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**ATOMIC SCATTERING IN THE PRESENCE OF A LOW-FREQUENCY
LASER**

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ATOMIC SCATTERING IN THE PRESENCE OF A LOW-FREQUENCY LASER

by

JAGANNATH BANERJI

**A dissertation submitted to the Graduate Faculty in
Physics in partial fulfillment of the requirements
for the degree of Doctor of Philosophy, The City
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1982

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**This work is dedicated
to
my beloved parents**

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CHAPTER 1

GENERAL INTRODUCTION

1.1 Introduction:

In the last decade or so, there has been a surge of interest in laser-modified atomic scattering motivated in part by the need for such information in applied problems¹. But laser-modified atomic scattering may also be a way of observing atomic scattering processes² which are, in principle, not observable in ordinary scattering. The analogue of this statement was recognized long ago in nuclear physics³ where attempts were made to observe off-shell T-matrices in nuclear bremsstrahlung. In both cases, the photons involved play the role of a "third body" in the scattering and carry away energy thereby making the remaining T-matrix off-shell. The advantage of the use of the laser over the bremsstrahlung process is that it is (one hopes) controllable in both frequency and intensity so that its effects can be more pronounced.

The laser introduces new parameters in the scattering problem such as the laser photon energy, its intensity, its polarization, its spatial and temporal distributions and its statistics. Of these, the first two seem to be the most important. But only a very limited range of these parameters have been investigated. The topic of this dissertation is some of the work that has been done on potential and electron-atom scattering in the presence of a low-frequency laser field.

1.2 Basics of the laser and laser-modified scattering:

We are interested in the interaction of lasers, a very special kind of radiation, with matter. Laser radiation is highly coherent⁴, intense and directional. For our purposes, however, the crucial point which distinguishes lasers from other radiation is their high intensity. More specifically, it is the large number of photons in a laser mode. For example, a laser with 1 ev. photons with a single mode flux of 1 mW/cm² in a volume of 1 cm³ has 2×10^5 photons in the field:

$$\begin{array}{l} \text{Number of} \\ \text{photons} = \frac{10^{-3} \text{ W/cm}^2 \times 1 \text{ cm}^3}{1 \text{ ev.} \times 3 \times 10^{10} \text{ cm/sec.}} \approx 2 \times 10^5 \end{array}$$

This very high quantum number makes it possible for the laser to be described as a classical electromagnetic field⁵. The laser has also been treated quantum electrodynamically as a collection of photons⁶. It has been shown⁷ that the two descriptions are equivalent provided that the number of photons in the laser modes are large. This being always the case, we shall limit ourselves to the simpler classical description here. The laser will be described as a single mode, linearly polarized, classical electromagnetic field in the dipole approximation. (All of these are approximations to the

true situation, which can be readily modified⁸).

Thus the vector potential $\vec{A}(t)$ is written as⁹

$$\vec{A}(t) = \frac{\vec{E}_0}{\omega} \cos \omega t \quad (1.2.1)$$

where $\vec{E}_0$ is the amplitude of the electric field of the laser. We shall not be interested in relativistic effects so the matter field will be described by a conventional Schrodinger wave function. In the following, we present the modified Schrodinger equation in the presence of a laser and the different gauge transformations that can be made to simplify it.

The Schrodinger equation for a particle of mass m and charge e in the presence of a scalar potential $V(r)$ and a laser with the vector potential $\vec{A}(t)$ is

$$i \frac{\partial \psi_i}{\partial t} = H_i(t) \psi_i \quad (1.2.2)$$

where

$$H_i(t) = \frac{1}{2m} (\vec{p} + e\vec{A})^2 + V(r) \quad (1.2.3)$$

With the gauge transformation

$$\psi_i = e^{-i\phi_i(t)} \psi_i \quad (1.2.4)$$

equation (1.2.2) transforms into

$$i \frac{\partial \psi_j}{\partial t} = H_j(t) \psi_j \quad (1.2.5)$$

with

$$H_j(t) = e^{i\phi_j} H_1(t) e^{-i\phi_j} - \dot{\phi}_j \quad (1.2.6)$$

The Hamiltonian in the absence of the laser is

$$H_0 = \frac{\vec{p}^2}{2m} + V(r) \quad (1.2.7)$$

So, the interaction of the laser with the particle is

$$H_j' = H_j - H_0 \quad (1.2.8)$$

ϕ_j can be freely chosen and the new wave function will be as physically meaningful as the old.

Our first choice will be designed to remove the A^2 term of (1.2.3):

$$\phi_2 = \frac{e^2}{2m} \int_0^t dt' A^2(t') \quad (1.2.9)$$

The result is

$$H_2 = H_0 + \frac{e}{m} \vec{p} \cdot \vec{A}(t) \quad (1.2.10)$$

so that the interaction of the laser with the particle is

$$H_2' = \frac{e}{m} \vec{p} \cdot \vec{A}(t) \quad (1.2.11)$$

This is the celebrated $\vec{p} \cdot \vec{A}$ gauge.

Another choice of ϕ_j can be made to remove the $\vec{p} \cdot \vec{A}$ term from $H_1(t)$. If we define

$$\phi_3 = e \vec{r} \cdot \vec{A}(t) \quad (1.2.12)$$

then,

$$H_3 = H_0 + e \vec{r} \cdot \vec{E}(t) \quad (1.2.13)$$

where $\vec{E}(t) = -\dot{\vec{A}}$ is the electric field of the laser. In this gauge, the interaction term is

$$H'_3 = e \vec{r} \cdot \vec{E}(t) \quad (1.2.14)$$

There is another gauge transformation due to Kramers¹⁰ in which the choice of ϕ_i is

$$\phi_4 = \frac{e}{m} \vec{p} \cdot \int_0^t dt' \vec{A}(t') + \frac{e^2}{2m} \int_0^t dt' A^2(t') \quad (1.2.15)$$

The corresponding transformed Hamiltonian is

$$H_4 = \frac{p^2}{2m} + V(\vec{r} - \vec{\alpha}(t)) \quad (1.2.16)$$

where

$$\vec{\alpha}(t) = \frac{e}{m} \int_0^t dt' \vec{A}(t') \quad (1.2.17)$$

by a suitable combination of the different ϕ_j 's or the use of other ϕ 's.
by a suitable combination of the different ϕ_j 's or the use of other ϕ 's.

1.3 Scattering in the presence of a low-frequency laser:

The different gauges are useful in different cases. For low-frequency lasers, calculations become simplified if we use the $\vec{p} \cdot \vec{A}$ gauge. In that case the Schrodinger equation is

$$i \frac{\partial \psi}{\partial t} = \left(\frac{p^2}{2m} + V(r) + \frac{e}{m} \vec{p} \cdot \vec{A}(t) \right) \psi \quad (1.3.1)$$

and the laser interaction is

$$\frac{e}{m} \vec{p} \cdot \vec{A} = \omega \vec{\alpha}_0 \cdot \vec{p} \cos \omega t \quad (1.3.2)$$

where $\vec{\alpha}_0 = e\vec{E}_0/m\omega^2$ is the amplitude of the oscillation that a classical electron would have in the potential $\vec{A}(t)$. One can write

$$\alpha_0/a_0 = 2 \times \frac{E_0/E_a}{(\omega/Ry)^2} \quad (1.3.3)$$

where $E_a = e/2a_0^2$ is the atomic unit for electric field and $Ry = e^2/2a_0 \approx 13.6$ ev. a_0 is the Bohr radius. Since the intensity is proportional to the square of the electric field,

$$\alpha_0/a_0 = 2 \times \frac{\sqrt{I/I_a}}{(\omega/Ry)^2} \quad (1.3.4)$$

where I is the intensity of the laser and $I_a \approx 1.75 \times 10^{16}$ W/cm² is the atomic unit for intensity.

We choose a low-frequency laser whose intensity and frequency are such that α_0 is held fixed at a value near a_0 .

The words "low-frequency laser" require some definition here. For a structureless target that can be

represented by a short range or non-Coulomb potential we require that the photon energy ω be small compared to the projectile energy ϵ_{pi} :

$$\omega \ll \epsilon_{pi} \quad (1.3.5)$$

When the target is an atom, the above definition must be enlarged to include

$$\omega \ll |W_n - W_{n'}| \quad (1.3.6)$$

where atomic states n and n' are relevant in the scattering.

A quintessential low-frequency laser is the CO_2 laser for which $\omega \approx 0.12$ ev. Substituting for the values of I_a , Ry and ω in (1.3.4), one finds that for a CO_2 laser

$$\alpha_s / a_0 \approx 1 \quad (1.3.7)$$

if $I \approx 10^7$ W/cm². In the present work we will assume that the condition (1.3.7) is satisfied. Then the interaction (1.3.2) is of order ω and so is small. We can, therefore, carry out a straightforward perturbation theory in powers of (1.3.2) except in the case in which it acts on an otherwise free particle for a long time. In that case, one has to integrate (1.3.2) over the interaction time. This will remove the factor of ω from the front. The resulting integrated interaction is no longer small and can not be treated perturbatively.

1.4 Outline of the thesis:

It has long been known from perturbation theory that the interaction of charged particles with low-frequency radiation modifies a scattering process in a simple way so that the amplitude for scattering with the radiation present is a multiple of the amplitude in the absence of the radiation¹¹. Kroll and Watson¹² extended this result in a seminal paper in 1973. They treated the potential scattering problem in which the projectile is an electron and found that the T-matrix for the absorption of l photons by the electron is

$$T_l(\vec{p}_f, \vec{p}_i) = J_{-l}(\vec{\alpha}_0 \cdot \Delta \vec{p}) \langle \vec{p}_f | T(\epsilon_{\vec{p}}) | \vec{p}_i \rangle + O(\omega^2) \quad (1.4.1)$$

where $\vec{p}_i$ and $\vec{p}_f$ are respectively the initial and final momenta of the electron. The energy-conservation condition is, therefore,

$$p_f^2/2m = p_i^2/2m + l\omega \quad (1.4.2)$$

$\Delta \vec{p} = \vec{p}_f - \vec{p}_i$ is the momentum transfer. The T-matrix appearing on the right side of (1.4.1) is that in the absence of the laser, but the momenta $\vec{p}_f$, $\vec{p}_i$ are shifted from $\vec{p}_f$, $\vec{p}_i$ in the following way:

$$\begin{aligned} \vec{p}_f &= \vec{p}_f - m\omega l \vec{\alpha}_0 / \vec{\alpha}_0 \cdot \Delta \vec{p} \\ \vec{p}_i &= \vec{p}_i - m\omega l \vec{\alpha}_0 / \vec{\alpha}_0 \cdot \Delta \vec{p} \end{aligned} \quad (1.4.3)$$

Also,

$$\epsilon_p = \frac{p^2}{2m} \quad (1.4.4)$$

It can be shown that

$$\frac{p_f^2}{2m} = \frac{p_i^2}{2m} \quad (1.4.5)$$

Hence the T-matrix is evaluated on-shell. For future reference we would like to rewrite the above result in the following way. We define a quantity

$$\lambda_0 = \ell \vec{\alpha}_0 \cdot \vec{p}_i / \vec{\alpha}_0 \cdot \vec{p} \quad (1.4.6)$$

Then, it is easy to show that

$$\frac{m\omega \ell \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}} = \frac{m\omega \lambda_0 \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} = \frac{m\omega (\lambda_0 + \ell) \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} = \vec{d}_{\lambda_0} \text{ (say)} \quad (1.4.7)$$

Also,

$$\epsilon_{p_i} = \epsilon_{p_i} - \lambda_0 \omega = \epsilon_{\vec{p}_i - \vec{d}_{\lambda_0}} + O(\omega^2) \quad (1.4.8)$$

Thus the Kroll-Watson result may also be written as

$$T_\ell(\vec{p}_f, \vec{p}_i) = T_\ell(\vec{\alpha}_0 \cdot \vec{p}) \langle \vec{p}_f - \vec{d}_{\lambda_0} | T(\epsilon_{\vec{p}_i - \vec{d}_{\lambda_0}}) | \vec{p}_i - \vec{d}_{\lambda_0} \rangle + O(\omega^2) \quad (1.4.9)$$

One can use (1.4.9) to obtain sum rules on cross sections.

For example, Kruger and Jung¹³ showed that

$$\sum_{\ell=-\infty}^{\infty} \frac{d\sigma_\ell}{d\Omega}(\vec{p}_f, \vec{p}_i) = \left. \frac{d\sigma}{d\Omega}(\vec{p}_f, \vec{p}_i) \right|_{\alpha=0} + O(\omega^2) \quad (1.4.10)$$

which says that the cross section for all possible values of ℓ sums to the cross section in the absence of the laser. This would be so only if the scattering is non-resonant.

and the corresponding T-matrix is evaluated to order ω

A quantity that is of practical importance in the problem of plasma heating is the average number of photons $\langle \ell \rangle$ transferred during a scattering event. For potential scattering it is defined as

$$\langle \ell \rangle = \frac{\sum_i e \int d\Omega_{pf} \frac{d\sigma_e}{d\Omega} (\vec{p}_f, \vec{p}_i)}{\sum_i \int d\Omega_{pf} \frac{d\sigma_e}{d\Omega} (\vec{p}_f, \vec{p}_i)} \quad (1.4.11)$$

and is given by¹⁴

$$\langle \ell \rangle = 1/\sigma_T \left[\frac{m\omega\alpha_0^2}{2} (1 + (\hat{q}_0 \cdot \hat{p}_i)^2) \sigma_\delta + m\omega\alpha_0^2 (\hat{q}_0 \cdot \hat{p}_i)^2 \epsilon_{pi} \frac{\partial \sigma_\delta}{\partial \epsilon} \right] \quad (1.4.12)$$

where σ_T and σ_δ are respectively the total and momentum transfer cross sections.

The Kroll-Watson result and its extension to higher orders in ω are valid for non-resonant potential scattering which is the topic of chapter 2. The Kroll-Watson result has been extended for resonant potential scattering^{13,15,16} and also for non-resonant electron-atom scattering¹⁴. In other words, either the effect of resonance or the effect of the degrees of freedom of the target was included. These are described in chapters 3 and 4 respectively. In chapter 5, we have treated these effects simultaneously and carried out calculations for resonant electron-atom scattering in the presence of a low frequency laser.

Experiments have been performed¹⁷ with electrons scattered on Argon in the presence of a CO₂ laser. The experiments were qualitative the only quantitative result being the confirmation of the sum rule(1.4.10). Chapter 6 is devoted to the experimental aspects of laser-modified electron-atom scattering.

An interesting laser-assisted atomic process is the electron-impact ionization of atoms^{18,19,20}. We have discussed this process in chapter 7 and found the following result for the triple differential cross section:²⁰

$$\frac{d^3\sigma_e(\vec{p}_f, \vec{q}; \vec{p}_i, 0)}{d\Omega_{p_f} d\Omega_q d\epsilon_q} \quad (1.4.13)$$

$$= J_e^2(\vec{\alpha}_0 \cdot \vec{Q}) \frac{|\vec{p}_f|}{|\vec{p}_f - \vec{d}|} \frac{|\vec{q}|}{|\vec{q} - \vec{d}|} \frac{|\vec{p}_i - \vec{d}|}{|\vec{p}_i|} \left(\frac{d^3\sigma(\vec{p}_f - \vec{d}, \vec{q} - \vec{d}; \vec{p}_i - \vec{d}, 0)}{d\Omega_{\vec{p}_f - \vec{d}} d\Omega_{\vec{q} - \vec{d}} d\epsilon_{\vec{q} - \vec{d}}} \right)_{d=0}$$

where $\vec{p}_f$, $\vec{q}$ and $\vec{p}_i$ are the momenta of the scattered, ejected and incident electron respectively.

$$\vec{d} = \frac{m\omega\ell\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{Q}} \quad (1.4.14)$$

with

$$\vec{Q} = \vec{p}_f + \vec{q} - \vec{p}_i \quad (1.4.15)$$

The cross section on the right side of (1.4.13) is the e-2e cross section in the absence of the laser but the

derivation requires that it be non-resonant in both initial and final states. This restriction has been lifted in section 7.3. However, the long range effects of the Coulomb potential have been ignored.

In chapter 8, the effect of the Coulomb tail has been accounted for. For simplicity, we have considered Coulomb potential scattering in the presence of a low-frequency laser. The result is in sharp distinction from the Kroll-Watson result (1.4.9) which is valid for short range potentials only.

Finally, we present a brief summary of the thesis with a particular emphasis on the specific contributions made by this author. This is the content of chapter 9.

CHAPTER 2

NON-RESONANT POTENTIAL SCATTERING IN THE PRESENCE OF A LOW-FREQUENCY LASER:

2.1 Derivation of the T-matrix:

The starting point is the exact S-matrix:

$$S = -i \int d^3r dt \chi_f^* (r, t) V(r) \psi_i^{(+)} (r, t) \quad (2.1.1)$$

where $\psi_i^{(+)}$ is the exact wave function which evolves out of the initial state at $t = -\infty$, V is the scattering potential and χ_f is the final state. It satisfies the Schrodinger equation (1.3.1) with V set equal to zero:

$$i \frac{\partial \chi_f}{\partial t} = \left(\frac{p^2}{2m} + \frac{e}{m} \vec{p} \cdot \vec{A}(t) \right) \chi_f \quad (2.1.2)$$

Solving,

$$\begin{aligned} |\chi_f\rangle &= e^{-i \vec{\alpha}_0 \cdot \vec{p}_f \sin \omega t - i \epsilon_{p_f} t} |\vec{p}_f\rangle \\ &= \sum_{\lambda'=-\infty}^{\infty} J_{\lambda'} (\vec{\alpha}_0 \cdot \vec{p}_f) e^{-i (\epsilon_{p_f} + \lambda' \omega) t} |\vec{p}_f\rangle \end{aligned} \quad (2.1.3)$$

where $\langle \vec{r} | \vec{p}_f \rangle = e^{i \vec{p}_f \cdot \vec{r}}$

The Lippman-Schwinger equation for $\psi_i^{(+)}$ is

$$|\psi_i^{(+)}\rangle = |\chi_i\rangle + \bar{G}^{(+)} V |\chi_i\rangle \quad (2.1.4)$$

where χ_i is the initial state given by

$$|\chi_i\rangle = \sum_{\lambda} J_{\lambda} (\vec{\alpha}_0 \cdot \vec{p}_i) e^{-i (\epsilon_{p_i} + \lambda \omega) t} |\vec{p}_i\rangle$$

or, by changing the sign of λ ,

$$|\chi_i\rangle = \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{-i (\epsilon_{p_i} - \lambda \omega) t} |\vec{p}_i\rangle \quad (2.1.5)$$

$\bar{G}^{(+)}$ is the full Green's function which we expand as

$$\bar{G}^{(+)} = G^{(+)} + G^{(+)} H' G^{(+)} + G^{(+)} H' G^{(+)} H' G^{(+)} + \dots \quad (2.1.6)$$

where

$$H' = \frac{e}{m} \vec{p} \cdot \vec{A} = \omega \vec{\alpha}_0 \cdot \vec{p} \cos \omega t \quad (1.3.2)$$

$G^{(+)}$ is the Green's function in the absence of the laser:

$$G^{(+)}(r, t; r', t') = -i \theta(t-t') \int_{\eta \rightarrow 0^+}^{\infty} dt \sum_n \phi_n(r) \phi_n^*(r') e^{-(iW_n + \eta)(t-t')} \quad (2.1.7)$$

where the ϕ_n are the stationary states (with energies W_n) in the presence of the potential V and η is a positive infinitesimal. Substitution of (2.1.4) and (2.1.6) into (2.1.1) results in a series for S :

$$S = \sum_{n=0}^{\infty} S^{(n)} \quad (2.1.8)$$

The individual terms can then be simplified by the use of (2.1.3), (2.1.5) and (2.1.7). An energy-delta function appears in all the terms so that (2.1.8) can be rewritten

$$S = -2\pi i \sum_{l=-\infty}^{\infty} \delta(\epsilon_{p_f} - \epsilon_{p_i} - l\omega) \sum_{n=0}^{\infty} T_l^{(n)}(\vec{p}_f, \vec{p}_i) \quad (2.1.9)$$

thereby defining the T-matrix for scattering with the net absorption of l quanta of energy from the laser:

$$T_l(\vec{p}_f, \vec{p}_i) = \sum_{n=0}^{\infty} T_l^{(n)}(\vec{p}_f, \vec{p}_i) \quad (2.1.10)$$

The $n = 0$ term is just

$$T_L^{(0)} = \sum_{\lambda=-\infty}^{\infty} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | V | \vec{p}_i \rangle \quad (2.1.11)$$

and the $n = 1$ term is

$$T_L^{(1)} = \sum_{\lambda=-\infty}^{\infty} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | V G^{(+)}(\epsilon_{p_i} - \lambda\omega) V | \vec{p}_i \rangle \quad (2.1.12)$$

where

$$\langle \vec{r} | G^{(+)}(E) | \vec{r}' \rangle = \sum_n \frac{\phi_n(r) \phi_n^*(r')}{E + i\eta - W_n} \quad (2.1.13)$$

is the time independent Green's function with the potential V .

The $n = 2$ term can be evaluated with only a little more effort, yielding

$$T_L^{(2)} = \frac{\omega}{2} \sum_{\lambda} \sum_{x_1=\pm 1} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-x_1-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \langle \vec{p}_f | V G^{(+)}(\epsilon_{p_i} - \omega(\lambda+x_1)) \vec{\alpha}_0 \cdot \vec{p} G^{(+)}(\epsilon_{p_i} - \lambda\omega) V | \vec{p}_i \rangle \quad (2.1.14)$$

The index x_1 arises from the $\cos \omega t$ in $\bar{A}(t)$ by the use of

$$\cos \omega t = \frac{1}{2} \sum_{x_1=\pm 1} e^{ix_1 \omega t} \quad (2.1.15)$$

With this result, it is now a simple matter to obtain the general term in the series which is¹⁶

$$\begin{aligned}
 T_L^{(n+1)} &= \left(\frac{\omega}{2}\right)^n \sum_{\lambda, x_1 \dots x_n} J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L - \sum_j x_j - \lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\times \langle \vec{p}_f | \vee G^{(+)} (\epsilon_{p_i} - \omega (\lambda + \sum_j x_j)) \times \\
 &\times \vec{\alpha}_0 \cdot \vec{p} G^{(+)} (\epsilon_{p_i} - \omega (\lambda + \sum_j x_j)) \vec{\alpha}_0 \cdot \vec{p} \dots \vec{\alpha}_0 \cdot \vec{p} G^{(+)} (\epsilon_{p_i} - \omega \lambda) \vee | \vec{p}_i \rangle
 \end{aligned} \tag{2.1.16}$$

Adding the expressions for $T_L^{(0)}$, $T_L^{(1)}$ and $T_L^{(n+1)}$ for all $n \gg 1$, one may write

$$\begin{aligned}
 T_L &= \sum_{\lambda} J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L - \lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T (\epsilon_{p_i} - \omega \lambda) | \vec{p}_i \rangle \\
 &+ \sum_{n=1}^{\infty} \left(\frac{\omega}{2}\right)^n \sum_{\lambda, x_1 \dots x_n} J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L - \sum_j x_j - \lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\times \langle \vec{p}_f | \vee G^{(+)} (\epsilon_{p_i} - \omega (\lambda + \sum_j x_j)) \times \\
 &\times \vec{\alpha}_0 \cdot \vec{p} G^{(+)} (\epsilon_{p_i} - \omega (\lambda + \sum_j x_j)) \vec{\alpha}_0 \cdot \vec{p} \dots \vec{\alpha}_0 \cdot \vec{p} G^{(+)} (\epsilon_{p_i} - \omega \lambda) \vee | \vec{p}_i \rangle
 \end{aligned} \tag{2.1.17}$$

If the Green's functions are not resonant, they can be expanded in powers of ω . The leading term inside the summation sign is of order ω^0 . Therefore, the leading term in $T_L^{(n+1)}$ is of order ω^n . In the derivation of the Kroll-Watson result, one keeps terms upto order ω only which means dropping $T_L^{(3)}$ and all higher order terms.

To simplify $T_L^{(2)}$ we proceed as follows:

The operator $G^{(+)}(E)$ is

$$G^{+}(E) = \frac{1}{E^{(+)} - H_0} \quad (2.1.18)$$

where

$$H_0 = \frac{p^2}{2m} + V(r) \quad (1.2.7)$$

and $E^{(+)} = E + i\eta$

Hence,

$$H_0 G^{(+)}(E) = G^{(+)}(E) H_0 = E^{(+)} G^{(+)}(E) - \bar{I} \quad (2.1.19)$$

where $\bar{I}$ is the identity operator.

Also,

$$\vec{p} = -i\omega [\vec{r}, H_0] \quad (2.1.20)$$

Using (2.1.19) and (2.1.20) we arrive at the identity

$$\begin{aligned} & G^{(+)}(E_1) \vec{\alpha}_0 \cdot \vec{p} G^{(+)}(E_2) \\ &= -i\omega \left[(E_2 - E_1) G^{(+)}(E_1) \vec{\alpha}_0 \cdot \vec{r} G^{(+)}(E_2) \right. \\ & \quad \left. + \vec{\alpha}_0 \cdot \vec{r} G^{(+)}(E_2) - G^{(+)}(E_1) \vec{\alpha}_0 \cdot \vec{r} \right] \end{aligned} \quad (2.1.21)$$

In (2.1.14),

$$E_2 = E_{p_i} - \omega\lambda$$

$$E_1 = E_{p_i} - \omega(\lambda + x_1) \quad (2.1.22)$$

So, $E_2 - E_1$ is of order ω . Therefore, when (2.1.21) is used in (2.1.14) the term containing $E_2 - E_1$ will be of order ω^2 , and hence, can be dropped. Thus,

$$T_e^{(2)} = -im\omega/2 \sum_{\lambda, x_1 = \pm 1} J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-x_1-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \quad (2.1.23)$$

$$\begin{aligned} & \langle \vec{p}_f | V [\vec{\alpha}_0 \cdot \vec{r} G^{(+)} (\epsilon p_i - \omega \lambda) \cdot \\ & - G^{(+)} (\epsilon p_i - \omega (\lambda + x_1)) \vec{\alpha}_0 \cdot \vec{r}] V | \vec{p}_i \rangle + O(\omega^2) \end{aligned}$$

For convenience, we would like to have the same argument for the two Green's functions. This we achieve by simply shifting λ to $\lambda - x_1$ in the second term on the right side of (2.1.23). Then,

$$T_e^{(2)} = -im\omega/2 \sum_{\lambda} \left[J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) \sum_{x_1} J_{-\ell-x_1-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \right. \quad (2.1.24)$$

$$\left. \langle \vec{p}_f | \vec{\alpha}_0 \cdot \vec{r} V G^{(+)} (\epsilon p_i - \omega \lambda) V | \vec{p}_i \rangle \right.$$

$$\left. - \sum_{x_1} J_{\lambda-x_1} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \right.$$

$$\left. \langle \vec{p}_f | V G^{(+)} (\epsilon p_i - \omega \lambda) V \vec{\alpha}_0 \cdot \vec{r} | \vec{p}_i \rangle \right] + O(\omega^2)$$

In arriving at (2.1.24) we have made the tacit assumption that $[V, r] = 0$ that is, V is a local potential. After performing the x_1 sums by the use of the formula

$$\sum_{a=\pm 1} J_{n+a}(x) = \frac{2n}{x} J_n(x) \quad (A-6)$$

we get,

$$\begin{aligned}
 T_L^{(2)} &= i \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\times \langle \vec{p}_f | \frac{m\omega (\lambda+L) \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_f} \vee G^{(+)} (\epsilon_{p_i} - \omega\lambda) \vee \\
 &- \vee G^{(+)} (\epsilon_{p_i} - \omega\lambda) \vee \frac{m\omega\lambda \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_i} | \vec{p}_i \rangle + O(\omega^2)
 \end{aligned} \quad (2.1.25)$$

Equations (2.1.12) and (2.1.25) can be combined to give

$$\begin{aligned}
 T_L^{(1)} + T_L^{(2)} &= \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\times \langle \vec{p}_f | \left(1 + i \frac{m\omega (\lambda+L) \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_f} \right) \vee G^{(+)} (\epsilon_{p_i} - \omega\lambda) \vee \times \\
 &\left(1 - \frac{i m\omega\lambda \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_i} \right) | \vec{p}_i \rangle + O(\omega^2)
 \end{aligned} \quad (2.1.26)$$

The factors appearing at the flanks can be incorporated with the initial and final plane wave states to give the shifted momenta with the result

$$\begin{aligned}
 &T_L^{(1)} + T_L^{(2)} \\
 &= \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\times \langle \vec{p}_f - \frac{m\omega (\lambda+L) \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} | \vee G^{(+)} (\epsilon_{p_i} - \omega\lambda) \vee | \vec{p}_i - \frac{m\omega\lambda \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \rangle + O(\omega^2)
 \end{aligned} \quad (2.1.27)$$

Also,

$$\begin{aligned}
 & \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \left\langle \vec{p}_f - \frac{m\omega(\lambda+\ell)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} \middle| V \middle| \vec{p}_i - \frac{m\omega\lambda\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \right\rangle \\
 &= \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | V | \vec{p}_i \rangle + O(\omega^2) \\
 &= T_{\ell}^{(0)}
 \end{aligned} \tag{2.1.28}$$

simply because of the identity

$$\begin{aligned}
 & \sum_{\lambda} \lambda J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \\
 &= \frac{\vec{\alpha}_0 \cdot \vec{p}_i}{\vec{\alpha}_0 \cdot \vec{p}_f} \sum_{\lambda} (\lambda+\ell) J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f)
 \end{aligned} \tag{2.1.29}$$

Adding (2.1.27) and (2.1.28) we get

$$\begin{aligned}
 T_{\ell}(\vec{p}_f, \vec{p}_i) &= T_{\ell}^{(0)} + T_{\ell}^{(1)} + T_{\ell}^{(2)} + O(\omega^2) \\
 &= \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\times \left\langle \vec{p}_f - \frac{m\omega(\lambda+\ell)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} \middle| T(\epsilon_{p_i} - \omega\lambda) \middle| \vec{p}_i - \frac{m\omega\lambda\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \right\rangle + O(\omega^2)
 \end{aligned} \tag{2.1.30}$$

By the use of (1.4.2) and (1.4.4),

$$\begin{aligned}
 & \frac{1}{2m} \left| \vec{p}_f - \frac{m\omega(\lambda+\ell)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} \right|^2 \\
 &= \epsilon_{p_i} - \omega\lambda + O(\omega^2) \\
 &= \frac{1}{2m} \left| \vec{p}_i - \frac{m\omega\lambda\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \right|^2 + O(\omega^2)
 \end{aligned} \tag{2.1.31}$$

Hence the T-matrices appearing in the summand are on-shell.

In order to perform the sum over λ , it is necessary to expand the matrix elements in 'powers of $\omega\lambda$ '. Since λ occurs there only as $\omega\lambda$, one can write

$$\begin{aligned} & \langle \vec{p}_f - m\omega \frac{(\lambda+1)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} \mid T(\epsilon_{p_i} - \omega\lambda) \mid \vec{p}_i - m\omega \frac{\lambda\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \rangle \\ &= A_0 + \omega\lambda A_1 + O(\omega^2) \end{aligned} \quad (2.1.32)$$

where A_0 and A_1 are independent of λ . Substituting (2.1.32) in (2.1.30) and using the sum rules

$$\sum_{n=-\infty}^{\infty} J_n(\gamma) J_{N-n}(\gamma') = J_N(\gamma+\gamma') \quad (A-1)$$

$$\sum_{n=-\infty}^{\infty} n J_n(\gamma) J_{N-n}(\gamma') = \frac{N\gamma}{\gamma+\gamma'} J_N(\gamma+\gamma') \quad (A-2)$$

the λ sums can be performed, yielding

$$T_e(\vec{p}_f, \vec{p}_i) = J_{-e}(\vec{\alpha}_0 \cdot d\vec{p}) (A_0 + \omega\lambda_0 A_1) \quad (2.1.33)$$

where

$$\lambda_0 = \frac{L \vec{\alpha}_0 \cdot \vec{p}_i}{\vec{\alpha}_0 \cdot d\vec{p}} \quad (1.4.6)$$

Finally, using (2.1.32) we can write immediately,

$$T_e(\vec{p}_f, \vec{p}_i) = J_{-e}(\vec{\alpha}_0 \cdot d\vec{p}) \times$$

$$\langle \vec{p}_f - m\omega \frac{(\lambda_0+1)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} \mid T(\epsilon_{p_i} - \omega\lambda_0) \mid \vec{p}_i - m\omega \frac{\lambda_0\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \rangle + O(\omega^2)$$

$$= J_{-1} (\vec{a}_0 \cdot \vec{p}) \langle \vec{p}_f - \vec{a}_{\lambda_0} | T (\epsilon_{fi} - \vec{a}_{\lambda_0}) | \vec{p}_i - \vec{a}_{\lambda_0} \rangle + O(\omega^2) \quad (2.1.34)$$

Since the matrix element (2.1.30) is on-shell for all values of λ , it is so for $\lambda = \lambda_0$ even though λ_0 may not be an integer.

2.2 Extension to higher orders:

Since its publication in 1973, the result of Kroll and Watson has been reproduced by many authors. It is not obvious how to generalize the method of Kroll and Watson to higher powers of ω . Mittleman²¹ used a different technique to extend it one order of ω further. Essentially, his method was to expand the full Green's function $\bar{G}^{(+)}$ in powers of V . However, it is simpler to work with equation (2.1.6) as has been pointed out by Kruger and Jung.

As is clear from our analysis in the previous sections, the prescription for extending the results therein to higher orders in ω runs like this:

- (a) One includes more terms on the right side of (2.1.10). Expressions for those higher order terms are borrowed from (2.1.16).
- (b) The matrix elements in each term are expanded to higher orders in ω .
- (c) One derives sum rules of the form

$\sum_n^m J_n(y) J_{N-n}(y')$ for $m > 1$ and uses them to perform summation over λ .

(d) Finally, one may make a judicious regrouping of terms in order to get a more compact result. This, however, is not always possible.

However straightforward the above approach is, it involves very messy algebra mainly because of steps (b) and (c). In view of this an alternative is suggested in which the task of summing over λ is greatly simplified by a suitable gauge transformation.

We start with the Schrodinger equation in the $\vec{p} \cdot \vec{A}$ gauge:

$$i \frac{\partial \psi_2}{\partial t} = H_2 \psi_2 \quad (2.2.1)$$

$$H_2 = H_0 + \frac{e}{m} \vec{p} \cdot \vec{A}(t) \quad (1.2.10)$$

$$H_0 = \frac{p^2}{2m} + V(r) \quad (1.2.7)$$

Writing

$$\psi_2 = e^{-i\phi(t)} \psi \quad (2.2.2)$$

one gets the equation for ψ :

$$i \frac{\partial \psi}{\partial t} = H \psi \quad (2.2.3)$$

$$H = e^{i\phi} H_2 e^{-i\phi} - \dot{\phi} \quad (2.2.4)$$

and the interaction of the laser with the particle is

$$H' = H - H_0 \quad (2.2.5)$$

If the momenta of the particle in its initial and final asymptotic states are $\vec{p}_i$ and $\vec{p}_f$ respectively, we choose

$$\phi = \frac{e}{m} (a\vec{p}_i + (1-a')\vec{p}_f) \cdot \int^t \vec{A}(t') dt' \quad (2.2.6)$$

where a and a' are arbitrary constants for the moment.

$$\dot{\phi} = \frac{e}{m} (a\vec{p}_i + (1-a')\vec{p}_f) \cdot \vec{A}(t) \quad (2.2.7)$$

Since ϕ commutes with H_2 , $H = H_2 - \dot{\phi}$ whereby the interaction term is

$$H' = \frac{e}{m} [\vec{p} - a\vec{p}_i - (1-a')\vec{p}_f] \cdot \vec{A}(t) \quad (2.2.8)$$

or writing

$$\vec{P} = \vec{p} - a\vec{p}_i - (1-a')\vec{p}_f \quad (2.2.9)$$

we get H' in the familiar form

$$H' = \frac{e}{m} \vec{P} \cdot \vec{A}(t) \quad (2.2.10)$$

Asymptotic states are solutions of the equation (2.2.3)

with V set equal to zero. The initial asymptotic state is

$$\begin{aligned} |x_{p_i}\rangle &= e^{-i\{(1-a)\vec{p}_i - (1-a')\vec{p}_f\} \cdot \int_0^t \frac{e}{m} \vec{A}(t') dt' - i\epsilon_{p_i} t} |p_i\rangle \\ &= \sum_n J_n(-\vec{a}_0 \cdot \vec{Q}) e^{-i(\epsilon_{p_i} - n\omega)t} |\vec{p}_i\rangle \end{aligned} \quad (2.2.11)$$

where

$$\vec{Q} = (1-a) \vec{p}_i - (1-a') \vec{p}_f \quad (2.2.12)$$

Similarly, the final asymptotic state is

$$|p_f\rangle = \sum_n J_n(\vec{\alpha}_0 \cdot \vec{Q}') e^{-i(\epsilon_{p_f} + n'\omega)t} |p_f\rangle \quad (2.2.13)$$

where

$$\vec{Q}' = a' \vec{p}_f - a \vec{p}_i \quad (2.2.14)$$

Comparing equations (2.2.10), (2.2.11) and (2.2.13) with the corresponding equations (1.3.2), (2.1.5) and (2.1.3) we observe that the effect of the gauge transformation (2.2.2) (with ϕ given by (2.2.6)) is to replace $\vec{p}_i$ and $\vec{p}_f$ in the arguments of the Bessel functions by $\vec{Q}$ and $\vec{Q}'$ respectively and to replace the $\vec{p}$ operator in the laser interaction by the $\vec{P}$ operator. $\vec{Q}$, $\vec{Q}'$ and $\vec{P}$ are given by (2.2.12), (2.2.14) and (2.2.9) respectively. Thus the equation analogous to (2.1.17) will be

$$\begin{aligned} T_L = & \sum_{\lambda} J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{Q}) J_{\ell-\lambda}(\vec{\alpha}_0 \cdot \vec{Q}') \langle \vec{p}_f | T(\epsilon_{p_i} - \lambda\omega) | \vec{p}_i \rangle \\ & + \sum_{n=1}^{\infty} \left(\frac{\omega}{2}\right)^n \sum_{\lambda, x_1 \dots x_n} J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{Q}) J_{\ell-\sum x_j - \lambda}(\vec{\alpha}_0 \cdot \vec{Q}') \times \\ & \times \langle \vec{p}_f | V G^{(+)}(\epsilon_{p_i} - \omega(\lambda + \sum_{j=1}^n x_j)) \vec{\alpha}_0 \cdot \vec{P} G^{(+)}(\epsilon_{p_i} - \omega(\lambda + \sum_{j=1}^{n-1} x_j)) \times \\ & \vec{\alpha}_0 \cdot \vec{P} \dots \vec{\alpha}_0 \cdot \vec{P} G^{(+)}(\epsilon_{p_i} - \omega\lambda) V | \vec{p}_i \rangle \end{aligned} \quad (2.2.15)$$

By assigning different values to a and a' in (2.2.6) we

can generate different gauge transformations. In particular for $a = 0$ and $a' = 1$, we recover the $\vec{p} \cdot \vec{A}$ gauge and the results of section 1.1. If $a = a' = 1$, or more generally, if a and a' are such that

$$(1-a) \vec{\alpha}_0 \cdot \vec{p}_i = (1-a') \vec{\alpha}_0 \cdot \vec{p}_f$$

then $\vec{\alpha}_0 \cdot \vec{Q} = 0$. Therefore,

$$J_\lambda (-\vec{\alpha}_0 \cdot \vec{Q}) = J_\lambda (0) = \delta_{\lambda,0}$$

Also,

$$\vec{\alpha}_0 \cdot \vec{Q}' = \vec{\alpha}_0 \cdot d\vec{p}$$

and

$$\vec{\alpha}_0 \cdot \vec{p} = \vec{\alpha}_0 \cdot (\vec{p} - \vec{p}_i)$$

Thus only the $\lambda = 0$ term will contribute in (2.2.15). The result is

$$\begin{aligned} T_L &= J_{-L} (\vec{\alpha}_0 \cdot d\vec{p}) \langle \vec{p}_f | T(\epsilon_{p_i}) | \vec{p}_i \rangle \\ &+ \sum_{n=1}^{\infty} \left(\frac{\omega}{2} \right)^n \sum_{\lambda, x_1 \dots x_n} J_{-L - \sum_{i=1}^n x_i} (\vec{\alpha}_0 \cdot d\vec{p}) \times \\ &\langle \vec{p}_f | \vee G^{(+)} (\epsilon_{p_i} - \omega \sum_{i=1}^n x_i) \vec{\alpha}_0 \cdot (\vec{p} - \vec{p}_i) \times \\ &\times G^{(+)} (\epsilon_{p_i} - \omega \sum_{i=1}^{n-1} x_i) \vec{\alpha}_0 \cdot (\vec{p} - \vec{p}_i) \dots \vec{\alpha}_0 \cdot (\vec{p} - \vec{p}_i) \times \\ &\times G^{(+)} (\epsilon_{p_i}) \vee | \vec{p}_i \rangle \end{aligned} \quad (2.2.16)$$

If on the other hand, $a = a' = 0$ or generally speaking, if

then $\vec{\alpha}_0 \cdot \vec{Q}' = 0$. In this case

$$J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{Q}') = \delta_{\lambda, -\ell}$$

and,

$$J_{-\ell - \sum_i \tilde{x}_i - \lambda} (\vec{\alpha}_0 \cdot \vec{Q}') = \delta_{\lambda, -\ell - \sum_i \tilde{x}_i}$$

Also,

$$\vec{\alpha}_0 \cdot \vec{Q} = -\vec{\alpha}_0 \cdot \vec{p}$$

and

$$\vec{\alpha}_0 \cdot \vec{p} = \vec{\alpha}_0 \cdot (\vec{p} - \vec{p}_f)$$

Substituting in (2.2.15) one would get the following expression for T_ℓ :

$$\begin{aligned} T_\ell &= J_{-\ell} (\vec{\alpha}_0 \cdot \vec{p}) \langle \vec{p}_f | T(\epsilon_{p_f}) | \vec{p}_i \rangle \\ &+ \sum_{n=1}^{\infty} \left(\frac{\omega}{2} \right)^n \sum_{x_1 \dots x_n} J_{-\ell - \sum_i \tilde{x}_i} (\vec{\alpha}_0 \cdot \vec{p}) \\ &\times \langle \vec{p}_f | \vee G^{(+)}(\epsilon_{p_f}) \vec{\alpha}_0 \cdot (\vec{p} - \vec{p}_f) G^{(+)}(\epsilon_{p_f} + \omega x_n) \\ &\times \vec{\alpha}_0 \cdot (\vec{p} - \vec{p}_f) \dots \vec{\alpha}_0 \cdot (\vec{p} - \vec{p}_f) G^{(+)}(\epsilon_{p_f} + \omega \sum_i \tilde{x}_i) \vee | \vec{p}_i \rangle \end{aligned} \quad (2.2.17)$$

where

$$\epsilon_{p_f} = \epsilon_{p_i} + \ell \omega \quad (2.2.18)$$

The expressions (2.1.17), (2.2.16) and (2.2.17) are

equivalent so long as we prohibit any of the Green's functions from being resonant. A specific choice of a and a' translates into an expansion of the different Green's functions around specific energy values. For example, when $a = a' = 1$ one arrives at equation (2.2.16) from (2.2.17) essentially by expanding the Green's function in the T-operator around ϵ_{p_i} , the next one around $\epsilon_{p_i} - \omega \sum_j x_j$ and so on. These expansions are valid provided the Green's function is not a rapidly varying function of energy. However, for non-resonant potential scattering, the expression (2.2.16) or (2.2.17) is one step ahead of the expression (2.1.17) in that the summation over λ has already been performed in the former. The shift in the argument of the Green's function is now strictly due to the laser interaction in the intermediate states. In (2.1.17) all the Green's functions contain additional energy shifts $\omega \lambda$ originating from the initial asymptotic state. Using (2.2.16) we will perform some higher order calculations. This is the content of the next two sections.

2.3 Derivation of the T-matrix to order ω^3

We expand the Green's function in powers of ω and sum over the indices x_j to get

$$T_L(\vec{p}_f, \vec{p}_i) = \sum_{i,j=0}^3 \frac{\omega^j}{z^i} \left[T_L(z) v_{ij} + T_L'(z) v'_{ij} \right] \quad (2.3.1)$$

with

$$Z = - \vec{\alpha}_0 \cdot d\vec{p}.$$

$\{m_{ij}\}$ are the elements of the matrix

$$\begin{bmatrix} m_{00} & 0 & 0 & 0 \\ 0 & m_{11} & 0 & m_{13} \\ 0 & 0 & m_{22} & m_{23} \\ 0 & 0 & 0 & m_{33} \end{bmatrix}$$

$$m_{00} = \langle \vec{p}_f | T(\epsilon_{pi}) | \vec{p}_i \rangle$$

$$m_{11} = \ell \langle \vec{p}_f | V G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} V | \vec{p}_i \rangle$$

$$m_{13} = \ell/2 \langle \vec{p}_f | V G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} V | \vec{p}_i \rangle$$

$$m_{22} = \ell^2 \langle \vec{p}_f | V G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} V | \vec{p}_i \rangle$$

$$\begin{aligned} m_{23} = & -\ell \langle \vec{p}_f | V \left(G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} + \right. \\ & + G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} + \\ & \left. + G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \right) V | \vec{p}_i \rangle \end{aligned} \quad (2.3.2)$$

$$\begin{aligned} m_{33} = & (\ell^3 + 2\ell) \langle \vec{p}_f | V G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \vec{\alpha}_0 \cdot \vec{p} \times \\ & \times G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} V | \vec{p}_i \rangle \end{aligned}$$

$$\vec{p} = \vec{p} - \vec{p}_i$$

$\{n_{ij}\}$ are the elements of the matrix

$$\begin{bmatrix} 0 & 0 & n_{02} & 0 \\ 0 & 0 & n_{12} & n_{13} \\ 0 & 0 & 0 & n_{23} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$n_{02} = \langle \vec{p}_f | V G^{(+)} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} V | \vec{p}_i \rangle$$

$$n_{12} = - \frac{n_{22}}{\lambda^2} \quad (2.3.3)$$

$$n_{13} = -n_{23}$$

$$n_{23} = - \frac{3\ell}{\ell^3 + 2\ell} n_{33}$$

The Green's functions in the above expressions have ϵ_{p_i} as their argument. Also, $G^{(+)}'$ and $G^{(+)}''$ are abbreviated notations for $\frac{\partial G^{(+)}(\epsilon_{p_i})}{\partial \epsilon_{p_i}}$ and $\frac{\partial^2 G^{(+)}(\epsilon_{p_i})}{\partial \epsilon_{p_i}^2}$ respectively.

It appears from the above expression for T_ℓ that as $z \rightarrow 0$, a singularity may develop in T_ℓ . That this is not so can be seen from the master expression (2.2.16) in which z

occurs only as arguments of Bessel functions. Bessel functions are non-singular in their arguments. However, it is well to note here that the technique of grouping some off-shell matrix elements to form an on-shell one (a la Kroll and Watson) may not always be justified for sufficiently small values of z . We support this statement with the following analysis: We pick out only a select group of terms from the right side of (2.3.1) and sum them. Let us write

$$\sum_{i=0}^3 \frac{\omega^i}{z^i} J_L(z) m_{ii} = X \quad (2.3.4)$$

with the understanding that in m_{33} only the term with the coefficient ℓ^3 be kept. Then,

$$\begin{aligned} X &= J_L(z) \left[\langle \vec{p}_f | T(\epsilon_{pi}) | \vec{p}_i \rangle + \right. \\ &\quad + \langle \vec{p}_f | \vee G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \vee | \vec{p}_i \rangle + \\ &\quad + \langle \vec{p}_f | \vee G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \vee | \vec{p}_i \rangle \\ &\quad + \langle \vec{p}_f | \vee G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \vee | \vec{p}_i \rangle \\ &= J_L(z) \langle \vec{p}_f | \vee | \vec{p}_i \rangle + \\ &\quad + J_L(z) \langle \vec{p}_f | \vee \left[G^{(+)} + G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} + \right. \\ &\quad \left. + G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} \frac{\omega \ell}{z} \vec{\alpha}_0 \cdot \vec{p} G^{(+)} + \right. \end{aligned}$$

$$J_L(z) \langle \vec{p}_f - \vec{a}_{\lambda_0} | \vee G^{(+)}(\epsilon_{\vec{p}_i - \vec{a}_{\lambda_0}}) \vee | \vec{p}_i - \vec{a}_{\lambda_0} \rangle$$

Also, the first term can be written as

$$J_L(z) \langle \vec{p}_f - \vec{a}_{\lambda_0} | \vee | \vec{p}_i - \vec{a}_{\lambda_0} \rangle$$

Adding the two we get

$$X = J_L(z) \langle \vec{p}_f - \vec{a}_{\lambda_0} | T(\epsilon_{\vec{p}_i - \vec{a}_{\lambda_0}}) | \vec{p}_i - \vec{a}_{\lambda_0} \rangle \quad (2.3.8)$$

In obtaining the above result one has to assume that $G^{(+)}(E)$ appearing in (2.3.6) can be expanded in powers of $\frac{\omega}{z} \vec{a}_0 \cdot \vec{p}$ (see equation (2.3.5)). This will be so if $\frac{\omega}{z} \vec{a}_0 \cdot \vec{p}$ is a small perturbation which is not the case when z goes to zero.

2.4 Evaluation of $\sum \frac{d\sigma_i}{d\Omega}(\vec{p}_f, \vec{p}_i)$ to order ω^2 :

We can still use the expression (2.3.1) for T_L . However, the indices i and j will now range from 0 to 2. This means that the quantities m_{13} , m_{23} , m_{33} , n_{13} and n_{23} will not appear in the expression for T_L . The surviving m_{ij} 's are diagonal, that is, $m_{ij} = m_{ii} \delta_{ij}$.

Hence,

$$T_L(\vec{p}_f, \vec{p}_i) = \sum_{i=0}^2 \frac{\omega^i}{z^i} J_L(z) m_{ii} + \sum_{i,j=0}^2 \frac{\omega^j}{z^i} J_L'(z) v_{ij} \quad (2.4.1)$$

Proceeding as in the previous section one can perform the sum over i in the first term on the right side of (2.4.1) to write

$$\sum_{i=0}^2 \frac{\omega_i}{z^i} J_L(z) m_{ii} = J_L(z) \langle \vec{Q}' | T(\epsilon_Q) | \vec{Q} \rangle \quad (2.4.2)$$

where

$$\begin{aligned} \vec{Q}' &= \vec{p}_f - \vec{d}_{\lambda_0} \\ \vec{Q} &= \vec{p}_i - \vec{d}_{\lambda_0} \end{aligned} \quad (2.4.3)$$

Also,

$$\sum_{i,j=0}^2 \frac{\omega_j}{z^i} J_L'(z) m_{ij} = \omega^2 J_L'(z) \left[\eta_{02} + \frac{\eta_{12}}{2} \right] \quad (2.4.4)$$

Substituting back in (2.4.1) one gets

$$\begin{aligned} T_L(\vec{p}_f, \vec{p}_i) &= J_L(z) \langle \vec{Q}' | T(\epsilon_Q) | \vec{Q} \rangle + \\ &+ \omega^2 J_L'(z) \left[\eta_{02} + \frac{\eta_{12}}{2} \right] + O(\omega^3) \end{aligned} \quad (2.4.5)$$

With the use of the energy-conservation equation

$$\epsilon_{p_f} = \epsilon_{p_i} + \lambda \omega \quad (2.2.18)$$

and the definitions (2.4.3) one can show that

$$\frac{Q'^2}{2m} = \epsilon_Q = \frac{Q^2}{2m} \quad (2.4.6)$$

Being thus on-shell in nature, the matrix element

$\langle \vec{Q}' | T(\epsilon_Q) | \vec{Q} \rangle$ is a function only of ϵ_Q and $\hat{Q}' \cdot \hat{Q}$ (for a central potential) and can be written in short hand notation as $t(\epsilon_Q, \hat{Q}' \cdot \hat{Q})$. Therefore,

$$\begin{aligned} T_L(\vec{p}_f, \vec{p}_i) &= J_L(z) t(\epsilon_Q, \hat{Q}' \cdot \hat{Q}) + \\ &+ \omega^2 J_L'(z) \left[\eta_{02} + \frac{\eta_{12}}{2} \right] + O(\omega^3) \end{aligned} \quad (2.4.7)$$

$$\frac{d\sigma_\ell}{d\Omega}(\vec{p}_f, \vec{p}_i) = \left(\frac{p_f}{p_i}\right) |T_\ell(\vec{p}_f, \vec{p}_i)|^2 \quad (2.4.8)$$

Using (2.2.18), we get

$$\frac{p_f}{p_i} = 1 + \frac{\ell\omega}{2\epsilon_{p_i}} - \frac{\ell^2\omega^2}{8\epsilon_{p_i}^2} + O(\omega^3) \quad (2.4.9)$$

Also,

$$Z = Z_0 + Y_0 \left(\frac{\ell^2\omega^2}{8\epsilon_{p_i}^2} - \frac{\ell\omega}{2\epsilon_{p_i}} \right) \quad (2.4.10)$$

where,

$$Y_0 = \vec{\alpha}_0 \cdot \vec{p}_f(0) ; \quad \vec{p}_f(0) = \vec{p}_f|_{\ell=0}$$

$$Z_0 = X_0 - Y_0 \quad (2.4.11)$$

$$X_0 = \vec{\alpha}_0 \cdot \vec{p}_i$$

The matrix element $t(\epsilon_q, \hat{Q}' \cdot \hat{Q})$ can be expanded in powers of ω . The leading term is $t(\epsilon_{p_i}, \mu)$ where

$$\mu = \hat{p}_f \cdot \hat{p}_i \quad (2.4.12)$$

The contribution of the second term on the right side of

(2.4.7) to $\sum_\ell \frac{d\sigma_\ell}{d\Omega}(\vec{p}_f, \vec{p}_i)$ is

$$2\omega^2 \sum_\ell J_\ell(Z_0) J'_\ell(Z_0) \text{Re} \left[t^*(\epsilon_{p_i}, \mu) \left(n_{02} + \frac{n_{12}}{Z_0} \right) \right] \quad (2.4.13)$$

These are off-shell terms. But the quantities $t^*(\epsilon_{p_i}, \mu)$,

n_{02} and n_{12} are independent of ℓ . So the expression

(2.4.13) vanishes because of the sum rule

$$\sum_l J_l(z_0) J_l'(z_0) = 0$$

In evaluating $\sum_l \frac{d\sigma_l}{d\Omega}(\vec{p}_f, \vec{p}_i)$, therefore, we can drop the second term from the right side of (2.4.7). Thus we write,

$$\sum_l \frac{d\sigma_l}{d\Omega}(\vec{p}_f, \vec{p}_i) = \sum_l \left(\frac{p_f}{p_i} \right) J_l^2(z) \frac{d\sigma}{d\Omega}(\epsilon_Q, \hat{Q}', \hat{Q}) \quad (2.4.14)$$

where the differential cross section appearing on the right side of (2.4.14) is that in the absence of the laser but the momenta $\vec{Q}$ and $\vec{Q}'$ are shifted from the initial and final momenta (see equation (2.4.3)). Since each term on the right side of (2.4.14) is on-shell, the expression for the resulting sum rule will contain only on-shell terms. Using equation (2.4.10) one writes

$$\begin{aligned} J_l^2(z) &= J_l^2(z_0) - \frac{\omega_l \gamma_0}{\epsilon_{pi}} J_l(z_0) J_l'(z_0) + \\ &+ \omega_l^2 z^2 \left[\frac{\gamma_0^2}{4 \epsilon_{pi}^2} \left\{ (J_l'(z_0))^2 + J_l(z_0) J_l''(z_0) \right\} \right. \\ &\left. - \frac{\gamma_0}{4 \epsilon_{pi}^2} J_l(z_0) J_l'(z_0) \right] \end{aligned} \quad (2.4.15)$$

Also,

$$\begin{aligned} &\frac{d\sigma}{d\Omega}(\epsilon_Q, \hat{Q}', \hat{Q}) \\ &= \frac{d\sigma}{d\Omega}(\epsilon_{pi}, \mu) + \omega_l \left[\frac{x_0}{z_0} \frac{\partial}{\partial \epsilon_{pi}} \frac{d\sigma}{d\Omega}(\epsilon_{pi}, \mu) + \right. \\ &\left. + \frac{(1-\mu)}{2 \epsilon_{pi}} \frac{(x_0 + \gamma_0)}{z_0} \frac{\partial}{\partial \mu} \frac{d\sigma}{d\Omega}(\epsilon_{pi}, \mu) \right] + \end{aligned}$$

$$\begin{aligned}
 & + \omega^2 \ell^2 \left[\frac{1}{2z_0^2} \left(m\alpha_0^2 + \frac{x_0 \gamma_0}{\epsilon_{pi}} \right) \frac{\partial}{\partial \epsilon_{pi}} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) \right. \\
 & + (1-\mu) \left\{ \frac{1}{8\epsilon_{pi}^2} + \left(m\alpha_0^2 + \frac{x_0 \gamma_0}{\epsilon_{pi}} \right) \frac{1}{2\epsilon_{pi} z_0^2} + \right. \\
 & + \left. \frac{x_0}{2\epsilon_{pi}^2 z_0} - \frac{x_0^2}{\epsilon_{pi}^2 z_0^2} \right\} \frac{\partial}{\partial \mu} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) \\
 & - \frac{1}{8\epsilon_{pi}^2} \frac{\partial}{\partial \mu} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) + \frac{x_0^2}{2z_0^2} \frac{\partial^2}{\partial \epsilon_{pi}^2} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) \\
 & + \frac{(1-\mu)(x_0 + \gamma_0)x_0}{2\epsilon_{pi} z_0^2} \frac{\partial^2}{\partial \epsilon_{pi} \partial \mu} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) + \\
 & \left. + \frac{(1-\mu)^2 (x_0 + \gamma_0)^2}{8\epsilon_{pi}^2 z_0^2} \frac{\partial^2}{\partial \mu^2} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) \right] \quad (2.4.16)
 \end{aligned}$$

Substituting the expansions (2.4.9), (2.4.15) and (2.4.16) back into (2.4.14), keeping terms to order ω^2 and summing over ℓ by the use of the Bessel function sum rules

$$\sum_{\ell} \begin{pmatrix} 1 \\ \ell \\ \ell^2 \end{pmatrix} J_{\ell}^2(z_0) = \begin{pmatrix} 1 \\ 0 \\ z_0^2/2 \end{pmatrix} \quad (A-5)$$

we get,

$$\sum_{\ell} \frac{d\sigma_{\ell}}{d\Omega} (\vec{p}_f, \vec{p}_i) = \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) + \beta \omega^2 \quad (2.4.17)$$

where,

$$\begin{aligned}
 \beta = & \frac{(3\gamma_0^2 - x_0^2)}{16 \epsilon_{pi}^2} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) + \\
 & \left[\frac{m\alpha_0^2}{4} + \frac{x_0 (x_0 - 2\gamma_0)}{4 \epsilon_{pi}} \right] \frac{\partial}{\partial \epsilon_{pi}} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu) + \\
 & \left\{ \frac{(1-\mu)}{4 \epsilon_{pi}} \left[m\alpha_0^2 - \frac{(x_0 + \gamma_0)(x_0 + 5\gamma_0)}{4 \epsilon_{pi}} \right] - \frac{z_0^2}{16 \epsilon_{pi}^2} \right\} \frac{\partial}{\partial \mu} \frac{d\sigma}{d\Omega} \\
 & + \frac{x_0^2}{4} \frac{\partial^2}{\partial \epsilon_{pi}^2} \frac{d\sigma}{d\Omega} + \frac{x_0 (x_0 + \gamma_0) (1-\mu)}{4 \epsilon_{pi}} \frac{\partial^2}{\partial \epsilon_{pi} \partial \mu} \frac{d\sigma}{d\Omega} \\
 & + \frac{(x_0 + \gamma_0)^2}{16 \epsilon_{pi}^2} (1-\mu)^2 \frac{\partial^2}{\partial \mu^2} \frac{d\sigma}{d\Omega} (\epsilon_{pi}, \mu)
 \end{aligned}
 \tag{2.4.18}$$

This is an extension of the sum rule (2.4.10) previously obtained by Kruger and Jung.

The ω^2 term on the right side of (2.4.17) deserves some comment. If β as given by (2.4.20) is greater (less) than zero, there would be more (less) scattering in the presence of the laser than in the absence of the laser. The occurrence of this term can be explained as follows:

Electrons with a specific energy ϵ have a definite probability of getting scattered by an angle θ . This is proportional to $\frac{d\sigma}{d\Omega}(\epsilon, \mu)$ where $\mu = \cos \theta$. In the presence of the laser, electrons absorb or emit photons. This changes their energies and hence their probabilities for scattering. The net change need not amount to zero. The details depend on $\frac{d\sigma}{d\Omega}(\epsilon, \mu)$ and its derivatives with

respect to ϵ and μ . We may integrate the above result over the final direction of the scattering. After some algebra, this would give the following sum rule:

$$\sum_L \sigma_L = \sigma_T(\epsilon_{pi}) + \frac{m\omega^2 \alpha_0^2}{4} \left[2 (\hat{\alpha}_0 \cdot \hat{p}_i)^2 \epsilon_{pi}^{1/2} \frac{\partial^2}{\partial \epsilon_{pi}^2} + \epsilon_{pi}^{-1/2} \frac{\partial}{\partial \epsilon_{pi}} \right] \left(\epsilon_{pi}^{1/2} \sigma_T(\epsilon_{pi}) \right) \quad (2.4.19)$$

CHAPTER 3

RESONANT POTENTIAL SCATTERING IN THE PRESENCE OF A LOW-FREQUENCY LASER

3.1 Previous result:

The result of Kroll and Watson has been extended to allow for the presence of scattering resonances in the absence of the laser^{13,15,16}. Since the rapid variation of the T-matrix with energy is characteristic of resonance the expansion of the T-matrices in powers of ω must be performed carefully. This was originally emphasized by Kruger and Jung¹³. The problem can be handled in the following way.

Let there be an isolated pole (resonance) in $G^{(+)}$ at the energy $E_R = \xi_R - i\Gamma/2$ such that $\xi_R \approx \epsilon_0 = \epsilon_{pi} + m_0 \omega$ where m_0 is an integer. (The approximate equality is taken as an equality within the width of the resonance). The Green's function $G^{(+)}(E)$ can be written as²²

$$G^{(+)}(E) = G_{NR}^{(+)}(E) + G_R^{(+)}(E)$$

where $G_{NR}^{(+)}(E)$ is a slowly varying function of E and

$$G_R^{(+)}(E) = \frac{\bar{U}_R \langle U_R}{E - E_R}.$$

U_R is interpreted as the resonance wave function.

Consequently, the T-matrix breaks into resonant and non-resonant parts:

$$T(E) = T_{NR}(E) + T_R(E)$$

$$T_{NR}(E) = V + V G_{NR}^{(+)}(E) V$$

$$T_R(E) = V G_R^{(+)}(E) V$$

(3.1.1)

Resonance introduces a new parameter Γ , the resonance width. For CO_2 laser $\omega \approx 0.1$ ev and a typical Feshbach resonance²³ width is of order 5×10^{-3} ev so that $\frac{\Gamma}{\omega} \approx 5 \times 10^{-2}$. In this chapter we will treat both ω/ϵ_0 and Γ/ω as quantities of first order of smallness. If

$$E - \epsilon_0 = M\omega, \quad M \neq 0$$

(3.1.2)

then it is easy to show that

$$G_{NR}^{(+)}(E) = \left(1 + M\omega \frac{\partial}{\partial \epsilon_0}\right) G_{NR}^{(+)}(\epsilon_0) + O(\omega^2)$$

(3.1.3)

For convergence of the above expansion M should be a reasonably small number. Specifically, M should not be of the order of $\frac{\epsilon_0}{\omega}$. Also

$$G_R^{(+)}(E) = \left(\frac{\epsilon_0 - E_R}{E - E_R} \right) \frac{\bar{u}_R \rangle \langle u_R}{\epsilon_0 - E_R}$$

$$\approx \frac{i\Gamma/2}{M\omega + i\Gamma/2} G_R^{(+)}(\epsilon_0)$$

$$= \frac{i\Gamma}{2M\omega} G_R^{(+)}(\epsilon_0) + O\left(\frac{\Gamma^2}{M^2\omega^2}\right)$$

(3.1.4)

Therefore,

$$T_{NR}(E) = T_{NR}(\epsilon_0) + M\omega \frac{\partial}{\partial \epsilon_0} T_{NR}(\epsilon_0) + O(\omega^2) \quad (3.1.5)$$

and

$$T_R(E) = \frac{i\Gamma}{2M\omega} T_R(\epsilon_0) + O\left(\frac{\Gamma^2}{M^2\omega^2}\right) \quad (3.1.6)$$

Adding,

$$T(E) = T_{NR}(\epsilon_0) + M\omega \frac{\partial}{\partial \epsilon_0} T_{NR}(\epsilon_0) + \frac{i\Gamma}{2M\omega} T_R(\epsilon_0) + O(\omega^2) + O\left(\frac{\Gamma^2}{M^2\omega^2}\right) \quad (3.1.7)$$

where E is given by (3.1.2)

The T-matrix for scattering with the net absorption of ℓ photons is given by

$$T_\ell(\vec{p}_f, \vec{p}_i) = \sum_{n=0}^{\infty} T_\ell^{(n)} \quad (2.1.10)$$

To observe the effect of resonance in the lowest order, it is sufficient to keep only the $n = 0$ and the $n = 1$ terms.

$$T_\ell^{(0)} = \sum_{\lambda=-\infty}^{\infty} J_\lambda(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | V | \vec{p}_i \rangle \quad (2.1.11)$$

$$T_e^{(1)} = \sum_{\lambda=-\infty}^{\infty} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \times \langle \vec{p}_f | \vee U^{(+)} (\epsilon_{pi} - \omega\lambda) \vee | \vec{p}_i \rangle \quad (2.1.12)$$

Adding,

$$T_e (\vec{p}_f, \vec{p}_i) = \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \times \langle \vec{p}_f | T (\epsilon_{pi} - \lambda\omega) | \vec{p}_i \rangle \quad (3.1.8)$$

For $\lambda = -m_0$, the T-matrix is rapidly varying in ω and no expansion is possible. We therefore, separate out the $\lambda = -m_0$ term and write

$$T_e (\vec{p}_f, \vec{p}_i) = J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-\ell} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T(\epsilon_0) | \vec{p}_i \rangle \\ + \sum_{\lambda \neq -m_0} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T(\epsilon_{pi} - \lambda\omega) | \vec{p}_i \rangle \quad (3.1.9)$$

For $\lambda \neq -m_0$, $T(\epsilon_{pi} - \omega\lambda)$ can be expanded by the use of (3.1.7). If one retains only the leading term, then

$$T(\epsilon_{pi} - \omega\lambda) \simeq T_{NR}(\epsilon_0) \quad \text{for } \lambda \neq -m_0 \quad (3.1.10)$$

Substituting in (3.1.9), performing the summation over λ and using the relation (3.1.1), one gets

$$T_e (\vec{p}_f, \vec{p}_i) = J_{-\ell} (\vec{\alpha}_0 \cdot \vec{p}_i) \langle \vec{p}_f | T_{NR}(\epsilon_0) | \vec{p}_i \rangle \\ + J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-\ell} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R(\epsilon_0) | \vec{p}_i \rangle \quad (3.1.11)$$

This is the lowest order result of Kruger and Jung. The

cross section is proportional to the absolute square of the T-matrix (3.1.11) which evidently contains interference between the resonant and non-resonant terms. This interference is present even in the absence of the laser:

$$\left. \frac{d\sigma}{d\Omega} (\vec{p}_f, \vec{p}_i) \right|_{\alpha=0} = \left| \langle \vec{p}_f | T_{NR}(\epsilon_0) + T_R(\epsilon_0) | \vec{p}_i \rangle \right|^2 \quad (3.1.12)$$

Using (3.1.11) and (3.1.12) one can get the following sum rule¹⁶

$$\begin{aligned} \sum_L \frac{d\sigma_L}{d\Omega} (\vec{p}_f, \vec{p}_i) &= \left[1 - J_{m_0}^2 (\vec{\alpha}_0 \cdot \vec{p}_i) \right] \left. \frac{d\sigma}{d\Omega} (\vec{p}_f, \vec{p}_i) \right|_{NR} \\ &+ J_{m_0}^2 (\vec{\alpha}_0 \cdot \vec{p}_i) \left. \frac{d\sigma}{d\Omega} (\vec{p}_f, \vec{p}_i) \right|_{\alpha=0} \end{aligned} \quad (3.1.13)$$

where

$$\left. \frac{d\sigma}{d\Omega} (\vec{p}_f, \vec{p}_i) \right|_{NR} = \left| \langle \vec{p}_f | T_{NR}(\epsilon_0) | \vec{p}_i \rangle \right|^2 \quad (3.1.14)$$

This result already provides a new tool since in the absence of the laser only (3.1.12) is measureable whereas the laser, through the parameter $\vec{\alpha}_0 \cdot \vec{p}_i$ allows for an additional measurement of the resonant part of the cross section. In the next sections we present a systematic extension of the previous work.

3.2 Derivation of the T- matrix:

Proceeding as in the previous chapter, we find

$$T_e(\vec{p}_f, \vec{p}_i) = \sum_{n=0}^{\infty} T_e^{(n)}(\vec{p}_f, \vec{p}_i) \quad (2.1.10)$$

where

$$T_e^{(0)} = \sum_{\lambda} J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | v | \vec{p}_i \rangle \quad (2.1.11)$$

$$T_e^{(1)} = \sum_{\lambda} J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | v G^{(+)}(\epsilon_{p_i} - \omega \lambda) v | \vec{p}_i \rangle \quad (2.1.12)$$

$$T_e^{(2)} = \frac{U}{2} \sum_{\lambda} \sum_{x_1=\pm 1} J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-x_1-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \times \quad (2.1.14)$$

$$\times \langle \vec{p}_f | v G^{(+)}(\epsilon_{p_i} - \omega(\lambda + x_1)) \vec{\alpha}_0 \cdot \vec{p} G^{(+)}(\epsilon_{p_i} - \omega \lambda) v | \vec{p}_i \rangle$$

The general term is

$$T_e^{(n+1)} = \left(\frac{\omega}{2}\right)^n \sum_{\lambda, x_1, \dots, x_n} J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\sum_1^n x_j - \lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \times$$

$$\times \langle \vec{p}_f | v G^{(+)}(\epsilon_{p_i} - \omega(\lambda + \sum_1^n x_j)) \vec{\alpha}_0 \cdot \vec{p} \times$$

$$\times G^{(+)}(\epsilon_{p_i} - \omega(\lambda + \sum_1^{n-1} x_j)) \vec{\alpha}_0 \cdot \vec{p} \dots \dots \quad (2.1.16)$$

$$\dots \vec{\alpha}_0 \cdot \vec{p} G^{(+)}(\epsilon_{p_i} - \omega \lambda) v | \vec{p}_i \rangle$$

In the sequence of G's in (2.1.16) it is clear that successive ones may not be resonant for a given set of the parameters $(\lambda, x_1, x_2, \dots, x_n)$ since the arguments

of any successive pair will differ by one of the x 's times ω which can not be zero. This is an expression of the fact that the $\vec{p} \cdot \vec{A}$ interaction changes the energy by $\pm \omega$. That means that at most $\left[\frac{\nu+2}{2} \right]$ of the G 's can be resonant where $\left[\frac{\nu+2}{2} \right]$ is the largest integer $\leq \frac{\nu+2}{2}$. This gives the number of resonant energy denominators that can occur in (2.1.16).

Now, let us turn to the successive terms in the series for $T_L(p_f, p_i)$. We write $T_L^{(1)}$ as

$$\begin{aligned} T_L^{(1)} = & \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | V_{GNR}^{(H)}(E_{p_i} - \lambda \omega) | \vec{p}_i \rangle \\ & + J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-L} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R(E_0) | \vec{p}_i \rangle \\ & + \sum_{\lambda \neq -m_0} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R(E_{p_i} - \lambda \omega) | \vec{p}_i \rangle \end{aligned}$$

(3.2.1)

$T_L^{(2)}$ can be simplified by the use of the identity (2.1.21). For non-resonant potential scattering the first term on the right side of (2.1.21) was ignored because it would introduce a term of order ω^2 in $T_L^{(2)}$. For resonant potential scattering, only one of the two Green's functions can be resonant. If for example, $G^{(+)}(E_2)$ is resonant (see (2.1.22)), then the corresponding matrix element in $T_L^{(2)}$ that results from the term in question, is of the form

$$\begin{aligned}
 & \langle \vec{p}_f | V \left[G^{(+)}(E_1) m \omega^2 \vec{\alpha}_0 \cdot \vec{r} \right] G^{(+)}(E_0) V | \vec{p}_i \rangle \\
 & \sim \langle \vec{p}_f | V \left[G^{(+)}(E_1) m \omega^2 \vec{\alpha}_0 \cdot \vec{r} \right] G_R^{(+)}(E_0) V | \vec{p}_i \rangle \quad (3.2.2) \\
 & \quad + O(\omega^2)
 \end{aligned}$$

This differs from the leading resonant matrix element of $T_L^{(1)}$ by the factor inside the square bracket. Since E_1 is a non-resonant energy we can approximate

$\langle \vec{r} | G^{(+)}(E_1) | \vec{r}' \rangle$ in the following way: We use the expression (2.1.13) for $\langle \vec{r} | G^{(+)}(E_1) | \vec{r}' \rangle$ and write $E_1^{(+)} - W_n = \bar{W} (1 + X)$ where $E_1^{(+)} = E_1 + i\eta$ and the absolute value of $X = (E_1^{(+)} - W_n - \bar{W})/\bar{W}$ is assumed to be less than 1. In that case $\bar{W}$ is to be identified as some average excitation energy. This is the closure approximation²⁴. Then

$$\langle \vec{r} | G^{(+)}(E_1) | \vec{r}' \rangle \simeq \sum_n \frac{\phi_n(r) \phi_n^*(r') (1-X)}{\bar{W}}$$

One determines $\bar{W}$ by claiming that $\sum_n \phi_n(r) \phi_n^*(r') X = 0$. This gives

$$\bar{W} = \frac{\sum_n (E_1^{(+)} - W_n) \phi_n(r) \phi_n^*(r')}{\sum_n \phi_n(r) \phi_n^*(r')} \quad (3.2.3)$$

So,

$$\langle \vec{r} | G^{(+)}(\epsilon_1) | \vec{r}' \rangle \simeq \frac{1}{\bar{W}} \sum_n \phi_n(r) \phi_n^*(r')$$

Therefore,

$$| G^{(+)}(\epsilon_1) | \simeq \frac{1}{\bar{W}} | \sum_n \phi_n \rangle \langle \phi_n | = \frac{1}{\bar{W}} \quad (3.2.4)$$

Also, $\langle \vec{r} \rangle \sim$ range of the potential $\sim a_0$ and $\epsilon_0 \sim a_0$.

Also, $Ry = e^2/2a_0$ and $a_0 = 1/me^2$. Using these values,

we get

$$| G^{(+)}(\epsilon_1) m\omega^2 \vec{\alpha}_0 \cdot \vec{r} | \sim \left(\frac{Ry}{\bar{W}} \right) \left(\omega/Ry \right)^2$$

For short range potentials typical value of $\bar{W}$ is of the order of one Rydberg. So the presence of the bracketted term introduces an over-all factor of order $(\omega/Ry)^2$ which is much too small to be considered. The upshot of the above analysis is that we can ignore the first term on the right side of (2.1.21) for resonant potential scattering.

Following chapter 2 we write

$$\begin{aligned} T_e^{(2)} &= i \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ &\times \langle \vec{p}_f | \frac{m\omega(\lambda+2)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{r} \vee G^{(+)}(\epsilon_{pi} - \lambda\omega) \vee \\ &- \vee G^{(+)}(\epsilon_{pi} - \lambda\omega) \vee \frac{m\omega\lambda \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_i} | \vec{p}_i \rangle \end{aligned} \quad (2.1.25)$$

The factors appearing at the flanks of the matrix element are of order $(\omega/R\gamma)/(\epsilon_0/R\gamma)^{1/2}$. Also, a typical value of ϵ_0 is the aforementioned factors are of order $\omega/R\gamma$. So, for λ the $\vec{p} = m_0$, we can ignore $G_R^{(+)}(\epsilon_{p_i} - \lambda\omega)$ for that would otherwise introduce an additional factor of order Γ/ω .

One can, therefore, write

$$\begin{aligned}
 T_\ell^{(2)} &= i \sum_\lambda J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\quad \langle \vec{p}_f | \frac{m\omega(\lambda+\ell) \vec{\alpha}_0 \cdot \vec{Y}}{\vec{\alpha}_0 \cdot \vec{p}_f} \vee G_{NR}^{(+)}(\epsilon_{p_i} - \lambda\omega) \vee \\
 &\quad - \vee G_{NR}^{(+)}(\epsilon_{p_i} - \lambda\omega) \vee \frac{m\omega\lambda \vec{\alpha}_0 \cdot \vec{Y}}{\vec{\alpha}_0 \cdot \vec{p}_i} | \vec{p}_i \rangle \\
 &\quad + i J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-\ell} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\quad \langle \vec{p}_f | T_R(\epsilon_0) \frac{m\omega m_0 \vec{\alpha}_0 \cdot \vec{Y}}{\vec{\alpha}_0 \cdot \vec{p}_i} \\
 &\quad - \frac{m\omega(m_0-\ell) \vec{\alpha}_0 \cdot \vec{Y}}{\vec{\alpha}_0 \cdot \vec{p}_f} T_R(\epsilon_0) | \vec{p}_i \rangle
 \end{aligned} \tag{3.2.5}$$

For $T_\ell^{(3)}$ and higher terms, only resonant contributions need be considered. We will show that the effect of including these terms in the series for T_ℓ is to replace $T_R(\epsilon_0)$ by $T_R(\epsilon_0 - D)$ in (3.2.1) and (3.2.5). D is defined in (3.2.10). From the general expression (2.1.16) we get

$$\begin{aligned}
 T_L^{(3)} &= \left(\frac{\omega}{2}\right)^3 \sum_{\lambda, x_1, x_2} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\lambda - x_1 - x_2 - \lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}_f | \vee G^{(+)}(\epsilon_{p_i} - \omega\lambda - \omega x_1 - \omega x_2) \vec{\alpha}_0 \cdot \vec{p} G^{(+)}(\epsilon_{p_i} - \omega\lambda - \omega x_1) \times \\
 &\vec{\alpha}_0 \cdot \vec{p} G^{(+)}(\epsilon_{p_i} - \omega\lambda) \vee | \vec{p}_i \rangle
 \end{aligned}
 \tag{3.2.6}$$

If only the last Green's function is resonant, the leading term of (3.2.6) is of the following form:

$$\langle \vec{p}_f | \vee \left[\omega^2 G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} \right] G_R^{(+)}(\epsilon_0) \vee | \vec{p}_i \rangle
 \tag{3.2.7}$$

In arriving at (3.2.7) we have expanded the Green's functions by the use of (3.1.3) and (3.1.4). $\langle \vec{r} \rangle \sim a_0$. Therefore, $\langle \vec{p} \rangle \sim 1/a_0$ by virtue of the uncertainty principle. Then

$$| \omega^2 G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | \sim \left(\omega/R_Y \right)^2$$

So, we drop the term (3.2.7) for the same reason that we ignored the term (3.2.2).

The same argument follows if the first Green's function is resonant. Also, two successive Green's functions can not be resonant simultaneously. Exploiting the remaining possibilities and keeping only the resonant contributions in (3.2.6) we would get

$$T_L^{(3)} = \frac{\omega^2}{2} \frac{J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f)}{\epsilon_0 - E_R} \times$$

$$\left[\frac{\langle \vec{p}_f | V | U_R \rangle \langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \langle U_R | V | \vec{p}_i \rangle +}{\epsilon_0 - E_R} \right.$$

$$\frac{2 m_0 (m_0 - l)}{(\vec{\alpha}_0 \cdot \vec{p}_i) (\vec{\alpha}_0 \cdot \vec{p}_f)} \langle \vec{p}_f | V G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \times$$

$$\left. \langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) V | \vec{p}_i \rangle \right]$$

(3.2.8)

Looking at the expression for $T_L^{(3)}$ we can immediately write down (for $j \gg 1$)

$$T_L^{(2j+1)} = \left(\frac{\omega^2}{2} \right)^j \frac{J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f)}{\epsilon_0 - E_R} \times$$

$$\left[\langle \vec{p}_f | V | U_R \rangle \left\{ \frac{\langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle}{\epsilon_0 - E_R} \right\}^j \langle U_R | V | \vec{p}_i \rangle \right.$$

$$+ \frac{2 m_0 (m_0 - l)}{(\vec{\alpha}_0 \cdot \vec{p}_i) (\vec{\alpha}_0 \cdot \vec{p}_f)} \langle \vec{p}_f | V G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \times$$

$$\left. \left\{ \frac{\langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle}{\epsilon_0 - E_R} \right\}^{j-1} \langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) V | \vec{p}_i \rangle \right]$$

(3.2.9)

Next, we define two quantities β and D

$$D = \frac{\omega^2}{2} \langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle$$

(3.2.10)

and

$$\beta = \frac{D}{\epsilon_0 - E_R}$$

(3.2.11)

Then ,

$$\begin{aligned} \sum_{j=1}^{\infty} T_L^{(2j+1)} &= J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f) \lambda \left[\langle \vec{p}_f | V | U_R \rangle \times \right. \\ &\left. \left(\sum_{j=1}^{\infty} \beta^j \right) \frac{\langle U_R | V | \vec{p}_i \rangle}{\epsilon_0 - E_R} + \frac{\omega^2 m_0 / m_0 - l}{(\vec{\alpha}_0 \cdot \vec{p}_i) (\vec{\alpha}_0 \cdot \vec{p}_f)} \times \right. \\ &\left. \langle \vec{p}_f | V G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \times \right. \\ &\left. \left(\sum_{j=1}^{\infty} \beta^{j-1} \right) \frac{\langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) V | \vec{p}_i \rangle}{\epsilon_0 - E_R} \right] \end{aligned} \quad (3.2.12)$$

Note that

$$|D| \sim \frac{\omega^2}{\epsilon_0} |\vec{\alpha}_0 \cdot \vec{p}|^2 \sim \left(\frac{\omega}{\epsilon_0} \right) \omega$$

Hence

$$|\beta| \sim \left(\frac{\omega}{\epsilon_0} \right) \left(\frac{\omega}{\Gamma} \right) = \frac{(\omega/\epsilon_0)}{(\Gamma/\omega)}$$

To perform the summation over j , it is necessary to assume that $|\beta| < 1$. This means that the resonance energy must satisfy the inequality

$$\epsilon_0 > \left(\frac{\omega}{\Gamma} \right) \omega$$

Because of this condition, one can not apply the present work to very low-energy resonances. For $|\beta| < 1$

$$\sum_{j=1}^{\infty} \beta^j = \frac{1}{1-\beta} - 1$$

and,

$$\sum_{j=1}^{\infty} \beta^{j-1} = \frac{1}{1-\beta}$$

Also,

$$\begin{aligned} & \frac{1}{1-\beta} \frac{U_R \rangle \langle U_R}{\epsilon_0 - E_R} \\ &= \frac{U_R \rangle \langle U_R}{\epsilon_0 - D - E_R} \\ &= G_R^{(+)} (\epsilon_0 - D) \end{aligned}$$

Substituting back in (3.2.12), we write

$$\begin{aligned} \sum_{j=1}^{\infty} T_{\ell}^{(2j+1)} &= J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-\ell} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ & \left[\langle \vec{p}_f | T_R (\epsilon_0 - D) - T_R (\epsilon_0) | \vec{p}_i \rangle + \right. \\ & \frac{\omega^2 m_0 (m_0 - \ell)}{(\vec{\alpha}_0 \cdot \vec{p}_i) (\vec{\alpha}_0 \cdot \vec{p}_f)} \langle \vec{p}_f | \vee G_{NR}^{(+)} (\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} G_R^{(+)} (\epsilon_0 - D) \times \\ & \left. \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)} (\epsilon_0) \vee | \vec{p}_i \rangle \right] \end{aligned} \quad (3.2.13)$$

Adding this to (3.2.1), we get

$$\begin{aligned} & \sum_{j=0}^{\infty} T_{\ell}^{(2j+1)} \\ &= \sum_{\lambda=-\infty}^{\infty} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | \vee G_{NR}^{(+)} (\epsilon_{p_i} - \lambda \omega) \vee | \vec{p}_i \rangle \\ &+ \sum_{\lambda \neq -m_0} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R (\epsilon_{p_i} - \lambda \omega) | \vec{p}_i \rangle \\ &+ J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-\ell} (\vec{\alpha}_0 \cdot \vec{p}_f) \left[\langle \vec{p}_f | T_R (\epsilon_0 - D) | \vec{p}_i \rangle + \right. \end{aligned}$$

$$\frac{\omega^2 m_0 (m_0 - \epsilon)}{(\vec{\alpha}_0 \cdot \vec{p}_i) (\vec{\alpha}_0 \cdot \vec{p}_f)} \langle \vec{p}_f | \vee G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} G_{TR}^{(+)}(\epsilon_0 - D) \times$$

$$\vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) \vee | \vec{p}_i \rangle]$$

(3.2.14)

The last term in the above expression is 'smaller' than the preceding one by a factor

$$\frac{\omega^2}{(\vec{\alpha}_0 \cdot \vec{p}_i) (\vec{\alpha}_0 \cdot \vec{p}_f)} | G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} |^2 \sim \left(\frac{\omega}{\epsilon_0} \right)^2$$

So we drop the last term in (3.2.14). Thus

$$\sum_{j=0}^{\infty} T_L^{(2j+1)}$$

$$= \sum_{\lambda=-\infty}^{\infty} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | \vee G_{NR}^{(+)}(\epsilon_{pi} - \lambda\omega) \vee | \vec{p}_i \rangle +$$

$$\sum_{\lambda \neq -m_0} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R(\epsilon_{pi} - \lambda\omega) | \vec{p}_i \rangle +$$

$$J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R(\epsilon_0 - D) | \vec{p}_i \rangle$$

(3.2.15)

In order to write the general term of the series

$T_L^{(2j+2)}$, we have to evaluate $T_L^{(4)}$. From (2.1.16)

$$T_L^{(4)} = \frac{\omega^3}{8} \sum_{\lambda, x_1, x_2, x_3} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-x_1-x_2-x_3-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times$$

$$\langle \vec{p}_f | \vee G^{(+)}(\epsilon_{pi} - \omega(\lambda + x_1 + x_2 + x_3)) \vec{\alpha}_0 \cdot \vec{p} \times$$

$$G^{(+)}(\epsilon_{pi} - \omega(\lambda + x_1 + x_2)) \vec{\alpha}_0 \cdot \vec{p} G^{(+)}(\epsilon_{pi} - \omega(\lambda + x_1)) \times$$

$$\vec{\alpha}_0 \cdot \vec{p} \ G^{(+)}(\epsilon_{pi} - \omega\lambda) \vee | \vec{p}_i \rangle \quad (3.2.16)$$

Maximum contribution to $T_L^{(4)}$ occurs when either both the first and the third Green's functions are resonant or when both the second and the last Green's functions are resonant. In the former case $-\lambda = m_0 + x_1$ and $x_2 + x_3 = 0$ while in the latter case $-\lambda = m_0$ and $x_1 + x_2 = 0$. Thus

$$\begin{aligned} T_L^{(4)} &\sim 2 \left(\frac{\omega}{2} \right)^3 \frac{J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f)}{\epsilon_0 - E_R} \times \\ &\left[\left(\frac{2m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \right) \langle \vec{p}_f | V | U_R \rangle \langle U_R | \vec{\alpha}_0 \cdot \vec{p} \ G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \times \right. \\ &\left. \langle U_R | \vec{\alpha}_0 \cdot \vec{p} \ G_{NR}^{(+)}(\epsilon_0) \vee | \vec{p}_i \rangle \right. \\ &+ 2 \frac{(m_0 - l)}{\vec{\alpha}_0 \cdot \vec{p}_f} \langle \vec{p}_f | V \ G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \times \\ &\left. \langle U_R | \vec{\alpha}_0 \cdot \vec{p} \ G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \langle U_R | V | \vec{p}_i \rangle \right] \quad (3.2.17) \end{aligned}$$

In an analogous way,

$$\begin{aligned} T_L^{(2j+2)} &\sim 2^j \left(\frac{\omega}{2} \right)^{2j+1} \frac{J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f)}{\epsilon_0 - E_R} \times \\ &\left[\frac{2m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \langle \vec{p}_f | V | U_R \rangle \left\{ \frac{\langle U_R | \vec{\alpha}_0 \cdot \vec{p} \ G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle}{\epsilon_0 - E_R} \right\}^j \times \right. \\ &\left. \langle U_R | \vec{\alpha}_0 \cdot \vec{p} \ G_{NR}^{(+)}(\epsilon_0) \vee | \vec{p}_i \rangle \right. \\ &+ 2 \frac{(m_0 - l)}{\vec{\alpha}_0 \cdot \vec{p}_f} \langle \vec{p}_f | V \ G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \times \end{aligned}$$

$$\begin{aligned}
 & \left\{ \frac{\langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle}{\epsilon_0 - E_R} \right\}^j \langle U_R | V | \vec{p}_i \rangle \Big] \\
 &= \left(\frac{\omega}{2} \right) J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-2} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \left[\frac{2m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \langle \vec{p}_f | V | U_R \rangle \frac{\beta^j}{\epsilon_0 - E_R} \langle U_R | \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) V | \vec{p}_i \rangle \right. \\
 & \left. + \frac{2(m_0-2)}{\vec{\alpha}_0 \cdot \vec{p}_f} \langle \vec{p}_f | V G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} | U_R \rangle \times \right. \\
 & \left. \frac{\beta^j}{\epsilon_0 - E_R} \langle U_R | V | \vec{p}_i \rangle \right] \quad (3.2.18)
 \end{aligned}$$

Summing over j we find

$$\begin{aligned}
 \sum_{j=1}^{\infty} T_L^{(2j+2)} &= \left(\frac{\omega}{2} \right) J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-2} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \left[\frac{2m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \langle \vec{p}_f | V \left\{ G_R^{(+)}(\epsilon_0 - D) - G_R^{(+)}(\epsilon_0) \right\} \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) V | \vec{p}_i \rangle \right. \\
 & \left. + \frac{2(m_0-2)}{\vec{\alpha}_0 \cdot \vec{p}_f} \langle \vec{p}_f | V G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} \left\{ G_R^{(+)}(\epsilon_0 - D) - G_R^{(+)}(\epsilon_0) \right\} V | \vec{p}_i \rangle \right] \quad (3.2.19)
 \end{aligned}$$

(3.2.19) can be rewritten as

$$\begin{aligned}
 & \sum_{j=1}^{\infty} T_e^{(2j+2)} \\
 &= \omega J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \left[\frac{m\omega}{\vec{\alpha}_0 \cdot \vec{p}_i} \langle \vec{p}_f | \vee \left\{ G^{(+)}(\epsilon_0 - D) - G^{(+)}(\epsilon_0) \right\} \vec{\alpha}_0 \cdot \vec{p} G^{(+)}(E) \vee | \vec{p}_i \rangle \right. \\
 & \left. + \frac{(m_0 - l)}{\vec{\alpha}_0 \cdot \vec{p}_f} \langle \vec{p}_f | \vee G^{(+)}(E) \vec{\alpha}_0 \cdot \vec{p} \left\{ G^{(+)}(\epsilon_0 - D) - G^{(+)}(\epsilon_0) \right\} \vee | \vec{p}_i \rangle \right] \\
 & \hspace{25em} (3.2.20)
 \end{aligned}$$

where $E = \epsilon_0 + M\omega$, $M \neq 0$, $|M| \ll \epsilon_0/\omega$

The difference between (3.2.19) and (3.2.20) is too small to be of any significance but the latter form allows us to use the identity (2.1.21). Keeping only the leading terms we would get

$$\begin{aligned}
 & \sum_{j=1}^{\infty} T_e^{(2j+2)} = i J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \left[\langle \vec{p}_f | \left\{ T(\epsilon_0 - D) - T(\epsilon_0) \right\} \frac{m\omega m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \vec{\alpha}_0 \cdot \vec{r} | \vec{p}_i \rangle \right. \\
 & \left. - \langle \vec{p}_f | \frac{m\omega(m_0 - l)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{r} \left\{ T(\epsilon_0 - D) - T(\epsilon_0) \right\} | \vec{p}_i \rangle \right] \\
 & \approx i J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \left[\langle \vec{p}_f | \left\{ T_R(\epsilon_0 - D) - T_R(\epsilon_0) \right\} \frac{m\omega m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \vec{\alpha}_0 \cdot \vec{r} | \vec{p}_i \rangle - \right. \\
 & \left. \langle \vec{p}_f | \frac{m\omega(m_0 - l)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{r} \left\{ T_R(\epsilon_0 - D) - T_R(\epsilon_0) \right\} | \vec{p}_i \rangle \right] \\
 & \hspace{25em} (3.2.21)
 \end{aligned}$$

Adding (3.2.21) to (3.2.5) we get

$$\begin{aligned}
 \sum_{j=0}^{\infty} T_e^{(2j+2)} &= i \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}_f | \frac{m\omega (\lambda+1) \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_f} \vee \sigma_{NR}^{(+)} (\epsilon_{p_i} - \lambda\omega) \vee - \\
 &\vee \sigma_{NR}^{(+)} (\epsilon_{p_i} - \lambda\omega) \vee \frac{m\omega \lambda \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_i} | \vec{p}_i \rangle + \\
 &i J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-1} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}_f | T_R (\epsilon_0 - D) \frac{m\omega m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \vec{\alpha}_0 \cdot \vec{r} - \\
 &\frac{m\omega (m_0-1) \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_f} T_R (\epsilon_0 - D) | \vec{p}_i \rangle
 \end{aligned}
 \tag{3.2.22}$$

Finally,

$$\begin{aligned}
 T_e (\vec{p}_f, \vec{p}_i) &= \sum_{n=0}^{\infty} T_e^{(n)} \\
 &= (2.1.11) + (3.2.15) + (3.2.22) \\
 &= \sum_{\lambda=-\infty}^{\infty} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \langle \vec{p}_f | T_{NR} (\epsilon_{p_i} - \lambda\omega) + \\
 &\frac{i m\omega (\lambda+1) \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_f} \vee \sigma_{NR}^{(+)} (\epsilon_{p_i} - \lambda\omega) \vee - \\
 &i \vee \sigma_{NR}^{(+)} (\epsilon_{p_i} - \lambda\omega) \vee \frac{m\omega \lambda \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_i} | \vec{p}_i \rangle \\
 &+ J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-1} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R (\epsilon_0 - D) +
 \end{aligned}$$

$$\begin{aligned}
 & + i T_R (\epsilon_0 - D) \frac{m\omega m_0 \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_i} \\
 & - i \frac{m\omega (m_0 - l) \vec{\alpha}_0 \cdot \vec{r}}{\vec{\alpha}_0 \cdot \vec{p}_f} T_R (\epsilon_0 - D) |\vec{p}_i\rangle \\
 & + \sum_{\lambda \neq -m_0} J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R (\epsilon_{p_i} - \lambda\omega) |\vec{p}_i\rangle
 \end{aligned}
 \tag{3.2.23}$$

The operator $\vec{\alpha}_0 \cdot \vec{r}$ acting on the plane wave states shifts their momenta thereby yielding

$$\begin{aligned}
 T_R (\vec{p}_f, \vec{p}_i) &= \sum_{\lambda} J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \langle \vec{p}_f - \frac{m\omega (\lambda + l) \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} | T_{NR} (\epsilon_{p_i} - \lambda\omega) | \vec{p}_i - \frac{m\omega \lambda \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \rangle \\
 & + J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \langle \vec{p}_f + \frac{m\omega (m_0 - l) \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} | T_R (\epsilon_0 - D) | \vec{p}_i + \frac{m\omega m_0 \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \rangle \\
 & + \sum_{\lambda \neq -m_0} J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | T_R (\epsilon_{p_i} - \lambda\omega) |\vec{p}_i\rangle
 \end{aligned}
 \tag{3.2.24}$$

The summation over λ in the first term on the right side of (3.2.24) can be performed by the method adopted in chapter 2. (see equations (2.1.30), (2.1.32), (2.1.33) and (2.1.34)). Then,

$$\begin{aligned}
 T_L(\vec{p}_f, \vec{p}_i) &= J_{-L}(\vec{\alpha}_0 \cdot \vec{\delta} \vec{p}) \langle \vec{Q}' | T_{NR}(\epsilon_Q) | \vec{Q} \rangle \\
 &+ J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-L}(\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{P}' | T_R(\epsilon_0 - D) | \vec{P} \rangle \\
 &+ \sum_{\lambda \neq -m_0} J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-L-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{P}_f | T_R(\epsilon_{p_i} - \lambda \omega) | \vec{P}_i \rangle
 \end{aligned}
 \tag{3.2.25}$$

where,

$$\begin{aligned}
 \vec{Q}' &= \vec{p}_f - \vec{\delta}_{\lambda_0} \\
 \vec{Q} &= \vec{p}_i - \vec{\delta}_{\lambda_0} \\
 \vec{P}' &= \vec{p}_f + \frac{m\omega(m_0 - L)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} \\
 \vec{P} &= \vec{p}_i + \frac{m\omega m_0 \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i}
 \end{aligned}
 \tag{3.2.26}$$

The first term on the right side of (3.2.25) is the sum of non-resonant contributions up to $T_L^{(2)}$. The second term is the sum of resonant contributions. The shift of resonance energy from ϵ_0 to $\epsilon_0 - D$ is due to contributions from terms higher than $T_L^{(2)}$. The last term on the right side of (3.2.25) is the resonant contributions for $\lambda \neq -m_0$. In the absence of any resonance only the first term on the right side of (3.2.25) survives which is essentially the Kroll-Watson result. The physical meaning of the Stark shift D is the following: D , as given by (3.2.10) is, in general, a

complex quantity. Then the denominator of $T_R(\epsilon_0 - D)$ is

$$\epsilon_0 - D - E_R$$

$$= \epsilon_0 - (\text{Re } D + i \text{Im } D) - (\epsilon_R - i\Gamma/2)$$

$$= \epsilon_0 - [(\epsilon_R + \text{Re } D) - i(\Gamma/2 - \text{Im } D)]$$

Thus the effect of the presence of D is to shift the resonance energy from ϵ_R to $\epsilon_R + \text{Re } D$ and the half width from $\Gamma/2$ to $\Gamma/2 - \text{Im } D$. If one plots $\frac{d\sigma}{d\Omega}$ versus energy, one gets a narrower resonance for $\text{Im } D > 0$ and a broader resonance for $\text{Im } D < 0$. The resonance peak is shifted to the right for $\text{Re } D > 0$ and to the left for $\text{Re } D < 0$ (see fig. 1). Clearly, if the values of $\text{Re } D$ and $\text{Im } D$ are sufficiently large, an electron with energy will go out of resonance. The condition for remaining on resonance is

$$|\epsilon_0 - \epsilon_R - \text{Re } D| < |\Gamma/2 - \text{Im } D|$$

Previously the condition was

$$|\epsilon_0 - \epsilon_R| < \Gamma/2$$

For $\lambda \neq m_0$, one can write

$$T_R(\epsilon_{pi} - \lambda\omega) = \sqrt{\frac{u_R}{\epsilon_0 - E_R}} \sqrt{\left(\frac{\epsilon_0 - E_R}{\epsilon_{pi} - \lambda\omega - E_R} \right)}$$

Also,

$$E_R = \Sigma_R - i\Gamma/2$$

and

$$\Sigma_R \approx \epsilon_0 = \epsilon_{pi} + m_0 \omega$$

So,

$$\frac{\epsilon_0 - E_R}{\epsilon_{pi} - \lambda \omega - E_R} \approx \frac{i\Gamma/2}{i\Gamma/2 - (\lambda + m_0)\omega} \approx \frac{-i\Gamma}{2\omega(\lambda + m_0)} + O\left(\frac{\Gamma^2}{\omega^2}\right)$$

Hence,

$$T_R(\epsilon_{pi} - \lambda \omega) \approx \frac{-i\Gamma}{2\omega(\lambda + m_0)} T_R(\epsilon_0)$$

Similarly,

$$T_R(\epsilon_0 - D) = \left(\frac{\epsilon_0 - E_R}{\epsilon_0 - D - E_R} \right) \vee \frac{U_R(\epsilon_0 - E_R)}{\epsilon_0 - E_R} \vee$$

$$\frac{\epsilon_0 - E_R}{\epsilon_0 - D - E_R} \approx \frac{i\Gamma/2}{i\Gamma/2 - D} = \frac{1}{1 + 2iD/\Gamma}$$

Therefore,

$$T_R(\epsilon_0 - D) \approx \left(1 + \frac{2iD}{\Gamma} \right)^{-1} T_R(\epsilon_0)$$

Substituting back into (3.2.25) we get

$$\begin{aligned} T_e(\vec{p}_f, \vec{p}_i) &= J_{-e}(\vec{\alpha}_0 \cdot \vec{\alpha}_f) \langle \vec{Q}' | T_{NR}(\epsilon_a) | \vec{Q} \rangle \\ &+ \left(1 + \frac{2iD}{\Gamma} \right)^{-1} J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-e}(\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{P}' | T_R(\epsilon_0) | \vec{P} \rangle \\ &- i\Gamma/2\omega J_{m_0,e} \langle \vec{p}_f | T_R(\epsilon_0) | \vec{p}_i \rangle \end{aligned} \quad (3.2.27)$$

where,

$$g_{m_0, l} = \sum_{\lambda \neq -m_0} \frac{J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f)}{\lambda + m_0} \quad (3.2.28)$$

For $m_0 = l = 0$, $g_{0,0} = 0$. In general, however, no analytic expression for $g_{m_0, l}$ is known.

Using the definitions (3.2.26) it is easy to show that

$$Q'^2_{/2m} = \epsilon_Q = Q^2_{/2m} + O(\omega^2)$$

$$P'^2_{/2m} = \epsilon_0 = P^2_{/2m} + O(\omega^2)$$

So the T-matrices in the first two terms on the right side of (3.2.27) are evaluated on-shell. Since the last term is of order Γ/ω , $\langle \vec{p}_f | T_R(\epsilon_0) | \vec{p}_i \rangle$ will also be treated as an on-shell matrix. For central potentials an on-shell T-matrix $\langle \vec{p}_f | T_{NR}(E) | \vec{p}_i \rangle$ is a function only of E and $\hat{p}_f \cdot \hat{p}_i$. So we use the short-hand notation

$$\langle \vec{p}_f | T_{NR} (E) | \vec{p}_i \rangle = t_{NR} (E, \hat{p}_f, \hat{p}_i) \quad (3.2.29)$$

The corresponding differential cross sections are

$$\frac{d\sigma}{d\Omega}^{NR} (\vec{p}_f, \hat{p}_i) = | t_{NR} (E, \hat{p}_f, \hat{p}_i) |^2 \quad (3.2.30)$$

and

$$\begin{aligned}
 T_L(\vec{p}_f, \vec{p}_i) &= J_{-L}(\vec{\alpha}_0 \cdot \vec{\Delta}\vec{p}) t_{NR}(\epsilon_Q, \hat{Q}, \hat{Q}) \\
 &+ (1 + 2iD_r)^{-1} J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-L}(\vec{\alpha}_0 \cdot \vec{p}_f) t_R(\epsilon_0, \hat{p}, \hat{p}) \\
 &- \frac{i\Gamma}{2\omega} g_{m_0, L} t_R(\epsilon_0, \mu)
 \end{aligned}
 \tag{3.2.31}$$

where

$$\mu = \hat{p}_f \cdot \hat{p}_i
 \tag{2.4.12}$$

This is the T-matrix for laser-assisted scattering with the absorption of L photons in the presence of an isolated resonance at $\epsilon_0 = \epsilon_{p_i} + m_0\omega$.

The first term on the right side of (3.2.31) is the background scattering. For $L \neq 0$, this can be suppressed²⁵ by choosing the laser direction such that $\hat{\alpha}_0 \perp \vec{\Delta}\vec{p}$. This means turning the electric field of the laser perpendicular to the momentum transfer $\vec{\Delta}\vec{p} = \vec{p}_f - \vec{p}_i$.

In that case

$$J_{-L}(\vec{\alpha}_0 \cdot \vec{\Delta}\vec{p}) = J_{-L}(0) = 0 \text{ for } L \neq 0.$$

For $L=0$, one could choose scattering angles and energies such that a zero of $J_0(\vec{\alpha}_0 \cdot \vec{\Delta}\vec{p})$ is found²⁶. In this case also, the non-resonant background scattering can be suppressed.

In the next section we will use (3.2.31) to obtain corrections to the sum rule (3.1.13) and also to

evaluate $\langle l \rangle$.

3.3 Evaluation of the sum rules:

The differential cross section for scattering with the transfer of l photons is

$$\frac{d\sigma_l(\vec{p}_f, \vec{p}_i)}{d\Omega} = \frac{p_f}{p_i} |T_l(\vec{p}_f, \vec{p}_i)|^2 \quad (3.3.1)$$

$T_l(\vec{p}_f, \vec{p}_i)$ can be written as

$$\begin{aligned} T_l(\vec{p}_f, \vec{p}_i) = & J_l(z) t_{NR}(\epsilon_q, \hat{q}; \hat{q}) \\ & + (1 + 2iD/r)^{-1} J_{m_0}(x) J_{m_0-l}(y) t_R(\epsilon_0, \hat{p}; \hat{p}) \\ & - \frac{i\Gamma}{2\omega} g_{m_0, l} t_R(\epsilon_0, \mu) \end{aligned} \quad (3.3.2)$$

where

$$\begin{aligned} x = & \vec{\alpha}_0 \cdot \vec{p}_i \\ y = & \vec{\alpha}_0 \cdot \vec{p}_f \\ z = & x - y = -\vec{\alpha}_0 \cdot \vec{p} \end{aligned}$$

(3.3.3)

We define

$$\begin{pmatrix} \vec{q}_f \\ \vec{q}_i \end{pmatrix} = k \begin{pmatrix} \hat{p}_f \\ \hat{p}_i \end{pmatrix}; \quad k^2 / 2m = \epsilon_0$$

$$x_0 = \vec{\alpha}_0 \cdot \vec{q}_i$$

$$y_0 = \vec{\alpha}_0 \cdot \vec{q}_f$$

$$z_0 = x_0 - y_0$$

(3.3.4)

We expand $\vec{p}_f$ and $\vec{p}_i$ around $\vec{q}_f$ and $\vec{q}_i$ respectively.

$$\vec{p}_f = \vec{q}_f \left(1 + \frac{(l-m_0)\omega}{2\epsilon_0} \right) + O(\omega^2)$$

$$\vec{p}_i = \vec{q}_i \left(1 - \frac{m_0\omega}{2\epsilon_0} \right) + O(\omega^2)$$

(3.3.5)

Then

$$x = x_0 \left(1 - \frac{m_0\omega}{2\epsilon_0} \right)$$

$$y = y_0 \left(1 + \frac{(l-m_0)\omega}{2\epsilon_0} \right)$$

$$z = z_0 - \frac{\omega}{2\epsilon_0} (m_0 z_0 + l y_0)$$

(3.3.6)

Using (3.3.6) the Bessel functions can be expanded as

$$J_l(z) = J_l(z_0) - \frac{\omega}{2\epsilon_0} (m_0 z_0 + l y_0) J'_l(z_0)$$

$$J_{m_0}(x) = J_{m_0}(x_0) - \frac{m_0\omega}{2\epsilon_0} x_0 J'_{m_0}(x_0)$$

$$J_{m_0-l}(\gamma) = J_{m_0-l}(\gamma_0) + \frac{(l-m_0)\omega}{2\epsilon_0} \gamma_0 J'_{m_0-l}(\gamma_0) \quad (3.3.7)$$

Then

$$J_{m_0,l} \simeq \sum_{\lambda \neq -m_0} \frac{J_\lambda(-x_0) J_{-l-\lambda}(\gamma_0)}{\lambda + m_0} + O(\omega) \quad (3.3.8)$$

Also

$$\begin{aligned} \epsilon_Q &= \epsilon_{ti} + \frac{\omega l x}{z} = \epsilon_0 + \omega \left(\frac{l x_0}{z_0} - m_0 \right) + O(\omega^2) \\ \hat{Q}' : \hat{Q} &= \mu + \frac{\omega l (x_0 + \gamma_0) (1-\mu)}{2 \epsilon_0 z_0} \\ \hat{P}' : \hat{P} &= \mu + \frac{\omega}{2 \epsilon_0} \left(\mu (l - 2m_0) + (m_0 - l) \frac{x_0}{\gamma_0} + m_0 \frac{\gamma_0}{x_0} \right) \end{aligned} \quad (3.3.9)$$

Using (3.3.9) we expand the T-matrices as

$$\begin{aligned} t_{NR}(\epsilon_Q, \hat{Q}' : \hat{Q}) &= t_{NR}(\epsilon_0, \mu) \\ &+ \omega \left(\frac{l x_0}{z_0} - m_0 \right) \frac{\partial}{\partial \epsilon_0} t_{NR}(\epsilon_0, \mu) \\ &+ \frac{\omega l (x_0 + \gamma_0) (1-\mu)}{2 \epsilon_0 z_0} \frac{\partial}{\partial \mu} t_{NR}(\epsilon_0, \mu) \\ t_R(\epsilon_0, \hat{P}' : \hat{P}) &= t_R(\epsilon_0, \mu) + \\ \frac{\omega}{2 \epsilon_0} \left[\mu (l - 2m_0) + (m_0 - l) \frac{x_0}{\gamma_0} + m_0 \frac{\gamma_0}{x_0} \right] \frac{\partial}{\partial \mu} t_R(\epsilon_0, \mu) \end{aligned} \quad (3.3.10)$$

Using the above expansions in (3.3.2) one can write

$$|T_L(\vec{p}_f, \vec{p}_i)|^2 \text{ as}$$

$$|T_L(\vec{p}_f, \vec{p}_i)|^2 = a_0 + a_1 \quad (3.3.11)$$

where a_0 and a_1 are respectively quantities of the zeroth and first order of smallness in the expansion parameters $\frac{\omega}{\epsilon_0}$, $\frac{r}{\omega}$ and $\frac{|D|}{r}$. The evaluation of a_0 and a_1 requires straight-forward algebra. After some effort one gets

$$a_0 = J_L^2(z_0) \frac{d\sigma_{NR}(\epsilon_0, \mu)}{d\Omega}$$

$$+ J_{m_0}^2(x_0) J_{m_0-L}^2(y_0) \frac{d\sigma_R(\epsilon_0, \mu)}{d\Omega}$$

$$+ 2 J_L(z_0) J_{m_0}(x_0) J_{m_0-L}(y_0) \text{Re}(t_{NR}^* t_R)$$

(3.3.12)

and

$$a_1 = J_L^2(z_0) \left[\left(\frac{\omega L x_0}{z_0} - m_0 \omega \right) \frac{\partial}{\partial \epsilon_0} + \right.$$

$$\left. \frac{\omega L (x_0 + y_0) (1 - \mu)}{2 \epsilon_0 z_0} \frac{\partial}{\partial \mu} \right] \frac{d\sigma_{NR}(\epsilon_0, \mu)}{d\Omega} -$$

$$\frac{\omega}{2 \epsilon_0} (m_0 z_0 + L y_0) \left(\frac{\partial}{\partial z_0} J_L^2(z_0) \right) \frac{d\sigma_{NR}(\epsilon_0, \mu)}{d\Omega}$$

$$+ \frac{\omega}{2 \epsilon_0} J_{m_0}^2(x_0) J_{m_0-L}^2(y_0) \left\{ L \left(\mu - \frac{x_0}{y_0} \right) + \right.$$

$$\begin{aligned}
 & + m_0 \left(\frac{x_0}{y_0} + \frac{y_0}{x_0} - 2\mu \right) \left\{ \frac{\partial}{\partial \mu} \frac{d\sigma_R}{d\Omega} (\epsilon_0, \mu) \right. \\
 & - \frac{\omega}{2\epsilon_0} \left\{ (m_0 - l) y_0 J_{m_0}^2(x_0) 2 J_{m_0-l}(y_0) J_{m_0-l}'(y_0) \right. \\
 & + m_0 x_0 2 J_{m_0}(x_0) J_{m_0}'(x_0) J_{m_0-l}^2(y_0) \left. \right\} \frac{d\sigma_R}{d\Omega} (\epsilon_0, \mu) \\
 & + \left(\frac{4 \frac{|D|^2}{r^2} - \frac{4}{r} \operatorname{Im} D}{1 + 4 \frac{|D|^2}{r^2} - \frac{4}{r} \operatorname{Im} D} \right) J_{m_0}^2(x_0) J_{m_0-l}^2(y_0) \frac{d\sigma_R}{d\Omega} (\epsilon_0, \mu) \\
 & - \frac{\omega}{2\epsilon_0} \left[(m_0 z_0 + l y_0) J_l'(z_0) J_{m_0}(x_0) J_{m_0-l}(y_0) \right. \\
 & + m_0 x_0 J_l(z_0) J_{m_0}'(x_0) J_{m_0-l}(y_0) \\
 & + (m_0 - l) y_0 J_l(z_0) J_{m_0}(x_0) J_{m_0-l}'(y_0) \left. \right] 2 \operatorname{Re}(t_{NR}^* t_R) \\
 & + 2 J_l(z_0) J_{m_0}(x_0) J_{m_0-l}(y_0) \times \\
 & \left[\omega \left(\frac{l x_0}{z_0} - m_0 \right) \operatorname{Re} \left\{ \left(\frac{\partial t_{NR}^*}{\partial \epsilon_0} \right) t_R \right\} \right. \\
 & + \frac{\omega l (x_0 + y_0) (1 - \mu)}{2 \epsilon_0 z_0} \operatorname{Re} \left\{ \left(\frac{\partial t_{NR}^*}{\partial \mu} \right) t_R \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\omega}{2\epsilon_0} \left\{ \mu (e - 2m_0) + (m_0 - e) \frac{x_0}{y_0} + m_0 \frac{y_0}{x_0} \right\} x \\
 & R_L \left(t_{NR}^x \frac{\partial t_R}{\partial \mu} \right) \Big] \\
 & + 2 \operatorname{Im} \left[\left\{ \left(\frac{2D}{\Gamma} \right) \left(1 + \frac{2iD}{\Gamma} \right)^{-1} x \right. \right. \\
 & J_L(z_0) J_{m_0}(x_0) J_{m_0-e}(y_0) \\
 & \left. \left. + \left(\frac{\Gamma}{2\omega} \right) J_L(z_0) J_{m_0,e} \right\} \left(t_{NR}^x t_R \right) \right] \\
 & \hspace{25em} (3.3.13)
 \end{aligned}$$

Also,

$$\frac{p_f}{p_i} = 1 + \frac{\omega}{2\epsilon_0} + O(\omega^2) \quad (3.3.14)$$

Using (3.3.11) and (3.3.14) in (3.3.1) and dropping terms of second order of smallness in the expansion parameters, we get

$$\frac{d\delta e}{d\Omega} (\tilde{p}_f, \tilde{p}_i) = a_0 + \frac{\omega}{2\epsilon_0} a_0 + a_1 \quad (3.3.15)$$

The corresponding sum rule is

$$\sum_l \frac{d\sigma_l}{d\Omega}(\vec{p}_f, \vec{p}_i) = \sum_l a_0 + \frac{\omega}{2\epsilon_0} \sum_l \epsilon a_0 + \sum_l a_1 \quad (3.3.16)$$

The leading term on the right side of (3.3.16) is

$$\begin{aligned} \sum_l a_0 = & \frac{d\sigma_{NR}}{d\Omega}(\epsilon_0, \mu) \\ & + J_{m0}^2(x_0) \left[\frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) + 2 \operatorname{Re} t_{NR}^* t_R \right] \end{aligned} \quad (3.3.17)$$

From (3.1.12) one gets

$$2 \operatorname{Re}(t_{NR}^* t_R) = \frac{d\sigma}{d\Omega}(\epsilon_0, \mu) - \frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) - \frac{d\sigma_{NR}}{d\Omega}(\epsilon_0, \mu) \quad (3.3.18)$$

Substituting in (3.3.17) and simplifying one can also

write $\sum_l a_0$ in the form

$$\begin{aligned} \sum_l a_0 = & \left[1 - J_{m0}^2(x_0) \right] \frac{d\sigma_{NR}}{d\Omega}(\epsilon_0, \mu) \\ & + J_{m0}^2(x_0) \frac{d\sigma}{d\Omega}(\epsilon_0, \mu) \end{aligned} \quad (3.3.19)$$

If we keep only this term on the right side of (3.3.16) we

will recover the lowest order sum rule (3.1.13). The

correction terms in (3.3.16) can be evaluated by extensive

use of appendix A. After some effort one would get

$$\begin{aligned} \sum_l \epsilon a_0 = & m_0 J_{m0}^2(x_0) \left[\frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) \right. \\ & \left. + \frac{z_0}{x_0} (2 \operatorname{Re} t_{NR}^* t_R) \right] \end{aligned} \quad (3.3.20)$$

and

$$\begin{aligned}
 \sum_l a_l = & -m_0 \omega \frac{\partial}{\partial \epsilon_0} \frac{d\sigma_{NR}}{d\Omega}(\epsilon_0, \mu) \\
 & + \frac{m_0 \omega}{2\epsilon_0} J_{m_0}^2(x_0) \left(\frac{y_0}{x_0} - \mu \right) \frac{\partial}{\partial \mu} \frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) \\
 & - \frac{m_0 \omega}{2\epsilon_0} x_0 2 J_{m_0}(x_0) J'_{m_0}(x_0) \frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) \\
 & + \left(\frac{4 \frac{|D|^2}{r^2} - \frac{4}{r} \text{Im} D}{1 + 4 \frac{|D|^2}{r^2} - \frac{4}{r} \text{Im} D} \right) J_{m_0}^2(x_0) \frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) \\
 & - \left[\frac{m_0 \omega y_0}{2\epsilon_0 x_0} J_{m_0}^2(x_0) + \frac{m_0 \omega x_0}{2\epsilon_0} \cdot 2 J_{m_0}(x_0) J'_{m_0}(x_0) \right] \times \\
 & \left\{ 2 \text{Re}(t_{NR}^* t_R) \right\} \\
 & + \frac{m_0 \omega}{2\epsilon_0} J_{m_0}^2(x_0) \left(1 + \frac{y_0}{x_0} \right) (1 - \mu) \frac{\partial}{\partial \mu} \left\{ 2 \text{Re}(t_{NR}^* t_R) \right\} \\
 & + \text{Im} \left[\left\{ \left(2D/r \right) \left(1 + \frac{2iD}{r} \right)^{-1} J_{m_0}^2(x_0) \right. \right. \\
 & \left. \left. + \left(\frac{r}{2\omega} \right) \sum_{\lambda \neq -m_0} \frac{J_{\lambda}^2(-x_0)}{\lambda + m_0} \right\} \left\{ 2 \text{Re}(t_{NR}^* t_R) \right\} \right]
 \end{aligned}$$

It is easy to show that

$$\begin{aligned}
 & \text{Im} \left[\left\{ \left(2D/\Gamma \right) \left(1 + 2\frac{iD}{\Gamma} \right)^{-1} J_{m_0}^2(x_0) \right. \right. \\
 & \left. \left. + \left(\frac{\Gamma}{2\omega} \right) \sum_{\lambda \neq -m_0} \frac{J_{\lambda}^2 \frac{1-x_0}{\lambda+m_0}}{\lambda+m_0} \right\} \left\{ 2 \text{Re} (t_{NR}^* t_R) \right\} \right] \\
 & = \left(\frac{\frac{2}{\Gamma} \text{Im} D - \frac{4}{\Gamma^2} |D|^2}{1 + 4 \frac{|D|^2}{\Gamma^2} - \frac{4}{\Gamma} \text{Im} D} \right) J_{m_0}^2(x_0) \left\{ 2 \text{Re} (t_{NR}^* t_R) \right\} \\
 & + \left[\frac{\frac{2}{\Gamma} \text{Re} D}{1 + 4 \frac{|D|^2}{\Gamma^2} - \frac{4}{\Gamma} \text{Im} D} J_{m_0}^2(x_0) \right. \\
 & \left. + \left(\frac{\Gamma}{2\omega} \right) \sum_{\lambda \neq -m_0} \frac{J_{\lambda}^2 \frac{(-x_0)}{\lambda+m_0}}{\lambda+m_0} \right] \left\{ 2 \text{Im} (t_{NR}^* t_R) \right\}.
 \end{aligned}$$

(3.3.22)

Substituting in (3.3.21) and rearranging the different terms we get

$$\begin{aligned}
 \sum_L a_L &= -m_0 \omega \frac{\partial}{\partial \epsilon_0} \left(\frac{d\sigma_{NR}}{d\Omega}(\epsilon_0, \mu) \right) \\
 &+ \left[J_{m_0}^2(x_0) \left\{ \left(\frac{4 |D|^2 / \Gamma^2 - \frac{4}{\Gamma} \text{Im} D}{1 + 4 |D|^2 / \Gamma^2 - \frac{4}{\Gamma} \text{Im} D} \right) + \right. \right. \\
 &\left. \left. \frac{m_0 \omega}{2 \epsilon_0} \left(\frac{y_0}{x_0} - \mu \right) \frac{\partial}{\partial \mu} \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{m_0 \omega x_0}{2 \epsilon_0} \left[2 J_{m_0}(x_0) J'_{m_0}(x_0) \right] \frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) \\
 & + \left[J_{m_0}^2(x_0) \left\{ \left(\frac{\frac{2}{\Gamma} \operatorname{Im} D - \frac{4}{\Gamma^2} |D|^2}{1 + 4 \frac{|D|^2}{\Gamma^2} - \frac{4}{\Gamma} \operatorname{Im} D} \right) \right. \right. \\
 & \quad \left. \left. - \frac{m_0 \omega y_0}{2 \epsilon_0 x_0} + \frac{m_0 \omega}{2 \epsilon_0} \left(1 + \frac{y_0}{x_0} \right) (1 - \mu) \frac{\partial}{\partial \mu} \right\} \right. \\
 & \quad \left. - \frac{m_0 \omega x_0}{2 \epsilon_0} \left[2 J_{m_0}(x_0) J'_{m_0}(x_0) \right] \left\{ 2 \operatorname{Re} (t_{NR}^* t_R) \right\} \right. \\
 & \quad \left. + \left[\left(\frac{\frac{2}{\Gamma} \operatorname{Re} D}{1 + 4 \frac{|D|^2}{\Gamma^2} - \frac{4}{\Gamma} \operatorname{Im} D} \right) J_{m_0}^2(x_0) \right. \right. \\
 & \quad \left. \left. + \left(\frac{\Gamma}{2\omega} \right) \sum_{\lambda \neq -m_0} \frac{J_{\lambda}^2(-x_0)}{\lambda + m_0} \right] \left\{ 2 \operatorname{Im} (t_{NR}^* t_R) \right\} \right]
 \end{aligned}$$

(3.3.23)

Using these expressions for $\sum_{\ell} a_0$, $\sum_{\ell} \ell a_0$ and $\sum_{\ell} a_1$ in (3.3.16) and simplifying, we get

$$\begin{aligned}
 \sum_{\vec{p}} \frac{d\sigma_0}{d\Omega} (\vec{p}_f, \vec{p}_i) &= (1 - m_0 \omega \frac{\partial}{\partial \epsilon_0}) \frac{d\sigma_{NR}}{d\Omega} (\epsilon_0, \mu) \\
 &+ \left[J_{m_0}^2(x_0) \left\{ 1 + \frac{m_0 \omega}{2\epsilon_0} + \frac{4|D|^2/\Gamma^2 - \frac{4}{\Gamma} \text{Im} D}{1 + 4|D|^2/\Gamma^2 - \frac{4}{\Gamma} \text{Im} D} \right. \right. \\
 &\quad \left. \left. + \frac{m_0 \omega}{2\epsilon_0} \left(\frac{\gamma_0}{x_0} - \mu \right) \frac{\partial}{\partial \mu} \right\} \right. \\
 &\quad \left. - \frac{m_0 \omega x_0}{2\epsilon_0} 2 J_{m_0}(x_0) J'_{m_0}(x_0) \right] \frac{d\sigma_R}{d\Omega} (\epsilon_0, \mu) \\
 &+ \left[J_{m_0}^2(x_0) \left\{ 1 + \frac{m_0 \omega}{2\epsilon_0} \left(1 - \frac{2\gamma_0}{x_0} \right) + \frac{\frac{2}{\Gamma} \text{Im} D - \frac{4|D|^2}{\Gamma^2}}{1 + \frac{4|D|^2}{\Gamma^2} - \frac{4}{\Gamma} \text{Im} D} \right. \right. \\
 &\quad \left. \left. + \frac{m_0 \omega}{2\epsilon_0} \left(1 + \frac{\gamma_0}{x_0} \right) (1 - \mu) \frac{\partial}{\partial \mu} \right\} \right. \\
 &\quad \left. - \frac{m_0 \omega x_0}{2\epsilon_0} 2 J_{m_0}(x_0) J'_{m_0}(x_0) \right] \left\{ 2 \text{Re} (t_{NR}^* t_R) \right\} \\
 &+ \left[\frac{(2/\Gamma) \text{Re} D}{1 + \frac{4|D|^2}{\Gamma^2} - \frac{4}{\Gamma} \text{Im} D} J_{m_0}^2(x_0) + \left(\frac{\Gamma}{2\omega} \right) \sum_{\lambda \neq -m_0} \frac{J_{\lambda}^2(-x_0)}{\lambda + m_0} \right] \times \\
 &\quad \left\{ 2 \text{Im} (t_{NR}^* t_R) \right\}. \tag{3.3.24}
 \end{aligned}$$

One of the novel features of the above result is the occurrence of both the real and imaginary parts of the complex quantity $t_{NR}^* t_R$. In the absence of the laser only the real part of $t_{NR}^* t_R$ appears (see equation

(3.1.12)). The coefficient of $\text{Im } t_{NR}^* t_R$ in (3.3.24) has its origin in the inclusion of resonant contributions (a) from $T_\ell^{(3)}$ and higher terms in the series for T_ℓ and (b) from $\lambda \neq -m_0$ terms in $T_\ell^{(1)}$.

It may be noticed that all the correction terms (including the one containing $\text{Im } t_{NR}^* t_R$ vanish if $\text{Re } D = \text{Im } D = 0$ and $m_0 = 0$.

We may integrate the above result over the final direction of scattering by the use of appendix B. The result is

$$\begin{aligned}
 \sum_{\ell} \sigma_{\ell} = & (1 - m_0 \omega \frac{\partial}{\partial \epsilon_0}) \sigma_{NR}(\epsilon_0) + \\
 & \left[J_{m_0}^2(x_0) \left\{ 1 + \frac{m_0 \omega}{2\epsilon_0} + \frac{4|D|^2/\Gamma^2 - \frac{4}{\Gamma} \text{Im } D}{1 + 4|D|^2/\Gamma^2 - \frac{4}{\Gamma} \text{Im } D} \right\} \right. \\
 & - \left. \frac{m_0 \omega x_0}{2\epsilon_0} 2 J_{m_0}(x_0) J'_{m_0}(x_0) \right] \sigma_R(\epsilon_0) + \\
 & \left[J_{m_0}^2(x_0) \left\{ 1 + \frac{m_0 \omega}{2\epsilon_0} + \frac{\frac{2}{\Gamma} \text{Im } D - \frac{4}{\Gamma^2} |D|^2}{1 + \frac{4}{\Gamma^2} |D|^2 - \frac{4}{\Gamma} \text{Im } D} \right\} \right. \\
 & - \left. \frac{m_0 \omega x_0}{2\epsilon_0} 2 J_{m_0}(x_0) J'_{m_0}(x_0) \right] \int 2 \text{Re}(t_{NR}^* t_R) d\Omega. \\
 & + \left[\frac{\frac{2}{\Gamma} \text{Re } D}{1 + \frac{4}{\Gamma^2} |D|^2 - \frac{4}{\Gamma} \text{Im } D} J_{m_0}^2(x_0) + \left(\frac{\Gamma}{2\omega} \right) \sum_{\lambda \neq -m_0} \frac{J_{\lambda}^2(-x_0)}{\lambda + m_0} \right] \times \\
 & \int 2 \text{Im}(t_{NR}^* t_R) d\Omega.
 \end{aligned}$$

(3.3.25)

Finally, we will evaluate $\langle l \rangle$ to the lowest order in the expansion parameters. $\langle l \rangle$ is the average number of photons absorbed during a scattering event. It is defined as

$$\langle l \rangle = \frac{\sum_l l \int d\Omega_{\vec{p}_f} \frac{d\sigma_l}{d\Omega} (\vec{p}_f, \vec{p}_i)}{\sum_l \int d\Omega_{\vec{p}_f} \frac{d\sigma_l}{d\Omega} (\vec{p}_f, \vec{p}_i)} \quad (1.4.11)$$

Using (3.3.15) and retaining only the leading term we get

$$\langle l \rangle = \frac{\int d\Omega_{\vec{p}_f} \sum_l l a_0}{\int d\Omega_{\vec{p}_f} \sum_l a_0} \quad (3.3.26)$$

The quantities $\sum_l l a_0$ and $\sum_l a_0$ are given by equations (3.3.20) and (3.3.12) respectively. Substituting in (3.3.26) and integrating over the final direction of scattering we get

$$\langle l \rangle = \frac{n_{m0} J_{m0}^2(x_0) \left[\sigma_R + 2 \operatorname{Re} \int (1-\mu) t_{NR}^* t_R d\Omega \right]}{\sigma_{NR} + J_{m0}^2(x_0) \left[\sigma_R + 2 \operatorname{Re} \int t_{NR}^* t_R d\Omega \right]} \quad (3.3.27)$$

where

$$\sigma_{NR} = \int \frac{d\sigma_{NR}}{d\Omega} d\Omega_{pf}$$

By the use of (3.3.18) the above result for $\langle l \rangle$ can be rewritten in the following form :

$$\langle l \rangle = \frac{m_0 J_{m_0}^2(x_0) [\sigma_R + \delta_{tot.} - \delta_{NR} - \delta_R]}{[1 - J_{m_0}^2(x_0)] \sigma_{NR} + J_{m_0}^2(x_0) \sigma_{tot.}} \quad (3.3.28)$$

where δ_{NR} and δ_R can be identified as momentum transfer cross sections for 'pure' field-free non-resonant scattering and 'pure' field-free resonant scattering respectively.

$$\delta_{NR} = \int (1-\mu) \frac{d\sigma_{NR}}{d\Omega} d\Omega_{pf} \quad (3.3.29)$$

$\sigma_{tot.}$ and $\delta_{tot.}$ are respectively the total and momentum transfer cross sections in the absence of the laser:

$$\begin{aligned} \sigma_{tot.} &= \int \frac{d\sigma}{d\Omega} d\Omega_{pf} \\ \delta_{tot.} &= \int (1-\mu) \frac{d\sigma}{d\Omega} d\Omega_{pf} \end{aligned} \quad (3.3.30)$$

One recalls that m_0 is the number of photons the

incident electron must absorb in order to attain the resonance energy $\epsilon_0 = \epsilon_{pi} + m_0 \omega$. It is not surprising, therefore, that $\langle l \rangle$ is directly proportional to m_0 . For a background-free resonance t_{NR}^* and σ_{NR} vanish from (3.3.27). In that case $\langle l \rangle$ is strictly equal to m_0 . It may be mentioned here that for non-resonant potential scattering, the lowest order value of $\langle l \rangle$ is zero. The origin of this result can be traced to the fact that $\sum_l l J_l^2(z_0) = 0$. Physically, it means that for an electron with a given energy the probability for absorbing l photons is the same as the probability for emitting l photons to the lowest order in ω .

CHAPTER 4

NON-RESONANT ELECTRON-ATOM SCATTERING IN THE PRESENCE OF A LOW-FREQUENCY LASER

4.1 Derivation of the T-matrix:

Our starting point is the generalization of the S-matrix (2.1.1) to the case in which the target is an atom with z electrons. We write

$$S = -i \langle \Omega_f, \psi_i^{(+)} \rangle \quad (4.1.1)$$

The j -th electron is defined by the position coordinates $\vec{r}_j$ and has the momentum $\vec{p}_j$. We label the electrons such that $j = 0$ for the projectile and ranges from 1 to z for the atomic electrons. Ω_f is the final state of the (atom + projectile) system. It is the product of the free electron state $\chi_f(\vec{r}_0, t)$ and the atomic state

$$\hat{\phi}_f(r_1, \dots, r_z; t)$$

$$|\Omega_f\rangle = \chi_f \hat{\phi}_f \quad (4.1.2)$$

$$|\chi_f\rangle = \sum_{n'} J_{n'}(\vec{\alpha}_0 \cdot \vec{p}_f) e^{-i(\epsilon_{p_f} + n'\omega)t} |\vec{p}_f(0)\rangle \quad (4.1.3)$$

where

$$\langle \vec{r} | \vec{p}_f(0) \rangle = e^{i\vec{p}_f \cdot \vec{r}_0}$$

The atomic state $\hat{\phi}_f$ can be written as

$$\hat{\phi}_f(r_1, \dots, r_z; t) = e^{-i\bar{W}_f t} \phi_f(r_1, \dots, r_z; \omega t) \quad (4.1.4)$$

where $\bar{W}_f$ is the dressed energy level and ϕ_f is a periodic function of ωt . (It has been pointed out by Shirley²⁷ that this is a general property of states in a periodic

field). It can, therefore, be expanded in a Fourier series:

$$\phi_f(r_1, \dots, r_2; \omega t) = \sum_{s'} \phi_f^{(s')}(r_1, \dots, r_2) e^{-is'\omega t} \quad (4.1.5)$$

Thus

$$\begin{aligned} |\Omega_f\rangle &= \sum_{\lambda', s'} J_{\lambda', -s'} (\vec{\alpha}_0 \cdot \vec{p}_f) e^{-i(\epsilon_{p_f} + \bar{w}_f + \lambda'\omega)t} \\ &\times |\vec{p}_f(0), \phi_f^{(s')}\rangle \end{aligned} \quad (4.1.6)$$

or taking the complex conjugate

$$\begin{aligned} \langle \Omega_f | &= \sum_{\lambda', s'} J_{\lambda', -s'} (\vec{\alpha}_0 \cdot \vec{p}_f) e^{i(\epsilon_{p_f} + \bar{w}_f + \lambda'\omega)t} \\ &\times \langle \vec{p}_f(0), \phi_f^{(s')} | \end{aligned} \quad (4.1.7)$$

V is the Coulomb interaction potential of the projectile with all the other charged particles. $\psi_i^{(+)}$ is the full wave function which can be written as

$$\psi_i^{(+)} = A_0 (1 + \bar{G}^{(+)} V) |\Omega_i\rangle \quad (4.1.8)$$

$\bar{G}^{(+)}$ is the full Green's function of the problem and

A_0 is the anti-symmetrization operator:

$$A_0 = 1 - \sum_{j=1}^Z X_{j0} \quad (4.1.9)$$

where X_{j0} exchanges indices '0' and 'j'. It is included in

(4.1.8) to account for the Pauli principle. $|\Omega_i\rangle$ is the

initial state of the (atom + projectile) system:

$$\begin{aligned}\Omega_i &= \chi_i \hat{\phi}_i \\ \chi_i &= \sum_n J_n (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{-i(\epsilon_{p_i} - n\omega)t} |\vec{p}_i(0)\rangle \\ \hat{\phi}_i &= e^{-i\vec{W}_i t} \phi_i(\omega t)\end{aligned}\tag{4.1.10}$$

$$\phi_i(\omega t) = \sum_s \phi_i^{(s)} e^{-is\omega t}\tag{4.1.11}$$

Thus Ω_i can be written as

$$\begin{aligned}\Omega_i &= \sum_{\lambda, s} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{-i(\epsilon_{p_i} + \vec{W}_i - \lambda\omega)t} \\ &\times |\vec{p}_i(0), \phi_i^{(s)}\rangle\end{aligned}\tag{4.1.12}$$

The time dependent Green's function $\bar{G}^{(+)}$ can be expanded in powers of the laser interaction with the result

$$\bar{G}^{(+)} = G^{(+)} + G^{(+)} H' G^{(+)} + G^{(+)} H' G^{(+)} H' G^{(+)} + \dots\tag{4.1.13}$$

where $G^{(+)}$ is the time dependent Green's function for all the particles in the absence of the laser and

$$H' = \omega \vec{\alpha}_0 \cdot \vec{P} \cos \omega t = \frac{\omega}{2} \vec{\alpha}_0 \cdot \vec{P} \sum_{a=\pm 1} e^{ia\omega t}\tag{4.1.14}$$

$$\vec{P} = \sum_{j=0}^{\infty} \vec{p}_j\tag{4.1.15}$$

Substitution of these results back into the S-matrix

(4.1.1) yields

$$S = S_0 + S_1 + S_2 + \dots \quad (4.1.16)$$

where

$$S_0 = -i \langle \Omega_f, \sqrt{A_0} \Omega_i \rangle \quad (4.1.17)$$

$$S_1 = -i \langle \Omega_f, \sqrt{A_0} G^{(+)} \sqrt{A_0} \Omega_i \rangle \quad (4.1.18)$$

$$S_2 = -i \langle \Omega_f, \sqrt{A_0} G^{(+)} H' G^{(+)} \sqrt{A_0} \Omega_i \rangle \quad (4.1.19)$$

and so on. We shall evaluate them individually.

Using (4.1.7) and (4.1.12) in (4.1.17) and performing the time integral we get

$$S_0 = -2\pi i \sum_e \delta(\epsilon_{pf} + \bar{w}_f - \epsilon_{pi} - \bar{w}_i - l\omega) T_e^{(0)} \quad (4.1.20)$$

where

$$T_e^{(0)} = \sum_{\lambda, s, s'} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-e-s'-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \langle \vec{p}_f(0), \phi_f^{(s')} | \sqrt{A_0} | \phi_i^{(s)}, \vec{p}_i(0) \rangle \quad (4.1.21)$$

The intermediate time integral in S_1 can be evaluated with the aid of the representation

$$G^{(+)}(t, t') = -i\theta(t-t') \lim_{\eta \rightarrow 0^+} \sum_m \psi_m \psi_m^* e^{-(\eta + i\varepsilon_m)(t-t')} \quad (4.1.22)$$

where $\{\psi_m\}$ is a complete set of normalized wave functions of the $(z+1)$ -electron problem and $\{\varepsilon_m\}$ are their eigenvalues.

Also

$$\begin{aligned}
 & \int G^{(+)}(t, t') e^{-iEt'} dt' \\
 &= -i \lim_{\eta \rightarrow 0+} \int_{-\infty}^t \sum_m \psi_m \psi_m^* e^{-(\eta + i\varepsilon_m)t} \int_{-\infty}^t e^{(\eta + i\varepsilon_m - iE)t'} dt' \\
 &= G^{(+)}(E) e^{-iEt}
 \end{aligned}
 \tag{4.1.23}$$

where

$$G^{(+)}(E) = \lim_{\eta \rightarrow 0+} \sum_m \frac{\psi_m \psi_m^*}{E + i\eta - \varepsilon_m}
 \tag{4.1.24}$$

is the time independent form of (4.1.22). Using the identity (4.1.23) we write

$$\begin{aligned}
 & G^{(+)} \vee \Omega_i \rangle \\
 &= \sum_{\lambda, s} T_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{-i(\varepsilon_{p_i} + \bar{w}_i - \lambda\omega)t} \\
 &\times G^{(+)}(\varepsilon_{p_i} + \bar{w}_i - \lambda\omega) \vee | \phi_i^{(s)}, \vec{p}_i(0) \rangle
 \end{aligned}
 \tag{4.1.25}$$

This is of the same form as (4.1.12). Thus we can immediately write down

$$S_1 = -2\pi i \sum_l \delta(\varepsilon_{p_f} + \bar{w}_f - \varepsilon_{p_i} - \bar{w}_i - l\omega) T_l^{(1)}
 \tag{4.1.26}$$

where

$$T_e^{(1)} = \sum_{\lambda, s, s'} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-s'-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \langle \vec{p}_f(0), \phi_f^{(s')} | \vee A_0 G^{(+)} (\epsilon_{p_i} + \bar{w}_i - \lambda\omega) \vee | \phi_i^{(s)}, \vec{p}_i(0) \rangle \quad (4.1.27)$$

Combining (4.1.21) and (4.1.27) we get

$$T_e^{(0)} + T_e^{(1)} = \sum_{\lambda, s, s'} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-s'-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \langle \vec{p}_f(0), \phi_f^{(s')} | T (\epsilon_{p_i} + \bar{w}_i - \lambda\omega) | \phi_i^{(s)}, \vec{p}_i(0) \rangle \quad (4.1.28)$$

where we have introduced the usual T-operator

$$T(E) = \vee A_0 (1 + G^{(+)}(E) \vee) \quad (4.1.29)$$

To evaluate S_2 use can be made of (4.1.25), (4.1.14) and (4.1.23). Thus

$$G^{(+)} H' G^{(+)} \vee \Omega i \rangle \\ = \left(\frac{\omega}{2}\right) \sum_{\substack{\lambda, s \\ x_1 = \pm 1}} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{-i(\epsilon_{p_i} + \bar{w}_i - \lambda\omega - x_1\omega)t} \\ \times G^{(+)} (\epsilon_{p_i} + \bar{w}_i - \lambda\omega - x_1\omega) \vec{\alpha}_0 \cdot \vec{p} G^{(+)} (\epsilon_{p_i} + \bar{w}_i - \lambda\omega) \vee \times \\ | \phi_i^{(s)}, \vec{p}_i(0) \rangle \quad (4.1.30)$$

Hence, finally

$$S_2 = -2\pi i \sum_{\ell} \delta(\epsilon_{p_f} + \bar{w}_f - \epsilon_{p_i} - \bar{w}_i - \ell\omega) T_e^{(2)} \quad (4.1.31)$$

and

$$\begin{aligned}
 T_L^{(2)} &= \left(\frac{\omega}{2}\right) \sum_{\lambda, s, s'} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-s'-x_1-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}_f(0), \phi_f^{(s')} | V A_0 G_L^{(+)} (\epsilon_{p_i} + \bar{W}_i - \lambda\omega - x_1\omega) \times \\
 &\vec{\alpha}_0 \cdot \vec{p} G_L^{(+)} (\epsilon_{p_i} + \bar{W}_i - \lambda\omega) V | \phi_i^{(s)}, \vec{p}_i(0) \rangle
 \end{aligned}
 \tag{4.1.32}$$

Obviously the general term is

$$\begin{aligned}
 T_L^{(n+1)} &= \left(\frac{\omega}{2}\right)^n \sum_{\substack{\lambda, s, s' \\ x_1 \dots x_n}} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-s'-\sum_1^n x_j - \lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}_f(0), \phi_f^{(s')} | V A_0 G_L^{(+)} (\epsilon_{p_i} + \bar{W}_i - \lambda\omega - \sum_1^n x_j \omega) \vec{\alpha}_0 \cdot \vec{p} \times \\
 &G_L^{(+)} (\epsilon_{p_i} + \bar{W}_i - \lambda\omega - \sum_1^{n-1} x_j \omega) \vec{\alpha}_0 \cdot \vec{p} \dots \vec{\alpha}_0 \cdot \vec{p} \times \\
 &G_L^{(+)} (\epsilon_{p_i} + \bar{W}_i - \lambda\omega) V | \phi_i^{(s)}, \vec{p}_i(0) \rangle
 \end{aligned}
 \tag{4.1.33}$$

It may be noted that the expressions for $T_L^{(0)}$, $T_L^{(1)}$ and $T_L^{(n+1)}$ are similar in form to their counterparts in potential scattering.

In what follows we will evaluate the T-matrix for the absorption of l photons to order ω . This would allow us to replace the dressed energies $\bar{W}_i$ by the corresponding bare energies W_i since the shift, the dynamical Stark effect is of order ω^2 . Expressions for the states ϕ_i can be obtained (see appendix C) in terms of the bare states U_i by using first order time dependent perturbation theory treating the laser-atom coupling as the perturbing term. The result

is

$$\begin{aligned} & \phi_f^i(r_1, \dots, r_z, \omega t) \\ &= (1 - i m \omega \vec{\alpha}_0 \cdot \vec{Q} \cos \omega t) \psi_f^i(r_1, \dots, r_z) + O(\omega^2) \end{aligned} \quad (4.1.34)$$

where

$$\vec{Q} = \sum_{j=1}^z \vec{r}_j \quad (4.1.35)$$

Substituting this into (4.1.5) and (4.1.11) and inverting those relations we get

$$\begin{aligned} & \phi_f^i(s) \\ &= \left(\delta_{s,0} - \frac{i m \omega}{2} \vec{\alpha}_0 \cdot \vec{Q} \sum_{a=\pm 1} \delta_{s+a,0} \right) \psi_f^i + O(\omega^2) \end{aligned} \quad (4.1.36)$$

Because of the Kronecker delta functions in the above expression the sum over s and s' are trivial. Thus (4.1.28) would reduce to

$$\begin{aligned} T_\ell^{(0)} + T_\ell^{(1)} &= \sum_\lambda J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ & \langle \vec{p}_f(0), \psi_f | T(\epsilon_{p_i} + W_i - \lambda \omega) + \\ & i T(\epsilon_{p_i} + W_i - \lambda \omega) \frac{m \omega \lambda \vec{\alpha}_0 \cdot \vec{Q}}{\vec{\alpha}_0 \cdot \vec{p}_i} \\ & - \frac{i m \omega (\lambda + \ell) \vec{\alpha}_0 \cdot \vec{Q}}{\vec{\alpha}_0 \cdot \vec{p}_f} T(\epsilon_{p_i} + W_i - \lambda \omega) | \psi_i, \vec{p}_i(0) \rangle \\ & + O(\omega^2) \end{aligned} \quad (4.1.37)$$

Since $T_\ell^{(2)}$ has a factor of ω in front, we can

approximate $\phi_i^{(s)}$ as

$$\phi_f^{(s)} = \delta_{s,0} U_f + O(\omega) \quad (4.1.38)$$

Substituting and performing the sum over s and s' in (4.1.32) we get

$$\begin{aligned} T_2^{(2)} &= \frac{\omega}{2} \sum_{\lambda, x_1 = \pm 1} J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-2-x_1-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ &< \vec{p}_f(0), U_f | \vee A_0 G^{(+)} (\epsilon_{p_i} + W_i - \lambda \omega - x_1 \omega) \times \\ &\vec{\alpha}_0 \cdot \vec{p} G^{(+)} (\epsilon_{p_i} + W_i - \lambda \omega) \vee | U_i, \vec{p}_i(0) \rangle + O(\omega^2) \end{aligned} \quad (4.1.39)$$

This can be simplified by using a generalization of the identity (2.1.21):

$$\begin{aligned} &G^{(+)}(E_1) \vec{\alpha}_0 \cdot \vec{p} G^{(+)}(E_2) \\ &= i m \left[(E_1 - E_2) G^{(+)}(E_1) \vec{\alpha}_0 \cdot \vec{R} G^{(+)}(E_2) \right. \\ &\quad \left. + G^{(+)}(E_1) \vec{\alpha}_0 \cdot \vec{R} - \vec{\alpha}_0 \cdot \vec{R} G^{(+)}(E_2) \right] \end{aligned} \quad (4.1.40)$$

where

$$\vec{R} = \sum_{j=0}^Z \vec{r}_j = \vec{Q} + \vec{r}_0 \quad (4.1.41)$$

Obviously $\vec{R}$ commutes with A_0 :

$$[A_0, \vec{R}] = 0 \quad (4.1.42)$$

We drop the term containing the product of two Green's

functions from the right side of (4.1.40) (see the discussion following (2.1.21)). Using (4.1.40) in (4.1.39) and proceeding as in chapter 2 we get

$$\begin{aligned}
 T_l^{(2)} = & \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \langle \vec{p}_f(0), U_f | \frac{i m \omega (\lambda + l)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{R} T(\epsilon_{p_i} + \omega_i - \lambda \omega) \\
 & - \frac{i m \omega \lambda}{\vec{\alpha}_0 \cdot \vec{p}_i} T(\epsilon_{p_i} + \omega_i - \lambda \omega) \vec{\alpha}_0 \cdot \vec{R} | U_i, \vec{p}_i(0) \rangle \\
 & + O(\omega^2)
 \end{aligned} \tag{4.1.43}$$

In arriving at (4.1.43) we used the relation (4.1.29) to write

$$V A_0 \mathcal{U}^{(+)}(E) V = T(E) - V A_0$$

and exploited (4.1.42) along with the fact that

$$[V, \vec{R}] = 0 \tag{4.1.44}$$

since V is a local potential.

For non-resonant electron-atom scattering to order ω , the T-matrix for the absorption of l photons is

$$T_l = T_l^{(0)} + T_l^{(1)} + T_l^{(2)} + O(\omega^2) \tag{4.1.45}$$

Adding (4.1.37) and (4.1.43) and observing that $\vec{R} - \vec{Q} = \vec{r}_0$, we get

$$\begin{aligned}
 T_l &= \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}_f(0), U_f | T (\epsilon_{p_i} + W_i - \lambda \omega) + \\
 &\quad \frac{i m \omega (\lambda + l)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{r}_0 T (\epsilon_{p_i} + W_i - \lambda \omega) \\
 &\quad - \frac{i m \omega \lambda}{\vec{\alpha}_0 \cdot \vec{p}_i} T (\epsilon_{p_i} + W_i - \lambda \omega) \vec{\alpha}_0 \cdot \vec{r}_0 | U_i, \vec{p}_i(0) \rangle + O(\omega^2) \\
 &= \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\quad \langle \vec{p}_f - \frac{m \omega (\lambda + l) \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f}, U_f | T (\epsilon_{p_i} + W_i - \lambda \omega) | U_i, \vec{p}_i - \frac{m \omega \lambda \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \rangle \\
 &\quad + O(\omega^2) \tag{4.1.46}
 \end{aligned}$$

The energy conserving delta function with the relation

$\bar{W}_i = W_i + O(\omega^2)$ gives

$$\epsilon_{p_f} + W_f = \epsilon_{p_i} + W_i + l \omega + O(\omega^2) \tag{4.1.47}$$

With the use of (4.1.47) one can easily show the on-shell character of the T-matrix in (4.1.46). That is

$$\begin{aligned}
 &\frac{1}{2m} \left| \vec{p}_f - \frac{m \omega (\lambda + l) \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f} \right|^2 + W_f \\
 &= \epsilon_{p_i} + W_i - \lambda \omega + O(\omega^2) \\
 &= \frac{1}{2m} \left| \vec{p}_i - \frac{m \omega \lambda \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \right|^2 + W_i + O(\omega^2) \tag{4.1.48}
 \end{aligned}$$

The λ sums can be performed as in chapter 2 yielding

$$T_e = J_{-L} (\vec{\alpha}_0 \cdot \vec{\delta p}) \times$$

$$\left\langle \vec{p}_f - \frac{m\omega(\lambda_0 + L)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f}, u_f \right| T (W_i + \epsilon_{p_i} - \lambda_0 \omega) \left| u_i, \vec{p}_i - \frac{m\omega\lambda_0\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \right\rangle$$

$$+ O(\omega^2)$$

(4.1.49)

where

$$\lambda_0 = \frac{L \vec{\alpha}_0 \cdot \vec{p}_i}{\vec{\alpha}_0 \cdot \vec{\delta p}}$$

(1.4.6)

Since the matrix element

$$\left\langle \vec{p}_f - \frac{m\omega(\lambda + L)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f}, u_f \right| T (W_i + \epsilon_{p_i} - \lambda \omega) \left| u_i, \vec{p}_i - \frac{m\omega\lambda\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \right\rangle$$

is on-shell for all values of λ (see equation (4.1.48)), it is so for $\lambda = \lambda_0$ also. Equation (4.1.49) can be rewritten as

$$T_e (\vec{p}_f, u_f; \vec{p}_i, u_i)$$

$$= J_{-L} (\vec{\alpha}_0 \cdot \vec{\delta p}) \left\langle \vec{p}_f - \vec{\delta}_{\lambda_0}, u_f \right| T (W_i + \epsilon_{\vec{p}_i - \vec{\delta}_{\lambda_0}}) \left| u_i, \vec{p}_i - \vec{\delta}_{\lambda_0} \right\rangle$$

$$+ O(\omega^2)$$

(4.1.50)

where

$$\vec{\delta}_{\lambda_0} = \frac{m\omega L \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{\delta p}}$$

(1.4.7)

This is the appropriate generalization of the Kroll-Watson result (1.4.9) to the case in which the target is an atom and undergoes a transition from state 'i' to state 'f' during the scattering process.

In view of the simplicity of the above result, the following comment can be made. When one goes from potential scattering to electron-atom scattering, two additional sources of correction of order ω do in fact, arise in the T-matrix. The first comes from the correction to the exchange potential of order ω (there is no correction to the direct potential of this order). The second comes from the translation of momenta that was necessary to get the Kroll-Watson result. It is well known that optical potentials are non-local. When the momenta are shifted to get the Kroll-Watson result, the zero order (in ω) part of the optical potential introduces a correction term of order ω . These two correction terms exactly cancel each other¹⁴ even though it is not manifest in the above treatment.

4.2 Evaluation of the sum rules:

Using the T-matrix given by (4.1.50) one constructs the following differential cross section

$$\frac{d\sigma_{fi}(p_f, p_i, l)}{d\Omega} = J_e^2(z) \frac{|\vec{p}_f| |\vec{Q}|}{|\vec{Q}'| |\vec{p}_i|} \frac{d\sigma_{fi}(\vec{Q}', \vec{Q})}{d\Omega} \quad (4.2.1)$$

The subscripts 'i' and 'f' in the above equation stand for the initial and final state respectively of the atomic target. The cross section appearing on the left side of (4.2.1) is that in the presence of the laser in which $\vec{p}_i$ and $\vec{p}_f$ are respectively the initial and final momenta of the projectile and l is the net number of photons absorbed during the scattering process. The cross section on the right side of (4.2.1) is that in the absence of the laser, but the momenta $\vec{Q}$ and $\vec{Q}'$ are shifted from $\vec{p}_i$ and $\vec{p}_f$ by a vector of order ω :

$$\begin{aligned} \vec{Q} &= \vec{p}_i + m\omega l \vec{\alpha}_0 / z \\ \vec{Q}' &= \vec{p}_f + m\omega l \vec{\alpha}_0 / z \end{aligned} \quad (4.2.2)$$

$$z = x_0 - \gamma; \quad x_0 = \vec{\alpha}_0 \cdot \vec{p}_i; \quad \gamma = \vec{\alpha}_0 \cdot \vec{p}_f \quad (4.2.3)$$

The energy conservation equation is

$$p_f^2(l)/2m = p_i^2/2m + W_i - W_f + l\omega \quad (4.2.4)$$

Therefore,

$$p_f^2(0)/2m = p_i^2/2m + W_i - W_f \quad (4.2.5)$$

(Note: This definition of $p_f(0)$ should not be confused

with the definition for the ket vector $|\vec{p}_f(0)\rangle$ (see equation (4.1.3)). We define

$$\vec{p}_f(0) = \hat{p}_f \quad ; \quad \gamma_0 = \vec{\alpha}_0 \cdot \vec{p}_f(0) \quad (4.2.6)$$

$$Z_0 = X_0 - \gamma_0 \quad (4.2.7)$$

Using (4.2.2) and (4.2.4) it can be shown that

$$Q_{/2m}^2 + W_i = Q_{/2m}'^2 + W_f = E \text{ (say)} \quad (4.2.8)$$

Also, (4.2.5) can be written as

$$p_i^2 / 2m + W_i = p_f^2(0) / 2m + W_f = E \text{ (say)} \quad (4.2.9)$$

This is the energy conservation equation for field-free scattering in which the momenta of the incident and scattered electrons are $\vec{p}_i$ and $\vec{p}_f(0) = \hat{p}_f p_f(0)$ respectively and the atom undergoes a transition from state 'i' to state 'f'. Rewriting $\frac{d\sigma_{fi}}{d\Omega}(\vec{q}', \vec{q})$ as $\frac{d\sigma_{fi}}{d\Omega}(\epsilon, \hat{q}', \hat{q})$ and expanding in powers of ω , we obtain

$$\begin{aligned} \frac{d\sigma_{fi}}{d\Omega}(\vec{q}', \vec{q}) &= \frac{d\sigma_{fi}}{d\Omega}(\epsilon, \mu) + \\ &+ \frac{m\omega\ell}{Z_0} \left[\frac{X_0 + \gamma_0}{p_i \cdot p_f(0)} - \mu \left(\frac{X_0}{p_i^2} + \frac{\gamma_0}{p_f^2(0)} \right) \right] \frac{\partial}{\partial \mu} \frac{d\sigma_{fi}}{d\Omega}(\epsilon, \mu) \\ &+ \frac{\omega\ell\vec{\alpha}_0 \cdot \vec{p}_i}{Z} \frac{\partial}{\partial \epsilon} \frac{d\sigma_{fi}}{d\Omega}(\epsilon, \mu) + O(\omega^2) \end{aligned} \quad (4.2.10)$$

where

$$\mu = \hat{p}_f \cdot \hat{p}_i \quad (2.4.12)$$

Also,

$$J_\ell^2(z) = J_\ell^2(z_0) - \frac{2m\omega\ell}{p_f^2(0)} \gamma_0 J_\ell(z_0) J_\ell'(z_0) + O(\omega^2) \quad (4.2.11)$$

and

$$\frac{|\vec{p}_f||\vec{q}|}{|\vec{q}'||\vec{p}_i|} = 1 + \frac{m\omega\ell}{z_0} \left(\frac{\chi_0}{p_i^2} - \frac{\gamma_0}{p_f^2(0)} \right) + O(\omega^2) \quad (4.2.12)$$

Substituting in (4.2.1) and performing the sum over ℓ by the use of the Bessel function sum rules

$$\sum_\ell \begin{pmatrix} 1 \\ \ell \\ \ell^2 \end{pmatrix} J_\ell^2(z_0) = \begin{pmatrix} 1 \\ 0 \\ z_0^2/2 \end{pmatrix} \quad (A-5)$$

we arrive at the following sum rule for the differential cross sections

$$\sum_\ell \frac{d\sigma_{fi}}{d\Omega}(\vec{p}_f, \vec{p}_i, \ell) = \frac{d\sigma_{fi}}{d\Omega}(\epsilon, \mu) + O(\omega^2) \quad (4.2.13)$$

This is a simple generalization of a previous sum rule (1.4.10) for non-resonant potential scattering. This means that for an electron-impact excitation of the atomic target from the initial state 'i' to a final state 'f' in the presence of a laser, the cross section for all

possible values of ℓ sums to the cross section in the absence of the laser. If the final state of the target is not monitored, both sides of the equation (4.2.13) will have to be summed over 'f'.

For electron-atom scattering, $\langle \ell \rangle$ is defined as

$$\langle \ell \rangle = \frac{\sum_f \sum_{\ell} \ell \int \frac{d\sigma_{fi}}{d\Omega} (\vec{p}_f, \vec{p}_i, \ell) d\Omega_{p_f}}{\sum_f \sum_{\ell} \int \frac{d\sigma_{fi}}{d\Omega} (\vec{p}_f, \vec{p}_i, \ell) d\Omega_{p_f}} \quad (4.2.14)$$

It can be shown that

$$\begin{aligned} & \sum_{\ell} \ell \frac{d\sigma_{fi}}{d\Omega} (\vec{p}_f, \vec{p}_i, \ell) \\ &= \frac{\omega}{2} z_0 x_0 \frac{\partial}{\partial \epsilon} \frac{d\sigma_{fi}}{d\Omega} (\epsilon, \mu) + \\ & \quad \frac{m\omega}{2} z_0 \left[x_0 \left(\frac{1}{p_i p_f(0)} - \frac{\mu}{p_i^2} \right) + \right. \\ & \quad \left. \gamma_0 \left(\frac{1}{p_i p_f(0)} - \frac{\mu}{p_f^2(0)} \right) \right] \frac{\partial}{\partial \mu} \frac{d\sigma_{fi}}{d\Omega} (\epsilon, \mu) \\ & + \frac{m\omega}{2} z_0 \left[\frac{x_0}{p_i^2} - \frac{3\gamma_0}{p_f^2(0)} \right] \frac{d\sigma_{fi}}{d\Omega} (\epsilon, \mu) \end{aligned} \quad (4.2.15)$$

Substituting and performing the integral over the final direction of the scattering by the use of appendix B, we get

$$\begin{aligned} \langle L \rangle = & \left(\sum_f \sigma_{fi}(\epsilon) \right)^{-1} \left[m\omega\alpha_0^2 (\hat{\alpha}_0 \cdot \hat{p}_i)^2 \epsilon_{pi} \frac{\partial}{\partial \epsilon} \sum_f \sigma_{fi,\delta}(\epsilon) \right. \\ & + m\omega\alpha_0^2 \sum_f \left\{ (1 + (\hat{\alpha}_0 \cdot \hat{p}_i)^2) \sigma_{fi,\delta}(\epsilon) \right. \\ & \left. \left. + \left(\frac{W_{if}}{\epsilon_{pf(0)}} \right)^2 (\hat{\alpha}_0 \cdot \hat{p}_i)^2 (\sigma_{fi}(\epsilon) - \sigma_{fi,\delta}(\epsilon)) \right\} \right] \end{aligned} \quad (4.2.16)$$

where

$$\sigma_{fi}(\epsilon) = \int \frac{d\sigma_{fi}(\epsilon, \mu)}{d\Omega} d\Omega_{pf} \quad (4.2.17)$$

$$\sigma_{fi,\delta}(\epsilon) = \int (1 - \mu \frac{p_{f(0)}}{p_i}) \frac{d\sigma_{fi}(\epsilon, \mu)}{d\Omega} d\Omega_{pf} \quad (4.2.18)$$

If only elastic channels are open, then 'f'='i' and

$$\begin{aligned} \langle L \rangle_{\text{elastic}} = & \frac{1}{\sigma_{ii}(\epsilon)} \left[m\omega\alpha_0^2 (\hat{\alpha}_0 \cdot \hat{p}_i)^2 \epsilon_{pi} \frac{\partial}{\partial \epsilon} \sigma_{ii,\delta}(\epsilon) \right. \\ & \left. + \frac{m\omega\alpha_0^2}{2} \left\{ (1 + (\hat{\alpha}_0 \cdot \hat{p}_i)^2) \sigma_{ii,\delta}(\epsilon) \right\} \right] \end{aligned} \quad (4.2.19)$$

where

$$\begin{aligned} & \sigma_{ii,\delta}(\epsilon) \\ = & \int (1 - \mu) \frac{d\sigma_{ii}(\epsilon, \mu)}{d\Omega} d\Omega_{pf} \end{aligned} \quad (4.2.20)$$

and

$$\sigma_{ii}(\epsilon, \mu) = \int \frac{d\sigma_{ii}(\epsilon, \mu)}{d\Omega} d\Omega_{pf} \quad (4.2.21)$$

CHAPTER 5

RESONANT ELECTRON-ATOM SCATTERING IN THE PRESENCE OF A LOW-FREQUENCY LASER

5.1 Derivation of the T-matrix:

Proceeding as in the previous chapter we find

$$T_L^{(0)} + T_L^{(1)} = \sum_{\lambda, s, s'} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-s'-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \langle \vec{p}_f(0), \phi_f^{(s')} | T (\epsilon_{p_i} + \bar{W}_i - \lambda \omega) | \phi_i^{(s)}, \vec{p}_i(0) \rangle \quad (4.1.28)$$

and

$$T_L^{(n+1)} = \left(\frac{\omega}{2}\right)^n \sum_{\substack{\lambda, s, s' \\ x_1 \dots x_n}} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-s'-\sum_1^n x_j - \lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \langle \vec{p}_f(0), \phi_f^{(s')} | V A_0 G_L^{(+)} (\epsilon_{p_i} + \bar{W}_i - \lambda \omega - \sum_1^n x_j \omega) \vec{\alpha}_0 \cdot \vec{p} \times \\ G_L^{(+)} (\epsilon_{p_i} + \bar{W}_i - \lambda \omega - \sum_1^{n-1} x_j \omega) \vec{\alpha}_0 \cdot \vec{p} \dots \vec{\alpha}_0 \cdot \vec{p} \times \\ G_L^{(+)} (\epsilon_{p_i} + \bar{W}_i - \lambda \omega) V | \phi_i^{(s)}, \vec{p}_i(0) \rangle \quad (4.1.33)$$

The energy conservation equation is

$$\epsilon_{p_f} + \bar{W}_f = \epsilon_{p_i} + \bar{W}_i + \ell \omega \quad (5.1.1)$$

where $\vec{p}_i$ ($\vec{p}_f$) is the initial (final) momentum of the projectile, $\bar{W}_i$ ($\bar{W}_f$) is the dressed energy of the initial (final) state of the atomic target and ℓ is the net number of laser photons absorbed during the scattering process. $\phi_{i,f}^{(s)}$ are wave functions related to the bare atomic states $U_i^{(s)}$ in the following way:

$$\phi_{if}^{(s)} = (\delta_{f,0} - im\omega/2 \vec{\alpha}_0 \cdot \vec{Q} \sum_{x=\pm 1} \delta_{s+x,0}) U_{if} + O(\omega^2) \quad (4.1.36)$$

$$\vec{Q} = \sum_{j=1}^Z \vec{r}_j \quad (4.1.35)$$

$G^{(+)}(E)$ is the time independent Green's function of the projectile-target system in the absence of the laser.

Let us assume the presence of an isolated resonance in $G^{(+)}$ at the energy $E_R = \epsilon_R - i\Gamma/2$ such that $\epsilon_R \approx \epsilon_0 = \epsilon_{pi} + \bar{W}_i + m_0\omega$ where m_0 is an integer. Mittleman²⁸ has shown that the Green's function $G^{(+)}(E)$ can be written as $G^{(+)}(E) = G_R^{(+)}(E) + G_{NR}^{(+)}(E)$ where $G_{NR}^{(+)}(E)$ is a slowly varying function of E and

$$G_R^{(+)}(E) = \frac{U_R \langle U_R}{E - E_R}$$

U_R is interpreted as the resonance wave function of the electron-atom system. Using the above form for $G^{(+)}(E)$ one can write the total T-operator as

$$T(E) = T_R(E) + T_{NR}(E)$$

with

$$T_R(E) = V G_R^{(+)}(E) V$$

$$T_{NR}(E) = V + V G_{NR}^{(+)}(E) V$$

(5.1.2)

The definition (5.1.2) allows us to rewrite (4.1.28) as

$$\begin{aligned} T_R^{(0)} + T_R^{(1)} &= \sum_{\lambda, s, s'} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-s'-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ &\langle \vec{p}_f(0), \phi_f^{(s')} | T_{NR}(\epsilon_{p_i} + \vec{w}_i - \lambda \omega) | \phi_i^{(s)}, \vec{p}_i(0) \rangle + \\ &\sum_{s, s'} J_{m_0-s} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-s'} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ &\langle \vec{p}_f(0), \phi_f^{(s')} | T_R(E_0) | \phi_i^{(s)}, \vec{p}_i(0) \rangle \\ &+ \sum_{\lambda \neq -m_0} \sum_{s, s'} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-s'-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ &\langle \vec{p}_f(0), \phi_f^{(s')} | T_R(\epsilon_{p_i} + \vec{w}_i - \lambda \omega) | \phi_i^{(s)}, \vec{p}_i(0) \rangle \end{aligned} \quad (5.1.3)$$

The presence of resonance introduces a new quantity

Γ , the resonance width. For a typical Feshbach resonance Γ is of the order 5×10^{-3} ev. Also, for a CO_2 laser, $\omega \sim 0.1$ ev. In the following, therefore, we will consider both $\frac{\omega}{\epsilon_0}$ and $\frac{\Gamma}{\omega}$ as quantities of first order of smallness and drop all higher order terms.

We note that for $E - \epsilon_0 = M\omega$, $M \neq 0$, the following expansion can be made for the individual T-operators.

$$T_{NR}(E) = T_{NR}(\epsilon_0) + M\omega \frac{\partial}{\partial \epsilon_0} T_{NR}(\epsilon_0) + O(\omega^2) \quad (3.1.5)$$

$$T_R(E) = \frac{i\Gamma}{2M\omega} T_R(\epsilon_0) + O\left(\frac{\Gamma^2}{M^2\omega^2}\right) \quad (3.1.6)$$

Also the expectation values of the atomic coordinates $\bar{r}_j$ are of order a_0 , the Bohr radius. So, $|\vec{Q}| \approx za_0$. α_0 is fixed at a value near a_0 . Thus the quantity $m\omega \vec{\alpha}_0 \cdot \vec{Q}$ appearing in the expressions for $\phi_i^{(s)}$ and $\phi_f^{(s')}$ (see equation (4.1.36)) contributes an effective factor of order $\omega/R\gamma$.

We use (4.1.36) in (5.1.3) and to first order of smallness in the expansion parameters write (5.1.3) as

$$\begin{aligned} T_e^{(0)} + T_e^{(1)} &= \sum_{\lambda, s, s'} J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\lambda-s'-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ &\left[\delta_{s',0} \delta_{s,0} \langle \vec{p}_f(0), U_f | T_{NR}(\epsilon_{p_i} + \vec{w}_i - \lambda\omega) | U_i, \vec{p}_i(0) \rangle \right. \\ &+ (im\omega/2) \sum_{x_1=\pm 1} \delta_{s'+x_1,0} \delta_{s,0} \times \\ &\langle \vec{p}_f(0), U_f | \vec{\alpha}_0 \cdot \vec{Q} T_{NR}(\epsilon_{p_i} + \vec{w}_i - \lambda\omega) | U_i, \vec{p}_i(0) \rangle \\ &- (im\omega/2) \sum_{x_2=\pm 1} \delta_{s+x_2,0} \delta_{s',0} \times \\ &\left. \langle \vec{p}_f(0), U_f | T_{NR}(\epsilon_{p_i} + \vec{w}_i - \lambda\omega) \vec{\alpha}_0 \cdot \vec{Q} | U_i, \vec{p}_i(0) \rangle \right] + \end{aligned}$$

$$\begin{aligned}
 & \sum_{s,s'} J_{m_0-s} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l-s'} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \left[\delta_{s',0} \delta_{s,0} \langle \vec{p}_f(0), U_f | T_R(\epsilon_0) | U_i, \vec{p}_i(0) \rangle \right. \\
 & + (i m \omega / 2) \sum_{\lambda_1 = \pm 1} \delta_{s'+\lambda_1,0} \delta_{s,0} \times \\
 & \langle \vec{p}_f(0), U_f | \vec{\alpha}_0 \cdot \vec{Q} T_R(\epsilon_0) | U_i, \vec{p}_i(0) \rangle \\
 & - (i m \omega / 2) \sum_{\lambda_2 = \pm 1} \delta_{s+\lambda_2,0} \delta_{s',0} \times \\
 & \left. \langle \vec{p}_f(0), U_f | T_R(\epsilon_0) \vec{\alpha}_0 \cdot \vec{Q} | U_i, \vec{p}_i(0) \rangle \right] \\
 & - (i \Gamma / 2 \omega) \sum_{\substack{\lambda \neq -m_0 \\ s, s'}} \frac{J_{\lambda+s} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-s'-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f)}{m_0 + \lambda} \times \\
 & \delta_{s',0} \delta_{s,0} \langle \vec{p}_f(0), U_f | T_R(\epsilon_0) | U_i, \vec{p}_i(0) \rangle \quad (5.1.4)
 \end{aligned}$$

Performing the sum over all indices but λ , we get

$$\begin{aligned}
 T_e^{(0)} + T_e^{(1)} &= \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \langle \vec{p}_f(0), U_f | \left\{ T_{NR}(\epsilon_{p_i} + \bar{w}_i - \lambda \omega) \right. \\
 & + i T_{NR}(\epsilon_{p_i} + \bar{w}_i - \lambda \omega) \frac{m \omega \lambda \vec{\alpha}_0 \cdot \vec{Q}}{\vec{\alpha}_0 \cdot \vec{p}_i} \\
 & - \frac{i m \omega (\lambda + l) \vec{\alpha}_0 \cdot \vec{Q}}{\vec{\alpha}_0 \cdot \vec{p}_f} T_{NR}(\epsilon_{p_i} + \bar{w}_i - \lambda \omega) \left. \right\} | U_i, \vec{p}_i(0) \rangle \\
 & + J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l} (\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f(0), U_f | \left\{ T_R(\epsilon_0) \right. \\
 & - i T_R(\epsilon_0) \frac{m \omega m_0 \vec{\alpha}_0 \cdot \vec{Q}}{\vec{\alpha}_0 \cdot \vec{p}_i} +
 \end{aligned}$$

$$i m \omega (m_0 - e) \frac{\vec{\alpha}_0 \cdot \vec{Q}}{\vec{\alpha}_0 \cdot \vec{p}_f} T_R(\epsilon_0) \left\{ |U_i, \vec{p}_i(0)\rangle \right. \\ \left. - (i \tau_{1/2} \omega) g_{m_0, e} \langle \vec{p}_f(0), U_f | T_R(\epsilon_0) | U_i, \vec{p}_i(0) \rangle \right\} \quad (5.1.5)$$

where

$$g_{m_0, e} \\ = \sum_{\lambda \neq -m_0} \frac{J_\lambda(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-e-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f)}{\lambda + m_0} \quad (3.2.28)$$

From the general expression (4.1.33)

$$T_L^{(2)} = \left(\frac{\omega}{2}\right) \sum_{\substack{\lambda, s, s' \\ \lambda_1 = \pm 1}} J_{\lambda+s}(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-e-s'-\lambda_1-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \langle \vec{p}_f(0), \phi_f^{(s')} | \sqrt{A_0} G^{(+)}(\epsilon_{p_i} + \bar{w}_i - \lambda\omega - \chi_1\omega) \vec{\alpha}_0 \cdot \vec{p} \times \\ G^{(+)}(\epsilon_{p_i} + \bar{w}_i - \lambda\omega) \sqrt{1} \phi_i^{(s)}, \vec{p}_i(0) \rangle + O(\omega^2) \quad (5.1.6)$$

Since $T_L^{(2)}$ has a factor of ω in front, $\phi_i^{(s)}$ can be approximated as

$$\phi_f^{(s)} = \delta_{s,0} U_f^{(s)} + O(\omega) \quad (4.1.38)$$

Substituting and performing the sum over s and s' in

(5.1.6) one gets

$$T_L^{(2)} = \left(\frac{\omega}{2}\right) \sum_{\lambda, \lambda_1 = \pm 1} J_\lambda(-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-e-\lambda_1-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \langle \vec{p}_f(0), U_f | \sqrt{A_0} G^{(+)}(\epsilon_{p_i} + \bar{w}_i - \lambda\omega - \chi_1\omega) \vec{\alpha}_0 \cdot \vec{p} \times \\ G^{(+)}(\epsilon_{p_i} + \bar{w}_i - \lambda\omega) \sqrt{1} U_i, \vec{p}_i(0) \rangle + O(\omega^2) \quad (5.1.7)$$

This can be simplified by the use of the identity (4.1.40). However, the term containing the product of two Green's functions can be dropped from the right side of (4.1.40) for reasons analogous to those used in the case of resonant potential scattering (see the discussion following equation (3.2.1)). After some algebra one can write

$$\begin{aligned}
 T_e^{(2)} = & \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \langle \vec{p}_f(0), U_f | \frac{i m \omega (\lambda + \ell)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{R} T (\epsilon_{p_i} + \bar{w}_i - \lambda \omega) \\
 & - T (\epsilon_{p_i} + \bar{w}_i - \lambda \omega) \frac{i m \omega \lambda}{\vec{\alpha}_0 \cdot \vec{p}_i} \vec{\alpha}_0 \cdot \vec{R} | U_i, \vec{p}_i(0) \rangle
 \end{aligned}
 \tag{5.1.8}$$

$$\vec{R} = \sum_{j=0}^{\infty} \vec{r}_j = \vec{a} + \vec{r}_0
 \tag{4.1.41}$$

Since the factors appearing at the flanks in (5.1.8) are of order ω/k_f , one can ignore $T_R (\epsilon_{p_i} + \bar{w}_i - \lambda \omega)$ for $\lambda \neq -m_0$. Thus $T_e^{(2)}$ can be written as

$$\begin{aligned}
 T_e^{(2)} = & \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \langle \vec{p}_f(0), U_f | \frac{i m \omega (\lambda + \ell)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{R} T_{NR} (\epsilon_{p_i} + \bar{w}_i - \lambda \omega) \\
 & - \frac{i m \omega \lambda}{\vec{\alpha}_0 \cdot \vec{p}_i} T_{NR} (\epsilon_{p_i} + \bar{w}_i - \lambda \omega) \vec{\alpha}_0 \cdot \vec{R} | U_i, \vec{p}_i(0) \rangle \\
 & + J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-\ell} (\vec{\alpha}_0 \cdot \vec{p}_f) \times
 \end{aligned}$$

$$\begin{aligned} & \langle \vec{p}_f(0), U_f | \frac{i m \omega m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} T_R(\epsilon_0) \vec{\alpha}_0 \cdot \vec{R} \\ & - \frac{i m \omega (m_0 - l)}{\vec{\alpha}_0 \cdot \vec{p}_f} T_R(\epsilon_0) | U_i, \vec{p}_i(0) \rangle \end{aligned} \quad (5.1.9)$$

Of course for $n \geq 2$ only resonant contributions of $T_L^{(n+1)}$ are to be considered. We simply quote results. For details the reader is referred to similar calculations for resonant potential scattering. Thus for electron-atom scattering, equations analogous to (3.2.10) and (3.2.13) are

$$D = \frac{\omega^2}{2} \langle U_R | \vec{\alpha}_0 \cdot \vec{P} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{P} | U_R \rangle \quad (5.1.10)$$

and

$$\sum_{j=1}^{\infty} T_L^{(2j+1)} = J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f) \times$$

$$\begin{aligned} & \left[\langle \vec{p}_f(0), U_f | T_R(\epsilon_0 - D) - T_R(\epsilon_0) | U_i, \vec{p}_i(0) \rangle \right. \\ & + \frac{\omega^2 m_0 (m_0 - l)}{(\vec{\alpha}_0 \cdot \vec{p}_i)(\vec{\alpha}_0 \cdot \vec{p}_f)} \langle \vec{p}_f(0), U_f | \sqrt{A_0} G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{P} \times \\ & \left. G_{NR}^{(+)}(\epsilon_0 - D) \vec{\alpha}_0 \cdot \vec{P} G_{NR}^{(+)}(\epsilon_0) V | U_i, \vec{p}_i(0) \rangle \right] \end{aligned} \quad (5.1.11)$$

Similarly the equation analogous to (3.2.21) is

$$\begin{aligned} & \sum_{j=1}^{\infty} T_L^{(2j+2)} = J_{m_0}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ & \left[\frac{i m \omega m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \langle \vec{p}_f(0), U_f | (T_R(\epsilon_0 - D) - T_R(\epsilon_0)) \vec{\alpha}_0 \cdot \vec{R} | U_i, \vec{p}_i(0) \rangle \right. \end{aligned}$$

$$\frac{-i m \omega (m_0 - 1)}{\vec{\alpha}_0 \cdot \vec{p}_f} \langle \vec{p}_f(0), U_f | \vec{\alpha}_0 \cdot \vec{R} (T_R(\epsilon_0 - D) - T_R(\epsilon_0)) | U_i, \vec{p}_i(0) \rangle \quad (5.1.12)$$

Finally,

$$T_e(\vec{p}_f, U_f; \vec{p}_i, U_i) = T_e^{(0)} + T_e^{(1)} + T_e^{(2)} + \sum_{j=1}^{\infty} T_e^{(2j+1)} + \sum_{j=1}^{\infty} T_e^{(2j+2)} \quad (5.1.13)$$

Using (5.1.5), (5.1.9), (5.1.11) and (5.1.12) one finds that the coefficient of $J_{m_i}(\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_f}(\vec{\alpha}_0 \cdot \vec{p}_f)$ in (5.1.13) is

$$\begin{aligned} & \langle \vec{p}_f(0), U_f | T_R(\epsilon_0 - D) - i T_R(\epsilon_0) \frac{m \omega m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \vec{\alpha}_0 \cdot \vec{q} \\ & + \frac{i m \omega (m_0 - 1)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{q} T_R(\epsilon_0) \\ & + \frac{i m \omega m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} T_R(\epsilon_0 - D) \vec{\alpha}_0 \cdot \vec{R} \\ & - i m \omega \frac{(m_0 - 1)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{R} T_R(\epsilon_0 - D) \\ & + \frac{\omega^2 m_0 (m_0 - 1)}{(\vec{\alpha}_0 \cdot \vec{p}_i) (\vec{\alpha}_0 \cdot \vec{p}_f)} v A_0 G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p} G_R^{(+)}(\epsilon_0 - D) \times \\ & \vec{\alpha}_0 \cdot \vec{p} G_{NR}^{(+)}(\epsilon_0) v | U_i, \vec{p}_i(0) \rangle \end{aligned} \quad (5.1.14)$$

The leading term in the above equation is

$$\langle \vec{p}_f(0), U_f | T_R(\epsilon_0 - D) | U_i, \vec{p}_i(0) \rangle$$

The last term in (5.1.14) is 'smaller than' the leading one by a factor

$$\frac{\omega^2}{(\vec{\alpha}_0 \cdot \vec{p}_i)(\vec{\alpha}_0 \cdot \vec{p}_f)} |G_{NR}^{(+)}(\epsilon_0) \vec{\alpha}_0 \cdot \vec{p}|^2 \sim \left(\frac{\omega}{\epsilon_0}\right)^2$$

and hence can be dropped. Also replacing $T_R(\epsilon_0)$ by $T_R(\epsilon_0 - D)$ in the second and third terms introduce an error of second order of smallness and hence is permissible. The resulting expression is simplified further by the use of (4.1.41). Finally, therefore, one can write the expression (5.1.14) as

$$\begin{aligned} & \langle \vec{p}_f(0), U_f | T_R(\epsilon_0 - D) + i \frac{m\omega m_0}{\vec{\alpha}_0 \cdot \vec{p}_i} T_R(\epsilon_0 - D) \vec{\alpha}_0 \cdot \vec{r}_0 \\ & - i m\omega \frac{(m_0 - e)}{\vec{\alpha}_0 \cdot \vec{p}_f} \vec{\alpha}_0 \cdot \vec{r}_0 T_R(\epsilon_0 - D) | U_i, \vec{p}_i(0) \rangle \\ & = \left\langle \vec{p}_f + \frac{m\omega(m_0 - e)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f}, U_f | T_R(\epsilon_0 - D) | U_i, \vec{p}_i + \frac{m\omega m_0 \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \right\rangle \end{aligned} \quad (5.1.15)$$

The remainder of (5.1.13) is made of non-resonant terms in (5.1.5) and (5.1.9) for all values of λ and resonant terms for values of λ other than $-m_0$. They too can be simplified by the use of (4.1.41) and the idea of momentum shifts in the initial and final states of the projectile. Eventually,

$$\begin{aligned}
 T_e (\vec{p}_f, u_f; \vec{p}_i, u_i) &= \sum_{\lambda} J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) J_{-l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}_f - \frac{m\omega(\lambda+l)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f}, u_f | T_{NR} (\epsilon_{pi} + \bar{w}_i - \lambda\omega) | u_i, \vec{p}_i - \frac{m\omega\lambda\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \rangle \\
 &+ J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}_f + \frac{m\omega(m_0-l)\vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_f}, u_f | T_R (\epsilon_0 - D) | u_i, \vec{p}_i + \frac{m\omega m_0 \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{p}_i} \rangle \\
 &- \left(\frac{i\Gamma}{2\omega} \right) \mathcal{J}_{m_0, l} \langle \vec{p}_f(0), u_f | T_R (\epsilon_0) | u_i, \vec{p}_i(0) \rangle + O(\omega^2)
 \end{aligned}
 \tag{5.1.16}$$

Summation over λ is performed in the first term on the right side of (5.2.16) by using the method of chapter 2 (see equations (2.1.30), (2.1.32) (2.3.33) and (2.1.34)). Then

$$\begin{aligned}
 T_e (\vec{p}_f, u_f; \vec{p}_i, u_i) &= J_{-l} (\vec{\alpha}_0 \cdot \vec{p}) \langle \vec{q}', u_f | T_{NR} (\epsilon_q + \bar{w}_i) | u_i, \vec{q} \rangle \\
 &+ J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-l} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 &\langle \vec{p}', u_f | T_R (\epsilon_0 - D) | u_i, \vec{p} \rangle \\
 &- \left(\frac{i\Gamma}{2\omega} \right) \mathcal{J}_{m_0, l} \langle \vec{p}_f(0), u_f | T_R (\epsilon_0) | u_i, \vec{p}_i(0) \rangle
 \end{aligned}
 \tag{5.1.17}$$

where the quantities $\bar{Q}$, $\bar{Q}'$, $\bar{P}$, $\bar{P}'$ are given by (3.2.26).

It was shown in chapter 3 (p.65) that

$$T_R (\epsilon_0 - D) \simeq (1 + 2iD/\Gamma)^{-1} T_R (\epsilon_0)$$

Substituting in (5.1.17), we finally write

$$\begin{aligned} & T_e (\vec{p}_f, u_f; \vec{p}_i, u_i) \\ &= J_{-e} (\vec{\alpha}_0 \cdot \vec{d}\vec{p}) \langle \vec{Q}', u_f | T_{NR} (\epsilon_Q + \vec{W}_i) | \vec{Q}, u_i \rangle \\ &+ \frac{1}{(1 + 2iD/\Gamma)} J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-e} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ &\langle \vec{P}', u_f | T_R (\epsilon_0) | u_i, \vec{P} \rangle \\ &- \left(\frac{i\Gamma}{2\omega} \right) J_{m_0, e} \langle \vec{p}_f(0), u_f | T_R (\epsilon_0) | u_i, \vec{p}_i(0) \rangle \end{aligned} \quad (5.1.18)$$

With the help of the energy conservation equation (5.1.1)

and the definitions (3.2.26) it is easy to show that

$$\begin{aligned} \frac{Q'^2}{2m} + \vec{W}_f &= \epsilon_Q + \vec{W}_i \\ \frac{P'^2}{2m} + \vec{W}_f &= \epsilon_0 + O(\omega^2) = \frac{P^2}{2m} + \vec{W}_i \end{aligned}$$

So the T-matrix in the first two terms on the right side of (5.1.18) are evaluated on-shell. Since the last term is of order Γ/ω , the matrix element therein will also be

treated as an on-shell matrix. We can, therefore, use the notation (3.2.29) to write (5.1.18) as

$$\begin{aligned}
 & T_e (\vec{p}_f, u_f; \vec{p}_i, u_i) \\
 &= J_{-e} (\vec{\alpha}_0 \cdot \vec{\sigma}_p) t_{NR} (\epsilon_0 + \bar{w}_i, \hat{Q}' \cdot \hat{Q}) \\
 &+ (1 + 2iD_f)^{-1} J_{m_0} (\vec{\alpha}_0 \cdot \vec{p}_i) J_{m_0-e} (\vec{\alpha}_0 \cdot \vec{p}_f) t_R (\epsilon_0, \hat{P}' \cdot \hat{P}) \\
 &- \frac{i\Gamma}{2\omega} J_{m_0, e} t_R (\epsilon_0, \mu)
 \end{aligned} \tag{5.1.19}$$

where

$$\mu = \hat{p}_f \cdot \hat{p}_i \tag{2.4.12}$$

5.2 Evaluation of the sum rules:

Let us define two momenta $\vec{q}_i$ and $\vec{q}_f$ such that

$$\begin{pmatrix} \hat{q}_i \\ \hat{q}_f \end{pmatrix} = \begin{pmatrix} \hat{p}_i \\ \hat{p}_f \end{pmatrix} \tag{5.2.1}$$

$$\frac{q_i^2}{2m} + \bar{w}_i = \epsilon_0 = \frac{q_f^2}{2m} + \bar{w}_f \tag{5.2.2}$$

This is the energy conservation equation for a field-free scattering in which an incident electron of momentum $\vec{q}_i = \hat{p}_i q_i$ excites the atomic target from state 'i' to state 'f' and scatter out with momentum $\vec{q}_f = \hat{p}_f q_f$. The

energy conservation equation in the presence of the laser is

$$\frac{p_i^2}{2m} + \bar{w}_i + l\omega = \frac{p_f^2}{2m} + \bar{w}_f \quad (5.1.1)$$

Also

$$\epsilon_0 = \frac{p_i^2}{2m} + \bar{w}_i + m_0\omega \quad (5.2.3)$$

From (5.2.2) and (5.2.3) it is clear that

$$\frac{q_i^2}{2m} = \frac{p_i^2}{2m} + m_0\omega \quad (5.2.4)$$

Also, from (5.1.1) and (5.2.3)

$$\bar{w}_f = \epsilon_0 + (l - m_0)\omega - \frac{p_f^2}{2m} \quad (5.2.5)$$

Substituting into the second half of (5.2.2) we get

$$\frac{q_f^2}{2m} = \frac{p_f^2}{2m} + (m_0 - l)\omega \quad (5.2.6)$$

Therefore,

$$\begin{pmatrix} p_i \\ p_f \end{pmatrix} = \begin{pmatrix} q_i \\ q_f \end{pmatrix} + O(\omega) \quad (5.2.7)$$

To lowest order in the expansion parameter $\frac{\omega}{\epsilon_0}$ and $\frac{\Gamma}{\omega}$ we can write

$$\begin{aligned} & T_e(\vec{p}_f, u_f; \vec{p}_i, u_i) \\ &= J_e(\vec{\alpha}_0 \cdot \vec{q}) t_{NR}(\epsilon_0, u) + \end{aligned}$$

$$J_{m0}(\vec{\alpha}_0 \cdot \vec{q}_i) J_{m0-l}(\vec{\alpha}_0 \cdot \vec{q}_f) t_R(\epsilon_0, \mu) \quad (5.2.8)$$

The differential cross section for scattering with the net transfer of l photons is

$$\frac{d\sigma_e}{d\Omega}(\vec{p}_f, u_f; \vec{p}_i, u_i) = \left(\frac{p_f}{p_i}\right) |T_e(\vec{p}_f, u_f; \vec{p}_i, u_i)|^2 \quad (5.2.9)$$

Using (5.2.7) and (5.2.8) we get

$$\begin{aligned} & \frac{d\sigma_e}{d\Omega}(\vec{p}_f, u_f; \vec{p}_i, u_i) \\ &= \left(\frac{q_f}{q_i}\right) \left[J_{-l}^2(\vec{\alpha}_0 \cdot \vec{q}) |t_{NR}(\epsilon_0, \mu)|^2 \right. \\ &+ J_{m0}^2(\vec{\alpha}_0 \cdot \vec{q}_i) J_{m0-l}^2(\vec{\alpha}_0 \cdot \vec{q}_f) |t_R(\epsilon_0, \mu)|^2 \\ &+ 2 J_{-l}(\vec{\alpha}_0 \cdot \vec{q}) J_{m0}(\vec{\alpha}_0 \cdot \vec{q}_i) J_{m0-l}(\vec{\alpha}_0 \cdot \vec{q}_f) \times \\ &\quad \left. \text{Re } t_{NR}^*(\epsilon_0, \mu) t_R(\epsilon_0, \mu) \right] \quad (5.2.10) \end{aligned}$$

The differential cross sections in the absence of the laser are defined as

$$\frac{d\sigma}{d\Omega}(\epsilon_0, \mu) = \left(\frac{q_f}{q_i}\right) |t_{NR}(\epsilon_0, \mu) + t_R(\epsilon_0, \mu)|^2 \quad (5.2.11)$$

$$\frac{d\sigma_{NR}^R(\epsilon_0, \mu)}{d\Omega} = \left(q_f/q_i \right) |t_{NR}^R(\epsilon_0, \mu)|^2 \quad (5.2.12)$$

Using (5.2.12) in (5.2.11) one can write

$$2 \left(q_f/q_i \right) \text{Re } t_{NR}^* t_R \\ = \frac{d\sigma}{d\Omega}(\epsilon_0, \mu) - \frac{d\sigma_{NR}}{d\Omega}(\epsilon_0, \mu) - \frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) \quad (5.2.13)$$

Substituting in (5.2.10) and simplifying we get

$$\begin{aligned} & \frac{d\sigma_e}{d\Omega}(\vec{p}_f, u_f; \vec{p}_i, u_i) \\ &= \left[J_{-l}^2(\vec{\alpha}_0 \cdot \vec{q}) - J_{-l}(\vec{\alpha}_0 \cdot \vec{q}) J_{m_0}(\vec{\alpha}_0 \cdot \vec{q}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{q}_f) \right] \\ & \times \frac{d\sigma_{NR}}{d\Omega}(\epsilon_0, \mu) + \left[J_{m_0}^2(\vec{\alpha}_0 \cdot \vec{q}_i) J_{m_0-l}^2(\vec{\alpha}_0 \cdot \vec{q}_f) \right. \\ & \left. - J_{-l}(\vec{\alpha}_0 \cdot \vec{q}) J_{m_0}(\vec{\alpha}_0 \cdot \vec{q}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{q}_f) \right] \times \\ & \frac{d\sigma_R}{d\Omega}(\epsilon_0, \mu) + J_{-l}(\vec{\alpha}_0 \cdot \vec{q}) J_{m_0}(\vec{\alpha}_0 \cdot \vec{q}_i) J_{m_0-l}(\vec{\alpha}_0 \cdot \vec{q}_f) \frac{d\sigma(\epsilon_0, \mu)}{d\Omega} \end{aligned} \quad (5.2.14)$$

Summing over l by the use of appendix A, we get the following sum rule

$$\begin{aligned} & \sum_l \frac{d\sigma_e}{d\Omega}(\vec{p}_f, u_f; \vec{p}_i, u_i) \\ &= \left[1 - J_{m_0}^2(\vec{\alpha}_0 \cdot \vec{q}_i) \right] \frac{d\sigma_{NR}}{d\Omega}(\epsilon_0, \mu) \\ & + J_{m_0}^2(\vec{\alpha}_0 \cdot \vec{q}_i) \frac{d\sigma}{d\Omega}(\epsilon_0, \mu) \end{aligned} \quad (5.2.15)$$

This result is analogous to that obtained by Mittleman¹⁶ for the case of resonant potential scattering (see equation (3.1.13)). However the kinematic factor q_f/q_i in the definitions (5.2.11) and (5.2.12) is characteristic of atomic scattering. Only for elastic scattering (that is for $W_i = W_f$) does q_f become equal to q_i (see equation (5.2.2)).

To evaluate $\langle l \rangle$ to the lowest order in the relevant expansion parameters, we use the definition (4.2.14). Substituting (5.2.14) in (4.2.14), performing the sum over ℓ by the use of appendix A and the angular integrations by the use of appendix B we get

$$\langle l \rangle = \frac{m_0 J_{m0}^2 (\vec{a}_0 \cdot \vec{q}_i) \sum_f [\sigma_R + \delta_{tot.} - \delta_{NR} - \delta_R]}{[1 - J_{m0}^2 (\vec{a}_0 \cdot \vec{q}_i)] \sum_f \sigma_{NR} + J_{m0}^2 (\vec{a}_0 \cdot \vec{p}_i) \sum_f \sigma_{tot.}} \quad (5.2.16)$$

where

$$\sigma_{NR} = \int \frac{d\sigma_{NR}}{d\Omega} d\Omega q_f \quad (5.2.17)$$

$$\sigma_{tot.} = \int \frac{d\sigma}{d\Omega} d\Omega q_f \quad (5.2.18)$$

The quantities δ_R , δ_{NR} and $\delta_{tot.}$ are the corresponding momentum transfer cross sections.

$$\delta_{NR}^R = \int (1-\mu) \frac{d\sigma_{NR}^R}{d\Omega} d\Omega_{q_f} \quad (5.2.19)$$

and

$$\delta_{tot.} = \int (1-\mu) \frac{d\sigma}{d\Omega} d\Omega_{q_f} \quad (5.2.20)$$

(Note: The summation over the index 'f' implies a summation over the final atomic states only)

Again, the result is similar to an earlier one for resonant potential scattering (see equation (3.3.28) and the comments that follow).

CHAPTER 6

EXPERIMENTAL ASPECTS OF LASER-MODIFIED ATOMIC SCATTERING

6.1 Survey of experiments:

This chapter is devoted to the experimental aspects of laser-modified electron-atom scattering. We start with a brief survey of experiments performed in this field.

The first experiment was that of Andrick and Langhans²⁹ in which electrons of energy 11 ev. were scattered from Ar in the presence of a 50 W CW CO₂ laser and absorbed or emitted one laser photon during the scattering process. The observable quantity is the differential cross section:

$$\frac{d\sigma_{fi}}{d\Omega}(\vec{p}_f, \vec{p}_i, \ell) = J_\ell^2(z) \frac{|\vec{p}_f||\vec{Q}|}{|\vec{Q}'||\vec{p}_i|} \frac{d\sigma_{fi}}{d\Omega}(\vec{Q}', \vec{Q}) \quad (4.2.1)$$

The quantities appearing in the above equation are defined in section 4.2.

To obtain a large value (for $\ell \neq 0$) of the differential cross section one has to choose the experimental parameters judiciously. Andrick and Langhans considered elastic scattering in which case $|\vec{Q}'| = |\vec{Q}|$. Also for $\ell = \pm 1$, the leading term in $J_\ell^2(z)$ is

$$\frac{z^2}{4} = \frac{e^2 E_0^2}{4 m^2 \omega^4} (\hat{\alpha}_0 \cdot \vec{p})^2 \quad (6.1.1)$$

The dependence of z^2 on the fourth power of the photon wavelength suggests the use of infrared light and hence of

CO₂ laser. Also z^2 is proportional to the square of the momentum transfer $|\vec{q}|$ which is maximal for backward scattering. But as the elastic backward scattering cross section drops severely with energy, the electron energies used should not be too high in spite of the growth of $|\vec{q}|$ with energy. This is the reason for choosing the electron energy around 10 ev. The specific choice of 11.55 ev for electron energies may have been motivated by the fact that it is a known resonance energy of argon. Therefore, the differential cross section on the right side of (4.2.1) will be peaked at that energy. Physically, an electron in the field of an atom accelerates and emits photons. In resonance scattering electrons spend more time (characterized by their time delay) in the field of the atom. So they accelerate more, which in turn, couples them more strongly with the electromagnetic wave.

In spite of the above considerations, the experimental results were less than satisfactory. Fig. 2 shows the predominant problem of the experiment: Though it is normally quite easy to resolve an energy difference of about 120 mev in a standard electron spectrometer, this becomes difficult when a weak effect has to be detected close to a very strong one. It is not the instrumental profile's width at half height which is important in such measurements but the behaviour of the wings.

A more accurate experiment and one that involves

multiphoton absorption and emission was presented by Weingartshofer et al³⁰. They repeated the experiment of Andrick and Langhans by using a power level of about 10^7 W/cm^2 and obtained l values between ± 3 .

6.2 Results and discussions:

The experiment of Weingartshofer et al was qualitative, the only quantitative result being the confirmation of the sum rule:

$$\sum_{l=-\infty}^{\infty} \frac{d\sigma}{d\Omega} (\bar{F}_f, \bar{F}_i, l) = \left. \frac{d\sigma}{d\Omega} (\bar{F}_f, \bar{F}_i) \right|_{\alpha_0 = 0} \quad (6.2.1)$$

which says that the cross section for all possible values of l sums to the cross section in the absence of the laser. The Experimental results are shown in fig. 3. The central peak shows a depletion of some 45% and the intensities of the electrons that have gained or lost energy display a symmetrical pattern in accordance with equation (4.2.1) and, as required by (6.2.1) the area under the peaks adds up to the area under the peak without the laser field.

Since the experiments of Weingartshofer et al were performed at a resonant energy ($m_0=0$, in the notation of chapter 5), they should confirm the sum rule (5.2.15) rather than (6.2.1). The apperent discrepancy could be

resolved if $\vec{\omega}_0, \vec{q}_i \ll 1$, in which case the first term of (5.2.15) would vanish for $m_0=0$ and the second term would go over to the right side of (6.2.1). However, as pointed out by Mittleman³¹, this condition $\vec{\omega}_0, \vec{q}_i \ll 1$ would make it hard to understand how the cross section (4.2.1) for ℓ as high as 3 (which they saw) could be seen. It is clear, though, that the difference between (6.2.1) and (5.2.15) should be observable at higher powers of the laser. Additional experiments have been performed^{17,32}. But these are still only qualitative in nature.

In the previous chapters, the electrons in the incident beam were assumed to have a definite energy ϵ_{p_i} . In reality, however, the beam has a finite width in energy. So all the observable results obtained in the previous chapters will have to be averaged over the electron energy distribution function. In the experiment of Weingartshofer et al¹⁷, the energy width of the incoming electron beam was 60-150 mev. This is much broader than the width of the resonances and, therefore, all resonance effects will be smeared out when the results are averaged over the electron energy distribution function. This may well be the reason why their experiment confirmed the sum rule (6.2.1) rather than (5.2.15).

Finally we make some comments about the theoretical treatment of the laser. In the previous chapters the laser has been treated as a classical single mode

electromagnetic field in the dipole approximation.

In a typical experiment, electrons will enter the laser field from vacuum and will move many laser wavelengths in the field before it scatters from the target. Therefore, the dipole approximation must be made with great care². The equation of an otherwise free electron moving in a linearly polarised laser potential

$$\bar{A}(\vec{r}, t) = \frac{\bar{E}(r)}{\omega} \cos(\vec{k} \cdot \vec{r} - \omega t) \quad (6.2.2)$$

is

$$\left[i \frac{\partial}{\partial t} - \frac{1}{2m} (\vec{p} + e\vec{A})^2 \right] \chi_q(\vec{r}, t) = 0 \quad (6.2.3)$$

where $\vec{q}$ is the wave number vector of the electron outside the laser. In the realistic situation the laser amplitude $\bar{E}(r)$ changes slowly over the laser wavelength (this is satisfied except for the case when the laser is very highly focussed) and the laser wavelength is large compared to the electron wavelength. This is described by the inequalities

$$\frac{1}{E} \frac{dE}{dr} \ll k, \quad k \ll q \quad (6.2.4)$$

The equation (6.2.3) can be solved in a power series in the small parameters implied by these inequalities and to lowest order the solution is

$$\begin{aligned} \chi_q(\vec{r}, t) = & \exp i \left[\vec{q} \cdot \vec{r} - \frac{q^2}{2m} t \right. \\ & + \frac{e^2}{4q\omega^2} \int_{-\infty}^{\vec{q} \cdot \vec{r}} \vec{q} \cdot d\vec{r}' E^2(r') + \\ & \left. \frac{e \vec{q} \cdot \vec{E}(r)}{m\omega^2} \sin(\vec{k} \cdot \vec{r} - \omega t) + \frac{e^2 E^2(r)}{8m\omega^2} \sin^2(\vec{k} \cdot \vec{r} - \omega t) \right] \end{aligned}$$

(6.2.5)

The interpretation of the first two terms in the exponential are obvious, the third is an eikonal treatment of that part of the A^2 term which survives a time average. The order of magnitude of its contribution to the phase accumulation during scattering is

$$\left(\frac{E}{E_0} \right)^2 \left(\frac{R\gamma}{\hbar\omega} \right)^2 \left(\frac{R}{a_0} \right) \left(\frac{e^2}{\hbar v} \right)$$

where R is the range of the scattering potential, $\vec{v}$ is the projectile velocity and $E_0 = e/2a_0^2$ is the atomic unit of electric field. This is typically very small so that it is safe to drop this term. The last term comes from the oscillating part of the A^2 term and is negligible because it is small and because it is state independent (that is, does not depend upon $\vec{q}$) and essentially constant over the range of the scattering potential. Then the asymptotic states $\tilde{\chi}_q$ are

approximately given by

$$\chi_q(\vec{r}, t) = \exp i \left[\vec{q} \cdot \vec{r} - \frac{q^2}{2m} t + \frac{e \vec{q} \cdot \vec{E}(r)}{m \omega^2} \sin(\vec{k} \cdot \vec{r} - \omega t) \right] \quad (6.2.6)$$

Finally, we anticipate that the wave functions χ_q will be used in an integral which contains $V(r)$ whose range we take to be R . If, as is usually the case, we have

$$\frac{R}{E} \frac{dE}{dr} \ll 1, \quad kR \ll 1 \quad (6.2.7)$$

then the last term in the exponential on the right side of (6.2.6) can be evaluated at $\vec{r} = 0$ to give

$$\chi_q = \exp i \left(\vec{q} \cdot \vec{r} - \frac{q^2}{2m} t - \vec{\alpha}_0 \cdot \vec{q} \sin \omega t \right) \quad (6.2.8)$$

with

$$\vec{\alpha}_0 = \frac{e \vec{E}(0)}{m \omega^2}$$

This result coincides with that obtained under the dipole approximation (see for example equation (2.1.3)). Thus the dipole approximation is valid when the inequalities (6.2.4) and (6.2.7) are satisfied.

The effect of a multimode laser in atomic processes has been investigated by Zoller³³ and by Gersten and Mittleman³⁴. Zoller represented the laser by a chaotic field, that is a field whose amplitude undergoes Gaussian

amplitude fluctuations³⁵. The process of averaging over the laser statistics smoothes out the peaks given by (4.2.1) and for elastic scattering in the first Born approximation, Zoller gets

$$\frac{d\sigma}{d\Omega}(\vec{p}_f, \vec{p}_i, \ell) = \left(\frac{p_f}{p_i}\right) e^{-\frac{1}{2} z^2} I_\ell\left(\frac{1}{2} z^2\right) \frac{d\sigma_{el.}}{d\Omega}(\vec{p}_f, \vec{p}_i) \quad (6.2.9)$$

Gersten and Mittleman describes the electric field of the multimode laser as $E = \sum_i E_i \cos(\omega_i t + \phi_i)$ where the sum runs over all modes. Their result is the following:

Suppose that a single mode laser induces some atomic process with a probability $F(E_0^2)$ where E_0 is the amplitude of the electric field in that mode. Let the transition probability for the same process be denoted by $\bar{F}_N$ when the laser is in N-mode operation with the same total power (that is, $\sum_{i=1}^N E_i^2 = E_0^2$). Then for square pulses (that is, pulses for which the intensity is constant with respect to the frequency)

$$\bar{F}_N = \left(\frac{N-1}{N}\right) \int_0^N dt F(E_0^2 t) \left(1 - \frac{t}{N}\right)^{N-2} \quad (6.2.10)$$

For $N \rightarrow \infty$,

$$\frac{N-1}{N} \rightarrow 1$$

Also,

$$\lim_{N \rightarrow \infty} (1 - t/N)^{N-2} = e^{-t}$$

Therefore,

$$\bar{F}_\infty = \int_0^\infty dt F(E_0^2 t) e^{-t} \quad (6.2.11)$$

In the first Born approximation, the Kroll-Watson result for a single mode laser is

$$\frac{d\sigma}{d\Omega}(\vec{p}_f, \vec{p}_i, e) = \left(\frac{p_f}{p_i}\right) J_e^2(z) \frac{d\sigma_{el}}{d\Omega}(\vec{p}_f, \vec{p}_i)$$

where

$$\frac{p_f^2}{2m} = \frac{p_i^2}{2m} + \hbar\omega$$

$$z = \frac{eE_0}{m\omega^2} (\hat{\alpha}_0 \cdot \Delta\vec{p})$$

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

$\hat{\alpha}_0$ is the direction of the electric field of the laser.

Thus

$$\begin{aligned} & F(E_0^2) \\ &= \left(\frac{p_f}{p_i}\right) \frac{d\sigma_{el}}{d\Omega}(\vec{p}_f, \vec{p}_i) \times J_e^2\left(\frac{e(\hat{\alpha}_0 \cdot \Delta\vec{p})}{m\omega^2} \sqrt{E_0^2}\right) \end{aligned}$$

Therefore,

$$F(E_0^2 t) = \left(\frac{p_f}{p_i} \right) \frac{d\sigma_{el}}{d\Omega}(\vec{p}_f, \vec{p}_i) \times J_e^2 \left(\frac{e}{m\omega^2} (\vec{r}_0 \cdot d\vec{p}) \sqrt{E_0^2 t} \right) \quad (6.2.12)$$

Substituting in (6.2.11) and evaluating the integral by the use of the formula³⁶

$$\int_0^\infty e^{-at} J_\nu(2\beta\sqrt{t}) J_\nu(2\gamma\sqrt{t}) dt = \frac{1}{a} I_\nu\left(2\frac{\beta\gamma}{a}\right) \exp\left(-\frac{\beta^2 + \gamma^2}{a}\right)$$

we get

$$\bar{F}_\infty = \left(\frac{p_f}{p_i} \right) \frac{d\sigma_{el}}{d\Omega}(\vec{p}_f, \vec{p}_i) \times I_e\left(\frac{1}{2} z^2\right) \exp\left(-\frac{1}{2} z^2\right)$$

This is the result (6.2.9). However, for finite values of N , $\bar{F}_N$ retains a peaking structure. In particular, for $N = 2$

$$\bar{F}_2 = \frac{1}{2} \int_0^2 dt F(E_0^2 t) \quad (6.2.13)$$

Using (6.2.12) in (6.2.13) we can rewrite the latter in the following form:

$$\bar{F}_2 = \left(\frac{p_f}{p_i} \right) \frac{d\sigma_{el}}{d\Omega}(\vec{p}_f, \vec{p}_i) \int_0^{\sqrt{2}} x J_e^2(x) dx$$

Evaluating the integral³⁶ we get

$$\begin{aligned} \bar{F}_2 &= \frac{d\sigma}{d\Omega} (\vec{p}_f, \vec{p}_i, \ell) \Big|_{N=2} \\ &= \left(\frac{p_f}{p_i} \right) \left[\left(1 - \frac{\ell^2}{2 (\vec{\alpha}_0 \cdot \vec{d}\vec{p})^2} \right) J_\ell^2 (\sqrt{2} \vec{\alpha}_0 \cdot \vec{d}\vec{p}) \right. \\ &\quad \left. + J_\ell'^2 (\sqrt{2} \vec{\alpha}_0 \cdot \vec{d}\vec{p}) \right] \frac{d\sigma}{d\Omega} (\vec{p}_f, \vec{p}_i) \Big|_{\omega_0=0} + o(\omega) \end{aligned} \tag{6.2.14}$$

Recently, Jung³⁷ has suggested that the problem can be turned around so that the measured cross section could be used as a probe of laser beam properties.

CHAPTER 7

ELECTRON-ATOM IONISING COLLISIONS IN THE PRESENCE OF A LOW-FREQUENCY LASER

7.1 Introduction:

The results described in the previous chapters do not apply to ionisation. The reason is that the ionised electron state is modified by the laser in much the same way that the initial and final electron states are. The problem of ionising collisions in the presence of a low-frequency has been treated by Mohan¹⁹ in the first Born approximation, by Cavaliere et al¹⁸ in the Coulomb-Born approximation and by Banerji and Mittleman²⁰ in the low-frequency approximation. We will discuss the last of these in the following section.

7.2 Low-frequency theory:

For simplicity we deal with an atomic hydrogen target. More complex targets yield no additional physics provided we deal with single ionisation only. The Schrodinger equation is

$$\left(i \frac{\partial}{\partial t} - T_1 - T_2 - \frac{e}{m} (\vec{p}_1 + \vec{p}_2) \cdot \vec{A}(t) - V_1 - V_2 - V_{12} \right) \psi = 0 \quad (7.2.1)$$

where V_1 is the interaction of the i -th electron with the (infinitely heavy) nucleus and V_{12} is the inter-electron interaction. The S-matrix for the process

can be written

$$S = -i \langle \chi_{pf}^{(+)} \phi_q^{(-)}(2) | V_1 + V_{12} | \psi_{pi}^{(+)} \rangle \quad (7.2.2)$$

where the brackets indicate both space and time integrals.

$\psi_{pi}^{(+)}$ satisfies equation (7.2.1) with initial conditions given at $t=-\infty$ where it is an anti-symmetrized laser distorted plane wave of momentum $\vec{p}_i$ for one electron and the laser-distorted hydrogen ground state for the other. The final state is made up of a laser-distorted plane wave for one electron and the laser-distorted continuum state for the other:

$$\left(i \frac{\partial}{\partial t} - T_2 - \frac{e}{m} \vec{p}_2 \cdot \vec{A}(t) - V_2 \right) \phi_q^{(-)}(2) = 0 \quad (7.2.3)$$

The picture described by (7.2.3) is analogous to that given by Geltman³⁸ for ionisation in the absence of the laser: The incident electron collides and ionises the atom and leaves with greater energy than the ejected electron. The fast electron in the final state is then screened by the slow one which feels the residual Coulomb field of the proton. It is well known³⁹ that this picture gives a poor description of the process near threshold where the long range nature of the potentials becomes important. We shall not discuss this energy range and so the potentials can be taken to be screened at large distances (see, however, the remarks made at the end of this section).

No exact solution of (7.2.3) exists. However, one can

obtain a perturbative solution of (7.2.3) in the following way. We write

$$\phi_q^{(-)}(z) = \chi_q + G_2^{(-)} V_2 | \chi_q \rangle \quad (7.2.4)$$

where

$$\left(i \frac{\partial}{\partial t} - T_2 - \frac{e}{m} \vec{p}_2 \cdot \vec{A}(t) - V_2 \right) G_2^{(-)} = I \quad (7.2.5)$$

In the low-frequency approximation one can expand $G_2^{(-)}$ as

$$G_2^{(-)} = \Gamma^{(-)} + \Gamma^{(-)} \frac{e}{m} \vec{p}_2 \cdot \vec{A} \Gamma^{(-)} + \dots \quad (7.2.6)$$

where

$$\left(i \frac{\partial}{\partial t} - T_2 - V_2 \right) \Gamma^{(-)} = I \quad (7.2.7)$$

$\Gamma^{(-)}$ is the time dependent cut-off Coulomb Green's function.

Explicitly

$$\begin{aligned} & \Gamma^{(-)}(t, t') \\ &= +i \theta(t' - t) \lim_{\eta \rightarrow 0^+} \int_S u_s u_s^* e^{(\eta - iE_s)(t - t')} \end{aligned} \quad (7.2.8)$$

where $\{u_s\}$ is a complete set of normalised Coulomb wave functions and $\{E_s\}$ are their eigenvalues. Substituting (7.2.6) into (7.2.4) we get

$$\begin{aligned} & \phi_q^{(-)}(z) \\ &= \left[1 + \Gamma^{(-)} v_2 + \Gamma^{(-)} \frac{e}{m} \vec{p}_2 \cdot \vec{A} \Gamma^{(-)} v_2 + \dots \right] |x_q\rangle \end{aligned} \quad (7.2.9)$$

With the aid of (7.2.8) the time integrals implicit in (7.2.9) can be performed. The individual terms are

$$|x_q\rangle = \sum_{\lambda=-\infty}^{\infty} e^{-i(\epsilon_q + \lambda\omega)t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) |\vec{q}\rangle \quad (7.2.10)$$

$$\begin{aligned} \Gamma^{(-)} v_2 |x_q\rangle &= \sum_{\lambda} e^{-i(\epsilon_q + \lambda\omega)t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) \times \\ &\quad \Gamma^{(-)} (\epsilon_q + \lambda\omega) v_2 |\vec{q}\rangle \end{aligned} \quad (7.2.11)$$

Assuming that the expansion of the cut-off Coulomb T-matrix in powers of ω is valid, one gets

$$\begin{aligned} \Gamma^{(-)} v_2 |x_q\rangle &= \sum_{\lambda} e^{-i(\epsilon_q + \lambda\omega)t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) \times \\ &\left[\Gamma^{(-)}(\epsilon_q) + \lambda\omega \frac{\partial \Gamma^{(-)}(\epsilon_q)}{\partial \epsilon_q} \right] v_2 |\vec{q}\rangle + O(\omega^2) \end{aligned} \quad (7.2.12)$$

and

$$\begin{aligned} & \Gamma^{(-)} \frac{e}{m} \vec{p}_2 \cdot \vec{A} \Gamma^{(-)} v_2 |x_q\rangle \\ &= \sum_{\lambda} e^{-i(\epsilon_q + \lambda\omega)t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) \frac{\omega \lambda}{\vec{\alpha}_0 \cdot \vec{q}} \times \\ & \quad \Gamma^{(-)}(\epsilon_q) \vec{\alpha}_0 \cdot \vec{p}_2 \Gamma^{(-)}(\epsilon_q) v_2 |\vec{q}\rangle + O(\omega^2) \end{aligned} \quad (7.2.13)$$

$\Gamma^{(-)}(\epsilon_q)$ is the time independent form of (7.2.8)

$$\Gamma^{(-)}(\epsilon) = \lim_{\eta \rightarrow 0+} \sum_s \frac{u_s u_s^*}{\epsilon - i\eta - E_s}$$

Finally,

$$\begin{aligned} \phi_q^{(-)}(z) &= \sum_{\lambda} e^{-i(\epsilon_q + \lambda\omega)t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) \times \\ &\left[u_q^{(-)}(z) + \lambda\omega \left\{ \frac{\partial \Gamma^{(-)}(\epsilon_q)}{\partial \epsilon_q} + \right. \right. \\ &\left. \left. \frac{1}{\vec{\alpha}_0 \cdot \vec{q}} \Gamma^{(-)}(\epsilon_q) \vec{\alpha}_0 \cdot \vec{p}_2 \Gamma^{(-)}(\epsilon_q) \right\} v_2 | \vec{q} \rangle + O(\omega^2) \right] \end{aligned} \quad (7.2.14)$$

$$u_q^{(-)}(z) = [1 + \Gamma^{(-)}(\epsilon_q) v_2] | \vec{q} \rangle \quad (7.2.15)$$

is the time independent Coulomb continuum solution with appropriate boundary conditions. We combine $\phi_q^{(-)}$ with χ_{pf} and use (7.2.10) to obtain

$$\begin{aligned} &\langle \chi_{pf}^{(1)} | \phi_q^{(-)}(z) | \\ &= \sum_{\lambda, n} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) J_{n-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) e^{i(\epsilon_{pf} + \epsilon_q + n\omega)t} \times \\ &\left[\langle p_f^{(1)} | u_q^{(-)}(z) | + \lambda\omega \langle \vec{p}_f, \vec{q} | v_2 \left\{ \frac{\partial \Gamma^{(+)}(\epsilon_q)}{\partial \epsilon_q} + \right. \right. \\ &\left. \left. \frac{1}{\vec{\alpha}_0 \cdot \vec{q}} \Gamma^{(+)}(\epsilon_q) \vec{\alpha}_0 \cdot \vec{p}_2 \Gamma^{(+)}(\epsilon_q) \right\} \right] \end{aligned} \quad (7.2.16)$$

Returning to (7.2.2) we replace the initial wave function by its formal solution:

$$\psi_{pi}^{(+)} = A_0 \left[1 + \bar{G}^{(+)} V(1) \right] \chi_{pi}(1) \phi_0(2) \quad (7.2.17)$$

where $\bar{G}^{(+)}$ is the exact Green's function of the full Schrodinger equation (7.2.1). Here, χ_{pi} is the laser-distorted plane wave of the form (7.2.10) and ϕ_0 is the laser-distorted hydrogenic ground state which is obtained as a power series in ω as

$$\phi_0(\vec{r}_2, t) = (1 - im\omega \vec{a}_0 \cdot \vec{r}_2 \cos \omega t) u_0(\vec{r}_2, t) + O(\omega^2) \quad (7.2.18)$$

where $u_0(\vec{r}_2, t) = e^{-iW_0 t} u_0(\vec{r}_2)$ is the usual hydrogenic ground state and W_0 is its energy. Finally in (7.2.2)

$$V(1) = V_1 + V_{12} \quad (7.2.19)$$

and A_0 is the anti symmetrization operator

$$A_0 = 1 - X_{12} \quad (7.2.20)$$

where X_{12} is the exchange operator for the two electrons.

Using (7.2.10) and (7.2.18) we write the initial state in the form

$$\chi_{pi}(1) \phi_0(2) = \sum_{n=-\infty}^{\infty} e^{-i(\epsilon_{pi} + \omega_0 + n\omega)t} \frac{1}{\sqrt{n}} (\vec{\alpha}_0 \cdot \vec{p}_i) \times$$

$$\left(1 - \frac{i m \omega n \vec{\alpha}_0 \cdot \vec{r}_2}{\vec{\alpha}_0 \cdot \vec{p}_i} \right) | \vec{p}_i(1) \phi_0(2) \rangle$$

(7.2.21)

The Green's function occurring in (7.2.17) can be expanded as

$$\bar{G}^{(+)} = G^{(+)} + G^{(+)} H' G^{(+)} + \dots$$

(7.2.22)

where H' is the interaction of the electrons with the laser:

$$H' = \omega \vec{\alpha}_0 \cdot (\vec{p}_1 + \vec{p}_2) \cos \omega t$$

(7.2.23)

and $G^{(+)}$ is the full Green's function in the absence of the laser ($\alpha_0 = 0$). Using the above expansions we obtain

$$S = S_0 + S_1 + S_2 + O(\omega^2)$$

(7.2.24)

where

$$S_0 = -i \langle \chi_{pf}(1) \phi_q^{(-)}(2) | V(1) A_0 | \chi_{pi}(1) \phi_0(2) \rangle$$

(7.2.25)

$$S_1 = -i \langle \chi_{pf}(1) \phi_q^{(-)}(2) | V(1) A_0 G^{(+)} \times$$

$$V(1) | \chi_{pi}(1) \phi_0(2) \rangle$$

(7.2.26)

$$S_2 = -i \langle \chi_{pf}(1) \phi_q^{(-)}(2) | V(1) A_0 G^{(+)} \times \\ H' G^{(+)} V(1) | \chi_{pi}(1) \phi_0(2) \rangle$$

(7.2.27)

We shall evaluate these individually

Substitution of (7.2.16) and (7.2.21) into (7.2.25) ,
the performance of the single time integral and the use of
the sum rules

$$\sum_{n=-\infty}^{\infty} J_n(\gamma) J_{N-n}(\gamma') = J_N(\gamma + \gamma') \quad (A-1)$$

$$\sum_{n=-\infty}^{\infty} n J_n(\gamma) J_{N-n}(\gamma') = \frac{N\gamma}{\gamma + \gamma'} J_N(\gamma + \gamma') \quad (A-2)$$

results in

$$S_0 = -2\pi i \sum_L \delta(\epsilon_{pf} + \epsilon_q - \epsilon_{pi} - W_0 - L\omega) T_0(L) \quad (7.2.28)$$

where

$$T_0(L) = J_{-L}(\vec{\alpha}_0 \cdot \vec{Q}) \times \\ \left[\langle \vec{p}_f(1) u_q^{(-)}(2) | V(1) A_0 (1 + i \vec{\alpha} \cdot \vec{v}_2) | \vec{p}_i(1) u_0(2) \rangle \right. \\ \left. - \frac{\vec{\alpha} \cdot \vec{q}}{m} \langle \vec{p}_f(1), \vec{q}(2) | V_2 \left\{ \frac{\partial \Gamma^{(+)}(\epsilon_q)}{\partial \epsilon_q} + \right. \right. \right.$$

$$\frac{1}{\vec{\alpha}_0 \cdot \vec{q}} \Gamma^{(+)}(\epsilon q) \vec{\alpha}_0 \cdot \vec{p}_2 \Gamma^{(+)}(\epsilon q) \left\{ v(1) A_0 \left| \vec{p}_i(1) u_0(2) \right. \right\}$$

$$\vec{Q} = \vec{p}_f + \vec{q} - \vec{p}_i \quad (7.2.29)$$

$$\vec{d} = \frac{m \omega e \vec{\alpha}_0}{\vec{\alpha}_0 \cdot \vec{Q}} \quad (7.2.30)$$

$$(7.2.31)$$

Using the expansions

$$\begin{aligned} \Gamma^{(+)}(\epsilon q - \vec{d}) &= \Gamma^{(+)}(\epsilon q) \\ &- \vec{d} \cdot \vec{q} / m \frac{\partial \Gamma^{(+)}(\epsilon q)}{\partial \epsilon q} + O(\omega^2) \end{aligned} \quad (7.2.32)$$

$$\langle \vec{q} - \vec{d} | = \langle \vec{q} | (1 + i \vec{d} \cdot \vec{y}_2) + O(\omega^2) \quad (7.2.33)$$

, the identity

$$\begin{aligned} &\Gamma^{(+)}(\epsilon q) \vec{\alpha}_0 \cdot \vec{p}_2 \Gamma^{(+)}(\epsilon q) \\ &= i m \left[\Gamma^{(+)}(\epsilon q) \vec{\alpha}_0 \cdot \vec{y}_2 - \vec{\alpha}_0 \cdot \vec{y}_2 \Gamma^{(+)}(\epsilon q) \right] \end{aligned} \quad (7.2.34)$$

and the equality

$$\begin{aligned}
 \langle u_{\vec{q}-\vec{s}}^{(-)} &= \langle u_q^{(-)} | i \vec{s} \cdot \vec{r}_2 \\
 &= \langle u_{\vec{q}-\vec{s}}^{(-)} | (1 - i \vec{s} \cdot \vec{r}_2) + O(\omega^2)
 \end{aligned}
 \tag{7.2.35}$$

one can rewrite $T_0(\ell)$ as

$$\begin{aligned}
 T_0(\ell) &= T_{-e}(\vec{\alpha}_0, \vec{q}) \left[\langle \vec{p}_f(1) u_{\vec{q}-\vec{s}}^{(-)}(2) | V(1) A_0 i \vec{s} \cdot \vec{r}_2 \right. \\
 &\quad \left. + (1 - i \vec{s} \cdot \vec{r}_2) V(1) A_0 | \vec{p}_i(1) u_0(2) \rangle \right]
 \end{aligned}
 \tag{7.2.36}$$

The time integrals in S_1 can be evaluated with the aid of the representation

$$\begin{aligned}
 G^{(+)}(t, t') &= -i \theta(t-t') \lim_{\eta \rightarrow 0^+} \int_S \psi_S \psi_S^* e^{-(\eta + i \epsilon_S)(t-t')}
 \end{aligned}
 \tag{7.2.37}$$

where $\{\psi_S\}$ is a complete set of normalised wave functions of the two-electron problem and $\{\epsilon_S\}$ are their eigenvalues. Thus

$$S_1 = -2\pi i \sum_{\ell} \delta(\epsilon_{p_f} + \epsilon_q - \epsilon_{p_i} - \omega_0 - \ell\omega) T_1(\ell)
 \tag{7.2.38}$$

where

$$\begin{aligned}
 T_1(\ell) = & \sum_{\lambda, n} J_{\lambda} (\vec{\alpha}_0 \cdot \vec{q}) J_n (\vec{\alpha}_0 \cdot \vec{p}_i) J_{n-\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) \times \\
 & \left[\langle \vec{p}_f(1) | u_q^{(-)}(2) | + \lambda \omega \langle \vec{p}_f(1), \vec{q}(2) | v_2 \left\{ \frac{\partial \Gamma^{(+)}(\epsilon_q)}{\partial \epsilon_q} \right. \right. \\
 & \left. \left. + \frac{1}{\vec{\alpha}_0 \cdot \vec{q}} \Gamma^{(+)}(\epsilon_q) \vec{\alpha}_0 \cdot \vec{p}_2 \Gamma^{(+)}(\epsilon_q) \right\} \right] v(1) A_0 \times \\
 & G^{(+)}(\epsilon_{p_i} + W_0 + n\omega) v(1) \times \\
 & \left(1 - \frac{i\omega m n}{\vec{\alpha}_0 \cdot \vec{p}_i} \vec{\alpha}_0 \cdot \vec{r}_2 \right) | \vec{p}_i(1) | u_0(2) \rangle
 \end{aligned} \tag{7.2.39}$$

$G^{(+)}(E)$ is the time independent form of (7.2.37)

$$G^{(+)}(E) = \alpha t \cdot \lim_{\eta \rightarrow 0^+} \sum_s \frac{\psi_s \psi_s^*}{E + i\eta - \epsilon_s} \tag{7.2.40}$$

The ω dependence of $G^{(+)}$ in (7.2.39) can be expanded assuming that there is no rapid variation as a function of energy. The sum over n and λ can then be performed. After some algebra one gets

$$\begin{aligned}
 T_1(\ell) = & J_{-\ell} (\vec{\alpha}_0 \cdot \vec{q}) \left[\langle \vec{p}_f(1) | u_{\vec{q}-\vec{\delta}}^{(-)}(2) | (1 - i\vec{\delta} \cdot \vec{r}_2) \times \right. \\
 & v(1) A_0 G^{(+)}(\epsilon_{\vec{p}_i-\vec{\delta}} + W_0) v(1) + \\
 & \left. i v(1) A_0 G^{(+)}(\epsilon_{\vec{p}_i-\vec{\delta}} + W_0) v(1) \vec{\delta} \cdot \vec{r}_2 | \vec{p}_i(1) | u_0(2) \right] \\
 & + O(\omega^2)
 \end{aligned} \tag{7.2.41}$$

Adding (7.2.36) and (7.2.41) we write

$$\begin{aligned}
 & T_0(\ell) + T_1(\ell) \\
 &= J_{-e} (\vec{\alpha}_0 \cdot \vec{q}) \langle \vec{p}_f(1) | U_{q-\delta}^{(-)}(2) | (1-i\delta \cdot \vec{r}_2) \times \\
 & T (\epsilon_{\vec{p}_i-\delta} + W_0) (1+i\delta \cdot \vec{r}_2) | \vec{p}_i(1) | U_0(2) \rangle + o(\omega^2)
 \end{aligned}
 \tag{7.2.42}$$

where we have introduced the usual T-operator:

$$T(E) = V(1) A_0 (1 + G^{(+)}(E) V(1))
 \tag{7.2.43}$$

Finally S_2 can be evaluated by noting that H' is of order ω so that the ω corrections in (7.2.16) and (7.2.21) can be dropped. The time integrals can be performed by using (7.2.37). The result is

$$S_2 = -2\pi i \sum_{\lambda} \delta(\epsilon_{p_f} + \epsilon_q - \epsilon_{p_i} - W_0 - \ell\omega) T_2(\ell)
 \tag{7.2.44}$$

where

$$\begin{aligned}
 & T_2(\ell) \\
 &= \frac{\omega}{2} \sum_{\substack{\lambda, n = -\infty \\ a = \pm 1}}^{\infty} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) J_{n-a-\ell-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) J_n(\vec{\alpha}_0 \cdot \vec{p}_i) \\
 & \times \langle \vec{p}_f(1) | U_q^{(-)}(2) | V(1) A_0 G^{(+)}(\epsilon_{p_i} + W_0 + n\omega - a\omega) \\
 & \times \vec{\alpha}_0 \cdot (\vec{p}_1 + \vec{p}_2) G^{(+)}(\epsilon_{p_i} + W_0 + n\omega) V(1) | \vec{p}_i(1) | U_0(2) \rangle
 \end{aligned}
 \tag{7.2.45}$$

The ω dependence in each of the Green's functions can be neglected since (7.2.45) is of order ω . Then the sums can be performed and the identity

$$\begin{aligned} G^{(+)}(E) \vec{\alpha}_0 \cdot (\vec{p}_1 + \vec{p}_2) G^{(+)}(E) \\ = i m \left[G^{(+)}(E) \vec{\alpha}_0 \cdot (\vec{v}_1 + \vec{v}_2) - \vec{\alpha}_0 \cdot (\vec{v}_1 + \vec{v}_2) G^{(+)}(E) \right] \end{aligned} \quad (7.2.46)$$

can be used to give

$$\begin{aligned} T_2(l) = \left(i m \omega \vec{\alpha}_0 \cdot \vec{Q} \right) J_{-l}(\vec{\alpha}_0 \cdot \vec{Q}) \times \\ \langle \vec{p}_f(1) u_q^{(-)}(2) | V(1) A_0 \left[\vec{\alpha}_0 \cdot (\vec{v}_1 + \vec{v}_2) G^{(+)}(\epsilon_{p_i} + W_0) \right. \\ \left. - G^{(+)}(\epsilon_{p_i} + W_0) \vec{\alpha}_0 \cdot (\vec{v}_1 + \vec{v}_2) \right] V(1) | \vec{p}_i(1) u_0(2) \rangle \end{aligned} \quad (7.2.47)$$

Since this is already of order ω , it can be rewritten as

$$\begin{aligned} T_2(l) = J_{-l}(\vec{\alpha}_0 \cdot \vec{Q}) \times \\ \langle \vec{p}_f(1) u_{\vec{q}-\vec{d}}^{(-)}(2) | i \vec{d} \cdot (\vec{v}_1 + \vec{v}_2) T(\epsilon_{\vec{p}_i-\vec{d}} + W_0) \\ - T(\epsilon_{\vec{p}_i-\vec{d}} + W_0) i \vec{d} \cdot (\vec{v}_1 + \vec{v}_2) | \vec{p}_i(1) u_0(2) \rangle \\ + O(\omega^2) \end{aligned} \quad (7.2.48)$$

where we have used the fact that $V(1)$ is a local potential and so commutes with $\vec{r}_1 + \vec{r}_2$. (7.2.48) can be added to (7.2.42) to give the total T-matrix with the result

$$\begin{aligned}
 & T_e (\vec{p}_f, \vec{q}; \vec{p}_i, 0) \\
 &= \langle \vec{p}_f (1) u_{\vec{q}-\vec{\delta}}^{(-)} (2) | (1 + i \vec{\delta} \cdot \vec{\gamma}_1) T (E_{\vec{p}_i-\vec{\delta}} + W_0) \times \\
 & \quad (1 - i \vec{\delta} \cdot \vec{\gamma}_1) | \vec{p}_i (1) u_0 (2) \rangle + O(\omega^2)
 \end{aligned}
 \tag{7.2.49}$$

The factors $(1 \pm i \vec{\delta} \cdot \vec{\gamma}_1)$ appearing in (7.2.49) can be incorporated with the initial and final plane wave states to give the shifted momenta with the result

$$\begin{aligned}
 & T_e (\vec{p}_f, \vec{q}; \vec{p}_i, 0) \\
 &= \langle \vec{p}_f - \vec{\delta} (1), u_{\vec{q}-\vec{\delta}}^{(-)} (2) | T (E_{\vec{p}_i-\vec{\delta}} + W_0) | \vec{p}_i - \vec{\delta} (1), u_0 (2) \rangle \\
 & \quad + O(\omega^2)
 \end{aligned}
 \tag{7.2.50}$$

In addition, the energy delta function can be used to show that

$$E_{\vec{p}_f-\vec{\delta}} + E_{\vec{q}-\vec{\delta}} = W_0 + E_{\vec{p}_i-\vec{\delta}}
 \tag{7.2.51}$$

which means that (7.2.50) is an on-shell matrix element of the conventional T-operator. This implies that the ionisation cross section with the absorption of L photons from the laser can be expressed in terms of the ionisation cross section in the absence of the laser:

$$\begin{aligned}
 & \frac{d^3 \sigma_e (\vec{p}_f, \vec{q}; \vec{p}_i, 0)}{d\Omega_{p_f} d\Omega_q d\epsilon_q} \\
 &= J_e^2 (\vec{\alpha}_0 \cdot \vec{q}) \frac{|\vec{p}_f|}{|\vec{p}_f - \vec{s}|} \frac{|\vec{q}|}{|\vec{q} - \vec{s}|} \frac{|\vec{p}_i - \vec{s}|}{|\vec{p}_i|} \times \\
 & \frac{d^3 \sigma (\vec{p}_f - \vec{s}, \vec{q} - \vec{s}; \vec{p}_i - \vec{s}, 0)}{d\Omega_{\vec{p}_f - \vec{s}} d\Omega_{\vec{q} - \vec{s}} d\epsilon_{\vec{q} - \vec{s}}} \Big|_{\alpha_s=0} \quad (7.2.52)
 \end{aligned}$$

This is the generalisation of the result (4.2.1) in which the atom is left in a neutral state. In that case the argument of the Bessel function was $\vec{\alpha}_0 \cdot (\vec{p}_f - \vec{p}_i)$ instead of $\vec{\alpha}_0 \cdot (\vec{p}_f + \vec{q} - \vec{p}_i)$. It would then appear that the results change discontinuously as one goes from the case in which the atom is left in a very highly excited bound state to the case in which it is left in a low-energy continuum state. This is not so, since we have been forced to exclude the latter case from our considerations.

Proceeding as in chapter 4 one can easily show that

$$\sum_e \frac{d^3 \sigma_e}{d\Omega_q d\Omega_{p_f} d\epsilon_q} = \frac{d^3 \sigma}{d\Omega_q d\Omega_{p_f} d\epsilon_q} \Big|_{\alpha_s=0} + O(\omega^2) \quad (7.2.53)$$

In conclusion, we have shown that if one neglects terms of order ω^2 and assumes that resonance scattering

is not significant the ionisation of atoms by electrons in the presence of a low-frequency laser with the absorption (or emission) of ℓ photons can be expressed in terms of the ionisation cross section in the absence of the laser. This result does not depend upon an approximation scheme for calculating ionisation cross sections (except for the fact that we have replaced the pure Coulomb potential in the final channel by a screened Coulomb potential. Subsequent works^{40,41} have shown that unless the collision energy is high enough for the first Born treatment to be valid, the effect of the Coulomb tail is, in fact, significant).

7.3 Resonances in electron-atom ionising collisions:

In the previous section we excluded the case in which there are scattering resonances in the absence of the laser in initial or/and final channel/s. Resonance introduces additional complications in the theory due mainly to the fact that the expansion of the various Green's functions in powers of ω has to be made carefully. This is because the Green's functions vary rapidly with energy near a resonance. We will highlight the salient features of an ionisation process in which there are scattering resonances.

In electron-atom ionising collisions two kinds of

resonances may occur. The incident electron may resonate with the atom. Also, the ejected electron may resonate with the positive ion. The former is called a negative resonance while the latter is a neutral one. The nomenclature simply reflects the total charge of the resonant state. Although it is possible for an incident electron to form a negative resonance with the hydrogen atom, the Coulomb potential between the proton and the ejected electron cannot support a neutral resonance. For this reason the treatment of the previous section has to be extended to the case of a many-electron atom. Thus the energy conservation equation will be

$$\epsilon_{pf} + \epsilon_q + W_n^+ = \epsilon_{pi} + W_0 + \ell \omega \quad (7.3.1)$$

ϵ_{pi} , ϵ_{pf} and ϵ_q are the energies of the incident, scattered and ejected electrons respectively. W_0 is the ground state energy of the atom. The incident electron ionises the atom and leaves it at the n th excited state of the positive ion with energy W_n^+ . ℓ is the total number of photons absorbed during scattering.

For simplicity, we will restrict our calculations to the lowest order in ω . Then the laser-distorted atomic state ϕ_0 and the laser-distorted ionic state ϕ_+ can be replaced by their unperturbed values:

$$\phi_0 = e^{-iW_0 t} u_0 + o(\omega) \quad (7.3.2)$$

$$\phi_+ = e^{-i W_n^+ t} u_+ + O(\omega) \quad (7.3.3)$$

The wave function for the (ejected electron + ion) system can be expanded as

$$\phi_2^{(-)} = [1 + \Gamma^{(-)} V_2 + O(\omega)] |\chi_q, \phi_+\rangle \quad (7.3.4)$$

Here, $\Gamma^{(-)}$ is the time dependent Green's function for the atom in the absence of the laser and has the representation (7.2.8). V_2 is the interaction potential between the electron and the positive ion. For a one-electron atom (7.3.4) has its analogue in (7.2.9). Combining (7.3.3) and (7.2.10) we get

$$\begin{aligned} & |\chi_q, \phi_+\rangle \\ &= \sum_{\lambda} e^{-i (\epsilon_q + W_n^+ + \lambda \omega) t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) |\vec{q}, u_+\rangle \end{aligned} \quad (7.3.5)$$

Also, by the use of (7.2.8) and (7.3.5) the implicit time integral in the second term on the right side of (7.3.4) can be performed, yielding

$$\begin{aligned} & \Gamma^{(-)} V_2 |\chi_q, \phi_+\rangle \\ &= \sum_{\lambda} e^{-i (\epsilon_q + W_n^+ + \lambda \omega) t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) \times \\ & \quad \Gamma^{(-)} (\epsilon_q + W_n^+ + \lambda \omega) V_2 |\vec{q}, u_+\rangle \end{aligned} \quad (7.3.6)$$

Then

$$|\phi_q^{(-)}\rangle = \sum_{\lambda} e^{-i(\epsilon_q + \omega_n^+ + \lambda\omega)t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) \times \\ \left[1 + \Gamma^{(-)}(\epsilon_q + \omega_n^+ + \lambda\omega) v_2 \right] |\vec{q}, u_+\rangle$$

or taking the complex conjugate

$$\langle \phi_q^{(-)} | = \sum_{\lambda} e^{i(\epsilon_q + \omega_n^+ + \lambda\omega)t} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) \times \\ \langle \vec{q}, u_+ | \left[1 + v_2 \Gamma^{(+)}(\epsilon_q + \omega_n^+ + \lambda\omega) \right] \quad (7.3.7)$$

Also,

$$\langle \chi_{p_f} | = \sum_{\lambda'} e^{i(\epsilon_{p_f} + \lambda'\omega)t} J_{\lambda'}(\vec{\alpha}_0 \cdot \vec{p}_f) \langle \vec{p}_f | \quad (7.3.8)$$

Therefore, the final state can be written as

$$\langle \chi_{p_f} \phi_q^{(-)} | = \sum_{s'} e^{i(\epsilon_{p_f} + \epsilon_q + \omega_n^+ + s'\omega)t} \\ \times \sum_{\lambda} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) J_{s'-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) \times \\ \left[\langle \vec{p}_f, \vec{q}, u_+ | + \langle \vec{p}_f, \vec{q}, u_+ | v_2 \Gamma^{(+)}(\epsilon_q + \omega_n^+ + \lambda\omega) \right] \quad (7.3.9)$$

The initial state at $t = -\infty$ is

$$|\chi_{p_i} \phi_0\rangle \\ = \sum_s e^{-i(\epsilon_{p_i} + \omega_0 + s\omega)t} J_s(\vec{\alpha}_0 \cdot \vec{p}_i) |\vec{p}_i, u_0\rangle \quad (7.3.10)$$

The S-matrix for the process is

$$S = -i \langle \chi_{pf} \phi_q^{(-)} | V(1) | \psi_{pi}^{(+)} \rangle \quad (7.3.11)$$

where $V(1)$ is the interaction potential between the atom and the incident electron. $\psi_{pi}^{(+)}$ can be replaced by its formal solution:

$$\psi_{pi}^{(+)} = A_0 \left(1 + \bar{G}^{(+)} V(1) \right) | \chi_{pi} \phi_0 \rangle \quad (7.3.12)$$

where A_0 is the appropriate anti-symmetrisation operator and $\bar{G}^{(+)}$ is the Green's function of the (atom + incident electron system) in the presence of the laser.

One can expand $\bar{G}^{(+)}$ in powers of the laser interaction as in (7.2.22). The leading term is $G^{(+)}$, which is the Green's function of the system in the absence of the laser and is given by (7.2.37).

Replacing $\bar{G}^{(+)}$ by $G^{(+)}$ in (7.3.12) and performing the time integral by the use of (7.3.10) and (7.3.37) we get

$$\psi_{pi}^{(+)} = \sum_s e^{-i (\epsilon_{pi} + \omega_0 + s\omega) t} J_s (\vec{\alpha}_0 \cdot \vec{p}_i) \times A_0 \left[1 + G^{(+)} (\epsilon_{pi} + \omega_0 + s\omega) V(1) \right] | \bar{p}_i, u_0 \rangle \quad (7.3.13)$$

where $G^{(+)}(E)$ is the time independent form of (7.2.37) and is given by (7.2.40). Substituting (7.3.9) and (7.3.13) in (7.3.11), performing the time integral and simplifying we get

$$S = -2\pi i \sum_{\ell} \delta(\epsilon_{pf} + \epsilon_q + W_n^+ - \epsilon_{pi} - W_0 - \ell\omega) T_{\ell} \quad (7.3.14)$$

where T_{ℓ} is the T-matrix for the process and is given by

$$\begin{aligned} & T_{\ell}(\vec{p}_f, \vec{q}, u_+; \vec{p}_i, 0, u_0) \\ &= \sum_{\lambda, s} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{q}) J_{s-\lambda-\ell}(\vec{\alpha}_0 \cdot \vec{p}_f) J_s(\vec{\alpha}_0 \cdot \vec{p}_i) \times \\ & \quad [\langle \vec{p}_f, \vec{q}, u_+ | V^{(1)} A_0 | \vec{p}_i, u_0 \rangle \\ & \quad + \langle \vec{p}_f, \vec{q}, u_+ | V^{(1)} A_0 G^{(+)}(\epsilon_{pi} + W_0 + s\omega) V^{(1)} | \vec{p}_i, u_0 \rangle \\ & \quad + \langle \vec{p}_f, \vec{q}, u_+ | V^{(2)} \Gamma^{(+)}(\epsilon_q + W_n^+ + \lambda\omega) V^{(1)} A_0 | \vec{p}_i, u_0 \rangle \\ & \quad + \langle \vec{p}_f, \vec{q}, u_+ | V^{(2)} \Gamma^{(+)}(\epsilon_q + W_n^+ + \lambda\omega) V^{(1)} \times \\ & \quad A_0 G^{(+)}(\epsilon_{pi} + W_0 + s\omega) V^{(1)} | \vec{p}_i, u_0 \rangle] \end{aligned} \quad (7.3.15)$$

Now let us suppose that in the absence of the laser $G^{(+)}$ becomes resonant at the energy $\epsilon_R^{(1)} - i\Gamma_1/2$ such that $\epsilon_R^{(1)} \simeq \epsilon_1 = \epsilon_i + W_0 + s_0\omega$ and $\Gamma^{(+)}$ becomes resonant at the energy $\epsilon_R^{(0)} - i\Gamma_0/2$ such that $\epsilon_R^{(0)} \simeq \epsilon_0 = \epsilon_q + W_n^+ + \lambda_0\omega$. s_0 and λ_0 are integers. Let the corresponding resonant wave functions be denoted by $U_R^{(1)}$ and $U_R^{(0)}$ respectively. Near the resonance energies the Green's functions can be written as

$$G^{(+)}(E) = G_{NR}^{(+)}(E) + \frac{\bar{U}_R^{(1)} \rangle \langle U_R^{(1)}}{(E - \epsilon_1) + i\Gamma_1/2}$$

$$\Gamma^{(+)}(E) = \Gamma_{NR}^{(+)}(E) + \frac{\bar{U}_R^{(0)} \rangle \langle U_R^{(0)}}{(E - \epsilon_0) + i\Gamma_0/2} \quad (7.3.16)$$

In the above form for the Green's functions, $G_{NR}^{(+)}$ and $\Gamma_{NR}^{(+)}$ are slowly varying functions of E .

Substituting in (7.3.15), separating out the $\lambda = \lambda_0$ and $S = S_0$ terms and simplifying, we get

$$T_\ell = A \langle T_{NR} \rangle + B \langle T_R^{(1)} \rangle + C \langle T_R^{(0)} \rangle + D \langle T_R^{(01)} \rangle \quad (7.3.17)$$

where

$$A = J_{-e} (\bar{\alpha}_0 \cdot \bar{Q})$$

$$B = J_{S_0} (\bar{\alpha}_0 \cdot \bar{p}_i) J_{S_0 - e} (\bar{\alpha}_0 \cdot (\bar{q} + \bar{p}_f))$$

$$C = J_{\lambda_0} (\bar{\alpha}_0 \cdot \bar{q}) J_{e + \lambda_0} (\bar{\alpha}_0 \cdot (\bar{p}_i - \bar{p}_f))$$

$$D = J_{\lambda_0} (\bar{\alpha}_0 \cdot \bar{q}) J_{S_0} (\bar{\alpha}_0 \cdot \bar{p}_i) J_{S_0 - e - \lambda_0} (\bar{\alpha}_0 \cdot \bar{p}_f)$$

T_ℓ accounts for the following scattering processes between the electron and the atom (denoted by A):

$$\textcircled{1} \quad e + A \rightarrow 2e + A^+$$

$$\textcircled{2} \quad e + A \rightarrow (A^-)^* \rightarrow 2e + A^+$$

$$\textcircled{3} \quad e + A \rightarrow e + (A)^* \rightarrow 2e + A^+$$

$$\textcircled{4} \quad e + A \rightarrow (A^-)^* \rightarrow e + (A)^* \rightarrow 2e + A^+$$

$$\langle T_{NR} \rangle = \langle \bar{p}_f, \bar{q}, u_+ | T(E_{NR}) | \bar{p}_i, u_0 \rangle$$

= the T-matrix for process (1).

$$\langle T_R^{(1)} \rangle = \frac{\langle \bar{p}_f, \bar{q}, u^+ | V^{(1)} A_0 | \bar{u}_R^{(1)} \rangle \langle \bar{u}_R^{(1)} | V^{(1)} | \bar{p}_i, u_0 \rangle}{(\epsilon_{p_i} + W_0 + S_0 \omega - \epsilon_R^{(1)}) + i\Gamma_1/2}$$

which is the T-matrix for process (2)

$$\langle T_R^{(0)} \rangle = \frac{\langle \bar{q}, u^+ | V_2 | \bar{u}_R^{(0)} \rangle \langle u_R^{(0)}, \bar{p}_f | T(E_{NR}) | \bar{p}_i, u_0 \rangle}{(\epsilon_q + W_n^+ + \lambda_0 \omega - \epsilon_R^{(0)}) + i\Gamma_0/2}$$

This is the T-matrix for process (3)

Finally,

$$\langle T_R^{(0)} \rangle = \frac{\langle \vec{q}, u^+ | v_2 | \bar{u}_R^0 \rangle \langle u_R^0, \bar{p}_f | v(1) \rangle \times}{(\epsilon_q + W_n^+ + \lambda_0 \omega - \epsilon_R^{(0)}) + i\Gamma_0/2}$$

$$\frac{A_0 | \bar{u}_R^{(1)} \rangle \langle u_R^{(1)} | v(1) | \bar{p}_i, u_0 \rangle}{(\epsilon_{p_i} + W_0 + s_0 \omega - \epsilon_R^{(1)}) + i\Gamma_1/2}$$

This is the T-matrix for ionisation with the formation of both the neutral and the negative resonances (process (4)). All these amplitudes add coherently in T_ℓ the absolute square of which will contain a host of interference terms.

$$\frac{d^3\sigma}{d\Omega_{p_f} d\Omega_q d\epsilon_q} = \frac{p_f q}{p_i} |T_\ell|^2 \quad (7.3.18)$$

Since the incident beam is not exactly monoenergetic, we have to average the above result over the energy profile of the incident beam. The averaged triple differential cross section is given by

$$\left\langle \frac{d^3\sigma}{d\Omega_{p_f} d\Omega_q d\epsilon_q} \right\rangle = \int_{\epsilon_{\min.}}^{\infty} \frac{p_f q}{p_i} |T_\ell|^2 f(\epsilon) d\epsilon \quad (7.3.19)$$

where $\epsilon_{\min.}$ is the minimum value of the energy parameter in the incident beam. In $|T_\ell|^2$ on the right side of (7.3.19) ϵ_{p_i} has been replaced by the integration

variable ϵ . $f(\epsilon)$ is a function which is peaked at $\epsilon = \epsilon_p$, with a variance σ . Since $f(\epsilon)$ is a peaked function, one can, to a good approximation, replace $\epsilon_{\min.}$ by $-\infty$. Also, with only the mean and variance of a distribution known, one can approximate it by a Gaussian. For simplicity, therefore, we choose the following normalised form for $f(\epsilon)$:

$$f(\epsilon) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\epsilon - \epsilon_i)^2}{2\sigma^2}} \quad (7.3.20)$$

Next, we argue that the energy spread of the incident beam will not affect the formation of the neutral resonance. The neutral resonance is formed at a definite energy $\epsilon_R^{(0)}$. Therefore, when the neutral resonance is produced along with a free electron, the energy distribution in the incident beam will manifest itself only in the energy distribution of the outgoing free electron. Also, when the neutral resonance is formed as a decay product of the negative resonance one can argue that the negative resonance "forgets" its mode of formation before it decays. This argument has its analogue in Bohr's compound nucleus model.

The energy spread of the incident beam does however affect the formation of the negative resonance. Two possibilities may arise. If $\sigma \gg \Gamma_1$, that is, if the spread in the incident beam is much greater than the width

of the negative resonance, then the integration over the energy parameter in (7.3.19) will completely wipe out the peaking structure of the negative resonance. In the opposite unrealistic limit, $\sigma \ll \Gamma_1$, the resonant structure will survive the integration. The relevant integral in (7.3.19) is of the form:

$$I = \int \frac{F(\epsilon) f(\epsilon) d\epsilon}{(\epsilon + W_0 + S_0 \omega - \epsilon_R^{(1)})^2 + \Gamma_1^2/4} \quad (7.3.21)$$

where $F(\epsilon)$ is a slowly varying function of ϵ .

Substituting for $f(\epsilon)$ in (7.3.21) we get

$$I = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(\epsilon) e^{-\frac{(\epsilon - \epsilon_{pi})^2}{2\sigma^2}} d\epsilon}{(\epsilon - \epsilon_1)^2 + \Gamma_1^2/4} \quad (7.3.22)$$

where

$$\epsilon_1 = \epsilon_R^{(1)} - W_0 - S_0 \omega \quad (7.3.23)$$

For $\sigma \ll \Gamma_1$, the exponential $e^{-\frac{(\epsilon - \epsilon_{pi})^2}{2\sigma^2}}$ is a more rapidly varying function of ϵ than the factor $\frac{F(\epsilon)}{(\epsilon - \epsilon_1)^2 + \Gamma_1^2/4}$. We, therefore, expand the latter around $\epsilon = \epsilon_{pi}$. Keeping only the leading term, we get

$$\begin{aligned}
 I(\sigma \ll \Gamma_1) &\approx \frac{1}{\sigma \sqrt{2\pi}} \frac{F(\epsilon_{pi})}{(\epsilon_{pi} - \epsilon_1)^2 + \Gamma_1^2/4} \int_{-\infty}^{\infty} e^{-(\epsilon - \epsilon_{pi})^2/2\sigma^2} d\epsilon \\
 &= \frac{F(\epsilon_{pi})}{(\epsilon_{pi} - \epsilon_1)^2 + \Gamma_1^2/4}
 \end{aligned}
 \tag{7.3.24}$$

For $\sigma \gg \Gamma_1$, however, the factor $\frac{1}{(\epsilon - \epsilon_1)^2 + \Gamma_1^2/4}$ varies more rapidly as a function of ϵ than $F(\epsilon) \exp[-(\epsilon - \epsilon_{pi})^2/2\sigma^2]$. In this case, we evaluate the latter at $\epsilon = \epsilon_1$ and take out of the integral sign. Thus

$$\begin{aligned}
 I(\sigma \gg \Gamma_1) &\approx \frac{F(\epsilon_1)}{\sigma \sqrt{2\pi}} e^{-(\epsilon_1 - \epsilon_{pi})^2/2\sigma^2} \int_{-\infty}^{\infty} \frac{d\epsilon}{(\epsilon - \epsilon_1)^2 + \Gamma_1^2/4} \\
 &= \frac{2\pi}{\Gamma_1} \frac{F(\epsilon_1)}{\sigma \sqrt{2\pi}} e^{-(\epsilon_1 - \epsilon_{pi})^2/2\sigma^2}
 \end{aligned}
 \tag{7.3.25}$$

It is clear that I varies much more smoothly with the incident energy for $\sigma \gg \Gamma_1$ than for the opposite inequality. Moreover it is easily seen that

$$I_{max}(\sigma \gg \Gamma_1) = \sqrt{\frac{\pi}{2}} \left(\frac{\Gamma_1}{\sigma} \right) I_{max}(\sigma \ll \Gamma_1)
 \tag{7.3.26}$$

Thus for $\sigma \gg \Gamma_1$ the resonant structure due to the formation of the negative resonance is all but lost (see

fig.4).

Finally we should say a few words on how the laser can be exploited to detect the resonances. For simplicity, let us assume that the energy of the incident electrons is such as to preclude the formation of the negative resonance. In that case

$$\begin{aligned}
 T_e &= A \langle T_{NR} \rangle + C \langle T_R^{(0)} \rangle \\
 &= J_{-e} (\vec{\alpha}_0 \cdot \vec{Q}) \langle \vec{p}_f, \vec{q}, u+1 | T(\epsilon_{pi} + W_0) | \vec{p}_i, u_0 \rangle \\
 &+ J_{\lambda_0} (\vec{\alpha}_0 \cdot \vec{q}) J_{-e-\lambda_0} (\vec{\alpha}_0 \cdot (\vec{p}_f - \vec{p}_i)) \times \\
 &\frac{\langle \vec{q}, u+1 | V_2 | \vec{u}_R^{(0)} \rangle \langle \vec{u}_R^{(0)}, \vec{p}_f | T(\epsilon_{pi} + W_0) | \vec{p}_i, u_0 \rangle}{(\epsilon_q + W_n + \lambda_0 \omega - \epsilon_R^{(0)}) + i\Gamma_0/2} \\
 &= X + \frac{Y}{\epsilon_q - E + i\Gamma_0/2} \quad (7.3.27)
 \end{aligned}$$

where

$$\begin{aligned}
 X &= J_{-e} (\vec{\alpha}_0 \cdot \vec{Q}) \langle \vec{p}_f, \vec{q}, u+1 | T(\epsilon_{pi} + W_0) | \vec{p}_i, u_0 \rangle \\
 Y &= J_{\lambda_0} (\vec{\alpha}_0 \cdot \vec{q}) J_{-e-\lambda_0} (\vec{\alpha}_0 \cdot (\vec{p}_f - \vec{p}_i)) \times \\
 &\langle \vec{q}, u+1 | V_2 | \vec{u}_R^{(0)} \rangle \langle \vec{u}_R^{(0)}, \vec{p}_f | T(\epsilon_{pi} + W_0) | \vec{p}_i, u_0 \rangle
 \end{aligned}$$

$$E = \epsilon_R^{(\omega)} - W_n^{(+)} - \lambda_0 \omega$$

(7.3.28)

Since the formation of the neutral resonance is insensitive to the energy profile of the incident electrons, we write

$$\left\langle \frac{d^2\sigma}{d\Omega_q d\epsilon_q} \right\rangle = \int \frac{p_f q}{p_i} |T_e|^2 d\Omega_{pf} \quad (7.3.29)$$

where

$$|T_e|^2 = |X|^2 + \frac{|Y|^2}{(\epsilon_q - E)^2 + \Gamma_0^2/4} + 2 \operatorname{Re} \left[\frac{X^* Y}{\epsilon_q - E + i\Gamma_0/2} \right] \quad (7.3.30)$$

Note that $\left\langle \frac{d^2\sigma}{d\Omega_q d\epsilon_q} \right\rangle$ is a rapidly varying function of $\epsilon_q - E$. Since E is a function of ω (for $\lambda_0 \neq 0$) this means that the double differential cross section varies rapidly with ω .

Now

$$\frac{d}{d\omega} \left\langle \frac{d^2\sigma}{d\Omega_q d\epsilon_q} \right\rangle \approx \frac{p_f q}{p_i} \int \frac{d}{d\omega} |T_e|^2 d\Omega_{pf} \quad (7.3.31)$$

and

$$\frac{\partial}{\partial \omega} |T_e|^2 = \frac{-2|Y|^2 (\epsilon_q - E) \lambda_0}{[(\epsilon_q - E)^2 + \Gamma_0^2/4]^2} - 2 \operatorname{Re} \left\{ \frac{x^* Y \lambda_0}{(\epsilon_q - E + i\Gamma_0/2)^2} \right\} \quad (7.3.32)$$

Suppose

$$\frac{\partial}{\partial \omega} \left\langle \frac{d^2 \sigma}{d\Omega_q d\epsilon_q} \right\rangle = 0 \text{ for } \omega = \omega_R \quad (7.3.33)$$

We write

$$\epsilon_q - E_R = z \quad (7.3.34)$$

where

$$E_R = E \Big|_{\omega = \omega_R} = \epsilon_R^{(0)} - W_n^+ - \lambda_0 \omega_R \quad (7.3.35)$$

Thus

$$\omega_R = \frac{1}{\lambda_0} \left[z + \epsilon_R^{(0)} - \epsilon_q - W_n^+ \right] \quad (7.3.36)$$

From (7.3.31) and (7.3.32) one sees that (7.3.33) will be satisfied if

$$z^2 \operatorname{Re} (x^* Y) + z \left(|Y|^2 + \Gamma_0 \operatorname{Im} (x^* Y) \right) - \left(\Gamma_0^2/4 \right) \operatorname{Re} (x^* Y) = 0 \quad (7.3.37)$$

Solving for z and substituting in (7.3.36) one would, in principle, get the values of ω for which the double

differential cross section attains extremum values. If, therefore, one repeats the experiment for different values of ω (this necessitates the use of a tunable laser) while keeping the energy of the incident beam fixed and counts the number N_e of ejected electrons one would get a peaking structure in the N_e vs. ω curve. This way we can measure the energy and the width of the neutral resonance to the accuracy dictated by the tunability of the laser employed.

CHAPTER 8

COULOMB SCATTERING

IN THE PRESENCE OF A LOW-FREQUENCY LASER

8.1 Introduction:

The original work of Kroll and Watson¹² and almost all subsequent papers⁴² on scattering in the presence of a low-frequency laser field ignored the long range effects of the Coulomb potential. They all arrive at a cross section which can be written

$$\frac{d\sigma_\ell(\vec{p}_f, \vec{p}_i)}{d\Omega} = \mathcal{T}_\ell^2(\vec{\alpha}_0 \cdot (\vec{p}_f - \vec{p}_i)) \left. \frac{d\sigma}{d\Omega}(\vec{p}_f, \vec{p}_i) \right|_{\alpha_0=0} + O(\omega) \quad (8.1.1)$$

where the cross section on the left describes scattering with the transfer of ℓ photons so that⁹

$$p_f^2/2m = p_i^2/2m - \ell\omega \quad (8.1.2)$$

The cross section on the right side of (8.1.1) is that in the absence of the laser. The field strength parameter $\vec{\alpha}_0 = e\vec{E}_0/m\omega$ is held fixed as the limit $\omega \rightarrow 0$ is taken. In ionising collisions or the case where the target is an ion with a net charge, additional complications arise as a result of the long range of the Coulomb potential. Our purpose here is to account for the effects of the Coulomb tail.

For simplicity, we ignore the internal structure of the target representing the scatterer by a local potential. The T-operator corresponding to a potential V which satisfies the condition

$$\lim_{\gamma \rightarrow 0^+} \gamma |V(\gamma)| = 0 \quad (8.1.3)$$

is

$$T(k) = V (1 + G^{(+)} V) \quad (8.1.4)$$

with

$$G^{(+)} = \frac{1}{\frac{k^2}{2m} - H_0 - V} \quad (8.1.5)$$

and $k^2 = 2m(E + i\gamma)$ where E is a real energy and $\gamma \rightarrow 0^+$.

The pure Coulomb potential does not satisfy the condition (8.1.3). We, however, take the point of view that the Coulomb potential is an idealization of a real potential which is cut off at some large radius R

$$V_R(\gamma) = \begin{cases} V_0(\gamma) = \frac{Ze^2}{\gamma}, & \gamma \leq R \\ 0, & \gamma > R \end{cases} \quad (8.1.6)$$

where V_0 is the pure Coulomb potential. The transition matrix element for the cut-off potential takes the form

$$\langle \vec{p}_f | T_R\left(\frac{k^2}{2m}\right) | \vec{p}_i \rangle = T_R(\vec{p}_f, \vec{p}_i; k^2/2m) \quad (8.1.7)$$

where $|\vec{p}_f\rangle$ and $|\vec{p}_i\rangle$ are the free particle wave functions (plane waves) and the subscript R indicates that the quantity is evaluated for the cut-off potential. For real k , the pure Coulomb transition matrix element can be

obtained from (8.1.7) by taking the limit $R \rightarrow \infty$ and $\eta \rightarrow 0^+$.

Ford⁴³ has shown that for a totally off-shell cut-off Coulomb T-matrix the order in which the two limits are taken is unimportant. After the limits are taken one gets the off-shell pure Coulomb T-matrix evaluated at real k . However, if k approaches p_i or p_f , the limit $\eta \rightarrow 0^+$ should be taken first⁴⁴. In his work, Ford has excluded the case $p_f = p_i$ which in the context of laser-modified scattering becomes the $\ell = 0$ case (see equation (8.1.2)). We, therefore, start with the derivation of the T-matrix for $\ell \neq 0$. In section 8.3, we will get an expression for the screened coulomb T-matrix $T_R(k_2, k_1; \vec{k}/2m)$ when $k_2 = k_1 \neq k$. This will be used in section 8.4 for the derivation of the T-matrix for $\ell = 0$.

8.2 Derivation of the T-matrix for $\ell \neq 0$:

Since the potential $V_R(r)$ is of finite range, previous results for finite range potential scattering in a low-frequency laser field may be used. The T-matrix for the transfer of ℓ photons can be expanded in powers of the laser-projectile interaction in intermediate states which is also a power series in ω . The first three terms are⁴⁵

$$T_e^{(0)} + T_e^{(1)} = \sum_{\lambda=-\infty}^{\infty} J_{e-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) \times \\ \langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda \omega) | \vec{p}_i \rangle \quad (8.2.1)$$

$$T_e^{(2)} = \frac{\omega}{2} \sum_{\lambda=-\infty}^{\infty} \sum_{x_1=\pm 1} J_{e-\lambda+x_1}(\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) \times \\ \langle \vec{p}_f | V_R G_R^{(+)}(\epsilon_{p_f} - (e-\lambda+x_1)\omega) \vec{\alpha}_0 \cdot \vec{p} \times \\ G_R^{(+)}(\epsilon_{p_i} - \lambda \omega) V_R | \vec{p}_i \rangle \quad (8.2.2)$$

where

$$\epsilon_p = \frac{p^2}{2m} \quad (1.4.4)$$

Following the notation (8.1.7) we write the matrix element in (8.2.1) in the form

$$\langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda \omega) | \vec{p}_i \rangle = T_R(\vec{p}_f, \vec{p}_i; k^2/2m) \quad (8.2.3)$$

where

$$k^2 = 2m(\epsilon_{p_i} - \lambda \omega + i\eta) \quad (8.2.4)$$

For $\lambda = 0$ (or ℓ) k will approach p_i or (p_f) . So, we single out the $\lambda = 0, \ell$ terms in (8.2.1) and write

$$\begin{aligned}
 T_e^{(0)} + T_e^{(1)} &= J_e (\vec{\alpha}_0 \cdot \vec{p}_f) J_0 (-\vec{\alpha}_0 \cdot \vec{p}_i) \langle \vec{p}_f | T_R (\epsilon_{p_i}) | \vec{p}_i \rangle \\
 &+ J_0 (\vec{\alpha}_0 \cdot \vec{p}_f) J_e (-\vec{\alpha}_0 \cdot \vec{p}_i) \langle \vec{p}_f | T_R (\epsilon_{p_f}) | \vec{p}_i \rangle \\
 &+ \sum_{\lambda \neq 0, l} J_{l-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) J_\lambda (-\vec{\alpha}_0 \cdot \vec{p}_i) T_R (\vec{p}_f, \vec{p}_i; \frac{k^2}{2m})
 \end{aligned}
 \tag{8.2.5}$$

For a cut-off Coulomb potential, the off-shell matrix element can be written as⁴⁶

$$\begin{aligned}
 T_R (\vec{p}_f, \vec{p}_i; k^2/2m) &= T_\infty (\vec{p}_f, \vec{p}_i; k^2/2m) \\
 &+ O(1/R) + O\left(\frac{1}{(k^2 - p_i^2)R}\right) + O\left(\frac{1}{(k^2 - p_f^2)R}\right)
 \end{aligned}
 \tag{8.2.6}$$

In the last term of (8.2.5) k is not equal to $\pm p_i$ or $\pm p_f$. So, we can write

$$T_R (\vec{p}_f, \vec{p}_i; k^2/2m) = T_\infty (\vec{p}_f, \vec{p}_i; k^2/2m) + O(1/R) \tag{8.2.7}$$

where $T_\infty (\vec{p}_f, \vec{p}_i; k^2/2m)$ is the off-shell pure Coulomb T-matrix evaluated at the complex energy

$$k^2/2m = \epsilon_{p_i} - \lambda \omega + i\eta \tag{8.2.4}$$

and has the following expression⁴⁷

$$T_0 (\vec{p}_f, \vec{p}_i; k^2/2m) = \frac{Ze^2}{2\pi^2 (\Delta p)^2} \times$$

$$\left\{ \Gamma_a (\vec{p}_f, \vec{p}_i; k^2/2m) + \Gamma_b (\vec{p}_f, \vec{p}_i; k^2/2m) \right\}$$

(8.2.8)

where

$$\Delta p = |\vec{p}_f - \vec{p}_i|$$

The quantities Γ_a and Γ_b are defined as

$$\Gamma_a (\vec{p}_f, \vec{p}_i; k^2/2m)$$

$$= 1 - (1+\varepsilon)^{-1/2} \left\{ 1 + \sum_{n=1}^{\infty} \frac{2\nu_\lambda^2}{n^2 + \nu_\lambda^2} t_+^n \right\} \quad (8.2.9)$$

$$\Gamma_b (\vec{p}_f, \vec{p}_i; k^2/2m)$$

$$= (1+\varepsilon)^{-1/2} \Gamma(1-i\nu_\lambda) \Gamma(1+i\nu_\lambda) (-t_+)^{-i\nu_\lambda} \quad (8.2.10)$$

where

$$\nu_\lambda = -Ze^2 m / |k| = \frac{-Z}{a_0 |k|} \quad (8.2.11)$$

$$\varepsilon = \frac{(k^2 - p_f^2)(k^2 - p_i^2)}{k^2 (\Delta p)^2} \quad (8.2.12)$$

$$t_+ = (1 + \frac{2}{\varepsilon}) - \frac{2}{\varepsilon} (1 + \varepsilon)^{1/2} \quad (8.2.13)$$

$$|\arg. (-t_+)| < \pi \quad (8.2.14)$$

With the use of (8.1.2), (1.4.4) and (8.2.4) we get

$$\varepsilon = \frac{-2m\omega^2 [\lambda(\ell-\lambda) + \frac{2i\eta}{\omega} (\lambda - \ell/2)]}{(\Delta p)^2 (\varepsilon_{pi} - \lambda\omega)} \quad (8.2.15)$$

This is of order ω^2 and, therefore, a small quantity.

Expanding in powers of ω and keeping only the leading terms, we obtain

$$\begin{aligned} T_\omega (\bar{p}_f, \bar{p}_i; k^2/2m) \\ = \frac{Ze^2}{2\pi^2 (\Delta p)^2} \frac{\pi \nu_\lambda}{\sinh \pi \nu_\lambda} (-\varepsilon/4)^{-i\nu_\lambda} \end{aligned} \quad (8.2.16)$$

Using (8.2.15) we write

$$(-\varepsilon/4)^{-i\nu_\lambda} = \left[\frac{2(\Delta p)^2 (\varepsilon_{pi} - \lambda\omega)}{m\omega^2} \right]^{i\nu_\lambda} (z_{\lambda,e})^{-i\nu_\lambda} \quad (8.2.17)$$

with

$$z_{\lambda,e} = \lambda(\ell-\lambda) + 2i\eta/\omega (\lambda - \ell/2)$$

This is a complex quantity. Writing

$$z_{\lambda, \ell} = |z_{\lambda, \ell}| e^{i\theta}$$

one gets

$$(z_{\lambda, \ell})^{-i\nu_\lambda} = |z_{\lambda, \ell}|^{-i\nu_\lambda} e^{\theta \nu_\lambda} \quad (8.2.18)$$

Substituting into (8.2.16) we get

$$T_\infty (\vec{p}_f, \vec{p}_i; k^2/2m) = \frac{ze^2}{2\pi^2 (\Delta p)^2} \frac{i\pi \nu_\lambda}{\sinh \pi \nu_\lambda} \\ \times \left[\frac{z (\Delta p)^2 (\epsilon_{p_i} - \lambda \omega)}{m \omega^2} \right]^{i\nu_\lambda} |z_{\lambda, \ell}|^{-i\nu_\lambda} e^{\theta \nu_\lambda} \quad (8.2.19)$$

To obtain the T-matrix for real k , one has to take the limit $\eta \rightarrow 0^+$. In the $\eta \rightarrow 0^+$ limit, (8.2.4) gives

$$k^2/2m = \epsilon_{p_i} - \lambda \omega$$

Then ν_λ (see (8.2.11)) is given by

$$\nu_\lambda = - \frac{z}{a_0} \left(\frac{1}{2m (\epsilon_{p_i} - \lambda \omega)} \right)^{1/2} \quad (8.2.20)$$

Also

$$\lim_{\eta \rightarrow 0^+} |z_{\lambda, \ell}| = |\lambda (\ell - \lambda)| \quad (8.2.21)$$

However, as $\eta \rightarrow 0^+$, θ assumes different values for different regions of λ, ℓ . We note that $\text{Re } z_{\lambda, \ell} = \lambda (\ell - \lambda)$ and

$\text{Im } z_{\lambda, \ell} = \frac{2\gamma}{\omega} (\lambda - \ell/2)$. Three possible cases can arise:

(1) If $\text{Re } z_{\lambda, \ell} < 0$ and $\text{Im } z_{\lambda, \ell} > 0$, then as $\gamma \rightarrow 0^+$, $z_{\lambda, \ell}$ approaches the negative real axis from above in which case $\theta = \bar{\pi}$. (2) If $\text{Re } z_{\lambda, \ell} < 0$ and $\text{Im } z_{\lambda, \ell} < 0$, then the approach is from below and $\theta = -\bar{\pi}$. (3) If $\text{Re } z_{\lambda, \ell} > 0$ then as $\gamma \rightarrow 0^+$, $z_{\lambda, \ell}$ approaches the positive real axis and $\theta = 0$. In the last case, if the approach is from below, θ does not become equal to $2\bar{\pi}$. This is because of the condition (8.2.14)

With the above prescription for θ , it is easy to show that (in the $\gamma \rightarrow 0^+$ limit)

$$e^{\theta \nu_{\lambda}} = \begin{cases} e^{-\pi \nu_{\lambda}} & , \lambda < 0 \\ 1 & 0 < \lambda < \ell \\ e^{+\pi \nu_{\lambda}} & \lambda > \ell \end{cases} ; \ell > 0 \quad (8.2.22)$$

$$e^{\theta \nu_{\lambda}} = \begin{cases} e^{+\pi \nu_{\lambda}} & , \lambda > 0 \\ 1 & \ell < \lambda < 0 \\ e^{-\pi \nu_{\lambda}} & \lambda < \ell \end{cases} ; \ell < 0 \quad (8.2.23)$$

Substituting into (8.2.19), we write

$$\begin{aligned} & T_{\omega} (\bar{p}_f, \bar{p}_i; \epsilon_{pi} - \lambda \omega) \\ &= \frac{ze^2}{2\pi^2 (\delta p)^2} \frac{\pi \nu_{\lambda}}{\sinh \pi \nu_{\lambda}} \left[\frac{2 (\delta p)^2 (\epsilon_{pi} - \lambda \omega)}{m \omega^2} \right]^{i \nu_{\lambda}} \\ & \times |\lambda (\ell - \lambda)|^{-i \nu_{\lambda}} e^{\theta \nu_{\lambda}} \end{aligned} \quad (8.2.24)$$

with $e^{\theta \nu_\lambda}$ given by (8.2.22) and (8.2.23).

Next we expand the quantities appearing in (8.2.24) about a reference energy which is the average of the initial and final energies.

$$\epsilon = \frac{1}{2} (\epsilon_{pf} + \epsilon_{pi}) = \frac{p^2}{2m} = \epsilon_{pi} - \ell \omega / 2 \quad (8.2.25)$$

We define

$$\nu = -\frac{z}{p a_0} = -\frac{z}{a_0} \left(\frac{1}{2m (\epsilon_{pi} - \ell \omega / 2)} \right)^{1/2} \quad (8.2.26)$$

Then

$$\nu_\lambda = \nu \left[1 + \frac{\omega}{2\epsilon} (\lambda - \ell/2) \right] + O(\omega^2) \quad (8.2.27)$$

$$\begin{aligned} & \sinh \pi \nu_\lambda \\ &= \sinh \pi \nu \left[1 + \frac{\pi \omega \nu}{2\epsilon} (\lambda - \ell/2) \coth \pi \nu \right] + O(\omega^2) \end{aligned} \quad (8.2.28)$$

Also,

$$\begin{aligned} (\delta p)^2 &= |\vec{p}_f - \vec{p}_i|^2 \\ &= 4 m \epsilon (1 - \mu) + O(\omega) \end{aligned}$$

and

$$\mu = \hat{p}_f \cdot \hat{p}_i \quad (8.2.29)$$

To order ω^0 we can replace ν_λ by ν in $|\lambda(1-\lambda)| e^{-i\nu_\lambda} e^{\theta \nu_\lambda}$. Finally,

$$\begin{aligned}
& \left(\frac{2 (\delta p)^2 (\epsilon_{pi} - \lambda \omega)}{m \omega^2} \right)^{i \nu \lambda} \\
&= \left(\frac{\omega^2}{8 \epsilon^2 (1-\mu)} \right)^{-i \nu} \left(1 + \frac{i \nu \omega}{2 \epsilon} (\lambda - \ell/2) \ln \left\{ \frac{8 \epsilon^2 (1-\mu)}{\omega^2} \right\} \right) \\
&+ o(\omega)
\end{aligned}$$

(8.2.30)

Using these expansions in (8.2.24) we get

$$\begin{aligned}
& T_{\omega} (\vec{p}_f, \vec{p}_i; \epsilon_{pi} - \lambda \omega) \\
&= \frac{2 e^2}{8 \pi^2 m \epsilon (1-\mu)} \frac{\pi \nu}{\sinh \pi \nu} \left[\lambda (\ell - \lambda) \right]^{-i \nu} e^{\theta \nu} \\
&\times \left(\frac{\omega^2}{8 \epsilon^2 (1-\mu)} \right)^{-i \nu} \left(1 + \frac{i \nu \omega}{2 \epsilon} (\lambda - \ell/2) \ln \left\{ \frac{8 \epsilon^2 (1-\mu)}{\omega^2} \right\} \right) \\
&+ o(\omega)
\end{aligned}$$

(8.2.31)

For large R , one can neglect the correction terms of order

$1/R$ on the right side of (8.2.7). $T_{\omega}(\vec{p}_f, \vec{p}_i; \epsilon_{pi} - \lambda \omega)$

as given by (8.2.31) will then be the value of

$T_R(\vec{p}_f, \vec{p}_i; k^2/2m)$ for real energy. Substituting in

(8.2.5) one gets

$$\begin{aligned}
 & \sum_{\lambda \neq 0, \ell} J_{\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) T_R (\vec{p}_f, \vec{p}_i; k^2/2m) \\
 &= \frac{ze^2}{8\pi^2 m \epsilon (1-\mu)} \frac{\pi \nu}{\sinh \pi \nu} \left(\frac{\omega^2}{8\epsilon^2 (1-\mu)} \right)^{-i\nu} \times \\
 & \left(K_1 + \frac{i\nu\omega}{2\epsilon} \left(K_2 - \frac{\ell}{2} K_1 \right) \ln \left\{ \frac{8\epsilon^2 (1-\mu)}{\omega^2} \right\} \right)
 \end{aligned}
 \tag{8.2.32}$$

where

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \sum_{\lambda \neq 0, \ell} J_{\ell-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) |\lambda(\ell-\lambda)|^{-i\nu} e^{\theta \nu} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}
 \tag{8.2.33}$$

Next we evaluate the first two terms on the right side of (8.2.5). Following Ford, we write⁴⁷

$$\begin{aligned}
 \langle \vec{p}_f | T_R (\epsilon p_i) | \vec{p}_i \rangle &= \frac{ze^2}{2\pi^2} \Gamma(1-i\nu_0) \times \\
 & \frac{e^{\pi\nu_0/2} (p_f^2 - k^2)^{-i\nu_0} e^{i\delta_k(R)}}{((\epsilon p)^2)^{1-i\nu_0}} + O(1/R)
 \end{aligned}
 \tag{8.2.34}$$

$$\delta_k(R) = \nu_0 \ln(2kR) + O(1/R)
 \tag{8.2.35}$$

$$k^2 = 2m (\epsilon p_i + i\gamma)
 \tag{8.2.36}$$

$$p_f^2 - k^2 = -2m\ell\omega \left(1 + \frac{i\eta}{\ell\omega}\right) = |p_f^2 - k^2| e^{i\phi} \text{ (say)}$$

(8.2.37)

Substituting in (8.2.34) we get

$$\begin{aligned} \langle \tilde{p}_f | T_R (E_{pi}) | \tilde{p}_i \rangle &= \frac{Ze^2}{2\pi^2} \Gamma(1-i\nu_0) \times \\ &\times \frac{e^{i\pi\nu_0/2} |p_f^2 - k^2|^{-i\nu_0}}{((\delta p)^2)^{1-i\nu_0}} e^{i\phi\nu_0 + i\nu_0 \ln |2kR|} + O(1/R) \end{aligned}$$

(8.2.38)

In the $\eta \rightarrow 0^+$ limit, ν_0 is given by

$$\nu_0 = -\frac{Z}{a_0} \left(\frac{1}{2mE_{pi}} \right)^{1/2}$$

(8.2.39)

(Note that ν_0 can be obtained by putting $\lambda = 0$ in the expression for ν_λ (see equation (8.2.20))

Also,

$$|p_f^2 - k^2| = 2m\omega |e|$$

(8.2.40)

However, ϕ is multivalued. In the $\eta \rightarrow 0^+$ limit

$$\phi = -\pi/2 - \pi/2 \operatorname{sgn} \ell$$

(8.2.41)

Substitution into (8.2.38) yields

$$\begin{aligned}
 & \lim_{\eta \rightarrow 0^+} \langle \vec{p}_f | T_R(\epsilon_{fi}) | \vec{p}_i \rangle \\
 &= \frac{ze^2}{2\pi^2 (\epsilon p)^2} \Gamma(1-i\nu_0) \left[\frac{(\epsilon p)^2 \kappa R}{m\omega |e|} \right]^{i\nu_0} e^{-\frac{\pi\nu_0}{2} \text{sgn} \ell}.
 \end{aligned}
 \tag{8.2.42}$$

As before we expand the quantities appearing in (8.2.42) about the reference energy ϵ given by (8.2.25) and obtain

$$\begin{aligned}
 & \lim_{\eta \rightarrow 0^+} \langle \vec{p}_f | T_R(\epsilon_{fi}) | \vec{p}_i \rangle \\
 &= \frac{ze^2}{8\pi^2 m \epsilon (1-\mu)} \Gamma(1-i\nu) e^{i\eta_e(R) - \frac{\pi\nu}{2} \text{sgn} \ell} \\
 & \times \left(1 - \frac{i\ell\omega}{4\epsilon} \eta_e(R) \right)
 \end{aligned}
 \tag{8.2.43}$$

where

$$\eta_e(R) = \nu \ln \left\{ \frac{4 p R \epsilon (1-\mu)}{\omega |e|} \right\}
 \tag{8.2.44}$$

In his paper Ford has shown that

$$\langle \vec{k}_1 | T(\kappa) | \vec{k}_2 \rangle = \langle \vec{k}_2 | T(\kappa) | \vec{k}_1 \rangle$$

Using this symmetry property in (8.2.34) one can get the following expression for $\langle \vec{p}_f | T_R(\epsilon_{fi}) | \vec{p}_i \rangle$.

$$\begin{aligned}
 & \langle \bar{p}_f | T_R (\epsilon_{p_f}) | \bar{p}_i \rangle \\
 &= \frac{Ze^2}{2\pi^2} \Gamma(1-i\nu_e) \frac{e^{\pi\nu_e/2 + i\delta_K(R)}}{((\delta p)^2)^{1-i\nu_e}} \times (p_i^2 - k^2)^{-i\nu_e} \\
 &+ O(1/R) \\
 &k^2 = 2m (\epsilon_{p_f} + i\eta) \\
 &\delta_K(R) = \nu_e \ln(2kR) + O(1/R)
 \end{aligned}
 \tag{8.2.45}$$

Proceeding as before we would get

$$\begin{aligned}
 & \lim_{\eta \rightarrow 0^+} \langle \bar{p}_f | T_R (\epsilon_{p_f}) | \bar{p}_i \rangle \\
 &= \frac{Ze^2}{8\pi^2 m \epsilon (1-\mu)} \Gamma(1-i\nu) e^{i\eta_e(R) + \frac{\pi\nu}{2} \text{sgn} \ell} \\
 &\times \left(1 + \frac{i\ell\omega}{4\epsilon} \eta_e(R) \right)
 \end{aligned}
 \tag{8.2.46}$$

As can be seen we need only reverse the sign of ℓ in (8.2.43) to obtain (8.2.46). This is because ϵ_{p_f} and ϵ_{p_i} are situated symmetrically about the reference energy ϵ . For real energy, therefore, the first two terms in (8.2.5) can be written as

$$\begin{aligned}
 & J_e (\vec{\alpha}_0 \cdot \vec{p}_f) J_e (-\vec{\alpha}_0 \cdot \vec{p}_i) \langle \vec{p}_f | T_R (\epsilon_{p_i}) | \vec{p}_i \rangle \\
 & + J_0 (\vec{\alpha}_0 \cdot \vec{p}_f) J_e (-\vec{\alpha}_0 \cdot \vec{p}_i) \langle \vec{p}_f | T_R (\epsilon_{p_f}) | \vec{p}_i \rangle \\
 & = \frac{Ze^2 \Gamma (1-i\nu)}{8\pi^2 m \epsilon (1-\mu)} e^{i\eta_e(R)} \times \\
 & \left[J_e (\vec{\alpha}_0 \cdot \vec{p}_f) J_0 (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{-\frac{\pi\nu}{2} \text{sgn } \ell} \right. \\
 & \left. \left(1 - \frac{i\omega\ell}{4\epsilon} \eta_e(R) \right) + \right. \\
 & \left. J_0 (\vec{\alpha}_0 \cdot \vec{p}_f) J_e (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{\frac{\pi\nu}{2} \text{sgn } \ell} \left(1 + \frac{i\omega\ell}{4\epsilon} \eta_e(R) \right) \right] \\
 & \hspace{15em} (8.2.47)
 \end{aligned}$$

Adding (8.2.32) and (8.2.47) we finally obtain

$$\begin{aligned}
 T_e^{(0)} + T_e^{(1)} &= \frac{Ze^2}{8\pi^2 m \epsilon (1-\mu)} \left[\left(\frac{\pi\nu}{\sinh \pi\nu} \right) \left(\frac{\omega^2}{8\epsilon^2 (1-\mu)} \right)^{-i\nu} \right. \\
 & \left. \left(K_1 + \frac{i\nu\omega}{2\epsilon} \left(K_2 - \frac{1}{2} K_1 \right) \ln \left\{ \frac{8\epsilon^2 (1-\mu)}{\omega^2} \right\} \right) + \right. \\
 & \Gamma (1-i\nu) e^{i\eta_e(R)} \left\{ J_e (\vec{\alpha}_0 \cdot \vec{p}_f) J_0 (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{-\frac{\pi\nu}{2} \text{sgn } \ell} \right. \\
 & \times \left(1 - \frac{i\omega\ell}{4\epsilon} \eta_e(R) \right) + J_0 (\vec{\alpha}_0 \cdot \vec{p}_f) J_e (-\vec{\alpha}_0 \cdot \vec{p}_i) e^{\frac{\pi\nu}{2} \text{sgn } \ell} \\
 & \times \left(1 + \frac{i\omega\ell}{4\epsilon} \eta_e(R) \right) \left. \right] \\
 & \hspace{15em} (8.2.48)
 \end{aligned}$$

The $T_L^{(2)}$ term is given by (8.2.2). It can be evaluated in the $R \rightarrow \infty$ limit. We show that it is finite and $O(\omega)$ and so can be neglected here. We can do this by allowing both Green's functions in (8.2.2) to be on-shell. That is, we set $\omega=0$ in the arguments. Then the matrix element in $T_L^{(2)}$ reduces to

$$\langle \vec{p}_f | V_R G_R^{(+)}(\epsilon_{p_f}) \vec{\alpha}_0 \cdot \vec{p} G_R^{(+)}(\epsilon_{p_i}) V_R | \vec{p}_i \rangle \quad (8.2.49)$$

Next we use the expressions

$$\langle \vec{p}_f | V_R G_R^{(+)}(\epsilon_{p_f}) = \langle \phi_{\vec{p}_f, R}^{(-)} - \langle \vec{p}_f \quad (8.2.50)$$

$$G_R^{(+)}(\epsilon_{p_i}) V_R | \vec{p}_i \rangle = | \phi_{\vec{p}_i, R}^{(+)} \rangle - | \vec{p}_i \rangle \quad (8.2.51)$$

where $\phi_{\vec{p}_f, R}^{(-)}$ and $\phi_{\vec{p}_i, R}^{(+)}$ are the in and out scattering states respectively. The plane wave parts are subtracted off in (8.2.50) and (8.2.51) so that all that remains is the scattered waves. Only the asymptotic forms of the scattered waves are needed to prove that the integral in (8.2.49) is finite. At large r the expressions (8.2.50) and (8.2.51) depend on r as $e^{i\vec{p}_f \cdot \vec{r}/r}$ and $e^{i\vec{p}_i \cdot \vec{r}/r}$ respectively. The parameter R enters as a phase factor and, therefore, does not affect the outcome. Also,

$$\vec{\alpha}_0 \cdot \vec{p} \left(\frac{e^{i\vec{p}_i \cdot \vec{r}}}{r} \right) = \frac{e^{i\vec{p}_i \cdot \vec{r}}}{r} \left(\vec{p}_i \frac{\vec{\alpha}_0 \cdot \vec{r}}{r} + i \frac{\vec{\alpha}_0 \cdot \vec{r}}{r^3} \right) \quad (8.2.52)$$

Thus the integral in (8.2.49) is of the form

$$\int d^3r \frac{e^{i(p_f + p_i)r}}{r^2} \left(p_i \frac{\vec{\alpha}_0 \cdot \vec{r}}{r} + i \frac{\vec{\alpha}_0 \cdot \vec{r}}{r^3} \right) \quad (8.2.53)$$

All the angular integrals in (8.2.53) are finite. The leading term in the radial integral is

$$\int r^2 dr \frac{e^{i(p_f + p_i)r}}{r^2}$$

which is finite as well. Thus the matrix element in $T_\ell^{(2)}$ is finite. Since $T_\ell^{(2)}$ contains a factor of ω it is of order ω and so can be dropped. A similar argument applies for higher terms. Therefore, (8.2.48) is the result for the T-matrix for the transfer of ℓ photons.

If we set $\alpha_0 = 0$ then $J_n(\vec{\alpha}_0 \cdot \vec{p}) = \delta_{n,0}$. Using this one can see that in the absence of the laser $T_\ell^{(0)} + T_\ell^{(1)}$ vanishes for $\ell \neq 0$. This is hardly surprising. The projectile can not absorb a non-zero number of photons without the presence of the external field.

In the high energy limit, ν goes to zero. Then,

$$\frac{\pi \nu}{\sinh \pi \nu} \rightarrow 1$$

$$e^{\pm \theta \nu} \rightarrow 1$$

$$\Gamma(1-i\nu) \rightarrow 1$$

$$\eta_C(R) \rightarrow 0$$

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \rightarrow \sum_{\lambda \neq 0,1} J_{\ell-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

Therefore,

$$\begin{aligned} & \lim_{\nu \rightarrow 0} \left(T_e^{(0)} + T_e^{(1)} \right) \\ &= \frac{ze^2}{8\pi^2 m \epsilon (1-\mu)} \left\{ K_1 + J_e(\vec{\alpha}_0 \cdot \vec{p}_f) J_0(-\vec{\alpha}_0 \cdot \vec{p}_i) \right. \\ & \quad \left. + J_0(\vec{\alpha}_0 \cdot \vec{p}_f) J_e(-\vec{\alpha}_0 \cdot \vec{p}_i) \right\} \\ &= J_e(\vec{\alpha}_0 \cdot (\vec{p}_f - \vec{p}_i)) \frac{ze^2}{8\pi^2 m \epsilon (1-\mu)} \end{aligned}$$

(8.2.54)

The quantity $\frac{ze^2}{8\pi^2 m \epsilon (1-\mu)}$ is $(-1/4m\pi^2)$ times the field-free Coulomb scattering amplitude in the first Born approximation for a projectile with energy $\epsilon = \epsilon_{p_i} - \frac{\ell\omega}{2}$. In this high energy limit, therefore, one recovers the Kroll-Watson result.

In general, however, the expression (8.2.48) does not reproduce the form of the Kroll-Watson result. There is a

Coulomb distortion phase factor $e^{i\eta_\ell(R)}$ in the last two terms of (8.2.48). The absolute square of $T_\ell^{(0)} + T_\ell^{(1)}$ is an observable and this phase factor will affect it. To the lowest order in ω $|T_\ell^{(0)} + T_\ell^{(1)}|^2$ will contain R in the form $e^{\pm i\eta_\ell(R)}$. In the spirit of Ford's suggestion we have treated R as physical parameter whose value is dictated by the experiment to be described. It could describe the box in which the scattering takes place. It could also arise from a model which attempts to describe the effect of neighboring atoms. In many cases it will not be known for each scattering event and then an ensemble average over R must be performed when a comparison with experiment is made.

Using the expression for $\eta_\ell(R)$ (see (8.2.44)) one sees that the factor $e^{i\eta_\ell(R)}$ will average to zero if

$$\int_{\nu \ln R_{\min}}^{\nu \ln R_{\max}} e^{i\nu \ln R} d(\nu \ln R) = 0$$

This is so only if

$$\nu \ln R_{\max} - \nu \ln R_{\min} = 2\pi n, \quad n = 1, 2, \dots$$

that is if

$$\frac{R_{\max}}{R_{\min}} = e^{2\pi n / \nu}$$

(8.2.55)

If this condition is satisfied then an effective

inherence will be introduced between the first and the last two terms of (8.2.48). It is, however, not obvious how the experimentalist will assign bounds on the values of R . This will depend on the experimental situation.

8.3 Evaluation of $T_R(\vec{k}_2, \vec{k}_1; k_{2m}^2)$ when $k_2 = k_1 \neq k$:

The calculation follows closely that given by Ford⁴⁶ for $k_2 \neq k_1$. Ford starts with an expansion of the T-matrix in Legendre polynomials. We instead expand only part of the T-matrix:

$$\begin{aligned} & \langle \vec{k}_2 | T_R(k_{2m}^2) - V_R | \vec{k}_1 \rangle \\ &= \frac{1}{4\pi^2 m} \sum_{l=0}^{\infty} (2l+1) P_l(\mu) M_l(k_2, k_1) \\ & \quad \mu = \hat{k}_2 \cdot \hat{k}_1 \end{aligned} \tag{8.3.1}$$

$$\begin{aligned} & M_l(k_2, k_1) \\ &= 2\pi^2 m \int_{-1}^{+1} \langle \vec{k}_2 | T_R(k_{2m}^2) - V_R | \vec{k}_1 \rangle P_l(\mu) d\mu \end{aligned} \tag{8.3.2}$$

$$= M_l(k_2, k_1) + m_l(k_1, k_2) \tag{8.3.3}$$

$$\begin{aligned} & i k m_l(k_2, k_1) \\ &= \int_0^R u_l(r, k_2) \int_r^R v_l(r', k_1) dr' dr \end{aligned} \tag{8.3.4}$$

$$\begin{aligned}
 &= \int_0^{\infty} u_e(r, k_2) \int_r^{\infty} v_e(r', k_1) dr' dr \\
 &- \int_0^R u_e(r, k_2) dr \int_R^{\infty} v_e(r, k_1) dr \\
 &- \int_R^{\infty} u_e(r, k_2) \int_r^{\infty} v_e(r', k_1) dr' dr
 \end{aligned}
 \tag{8.3.5}$$

$$u_e(r, k) = r j_e(kr) W(r) F_e^c(r)
 \tag{8.3.6}$$

$$v_e(r, k) = r j_e(kr) W(r) H_e^c(r)
 \tag{8.3.7}$$

$F_e^c(r)$ and $H_e^c(r)$ are the pure Coulomb functions.

$$W(r) = 2 m V_R(r)$$

The first term on the right side of (8.3.5) can be obtained from (8.3.4) if $R \rightarrow \infty$. This is what $ikm_e(k_2, k_1)$ would be for the pure Coulomb case. To emphasize this we write

$$\begin{aligned}
 &m_e(k_2, k_1, R) \\
 &= m_e(k_2, k_1, \infty) - \frac{1}{ik} U_e(R) V_e(R) \\
 &- \int_R^{\infty} u_e(r) v_e(r) dr
 \end{aligned}
 \tag{8.3.9}$$

where

$$U_e(R) = \int_0^R u_e(r) dr
 \tag{8.3.10}$$

and

$$V_L(R) = \int_R^{\infty} V_L(r) dr \quad (8.3.11)$$

We will show that the last two terms in (8.3.9) can be neglected for large R. Then

$$m_L(k_2, k_1, R) \rightarrow m_L(k_2, k_1, \infty)$$

which after the summation over $\mathcal{L}$ has been performed would give

$$\begin{aligned} & \langle \tilde{k}_2 | T_R (k^2/2m) - V_R | \tilde{k}_1 \rangle \\ &= \langle \tilde{k}_2 | T_{\infty} (k^2/2m) - V_{\infty} | \tilde{k}_1 \rangle \end{aligned} \quad (8.3.12)$$

Rearranging and using the notation (8.1.7) we would get

$$\begin{aligned} & T_R (\tilde{k}_2, \tilde{k}_1, k^2/2m) \\ &= T_{\infty} (\tilde{k}_2, \tilde{k}_1, k^2/2m) + \langle \tilde{k}_2 | V_R - V_{\infty} | \tilde{k}_1 \rangle \end{aligned} \quad (8.3.13)$$

The calculation so far has been general enough to include the case $k_2 = k_1$. The reason Ford excluded this case is the following: Ford has found that as $R \rightarrow \infty$

$$U_L(R) V_L(R) = O \left[\frac{1}{(k^2 - k_1^2) R} \right] \quad (8.3.14)$$

Also,

$$\int_R^{\infty} U_L(r) V_L(r) dr = O \left[\frac{1}{(k^2 - k_2^2) R} \right] \text{ if } k^2 \rightarrow k_1^2 \quad (8.3.15)$$

$$\int_R^\infty u_e(r) V_e(r) dr = O\left[\frac{1}{(k^2 - k_1^2)R}\right] \text{ if } k^2 \not\rightarrow k_1^2 \quad (8.3.16)$$

Thus if $k_2^2 = k_1^2$, the expression (8.3.15) is no longer negligible as $k^2 \rightarrow k_1^2$. Similarly the expressions (8.3.14) and (8.3.16) are not negligible as $k^2 \rightarrow k_2^2$. We, however, are considering the case when k^2 approaches neither k_1^2 nor k_2^2 (k_1^2 and k_2^2 are, in fact, equal to each other. But that is beside the point). Thus the last two terms in (8.3.9) can indeed be neglected as $R \rightarrow \infty$. This gives the result (8.3.13).

One may ask why we subtracted off V_R from the T-operator in (8.3.1). The answer is the following. One may write

$$\langle \vec{k}_2 | V_R | \vec{k}_1 \rangle = \frac{1}{4\pi^2 m} \sum_{\lambda=0}^{\infty} (2\lambda+1) B_\lambda(R) P_\lambda(\mu) \quad (8.3.17)$$

Therefore,

$$B_\ell(R) = 2\pi^2 m \int_{-1}^{+1} \langle \vec{k}_2 | V_R | \vec{k}_1 \rangle P_\ell(\mu) d\mu \quad (8.3.18)$$

This can be written as

$$B_\ell(R) = B_\ell(\infty) + C_\ell(R) \quad (8.3.19)$$

where

$$C_\ell(R) = 2\pi^2 m \int_{-1}^{+1} \langle \vec{k}_2 | V_R - V_\infty | \vec{k}_1 \rangle P_\ell(\mu) d\mu \quad (8.3.20)$$

The quantity $\langle \vec{k}_2 | V_R - V_\infty | \vec{k}_1 \rangle$ is not absolutely convergent. The integration can be performed by inserting a convergence factor $e^{-\lambda r}$. Thus

$$\begin{aligned} & \langle \vec{k}_2 | V_R - V_\infty | \vec{k}_1 \rangle \\ &= -\frac{ze^2}{2\pi^2 q} \lim_{\lambda \rightarrow 0^+} \int_R^\infty dr e^{-\lambda r} \sin qr \\ &= -\frac{ze^2}{2\pi^2 q^2} \cos qR \end{aligned} \quad (8.3.21)$$

Substituting in (8.3.20) we get

$$C_\ell(R) = -ze^2 m \int_{-1}^{+1} \frac{\cos qR}{q^2} P_\ell(\mu) d\mu \quad (8.3.22)$$

where

$$q^2 = k_1^2 + k_2^2 - 2k_1 k_2 \mu \quad (8.3.23)$$

The integral in (8.3.22) becomes singular if $q^2 = 0$. This is so if

$$\mu = \frac{k_1^2 + k_2^2}{2k_1 k_2} = \mu_0 \text{ (say)} \quad (8.3.24)$$

If $k_1 \neq k_2$ then one can write

$$\mu_0 = \frac{1 + \chi^2}{1 - \chi^2}$$

where

$$\chi^2 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

is less than unity. This value of μ_0 is greater than 1 and hence outside the domain of integration. The integral is finite and $O(1/R)$ as $R \rightarrow \infty$. This confirms Ford's assertion that for $k_1 \neq k_2$

$$B_\ell(R) = B_\ell(\infty) + O(1/R) \quad (8.3.25)$$

If, however, we set $k_1 = k_2$ then $\mu_0 = 1$ which is the upper limit of the integral in (8.3.22). then $C_\ell(R)$ is no longer a small quantity.

8.4 Derivation of the T-matrix for $\ell=0$:

For $\ell=0$, (8.2.1) gives

$$\begin{aligned} & T_c^{(0)} + T_c^{(1)} \\ &= \sum_{\lambda=-\infty}^{\infty} J_{-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) \langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda \omega) | \vec{p}_i \rangle \end{aligned} \quad (8.4.1)$$

We separate out the $\lambda=0$ term and write

$$\begin{aligned}
 & T_L^{(0)} + T_L^{(1)} \\
 &= J_{-0}(\vec{\alpha}_0 \cdot \vec{p}_f) J_0(-\vec{\alpha}_0 \cdot \vec{p}_i) \langle \vec{p}_f | T_R(\epsilon_{p_i}) | \vec{p}_i \rangle \\
 &+ \sum_{\lambda \neq 0} J_{-\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) \langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda \omega) | \vec{p}_i \rangle
 \end{aligned}
 \tag{8.4.2}$$

In the second term on the right side of (8.4.2) we can apply (8.3.13) to get

$$\begin{aligned}
 & \langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda \omega) | \vec{p}_i \rangle \\
 &= T_R(\vec{p}_f, \vec{p}_i, k^2/2m) \\
 &= T_{\omega}(\vec{p}_f, \vec{p}_i, \frac{k^2}{2m}) + \langle \vec{p}_f | V_R - V_{\omega} | \vec{p}_i \rangle
 \end{aligned}
 \tag{8.4.3}$$

where

$$k^2/2m = \epsilon_{p_i} - \lambda \omega + i\eta
 \tag{8.4.4}$$

The expression for the first term on the right side of (8.4.3) can be obtained by putting $\ell=0$ in the expression (8.2.31). For $\ell=0$,

$$\epsilon = \epsilon_{p_i}, \quad \nu = \nu_0$$

$$| \lambda (1-\lambda) | = | -\lambda^2 | = | \lambda^2 | = \lambda^2$$

Also,

$$e^{i\theta\gamma} = e^{i\pi\gamma_0 \operatorname{sgn} \lambda}$$

Therefore, for real energy,

$$\begin{aligned} T_{\omega}(\vec{p}_f, \vec{p}_i, \epsilon_{pi} - \lambda\omega) &= \frac{Ze^2}{8\pi^2 m \epsilon_{pi} (1-\mu)} \times \\ &\frac{\pi\gamma_0}{\sinh \pi\gamma_0} (\lambda^2)^{-i\gamma_0} e^{i\pi\gamma_0 \operatorname{sgn} \lambda} \left(\frac{\omega^2}{8\epsilon_{pi}^2 (1-\mu)} \right)^{-i\gamma_0} \\ &\times \left(1 + \frac{i\gamma_0 \omega \lambda}{2\epsilon_{pi}} \ln \left\{ \frac{8\epsilon_{pi}^2 (1-\mu)}{\omega^2} \right\} \right) \end{aligned} \quad (8.4.5)$$

Also,

$$\langle \vec{p}_f | V_R - V_{\infty} | \vec{p}_i \rangle = - \frac{Ze^2 \cos[(\Delta p)R]}{2\pi^2 (\Delta p)^2} \quad (8.4.6)$$

where

$$\begin{aligned} (\Delta p)^2 &= p_f^2 + p_i^2 - 2\mu p_f p_i \\ &= 4m \epsilon_{pi} (1-\mu) \end{aligned}$$

Hence,

$$\langle \vec{p}_f | V_R - V_{\infty} | \vec{p}_i \rangle = \frac{-Ze^2 \cos[(\Delta p)R]}{8\pi^2 m \epsilon_{pi} (1-\mu)} \quad (8.4.7)$$

Substituting into (8.4.3) we get

$$\begin{aligned} \langle \vec{p}_f | T_R (\epsilon_{pi} - \lambda \omega) | \vec{p}_i \rangle &= \frac{ze^2}{8\pi^2 m \epsilon_{pi} (1-\mu)} \times \\ &\left[\frac{\pi \nu_0}{\sinh \pi \nu_0} (\lambda^2)^{-i\nu_0} e^{\pi \nu_0 \operatorname{sgn} \lambda} \times \left(\frac{\omega^2}{8 \epsilon_{pi}^2 (1-\mu)} \right)^{-i\nu_0} \right. \\ &\left. \left(1 + \frac{i\nu_0 \omega \lambda}{2 \epsilon_{pi}} \ln \left\{ \frac{8 \epsilon_{pi}^2 (1-\mu)}{\omega^2} \right\} \right) - \cos [(4p)R] \right] \end{aligned}$$

(8.4.8)

The quantity $\langle \vec{p}_f | T_R (\epsilon_{pi}) | \vec{p}_i \rangle$ is the on-shell T-matrix for the cut-off Coulomb potential. The corresponding scattering amplitude is given by⁴⁸

$$f_R (\vec{p}_f \leftarrow \vec{p}_i) = f_{osc.} (\vec{p}_f \leftarrow \vec{p}_i) + e^{2i\nu_0 \ln 2p_i R} f_{\infty} (\vec{p}_f \leftarrow \vec{p}_i) \quad (8.4.9)$$

where $f_{\infty} (\vec{p}_f \leftarrow \vec{p}_i)$ is the usual Coulomb scattering amplitude. f_{osc} oscillates rapidly like $\cos (p_i R \theta)$ when θ is larger than $1/p_i R$.

$$\begin{aligned} \langle \vec{p}_f | T_R (\epsilon_{pi}) | \vec{p}_i \rangle &= -\frac{1}{4\pi^2 m} f_R (\vec{p}_f \leftarrow \vec{p}_i) \\ &= -\frac{1}{4\pi^2 m} \left[f_{osc.} (\vec{p}_f \leftarrow \vec{p}_i) + e^{2i\nu_0 \ln 2p_i R} f_{\infty} (\vec{p}_f \leftarrow \vec{p}_i) \right] \end{aligned} \quad (8.4.10)$$

Using (8.4.8) and (8.4.10) in (8.4.2) we get

$$\begin{aligned}
 T_{\ell}^{(0)} + T_{\ell}^{(1)} = & -J_0(\vec{\alpha}_0 \cdot \vec{p}_f) J_0(\vec{\alpha}_0 \cdot \vec{p}_i) \frac{1}{4\pi^2 m} \times \\
 & \left[f_{osc} + e^{2i\nu_0 \ln 2\pi R} f_{\infty} \right] + \\
 & \frac{Ze^2}{8\pi^2 m \epsilon_{pi} (1-\mu)} \left\{ \frac{\pi \nu_0}{\sinh \pi \nu_0} \left(\frac{\omega^2}{8\epsilon_{pi}^2 (1-\mu)} \right)^{-i\nu_0} \right. \\
 & \left. \left[I_1 + \frac{i\nu_0 \omega}{2\epsilon_{pi}} I_2 \ln \left\{ \frac{8\epsilon_{pi}^2 (1-\mu)}{\omega^2} \right\} \right] \right. \\
 & \left. - \left(J_0(\vec{\alpha}_0 \cdot (\vec{p}_f - \vec{p}_i)) - J_0(\vec{\alpha}_0 \cdot \vec{p}_f) J_0(-\vec{\alpha}_0 \cdot \vec{p}_i) \right) \cos[(\vec{q}R)] \right\}
 \end{aligned}
 \tag{8.4.11}$$

where

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \sum_{\substack{\lambda=-\infty \\ \lambda \neq 0}}^{\infty} J_{\lambda}(\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda}(-\vec{\alpha}_0 \cdot \vec{p}_i) (\lambda^2)^{-i\nu_0} e^{i\pi \nu_0 \operatorname{sgn} \lambda} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}
 \tag{8.4.12}$$

As can be seen the R-dependence of $T_{\ell}^{(0)} + T_{\ell}^{(1)}$ is much more involved for $\ell = 0$ than for $\ell \neq 0$. Unlike the $\ell \neq 0$ case, there is no unique condition (such as (8.2.55)) for the introduction of an effective incoherence among the terms of $T_{\ell}^{(0)} + T_{\ell}^{(1)}$. Thus the observable result will depend upon R. One will of course get a unique result for a unique value of R. But since the experimental result depends on R and the values of R vary from laboratory to laboratory, this would seem to suggest that the experiments are not reproducible. While we do not know the

way out of this situation we would like to point out that an analogous situation occurs even for the field-free case (see equation (8.4.10))

To see the significance of this we use the wave packet description of scattering⁵⁰. The scattering cross section is defined in terms of a large number of collisions with incident wave packets suitably randomized as to shape and impact parameter. The first step in the calculation is to evaluate the out wave function

$\psi_{out}(p)$ for any given in wave packet $\psi_{in}(p)$:

$$\psi_{in}(\vec{p}) = \int d^3p' \phi(\vec{p}) |\vec{p}'\rangle \quad (8.4.13)$$

$$\begin{aligned} \psi_{out}(\vec{p}) &= \int d^3p' \langle \vec{p} | S | \vec{p}' \rangle \psi_{in}(\vec{p}') \\ &= \psi_{in}(\vec{p}) + \frac{i}{2\pi} \int d\Omega_{p'} f(\vec{p} \leftarrow \vec{p}') \psi_{in}(\vec{p}') \end{aligned} \quad (8.4.14)$$

Here the first term is the unscattered incident wave and to avoid seeing this one restricts measurements to non-forward directions. The integral represents the scattered wave which we wish to measure. To evaluate the integral one tacitly assumes that $\psi_{in}(p')$ is sufficiently peaked in momentum so that $f(\vec{p} \leftarrow \vec{p}')$ varies by only a small amount over the spread of the wave packet. This will not in general be true for the amplitude f_{osc} when $p_i R \theta \gg 1$. Unless the wave packet spread in momentum space is very small, f_{osc} will oscillate and

contribute little to the integral in (8.4.14). For example, with a Gaussian wave packet of width Δk , one can find⁴⁹

$$\langle f_{osc} \rangle \simeq e^{-[\theta (\Delta k) R]^2} \quad (8.4.15)$$

for the mean value of f_{osc} averaged over the wave packet. We conclude from this that the contribution of f_{osc} to (8.4.14) is effectively negligible if the wave packet has a width $\Delta(k\theta)$ corresponding to the range of $k\theta$ values such that $\Delta(k\theta) \gg 1/R$. In that case only the second term on the right side of (8.4.10) contributes. Since the observable quantity is the absolute square of the T-matrix, the phase factor in front disappears and one gets an R-independent result. Thus the condition on R is

$$R \gg \frac{1}{\Delta(k\theta)} \quad (8.4.16)$$

Note that

$$(\Delta p) R = 2 p R \sin \theta/2$$

Precisely the analysis applied to f_{osc} can be used for the term containing $\cos[(\Delta p) R]$ in (8.4.11).

Assuming that R satisfies the condition (8.4.16) we can drop f_{osc} and $\cos[(\Delta p) R]$ from the right side of (8.4.11). Unlike the field free case the scattering cross

section will still depend on R because of the interference of the term $e^{2i\nu_0 \ln 2\pi i R} f_0$ with the remaining R-independent terms. The condition for an effective incoherence will be

$$\int_{2\nu_0 \ln R_{\min}}^{2\nu_0 \ln R_{\max}} e^{2i\nu_0 \ln R} d(2\nu_0 \ln R) = 0$$

This is so if

$$\frac{R_{\max}}{R_{\min}} = e^{\pi n / \nu}, \quad n = 1, 2, \dots \quad (8.4.17)$$

Finally we consider two limiting cases. In the absence of the laser ($\alpha_0 = 0$)

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \sum_{\lambda \neq 0} \delta_{\lambda,0} \delta_{\lambda,0} (\lambda^2)^{-i\nu_0} e^{\pi \nu_0 \operatorname{sgn} \lambda} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Also,

$$J_{-0}(\bar{\alpha}_0 \cdot \bar{p}_f) J_0(-\bar{\alpha}_0 \cdot \bar{p}_i) = 1 = J_0(\bar{\alpha}_0 \cdot (\bar{p}_f - \bar{p}_i))$$

Thus for $\alpha_0 = 0$

$$T_e^{(0)} + T_e^{(1)} = -\frac{1}{4\pi^2 m} \left(f_{osc} + e^{2i\nu_0 \ln 2\pi i R} f_0 \right)$$

which is the field-free result.

In the high energy limit, the condition (8.4.16) is

satisfied. So we drop f_{osc} and $\cos[(\mathbf{q} \cdot \mathbf{p})R]$ from the right side of (8.4.11). Also, γ_0 goes to zero in which case

$$e^{2i\gamma_0} \ln 2\pi i R f_{\omega} \rightarrow -\frac{2mZe^2}{(\mathbf{q} \cdot \mathbf{p})^2} = -\frac{Ze^2}{2\epsilon_{pi}(1-\mu)} \quad (8.4.18)$$

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \rightarrow \sum_{\lambda \neq 0} J_{-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

Therefore,

$$\begin{aligned} & \mathcal{L}_{\gamma_0 \rightarrow 0} \left(T_e^{(0)} + T_e^{(1)} \right) \\ &= \frac{Ze^2}{8\pi^2 m \epsilon_{pi}(1-\mu)} \left[J_{-0} (\vec{\alpha}_0 \cdot \vec{p}_f) J_0 (-\vec{\alpha}_0 \cdot \vec{p}_i) \right. \\ & \quad \left. + \sum_{\lambda \neq 0} J_{-\lambda} (\vec{\alpha}_0 \cdot \vec{p}_f) J_{\lambda} (-\vec{\alpha}_0 \cdot \vec{p}_i) \right] \\ &= J_0 (\vec{\alpha}_0 \cdot (\vec{p}_f - \vec{p}_i)) \frac{Ze^2}{8\pi^2 m \epsilon_{pi}(1-\mu)} \quad (8.4.19) \end{aligned}$$

Thus we again recover the Kroll-Watson result in the high energy limit.

The results obtained in this chapter may of more than just mathematical interest. The laser-modified T-matrix for the Coulomb case contains terms other than the T-matrix in the absence of the laser. In the more

interesting physical case we would have an additional short range potential. These results seem to indicate that the laser-modified T-matrix with an additional short range potential would contain scattering information which is not available in the absence of the laser. This is in sharp distinction from the Kroll-Watson result (8.1.1) for short range potentials where this is not so.

We conclude this chapter by posing a paradoxical question and then resolving the paradox. If one could proceed as in chapter 2, one would have gotten the Kroll-Watson result (8.1.1) for the cut-off Coulomb potential. The results of this chapter show, however, that the Kroll-Watson result can be recovered only in the limit of high energy. What (if any) went wrong? The answer to this question lies in the fact that V_R is a long (albeit finite) range potential, that is that R is a large number.

The first step in getting the Kroll-Watson result is the expansion of the matrix elements in powers of ω .

$$\begin{aligned} & \langle \vec{p}_f | T(\epsilon_{p_i} - \lambda\omega) | \vec{p}_i \rangle \\ &= \langle \vec{p}_f | T(\epsilon_{p_i}) | \vec{p}_i \rangle - \lambda\omega \langle \vec{p}_f | \frac{\partial T(\epsilon_{p_i})}{\partial \epsilon_{p_i}} | \vec{p}_i \rangle + o(\omega^2) \end{aligned} \quad (8.4.20)$$

This requires that $\langle \vec{p}_f | T(\epsilon_{p_i} - \lambda\omega) | \vec{p}_i \rangle$ be analytic in ω at $\omega = 0$. In what follows we will show that for the cut-off Coulomb T-matrix $\langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda\omega) | \vec{p}_i \rangle$ the analyticity in ω breaks down in the limit $R \rightarrow \infty$.

For $\lambda = 0, \ell$, $\langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda \omega) | \vec{p}_i \rangle$ has the expressions (8.2.43) and (8.2.46) respectively. These are functions of $\gamma_\ell(R)$ (see equation (8.2.44)) which is not analytic in ω . If $\lambda \neq 0, \ell$, $\langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda \omega) | \vec{p}_i \rangle$ is totally off-shell and has the following expression for large R

$$\begin{aligned} T_R(\vec{p}_f, \vec{p}_i, k^2/2m) &= T_\infty(\vec{p}_f, \vec{p}_i; k^2/2m) \\ &+ Q[(R - p_i)R] T_R(\vec{p}_f, \vec{p}_i; p_i^2/2m) \\ &+ Q[(R + p_i)R] T_R(\vec{p}_f, \vec{p}_i; (-p_i)^2/2m) \\ &+ Q[(R - p_f)R] T_R(\vec{p}_f, \vec{p}_i; p_f^2/2m) \\ &+ Q[(R + p_f)R] T_R(\vec{p}_f, \vec{p}_i; (-p_f)^2/2m) \\ &+ O(1/R) \end{aligned} \quad (8.4.21)$$

with⁴⁷

$$Q[z] = -i\nu \int_1^\infty t^{i\nu-1} e^{izt} dt \quad (8.4.22)$$

and

$$k^2 = 2m(\epsilon_{p_i} - \lambda \omega) \quad (8.4.23)$$

The first term on the right side of (8.4.21) is the totally off-shell pure Coulomb T-matrix (see equation

(8.2.24)). This is not analytic in ω . The same statement applies to the Q functions also. We show this explicitly for $Q[(k-p_i)R]$. A similar analysis holds for the others.

Using (1.4.4) and (8.4.23) one gets

$$z = (k - p_i)R \approx \frac{-\lambda\omega p_i R}{2\epsilon_{p_i}} \quad (8.4.24)$$

For $\lambda\omega \neq 0$, this quantity is large because R is a large quantity, in which case $Q[z]$ has the limiting behavior⁴⁷

$$Q[z] \xrightarrow{z \rightarrow \infty} \nu \left(\frac{e^{i^2 z}}{z} \right) \quad (8.4.25)$$

Substitution of (8.4.24) into (8.4.25) gives

$$Q[(k-p_i)R] \xrightarrow{R \rightarrow \infty} \frac{-2\nu \epsilon_{p_i}}{\lambda\omega p_i R} e^{-i\lambda\omega \frac{p_i R}{2\epsilon_{p_i}}} \quad (8.4.26)$$

This asymptotic value of $Q[(k-p_i)R]$ is non-analytic in ω at $\omega = 0$.

The upshot of the above discussion is that the quantity $\langle \vec{p}_f | T_R(\epsilon_{p_i} - \lambda\omega) | \vec{p}_i \rangle$ cannot be expanded in the form (8.4.20) for any value of λ and, therefore, one is not guaranteed a result in the Kroll-Watson form.

The physical reason for the deviation of the results of this chapter from the Kroll-Watson result is the following: The Kroll-Watson result is derived in the low-frequency approximation under the assumption that the scattering potential is not long range. Therefore, the contribution of that part of the potential for which the

potential energy is less than the laser photon energy falls off rapidly. This evidently is not so for a long range potential. In that case the electron is influenced by the potential for a long time and for much of that time the potential energy is less than the laser photon energy.

On the basis of this we make the following remark on the results of chapter 7. Those results are valid only for short range potentials in the low frequency approximation. But the potentials involved in ionising collisions are long range in nature in which case the results of chapter 7 lose relevance. In the limit of high incident energy $E \gg Ry$ one will of course obtain results in the Kroll-Watson form. But in that case, it is not necessary to make the low frequency approximation and the first Born treatment will suffice.

CHAPTER-9

CONCLUSION

In order to put the work of the present author in the proper perspective it was necessary to discuss the contribution of many other workers. This may have enshrouded the specific contributions made by the present author in this dissertation. It is, therefore, appropriate to point them out on a chapter by chapter basis.

The central result of chapter 2 is that of Kroll and Watson for non-resonant potential scattering in the presence of a low-frequency laser (see equations (1.4.1)-(1.4.5)). (a) We noted that the extension of the Kroll-Watson result to higher orders in ω is straightforward but tedious (see equation (2.1.17) and the discussion in p.21). Major effort is involved in the summation over the index λ for which one (i) expands the matrix elements to higher orders in ω (ii) derives sum rules of the form $\sum_{n=-\infty}^{\infty} n^m J_n(y) J_{N-n}(y')$ for $m > 1$ and then (iii) uses them to perform the summation over λ . In view of this we have invented a gauge transformation (see equation (2.2.6)) that reduces the summation over λ to a trivial job. (b) Using this simplified technique we have calculated T_L to order ω^3 and (c) $\sum_L \frac{d\sigma_L}{d\Omega}$ to order ω^2 . An interesting finding is that

$$\sum_{\ell=-\infty}^{\infty} \frac{d\sigma_{\ell}}{d\Omega} (\vec{p}_f, \vec{p}_i) = \frac{d\sigma}{d\Omega} (\epsilon_{p_i}, \hat{p}_f, \hat{p}_i) + \beta \omega^2 \quad (2.4.17)$$

where β as given by (2.4.18) is a sum of on-shell terms.

This is an extension of the sum rule (1.4.10) previously obtained by Kruger and Jung. If β is positive (negative) there would be more (less) scattering in the presence of the laser than in the absence of the laser. The occurrence of this term can be explained as follows:

Electrons with a specific energy ϵ have a definite probability of getting scattered by an angle θ . This is proportional to $\frac{d\sigma}{d\Omega}(\epsilon, \mu)$ where $\mu = \cos\theta$. In the presence of the laser, electrons absorb or emit photons. This changes their energies and hence their probabilities for scattering. The net change need not amount to zero. The details depend on $\frac{d\sigma}{d\Omega}(\epsilon, \mu)$ and its derivatives with respect to ϵ and μ . Integration of the above result over the final direction of the scattering would result in a simpler sum rule (see Equation (2.4.19)). (c) Finally, we have pointed out that the technique of grouping some off-shell matrix elements to form an on-shell one (a la Kroll and Watson) may not always be justified for sufficiently small values of the quantity $\vec{\alpha}_0 \cdot (\vec{p}_f - \vec{p}_i)$.

Chapter 3 deals with resonant potential scattering in the presence of a low frequency laser. Resonance introduces a new energy parameter Γ , the resonance width. For CO_2 laser $\omega \sim 0.1 \text{ ev}$ and a typical Feshbach resonance width is of order $5 \times 10^{-3} \text{ ev}$ so that $\frac{\Gamma}{\omega} \sim 5 \times 10^{-2}$. Let us consider the presence of a scattering resonance in the absence of the laser at the energy $E_R \approx E_K - \frac{i\Gamma}{2}$ such

that $\epsilon_R \simeq \epsilon_0 = \epsilon_{p_i} + m_0 \omega$ where ϵ_{p_i} is the energy of the incident electron and m_0 is an integer. Treating both Γ/ω and ω/ϵ_0 as quantities of first order of smallness we systematically extended the previous work of Kruger and Jung¹³ and Mittleman¹⁶ to higher orders in the expansion parameters Γ/ω and ω/ϵ_0 .

The T-matrix for scattering in which the momentum of the electron changes from $\vec{p}_i$ to $\vec{p}_f$ with the net absorption of ℓ photons is given by

$$T_\ell(\vec{p}_f, \vec{p}_i) = \sum_{n=0}^{\infty} T_\ell^{(n)}(\vec{p}_f, \vec{p}_i) \quad (2.1.10)$$

where the expressions for the terms on the right side of (2.1.10) are given in (2.1.11), (2.1.12), (2.1.14) and (2.1.16). Carefully separating out the resonant and non-resonant parts of each term, we get the expression (3.2.25) for T_ℓ . This result is more compact and elegant than that of the previous workers and it reveals some interesting new features. The first term on the right side of (3.2.25) is the sum of non-resonant contributions upto $T_\ell^{(2)}$, the second term is the sum of resonant contributions. The shift of the resonance energy from ϵ_0 to $\epsilon_0 - D$ is due to contributions from terms higher than $T_\ell^{(2)}$. The last term on the right side of (3.2.25) is the resonant contributions from off-resonance energies. D , the dynamical Stark shift, is in general, a complex quantity. The effect of the presence of D is to shift the resonance

energy from ϵ_R to $\epsilon_R + \text{Re } D$ and the half-width from $\Gamma/2$ to $\Gamma/2 - \text{Im } D$. If one plots $\frac{d\sigma_\ell}{d\Omega}$ versus energy, one gets a narrower resonance for $\text{Im } D > 0$ and a broader resonance for $\text{Im } D < 0$. The resonance peak is shifted to the right for $\text{Re } D > 0$ and to the left for $\text{Re } D < 0$ (see fig. 1). Clearly, if the values of $\text{Re } D$ and $\text{Im } D$ are sufficiently large, an electron with energy ϵ_0 will go out of resonance. The condition for remaining on resonance is

$$|\epsilon_0 - \epsilon_R - \text{Re } D| < |\Gamma/2 - \text{Im } D|$$

Previously, the condition was

$$|\epsilon_0 - \epsilon_R| < \Gamma/2$$

In (3.2.27), we have an alternative expression for T_ℓ in which all the matrix elements are evaluated on-shell.

In order to sum the resonant contributions from terms higher than $T_\ell^{(2)}$ on the right side of (2.1.10), we had to impose the condition

$$\epsilon_0 > \left(\frac{\omega}{\Gamma}\right) \omega$$

Because of this inequality, one cannot apply the present work to very low-energy resonances.

Using the expression for T_ℓ , we have evaluated $\sum \frac{d\sigma_\ell}{d\Omega}$ (see equation (3.3.24)). This is an extension of the previous result of Mittleman¹⁶ (see equation (3.1.13)) to higher orders in the expansion parameters. In Mittleman's

result only the real part of the interference between the resonant and the non-resonant terms is present whereas in (3.3.24) both the real and the imaginary parts of the interference terms occur.

Finally, we have evaluated $\langle l \rangle$, the average number of photons absorbed during a scattering event. To the lowest order in the expansion parameters, we found (see equation (3.3.28)) that $\langle l \rangle$ is proportional to m_0 , where m_0 is the number of photons the incident electron must absorb in order to attain the resonance energy $\epsilon_0 = \epsilon_{p_i} + m_0 \omega$. For a background free (that is, pure Breit-Wigner) resonance, $\langle l \rangle$ is strictly equal to m_0 .

The results of chapter 4 have been obtained previously by Mittleman¹⁴. However, the method adopted in chapter 4 is simpler than Mittleman's and a minor error in his evaluation of $\langle l \rangle$ has been corrected here.

The real purpose of chapter 4 was to define and build the machinery for solving the problem of resonant electron-atom scattering in the presence of a low-frequency laser. This is the content of chapter 5. We have obtained the T-matrix (see (5.1.17) or (5.1.18)) for resonant electron-atom scattering with the net absorption of l photons. We have also evaluated $\sum_l \frac{d\sigma_l}{d\Omega}$ and $\langle l \rangle$ (see equations (5.2.15) and (5.2.16) respectively). The results are analogous to those obtained for resonant potential scattering.

Weingartshofer et al^{17, 30, 32} performed a series of experiments on resonant electron-atom scattering in the presence of a low-frequency laser. But the theory for the above scattering process was lacking. We must stress here that the results of chapter 5 would, for the first time, enable one to compare the theory with the experiments of Weingartshofer et al and vice versa.

Chapter 6 deals with the experimental aspects of laser-modified electron-atom scattering. After a brief survey of experiments performed in this field we have analyzed the experimental results and pointed out the difficulties encountered by the experimentalist. We also commented on the approximations inherent in the theory and how they can be remedied.

In chapter 7 we discussed the problem of laser-assisted electron-atom ionising collisions in the low-frequency approximation. In section 7.2 we obtained the expression for the triple differential cross section in the absence of any scattering resonance. The subsequent section deals with the effect of resonance in laser-assisted ionising collisions.

The original work of Kroll and Watson¹² and almost all subsequent papers⁴² on scattering in the presence of a low-frequency laser field ignored the long range effects of the Coulomb potential. The purpose of chapter 8 is to account for the effects of the Coulomb tail. For

simplicity we have considered cut-off Coulomb potential scattering in the presence of a low-frequency laser. Our conclusion is that the Kroll-Watson result is not valid for cut-off Coulomb potentials unless the collision energy is very high. For high energy, however, it is unnecessary to make the low-frequency approximation and the first Born treatment will be sufficient. Moreover, we have suggested that the experiments are not reproducible unless the details of the cut-off Coulomb potential are spelled out.

APPENDIX

A. Some Bessel function sum rules:

$$\sum_{n=-\infty}^{\infty} J_n(\gamma) J_{N-n}(\gamma') = J_N(\gamma + \gamma') \quad (\text{A-1})$$

$$\sum_{n=-\infty}^{\infty} n J_n(\gamma) J_{N-n}(\gamma') = \frac{N\gamma}{\gamma + \gamma'} J_N(\gamma + \gamma') \quad (\text{A-2})$$

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} n^2 J_n(\gamma) J_{N-n}(\gamma') \\ &= \frac{\gamma}{\gamma + \gamma'} \left[\gamma' J'_N(\gamma + \gamma') + \frac{N^2 \gamma}{\gamma + \gamma'} J_N(\gamma + \gamma') \right] \end{aligned} \quad (\text{A-3})$$

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} n^3 J_n(\gamma) J_{N-n}(\gamma') \\ &= \left[\frac{N^3 \gamma^3}{(\gamma + \gamma')^3} - \frac{N\gamma\gamma'(\gamma - \gamma')}{(\gamma + \gamma')^3} \right] J_N(\gamma + \gamma') \\ &+ \frac{3N\gamma^2\gamma'}{(\gamma + \gamma')^2} J'_N(\gamma + \gamma') \end{aligned} \quad (\text{A-4})$$

$$\sum_{\ell=-\infty}^{\infty} \begin{pmatrix} 1 \\ \ell \\ \ell^2 \\ 3 \end{pmatrix} J_{\ell}^2(x) = \begin{pmatrix} 1 \\ 0 \\ x^2/2 \\ 0 \end{pmatrix} \quad (\text{A-5})$$

$$\sum_{a=\pm 1} J_{n+a}(x) = \frac{2n}{x} J_n(x) \quad (\text{A-6})$$

B. Some formulas for integration over the final direction of the scattering:

Suppose the unit vectors along the initial and final directions of scattering are $\hat{p}_i$ and $\hat{p}_f$ respectively. We set up a coordinate system such that $\hat{p}_i$ is along the Z axis. Furthermore, the X axis is drawn (perpendicular to the Z axis, of course) in such a way that the XZ plane contains $\hat{\alpha}_0$ where $\hat{\alpha}_0$ is the unit vector along the direction of the laser (see fig.5).

$$\begin{aligned}\hat{p}_f &= \hat{i} \sin \theta \cos \phi + \hat{j} \sin \theta \sin \phi + \hat{k} \cos \theta \\ \hat{\alpha}_0 &= \hat{i} \sin \psi + \hat{k} \cos \psi \\ \hat{\alpha}_0 \cdot \hat{p}_f &= \sin \theta \sin \psi \cos \phi + \cos \theta \cos \psi\end{aligned}\tag{B-1}$$

We define

$$\begin{aligned}\nu &= \hat{\alpha}_0 \cdot \hat{p}_i \\ \mu &= \hat{p}_f \cdot \hat{p}_i\end{aligned}\tag{B-2}$$

Then

$$\begin{aligned}\nu &= \cos \psi \\ \mu &= \cos \theta\end{aligned}\tag{B-3}$$

and

$$\hat{\alpha}_0 \cdot \hat{p}_f = (1-\mu^2)^{1/2} (1-\nu^2)^{1/2} \cos \phi + \mu \nu\tag{B-4}$$

Integrating over ϕ we get

$$\int_0^{2\pi} \hat{\alpha}_0 \cdot \hat{p}_f d\phi = 2\pi \mu \nu = \gamma_1 \text{ (say)}$$

$$\int_0^{2\pi} (\hat{\alpha}_0 \cdot \hat{p}_f)^2 d\phi = \pi (3\mu^2 \nu^2 + 1 - \mu^2 - \nu^2) = \gamma_2 \text{ (say)}$$

(B-5)

So,

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = 2\pi \begin{pmatrix} \nu \\ \nu^2 \end{pmatrix} \text{ at } \mu = 1$$

(B-6)

and

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = 2\pi \begin{pmatrix} -\nu \\ \nu^2 \end{pmatrix} \text{ at } \mu = -1$$

(B-7)

C. Atomic states in low-frequency laser fields:

The equation for an atom in the presence of a low frequency laser is

$$i \frac{\partial \hat{\phi}}{\partial t} = (H_0 + H') \hat{\phi}$$

(C-1)

where H_0 is the unperturbed Hamiltonian with eigenfunctions $\{U_n\}$ and corresponding eigenvalues $\{E_n\}$:

$$H_0 U_n = E_n U_n$$

(C-2)

H' is the laser-atom interaction

$$H' = \omega \vec{\alpha}_0 \cdot \vec{P} \cos \omega t$$

(C-3)

where

$$\vec{p} = \sum_{j=1}^z \vec{p}_j \quad (C-4)$$

$\vec{p}_j$ is the momentum of the j-th electron and z is the atomic number. $\vec{\alpha}_1 = e\vec{E}_0/m\omega^2$ where $\vec{E}_0$ is the amplitude of the electric field of the laser.

The standard procedure is to expand $\hat{\phi}$ in terms of the eigenfunctions $U_n e^{-iE_n t}$ of the unperturbed time dependent wave equation:

$$\hat{\phi} = \sum_s a_s(t) U_s e^{-iE_s t} \quad (C-5)$$

Substituting (C-5) into (C-1) we get

$$\begin{aligned} & \sum_s i \dot{a}_s U_s e^{-iE_s t} + \sum_s a_s E_s U_s e^{-iE_s t} \\ &= \sum_s a_s (H_0 + H') U_s e^{-iE_s t} \end{aligned} \quad (C-6)$$

Replacing $H_0 U_s$ by $E_s U_s$ on the right side, multiplying through on the left by U_k^* and integrating over all space, we get

$$\dot{a}_k = -i \sum_s \langle U_k | H' | U_s \rangle a_s e^{iE_{ks} t} \quad (C-7)$$

Next, one replaces H' by $\lambda H'$ and expresses the a 's as power series in λ :

$$a_s = a_s^{(0)} + \lambda a_s^{(1)} + \lambda^2 a_s^{(2)} + \dots \quad (C-8)$$

We assume that this series for a_s is a continuous

analytic function of λ for λ between 0 and 1. We, therefore, substitute (C-8) into (C-7), equate coefficients of corresponding powers of λ and set $\lambda = 1$ in the final results. The substitution yields the set of equations

$$\dot{a}_k^{(0)} = 0 \quad (C-9)$$

$$\begin{aligned} \dot{a}_k^{(1)} &= -i \sum_s \langle u_k | H' | u_s \rangle a_s^{(0)} e^{iE_{ks}t} \\ &\vdots \end{aligned} \quad (C-10)$$

Equation (C-9) shows that the zero-order coefficients $a_k^{(0)}$ are constant in time. Their values are dictated by the initial conditions of the problem. If initially, (that is in the absence of the laser) the atom is in the n -th unperturbed state, then

$$a_k^{(0)} = \delta_{n,k} \quad (C-11)$$

Substituting the values of $a_s^{(0)}$ and H' on the right side of (C-10) and integrating, we get

$$\begin{aligned} a_k^{(1)} &= -\frac{\omega}{2} \bar{\alpha}_0 \cdot (\vec{P})_{kn} e^{iE_{kn}t} \left(\frac{e^{i\omega t}}{E_{kn} + \omega} + \frac{e^{-i\omega t}}{E_{kn} - \omega} \right) \\ &\approx -\omega \bar{\alpha}_0 \cdot \frac{(\vec{P})_{kn}}{E_{kn}} \cos \omega t e^{iE_{kn}t} \end{aligned} \quad (C-12)$$

Also,

$$\vec{P} = -im [\vec{Q}, H_0]; \quad \vec{Q} = \sum_{j=1}^Z \vec{r}_j \quad (C-13)$$

Therefore,

$$\begin{aligned} (P)_{kn} &= -im \langle U_k | [\vec{Q}, H_0] | U_n \rangle \\ &= im E_{kn} (\vec{Q})_{kn} \end{aligned} \quad (C-14)$$

Hence,

$$a_k^{(1)} = -im \omega \vec{\alpha}_0 \cdot (\vec{Q})_{kn} \cos \omega t e^{iE_{kn}t} \quad (C-15)$$

Substituting for $a_k^{(0)}$ and $a_k^{(1)}$ in (C-5) we finally get

$$\begin{aligned} \hat{\phi} &= \sum_k a_k(t) U_k e^{-iE_k t} \\ &= e^{-iE_n t} (1 - im \omega \vec{\alpha}_0 \cdot \vec{Q} \cos \omega t) U_n \end{aligned} \quad (C-16)$$

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47. The eqs. (8.2.8), (8.2.9), (8.2.10), (8.2.12), (8.2.13), (8.2.14), (8.2.34), (8.4.21), (8.4.22) and (8.4.25) correspond respectively to eqs. (135), (136), (137), (49), (70), (72), (215), (213), (214) and (218) of ref. 44.
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FIG. 1

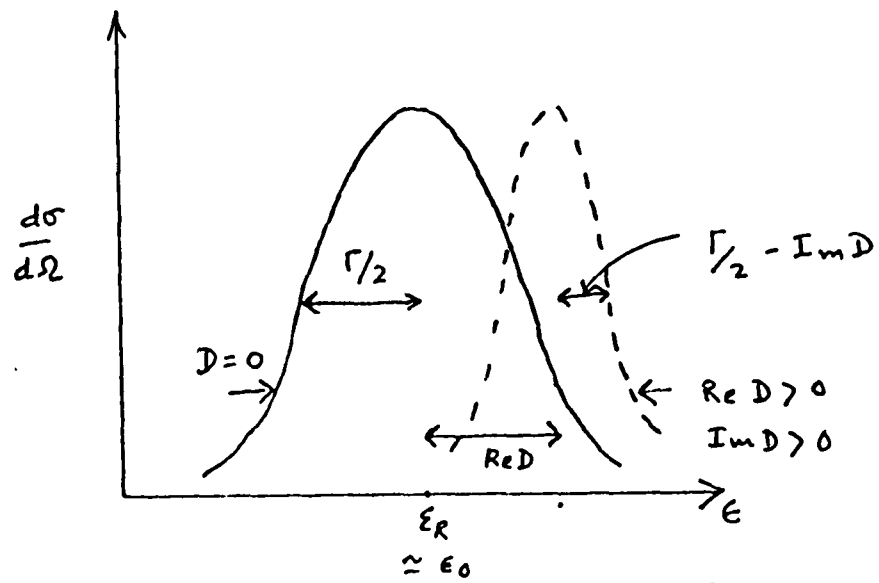


FIG. 2

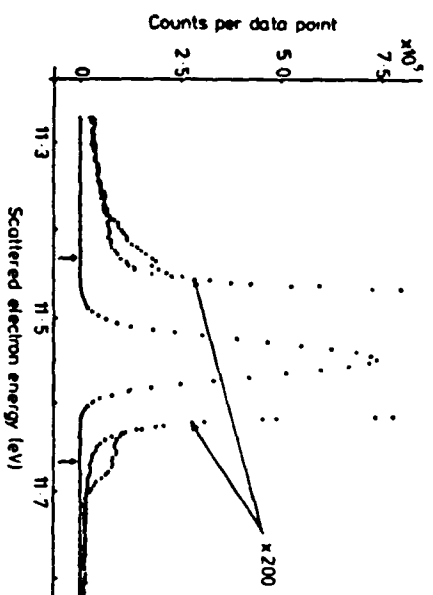


FIG. 3

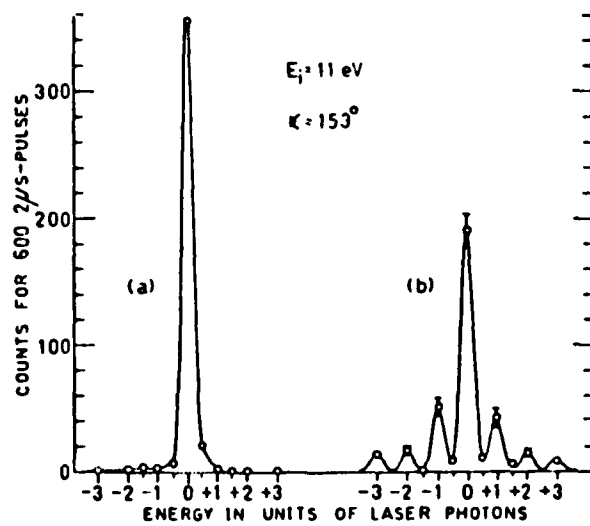


FIG. 4

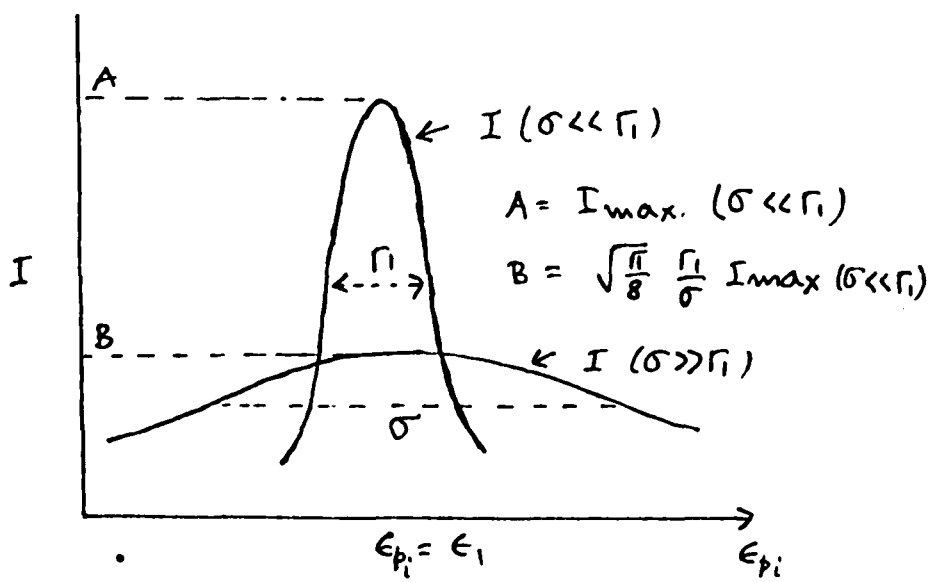


FIG. 5

