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A real business cycle model with government

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A REAL BUSINESS CYCLE MODEL

WITH GOVERNMENT

by

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Abstract

A REAL BUSINESS CYCLE MODEL WITH GOVERNMENT

by

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The purpose of this paper is to analyze the impact of an exogenous government sector on a traditional real business cycle model. Since an exact solution for the specified artificial economy does not exist, the model is calibrated and a linear-quadratic approximation around its steady state is considered. Linear decision rules for this approximated economy are then derived and utilized to simulate time paths for the relevant aggregate variables. The results will show that the presence of a government sector contributes to enhance the performance of the model, as measured by its ability to mimic the fluctuations observed from U.S. real time series.

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I. INTRODUCTION

The term Business Cycle is used to refer to the short-run fluctuations commonly observed in major macroeconomic variables, particularly output and employment. As suggested by Long and Plosser [1983], two key features seem to characterize all business cycles:

(1) when measured as deviations from their trend, economic time series display substantial persistence in their movements, in the sense that if a variable is currently above (below) its trend value, it tends to remain above (below) its trend for some time; and

(2) most real economic time series tend to move together, i.e. when a variable is above (below) its trend value, other variables also tend to be above (below) their trend value.

The trend of an aggregate time series is defined as the slowly varying path around which fluctuations are measured. In turn, the measure of this path is defined by the method used to fit a smooth curve through the data.

Business Cycles represent a phenomenon which characterizes virtually all market economies. As a result, they have been a major topic of discussion for many years. Several theories have been put forward to explain what is the driving force behind these recurring, stationary fluctuations. In line with the major developments of general macroeconomic analysis, three basic approaches to the study of business cycles can be discerned. They are the Classical, the Keynesian and the Rational Expectations, or modern, approach.

1.1 Theories of Business Cycles

Until the 1930s, classical economics represented the dominant economic thinking. Because of the emphasis placed on long run equilibria, in general, and on full-employment growth processes which would culminate in the emergence of "stationary states", in particular, cyclical disturbances were largely ignored. However, the presence of persistent cyclical fluctuations in the economy led some reputed economists to concentrate on business cycles' issues. A variety of theories was thus developed, including Hawthrey's "Purely Monetary Theory", Hayek's "Over-Investment Theory", Schumpeter and Hansen's "Innovational Investment Theories", Marshall's "Psychological Theory", Pigou, Robertson and Clark's "Harvest Theories" and Mitchell's "Profit Theory". Most of these theories were based upon self-sustaining cycles' models, that is models in which booms contained the seeds for subsequent slumps and vice-versa. In other words, these models did not provide a theoretical framework within which to analyze the occurrence of business cycles. Instead, they focused on explaining certain specific empirical regularities. It is in this spirit that Mitchell and Burns began a systematic collection of business cycles facts, which culminated in 1946 in the publication of the book "Measuring Business Cycles". The major purpose of this volume was to provide a description of the behavior of a large number of price and quantity variables relative to the stages of the cycle.

Around the same time, Keynesian economics made its introduction and established itself as the new economic way of thinking. Keynesian economics did not contain *per se* a theory of business cycles. In fact, the emphasis placed on short run equilibrium analysis essentially made it a theory which was static in nature and, as such, concerned with neither growth nor fluctuations issues. Obviously, certain dynamic elements were present, since the movements of the economy from one short run equilibrium to the next had to be explained. However, these dynamic elements were always introduced using behavioral rules which resulted from best-fitting analysis rather than microeconomic considerations. Modern business cycle theorists believe this to be a major drawback of Keynesian economics. Charles Plosser [1989], for instance, claims that "...the absence of a foundation based on the choice theoretic framework of microeconomics is the major flaw in the Keynesian approach to macroeconomic phenomena".

As with classical economists, some Keynesian scholars did focus their attention on the study of business cycles. Their emphasis, however, was no longer on the analysis of stylized facts (i.e. the Burns-Mitchell approach) but rather on the development of structural models. In particular, the aim of these researchers was to specify and estimate individual blocks of general equilibrium models, which, once disturbed, could give rise to aggregate fluctuations similar to those encountered historically. This was the core of the work conducted, for instance, by Adelman and Adelman in 1959, whereby they proved that, under certain conditions, the replication of cyclical behavior was indeed possible.

In the 1970s, the development of Rational Expectations theory led to a forceful criticism of the Keynesian approach to macroeconomics. With respect to the study of business cycles' phenomena, rational expectations theorists advocated a return to small general equilibrium models aimed at explaining the major features of cycles. They argued that the validation of these models had to focus on their stochastic properties rather than the empirical fit of individual equations. Unlike early classical business cycles theorists who, being unclear as to the sources of the fluctuations, had primarily developed self-sustaining cycles' models, rational expectations theorists emphasized a new approach, based on the concepts of impulse and propagation mechanisms. In this approach, serially uncorrelated shocks (impulses) affect output through distributed lag relations (propagation mechanism) thereby leading to serially correlated fluctuations in output. While this general framework of analysis is accepted by most rational expectations theorists, there are widespread disagreements as to the source of the disturbances and the nature of the propagation mechanism. In fact, two major directions of research have developed. The first has its roots in Keynesian macroeconomics, in the sense that it emphasizes the role played by Aggregate Demand shocks, as generated by market imperfections, in explaining aggregate fluctuations. The second and more recent approach, which has become known as the Real Business Cycle (RBC) Theory, emphasizes the role played by real shocks, in particular technological shocks, in generating aggregate fluctuations in a competitive environment. The assumption of technological shocks finds justification in the observation that U.S. post-war time series indicate that the change in GNP can not be entirely accounted for by changes in inputs, specifically labor and capital. Following Solow [1957], the residual, that is the unexplained portion of the change in GNP, has been interpreted as representing a technological shock.

The serial correlation and the co-movement of major aggregate economic variables are, therefore, explained by RBC theorists with the following argument: whenever a temporary increase in productivity occurs, agents react by consuming more (since output is above its normal level) and by investing more (since agents value future consumption as well, and investment represents the intertemporal channel for higher future production and consumption). It is furthermore assumed that work effort will increase, that is the substitution effect (which is the result of higher productivity) dominates the wealth effect. Because the effect of a temporary shock propagates in the future, it is easy to explain why variables like output, consumption, investment, etc. are likely to be serially correlated even though the shocks themselves are not.

1.2 Real Business Cycle Theory

RBC theory employs general equilibrium models in which identical individual agents maximize their utility function subject to production possibilities and resource constraints to determine how major macroeconomic variables respond to changes in the economic environment. The effect of these changes is assessed by including them as part of the rules of the model and computing equilibria under the assumption that agents make their best choices given the rules. In principle, then, this allows to evaluate alternative policies by comparing the utility levels attained under each of them.

Two distinctive advantages can be ascribed to the RBC approach:

(1) Aggregate phenomena are the result of decisions made by individual agents. In other words, the restrictions imposed on aggregate behavior are the result of aggregating the restrictions imposed on individual behavior; and

(2) Simple structures can generate behaviors that not only are similar to those of actual time series but also can be viewed as best responses to exogenous

shocks.

RBC models have been criticized on several grounds, particularly for (i) limiting their scope of analysis to real phenomena only, (ii) assuming that markets are competitive and complete and (iii) relying exclusively on supply shocks.

With respect to the first criticism, it should be noted that proponents of RBC theory do not claim that monetary phenomena are irrelevant. It is only for reasons of models' tractability that they have chosen to leave monetary issues aside.

In terms of the second criticism, a legitimate explanation for the assumption that markets be competitive and complete is provided by Plosser [1989] when he argues that "...it is logically impossible to attribute an important portion of fluctuations to market failures without an understanding of the sorts of fluctuations that would be observed in the absence of the hypothesized market failure...".

As for the last criticisms, two remarks should be considered. First, technological shocks are not purely supply shocks. Indeed, they affect supply through shifts in the production function and demand through the wealth effect and the labor/leisure decision. Second, while so far the emphasis has been placed on technological shocks exclusively, RBC theorists do not exclude the possibility that shocks may arise from changes in tastes and preferences or in government spending, which are examples of demand shocks. As a matter of fact, the core of this analysis is to explore the impact of government spending shocks in an otherwise traditional RBC model.

1.3 Major Contributions to RBC Theory

Most of the existing literature on RBC theory focuses on modified versions of the general equilibrium standard growth model to explain general features of business cycles by simulating the fluctuations of major macroeconomic variables. The most significant contributions in this area of research have been provided by Kydland and Prescott [1982], Long and Plosser [1983], Hansen [1985], Cho and Rogerson [1988], Greenwood, Hercowitz and Huffman [1988], Hansen and Cooley [1989], Benhabib, Rogerson and Wright [working paper, 1990], among others.

Kydland and Prescott's "Time to Build and Aggregate Fluctuations" [1982] undoubtedly represents the seminal contribution to RBC theory. Their model features several important "deviations" from the standard growth model, including. In particular,

- A non-time separable utility function is employed to specify individual agents' preferences;

- Inventories of finished goods are added to capital and labor as inputs in the production process;
- Multiple periods are introduced to obtain productive capital;
- The shock to technology is specified as having a highly persistent as well as a transitory component, which cannot be directly distinguished by producers and/or consumers.

The first assumption is perhaps the most important because it addresses a very serious limitation of the standard growth model, namely its inability to generate sufficient volatility in aggregate hours worked relative to average productivity fluctuations (which are viewed as a proxy for wage fluctuations). The introduction of a non-time separable utility function enhances the intertemporal substitution response of leisure to a productivity shock, thereby increasing, at least theoretically, the relative volatility of aggregate hours worked and productivity. Indeed, Kydland and Prescott find that the ability of the model to mimic results from U.S. Post War data improves, even though, in their own admission, it is still far from being satisfactory.

The dependence on individuals' willingness to substitute leisure intertemporally in response to wage changes in order to generate sufficient volatility in hours worked has been harshly criticized because it requires an elasticity of substitution whose value is far above the one suggested by micro panel studies. This criticism has led Hansen to address the employment/productivity issue using a different approach. In "Indivisible Labor and the Business Cycle" [1985], he introduces the concept of labor indivisibility, whereby individuals' preferences are defined at two levels of leisure only - one consistent with working full time and the other with not working at all. With labor indivisibility, then, fluctuations in hours worked are the result of individuals' entering and exiting the labor force at different time periods as opposed to employed individuals adjusting continuously the number of hours worked. This assumption is more consistent with the observation that a large portion of the observed fluctuations in aggregate hours is associated with variations in the number of individuals employed, since most people do not really have a choice as to how many hours they want to work. With labor indivisibility, the distinction between the utility function of an individual and that of a "representative agent" becomes crucial. In fact, the utility function of the latter implies an intertemporal elasticity of substitution for leisure which is infinite, and thus capable of generating high volatility in aggregate hours worked. However, this result is independent of the elasticity of substitution implied by the preferences of the individual and, as such, it is in principle consistent with the low estimates of micro panel studies.

Another interesting approach to the analysis of the employment/productivity issue is considered by Cho and Rogerson in "Family Labor Supply and Aggregate Fluctuations" [1988]. Relying on the evidence that labor supply responses to changes in wages vary across demographic and skill groups, they introduce a simple form of heterogeneity in the labor force, by assuming that the economy is populated by identical households, each consisting of two heterogeneous members. As in Hansen's contribution, it is shown that large aggregate elasticities are still consistent with the fact that the majority of the labor force has an elasticity close to zero. Here, however, large aggregate elasticities are produced by a different type of non-convexity. In particular,it is assumed that households face a fixed cost when both members work simultaneously. Since, by assumption, women have lower wages, they bear the burden of the fixed cost, thereby displaying much higher elasticities of labor supply than their male counterpart, even though preferences for leisure are identical.

Recognizing that the ability of the standard growth model to reproduce the actual aggregate fluctuations may have been partially impaired by the simplicity of its structure, several efforts have been made to analyze the impact of assumptions more "consistent" with stylized facts. The remaining contributions reviewed here represent research efforts in this particular direction.

For instance, in "Real Business Cycles" [1983], Long and Plosser consider a multisector extension of the standard growth model by assuming that the economy is characterized by a vector of commodities which serve a dual purpose. In fact, they can be either consumed directly or they can be employed as inputs in the production of other commodities. Their analysis shows that, in conformity with stylized facts, consumer goods tend to be more volatile than producer goods. In addition, the analysis has useful implications for determining the leading and the lagging sectors in business cycles.

Benhabib, Rogerson and Wright's "Homework in Macroeconomics II: Aggregate Fluctuations" [1990] explores a different type of multisector extension, namely the introduction of a home, or nonmarket, production function. Both market and household productions use labor and capital as inputs to produce outputs according to a stochastic technology. This implies that fluctuations in aggregate market variables now depend on relative productivity shocks which induce individual agents to substitute between market and household activities.

In response to the familiar argument that RBC theory denies the relevance of money as a source of aggregate fluctuations, in "The Inflation Tax in a Real Business Cycle Model" [1989], Cooley and Hansen develop a model in which money is incorporated by using a cash-in-advance constraint. Money is introduced in a way that emphasizes exclusively the influence of anticipated inflation on real variables. In fact, the role played by unanticipated inflation, which some economists regard as the most important channel of influence on real variables, is not discussed. The model shows that anticipated inflation does not seem to alter drastically the cyclical properties of the economy. It does, however, have a significant impact on the long run values of aggregate real variables. In fact, the welfare cost of the inflation tax, which they measure by comparing steady state solutions, is positively related to the annual rates of growth of money.

Lastly, in "Investment, Capacity Utilization, and the Real Business Cycle" [1988], Greenwood, Hercowitz and Huffman emphasize Keynes' argument that output fluctuations are generated by shocks to the marginal efficiency of investment (MEI). The authors incorporate these shocks in a standard growth model in which the rate of capacity utilization is assumed to be endogenous. In particular, the model is specified so as to have positive shocks to the MEI inducing both the formation of new capital as well as a more intensive utilization and accelerated depreciation of old capital. The assumptions that (i) the rate of capacity utilization is endogenous and that (ii) the shocks do not affect the productivity of existing capital are crucial to generating co-movements of major aggregate economic variables. In fact, if the transmission mechanism did not operate through an optimal utilization of capital, shocks to the MEI would simply stimulate employment and output through an intertemporal substitution effect on leisure and consumption. This would imply that consumption would move countercyclically, thus contradicting the empirical evidence. Labor productivity would also be moving in the wrong direction in the sense that, in the short run, given a fixed capital stock, an expansion in employment would cause labor productivity to decline. With an endogenous rate of capacity utilization, instead, shocks to the MEI increase the rate of capacity utilization. Since labor and capital are complements, labor productivity increases thereby inducing an expansion in labor effort. Now, however, the change in labor productivity generates an intratemporal substitution effect away from leisure and toward consumption, so that the procyclicality of both consumption and productivity is maintained. As for the second assumption, if the shocks to the MEI were to affect old capital as well as new one, it would not pay to depreciate old capital through higher rates of capacity utilization. Therefore, the positive effects of a shock on utilization, labor, output, productivity and consumption would disappear all together.

II. MOTIVATION

The model presented in this paper capitalizes on the contribution provided by Hansen in "Indivisible Labor and the Business Cycle" [1985]. In fact, it is a direct outgrowth of Hansen's model, the major difference being that an exogenous government sector is being introduced at this time. The incorporation of a government sector is motivated by two observations. First, U.S. data indicate that, since the beginning of the century, government purchases of goods and services have represented a sizable portion of the total U.S. economic activity. Second and perhaps more important, while there is no evidence that the government sector plays a direct role in generating aggregate fluctuations, failure to consider its existence may have repercussions on the model's ability to reproduce the cyclical fluctuations which characterize other components of economic activity.

In a two-sector closed economy, the feasibility condition $y_t = c_t + i_t$ implies that

$$var\ (y_t) = var\ (c_t) + var\ (i_t) + 2cov\ (c_t, i_t) \tag{2.1}$$

Similarly, in a three-sector closed economy, the feasibility condition $y_t = c_t + i_t + G_t$ implies that

$$var\ (y_t) = var\ (c_t) + var\ (i_t) + var\ (G_t) +$$

$$+ 2cov\ (c_t, i_t) + 2cov\ (c_t, G_t) + 2cov\ (i_t, G_t) \tag{2.2}$$

As will be shown later, when conducting simulations of artificial economies, one of the restrictions imposed on the model is to calibrate parameters so as to obtain a standard deviation for the output series that equal the standard deviation of GNP for the U.S. economy. The rationale for imposing this restriction is the need to establish a benchmark against which to measure the relative performance of an artificial economy *vis a vis* the actual economy. This restriction implies that regardless of whether the model is specified as consisting of two or three sectors, the magnitude of the variance of the output series will be the same. However, since the actual economy does contain a government sector, if the model is specified otherwise, the individual components of the output variance will, in general, turn out to be biased, unless it can be shown that

$$var\ (G_t) + 2cov\ (c_t, G_t) + 2cov\ (i_t, G_t) = 0 \tag{2.3}$$

Two observations contribute to suggest that such bias exists. In particular, they seem to indicate that the bias is in a downward direction (i.e. the above term has a negative sign):

(1) The simulation results reported in Hansen's paper indicate that the second

moment statistics for both the consumption and the investment series are lower than their actual counterparts.

(2) A study conducted by Barro in 1981, in which he analyzes the impact of government purchases on output, suggests the existence of a crowding out effect in the U.S. economy. In particular, it is shown that, during wartime, crowding out is only partial and affects mainly private investment whereas, during peacetime, public spending replaces almost completely private spending and affects mainly consumption. Such finding suggests that both $cov(c_t, G_t)$ and $cov(i_t, G_t)$ should have a negative sign, thereby ruling out the possibility that the above term be positive with certainty.

Indeed, a measurement of the individual components of equation (2.3) above, obtained using quarterly data for the period 1955.3 - 1984.1, indicates that their sum is negative[1] This evidence, however, is not necessarily conclusive. In fact, the use of a limited sample, which was motivated by the fact that it is consistent with time frame utilized later to run simulations, may introduce some measurement bias. As for Hansen's results, the underestimation of the fluctuations in the consumption and investment series could be due to model misspecifications other than those associated with the absence of a government sector.

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[1] Using logged and detrended data, $var(G_t) = 0.0003387$, $cov(G_t, c_t) = -0.0000172$, and $cov(i_t, G_t) = -0.0002335$.

III. THE MODEL

The economy analyzed here is composed of three sectors, namely households, business firms and government. The mathematical model underlying the dynamic evolution of this economy is specified so as to be consistent with both growth and micro observations.

It is assumed that firms have access to a technology described by the following Cobb-Douglas production function

$$f(z_t, k_t, h_t) = z_t k_t^{\theta} h_t^{1-\theta} \tag{3.1}$$

where h_t is the labor input at time t;

k_t is the capital input at time t;

z_t is the technology shock at time t; and

$\Theta \in (0,1)$ is a parameter which measures the share of capital in the production process.

The assumption of a Cobb-Douglas production function is consistent with the observations that (i) in the United States, capital and labor shares of output have remained fairly constant, and (ii) while the rental price of capital, q, has remained approximately constant, the average real wage rate, w, has increased greatly.

It is further assumed that capital accumulation follows a law of motion given by

$$k_{t+1} = (1-\delta)k_t + i_t \tag{3.2}$$

where i_t is the level of investment at time t; and

$\delta \in (0,1)$ is a parameter which measures the rate of capital depreciation.

The technology shock z_t, which is assumed to be observed before any period t decision is made, is a random shock which follows a first-order Markov process. In particular,

$$z_{t+1} = \gamma z_t + \epsilon_{1,t+1} \tag{3.3}$$

where γ is a technology parameter; and

$\varepsilon_{1,t}$ is i.i.d. normal with mean $1-\gamma$ and standard deviation σ_1. Note that by requiring that the mean be $1-\gamma$, the unconditional mean of z_t is equal to 1.

In each period t, firms maximize real profits which can be written as[2]:

$$\Pi = z_t k_t^{\theta} h_t^{1-\theta} - w_t h_t - q_t k_t \tag{3.4}$$

[2] The price of output, p_t, is not shown in the equation, since it is normalized to be 1 in each time period.

where w_t is the real wage rate at time t; and

q_t is the real rental rate at time t.

The first-order conditions for profit maximization imply that, in each time period, the real wage rate and the real rental rate are respectively equal to the marginal products of labor and capital, thereby indicating that in equilibrium firms will earn zero economic profits. In fact,

$$\frac{\partial \Pi}{\partial k_t} = \theta z_t k_t^{\theta - 1} h_t^{1-\theta} - q_t = 0$$

implies

$$\theta z_t k_t^{\theta - 1} h_t^{1-\theta} = q_t \tag{3.5}$$

and

$$\frac{\partial \Pi}{\partial h_t} = (1-\theta) z_t k_t^{\theta} h_t^{-\theta} - w_t = 0$$

implies

$$(1-\theta) z_t k_t^{\theta} h_t^{-\theta} = w_t \tag{3.6}$$

Therefore, without loss of generality, it is possible to assume that a single representative firm exists in the economy.

The per capita output produced in the economy can be sold either to the government or to households, in which case it can be either consumed or invested. Feasibility then requires that the following constraint be satisfied:

$$c_t + i_t + G_t \leq f(z_t, k_t, h_t) \tag{3.7}$$

where c_t is consumption at time t; and

G_t is government purchases at time t.

Government faces a per capita budget constraint given by:

$$T_t + (b_t - b_{t-1}) = G_t + r_t b_{t-1} \tag{3.8}$$

where T_t is the lump-sum tax levied at time t;

b_t is the stock of bonds issued at time t; and

r_t is the real interest rate on bonds at time t.

The observation that most government programs are implemented over time periods longer than just a quarter leads to postulate a law of motion for Government purchases that follows an autoregressive scheme of the form

$$G_{t+1} = \Phi + \Psi G_t + \epsilon_{2,t+1} \tag{3.9}$$

where $\Phi > 0$ and $\Psi \in (0,1)$ are government parameters; and

$\varepsilon_{2,t}$ is the shock to government spending, which is assumed to be i.i.d. normal with mean 0 and standard deviation σ_2.

The economy is assumed to be populated by a large number of identical, infinitely-lived households facing a utility function which is defined over two variables, consumption (c_t) and leisure (l_t). At time t, each household maximizes the expected value of

$$\sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \tag{3.10}$$

where $\beta \in (0,1)$ is the discount factor.

The endowment of time is normalized to be 1, so that, when defining hours worked as h_t, it turns out that $l_t=1-h_t$. In the standard growth model, the utility function is assumed to have the following log-linear specification

$$U(c_t, h_t) = \log c_t + \Gamma \log (1-h_t) \tag{3.11}$$

where Γ is a utility parameter.

This specification is consistent with the observation that, since over time the choice of leisure has not shown any trend, whereas real wage has increased substantially, (i) the intratemporal elasticity of substitution between leisure and consumption should be approximately one and (ii) the intertemporal elasticity of substitution of the composite commodity (c,l) should be constant (with the above specification, it has a value of one).

When labor indivisibility is introduced, households can no longer choose h_t in the interval [0,1]. In fact, they now face two possible alternatives only, either work full-time (a choice denoted by setting $h_t = h_0$) or do not work at all (a choice denoted by setting $h_t = 0$). The assumption of labor indivisibility, therefore, introduces a non-convexity in the consumption possibility set which potentially poses a major problem, in the sense that the solution to the representative household's problem can no longer be supported as a competitive equilibrium. However, following Hansen [1985], convexity in the consumption possibility set is restored by assuming that households do not choose hours worked. Instead, they choose a lottery which will then determine whether they work or not. In other words, in each period, households choose a probability of working, α_t.

The expected utility of a representative agent in period t, then, is given by

$$U(c_t, \alpha_t) = \alpha_t U(c_t, h_0) + (1-\alpha_t) U(c_t, 0) \tag{3.12}$$

In particular, given a log-linear specification, the expected utility function takes the form

$$U(c_t, \alpha_t) = \alpha_t [\log c_t + \Gamma \log (1 - h_0)] + (1 - \alpha_t)[\log c_t + \Gamma \log 1] \tag{3.13}$$

which reduces to

$$U(c_t, \alpha_t) = \log c_t + \Gamma \alpha_t \log (1 - h_0) \tag{3.14}$$

Since at any time t a portion α_t of the households works h_0 hours, average hours worked by each household is given by $h_t = \alpha_t h_0$. This implies that the expected utility function can be rewritten as

$$U(c_t, h_t) = \log c_t - \Delta h_t \tag{3.15}$$

where

$$\Delta = -\frac{\Gamma \log (1 - h_0)}{h_0} \tag{3.16}$$

Each household maximizes the utility function defined by (3.15) subject to the following budget constraint:

$$c_t + i_t + T_t + (b_t - b_{t-1}) \leq w_t h_t + q_t k_t + r_t b_{t-1} \tag{3.17}$$

When the government's budget constraint is substituted into the household's budget constraint, the latter can be rewritten as:

$$c_t + i_t + G_t \leq w_t h_t + q_t k_t \tag{3.18}$$

Notice that this constraint is identically equal to the per capita feasibility constraint defined by equation (3.7), since it has been shown that in equilibrium real profits are zero.

An important observation should be made at this point. Neither taxes nor government debt appear in the household's budget constraint, thus implying that only government spending matters. In other words, for a given path of government spending, the method of financing is irrelevant because neither lump-sum taxation nor debt financing will affect the allocation of resources. If the government decides to borrow, households recognize the offsetting obligation to pay off today's debt with higher future taxes. This induces households to allocate resources just as if they had been taxed in the present. However, if taxes were levied on return to capital or, more generally, on productive activity, distortions in the allocation of resources would result.

The absence of distortions in this economy implies that the Pareto optimum can be supported as a competitive equilibrium. Since individuals in this economy are ex-ante homogeneous (even though ex-post they are not, because a fraction α_t will work while the rest will not), the Pareto optimum solution is the one which solves the problem of maximizing the expected welfare of a representative household subject to the technology constraints. In particular, it is the solution to the following problem:

$$Max. \quad E_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_t , h_t) \right] \tag{3.19}$$

subject to (3.1)-(3.3),(3.7) and (3.9), given k_0,z_0 and G_0.

At each time t, agents make decisions based upon all the information available at time t. It is assumed that their expectations are rational in the sense that their forecasts of future variables are the same as those implied by the laws of motion, i.e. by the structure of the economy as represented by the model.

IV. STATE SPACE LINEAR SYSTEMS

When taking appropriate approximations (discussed in a later section), the model described in Section III can be viewed as an application of a discrete time linear quadratic regulator problem, which is a type of optimal control problem for linear systems.

This section provides a review of linear systems and focuses, in particular, on state space linear systems. It should be emphasized that the purpose of this section is not to provide a rigorous analysis of state space linear systems, rather to present an informal and intuitive exposition of the major concepts involved.

In the context of this discussion, linear systems should be viewed as mathematical models for real world time series processes in which input time functions give rise to output time functions in some linear fashion. The inputs, also known as decision or control variables, represent the driving force behind the process, while the outputs constitute the observable manifestation of what goes on in the process and of how the internal behavior of the process is affected by the inputs.

4.1 Definition of State Space Linear Systems

State space linear systems represent a subclass of linear systems. A state space linear system is a model for an input-output process which has associated with it a well-defined notion of internal situation or state. Like the inputs and the outputs, the evolution of the state of the process is modelled as a time function. This function contains a great deal of information about the process; in fact, knowledge of the value of the state at any time t allows to determine the entire future evolution of the state and of the output functions, once the future time paths of the input functions are specified.

To be defined as such, a state space linear system must contain:

(1) a mathematical specification of the input and output time functions that are permissible, i.e. a vector space U of m input functions and a vector space Y of p output functions.

(2) a finite n-dimensional real vector space X, which defines the state space of the system, in the sense that the elements of X, or states, correspond to possible internal situations of the process.

(3) a mathematical specification of how the current state and future inputs determine the future state, i.e. a state transition mapping

$$(x_0, u) \rightarrow \omega(t_1, t_0, x_0, u) \tag{4.1.1}$$

for $x_0 \in X$, $u \in U$, $t_1 \geq t_0$, and

(4) a mathematical specification of how the current state and current inputs determine the current output, i.e. a readout mapping

$$(x, u) \rightarrow \rho(t, x, u)$$

for $x \in X$, $u \in R^m$ (where R^m is the mapping for vector space U).

Both (3) and (4) must satisfy the requirement of linearity. In other words, the state at some time t_1 must depend linearly on the state at time t_0 ($t_1 > t_0$) and on the input function applied between t_0 and t_1. Similarly, the value of the output at any time t must depend linearly on the state of the system and on the value of the input at time t.

In addition, (3) must satisfy the constraints of consistency and causality. Consistency implies that the state "now" determines the state "now" regardless of what is happening to the input. Causality implies that, given the state of the system at time t_0, the evolution of the state between t_0 and t_1 depends exclusively on the inputs applied between t_0 and t_1. Inputs applied before t_0 are irrelevant to the evolution of the state in the interval $[t_0, t_1]$ once the state at t_0 is known. Similarly, inputs to be applied after t_1 are irrelevant to the evolution of the state before time t_1. This amounts to saying that the state of the system at a given time summarizes the entire history of the system up to that time and constitutes everything that is required to determine future states of the system, given future inputs.

Using (4), it is possible to define the overall response function S of a state space linear system as the value of output at time $t_1 \geq t_0$, given the state of the system at t_0 and the value of the input applied between t_0 and t_1. It is easy to show that output values at any given time depend exclusively on the state of the system at that time and on input values at that time. In other words, as far as output is concerned, the present state summarizes the entire history of the system, and only the present value of the input function has additional influence on the present value of output.

Thus far, state space linear systems have been described in an abstract form; however, they have a well-defined mathematical representation, which depends on whether the system is assumed to be continuous or discrete. Since the model presented in the previous section is in a discrete form, for the purpose of this analysis, only the representation of discrete state space linear systems is considered. For all discrete state space linear systems, it is possible to find matrix functions (A,B,C,D) such that the system can be represented by the following set of equations:

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t u_t \\ y_t &= C_t x_t + D_t u_t \end{aligned} \qquad (4.1.3)$$

where x_0 is given.

The first equation, known as the state space equation, specifies how the state vector evolves over time; it is, therefore, the analogous of the state transition mapping defined in equation (4.1.1). The second equation, known as the observation equation, specifies the relationship between the state vector and the outputs; it is, in other words, the analogous of the readout mapping defined in equation (4.1.2). The set of matrix functions A_t, B_t, C_t and D_t, which have dimensions $(n \times n)$, $(n \times m)$, $(p \times n)$ and $(p \times m)$, respectively, is called the realization of the system. If the system is time invariant, this set of matrix functions will be constant. For a system to be time invariant, the following conditions must be satisfied:

- Both U and Y spaces must be invariant under time shifting;
- If the system is started at some time t_0 and is forced for a certain length of time with some input function, then the same thing must happen to the system that would happen if the same procedure were followed starting at some other time t_1.
- The way in which the present state and the present input values determine the present output values must not change over time.

4.2 Properties of Time Invariant State Space Linear Systems

Recalling that the state transition mapping for a time invariant discrete state space linear system can be represented by the equation $x_{t+1} = Ax_t + Bu_t$, recursive substitution indicates that knowledge of A^t for $t > 0$ is essentially equivalent to knowing everything about the state transition behavior of the system. This information about the system is fundamental because, as will be seen later, it reflects the property of stability. There are, however, two additional features of the system which warrant some discussion:

(1) the degree to which input functions determine the state of the system and may be used to influence its evolution, and

(2) the degree to which the state evolution influences the output of the system.

The first feature is reflected in the properties of reachability and controllability; the second, in the properties of observability and constructability. These properties are desirable because they assure minimality of the state vector dimensions; in other words, they guarantee a parsimonious representation of the state space linear system.

<u>Reachability</u>

Given an abstract time invariant discrete state space linear system, a state $x_1 \in X$ is reachable if the initial state $x_0 = 0$ at the initial time t_0 may be "forced" to x_1 in finite time T assuming that a suitable input sequence is applied, i.e.

$$\omega(T, 0, 0, u) = x_1 \tag{4.2.1}$$

If every state $x_1 \in X$ is reachable, then the system is said to be reachable or to have a reachable space.

Given the representation of a time invariant discrete state space linear system with realization (A,B,C,D), a vector $x_1 \in R^n$ is reachable with respect to the pair (A,B) if there exist an integer $T > 0$ and a sequence of real m-vectors $u_0, u_1, \ldots, u_{T-1}$, such that

$$x_1 = \sum_{t=0}^{T-1} A^{T-1-t} B u_t \tag{4.2.2}$$

If every $x_1 \in R^n$ is reachable with respect to the pair (A,B), then (A,B) is said to be a reachable pair.

Condition for Reachability:

Let the reachability matrix be defined by

$$Q_r(A, B) = [B \;\; AB \;\; A^2B \;\; \ldots A^{n-1}B] \tag{4.2.3}$$

It can be shown that the pair (A,B) is reachable if and only if $Q_r(A,B)$ has full rank n, where n is the dimension of the state vector.

Controllability

A time invariant discrete state space linear system is controllable if an arbitrary initial state x_0 may be "forced to zero" in finite time T given that a suitable input sequence is applied, i.e.

$$0 = \omega(T, 0, x_0, u) \tag{4.2.4}$$

Given a system representation with realization (A,B,C,D), the pair (A,B) is controllable if for every $x_0 \in R^n$, there exists an integer T and a sequence of real m-vectors $u_0, u_1, \ldots u_{T-1}$, such that

$$A^T x_0 + \sum_{t=0}^{T-1} A^{T-1-t} B u_t = 0 \tag{4.2.5}$$

It can be shown that a time invariant discrete state space system is controllable if it is reachable. Since the converse is not necessarily true, controllability is a weaker property than reachability. Notice, however, that when the matrix A is nonsingular, the distinction between reachability and controllability disappears.

Observability

Given an abstract time invariant discrete state space linear system, a state $x_0 \in X$ is unobservable if it is undistinguishable as an initial condition at an initial time t_0 from the initial condition $x_0 = 0$. In other words, x_0 is unobservable if any input choice applied to the system at time $t \geq t_0$ will give rise to the same output regardless of whether at time t_0 the system was started in any state x_0 or in state zero. This condition can be expressed as follows

$$\rho(t, \omega(t, 0, x_0, u), u(t)) = \rho(t, \omega(t, 0, 0, u), u(t)) \tag{4.2.6}$$

If the only $x_0 \in X$ which is unobservable is the state $x_0 = 0$, then the system is said to be observable or to have an observable space.

Given the representation of a time invariant discrete state space linear system with realization (A,B,C,D), a vector $x_0 \in R^n$ is unobservable with respect to the pair (A,C) if, $\forall\ T \geq 0$, the following equality is satisfied

$$CA^T x_0 = 0 \tag{4.2.7}$$

In fact, in general

$$y_T = CA^T x_0 + \sum_{t=0}^{T-1} CA^{T-1-t} Bu_t + Du_T \tag{4.2.8}$$

If x_0 is unobservable (i.e. undistinguishable from x_0=0), (4.2.8) reduces to

$$y_T = \sum_{t=0}^{T-1} CA^{T-1-t} Bu_t + Du_T \tag{4.2.9}$$

which is consistent with condition (4.2.7) above.

Equation (4.2.9) implies that, for all input functions u, output will be the same regardless of whether the initial state was x_0 or 0. If the only $x_0 \in R^n$ which is unobservable with respect to the pair (A,C) is $x_0 = 0$, then (A,C) is said to be an observable pair.

Condition for observability:

Let the observability matrix be defined by:

$$Q_0(A, C) = [\,C' \;\; A'C' \;\; A^{2\prime}C' \ldots A^{n-1\prime}C'\,]' \tag{4.2.10}$$

It can be shown that the pair (A,C) is observable if and only if $Q_0(A,C)$ has full rank n.

Constructibility

A time invariant discrete state space system is constructible if every unobservable initial state x_0 can be "forced to zero" in finite time T given that the zero input is applied to the system. This implies that the long run behavior of a constructible system is unaffected by its unobservable states, i.e.

$$\omega(T, 0, x_0, 0) = 0 \tag{4.2.11}$$

Given a system representation with realization (A,B,C,D), the pair (A,C) is constructible if for every $x_0 \in R^n$ which is unobservable with respect to (A,C), there exists $T > 0$ such that

$$A^T x_0 = 0 \tag{4.2.12}$$

It is easy to show that if (A,C) is observable, then it is constructible. In fact, observability implies that $x_0 = 0$ is the only unobservable vector with respect to

(A,C) and this particular x_0 does indeed satisfy the constructibility condition (4.2.12). The converse, however, is not necessarily true; for this reason, constructibility is considered a weaker property than observability.

Stability

Stability theory for discrete state space linear systems is identical to stability theory for difference equations. The notion of equilibrium is essential to the understanding of stability.

Given the difference equation $x_{t+1} = A_t x_t$, $x^* \in R^n$ is defined as an equilibrium of the difference equation if the unique solution to the equation starting from the initial condition x^* at any time t_0 is identically equal to x^*. This means that when the equation is started at x^*, the solution will stay at x^* forever. Intuitively, then, an equilibrium is a stationary condition or state of rest for the real world process that the difference equation is supposed to model. If an unforced process is started in a state of rest, it will remain forever in a state of rest.

When dealing with real world processes, however, it is unlikely, and therefore implausible to assume, that a process will start from a state of rest. The concept of stability, then, becomes relevant. An equilibrium x^* is defined as stable if an initial condition which is close to x^* leads to an x function which is forever close to x^*. A stronger definition of stability is provided by the concept of asymptotic stability. An equilibrium x^* is asymptotically stable if, as $T \to \infty$, the process approaches (without ever reaching it) a stable state of rest. Observe that asymptotic stability implies stability, i.e. for x^* to be asymptotically stable, x^* must be stable. It turns out that, given the above difference equation, the only possible asymptotically stable equilibrium is $x^* = 0$. Also, if $x^* = 0$ is an asymptotically stable equilibrium, no other equilibrium can exist.

Consider now the difference equation which arises from a discrete state space linear model, that is $x_{t+1} = A_t x_t + B_t u_t$. If the initial condition x_0 is specified and the identically zero input is forced through the process for all $t \geq t_0$, then the above equation reduces to the unforced difference equation discussed in the previous paragraph. It is in this sense that it is claimed that stability theory for discrete state space linear systems is identical to that of difference equations. Limiting the discussion to time invariant discrete state space linear systems, it can be shown that stability properties are determined by the time behavior of A^t for $t \geq 0$. This behavior, in turn, depends on the eigenvalues of the matrix A. For this reason, stability conditions are expressed in terms of the eigenvalues of A. In particular, the equilibrium x^* of $x_{t+1} = Ax_t + Bu_t$ is asymptotically stable if all the eigenvalues of A have magnitude less than one; on the other hand, the same equilibrium will be stable if no eigenvalue of A has magnitude greater than one and, in addition, whenever λ_0 is an eigenvalue of A with magnitude equal to one, the algebraic multiplicity of λ_0 is equal to its geometric multiplicity.

4.3 Discrete Time Linear Quadratic Regulator Problem

As indicated in the introductory paragraph of this section, the Linear Quadratic Regulator Problem is an optimal control problem for linear systems. It is a problem of a prescriptive, as opposed to descriptive, nature. Problems which are descriptive in nature are concerned with what happens in a system when a given input function is applied; problems which are prescriptive in nature, on the other hand, focus on what can be made to happen under a given set of operating conditions about the system and what is our ability to control it.

The Linear Quadratic Regulator Problem requires the maximization of a quadratic objective (or performance) function or the minimization of a cost function through a choice of controls. The problem may be solved either for a finite or for an infinite time horizon. In either case, the maximization (minimization) of the objective (cost) function is subject to the constraints imposed by a state space linear system of the type discussed in this section.

Linear Quadratic Regulator Problems can be solved using dynamic programming techniques. The basic principle of dynamic programming, which in Optimal Control Theory is known as the Principle of Optimality, can be formulated as follows. Given an optimal path from point A to point C, the portion of the path from any intermediate point B to point C must be the optimal path from B to C. More formally, Bellman [1957], the pioneer of dynamic programming techniques, has defined the Principle of Optimality in the following terms: "An optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

The use of dynamic programming greatly simplifies the solution of linear quadratic regulator problems (and, in general, of optimal control policy problems), since it provides the flexibility of solving a problem recursively.

To illustrate this principle, assume that the objective (cost) function to be maximized (minimized) is generically defined as

$$J = \sum_{t=0}^{T} L(x_t, u_t, t) \tag{4.3.1}$$

Assume, further, that the constraints are expressed as

$$x_{t+1} = g(x_t, u_t, t) \qquad t = 0, 1, \ldots, T-1 \tag{4.3.2}$$

The above maximization (minimization) problem requires the choice of an optimal sequence $\{u_t\}$ and can be solved iteratively as follows.

Define the maximum objective (minimum cost) function $I(x_t,t)$ as the maximum performance (minimum cost) that can be attained by using an admissible decision sequence for the remainder of the process starting from an arbitrary admissible state $x \in X$ at an arbitrary time τ, for $0 \leq \tau \leq T$. This function can be written as

$$I(x, \tau) = \min_{u_\tau, u_{\tau+1}, \ldots, u_T} \left[\sum_{t=\tau}^{T} L(x_t, u_t, t) \right] \tag{4.3.3}$$

The summation inside the parenthesis can be split into two components: the term at time τ, and the summation from time $\tau+1$ to T, so that the above equation can be rewritten as

$$I = \min_{u_\tau, \ldots, u_T} \left[L(x_\tau, u_\tau, \tau) + \sum_{t=\tau+1}^{T} L(x_t, u_t, t) \right] \tag{4.3.4}$$

The first term inside the parenthesis depends exclusively on u_τ and not on any u_t, for $t=\tau+1,\tau+2,\ldots,T$. Therefore,

$$\min_{u_\tau, \ldots, u_T} [L(x_\tau, u_\tau, \tau)] = \min_{u_\tau} [L(x_\tau, u_\tau, \tau)] \tag{4.3.5}$$

The second term of equation (4.3.4) depends implicitly on u_τ, as u_τ determines the state $x_{\tau+1}$ through the state transition equation $x_{\tau+1} = g(x_\tau, u_\tau, \tau)$. This implies that

$$\begin{aligned} &\min_{u_\tau, u_{\tau+1}, \ldots, u_T} \left[\sum_{t=\tau+1}^{T} L(x_t, u_t, t) \right] \\ &= \min_{u_\tau} [I(g(x_\tau, u_\tau, \tau), \tau+1)] \\ &= \min_{u_\tau} [I(x_{\tau+1}, \tau+1)] \end{aligned} \tag{4.3.6}$$

Then, the functional equation for the maximization (minimization) problem can be rewritten as

$$I(x_\tau, \tau) = \min_{u_\tau} [\, L(x_\tau, u_\tau, \tau) + I(x_{\tau+1}, \tau+1) \,] \tag{4.3.7}$$

The above result shows that the best that one can do with regard to the maximization (minimization) of the objective (cost) function starting in period τ from some state x_τ is to plan to proceed optimally from $\tau+1$ onward, regardless of what state $x_{\tau+1}$ has been achieved because of the input u_τ, and then choose u_τ so as to maximize (minimize) the sum of the performance (cost) due to $L(x_\tau,u_\tau,\tau)$ and the performance (cost) $I(x_{\tau+1},\tau+1)$, associated with the rest of the optimal strategy which depends, through $x_{\tau+1}$, on u_τ.

It can be shown that as long as the linear quadratic regulator problem is solved for a finite time horizon, no assumptions need to be made about the reachability, controllability, etc. of the state space linear system. However, when maximizing (minimizing) an objective (cost) function over an infinite time horizon, convergence requirements demand that such conditions be imposed on the linear system.

For example, if the pair (A,B) is not controllable (weaker property than reachability), there might be an initial condition x_0 for which the value (cost) function does not converge for any choice of u. On the other hand, if the pair

(A,C) is not observable , some initial condition x_0 will lead to zero value (cost) if a zero input is applied. Such uncontrolled evolution of x_t might allow instabilities in x which are not revealed in $t \rightarrow y_t$.

As is explained in more details in the section on linear-quadratic approximations, if the pair (A,B) is reachable and the pair (A,C) is observable, then maximization (minimization) of a value (cost) function is achieved by setting

$$u_t = -F x_t \qquad\qquad t \geq 0 \tag{4.3.8}$$

where

$$F = [\, I_m + B^T P B \,]^{-1} B^T P A \tag{4.3.9}$$

and P is the solution to the algebraic Riccati equation

$$P = F^T F + C^T C + [\, A - BF \,]^T P [\, A - BF \,] \tag{4.3.10}$$

Notice that the algebraic definitions of F and P provided in the section on linear-quadratic approximations are not identically equal to equations (4.3.9) and (4.3.10), respectively. This is because those are solutions which refer to a problem that has been approximated around its steady state in order to yield a linear quadratic regulator problem.

V. SOLUTION OF THE MODEL

In terms of the model described in Section III, the state of the economy at time t is fully described by the variables k_t, z_t and G_t. The decision (or, input) variables at time t are provided by h_t, c_t and i_t. The equations specifying the model equilibrium solution, then, are of the form:

$$\begin{aligned} h_t &= h(z_t, k_t, G_t) \\ c_t &= c(z_t, k_t, G_t) \\ i_t &= i(z_t, k_t, G_t) \end{aligned} \tag{5.1}$$

Analytical solutions for the above equilibrium decision rules do not exist since the objective function is not quadratic and some of the constraints are not linear. Therefore, approximations are required before these decision rules can be computed. Following Kydland and Prescott [1982], steady state solutions for the non-stochastic version of the model (i.e. the model with no shocks to technology and a non-stochastic government sector) are derived. Their numerical value is then computed by "calibrating" the parameters of the model. Next, a quadratic approximation of the non-linear objective function around the steady state values is taken. Finally, linear equilibrium decision rules for this approximated economy are computed.

5.1 Steady State Solutions

To determine the steady state solutions of the model under study, the non-stochastic version of the model is considered. To this end, the shock to technology is set equal to its unconditional mean, i.e. $z_t=1$, and the stochastic component of government spending is set equal to 0, i.e. $\varepsilon 2_{,t}=0$. It is easy to show that in steady state, where variables are denoted without a time subscript,

$$i = \delta K$$

and

$$G = \frac{\phi}{(1 - \psi)} \qquad (5.1.2)$$

When substituting the model's constraints into the representative agent's utility function, the objective function becomes:

$$L=\sum_{t=0}^{\infty}\beta^t U[f(k_t, h_t) - k_{t+1} + (1-\delta)k_t - G_t, h_t] \qquad (5.1.3)$$

The first order conditions for maximizing this objective are

$$\frac{\partial L}{\partial k_t} = U_1(t)[f_1(t) - \beta^{-1} + 1 - \delta] = 0 \qquad (5.1.4)$$

and

$$\frac{\partial L}{\partial h_t} = U_1(t) f_2(t) + U_2(t) = 0 \tag{5.1.5}$$

In particular, given the utility and production functions specified in the previous section

$$L = \sum_{t=0}^{\infty} \beta^t [\log (k_t^{\theta} h_t^{1-\theta} - k_{t+1} + +(1-\delta) k_t - G_t) - \Delta h_t] \tag{5.1.6}$$

In steady state, equation (5.1.4) implies:

$$\theta k^{\theta-1} h^{1-\theta} = \beta^{-1} - 1 + \delta \tag{5.1.7}$$

or

$$k = \pi h$$

where

$$\pi = \left[\frac{(\beta^{-1} - 1 + \delta)}{\theta}\right]^{\frac{1}{(\theta-1)}} \tag{5.1.9}$$

Furthermore, equation (5.1.5) together with (5.1.1) and (5.1.2) implies:

$$(1-\theta) k^{\theta} h^{-\theta} = \Delta[k^{\theta} h^{1-\theta} - \delta k - G] \tag{5.1.10}$$

or

$$h = \frac{[\Delta G + (1-\theta)\pi^{\theta}]}{[\Delta(\pi^{\theta} - \delta\pi)]} \tag{5.1.11}$$

Given the steady state solutions for *k* and *h*, as represented by equations (5.1.8) and (5.1.11), respectively, the production function can be used to determine the steady state solution for output *y*. Furthermore, substituting *y*, *i*, and *G* into the feasibility constraint, the steady state solution for consumption, *c*, can be derived.

5.2 Calibration of Parameters

In order to compute quadratic approximations and solve for the linear equilibrium decision rules, it is necessary to specify the value of all the parameters appearing in the various equations. There are several estimation procedure which could be employed to this end.

For once, traditional econometric methods could be used. As a general rule, this approach requires writing a set of structural equations, jointly estimating the relevant parameters and testing any restriction not necessary to identify parameters. In the context of a dynamic growth model, this procedure amounts to derive the optimal decision rules for the consumption, investment and work effort variables, jointly estimate the technology and preferences parameters and then test the overidentifying restrictions imposed on these decision rules.

As a second possibility, the Generalized Method of Moments (GMM) approach could be employed. GMM is an informal econometric method which estimates parameters by setting the first moment conditions of the model equal to their sample counterpart. An advantage of this method is the fact that it allows to quantify the degree of uncertainty in the estimates of the model's parameters.

Calibration represents a third method of dealing with parameters' estimation. The major advantage of calibration, which makes it a more attractive approach relative to the other two suggested methods, is the fact that parameter values are selected so as to make the model consistent with growth considerations, micro panel studies and stylized facts. This approach greatly reduces the number of free parameters. The remaining free parameters are then chosen so as to yield a close correspondence between the second moments predicted by the model and those associated with the sample data. Calibration, which is the procedure adopted here, is conducted assuming that the economy is in steady state. In general, for a given set of parameters, Ω, it is possible to determine a unique set of steady state values for the vector of variables, S. In other words, $S = S(\Omega)$. With calibration, however, we are interested in the inverse mapping, i.e. mapping from part of the vector S, or functions of it, to a subset of parameters. The vector S, in fact, consists of variables for which there exist long run relationships in the data which have remained approximately constant throughout the sample period. Examples of such relationships are the ratio of consumption to total output, the ratio of capital to total output and the amount of work effort.

In order to facilitate the comparison of results across models, whenever feasible, the values of the parameters have been taken directly from previous studies, particularly Kydland and Prescott [1982] and Hansen [1985]. For the remaining variables, the same criteria adopted in previous studies have been followed; yet, the values of the parameters have been selected so as to reflect the different specifications of the model at hand.

The parameter Θ, which represents the share of capital in production, is set equal to 0.36, based upon a time-series study conducted by Kydland and Prescott in which they show this value to be approximately the percentage of GNP received by owners of capital.

The rate of time preference or discount rate, β, is assumed to be 0.99. This value implies an annual real interest rate of approximately 4 percent, which is consistent with the historical performance of interest rates in the U.S. post-war economy.

The depreciation rate δ is set equal to 0.025 so as to yield an annual depreciation rate of 10 percent. While it is acknowledged that different assets depreciate at different rates, the choice of an average annual depreciation rate of 10 percent seems appropriate because, together with an average annual real interest rate of 4 percent, it implies steady state capital-output and investment-output ratios which are close to the historical averages.

The parameter γ is set equal to 0.95, a value consistent with the autoregressive properties of the production function residual, as computed by Solow [1957].

The parameters Φ and Ψ, which fully determine the value of government purchases in steady state, have been chosen so as to satisfy the conditions that $\Phi > 0$ and $0 < \Psi < 1$. These restrictions are required to obtain a positive and stationary series for government expenditures. In particular, the value for the parameter Φ has been determined by fitting a simple first-order autoregressive equation to the government expenditures time series. This exercise leads to set Φ at a value of 0.98. The value for Ψ is selected by imposing the constraint that, in steady state, government purchases be approximately 22.5 percent of total output. In fact, analysis of U.S. post-war data reveals this to be the average contribution of the government sector to the total U.S. economic activity. This constraint, coupled with the value selected for Φ, implies a value for Ψ equal to

0.005.

The parameter Γ is set equal to 2. In the standard growth model, where the steady state value for hours worked is fully determined by Θ, β, δ and Γ, a value of 2 for the parameter Γ implies that the steady state value for hours worked is close to 0.33. This value is consistent with the observation that individuals spend approximately 1/3 of their time in market activities and 2/3 of their time in non-market activities. In the two-sector model with indivisible labor where, in addition to the above mentioned parameters, h_0 also contributes to the determination of the steady state value of hours worked, the imposition of the above restriction implies a unique value for h_0 equal to 0.41. In a three sector model with indivisible labor, G (i.e. the steady state value for government purchases) also enters the picture. In the presence of a government sector, the restriction on the steady state value for hours worked implies setting h_0 equal to 0.81, approximately.

The parameters that remain to be discussed are σ_1 and σ_2. Following Kydland and Prescott [1982], and Hansen [1985], the selection of these two standard deviation parameters is restricted by imposing the condition that the values of σ_1 and σ_2 yield a mean standard deviation for output in the model that equals the standard deviation of GNP for the U.S. economy. In Hansen's model, where σ_1 is the only relevant standard deviation, the above restriction implies setting σ_1 equal to 0.00728. This same value has been maintained for the three sector model. It can be shown, then, that the implied value for σ_2 is 0.0063.

5.3 Linear-Quadratic Approximation

The model described in section III can be written in the following general form:

$$\begin{aligned} &Max.\ E_0 \left[\sum_{t=0}^{\infty} \beta^t\, s\,(x_t, u_t) \right] \\ &s.t.\ x_{t+1} = g\,(x_t, u_t, \epsilon_{t+1}), \qquad x_0 \text{ given} \end{aligned} \tag{5.3.1}$$

where β is the discount factor;

x_t is an m x 1 vector of state variables at time t;

u_t is an n x 1 vector of control variables at time t;

ε_t is an m x 1 vector of random variables that is iid through time;

s is a scalar-valued function; and

g is a vector-valued function.

As mentioned earlier, for most specifications of s and g, it is virtually impossible to find the control policy function f, such that $u_t = f(x_t)$. Therefore, alternative estimation procedures must be considered.

One possibility is to resort to numerical methods, such as Sims's backward solution method [1985] or Tauchen's quadrature solution method [1986,1987].

Sims's backward solution method requires positing simple time series representations for linearly independent functions of the endogenous variables and using these equations, together with the budget constraints, to solve backward for the decision variables and the exogenous variables of the model. The three major drawbacks of this approach are the following: (a) there are no general algorithms for selecting the functions of the endogenous variables; (b) the solutions for the decision variables may not be measurable with respect to the set of information generated by the exogenous variables impinging on agents' decisions; and, (c) the functions of the endogenous variables may not be policy-invariant, thus making it difficult to assess the consequences of an important class of policy interventions.

Tauchen's quadrature method discretizes the probability space for the exogenous variables and solves the discrete state space model for an exact analytical solution. The major disadvantage of this method is that it does not allow to handle very easily the presence of endogenous state variables (such as capital in a capital-accumulation model). Furthermore, the validity of its solution depends on the specification of the laws of motion of the unobserved taste and/or technology shocks, and the validity of these specifications is not readily verifiable.

In addition to numerical methods, there is another approach which involves finding the optimal control policy function for an approximated problem. This approach, suggested by Kydland and Prescott [1982], requires approximating the non-linear model with a quadratic objective function and linear constraints. The task is accomplished by taking second-order and first-order Taylor expansions of the corresponding non-linear functions around the steady state of the system. Singleton [1987] claims that the quality of the approximation depends entirely on the nature of the agents' objective function, the degree of nonlinearity in the constraints, and the distributions of the shocks. He concludes that for business cycle models with smooth sources of endogenous dynamics, such as the model at hand, the linear-quadratic approximation method is, for most purposes, fairly accurate. For this reason, this approach has been adopted in this analysis.

The general problem described in (5.3.1) can be recast in the following form:

$$\begin{aligned} &Max.\ E_0\left[\sum_{t=0}^{\infty}\beta^t\,(x_t'Qx_t + u_t'Ru_t + 2x_t'Wu_t)\right] \\ &s.t.\ \ x_{t+1} = Ax_t + Bu_t + E\epsilon_t\ , \qquad x_0 given \end{aligned} \tag{5.3.2}$$

where

$$\begin{aligned} &s(x_t, u_t) \approx x_t'Qx_t + u_t'Ru_t + 2x_t'Wu_t \\ &g(x_t, u_t, \epsilon_{t+1}) \approx Ax_t + Bu_t + E\epsilon_{t+1} \end{aligned} \tag{5.3.3}$$

and Q and R are symmetric matrices. Notice that if the constraints are already linear (as in the case of the economy under study), a second-order Taylor expansion to approximate the objective function is all that is required.

The non-stochastic version of the model can be written as:

$$\begin{aligned} &Max.\ E_0\left[\sum_{t=0}^{\infty} \beta^t\ s\,(x_t\,,\,u_t)\right] \\ &s\,.\,t\,.\ \ x_{t+1} = g\,(x_t\,,\,u_t\,,\,0)\ ,\qquad x_0 \text{ given} \end{aligned} \tag{5.3.4}$$

where, without loss of generality, it is assumed that the unconditional mean of ε_t is zero.

The Lagrangian function, then, becomes:

$$L=\sum_{t=0}^{\infty}\beta^t s\,(x_t,u_t)\,-\lambda'_{t+1}(x_{t+1}-g\,(x_t,u_t,0)) \tag{5.3.5}$$

Taking the derivatives with respect to x_{t+1}, u_t and λ_{t+1} and eliminating the time subscripts, the following set of nonlinear equations is obtained:

$$\begin{aligned} &\frac{\partial s\,(x\,,\,u)}{\partial u} + \frac{\partial g\,(x\,,\,u\,,\,0)'}{\partial u} = 0 \\ &\beta\,\frac{\partial s\,(x\,,\,u)}{\partial x} - \lambda + \beta\,\frac{\partial g\,(x\,,\,u\,,\,0)'}{\partial x}\,\lambda = 0 \\ &x - g\,(x\,,\,u\,,\,0) = 0 \end{aligned} \tag{5.3.6}$$

This is a system of $2m + n$ equations in $2m + n$ unknowns. The fixed point of the system is the steady state, x, u, and λ, around which the approximation is taken.

Define:

$$\begin{aligned} \tilde{B} &= \sqrt{\beta}B \\ \tilde{A} &= \sqrt{\beta}(A - BR^{-1}W') \\ \tilde{Q} &= Q - WR^{-1}W' \\ \tilde{x}_t &= \beta^{\frac{t}{2}} x_t \\ \tilde{u}_t &= \beta^{\frac{t}{2}} (u_t + R^{-1}W'x_t) \end{aligned} \tag{5.3.7}$$

and define M as some matrix satisfying

$$\tilde{Q} = M'\Omega M \tag{5.3.8}$$

for some $\Omega < 0$.

It has been shown that if $R < 0$ and the system

$$\begin{aligned} \tilde{x}_{t+1} &= \tilde{A}\tilde{x}_t + \tilde{B}\tilde{u}_t \\ Y_t &= M\tilde{x}_t \end{aligned} \tag{5.3.9}$$

is stabilizable and detectable, then the optimal policy function for the approximated problem described above is the time invariant linear rule

$$u_t = -Fx_t \tag{5.3.10}$$

where

$$
\begin{aligned}
F &= (R + \beta B'PB)^{-1}(\beta B'PA + W) \\
&= (R + B'P\tilde{B})^{-1}\tilde{B}P\tilde{A} + R^{-1}W
\end{aligned}
\tag{5.3.11}
$$

For a proof of (5.3.11), see Appendix I.

The matrix P in (5.3.11) is the steady state solution to the matrix Riccati difference equation

$$
\begin{aligned}
P_t &= Q+\beta A'P_{t+1}A-(\beta A'P_{t+1}B+W)\cdot \\
&\quad (R+\beta B'P_{t+1}B)^{-1}(\beta B'P_{t+1}A+W) \\
&= Q+\tilde{A}'P_{t+1}\tilde{A}-\tilde{A}'P_{t+1}B(R+B'P_{t+1}B)^{-1}B'P_{t+1}\tilde{A}
\end{aligned}
\tag{5.3.12}
$$

as $t \rightarrow -\infty$ with $P_T \leq 0$.

For a proof of (5.3.12), see Appendix II.

Many algorithms have been developed for the solution of the discrete-time Riccati equations. Given the matrices *A, B, Q, R, W*, the scalar β, the tolerance criteria γ_1 and γ_2 , and a matrix norm $\|\cdot\|$, all the following are feasible algorithms.

Direct Iteration

Set an initial symmetric Riccati matrix, $P^0 \leq 0$.

1. At iteration n, compute P^{n+1} and F^n to be:

$$P^{n+1}=Q+A'P^nA-A'P^nB(R+B'P^nB)^{-1}B'P^nA$$
$$F^n = (R + B'P^nB)^{-1}B'P^nA \tag{5.3.13}$$

2. If

$$\|P^{n+1} - P^n\| < \gamma_1 \|P^n\| \tag{5.3.14}$$

and

$$\|F^{n+1} - F^n\| < \gamma_2 \|F^n\| \tag{5.3.15}$$

go to step 3; otherwise, increase n by 1 and go back to step 1.

3. Set

$$F = F^n + R^{-1}W$$
$$P = P^n \tag{5.3.16}$$

Doubling Algorithm

Set the following initial conditions:

$$a^0 = A, \quad b^0 = BR^{-1}B', \quad p^0 = Q$$

1. At iteration n, compute a^{n+1}, b^{n+1}, p^{n+1} and F^n to be:

$$
\begin{aligned}
a^{n+1} &= a^{n}(I + b^{n}p^{n})^{-1}a^{n} \\
b^{n+1} &= b^{n} + a^{n}(I + b^{n}p^{n})^{-1}b^{n}a^{n\prime} \\
p^{n+1} &= p^{n} + a^{n\prime}p^{n}(I + b^{n}p^{n})^{-1}a^{n} \\
F^{n} &= (R + \bar{B}'p^{n}\bar{B})^{-1}\bar{B}'p^{n}\bar{A}
\end{aligned}
\tag{5.3.17}
$$

2. If

$$\|p^{n+1} - p^{n}\| < \gamma_1 \|p^{n}\| \tag{5.3.18}$$

and

$$\|F^{n+1} - F^{n}\| < \gamma_2 \|F^{n}\| \tag{5.3.19}$$

go to step 3; otherwise, increase n by 1 and go back to step 1.

3. Set

$$
\begin{aligned}
F &= F^{n} + R^{-1}W' \\
P &= p^{n}
\end{aligned}
\tag{5.3.20}
$$

Vaughan's Algorithm

1. Find the eigenvalues and eigenvectors of the Hamiltonian matrix,

$$H = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B' \\ QA^{-1} & QA^{-1}BR^{-1}B' + A' \end{bmatrix}$$

$$H = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1} \tag{5.3.21}$$

where Λ is a diagonal matrix containing the eigenvalues of H that exceed 1 in absolute value.

2. Set

$$P = V_{21}V_{11}^{-1}$$
$$F = (R + B'PB)^{-1}B'PA + R^{-1}W' \tag{5.3.22}$$

Once the steady state solution for the Riccati matrix is obtained, it is possible to compute F. Given $u_t = -FX_t$, the law of motion for the state variables can be computed as

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + C\epsilon_t \\ &= Ax_t - BFx_t + C\epsilon_t \\ &= (A - BF)x_t + C\epsilon_t \end{aligned} \tag{5.3.23}$$

Also, given initial conditions for the state x_t and a realization of the shocks ε_t, it is possible to generate time series for x_t via (5.3.23) and for u_t via (5.3.10).

When applying the linear quadratic approximation to the model described in Section III, the following results are obtained. Define x_t and u_t as:

$$x_t = [\,1\ k_t\ z_t\ G_t\,]' \qquad u_t = [\,h_t\ i_t\,]' \tag{5.3.24}$$

Then, the matrices of the transition function for x_t are given by

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 \\ 1-\rho & 0 & \rho & 0 \\ d & 0 & 0 & g \end{bmatrix} \quad B=\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad C=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.3.25}$$

Since these constraints are already linear, only the objective function needs to be approximated. Taking a second-order Taylor expansion of the return function around the steady state, the following matrices for W, Q and R are obtained:

$$W=\frac{m}{2n}\begin{bmatrix} \frac{(1-\theta)}{h}(1+\frac{m}{n})-a_1\frac{n}{m} & -\frac{2}{m}-\frac{1}{n} \\ \frac{\theta(1-\theta)}{hk}(1-\frac{m}{n}) & \frac{\theta}{nk} \\ \frac{(1-\theta)}{h}(1-\frac{m}{n}) & \frac{1}{n} \\ \frac{(1-\theta)}{hn} & -\frac{1}{mn} \end{bmatrix} \tag{5.3.28}$$

$$Q = \frac{m}{2n} \cdot$$

$$\begin{bmatrix} \frac{nz}{m} - \frac{m}{n} - 2 & \frac{\theta}{k}(1 + \frac{m}{n}) & 1 + \frac{m}{n} & -\frac{2}{m} - \frac{1}{n} \\ \frac{\theta}{k}(1 + \frac{m}{n}) & \frac{\theta^2}{k^2}(1 - \frac{m}{n} - \frac{1}{\theta}) & \frac{\theta}{k}(1 - \frac{m}{n}) & \frac{\theta}{kn} \\ 1 + \frac{m}{n} & \frac{\theta}{k}(1 - \frac{m}{n}) & -\frac{m}{n} & \frac{1}{n} \\ -\frac{2}{m} - \frac{1}{n} & \frac{1}{n} & \frac{\theta}{kn} & -\frac{1}{mn} \end{bmatrix} \tag{5.3.26}$$

and

$$R = \frac{m}{2n} \begin{bmatrix} \frac{\theta(\theta - 1)}{h^2} - \frac{(1-\theta)^2}{h^2} \frac{m}{n} & \frac{(1-\theta)}{h} \frac{m}{n} \\ \frac{(1-\theta)}{h} \frac{m}{n} & -\frac{1}{mn} \end{bmatrix} \tag{5.3.27}$$

where

$$\begin{aligned} z &= 2\log n - 3 \\ n &= k^{\theta} h^{1-\theta} - i - G \\ m &= k^{\theta} h^{1-\theta} \end{aligned} \tag{5.3.29}$$

When substituting the relevant parameter values and steady state solutions, the above matrices become:

$$W = \begin{bmatrix} 3.9656 & -3.4080 \\ 0.0603 & 0.0528 \\ -1.9089 & 1.6728 \\ 3.5686 & -1.5054 \end{bmatrix} \tag{5.3.32}$$

$$Q=\begin{bmatrix} -5.8382 & 0.0892 & 2.8230 & -3.4080 \\ 0.0892 & -0.0036 & -0.0283 & 0.0528 \\ 2.8230 & -0.0283 & -1.8589 & 1.6728 \\ -3.4080 & 0.0528 & 1.6728 & -1.5054 \end{bmatrix} \tag{5.3.30}$$

and

$$R=\begin{bmatrix} -10.9277 & 3.9656 \\ 3.9656 & -1.5054 \end{bmatrix} \tag{5.3.31}$$

VI. INTERPRETATION OF STEADY STATE VALUES AND DECISION RULES

6.1 Analysis of Steady State

The vector of steady state values implied by any dynamic economic model is strictly a function of (a) the vector of parameter values selected when calibrating the model, and (b) the underlying structure of the model.

It is easy to verify that Hansen's indivisible labor model and the model considered in this analysis imply essentially the same vector of steady state values. Indeed, the only difference is represented by the value of consumption, which in the model with a government sector is reduced by the amount of the steady state value of government spending. In particular, both models imply the following common steady state solutions:

$$k = 11.3971$$

$$h = 0.3000$$

$$i = 0.2849$$

$$y = 1.1112$$

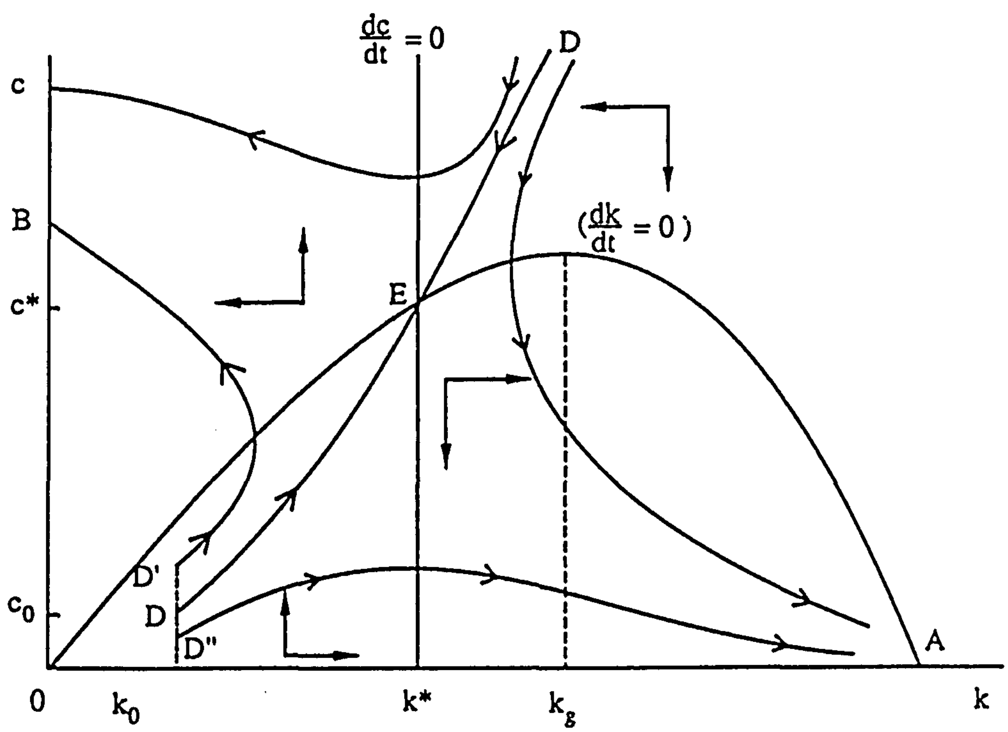
In addition, Hansen's model implies $c = 0.8263$, whereas the model with government implies $c = 0.5763$, since $G = 0.2500$.

The fact that both models are characterized by virtually the same vector of steady state values is not surprising. This result is partly attributable to the fact that, in order to facilitate comparison of relevant statistics across models, whenever feasible, the parameters associated with the model with government have been selected so as to be consistent with, if not identical to, those of previous studies. In particular, careful consideration has been given to the choice of parameters in Kydland and Prescott [1982] and Hansen [1985].

There is, however, another explanation for the above finding. It centers around the observation that neither model actually selects parameter values for the utility function. Instead, these parameters are implicitly determined by imposing the condition that the steady state value for work effort be approximately 1/3. This assumption is critical because it implies that, when analyzing the steady state equilibrium, the dynamics of the models can be addressed without any concern for work effort as a decision variable. In other words, both models can be treated as optimal economic growth models in which the only concern is the trade-off between consumption and investment, that is current versus future consumption. For these models, it has been shown that the equilibrium is consistent with a saddle point, as indicated in the two-variable phase diagram in Figure 6.1.1.

This phase diagram illustrates the intertemporal decision problem for an economy in which there is no government sector. Observe that all the pairs in the positive quadrant represent feasible combinations of capital accumulation and consumption. The only exception is represented by the points on the vertical axis above the origin. Without capital, in fact, output would be zero and so would consumption. The locus of $dk/dt = 0$, which is given by $c_t = f(k_t,h)$, starts from the origin, achieves a maximum at k_g where $f'(k_t,h) = 0$, and crosses the horizontal axis at A where $f(k_t,h) = 0$. The locus of $dc/dt = 0$, on the other hand, is vertical at k^* where $f'(k_t,h) = \beta$.

Figure 6.1.1 Dynamics of Capital and Consumption[3]

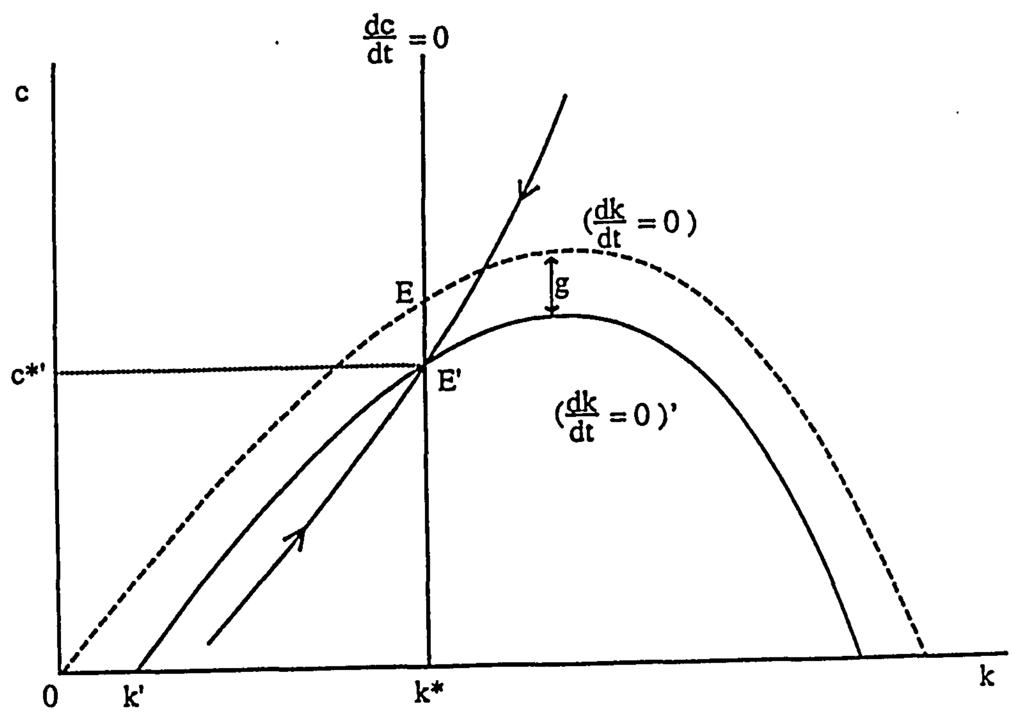


[3] Source: Blanchard, O.J. and Fisher, S., "*Lectures on Macroeconomics*", 1989. Reproduced with the permission of The MIT Press.

Anywhere above the $dk/dt = 0$ locus, capital accumulation is decreasing, i.e. consumption is above the level required to maintain capital constant. The opposite is true for pairs below the $dk/dt = 0$ locus. Similarly, consumption is decreasing to the right of the $dc/dt = 0$ locus, where $f'(k_t,h) < \beta$, while it is increasing to the right of $dc/dt = 0$. These intertemporal movements of consumption and capital accumulation are explained by the directional arrows in the phase diagram. It can be shown that only the trajectory DD (the saddle point path), which converges to E (the saddle point) satisfies all the model's conditions. It is in this sense that E is defined as the unique steady state solution of the model.

When a government sector is introduced, in which government spending is set exogenously, the dynamics of the model remain essentially the same, except that output available for the private sector is now reduced by a uniform amount G (the per capita government spending). This reduction explains the vertical, downward shift of the $dk/dt = 0$ locus to $(dk/dt = 0)'$ in Figure 2. Observe that now, for low levels of capital accumulation, no equilibrium exists. However, once there is enough capital to produce goods for the government (beyond k'), the economy will converge to the equilibrium E', where capital accumulation is at the same level as in the absence of a government sector, and consumption is smaller by the amount of government spending. In steady state, then, government crowds out completely private consumption but has no impact on capital stock and, therefore, investment. It is important to observe, however, that this conclusion holds only because it is assumed that government spending is financed either through lump-sum taxes or borrowing (deficit financing). If, on the other hand, income or capital taxation were incorporated in the model, allocation of resources would be altered because taxes would have a distortionary effect.

Figure 6.1.2 Effects of Government Spending[4]



[4]Source: Blanchard, O.J. and Fisher, S., "*Lectures on Macroeconomics*", 1989. Reproduced with the permission of The MIT Press.

6.2 Analysis of Decision Rules

For both Hansen's model and the model considered in this analysis, the linear decision rules and the intertemporal evolution rules for state variables, which result from the quadratic approximation of the economies around their respective steady states, are summarized by the following system of equations:

$$\begin{aligned} x_{t+1} &= AA\,x_t + \epsilon_t \\ u_t &= -FF\,x_t \end{aligned} \qquad (6.2.1)$$

Specifically, in Hansen's model, the matrices AA and FF are:

$$AA = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ -1.2296 & 0.9418 & 1.9667 \\ 0.0500 & 0.0000 & 0.9500 \end{bmatrix}$$

$$FF = \begin{bmatrix} -0.0016 & 0.0126 & -0.4908 \\ 1.2296 & 0.0332 & -1.9667 \end{bmatrix} \qquad (6.2.2)$$

In other words,

$$\lambda_{t+1} = 0.0500 + 0.9500\lambda_t + \varepsilon_t \qquad 6.2.2.a$$

$$k_{t+1} = -1.2296 + 0.9418k_t + 1.9667\lambda_t \qquad 6.2.2.b$$

$$h_t = 0.0016 - 0.0126k_t + 0.4908\lambda_t \qquad 6.2.2.c$$

$$i_t = -1.2296 - 0.0332k_t + 1.9667\lambda_t \qquad 6.2.2.d$$

In the model which incorporates a government sector, the matrices AA and FF are given by:

$$AA=\begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ -1.0343 & 0.9418 & 1.5944 & 0.4136 \\ 0.0500 & 0.0000 & 0.9500 & 0.0000 \\ 0.0050 & 0.0000 & 0.0000 & 0.9800 \end{bmatrix}$$

$$FF=\begin{bmatrix} -0.0252 & 0.0164 & -0.3460 & -0.4617 \\ 1.0343 & 0.0332 & -1.5944 & -0.4136 \end{bmatrix} \qquad (6.2.3)$$

In other words,

$$\lambda_{t+1} = 0.0500 + 0.9500\lambda_t + \varepsilon_{t+1} \qquad 6.2.3.a$$

$$G_{t+1} = 0.0050 + 0.9800G_t + \nu_{t+1} \qquad 6.2.3.b$$

$$k_{t+1} = -1.0343 + 0.9418k_t + 1.5944\lambda_t + 0.4136G_t \qquad 6.2.3.c$$

$$h_t = 0.0252 - 0.0164k_t + 0.3460\lambda_t + 0.4617G_t \qquad 6.2.3.d$$

$$i_t = -1.0343 - 0.0332k_t + 1.5944\lambda_t + 0.4136G_t \qquad 6.2.3.e$$

Some qualitative as well as quantitative observations are in order with respect to these linear solutions. The "qualitative" analysis will focus mainly on the algebraic signs of the coefficients of the linear decision rules. While the analysis applies equally to either model, it will be discussed with respect to the government model which is more comprehensive, given that it includes G as an additional state variable. The quantitative analysis will then focus on a comparison of the magnitude of the coefficients between the two models.

With respect to the "qualitative" analysis, it should be first mentioned that equations (6.3.2a) and (6.3.2b) do not really require any analytical discussion. This is because they are, in a sense, "trivial" equations, since they simply reflect conditions imposed *a priori* on the model. Equation (6.3.2c), on the other hand, warrants some discussion. At first sight, the presence of this equation may seem to imply that the model is overidentified, since the evolution of capital accumulation over time is elsewhere specified by the equation

$$k_{t+1} = (1 - \delta) k_t + i_t \tag{6.3.2c'}$$

However, a close examination of equations (6.3.2c) and (6.3.2e) reveals that both (6.3.2c) and (6.3.2c') have exactly identical implications for the model, thereby eliminating any potential concern on the issue of overidentification. In particular, the coefficients of equations (6.3.2c) and (6.3.2e) indicate that any change in either G_t, λ_t , or the constant term will have the same effect on i_t and k_{t+1}. This is consistent with specification (6.3.2c') which clearly implies a one-to-one relationship between i_t and k_{t+1}. Changes in k_t, on the other hand, will not have the same impact on i_t and k_{t+1}. This is because the total impact of a change in k_t on k_{t+1} is the result of the sum of two effects: a direct effect, which operates exclusively between k_t and k_{t+1}, and an indirect effect, which works on k_{t+1} via i_t. The direct effect, as indicated by equation (6.3.2c'), is given by $1 - \delta =$ 0.975. The indirect effect, on the other hand, is given, according to equation

(6.3.2e), by a coefficient value of -0.0332. This explains why the k_t coefficient in equation (6.3.2c) has a numerical value of 0.9418 (0.975 - 0.0332).

The second important qualitative observation refers to the fact that most of the coefficients of the linear solutions have signs which are consistent with theoretical expectations, as indicated in the analysis that follows.

(a) Effect of a change in λ_t on h_t and i_t[5]

A positive shock to productivity produces an upward shift in the production function which causes output to be, at least temporarily, above "normal" levels. It is possible to assume that, in response to this occurrence, individual households decide to increase current consumption, while holding current investment and work effort constant. This assumption would imply that the technological shock is completely absorbed within the current time period, so that no implications for future decisions would arise. In general, however, economic agents tend to value both future consumption and leisure, in addition to current consumption, which means that they want to spread the currently high output level to future time periods, as well. Since the only channel through which this intertemporal transfer can be made possible is capital accumulation,

[5] The effect of a change in λ_t on k_{t+1} is not considered, since it is identical, for reasons already discussed, to the effect on i_t.

investment must respond positively to an increase in productivity. This theoretical result is indeed confirmed, for the economy under study, by the finding of a positive coefficient relating technological shocks to investment.

The effect of a technological shock on work effort is more difficult to assess, since it is the result of two forces working in opposite directions. On one side, there is a substitution effect which tends to increase work effort. In fact, the temporarily above normal productivity induces both an intertemporal substitution of current for future work effort and an intratemporal substitution of current consumption for leisure. Simultaneously, however, there is negative wealth effect which operates so as to reduce both current and future work effort. Several economists, after analyzing this issue in detail, have indicated that for plausible models' parameterizations the substitution effect dominates the wealth effect. In other words, they have concluded that, in response to a positive productivity shock, work effort will increase. For the economy under study, this general opinion seems to be confirmed, as indicated by the finding of a positive coefficient relating work effort to technological shocks.

(b) Effect of a change in G_t on h_t and i_t[6]

An increase in government purchases of goods and services causes a negative wealth effect, which induces households to reduce current consumption

---

[6] Again, the effect of a change in G_t on k_{t+1} is not discussed because redundant.

and increase work effort, so as to increase the output available for private consumption. This intuitive explanation is consistent with the model's finding of a positive response of work effort to an increase in government spending.

The model also predicts a positive response of investment to an increase in government expenditures. This result, however, does not seem to conform to expectations. Several authors, including Hall [1980], Barro [1981], Barro & King [1984], Baxter & King [1988], have analyzed the quantitative impact of permanent and temporary changes in government spending on aggregate output and other macroeconomic time series. They have all concluded that (i) temporary changes in government purchases have larger multiplier effects on investment than permanent changes do, and (ii) the multipliers for both the temporary and the permanent changes are negative. The decision rules obtained for the economy under study indicate that increases in government spending have a positive impact on investment, thereby yielding a result which is inconsistent with the finding of other scholars. A possible explanation for this result lies in the strong degree of autocorrelation imposed, when calibrating the model, on the law of motion for government spending (recall that the parameter Ψ was set at a value of 0.98). In fact, the tendency for government expenditures to remain above normal for quite some time may induce agents to increase current investment so as to be able to sustain higher levels of future private consumption (which would, otherwise, be eroded by public consumption). This hypothesis was tested by carrying out a very simple exercise. Linear decision rules were derived for an otherwise identical model, in which the parameter Ψ was, however, assigned lower values. It was found that, for Ψ less than or equal to 0.94, the sign of the coefficient did indeed become negative. This quantitative result suggests that further research on the calibration of government parameters is perhaps warranted.

(c) Effect of a change in k_t on h_t, i_t and k_{t+1}

As expected, the capital series is highly positively correlated. Depreciation and the negative indirect effect of k_t on k_{t+1}, via i_t, imply, however, that the coefficient of autocorrelation has a value less than one.

The negative impact of k_t on i_t, implied by the linear decision rules, can be viewed as the result of a pure wealth effect; as k_t increases, it becomes possible to sustain higher future output levels, so that the incentive to invest is reduced. In other words, agents start trading future for current consumption. A similar explanation justifies the negative effect of an increase in k_t on work effort. Since an increase in k_t results in higher current and future output levels, leisure, which is a normal good, increases. In addition, the lower level of work effort is attributable to the substitutability of capital and labor in production.

Turning now the attention to a quantitative type of analysis, a comparison of the relative magnitude of the coefficients across the two models suggests that two observations are worth noticing.

(a) Effect of a change in k_t on h_t, i_t and k_{t+1}

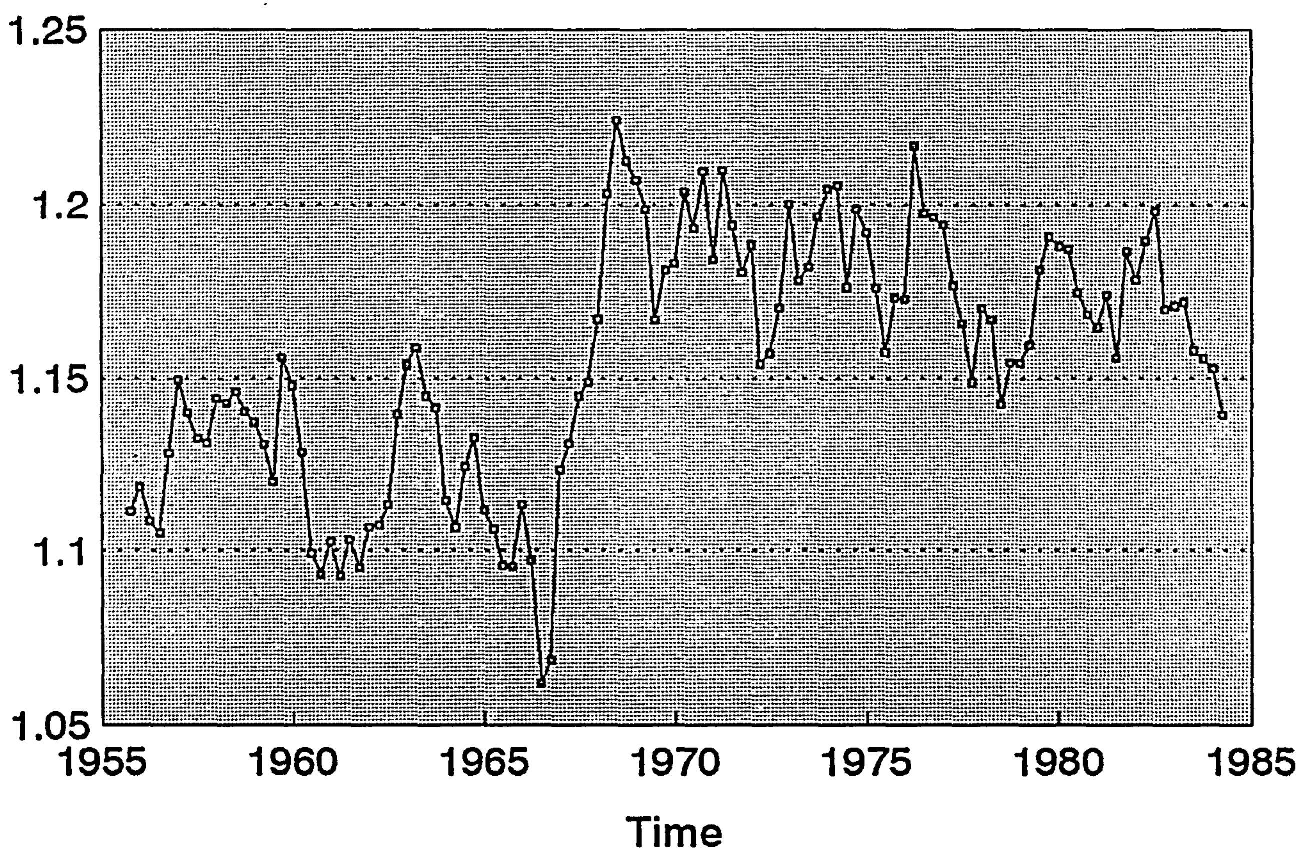
The presence of an exogenous government sector does not seem to have much impact on the k_t multipliers. In other words, the effects of a change in k_t on h_t, i_t and k_{t+1}, respectively, are not much different across the two models. As a matter of fact, while the effect on h_t is slightly magnified (it increases in absolute value from 0.0126 to 0.0164), the effects on i_t and k_{t+1} are identical for both models.

(b) Effect of a change in λ_t on h_t, i_t and k_{t+1}

The introduction of an exogenous government sector substantially dampens the impact of an increase in λ_t on both h_t and i_t (and, consequently, k_{t+1}) . In fact, the λ_t coefficient in the decision rule for work effort declines from 0.4908 to 0.3460, while the coefficient in the investment decision rules (or, equivalently, in the capital stock law of motion) drops from 1.9667 to 1.5944. This result is not difficult to interpret when considering that, in a framework in which government purchases are financed by either borrowing or lump-sum taxation, positive changes in government expenditures can be viewed as negative shocks to production which enter additively.

The linear decisions rules discussed in this section are used to generate simulated time paths for the various graphs provide examples of such time paths. For each variable, two types of paths have been considered, one which uses "raw" data, and the other which uses "detrended" data. A detailed description of the detrending procedure utilized is provided in the next section.

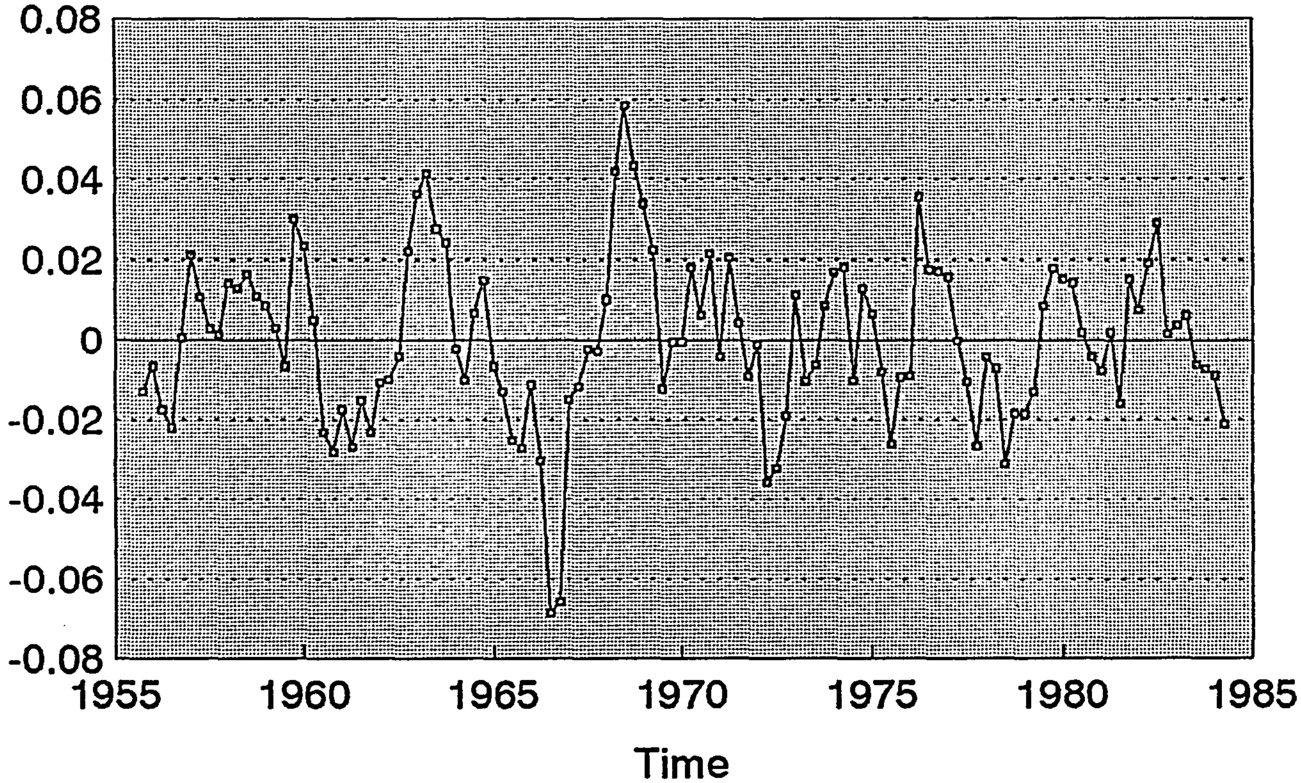
Graph 6.2.1 Total Output



Simulated Time Series Path

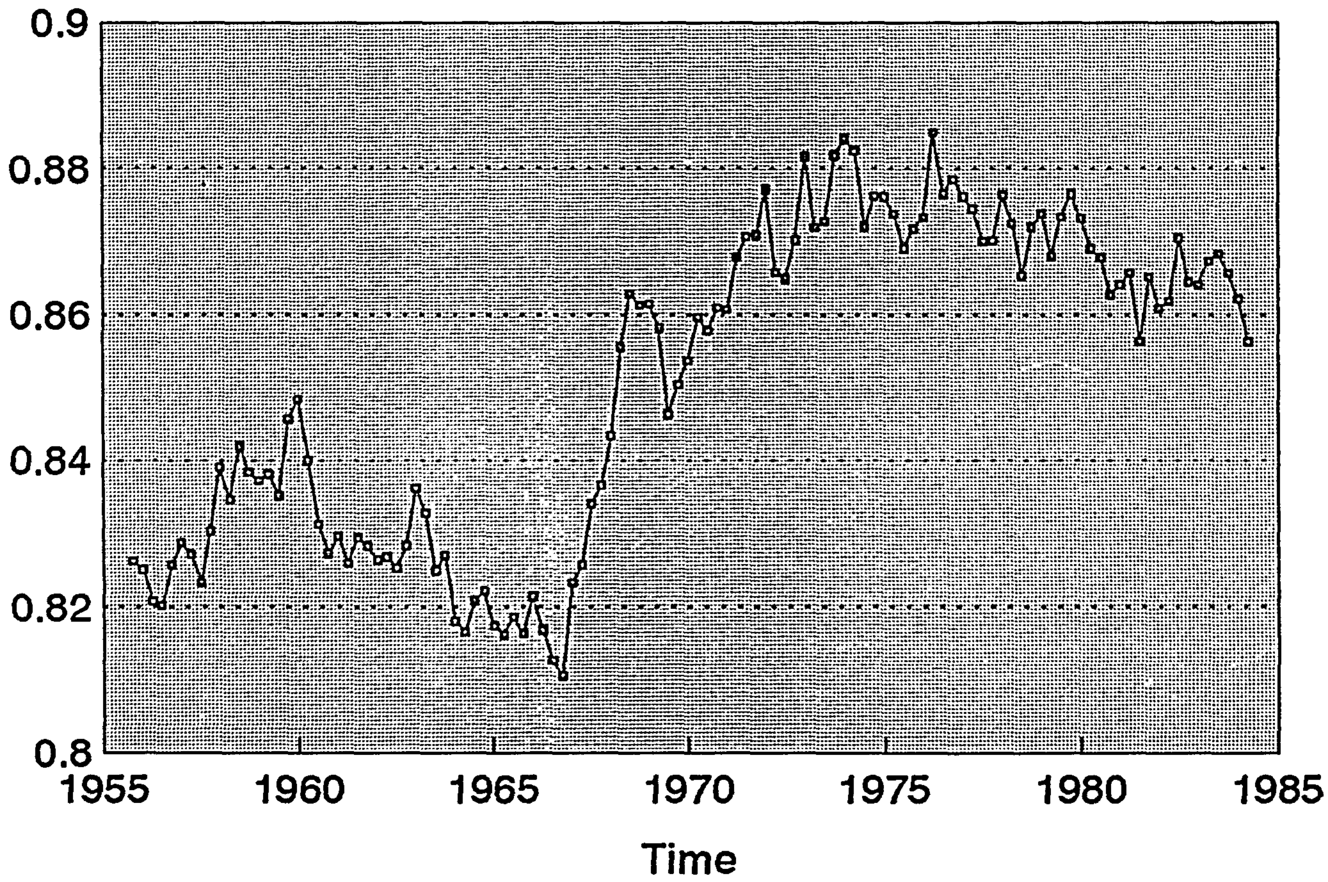
Graph 6.2.2 Total Output

Detrended



Simulated Time Series Path

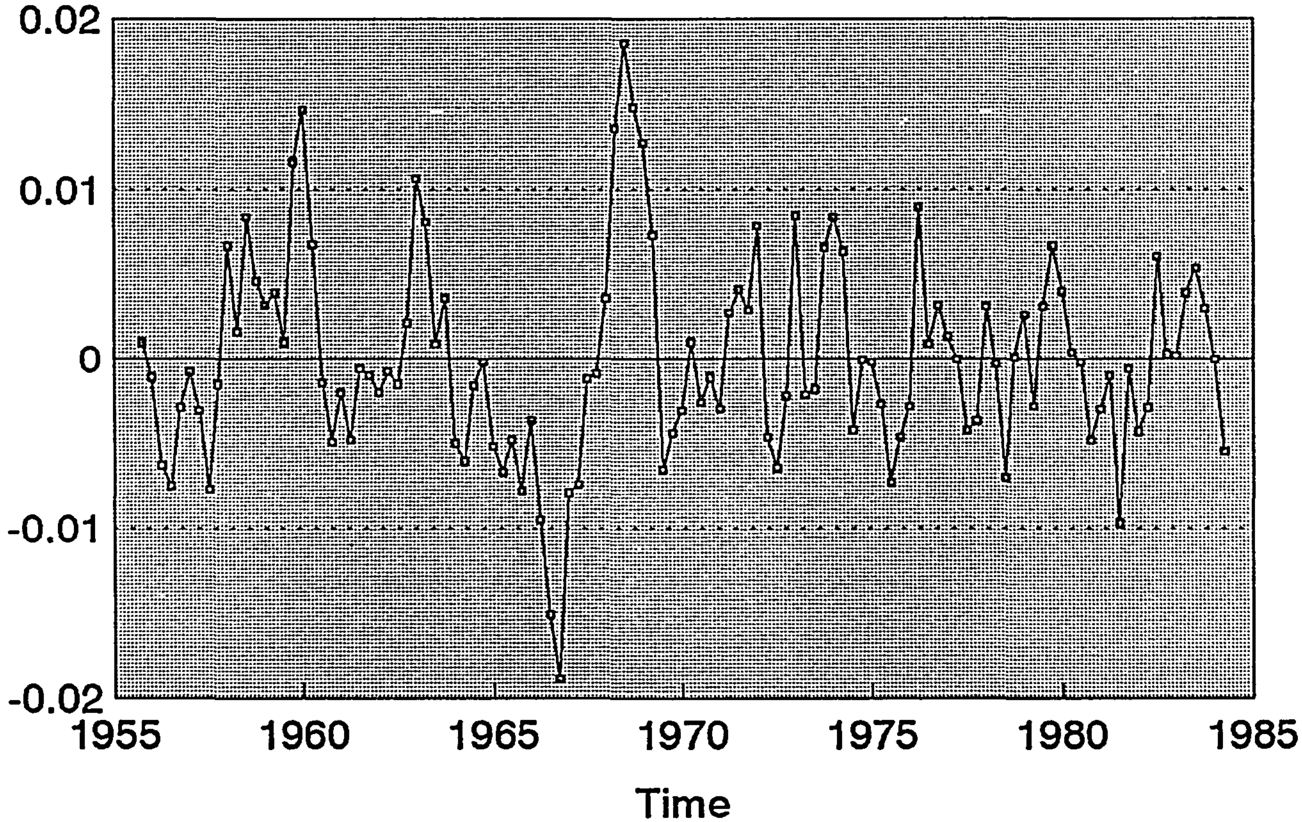
Graph 6.2.3   Consumption Expenditures



Simulated Time Series Path

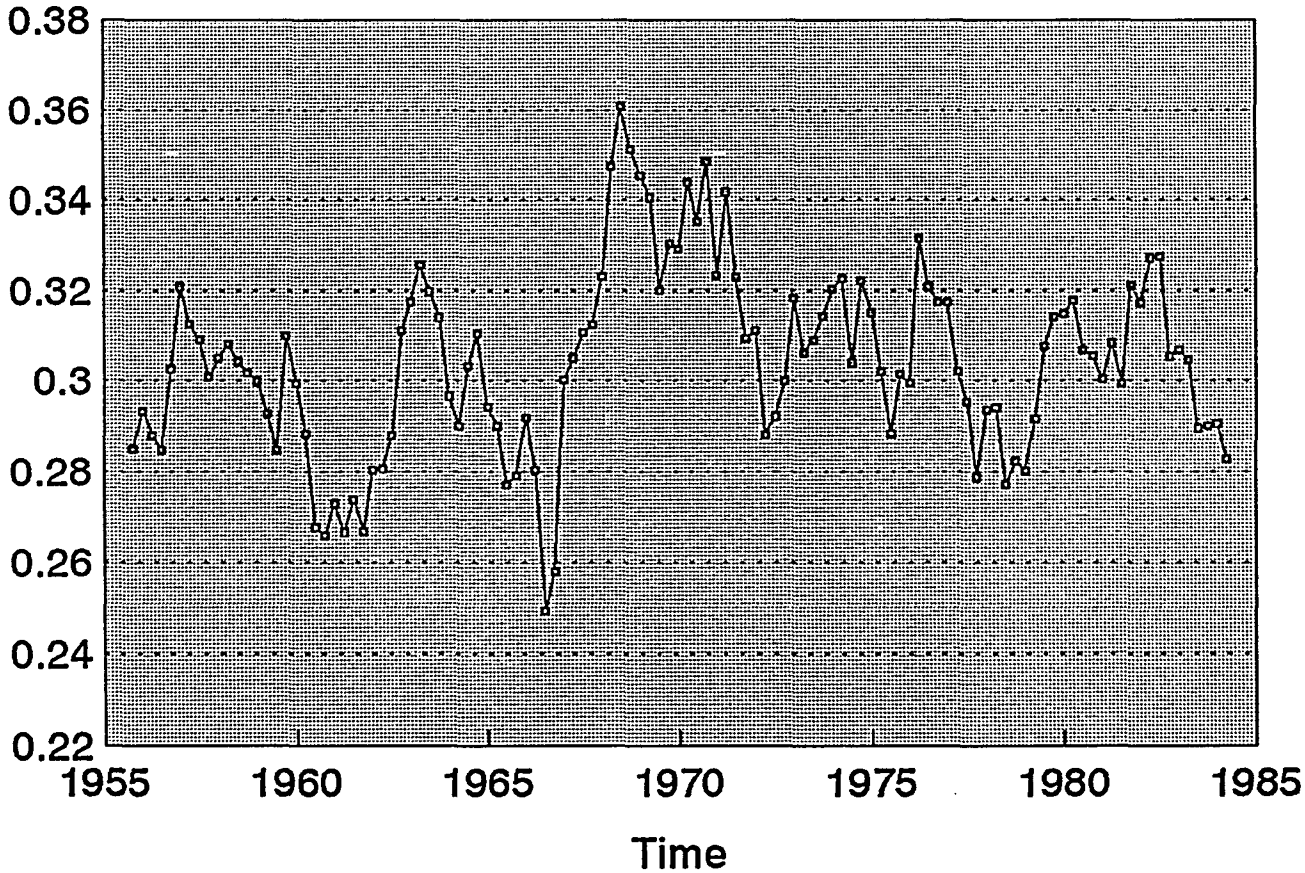
Graph 6.2.4 Consumption Expenditures

Detrended



Simulated Time Series Path

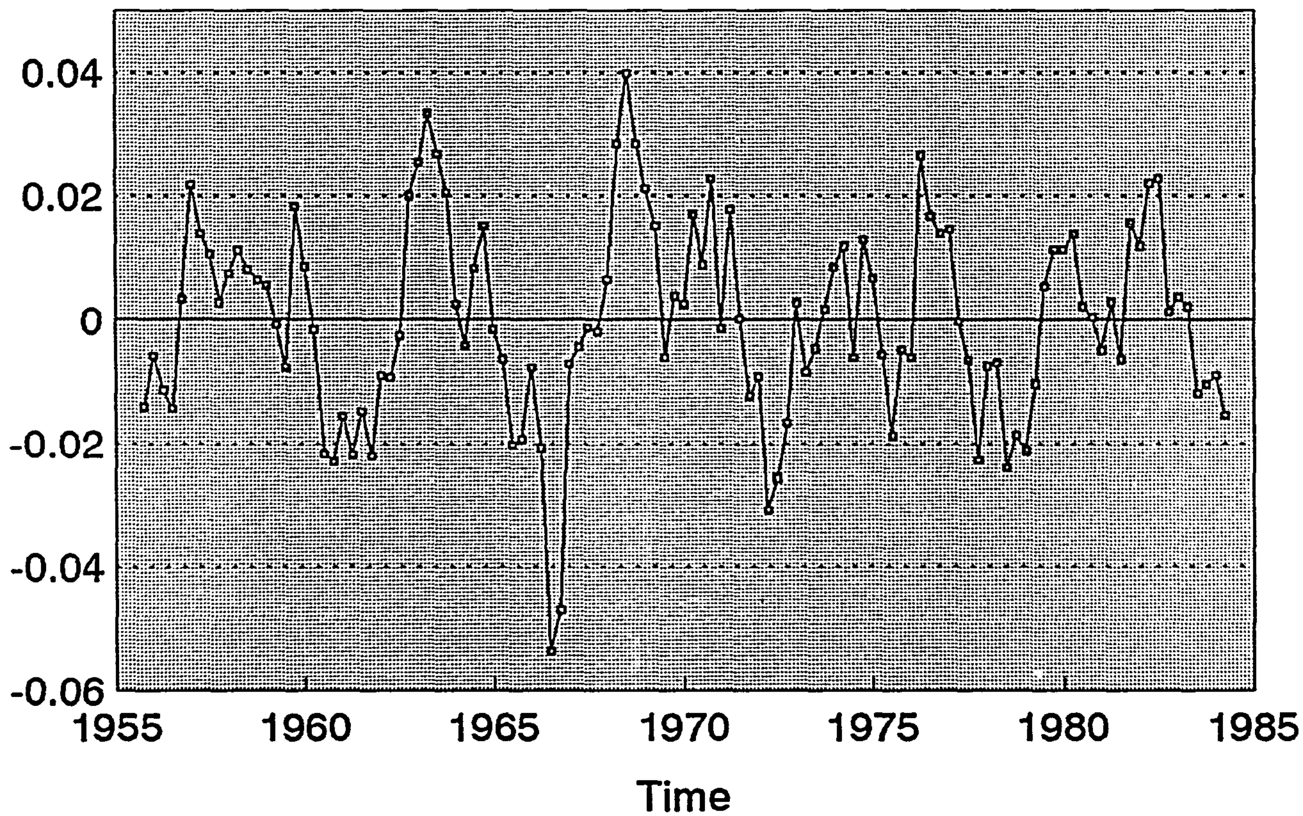
Graph 6.2.5 Investment Expenditures



Simulated Time Series Path

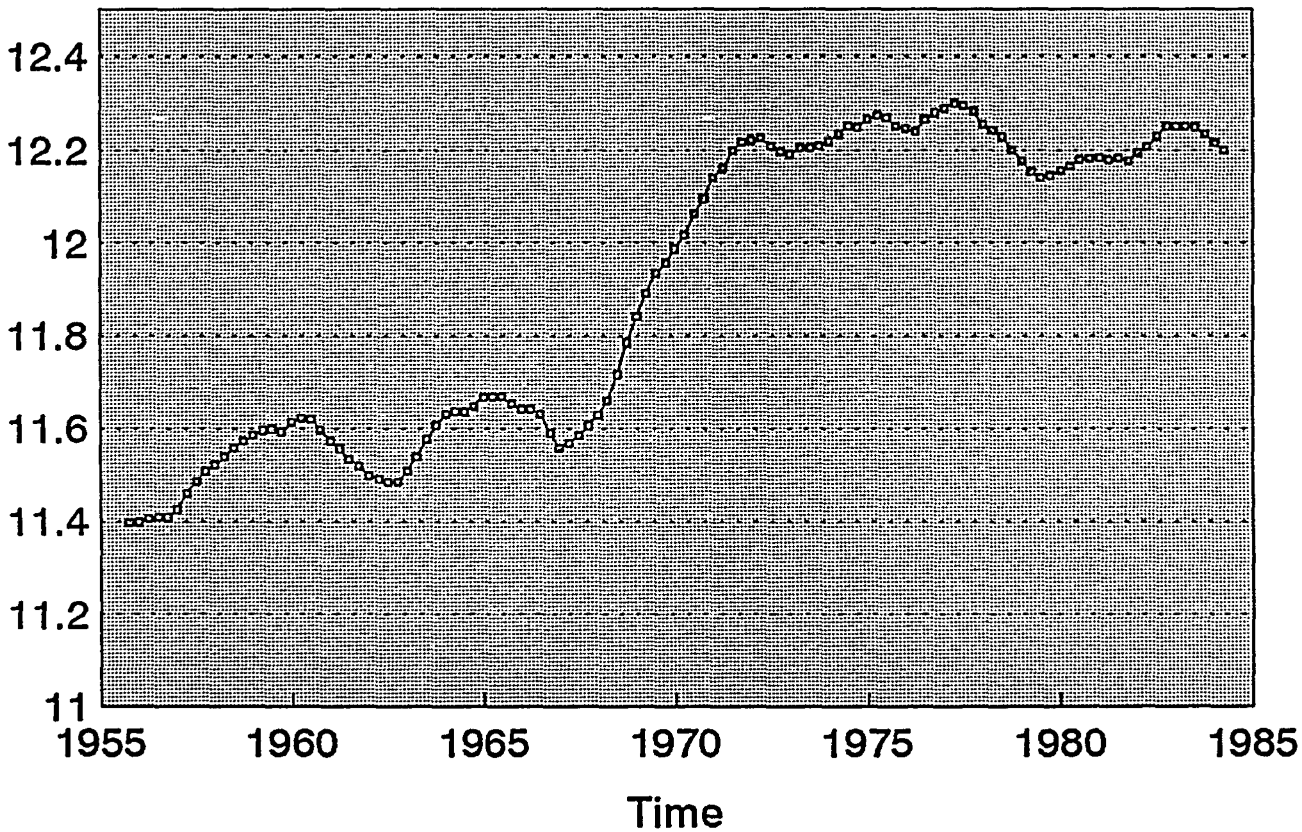
Graph 6.2.6 Investment Expenditures

Detrended



Simulated Time Series Path

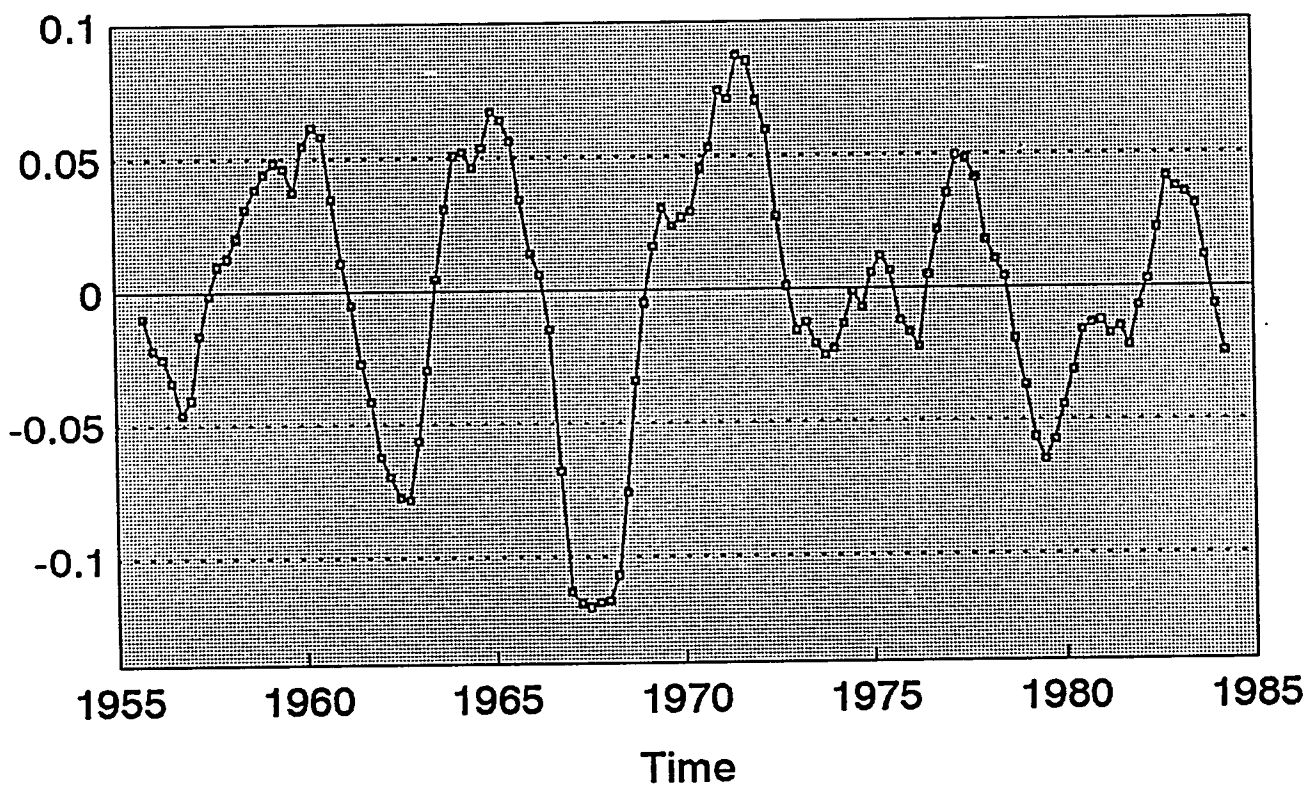
Graph 6.2.7 Capital Stock



Simulated Time Series Path

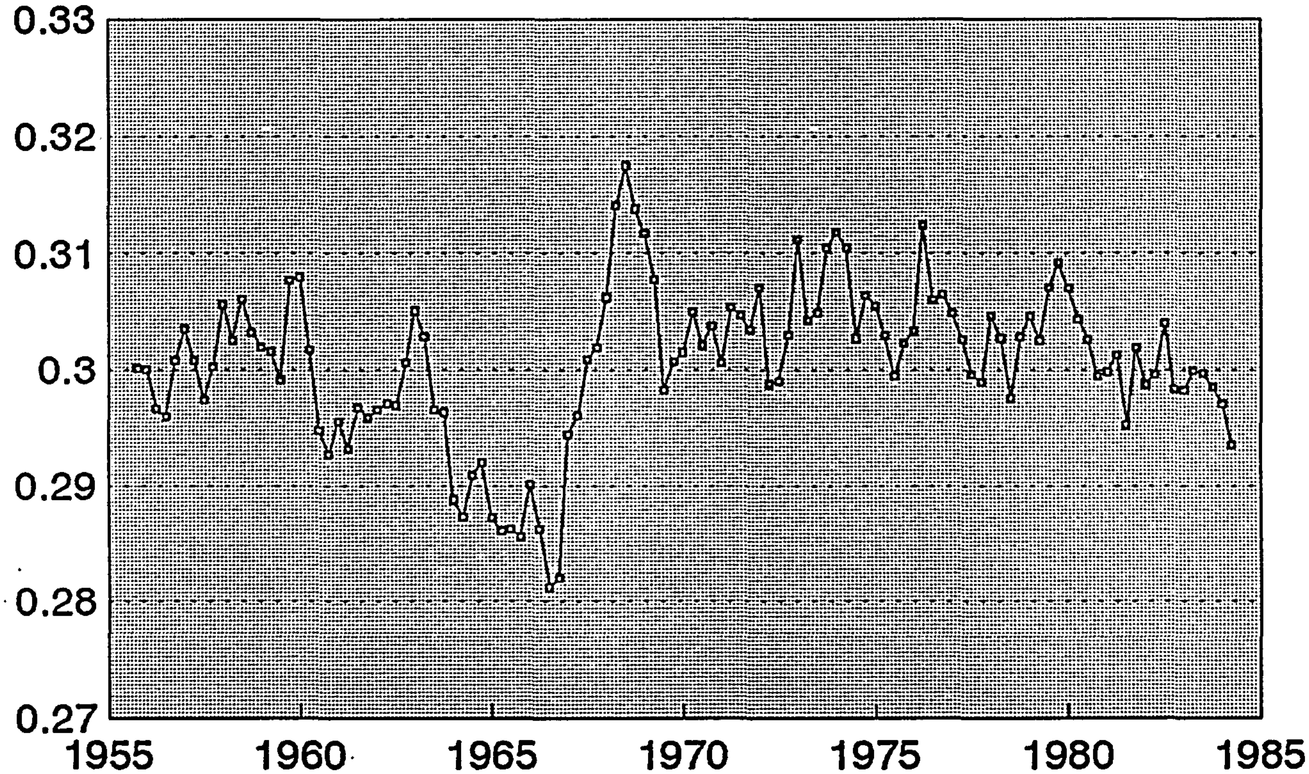
Graph 6.2.8 Capital Stock

Detrended



Simulated Time Series Path

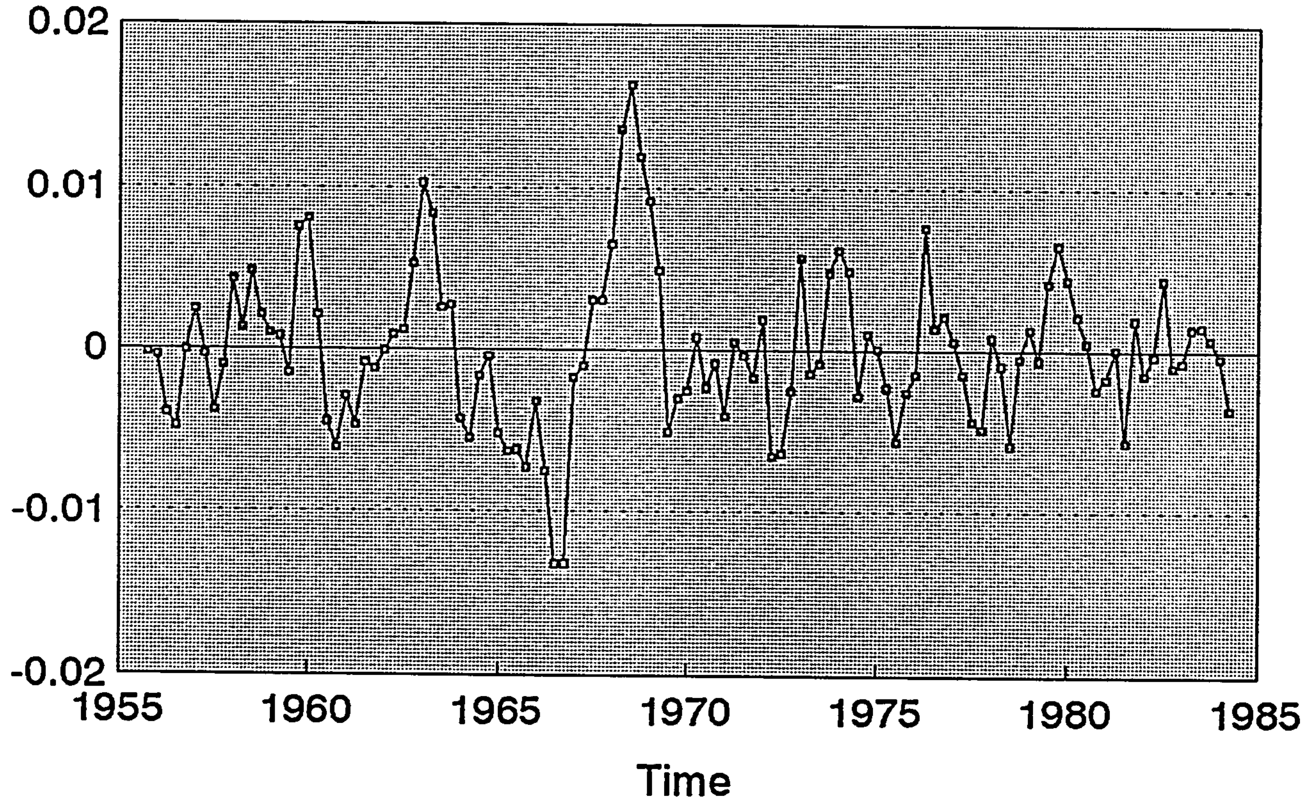
Graph 6.2.9 Hours Worked



Simulated Time Series Path

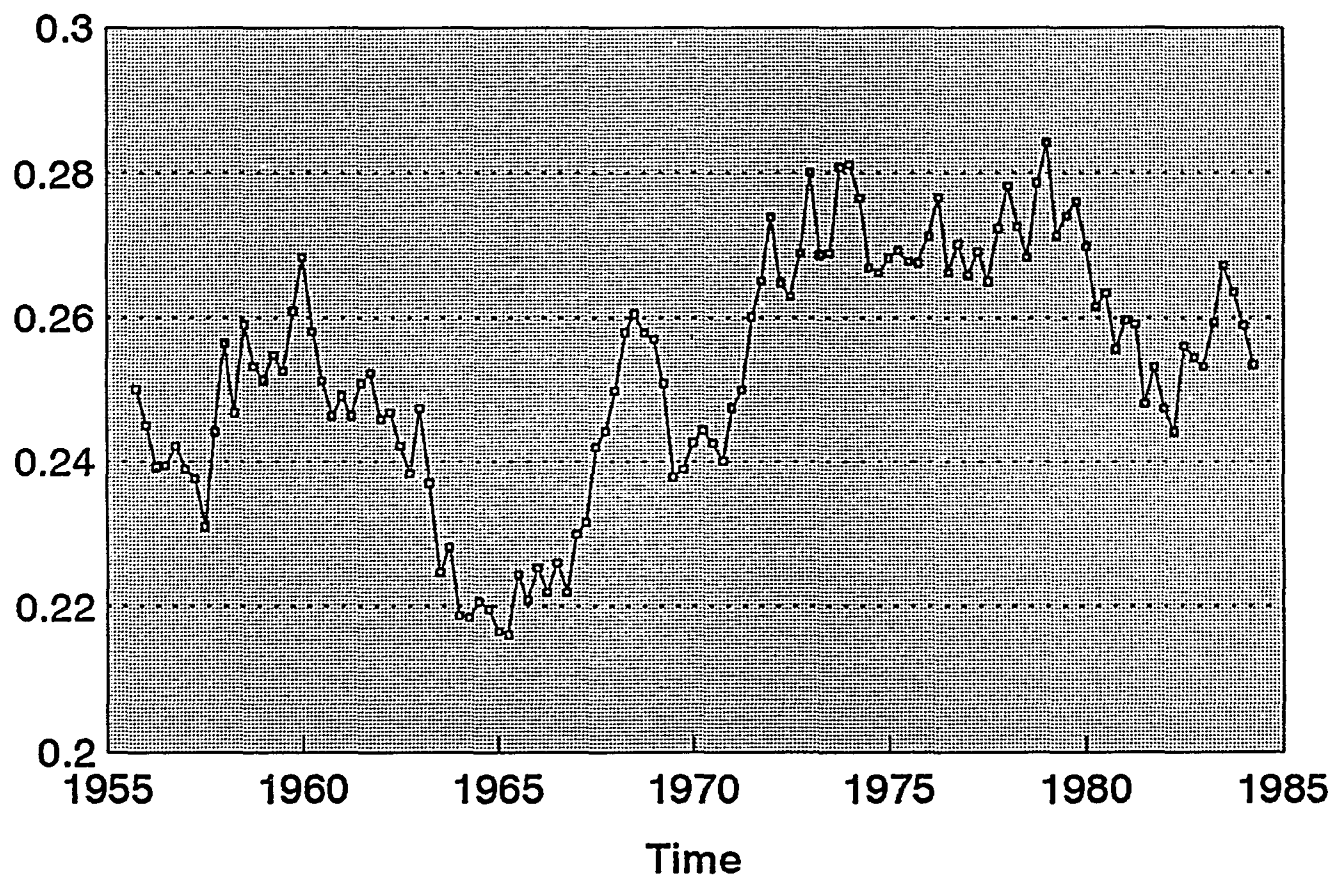
Graph 6.2.10 Hours Worked

Detrended



Simulated Time Series Path

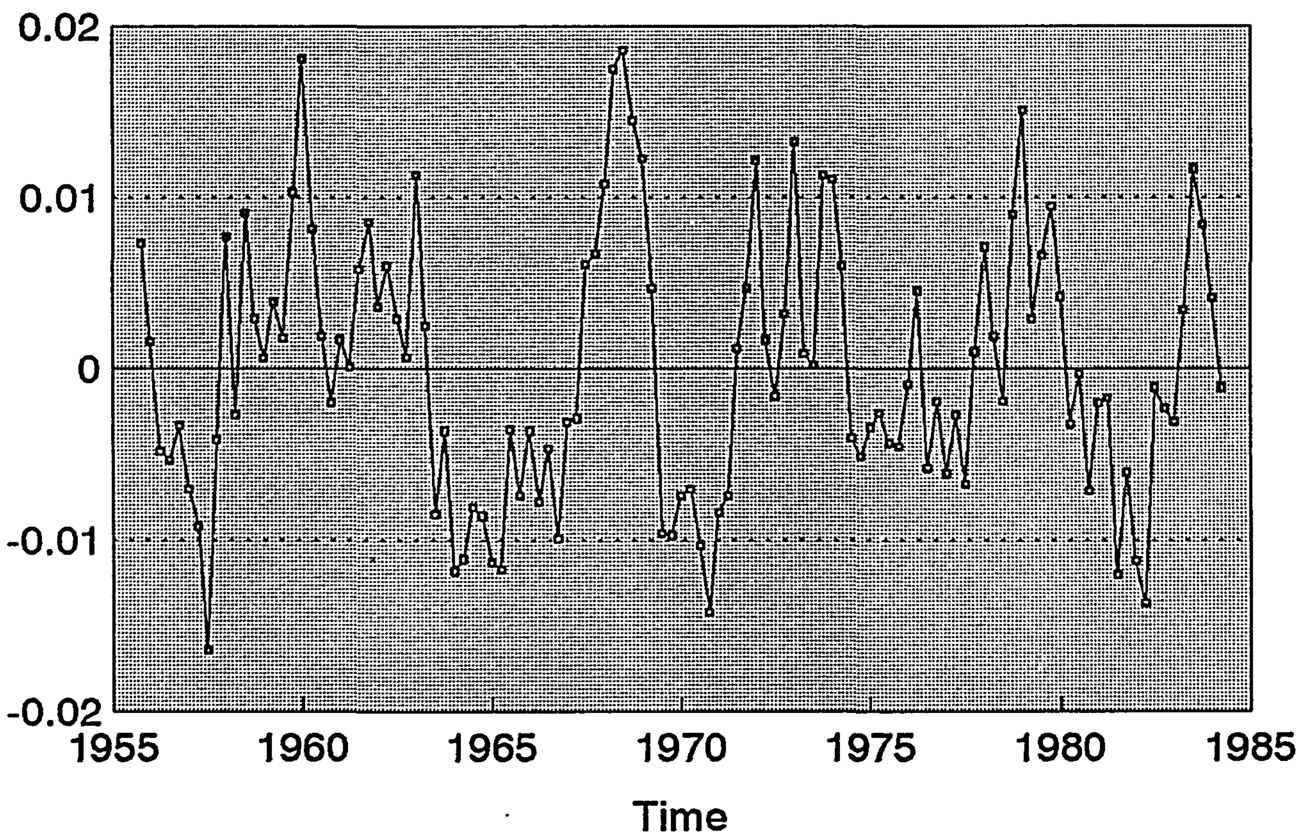
Graph 6.2.11 Government Expenditures



Simulated Time Series Path

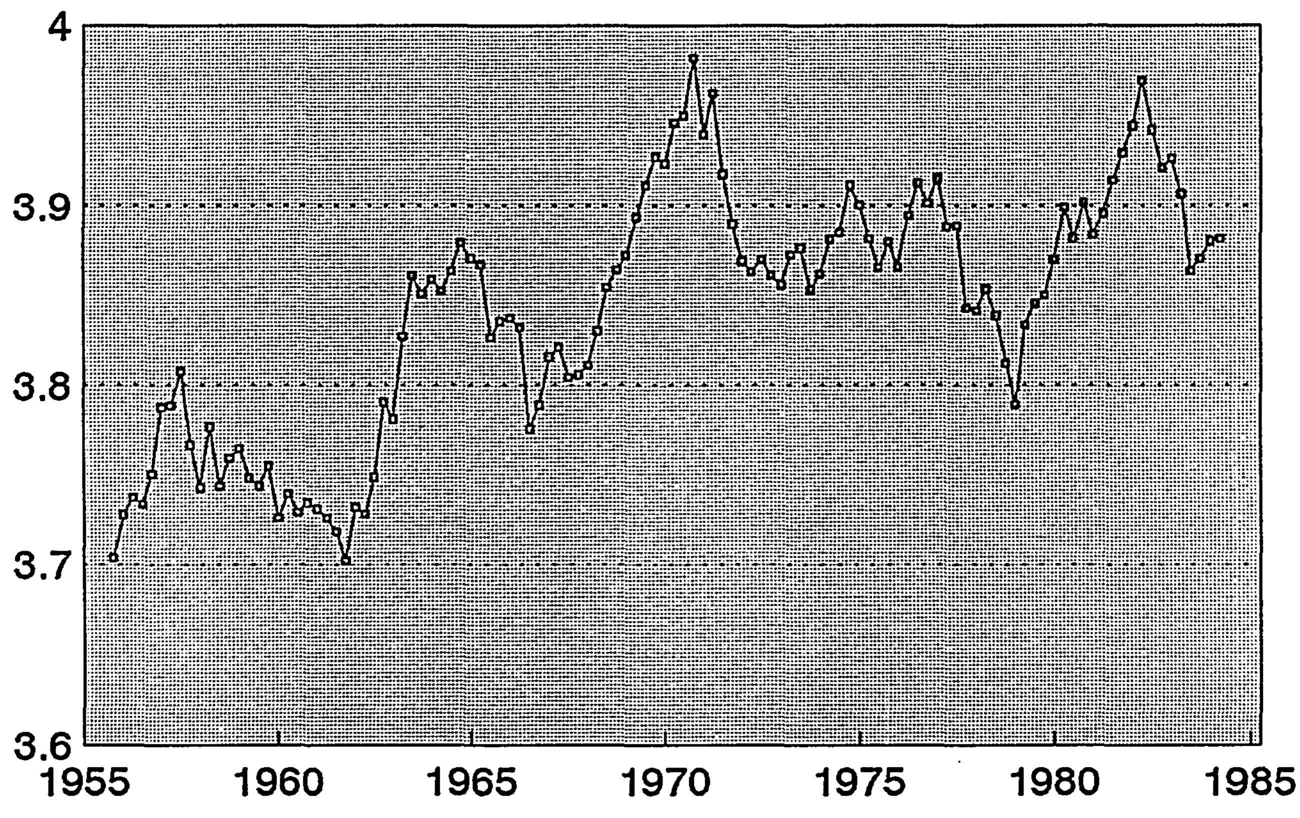
Graph 6.2.12 Government Expenditures

Detrended



Simulated Time Series Path

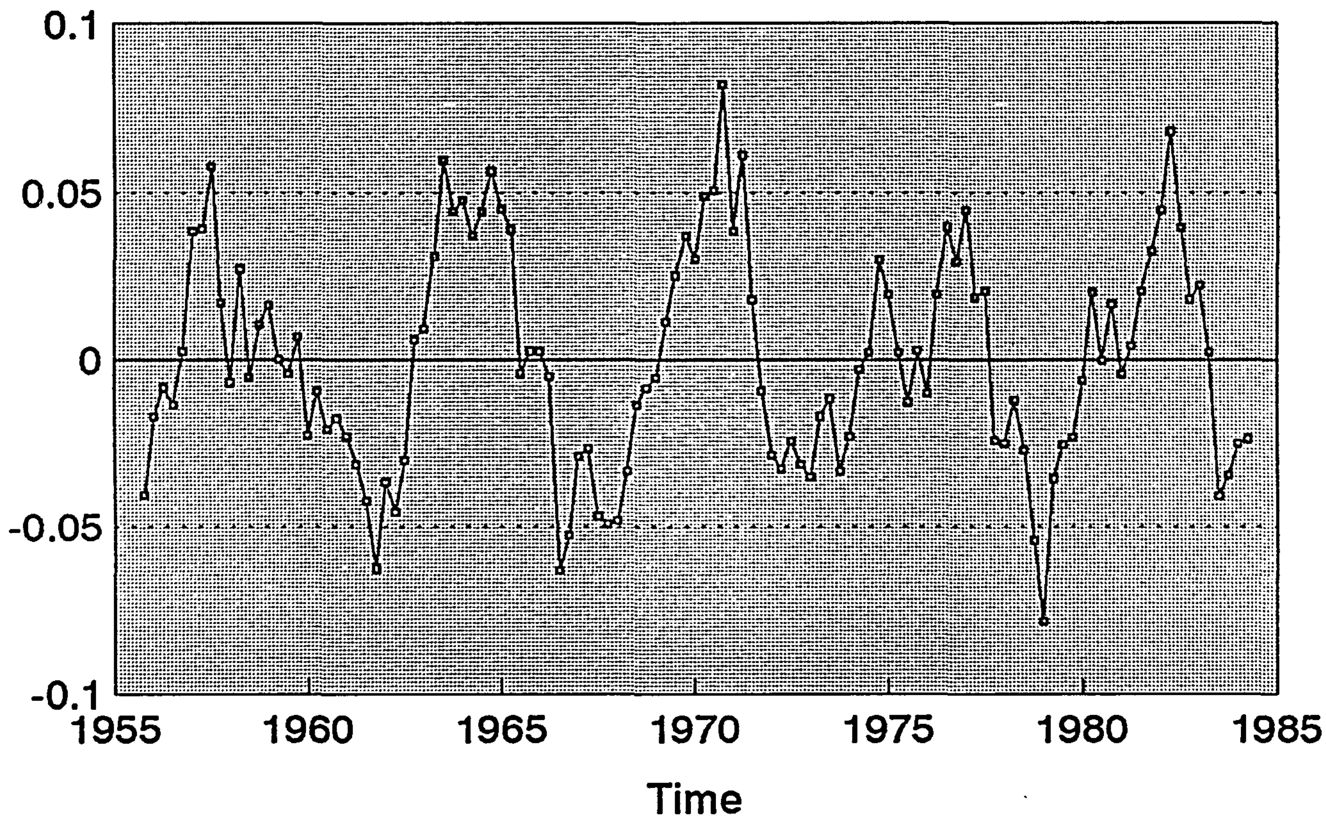
Graph 6.2.13 Average Productivity



Simulated Time Series Path

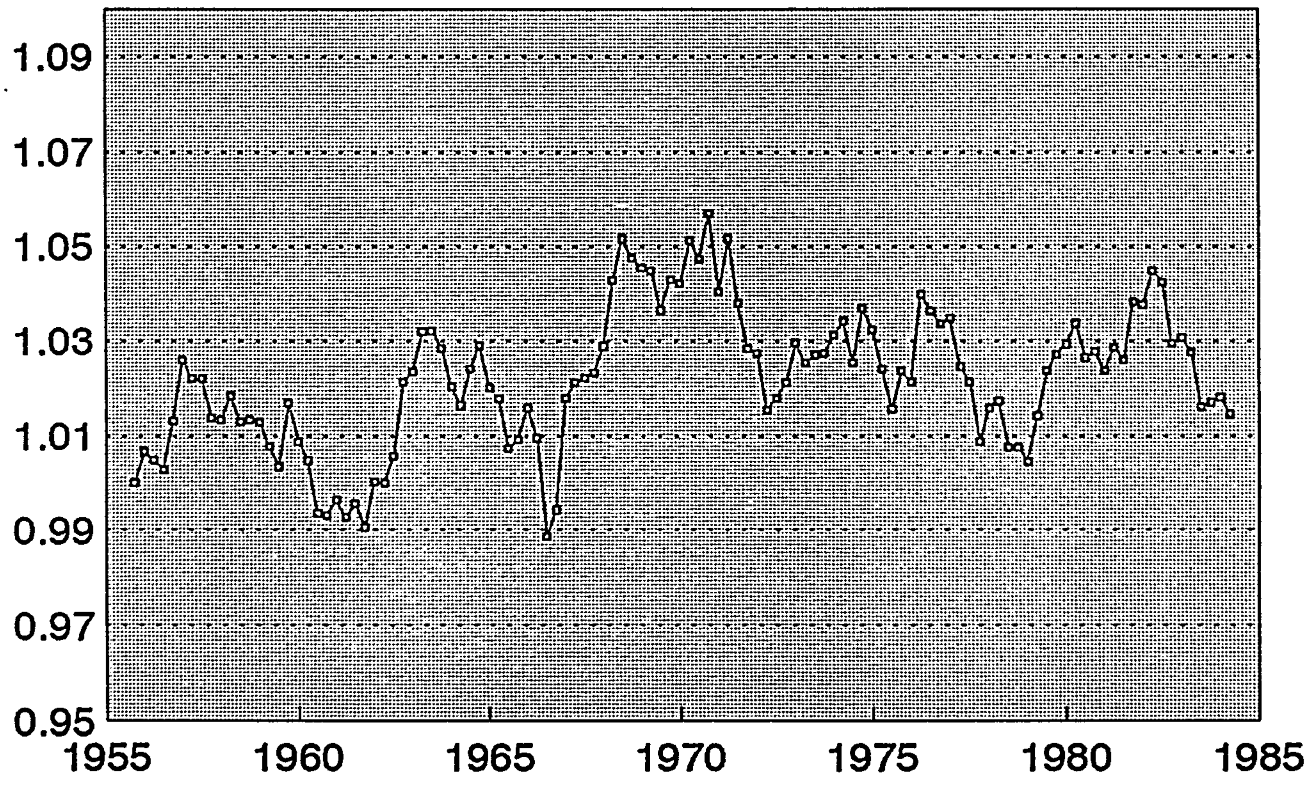
Graph 6.2.14 Average Productivity

Detrended



Simulated Time Series Path

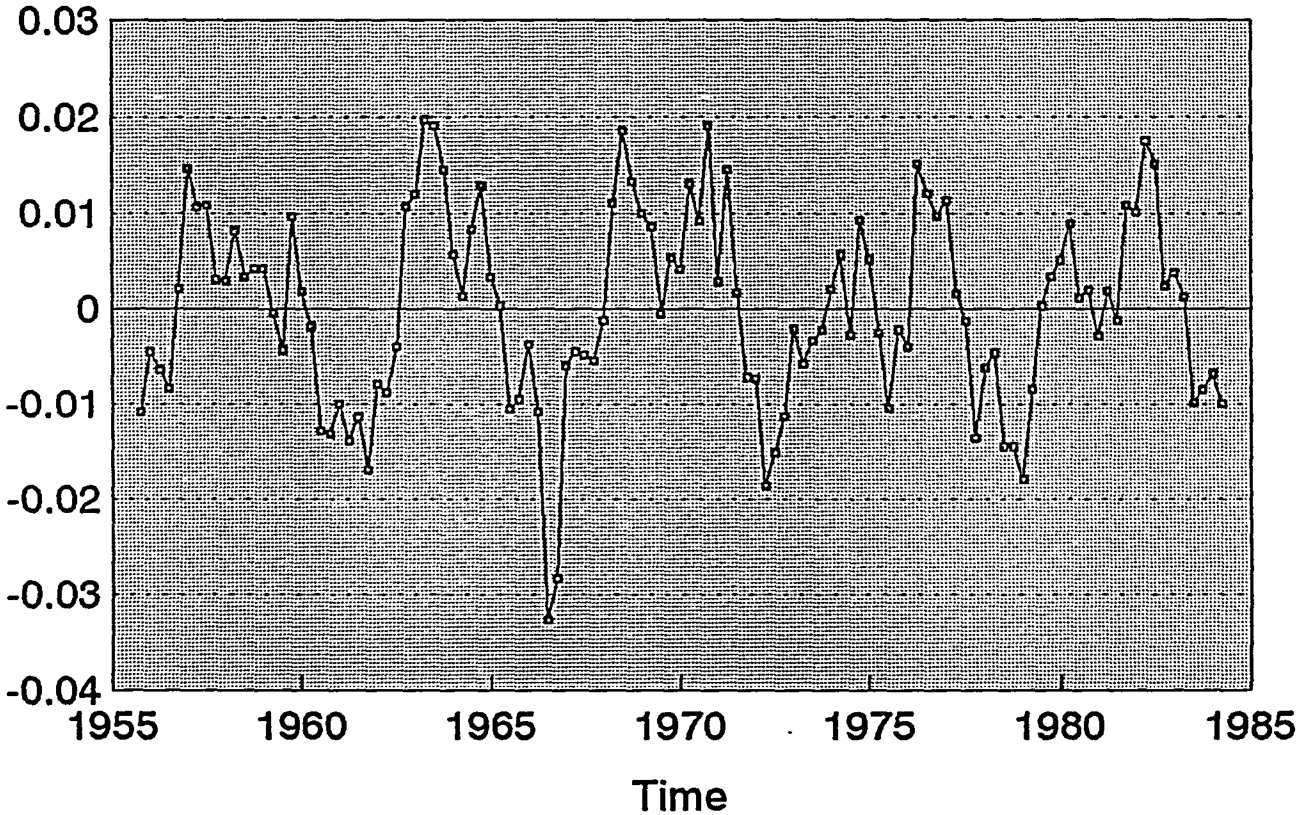
Graph 6.2.15 Technology Shock



Simulated Time Series Path

Graph 6.2.16 Technology Shock

Detrended



Simulated Time Series Path

VII. RESULTS OF SIMULATIONS

In this section, the statistical properties of the economies under study are compared with those of the U.S. post-war economy. In particular, the summary statistics considered are (a) the standard deviations of the individual time series and (b) the correlation of the various time series with output.

Table 7.1 reports the above statistics for the U.S. economy, which have been computed using quarterly data for the period 1955.3 - 1984.1. All variables expressed in dollar amounts have been measured using 1972 dollars. In particular, the time series used are real GNP, total consumption expenditures, gross private domestic investment, capital stock (which includes nonresidential equipment and structures), government purchases of goods and services, hours worked (which is measured by total hours for persons at work in non-agricultural industries as derived from the *Current Population Survey*) and productivity (which is output divided by hours). All statistics but those for the government series have been taken from Hansen [1985]. The government statistics have been computed using Citibase data. Before computing these statistics, the data have been logged and detrended so as to eliminate growth considerations and obtain standard deviations that can be interpreted as percentage deviations from their respective trends.

In conformity with previous studies, the detrending procedure used is the one developed by Hodrick and Prescott. For each time series $\{x_t\}$, this procedure involves choosing the smoothed series $\{s_t\}$ which minimizes

$$T^{-1} \sum_{t=1}^{T} (x_t - s_t)^2 +$$

$$\mu T^{-1} \sum_{t=2}^{T-1} [(s_{t+1} - s_t) - (s_t - s_{t-1})]^2 \qquad (7.1)$$

where $\mu > 0$ represents the chosen degree of smoothness. The larger the value of μ, the smoother the $\{s_t\}$ series is. Following previous studies, the value of μ is set equal to 1600. The detrended time series is then obtained by computing $d_t = x_t - s_t$.

Table 7.3 reports the same statistics for the two-sector model with indivisible labor (i.e. Hansen's model). A total of 100 simulations have been conducted and, for each simulation, sample statistics have been computed. The numerical values shown in the table are, therefore, the means of the standard deviations and the means of the correlations with output. Before computing the relevant statistics, each set of artificially generated time series has been logged and detrended using the same procedure that has been adopted for the U.S. time series. For purposes of consistency, the length of the simulated time series has been chosen so as to match the length of the sample for the U.S. economy data (that is, 115 observations).

Table 7.4 reports the results obtained from the simulation of the three-sector model with indivisible labor. The computational procedure utilized for the two-sector model applies here as well.

A comparison of the summary statistics for the artificial economies with those for the U.S. economy reveals that both models are, to a large extent, successful at replicating major stylized facts about business cycles. For example, the basic finding that investment is more volatile than output while consumption is less volatile than output is supported by both the two-sector and the three-sector models.

However, the results reported in the tables also indicate that both models fail to generate sufficient volatility for most variables. There is a very simple technique that could be used to solve this problem. It requires increasing the value of the standard deviation of the shock to technology while simultaneously reducing the value of the shock to government expenditures (so that the standard deviation of output, which is used as a benchmark for comparisons of results, be maintained constant at 1.76). The use of this technique, however, is not advisable since it would make calibration a questionable method for selecting parameter values.

As a matter of fact, the introduction of a government sector partially eliminates the problem. In fact, when comparing the overall properties of the three-sector model with those of the two-sector model (Table 7.4 and Table 7.3 respectively), it can be seen that the presence of a government sector contributes to generate stronger cyclical fluctuations for several variables. For instance, the volatility of both consumption and productivity increases by virtually 50 percent, while the volatility of hours worked increases by 20 percent. Indeed, the standard deviation of hours worked for the artificial three-sector economy is very close in value to its counterpart for the actual U.S. economy (1.62 versus 1.66).

At the same time, the introduction of a government sector provides disappointing results for both the investment and the government expenditures series, whose volatilities are, in fact, largely underestimated and overestimated, respectively. In particular, as indicated in section II, reliance on the impact of the crowding out effect of government expenditures on investment (quantitatively measured by Barro), would lead to predictions of stronger cyclical fluctuations for this series than the two-sector model with indivisible labor would suggest. Unfortunately, the results obtained do not seem to conform with the expectations.

When the correlation statistics are considered, improvements with respect to the two-sector model are observed in relation to the hours, investment, capital stock and productivity series. In terms of the latter, it should be noted that while the two-sector model overestimates its contemporaneous correlation with output, the three-sector model underestimates it. Yet, since in absolute value the extent of the deviation from the true correlation is by far less in the three-sector model, it can be concluded that the correlation for this series is better estimated by the three-sector model. As it turns out, consumption is the only series for which the two-sector model outperforms the three-sector one.

Several factors contribute to explain, at least partially, the findings described above.

Measurement errors, particularly with respect to the empirical estimate of the production function residual and the hours worked series, have already been described by Hansen as the single most important source of discrepancy between the volatility of the time series implied by the artificial economies *vis a vis* the U.S. economy.

The parsimonious modeling of the artificial economies (which is essential to the attainment of tractable mathematical structures) is a second important factor in explaining some of the results obtained. For instance, the economies analyzed here do not allow to make distinctions between durable and non-durable consumption expenditures, between capital investment and inventories, or between government investment and consumption. The models are just too simple to capture these sophisticated subtleties which, however, greatly impact the values of the associated summary statistics.

With respect to the government expenditures series, a third consideration seems to be relevant. The modeling of government behavior may be impaired by the fact that the decision making process by which the government sets the actual level of spending for the economy is often motivated by political considerations in addition to economic ones. In particular, government decision makers are usually concerned with avoiding huge cuts in the spending levels. If this observation were indeed true, the government expenditure time series would be characterized by relatively few and small negative fluctuations as compared to positive ones. When the behavioral rule for government expenditures in an artificial economy is specified by the assumption that its stochastic component be drawn from a normal distribution with mean zero and constant variance, the above stylized fact is obviously overlooked. This may explain why the standard deviation of government expenditures obtained from the artificial economy has a much larger value than the standard deviation computed using U.S. time series data. Future research should be aimed at exploring this issue in more details. A possible method of integrating this observation in an artificial economy would be to construct a model in which the stochastic component of the government expenditures series is bounded from below. This could be accomplished, for instance, by assuming that the stochastic component of government expenditures be drawn from a truncated distribution.

TABLE 7.1

Percentage standard deviations and correlations with output for the U.S. economy[a]

Variable	Standard deviations	Correlations with output
Output[b]	1.76	1.00
Consumption[b]	1.29	0.85
Investment[b]	8.60	0.92
Capital stock[b]	0.63	0.04
Hours[b]	1.66	0.76
Productivity[b]	1.18	0.42
Government purchases[c]	1.84	0.08

a. The U.S. time series used are real GNP, total consumption expenditures, gross private domestic investment and government purchases of goods and services. The capital stock series includes nonresidential equipment and structures. The hours series includes total hours worked in non-agricultural industries. Productivity is output divided by hours.

b. Quarterly data, 1955.3-1984.1, from Hansen (1985)

c. Quarterly data, 1955.3-1984.1, from Citibase data bank.

TABLE 7.2

Values of Parameters Used in the Artificial Economies

	Two-sector Economy	Three-sector Economy
β	0.99	0.99
Γ	2	2
h_0	0.41	0.81
δ	0.025	0.025
Θ	0.36	0.36
γ	0.95	0.95
Φ	-	0.98
Ψ	-	0.005
σ_1	0.00728	0.00728
σ_2	-	0.0063

TABLE 7.3

Percentage standard deviations and correlations with output for the two-sector economy with indivisible labor[a]

Variable	Standard Deviations	Correlations with output
Output	1.76 (0.21)	1.00 (0.00)
Consumption	0.51 (0.08)	0.87 (0.03)
Investment	5.62 (0.69)	0.99 (0.00)
Capital Stock	0.48 (0.09)	0.07 (0.06)
Hours	1.34 (0.16)	0.98 (0.00)
Productivity	0.51 (0.07)	0.87 (0.02)

a. The standard deviations and correlations with output are sample means of statistics computed for each of 100 simulations. The numbers in parenthesis are the sample standard deviations of these statistics.

TABLE 7.4

Percentage standard deviations and correlations with output for the three-sector economy with indivisible labor[a]

Variable	Standard Deviations	Correlations with output
Output	1.76 (0.23)	1.00 (0.00)
Consumption	0.78 (0.11)	0.80 (0.07)
Investment	5.23 (0.68)	0.97 (0.01)
Capital Stock	0.45 (0.10)	0.04 (0.08)
Hours	1.62 (0.20)	0.90 (0.03)
Government Purchases	3.18 (0.41)	0.43 (0.15)
Productivity	0.76 (0.12)	0.38 (0.15)

a. The standard deviations and correlations with output are sample means of statistics computed for each of 100 simulations. The numbers in parenthesis are the sample standard deviations of these statistics.

VIII. CONCLUDING REMARKS

Most of the existing literature on Real Business Cycle Theory relies on technological shocks as the unique source for the cyclical fluctuations typically observed in major macroeconomic time series. The justification for this assumption comes from the observation that, at least for U.S. data, changes in inputs cannot explain entirely the change in output. Using the argument put forward by Solow [1957], the unexplained portion of the change in GNP is then attributed to technological shocks.

While it is recognized that these technological shocks are not purely supply shocks, since they indirectly affect demand through a wealth effect and through the labor/leisure decision made by economic agents, no explicit emphasis has been placed so far on the role played by direct demand shocks in generating fluctuations of aggregate variables.

The major contribution provided by this paper is the acknowledgment of the potential importance of demand shocks through the incorporation of a stochastic government sector in an otherwise traditional Real Business Cycle model.

The basic finding of this paper is that government does indeed play a role in generating aggregate fluctuations, as indicated by the observation that the presence of a government sector enhances the ability of the artificial economy to mimic the fluctuations associated with U.S. post-war time series, even though the model is still unable to generate sufficient volatility for several variables.

This last observation indicates that further research on the subject is warranted. In particular, as pointed out elsewhere in this analysis, future research should be aimed at addressing two main issues.

In the first place, alternative methods of calibrating the parameters associated with the law of motion for government spending should be considered. In fact, it has been shown that, if it were possible to justify empirically the use of a lower numerical value for the autoregressive coefficient of the government spending equation, the linear solutions obtained for the model would have algebraic properties entirely consistent with theoretical expectations.

In addition, future research should address the issue of alternative specifications for the stochastic component of the law of motion of government spending. With the current specification of a shock drawn from a normal distribution with mean zero and constant variance, the artificial economy generates second moment statistics for the government expenditure series which far exceed those computed from U.S. data. Intuitively, the assumption of a government shock drawn from a truncated distribution should contribute to reduce this discrepancy.

APPENDIX

Appendix I

$$
\begin{aligned}
F &= (R + \beta B'PB)^{-1}(\beta B'PA + W') \\
&= (R + \tilde{B}'P\tilde{B})^{-1}(\tilde{B}'P\tilde{A} + \tilde{B}'P\tilde{B}R^{-1}W' + W') \\
&= (R + \tilde{B}'P\tilde{B})^{-1}\tilde{B}'P\tilde{A} + (R + \tilde{B}'P\tilde{B})^{-1}(\tilde{B}'P\tilde{B}R^{-1} + I)W' \\
&= (R + \tilde{B}'P\tilde{B})^{-1}\tilde{B}'P\tilde{A} + (R + \tilde{B}'P\tilde{B})^{-1}(\tilde{B}'P\tilde{B} + R)R^{-1}W' \\
&= (R + \tilde{B}'P\tilde{B})^{-1}\tilde{B}'P\tilde{A} + R^{-1}W'
\end{aligned}
$$

Q. E. D.

Appendix II

$$
\begin{aligned}
P_t = {} & Q + \beta A'P_{t+1}A - (\beta A'P_{t+1}B + W) \cdot \\
& (R + \beta B'P_{t+1}B)^{-1}(\beta B'P_{t+1}A + W')
\end{aligned}
$$

The proof from Appendix I implies:

$$
\beta B'P_{t+1}A + W' = \tilde{B}'P_{t+1}\tilde{A} + (R + \tilde{B}'P_{t+1}\tilde{B})R^{-1}W'
$$

In addition,

$$\beta A'P_{t+1}A = (A' + WR^{-1}B')P_{t+1}(A + BR^{-1}W')$$

$$= A'P_{t+1}A + A'P_{t+1}BR^{-1}W' + WR^{-1}B'P_{t+1}A + $$

$$+ WR^{-1}B'P_{t+1}BR^{-1}W'$$

Substituting into P_t, we obtain:

$$P_t = Q + WR^{-1}W' + A'P_{t+1}A + A'P_{t+1}BR^{-1}W' + WR^{-1}B'P_{t+1}A +$$

$$+ WR^{-1}B'P_{t+1}BR^{-1}W' - [A'P_{t+1}B + WR^{-1}(R + B'P_{t+1}B)] \cdot$$

$$[R + B'P_{t+1}B]^{-1}[B'P_{t+1}A + (R + B'P_{t+1}B)R^{-1}W']$$

$$P_t = Q + WR^{-1}W' + A'P_{t+1}A + A'P_{t+1}BR^{-1}W' + WR^{-1}B'P_{t+1}A +$$

$$+ WR^{-1}B'P_{t+1}BR^{-1}W' - A'P_{t+1}B(R + B'P_{t+1}B)^{-1}B'P_{t+1}A +$$

$$- A'P_{t+1}BR^{-1}W' - WR^{-1}B'P_{t+1}A - WR^{-1}(R + B'P_{t+1}B)R^{-1}W'$$

$$P_t = Q + A'P_{t+1}A - A'P_{t+1}B(R + B'P_{t+1}B)^{-1}B'P_{t+1}A + WR^{-1}W'$$

$$+ WR^{-1}B'P_{t+1}BR^{-1}W' - WR^{-1}(R + B'P_{t+1}B)R^{-1}W'$$

$$P_t = Q + A'P_{t+1}A - A'P_{t+1}B(R + B'P_{t+1}B)^{-1}B'P_{t+1}A +$$

$$+WR^{-1}(R + B'P_{t+1}B)R^{-1}W' - WR^{-1}(R + B'P_{t+1}B)R^{-1}W'$$

$$P_t = Q + A'P_{t+1}A - A'P_{t+1}B(R + B'P_{t+1}B)^{-1}B'P_{t+1}A$$

Q. E. D.

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