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THERMOELECTRIC CONVERSIONS BASED ON NOISE RECTIFICATION

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THERMOELECTRIC CONVERSIONS BASED ON NOISE RECTIFICATION

by

ANDREI CERNASOV

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Phylosophy, The City University of New York.

1981

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

INTRODUCTION

A system in contact with heat reservoirs that, if carried through a cyclic process, will convert part of the heat received from the hotter reservoirs into work or available energy, while rejecting the rest to the colder reservoirs is, by definition, a heat engine.

Depending upon those properties of the working substance most important to their operation, we can distinguish two types of heat engines, mean value engines and fluctuation engines.

In the case of mean value engines most important are the average values of such thermodynamic parameters as temperature, pressure or volume. In this category we find all "normal" heat engines like Otto, Diesel, Stirling or the Steam engines. We can also classify as mean value engines, although this is not immediately apparent, more subtle devices like the thermoionic or the photoelectric converters.

For fluctuation engines the driving forces are the deviations of the appropriate thermodynamic parameters of the working substance from their average values.

If the importance of the mean value engines previously mentioned is beyond any doubt, fluctuation engines are, at this time, significant only as tools of understanding the physical concepts their operation is based on.

This is true because, as this thesis will prove, using today's devices and technologies, reasonable efficiencies and output powers can be obtained only at sizes and configurations totally impractical.

As detailed analysis will show, they do not represent a loophole in the Second Law of thermodynamics that would reveal an inexhaustible energy source, as some had hoped. Our overall goal is to study the physics of fluctuation engines using rectification mechanisms and answer the obvious questions related to their theoretical performance and feasibility.

This thesis is not the first attempt to analyze the fluctuation engines from a thermodynamically correct viewpoint, but it is believed to be more complete and rigorous than most. Also, a number of new results relevant to the problem are derived and analyzed.

. SUMMARY

From the study of Maxwell's demon, to Feynman's educationally valuable analysis of the Ratchet-Pawl mechanism [1] and to the optimistic evaluation of diode-diode fluctuation engines given by Yater [2], the Second Law in the Kelvin-Planck sense was always found to be valid. If the system operates using only one heat reservoir, or equivalently, if all the elements of the system are at the same temperature, there is no macroscopic heat to work conversion. But if the system is in contact with two or more reservoirs at different temperatures, heat to work conversion is possible. Also, if the system does not include adiabatic partitions, heat flows from the hotter to the colder reservoirs will occur.

Since our study will mainly involve electrical fluctuation engines we will adopt the term "noise" to describe the stochastic signals encountered in electrical systems.

SECTION 2

A classical quantitative calculation of the thermal noise, or fluctuations, spontaneously generated within a linear resistor is provided by Nyquist [3]. The results obtained are only a special case of a more powerful statement known as the Fluctuation-Dissipation Theorem but it is, nevertheless, instructive to repeat their derivation here.

SECTION 3.

The details of the behavior of a nonlinear resistor, including the generated thermal noise, under arbitrary operating conditions, are very difficult to derive. However, when the resistor operates close to thermodynamic equilibrium important general results can be obtained. In this section a derivation of such results, due to Gunn [15], is presented in detail.

SECTION 4.

A general thermodynamic argument can also reveal the expected behavior of a fluctuation heat engine when operated close to thermodynamic equilibrium.

SECTION 5.

In general, a fluctuation heat engine will consist of noise generating dissipative devices at different temperatures. At least one of the devices to be used will be nonlinear, and thus will have the ability to generate a nonzero average flux when driven by zero average fluctuations.

We will now study a fluctuation heat engine employing at least one diode whose properties can be described by Gunn's model (see Section 3.). This study reveals the general properties of fluctuation heat engines when operating close to thermodynamic equilibrium.

SECTION 6.

However, Gunn's model for nonlinear elements is valid when the system is only weakly nonlinear. If results concerning strongly nonlinear systems are desired, the dissipative elements we use have to be described by different models.

In this section we will study a fluctuation heat engine using a simplified model of the vacuum diode due to Alkemade and Van Kampen [16,17,18], and a novel kinetic model of a linear resistor. The results obtained are checked against results based on Gunn's model, when operating close to thermodynamic equilibrium.

Nevertheless the results obtained here are valid under any operating conditions.

SECTION 7.

If the temperature of the cold element can be neglected when compared to the temperature of the hot element, analytical results concerning the efficiency and output power of the engine can be derived.

SECTION 8.

Finally, general conclusions about the performance and feasibility of fluctuation heat engines, based on the results of Sections 5.,6. and 7. are presented. Also the very weak nonlinearity of practical diodes, when compared to thermodynamic limitations, is now pointed out.

2. NOISE IN LINEAR DISSIPATIVE ELEMENTS

2.1 Nyquist Type Analysis

Let us consider a segment of a lossless transmissian line of real characteristic impedance Z_o, terminated at both ends with ideal resistors of equal resistance (see Fig.2.1) If R is chosen equal to Z_o, the line will be matched at both ends with no reflections occuring or standing waves developing on the line. We assume that the whole system had been in contact with a thermal reservoir of temperature T, for a time long enough to insure the establishment of thermal equilibrium.

Due to the thermal "agitation" of the electrons in the two resistors, current fluctuations through the system are generated. The current through each point on the transmission line is then given by the solution of the appropriate telegraph equations.

$$i_\ell = I_\ell \, e^{j(\omega t - \beta z)} \tag{2.1}$$

$\omega \equiv$ angular frequency

$I_\ell \equiv$ current amplitude

$t \equiv$ time

$\beta \equiv \frac{\omega}{c}$

$c \equiv$ propagation velocity

If we impose the boundary condition,

$$i_e(t,0) = i_e(t,L)$$

on the possible propagating waves then:

$$\beta L = 2\pi n \tag{2.2}$$

L ≡length of the line

n ≡positive integer

Within a bandwidth $\Delta\omega$ a number Δn of such allowed "modes" will propagate in each direction, with:

$$\Delta n = \frac{L}{2\pi c}\Delta\omega \tag{2.3}$$

According to the principle of equipartition of energy, the mean energy in each mode will be, in the classical limit, . This value is due to the terms quadratic in the electric and magnetic field intensities, present in the expression of the electromagnetic energy associated with each mode. The average energy that will be present on the line, taking into consideration the bidirectional character of the modes is:

$$\Delta_\omega \mathcal{E} = 2kT\Delta n = \frac{kT}{\pi c}\Delta\omega \tag{2.4}$$

k ≡Boltzmann constant

Within a time interval

$$(2.5) \qquad \Delta t = \frac{L}{c}$$

this energy leaves the line and is replaced by the energy emitted by the two resistors during Δt. The average thermal power generated is therefore:

$$(2.6) \qquad P_g = \frac{\Delta_\omega \mathcal{E}}{\Delta t} = \frac{kT\Delta\omega}{\pi}$$

Since the system is in thermal equilibrium the average power absorbed by the resistors has to be equal to the average power emitted by them.

$$(2.7) \qquad P_d = 2R\left\langle\left(\frac{i_1}{2}+\frac{i_2}{2}\right)^2\right\rangle = \frac{kT\Delta\omega}{\pi}$$

Taking into account the fact that the currents i_1 and i_2, generated by the equivalent thermal current generators of the two resistors, are uncorrelated:

$$(2.8) \qquad P_d = 2R\left[\left\langle\left(\frac{i_1}{2}\right)^2\right\rangle + \left\langle\left(\frac{i_2}{2}\right)^2\right\rangle\right] = \frac{kT\Delta\omega}{\pi}$$

The resistors being identical and at the same temperature it is reasonable to assume:

$$(2.9) \qquad \langle i_1^2 \rangle = \langle i_2^2 \rangle \equiv \langle i^2 \rangle$$

Then:

$$(2.10) \qquad R\frac{\langle i^2\rangle}{2} = \frac{kT\Delta\omega}{\pi}$$

Replacing $\Delta\omega$ in the above equation by $2\pi\Delta f$, and solving for $\langle i^2\rangle$:

$$(2.11) \qquad \langle i^2\rangle = \frac{4kT}{R}\Delta f$$

Or equivalently:

$$(2.12) \qquad \langle v^2\rangle = R^2\langle i^2\rangle = 4kTR\Delta f$$

The power spectrum of these fluctuations is frequency independent (white noise spectrum) and has an amplitude:

$$(2.13) \qquad S_i = \frac{4kT}{R}$$

Or:

$$(2.14) \qquad S_v = 4kTR$$

2.2 Approach to Equilibrium

If the two resistors are initially at different temperatures $T_1 > T_2$, the system will reach thermal equilibrium via a heat flow from the hotter resistor to the colder resistor. To prove it let us go back and modify our previous analysis.

Since the line is matched at both ends all the power emitted by one resistor will be absorbed by the other. Therefore, all the power traveling rightward will be generated by the hotter resistor and absorbed by the colder one.

This power is:

$$(2.15) \qquad P_1 = \frac{k T_1 \Delta n}{\Delta t} = k T_1 \Delta f$$

The power traveling leftward, generated by the colder resistor at temperature is then:

$$(2.16) \qquad P_2 = \frac{k T_2 \Delta n}{\Delta t} = k T_2 \Delta f$$

The kT_1, and respectively kT_2 terms are the average energies for each propagating mode, referred to the temperatures at the point of generation.

The net power flow from the hot to the cold resistor is then:

$$(2.17) \qquad \Delta_T P \equiv P_1 - P_2 = k (T_1 - T_2) \Delta f$$

If we employ bias dependent resistors the noise they will generate will correspond to the value of their resistances at the operating point. This can be seen if we connect a bias dependent resistor on our transmission line through a filter designed to short circuit the line's DC component. If we bias the resistor as shown in Fig. 2.2, in view of the characteristics of the filter, the value of R_2 at the bias point V will determine our final results, which proves our initial statement.

A last point to be made at this time is that relation (2.17) is true even if our resistors are not equal. In this case for our transmission line argument to be valid we will connect the resistors to the line by appropriate impedance transformers (Fig. 2.3). These transformers will match the resistors to the line without interfering with the power flows through the system. As required by the Second Law, there is no net power flow when the two temperatures are equal. Since this power flow occurs only when there is a temperature difference and since it is not accompanied by performance of work, it must be identified with a heat flow.

It is important to underline the fact, that the origin of this heat flow is the amplitude difference between the noise signals developed in resistors at different temperatures.

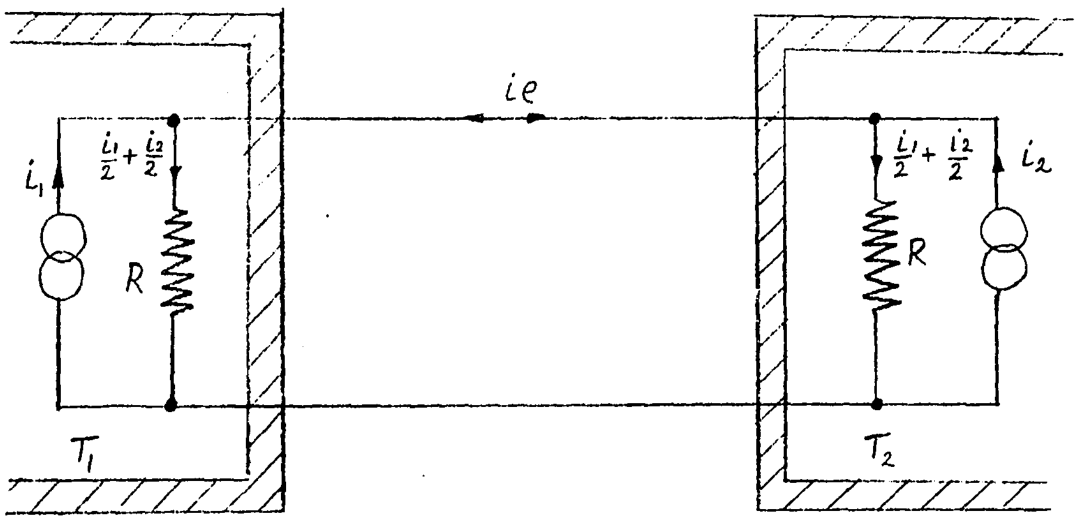


Fig 2.1.

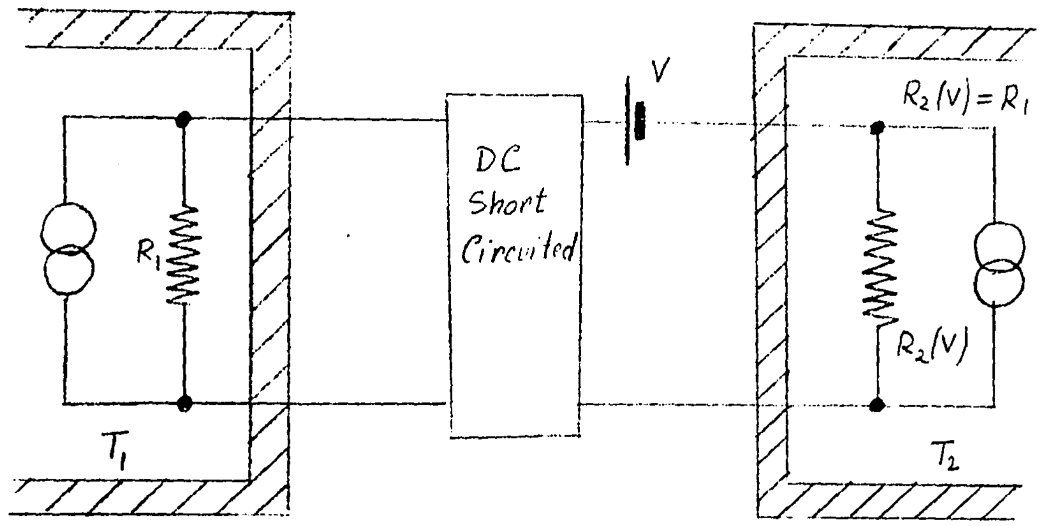


Fig 2.2.

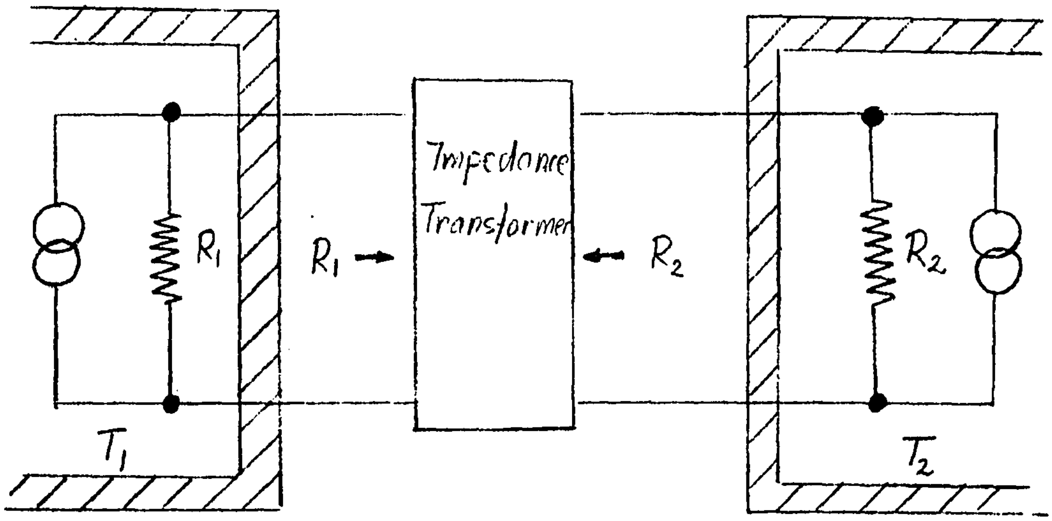


Fig. 2.3.

3. NOISE IN NONLINEAR DISSIPATIVE ELEMENTS

3.1 Historical Notes

The above method is not valid if the resistors are nonlinear, since there is no way of matching them to the transmission line. A vast amount of experimental data [4,5,6,7,8,9] indicates that the results we obtained are also invalid for this case. If the resistors are nonlinear new physical processes seem to play important roles in noise generation. For example, as early as 1918, Schottky [10,11] pointed out that we should expect additional contributions to the electric fluctuations within certain devices, arising from the "grainy" structure of electricity. He also showed that thermoionic diodes are one kind of generator of this type of noise.

As experimental evidence accumulated it was found that additional white noise occurs in nonlinear systems only when significant energy fluxes are present or, in other words, when the system is <u>not</u> in thermodynamic equilibrium. We will term this type of flux dependent white noise fluctuations "shot noise", in accordance with traditional electrical engineering. By now it was clear that these new fluctuations are not essentially related to the irreversible interactions with the thermal environment, as the Nyquist fluctuations are.

With the development of new nonlinear electronic devices the study of the associated shot noise also intensified.

Experimental and theoretical studies based on detailed analysis of carrier flows through PN junctions showed that shot noise is present, and is proportional to the current, a result similar to that obtained by Shottky for thermionic diodes. [7,8,9,12,13].

Nevertheless, a general theory describing from a thermodynamic standpoint the general connection between the noise and the nonlinearity of dissipative elements was not in place until 1959, when Bernard and Callen published a benchmark paper on the subject 14 .

Because of the complexity of their analysis, an alternate very elegant, derivation due to Gunn (even if somewhat less general) will be used here [15]. A very interesting feature of Gunn's work is that without assuming the charge quantized, as Shottky did, he proves that nonlinear elements have to exhibit shot noise if they are to obey the laws of thermodynamics.

Because the validity of Gunn's results is limited to the continuous charge approximation, a different approach will have to be used for the case when this approximation does not hold. We will then use for our dissipative elements kinetic models of the type suggested by Alkemade and Van Kampen [16,17,18] and we will then proceed to solve the appropriate Master Equation.

3.2 Gunn Type Analysis

3.2.1 Direct Current Calculations

We will start by analysing the behaviour of a nonlinear electric element, under the influence of a voltage consisting of a bias value V_0 and a noise component of frequency f, $v_f(t)$:

$$(3.1) \qquad v_D = V_0 + v_f(t)$$

The current generated by such a voltage is:

$$(3.2) \qquad i_D = i_D(V_0) + g_1(f)\, v_f(t) + g_2(f)\, v_f^2(t)$$

We will assume that higher order terms can be neglected, this first order nonlinearity condition being the strongest approximation imposed by Gunn on his model. Nevertheless, the model holds well as long as the amplitude of the noise is small enough to justify a quadratic approximation of the nonlinearity.

The DC part of this current, per unit frequency, is

$$(3.3) \qquad i_D^0 = i_D(V_0) + g_2(f) \langle v_f^2(t) \rangle$$

The total DC current obtained, if the input noise voltage is allowed to have a full frequency spectrum, will then be:

$$(3.4) \qquad I^0 = i_D(V_0) + \int_0^\infty g_2(f) \langle v_f^2(t) \rangle$$

The two parameters $g_1(f)$ and $g_2(f)$ can be identified as the differential conductance of the element and respectively half of the slope of the conductance with respect to the input voltage:

$$(3.5) \qquad g_1(f) = \frac{\partial i_D}{\partial v_D}$$

$$(3.6) \qquad g_2(f) = \frac{1}{2} \frac{\partial g_1(f)}{\partial v_D}$$

Besides frequency these parameters will generally also be functions of the applied voltage v_D ,

The expression of the direct current I^0 can be easily

s in Fig.3.1, where g_1 and g_2 are the assumed frequency

in Fig.3.1, where g_1 and g_2 are the assumed frequency

independent parameters of a diode .

If the circuit is the thermal equilibrium at temperature the principle of equipartition of energy will give for the total average energy, on the capacitor, due to spontaneous voltage fluctuations.

$$\mathcal{E}_c = \frac{C}{2} \int_0^\infty \langle v_f^2 \rangle \, df = \frac{kT}{2}$$

These voltage fluctuations, according to (3.4) , will be rectified by the diode giving a net DC current I^0 that could conceivably drive the small DC motor M, where:

$$I^{o} = g_2 \frac{kT}{C}$$

Thus work could be obtained using only one heat reservoir, in contradiction to the Kelvin-Planck statement of the Second Law. Since this cannot be allowed we will postulate the existence of a reverse current I_0, opposite to the calculated current I^{o}, such that when a system containing the nonlinear element is in thermal equilibrium:

$$I = I_0 + I^{o} = 0$$

For the example of Fig.3.1, I_0 takes the value:

$$I_0 = -I^{o} = -g_2 \frac{kT}{C} \tag{3.7}$$

In conclusion, the general expression for the DC current flowing through the element is:

$$I = i_D(V_0) + I_0 + \int_0^{\infty} g_2(f) \langle v_f^2 \rangle \, df$$

Or:

$$I = i_D(V_0) + I_0 + \int_0^{\infty} g_2(f) S_v(f) \, df \tag{3.8}$$

Where S_v is the power spectrum of the resulting fluctuating voltage across the nonlinear element.

3.2.2 Noise Power Spectrum Calculations

Let us now analyze the behaviour of a nonlinear resistor connected through a lossless bandpass filter to a linear resistor at a slightly different temperature (Fig.3.2). Each resistor is modeled by a noiseless conductance $g_{ij}(f), (j=1,2)$ in parallel with an ideal current source of fluctuating amplitude i_j . The nonlinear element, which we will assume to be a diode, is also biased by an ideal battery of voltage V. The filter is designed to have the following properties:

a - Outside its bandwidth δf it presents a zero impedance in both direction.

b - Within the bandwidth, it acts as an impedance transformer to match the two resistors at the center frequency f and given battery voltage V.

One immediate consequence of the properties of the filter is that the voltage across the diode is equal to the battery voltage, as seen by applying Kirchhoff's Voltage Law in the diode loop.

If $\mathring{Q}$ is the net AC heat flow through the filter, and is the DC current flowing through the diode, if $T_1 = T_2$ then both have to be zero to satisfy the Second Law.

If $T_1 > T_2$ a net heat flow from the hotter element to the colder one will take place, with a net rate of entropy production given by:

$$(3.9)\quad \mathring{S} = \frac{\mathring{Q}}{T_2} - \frac{\mathring{Q}}{T_1} + \frac{V \cdot I}{T_1}$$

With the assumption of small temperature differences, i.e.

$$(3.10)\quad \frac{T_1 - T_2}{T_2} \ll 1$$

we can define $\Delta T = T_1 - T_2$ and $T \equiv T_2 \lessgtr T_1$. Then:

$$(3.11)\quad T\mathring{S} \cong \mathring{Q}\ \frac{\Delta T}{T} + V \cdot I$$

Now we will choose $\frac{\Delta T}{T}$ and V to be our thermodynamic forces, with $\mathring{Q}$ and I being the corresponding thermodynamic fluxes. If is also small enough, the system will be sufficiently close to thermodynamic equilibrium to warrant a linear expansion of the fluxes in terms of the chosen forces.

$$(3.12)\quad I = \alpha_{11} V + \alpha_{12} \frac{\Delta T}{T}$$

$$(3.13)\quad \mathring{Q} = \alpha_{21} V + \alpha_{22} \frac{\Delta T}{T}$$

Our next goal is to determine the four phenomenological parameters $\alpha_{ij}\,(i,j = 1,2)$ in the above equations.

One immediate equality can be established by imposing the Onsager reciprocal relations on this system [19,20]

$$(3.14) \qquad \alpha_{12} = \alpha_{21}$$

It is obvious that the presence of a rectified current of thermal origin, $\alpha_{12}\frac{\Delta T}{T}$, will be accompanied by an additional noise heat flow of magnitude $\alpha_{12} V$.

3.2.3 Calculation of α_{12} and α_{21}

Within the bandwidth δf , the equivalent circuit of our system is that of Fig.3.3, where g_{11} is the conductance of the linear resistor reflected through the filter in the diode loop as to match g_{12} . All the components of the two current sources outside δf , will be shortcircuited, in the view of the properties of the filter.

We will start by calculating the total DC current flowing through the diode. To achieve this let us first look at the power spectrums of the two current sources in our system.

The instantaneous AC voltage developed across the parallel combination of g_{11} and g_{12} is:

$$v_f = \frac{i_1 + i_2}{g_{11}(f) + g_{12}(f)} \tag{3.15}$$

The power spectrum of this voltage is then:

$$S_v = \langle v_f^2 \rangle = \frac{S_{i_1} + S_{i_2}}{\left[g_{11}(f) + g_{12}(f) \right]^2} \tag{3.16}$$

The resistor noise S_{i_1} is given by the Nyquist theorem. The diode noise S_{i_2} , on the other hand, will have to include the yet unknown expression of the additional noise due to the nonlinearity of the diode,

$$(3.17) \qquad S_{i_1} = 4kT_1 g_{11}(f)$$

$$(3.18) \qquad S_{i_2} = 4kT_2 g_{12}(f) + S.H.$$

where $S.H.$ is the noise term additional to the Nyquist value.

Recalling (3.8) the DC current through the diode will be:

$$(3.19) \qquad I = g_{12}(0)V + I_0 + g_2(f) S_v \delta f$$

Making the appropriate replacements:

$$(3.20) \quad I = g_{12}(0)V + I_0 + \frac{4kT_1 g_{11}(f) + 4kT_2 g_{12}(f)}{[g_{11}(f) + g_{12}(f)]^2} g_2(f)\delta f +$$

$$+ \frac{S.H.}{[g_{11}(f) + g_{12}(f)]^2} g_2(f)\delta f$$

Since $S.H.$ is only a function of T and V ,the only terms in (3.20) conceivably proportional to ΔT are:*

$$I_o + \frac{4kT_1 g_{11}(f) + 4kT_2 g_{12}(f)}{[g_{11}(f) + g_{12}(f)]^2} g_2(f)\,\delta f$$

When $T_1 = T_2$ these terms have to vanish and therefore:

$$(3.21) \qquad I_o = -\frac{4kT_2 g_{11}(f) + 4kT_2 g_{12}(f)}{[g_{11}(f) + g_{12}(f)} g_2(f)\,\delta f$$

Introducing I_o back into the expression for I we obtain:

$$(3.22)\quad I = g_{12}(0)V + \frac{4kT_2 g_{11}(f)\delta f}{[g_{11}(f) + g_{12}(f)]^2} g_2(f)\frac{\Delta T}{T} +$$

$$+ \frac{S.N.}{[g_{11}(f) + g_{12}(f)]^2} g_2(f)\,\delta f\,V$$

Using Onsager's reciprocal relations we also find:

$$(3.23) \qquad \alpha_{12} = \alpha_{21} = \frac{4kT g_{11}(f)}{[g_{11}(f) + g_{12}(f)]^2} g_2(f)$$

* Even if $S.N.$ would have a term $S.N._{\frac{\Delta T}{T}}$ linear in ΔT, and I_o a term linear in V, I_{oV}, our analysis would not be affected since we could always lump $S.N._{\frac{\Delta T}{T}}$ into the, yet undetermined, I_o, and I_{oV} into the still unknown $S.N.$.

3.2.4 Calculation of α_{12} and the Net Heat Flow

The AC thermal current generated by the resistor that will dissipate on $g_{12}(f)$ is given by:

$$(3.24) \qquad i_{12} = \frac{i_1 \, g_{12}(f)}{[g_{11}(f) + g_{12}(f)]^2}$$

The thermal power, or heat flow carried by this current is then:

$$(3.25) \qquad \mathring{Q}_{1\to 2} = \frac{\langle i_{12}^2 \rangle}{g_{12}(f)} = \frac{S_{i_1} \, g_{12}(f)}{[g_{11}(f) + g_{12}(f)]^2} \, \delta f$$

In a similar fashion the heat flow from the diode to the resistor, within δf is found to be:

$$(3.26) \qquad \mathring{Q}_{2\to 1} = \frac{S_{i_2} \, g_{11}(f)}{[g_{11}(f) + g_{12}(f)]^2} \, \delta f$$

If we now calculate the net heat flow from the resistor to the diode:

$$(3.27) \qquad \mathring{Q} = \mathring{Q}_{1\to 2} - \mathring{Q}_{2\to 1}$$

Or:

$$(3.28) \quad \mathring{Q} = \frac{4k\, g_{12}(f) T_1 - 4k\, g_{11}(f) T_2}{[g_{11}(f) + g_{12}(f)]^2} \, \delta f + \frac{S.H. \; g_{11}(f)}{[g_{11}(f) + g_{12}(f)]^2} \, \delta f$$

Since our analysis is valid only when no reflections occur within $\delta\rho$, looking back at the properties of the filter, we find that:

$$g_{11}(\rho) = g_{12}(\rho) \equiv g_1(\rho) \tag{3.29}$$

Then:

$$\mathring{Q} = \frac{kT}{g_1(\rho)} \delta\rho \frac{\Delta T}{T} - \frac{S.H.}{4\, g_1(\rho)} \delta\rho \tag{3.30}$$

Recalling equation (3.13) we immediately identify:

$$\alpha_{22} = \frac{kT}{g_1(\rho)} \delta\rho \tag{3.31}$$

$$\alpha_{21} V = - \frac{S.H.}{4\, g_1(\rho)} \delta\rho \tag{3.32}$$

If in the second relation we replace α_{21} by its already known expression (3.23) and using (3.29), we can solve for $S.H.$:

$$S.H. = - 4 k T g_2(\rho) V \tag{3.33}$$

Where:

$$g_2(\rho) \equiv g_{22}(\rho) \tag{3.34}$$

This relation is the central result of our analysis and is important because it directly relates the shot noise in a nonlinear dissipative element, to the magnitude of its nonlinearity as measured by .

Our derivation is slightly different than the original one given by Gunn, nevertheless, the answer obtained is the same.

Let us check our result by applying our model to the P-N diode. The current - voltage characteristic of the diode is given by the Shockley Equation [21]:

$$I = I_s \left[\exp(eV/kT) - 1 \right] \tag{3.35}$$

$I_s \equiv$ inverse saturation current

$e \equiv$ electronic charge

Calculation of $g_2(f)$ yields:

$$g_2(f) = \frac{1}{2} \frac{\partial^2 I}{\partial V^2} = \frac{1}{2} \frac{e^2 I_s}{k^2 T^2} \exp(eV/kT) \tag{3.36}$$

The spectrum of the current fluctuations will then be:

$$S_i = 4kTg_1(f) + \text{S.N.} \tag{3.37}$$

$$S_i = 4kTg_1(f) - 2 \frac{e^2 I_s}{kT} \exp(eV/kT) \tag{3.38}$$

If we recognize that the current through the PN diode, at V small enough for the Gunn model to be valid, is given by:

$$(3.39) \qquad I = g_1(\rho)V = \frac{\partial I}{\partial V} \cdot V = \frac{e I_s}{k T} exp(eV/kT) V$$

Then

$$(3.40) \qquad S_i = 4kTg_1(\rho) - 2eI$$

The above result is well known both from experimental observations and theoretical studies based on corpuscular approaches to the problem [9]

It is interesting to observe that S_i is also given by:

$$(3.41) \qquad S_i = 4kTg_1(\rho, V=0) + 2eI$$

For forward currents, the noise generated by diodes is less than the Nyquist noise associated with the value of the conductance at the operating point, but it is larger than the Nyquist noise at zero bias voltage.

3.2.5 Calculation of α_{11} and the DC Current

For the sake of completeness we will conclude with the derivation of the overall DC current I .

Replacement of the expresion found for S.H. into equation (3.22) will yield:

$$(3.42)\quad I = \frac{4kT g_{11}(f)}{[g_{11}(f) + g_{12}(f)]^2} g_2(f)\delta f \frac{\Delta T}{T} +$$

$$+ \left\{ g_{12}(0) - \frac{4kT g_2^2(f)}{[g_{11}(f) + g_{12}(f)]^2}\delta f \right\} V$$

It is now easy to identify α_{11} :

$$(3.43)\quad \alpha_{11} = g_{12}(0) - \frac{4kT g_2^2(f)}{[g_{11}(f) + g_{12}(f)]^2}$$

Refering again to the essential impedance matching characteristics of the filter:

$$(3.44)\quad g_{11}(f) = g_{12}(f) \equiv g_1(f)$$

We can now write the final form of the flux equations :

$$(3.45)\quad I = \left[g_1(0) - \frac{kT g_2^2(f)}{g_1^2(f)}\delta f \right] \cdot V +$$

$$+ \frac{kT g_2(f)}{g_1(f)}\delta f \frac{\Delta T}{T}$$

$$(3.46)\quad \mathring{Q} = \frac{kT g_2(f)}{g_1(f)}\,\delta f \cdot V \quad + \quad \frac{kT}{g_1(f)}\,\delta f\,\frac{\Delta T}{T}$$

Let us summarize our results. If a diode or, in general, a nonlinear dissipative element, has a voltage - current characteristic $i = F(v)$, that is well approximated within the voltage noise range by a quadratic expanssion, and if we define the parameters $g_1(f)$ and $g_2(f)$ by,

$$(3.47)\qquad g_1(f) = \frac{\partial i}{\partial v}$$

$$(3.48)\qquad g_2(f) = \frac{1}{2}\cdot\frac{\partial^2 i}{\partial v^2}$$

then the DC current and the AC noise power spectrum for the device are found to be

$$(3.49)\qquad I = g_1(0)V + I_0 + \int_0^\infty g_2(f)\,S_v\,df$$

$$(3.50)\qquad S_i = 4kT g_1(f) - 4kT g_2(f) V$$

$S_v \equiv$ the power spectrum of the resulting voltage fluctuations across the diode

$V \equiv$ the resulting DC voltage across the diode

$I_0 \equiv$ postulated current

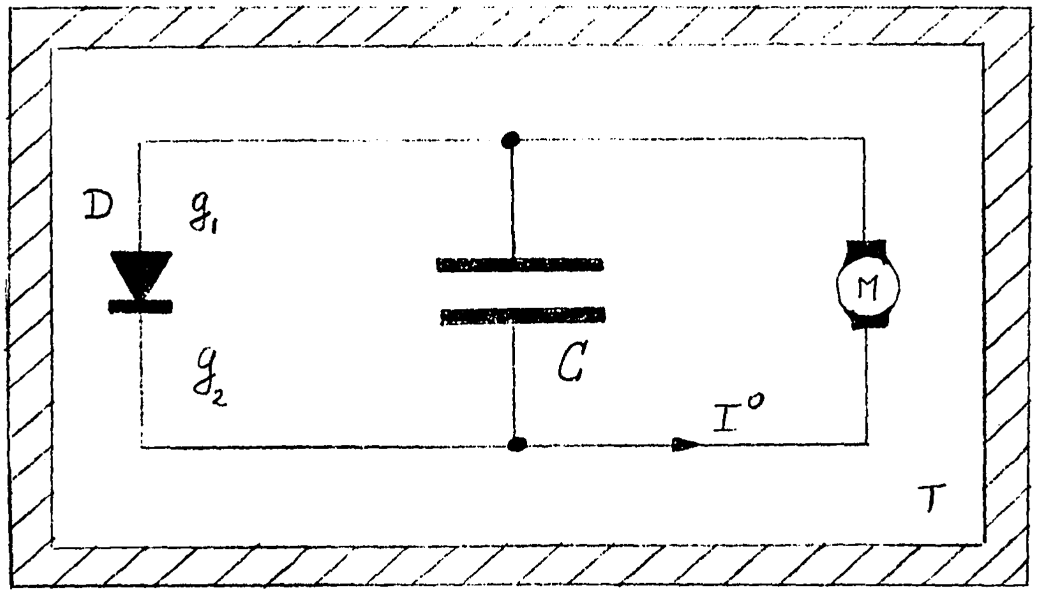


Fig 3.1

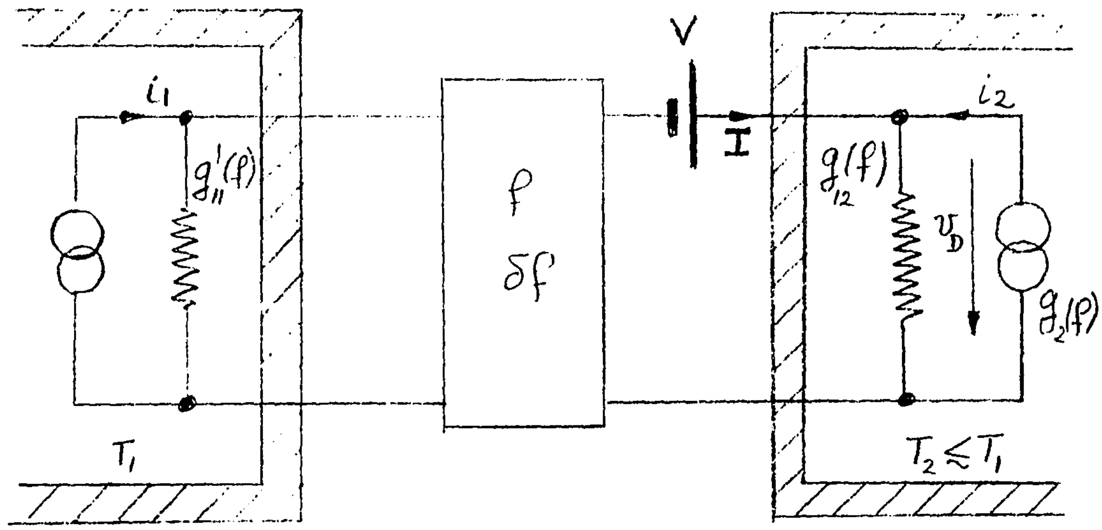


Fig 3.2.

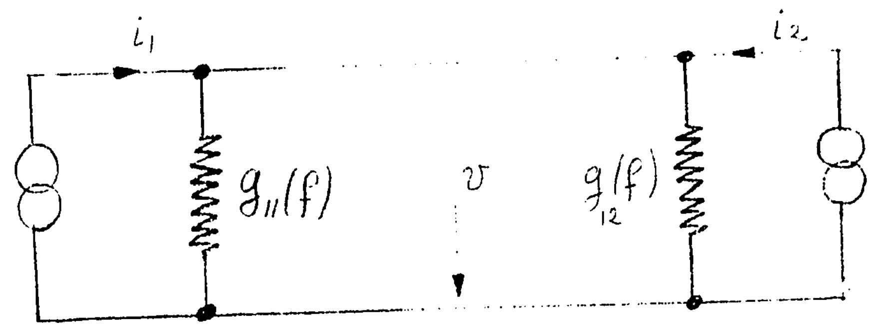


Fig 3.3.

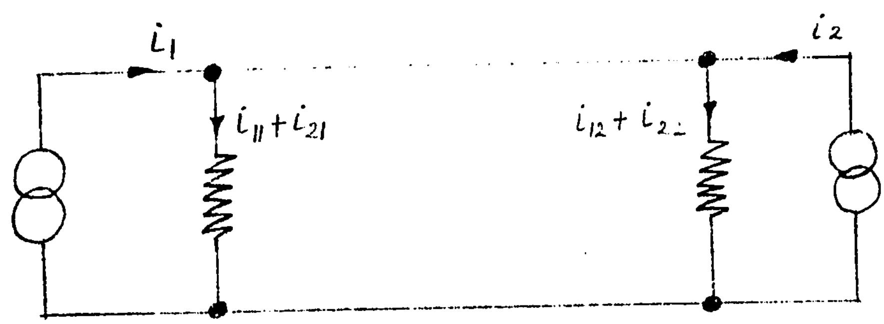


Fig 3.7.

4. THERMODYNAMICS OF FLUCTUATION HEAT ENGINES

We will study a heat fluctuation engine consisting of two dissipative electric devices (at least one of them nonlinear) in series with an ideal battery of voltage E (Fig.4.1). The two devices are in contact with two heat reservoirs of temperatures T_1 and respectively T_2 slightly lower than T_1. As a result of the temperature difference and the presence of the battery, heat and current flows are expected to develop.

If $\overset{o}{Q}$ is the rate of heat flow from device 1 to device 2 and I is the electric current flowing in the circuit, the rate of entropy production for our system is seen to be:

$$(4.1) \qquad \overset{o}{S} \cong \frac{\overset{o}{Q}}{T_2} - \frac{\overset{o}{Q}}{T_1} + \frac{I \cdot E}{T}$$

Where: $T \equiv T_2 \lesssim T_1$

In terms of the temperature difference ΔT, (4.1) reads:

$$(4.2) \qquad \overset{o}{S} \cong \frac{\overset{o}{Q}}{T}\frac{\Delta T}{T} + \frac{I \cdot E}{T}$$

At this point we will choose $\frac{\Delta T}{T}$ and E to be our thermodynamic forces with $\overset{o}{Q}$ and I the corresponding thermodynamic fluxes. If our engine is operating close enough to thermodynamic equilibrium the fluxes can be linearly expanded in terms of the forces. Therefore if $\frac{\Delta T}{T}$ and E are sufficiently small:

$$(4.3) \qquad I = A_{11} E + A_{12} \frac{\Delta T}{T}$$

$$(4.4) \qquad \overset{o}{Q} = A_{21} E + A_{22} \frac{\Delta T}{T}$$

According to Onsager's reciprocal relations:

$$(4.5) \qquad A_{12} = A_{21}$$

Substituting I and $\overset{o}{Q}$ from (4.3) and (4.4) into (4.2), and using (4.5), $\overset{o}{S}$ becomes:

$$(4.6) \quad \overset{o}{S} \cong \frac{1}{T} \frac{\Delta T}{T}\left(A_{12} E + A_{22} \frac{\Delta T}{T}\right) + \frac{1}{T} E\left(A_{11} E + A_{12} \frac{\Delta T}{T}\right)$$

After some algebra we obtain:

$$(4.7) \quad \frac{T\overset{o}{S}}{E^2} = A_{22}\left[\frac{1}{E}\left(\frac{\Delta T}{T}\right)\right]^2 + 2A_{12}\left[\frac{1}{E}\left(\frac{\Delta T}{T}\right)^2\right] + A_{11}$$

According to the Second Law of thermodynamics the rate of entropy production cannot be negative:

$$(4.8) \qquad \overset{o}{S} \geqslant 0$$

This condition written in terms of the phenomenological coefficients A_{ij} translates into:

$$A_{11} \geq 0 \tag{4.9}$$

$$A_{22} \geq 0 \tag{4.10}$$

$$A_{12}^2 \leq A_{11} \cdot A_{22} \tag{4.11}$$

The rate of entropy production is zero only if our engine operates in a reversible mode. Then:

$$A_{22}\left[\frac{1}{E}\left(\frac{\Delta T}{T}\right)\right]^2 + 2A_{12}\left[\frac{1}{E}\left(\frac{\Delta T}{T}\right)\right] + A_{11} = 0 \tag{4.12}$$

The real solution of this equation, compatible with (4.9), (4.10) and (4.11) is given by:

$$\frac{1}{E}\,\frac{\Delta T}{T} = -\frac{A_{12}}{A_{22}} = -\frac{A_{11}}{A_{12}} \tag{4.13}$$

Thus if we could design our engine so that $A_{12}^2 = A_{11} \cdot A_{22}$, when the battery voltage E becomes equal to $-\frac{A_{12}}{A_{11}}\frac{\Delta T}{T}$ the system becomes reversible. Then, according to Carnot's

Theorem, the efficiency η of the engine will be maximum at:

$$(4.14) \qquad \eta = \eta_c = \frac{\Delta T}{T}$$

To verify this result we first observe that the system will behave as a heat engine when the current I is negative in order to charge the battery. The output power delivered will then be:

$$(4.15) \qquad P_o = -E\,I$$

With the help of (4.3) the above relation becomes:

$$(4.16) \qquad P_o = -E\left(A_{11}E + A_{12}\frac{\Delta T}{T}\right)$$

By definition the efficiency of our engine is given by:

$$(4.17) \qquad \eta = -\frac{E\,I}{\dot{Q} + E\,I} \cong -\frac{E\,I}{\dot{Q}}$$

If we use (4.3) , (4.4) and (4.5) we arrive at:

$$(4.18) \qquad \eta = -\frac{E\left(A_{11}E + A_{12}\frac{\Delta T}{T}\right)}{A_{12}E + A_{22}\frac{\Delta T}{T}}$$

Equations (4.16) and (4.18) describe the performance of our system for any values of the parameters A_{ij} and any choice of forces $\frac{\Delta T}{T}$ and E , as long as we operate close to thermodynamic equilibrium.

If we plot equations (4.16) and (4.18), for various voltages E, at a given temperature gradient $\frac{\Delta T}{T}$, the graphs of Figs. 4.1, 4.2 and 4.3 are obtained.

The graphs are scaled in such a way as to have the I and $\mathring{Q}$ lines intersect at $E=0$.

According to 4.11 the $\mathring{Q}$ line will always intersect the horizontal axis at the left of the I line unless the equality holds.

If the voltage E is between zero and $-\frac{A_{12}}{A_{11}}\frac{\Delta T}{T}$ the system will function as an engine with a positive output power P_0. The efficiency will be maximum when the two lines $\mathring{Q}$ and I overlap. At that point the efficiency will be equal to the Carnot limit. However, as it will be determined later, physical engines will never reach it.

The output power delivered, as seen in Figs. 4.1, 4.2 and 4.3, will always have its maximum midrange between $E=0$ and $E=-\frac{A_{12}}{A_{11}}$.

When E is more negative than $-\frac{A_{22}}{A_{21}}\frac{\Delta T}{T}$ the system will operate in a refrigerating mode with both P_0 and the heat flux $\mathring{Q}$ negative.

Another important question is if our system CAN reach a state of zero rate of entropy production. Such a state would imply that if we start with a slight thermodynamic nonequilibrium condition, it will persist forever, since there are no fluxes to thermalize the system. Therefore, we will have a system with no adiabatic partitions which is unable to reach thermodynamic equilibrium.

The existence of such a system would contradict the principles of basic thermodynamics and thus we conclude that the rate of entropy production can approach but never reach zero. As a consequence the efficiency of the engine can approach but not reach the Carnot limit.

Finally let us go back to equations (4.3) and (4.4) and identify the physical significance of the phenomenological coefficients .

A_{11} ≡ the equivalent conductance of the system

A_{12} ≡ term due to the rectification of the thermal noise noise present in the system at nonzero temperatures

A_{21} ≡ additional shot noise term due to the nonlinear character of the system

A_{22} ≡ Nyquist thermal heat flow

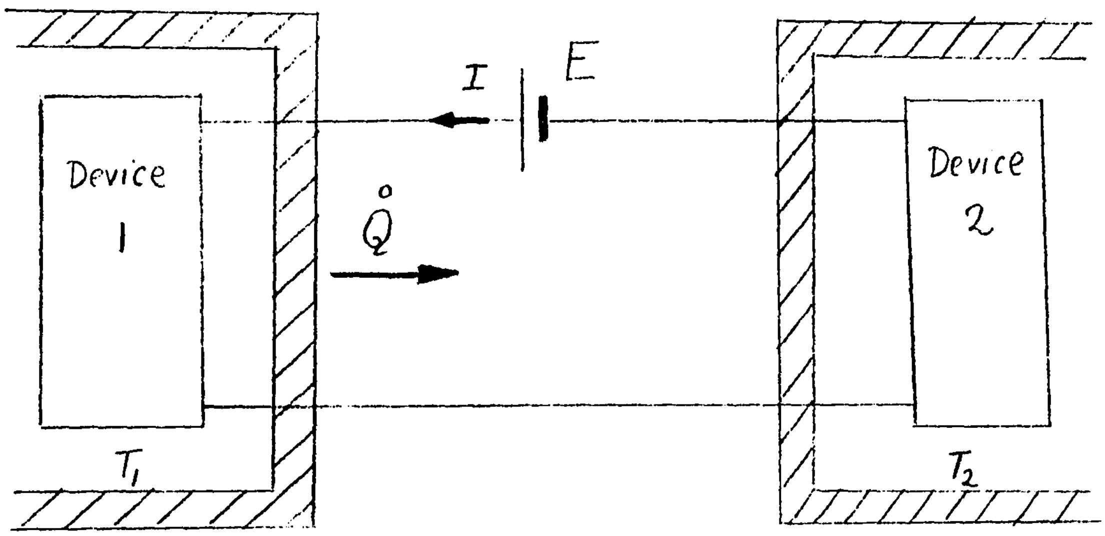


Fig 4.1.

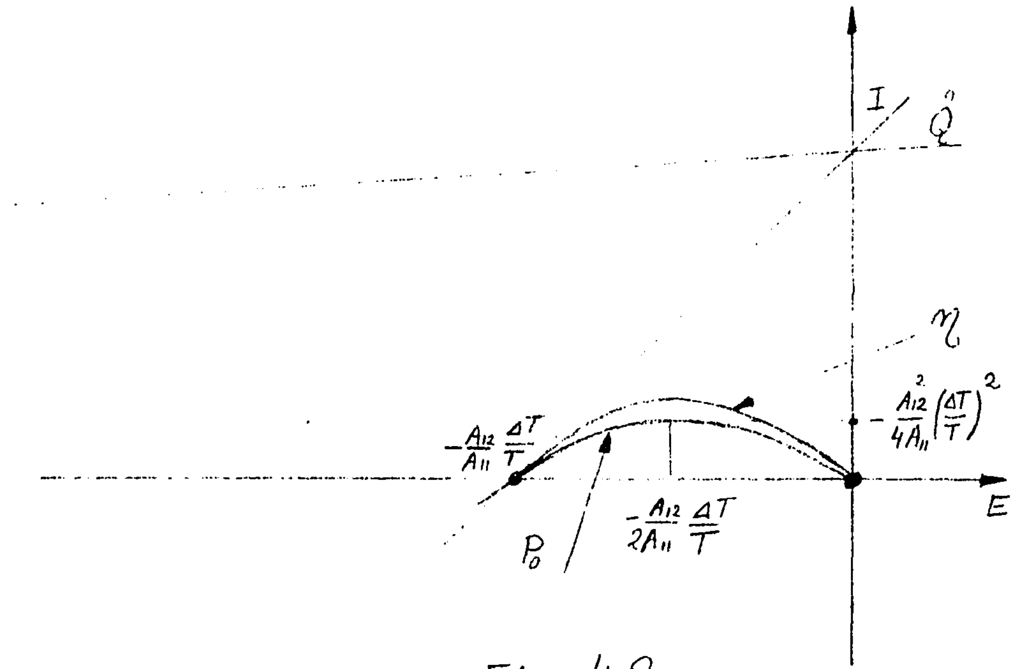


Fig 4.2.

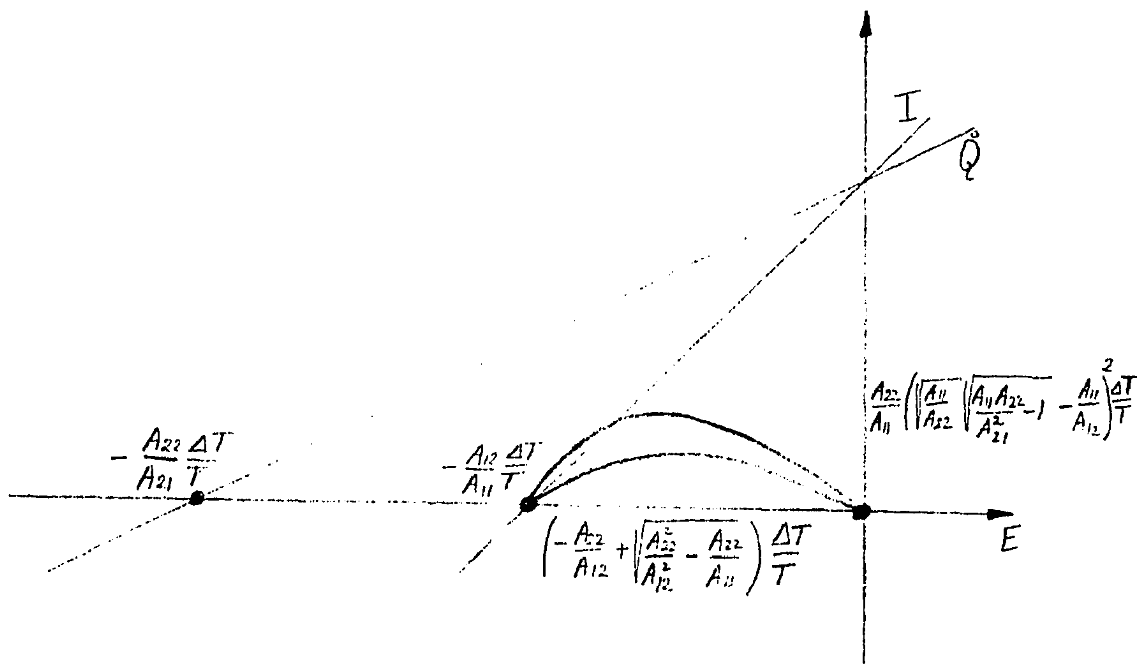
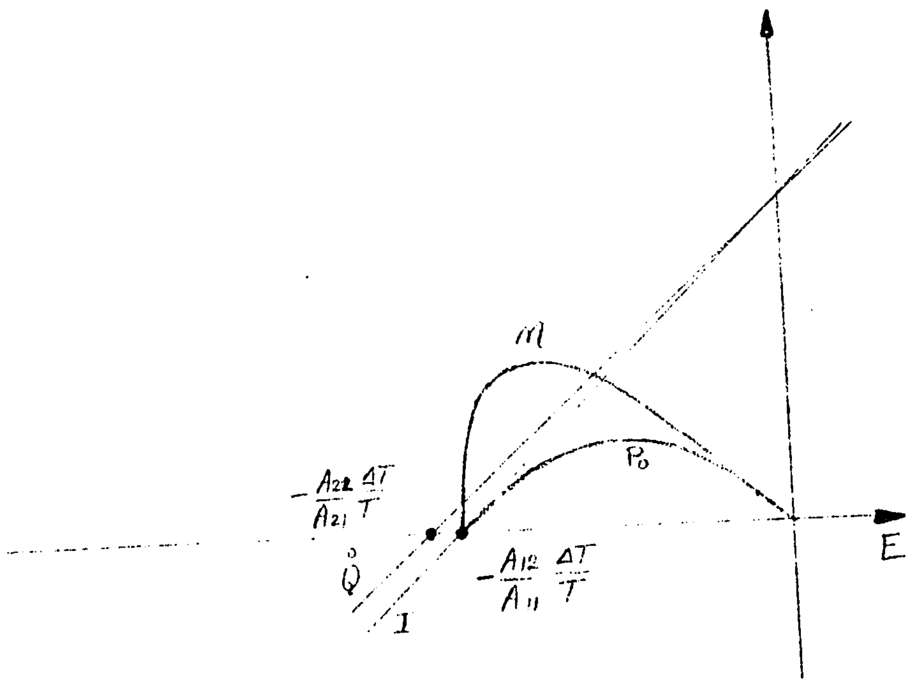


Fig 4.3.



m
P_0
$-\frac{A_{22}}{A_{21}}\frac{\Delta T}{T}$
$\mathring{Q}$
I
$-\frac{A_{12}}{A_{11}}\frac{\Delta T}{T}$
E

Fig 4.4.

5. FLUCTUATION ENGINES CLOSE TO THERMODYNAMIC EQUILIBRIUM

5.1 Analysis Based on Gunn's Model for Nonlinear Devices

At this stage we are ready to employ the Gunn model for diodes in more complex systems, specifically in electrical heat engine configurations.

Let us look at the behaviour of two diodes at slightly different temperatures connected in series with an ideal battery of voltage E (Fig.5.1)

In accordance with Gunn's results we will model each diode as a conductance g_{1j} in parallel to an ideal current source of value i_j. For each diode i_j will have a direct current component I_j and alternating current components i_{jf} characterized by a power spectrum S_{ij}. We also have to recognize that each diode will present an equivalent capacitance which we will lump into the discrete capacitors C_j. Without much loss of generality and for the purpose of simplifying our calculations we will assume that g_{1j} and g_{2j} are frequency independent parameters. We thus arrive at Fig.5.2.

Another assumption that we will use is that the length of the connecting wires is small enough for transmission line effects to be either negligible or beyond the cutoff frequency of the circuit.

To better analyze our system let us separately draw the DC and the AC equivalent circuits as shown in Fig. and respectively Fig. , where $C = C_1 + C_2$.

The expressions of the direct currents I_j can be found by subtracting from the overall DC diode currents, as given by equation (3.8) , the terms corresponding to the flows through g_{ij} .

(5.1) $$I_1 = I_{01} + g_{21} \int_0^\infty S_v \, df$$

(5.2) $$I_2 = I_{02} + g_{22} \int_0^\infty S_v \, df$$

S_v ≡the power spectrum of the resulting AC voltage across the two diodes

V_j ≡the resulting DC voltage biases across the given diodes

The expressions for the noise sources S_{i_j} are given by (3.50) which if rewritten for our two diodes read:

(5.3) $$S_{i_1} = 4kT_1 g_{11} - 4kT_1 g_{21} V_1$$

(5.4) $$S_{i_2} = 4kT_2 g_{12} - 4kT_2 g_{22} V_2$$

Let us first calculate $\int_0^\infty S_v \, df$. The AC voltage drop across the diodes at frequency f is found to be:

$$(5.5) \qquad v_f = \frac{i_{1f} + i_{2f}}{g_{11} + g_{12} + j\,2\pi f C}$$

$i_{jf} \equiv$ the frequency f components of the diode currents i_j.

The power spectrum associated with this voltage is then:

$$(5.6) \qquad S_v = \left| \langle v_f \cdot v_f^* \rangle \right|$$

$$(5.7) \qquad S_v = \frac{\left| \langle (i_{1f} + i_{2f})(i_{1f}^* + i_{2f}^*) \rangle \right|}{(g_{11} + g_{12})^2 + (2\pi f)^2 C^2}$$

Since i_1 and i_2 are uncorrelated currents:

$$(5.8) \qquad S_v = \frac{\left| \langle i_{1f} \cdot i_{1f}^* \rangle \right| + \left| \langle i_{2f} \cdot i_{2f}^* \rangle \right|}{(g_{11} + g_{12})^2 + (2\pi f)^2 C^2}$$

Or:

$$(5.9) \qquad S_v = \frac{S_{i_1} + S_{i_2}}{(g_{11} + g_{12})^2 + (2\pi f)^2 C^2}$$

Replacing S_{i_1} and S_{i_2} by their corresponding expressions and performing the integral of S_v :

$$(5.10) \qquad \int_0^\infty S_v \, df = \frac{k}{C(g_{11} + g_{12})} \left[g_{11} T_1 + g_{12} T_2 - (g_{21} T_1 V_1 + g_{22} T_2 V_2) \right]$$

Substituting the above integral into the expressions for I_1 and I_2 we obtain:

$$(5.11)\quad I_1 = I_{01} + \frac{k\, g_{21}}{C(g_{11}+g_{12})}\left[g_{11} T_1 + g_{12} T_2 - \left(g_{21} T_1 V_1 + g_{22} T_2 V_2\right)\right]$$

$$(5.12)\quad I_2 = I_{02} + \frac{k\, g_{22}}{C(g_{11}+g_{12})}\left[g_{11} T_1 + g_{12} T_2 - \left(g_{21} T_1 V_1 + g_{22} T_2 V_2\right)\right]$$

We can also make a first evaluation of the rate of heat transfer between the two diodes. This rate is equal to the net real AC power flow in the system. Since the average real AC power generated by the first diode is $\int_0^\infty |\langle v_f\, i_{1f}^* \rangle|\, df$ while the average real AC power dissipated by the same diode is $\int_0^\infty g_{11} |\langle v_{1f} \cdot v_{1f}^* \rangle|\, df$, the net average real AC power leaving the diode is found to be:

$$(5.13)\qquad \dot{Q} = \int_0^\infty \left[|\langle v_f \cdot i_f^* \rangle| - g_{11} S_f\right] df$$

Using (5.5) , (5.9) and the fact that i_{1f} and i_{2f} are uncorrelated:

$$(5.14)\quad \dot{Q} = \int_0^\infty \left[\frac{S_{i_1}(g_{11}+g_{12})}{(g_{11}+g_{12})^2+(2\pi f)^2 C^2} - \frac{(S_{i_1}+S_{i_2})\, g_{11}}{(g_{11}+g_{12})^2+(2\pi f)^2 C^2}\right] df$$

After simplifying and performing the integration:

$$(5.15)\qquad \dot{Q} = \frac{S_{i_1} g_{12} - S_{i_2} g_{11}}{4C(g_{11}+g_{12})}$$

Substitution of S_{i_1} and S_{i_2} yields:

$$(5.16)\quad \overset{\circ}{Q} = \frac{k}{C(g_{11}+g_{12})}\left[g_{11}g_{12}(T_1 - T_2) - (g_{12}g_{21}T_1V_1 - g_{11}g_{22}T_2V_2)\right]$$

We will next calculate the expressions for the DC bias voltages V_1 and V_2. By repeatedly applying Kirchhoff's and Ohm's Laws to the circuit of Fig. 5.3, we find:

$$(5.17)\qquad V_1 = V_2 + E$$

$$(5.18)\qquad I_1 = I - g_{11}V_1$$

$$(5.19)\qquad I_2 = -I - g_{12}V_2$$

Adding the last two equations we obtain:

$$(5.20)\qquad I_1 + I_2 = -g_{11}V_1 - g_{12}V_2$$

Solving equations (5.17) and (5.20) we get:

$$(5.21)\qquad V_1 = -\frac{I_1 + I_2 - E g_{12}}{g_{11} + g_{12}}$$

$$(5.22)\qquad V_2 = -\frac{I_1 + I_2 + E g_{11}}{g_{11} + g_{12}}$$

The net current I flowing through the system can easily be found using, for example, equation (5.17):

$$(5.23) \qquad I = I_1 + g_{11} V_1$$

Replacing V_1 from (5.21) gives:

$$(5.24) \qquad I = \frac{I_1 g_{12} - I_2 g_{11} + E g_{11} g_{12}}{g_{11} + g_{12}}$$

If the expressions for I_1 and I_2, (5.11) and respectively (5.12), are introduced in equations (5.21), (5.22) and (5.23):

$$(5.25) \; V_1 = -\frac{I_{01} + I_{02}}{g_{11} + g_{12}} - \frac{k(g_{21} + g_{22})}{C(g_{11} + g_{12})^2} \Big[g_{11} T_1 + g_{12} T_2 - \left(g_{21} T_1 V_1 + g_{22} T_2 V_2 \right) \Big] + \frac{E g_{12}}{g_{11} + g_{12}}$$

$$(5.26) \; V_2 = -\frac{I_{01} + I_{02}}{g_{11} + g_{12}} - \frac{k(g_{21} + g_{22})}{C(g_{11} + g_{12})^2} \Big[g_{11} T_1 + g_{12} T_2 - \left(g_{21} T_1 V_1 + g_{22} T_2 V_2 \right) \Big] - \frac{E g_{11}}{g_{11} + g_{12}}$$

$$(5.27) \; I = \frac{I_{01} g_{12} - I_{02} g_{11}}{g_{11} + g_{12}} + \frac{k(g_{12} g_{21} - g_{11} g_{22})}{C(g_{11} + g_{12})^2} \times$$

$$\times\left[g_{11}T_1 + g_{12}T_2 - \left(g_{21}T_1V_1 + g_{22}T_2V_2\right)\right] + \frac{E\,g_{11}\,g_{12}}{g_{11}+g_{12}}$$

We now have to evaluate the reverse currents I_{oj}. To do so, we will impose the condition of no thermodynamic fluxes when thermodynamic equilibrium prevails. For our system the conditions for thermodynamic equilibrium can be found by setting the rate of entropy production to zero.

Similar to our previous calculation of the rate of entropy production for a general electric fluctuation heat engine, we find for our new system:

$$\overset{\circ}{S} = \frac{\overset{\circ}{Q}}{T}\,\frac{\Delta T}{T} + \frac{E\cdot I}{T} \tag{5.28}$$

$$\Delta T \equiv T_1 - T_2$$

$$T \equiv T_2 \lessapprox T_1$$

If we choose $\frac{\Delta T}{T}$ and E to be our thermodynamic forces, $\overset{\circ}{Q}$ and I will be the corresponding fluxes. If our system is close to thermodynamic equilibrium:

$$I = A_{11}E + A_{12}\frac{\Delta T}{T} \tag{5.29}$$

$$\overset{\circ}{Q} = A_{21}E + A_{22}\frac{\Delta T}{T} \tag{5.30}$$

$A_{ij} \equiv$ phenomenological parameters

It is clear that we will have thermodynamic equilibrium when the forces are zero, or $T_1 = T_2$ and $E = 0$. Under such conditions the identified fluxes have also to be zero.

If $T_1 = T_2$ and $E = 0$ then $V_1 = V_2$ and (5.16) becomes

$$(5.31) \qquad \mathring{Q} = \frac{kT_1 V_1}{C(g_{11} + g_{12})}\left(g_{11} g_{22} - g_{12} g_{21}\right) = 0$$

Since the diodes do not have to be identical, $g_{11} g_{22} \neq g_{12} g_{21}$, and the solution of this equation is $V_1 = 0$, or from (5.25):

$$(5.32) \qquad I_{01} + I_{02} + \frac{kT_1}{C}\left(g_{21} + g_{22}\right) = 0$$

The condition that the current flux be also zero gives us a second equation in I_{01} and I_{02}. From (5.27):

$$(5.33) \qquad I_{01} g_{12} - I_{02} g_{11} + \frac{kT_1}{C}\left(g_{12} g_{21} - g_{11} g_{22}\right) = 0$$

Solving (5.32) and (5.33):

$$(5.34) \qquad I_{01} = -\frac{kT_1}{C} g_{21}$$

In a similar fashion

$$(5.35) \qquad I_{02} = -\frac{k T_2}{C} g_{22}$$

The reverse currents thus obtaied will also cancel any other fluxes present in the system such as ,for example, I_1 and I_2 . This can be checked by direct substitution (see derivation of (3.7))

It is clear that the reverse current of the diode is $-\frac{kT}{C} g_2$ regardless of the configuration of the outside circuit.

We will proceed towards our final goal, the calculation of the fluxes I and $\overset{o}{Q}$, by first finding the expression of V_1 .

Replacing (5.34) and (5.35) into (5.25) we obtain:

$$(5.36) \qquad V_1 = \frac{k}{C(g_{11}+g_{12})}\left\{g_{21}T_1 + g_{22}T_2 - \frac{g_{21}+g_{22}}{g_{11}+g_{12}}\left[g_{11}T_1 + g_{12}T_2 - \left(g_{21}T_1V_1 + g_{22}T_2V_2\right)\right]\right\} + \frac{E\,g_{12}}{g_{11}+g_{12}}$$

If we solve (5.17) for V_2 and replace it into the above equation:

$$(5.37) \qquad V_1 = \frac{k}{C(g_{11}+g_{12})}\left\{g_{21}T_1 + g_{22}T_2 - \frac{g_{21}+g_{22}}{g_{11}+g_{12}}\left[g_{11}T_1 + \right.\right.$$

$$+ g_{12}T_2 - (g_{21}T_1 + g_{22}T_2)V_1 + g_{22}T_2E\Big]\Big\} + \frac{E\,g_{12}}{g_{11}+g_{12}}$$

We can now solve for V_1 :

$$(5.38)\quad V_1 = \frac{\dfrac{k}{C(g_{11}+g_{12})}\Big[g_{21}T_1 + g_{22}T_2 - \dfrac{g_{21}+g_{22}}{g_{11}+g_{12}}\big(g_{11}T_1 + g_{12}T_2 + g_{22}T_2E\big)\Big] + \dfrac{E\,g_{12}}{g_{11}+g_{12}}}{1 - \dfrac{k(g_{21}+g_{22})}{C(g_{11}+g_{12})^2}(g_{21}T_1 + g_{22}T_2)}$$

After some algebra we finally get:

$$(5.39)\quad V_1 = -\frac{k(T_1-T_2)(g_{11}g_{22}-g_{12}g_{21}) + E\big[kT_2g_{22}(g_{21}+g_{22}) - Cg_{12}(g_{11}+g_{12})\big]}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)}$$

Introducing I_{01} and I_{02} from (5.34) and (5.35) into (5.27) , and replacing V_2 by $V_1 - E$ we obtain for I :

$$(5.40)\quad I = \frac{k}{C(g_{11}+g_{12})}\Big\{-g_{12}g_{21}T_1 + g_{11}g_{22}T_2 + \frac{g_{12}g_{21}-g_{11}g_{22}}{g_{11}+g_{12}} \times$$

$$\times\left[g_{11}T_1 + g_{12}T_2 - (g_{21}T_1 + g_{22}T_2)V_1 + g_{22}T_2 E\right]\Big\} +$$

$$+ \frac{E\, g_{11} g_{12}}{g_{11} + g_{12}}$$

Substituting V_1 in (5.40) by the expression we found at (5.39) :

$$(5.41)\quad I = \frac{k}{C(g_{11} + g_{12})}\Big\{ - g_{12} g_{21} T_1 + g_{11} g_{22} T_2 + \frac{g_{12} g_{21} -}{g_{11} +}$$

$$\frac{- g_{11} g_{22}}{+ g_{12}}(g_{11}T_1 + g_{12}T_2) + \frac{(g_{12}g_{21} - g_{11}g_{22})(g_{21}T_1 + g_{22}T_2)}{g_{11} + g_{12}} \times$$

$$\times \frac{k(T_1 - T_2)(g_{11}g_{22} - g_{12}g_{21}) + E\left[kT_2 g_{22}(g_{21} + g_{22}) - Cg_{12}(g_{11} + g_{12})\right]}{}\Big\} +$$

$$+ \frac{kT_2 g_{22}(g_{12}g_{21} - g_{11}g_{22})}{C(g_{11} + g_{12})^2} E + \frac{g_{11}g_{12}}{g_{11} + g_{12}} E$$

If we collect all the terms proportional to E and name the sum I_E :

$$(5.42)\quad I_E = \frac{k(g_{12}g_{21} - g_{11}g_{22})(g_{21}T_1 + g_{22}T_2)}{C(g_{11} + g_{12})^2} \times \frac{kT_2 g_{22}(g_{21} + g_{22}) -}{C(g_{11} + g_{12})^2 - k(g_{21} + g_{22}) \times}$$

$$\frac{- Cg_{12}(g_{11} + g_{12})}{\times (g_{21}T_1 + g_{22}T_2)} E + \frac{kT_2 g_{22}(g_{12}g_{21} - g_{11}g_{22})}{C(g_{11} + g_{12})^2} E + \frac{g_{11}g_{12}}{g_{11} + g_{12}} E$$

After lengthly algebra this expression can be reduced to:

$$(5.43)\quad I_E = \frac{-k(g_{12}g_{21}^2T_1 + g_{11}g_{22}^2T_2) + C\,g_{11}g_{12}(g_{11}+g_{12})}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1 + g_{22}T_2)}\,E$$

If we call the rest of the terms in (5.41), $I_{\Delta T}$ then:

$$(5.44)\quad I_{\Delta T} = \frac{k}{C(g_{11}+g_{12})}\Big[-g_{12}g_{21}T_1 + g_{11}g_{22}T_2 + \frac{(g_{12}g_{21}-g_{11}g_{22})(g_{11}T_1+g_{12}T_2)}{g_{11}+g_{12}} +$$

$$+\frac{(g_{12}g_{21}-g_{11}g_{22})(g_{21}T_1+g_{22}T_2)}{g_{11}+g_{12}} \times \frac{k(T_1-T_2)(g_{11}g_{22}-g_{12}g_{21})}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)}$$

This can also be considerably simplified and as a result (5.44) becomes:

$$(5.45)\quad I_{\Delta T} = \frac{k(T_1-T_2)\left[\frac{k\,g_{21}g_{22}}{C}(g_{21}T_1+g_{22}T_2) - (g_{12}^2g_{21} + g_{11}^2g_{22})\right]}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)}$$

We can now identify the two phenomenological parameters A_{11}, A_{12} appearing in the current flux equations (4.3):

$$(5.46)\qquad A_{11} = \frac{-k(g_{12}g_{21}^2T_1 + g_{11}g_{22}^2T_2) + C\,g_{11}g_{12}(g_{11}+g_{12})}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1 + g_{22}T_2)}$$

$$(5.47)\qquad A_{12} = \frac{kT_2\left[k\,g_{21}g_{22}(g_{21}T_1+g_{22}T_2) - C(g_{12}^2g_{21} + g_{11}^2g_{22})\right]}{C\left[C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)\right]}$$

In the view of our "close to thermodynamic equilibrium" assumption $T_1 \gtrsim T_2 \equiv T$, and the above formulas can be reduced to:

$$(5.48) \quad A_{11} \cong \frac{-kT\left(g_{12}g_{21}^{2}+g_{11}g_{22}^{2}\right)+C\,g_{11}g_{12}\left(g_{11}+g_{12}\right)}{C\left(g_{11}+g_{12}\right)^{2}-kT\left(g_{21}+g_{22}\right)^{2}}$$

$$(5.49) \quad A_{12} \cong \frac{kT\left[kT\,g_{21}g_{22}\left(g_{21}+g_{22}\right)-C\left(g_{12}^{2}g_{21}+g_{11}^{2}g_{22}\right)\right]}{C\left[C\left(g_{11}+g_{12}\right)^{2}-kT\left(g_{21}+g_{22}\right)^{2}\right]}$$

Our next task is the evaluation of the rate of heat flow $\mathring{Q}$. Starting with (5.16) by replacing V_2 with $V_1 - E$ we obtain:

$$(5.50) \quad \mathring{Q}=\frac{k}{C\left(g_{11}+g_{12}\right)}\left[g_{11}g_{12}\left(T_1-T_2\right)-\left(g_{12}g_{21}T_1-g_{11}g_{22}T_2\right)V_1-g_{11}g_{22}E\right]$$

If we substitute V_1 from (5.39) in the above expression:

$$(5.51) \quad \mathring{Q}=\frac{k}{C\left(g_{11}+g_{12}\right)}\left[g_{11}g_{12}\left(T_1-T_2\right)+\left(g_{12}g_{21}T_1-g_{11}g_{22}T_2\right)\times\right.$$

$$\times\frac{k\left(T_1-T_2\right)\left(g_{11}g_{22}-g_{12}g_{21}\right)}{C\left(g_{11}+g_{12}\right)^{2}-k\left(g_{21}+g_{22}\right)\left(g_{21}T_1+g_{22}T_2\right)}+\left(g_{12}g_{21}T_1-g_{11}g_{22}T_2\right)\times$$

$$\left.\times\frac{kT_2\,g_{22}\left(g_{21}+g_{22}\right)-C\,g_{12}\left(g_{11}+g_{12}\right)}{C\left(g_{11}+g_{12}\right)^{2}-k\left(g_{21}+g_{22}\right)\left(g_{21}T_1+g_{22}T_2\right)}E-g_{11}g_{22}T_2E\right]$$

If we call $\mathring{Q}_E$ the sum of the terms proportional to E then:

$$(5.52)\quad \mathring{Q}_E = \frac{k}{C(g_{11}+g_{12})}\left[\left(g_{12}g_{21}T_1 - g_{11}g_{22}T_2\right)\times\frac{kT_2 g_{22}(g_{21}+g_{22}) - C g_{12}(g_{11}+g_{12})}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)}E - g_{11}g_{22}T_2 E\right]$$

After bringing the expression to a common denominator and simplifying:

$$(5.53)\quad \mathring{Q}_E = \frac{kE}{C}\left[\frac{k g_{21}g_{22}(g_{21}+g_{22})T_1T_2 - C\left(g_{12}^2 g_{21}T_1 + g_{11}^2 g_{22}T_2\right)}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)}\right]$$

Collecting the rest of the terms in (5.51) and calling the sum $\mathring{Q}_{\Delta T}$:

$$(5.54)\quad \mathring{Q}_{\Delta T} = \frac{k}{C(g_{11}+g_{12})}\left[g_{11}g_{12}(T_1-T_2) + \left(g_{12}g_{21}T_1 - g_{11}g_{22}T_2\right)\times\frac{k(T_1-T_2)(g_{11}g_{22}-g_{12}g_{21})}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)}\right.$$

The term $T_1 - T_2$ can be factored out:

$$(5.55)\quad \dot{Q}_{\Delta T} = \frac{k(T_1 - T_2)}{C}\left[g_{11} g_{12} + \frac{k(g_{12} g_{21} T_1 - g_{11} g_{22} T_2)(g_{11} g_{22} - g_{12} g_{21})}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1 + g_{22}T_2)}\right]$$

A final reduction yields:

$$(5.56)\quad \dot{Q}_{\Delta T} = \frac{k(T_1 - T_2)}{C} \cdot \frac{C g_{11} g_{12}(g_{11}+g_{12}) - k(g_{12} g_{21}^2 T_1 + g_{11} g_{22}^2 T_2)}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1 + g_{22}T_2)}$$

Identifying A_{21} and A_{22} of equation (4.4) :

$$(5.57)\quad A_{21} = \frac{k}{C} \times \frac{k g_{21} g_{22}(g_{21}+g_{22}) T_1 T_2 - C(g_{12}^2 g_{21} T_1 + g_{11}^2 g_{22} T_2)}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1 + g_{22}T_2)}$$

$$(5.58)\quad A_{22} = \frac{k T_2}{C} \times \frac{C g_{11} g_{12}(g_{11}+g_{12}) - k(g_{12} g_{21}^2 T_1 + g_{11} g_{22}^2 T_2)}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1 + g_{22}T_2)}$$

Since $T_1 \simeq T_2 \equiv T$, A_{21} and A_{22} can be rewritten:

$$(5.59)\quad A_{21} \cong \frac{kT}{C} \cdot \frac{kT g_{21} g_{22}(g_{21}+g_{22}) - C(g_{12}^2 g_{21} + g_{11}^2 g_{22})}{C(g_{11}+g_{12})^2 - kT(g_{21}+g_{22})^2}$$

$$(5.60)\quad A_{22} \cong \frac{kT}{C} \cdot \frac{-kT(g_{12} g_{21}^2 + g_{11} g_{22}^2) + C g_{11} g_{12}(g_{11}+g_{12})}{C(g_{11}+g_{12})^2 - kT(g_{21}+g_{22})^2}$$

We immediately see that $A_{12} = A_{21}$ as required by the Onsager's reciprocal relations. This is not at all surprising since compliance with the Onsager's results is built in the

Gunn's model for nonlinear dissipative devices.

In addition A_{11} and A_{22} are also simply related:

$$(5.61) \quad A_{22} = \frac{kT}{C} A_{11}$$

If we define $A_1 \equiv A_{11}$ and $A_2 \equiv A_{12} = A_{21}$ the two flux equations (4.3) and (4.4) become:

$$(5.62) \quad I = A_1 E + A_2 \frac{\Delta T}{T}$$

$$(5.63) \quad \mathring{Q} = A_2 E + \frac{kT}{C} A_1 \frac{\Delta T}{T}$$

Before we continue with a detailed analysis of the above results let us go back and calculate the averge mean square value of the fluctuation voltage across the capacitor C.

$$(5.64) \quad \langle v_c^2 \rangle = \int_0^{\infty} S_v \, df$$

Using (5.10) and replacing V_2 by $V_1 - E$:

$$(5.65) \quad \langle v_c^2 \rangle = \frac{k}{C(g_{11}+g_{12})}\left[g_{11} T_1 + g_{12} T_2 - (g_{21} T_1 + g_{22} T_2) V_1 + g_{22} T_2 E \right]$$

If we now subsitute V_1 by its expression as given by (5.39) :

$$(5.66)\quad \langle v_c^2 \rangle = \frac{k}{C(g_{11}+g_{12})}\Big[g_{11}T_1 + g_{12}T_2 + (g_{21}T_1 + g_{22}T_2)\times$$

$$\times\frac{k(T_1-T_2)(g_{11}g_{22}-g_{12}g_{21})}{C(g_{11}+g_{12})^2-k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)} + (g_{21}T_1 + g_{22}T_2)\times$$

$$\times\frac{kT_2g_{22}(g_{21}+g_{22})-Cg_{12}(g_{11}+g_{12})}{C(g_{11}+g_{12})^2-k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2}E + g_{22}T_2E\Big]$$

Again after some algebra:

$$(5.67)\quad \langle v_c^2 \rangle = \frac{k}{C}\times\frac{C(g_{11}+g_{12})(g_{11}T_1+g_{12}T_2) - k(g_{21}T_1+g_{22}T_2)^2 + E\cdot C(g_{11}g_{22}T_2 - g_{12}g_{21}T_1)}{C(g_{11}+g_{12})^2 - k(g_{21}+g_{22})(g_{21}T_1+g_{22}T_2)}$$

At thermodynamic equilibrium $E=0$ and $T_1 = T_2 \equiv T$, and

$$(5.68)\quad \langle v_c^2 \rangle = \frac{k}{C}\times\frac{C(g_{11}+g_{12})(g_{11}T+g_{12}T)-k(g_{21}T+g_{22}T)^2}{C(g_{11}+g_{12})^2-k(g_{21}+g_{22})(g_{21}T+g_{22}T)}$$

Or:

$$(5.69) \quad \langle v_c^2 \rangle = \frac{kT}{C}$$

The above result is, as expected, consistant with the principle of equipartition of energy.

If the dissipative devices in our system are linear, $g_{21} = g_{22} = 0$ and:

$$(5.70) \quad \langle v_c^2 \rangle = \frac{k(g_{11}T_1 + g_{12}T_2)}{C(g_{11} + g_{12})}$$

Thus the voltage fluctuations across the capacitor behave as if the system is in thermal equilibrium at an effective temperature T_{eff} , independent of the bias battery voltage E .

$$(5.71) \quad T_{eff} = \frac{g_{11}T_1 + g_{12}T_2}{g_{11} + g_{12}}$$

This result is well known from the simpler analysis of two linear resistors at different temperatures [22].

If only thermal equilibrium exists, i.e. $T_1 = T_2 \equiv T$, $E \neq 0$, the excess mean square value of v_c is directly proportional to the bias battery voltage E.

$$(5.72) \quad \langle v_c^2 \rangle = \frac{kT}{C} + \frac{kT(g_{11}g_{22} - g_{12}g_{21})}{C(g_{11} + g_{12})^2 - kT(g_{21} + g_{22})^2} E$$

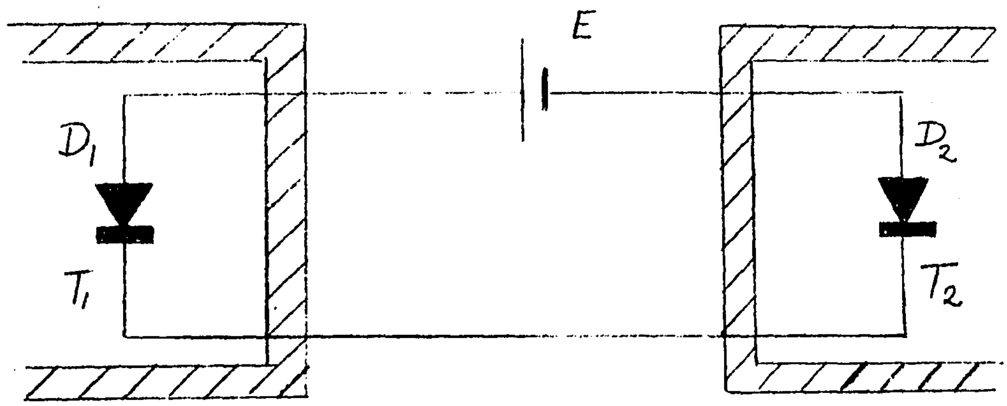


Fig. 5.1.

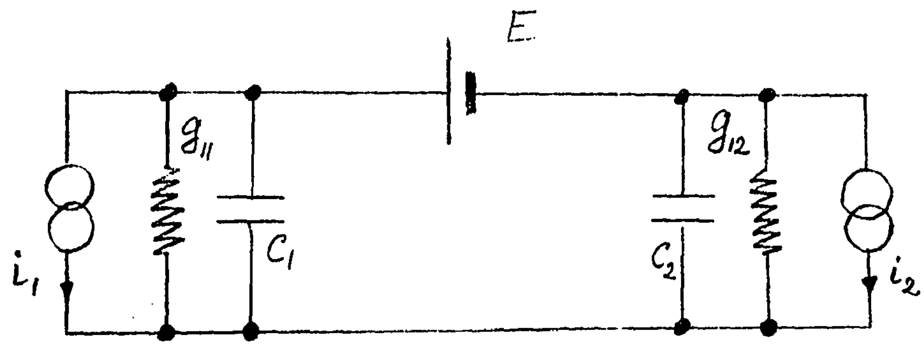


Fig. 5.2.

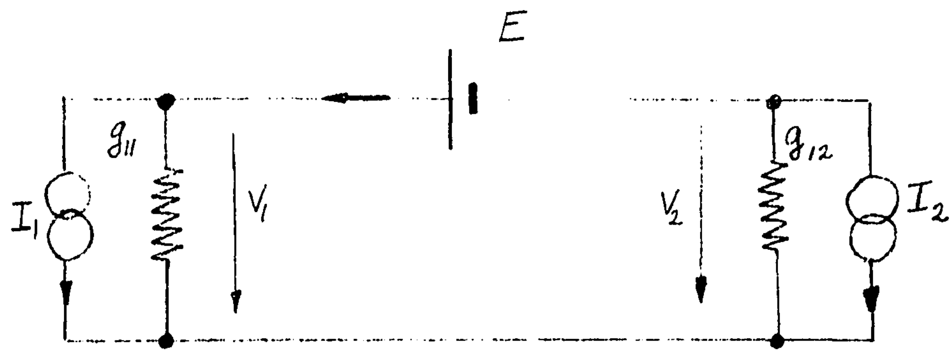


Fig 5.3

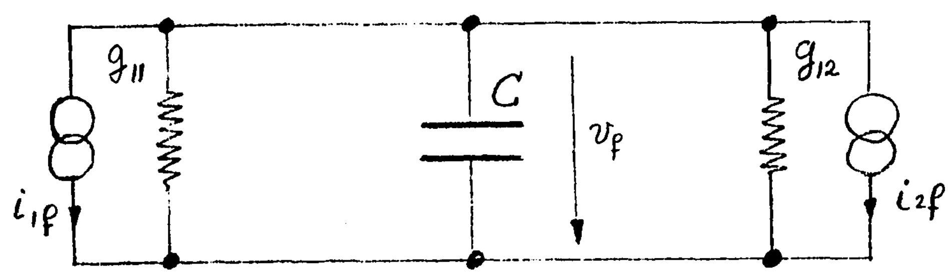


Fig 5.4

5.2 Analysis of Results

5.2.1 General Discussion

Let us summarize the results of the previous section:

Rate of entropy production $\overset{o}{S}$

$$(4.2) \quad \overset{o}{S} = \frac{\overset{o}{Q}}{T}\frac{\Delta T}{T} + \frac{E \cdot I}{T}$$

Thermodynamic flux equations

$$(5.62) \quad I = A_1 E + A_2 \frac{\Delta T}{T}$$

$$(5.63) \quad \overset{o}{Q} = A_2 E + \frac{kT}{C} A_1 \frac{\Delta T}{T}$$

Phenomenological coefficients:

$$(5.48) \quad A_1 = \frac{-kT(g_{12} g_{21}^2 + g_{11} g_{22}^2) + C g_{11} g_{12}(g_{11} + g_{12})}{C(g_{11} + g_{12})^2 - kT(g_{21} + g_{22})^2}$$

$$(5.60) \quad A_2 = \frac{kT}{C} \cdot \frac{kT g_{21} g_{22}(g_{21} + g_{22}) - C(g_{12}^2 g_{21} + g_{11}^2 g_{22})}{C(g_{11} + g_{12})^2 - kT(g_{21} + g_{22})^2}$$

We will first briefly analyze the behaviour of the system when the two diodes are replaced by linear resistors. If we let $g_{21} = g_{22} = 0$:

$$(5.73) \quad A_1 = \frac{g_{11}\, g_{12}}{g_{11} + g_{12}}$$

$$(5.74) \quad A_2 = 0$$

As expected, there is no net current flow due to the temperature difference between the resistors, and there is no additional "shot noise" heat flow:

$$(5.75) \quad I = \frac{g_{11}\, g_{12}}{g_{11} + g_{12}} E$$

$$(5.76) \quad \mathring{Q} = \frac{kT}{C} \cdot \frac{g_{11}\, g_{12}}{g_{11} + g_{12}} \cdot \frac{\Delta T}{T}$$

The expression for the current is simply a restatement of Ohm's Law. If we observe that $B \equiv \dfrac{g_{11}\, g_{12}}{C(g_{11} + g_{12})}$ is nothing else but the natural bandwidth of our circuit it is apparent that the expression for the heat flow is just the Nyquist noise based heat flow within the bandwidth B. We can also point out the fact that, around thermodynamic equilibrium the net current flow and the net heat flow in our linear system are related, through the relation:

$$(5.77) \quad \frac{\mathring{Q}}{I} = \frac{k \Delta T}{C E}$$

Going back to the nonlinear two diodes system, let us calculate the rate of entropy production by replacing I and $\mathring{Q}$ from (5.62) and (5.63) into (4.2) .

$$(5.78)\quad \mathring{S} = \frac{1}{T} \cdot \frac{\Delta T}{T}\left(A_2 E + \frac{kT}{C} A_1 \frac{\Delta T}{T}\right) + \frac{E}{T}\left(A_1 E + A_2 \frac{\Delta T}{T}\right)$$

Multiplying by T and expanding:

$$T\mathring{S} = \frac{kT}{C} A_1 \left(\frac{\Delta T}{T}\right)^2 + 2A_2 \frac{\Delta T}{T} E + A_1 E^2$$

Dividing by E^2 and introducing $X \equiv \dfrac{\Delta T}{T\,E}$:

$$(5.79)\quad \frac{T\mathring{S}}{E^2} = \frac{kT}{C} A_1 X^2 + 2A_2 X + A_1$$

If our system is to obey the Second Law the rate entropy production has to be positive, $\mathring{S} > 0$, unless, we are in thermodynamic equilibrium when $\mathring{S}$ becomes zero.

$$(5.80)\quad \frac{kT}{C} A_1 X^2 + 2A_2 X + A_1 > 0$$

This inequality is satisfied for any X if and only if :

$$(5.81)\quad A_1 > 0$$

and

$$(5.82)\quad A_2^2 < \frac{kT}{C} A_1^2$$

If we introduce the dimensionless quantity β, (5.82) translates into:

$$(5.83)\quad \beta \equiv \frac{C}{kT} \cdot \frac{A_2^2}{A_1^2} < 1$$

Substituting A_1 and A_2 from (5.48) and (5.60) we obtain:

$$(5.84)\quad \frac{-kT(g_{12}g_{21}^2 + g_{11}g_{22}^2) + C g_{11}g_{12}(g_{11}+g_{12})^2}{C(g_{11}+g_{12})^2 - kT(g_{21}+g_{22})^2} > 0$$

$$(5.85)\quad \left\{\frac{kT}{C}\left[kT g_{21}g_{22}(g_{21}+g_{22}) - C(g_{12}^2 g_{21} + g_{11}^2 g_{22})\right]\right\}^2 <$$

$$< \frac{kT}{C}\left[-kT(g_{12}g_{21}^2 + g_{11}g_{22}^2) + C g_{11}g_{12}(g_{11}+g_{12})\right]^2$$

Let us simplify our task by choosing similar diodes with $g_{11} \cong g_{12} \equiv g_1$ and $g_{21} \cong g_{22} \equiv g_2$.

Then:

$$(5.86)\quad A_1 = \frac{g_1}{2}$$

$$(5.87)\quad A_2 = -\frac{kT}{2C} g_2$$

Our inequalities, for the two diodes system, become:

$$(5.88)\quad g_1 > 0$$

$$(5.89)\quad \beta = \frac{kT}{C}\left(\frac{g_2}{g_1}\right)^2 < 1$$

The first inequality forbids, on thermodynamical grounds, the existence of purely dissipative devices exhibiting negative conductances when operating close to thermodynamic equilibrium. This statement does not contradict the existance of such negative impedance devices, like Gunn diodes or triacs, since the mentioned devices have negative conductances only when they function under conditions far from thermodynamic equilibrium such as under large applied bias voltages.

The second inequality brings a second limitation on our engine. If the system contains nonlinear elements, the quantity β depending on the temperature, conductance and the degree of nonlinearity of the system, has to be smaller than one. For our circuit C is the equivalent capacitance of the two diodes in parallel, and therefore we obtain a limit for the β_D quantity associated with a single diode:

$$(5.90) \quad \beta_D = \frac{kT}{C_D}\left(\frac{g_2}{g_1}\right)^2 < 2,$$

$C_D \equiv$ capacitance of a single diode

Again this is the case as long as we do not leave the neighborhood of the state of thermodynamic equilibrium.

It will be later shown that the condition on β is automatically satisfied in the range of validity of Gunn's model. Thus violation of this condition does not imply the violation of the Second Law, but just that our system is beyond the validity range of our results.

We will next study the behaviour of our system if we replace one of the diodes by a linear resistor.

If for convenience we choose our resistor such that $g_{11} = g_{12}$ at the bias voltage and define $g_1 \equiv g_{11} = g_{12}$, $g_2 \equiv g_{22}$ then:

$$(5.91) \quad A_1 = \frac{-kTg_1g_2^2 + 2Cg_1^3}{4Cg_1^2 - kTg_2^2}$$

$$(5.92) \quad A_2 = \frac{-kTg_1^2g_2}{4Cg_1^2 - kTg_2^2}$$

The inequalities (5.81) and (5.82) become:

$$(5.93) \qquad \frac{g_1\left(2Cg_1^2 - kTg_2^2\right)}{4Cg_1^2 - kTg_2^2} > 0$$

$$(5.94) \qquad \left[kTg_1^2 g_2\right]^2 < \frac{kT}{C}\left[-kTg_1 g_2^2 + 2Cg_1^3\right]^2,$$

We can rearrange (5.94) into the more convenient form:

$$(5.95) \qquad \left[\frac{kT}{C}\left(\frac{g_2}{g_1}\right)^2\right]^2 - 5\left[\frac{kT}{C}\left(\frac{g_2}{g_1}\right)\right] + 4 > 0$$

This inequality is also automatically satisfied in the range of validity of Gunn's model.

In the limit of small $\beta \equiv \frac{kT}{C}\left(\frac{g_2}{g_1}\right)^2$, A_1 and A_2 reduce to:

$$(5.96) \qquad A_1 = \frac{g_1}{2}$$

$$(5.97) \qquad A_2 = -\frac{kTg_2}{4C}$$

In this approximation the flux equations become:

$$(5.98)\quad I = E\frac{g_1}{2} - \frac{kTg_2}{4C}\,\frac{\Delta T}{T}$$

$$(5.99)\quad \mathring{Q} = -\frac{kTg_2}{4C}E + \frac{kTg_1}{2C}\,\frac{\Delta T}{T}$$

The above relations will be later used to compare our results with the rigorous answers provided by kinetic models of a special type of electric fluctuation heat engine.

5.2.2 Engine Efficiency, Power Output

Lets us start by rewriting the equations (4.2) , (5.62) and (5.63):

$$\overset{\circ}{S} = \frac{\overset{\circ}{Q}}{T}\frac{\Delta T}{T} + \frac{E \cdot I}{T} \tag{4.2}$$

$$I = A_1 E + A_2 \frac{\Delta T}{T} \tag{5.62}$$

$$\overset{\circ}{Q} = A_2 E + \frac{kT}{C} A_1 \frac{\Delta T}{T} \tag{5.63}$$

If the system is to function as an engine, considering the sign convention we adopted, the direct current has to be negative as to charge the battery. Then the efficiency and the power output will be given by:

$$\eta = \frac{-IE}{\overset{\circ}{Q}} = \frac{-E\left(A_1 E + A_2 \frac{\Delta T}{T}\right)}{A_2 E + \frac{kT}{C} A_1 \frac{\Delta T}{T}} \tag{5.100}$$

$$P_o = -I \cdot E = -E\left(A_1 E + A_2 \frac{\Delta T}{T}\right) \tag{5.101}$$

The voltages E_η and E_P that will maximize the efficiency and respectively the output power of the engine are found to be:

$$(5.102) \quad E_{\eta} = \sqrt{\frac{kT}{C}} \left[\frac{-1 + \sqrt{1-\beta}}{\sqrt{\beta}} \right]^2 \frac{\Delta T}{T}$$

$$(5.103) \quad E_P = -\frac{A_2}{2A_1} \cdot \frac{\Delta T}{T}$$

The efficiencies and powers at these two important operating points are found by replacing E in (5.100) and (5.101) succesively by E_{η} and E_P. We then obtain:

$$(5.104) \quad \eta_{mE} = \left[\frac{-1 + \sqrt{1-\beta}}{\sqrt{\beta}} \right]^2 \frac{\Delta T}{T}$$

$$(5.105) \quad P_{\eta E} = -\frac{A_2}{\beta} \left(\frac{\Delta T}{T}\right)^2 \sqrt{1-\beta} \left[\frac{-1 + \sqrt{1-\beta}}{\sqrt{\beta}} \right]^2$$

$$(5.106) \quad P_{mE} = \frac{A_2^2}{4A_1} \left(\frac{\Delta T}{T}\right)^2$$

$$(5.107) \quad \eta_{PE} = \frac{\Delta T}{2T} \; \frac{\beta}{2-\beta}$$

η_{mE} ≡maximum efficiency with respect to E

P_{η_E} ≡output power at maximum efficiency with respect to E

P_{mE} ≡maximum output power with respect to E

η_{P_E} ≡efficiency at maximum output power with respect to E

If we look at the expressions of A_1 and A_2 it becomes clear that in the range of validity of Gunn's model the efficiency is lower than Carnot's limit

We could observe that η_{mE} and η_{PE} are monotonically decreasing functions of β on the real domain $[1, \infty)$, and therefore maximum efficiencies will be obtained at $\beta = 1$. Then:

$$E_\eta = \sqrt{\frac{kT}{C}}\,\frac{\Delta T}{T} \tag{5.108}$$

$$E_P = \frac{1}{2}\sqrt{\frac{kT}{C}}\,\frac{\Delta T}{T} \tag{5.109}$$

$$\eta_M = \frac{\Delta T}{T} \tag{5.110}$$

$$P_\eta = 0 \tag{5.111}$$

$$P_M = -\frac{A_2}{4}\sqrt{\frac{kT}{C}}\left(\frac{\Delta T}{T}\right)^2 \tag{5.112}$$

$$(5.113) \qquad \eta_P = \frac{\Delta T}{2T}$$

$\eta_M \equiv$ maximum efficiency

$P_\eta \equiv$ power at maximum efficiency

$P_M \equiv$ maximum output power

$\eta_P \equiv$ efficiency at maximum power

The interesting feature of the above analysis is that the relative relations among the phenomenological coefficients of equations (5.62) and (5.63) are sufficient to derive the Carnot limit.

However, by letting β approach 1 we already are outside our validity range which requires $\beta \ll 1$.

5.2.3 Range of Validity

Let us start by analysing the open circuit behaviour of a diode using Gunn's model as shown in Fig. 5.5 .

Since the diode is assumed to be in thermodynamic equilibrium at temperature T no flux is expected to develop and therefore the DC current component of the current source is expected to be zero.

The resulting DC voltage across the capacitor, $V = I/g_1$, will also be zero. Then according to Gunn the power spectrum of the current source is given by:

$$S_i = 4kTg_1 \tag{5.114}$$

The power spectrum of the current i_D passing through the capacitor can then be easily calculated.

$$S_{i_D} = 4kTg_1 \frac{\omega^2 C^2}{g_1^2 + \omega^2 C^2} \tag{5.115}$$

If we compare this result with the more rigorous answers obtained by Van Kampen [17,18] for the Alkemade diode we conclude that Gunn's model is definitely valid when the quantity β defined by (5.83) is much smaller than one.

$$\beta \ll 1$$

For β approaching one Van Kampen's results vary drastically from ours, suggesting that Gunn's model is valid only for modest nonlinearity or large capacitance. Under such circumstances the meaning of the limit value of β would simply be to define the region of validity for our results. The corresponding capacitance is then limited to:

$$(5.116) \quad C \gg kT\left(\frac{g_2}{g_1}\right)^2$$

Since all technologically feasible diodes have capacitances very much larger than the limit, and therefore $\beta \ll 1$, our general conclusions apply and we find the efficiency and output power of such fluctuation engines totally negligible

What should we expect to happen if we could build a diode with a capacitance small enough to approach the lower limit ?

Since our analysis is valid to the first order in β our results will be valid until C approaches the limit within one order of magnitude. Beyond that point, as suggested by Van Kampen's work, the power spectrum of the current source will no longer be white and therefore our conclusions will no longer be correct.

Nevertheless, if we want to design efficient fluctuation engines our analysis gives us the right direction for our effort.

We expect to reach higher efficiencies as the capacitance approaches the range of values around the limit, even if our previous results are no longer rigorously valid.

As an example, if we choose for our engine PN diodes then:

$$(5.117) \quad g_{1PN} = \frac{e}{kT} I_s e^{\frac{eV}{kT}}$$

$$(5.118) \quad g_{2PN} = \frac{e^2}{2k^2T^2} I_s e^{\frac{eV}{kT}}$$

High efficiencies are expected to occur when the capacitance of the diode approaches its limit value which in this case is given by:

$$(5.119) \quad C = \frac{e^2}{4kT}$$

Such a small capacitance ($\sim 10^{-18} F$) is attainable only if the physical volume of the diode is of the order of 10^{-15} cm^3 !!

Therefore, high efficiency fluctuation engines are, at least for the time being, technologically absurd.

However, if we want to theoretically study engines beyond the validity limit for our Gunn type analysis, a different approach based on kinetic models of our dissipative elements must be used.

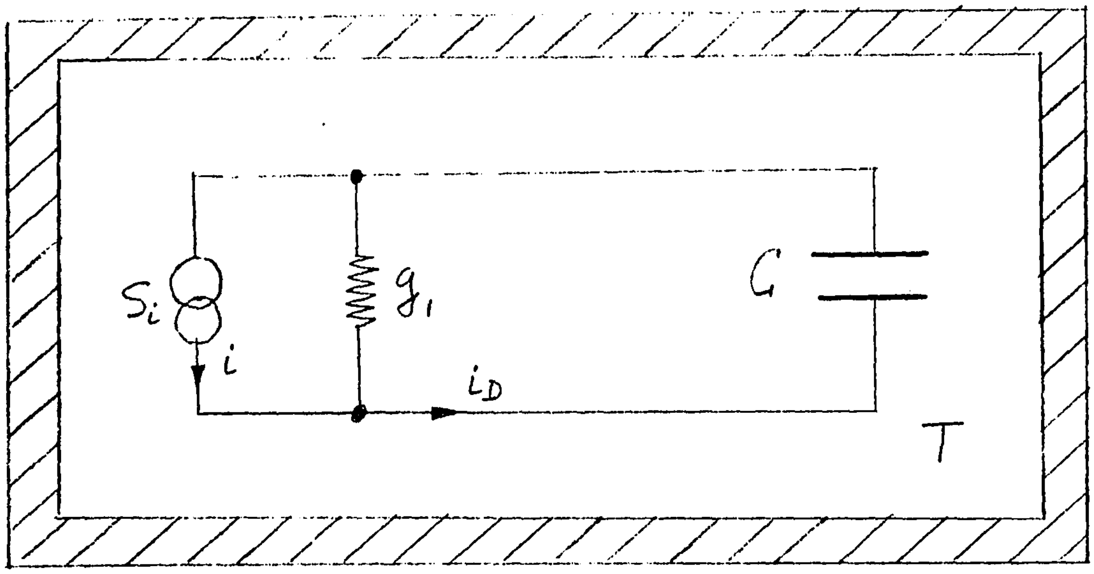


Fig 5.5.

6. KINETIC ANALYSIS OF FLUCTUATION HEAT ENGINES

We will study in detail an electrical fluctuation engine employing a linear resistor at temperature T_1 , connected in series with an ideal battery of voltage E and a nonlinear diode at a slightly different temperature $T_2 \lessgtr T_1$ (Fig.6.1). The two terminal leads of the diode are made of materials a and b characterized by work functions W_a and respectively W_b . They connect through the junctions A and B to the rest of the circuit, which is made of the different material c characterized by the work function W_c .

We neglect thermoelectric effects and consider only the fluxes generated by the battery, and the thermal current fluctuations

Since kinetic models of practical diodes are very difficult to devise we will use for D a simplified discription of a vacuum diode initially proposed by Alkemade [16]. The resistor will also be kinetically modeled using a novel approach based on the principle of detailed balance. If our study will be correct we expect it to yield results compatible, in the limit of large capacitance, with the Gunn type analysis.

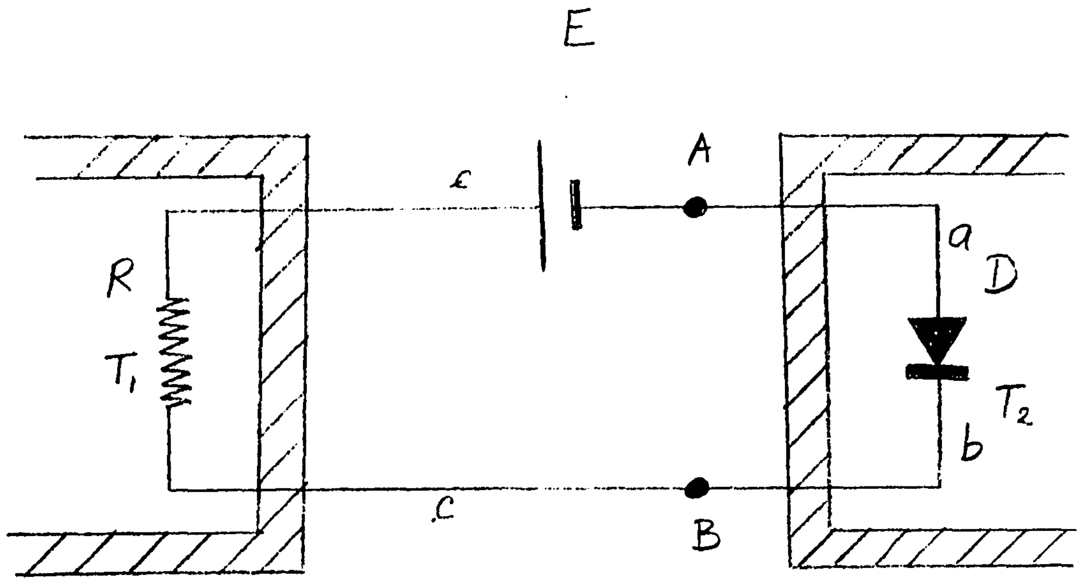


Fig 6.1

6.1 The Alkemade Diode [17]

We will consider a vacuum diode with two plane, parallel, electrodes consisting of two different metals a and b presenting different work functions W_a and W_b. The two electrodes are located very close to each other such as to minimize the effects of the finite time of flight of the electrons. The parallel plates present a capacitance C and the whole diode system is in thermal equilibrium with a heat bath at temperature T. We also assume that the plate charged with excess electrons will operate under saturated conditions, with the rate of electron emissions independent of the potential difference between the plates.

Let us start with n electrons on the plate a. The rate of electron transitions from plate a to plate b is then given by Richardson's formula.

$$a_D(n) = \frac{4\pi m_e}{h^3} k^2 T^2 S \, e^{-\frac{W_a}{kT}} \tag{6.1}$$

$m_e \equiv$ electronic mass

$h \equiv$ Planck's constant

$k \equiv$ Boltzmann's constant

$S \equiv$ area of the elctrodes

The rate of electrons leaving the electrode b is similarly given by:

$$(6.2) \qquad b_D'(m) = \frac{4\pi m_e}{h^3} k^2 T^2 S\, e^{-\frac{W_b}{kT}}$$

But not all electrons leaving b will reach the electrode a because of the electrostatic potential difference between the plates.

The electrostatic energy of the capacitor is:

$$(6.3) \qquad E_m = \frac{e^2 m^2}{2C}$$

Then the potential barrier seen by the electrons leaving is:

$$\Delta E_m = E_{m+1} - E_m = \frac{e^2}{C}\left(m + \frac{1}{2}\right)$$

We can now calculate the rate of electron transitions from b to a :

$$b_D(m) = b_D'(m) \cdot e^{-\frac{\Delta E_m}{kT}}$$

$$(6.4) \qquad b_D(m) = \frac{4\pi m_e}{h^3} k^2 T^2 S\, e^{-\frac{W_b}{kT}}\, e^{-\frac{e^2}{kTC}\left(m+\frac{1}{2}\right)}$$

Observing that,

$$(6.5) \qquad b_D'(m) = a_D(m) \cdot e^{\frac{W_a}{kT}}$$

and defining the quantity $\mathcal{E}$ by:

$$(6.6) \qquad \mathcal{E} \equiv \frac{e^2}{kTC}$$

then (6.6) becomes:

$$(6.7) \qquad b_D(m) = a_D(m) \cdot e^{-\frac{W_b - W_a}{kT}} \, e^{-\mathcal{E}(m+\frac{1}{2})}$$

If $P_D(m)$ is the probability of having at time t, m excess electrons on the electrode a, using (6.1) and (6.7) we can write the associated Master Equation:

$$(6.8) \qquad \frac{\partial P_D(m)}{\partial t} = a_D(m+1)\, P_D(m+1) + b_D(m-1)\, P_D(m-1) -$$

$$- a_D(m)\, P_D(m) - b_D(m)\, P_D(m)$$

The first two terms correspond to gains due to transitions down from the $m+1$ electrons and up from the $m-1$ electrons states. The last two terms correspond to losses because of transitions away from the m electrons state.

We have assumed that the probability of two or more simultaneous electron transitions is negligible.

Since we are interested only in the steady state behaviour of the diode we will seek the stationary solution of the derived Master Equation, $P_{DS}(m)$:

$$(6.9)\quad a_D(m+1)P_{DS}(m+1) + b_D(m-1)P_{DS}(m-1) - a_D(m)P_{DS}(m) - b_D(m)P_{DS}(m) = 0$$

If we group the first and last terms and rearrange we obtain:

$$(6.10)\quad a_D(m+1)P_{DS}(m+1) - b_D(m)P_{DS}(m) = a_D(m)P_{DS}(m) - b_D(m-1)P_{DS}(m-1)$$

Therefore,

$$a_D(m+1)P_{DS}(m+1) - b_D(m)P_{DS}(m)$$

is a constant independent of m.

In the limit of infinite m the probability $P_{DS}(m)$ has to be zero and therefore:

$$(6.11)\qquad a_D(m+1)P_{DS}(m+1) - b_D(m)P_{DS}(m) = 0$$

We can now write the recursion relation that defines the stationary solution of our Master Equation:

$$(6.12)\qquad P_{DS}(m+1) = \frac{b_D(m)}{a_D(m+1)}\,P_{DS}(m)$$

If we replace $a_D(m)$ and $b_D(m)$ by their corresponding expressions as given by (6.1) and (6.7) :

$$(6.13) \quad P_{DS}(m+1) = e^{-\frac{W_b - W_a}{kT} - \varepsilon(m+\frac{1}{2})} P_{DS}(m)$$

Starting with $m = 0$:

$$(6.14) \quad P_{DS}(1) = e^{-\frac{W_b - W_a}{kT} - 1 \cdot \frac{\varepsilon}{2}} \cdot e^{-0 \cdot \varepsilon} P_{DS}(0)$$

Then for $m = 1$:

$$(6.15) \quad P_{DS}(2) = e^{-2\frac{W_b - W_a}{kT} - 2 \cdot \frac{\varepsilon}{2}} \cdot e^{-(0+1)\varepsilon} P_{DS}(0)$$

In general:

$$(6.16) \quad P_{DS}(m) = e^{-m\frac{W_b - W_a}{kT} - m\frac{\varepsilon}{2}} \cdot e^{-(0+1+\ldots+(m-1))\varepsilon} P_{DS}(0)$$

Or writing the sum in closed form:

$$(6.17) \quad P_{DS}(m) = e^{-\frac{W_b - W_a}{kT} - m\frac{\varepsilon}{2}} e^{-\frac{m(m-1)}{2}} P_{DS}(0)$$

By rearraging the exponent we find:

$$(6.18) \quad P_{DS}(m) = e^{\frac{1}{2\varepsilon}\left(\frac{W_a - W_b}{kT}\right)^2} P_{DS}(0) \cdot e^{-\frac{\varepsilon}{2}\left(m - \frac{W_a - W_b}{\varepsilon kT}\right)^2}$$

Thus the stationary solution of the Master Equation is a (discrete) Gaussian centered at $\frac{W_a - W_b}{\varepsilon kT}$, and can be written as

$$(6.19) \quad P_{DS}(m) = \frac{e^{-\frac{\varepsilon}{2}(m - m_D)^2}}{Z_D}$$

where

$$(6.20) \quad Z_D = \sum_{m=-\infty}^{\infty} e^{-\frac{\varepsilon}{2}(m - m_D)^2}$$

and

$$(6.21) \quad m_D = \frac{W_a - W_b}{e^2/C}$$

If ε is small compared with unity, the situation is essentialy continuous, and standard results for continuous Gaussians hold. These are:

$$(6.22) \quad \langle m \rangle = m_D$$

$$(6.23) \quad \langle (m - m_D)^2 \rangle = \frac{1}{\varepsilon}$$

$$(6.24) \quad \langle V_c^2 \rangle = \frac{e^2}{C^2} \frac{1}{\varepsilon} = \frac{kT}{C}$$

$$(6.25) \quad \langle V_c \rangle = -\frac{e}{C} \langle m \rangle = \frac{W_a - W_b}{-e}$$

In is seen that, for small ε , $\langle v_c \rangle$ is nothing else but the contact potential due to the difference between the work functions of the electrodes.

If ε is not small compared to unity, mean values and mean square values depart from those of the continuous Gaussian. Thus, for example, the mean square voltage $\langle v_c^2 \rangle$ is no longer to the contact potential of (6.25) , because $\langle m \rangle$ is no longer equal to m_c . However, as seen from the detailed analysis provided in Appendix A, such a voltage will generate a zero average current, thus preserving the validity of the Second Law.

We note that the charge and voltage offsets are also zero, for any ε , in the special cases when m_D is either an integer or a half integer.

Finally we have to prove that the Alkemade diode has a nonlinear asymmetric current-voltage characteristic.

To obtain the current through the diode we multiply the Master Equation (6.8) by $-me$ and sum over the full range of m values:

$$(6.26) \quad (-e)\sum_{m=-\infty}^{\infty} m \frac{\partial P_D(m)}{\partial t} = (-e)\sum_{m=-\infty}^{\infty} a_D(m+1)\, m\, P_D(m+1) +$$

$$+(-e)\sum_{m=-\infty}^{\infty} b_D(m-1)\, m\, P_D(m-1) - (-e)\sum_{m=-\infty}^{\infty} a_D(m)\, m\, P_D(m) -$$

$$-(-e)\sum_{m=-\infty}^{\infty} b_D(m)\, m\, P_D(m)$$

For our present diode model $a_D(m)$ is assumed to have a value independent of m which we will call a_D . Then using (6.9) and a_D , (6.27) becomes:

$$(6.27)\quad (-e)\frac{\partial}{\partial t}\left[\sum_{m=-\infty}^{\infty} m\, P_D(m)\right] = (-e)a_D\sum_{m=-\infty}^{\infty} m\, P_D(m+1) +$$

$$+(-e)a_D\, e^{-\frac{W_b-W_a}{kT}}\sum_{m=-\infty}^{\infty} e^{-\varepsilon(m-\frac{1}{2})}\, m\, P(m-1) -$$

$$-(-e)a_D\sum_{m=-\infty}^{\infty} m\, P_D(m) - (-e)a_D e^{-\frac{W_b-W_a}{kT}}\sum_{m=-\infty}^{\infty} e^{-\varepsilon(m+\frac{1}{2})}\, m\, P_D(m)$$

If we proceed to calculate each of the terms in the above equation we succesively obtain:

$$(6.28)\quad \sum_{m=-\infty}^{\infty} m\, P_D(m) = \langle m\rangle$$

$$(6.29)\quad \sum_{m=-\infty}^{\infty} m\, P_D(m+1) = \sum_{m=-\infty}^{\infty}(m+1)\, P_D(m+1) - \sum_{m=-\infty}^{\infty} P_D(m+1) = \langle m\rangle - 1$$

$$(6.30)\quad \sum_{m=-\infty}^{\infty} e^{-\varepsilon(m-\frac{1}{2})} m P_D(m-1) = e^{-\frac{\varepsilon}{2}} \sum_{m=-\infty}^{\infty} e^{-\varepsilon(m-1)} (m-1) P_D(m-1) +$$

$$+ e^{-\frac{\varepsilon}{2}} \sum_{m=-\infty}^{\infty} e^{-\varepsilon(m-1)} P_D(m-1) = e^{-\frac{\varepsilon}{2}} \langle m e^{-\varepsilon m} \rangle + e^{-\frac{\varepsilon}{2}} \langle e^{-\varepsilon m} \rangle$$

$$(6.31)\quad \sum_{m=-\infty}^{\infty} e^{-\varepsilon(m+\frac{1}{2})} m P_D(m) = e^{-\frac{\varepsilon}{2}} \sum_{m=-\infty}^{\infty} e^{-\varepsilon m} m P_D(m) =$$

$$= e^{-\frac{\varepsilon}{2}} \langle m e^{-\varepsilon m} \rangle$$

The Master Equation can then be rewritten:

$$(6.32)\quad (-e) \frac{\partial \langle m \rangle}{\partial t} = (-e)\, a_D \left[\langle m \rangle - 1 \right] +$$

$$+ (-e)\, a_D\, e^{-\frac{W_b - W_a}{kT}} \left[e^{-\frac{\varepsilon}{2}} \langle m e^{-\varepsilon m} \rangle + e^{-\frac{\varepsilon}{2}} \langle e^{-\varepsilon m} \rangle \right] -$$

$$- (-e)\, a_D \langle m \rangle - (-e)\, a_D\, e^{-\frac{W_b - W_a}{kT}}\, e^{-\frac{\varepsilon}{2}} \langle m e^{-\varepsilon m} \rangle$$

But since the term at the left hand side is the current through the diode the Master Equation finally becomes:

$$(6.33) \quad \langle I \rangle = (-e)\, a_D \left[e^{-\frac{\varepsilon}{2}} \left\langle e^{-\varepsilon(m - m_D)} \right\rangle - 1 \right]$$

The above expression represents the characteristic function of the Alkemade diode.

To find the macroscopic current-voltage characteristic we have to let C, m and m_D become very large in such a way as to maintain $\frac{m - m_c}{C}$ constant. But since

$$(6.34) \quad V = \frac{-e(m - m_D)}{C}$$

can be identified with the externally applied voltage, we find that the macroscopic Alkemade diode satisfies the same type of exponential law as most known diodes (see for comparison the Shockley equation(3.35))

$$(6.35) \quad I = (-e)\, a_D \left(e^{\frac{eV}{kT}} - 1 \right)$$

We can now see, by calculating $g_1 \equiv \frac{\partial I}{\partial V}$, $g_2 \equiv \frac{1}{2}\frac{\partial^2 I}{\partial V^2}$, that the quantity ε previously introduced is just the discrete counterpart of the continuous parameter β defined in our Gunn type analysis.

Given the above description of the Alkemade diode, the physical meaning of the reverse current postulated by Gunn (

see Section 3.2.1) can be easily understood.

If the diode is operated close enough to thermodynamic equilibrium, and also within the range of validity of Gunn's model, then $\varepsilon \ll 1$ and the exponential of equation (6.34) can be approximated by its quadratic Taylor expansion:

$$(6.36) \quad \langle I \rangle = (-e) a_D \left[-\frac{\varepsilon}{2} - \varepsilon \langle m - m_D \rangle + \frac{\varepsilon^2}{8} + \frac{\varepsilon^2}{2} \langle (m - m_D)^2 \rangle + \ldots \right]$$

It is important to observe that the parameter ε of this equation is calculated at the temperature T of the diode, while the mean square deviation of the distribution of electrons on the equivalent capacitance of the diode is given by a different parameter, ε_e:

$$(6.37) \quad \langle (m - m_D)^2 \rangle = \frac{1}{\varepsilon_e} \equiv \frac{k T_e C}{e^2}$$

where T_e is an equivalent temperature depending both on the temperature of the diode and also on the temperature of the outside system the diode is connected to. Therefore the current $\langle I \rangle$ can be rewritten in the form:

$$(6.38) \quad \langle I \rangle = (-e) a_D \left[-\frac{\varepsilon}{2} - \varepsilon \langle m - m_D \rangle + \frac{\varepsilon^2}{8} + \frac{\varepsilon^2}{2 \varepsilon_e} \right]$$

Since under the given conditions,

$$\frac{\varepsilon^2}{8} \simeq 0 \quad , \quad \langle n - n_D \rangle = 0$$

the current through the diode can be put in the form:

$$\langle I \rangle = (-e) a_D \left[-\frac{\varepsilon}{2} + \frac{\varepsilon^2}{2 \varepsilon_e} \right] \tag{6.39}$$

The second term in the above expression is the expected rectification current of the diode, while the first term is the reverse current postulated by Gunn in order to satisfy the Second Law at thermodynamic equilibrium. Indeed, when $T = T_e$, $\varepsilon = \varepsilon_e$ and the overall current is zero as required by the Second Law.

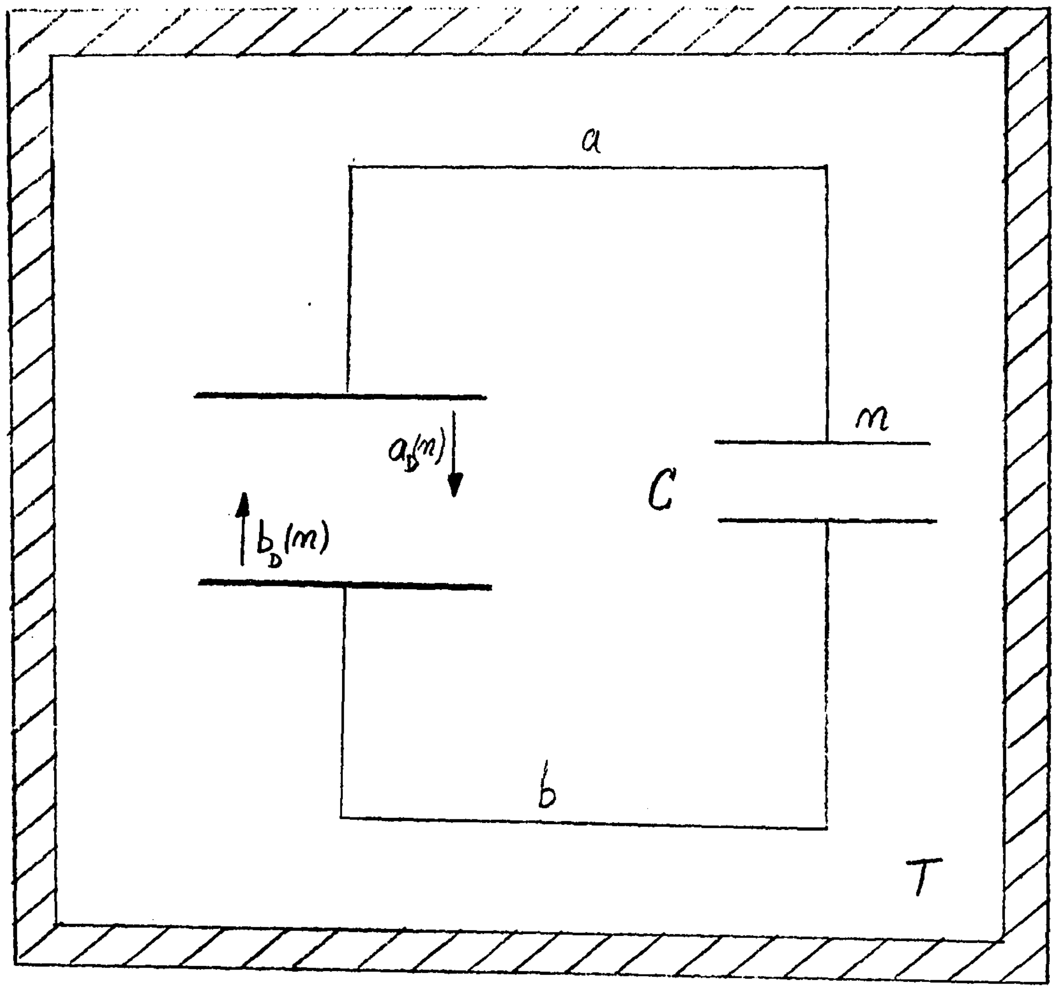


Fig 6.2

6.2 Kinetic Model of the Linear Resistor

Let us study the behavior of a linear resistor R when connected in series with an ideal battery of voltage E and a capacitor of capacitance C all in thermal equilibrium with a heat bath at temperature T (see Fig. 6.3). The resistor, its lead wires and the battery's lead wires are made of a material c with work function W_c , while the two plates of the capacitor and the corresponding leads are made of materials a and b with work functions W_a and W_b . The junctions A and B between the material c and the materials a and b are kept at the same temperature.

Our basic assumption will be that the transition rates in both directions through the resistor are independent of each other.

If we start our analysis with n excess electrons on the plate a of the capacitor, we can calculate the rates of the electron transitions between the plates a and b via the resistor-battery loop. If we replace our resistor by a short circuit element, an electron on a , in order to reach b , will have to overcome a potential barrier of height:

$$W_a - W_c + eE$$

We can then expect a transition rate given by the Boltzmann factor. We will now assume that the presence of a nonzero resistor in the circuit will affect the transition rate through a, yet unknown, multiplicative factor $k(m,R)$. Then the rate of electronic transitions from a to b can be formally written as:

$$(6.40) \qquad a_R(m) = k(m,R)\, e^{-\frac{W_a - W_c}{kT} - \frac{eE}{kT}}$$

The potential barrier seen by an electron on b ,when the resistor is short circuited, is $W_b - W_c + \Delta E_m$, where ΔE_m is the additional electrostatic barrier due to the electrons on a :

$$(6.41) \qquad \Delta E_m = \frac{e^2(m+1)^2}{2C} - \frac{e^2 m^2}{2C} = \frac{e^2}{C}\left(m + \tfrac{1}{2}\right)$$

Formally we then write the transition rate from b to a :

$$(6.42) \qquad b_R(m) = k(m,R)\, e^{-\frac{W_b - W_a}{kT} - \frac{e^2}{kTC}\left(m+\frac{1}{2}\right)}$$

We can now set up the Master Equation for the probability $P_R(m)$ of having m electrons on the plate a :

$$(6.43) \qquad \frac{\partial P_R(m)}{\partial t} = a_R(m+1) P_R(m+1) + b_R(m-1) P_R(m-1) -$$

$$- a_R(m) P_R(m) - b_R(m) P_R(m)$$

We are interested in the steady state solution of this equation and therefore we let $\partial P_{RS}(m)/\partial t = 0$. Then:

$$a_R(m+1) P_{RS}(m+1) - b_R(m) P_{RS}(m) =$$

$$= a_R(m) P_{RS}(m) - b_R(m-1) P_{RS}(m-1)$$

It results that $a_R(m+1) P_{RS}(m+1) - b_R(m) P_{RS}(m)$ is a constant independent of m , and since we must require that $P_{RS}(m)$ goes to zero in the limit of large m , the mentioned constant has to be zero. This yields the recursion relation:

$$(6.44) \qquad P_{RS}(m+1) = \frac{b_R(m)}{a_R(m+1)} P_{RS}(m)$$

It is evident that the stationary solution of our Master

Equation is just a restatement of the principle of detailed balance. Using (6.40) and (6.42) we find:

$$(6.45) \qquad P_{RS}(m+1) = e^{-\left(\frac{W_b - W_a}{kT} - \frac{eE}{kT}\right) - \varepsilon\left(m+\frac{1}{2}\right)} P_{RS}(m)$$

Where: $\varepsilon \equiv \dfrac{e^2}{kTC}$

If we repeat the steps leading to (6.22), we find that $P_{RS}(m)$ is again a Boltzmann or (discrete) Gaussian distribution:

$$(6.46) \qquad P_{RS}(m) = \frac{e^{-\frac{\varepsilon}{2}(m-m_R)^2}}{Z_R}$$

where

$$(6.47) \qquad Z_R = \sum_{m=-\infty}^{\infty} e^{-\frac{\varepsilon}{2}(m-m_R)^2}$$

and

$$(6.48) \qquad m_R \equiv \frac{W_a - W_b}{e^2/C} + \frac{E \cdot C}{e}$$

The derived m_R is simply the macroscopic equilibrium value for the number of electrons on the capacitor (the equilibrium charge being $-em_R$).

Note that we could have started by assuming the validity of (6.46) directly, by a thermodynamic argument, without going through the preceding motivational arguments.

In the continuum limit, $\langle m \rangle = m_R$,

$$(6.49) \quad \langle v_c^2 \rangle = \frac{e^2}{C^2} \langle (m - m_R)^2 \rangle = \frac{e^2}{C^2} \frac{1}{\varepsilon} = \frac{kT}{C}$$

and

$$(6.50) \quad \langle v_c \rangle = -\frac{e}{C} \langle m \rangle = \frac{W_a - W_b}{-e} - E$$

When ε is not small, $\langle m \rangle = m_R$ is still valid when m_R is an integer or half integer. Results for other values of m_R are presented in Appendix A .

Expressions for $a_R(m)$ and $b_R(m)$ can be obtained by using the detailed balance condition (6.44) together with the Boltzmann distribution (6.46) for m , and a further condition that specifies the linear nature of the resistance. This linearity condition is :

$$(6.51) \quad I = (-e)\left[b_R(m) - a_R(m) \right] = e \frac{m - \langle m \rangle}{R \cdot C}$$

which states that the net rate of decrease of the electron number is equal to the excess electron number $m - \langle m \rangle$ devided by the time constant $R \cdot C$. This condition can also be expressed in terms of currents and voltages, and simply represents Ohm's Law for a linear resistance.

Expressions for $a_R(m)$ and $b_R(m)$ can be obtained by using (6.51) together with

$$(6.52) \quad a_R(m) = b_R(m-1)\,\frac{P_{RS}(m-1)}{P_{RS}(m)} = b_R(m-1)\, e^{\varepsilon(m-m_R-\frac{1}{2})}$$

which follows from (6.44) and (6.46).

Replacing $a_R(m)$ in the equation (6.51) by its expression from (6.52) we obtain a recursion relation in $b_R(m)$:

$$(6.53) \quad b_R(m)\,P_{RS}(m) = b_R(m-1)\,P_{RS}(m-1) - \frac{m-\langle m\rangle}{RC}\,P_{RS}(m)$$

If we define the function $B_R(m)$ by,

$$(6.54) \quad B_R(m) \equiv b_R(m)\,P_{RS}(m)\,Z_R$$

and recall, from (6.46), the expression of the stationary distribution function $P_{RS}(m)$, relation (6.53) yields:

$$(6.55) \quad B_R(m) = B_R(m-1) - \frac{m-\langle m\rangle}{RC}\, e^{-\frac{\varepsilon}{2}(m-m_R)^2}$$

By summing we obtain the solution of the above equation:

$$(6.56)\quad B_R(m) = B_R(m-1) - \sum_{m=-\infty}^{\infty} \frac{m-\langle m\rangle}{RC} e^{-\frac{\varepsilon}{2}(m-m_R)^2}$$

Writing this solution in terms of the transition rate $b_R(m)$ we obtain:

$$(6.57)\quad b_R(m)\, e^{-\frac{\varepsilon}{2}(m-m_R)^2} = b_R(0)\, e^{-\frac{\varepsilon}{2}m_R^2} -$$

$$-\sum_{m=1}^{\infty} \frac{m-\langle m\rangle}{RC}\, e^{-\frac{\varepsilon}{2}(m-m_R)^2}$$

If we now solve for $b_R(m)$:

$$(6.58)\quad b_R(m) = e^{\frac{\varepsilon}{2}(m-m_R)^2}\left[b_R'(0) - \sum_{m=1}^{\infty} \frac{m-\langle m\rangle}{RC}\, e^{-\frac{\varepsilon}{2}(m-m_R)^2}\right]$$

Where $b_R'(0) = b_R(0)\, e^{-\frac{\varepsilon}{2}m_R^2}$

Since when m goes to infinity, the transition rate from plate b to plate a has to vanish, $b_R'(0)$ is given by:

$$(6.59)\quad b_R'(0) = e^{\frac{\varepsilon}{2}(m-m_R)^2}\sum_{m=m+1}^{\infty} \frac{m-\langle m\rangle}{RC}\, e^{-\frac{\varepsilon}{2}(m-m_R)^2}$$

Then $b_R(n)$ takes the form:

$$(6.60)\qquad b_R(n) = e^{\frac{\varepsilon}{2}(n-n_R)^2} \sum_{m=n+1}^{\infty} \frac{m-\langle n\rangle}{RC}\, e^{-\frac{\varepsilon}{2}(m-n_R)^2}$$

Going back to (6.51) we can calculate $a_R(n)$:

$$(6.61)\qquad a_R(n) = b_R(n) + \frac{n-\langle n\rangle}{RC}$$

We can now write the final expressions for the transition rates $a_R(n)$ and $b_R(n)$ of our linear resistor model:

$$(6.62)\qquad a_R(n) = \frac{n-\langle n\rangle}{RC} + \sum_{m=n+1}^{\infty} \frac{m-\langle n\rangle}{RC}\, e^{\frac{\varepsilon}{2}(n-m)(n+m-2n_R)}$$

$$(6.63)\qquad b_R(n) = \sum_{m=n+1}^{\infty} \frac{m-\langle n\rangle}{RC}\, e^{\frac{\varepsilon}{2}(n-m)(n+m-2n_R)}$$

We mentioned before that with the chosen $a_R(n)$ and $b_R(n)$ Ohm's Law is automatically satisfied in the limit of small ε . We will further test our results by calculating in the continous limit, the power spectrum of the noise generated within our system.

It can be shown [35] that the power spectrum of the current through the resistor is given by:

$$(6.64) \qquad S_i = 2e^2 \langle a_R(n) + b_R(n) \rangle$$

$$(6.65) \qquad S_i = 2e^2 \left\langle 2\, b_R(n) + \frac{n - \langle n \rangle}{RC} \right\rangle$$

In the continuum limit the sum in (6.63) can be replaced by an integral

$$(6.66) \qquad b_R(n) = \int_{n+\frac{1}{2}}^{\infty} \frac{m - \langle n \rangle}{RC} \, e^{\frac{\varepsilon}{2}(n-m)(n+m-2n_R)} \, dm$$

This yields in the limit of $\varepsilon \to 0$:

$$(6.67) \qquad b_R(n) = \frac{kT}{e^2 R}$$

Since in this limit $\dfrac{n - \langle n \rangle}{RC}$ becomes zero,

$$(6.68) \qquad S_i = \frac{4kT}{R}$$

as expected from Nyquist's theorem.

Equations (6.62) and (6.63) can be numerically evaluated . Results of such calculations are illustrated in Table 6.1 .

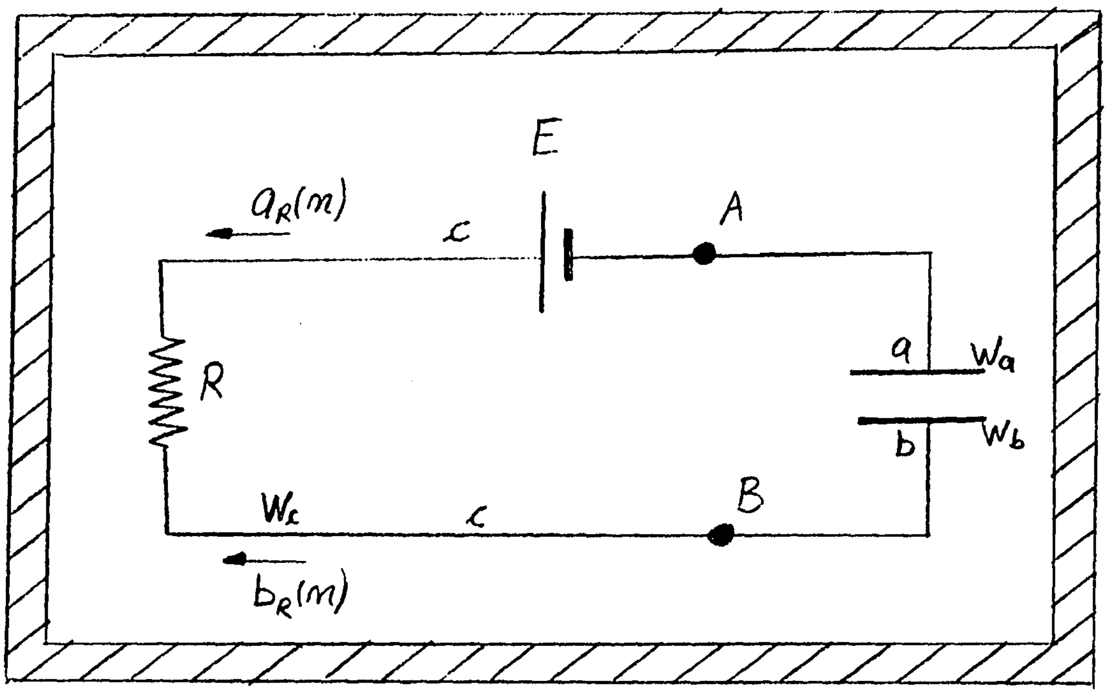


Fig 6.3

ε	n	$a_R(n)$	$b_R(n)$
10.0000	-4	-0.5882886E+25	-0.2284015E+27
10.0000	-3	-0.3709228E+10	-0.1440098E+12
10.0000	-2	-0.2051514E+01	0.0000000E+00
10.0000	-1	-0.9999994E+00	0.1000000E+01
10.0000	0	0.0000000E+00	0.6737952E-02
10.0000	1	0.1000001E+01	0.5960464E-06
10.0000	2	0.1948486E+01	-0.5151355E-01
10.0000	3	-0.3709228E+10	-0.3709228E+10
10.0000	4	-0.5882886E+25	-0.5882886E+25

1.0000	-13	0.6107139E+30	0.6107139E+30
1.0000	-12	0.2275919E+25	0.2275919E+25
1.0000	-11	0.2305528E+20	0.2305528E+20
1.0000	-10	0.6348605E+15	0.6348605E+15
1.0000	-9	0.4752048E+11	0.4752048E+11
1.0000	-8	0.9668906E+07	0.9668922E+07
1.0000	-7	0.5340729E+04	0.5354729E+04
1.0000	-6	0.2050509E+01	0.1405051E+02
1.0000	-5	-0.4942579E+01	0.5057421E+01
1.0000	-4	-0.3943817E+01	0.4056183E+01
1.0000	-3	-0.2877514E+01	0.3122486E+01
1.0000	-2	-0.1743691E+01	0.2256309E+01
1.0000	-1	-0.4965494E+00	0.1503451E+01
1.0000	0	0.0000000E+00	0.9118888E+00
1.0000	1	0.1503451E+01	0.5034506E+00
1.0000	2	0.2256309E+01	0.2563093E+00
1.0000	3	0.3122486E+01	0.1224861E+00
1.0000	4	0.4056183E+01	0.5618286E-01
1.0000	5	0.5057421E+01	0.5742121E-01
1.0000	6	0.1405051E+02	0.8050509E+01
1.0000	7	0.5354729E+04	0.5347729E+04
1.0000	8	0.9668922E+07	0.9668914E+07
1.0000	9	0.4752048E+11	0.4752048E+11
1.0000	10	0.6348605E+15	0.6348605E+15
1.0000	11	0.2305528E+20	0.2305528E+20
1.0000	12	0.2275919E+25	0.2275919E+25
1.0000	13	0.6107139E+30	0.6107139E+30

TABLE 6.1

ε	m	$a_R(m)$	$b_R(m)$
0.1000	-20	0.1266834E+04	0.1306834E+04
0.1000	-19	0.1669286E+03	0.2049286E+03
0.1000	-18	0.1422239E+02	0.5022239E+02
0.1000	-17	-0.8272659E+01	0.2572734E+02
0.1000	-16	-0.1105907E+02	0.2094093E+02
0.1000	-15	-0.1055533E+02	0.1944467E+02
0.1000	-14	-0.9438858E+01	0.1856114E+02
0.1000	-13	-0.8188204E+01	0.1781180E+02
0.1000	-12	-0.6896835E+01	0.1710316E+02
0.1000	-11	-0.5584509E+01	0.1641549E+02
0.1000	-10	-0.4255601E+01	0.1574440E+02
0.1000	-9	-0.2910995E+01	0.1508900E+02
0.1000	-8	-0.1550735E+01	0.1444926E+02
0.1000	-7	-0.1746511E+00	0.1382535E+02
0.1000	-6	0.1217465E+01	0.1321747E+02
0.1000	-5	0.2625815E+01	0.1262582E+02
0.1000	-4	0.4050574E+01	0.1205057E+02
0.1000	-3	0.5491896E+01	0.1149190E+02
0.1000	-2	0.6949898E+01	0.1094990E+02
0.1000	-1	0.8424664E+01	0.1042466E+02
0.1000	0	0.0000000E+00	0.9916245E+01
0.1000	1	0.1042466E+02	0.9424664E+01
0.1000	2	0.1094990E+02	0.8949898E+01
0.1000	3	0.1149190E+02	0.8491896E+01
0.1000	4	0.1205057E+02	0.8050574E+01
0.1000	5	0.1262582E+02	0.7625815E+01
0.1000	6	0.1321747E+02	0.7217465E+01
0.1000	7	0.1382535E+02	0.6825349E+01
0.1000	8	0.1444926E+02	0.6449265E+01
0.1000	9	0.1508900E+02	0.6089005E+01
0.1000	10	0.1574440E+02	0.5744399E+01
0.1000	11	0.1641549E+02	0.5415491E+01
0.1000	12	0.1710316E+02	0.5103165E+01
0.1000	13	0.1781180E+02	0.4811796E+01
0.1000	14	0.1856114E+02	0.4561142E+01
0.1000	15	0.1944467E+02	0.4444670E+01
0.1000	16	0.2094093E+02	0.4940933E+01
0.1000	17	0.2572734E+02	0.8727341E+01
0.1000	18	0.5022239E+02	0.3222239E+02
0.1000	19	0.2049286E+03	0.1859286E+03
0.1000	20	0.1306834E+04	0.1286834E+04

TABLE 6.1

ε	n	$a_R(n)$	$b_R(n)$
0.0100	-20	0.7024984E+02	0.1102498E+03
0.0100	-19	0.7171740E+02	0.1097174E+03
0.0100	-18	0.7318661E+02	0.1091866E+03
0.0100	-17	0.7465747E+02	0.1086575E+03
0.0100	-16	0.7612999E+02	0.1081300E+03
0.0100	-15	0.7760416E+02	0.1076042E+03
0.0100	-14	0.7908000E+02	0.1070800E+03
0.0100	-13	0.8055750E+02	0.1065575E+03
0.0100	-12	0.8203665E+02	0.1060367E+03
0.0100	-11	0.8351749E+02	0.1055175E+03
0.0100	-10	0.8499999E+02	0.1050000E+03
0.0100	-9	0.8648415E+02	0.1044842E+03
0.0100	-8	0.8796999E+02	0.1039700E+03
0.0100	-7	0.8945748E+02	0.1034575E+03
0.0100	-6	0.9094665E+02	0.1029466E+03
0.0100	-5	0.9243748E+02	0.1024375E+03
0.0100	-4	0.9392997E+02	0.1019300E+03
0.0100	-3	0.9542413E+02	0.1014241E+03
0.0100	-2	0.9691995E+02	0.1009200E+03
0.0100	-1	0.9841746E+02	0.1004175E+03
0.0100	0	0.0000000E+00	0.9991661E+02
0.0100	1	0.1004175E+03	0.9941746E+02
0.0100	2	0.1009200E+03	0.9891995E+02
0.0100	3	0.1014241E+03	0.9842413E+02
0.0100	4	0.1019300E+03	0.9792997E+02
0.0100	5	0.1024375E+03	0.9743748E+02
0.0100	6	0.1029466E+03	0.9694665E+02
0.0100	7	0.1034575E+03	0.9645748E+02
0.0100	8	0.1039700E+03	0.9596999E+02
0.0100	9	0.1044842E+03	0.9548415E+02
0.0100	10	0.1050000E+03	0.9499999E+02
0.0100	11	0.1055175E+03	0.9451749E+02
0.0100	12	0.1060367E+03	0.9403665E+02
0.0100	13	0.1065575E+03	0.9355750E+02
0.0100	14	0.1070800E+03	0.9308000E+02
0.0100	15	0.1076042E+03	0.9260416E+02
0.0100	16	0.1081300E+03	0.9212999E+02
0.0100	17	0.1086575E+03	0.9165747E+02
0.0100	18	0.1091866E+03	0.9118661E+02
0.0100	19	0.1097174E+03	0.9071740E+02
0.0100	20	0.1102498E+03	0.9024984E+02

TABLE 6.1

ε	m	$a_R(m)$	$b_R(m)$
0.0010	-20	0.9699493E+03	0.1009949E+04
0.0010	-19	0.9714460E+03	0.1009446E+04
0.0010	-18	0.9729429E+03	0.1008943E+04
0.0010	-17	0.9744401E+03	0.1008440E+04
0.0010	-16	0.9759374E+03	0.1007937E+04
0.0010	-15	0.9774349E+03	0.1007435E+04
0.0010	-14	0.9789326E+03	0.1006933E+04
0.0010	-13	0.9804302E+03	0.1006430E+04
0.0010	-12	0.9819280E+03	0.1005928E+04
0.0010	-11	0.9834262E+03	0.1005426E+04
0.0010	-10	0.9849245E+03	0.1004924E+04
0.0010	-9	0.9864229E+03	0.1004423E+04
0.0010	-8	0.9879216E+03	0.1003922E+04
0.0010	-7	0.9894205E+03	0.1003420E+04
0.0010	-6	0.9909194E+03	0.1002919E+04
0.0010	-5	0.9924186E+03	0.1002419E+04
0.0010	-4	0.9939177E+03	0.1001918E+04
0.0010	-3	0.9954171E+03	0.1001417E+04
0.0010	-2	0.9969167E+03	0.1000917E+04
0.0010	-1	0.9984164E+03	0.1000416E+04
0.0010	0	0.0000000E+00	0.9999164E+03
0.0010	1	0.1000416E+04	0.9994164E+03
0.0010	2	0.1000917E+04	0.9989167E+03
0.0010	3	0.1001417E+04	0.9984171E+03
0.0010	4	0.1001918E+04	0.9979177E+03
0.0010	5	0.1002419E+04	0.9974186E+03
0.0010	6	0.1002919E+04	0.9969194E+03
0.0010	7	0.1003420E+04	0.9964205E+03
0.0010	8	0.1003922E+04	0.9959216E+03
0.0010	9	0.1004423E+04	0.9954229E+03
0.0010	10	0.1004924E+04	0.9949245E+03
0.0010	11	0.1005426E+04	0.9944262E+03
0.0010	12	0.1005928E+04	0.9939280E+03
0.0010	13	0.1006430E+04	0.9934302E+03
0.0010	14	0.1006933E+04	0.9929326E+03
0.0010	15	0.1007435E+04	0.9924349E+03
0.0010	16	0.1007937E+04	0.9919374E+03
0.0010	17	0.1008440E+04	0.9914401E+03
0.0010	18	0.1008943E+04	0.9909429E+03
0.0010	19	0.1009446E+04	0.9904460E+03
0.0010	20	0.1009949E+04	0.9899493E+03

TABLE 6.1

6.3 Kinetic Analysis of Resistor-Diode Fluctuation Heat Engines

Now we will proceed to analyze the electrical fluctuation engine of Fig(6.6) employing an Alkemade type diode and a linear resistor which we will describe using the kinetic model introduced in the previous section.

When thermodynamic equilibrium prevails $(E=0,\ T_1=T_2)$ no heat or current flows will occur, the macroscopic equilibrium number of electrons on the electrode a will have the common equilibrium value $n_R = n_D \equiv n_0$ and their distribution will be the common equilibrium distribution function for the separate diode and resistor subsystems.

$$(6.69)\quad P_{S0}(n) = P_{DS}(n) = P_{RS}(n)$$

If we operate the system under a temperature difference $\Delta T \equiv T_1 - T_2 > 0$, with a nonzero battery voltage E, net current and heat fluxes are expected to develop. To determine these fluxes we'll have to first calculate the new nonequilibrium steady state distribution functions of the electrons on the electrode a. Increases in n will occur due to transitions from the b electrode both through the diode and through the resistor. The rate $b(n)$ of the electronic transitions from b, is given by the sum of the transition rates $b_D(m)$ and $b_R(m)$.

$$(6.70)\quad b(m) = b_D(m) + b_R(m)$$

Decreases in m are due to electronic transitions from a to b which have a rate equal to the sum of transition rates $a_D(m)$ and $a_R(m)$

$$(6.71)\quad a(m) = a_D(m) + a_R(m)$$

It is important to observe that $a_R(m)$ and $b_R(m)$ are calculated at the resistor temperature T_1 while $a_D(m)$ and $b_D(m)$ are computed at the diode temperature T_2.

We can now write the Master Equation describing our system

$$(6.72)\quad \frac{\partial P(m)}{\partial t} = a(m+1)\,P(m+1) + b(m-1)\,P(m-1) - a(m)\,P(m) - b(m)\,P(m)$$

If we look for the steady state solution of this equation then $\partial P_s(m)/\partial t = 0$ and therefore:

$$(6.73)\quad a(m+1)\,P_s(m+1) - b(m)\,P_s(m) - a(m)\,P_s(m) + b(m-1)\,P_s(m-1) = 0$$

If we group the terms as we have done before,

$$(6.74)\quad a(m+1)P_s(m+1) - b(m)P_s(m) = a(m)P_s(m) - b(m-1)P_s(m-1)$$

it becomes clear that $a(m+1)P_s(m+1) - b(m)P_s(m)$ is a constant independent of m. Since $P_s(m)$ has to become zero as m goes to infinity we find that the stationary solution of our Master Equation is again the detailed balance result:

$$(6.75)\quad P_s(m+1) = \frac{b(m)}{a(m+1)} P_s(m)$$

The above solution is valid for any relative departure from thermodynamic equilibrium and will be the basis for our numerical analysis of fluctuation engines operating under large thermodynamic forces.

The average electronic current flowing through our system is given by the mean value of the difference in transition rates, in the two opposite directions, through the resistor or diode. For the chosen sign convention:

$$(6.76)\quad I = (-e)\sum_{m=-\infty}^{\infty} P_s(m)\left[b_D(m) - a_D(m)\right]$$

Or:

$$(6.77)\quad I = (-e)\sum_{m=-\infty}^{\infty} P_s(m)\left[a_R(m) - b_R(m)\right]$$

We can rewrite (6.76) in the form:

$$(6.78) \quad I = (-e) \sum_{m=-\infty}^{\infty} \left[P_s(m) b(m) \frac{b_D(m)}{b(m)} - P_s(m) a(m) \right]$$

Using (6.75) to replace $P_s(m) b(m)$ by $P_s(m+1) a(m+1)$ we obtain:

$$(6.79) \quad I = (-e) \sum_{m=-\infty}^{\infty} \left[P_s(m+1) a(m+1) \frac{b_D(m)}{b(m)} - P_s(m) a_D(m) \right]$$

Since for the given m range,

$$(6.80) \quad \sum_{m=-\infty}^{\infty} P_s(m+1) a(m+1) \frac{b_D(m)}{b(m)} = \sum_{m=-\infty}^{\infty} P_s(m) a(m) \frac{b_D(m-1)}{b(m-1)}$$

we can write for the current:

$$(6.81) \quad I = (-e) \sum_{m=-\infty}^{\infty} P_s(m) a(m) \left[\frac{b_D(m-1)}{b(m-1)} - \frac{a_D(m)}{a(m)} \right]$$

In a similar fashion (6.77) will translate into:

$$(6.82) \quad I = (-e) \sum_{m=-\infty}^{\infty} P_s(m) a(m) \left[\frac{a_R(m)}{a(m)} - \frac{b_R(m-1)}{b(m-1)} \right]$$

With the intention of later applying perturbation techniques, we will rearrange the above expressions into more convenient forms. To do so let us expand the bracket in (6.81) by replacing $b(m-1)$ and $a(m)$ with their correspoding expressions (6.70) and (6.71) .

$$(6.83)\quad \frac{b_D(m-1)}{b(m-1)} - \frac{a_D(m)}{a(m)} = \frac{b_D(m-1)\left[a_R(m)+a_D(m)\right]}{a(m)\,b(m-1)} - $$

$$- \frac{a_D(m)\left[b_R(m-1)+b_D(m-1)\right]}{a(m)\,b(m-1)}$$

Or:

$$(6.84)\quad \frac{b_D(m-1)}{b(m-1)} - \frac{a_D(m)}{a(m)} = \frac{a_R(m)\,b_D(m-1) - a_D(m)\,b_R(m-1)}{a(m)\,b(m-1)}$$

Finally we obtain:

$$(6.85)\quad \frac{b_D(m-1)}{b(m-1)} - \frac{a_D(m)}{a(m)} = \frac{a_R(m)a_D(m)}{a(m)b(m-1)}\left[\frac{b_D(m-1)}{a_D(m)} - \frac{b_R(m-1)}{a_R(m)}\right]$$

Using a similar procedure the bracket in (6.82) yields also.

$$(6.86)\quad \frac{a_R(m)}{a(m)} - \frac{b_R(m-1)}{b(m-1)} = \frac{a_R(m)a_D(m)}{a(m)b(m-1)}\left[\frac{b_D(m-1)}{a_D(m)} - \frac{b_R(m-1)}{a_R(m)}\right]$$

As expected the two brackets in (6.81) and (6.82) are equal, and therefore the current through the resistor R is equal to the current through the diode D as required by the conservation of charge.

Going back to the expression of the current:

$$(6.87)\quad I=(-e)\sum_{m=-\infty}^{\infty}\frac{P_s(m)a_R(m)a_D(m)}{b(m-1)}\left[\frac{b_D(m-1)}{a_D(m)}-\frac{b_R(m-1)}{a_R(m)}\right]$$

Or if we use the detailed balance relation to replace $b(m-1)$ by $\dfrac{P_s(m)\,a(m)}{P_s(m-1)}$, in the view of the properties of the sum:

$$(6.88)\quad I=(-e)\sum_{m=-\infty}^{\infty}\frac{P_s(m)a_R(m+1)a_D(m+1)}{a(m+1)}\left[\frac{b_D(m)}{a_D(m+1)}-\frac{b_R(m)}{a_R(m+1)}\right]$$

To determine the heat flux we have to calculate the average energy being transported through electronic transitions between the resistor and the diode. For each transition through the resistor that increases m, the net change in the free energy of the system is given by:

$$(6.89)\quad \Delta FE_{\uparrow}=FE_R(m+1)-FE_R(m)$$

$$(6.90)\quad \Delta FE_{\uparrow}=\frac{e^2}{2C}(m-m_R+1)^2-\frac{e^2}{2C}(m-m_R)^2$$

$$(6.91)\quad \Delta FE_{\uparrow}=\frac{e^2}{C}\left(m-m_R+\frac{1}{2}\right)$$

The transitions through the resistor that decrease will bring a net free energy contribution:

$$(6.92)\quad \Delta FE_{\downarrow} = \frac{e^2}{2C}(m - m_R - 1)^2 - \frac{e^2}{2C}(m - m_R)^2$$

$$(6.93)\quad \Delta FE_{\downarrow} = -\frac{e^2}{C}\left(m - m_R - \frac{1}{2}\right)$$

The net rate of heat flow out of the resistor $\mathring{Q}_R$ is then equal to the average rate of change of the free energy, due to transitions through R alone:

$$(6.94)\quad \mathring{Q}_R = \frac{e^2}{C}\sum_{m=-\infty}^{\infty} P_s(m)\left[b_R(m)\left(m - m_R + \frac{1}{2}\right) - a_R(m)\left(m - m_R - \frac{1}{2}\right)\right]$$

Nevertheless, the net rate of change of the free energy also contains terms due to transitions through the diode. When all terms are counted the net rate of change is found to be zero, such as to yield the expected stationary character of the free energy.

If we perform the same type of analysis on (6.94), as we have done for the net electric current expression, we succesively obtain

$$(6.95)\quad \mathring{Q}_R = \frac{e^2}{C}\left[\sum_{n=-\infty}^{\infty} P_s(n) b(n) \frac{b_R(n)}{b(n)}\left(n - n_R + \frac{1}{2}\right) - \right.$$

$$\left. - \sum_{n=-\infty}^{\infty} P_s(n) a(n) \frac{a_R(n)}{a(n)}\left(n - n_R - \frac{1}{2}\right)\right.$$

$$(6.96)\quad \mathring{Q}_R = \frac{e^2}{C}\left[\sum_{n=-\infty}^{\infty} P_s(n+1) a(n+1) \frac{b_R(n)}{b(n)}\left(n - n_R + \frac{1}{2}\right) - \right.$$

$$\left. - \sum_{n=-\infty}^{\infty} P_s(n) a(n) \frac{a_R(n)}{a(n)}\left(n - n_R - \frac{1}{2}\right)\right.$$

$$(6.97)\quad \mathring{Q}_R = \frac{e^2}{C}\sum_{n=-\infty}^{\infty} P_s(n) a(n)\left(n - n_R - \frac{1}{2}\right)\left[\frac{b_R(n-1)}{b(n-1)} - \frac{a_R(n)}{a(n)}\right]$$

$$(6.98)\quad \mathring{Q}_R = -\frac{e^2}{C}\sum_{n=-\infty}^{\infty} \frac{P_s(n) a_R(n) a_D(n)\left(n - n_R - \frac{1}{2}\right)}{b(n-1)}\left[\frac{b_R(n)}{a_R(n+1)} - \frac{a_D(n)}{a_D(n+1)}\right]$$

$$(6.99\quad \mathring{Q}_R = \frac{e^2}{C}\sum_{n=-\infty}^{\infty} \frac{P_s(n) a_R(n+1) a_D(n+1)\left(n - n_R + \frac{1}{2}\right)}{a(n+1)}\left[\frac{b_R(n)}{a_R(n+1)} - \frac{b_D(n)}{a_D(n+1)}\right]$$

Let us now summarize the main results we have obtained thus far:

$$(6.100) \quad P_s(m+1) = \frac{b(m)}{a(m+1)} P_s(m)$$

$$(6.101) \quad I = (-e) \sum_{m=-\infty}^{\infty} \frac{P_s(m) a_R(m+1) a_D(m+1)}{a(m+1)} \left[\frac{b_R(m)}{a_R(m+1)} - \frac{b_D(m)}{a_D(m+1)} \right]$$

$$(6.102) \quad \overset{o}{Q}_R = \frac{e^2}{C} \sum_{m=-\infty}^{\infty} \frac{P_s(m) a_R(m) a_D(m+1)}{a(m+1)} \left(m - m_R + \frac{1}{2}\right) \left[\frac{b_R(m)}{a_R(m+1)} - \frac{b_D(m)}{a_D(m+1)} \right]$$

It is important to note that these results are valid for any applied thermodynamic forces E and $\frac{\Delta T}{T}$. They are also independent of the chosen diode and resistor models, as long as the transition rates $a(m)$ and $b(m)$ can be assumed to be independent.

When the system is in thermodynamic equilibrium:

$$(6.103) \quad P_{so}(m) = P_{DS}(m) = P_{RS}(m)$$

Using detailed balance:

$$(6.104)\quad \frac{b_R(m)}{a_R(m+1)} = \frac{P_{RS}(m+1)}{P_{RS}(m)} = \frac{P_{DS}(m+1)}{P_{DS}(m)} = \frac{b_D(m)}{a_D(m+1)}$$

If we introduce the above results in (6.101) and (6.102), I and $\mathring{Q}$ vanish. Therefore our system satisfies the Second Law requirement of having no fluxes in thermodynamic equilibrium.

The sums in (6.77) and (6.94) generally cannot be put in closed form and numerical methods have to be used for their evaluation.

The results of such a calculation are shown in Table B.1 .

However, when the system operates close to thermodynamic equilibrium significant conclusions about our engine can be drawn.

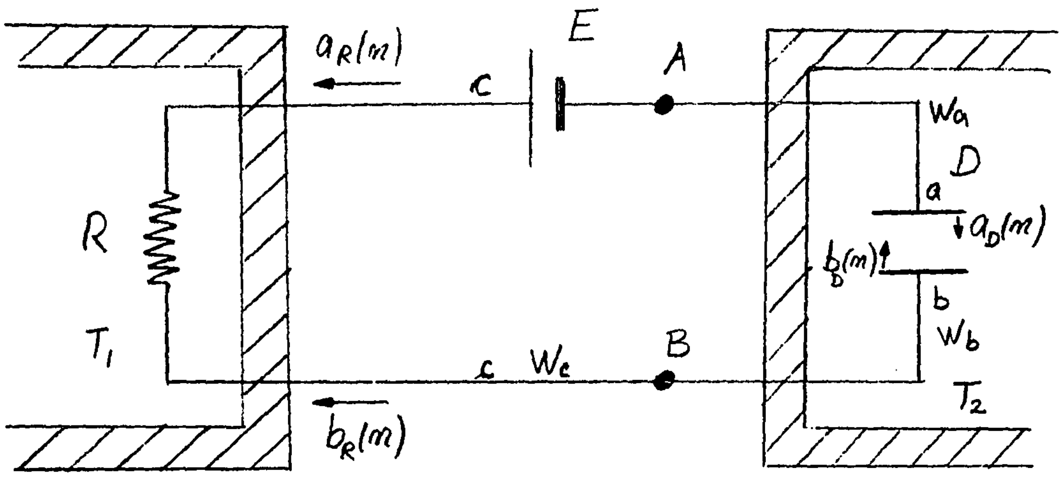


Fig 6.4

6.4 Resistor-Diode Fluctuation Heat Engine Near Thermodynamic Equilibrium

We will study the operation of our fluctuation heat engine when the applied thermodynamic forces E and $\frac{\Delta T}{T}$ are small enough to justify linear expanssions of I and $\mathring{Q}$, where T is the approximately uniform temperature of the system.

The electronic transition rates through the resistor and diode are related through the two local probability distribution functions $P_{RS}(n)$ and $P_{DS}(n)$:

$$(6.105) \qquad \frac{b_D(n)}{a_D(n+1)} = \frac{P_{DS}(n+1)}{P_{DS}(n)}$$

$$(6.106) \qquad \frac{b_R(n)}{a_R(n+1)} = \frac{P_{RS}(n+1)}{P_{RS}(n)}$$

Recalling the expresions of $P_{RS}(n)$ and $P_{DS}(n)$, (6.105) and respectively (6.106) become:

$$(6.107) \qquad \frac{b_D(n)}{a_D(n+1)} = e^{-\frac{e^2}{kT_2C}\left(n - n_D + \frac{1}{2}\right)}$$

$$(6.108) \qquad \frac{b_R(n)}{a_R(n+1)} = e^{-\frac{e^2}{kT_1C}\left(n - n_R + \frac{1}{2}\right)}$$

If we now expand the bracket of equation (6.101) in Taylor series around the equilibrium point $E=0$, $\Delta T=0$:

$$(6.109)\quad \frac{b_D(m)}{a_D(m+1)} - \frac{b_R(m)}{a_R(m+1)} = \left[\frac{b_D(m)}{a_D(m+1)} - \frac{b_R(m)}{a_R(m+1)}\right]_{\substack{\Delta T=0\\ E=0}} +$$

$$+\, e^{-\varepsilon(m-m_0+\frac{1}{2})}\left[\varepsilon(m_D-m_R) - \frac{e^2}{kC}\left(m-m_0+\frac{1}{2}\right)\left(\frac{1}{T_2}-\frac{1}{T_1}\right)\right]$$

Where: $\varepsilon = \left.\frac{e^2}{kTC}\right|_{\Delta T=0}$

$$m_0 = m_D\Big|_{\substack{\Delta T=0\\ E=0}} = m_R\Big|_{\substack{\Delta T=0\\ E=0}}$$

The first term of the above Taylor expansion vanishes as previously proven. Since at the equilibrium point,

$$(6.110)\quad \frac{b_{0D}(m)}{a_{0D}(m+1)} = \frac{b_{0R}(m)}{a_{0R}(m+1)} = \frac{b_0(m)}{a_0(m+1)} = e^{-\varepsilon(m-m_0+\frac{1}{2})}$$

we can write our Taylor expanssion in the form:

$$(6.111)\quad \frac{b_D(m)}{a_D(m+1)} - \frac{b_R(m)}{a_R(m+1)} \cong \frac{b_0(m)}{a_0(m+1)}\left[\varepsilon(m_D-m_R) - \varepsilon\left(m-m_0+\frac{1}{2}\right)\frac{\Delta T}{T}\right]$$

If we recall that,

$$n_D = \frac{W_a - W_b}{e^2/C} \qquad n_R = \frac{W_a - W_b}{e^2/C} + \frac{L\,C}{e}$$

we finally obtain:

$$(6.112) \quad \frac{b_D(n)}{a_D(n+1)} - \frac{b_R(n)}{a_R(n+1)} = \frac{b_0(n)}{a_0(n+1)}\left[\frac{-e}{kT}E - \varepsilon\left(n - n_0 + \tfrac{1}{2}\right)\frac{\Delta T}{T}\right]$$

To first order in E and $\frac{\Delta T}{T}$, I can be obtained from (6.101) by using the first order approximation (6.112) for the bracket and the zeroth order approximation for the coefficient of the bracket:

$$(6.113) \quad I = (-e)\sum_{n=-\infty}^{\infty}\left[\frac{a_R(n+1)\,a_D(n+1)}{a(n+1)}\right]_{\substack{\Delta T = 0\\ E=0}} P_{s0}(n)\times$$

$$\times\ \frac{b_0(n)}{a_0(n+1)}\left[\frac{-e}{kT}E - \varepsilon\left(n - n_0 + \tfrac{1}{2}\right)\frac{\Delta T}{T}\right]$$

Replacing $b_0(n)P_{s0}(n)$ by $a_0(n+1)P_{s0}(n+1)$ and offsetting $n+1$ to n in our infinite series we obtain:

$$(6.114)\quad I = (-e)\sum_{n=-\infty}^{\infty} \frac{a_{OR}(n)\,a_{OD}(n)}{a_{OR}(n) + a_{OD}(n)} P_{s}(n)\left[-\frac{e}{kT}E - \varepsilon\left(n - n_0 - \frac{1}{2}\right)\frac{\Delta T}{T}\right]$$

The Taylor expansion of the rate of heat flow (6.102) is found in a similar way:

$$(6.115)\quad \mathring{Q}_R = \frac{e^2}{C}\sum_{n=-\infty}^{\infty}\left[\frac{a_R(n+1)\,a_D(n+1)}{a(n+1)}\right]_{\substack{\Delta T=0\\ E=0}} \frac{P_{so}(n)\,b_o(n)}{a_o(n+1)}\left(n - n_0 + \frac{1}{2}\right)\times$$

$$\times\left[\frac{-e}{kT}E - \varepsilon\left(n - n_0 + \frac{1}{2}\right)\frac{\Delta T}{T}\right]$$

Again substituting $a_o(n+1)P_{so}(n+1)$ for $P_{so}(n)\,b_o(n)$:

$$(6.116)\quad \mathring{Q}_R = \frac{e^2}{C}\sum_{n=-\infty}^{\infty}\frac{a_{OR}(n)\,a_{OD}(n)}{a_{OR}(n) + a_{OD}(n)} P_{so}(n)\left(n - n_0 - \frac{1}{2}\right)\times$$

$$\times\left[\frac{e}{kT}E + \varepsilon\left(n - n_0 - \frac{1}{2}\right)\frac{\Delta T}{T}\right]$$

Let us define $A_o(n)$ by:

$$(6.117)\quad A_o(n) \equiv \frac{a_{OR}(n)\,a_{OD}(n)}{a_{OR}(n) + a_{OD}(n)}$$

Then the electric current and the rate of heat flow become:

$$(6.118)\quad I = \frac{e^2}{kT}\left[\sum_{n=-\infty}^{\infty} A_0(n)\,P_{so}(n)\right]\cdot E \;+$$

$$+\;\varepsilon\cdot e\left[\sum_{n=-\infty}^{\infty} A_0(n)\,P_{so}(n)\left(n-n_0-\tfrac{1}{2}\right)\right]\frac{\Delta T}{T}$$

$$(6.119)\quad \dot{Q} = \varepsilon e\left[\sum_{n=-\infty}^{\infty} A_0(n)\,P_{so}(n)\left(n-n_0-\tfrac{1}{2}\right)E \;+\right.$$

$$+\;\frac{\varepsilon e^2}{c}\left[\sum_{n=-\infty}^{\infty} A_0(n)\,P_{so}(n)\left(n-n_0-\tfrac{1}{2}\right)^2\right]\frac{\Delta T}{T}$$

If we now define M_{ij} by:

$$(6.120)\quad M_{11} \equiv \sum_{n=-\infty}^{\infty} A_0(n)\,P_{so}(n)$$

$$(6.121)\quad M_{2} \equiv \sum_{n=-\infty}^{\infty} A_0(n)\,P_{so}(n)\left(n-n_0-\tfrac{1}{2}\right)$$

$$(6.122)\quad M_{22} \equiv \sum_{n=-\infty}^{\infty} A_0(n)\,P_{so}(n)\left(n-n_0-\tfrac{1}{2}\right)^2$$

we can rewrite our flux equations in the form:

$$(6.123)\quad I = \frac{e^2}{kT} M_{11} E + \varepsilon e M_2 \frac{\Delta T}{T}$$

$$(6.124)\quad \overset{\circ}{Q} = \varepsilon e M_2 E + \frac{\varepsilon e^2}{C} M_{22} \frac{\Delta T}{T}$$

It is clear that the system satisfies the Onsager reciprocal relation for any value of the parameter ε .

Let us look next at the rate of entropy production exhibited by our fluctuation engine. With our sign conventions:

$$(6.125)\quad \overset{\circ}{S} = \frac{\overset{\circ}{Q}}{T} \frac{\Delta T}{T} + \frac{I E}{T}$$

If using (6.123), (6.124) we replace $\overset{\circ}{Q}$ and I , and impose, according to the Second Law, the condition of positive rate of entropy production we obtain:

$$(6.126)\quad M_{11} > 0$$

$$(6.127)\quad M_{22} > 0$$

$$(6.128)\quad M_2^2 < M_{11} \cdot M_{22}$$

Since $A_0(m)$, $P_{s0}(m)$ and $(m - m_0 - \frac{1}{2})^2$ are all positive quantities the first two inequalities are automatically satisfied.

To prove that the third inequality is also satisfied we will first define the unnormalized probability distribution function $P^0(m)$:

$$(6.129) \quad P^0(m) = A_0(m) P_{so}(m)$$

Then M_{11}^{-1} is equal to the normalization constant of $P^0(m)$:

$$(6.130) \quad M_{11} = \sum_{m=-\infty}^{\infty} A_0(m) P_{so}(m) = \sum_{m=-\infty}^{\infty} P^0(m)$$

If we rewrite M_2 and M_{22} in terms of $P^0(m)$ we get:

$$(6.131) \quad M_2 = \sum_{m=-\infty}^{\infty} P^0(m)(m - m_0 - \tfrac{1}{2})$$

$$(6.132) \quad M_{22} = \sum_{m=-\infty}^{\infty} P^0(m)(m - m_0 - \tfrac{1}{2})^2$$

In view of the definition of $P^0(m)$:

$$(6.133) \quad M_2 = M_{11} \langle m - m_0 - \tfrac{1}{2} \rangle$$

$$(6.134) \quad M_{22} = M_{11} \langle (m - m_0 - \tfrac{1}{2})^2 \rangle$$

Then (6.[illegible]) becomes:

$$\left\langle n - n_0 - \tfrac{1}{2} \right\rangle^2 < \left\langle \left(n - n_0 - \tfrac{1}{2} \right)^2 \right\rangle \tag{6.135}$$

The above relation is always satisfied (see for example the Schwartz Inequality). We therefore conclude that the rate of entropy production for our engine is always positive regardless of the value of ε .

Finally we will have to prove that in the continuum limit size our results translate into the answers given by the Gunn type analysis.

6.5 Resistor-Alkemade Diode Engine - Continuum Limit

The equilibrium probability distribution function for the system under consideration was found to be given by the Gaussian

$$(6.136)\quad P_{so}(n) = \frac{e^{-\frac{\varepsilon}{2}(n-n_0)^2}}{Z_0}$$

$$\text{Where:}\quad Z_0 = \sum_{n=-\infty}^{\infty} e^{-\frac{\varepsilon}{2}(n-n_0)^2}$$

In the continuum limit the infinite sum is the expression of the normalization constant Z_0, becomes an integral which if evaluated yields:

$$(6.137)\quad Z_0 \rightarrow \int_{-\infty}^{\infty} e^{-\frac{\varepsilon}{2}(n-n_0)^2}\, dn = \sqrt{\frac{2\pi}{\varepsilon}}$$

Therefore in the continuum limit:

$$(6.138)\quad P_{so}(n) = \sqrt{\frac{\varepsilon}{2\pi}}\, e^{-\frac{\varepsilon}{2}(n-n_0)^2}$$

It is also evident that in this approximation:

$$\langle n \rangle = n_0$$

Let us now proceed to evaluate the pivotal terms M_{ij}. To do so we will first calculate the continuum limit of the electronic transition rates a_{OD} and a_{OR} . The equilibrium transition rate through the diode, a_D , is a constant independent of n , and therefore will not be affected by the limiting process. However, the equilibrium transition rate through the resistor a_{OR} is a function of n , and in the continuum limit:

$$(6.139)\quad a_{OR}(n) = \frac{n - n_0}{RC} + \int_{n+\frac{1}{2}}^{\infty} \frac{m - n_0}{RC}\, e^{\frac{\varepsilon}{2}(n-m)(n+m-2n_R)}\, dm$$

If we perform the integration we obtain:

$$(6.140)\quad a_{OR}(n) = \frac{n - n_0}{R \cdot C} + \frac{1}{\varepsilon RC}\, e^{-\frac{\varepsilon(n-n_0)}{2} - \frac{\varepsilon}{8}}$$

Since $\varepsilon(n - n_0)$ is of the order of $\sqrt{\varepsilon}$ and, in the continuum limit, $\varepsilon \ll 1$, a linear expansion of the exponential is well justified. Then is we neglect $\varepsilon/8$:

$$(6.141)\quad a_{OR}(n) = \frac{1}{\varepsilon RC} + \frac{n - n_0}{2RC}$$

Replacing the derived expressions of $a_{OR}(n)$ and $a_{OD}(n)$ in (6.117) we find $A_0(n)$ to be:

$$(6.142)\quad A_0(n) = \frac{a_D\left(\frac{1}{\varepsilon RC} + \frac{n - n_0}{RC}\right)}{a_D + \frac{1}{\varepsilon RC} \quad \frac{n - n_0}{RC}}$$

After some algebra we approximate:

$$(6.143)\quad A_0(n) \cong \frac{a_D}{1 + a_D \varepsilon RC}\left[1 + \frac{\varepsilon}{2}(n - n_0)\left(1 - \frac{1}{1 + a_D \varepsilon RC}\right)\right]$$

Calculation of M_{ij} in the continuum limit yields:

$$(6.144)\quad M_{11} = \int_{-\infty}^{\infty} A_0(n)\, P_{s0}(n)\, dn = \frac{a_D}{1 + \varepsilon a_D RC}$$

$$(6.145)\quad M_2 = \int_{-\infty}^{\infty} A_0(n)\, P_{s0}(n)\left(n - n_0 - \tfrac{1}{2}\right) dn =$$

$$= \frac{a_D}{1 + a_D \varepsilon RC}\left[\frac{\varepsilon}{2}\left(1 - \frac{1}{1 + a_D \varepsilon RC}\right)\left\langle (n - n_0)^2 \right\rangle - \frac{1}{2}\right]$$

$$(6.146)\quad M_{22} = \int_{-\infty}^{\infty} A_0(n)\, P_{s0}(n)\left(n - n_0 - \tfrac{1}{2}\right)^2 dn =$$

$$= \frac{a_D}{1 + a_D \varepsilon RC}\left\{\left[1 - \frac{\varepsilon}{2}\left(1 - \frac{1}{1 + a_D \varepsilon RC}\right)\right]\left\langle (n - n_0)^2 \right\rangle + \frac{1}{4}\right\}$$

Observing that for our distribution function $\left\langle (n - n_0)^2 \right\rangle = \frac{1}{\varepsilon}$,

we can write:

$$(6.147)\quad M_{11} = \frac{a_D}{1 + a_D \varepsilon RC}$$

$$(6.148)\quad M_{2} \cong -\frac{a_D}{2(1 + a_D \varepsilon RC)}$$

$$(6.149)\quad M_{22} \cong \frac{a_D}{\varepsilon(1 + a_D \varepsilon RC)}$$

In terms of the resistance of the diode at the operating point:

$$(6.150)\quad a_D = \frac{kT}{e^2} \cdot \frac{1}{R_D}$$

Then, if we replace a_D in the expressions of M_{ij} we get:

$$(6.151)\quad M_{11} = \frac{kT}{e^2} \cdot \frac{1}{R + R_D}$$

$$(6.152)\quad M_{2} = -\frac{kT}{e^2} \cdot \frac{R_D}{(R + R_D)^2}$$

$$(6.153)\quad M_{22} = \frac{k^2T^2C}{e^4} \cdot \frac{1}{R + R_D}$$

Substituting M_{ij} into (6.118) and (6.119) we obtain for the fluxes I and $\mathring{Q}$ in the continuum limit:

$$(6.154) \quad I = \frac{E}{R + R_D} - \frac{eR}{C(R+R_D)^2} \frac{\Delta T}{T}$$

$$(6.155) \quad \mathring{Q} = -\frac{eR}{C(R+R_D)^2} + \frac{kT}{C(R+R_D)} \frac{\Delta T}{T}$$

If for simplicity we choose our engine such that $R = R_D = \frac{1}{g_1}$ and if we note that the nonlinearity coefficient g_2 for the Alkemade diode is given by $g_2 = \frac{e}{kT} g_1$ the flux equations (6.155) and (6.156) become:

$$(6.156) \quad I = \frac{E g_1}{2} - \frac{kT g_2}{4C} \frac{\Delta T}{T}$$

$$(6.157) \quad \mathring{Q} = -\frac{kT g_2}{4C} E + \frac{kT g_1}{2C} \frac{\Delta T}{T}$$

It is gratifying to observe that these equations are identical to the flux equations (5.98), (5.99) derived on the basis of Gunn's model for the diode-resistor fluctuation heat engine.

7. RESISTOR-DIODE FLUCTUATION HEAT ENGINE FAR FROM THERMODYNAMIC EQUILIBRIUM - THE CONTINUUM LIMIT

We will now explore the behaviour of our resistor-diode engine when operating under large temperature differences.

Generally such an analysis requires the use of numerical methods, as it will be done in Appendix B of this thesis.

However, if the temperature of the resistor is much larger than the temperature of the diode and the conductance of the resistor is, within the operating range, much larger than the conductance of the diode, analytical evaluations of the output power and efficiency of the engine are possible.

We will consider for our calculation a diode with an exponential characteristic, such as the PN or Alkemade diodes.

If we apply Kirchhoff's Voltage Law on the DC equivalent circuit of our system (Fig. 7.1), in view of the high conductance of the resistor, we find that the DC voltage drop across the diode is simply E .

As a consequence of the operating conditions of the engine, the noise generated by the diode can be neglected when compared to the noise originating in the resistor.

Because of the low relative conductance of the diode, the thermal coupling between the capacitor and the diode can also be neglected.

Under the above conditions the electrons on the capacitor will be in thermal equilibrium with the electrons in the resistor. Then the equilibrium probability distribution function of the voltage v across the diode is given by the appropriate Gaussian:

$$(7.1)\quad P(v) = \frac{1}{\sqrt{2\pi \frac{kT}{C}}}\, e^{-\frac{Cv^2}{2kT}}$$

We will characterize the diode by a Shockley type exponential equation, which for our sign convention reads:

$$(7.2)\quad i = -I_s\left(e^{\frac{eV}{kT}} - 1\right)$$

Where: $V = v - E$

The mean value of the output power supplied by our engine is given by:

$$(7.3)\quad P_o = \langle -i \cdot E \rangle$$

Using (7.1) , (7.2) and (7.3) the output power becomes:

$$(7.4)\quad P_o = \frac{E I_s}{\sqrt{2\pi \frac{kT_1}{C}}} \int_{-\infty}^{\infty} \left\{ \exp\left[\frac{e}{kT_2}(v - E)\right] - 1 \right\} e^{-\frac{Cv^2}{2kT_1}}\, dv$$

If we define ε_2 by,

$$(7.5) \qquad \varepsilon_2 \equiv \frac{e^2}{kT_2C}$$

the expression of the output power finally reads:

$$(7.6) \qquad P_o = E I_s \left\{ \exp\left[\frac{\varepsilon_2 T_1}{T_2} \left(\frac{1}{2} - \frac{eE}{kT_1 \varepsilon_2} \right) \right] - 1 \right\}$$

In a similar way we can calculate the power dissipated on the diode by evaluating the average:

$$(7.7) \qquad P_d = \langle -i(v-E) \rangle$$

Applying (7.1), (7.2) and $V = v - E$ to the above relation:

$$(7.8) \quad P_d = \frac{I_s}{\sqrt{2\pi \frac{kT_1}{C}}} \int_{-\infty}^{\infty} (v-E) \left\{ \exp\left[\frac{e}{kT_2}(v-E) \right] - 1 \right\} e^{-\frac{Cv^2}{2kT_1}} dv$$

Or after we perform the integration:

$$(7.9) \quad P_d = I_s \left[\left(\frac{e}{C} \cdot \frac{T_1}{T_2} - E \right) \exp\left(\frac{e^2}{2kT_2C} \cdot \frac{T_1}{T_2} - \frac{eE}{kT_2} \right) + E \right]$$

Recalling the definition of ε_2 :

$$(7.10) \quad P_d = I_s \left[\left(\frac{e}{C} \frac{T_1}{T_2} - E \right) \exp\left(\frac{\varepsilon_2}{2} \frac{T_1}{T_2} - \frac{eE}{kT_2} \right) + E \right]$$

We can now calculate the efficiency of our engine. By definition:

$$(7.11) \qquad \eta = \frac{P_0}{P_0 + P_d}$$

Substituting P_0 and P_d by their corresponding expressions (7.6) and (7.10) :

$$(7.12) \quad \eta = \frac{E\left[\exp\left(\frac{\varepsilon_2}{2}\frac{T_1}{T_2} - \frac{eE}{kT_2}\right) - 1\right]}{E\left[\exp\left(\frac{\varepsilon_2}{2}\frac{T_1}{T_2} - \frac{eE}{kT_2}\right) - 1\right] + \left[\left(\frac{e}{C}\frac{T_1}{T_2} - E\right)\exp\left(\frac{\varepsilon_2}{2}\frac{T_1}{T_2} - \frac{eE}{kT_2}\right) + E\right]}$$

After some algebra the efficiency is found to be:

$$(7.13) \quad \eta = \frac{eE}{kT_1\varepsilon_2}\left\{1 - \exp\left[\frac{\varepsilon_2 T_1}{T_2}\left(\frac{eE}{kT_1\varepsilon_2} - \frac{1}{2}\right)\right]\right\}$$

If we define the new parameters x and α by:

$$(7.14) \qquad x \equiv \frac{eE}{kT_1\varepsilon_2}$$

$$(7.15) \qquad \alpha \equiv \frac{\varepsilon_2 T_1}{T_2}$$

equations (7.6) and (7.13) can be translated into:

$$(7.16) \quad \frac{P_o}{E I_s} = e^{\alpha(\frac{1}{2}-x)} - 1$$

$$(7.17) \quad \eta = x\left[1 - e^{\alpha(x-\frac{1}{2})}\right]$$

The efficiency of the engine is maximum when α becomes very large and x increases towards $1/2$. In this limit the efficiency is seen to approach 50%, as the relative output power also becomes large. The general behaviour of the relative output power $P_o/E I_s$ and efficiency η with respect to x, for various values of the parameter α, are shown in Figs. 7.2 and 7.3 .

For our present description of the resistor-diode engine to be valid, ε_2 has to be much smaller than unity (the continuum limit), and therefore large values of α are rather unfeasible, unless the system is operated with a very low temperature cold reservoir. The small ε_2 operating range also coincides with the region which describes technologically conceivably diodes, and thus we conclude that that the maximum absolute efficiency of fluctuation engines could approach, under extreme conditions 50% .

Nevertheless, for all diodes I_s is highly dependent on temperature and when the diode is very cold, such as to allow for a large efficiency, I_s , and therefore P_o , become extremely small.

As an example, for most diodes the strongest temperature dependence comes from an exponential term of the form $e^{eV'/kT}$, where V' is a voltage of the order of volts. If at the limit of present technology, we could build a diode with,

$$\mathcal{E}_2 T_2 = \frac{e^2}{kC} = 1$$

operate the system with the resistor at $1000\,^\circ K$ and the diode at $10\,^\circ K$, such that $\alpha = 10$, the above mentioned exponential factor is then of the order of 10^{-100} , yielding a practically zero output power.

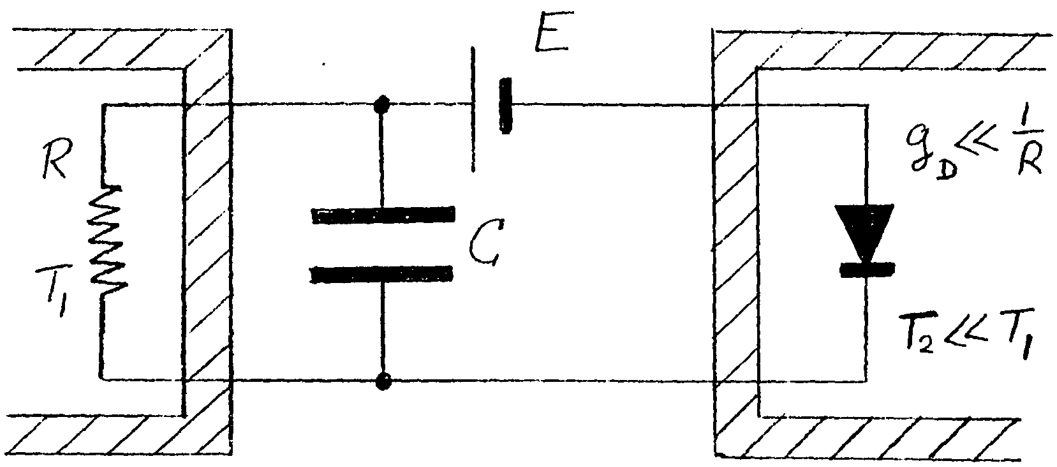


Fig 7.1

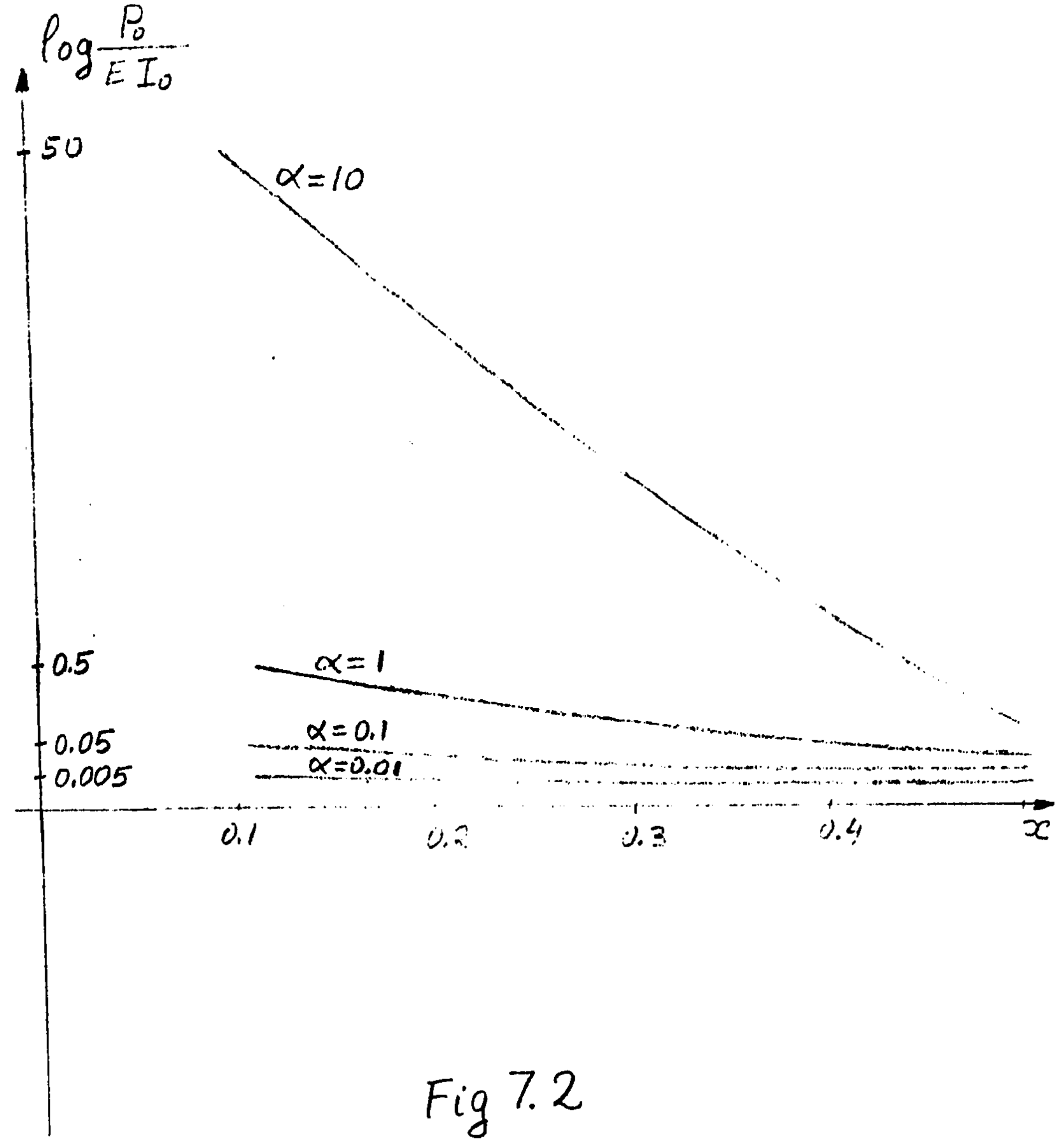


Fig 7.2

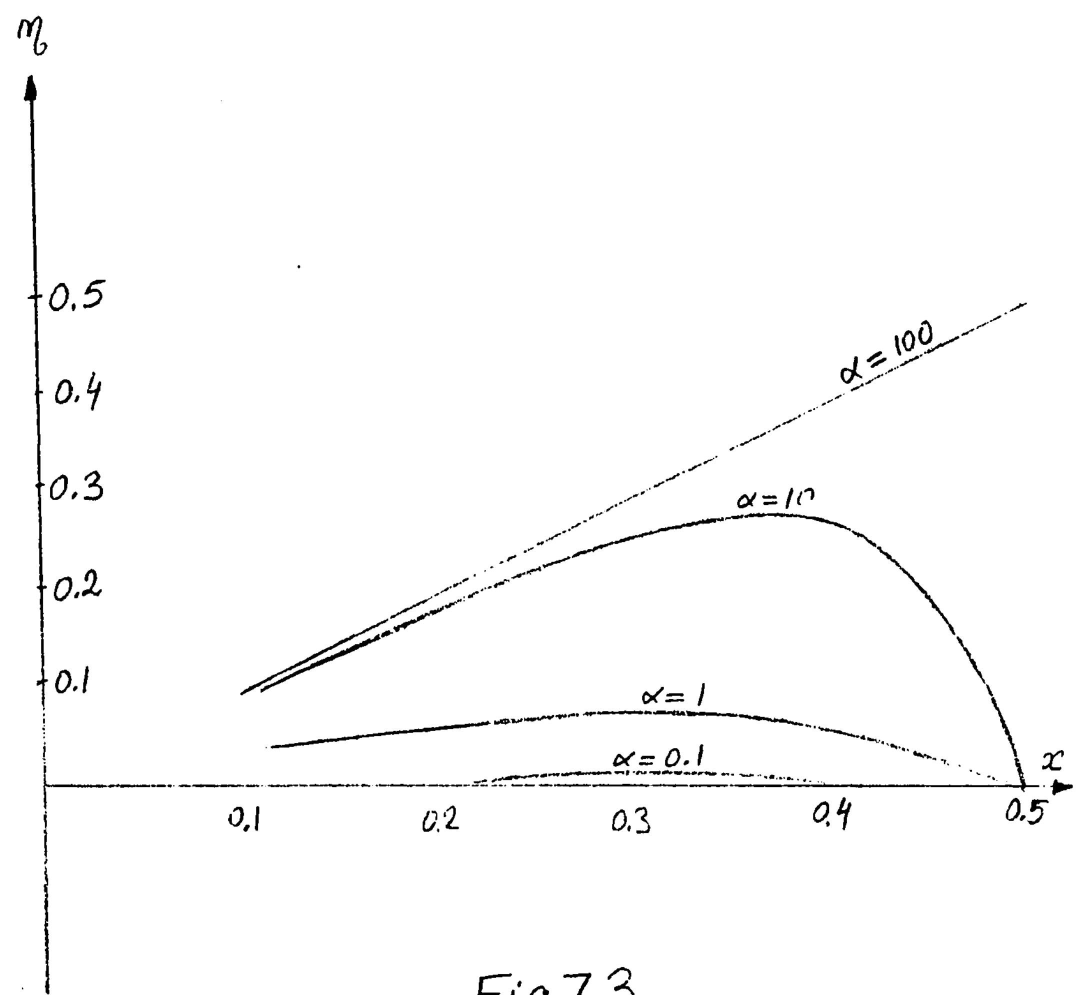


Fig 7.3.

8. CONCLUSIONS

8.1 General Thermodynamics of Fluctuation Heat Engines

We have studied in detail the thermodynamics of fluctuation heat engines both from a phenomenological as well as kinetic point of view.

At no time we have found any results in contradiction to the Second Law. Thus, speculations about the possibility of violating the Second Law by rectifying thermal noise, have no physical foundation.

We have also found that, when a fluctuation engine operates close to thermodynamic equilibrium, Onsager's reciprocal relations are always satisfied.

8.2 Efficiency and Power Output of Fluctuation Heat Engines

We have found that the efficiency and output power of fluctuation heat engines are strongly dependent on the capacitance of the system under consideration. Thus the physical size of the system will play an important role in any engine configuration.

Reasonable efficiencies and output powers were consistently found at system sizes beyond the reach of today's technology.

During our investigation we never found any size, configuration or operation conditions for which the Carnot efficiency was reached.

Also under any conditions we never found an efficiency higher than 53% .

Since standard thermoelectric, thermionic and photoelectric conversion mechanisms offer same or better efficiencies at vastly simpler configurations, it appears to us that efforts to develop the technology involved in manufacturing fluctuation heat engines are unwarranted.

The behaviour of the output power and efficiency of a resistor-diode fluctuation heat engine, with respect to various relevant parameters, are shown in Figs. 8.1, 8.2 and 8.3.

8.3 Maximum Nonlinearity of Diodes

When studying actual diode configurations we have found no thermodynamic reasons for their relatively weak nonlinearity.

If we recall our Gunn type analysis of fluctuation heat engines, macroscopic diodes will not violate the Second Law as long as, with the proper definitions of the various parameters:

$$(8.1) \qquad \beta \equiv \frac{kT}{C}\left(\frac{g_2}{g_1}\right)^2 < 1$$

Let us try to derive the characteristic equation of the "most nonlinear diode", yet to be invented, that will not violate the principles of thermodynamics.

In the limit of $\varepsilon \to 1$ the nonlinearity coefficient of the "most nonlinear diode" is found to be:

$$(8.2) \qquad g_2 = \sqrt{\frac{C}{kT}}\, g_1$$

Using the definition of g_2 ,

$$(8.3) \qquad g_2 \equiv \frac{1}{2} \frac{\partial g_1}{\partial v}$$

we find for the conductance of the diode:

$$(8.4) \qquad g_1 = C_1 e^{2\sqrt{\frac{C}{kT}}\, v}$$

$C_1 \equiv$ integration constant

Recalling the definition of g_1 , we find the characteristic function of the "most nonlinear diode" to be:

$$(8.5) \qquad i = \frac{C_1}{2}\sqrt{\frac{kT}{C}}\, e^{2\sqrt{\frac{C}{kT}}\, v} + C_2$$

$C_2 \equiv$ integration constant

If we require the current i through the diode to be zero when the voltage v across the diode is zero, with the obvious identification of the reverse current I_S , the characteristic function of the "most nonlinear diode" not to violate thermodynamics is found to be:

$$(8.6) \qquad i = I_s\left(e^{2\sqrt{\frac{C}{kT}}\,v} - 1\right)$$

Although we admit the above derivation to be highly speculative, it indicates the possibility of creating diodes with "extensive" nonlinearities, depending on the capacitance, and therefore the size, of the diode.

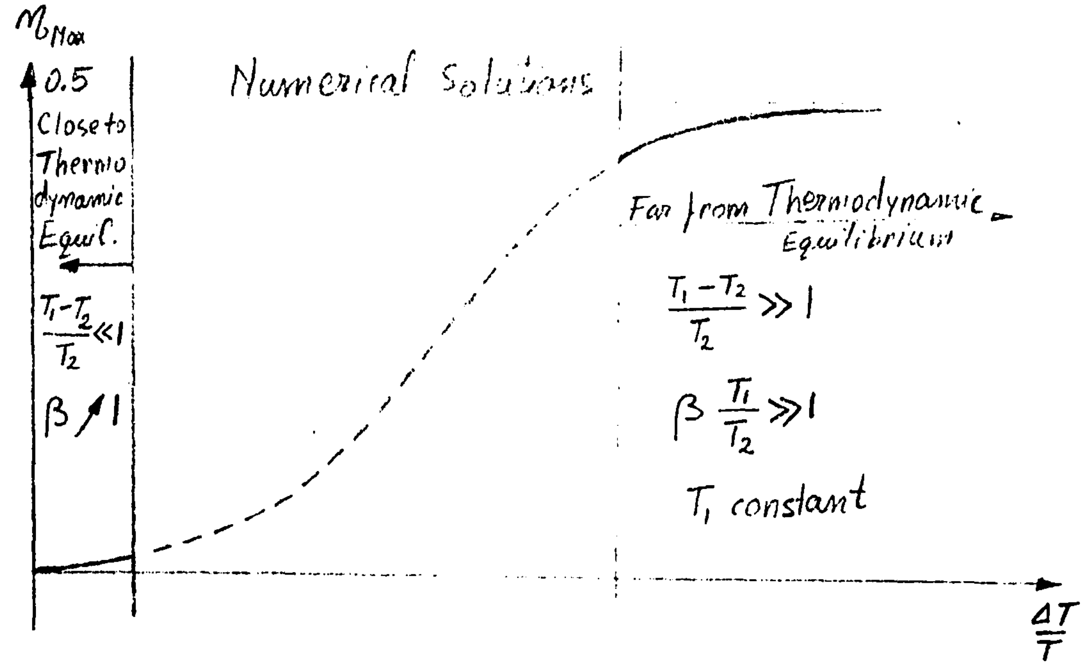

η Max
0.5
Close to Thermo dynamic Equil.
$\frac{T_1-T_2}{T_2} \ll 1$
β ↗ 1
Numerical Solutions
Far from Thermodynamic Equilibrium
$\frac{T_1-T_2}{T_2} \gg 1$
$\beta \frac{T_1}{T_2} \gg 1$
T_1 constant
$\frac{\Delta T}{T}$

Fig. 8.1.

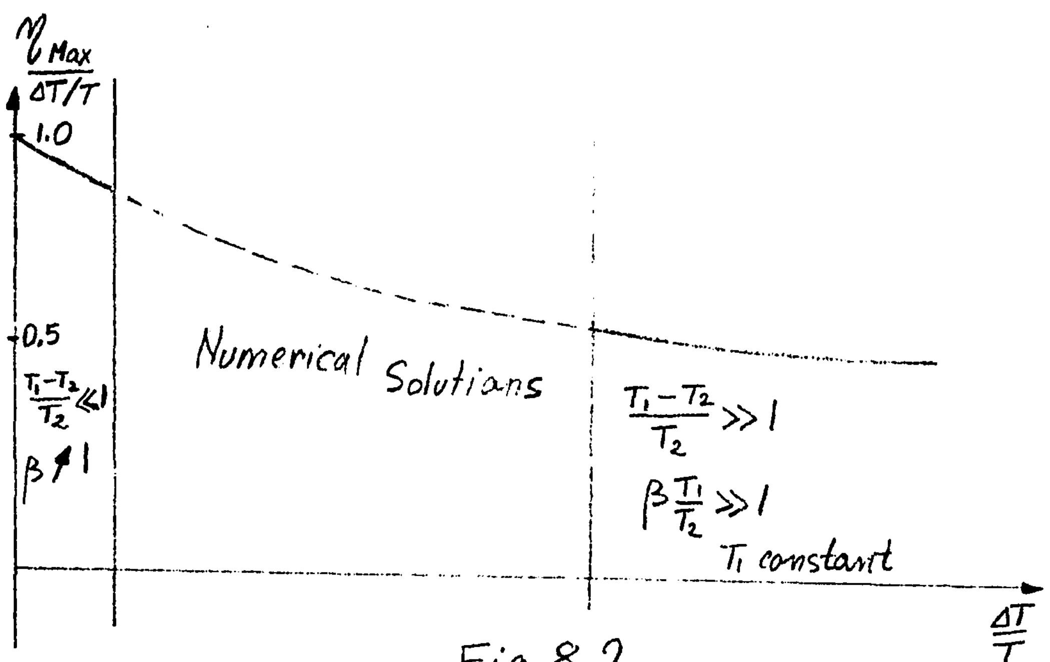
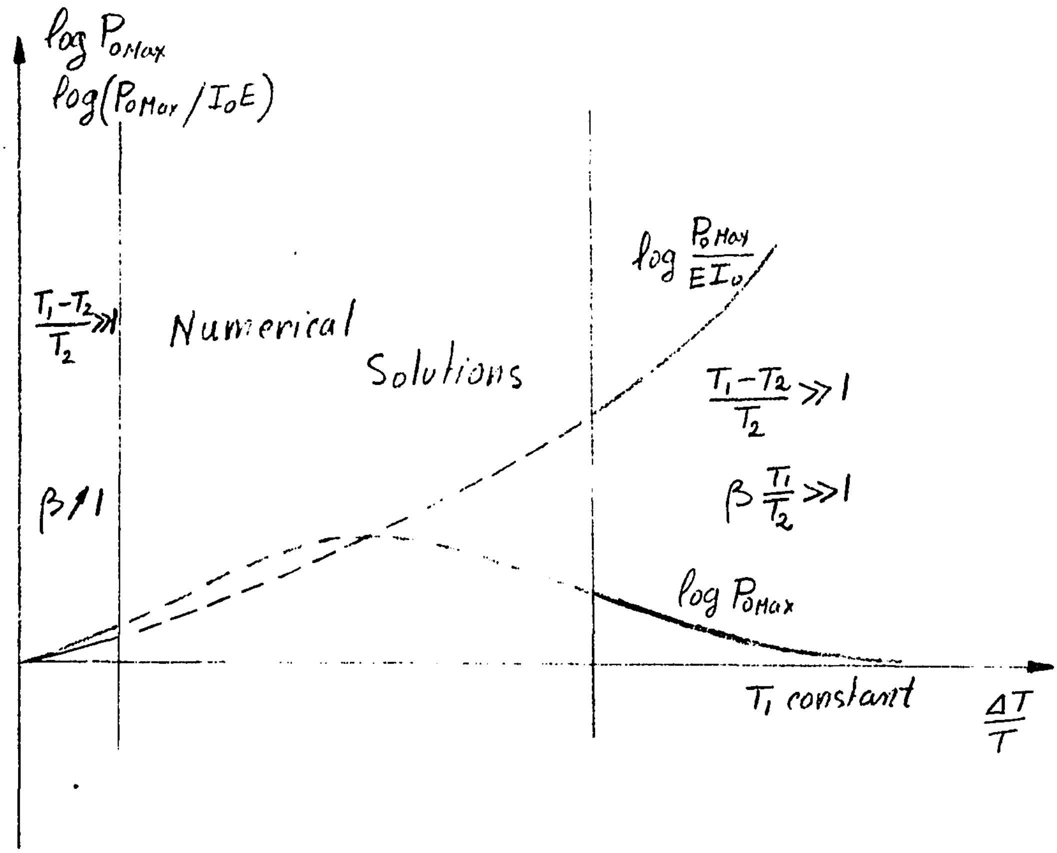

$\frac{\eta_{Max}}{\Delta T/T}$
1.0
0.5
$\frac{T_1-T_2}{T_2} \ll 1$
β ↗ 1
Numerical Solutions
$\frac{T_1-T_2}{T_2} \gg 1$
$\beta \frac{T_1}{T_2} \gg 1$
T_1 constant
$\frac{\Delta T}{T}$

Fig. 8.2.



log P_{0Max}
log(P_{0Max}/I_0E)
$\frac{T_1-T_2}{T_2} \gg 1$
$\beta \gg 1$
Numerical
Solutions
log $\frac{P_{0Max}}{EI_0}$
$\frac{T_1-T_2}{T_2} \gg 1$
$\beta \frac{T_1}{T_2} \gg 1$
log P_{0Max}
T_1 constant
$\frac{\Delta T}{T}$

Fig. 8.3.

9. APPENDIX A

Asymmetric Gaussian Distribution Functions

In our kinetic analysis of fluctuation heat engines we have repeatedly encountered discrete Gaussian distribution functions. In general such functions can be written in the form,

$$(A.1) \qquad P(n) = \frac{e^{-\frac{\varepsilon}{2}(n-n_0)^2}}{Z(n_0)}$$

where n is a discrete integer variable.

In this section we will study the behaviour of $P(n)$ and related quantities with respect to the value of the real parameter n_0 .

Let us start with the evaluation of the partition function $Z(n_0)$ which, by definition, is given by:

$$(A.2) \qquad Z(n_0) = \sum_{n=-\infty}^{\infty} e^{-\frac{\varepsilon}{2}(n-n_0)^2}$$

Because of the infinite range of the sum, $Z(n_0)$ is a periodic function with a period equal to one.

$$(A.3) \qquad Z(n_0+1) = Z(n_0)$$

We can therefore expand $Z(n_0)$ in Fourier series,

$$(A.4) \qquad Z(n_0) = \sum_{m=-\infty}^{\infty} C_m e^{2\pi i m n_0}$$

with C_m given by:

$$(A.5) \qquad C_m = \int_0^1 e^{-2\pi i m n_0} Z(n_0)\, dn_0$$

Using $(A.4)$ we can succesively find for the Fourier coefficients C_m :

$$(A.6) \qquad C_m = \int_0^1 e^{-2\pi i m n_0} \sum_{n=-\infty}^{\infty} e^{-\frac{\varepsilon}{2}(n-n_0)^2} dn_0$$

$$(A.7) \qquad C_m = \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} e^{2\pi i m y} e^{-\frac{\varepsilon}{2}y^2} dy$$

$$(A.8) \qquad C_m = \int_{-\infty}^{\infty} e^{2\pi i m y} e^{-\frac{\varepsilon}{2}y^2} dy = \sqrt{\frac{2\pi}{\varepsilon}}\, e^{-\frac{4\pi^2 m^2}{2\varepsilon}}$$

Then the partition function associated with our Gaussian can be written as the rapidly converging Fourier series:

$$(A.9)\quad Z(m_0)=\sqrt{\frac{2\pi}{\varepsilon}}\left[1+2e^{-\frac{4\pi^2}{2\varepsilon}\cdot 1^2}\cos 2\pi m_0+2e^{-\frac{4\pi^2}{2\varepsilon}\cdot 2^2}\cos 4\pi m_0+\dots\right]$$

By differentiating $(A.2)$ with respect to m_0 we find:

$$(A.10)\qquad \langle m\rangle = m_0+\frac{1}{\varepsilon Z(m_0)}\cdot\frac{dZ(m_0)}{dm_0}$$

In a similar fashion we can derive the expression of the mean square deviation of m :

$$(A.11)\qquad \langle (m-m_0)^2\rangle=\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\,\frac{1}{Z(m_0)}\,\frac{d^2Z(m_0)}{dm_0^{\,2}}$$

Finally if we substitute $Z(m_0)$ from $(A.9)$ into $(A.10)$ and $(A.11)$:

$$(A.12)\quad \langle m\rangle = m_0-\frac{4\pi}{3}\,\frac{e^{-\frac{4\pi^2}{2\varepsilon}\cdot 1^2}\sin 2\pi m_0+2e^{-\frac{4\pi^2}{2\varepsilon}\cdot 2^2}\sin 4\pi m_0+\dots}{1+2e^{-\frac{4\pi^2}{2\varepsilon}\cdot 1^2}\cos 2\pi m_0+2e^{-\frac{4\pi^2}{2\varepsilon}\cdot 2^2}\cos 4\pi m_0+\dots}$$

$$(A.13)\quad \langle (m-m_0)^2\rangle = \frac{1}{\varepsilon} - \frac{8\pi^2}{\varepsilon^2}\,\frac{e^{-\frac{4\pi^2}{2\varepsilon}\cdot 1^2}\cos 2\pi m_0 + 2^2 e^{-\frac{4\pi^2}{2\varepsilon}\cdot 2^2}\cos 4\pi m_0 + \dots}{1+2e^{-\frac{4\pi^2}{2\varepsilon}\cdot 1^2}\cos 2\pi m_0 + 2e^{-\frac{4\pi^2}{2\varepsilon}\cdot 2^2}\cos 4\pi m_0 + \dots}$$

For m_0 integer or half integer $(A.12)$ and $(A.13)$ become:

$$(A.14)\qquad \langle m\rangle = m_0$$

$$(A.15)\qquad \langle (m-m_0)^2\rangle = \frac{1}{\varepsilon}$$

If m_0 is neither an integer nor a half integer the probability distribution function $P(m)$ becomes asymmetrical as seen in Figs. $A.1$, $A.2$ and $A.3$.

Finally, if ε is less then 10, $(A.12)$ and $(A.13)$ can be very well approximated by:

$$(A.16)\qquad \langle m\rangle \cong m_0 - \frac{4\pi}{3}\, e^{-\frac{4\pi^2}{2\varepsilon}} \sin 2\pi m_0$$

$$(A.17)\qquad \langle (m-m_0)^2\rangle \cong m_0 - \frac{8\pi^2}{\varepsilon^2}\cdot e^{-\frac{4\pi^2}{2\varepsilon}} \cos 2\pi m_0$$

If we recall the kinetic models previously introduced, the average voltage across the open circuit diode (or resistor) is found to be:

$$\langle v \rangle = \frac{e}{C} \langle n - n_0 \rangle$$

Or using (A.16) :

$$\langle v \rangle = -\frac{4\pi}{3} \frac{e}{C} e^{-\frac{4\pi^2}{2\varepsilon}} \sin 2\pi n_0$$

It has been speculated that, since for n not equal to an integer or half integer the above relation yields a nonzero open circuit voltage across the diode, methods for obtaining electric energy from a diode in thermodynamic equilibrium can be devised [24] .

However tempting, such a conclusion is false since no fluxes, and therefore no electric currents, can develop in a system in thermodynamic equilibrium. Without generating an electric current the diode will not generate electric power, even if an open circuit voltage will be present.

Formally we can calculate the net current flow through a short circuited diode by evaluating the sum,

$$\langle I \rangle = \sum_{n=-\infty}^{\infty} P_{DS}(n) \left[b_D(n) - a_D(n) \right]$$

where $P_{DS}(n)$, $b_D(n)$ and $a_D(n)$ are defined in

Section 5.1. Then we can succesively write:

$$(A.18)\quad \langle I \rangle = \sum_{m=-\infty}^{\infty} P_{DS}(m)\, b_D(m) - \sum_{m=-\infty}^{\infty} P_{DS}(m)\, a_D(m)$$

$$(A.19)\quad \langle I \rangle = \sum_{m=-\infty}^{\infty} P_{DS}(m)\, b_D(m) - \sum_{m=-\infty}^{\infty} P_{DS}(m+1)\, a_D(m+1)$$

$$(A.20)\quad \langle I \rangle = \sum_{m=-\infty}^{\infty} \left[P_{DS}(m)\, b_D(m) - P_{DS}(m+1)\, a_D(m+1) \right]$$

According to the principle of detailed balance the bracket in the above expression vanishes.

$$(A.21)\qquad \langle I \rangle = 0$$

We thus proved that a diode in thermodynamic equilibrium cannot generate even a short circuit current, and therefore no violation of the Second Law occurs.

Finally, from (A.13) we see that the mean square voltage deviation across the diode,

$$(A.22)\qquad \langle v^2 \rangle = \frac{e^2}{C^2} \langle (m - m_0)^2 \rangle$$

takes the equipartition of energy value $\frac{kT}{C}$ only when n_0 is an integer or half integer.

In the continuum limit $\varepsilon \rightarrow 0$, and regardless of the value of ,

$$(A.23)\qquad \langle n \rangle = n_0$$

$$(A.24)\qquad \langle (n-n_0)^2 \rangle = \frac{1}{\varepsilon}$$

as expected from standard Statistical Mechanics.

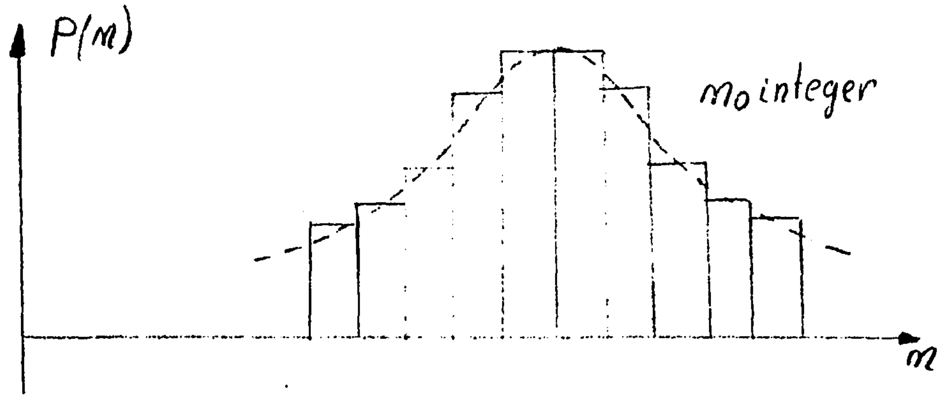


Fig. A.1

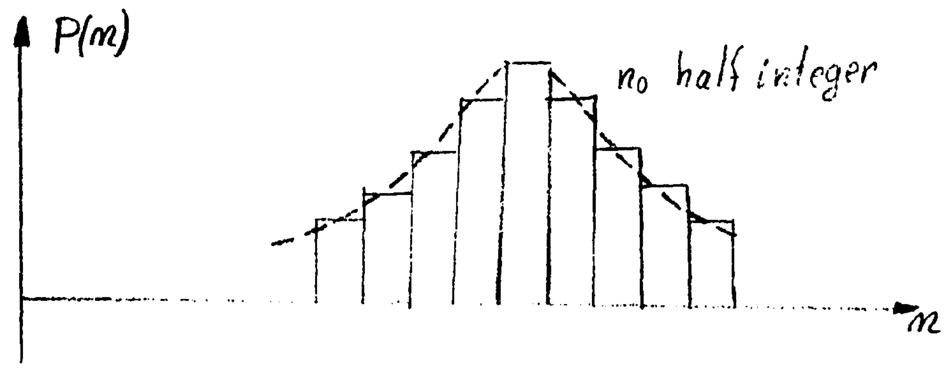


Fig. A.2

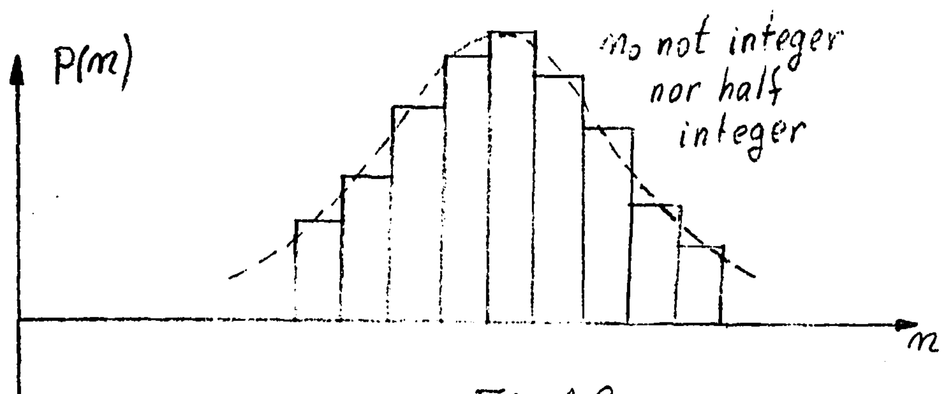


Fig. A.3

10. APPENDIX B

Resistor - Diode Fluctuation Heat Engines Far From Thermodynamic Equilibrium - Numerical Calculations

As previously mentioned, general results concerning the behavior of fluctuation heat engines when operating far from thermodynamic equilibrium are very difficult to obtain.

However, if kinetic models for the dissipative elements involved can be constructed, exact expressions for the fluxes developed within the engine can be derived.

If the engine employs an Alkemade diode and a linear resistor that can be described by the kinetic model introduced in Section 6.2, the exact expressions for the current and heat fluxes are given by equations (6.77) and (6.94).
Using (6.1), (6.4), (6.61), (6.63), (6.75), (6.77) and (6.94), we can numerically calculate the values of these fluxes, and the results of such calculations are illustrated in Table B.1.
The various quantities involved are:

R ≡ resistance of the linear resistor

R_D ≡ resistance of the diode at the operating point

$\frac{I}{e}$ ≡normalized output current per time constant $R_D C$

$\frac{Q}{kT_D}$ ≡normalized heat transfer per time constant $R_D C$

$\frac{P_o}{kT_D}$ ≡normalized output power per time constant $R_D C$

η ≡efficiency

These results confirm the fact that high efficiencies and output powers are obtained at large values of $\mathcal{E}$, or equivalently, at small engine sizes. As the size of the system increases, and therefore increases, the efficiency and output power are seen to drop fast.

It is also confirmed that higher efficiencies are obtained when the resistance of the diode at the operating point is larger than that of the linear resistor, with the largest calculated efficiency around 50% , in accordance with the continuum case of Section 7.

At thermodynamic equilibrium ($E=0$, $T_1=T_2$), allowing for computer tolerance, the current and heat fluxes are zero, as required by the Second Law.

$\frac{R}{R_D}$	$\mathcal{E}_D$	n_R	$\frac{T_1}{T_2}$	$\frac{I}{C}$	$\frac{\dot{Q}}{k T_D}$	$\frac{P_o}{k T_D}$	η
0.5	10.000	0	0.50	-0.55E-02	-0.25E-01	0.00E+00	0.0000
0.5	10.000	0	1.00	-0.47E-09	0.78E-07	0.00E+00	0.0000
0.5	10.000	0	2.00	0.57E-01	0.32E+00	0.00E+00	0.0000
0.5	10.000	0	4.00	0.19E+00	0.10E+01	0.00E+00	0.0000
0.5	10.000	0	8.00	0.35E+00	0.21E+01	0.00E+00	0.0000
0.5	10.000	0	16.00	0.58E+00	0.40E+01	0.00E+00	0.0000
0.5	10.000	0	32.00	0.92E+00	0.76E+01	0.00E+00	0.0000
0.5	10.000	0	64.00	0.14E+01	0.15E+02	0.00E+00	0.0000
0.5	10.000	1	0.50	-0.48E-01	-0.19E+00	-0.48E+00	0.0000
0.5	10.000	1	1.00	-0.47E-01	-0.23E+00	-0.47E+00	0.0000
0.5	10.000	1	2.00	-0.46E-01	-0.21E+00	-0.46E+00	0.0000
0.5	10.000	1	4.00	-0.34E-01	-0.77E-01	-0.34E+00	0.0000
0.5	10.000	1	8.00	0.29E-01	0.96E+00	0.29E+00	29.7221
0.5	10.000	1	16.00	0.19E+00	0.36E+01	0.19E+01	51.5888
0.5	10.000	1	32.00	0.47E+00	0.89E+01	0.47E+01	52.1754
0.5	10.000	1	64.00	0.91E+00	0.19E+02	0.91E+01	48.4959
0.5	1.000	0	0.50	-0.47E-01	-0.15E+00	0.00E+00	0.0000
0.5	1.000	0	1.00	-0.30E-07	0.13E-05	0.00E+00	0.0000
0.5	1.000	0	2.00	0.96E-01	0.33E+00	0.00E+00	0.0000
0.5	1.000	0	4.00	0.28E+00	0.10E+01	0.00E+00	0.0000
0.5	1.000	0	8.00	0.60E+00	0.24E+01	0.00E+00	0.0000
0.5	1.000	0	16.00	0.12E+01	0.52E+01	0.00E+00	0.0000
0.5	1.000	0	32.00	0.21E+01	0.10E+02	0.00E+00	0.0000
0.5	1.000	0	64.00	0.35E+01	0.19E+02	0.00E+00	0.0000
0.5	1.000	1	0.50	-0.28E+00	-0.26E+00	-0.28E+00	0.0000
0.5	1.000	1	1.00	-0.25E+00	-0.15E+00	-0.25E+00	0.0000
0.5	1.000	1	2.00	-0.18E+00	0.11E+00	-0.18E+00	0.0000
0.5	1.000	1	4.00	-0.33E-01	0.72E+00	-0.33E-01	0.0000
0.5	1.000	1	8.00	0.25E+00	0.21E+01	0.25E+00	11.9469
0.5	1.000	1	16.00	0.76E+00	0.49E+01	0.76E+00	15.4065
0.5	1.000	1	32.00	0.16E+01	0.10E+02	0.16E+01	15.6189
0.5	1.000	1	64.00	0.30E+01	0.20E+02	0.30E+01	14.8766
0.5	1.000	3	0.50	-0.46E+00	-0.43E+00	-0.14E+01	0.0000
0.5	1.000	3	1.00	-0.45E+00	-0.42E+00	-0.14E+01	0.0000
0.5	1.000	3	2.00	-0.43E+00	-0.33E+00	-0.13E+01	0.0000
0.5	1.000	3	4.00	-0.37E+00	-0.27E-01	-0.11E+01	0.0000
0.5	1.000	3	8.00	-0.21E+00	0.92E+00	-0.64E+00	0.0000
0.5	1.000	3	16.00	0.16E+00	0.35E+01	0.47E+00	13.5349
0.5	1.000	3	32.00	0.89E+00	0.92E+01	0.27E+01	28.8246
0.5	1.000	3	64.00	0.22E+01	0.20E+02	0.65E+01	31.7871
0.5	0.100	0	0.50	-0.55E-01	-0.17E+00	0.00E+00	0.0000
0.5	0.100	0	1.00	-0.24E-06	0.28E-05	0.00E+00	0.0000
0.5	0.100	0	2.00	0.11E+00	0.33E+00	0.00E+00	0.0000
0.5	0.100	0	4.00	0.33E+00	0.10E+01	0.00E+00	0.0000
0.5	0.100	0	8.00	0.75E+00	0.24E+01	0.00E+00	0.0000
0.5	0.100	0	16.00	0.16E+01	0.52E+01	0.00E+00	0.0000
0.5	0.100	0	32.00	0.31E+01	0.11E+02	0.00E+00	0.0000

TABLE B.1

$\frac{R}{R_D}$	ε_D	n_R	$\frac{T_1}{T_2}$	$\frac{I}{e}$	$\frac{\dot{Q}}{kT_D}$	$\frac{P_e}{kT_D}$	η
0.5	0.100	0	64.00	0.59E+01	0.22E+02	0.00E+00	0.0000
0.5	0.100	1	0.50	-0.38E+00	-0.18E+00	-0.38E-01	0.0000
0.5	0.100	1	1.00	-0.32E+00	-0.21E-01	-0.32E-01	0.0000
0.5	0.100	1	2.00	-0.22E+00	0.31E+00	-0.22E-01	0.0000
0.5	0.100	1	4.00	-0.54E-02	0.97E+00	-0.54E-03	0.0000
0.5	0.100	1	8.00	0.41E+00	0.23E+01	0.41E-01	1.7755
0.5	0.100	1	16.00	0.12E+01	0.51E+01	0.12E+00	2.3941
0.5	0.100	1	32.00	0.27E+01	0.11E+02	0.27E+00	2.5438
0.5	0.100	1	64.00	0.55E+01	0.22E+02	0.55E+00	2.4704
0.5	0.100	3	0.50	-0.98E+00	-0.27E+00	-0.29E+00	0.0000
0.5	0.100	3	1.00	-0.93E+00	-0.12E+00	-0.28E+00	0.0000
0.5	0.100	3	2.00	-0.83E+00	0.19E+00	-0.25E+00	0.0000
0.5	0.100	3	4.00	-0.63E+00	0.82E+00	-0.19E+00	0.0000
0.5	0.100	3	8.00	-0.23E+00	0.21E+01	-0.69E-01	0.0000
0.5	0.100	3	16.00	0.54E+00	0.48E+01	0.16E+00	3.4010
0.5	0.100	3	32.00	0.20E+01	0.10E+02	0.60E+00	5.8140
0.5	0.100	3	64.00	0.47E+01	0.22E+02	0.14E+01	6.4544
0.5	0.100	10	0.50	-0.26E+01	-0.90E+00	-0.26E+01	0.0000
0.5	0.100	10	1.00	-0.26E+01	-0.78E+00	-0.26E+01	0.0000
0.5	0.100	10	2.00	-0.25E+01	-0.54E+00	-0.25E+01	0.0000
0.5	0.100	10	4.00	-0.24E+01	-0.47E-01	-0.24E+01	0.0000
0.5	0.100	10	8.00	-0.21E+01	0.10E+01	-0.21E+01	0.0000
0.5	0.100	10	16.00	-0.14E+01	0.33E+01	-0.14E+01	0.0000
0.5	0.100	10	32.00	-0.16E+00	0.84E+01	-0.16E+00	0.0000
0.5	0.100	10	64.00	0.22E+01	0.20E+02	0.22E+01	11.4856
0.5	0.010	0	0.50	-0.55E-01	-0.17E+00	0.00E+00	0.0000
0.5	0.010	0	1.00	-0.15E-05	-0.23E-05	0.00E+00	0.0000
0.5	0.010	0	2.00	0.11E+00	0.33E+00	0.00E+00	0.0000
0.5	0.010	0	4.00	0.33E+00	0.10E+01	0.00E+00	0.0000
0.5	0.010	0	8.00	0.77E+00	0.23E+01	0.00E+00	0.0000
0.5	0.010	0	16.00	0.15E+01	0.53E+01	0.00E+00	0.0000
0.5	0.010	0	32.00	0.19E+01	0.14E+02	0.00E+00	0.0000
0.5	0.010	0	64.00	0.16E+01	0.40E+02	0.00E+00	0.0000
0.5	0.010	1	0.50	-0.39E+00	-0.17E+00	-0.39E-02	0.0000
0.5	0.010	1	1.00	-0.33E+00	-0.22E-02	-0.33E-02	0.0000
0.5	0.010	1	2.00	-0.22E+00	0.33E+00	-0.22E-02	0.0000
0.5	0.010	1	4.00	-0.57E-03	0.10E+01	-0.57E-05	0.0000
0.5	0.010	1	8.00	0.44E+00	0.23E+01	0.44E-02	0.1875
0.5	0.010	1	16.00	0.11E+01	0.53E+01	0.11E-01	0.2152
0.5	0.010	1	32.00	0.15E+01	0.14E+02	0.15E-01	0.1045
0.5	0.010	1	64.00	0.10E+01	0.40E+02	0.10E-01	0.0255
0.5	0.010	3	0.50	-0.10E+01	-0.18E+00	-0.31E-01	0.0000
0.5	0.010	3	1.00	-0.99E+00	-0.13E-01	-0.30E-01	0.0000
0.5	0.010	3	2.00	-0.88E+00	0.32E+00	-0.26E-01	0.0000
0.5	0.010	3	4.00	-0.66E+00	0.98E+00	-0.20E-01	0.0000
0.5	0.010	3	8.00	-0.23E+00	0.23E+01	-0.68E-02	0.0000
0.5	0.010	3	16.00	0.44E+00	0.53E+01	0.13E-01	0.2468

TABLE B.1

$\frac{R}{R_D}$	ε_D	n_R	$\frac{T_1}{T_2}$	$\frac{I}{e}$	$\frac{\dot{Q}}{kT_D}$	$\frac{P_o}{kT_D}$	η
0.5	0.010	3	32.00	0.56E+00	0.14E+02	0.17E-01	0.1184
0.5	0.010	3	64.00	-0.27E+00	0.39E+02	-0.81E-02	0.0000
0.5	0.010	10	0.50	-0.33E+01	-0.28E+00	-0.33E+00	0.0000
0.5	0.010	10	1.00	-0.33E+01	-0.12E+00	-0.33E+00	0.0000
0.5	0.010	10	2.00	-0.31E+01	0.21E+00	-0.31E+00	0.0000
0.5	0.010	10	4.00	-0.29E+01	0.86E+00	-0.29E+00	0.0000
0.5	0.010	10	8.00	-0.25E+01	0.22E+01	-0.25E+00	0.0000
0.5	0.010	10	16.00	-0.20E+01	0.52E+01	-0.20E+00	0.0000
0.5	0.010	10	32.00	-0.27E+01	0.14E+02	-0.27E+00	0.0000
0.5	0.010	10	64.00	-0.48E+01	0.38E+02	-0.48E+00	0.0000
0.5	0.010	30	0.50	-0.94E+01	-0.11E+01	-0.28E+01	0.0000
0.5	0.010	30	1.00	-0.93E+01	-0.90E+00	-0.28E+01	0.0000
0.5	0.010	30	2.00	-0.92E+01	-0.60E+00	-0.28E+01	0.0000
0.5	0.010	30	4.00	-0.90E+01	0.23E-01	-0.27E+01	0.0000
0.5	0.010	30	8.00	-0.87E+01	0.13E+01	-0.26E+01	0.0000
0.5	0.010	30	16.00	-0.92E+01	0.45E+01	-0.28E+01	0.0000
0.5	0.010	30	32.00	-0.13E+02	0.13E+02	-0.38E+01	0.0000
0.5	0.010	30	64.00	-0.19E+02	0.33E+02	-0.56E+01	0.0000
0.5	0.001	0	0.50	-0.55E-01	-0.17E+00	0.00E+00	0.0000
0.5	0.001	0	1.00	0.18E-04	0.16E-01	0.00E+00	0.0000
0.5	0.001	0	2.00	0.79E-01	0.49E+00	0.00E+00	0.0000
0.5	0.001	0	4.00	0.13E+00	0.19E+01	0.00E+00	0.0000
0.5	0.001	0	8.00	0.12E+00	0.54E+01	0.00E+00	0.0000
0.5	0.001	0	16.00	0.78E-01	0.13E+02	0.00E+00	0.0000
0.5	0.001	0	32.00	0.46E-01	0.29E+02	0.00E+00	0.0000
0.5	0.001	0	64.00	0.25E-01	0.61E+02	0.00E+00	0.0000
0.5	0.001	1	0.50	-0.39E+00	-0.17E+00	-0.39E-03	0.0000
0.5	0.001	1	1.00	-0.34E+00	0.16E-01	-0.34E-03	0.0000
0.5	0.001	1	2.00	-0.32E+00	0.49E+00	-0.32E-03	0.0000
0.5	0.001	1	4.00	-0.40E+00	0.19E+01	-0.40E-03	0.0000
0.5	0.001	1	8.00	-0.58E+00	0.53E+01	-0.58E-03	0.0000
0.5	0.001	1	16.00	-0.74E+00	0.13E+02	-0.74E-03	0.0000
0.5	0.001	1	32.00	-0.86E+00	0.29E+02	-0.86E-03	0.0000
0.5	0.001	1	64.00	-0.93E+00	0.60E+02	-0.93E-03	0.0000
0.5	0.001	3	0.50	-0.11E+01	-0.17E+00	-0.32E-02	0.0000
0.5	0.001	3	1.00	-0.10E+01	0.15E-01	-0.31E-02	0.0000
0.5	0.001	3	2.00	-0.11E+01	0.49E+00	-0.34E-02	0.0000
0.5	0.001	3	4.00	-0.15E+01	0.19E+01	-0.44E-02	0.0000
0.5	0.001	3	8.00	-0.20E+01	0.53E+01	-0.59E-02	0.0000
0.5	0.001	3	16.00	-0.24E+01	0.13E+02	-0.72E-02	0.0000
0.5	0.001	3	32.00	-0.27E+01	0.28E+02	-0.80E-02	0.0000
0.5	0.001	3	64.00	-0.28E+01	0.59E+02	-0.85E-02	0.0000
0.5	0.001	10	0.50	-0.34E+01	-0.18E+00	-0.34E-01	0.0000
0.5	0.001	10	1.00	-0.34E+01	0.60E-02	-0.34E-01	0.0000
0.5	0.001	10	2.00	-0.40E+01	0.48E+00	-0.40E-01	0.0000

TABLE B.1

$\frac{R}{R_D}$	ε_D	m_R	$\frac{T_1}{T_2}$	$\frac{I}{e}$	$\frac{\dot{Q}}{kT_D}$	$\frac{P_0}{kT_D}$	η
0.5	0.001	10	4.00	-0.53E+01	0.18E+01	-0.53E-01	0.0000
0.5	0.001	10	8.00	-0.69E+01	0.51E+01	-0.69E-01	0.0000
0.5	0.001	10	16.00	-0.82E+01	0.12E+02	-0.82E-01	0.0000
0.5	0.001	10	32.00	-0.91E+01	0.27E+02	-0.91E-01	0.0000
0.5	0.001	10	64.00	-0.95E+01	0.56E+02	-0.95E-01	0.0000
0.5	0.001	30	0.50	-0.10E+02	-0.26E+00	-0.30E+00	0.0000
0.5	0.001	30	1.00	-0.11E+02	-0.70E-01	-0.32E+00	0.0000
0.5	0.001	30	2.00	-0.13E+02	0.40E+00	-0.38E+00	0.0000
0.5	0.001	30	4.00	-0.17E+02	0.16E+01	-0.50E+00	0.0000
0.5	0.001	30	8.00	-0.22E+02	0.44E+01	-0.65E+00	0.0000
0.5	0.001	30	16.00	-0.25E+02	0.10E+02	-0.76E+00	0.0000
0.5	0.001	30	32.00	-0.28E+02	0.23E+02	-0.83E+00	0.0000
0.5	0.001	30	64.00	-0.29E+02	0.47E+02	-0.86E+00	0.0000
0.5	0.001	100	0.50	-0.44E+02	-0.13E+01	-0.44E+01	0.0000
0.5	0.001	100	1.00	-0.50E+02	-0.12E+01	-0.50E+01	0.0000
0.5	0.001	100	2.00	-0.59E+02	-0.97E+00	-0.59E+01	0.0000
0.5	0.001	100	4.00	-0.69E+02	-0.60E+00	-0.69E+01	0.0000
0.5	0.001	100	8.00	-0.80E+02	0.53E+00	-0.80E+01	0.0000
0.5	0.001	100	16.00	-0.89E+02	0.39E+01	-0.89E+01	0.0000
0.5	0.001	100	32.00	-0.95E+02	0.12E+02	-0.95E+01	0.0000
0.5	0.001	100	64.00	-0.97E+02	0.28E+02	-0.97E+01	0.0000
1.0	10.000	0	0.50	-0.56E-02	-0.22E-01	0.00E+00	0.0000
1.0	10.000	0	1.00	-0.93E-09	0.75E-07	0.00E+00	0.0000
1.0	10.000	0	2.00	0.59E-01	0.36E+00	0.00E+00	0.0000
1.0	10.000	0	4.00	0.19E+00	0.12E+01	0.00E+00	0.0000
1.0	10.000	0	8.00	0.37E+00	0.24E+01	0.00E+00	0.0000
1.0	10.000	0	16.00	0.62E+00	0.43E+01	0.00E+00	0.0000
1.0	10.000	0	32.00	0.98E+00	0.75E+01	0.00E+00	0.0000
1.0	10.000	0	64.00	0.15E+01	0.14E+02	0.00E+00	0.0000
1.0	10.000	1	0.50	-0.91E-01	-0.27E+00	-0.91E+00	0.0000
1.0	10.000	1	1.00	-0.90E-01	-0.45E+00	-0.90E+00	0.0000
1.0	10.000	1	2.00	-0.87E-01	-0.39E+00	-0.87E+00	0.0000
1.0	10.000	1	4.00	-0.70E-01	-0.11E+00	-0.70E+00	0.0000
1.0	10.000	1	8.00	0.55E-02	0.11E+01	0.55E-01	4.9654
1.0	10.000	1	16.00	0.19E+00	0.41E+01	0.19E+01	45.0319
1.0	10.000	1	32.00	0.49E+00	0.97E+01	0.49E+01	50.8840
1.0	10.000	1	64.00	0.97E+00	0.19E+02	0.97E+01	49.7058
1.0	1.000	0	0.50	-0.53E-01	-0.21E+00	0.00E+00	0.0000
1.0	1.000	0	1.00	-0.61E-09	0.13E-05	0.00E+00	0.0000
1.0	1.000	0	2.00	0.11E+00	0.45E+00	0.00E+00	0.0000
1.0	1.000	0	4.00	0.31E+00	0.13E+01	0.00E+00	0.0000
1.0	1.000	0	8.00	0.65E+00	0.30E+01	0.00E+00	0.0000
1.0	1.000	0	16.00	0.12E+01	0.61E+01	0.00E+00	0.0000
1.0	1.000	0	32.00	0.22E+01	0.11E+02	0.00E+00	0.0000
1.0	1.000	0	64.00	0.36E+01	0.21E+02	0.00E+00	0.0000
1.0	1.000	1	0.50	-0.45E+00	-0.45E+00	-0.45E+00	0.0000
1.0	1.000	1	1.00	-0.41E+00	-0.28E+00	-0.41E+00	0.0000

TABLE B.1

$\frac{R}{R_D}$	ε_D	n_R	$\frac{T_1}{T_2}$	$\frac{I}{e}$	$\frac{\dot{Q}}{kT_D}$	$\frac{P_e}{kT_D}$	η
1.0	1.000	1	2.00	-0.31E+00	0.15E+00	-0.31E+00	0.0000
1.0	1.000	1	4.00	-0.13E+00	0.10E+01	-0.13E+00	0.0000
1.0	1.000	1	8.00	0.20E+00	0.28E+01	0.20E+00	7.2944
1.0	1.000	1	16.00	0.76E+00	0.60E+01	0.76E+00	12.6534
1.0	1.000	1	32.00	0.17E+01	0.12E+02	0.17E+01	14.0661
1.0	1.000	1	64.00	0.31E+01	0.22E+02	0.31E+01	14.0489
1.0	1.000	3	0.50	-0.86E+00	-0.10E+01	-0.26E+01	0.0000
1.0	1.000	3	1.00	-0.85E+00	-0.11E+01	-0.25E+01	0.0000
1.0	1.000	3	2.00	-0.80E+00	-0.83E+00	-0.24E+01	0.0000
1.0	1.000	3	4.00	-0.69E+00	-0.21E+00	-0.21E+01	0.0000
1.0	1.000	3	8.00	-0.46E+00	0.13E+01	-0.14E+01	0.0000
1.0	1.000	3	16.00	0.16E-02	0.46E+01	0.49E-02	0.1068
1.0	1.000	3	32.00	0.82E+00	0.11E+02	0.25E+01	22.2280
1.0	1.000	3	64.00	0.22E+01	0.23E+02	0.65E+01	28.4330
1.0	0.100	0	0.50	-0.61E-01	-0.25E+00	0.00E+00	0.0000
1.0	0.100	0	1.00	0.81E-07	0.28E-05	0.00E+00	0.0000
1.0	0.100	0	2.00	0.12E+00	0.49E+00	0.00E+00	0.0000
1.0	0.100	0	4.00	0.37E+00	0.15E+01	0.00E+00	0.0000
1.0	0.100	0	8.00	0.84E+00	0.34E+01	0.00E+00	0.0000
1.0	0.100	0	16.00	0.17E+01	0.72E+01	0.00E+00	0.0000
1.0	0.100	0	32.00	0.34E+01	0.14E+02	0.00E+00	0.0000
1.0	0.100	0	64.00	0.64E+01	0.28E+02	0.00E+00	0.0000
1.0	0.100	1	0.50	-0.55E+00	-0.28E+00	-0.55E-01	0.0000
1.0	0.100	1	1.00	-0.49E+00	-0.36E-01	-0.49E-01	0.0000
1.0	0.100	1	2.00	-0.37E+00	0.46E+00	-0.37E-01	0.0000
1.0	0.100	1	4.00	-0.13E+00	0.14E+01	-0.13E-01	0.0000
1.0	0.100	1	8.00	0.35E+00	0.34E+01	0.35E-01	1.0242
1.0	0.100	1	16.00	0.12E+01	0.72E+01	0.12E+00	1.7397
1.0	0.100	1	32.00	0.29E+01	0.14E+02	0.29E+00	2.0169
1.0	0.100	1	64.00	0.58E+01	0.28E+02	0.58E+00	2.0866
1.0	0.100	3	0.50	-0.15E+01	-0.49E+00	-0.45E+00	0.0000
1.0	0.100	3	1.00	-0.14E+01	-0.24E+00	-0.43E+00	0.0000
1.0	0.100	3	2.00	-0.13E+01	0.25E+00	-0.39E+00	0.0000
1.0	0.100	3	4.00	-0.11E+01	0.12E+01	-0.32E+00	0.0000
1.0	0.100	3	8.00	-0.60E+00	0.32E+01	-0.18E+00	0.0000
1.0	0.100	3	16.00	0.29E+00	0.69E+01	0.87E-01	1.2544
1.0	0.100	3	32.00	0.19E+01	0.14E+02	0.58E+00	4.0852
1.0	0.100	3	64.00	0.49E+01	0.28E+02	0.15E+01	5.2044
1.0	0.100	10	0.50	-0.44E+01	-0.22E+01	-0.44E+01	0.0000
1.0	0.100	10	1.00	-0.43E+01	-0.20E+01	-0.43E+01	0.0000
1.0	0.100	10	2.00	-0.42E+01	-0.15E+01	-0.42E+01	0.0000
1.0	0.100	10	4.00	-0.40E+01	-0.61E+00	-0.40E+01	0.0000
1.0	0.100	10	8.00	-0.35E+01	0.12E+01	-0.35E+01	0.0000
1.0	0.100	10	16.00	-0.27E+01	0.49E+01	-0.27E+01	0.0000
1.0	0.100	10	32.00	-0.11E+01	0.12E+02	-0.11E+01	0.0000
1.0	0.100	10	64.00	0.17E+01	0.26E+02	0.17E+01	6.4018
1.0	0.010	0	0.50	-0.62E-01	-0.25E+00	0.00E+00	0.0000
1.0	0.010	0	1.00	0.20E-05	-0.23E-05	0.00E+00	0.0000

TABLE B.1

$\frac{R}{R_D}$	$\mathcal{E}_D$	M_R	$\frac{T_1}{T_2}$	$\frac{I}{e}$	$\frac{\dot{Q}}{RT_D}$	$\frac{P_0}{RT_D}$	η
1.0	0.010	0	2.00	0.12E+00	0.50E+00	0.00E+00	0.0000
1.0	0.010	0	4.00	0.37E+00	0.15E+01	0.00E+00	0.0000
1.0	0.010	0	8.00	0.87E+00	0.35E+01	0.00E+00	0.0000
1.0	0.010	0	16.00	0.18E+01	0.75E+01	0.00E+00	0.0000
1.0	0.010	0	32.00	0.28E+01	0.17E+02	0.00E+00	0.0000
1.0	0.010	0	64.00	0.27E+01	0.42E+02	0.00E+00	0.0000
1.0	0.010	1	0.50	-0.56E+00	-0.25E+00	-0.56E-02	0.0000
1.0	0.010	1	1.00	-0.50E+00	-0.37E-02	-0.50E-02	0.0000
1.0	0.010	1	2.00	-0.37E+00	0.50E+00	-0.37E-02	0.0000
1.0	0.010	1	4.00	-0.13E+00	0.15E+01	-0.13E-02	0.0000
1.0	0.010	1	8.00	0.37E+00	0.35E+01	0.37E-02	0.1065
1.0	0.010	1	16.00	0.13E+01	0.75E+01	0.13E-01	0.1706
1.0	0.010	1	32.00	0.22E+01	0.17E+02	0.22E-01	0.1286
1.0	0.010	1	64.00	0.20E+01	0.42E+02	0.20E-01	0.0488
1.0	0.010	3	0.50	-0.16E+01	-0.28E+00	-0.47E-01	0.0000
1.0	0.010	3	1.00	-0.15E+01	-0.26E-01	-0.45E-01	0.0000
1.0	0.010	3	2.00	-0.14E+01	0.47E+00	-0.41E-01	0.0000
1.0	0.010	3	4.00	-0.11E+01	0.15E+01	-0.34E-01	0.0000
1.0	0.010	3	8.00	-0.62E+00	0.35E+01	-0.19E-01	0.0000
1.0	0.010	3	16.00	0.28E+00	0.75E+01	0.84E-02	0.1119
1.0	0.010	3	32.00	0.11E+01	0.17E+02	0.33E-01	0.1907
1.0	0.010	3	64.00	0.69E+00	0.42E+02	0.21E-01	0.0495
1.0	0.010	10	0.50	-0.50E+01	-0.50E+00	-0.50E+00	0.0000
1.0	0.010	10	1.00	-0.49E+01	-0.26E+00	-0.49E+00	0.0000
1.0	0.010	10	2.00	-0.48E+01	0.24E+00	-0.48E+00	0.0000
1.0	0.010	10	4.00	-0.46E+01	0.12E+01	-0.46E+00	0.0000
1.0	0.010	10	8.00	-0.41E+01	0.32E+01	-0.41E+00	0.0000
1.0	0.010	10	16.00	-0.32E+01	0.73E+01	-0.32E+00	0.0000
1.0	0.010	10	32.00	-0.28E+01	0.17E+02	-0.28E+00	0.0000
1.0	0.010	10	64.00	-0.41E+01	0.41E+02	-0.41E+00	0.0000
1.0	0.010	30	0.50	-0.14E+02	-0.23E+01	-0.43E+01	0.0000
1.0	0.010	30	1.00	-0.14E+02	-0.21E+01	-0.43E+01	0.0000
1.0	0.010	30	2.00	-0.14E+02	-0.16E+01	-0.43E+01	0.0000
1.0	0.010	30	4.00	-0.14E+02	-0.63E+00	-0.42E+01	0.0000
1.0	0.010	30	8.00	-0.14E+02	0.14E+01	-0.41E+01	0.0000
1.0	0.010	30	16.00	-0.13E+02	0.55E+01	-0.39E+01	0.0000
1.0	0.010	30	32.00	-0.14E+02	0.15E+02	-0.43E+01	0.0000
1.0	0.010	30	64.00	-0.18E+02	0.36E+02	-0.55E+01	0.0000
1.0	0.001	0	0.50	-0.61E-01	-0.25E+00	0.00E+00	0.0000
1.0	0.001	0	1.00	-0.37E-05	0.16E-01	0.00E+00	0.0000
1.0	0.001	0	2.00	0.96E-01	0.61E+00	0.00E+00	0.0000
1.0	0.001	0	4.00	0.18E+00	0.21E+01	0.00E+00	0.0000
1.0	0.001	0	8.00	0.19E+00	0.55E+01	0.00E+00	0.0000
1.0	0.001	0	16.00	0.14E+00	0.13E+02	0.00E+00	0.0000
1.0	0.001	0	32.00	0.86E-01	0.29E+02	0.00E+00	0.0000
1.0	0.001	0	64.00	0.48E-01	0.61E+02	0.00E+00	0.0000
1.0	0.001	1	0.50	-0.56E+00	-0.25E+00	-0.56E-03	0.0000

TABLE B.1

$\frac{R}{R_D}$	ε_D	n_R	$\frac{T_1}{T_2}$	$\frac{I}{e}$	$\frac{\dot{Q}}{RT_D}$	$\frac{P_o}{RT_D}$	η
1.0	0.001	1	1.00	-0.51E+00	0.16E-01	-0.51E-03	0.0000
1.0	0.001	1	2.00	-0.44E+00	0.61E+00	-0.44E-03	0.0000
1.0	0.001	1	4.00	-0.43E+00	0.21E+01	-0.43E-03	0.0000
1.0	0.001	1	8.00	-0.54E+00	0.55E+01	-0.54E-03	0.0000
1.0	0.001	1	16.00	-0.69E+00	0.13E+02	-0.69E-03	0.0000
1.0	0.001	1	32.00	-0.82E+00	0.29E+02	-0.82E-03	0.0000
1.0	0.001	1	64.00	-0.90E+00	0.60E+02	-0.90E-03	0.0000
1.0	0.001	3	0.50	-0.16E+01	-0.25E+00	-0.47E-02	0.0000
1.0	0.001	3	1.00	-0.15E+01	0.14E-01	-0.46E-02	0.0000
1.0	0.001	3	2.00	-0.15E+01	0.60E+00	-0.45E-02	0.0000
1.0	0.001	3	4.00	-0.17E+01	0.20E+01	-0.50E-02	0.0000
1.0	0.001	3	8.00	-0.20E+01	0.54E+01	-0.60E-02	0.0000
1.0	0.001	3	16.00	-0.24E+01	0.13E+02	-0.71E-02	0.0000
1.0	0.001	3	32.00	-0.26E+01	0.28E+02	-0.79E-02	0.0000
1.0	0.001	3	64.00	-0.28E+01	0.59E+02	-0.84E-02	0.0000
1.0	0.001	10	0.50	-0.51E+01	-0.27E+00	-0.51E-01	0.0000
1.0	0.001	10	1.00	-0.51E+01	-0.92E-02	-0.51E-01	0.0000
1.0	0.001	10	2.00	-0.53E+01	0.58E+00	-0.53E-01	0.0000
1.0	0.001	10	4.00	-0.60E+01	0.20E+01	-0.60E-01	0.0000
1.0	0.001	10	8.00	-0.72E+01	0.53E+01	-0.72E-01	0.0000
1.0	0.001	10	16.00	-0.83E+01	0.12E+02	-0.83E-01	0.0000
1.0	0.001	10	32.00	-0.90E+01	0.27E+02	-0.90E-01	0.0000
1.0	0.001	10	64.00	-0.95E+01	0.56E+02	-0.95E-01	0.0000
1.0	0.001	30	0.50	-0.15E+02	-0.47E+00	-0.45E+00	0.0000
1.0	0.001	30	1.00	-0.15E+02	-0.21E+00	-0.46E+00	0.0000
1.0	0.001	30	2.00	-0.16E+02	0.37E+00	-0.49E+00	0.0000
1.0	0.001	30	4.00	-0.19E+02	0.17E+01	-0.56E+00	0.0000
1.0	0.001	30	8.00	-0.22E+02	0.46E+01	-0.67E+00	0.0000
1.0	0.001	30	16.00	-0.26E+02	0.11E+02	-0.77E+00	0.0000
1.0	0.001	30	32.00	-0.28E+02	0.23E+02	-0.83E+00	0.0000
1.0	0.001	30	64.00	-0.29E+02	0.48E+02	-0.86E+00	0.0000
1.0	0.001	100	0.50	-0.10E+03	-0.20E+02	-0.10E+02	0.0000
1.0	0.001	100	1.00	-0.56E+02	-0.25E+01	-0.56E+01	0.0000
1.0	0.001	100	2.00	-0.61E+02	-0.21E+01	-0.61E+01	0.0000
1.0	0.001	100	4.00	-0.70E+02	-0.13E+01	-0.70E+01	0.0000
1.0	0.001	100	8.00	-0.80E+02	0.33E+00	-0.80E+01	0.0000
1.0	0.001	100	16.00	-0.89E+02	0.43E+01	-0.89E+01	0.0000
1.0	0.001	100	32.00	-0.94E+02	0.13E+02	-0.94E+01	0.0000
1.0	0.001	100	64.00	-0.97E+02	0.29E+02	-0.97E+01	0.0000
2.0	10.000	0	0.50	-0.53E-02	-0.12E-01	0.00E+00	0.0000
2.0	10.000	0	1.00	0.47E-09	0.82E-07	0.00E+00	0.0000
2.0	10.000	0	2.00	0.56E-01	0.40E+00	0.00E+00	0.0000
2.0	10.000	0	4.00	0.19E+00	0.13E+01	0.00E+00	0.0000
2.0	10.000	0	8.00	0.37E+00	0.27E+01	0.00E+00	0.0000
2.0	10.000	0	16.00	0.63E+00	0.47E+01	0.00E+00	0.0000
2.0	10.000	0	32.00	0.10E+01	0.90E+01	0.00E+00	0.0000
2.0	10.000	0	64.00	0.15E+01	0.14E+02	0.00E+00	0.0000

TABLE B.1

$\frac{R}{R_D}$	ε_D	n_R	$\frac{T_1}{T_2}$	$\frac{I}{e}$	$\frac{\dot{Q}}{k T_D}$	$\frac{P_o}{k T_D}$	η
2.0	10.000	1	0.50	-0.17E+00	-0.16E+00	-0.17E+01	0.0000
2.0	10.000	1	1.00	-0.17E+00	-0.82E+00	-0.17E+01	0.0000
2.0	10.000	1	2.00	-0.16E+00	-0.72E+00	-0.16E+01	0.0000
2.0	10.000	1	4.00	-0.14E+00	-0.30E+00	-0.14E+01	0.0000
2.0	10.000	1	8.00	-0.47E-01	0.12E+01	-0.47E+00	0.0000
2.0	10.000	1	16.00	0.15E+00	0.47E+01	0.15E+01	32.4016
2.0	10.000	1	32.00	0.48E+00	0.11E+02	0.48E+01	45.0599
2.0	10.000	1	64.00	0.98E+00	0.21E+02	0.98E+01	47.5481
2.0	1.000	0	0.50	-0.48E-01	-0.28E+00	0.00E+00	0.0000
2.0	1.000	0	1.00	0.77E-07	0.13E-05	0.00E+00	0.0000
2.0	1.000	0	2.00	0.10E+00	0.59E+00	0.00E+00	0.0000
2.0	1.000	0	4.00	0.29E+00	0.17E+01	0.00E+00	0.0000
2.0	1.000	0	8.00	0.63E+00	0.38E+01	0.00E+00	0.0000
2.0	1.000	0	16.00	0.12E+01	0.74E+01	0.00E+00	0.0000
2.0	1.000	0	32.00	0.22E+01	0.14E+02	0.00E+00	0.0000
2.0	1.000	0	64.00	0.36E+01	0.24E+02	0.00E+00	0.0000
2.0	1.000	1	0.50	-0.63E+00	-0.68E+00	-0.63E+00	0.0000
2.0	1.000	1	1.00	-0.58E+00	-0.44E+00	-0.58E+00	0.0000
2.0	1.000	1	2.00	-0.48E+00	0.17E+00	-0.48E+00	0.0000
2.0	1.000	1	4.00	-0.29E+00	0.14E+01	-0.29E+00	0.0000
2.0	1.000	1	8.00	0.55E-01	0.36E+01	0.55E-01	1.5075
2.0	1.000	1	16.00	0.64E+00	0.76E+01	0.64E+00	8.4001
2.0	1.000	1	32.00	0.16E+01	0.14E+02	0.16E+01	10.9504
2.0	1.000	1	64.00	0.31E+01	0.26E+02	0.31E+01	11.7838
2.0	1.000	3	0.50	-0.14E+01	-0.20E+01	-0.43E+01	0.0000
2.0	1.000	3	1.00	-0.14E+01	-0.24E+01	-0.43E+01	0.0000
2.0	1.000	3	2.00	-0.14E+01	-0.19E+01	-0.41E+01	0.0000
2.0	1.000	3	4.00	-0.12E+01	-0.77E+00	-0.36E+01	0.0000
2.0	1.000	3	8.00	-0.88E+00	0.16E+01	-0.26E+01	0.0000
2.0	1.000	3	16.00	-0.34E+00	0.59E+01	-0.10E+01	0.0000
2.0	1.000	3	32.00	0.56E+00	0.14E+02	0.17E+01	12.2637
2.0	1.000	3	64.00	0.20E+01	0.27E+02	0.60E+01	21.9728
2.0	0.100	0	0.50	-0.55E-01	-0.33E+00	0.00E+00	0.0000
2.0	0.100	0	1.00	0.93E-07	0.27E-05	0.00E+00	0.0000
2.0	0.100	0	2.00	0.11E+00	0.66E+00	0.00E+00	0.0000
2.0	0.100	0	4.00	0.33E+00	0.20E+01	0.00E+00	0.0000
2.0	0.100	0	8.00	0.76E+00	0.45E+01	0.00E+00	0.0000
2.0	0.100	0	16.00	0.16E+01	0.95E+01	0.00E+00	0.0000
2.0	0.100	0	32.00	0.32E+01	0.19E+02	0.00E+00	0.0000
2.0	0.100	0	64.00	0.60E+01	0.36E+02	0.00E+00	0.0000
2.0	0.100	1	0.50	-0.71E+00	-0.38E+00	-0.71E-01	0.0000
2.0	0.100	1	1.00	-0.66E+00	-0.54E-01	-0.66E-01	0.0000
2.0	0.100	1	2.00	-0.55E+00	0.61E+00	-0.55E-01	0.0000
2.0	0.100	1	4.00	-0.33E+00	0.19E+01	-0.33E-01	0.0000
2.0	0.100	1	8.00	0.11E+00	0.45E+01	0.11E-01	0.2358
2.0	0.100	1	16.00	0.95E+00	0.95E+01	0.95E-01	0.9968
2.0	0.100	1	32.00	0.25E+01	0.19E+02	0.25E+00	1.3402

TABLE B.1

$\frac{R}{R_D}$	ε_D	n_R	$\frac{T_1}{T_2}$	$\frac{I}{e}$	$\frac{\dot{Q}}{kT_D}$	$\frac{P_0}{kT_D}$	η
2.0	0.100	1	64.00	0.54E+01	0.36E+02	0.54E+00	1.4999
2.0	0.100	3	0.50	-0.20E+01	-0.74E+00	-0.60E+00	0.0000
2.0	0.100	3	1.00	-0.20E+01	-0.41E+00	-0.59E+00	0.0000
2.0	0.100	3	2.00	-0.18E+01	0.26E+00	-0.55E+00	0.0000
2.0	0.100	3	4.00	-0.16E+01	0.16E+01	-0.48E+00	0.0000
2.0	0.100	3	8.00	-0.12E+01	0.42E+01	-0.35E+00	0.0000
2.0	0.100	3	16.00	-0.32E+00	0.93E+01	-0.96E-01	0.0000
2.0	0.100	3	32.00	0.13E+01	0.19E+02	0.38E+00	2.0344
2.0	0.100	3	64.00	0.42E+01	0.36E+02	0.13E+01	3.4437
2.0	0.100	10	0.50	-0.62E+01	-0.38E+01	-0.62E+01	0.0000
2.0	0.100	10	1.00	-0.62E+01	-0.39E+01	-0.62E+01	0.0000
2.0	0.100	10	2.00	-0.61E+01	-0.33E+01	-0.61E+01	0.0000
2.0	0.100	10	4.00	-0.59E+01	-0.18E+01	-0.59E+01	0.0000
2.0	0.100	10	8.00	-0.54E+01	0.95E+00	-0.54E+01	0.0000
2.0	0.100	10	16.00	-0.45E+01	0.63E+01	-0.45E+01	0.0000
2.0	0.100	10	32.00	-0.29E+01	0.16E+02	-0.29E+01	0.0000
2.0	0.100	10	64.00	0.93E-01	0.35E+02	0.93E-01	0.2640
2.0	0.010	0	0.50	-0.55E-01	-0.33E+00	0.00E+00	0.0000
2.0	0.010	0	1.00	0.19E-05	-0.22E-05	0.00E+00	0.0000
2.0	0.010	0	2.00	0.11E+00	0.67E+00	0.00E+00	0.0000
2.0	0.010	0	4.00	0.33E+00	0.20E+01	0.00E+00	0.0000
2.0	0.010	0	8.00	0.78E+00	0.47E+01	0.00E+00	0.0000
2.0	0.010	0	16.00	0.16E+01	0.99E+01	0.00E+00	0.0000
2.0	0.010	0	32.00	0.30E+01	0.21E+02	0.00E+00	0.0000
2.0	0.010	0	64.00	0.38E+01	0.46E+02	0.00E+00	0.0000
2.0	0.010	1	0.50	-0.72E+00	-0.34E+00	-0.72E-02	0.0000
2.0	0.010	1	1.00	-0.67E+00	-0.55E-02	-0.67E-02	0.0000
2.0	0.010	1	2.00	-0.55E+00	0.66E+00	-0.55E-02	0.0000
2.0	0.010	1	4.00	-0.33E+00	0.20E+01	-0.33E-02	0.0000
2.0	0.010	1	8.00	0.11E+00	0.47E+01	0.11E-02	0.0238
2.0	0.010	1	16.00	0.98E+00	0.10E+02	0.98E-02	0.0999
2.0	0.010	1	32.00	0.23E+01	0.21E+02	0.23E-01	0.1122
2.0	0.010	1	64.00	0.31E+01	0.46E+02	0.31E-01	0.0671
2.0	0.010	3	0.50	-0.21E+01	-0.38E+00	-0.62E-01	0.0000
2.0	0.010	3	1.00	-0.20E+01	-0.43E-01	-0.60E-01	0.0000
2.0	0.010	3	2.00	-0.19E+01	0.62E+00	-0.57E-01	0.0000
2.0	0.010	3	4.00	-0.17E+01	0.20E+01	-0.50E-01	0.0000
2.0	0.010	3	8.00	-0.12E+01	0.46E+01	-0.37E-01	0.0000
2.0	0.010	3	16.00	-0.34E+00	0.99E+01	-0.10E-01	0.0000
2.0	0.010	3	32.00	0.99E+00	0.21E+02	0.30E-01	0.1421
2.0	0.010	3	64.00	0.16E+01	0.46E+02	0.48E-01	0.1048
2.0	0.010	10	0.50	-0.67E+01	-0.78E+00	-0.67E+00	0.0000
2.0	0.010	10	1.00	-0.66E+01	-0.45E+00	-0.66E+00	0.0000
2.0	0.010	10	2.00	-0.65E+01	0.22E+00	-0.65E+00	0.0000
2.0	0.010	10	4.00	-0.63E+01	0.16E+01	-0.63E+00	0.0000
2.0	0.010	10	8.00	-0.58E+01	0.43E+01	-0.58E+00	0.0000
2.0	0.010	10	16.00	-0.50E+01	0.96E+01	-0.50E+00	0.0000

TABLE B.1

$\frac{R}{R_D}$	ε_y	n_K	$\frac{n_1}{n_2}$	$\frac{I}{C}$	$\frac{Q}{\varepsilon r_D}$	$\frac{P_2}{\varepsilon r_D}$	φ_2
2.0	0.010	10	32.00	-0.37E+01	0.21E+02	-0.37E+00	0.0000
2.0	0.010	10	64.00	-0.35E+01	0.45E+02	-0.36E+00	0.0000
2.0	0.010	30	0.50	-0.19E+02	-0.36E+01	-0.58E+01	0.0000
2.0	0.010	30	1.00	-0.20E+02	-0.39E+01	-0.59E+01	0.0000
2.0	0.010	30	2.00	-0.20E+02	-0.32E+01	-0.59E+01	0.0000
2.0	0.010	30	4.00	-0.19E+02	-0.18E+01	-0.58E+01	0.0000
2.0	0.010	30	8.00	-0.19E+02	0.91E+00	-0.57E+01	0.0000
2.0	0.010	30	16.00	-0.18E+02	0.61E+01	-0.54E+01	0.0000
2.0	0.010	30	32.00	-0.17E+02	0.18E+02	-0.52E+01	0.0000
2.0	0.010	30	64.00	-0.19E+02	0.41E+02	-0.56E+01	0.0000
2.0	0.001	0	0.50	-0.54E-01	-0.33E+00	0.00E+00	0.0000
2.0	0.001	0	1.00	-0.56E-05	0.16E-01	0.00E+00	0.0000
2.0	0.001	0	2.00	0.91E-01	0.73E+00	0.00E+00	0.0000
2.0	0.001	0	4.00	0.20E+00	0.23E+01	0.00E+00	0.0000
2.0	0.001	0	8.00	0.26E+00	0.58E+01	0.00E+00	0.0000
2.0	0.001	0	16.00	0.23E+00	0.13E+02	0.00E+00	0.0000
2.0	0.001	0	32.00	0.15E+00	0.29E+02	0.00E+00	0.0000
2.0	0.001	0	64.00	0.90E-01	0.61E+02	0.00E+00	0.0000
2.0	0.001	1	0.50	-0.72E+00	-0.33E+00	-0.72E-03	0.0000
2.0	0.001	1	1.00	-0.67E+00	0.16E-01	-0.67E-03	0.0000
2.0	0.001	1	2.00	-0.59E+00	0.73E+00	-0.59E-03	0.0000
2.0	0.001	1	4.00	-0.51E+00	0.23E+01	-0.51E-03	0.0000
2.0	0.001	1	8.00	-0.52E+00	0.58E+01	-0.52E-03	0.0000
2.0	0.001	1	16.00	-0.63E+00	0.13E+02	-0.63E-03	0.0000
2.0	0.001	1	32.00	-0.76E+00	0.29E+02	-0.76E-03	0.0000
2.0	0.001	1	64.00	-0.86E+00	0.60E+02	-0.86E-03	0.0000
2.0	0.001	3	0.50	-0.21E+01	-0.33E+00	-0.62E-02	0.0000
2.0	0.001	3	1.00	-0.20E+01	0.12E-01	-0.60E-02	0.0000
2.0	0.001	3	2.00	-0.20E+01	0.73E+00	-0.59E-02	0.0000
2.0	0.001	3	4.00	-0.19E+01	0.23E+01	-0.58E-02	0.0000
2.0	0.001	3	8.00	-0.21E+01	0.57E+01	-0.62E-02	0.0000
2.0	0.001	3	16.00	-0.23E+01	0.13E+02	-0.70E-02	0.0000
2.0	0.001	3	32.00	-0.26E+01	0.28E+02	-0.78E-02	0.0000
2.0	0.001	3	64.00	-0.28E+01	0.59E+02	-0.83E-02	0.0000
2.0	0.001	10	0.50	-0.67E+01	-0.37E+00	-0.67E-01	0.0000
2.0	0.001	10	1.00	-0.67E+01	-0.29E-01	-0.67E-01	0.0000
2.0	0.001	10	2.00	-0.68E+01	0.69E+00	-0.68E-01	0.0000
2.0	0.001	10	4.00	-0.70E+01	0.22E+01	-0.70E-01	0.0000
2.0	0.001	10	8.00	-0.76E+01	0.56E+01	-0.76E-01	0.0000
2.0	0.001	10	16.00	-0.84E+01	0.13E+02	-0.84E-01	0.0000
2.0	0.001	10	32.00	-0.90E+01	0.27E+02	-0.90E-01	0.0000
2.0	0.001	10	64.00	-0.95E+01	0.56E+02	-0.95E-01	0.0000
2.0	0.001	30	0.50	-0.20E+02	-0.73E+00	-0.60E+00	0.0000
2.0	0.001	30	1.00	-0.20E+02	-0.39E+00	-0.61E+00	0.0000
2.0	0.001	30	2.00	-0.20E+02	0.32E+00	-0.61E+00	0.0000
2.0	0.001	30	4.00	-0.21E+02	0.18E+01	-0.64E+00	0.0000
2.0	0.001	30	8.00	-0.23E+02	0.50E+01	-0.70E+00	0.0000
2.0	0.001	30	16.00	-0.26E+02	0.11E+02	-0.77E+00	0.0000

TABLE B.1

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