the Kindergarten students and the majority of first graders, regardless of teacher ratings, got a rating of one on verbal contrasts;while the majority of second graders got a rating of two. There was no significant relationship between teacher rating and either verbal contrasts or bigness explanations.

Performance By Task

Nonparametric sign tests[1] were done within grade and across grades in order to compare ratings between the two pictorial parts of the task, drawings and picture choices. The two verbal parts of the task, verbal contrasts and bigness explanations, were also compared. Comparisons were made of ratings on block construction with ratings on drawings, picture choice, verbal contrasts of number pairs, and bigness explanations. In addition, comparisons were also made between ratings on drawings and pictures, and ratings on verbal contrasts and bigness explanations.

Comparison of Pictorial and Verbal Ratings

Table 10 shows that students did equally well within and across grades, on drawings and pictures. That is, the number of students who got higher ratings on drawings did not outnumber the students who got higher ratings on pictures. The levels of quantitative

[1]Nonparametric sign tests compare ordinal rankings on two variables for a group of subjects by looking at how many subjects scored higher on one variable than another; how many scored lower, and how many had equal scores on both variables. The binomial distribution is used to determine whether this could have occurred by chance.(Siegel and Castellan, 1988)

TABLE 10

Comparison of Ratings Between Verbal And Pictorial Tasks

	Grade			
	K	1	2	Overall
Drawings and Pictures				
Drawings Higher	1	3	4	8
Pictures Higher	5	2	3	10
Ties[a]	13	12	11	36
probability	.2157	1.0000	1.0000	.4807
Verbal Contrasts and Bigness Explanations				
Bigness Higher	10	15	14	39
Verbal Higher	0	0	0	0
Ties	9	2	4	15
probability	.0020	.0001	.0001	<.0001

[a]The number of students who received equal ratings.

knowledge revealed in these measures remained in the same relationship across grades.

Table 10 also shows, however, that both within and across grades, students revealed more quantitative knowledge in their bigness explanations than they did in their verbal contrasts of number pairs. While none of the 54 students got a higher rating on verbal contrasts than bigness explanations, 39 students, or 72 percent, got higher ratings on bigness explanations. The level of quantitative knowledge revealed in these measures also remained in the same relationship across grades.

Comparison of Block Constructions With Other Measures

Table 11 compares ratings on block construction with ratings on drawings, pictures, verbal contrasts and bigness explanations. In general, more students got higher ratings on block constructions than they did on drawings, pictures, verbal contrasts of number pairs, and bigness explanations. However, the number of Kindergarten students who got higher ratings on blocks than they did on picture choices was not statistically significant. Nor was the number of students, both within grades and overall, who got higher ratings on blocks than they did on bigness explanations.

TABLE 11

Ratings on Blocks Compared With Other Ratings

	Grade			
	K	1	2	Overall
Drawings				
Blocks Higher	8	12	14	34
Drawings Higher	0	0	0	0
Ties	11	5	4	20
probability	.0078	.0005	.0001	<.0001
Pictures				
Blocks Higher	5	12	17	34
Pictures Higher	1	0	0	1
Ties	13	5	1	19
probability	.2187	.0005	<.0001	<.0001
Verbal Contrasts				
Blocks Higher	12	16	17	45
Verbal Higher	0	0	0	0
Ties	7	1	1	9
Probability	.0005	<.0001	<.0001	<.0001
Bigness Explanations				
Blocks Higher	5	5	5	15
Bigness Higher	3	3	0	6
Ties	11	9	13	33
probability	.7266	.7266	.0625	.0784

Except for Kindergarten picture choices, the quantitative knowledge revealed in block constructions was greater than for drawings, pictures, and verbal contrasts of number pairs. However, there was not a significant difference between block constructions and bigness explanations.

Table 11 also shows that with the exception of the comparison between blocks and bigness explanations, the number of students who were tied on two variables, that is, whose ratings were identical, decreased as grade increased. On the one hand, previous analyses showed that all ratings tended to increase with grade. However, when we look at the comparisons of blocks paired with each of the other ratings, we see that increases are not uniform across variables.

Pictorial Measures Compared With Verbal Contrasts and Bigness Explanations

Tables 12 and 13 show that both within and across grades, ratings on drawings and picture choice were generally higher than ratings on verbal contrasts of number pairs. The only exception was for the second grade, where the numbers of students with higher ratings on drawings and pictures were not statistically significant (p.=.3437 and .1797).

TABLE 12

Ratings on Drawings Compared With Ratings on Verbal Contrasts and Bigness Explanations

	Grade			
	K	1	2	Overall
Verbal Contrasts				
Drawings Higher	6	11	7	24
Verbal Higher	0	0	3	3
Ties	13	6	8	27
probability	.0312	.0010	.3437	.0001
Bigness Explanations				
Drawings Higher	1	0	1	2
Bigness Higher	6	11	12	29
Ties	12	6	5	23
probability	.1250	.0010	.0034	<.0001

TABLE 13

Ratings on Picture Choice Compared With Ratings on Verbal Contrasts and Bigness Explanations

	Grade			
	K	1	2	Overall
Verbal Contrasts				
Pictures Higher	10	10	7	27
Verbal Higher	0	1	2	3
Ties	9	6	9	24
probability	.0020	.0117	.1797	<.0001
Bigness Explanations				
Pictures Higher	3	0	1	4
Bigness Higher	6	9	13	28
Ties	10	8	4	22
probability	.5078	.0039	.0018	<.0001

However, there is a different pattern of results when drawings and picture choices are compared to bigness explanations. In general, students got higher ratings for their explanations of how they knew one number was bigger than another than they did for their drawings and picture choices. In both comparisons for Kindergarten students, the number of students with higher ratings on bigness explanations did not reach statistical significance. However, the comparisons for first and second grade, as well as the overall comparisons, were statistically significant. Overall, students revealed more quantitative knowledge when they explained why one number was bigger than another than they did when they drew or chose a picture of what a number meant.

Except for the verbal contrast-picture choice comparison, the number of ties for first and second grade were lower than for Kindergarten. This supports the data from the comparisons with block constructions, which shows that while ratings in general increased with grade, they did not increase uniformly across all variables.

Number of Students At Different Rating Levels

Since each of the tasks in this study was rated on a three point scale, a comparison of ratings on a pair of tasks could result in nine possible relationships. That is, the two tasks could be tied at one of three levels. One task could be rated at level one, and the other at level two or three. Or, that same task could be rated at level two and the other at level one or three. Finally, the first task could be rated at level three, and the other at level one or two. Tables 14, 15, and 16 show, for each grade, the number of students whose ratings on blocks, when compared with the other four measures, fell into each of these nine relationships.

Kindergarten. Table 14 shows that of the 13 Kindergarten students who completed the task, the majority of ties occurred at rating level two. These students showed knowledge of one-to-one correspondence in their block constructions, their drawings, and their picture choices; they evidenced no quantitative knowledge in their contrast of number pairs, and they used number sequence explanations for why one number is bigger than another. A larger number of Kindergarten

TABLE 14

The Relationship of Kindergarten Students'[a] Ratings On Blocks With Ratings On Four Other Tasks

	Comparisons With Block Ratings			
	Drawings	Pictures	Verbal[b]	Bigness
Tied at 1[c]	8	8	8	7
Tied at 2	4	7	0	5
Tied at 3	0	0	0	1
Ratings of 1/2	0	0	0	0
Ratings of 1/3	0	0	0	1
Ratings of 2/1	5	2	9	3
Ratings of 2/3	0	0	0	1
Ratings of 3/1	0	0	2	0
Ratings of 3/2	2	2	0	0

[a]The number of students out of 19 whose ratings on blocks and each of four other tasks formed the above relationships.

[b]Kindergarten students were all at level one.

[c]Six Kindergarten students who were discontinued are tied at rating level one for all tasks.

TABLE 15

The Relationship Of First Graders'* Ratings On Blocks With Ratings On Four Other Tasks

	Comparisons With Block Ratings			
	Drawings	Pictures	Verbal	Bigness
Tied at 1	1	1	1	1
Tied at 2	4	3	0	1
Tied at 3	0	1	0	7
Ratings of 1/2	0	0	0	0
Ratings of 1/3	0	0	0	0
Ratings of 2/1	0	1	4	0
Ratings of 2/3	0	0	0	3
Ratings of 3/1	2	3	9	0
Ratings of 3/2	10	8	3	5

*The number of students out of 17 whose ratings on blocks and each of four other tasks formed the above relationship.

TABLE 16

The Relationship Of Second Graders' [a]Ratings On Blocks With Ratings On Four Other Tasks

	Comparisons With Block Ratings			
	Drawings	Pictures	Verbal	Bigness
Tied at 1	0	0	0	0
Tied at 2	0	0	0	0
Tied at 3[b]	4	1	1	13
Ratings of 1/2	0	0	0	0
Ratings of 1/3	0	0	0	0
Ratings of 2/1	0	0	0	0
Ratings of 2/3	0	0	0	0
Ratings of 3/1	4	1	7	0
Ratings of 3/2	10	16	10	5

[a]The number of students out of 18 whose ratings on blocks and each of four other tasks formed the above relationship.

[b]All 18 second graders achieved a rating of 3 on block constructions.

students, however, were one or two levels apart in all four comparisons. The largest number of these students showed one-to-one correspondence in their block constructions, but no evidence of quantitative knowledge in drawings, pictures, or verbal contrasts. However, six out of 13 students altogether, used number sequence explanations for why one number is bigger than another.

First Grade. Table 15 shows that the majority of first graders were rated at level three for their block constructions. While a few students were tied at level 2 for comparisons of blocks with drawings and pictures, seven out of 17 students were tied at level 3 for blocks and bigness explanations. These students showed base ten knowledge in their block constructions, and place value or other quantitative knowledge in their explanations for why one number was greater than another.

The majority of first graders were one or two levels apart in all of the comparisons. A majority showed base ten knowledge in their block constructions, but one-to-one correspondence in their drawings and pictures, and no evidence of quantitative knowledge in their verbal contrasts. About half of these students gave number sequence explanations, and the rest evidenced place value or other quantitative knowledge in their explanations for why one number was bigger than another.

Second Grade. Table 16 shows that all 18 second graders showed base ten knowledge in their block constructions. The majority of them showed one-to-one correspondence in their drawings and pictures, knowledge of the size of numbers in their verbal contrasts, and place value or other quantitative knowledge in their bigness explanations.

Summary of Results

Ratings Increase With Grade

Kendall's tau showed a statistically significant positive relationship between grade and ratings on block constructions, drawings, picture choices, verbal contrasts of number pairs, and bigness explanations. At one end of the continuum, Kindergarten students were most likely either to demonstrate no quantitative knowledge, or to show knowledge of one-to-one correspondence and number sequence information. At the other end of the continuum, second graders were most likely to show knowledge of base ten information and place value.

Some Ratings Increase With Teacher Ratings

In the second grade, teacher ratings, except for a negative relationship with picture choice, were not significantly related to any of the performance measures. However, in Kindergarten and first grade, there was a significant positive relationship between teacher ratings and ratings on blocks; and in Kindergarten only, there was a significant positive relationship with drawings and pictures. There were no significant relationships between teacher ratings and verbal measures.

Flexible Use Of Base Ten Knowledge

Two Kindergarten students, and the majority of first and second graders, produced block constructions which used base ten knowledge in a variety of ways. That is, they did not merely construct numbers using unit blocks, or maximum tens constructions (i.e., three tens in 36). Instead, they also used smaller numbers of ten blocks, demonstrating knowledge of alternative number constructions.

Uneven Demonstration of Quantitative Knowledge Across Tasks

Blocks. In all grades, a statistically significant majority of students received higher ratings on blocks than they did on drawings, pictures, or verbal contrasts. However, the number of students who scored higher on blocks was not significantly different from the number of students who scored higher on bigness explanations.

Pictorial Measures. There was no significant difference, at any grade, between students who scored higher on drawings, and students who scored higher on picture choices.

Verbal Measures. A statistically significant majority of students got higher ratings on bigness explanations than they did on verbal contrasts of number pairs.

None of the Kindergarten students, three first graders, and 12 second graders were able to say or figure out how much bigger one number was than another.

Comparison of Pictorial and Verbal Measures. There was a significant majority of students in all three grades who scored higher on bigness explanations than they did on drawings or pictures. This pattern was reversed with verbal contrasts. More students got higher ratings on drawings and picture choices than they did on verbal contrasts.

Awareness of Number of Yellow Blocks

When students constructed numbers by adding on yellow blocks to red ten blocks, they did not know how many yellow blocks they ended up with, and had to count them.

Awareness of the Difference Between Two Numbers

None of the Kindergarten students, three first graders, and 12 second graders knew or could figure out (usually by counting on their fingers) how much bigger one number was from the other in the verbal contrasts. This is not surprising, since two-digit subtraction is not taught until the second grade.

CHAPTER V

Discussion

The data from this study indicate that number knowledge is multiply represented within a developing system, and that language, including but not limited to number words, is brought to bear in a variety of ways in it's development. The data also show uneven development within the domain of number knowledge, along the lines of Bruner's (1973) enactive, iconic, and symbolic representation. Finally, greater quantitative knowledge was expressed in verbal responses to a more specific question than to a less specific question; suggesting that specific questions may serve as attentional spotlights which help children link up and express knowledge.

The Mental Representation of Number Is More Than The Meaning of Number Words

When Miura et al.'s (1988) Asian first grade subjects, over two trials, produced more, and more kinds of base ten block constructions than American first graders, this was taken as evidence of the benefits conferred by Asian number words containing explicit base-ten information. The data from the English-speaking children in this study, however, suggest that the picture is more complex. Language plays a broader role, in both direct and indirect ways.

School Mathematics Instruction

While this study does not include a cross-cultural curriculum comparison, it is obvious that schooling impacts on mathematical understanding. Appendices D.1 through D.3 contain a sample of the topics included in New York City's mathematics curriculum, which is called the scope and sequence, for grades Kindergarten through two. For example, one-to-one correspondence is included in the Kindergarten and first grade curricula. Numbers and numeration topics are included for all three grades, with one through ten covered in Kindergarten, one through 99 covered in first grade, and numbers up to 1000 covered in the second grade. In a typical lesson which introduces place value to first graders, children are shown pictures of objects such as stars, flowers, pencils, or lollipops which number in the teens. Then they are asked to circle a group of ten objects and say how many ones there are. A subsequent lesson provides illustrations of one number each in the twenties, thirties, and forties.

Second grade place value lessons include larger numbers. Besides including illustrations of objects, the lessons include illustrations of arrays of blocks, tally marks, and the like; as well as explications of "the ones place", "the tens place", and "the hundreds place" in three and four digit numbers.

All of the topics included in the curricula for each grade are supposed to be taught during the school year in the given sequence. In practice, however, there is considerable diversity from school to school. Depending on students and teachers, there is also diversity between and within classrooms as to what topics students actually master.

Except for when a student reproduces an array of blocks or tally marks, or refers explicitly to material such as "the tens place", it is really not possible to draw direct connections between classroom instruction and the task performance of students in this study. Nevertheless, classroom instruction for these students did include a diverse range of topics. The majority of students were rated at medium or high ability in mathematics by their teachers, and standardized test scores for second-graders showed that they were all (even those rated by their teachers as low ability) achieving at or above grade level. Therefore it is fair to say that the students in this study had probably mastered a good part of the curriculum.

Number Words And Mathematical Understanding

If it were true that some groups of Chinese and Japanese children (having been matched on more variables than grade alone with their American counterparts) demonstrated higher mathematics achievement than some groups of American children, other factors besides the meaning of number words could explain the difference. For example, Miller and Stigler (1987) point out that Chinese culture places greater emphasis than American culture on mathematical skill. Stigler and Stevenson (1991) conducted an observational study of 120 classrooms in Taipei, Japan, and Minneapolis. They found that not only did Chinese and Japanese classes devote more time to mathematics instruction, but also there were qualitative differences between Asian and American instruction. The Asian classes were more coherently organized and freer from interruption than the American classes. The Asian schools did not practice tracking, and utilized instructional methodologies such as cooperative learning.

Miller and Stigler (1987) however,in their cross-cultural comparison of children's counting, did find different patterns of acquisition between Chinese and American children. They also found that Chinese children made fewer counting errors. These authors point to the work of Gelman and Gallistel (1978) which demonstrates the relationship between counting and the understanding of number. Insofar as number words are necessary to tag objects which are being counted, the words may in this way exert an indirect influence on the understanding of number.

However, Miller and Stigler state that linguistically-based differences in counting may reflect "temporary phenomena limited to early stages of acquisition.." It may be that American children learn number words over a longer course of time than the Asian children. (The New York City mathematics curriculum gets to 1000 in the second grade.) The data from this study showed an apparent ceiling effect for second graders on the block construction task, and also that the majority of first graders produced base ten block constructions. Since this was not a replication of Miura et al.'s study, it is not possible to compare Asian and American children on this task. However even if there were linguistically-based differences in block construction in the first grade, it is clear from this study that they would wash out by the second grade.

English-Speaking Children's Knowledge of Number

The students in this study demonstrated a continuum of knowledge about the number system. At one end of the continuum, 16 percent of the students in the main study either could not do the task, or put blocks into configural patterns. Throughout the continuum, 84 percent of all the students in the main study, showed knowledge of one-to-one correspondence. At the other end of the continuum, ten percent of Kindergarten students (two children), 71 percent of first graders, and all of the second graders in the main study did base ten block constructions. Moreover, one of these Kindergarten students, 59 percent of the first graders, and 94 percent of the second graders produced, in addition to units constructions and maximum-reds base ten constructions, a variety of other base ten constructions. In both the preliminary study and the main study, a few Kindergarten students, a majority of first graders, and most of the second graders, were able to demonstrate flexibly deployed knowledge of the base-ten system in block constructions without the benefit of Asian number words.

A Broader Role For Language In The Representation Of Number

These data highlight Nelson's (1985) point that representations are not merely internalizations, and that language mediates the representation of experience. The explanations which children gave for their picture choices, their verbal contrasts of number pairs, and their bigness explanations all suggest that language is brought to bear in a variety of ways as the representation of number develops.

Symbol Systems Developing In Tandem

Data from verbal contrasts and bigness explanations demonstrate that the symbol systems of language and number are being learned at the same time. Indeed, in grades Kindergarten through two, which this study included, reading is taught as a specific subject, as is mathematics.

Knowledge of Place Value in Verbal Contrasts and Bigness Explanations. In addition to using the base-ten system in block constructions, children in this study also showed knowledge of place value. When the children were asked to contrast pairs of two digit numbers, three first graders and 10 second graders volunteered size comparisons. The children who did not volunteer size comparisons contrasted the location of digits in the two digit numbers. All of them, however, answered correctly when asked which number was bigger. While the non-volunteers may have been referencing number sequence information to determine which number was bigger than another, the fact that they noted the location of digits suggests at least the beginnings of place value knowledge.

All of the children were asked how they knew that one number was bigger than another. Four Kindergarten students, eight first graders, and ten second graders either gave initial digit, or implicit place value explanations, or explicit place value explanations which referenced "the tens place" and "the ones place".

Counting and The Number Sequence. The number sequence explanations used by students in all three grades, and also the counting observed during picture choices and block constructions suggest that counting, which includes knowledge of the number sequence, contributes to quantitative understanding.

Number sequence explanations also provide an example of language and number developing in tandem. A typical Kindergarten number sequence explanation included simply reciting a string of numbers up to 21, and then saying, "21 comes last." In contrast, a more complex second grade sentence was, "Because if you want to reach 26, you reach 16 before you reach 26."

A second grade verbal contrast of number pairs provides another illustration. In order to contrast 16 and 26, the child said, " Um first like the first <u>letter</u> in 16 is one, and the first <u>letter</u> in 26 is two..and um one is higher and one is lower."

<u>Language in General</u>. In the course of doing the tasks in this study, children used words like bigger/smaller, more/less, and sequence words such as first, last, and next, and locational words such as in the front, and in the back. To the extent that quantitative knowledge can be defined as a specific domain, it is enriched by connections to more general knowledge.

<u>Event Representations</u>. For the children in this study who drew or chose pictures which used numbers as labels, birthday cakes were a popular choice both in the pilot study and the main study. In addition, a second grader in the pilot study drew a road sign in which the number stood for miles per hour, and talked about a family outing he had recently been on. Another child drew the speedometer in his family's car. For these children, the meanings of the numbers seemed embedded within events.

It is possible that some of the everyday objects children drew such as pencils, flowers, and the like which showed knowledge of one-to-one correspondence also originated in event representations. One child in the pilot study drew 14 "birthday balloons".

It is also possible that structured activities included in classroom mathematics instruction may be experienced as events by the children.

Other Combinatorial Systems. Other combinatorial systems besides the base ten system are also nessary for mathematical reasoning; and Chinese-based number words such as four 10s, two, may not be especially useful for alternative ways of combining numbers. For example, students in this study, referring to a group of six blocks, said, "three plus three equals six." Students used talley marks in groups of 5 to draw what a number meant, they used skip counting by twos and by fives, and they used math facts such as "26 take away 10 is 16" to express the difference between numbers. One student referred to 12 as "a dozen".

Mapping On: Matching And Linking Information

The tasks in this study were constructed to map on to Bruner's iconic, enactive, and symbolic modes of representation. The results can be viewed as an illustration of the importance of linking and matching processes in the development of a representational system. If we think of a representational system as a functional unit consisting of interacting groups of related components in an organized assemblage, then the larger the number of components and the better the organization, the more functional the system will be. Matching and linking processes in the development of a system will affect the relatedness or organization of the system.

The functionality of a representational system consists not only of the information it contains, but also of the ability to bring that information to bear on specific tasks. In the performance of a task, information in the system has to be matched to relevant features of the task. In this dynamic, mapping on is a two way process. If a learner has to map syntactic knowledge on to semantic knowledge, so also must a developing system map on or match relevant knowledge to features of a task. Depending on the degree to which a system has developed, the structure of some tasks may be easier than the structure of other tasks for the system to deal with.

Variability And Flexibility

When the same quantitative knowledge is expressed in a variety of ways, such as base ten information used in picture choices as well as block constructions, it indicates that a representational system is flexible and not merely variable. Flexibly deployed knowledge suggests a greater level of functionality.

Number Knowledge Develops Unevenly In Different Representational Modes

The data from this study show uneven development within the domain of mathematical knowledge. Student ratings of quantitative knowledge were not uniform across the block construction, pictorial, and verbal tasks, which were set up along the lines of Bruner's enactive, iconic, and symbolic modes of representation.

Overall, the highest ratings were achieved with block constructions and bigness explanations; the second highest ratings with drawings and pictures, and the lowest ratings were for verbal contrasts of number pairs. Most of the students demonstrated no quantitative knowledge in their verbal contrasts. This is interesting since all of the children had previously constructed those same numbers with blocks, as well as drawing and choosing pictures of what the numbers meant.

Modes of Information and Nodes of Information

When Chi(1983) talks about a well organized knowledge structure in connection with dinosaur knowledge, a brontosaurus and a tyrannosaurus are linked together by virtue of their membership in the class of dinosaurs; and also by virtue of physical characteristics which they have in common. When Chase and Simon (1973) talk about a well organized knowledge structure for chess, they include the a large number of chess positions and procedural rules which are related by virtue of the roles they play in games of chess. In both instances, the meaning of individual components of a structure can be seen as both derivative of and enhanced by the structure. The structure is considered coherent and organized if there are many well established linkages between nodes of information. The best way of establishing linkages is an interesting question for memory and learning.

It may be that translating information from one representational system to another provides useful linkages; not only between nodes of information, but between modes of information.

One way of determining whether a child has comprehended something she has read is to ask her to restate it in her own words. This technique is well known to reading teachers, and it both measures and enhances comprehension. It amounts to a kind of translation, and translation carries meaning as well as transforming it.

If a child is able to demonstrate base-ten knowledge in block constructions, pictures, and verbal contrasts of number pairs (which some of the second grade students in this study were able to do) then it is reasonable to say that that child's knowledge is both well integrated and flexibly deployed. Well integrated and flexibly deployed knowledge is necessary for quantitative reasoning.

The Role of Specific Questions As Attentional Spotlights

The data from this study showed a discrepancy between the levels of quantitative knowledge revealed in verbal contrasts of number pairs, and bigness explanations. It may be that the request to tell how one number was different from another was less specific than the request to explain how a child knew that one number was bigger than another. Numbers can be different in a variety of ways. The children in this study revealed greater quantitative information in their bigness explanations than they did in their verbal contrasts; in fact the overall ratings on bigness explanations were equal to the ratings on block constructions. The majority of verbal contrasts consisted of digit location explanations which attended to the surface appearance of the numeric symbols, and did not contain any quantitative meaning. This resembles the rote use of mathematical symbols to operate on numbers without understanding their meaning, which Resnick et al.(1987) called attention to. Resnick et al. suggested that "syntactic" mathematical information needs to be mapped on to semantic information. Perhaps this mapping on might be accomplished by translating information from one modality to another.

However, responses to a request for an explanation of how a child knew that one number was bigger than another were varied. Some children gave number sequence explanations. Others gave implicit or explicit place value explanations. Others gave math facts explanations.

Questions may serve as attentional spotlights which help children link up and express information. Children doing block constructions had the support of concrete objects, and children giving bigness explanations had the benefit of a specific question to which they could respond. The concreteness and the specificity involved in these two parts of the task may have been related to the successful deployment of quantitative information in two different modes.

<u>The Verbal Side Of A Pictorial Task</u>

This speculation is further supported by the reasons some of the children gave for their picture choices. When children were asked to choose the picture which best explained what a number meant, the majority of children chose pictures which illustrated one-to-one correspondence. When they were asked why that picture best explained what the number meant they focused in on the picture in order to justify their answer. Some of their reasons demonstrated quantitative knowledge which went beyond one-to-one correspondence. For example, some children utilized the layout of objects on the page to express base-ten information. A child who picked 21 dots said, "Because it's two groups of 10 and one one." A child who picked 26 cats said, "This you can count by fives and then you put one." Yet another child, who only did unit block constructions, picked 21 dots and said, "Because it has two 10s and a one..it has 20 um twenty and one more is 21."

These children drew upon knowledge of the base-ten system to justify choosing illustrations of one-to-one correspondence.

The Representation of Number: A Developing System

The data from this study suggest that the most useful way to think about a child's representation of number is that of a developing system in the process of integrating information from diverse sources. The data also suggest that information from diverse sources is processed in different modalities which agree with Bruner's enactive, iconic, and symbolic representation.

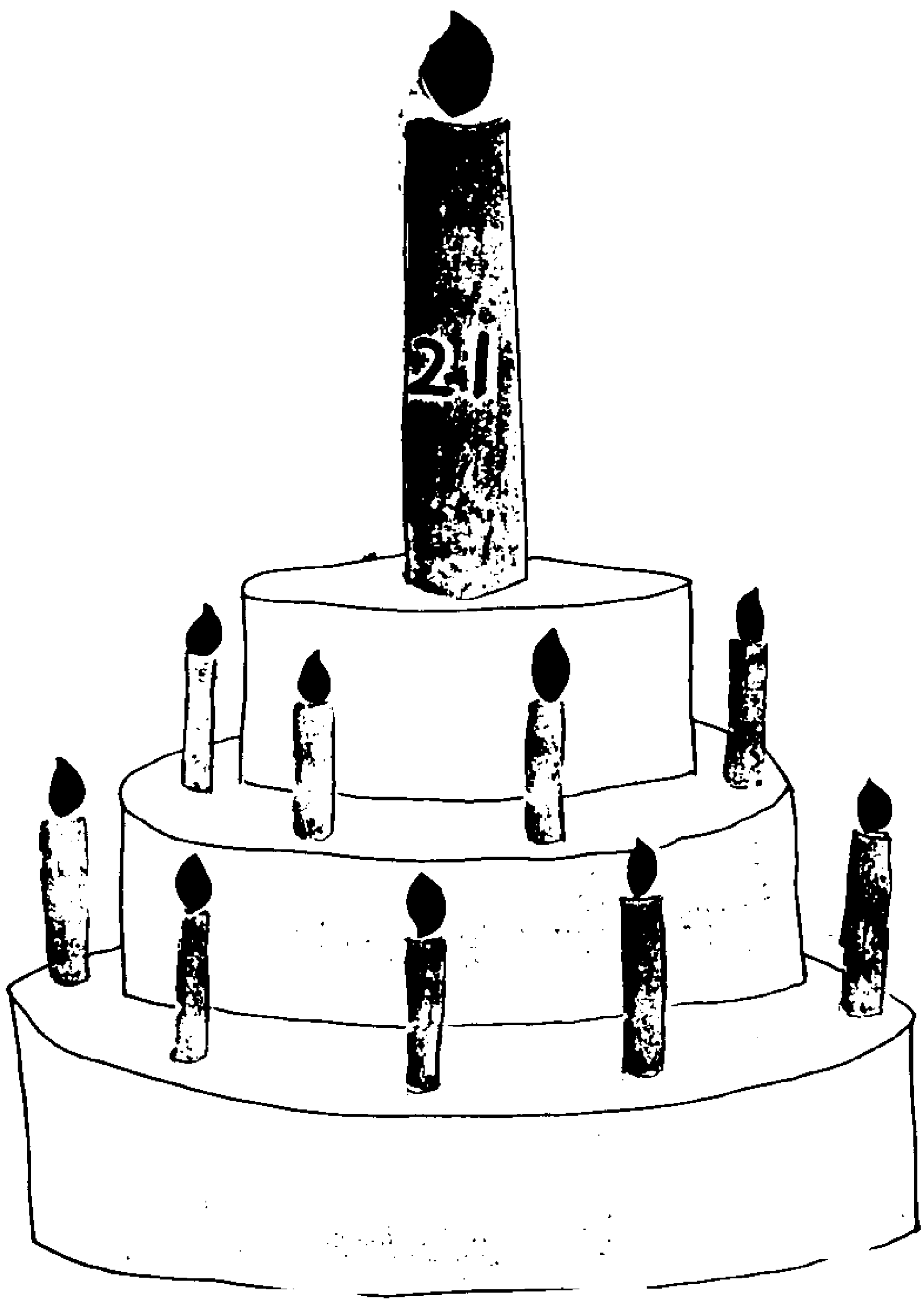
If this characterization of the representation of number is correct, then it is important not only to provide input to the system, but also to find ways to actively assist the integration process. Translating information from one modality to another may be one way to do this.

APPENDIX A.1

Picture Choice Task: Numbers As Labels

APPENDIX A.2

Picture Choice Task: Numbers As Labels



APPENDIX A.3

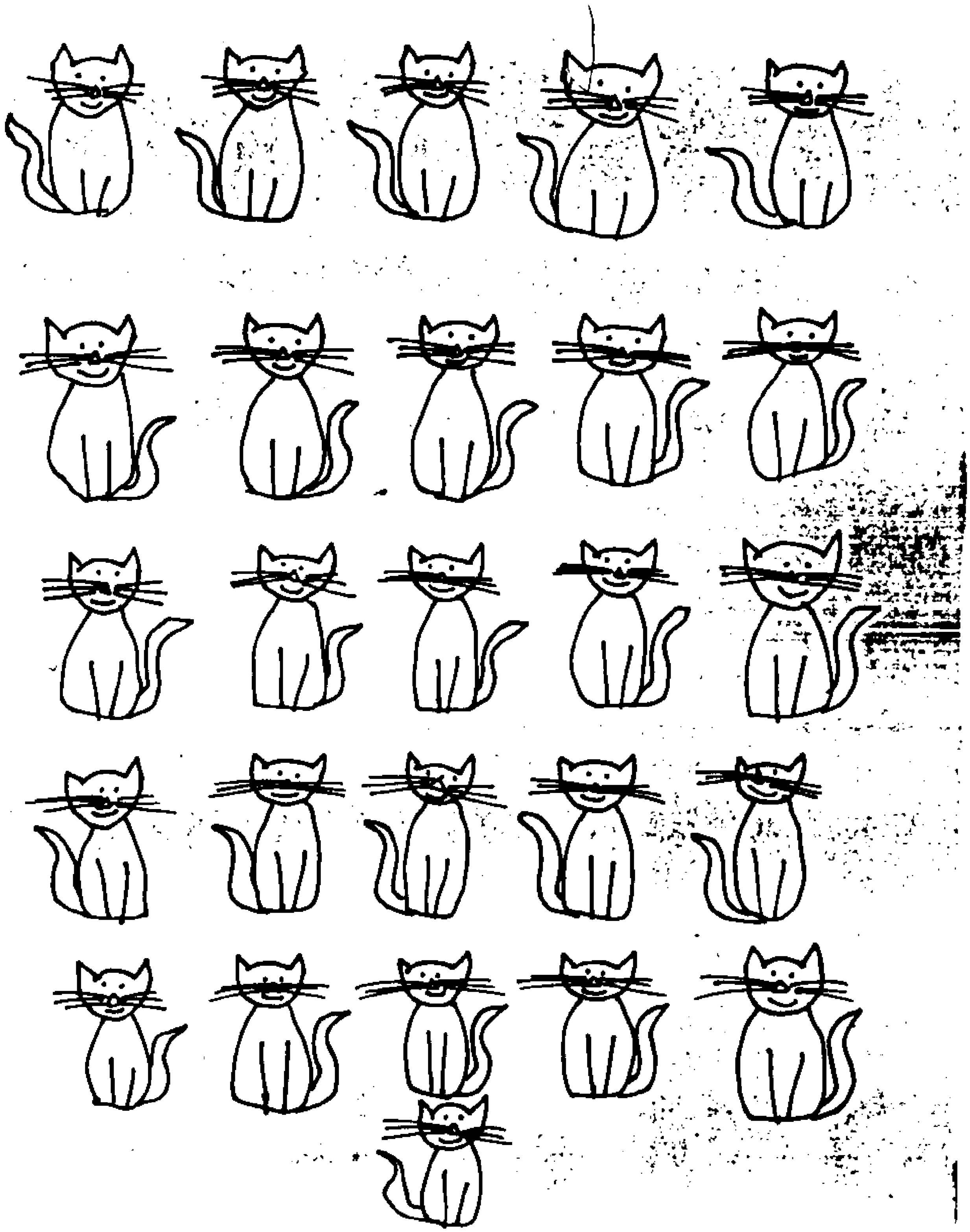
Picture Choice Task: Numbers As Labels

APPENDIX A.4

Picture Choice Task: Numbers As Labels

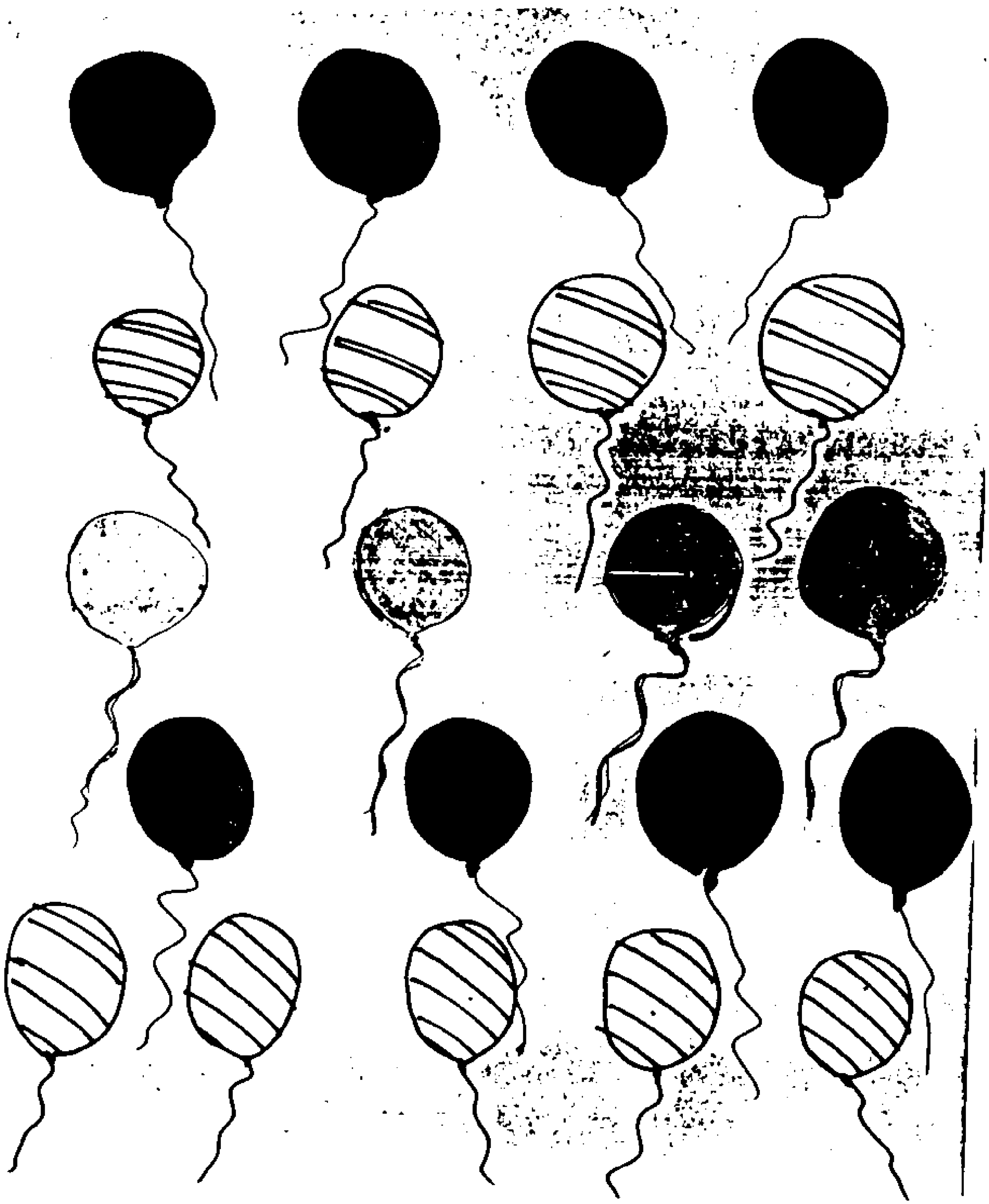
APPENDIX A.5

Picture Choice Task: One-To-One Correspondence



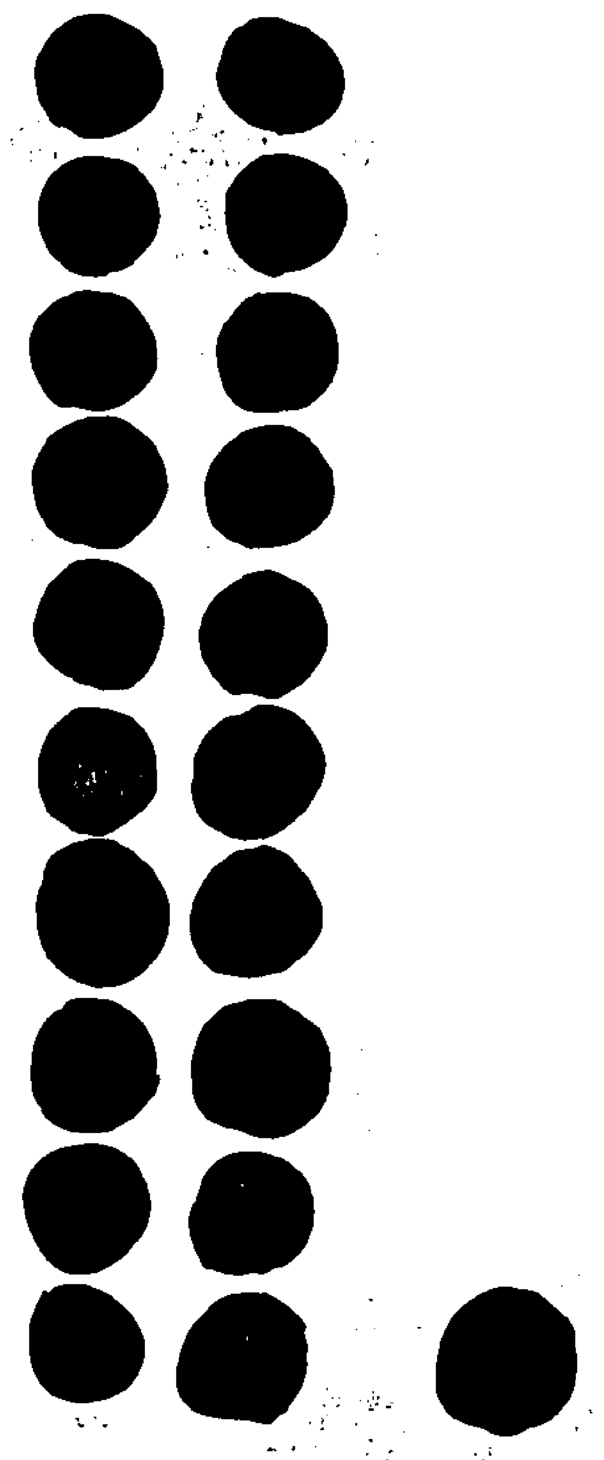
APPENDIX A.6

Picture Choice Task:One-To-One Correspondence



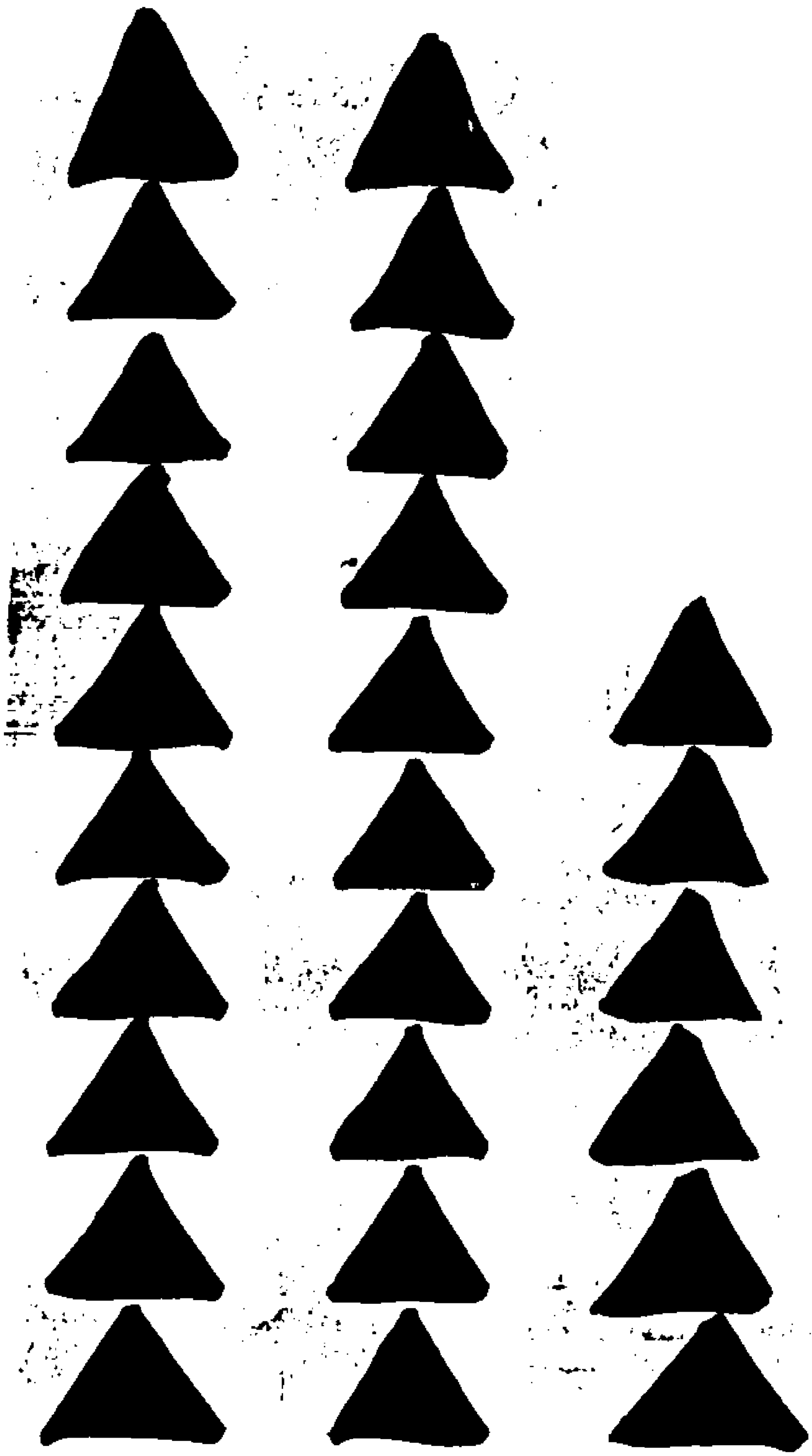
APPENDIX A.7

Picture Choice Task:One-To-One Correspondence



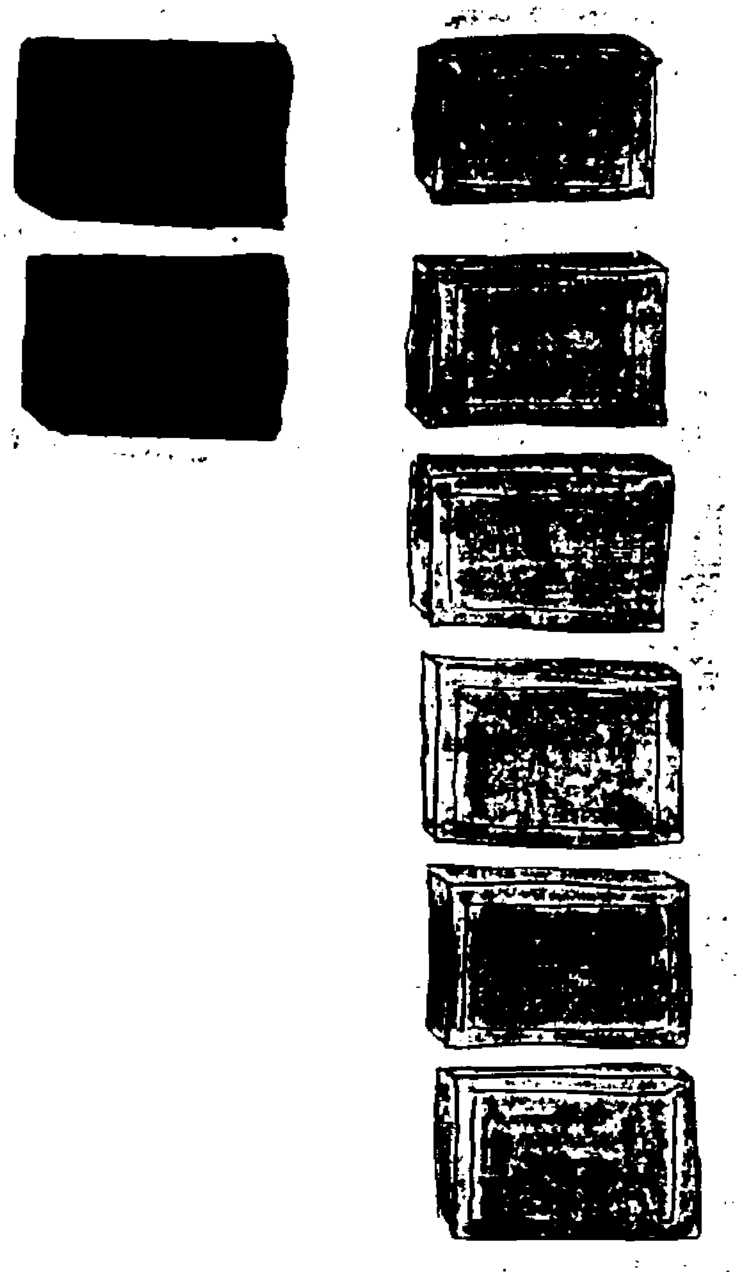
APPENDIX A.8

Picture Choice Task:One-To-One Correspondence



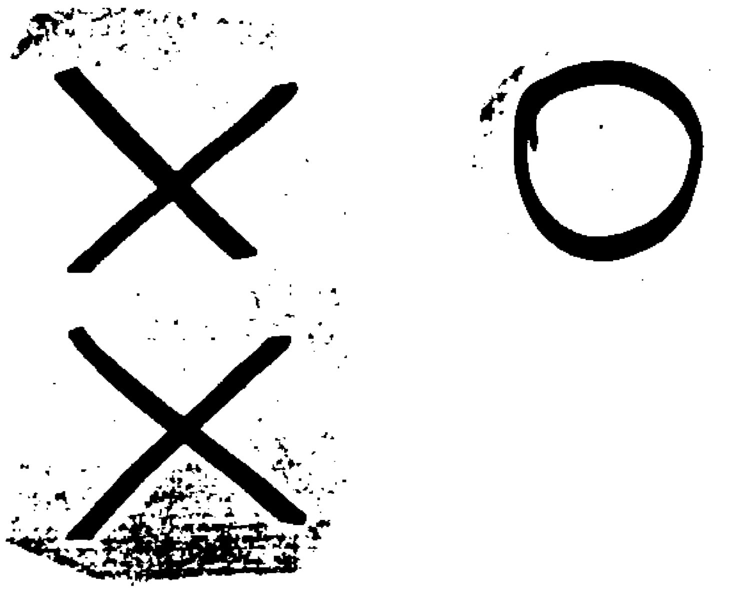
APPENDIX A.9

Picture Choice Task: Base Ten Knowledge



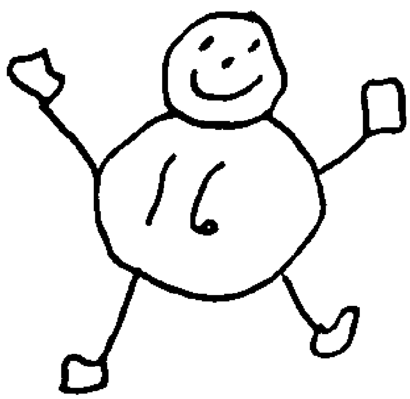
APPENDIX A.10

Picture Choice Task: Base Ten Knowledge



APPENDIX B.1

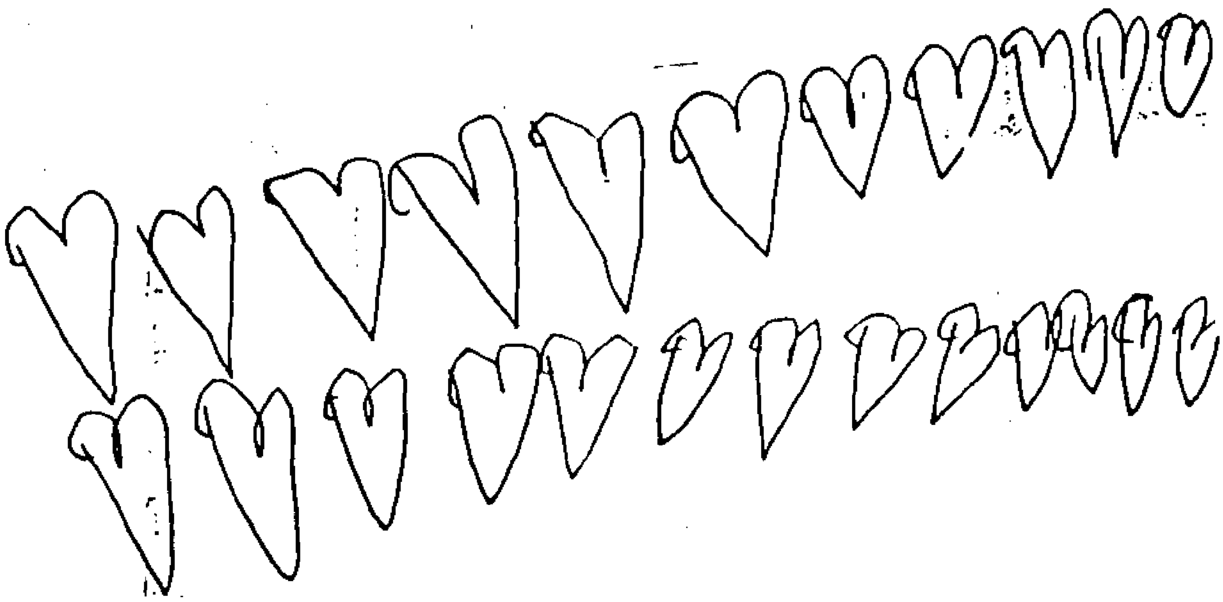
An Example Of A Kindergarten Drawing Rated At Level One



"A person who's 16"

APPENDIX B.2

An Example Of A Kindergarten Drawing Rated At Level Two



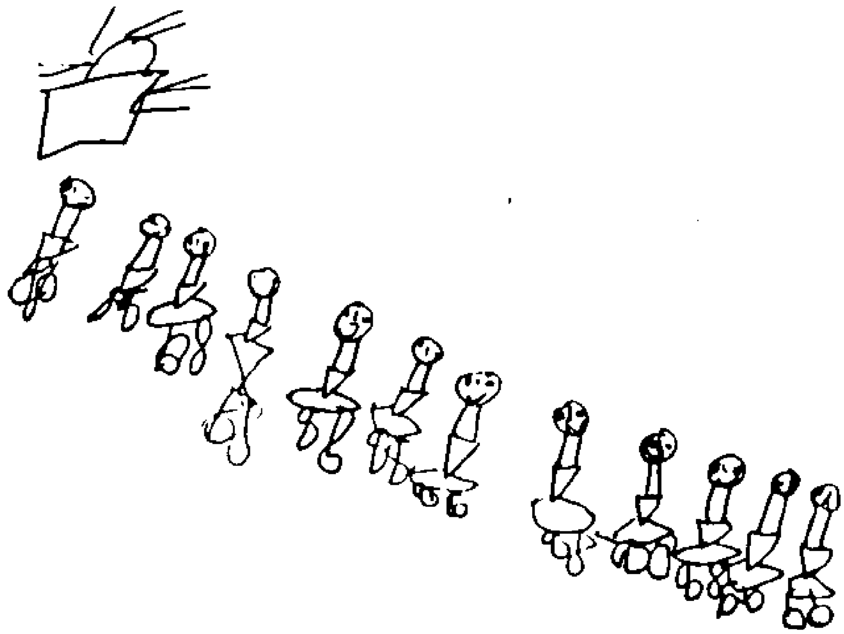
"Twenty four hearts"

APPENDIX B.3

An Example Of A First Grade Drawing Rated At Level One

APPENDIX B.4

An Example Of A First Grade Drawing Rated At Level Two



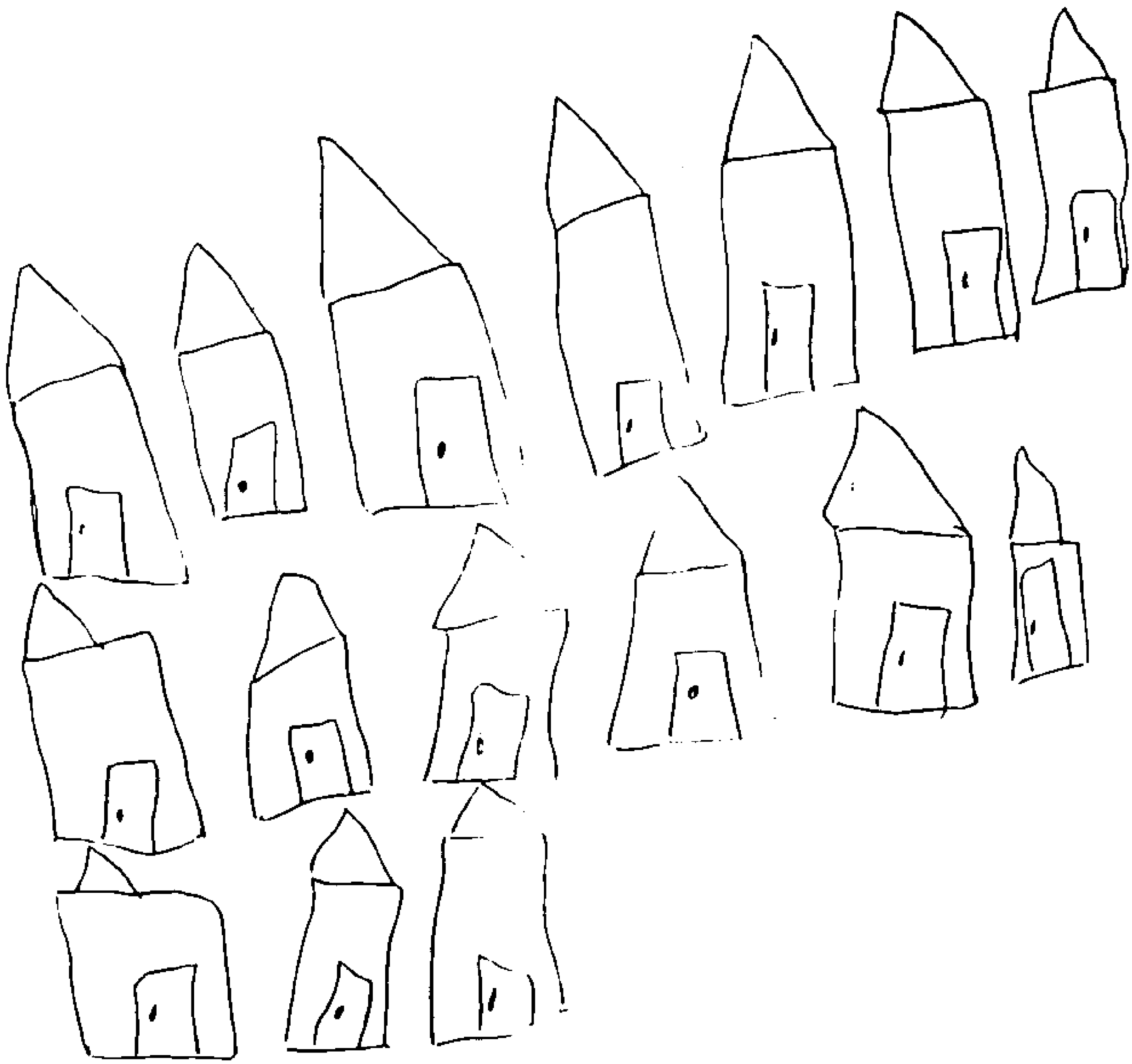
"Twelve people watching TV"

APPENDIX B.5

An Example Of A Second Grade Drawing Rated At Level One

APPENDIX B.6

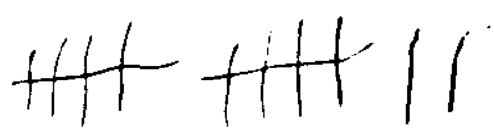
An Example Of A Second Grade Drawing Rated At Level Two



"Sixteen houses"

APPENDIX B.7

An Example Of A Second Grade Drawing Rated At Level Three



"Twelve"

APPENDIX C

Bigness Explanations

Subject	Explanation	Rating
01	Pointed to 21 as bigger. How did she know? "Because its big". Couldn't say how she knew, didn't know how much bigger.	1
	Pointed to 26 as bigger. How did she know? "Cause its higher". Couldn't say how she knew that, didn't know how much higher, but pointed to the 2 in 26 and the 1 in 16.	3
02	21 bigger because "Cause its..instead of the 11 is twice? (2 ones in 11) Well 2 is higher than 1. (the 2 in 21). Didnt know how much bigger, tried counting on fingers but "I can't just get it".	3
	26 is bigger than 16 because "16 is by the 12 and 26 is by the 24". Didn't know how much bigger.	2
03	Knew 21 was bigger but couldn't say how he knew (shrugged), and didn't know how much bigger.	1

Subject Explanation Rating

03	Pointed to 26 as the bigger #, but couldn't say how he knew. Didn't know how much bigger.	1
04	Pointed to 21 as bigger "Cause it doesn't um 12 doesn't come after um I mean um 21 comes after but not before 12". How much bigger? Spread arms out wide to show a lot bigger.	2
	26 is bigger "Because it doesnt come after no it comes after not before" How much bigger? "A LOT much bigger".	2
05	DISCONTINUED	1
06	Knew 21 was bigger because "1,2,3,4, 5,6,7,8,9,10,11,12,13,14,15,16,17, 18,19,20,21.. 21 COMES LAST! Didn't know how much bigger.	2
	Knew was 26 bigger "Cause it comes after". Didnt know how much bigger.	2
07	Pointed at 21 as bigger "Cause its farther than 12" (Couldn't explain farther from what) Didnt know how much bigger.	2
	Pointed at 26 as bigger but couldn't	

Subject	Explanation	Rating
07 (cont.)	explain how she knew. "Cause the uh the numbers..cause its its its..there is	

	no 26 in um..."Didn't know how much bigger.	1
08	Knew 21 is bigger because "I counted" (then counted aloud from 1 to 21). Didn't know how much bigger.	2
	26 is bigger "Cause I counted to it." Didn't know how much bigger.	2
09	DISCONTINUED	1
10	Knew 21 bigger but couldn't say how he knew and didn't know how much bigger. (NOTE: In verbal contrast, this child said that 12 comes before 21; but didn't make the connection to the size of the number.	1
	Knew 26 was bigger but couldn't say how he knew, didn't know how much bigger.	1
11	DISCONTINUED	1
12	Points to 21 as biggest, couldn't explain how she knew, didn't know how much bigger.	1
	Points to 26 as bigger, couldn't explain how she knew, didn't know how much bigger.	1

<u>Subject</u>	<u>Explanation</u>	<u>Rating</u>
13	21 is bigger "cause 12 comes after 21" What? "12 comes (pause) <u>before</u> 21."	

	Didn't know how much bigger.	2
	26 is bigger "Cause this number comes before 26" Didn't know how much bigger.	2
14	DISCONTINUED	1
15	DISCONTINUED	1
16	Points to 21 as bigger "cause the 2 is starting" Didnt know how much bigger.	3
	Points to 26 as higher "Cause its a 2 and the 1 is starting in this (12) so its (26) higher." Didn't know how much higher.	3
17	21 bigger "Because this one's (21) after this one (12). Didn't know how much bigger.	2
	Points to 26 as bigger. Knows its bigger "because of counting". Didn't know how much bigger.	2
18	Points to 21 as bigger "Cause it has a 2 and a 1" Didn't know how much bigger.	3
	Points to 26 as bigger. Cant say how he knows. How much? Stretches out arms.	3
<u>Subject</u>	<u>Explanation</u>	<u>Rating</u>
19	DISCONTINUED	1
20	Picked 21 as bigger. "Because uh because uh all the numbers like 19 and 18 and	

	16,17 they're lower and 21s the highest like 20, like 21..see 21 is bigger the higher number like 30 is..like 100 the biggest number and 21 is the lowest number a little" Didn't know how much bigger.	2
	Picked 26 as bigger "Because here's 26 and here's 16" (showing locations on an imaginary number line), "16,..20,21,22,23, 24,25,26" Didn't know how much bigger.	2
21	(After size comparison in verbal contrast) "Cause I count sometimes..I go up to 12 and I go up to 21" Didn't know how much higher.	2
	Picks 26 as bigger "Because it has a 2 and that one has a 1". Didn't know how much bigger.	3
22	Picked 21 as bigger. Because "I'd count 1,2,3,4,5,6,7,8,9,10,11,12,13, 14,15,16,17,18,19,20,21..yeah and I'd go past the 12". Guessed that 21 was 7 higher.	2

Subject	Explanation	Rating
22 (cont.)	Picked 26 as bigger. Because "By putting the 1 that counts as one single, but 2 counts as 2 singles" Guessed that 26 was 6 higher.	3

23	Points to 21 as bigger "Because its 2 and 1..well because first you say 11 then you say 12 and then you say 21" Didn't know how much bigger.	2
	"But if I count um if I count then um first you say 16 then you say 26" Didn't know how much bigger.	2
24	Picked 21 as bigger "Because you you count it after 12" Didn't know how much bigger.	2
	Picked 26 as bigger "When you count it comes after 16" Didn't know how much bigger.	2
25	(Volunteered that 21 was bigger in verbal contrast) "Because you go 12,13,14,15,16,17,18,19,20,21..21 is more further away." How much bigger? "Um like 5 or 6 or 7".	2
	Um like um same as 12 and 21..26	

Subject	Explanation	Rating
25 (cont.)	is more farther away..Because like from 16 it goes to 26..26 is like higher" How much higher? "Um 8".	2
26	Knew 21 was bigger "Cause its..the 1 is in front there and the 2 is in	

	front there. " How much higher? "9" (counted on fingers)	3
	26 is bigger "cause2 is more than 1". How much? "10" (counted on fingers.	3
27	Knew 21 is bigger. Why?(Confusion here) "Because 12 is more than 21.. because um cause the um because they're both reversed" Didn't know how much bigger.	1
	Knew 26 is bigger " Because these two (6s) are the same but the 1 is like smaller than the 2." Didn't know how much bigger.	3
28	Knew 21 was bigger but couldn't say how he knew. Didn't know how much bigger.	1
	Said 16 was bigger than 26	1
29	Knew 21 bigger "Cause 12 comes after 11 and 21 comes after 20" Thought 21 was 10 more than 12.	2

Subject	Explanation	Rating
29 (cont.)	Knew 26 was bigger "Because it comes after 16". Knew it was 10 bigger.	2
30	Knew 21 was bigger "Because um 12 is uh lesser than 21..because um when you count um cause 12 comes before 21".	

	Didn't know how much bigger.	2
	Knew that 26 was bigger "Because 16 comes before 26" Didn't know how much bigger.	2
31	Knew 21 bigger "because the 12 um is less" "Because if you have um a block and um if you want more um and if you want 21 its more higher" Didn't know how much higher. (quantity of blocks)	3
	Knew 26 was bigger "Because if you have um twen..um 16 um boxes and you need more for your other toys at home um and you have um 26 toys you know uh you'd need 26 boxes and thats more." Didn't know how much more.	3
32	Knew 21 bigger "Cause this is in the 10s and the 21 is in the 20s. How much bigger? "Wait I'm thinkin'..8 or 9"	3

Subject	Explanation	Rating
32 (cont.)	Knew 26 higher because "the 20s are higher than the 10s" Knew 26 was 10 higher, "cause 10 more makes 26".	3
33	(Volunteered that 21 was higher in verbal contrast.) "Because if you	

	counted the blocks then you would have..you would have more than the other pile". Didn't know how much higher.	3
	"Um somebody growing up could be 26 and some kid could be 16 and you can tell the higher and you could tell the difference how shorter and taller." Didn't know how much bigger 26 was.	3
34	Knew 21 was bigger "Because 12 is in is in um the teens and um and um 21 are in the 20s" Didn't know how much bigger.	3
	Knew 26 was bigger "Because its in the teens (16) and 26 is in the 20s." Didn't know how much bigger.	3

Subject	Explanation	Rating
35	Knew 21 was higher because "Well 12 comes before 21 and 21's the higher number." Didn't know how much higher.	2
	Knew 26 was bigger. "Wait I'm not sure..I think I know what you're doin?..Um I think the 10s collumn	

and the 1s collumn (Points to 1 and 6 in 16) (Got confused) Didn't know how much bigger. 3

36 Knew 21 bigger "Because um cause its in um it's in the 20s." How much bigger? "A lot bigger." 3

Knew 26 bigger "Because it's still in the 20s." Didn't know how much bigger. 3

37 Knew 21 was higher because "It's in the 20s collumn." Knew it was 9 higher after counting on fingers. 3

Knew 26 was 10 higher because "26 take away 10 is 16". 3

38 Knew 21 was bigger because "The first number is bigger." Knew it was 9 more, counted on fingers. 3

Knew 26 was bigger because "The first number is bigger. Knew it was 10 more by counting on fingers. 3

39 Knew 21 was bigger "Because if you count to 12, you won't have 21..you have to count to 20 then you have to count one more to get 21."

Knew it was 9 bigger after counting on fingers. 3

Knew 26 was bigger "Because if you count up to 16 you won't have 26..you have to count all the way to 20 then count six more." Knew it was 10 more after counting on fingers. 3

40 (Had volunteered size comparison in verbal contrast)"The one is before the two and the two is before the one in this one here." Guessed that 21 is 11 bigger than 12. 3

Knew 26 was bigger "Because if you want to reach 26 you reach 16 before you reach 26. Thought 26 was six more than 16. 2

41 Knew 21 was higher "Because when you count you pass 12 before you pass 21." Guessed 21 was 11 higher, but corrected to nine. 2

"Twenty six is 10 higher because you have a one over here and a one over here and one and one is two." 3

42 Knew 21 was bigger "Because the one (in 12) is less than the two (in 21), and the two is more than the one." Said it was 11 higher. 3

Knew 26 was higher "Because this has a two and a six, and this has a one and a six."

Said it was 12 higher. 3

43 "Um 21 has more than 12 cause 12 is..it has 12 and 21 has 21." Knew it was 9 more after counting on fingers. 1

"Um 26 is bigger than 16..cause two is more than one." Knew 10 more after counting on fingers. 3

44 Knew 21 was bigger "Because when you count up to 10, it's 11 and 12. When you count up to 19, it's 20,21. And 19 is higher than 11 and 21 is higher than 12." Didn't know how much higher. 2

Knew 26 was higher .."See I know cause you count 1,2,3,4,5,6,7,8,9,10, and you go up to 15 and after that it's 16..and after that is 16,17,18,19,20,21,22,23,24,25,26. It's higher." Didn't know how much higher. 2

45 Knew 21 was bigger, but couldn't say why, didn't know how much. 1

Knew 26 was bigger, "Because it's (the two in 26) higher than one." Didn't know how much bigger. 3

46 Knew 21 was bigger "Cause when you're..if you count um if you get to 12 and you haven't said 21, you know that it's bigger than the 12." Counted on fingers to discover 9 higher. 2

Knew 26 was bigger "Because um..26..a lot of people count 26 um they said 16 already so they know that 26 is bigger than 16." Knew that 26 was "10 numbers larger". 2

47 Knew 12 was less than 21 because "When you count, 12 comes first." Didn't know how much less. 2

Knew 16 was less than 26 "Cause it comes first in counting." Knew 26 was 10 more because "16 + 10 equals 26." 2

48 Knew 21 was higher because "Well the two (in 21) comes before the one." After counting on fingers, said it was 8 higher. 3

Knew 26 higher "Well because they're 10 away...cause um 16 and 10 is 26." 3

49 Knew 21 was higher "Because this (the 2 in 21) is a two, and this (the one in 12) is a one."

Reaches for pencil, does "12 take away 21" gets 9. 3

"Um um the 2 (in 26) is bigger than the one (in 16)."Used pencil to figure out that 26 was 10 more. 3

50 Knew 12 was lower "Because when you count, 12 is before 21." Knew it was 9 lower. 2

Knew 26 was bigger "Because when you count uh 26 is after." Knew it was 10 more. 2

51 Knew 21 was bigger "Cause first you come to 12 and then you come to 21." 2

Knew 26 was bigger "Cause here is a two (in 26) and here is a one (in 16) and its bigger." Counted on fingers to get 10 bigger. 3

52 Knew 21 was bigger because "The two is in the tens place and the one is in the ones place." Then, "Um you would have to take away..you would have to regroup...nine." 3

Knew 26 was bigger "Uh cause 16 is before." Knew it was 10 more. 2

53 Knew 21 was bigger "Because lets see there's a two in the tens place (in 21) and only a one

in the tens place (in 12)." Figured out 9 difference with pencil and paper. 3

Knew 26 was bigger "Cause 6 take away 6 is zero and two take away one is one..10" 3

54 Knew 21 was bigger "Cause two is in the tens collumn and in 12, two is in the ones collumn." Couldn't figure out how much. 3

Same for 26. 3

APPENDIX D.1

A Sample of Topics[a] Covered in the New York City Mathematics Curriculum For Kindergarten

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Problem solving: One-to-one correspondence

Numerals (1) through (4)

Numerals (5) through (9)

Numeral zero (0)

Numeral ten (10)

Numerals eleven (11) through fifteen (15)

Construct and count using a number line

---

[a]These topics represent seven instructional modules out of 51 for the school year.

APPENDIX D.2

A Sample of the Topics* Covered in the New York City Mathematics Curriculum For the First Grade

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One-to-one and more than one correspondence

Compare groups of one through five objects

Numerals one (1) through five (5)

Numerals six (6) through nine (9)

The numeral ten (10)

Count and group by tens through 50

Place Value from ten through 50

Order numbers from 10 through 50

Place value from 51 through 99

Order numerals from 51 through 99

---

*These topics represent 11 instructional modules out of 72 for the entire school year.

APPENDIX D.3

A Sample Of Topics[a] Covered In The New York City Mathematics Curriculum For The Second Grade

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Review

Recognize and name whole numbers through 100 by ones

Recognize and name whole numbers through 100 by tens

Addition and subtraction facts

Count hundreds through ten hundred (1000)

Place value through 999

---

[a]These topics represent six instructional modules out of 70 for the school year.

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