74-20,085

PLAKUN, Geraldine Taiani, 1948-
NON-EXISTENCE OF SOLUTIONS TO THE CAUCHY PROBLEM FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS.

The City University of New York, Ph.D., 1974
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

Non-existence of Solutions to the Cauchy Problem for First Order Partial Differential Equations

by

Geraldine Taiani Plakun

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1974

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

May 9, 1974
date

Stanley Kaplan
Stanley Kaplan, Chairman
Examining Committee

May 9, 1974
date

Richard Sacksteder
Richard Sacksteder
Acting Executive Officer

Professor Louis Auslander
Professor Edgar A. Feldman
Professor Richard Sacksteder

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

TABLE OF CONTENTS

Acknowledgements

I would like to thank Professor Stanley Kaplan for his assistance in my mathematical studies, and especially in the writing of this paper. For reassurance and moral support, I am grateful to Professor Stanley Kaplan, my husband Eric, and James Rand.

I also appreciate the financial support given me by The City University of New York, the National Defense Education Act, the National Science Foundation, and my husband.

Introduction

Let P be a partial differential operator in n real variables of the form $P = \sum_{j=1}^{n} A_j \frac{\partial}{\partial \alpha_j}$, where the coefficients A_j are analytic functions in a neighborhood of some point p_0. Let N be an $n-1$ dimensional surface containing p_0 which is non-characteristic with respect to P, i.e. if N is expressed locally as $N = \{\varphi(x) = 0, \varphi \text{ real}\}$ then $P\varphi \neq 0$. We know from the Cauchy-Kovalevsky theorem [4, Chapter 5] that given any function f defined and analytic on N in a neighborhood of p_0 we can always solve the analytic initial value problem locally, i.e. we can find a neighborhood $U \subset \mathbb{E}^n$ of p_0 and an analytic function u on U such that

$$Pu = 0 \quad \text{on} \quad U$$
$$u = f \quad \text{on} \quad N \cap U .$$

Suppose f is not analytic but merely C^∞ on N in a neighborhood of p_0. Can we find a C^∞ solution to the initial value problem? In fact, can we even find a C^∞ solution to the <u>one-sided</u> initial value problem i.e. for any given function f which is C^∞ on N can we find a neighborhood U of p_0 and a function $u \in C^\infty(U)$ such that

$$Pu = 0 \text{ on } (N \cup \{\text{one side of } N \text{ in } \mathbb{E}^n\}) \cap U$$
$$u = f \quad \text{on} \quad N \cap U .$$

In Section 1 of this paper, using the method of characteristic curves, we reduce the problem of solving the C^∞ initial value problem for the general operator with analytic coefficients $P = \sum_{j=1}^{n} A_j \frac{\partial}{\partial \alpha_j}$ at

p_0 with initial surface N, to one of solving the C^∞ initial value problem for an operator P' of the form $P' = \frac{\partial}{\partial t} + i \sum_{j=1}^{n-1} b_j(t,x_1,\ldots,x_{n-1}) \frac{\partial}{\partial x_j}$ at the origin with initial surface $t = 0$. Here all the b_j's are real and analytic. If all the coefficients of our original operator, P, have real ratio, i.e. $\frac{A_j}{A_k}$ is real for all j and k in a neighborhood of p_0, then our problem reduces to one of solving the initial problem of $P' = \frac{\partial}{\partial t}$ at the origin with initial surface $t = 0$. We can always solve the C^∞ initial value problem for this operator with the solution $u(t,x_1,\ldots,x_{n-1}) = f(x_1,\ldots,x_{n-1})$. Thus, as is well-known, we can always solve the C^∞ initial value problem when the coefficients of our operator have real ratio in a neighborhood of p_0. But, suppose this is not the case. Then we shall prove that in the 2 and, with one exception, 3 dimensional case and in specific higher dimensional cases no solution can be found to the C^∞ initial value problem. Although it seems that this is probably true in the general n-dimensional case, we have not yet been able to prove it. But, we shall discuss results of Andreotti and Hill [1] and [3] which strengthen our belief that the C^∞ initial value problem cannot be solved for an n-dimensional operator whose coefficients do not have real ratio.

In Section 2, we prove that the one-sided C^∞ initial value problem cannot be solved in the 2-dimensional case using an idea due to Lewy [4]. We need only look at operators of the form

$$P = \frac{\partial}{\partial t} + i\, b(t,x) \frac{\partial}{\partial x} \quad b \not\equiv 0 \text{ in a neighborhood of the origin.}$$

We see that if u_0 is a C^∞ solution of $Pu = 0$ such that

$\frac{\partial u_0}{\partial x}(0) \neq 0$ then u_0 maps a sufficiently small neighborhood U of $\mathbb{R}^2$ onto a neighborhood V of $\mathbb{C}$ invertibly. This gives rise to a one to one correspondence between functions on U and functions on V. Under this correspondence every C^∞ solution of $P\ddot{u} = 0$ is associated with an analytic function on V. Using this fact and the Schwarz Reflection Principle we prove that we cannot solve the one-sided C^∞ initial value problem in the 2-dimensional case whenever the initial value, f, is real valued, C^∞, but not analytic.

In Section 3, we look at the 3-dimensional case $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{2} b_j(t,x_1,x_2) \frac{\partial}{\partial x}$; where b_1 and b_2 are not both identically equal to zero in a neighborhood of the origin. If $[P,\bar{P}]$, the Lie bracket of P and its complex conjugate $\bar{P}$, is a linear combination of P and $\bar{P}$ in a neighborhood of the origin then, using Frobenius' Theorem, we find that our problem reduces to a problem in the 2-dimensional case. Thus, the one-sided initial value problem cannot be solved. If $[P,\bar{P}]$ is not a linear combination of P and $\bar{P}$ in $0 \times \Omega$ where Ω is any neighborhood of the origin in $\mathbb{R}^2$ we have, from a result of Hans Lewy [5], that if u_0 and v_0 are two independent solutions of $Pu = 0$ then the pair (u_0,v_0) maps a neighborhood U of $\mathbb{R}^3$ invertibly onto a 3-dimensional surface S of $\mathbb{C}^2$, and every solution u of $Pu = 0$ is in one to one correspondence with a function u' on S which can be extended to be analytic, as a function of $\mathbb{C}^2$, on one side of S. Using this fact and the Schwarz Reflection Principle we prove that the (2-sided) initial value problem cannot be solved when our initial function f is real valued, C^∞, but not analytic.

In Section 4, we look at operators of the form $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{n-1} b_j(t,x_1,\ldots,x_{n-1}) \frac{\partial}{\partial x_j}$ with not all b_j's $\equiv 0$ in a neighborhood of the origin, such that the Lie algebra generated by P and $\bar{P}$ has dimension ≤ 3. We use Frobenius' Theorem to reduce such operators to 3-dimensional ones. Thus, except when the Lie algebra generated by P and $\bar{P}$ does not have dimension 2 in a whole neighborhood of the origin but has dimension 2 on $0 \times \Omega$ for all neighborhoods of the origin in $\mathbb{R}^2$, we prove that the C^∞ initial value problem is not solvable for these operators.

In Section 5, we state results of Andreotti and Hill [1] and [3] in n-dimensions. We prove that Lewy's result follows as a special 3-dimensional case of the combined results of the two papers. It seems probable that [1] and [3] imply a result analagous to Lewy's in higher dimensions; in Section 6 we discuss the problem we encountered trying to prove the analogous result in the 4-dimensional case.

Before beginning with the main part of this work, we would like to discuss a method which can be found, for example, in Duff [2], for reducing the study of a linear partial differential equation with real coefficients to that of an ordinary differential equation. We use this method, the method of characteristic curves, often.

Suppose P is a linear partial differential operator with real C^∞ coefficients in n real variables, $P = \sum_{j=1}^{n} A_j(x_1,\ldots,x_n) \frac{\partial}{\partial x_j}$.

Let $T: (x_1,\ldots,x_n) \to (y_1,\ldots,y_n)$ be an invertible transformation of variables. Then $u(x_1,\ldots,x_n)$ is a solution of $Pu = 0$ if and only if $P'u' = 0$ where $u'(y_1,\ldots,y_n) = u(x_1,\ldots,x_n)$, $A_j'(y_1,\ldots,y_n) = \sum_{k=1}^{n} A_k(x_1,\ldots,x_n) \frac{\partial y_j}{\partial x_k}$, and $P' = \sum_{j=1}^{n} A_j'(y_1,\ldots,y_n) \frac{\partial}{\partial y_j}$. The vector $A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}$ of the coefficients is called the characteristic vector.

Any curve in E^n whose tangent is the characteristic vector at every point, i.e. any curve $(x_1(t),\ldots,x_n(t))$ such that $\frac{dx_j}{dt} = A_j(x_1(t),\ldots,x_n(t))$, is called a characteristic curve. From the existence and uniqueness theorems for solutions of ordinary differential equations we know that through each point in E^n there exists one and only one characteristic curve. A function u is a solution of the original partial differential equation if and only if $\sum_{j=1}^{n} A_j(x_1,\ldots,x_n)\frac{\partial u}{\partial x_j} = 0$.

Since $\frac{du}{dt} = \sum_{j=1}^{n} A_j(x_1,\ldots,x_n) \frac{\partial u}{\partial x_j}$ where $\frac{du}{dt}$ is the derivative of u along any characteristic curve, we have that u is a solution of $Pu = 0$ if and only if u is constant along every characteristic curve.

Locally, E^n can be described in terms of the characteristic curves and an $n-1$ dimensional hypersurface, V , which intersects each characteristic curve transversally. In fact, let $y_1,\ldots,y_{n-1}$ be local coordinates on V in a neighborhood of x_0 . We suppose V is given by the equation $x = x(y)$, with $x_0 = x(y_0)$. We define $x(y,t)$ for sufficiently small t , to be the solution of $\frac{dx}{dt} = A(x)$ with $x(y,0) = x(y)$. If, at $t = 0,\ y = y_0$, we have

$$\begin{vmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_{n-1}} \\ \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial t} & \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_{n-1}} \end{vmatrix} = \begin{vmatrix} A_1(x_1,\ldots,x_n) & \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_{n-1}} \\ \vdots & \vdots & & \vdots \\ A_n(x_1,\ldots,x_n) & \frac{\partial x_n}{\partial y_1} & & \frac{\partial x_n}{\partial y_{n-1}} \end{vmatrix} \neq 0$$

(which says only that A is not tangent to V), then the transformation $T: (t,y,\ldots,y_{n-1}) \to (x_1,\ldots,x_n)$ is invertible in a neighborhood of x_0. Thus, solving $Pu = 0$ is equivalent to solving $\frac{du}{dt} = 0$ in this neighborhood.

I. Preliminary Reduction of the Problem

Given an analytic $n-1$ dimensional hypersurface N in E^n, locally represented by $\varphi(x) = 0$ and a first order linear partial differential operator $P = X + iY = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + i \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$ with real valued analytic coefficients, and assuming that N is not characteristic for P, i.e. $P\varphi \neq 0$, can we locally solve the C^∞ initial value problem, that is given an arbitrary C^∞ function, f, on N can we find a locally defined function w such that $Pw = 0$ and $w = f$ on N ?

In this section we shall put P in a more convenient form. Let us assume that $\varphi(0) = 0$, otherwise a linear change of variable will produce this. We know there exists J, $1 \leq J \leq n$ such that $\frac{\partial y}{\partial x_J}(0) \neq 0$ and $a_J(0) + ib_J(0) \neq 0$. For simplicity of notation, assume $J = 1$. Due to the analyticity (C' is obviously sufficient) of φ and the coefficients $a_1 + ib_1$ we have $\frac{\partial y}{\partial x_1}(x) \neq 0$ and $a_1(x) + ib_1(x) \neq 0$ in some neighborhood U of the origin.

First, we shall perform an analytic change of variables

$$T: (x_1, x_2, \ldots, x_n) \to (\theta, \eta_1, \ldots, \eta_{n-1})$$

defined by $\theta = \varphi(x_1, \ldots, x_n)$

$$\eta_j = x_{j+1} \qquad j=1,\ldots,n-1 .$$

Since the jacobian associated with the transformation is:

$$J_T = \begin{pmatrix} \frac{\partial\varphi}{\partial x_1} & \frac{\partial\varphi}{\partial x_2} & & \frac{\partial\varphi}{\partial x_n} \\ & 1 & & \\ & & \ddots & 0 \\ 0 & & & \\ & & & 1 \end{pmatrix}$$

we have $|J_T| = |\frac{\partial\varphi}{\partial x_1}| \neq 0$ in U. Thus, the noncharacteristic plane, N, in the new variables is $\theta = 0$. We define $P' = d_0'(\theta,\eta_1,\ldots,\eta_{n-1})\frac{\partial}{\partial\theta} + \sum_{j=1}^{n-1} d_j'(\theta,\eta_1,\ldots,\eta_{n-1}) \frac{\partial}{\partial\eta_j}$ by $P'\, w'(\theta,\eta_1,\ldots,\eta_{n-1}) = Pw(x_1,\ldots,x_n)$ where $(\theta,\eta_1,\ldots,\eta_{n-1}) = T(x_1,\ldots,x_n)$ and $w' \circ T = w$, or, setting $v = w$, $P(v \circ T) = P'v \circ T$. Since

$$\frac{\partial}{\partial x_j}(v \circ T) = \begin{cases} \frac{\partial\varphi}{\partial x_1} \frac{\partial v}{\partial\theta} & j=1 \\ \frac{\partial\varphi}{\partial x_j} \frac{\partial v}{\partial\varphi} + \frac{\partial v}{\partial\eta_{j-1}} & j=2,\ldots,n \end{cases}$$

by the chain rule, a simple calculation gives

$$d_0' \circ T = \sum_{j=1}^{n} (a_j+ib_j) \frac{\partial\varphi}{\partial x_j}$$

$$d_j' \circ T = a_{j+1} + ib_{j+1} \qquad j=1,\ldots,n-1 .$$

Since y is not characteristic for P at 0 and φ and the coefficients of P are analytic, we have $d_0^1(\theta,\eta_1,\ldots,\eta_{n-1}) = P\varphi(x_1,\ldots,x_n) \neq 0$ in some neighborhood of the origin. Let $P'' = \frac{1}{d_1'} P'$. Then $\theta = 0$ is still noncharacteristic for P'' and any solution w of $P'w = 0$ is also a solution of $P''w = 0$. P'' is now of the form

$$P'' = \frac{\partial}{\partial\theta} + \sum_{j=1}^{n-1} (a_j'' + ib_j'') \frac{\partial}{\partial z_j}$$

where a_j'' and b_j'' are real valued analytic functions of $(\theta,\eta_1,\ldots,\eta_{n-1})$.

Now, we again perform an analytic change of variables $S: (\theta,\eta_1,\ldots,\eta_{n-1}) \to (t,y_1,\ldots,y_{n-1})$. This time we use the method of characteristic curves on the real part of the operator P'', that is, on $\frac{\partial}{\partial\theta} + \sum_{j=1}^{n-1} a_j'' \frac{\partial}{\partial\eta_j}$.

The variable t shall be our parameter for the characteristic curves.

Thus, $\frac{\partial\theta}{\partial t} = 1$, $\frac{\partial\eta_j}{\partial t} = a_j''(\theta,\eta_1,\ldots,\eta_{n-1})$. Our n-1 dimensional hypersurface which intersects characteristics transversally will be the surface $t = 0$, $y_j = \eta_j$, $j=1,\ldots,n-1$. We know that we can perform this change of variables in a neighborhood of the origin provided that the following determinant

$$\begin{vmatrix} \frac{\partial\theta}{\partial t} & \frac{\partial\theta}{\partial y_1} & \cdots\cdots & \frac{\partial\theta}{\partial y_{n-1}} \\ \frac{\partial\eta_1}{\partial t} & \frac{\partial\eta_1}{\partial y_1} & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial\eta_{n-1}}{\partial t} & \frac{\partial\eta_{n-1}}{\partial y_1} & \cdots & \frac{\partial\eta_{n-1}}{\partial y_{n-1}} \end{vmatrix}$$

does not vanish at the origin.

But, at the origin we have

$$\begin{vmatrix} a_1''(0) & 1 & & & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ a_{n-1}''(0) & 0 & & & 1 \end{vmatrix} = 1$$

Thus, this change of variables is well defined.

We define c_j , $j=1,\ldots,n-1$ real valued analytic functions of $(t,y_1,\ldots,y_{n-1})$ by

$$c_j(t,y_1,\ldots,y_{n-1}) = \sum_{k=1}^{n-1} b_k''(\theta,\eta_1,\ldots,\eta_{n-1}) \frac{\partial y_j}{\partial\eta_k} .$$

Let $\tilde{P} = \frac{\partial}{\partial t} + i \sum_{j=1}^{n-1} c_j \frac{\partial}{\partial y_j}$ and define w'' by $w''(t,y_1,\ldots,y_{n-1}) = w' \circ S^{-1}(t,y_1,\ldots,y_{n-1})$. Then we have $\tilde{P}w''(t,y_1,\ldots,y_{n-1}) =$

$P''w'(\theta,\eta_1,\ldots,\eta_{n-1})$ for $S(\theta,\eta_1,\ldots,\eta_{n-1}) = (t,y_1,\ldots,y_{n-1})$. Note that if the ratios of all the original coefficients were real , then $b_k'' = 0$ for all $k = 1,\ldots,n-1$ and thus $\tilde{P}$ would be of the form $\tilde{P} = \frac{\partial}{\partial t}$. The surface $t = 0$ is the noncharacteristic hypersufrace N in these new variables.

Thus, given any C^∞ function for N there exists w such that $Pw = 0$ and $w = f$ on N if and only if there exists w'' such that $\tilde{P}w'' = 0$ and $w'' = f''$ on $t = 0$, where $f''(y_1,\ldots,y_{n-1}) = f \circ T^{-1} \circ S^{-1}(y_1,\ldots,y_{n-1})$. Thus, without loss of generality, we may assume P is of the form

$$P = \frac{\partial}{\partial t} + i \sum_{j=1}^{n-1} b_j \frac{\partial}{\partial x_j}$$

where the b_j's are real and N is of the form $t = 0$.

II. The 2-dimensional Case

Let us now assume that P is an operator in two variables. From the preceding section we see that we need only consider operators of the form $P = \frac{\partial}{\partial t} + ib(t,x) \frac{\partial}{\partial x}$ where b is real analytic in a neighborhood of the origin. We shall prove the following:

<u>Theorem</u>. Given $P = \frac{\partial}{\partial t} + ib(t,x) \frac{\partial}{\partial x}$ where b is real analytic in a neighborhood of the origin and $b \not\equiv 0$ in any neighborhood of the origin. Then we cannot always solve the one-sided C^∞ initial value problem for P, i.e. there exists $f \in C^\infty(\Omega')$, $0 \in \Omega' \subset \mathbb{R}^1$, such that no T, Ω, w exist with $0 \in \Omega \subset \Omega'$, $T > 0$, $w \in C^\infty([0,T) \times \Omega)$ such that

$$Pw = 0 \quad \text{on} \quad [0,T) \times \Omega$$

$$w(0,x) = f(x) \qquad x \in \Omega .$$

Here $C^\infty([0,T) \times \Omega)$ is defined to be the set of functions $u \in C^\infty((0,T) \times \Omega)$ which may be extended, along with all their derivatives so as to be continuous on $[0,T) \times \Omega$.

Before beginning our proof we shall prove a fact observed by Hans Lewy [4]:

Suppose P is an operator of the form

$$P = A_1(\alpha_1,\alpha_2) \frac{\partial}{\partial \alpha_1} + A_2(\alpha_1,\alpha_2) \frac{\partial}{\partial \alpha_2}$$

where α_1,α_2 are real variables and A_1 and A_2 are two continuous functions on $\mathbb{R}^2$ with a non-real ratio at some point (α_1^0,α_2^0) i.e. $\frac{A_1}{A_2} = B_1 + iB_2$ with $B_2(\alpha_1^0,\alpha_2^0) \neq 0$. Let $u_0(\alpha_1,\alpha_2)$ be a continuously differentiable solution of the homogeneous equation $Pu_0 = 0$ in a neighborhood of (α_1^0, α_2^0) such that $\frac{\partial u_0}{\partial \alpha_1}(\alpha_1^0, \alpha_2^0) \neq 0$. Then, associated with every continuously differentiable solution $w(\alpha_1, \alpha_2)$ of $Pw = 0$ in a sufficiently small neighborhood U of (α_1^0, α_2^0) is a function w' which is defined on $u_0(U) \subset \mathbb{C}^1$ by

$$w'(z) = w \circ u_0^{-1}(t) \qquad z \in u_0(U)$$

and is analytic on $u_0(U)$. In other words, $w' \circ u_0 = w$ and if $u_0 = u_1 + iu_2$ then w' satisfies the Cauchy-Riemann equations with respect to u_1 and u_2 .

<u>Proof</u>: Let $P = A_1 \frac{\partial}{\partial \alpha_1} + A_2 \frac{\partial}{\partial \alpha_2}$ and u_0 be as in the hypothesis, with $u_0(\alpha_1, \alpha_2) = u_1(\alpha_1, \alpha_2) + iu_2(\alpha_1, \alpha_2)$. We can consider u_0 as a mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$. Considered in this way, the jacobian of u_0, J is

$$J = \begin{pmatrix} \frac{\partial u_1}{\partial \alpha_1} & \frac{\partial u_1}{\partial \alpha_2} \\ \frac{\partial u_2}{\partial \alpha_1} & \frac{\partial u_2}{\partial \alpha_2} \end{pmatrix}$$

since $Pu_0 = 0$, we have $A_1 \frac{\partial u_0}{\partial \alpha_1} + A_2 \frac{\partial u_0}{\partial \alpha_2} = 0$. Therefore,

$\frac{A_1}{A_2} \frac{\partial u_1}{\partial \alpha_1} + \frac{\partial u_0}{\partial \alpha_2} = 0$ and $\frac{A_1}{A_2} = B_1 + iB_2$ where B_1, B_2 are continuous in a neighborhood of (α_1^0, α_2^0) . $B_2(\alpha_1^0, \alpha_2^0) \neq 0$ and hence $B_2 \neq 0$ in a neighborhood of (α_1^0, α_2^0) . Thus we have $B_1 \frac{\partial u_1}{\partial \alpha_1} - B_2 \frac{\partial u_2}{\partial \alpha_1} + \frac{\partial u_1}{\partial \alpha_2} = 0$ and $B_2 \frac{\partial u_1}{\partial \alpha_1} + B_1 \frac{\partial u_2}{\partial \alpha_1} + \frac{\partial u_2}{\partial \alpha_2} = 0$. Therefore,

$$J = \begin{pmatrix} \frac{\partial u_1}{\partial \alpha_1} & -B_1 \frac{\partial u_1}{\partial \alpha_1} + B_2 \frac{\partial u_2}{\partial \alpha_1} \\ \frac{\partial u_2}{\partial \alpha_1} & -B_2 \frac{\partial u_1}{\partial \alpha_1} - B_1 \frac{\partial u_2}{\partial \alpha_1} \end{pmatrix}$$

$$|J| = -B_2\left(\frac{\partial u_1}{\partial \alpha_1}\right)^2 - B_1 \frac{\partial u_1}{\partial \alpha_1}\frac{\partial u_2}{\partial \alpha_1} + B_1 \frac{\partial u_2}{\partial \alpha_1}\frac{\partial u_1}{\partial \alpha_1} - B_2\left(\frac{\partial u_2}{\partial \alpha_1}\right)^2$$

$$= -B_2\left[\left(\frac{\partial u_1}{\partial \alpha_1}\right)^2 + \left(\frac{\partial u_2}{\partial \alpha_1}\right)^2\right].$$

Since $B_2(\alpha_1^0,\alpha_2^0) \neq 0$ and $\frac{\partial u_0}{\partial \alpha_1}(\alpha_1^0,\alpha_2^0) \neq 0$, $|J|(\alpha_1^0,\alpha_2^0) \neq 0$. Let w be any continuously differentiable solution of $Pw = 0$ in a neighborhood of (α_1^0,α_2^0). Then we can define a continuously differentiable function w' in a neighborhood of $u_0(\alpha_1^0,\alpha_2^0)$ by: $w'(z) = w \circ u_0^{-1}(z)$ i.e. $w(\alpha_1,\alpha_2) = w'(u_0(\alpha_1,\alpha_2)) = w'(u_1(\alpha_1,\alpha_2) + iu_2(\alpha_1,\alpha_2))$. We want to prove: $\frac{\partial w'}{\partial u_1} + i\frac{\partial w'}{\partial u_2} = 0$. We know that $Pw = 0$, thus

$$0 = (B_1+iB_2)\frac{\partial w}{\partial \alpha_1} + \frac{\partial w}{\partial \alpha_2} = (B_1+iB_2)\left[\frac{\partial w'}{\partial u_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{\partial w'}{\partial u_2}\frac{\partial u_2}{\partial \alpha_1}\right]$$

$$+ \frac{\partial w'}{\partial u_1}\frac{\partial u_1}{\partial \alpha_2} + \frac{\partial w'}{\partial u_2}\frac{\partial u_2}{\partial \alpha_2}$$

$$= \frac{\partial w'}{\partial u_1}\left[(B_1+iB_2)\frac{\partial u_1}{\partial \alpha_1} + \frac{\partial u_1}{\partial \alpha_2}\right] + i\frac{\partial w'}{\partial u_2}\left[(-iB_1+B_2)\frac{\partial u_2}{\partial \alpha_1} - i\frac{\partial u_2}{\partial \alpha_2}\right]$$

$$= \frac{\partial w'}{\partial u_1}\left[iB_2\frac{\partial u_1}{\partial \alpha_1} + B_2\frac{\partial u_2}{\partial \alpha_1}\right] + i\frac{\partial w'}{\partial u_2}\left[B_2\frac{\partial u_2}{\partial \alpha_1} + iB_2\frac{\partial u_1}{\partial \alpha_1}\right]$$

$$= \left[\frac{\partial w'}{\partial u_1} + i\frac{\partial w'}{\partial u_2}\right]\left[iB_2\frac{\partial u_1}{\partial \alpha_1} + B_2\frac{\partial u_2}{\partial \alpha_1}\right] = \left[\frac{\partial w'}{\partial u_1} + i\frac{\partial w'}{\partial u_2}\right][iB_2]\left[\frac{\partial \bar{u}_0}{\partial \alpha_1}\right].$$

But, B_2 does not vanish in a neighborhood of (α_1^0, α_2^0) and $\frac{\partial u_0}{\partial \alpha_1}$ and hence $\frac{\partial \overline{u_0}}{\partial \alpha_1}$ does not vanish in a neighborhood of (α_1^0, α_2^0). Therefore, $\frac{\partial w'}{\partial u_1} + i \frac{\partial w'}{\partial u_2} \equiv 0$ in a neighborhood of $u_0(\alpha_1^0, \alpha_2^0)$. Thus, w' is analytic when considered as a function of u_0 in a neighborhood of (α_1^0, α_2^0).

In fact, with P, u_0 as above then for any function f of $\mathbb{R}^2$ defined and continuously differentiable in a neighborhood of (α_1^0, α_2^0) we can define f' by $f' = f \circ u_o^{-1}$ and we will have

$$(B_1 + iB_2) \frac{\partial f}{\partial \alpha_1} + \frac{\partial f}{\partial \alpha_2} = \left[\frac{\partial f'}{\partial u_1} + i \frac{\partial f'}{\partial u_2} \right] \left[iB_2 \frac{\partial u_1}{\partial \alpha_1} + B_2 \frac{\partial u_2}{\partial \alpha_1} \right].$$

Therefore, if f' is an analytic function of u_0, then $Pf = 0$. Thus, given $P = A \frac{\partial}{\partial \alpha_1} + A_2 \frac{\partial}{\partial \alpha_2}$ as above and u_0 such that $Pu_0 = 0$ and $\frac{\partial u_0}{\partial \alpha_1} \neq 0$, a function w satisfies $Pw = 0$ if and only if the function $w' = w \circ u_0^{-1}$ is an analytic function of u_0.

We return to our original problem and first prove the following

<u>Theorem</u>. Let $P = \frac{\partial}{\partial t} + ib(t,x) \frac{\partial}{\partial x}$ where b is real analytic in a neighborhood of (0,0) and $b(0,0) \neq 0$. Suppose f is real valued, $f \in C^\infty(\Omega')$ and $\Omega' \subset \mathbb{R}^1$ is a neighborhood of 0 such that f is not analytic in any neighborhood of the origin. Then there does not exist Ω, T, w such that $0 \in \Omega \subset \Omega^1$, $w \in C^\infty([0,T) \times \Omega)$ and

$$(1) \quad \begin{cases} Pw = 0 & \text{on } [0,T) \times \Omega \\ w(0,x) = f(x) & \text{on } \Omega \end{cases}$$

<u>Proof:</u> The outline of the proof is as follows:

We shall argue by contradiction and assume that there exist

Ω, T, w such that (1) holds. Using the fact we just proved, we shall show that there exists a function w' which is analytic on some neighborhood in $\mathbb{C}^1$ bounded by the real axis and such that on the real axis $w'(x) = f(x)$. We shall then use the Schwarz Reflection Principle and extend w' to be analytic in a neighborhood of $\mathbb{C}^1$ containing the origin. But, this would imply that f is analytic in a neighborhood of the origin which would contradict the hypothesis.

We now begin the proof. From the Cauchy-Kovalevsky Theorem we know that there exist Ω'', ϵ', T' and u_0 such that $0 \in \Omega'' \subset \Omega'$, u_0 is analytic in $(-\epsilon',T') \times \Omega''$, $P_{u_0} = 0$ on $(-\epsilon',T') \times \Omega''$ and $u_0(0,x) = x$ on Ω''. Since $\frac{\partial u_0}{\partial x}(0,0) = 1$ and $b(0,0) \neq 0$ and both functions are analytic, we can choose Ω'', ϵ', T' small enough so that $b(t,x) \neq 0$ and $\frac{\partial u_0}{\partial x}(t,x) \neq 0$ in $(-\epsilon',T') \times \Omega''$. Now we suppose there does exist Ω, T, w such that $0 \in \Omega \subset \Omega'' \subset \Omega'$, $0 < T < T'$, $w \in C^\infty([0,T) \times \Omega)$ and (1) is satisfied. Then, in particular, $Pw = 0$ on $(0,T) \times \Omega$. Let $u_0 = u_1 + iu_2$. For each point $(t^0,x^0) \in (0,T) \times \Omega$, $b(t^0,x^0) \neq 0$, $\frac{\partial u_0}{\partial x}(t^0,x^0) \neq 0$, and $P = \frac{\partial}{\partial t} + ib(t,x)\frac{\partial}{\partial x}$. Thus, we are in a position to use the fact we just proved. That is, if we define w' on $u_0([0,T) \times \Omega) \subset C'$ by $w'(u_1+iu_2) = w \circ u_0^{-1}(u_1+iu_2), u_1+iu_2 \in u_0([0,T) \times \Omega)$ then w' is an analytic function of u_1+iu_2 in a neighborhood of $u_0(t^0,x^0)$ for each $(t^0,x^0) \in (0,T) \times \Omega$.

Since $u_1(0,0) = 0$, u_0 is analytic, and $\{z \in \mathbb{C}^1, \operatorname{Re} z \in \Omega''\}$ is a neighborhood of $0 \in \mathbb{C}^1$, we can shrink Ω, ϵ, T sufficiently so that Ω is connected and $u_0((-\epsilon,T) \times \Omega) \subset \{z \in \mathbb{C}^1, \operatorname{Re} z \in \Omega''\}$. Let $U^0 = u_0((0,T) \times \Omega)$. $U^0 \subset \mathbb{C}^1$ and is open since by the inverse function theorem, u_0^{-1} is analytic. w' is analytic on U^0.

Let $U = u_0([0,T) \times \Omega)$ $U^0 \subset U \subset \{z \in C', \text{Re } z \in \Omega''\}$. Since $\frac{\partial u_\Omega}{\partial x} \neq 0$ on $(-\epsilon', T') \times \Omega''$, u_0 is injective on $(-\epsilon', T') \times \Omega''$ and thus, for $(t,x) \in [0,T) \times \Omega$, $u_0(t,x) = y \in \Omega''$ if and only if $t = 0$ and $x = y$. Therefore $U - U^0 \subset \{z \in \mathbb{C}^1, z \text{ real}, z \in \Omega\}$ and $U^0 \cap \{\text{real axis}\} = \emptyset$. Since U^0 is connected and does not intersect the real axis either $U^0 \subset \{z \in \mathbb{C}^1: \text{Im } z > 0\}$ or $U^0 \subset \{z \in \mathbb{C}^1: \text{Im } z < 0\}$. Assume $U^0 \subset \{z: \text{Im } z > 0\}$. (A symmetric argument can be used to treat the case $U^0 \subset \{z: \text{Im } z < 0\}$.)

Let $\bar{U} = \{z \in \mathbb{C}^1: \bar{z} \in U\}$ and $U^0 = \{z \in \mathbb{C}^1: \bar{z} \in U^0\}$.

Let $V = U \cup \bar{U} = U^0 \cup (U-U^0) \cup \bar{U}^0$. Then V is a symmetric region with respect to the real axis. If we define V^+ by $V^+ = V \cap \{z: \text{Im } z \geq 0\}$ then $V^+ = U$. We claim that w' is continuous on $V^+ = U$. In fact, w' is analytic on U^0, and is, therefore, continuous there. Thus, we must only show that w' is continuous on $U - U^0$. Let $x \in U-U^0$, then $x \in \Omega$. Let $z_j \in U$, $z_j \to x$ $\lim_{j\to\infty} w'(z_j) = \lim_{j\to\infty} w \circ u_0^{-1}(z_j)$. Since u_0^{-1} is analytic, $\lim_{j\to\infty} u_0^{-1}(z_j) = u_0^{-1}(x) = (0,x) \in [0,T) \times \Omega$.

$w \in C^\infty([0,T) \times \Omega)$. Therefore, $\lim_{j\to\infty} w \circ u_0^{-1}(z_j) = w(\lim_{j\to\infty} u_0^{-1}(z_j)) =$

$w(0,x) = w' \circ u_0(0,x) = w'(x)$. Thus $\lim_{j\to\infty} w'(z_j) = w'(x)$ and w' is thus continuous on $U = V^+$. Moreover, we have from the above string of equalities, that $w'(x) = w(0,x) = f(x)$. $f(x)$ is real and thus w' takes on real values on $V \cap \{z: \text{Im } z = 0\} = U - U^0$.

Therefore, we can use the Schwarz Reflection Principle to extend

w' to be analytic on all of V. But then for x real, $x \in \Omega$ we have $x \in u_0([0,T) \times \Omega)$. Therefore, $x \in V$ and thus $w'(x)$ is analytic on Ω. But $w'(x) = f(x)$ for $x \in \Omega$ implying that f is analytic on Ω. This contradiction completes the proof.

We now turn to the case where P is again of the form $P = \frac{\partial}{\partial t} + ib(t,x)\frac{\partial}{\partial x}$ but with $b(0,0) = 0$, b real analytic in a neighborhood of the origin and $b \not\equiv 0$ in any neighborhood of the origin. We shall prove

Theorem. Let P be as above. Suppose f is real valued, $f \in C^\infty(\Omega')$ $\Omega' \subset \mathbb{R}^1$ a neighborhood of 0 such that f is not analytic in any subneighborhood of Ω' (see Appendix), then there does not exist Ω, T, w with $0 \in \Omega \subset \Omega'$ and $w \in C^\infty([0,T) \times \Omega)$ such that (1) holds.

Proof: This proof is also done by contradiction. We shall assume that Ω, T, w exist such that (1) holds. Since $b \not\equiv 0$ in any neighborhood of the origin, there exists some point $(t_0,x_0) \in (-t,T) \times \Omega$, $t > 0$, such that $b(t_0,x_0) \neq 0$. We shall consider this condition in two cases:

Case 1: $t_0 = 0$, Case 2: $t_0 \neq 0$. In Case 1 the same argument as was used for $b(0,0) \neq 0$ will be used to lead to the contradiction that f is analytic in a neighborhood of x_0. In Case 2, we shall find some derivative of b with respect to t, $B_k(t,x)$ such that $B_k(0,x_0) \neq 0$ for some $x_0 \in \Omega$. Then, after a change of variables, the proof of this case will follow as in Case 1.

We now begin the proof.

We assume there exists Ω, T, w such that (1) holds. By the Cauchy-Kovalevsky Theorem there exists Ω'', T', ε, u_0 such that

$0 \in \Omega'' \subset \Omega$, u_0 is analytic on $(-\epsilon,T') \times \Omega''$, $Pu_0 = 0$ $u_0(0,x) = x$ and $\frac{\partial u_0}{\partial x}(t,x) \neq 0$ on $(-\epsilon,T') \times \Omega''$.

Case 1. Suppose, first, that $b(0,x) \not\equiv 0$ on Ω'' . Thus, there exists $x_0 \in \Omega''$ such that $b(0,x_0) \neq 0$. From the analyticity of b we know that we can shrink ϵ, T' sufficiently and choose U a neighborhood of x_0 in $\mathbb{R}^1$, $U \subset \Omega''$, such that $b(t,x) \neq 0$ on $(-\epsilon,T') \times U$. Define w' on $u_0([0,T) \times U)$ by

$$w'(u_1+iu_2) = w \circ u_0^{-1}(u_1+iu_2) \qquad u_1+iu_2 \in u_0([0,T) \times U)$$

Applying Lewy's observation of [4] at the point $(0,x_0)$, we have that w' is analytic in $u((0,T) \times U)$. Since f is real valued and $w'(x) = w \circ u_0^{-1} = w(0,x) = f(x)$, we have that w' is real valued on the real axis. Using the Schwarz Reflection Principle as above, we can extend w' to be analytic in a whole complex neighborhood of x_0 . This implies that f is real analytic in U which contradicts the hypothesis.

Case 2. Now we suppose that $b(0,x) \equiv 0$ on Ω'' . Since b is analytic in a neighborhood of the origin, we can express b , in a neighborhood of the origin as $b(t,x) = b_0(x) + b_1(x)t + b_2(x)t^2 + \dots$.

Thus, $b_0(x) \equiv 0$ on Ω'' . But since $b \not\equiv 0$ in any neighborhood of 0 , there exists a positive integer k such that $b_j \equiv 0$ on Ω'' , $j < k$ but $b_k \not\equiv 0$ on Ω'' . Therefore, there exists $x_0 \in \Omega''$ such that $b_k(x_0) \neq 0$. Since b_k is analytic, we can shrink ϵ, T' sufficiently and choose U a neighborhood of x_0 in $\mathbb{R}^1$, $U \subset \Omega''$ such that

$$b(t,x) = b_k(x)t^k + b_{k+1}(x)t^{k+1} \dots \qquad (t,x) \in (-\epsilon,T') \times U$$

Thus, on $(-\epsilon,T')\times U$ we can express b as $b(t,x) = t^k B_k(t,x)$ where $B_k(0,x) \neq 0$ and B_k is real analytic.

Consider the change of variables $[0,T') \to [0, \frac{T'^{k+1}}{k+1})$ defined by $t \to s = \frac{t^{k+1}}{k+1}$. Let $\tilde{u}_0(s,x) = u_0(t,x)$, $\tilde{B}_\epsilon(t,x) = w(t,x)$. Since $Pu_0 = 0$, $u(0,x) = x$ and $\frac{\partial u_0}{\partial t} = \frac{\partial \tilde{u}_0}{\partial s} t^k$ we have

$\frac{\partial \tilde{u}_0}{\partial s} t^k + i t^k \tilde{B}_k(s,x) \frac{\partial \tilde{u}_0}{\partial x} = 0$ and $\tilde{u}_0(0,x) = x$. Thus, if $\tilde{P} = \frac{\partial}{\partial s} + i\tilde{B}_k(s,x)\frac{\partial}{\partial x}$, $\tilde{u}_0$ and $\tilde{w}$ satisfy $\tilde{P}\tilde{u}_0 = \tilde{P}\tilde{w} = 0$ on $(0, \frac{T'^{k+1}}{k+1}) \times U$. $\tilde{w}(0,x) = f(x)$, $\tilde{u}_0,\tilde{w} \in \mathcal{C}([0, \frac{T'^{k+1}}{k+1}) \times U)$ and $\tilde{u}_0,\tilde{w} \in \mathcal{C}^1((0,\frac{T'^{k+1}}{k+1}) \times U)$. On $\tilde{u}_0([0,\frac{T'^{k+1}}{k+1}) \times U)$ define $\tilde{w}'$ by $\tilde{w}'(u_1+iu_2) = \tilde{w}\circ\tilde{u}_0^{-1}$ $u_1+iu_2 \in \tilde{u}_0([0,\frac{T'^{k+1}}{k+1}) \times U)$. We again have, by Lewy's observation of [4] that $\tilde{w}'$ is analytic on $\tilde{u}_0((0,\frac{T'^{k+1}}{k+1}) \times U)$. Since $\tilde{w}'(x) = \tilde{w}\circ\tilde{u}_0^{-1} = \tilde{w}(0,x) = f(x)$, $\tilde{w}'(x)$ is real valued on the real axis. Again using the Schwarz Reflection Principle we arrive at the contradiction that f is real analytic in U.

III. The 3-dimensional Case

Now we look at the operator in 3 dimensions $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{2} b_j(t,x_1,x_2) \frac{\partial}{\partial x_j}$ with b_j real valued and analytic in a neighborhood of the origin in $\mathbb{R}^3$.

In the 2-dimensional case, $P_2 = \frac{\partial}{\partial t} + ib(t,x) \frac{\partial}{\partial x}$, we assumed $b \not\equiv 0$, i.e. given any neighborhood of the origin $U_2 \subset \mathbb{R}^2$ there exists a subneighborhood $U_2' \subset U_2$ on which P_2 and $\bar{P}_2$ (the conjugate of P) are linearly independent. We shall first prove that, in the 3-dimensional case, if b_1 and b_2 are not both identically 0 in a neighborhood $U \subset \mathbb{R}^3$ of $(0,0,0)$, but $[P,\bar{P}]$ (the Lie bracket of P and $\bar{P}$) is a linear combination of P and $\bar{P}$ in U then, by using the Frobenius Theorem, the 3-dimensional case reduces to the 2-dimensional one. Thus, we shall prove the following

<u>Theorem</u>. Let $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{2} b_j(t,x_1,x_2) \frac{\partial}{\partial x_j}$ with b_j real analytic and not both identically zero in a neighborhood of the origin in $\mathbb{R}^3$. Suppose $[P,\bar{P}]$ is a linear combination of P and $\bar{P}$ in a neighborhood of the origin. Then there exists a function $f \in C^\infty(\Omega')$, Ω' a neighborhood of the origin in $\mathbb{R}^2$ such that no T, Ω, w exist with $w \in C^\infty([0,T) \times \Omega)$, $0 \in \Omega \subset \Omega'$ and such that

$$(2) \quad \begin{cases} Pw = 0 & \text{on } [0,T) \times \Omega \\ w(0,x_1,x_2) = f(x_1,x_2) & \text{on } \Omega . \end{cases}$$

<u>Proof:</u> The Lie algebra generated by P and $\bar{P}$ is the same as the Lie algebra generated by the analytic real vector fields $X = \frac{\partial}{\partial t}$ and

$Y = \sum_{j=1}^{2} b_j(t,x_1,x_2) \frac{\partial}{\partial x_j}$. From the hypothesis we have that X and Y form an involutive system of real analytic vector fields in a neighborhood of the origin, i.e. all Lie products of X and Y are linear combinations of X and Y .

Frobenius' Theorem states: Suppose $X^1,\ldots,X^s$ is an involutive system of C^j (or analytic) vector fields on M . Then given any $p \in M$ there exists a neighborhood of p in which we can find local coordinates $\tau_1,\ldots,\tau_n$ such that

$$X^j = \sum_{j=1}^{s} a_j^k \frac{\partial}{\partial \tau_k} \quad j=1,\ldots,s .$$

Thus, in our case, we can find an analytic, invertible, change of coordinates $H(t,x_1,x_2) = (s,y_1,y_2)$ in a neighborhood U of $(0,0,0)$ such that $X = a_1(s,y_1,y_2) \frac{\partial}{\partial s} + a_2(s,y_1,y_2) \frac{\partial}{\partial y_1}$ and $Y = c_1(s,y_1,y_2) \frac{\partial}{\partial s} + c_2(s,y_1,y_2) \frac{\partial}{\partial y_1}$. Letting $P' = (a_1+ic_1) \frac{\partial}{\partial s} + (a_2+ic_2) \frac{\partial}{\partial y}$, we have $Pu = P'u'$ where $u' \circ H = u$.

Let (t^0,x_1^0,x_2^0) be any point in U at which $b_j(t^0,x_1^0,x_2^0) \neq 0$ for some j , $j=1,2$. Then, at such a point the dimension of the Lie algebra generated by P and $\bar{P}$ is 2 . Therefore, at $H(t^0,x_1^0,x_2^0)$ the dimension of the Lie algebra generated by P' and $\bar{P}'$ is 2 , and, thus, the coefficients a_1+ic_1 and a_2+ic_2 have non real ratio at such a point.

To complete our proof we assume that given any $f \in C^\infty(\Omega')$ there exist Ω, T, w with Ω, T taken sufficiently small so that $[0,T) \times \Omega \subset U$ and such that (2) holds. Let (t^0,x_1^0,x_2^0), $(x_1^0,x_2^0) \in \Omega$, $0 \le t^0 < T$, be a point at which $b_j(t^0,x_1^0,x_2^0) \neq 0$ for some j. Define $\tilde{f}$ on $[0,T) \times \Omega'$ by $\tilde{f}(t,x_1,x_2) = f(x_1,x_2)$. Define $f' \in C^\infty(H(0\times\Omega))$ by $f' = \tilde{f} \cdot H^{-1}$. Let $N = H(0\times\Omega)$. N is a non-characteristic hypersurface for P'. Let $w' = w \circ H^{-1}$ on $H([0,T)\times \Omega)$. $w' \in C^\infty(H([0,T) \times \Omega))$. Then $P'w' = 0$ on $H([0,T) \times \Omega)$ and $w' = f'$ on N. Let $V = \pi_1(H([0,T) \times \Omega)) \times \pi_2(H([0,T) \times \Omega))$ where $\pi_j\colon \mathbb{R}^3 \to \mathbb{R}^1$, $\pi_j(x_1,x_2,x_3) = x_j$ $j=1,2,3$. On V, let $P_2' = [a_1(s,y_1,y_2^0) + iC_1(s,y_1,y_2^0)] \frac{\partial}{\partial s} + [a_2(s,y_1,y_2^0) + iC_2(s,y_1,y_2^0)] \frac{\partial}{\partial y}$, and $w_2'(s,y_1) = w'(s,y_1,y_2^0)$. Let $f_2'(s,y) = f'(s,y_1,y_2^0)$ on $\pi_1 \times \pi_2(N)$. Then $P_2' w_2' = 0$ on V and $w_2' = f_2'$ on $\pi_1 \times \pi_2(N)$, a non-characteristic hypersurface for P_2'. But since the mapping $f \to f_2'$ from $C^\infty(\Omega)$ to $C^\infty(\pi_1 \times \pi_2(N))$ is onto, we have that we can always solve the C^∞ initial value problem for P_2'. But this contradicts the theorem we have already proven.

We now continue to look at the operator in 3 dimensions but, this time we shall assume that P, $\bar{P}$ and $[P,\bar{P}]$ are all linearly independent in a neighborhood of the origin. Our results will not be as strong as in the 2-dimensional case. We shall prove the following:

<u>Theorem</u>. Let $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{2} b_j(t,x_1,x_2) \frac{\partial}{\partial x_j}$ with b_j real valued and analytic in a neighborhood U' of the origin in $\mathbb{R}^3$. Suppose $b_2(0,0,0) \frac{\partial b_1}{\partial t}(0,0,0) \neq b_1(0,0,0) \frac{\partial b_2}{\partial t}(0,0,0)$. Then, there exists $f \in C^\infty(\Omega)$, Ω a neighborhood of the origin in $\mathbb{R}^2$ for which there exist no U and w with $(0,0,0) \in U \subset U'$, $\pi_2(U) \times \pi_3(U) \subset \Omega$, and

$w \in C^\infty(U)$ such that:

$$(3) \begin{cases} Pw = 0 \quad \text{on} \quad U \\ w(0,x_1,x_2) = f(x_1,x_2) \quad \text{on} \quad \pi_2(U) \times \pi_3(U) \end{cases}$$

i.e. such that w is a two-sided C^∞ solution to the initial value problem.

In the 2-dimensional case we have only a one-dimensional solution space, i.e. given w ,v such that $Pu_0 = Pv_0 = 0$ then ∇u_0 is proportional to ∇v_0 . If u_0 is a solution such that $\frac{\partial u_0}{\partial x}(0,0) \neq 0$ then u_0 maps a neighborhood of the origin in $\mathbb{R}^2$ onto a neighborhood of $\mathbb{C}^1$ invertibly. In order to prove our theorem in the 2-dimensional case, we used the fact that every solution w of $Pw = 0$ has a corresponding function w' which is an analytic function of u_0 . In the 3-dimensional case we have a 2-dimensional solution space. Let u_0,v_0 be two solutions such that ∇u_0 and ∇v_0 are linearly independent. Lewy has shown [5], that if P, $\bar{P}$ and $[P,\bar{P}]$ are all linearly independent in $U \subseteq \mathbb{R}^3$ then there is a result analagous to the fact used in the 2-dimensional case: Let $S = \{u_0(\alpha_1,\alpha_2,\alpha_3),\ v_0(\alpha_1,\alpha_2,\alpha_3) : (\alpha_1,\alpha_2,\alpha_3) \in U\}$ then $S \subset \mathbb{C}^2$, $\dim_{\mathbb{R}} S = 3$ and if w is any C^∞ solution of $Pw = 0$ then associated with w is a function w' on S which can be extended to be analytic (as a function of $\mathbb{C}^2$) in a neighborhood on one side of S , which side depends on S - not on w . We shall discuss Lewy's method of proof later. Now we shall use Lewy's results to prove our theorem.

<u>Proof:</u> We want to apply Lewy's result in a neighborhood of the origin. Therefore, we need that P, $\bar{P}$ and $[P,\bar{P}]$ are linearly independent in

some neighborhood of the origin, i.e. if $P = \sum_{j=1}^{3} a_j \frac{\partial}{\partial \alpha_j}$,
$\bar{P} = \sum_{j=1}^{3} \bar{a}_j \frac{\partial}{\partial \alpha_j}$, and $[P,\bar{P}] = \sum_{j=1}^{3} c_j \frac{\partial}{\partial \alpha_j}$ we need that (a_1,a_2,a_3), $(\bar{a}_1,\bar{a}_2,\bar{a}_3)$, and (c_1,c_2,c_3) Span $\mathbb{C}^3$ in a neighborhood of the origin. Thus for our operator, P, we need that $D \neq 0$ in a neighborhood of the origin where

$$D(t,x_1,x_2) \equiv \begin{vmatrix} 1 & 1 & 0 \\ ib_1(t,x_1,x_2) & -ib_1(t,x_1,x_2) & -2i \frac{\partial L_1}{\partial t}(t,x_1,x_2) \\ ib_2(t,x_1,x_2) & -ib_2(t,x_1,x_2) & -2i \frac{\partial b_2}{\partial t}(t,x_1,x_2) \end{vmatrix}$$

But, $D = -4b_1 \frac{\partial b_2}{\partial t} + 4b_2 \frac{\partial b_1}{\partial t}$. By assumption $b_2(0) \frac{\partial b_1}{\partial t}(0) \neq b_1(0) \frac{\partial b_2}{\partial t}(0)$ and since b_1, b_2 are analytic we have that $D \neq 0$ in a neighborhood of the origin. Thus we have P, $\bar{P}$, $[P,\bar{P}]$ are linearly independent in a neighborhood of the origin. Either $b_1(0) \neq 0$ or $b_2(0) \neq 0$. We shall assume $b_1(0) \neq 0$ (if not a simple change of variables $(x_1,x_2) \to (x_2,x_1)$ will yield this result).

By the Cauchy-Kovalevsky Theorem we know that there exist U'', u_0, v_0 such that $(0,0,0) \in U'' \subset U'$, $Pu_0 = Pv_0 = 0$ on U'', u_0, v_0 are both analytic in U'' and $u_0(0,x_1,x_2) = x_1, v_0(0,x_1,x_2) = x_2$ for $(x_1,x_2) \in \pi_2(U'') \times \pi_3(U'')$.

Let $u_0 = u_1 + iu_2$, $v_0 = v_1 + iv_2$.

$$\nabla u_0(0,0,0) = \begin{pmatrix} -ib_1(0,0,0) \\ 1 \\ 0 \end{pmatrix} \qquad \nabla v_0(0,0,0) = \begin{pmatrix} -ib_2(0,0,0) \\ 0 \\ 1 \end{pmatrix}$$

Therefore, ∇u_0, ∇v_0 are linearly independent at the origin.

Since ∇u_0, ∇v_0, b_1 and b_2 are all analytic, we can shrink U'' sufficiently so that ∇u_0 and ∇v_0 are linearly independent, $b_1(t,x_1,x_2) \neq 0$ and $D(t,x_1,x_2) \neq 0$ in U'' .

Let $S = \{(u_0(t,x_1,x_2),\ v_0(t,x_1,x_2)) \colon (t,x_1,x_2) \in U''\}$. We define the matrix $M(t,x_1,x_2)$ by:

$$M(t,x_1,x_2) \equiv \begin{pmatrix} u_{1t}(t,x_1,x_2) & u_{1x_1}(t,x_1,x_2) & u_{1x_2}(t,x_1,x_2) \\ u_{2t}(t,x_1,x_2) & u_{2x_1}(t,x_1,x_2) & u_{2x_2}(t,x_1,x_2) \\ v_{1t}(t,x_1,x_2) & v_{1x_1}(t,x_1,x_2) & v_{1x_2}(t,x_1,x_2) \\ v_{2t}(t,x_1,x_2) & v_{2x_1}(t,x_1,x_2) & v_{2x_2}(t,x_1,x_2) \end{pmatrix}$$

where $u_\alpha = \frac{\partial u}{\partial \alpha}$. Then, since $M(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ b_1(0,0,0) & 0 & 0 \\ 0 & 0 & 1 \\ b_2(0,0,0) & 0 & 0 \end{pmatrix}$

we have that the rank of M at the origin is 3 . We can shrink U'' further so that the rank of M is 3 on U'' . Thus S is a three dimensional surface in $\mathbb{C}^2$.

Consider the map $\varphi\colon U'' \to \mathbb{R}^3$ given by

$$\varphi(t,x_1,x_2) = (u_1(t,x_1,x_2),\ u_2(t,x_1,x_2),\ v_1(t,x_1,x_2)) .$$

Then the Jacobian, J_φ , associated with φ is

$$J_\varphi = \begin{pmatrix} u_{1t} & u_{1x_1} & u_{1x_2} \\ u_{2t} & u_{2x_1} & u_{2x_2} \\ v_{1t} & v_{1x_1} & v_{1x_2} \end{pmatrix}$$

and we have determinant $J_\varphi(0) = -b_1(0) \neq 0$. Shrinking again, if necessary, we may assume that $\det(J_\varphi) \neq 0$ on U'' .

By the inverse function theorem there exists an analytic function φ^{-1} , the inverse of φ on $V'' = \varphi(U'')$. We can characterize S in another way:

$$S = \{(z_1,z_2)\colon (\text{Re } z_1, \text{Im } z_1, \text{Re } z_2) \in V''$$

and

$$\text{Im } z_2 = v_2 \circ \varphi^{-1}(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2)\} .$$

Let $f \in C^\infty(\Omega)$ be such that $f(0,x_2)$ is real valued and is not analytic in any neighborhood of the origin. Suppose there exist U and w with $(0,0,0) \in U \subset U''$, $\pi_2(U) \times \pi_3(U) \subset \Omega$, $w \in C^\infty(U)$ and such that (3) holds. Using Lewy's results and the Schwarz Reflection Principle in a method analogous to the 2-dimensional case, we shall arrive at a contradiction.

Since φ, φ^{-1} are analytic and $\varphi(0,0,0) = (0,0,0)$ we can shrink U sufficiently so that $\varphi(U) = V$ is an open connected set in $\mathbb{R}^3$ such that $\pi_3 V \subset \pi_3 U''$. Define $S' \subset S$ by

$$S' = \{(z_1,z_2)\colon (\text{Re } z_1, \text{Im } z_1, \text{Re } z_2) \in V$$

and

$$\text{Im } z_2 = v_2 \circ \varphi^{-1}(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2)\} .$$

On S' we define w' by $w'(z_1,z_2) = w \circ \varphi^{-1}(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2)$. From Lewy's results we have that w' can be extended analytically on one side of S' .

For $(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2) \in U$ we have (z_1,z_2) is on S' if and only if $\text{Im } z_2 = v_2 \circ \varphi^{-1}(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2)$. Thus, the two sides of S' are $\text{Im } z_2 > v_2 \circ \varphi^{-1}(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2)$ and $\text{Im } z_2 < v_2 \circ \varphi^{-1}(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2)$.

We shall assume that w' can be extended analytically on the side $\text{Im } z_2 > v_2 \circ \varphi^{-1}(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2)$ (the other case can be handled in a symmetric manner). For $z_1 = 0$ and $\text{Re } z_2 \in \pi_2 U''$ we have $\varphi(0,0,\text{Re } z_2) = (0,0,\text{Re } z_2)$. Thus, for $(0,0,\text{Re } z_2) \in V$, $v_2 \circ \varphi^{-1}(0,0,\text{Re } z_2) = v_2(0,0,\text{Re } z_2) = 0$. Therefore, $w'(0,z_2)$ may be extended so as to be an analytic function of z_2 in a neighborhood of the form

$$\{z_2 : \text{Re } z_2 \in \pi_3 U,\ a > \text{Im } z_2 > 0\}$$

Let $W = \{z \in \mathbb{C}^1 : \text{Re } z \in \pi_3 U,\ -a < \text{Im } z < a\}$. W is a symmetric region. Let $W^+ = W \cap \{z \in \mathbb{C}^1 : a > \text{Im } z > 0\}$, $W^- = W \cap \{z \in \mathbb{C}^1 : -a < \text{Im } z < 0\}$, $\sigma_0 = W \cap \{z \in \mathbb{C}^1 : \text{Im } z = 0\}$. For $z \in W^+ \cup \sigma_0$ define w_0 by $w_0(z) = w'(0,z)$. w_0 is analytic on W^+. As $\text{Im } z \to 0^+$, $w_0(z) = w'(0,z) \to w'(0,\text{Re } z) = w \circ \varphi(0,0,\text{Re } z) = w(0,0,\text{Re } z) = f(0,\text{Re } z)$. Thus, w_0 is continuous on $W^+ \cup \sigma$, and since f is real valued, w_0 is real valued on σ_0. By the Schwarz Reflection Principle we can extend w_0 to all of W by $w_0(z) = \overline{w_0(\bar{z})}$ for $z \in W^-$. Defined in this way w_0 is analytic in W and, therefore, has a convergent Taylor series on the real line. But then f must be real analytic in its second variable. This contradicts the way f was chosen and the proof is complete.

There is one case remaining in 3 dimensions. That is the case where $b_1(0)\frac{\partial b_2}{\partial t}(0) - b_2(0)\frac{\partial b_1}{\partial t}(0) = 0$ but $b_1\frac{\partial b_2}{\partial t} - b_2\frac{\partial b_1}{\partial t}$ is not identically zero in any neighborhood of the origin. We do not have results for this case in general. But, we have

<u>Theorem</u>. Let $P = \frac{\partial}{\partial t} + ib_1\frac{\partial}{\partial x_1} + ib_2\frac{\partial}{\partial x_2}$ where b_1 and b_2 are real

valued analytic functions in a neighborhood of the origin. Suppose

$b_1(0,0,0)\ \frac{\partial b_1}{\partial t}\ (0,0,0) - b_2(0,0,0)\ \frac{\partial b_1}{\partial t}\ (0,0,0) = 0$ but

$b_1(0,x_1,x_2)\ \frac{\partial b_2}{\partial t}\ (0,x_1,x_2) - b_2(0,x_1,x_2)\ \frac{\partial b_2}{\partial t}\ (0,x_1,x_2) \not\equiv 0$ in any neighborhood of the origin in $\mathbb{R}^2$. Then there exists a function $f \in C^\infty(\Omega)$, Ω a neighborhood of $(0,0)$ in $\mathbb{R}^2$ for which no U,w exist with $(0,0,0) \in U \subset \mathbb{R}^3$, $\pi_2(U) \times \pi_3(U) \subset \Omega$, $w \in C^\infty(U)$ such that (3) holds.

<u>Proof:</u> Let $f \in C^\infty(\Omega)$ such that f is real valued and not analytic in either variable in any subneighborhood of Ω (see Appendix). We suppose that there exists U,w with $(0,0,0) \in U \subset \mathbb{R}^3$, $\pi_2(U) \times \pi_3(U) \subset \Omega$, $w \in C^\infty(U)$ such that (3) holds. Shrinking U, if necessary, we can find $u_0 = u_1+iu_2$, $v_0 = v_1+iv_2$ analytic solutions of $Pu = 0$ on U such that $u_0(0,x_1,x_2) = x_1$, $v_0(0,x_1,x_2) = x_2$ on $\pi_2(U) \times \pi_3(U)$ and ∇u_1, ∇u_2, ∇v_1 are all linearly independent on U.

We define $D(t,x_1,x_2)$ as above by:

$$D(t,x_1,x_2) = \begin{pmatrix} 1 & 1 & 0 \\ ib_1(t,x_1,x_2) & -ib_1(t,x_1,x_2) & -2i\ \frac{\partial b_1}{\partial t}(t,x_1,x_2) \\ ib_2(t,x_1,x_2) & -b_2(t,x_1,x_2) & -2i\ \frac{\partial b_2}{\partial t}(t,x_1,x_2) \end{pmatrix}$$

we have $D(0,0,0) = 0$ but there exists $(x_1^0,x_2^0) \in \pi_2(U) \times \pi_3(U)$ such that $D(0,x_1^0,x_2^0) \neq 0$. Our proof now follows as in the preceeding case with a neighborhood of $(0,x_1^0,x_2^0)$ in place of a neighborhood of the origin.

IV. Special n-dimensional Cases

Now we look at the n-dimensional case. $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{n-1} b_j(t,x_1,\ldots,x_{n-1}) \frac{\partial}{\partial x_j}$ with b_j, $j=1,\ldots,n-1$ real and analytic in a neighborhood of the origin in $\mathbb{R}^n$. For this operator can we solve the initial value problem (one or two-sided) when not all the b_j's are identically zero in a neighborhood of the origin? We cannot treat the general case. But, we can use Frobenius' Theorem to treat cases where P reduces to a lower dimensional operator. We have

Theorem. Let $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{n-1} b_j(t,x_1,\ldots,x_{n-1}) \frac{\partial}{\partial x_j}$ with b_j real and analytic and not all identically zero in a neighborhood of the origin in $\mathbb{R}^n$. Suppose the Lie algebra generated by P and $\bar{P}$ has dimension ≤ 2 in a neighborhood of the origin. Then there exists a function $f \in C^\infty(\Omega')$, Ω' a neighborhood of the origin in $\mathbb{R}^{n-1}$ such that we cannot solve the one-sided C^∞ initial value problem for P with f as the initial value.

Proof: This proof follows exactly as the proof in 3-dimensions. The Lie algebra generated by P and $\bar{P}$ is the same as the Lie algebra generated by the analytic real vector fields $X = \frac{\partial}{\partial t}$ and $Y = \sum b_j \frac{\partial}{\partial x_j}$. X and Y form an involutive system of analytic vector fields in a neighborhood of the origin. From Frobenius' Theorem we have that we can find an invertible, analytic change of coordinates $H(t,x_1,\ldots,x_{n-1}) = (s,y_1,\ldots,y_{n-1})$ in a neighborhood of the origin such that

$$X = a_1(s,y_1,\ldots,y_{n-1}) \frac{\partial}{\partial s} + a_2(s,y_1,\ldots,y_{n-1}) \frac{\partial}{\partial y_1}$$

and

$$Y = c_1(s,y_1,\dots,y_{n-1}) \frac{\partial}{\partial s} + c_2(s,y_1,\dots,y_{n-1}) \frac{\partial}{\partial y_1} .$$

Letting $P' = a_1 \frac{\partial}{\partial s} + a_2 \frac{\partial}{\partial y} + i(c_1 \frac{\partial}{\partial s} + c_2 \frac{\partial}{\partial y_1})$ we have $Pu = P'u'$ where $u' \circ H = u$.

Let $(t^0,x_1^0,\dots,x_{n-1}^0)$ be any point in the domain of H for which $b_k(t^0,x_1^0,\dots,x_{n-1}^0) \neq 0$ for some t . In a neighborhood of such a point the Lie algebra generated by P and $\bar{P}$, and, hence by X and Y , has dimension 2 . Let $(s^0,y_1^0,\dots,y_{n-1}^0) = H(t^0,x_1^0,\dots,x_{n-1}^0)$ and let P_2 be the operator on $C^\infty(U)$, U a sufficiently small neighborhood of (s^0,y_1^0) defined by

$$P_2' = a_1(s,y_1,y_2^0,\dots,y_{n-1}^0) \frac{\partial}{\partial s} + a_2(s,y_1,y_2^0,\dots,y_{n-1}^0) \frac{\partial}{\partial y_1}$$
$$+ ic_1(s,y_1,y_2^0,\dots,y_{n-1}^0) \frac{\partial}{\partial s} + ic_2(s,y_1,y_2^0,\dots,y_{n-1}^0) \frac{\partial}{\partial y_1} .$$

Any solution of the C^∞ initial value problem for P gives rise to a solution of the C^∞ initial value problem for P_2' (as in the 3-dimensional case). But P_2' does not always have such a solution. Therefore, P does not always have a solution.

In the same manner we can prove

__Theorem__. Let $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{n-1} b_j(t,x_1,\dots,x_{n-1}) \frac{\partial}{\partial x_j}$ with b_j , $j=1,\dots,n-1$ real and analytic and not all identically zero in a neighborhood of the origin in $\mathbb{R}^n$. Suppose the Lie algebra generated by P and $\bar{P}$ has $\dim \leq 3$ in a neighborhood of the origin. Suppose also that given any neighborhood $\Omega \subset \mathbb{R}^{n-1}$ of the origin there exists a point in $0 \times \Omega$ at which the dimension of the Lie algebra is 3 . Then there exists a function $f \in C^\infty(\Omega')$, Ω' a neighborhood of the origin in $\mathbb{R}^{n-1}$ such that we cannot solve the 2-sided C^∞ initial value problem for P with f as the initial value.

Note that in order to prove both of these theorems we must restrict our attention to a neighborhood on which the Lie algebra generated by P and $\bar{P}$ is constant and maximal.

V. Discussion of Results of Andreotti and Hill, Hill, and Lewy

We now discuss Lewy's paper [5] and generalizations of it which are found in Andreotti and Hill's work [1] and Hill's work [2].

Given $P = \sum_{j=1}^{n} A_j \frac{\partial}{\partial \alpha_j}$, $A_j \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$, then there are at most n-1 linearly independent complex valued solutions $\zeta_j(\alpha_1,\ldots,\alpha_n)$, $j=1,\ldots,n-1$ of the homogeneous problem. That is, there exist at most n-1 functions ζ_j such that $P\zeta_j = 0$ and $(\nabla\zeta_1,\ldots,\nabla\zeta_{n-1})$ form a linearly independent set. As in [1] we shall call such ζ_j's characteristic coordinates. If the coefficients of P , the A_j's , are all analytic, then by the Cauchy-Kovalevsky Theorem we know there exist analytic functions ζ_j , $j=1,\ldots,n-1$ such that

$$(4) \qquad \begin{cases} P\zeta_j = 0 \\ \zeta_j(0,\alpha_2,\ldots,\alpha_n) = \alpha_{j+1} . \end{cases}$$

Thus, in the analytic case all n-1 independent characteristic coordinates exist. This is not true in general. Nirenberg constructs an example [6] of an operator P in 3-dimensions with $P, \bar{P}, [P,\bar{P}]$ linearly independent in a neighborhood of the origin such that if w is any C^∞ solution of $Pw = 0$ in a neighborhood of the origin, then $\nabla w(0,0,0) = 0$. If there exist C^∞ functions ζ_j such that (4) holds for $j=1,\ldots,n-1$, then all n-1 independent characteristic coordinates exist. Since we are trying to prove that there exist C^∞ functions, f , for which there exist no C^∞ solution to the homogeneous problem with initial value f , we can assume all n-1 independent characteristic coordinates exist. Otherwise, for some k , $1 \le k \le n-1$, no C^∞ function, ζ , would exist such that $P\zeta = 0$ and $\zeta(0,\alpha_2,\ldots,\alpha_n) = \alpha_k$. Thus $f = \alpha_k$ would be the function whose existence we wished to establish.

Let $\{\zeta_1,\ldots,\zeta_{n-1}\}$ be the set of n-1 independent characteristic coordinates associated with $P = \sum_{j=1}^{n} A_j \frac{\partial}{\partial\alpha_j}$: Define a surface $S \subset \mathbb{C}^{n-1}$ by:

$$S = (\zeta_1(\alpha_1,\ldots,\alpha_n),\ \zeta_2(\alpha_1,\ldots,\alpha_n),\ \ldots,\ \zeta_{n-1}(\alpha_1,\ldots,\alpha_n)) .$$

What is the real dimension of the surface S ? Lewy [5] proves that for n=3 if P and $\bar{P}$ are linearly independent, then the real dimension of the surface S is 3 . Andreotti and Hill [1] generalize this and prove:

<u>Theorem.</u> Let $P = \sum_{j=1}^{n} A_j \frac{\partial}{\partial\alpha_j}$ where A_j are C^∞ functions. Let ζ_j, $1 \le j \le n-1$ be n-1 independent, C^∞, characteristic coordinates associated with P , and define $S \subset \mathbb{C}^{n-1}$ as above. If P and $\bar{P}$, the complex conjugate of P , are linearly independent, then $\dim_{\mathbb{R}} S = n$.

$$\mathrm{Dim}_{\mathbb{R}}\, S = \dim_{\mathbb{R}}(\mathrm{Re}\ \nabla\zeta_1, \mathrm{Im}\ \nabla\zeta_1,\ldots,\mathrm{Re}\ \nabla\zeta_{n-1}, \mathrm{Im}\ \nabla\zeta_{n-1})$$

where $\{\zeta_1,\ldots,\zeta_{n-1}\}$ are the characteristic coordinates. Thus we can find a small enough neighborhood $\Omega \subset \mathbb{R}^n$ such that there is a 1-1 correspondence between a point in Ω and a point in $\zeta(\Omega) \subset S \subset \mathbb{C}^{n-1}$.

Following is a discussion of some results of Andreotti and Hill [1].

Let $p \in S$, let U be a neighborhood of p in $\mathbb{C}^{n-1}$. We can express $S \cap U$ as the null set of the system of $2(n-1) - n = n-2$ real functions defined on $\mathbb{C}^{n-1}$, i.e.

$$S \cap U = \left(\begin{array}{l} (z_1,\ldots,z_{n-1}) \in U\colon\ f_j(z_1,\ldots,z_{n-1}) = 0,\ j=1,\ldots,n-2 \\ \text{where the } f_j\text{'s are real valued } C^\infty \text{ functions and} \\ (df_1 \cap \ldots \cap df_{n-2})_q \neq 0,\ q \in U \end{array} \right)$$

<u>Definition:</u> A complex valued tangent vector to S at a point $p \in S \cap U$ is a vector X of the form:

$$X \equiv \sum_1^{n-1} a_j \frac{\partial}{\partial z_j} + \sum_1^{n-1} b_j \frac{\partial}{\partial \bar{z}_j} \qquad a_j, b_j \in \mathbb{C}$$

such that $Xf_j(p) = 0,\ j=1,\ldots,n-2$.

The set of complex valued tangent vectors to S at a point $p \in S \cap U$ is independent of the choice of the functions f_j . Suppose $\{g_1,\ldots,g_{n-2}\}$ are another set of C^∞ real valued functions with the same properties as the f_j's , then for any $1 \le k \le n-2$

$$S \cap U = \left(\begin{array}{l} (z_1,\ldots,z_{n-1}) \in U\colon\ g_k(z_1,\ldots,z_{n-1}) = 0 \\ \text{and } f_j(z_1,\ldots,z_{n-1}) = 0\ ,\ j=1,\ldots,n-2\ . \end{array} \right)$$

Since $\dim_{\mathbb{R}}(S \cap U) = n$ we must have that the rank of

$$\begin{pmatrix} \frac{\partial g_k}{\partial z_1} & \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_{n-2}}{\partial z_1} \\ \vdots & \vdots & & \\ \frac{\partial g_k}{\partial z_{n-1}} & \frac{\partial f_1}{\partial z_{n-1}} & & \frac{\partial f_{n-2}}{\partial z_{n-1}} \\ \frac{\partial g_k}{\partial \bar{z}_1} & \frac{\partial f_1}{\partial \bar{z}_1} & & \frac{\partial f_{n-2}}{\partial \bar{z}_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_k}{\partial \bar{z}_{n-1}} & \frac{\partial f_1}{\partial \bar{z}_{n-1}} & & \frac{\partial f_{n-2}}{\partial \bar{z}_{n-1}} \end{pmatrix}$$

is $n-2$ at each $q \in S \cap U$.

But $(df_1 \cap \ldots \cap df_{n-2}) \neq 0$ and thus the rank of

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_{n-2}}{\partial z_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial z_{n-1}} & & \frac{\partial f_{n-2}}{\partial z_{n-1}} \\ \frac{\partial f_1}{\partial \bar{z}_1} & & \frac{\partial f_{n-2}}{\partial \bar{z}_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial \bar{z}_{n-1}} & & \frac{\partial f_{n-2}}{\partial \bar{z}_{n-1}} \end{pmatrix} \quad \text{is } n-2 .$$

Therefore for each k there exists real valued functions $c_1^k, \ldots, c_{n-2}^k$ on $S \cap U$ such that $\frac{\partial g_k}{\partial z_j}(p) = \sum_{\ell=1}^{n-2} c_\ell^k(p) \frac{\partial f_\ell}{\partial z_j}(p)$ and $\frac{\partial g_k}{\partial \bar{z}_j}(p) = \sum_{\ell=1}^{n-2} c_\ell^k(p) \frac{\partial f_\ell}{\partial \bar{z}_j}(p)$, $j=1,\ldots,n-1$, $p \in S \cap U$. Let $T_p^f(S)$, $T_p^g(S)$ denote the set of complex valued tangent vectors to S at a point p associated with the f_j's and g_j's respectively. Let $X \in T_p^f(S)$. Then, for each k,

$$\begin{aligned} X_{g_k}(p) &= \sum_{j=1}^{n-1} a_j \frac{\partial g_k}{\partial z_j}(p) + \sum_{1}^{n-1} b_j \frac{\partial g_k}{\partial \bar{z}_j}(p) \\ &= \sum_{\ell=1}^{n-2} c_\ell^k(p) \left[\sum_{j=1}^{n-1} a_j \frac{\partial f_\ell}{\partial \bar{z}_j}(p) + \sum_{1}^{n-1} b_j \frac{\partial f_\ell}{\partial \bar{z}_j}(p) \right] \\ &= \sum_{\ell=1}^{n-2} c_\ell^k(p)\, X_{f_\ell}(p) = 0 . \end{aligned}$$

Thus, $T_p^f(S) \subset T_p^g(S)$. The same argument shows that $T_p^g(S) \subset T_p^f(S)$. Therefore, $T_p^g(S) = T_p^f(S)$ and this space is denoted $T_p(S)$.

<u>Definition:</u> The holomorphic tangent vectors to S at a point p are the tangent vectors of the form $X = \sum_1^n a_j \frac{\partial}{\partial z_j}$, i.e. $b_j = 0$, $j=1,\ldots,n-1$.

The space of holomorphic tangent vectors to S at p is denoted $HT_p(S)$.

<u>Definition:</u> The antiholomorphic tangent vectors to S at p are the tangent vectors of the form $X = \sum_{j=1}^{n-1} b_j \frac{\partial}{\partial \bar{z}_j}$, i.e. $a_j = 0$, $j-1,\ldots,n-1$.

Since $f_\ell(\zeta_1(\alpha_1,\ldots,\alpha_n),\ldots,\zeta_{n-1}(\alpha_1,\ldots,\alpha_n)) \equiv 0$ for $1 \le \ell \le n-1$, we have $\frac{\partial f_\ell}{\partial \alpha_j} = 0$ for $1 \le \ell \le n-2$, $1 \le j \le n$. $\mathrm{Dim}_{\mathbb{C}}(T_p(S)) = n$.
Therefore, $\frac{\partial}{\partial \alpha_j} = \sum_{k=1}^{n-1} \left(\frac{\partial \zeta_k}{\partial \alpha_j}\right)_p \frac{\partial}{\partial z_k} + \sum_{k=1}^{n-1} \left(\frac{\partial \overline{\zeta_k}}{\partial \alpha_j}\right)_p \frac{\partial}{\partial \bar{z}_k}$, $1 \le j \le n$ span the full complexified tangent space to S at a point p . Every tangent vector to S at a point p is thus of the form:

$$X = \sum_{j=1}^n c_j \frac{\partial}{\partial \alpha_j} = \sum_{j=1}^n c_j \sum_{k=1}^{n-1} \left(\frac{\partial \zeta_k}{\partial \alpha_j}\right)_p \frac{\partial}{\partial z_k} + \sum_{j=1}^n c_j \sum_{k=1}^{n-1} \left(\frac{\partial \overline{\zeta_k}}{\partial \alpha_j}\right) \frac{\partial}{\partial \bar{z}_k}$$

where $c_j \in \mathbb{C}$, $j=1,\ldots,n$. A vector $X = \sum_{k=1}^{n-1} b_k \frac{\partial}{\partial \bar{z}_k}$ is an antiholomorphic tangent vector to S at p if and only if $b_k = \sum_{j=1}^n c_j \left(\frac{\partial \overline{\zeta_k}}{\partial \alpha_j}\right)_p$ and $0 = \sum_{j=1}^n c_j \left(\frac{\partial \zeta_k}{\partial \alpha_j}\right)_p$ for some vector $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{C}^n$. That is,

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{\zeta}_1}{\partial \alpha_1} & \cdots & \frac{\partial \bar{\zeta}_1}{\partial \alpha_n} \\ \vdots & & \vdots \\ \frac{\partial \bar{\zeta}_{n-1}}{\partial \alpha_1} & \cdots & \frac{\partial \bar{\zeta}_{n-1}}{\partial \alpha_n} \end{pmatrix}_p \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial\zeta_1}{\partial\alpha_1} & \cdots\cdots & \frac{\partial\zeta_1}{\partial\alpha_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial\zeta_{n-1}}{\partial\alpha_1} & \cdots\cdots & \frac{\partial\zeta_{n-1}}{\partial\alpha_n} \end{pmatrix}_p \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix}$$

Let $r(p) = \dim_{\mathbb{C}}$ (the antiholomorphic tangent vector space to S at p). Then

$$r(p) = n - \operatorname{rank}\begin{pmatrix} \frac{\partial\zeta_1}{\partial\alpha_1} & \cdots & \frac{\partial\zeta_1}{\partial\alpha_n} \\ & & \\ \frac{\partial\zeta_{n-1}}{\partial\alpha_1} & & \frac{\partial\zeta_{n-1}}{\partial\alpha_n} \end{pmatrix}_p = n - (n-1) = 1 .$$

Thus $r(p) = 1 \quad \forall\, p \in S \cap U$.

Since $\zeta_1,\ldots,\zeta_{n-1}$ are all characteristic coordinates, we have:

$$\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial\zeta_1}{\partial\alpha_1} & \cdots & \frac{\partial\zeta_1}{\partial\alpha_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial\zeta_{n-1}}{\partial\alpha_1} & \cdots & \frac{\partial\zeta_{n-1}}{\partial\alpha_n} \end{pmatrix} \begin{pmatrix} A_1(\alpha_1,\ldots,\alpha_n) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_n(\alpha_1,\ldots,\alpha_n) \end{pmatrix}$$

Therefore,

$$X = \sum_{j=1}^{n} A_j(\alpha_1,\ldots,\alpha_n) \sum_{k=1}^{n-1} \left(\frac{\partial\bar{\zeta}_k}{\partial\alpha_j}\right) \frac{\partial}{\partial\bar{z}_k} = \sum_{k=1}^{n-1} P(\bar{\zeta}_k) \frac{\partial}{\partial\bar{z}_k}$$

is an antiholomorphic tangent vector to S. Since $r(p) = 1$, X is the only antiholomorphic tangent vector to S up to a constant.

P operates on functions defined on $\mathbb{E}^n$. But given a sufficiently small neighborhood $\Omega \subset \mathbb{E}^n$ we have a one to one correspondence between functions defined on Ω and functions defined on $\zeta(\Omega) \subset S \subset \mathbb{C}^{n-1}$ given by

$$u(\alpha_1,\ldots,\alpha_n) = u'(\zeta(\alpha_1,\ldots,\alpha_n),\ldots,\zeta_{n-1}(\alpha_1,\ldots,\alpha_n)) .$$

We can define an operator P' which operates on functions defined on $\zeta(\Omega)$ by $P'u' = Pu$.

$$Pu = \sum_{j=1}^{n} A_j \frac{\partial u}{\partial \alpha_j} = \sum_{j=1}^{n} A_j \sum_{k=1}^{n-1} \frac{\partial u'}{\partial z_k} \frac{\partial \zeta_k}{\partial \alpha_j} + \sum_{j=1}^{n} A_j \frac{\partial u'}{\partial \bar{z}_k} \frac{\partial \bar{\zeta}_k}{\partial \alpha_j}$$

$$= \sum_{k=1}^{n-1} \sum_{j=1}^{n} A_j \frac{\partial \zeta_k}{\partial \alpha_j} \frac{\partial u'}{\partial z_k} + \sum_{k=1}^{n-1} \sum_{j=1}^{n} A_j \frac{\partial \bar{\zeta}_k}{\partial \alpha_j} \frac{\partial u'}{\partial \bar{z}_k}$$

$$= \sum_{k=1}^{n-1} P(\zeta_k) \frac{\partial u'}{\partial z_k} + \sum_{k=1}^{n-1} P(\bar{\zeta}_k) \frac{\partial u'}{\partial \bar{z}_k} = \sum_{k=1}^{n-1} P(\bar{\zeta}_k) \frac{\partial u'}{\partial \bar{z}_k} = P'u' .$$

Thus, P' is of the form $\sum_{k=1}^{n-1} P(\bar{\zeta}_k) \frac{\partial}{\partial \bar{z}_k}$ and is, therefore, the antiholomorphic tangent vector to S at every point of $\zeta(\Omega)$.

In general, if M is a submanifold of $\mathbb{C}^n$ of real dimension ℓ and $p \in M$ and U is a neighborhood of p with

$$M \cap U = \begin{cases} z = (z_1,\ldots,z_n):\ f_k(z_1,\ldots,z_n) = 0,\ k=1,2,\ldots,2n-\ell \\ \text{where } f_k\text{'s are real valued and } df_1 \cap \ldots \cap df_{2n-\ell} \neq 0 \\ \text{in } U \end{cases}$$

we define an antiholomorphic tangent vector to M at p to be a vector of the form $X = \sum_{j=1}^{n} b_j \frac{\partial}{\partial \bar{z}_j}$ such that $Xf_k(p) = 0,\ k=1,2,\ldots,2n-\ell$.

We denote this space by $\bar{H}T_p(M)$. Let $r(p) = \dim_{\mathbb{C}}(\bar{H}T_p(M))$. If $X \in \bar{H}T_p(M)$, then $\bar{X} \in HT_p(M)$ where $HT_p(M)$ is the set of holomorphic tangent vectors to M at p . Conversely, if $X \in HT_p(M)$, then $\bar{X} \in \bar{H}T_p(M)$. Thus $r(p) = \dim_{\mathbb{C}}(HT_p(M))$. $\bar{H}T_p(M) \cap HT_p(M) = 0$ and $\bar{H}T_p(M) \cup HT_p(M) \subset T_p(M)$, where $T_p(M)$ is the set of complex valued tangent vectors to M at p . Therefore, $2 \dim_{\mathbb{C}}(\bar{H}T_p(M)) \le \dim_{\mathbb{C}} T_p(M) = \ell$. On the other hand,

$$r(p) = n - \text{rank} \begin{pmatrix} \frac{\partial f_1}{\partial \bar{z}_1} & \cdots & \frac{\partial f_{2n-\ell}}{\partial \bar{z}_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial \bar{z}_n} & \cdots & \frac{\partial f_{2n-\ell}}{\partial \bar{z}_n} \end{pmatrix} \ge n - (2n-\ell) = \ell - n .$$

Thus $\ell - n \le r(p) \le \left[\frac{\ell}{2}\right]$. If $X \in \bar{H}T_p(M)$ is of the form

$X = \sum_{j=1}^{n} b_j' + b_j''$ with b_j', b_j'' real then $r(b) \equiv \sum_{j=1}^{n} b_j' \frac{\partial}{\partial x_j} - \sum_{j=1}^{n} b_j'' \frac{\partial}{\partial y_j}$

and $i(b) \equiv \sum_{j=1}^{n} b_j'' \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j' \frac{\partial}{\partial y_j}$ are real tangent vectors to M at p when M is considered as a subset of $\mathbb{R}^{2n}$. If $\sum_{j=1}^{n} c_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} d_j \frac{\partial}{\partial y_j}$

is a real tangent vector to M at p then $\sum_{j=1}^{n} (c_j - id_j) \frac{\partial}{\partial \bar{z}_j} \in \bar{H}T_p(M)$

if and only if $\sum_{j=1}^{n} - d_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} c_j \frac{\partial}{\partial y_j}$ is also a real tangent vector

to M at p . Thus, $\bar{H}T_p(M)$ is isomorphic to the maximal complex subspace contained in the real tangent space to M at p .

By the way $\bar{H}T_p(M)$ is defined, we have that the $2n-\ell$ vectors $\left(\frac{\partial f_1}{\partial \bar{z}_1}(p), \ldots, \frac{\partial f_1}{\partial \bar{z}_n}(p)\right), \ldots \left(\frac{\partial f_{2n-\ell}}{\partial \bar{z}_1}(p), \ldots, \frac{\partial f_{2n-\ell}}{\partial \bar{z}_n}(\bar{p})\right)$ in $\mathbb{C}^n$ span the space perpendicular to the set

$$L = \left\{ b \in \mathbb{C}^n : b = (b_1, \ldots, b_n), \sum_{j=1}^{n} b_j \frac{\partial}{\partial \bar{z}_j} \in \bar{H}T_p(M) \right\} .$$

Therefore, if $Xu(p) = 0$ for every $X \in \bar{H}T_p(M)$, then $\bar{\partial}u(p)$ $= \sum_{j=1}^{2n-\ell} c_j \bar{\partial} f_j(p)$, $c_j \in \mathbb{C}$, where we define $\bar{\partial} f \equiv \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$.

Let $U \in \mathbb{C}^n$ be an open neighborhood of p . Suppose $r(q) = r$ is constant for $q \in M \cap U$.

Definition 1: If $X_1, \ldots, X_r$ forms a basis for $\bar{H}T_q(M)$, $q \in M \cap U$, then the system of r equations in one unknown u :

$$(5) \quad X_j u = 0 \quad \text{in} \quad M \cap U, \ j=1,\ldots,r \qquad u \in C^\infty(U)$$

is called the system of tangential Cauchy-Riemann equations to M at each point in $M \cap U$.

We wish to give an equivalent definition of the system of tangential Cauchy-Riemann equations to M on $M \cap U$. First we need some definitions.

Definition: If X is a complex analytic manifold and $z_1, \ldots, z_n$ are local holomorphic coordinates, any differential 1-form may be written as linear combinations of the dz_j and $d\bar{z}_j$, $j=1,\ldots,n$. Differential 1-forms which are linear combinations of the dz_j alone are said to be of type (1,0) . Differential 1-forms which are linear combinations of the $d\bar{z}_j$ alone are said to be of type (0,1) [4,Chapter 2].

As in [3], we define the set $I^{0,1}(M\cap U)$ to be the ideal of C^∞ forms on U generated by f_j and $\bar{\partial} f_j$, $j=1,\ldots,2n-\ell$. That is:

$$I^{0,1}(M\cap U) = \sum_\alpha f_\alpha \varphi_\alpha + \sum_\alpha \beta_\alpha \bar{\partial} f_\alpha : \ \varphi_\alpha \text{ are forms of type } (0,1) \text{ on } U \text{ and } \beta_\alpha \in C^\infty(U) .$$

<u>Definition 2</u>: The system of tangential Cauchy-Riemann equations to M on $M \cap U$ is the system of equations in one unknown u which arises from the condition

$$(6) \quad \bar{\partial} u \equiv 0 \bmod I^{0,1}(M \cap U) \quad u \in C^{\infty}(U)$$

To prove Definition 1 is equivalent to Definition 2, we must prove that (5) is equivalent to (6) to show that (5) $\Longrightarrow$ (6) : Suppose $X_j u = 0$ in $M \cap U$, $j=1,\ldots,n$. Then $X_j u(p) = 0$ for each $p \in M \cap U$. Thus $Xu(p) = 0$ for all $X \in \bar{H}T_p(M)$, $p \in M \cap U$, which implies that $\bar{\partial} u(p) = \sum_{j=1}^{2n-\ell} c_j^p \, \bar{\partial} f(p)$ with $c_j^p \in \mathbb{C}$. Define β_j by $\beta_j(p) = c_j^p$, $p \in M \cap U$, and extend β_j so as to be C^{∞} in all of U. We have $\beta_j \in C^{\infty}(U)$ and $\bar{\partial} u - \sum_{j=1}^{2n-\ell} \beta_j \, \bar{\partial} f \,|_{M \cap U} = 0$. Thus, $\bar{\partial} u - \sum_{j=1}^{2n-\ell} \beta_j \, \bar{\partial} f$ is a (0,1) form which vanishes on $M \cap U$. Hence, there exist (0,1) forms, φ_j, on U such that $\bar{\partial} u - \sum_{j=1}^{2n-\ell} \beta_j \, \bar{\partial} f = \sum_{j=1}^{2n-\ell} f_j \, \varphi_j$. Therefore, u satisfies (6). (6) $\Longrightarrow$ (5) : Suppose u satisfies (6). Then there exist φ_j, (0,1) forms on U, and $\beta_j \in C^{\infty}(U)$ such that $\bar{\partial} u = \sum_{j=1}^{2n-\ell} f_j \, \varphi_j + \sum_{j=1}^{2n-\ell} \beta_j \, \bar{\partial} f_j$. Since φ_j is a (0,1) form for each j, we have $\varphi_j = \sum_{k=1}^{n} c_j^k \, d\bar{z}_k$. Thus, $\frac{\partial u}{\partial \bar{z}_k} = \sum_{j=1}^{2n-\ell} f_j \, c_j^k + \sum_{j=1}^{2n-\ell} \beta_j \frac{\partial f_j}{\partial \bar{z}_k}$. Let $X_1,\ldots,X_r$ be a basis for $\bar{H}T_q(M)$, $q \in M \cap U$, $X_m = \sum_{k=1}^{n} b_k^m \frac{\partial}{\partial \bar{z}_k}$, $1 \le m \le r$. Then $X_m u = \sum_{k=1}^{n} b_k^m \frac{\partial u}{\partial \bar{z}_k} = \sum_{k=1}^{n} b_k^m \left[\sum_{j=1}^{2n-\ell} f_j \, c_j^k + \sum_{j=1}^{2n-\ell} \beta_j \frac{\partial f_j}{\partial \bar{z}_k} \right]$

$$= \sum_{j=1}^{2n-\ell} f_j \sum_{k=1}^{n} b_k^m \, c_j^k + \sum_{j=1}^{2n-\ell} \beta_j \sum_{k=1}^{n} b_k^m \frac{\partial f_j}{\partial \bar{z}_k}$$

$$= \sum_{j=1}^{2n-\ell} f_j \sum_{k=1}^{n} b_k^m c_j^k + \sum_{j=1}^{2n-\ell} \beta_j X_m f_j .$$

Thus $X_m u|_{M \cap U} = \sum_{j=1}^{2n-\ell} \beta_j X_m f_j |_{M \cap U} = 0$ and u satisfies (5) .

The Cauchy-Riemann operator $\bar\partial$ on $\mathbb{C}^n$ induces an operator $\bar\partial_M$ on M .

<u>Definition:</u> We define an operator on M , called the tangential Cauchy-Riemann operator on M and denoted by $\bar\partial_M$, as follows:

If $u \in C^\infty(M)$, $\bar\partial_M u = 0$ on $M \cap U$ means that (5) is satisfied by any extension $\tilde u \in C^\infty(U)$ of $u|_{M \cap U}$.

The following Remarks can be found in [3].

<u>Remark:</u> If condition (5) (or (6)) holds for one such extension $\tilde u$ of $u|_{M \cap U}$ then it holds for all. In fact, if $\tilde u$ and $\tilde{\tilde u}$ are two such extensions, then $\tilde u - \tilde{\tilde u}|_{M \cap U} = 0$. Thus, $\tilde u - \tilde{\tilde u} = \sum_{j=1}^{2n-\ell} g_j f_j$,

$g_j \in C^\infty(U)$, and for $X_m \in \bar{H}T_q(M)$, $q \in M \cap U$. $X_m \tilde u|_{M \cap U} - X_m \tilde{\tilde u}|_{M \cap U} =$
$X_m(\tilde u - \tilde{\tilde u})|_{M \cap U} = \left[\sum_{j=1}^{2n-\ell} f_j X_m g_j + \sum_{j=1}^{2n-\ell} g_j X_m f_j \right]\Big|_{M \cap U} = 0$. Thus,

$X_m \tilde u|_{M \cap U} = 0$ if and only if $X_m \tilde{\tilde u}|_{M \cap U} = 0$.

<u>Remark 2:</u> For $u \in C^\infty(M)$, $\bar\partial_M u = 0$ on $M \cap U$ if and only if there exists an extension $\tilde u \in C^\infty(U)$ of $u|_{M \cap U}$ such that $\bar\partial \tilde u|_{M \cap U} = 0$.
In fact, by the definition of $\bar\partial_M u$ we have that for any extension $\tilde{\tilde u} \in C^\infty(U)$ of $u|_{M \cap U}$ $\bar\partial \tilde{\tilde u} = 0 \bmod I^{0,1}(M \cap U)$. Thus $\bar\partial \tilde{\tilde u} =$
$\sum_{j=1}^{2n-\ell} f_j \varphi_j + \sum_{j=1}^{2n-\ell} \beta_j \bar\partial f_j$ where φ_j are (0,1) forms on U and

$\beta_j \in C^\infty(U)$. Let $\tilde{u} = \tilde{\tilde{u}} - \sum_{j=1}^{2n-\ell} \beta_j f_j$. Then $\tilde{u}$ is another C^∞ extension of $u_{|M \cap U}$ and

$$\bar{\partial}\tilde{u} = \bar{\partial}\tilde{\tilde{u}} - \sum_{j=1}^{2n-\ell}\left[(\bar{\partial}\beta_j)f_j + \beta_j\ \bar{\partial}f_j\right] = \sum_{j=1}^{2n-\ell}\left[\varphi_j - \bar{\partial}\beta_j\right] f_j \ .$$

Thus $\bar{\partial}\tilde{u}_{|M \cap U} = 0$.

Remark 3: By using a partition of unity we have: If $u \in C^\infty(M)$ then $\bar{\partial}_M u = 0$ on M if and only if there exists an extension $\tilde{u} \in C^\infty(\text{nbh. } M)$ such that $\bar{\partial}\tilde{u}_{|M} = 0$.

Definition: A function $u \in C^\infty(M)$ which satisfies $\bar{\partial}_M u = 0$ on M is said to be a Cauchy-Riemann (or CR) function on M, and is denoted by $u \in CR(M)$.

Thus, we have the following:

Theorem. Given any partial differential equation of the form $P = \sum_{j=1}^{n} A_j \frac{\partial}{\partial \alpha_j}$ where $P, \bar{P}$ are linearly independent, suppose ζ_j , $j=1,\ldots,n-1$ are $n-1$ independent, smooth, characteristic coordinates associated with P . Then, for $\Omega \subset \mathbb{R}^n$ sufficiently small and $S = (\zeta_1(\Omega),\ldots,\zeta_n(\Omega))$, $\dim_{\mathbb{R}} S = n$ and S is in a one to one correspondence with Ω . There also exists a one to one correspondence between $C^\infty(\Omega)$ and $C^\infty(S)$ given by $u(\alpha_1,\ldots,\alpha_n) = u'(\zeta_1(\alpha_1,\ldots,\alpha_n),\ldots,\zeta_{n-1}(\alpha_1,\ldots,\alpha_n))$. We can define an operator P' on $C^\infty(S)$ by $(P'u')\circ\zeta = Pu$ on Ω . P' is a basis for $\bar{H}T_q(S)$ for all $q \in S$. For $u' \in C^\infty(S)$ $\bar{\partial}_S u' = 0$ if and only if $P'u' = 0$ on S . Thus, we have $P' = \bar{\partial}_S$ and u is a solution of $Pu = 0$ if and only if $u' \in CR(S)$, where $u'(\zeta_1,\ldots,\zeta_{n-1}) = u(\alpha_1,\ldots,\alpha_n)$.

In [3], Hill proves that under certain geometric conditions, every CR function on a manifold M_0 can be extended to be a CR function on a manifold M, where M is a manifold of one dimension higher than M_0 and M_0 is contained in M as part of its boundary. We shall define the geometric conditions that M_0 and M must satisfy. We shall see that in Lewy's special case, with S playing the role of M_0 and one side of S playing the role of M, these geometric conditions are satisfied. Thus, Lewy's result [5] is a special case of the results of the two papers [1] and [3].

We first need some definitions.

Definition: Let Q be a C^∞ n-dimensional manifold. A disconnected k-dimensional submanifold N of Q is called a foliation of Q if every point of Q is in N, and if around every point $p \in Q$ there is a coordinate system (x,U) with $x(U) = (-\epsilon,\epsilon) \times \dots \times (-\epsilon,\epsilon)$ such that the components of $N \cap U$ are the sets of the form

$$\{q \in U: x^{k+1}(q) = a^{k+1}, x^{k+2}(q) = a^{k+2}, \dots, x^n(q) = a^n\}$$

where $|a^i| < \epsilon$, $i=k+1,\dots,n$. Each component of N is called a leaf of the foliation N, and we say that Q is foliated by the leaves of N [8].

For what follows, let X be a complex analytic manifold of complex dimension n. Let M^0, M' be locally closed differentiable submanifolds of S, where a set is said to be locally closed if it is a relatively closed subset of some open set. Let $M^0 \subset M'$, M^0 open in M' and $\dim_{\mathbb{R}} M^0 = \dim_{\mathbb{R}} M' = \ell$. Let M be the closure of M^0 in M' and let $M_0 = M - M^0$. For any subset $A' \subset M'$. A^0 and A shall be defined by $A^0 = A' \cap M^0$, $A = A' \cap M$.

<u>Definition</u>: The pair (M,M') has a top hat foliation in X if and only if:

i) M' is foliated into connected differentiable submanifolds L' and each leaf L' is itself foliated into connected complex analytic submanifolds ℓ' of X with $\dim_{\mathbb{C}} \ell' = 1$.

ii) Each leaf L' has a neighborhood U in M' , which is a union of the L's and such that the foliation of U by the ℓ's can be defined by a single holomorphic coordinate system for X . That is, U has a neighborhood $\tilde{U}$ in X in which there is a holomorphic coordinate system $(z_1,\ldots,z_n)$ such that each ℓ' in U can be described as $\{$an open subset of the z_1-plane$\} \cap \{(z_2,\ldots,z_n) = \text{constant}\}$.

iii) $\ell \subset\subset \ell'$ for each leaf ℓ' .

iv) Each $L'-L$ is connected and contains a non-void open set in L' which is the union of certain leaves ℓ' from the foliation of L' .

v) Given any sufficiently smooth function u that satisfies $\bar{\partial}_{L'} u = 0$ in $c' \subset L'$, the vanishing of u is some open connected subset w of c' implies that u is identically zero in the connected component of w in c' . (This property is called the unique continuation property for $\bar{\partial}_{L'}$.)

<u>Definition</u>: Let $M_0 = M - M^0$, then M_0 is a holomorphically transverse hypersurface in M' if and only if:

1) M_0 is a closed C^∞ differentiable submanifold of M' , of real codimension one in M' , which has two sides in M' , one of them being M^0 .

2) $\dim_{\mathbb{C}} HT_p(M_0) = \dim_{\mathbb{C}} HT_p(M') - 1 \ \forall\ p \in M_0$ and $\dim_{\mathbb{C}} HT_p(M_0)$ and $\dim_{\mathbb{C}} HT_p(M')$ are both constant for $p \in M_0$.

If M_0 is a holomorphically transverse hypersurface in M', we have that for $p \in M_0$ there exists U, a neighborhood of p in X, in which there are C^∞ real valued functions f, f_k $1 \le k \le 2n-\ell$ such that:

$$
(7)\quad
\begin{aligned}
& M' \cap U = \{f_k = 0,\ 1 \le k \le 2n-\ell\} \\
& df_1 \wedge df_2 \wedge \dots \wedge df_{2n-\ell} \neq 0 \quad \text{on} \quad M' \cap U \\
& M_0 \cap U = \{f = 0,\ f_k = 0,\ 1 \le k \le 2n-\ell\} \\
& df \wedge df_1 \wedge \dots \wedge df_{2n-\ell} \neq 0 \quad \text{on} \quad M_0 \cap U
\end{aligned}
$$

The main result of Hill's paper [3] is the following:

<u>Theorem</u>. If (M, M') has a top hat foliation in X and $M_0 = M - M^0$ is a holomorphically transverse hypersurface in M' then the restriction map:

$$CR(M) \xrightarrow{\sim} CR(M_0)$$

is an isomorphism.

Now we want to show that Lewy's results, [5], follow from [1] and [3]. We have $P = \sum_{j=1}^{n} A_j \frac{\partial}{\partial \alpha_j}$, $A_j \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^3$ and we assume $P, \bar{P}$ and $[P, \bar{P}]$ are linearly independent in a neighborhood of some point in $\mathbb{R}^3$, which, for simplicity, we will assume to be the origin. Let u_0, v_0 be two independent characteristic coordinates for P such that $u_0(0,0,0) = v(0,0,0) = 0$. Define a surface $S \subset \mathbb{C}^2$ by:

$$S = \{(u_0(\alpha_1, \alpha_2, \alpha_3),\ v_0(\alpha_1, \alpha_2, \alpha_3)) : (\alpha_1, \alpha_2, \alpha_3) \in \Omega\}.$$

$\mathrm{Dim}_{\mathbb{R}} S = 3$. There exists a neighborhood of 0, $U' \subset \mathbb{C}^2$ and a real valued function $f \in C^\infty(U')$ with $df \neq 0$ on $U' \cap S$ such that we can express $S \cap U'$ by

$$S \cap U' = \{(z_1,z_2) \in U' : f(z_1,z_2) = 0\} .$$

Let u be a solution of $Pu = 0$. We can define a function u' on S associated with u by

$$u(\alpha_1,\alpha_2,\alpha_3) = u'(u_0(\alpha_1,\alpha_2,\alpha_3),\ v_0(\alpha_1,\alpha_2,\alpha_3)) .$$

We want to show that there exists $U \subset U'$, a neighborhood of the origin such that we can extend u' to $\bar{u}'$, an analytic function on $S^+ = \{(z_1,z_2) \in U: f(z_1,z_2) > 0\}$. From [1] we know that $u' \in CR(S)$. We shall show that there exists U such that if we define $\bar{S}+$ by $\bar{S}+ = \{(z_1,z_2) \in U: f(z_1,z_2) \geq 0\}$, then $(\bar{S}+,U)$ has a top hat foliation and $S \cap U$ is a holomorphically transverse hypersurface in U. The results of [3] would then immediately yield that u' can be extended to $\bar{u}' \in CR(S^+)$. S^+ is an open set in $\mathbb{C}^2$. Therefore, $CR(S^+)$ = set of all analytic functions on S^+. This leads to the conclusion that $\bar{u}'$ is an analytic extension of u' to one side of S.

We know that if P' is the operator on $C^\infty(S)$ associated with P, then $P' = \bar{\partial}_S$. Thus, from the definition of the tangential Cauchy-Riemann operator, we know that P' is of the form $P' = b_1 \frac{\partial}{\partial \bar{z}_1} + b_2 \frac{\partial}{\partial \bar{z}_2}$ and that $P'f = 0$ on $S \cap U$. Since $df \neq 0$ at the origin, either $\frac{\partial f}{\partial \bar{z}_1}(0) \neq 0$ or $\frac{\partial f}{\partial \bar{z}_2}(0) \neq 0$. Without loss of generality, we can assume that $\frac{\partial f}{\partial \bar{z}_2}(0) \neq 0$. Thus, on $S \cap U'$, $P'f = b_1 \frac{\partial f}{\partial \bar{z}_1} + b_2 \frac{\partial f}{\partial \bar{z}_2} = 0$, and, in a neighborhood of the origin, $\frac{b_2}{b_1} = \dfrac{-\frac{\partial f}{\partial \bar{z}_1}}{\frac{\partial f}{\partial \bar{z}_2}}$. Let $\sigma = \dfrac{-\frac{\partial f}{\partial \bar{z}_1}}{\frac{\partial f}{\partial \bar{z}_2}} =$

$$= \frac{\frac{\partial f}{\partial x_1} + i\,\frac{\partial f}{\partial y_1}}{\frac{\partial f}{\partial x_2} + i\,\frac{\partial f}{\partial y_2}}$$ where $z_j = x_j + iy_j$, $j=1,2$. P' is, therefore,

some multiple of the operator $H = \frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial y_1} + \sigma\left[\frac{\partial}{\partial x_2} + i\,\frac{\partial}{\partial y_2}\right]$ in a neighborhood of the origin. Let $\bar{P}'$ be the operator on $C^\infty(S)$ associated with $\bar{P}$. Then, $\bar{P}' = \bar{b}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{b}_2 \frac{\partial}{\partial \bar{z}_2}$ and is some multiple of $\bar{H}$ in a neighborhood of the origin. The condition that $P, \bar{P}$, $[P,\bar{P}]$ are independent at the origin is equivalent to the condition that P', $\bar{P}'$, $[P',\bar{P}']$ are independent at the origin. And, this is equivalent to the fact that the matrix $M_{P'}$ defined by

$$M_{P'} = \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ \sigma & \bar{\sigma} & H\bar{\sigma}-\bar{H}\sigma \\ \sigma i & -\bar{\sigma} i & -H\bar{\sigma}i-\bar{H}\sigma i \end{pmatrix}$$ has rank 3 at the origin.

$M_{P'}$ has rank 3 at the origin if and only if either $(H\bar{\sigma}-\bar{H}\sigma)(0) = (i\,\mathrm{Im}H\bar{\sigma})(0) \neq 0$ or $-H\bar{\sigma}i - \bar{H}\sigma i = (-i\mathrm{Re}\,H\bar{\sigma})(0) \neq 0$, i.e. if and only if $H\bar{\sigma}(0) \neq 0$. H is a multiple of $\bar{\partial}_S$. Thus, the condition that P, $\bar{P}$, $[P,\bar{P}]$ are independent at $(0,0,0)$ becomes the condition:

$$\bar{\partial}_S \left[\frac{-\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} \right](0) \neq 0 \ . \tag{8}$$

The above is true for any two independent characteristic coordinates of P , i.e. we could have chosen for u_0 and v_0 any two independent linear combinations of u_0 and v_0 . This would result in new variables z_1' and z_2' which would each be an independent linear combination of

z_1 and z_2 .

We shall define new variables, as in [5], in which Condition (8) becomes a simpler expression. Define $z_1' = z_1$. $z_2' =$ $[f_{x_1}(0) - i\, f_{y_1}(0)]z_1 + [f_{x_2}(0) - i\, f_{y_2}(0)]z_2$ where $f_\alpha = \frac{\partial f}{\partial \alpha}$. Since $\frac{\partial f}{\partial z_2}(0) \neq 0$, $f_{x_2}(0) - i\, f_{y_2}(0) \neq 0$, and the new variables are independent in a neighborhood of the origin. If we define H' by

$$H' = \frac{\partial}{\partial \bar{z}_1'} + \left[\frac{\dfrac{-\partial f}{\partial \bar{z}_1'}}{\dfrac{\partial f}{\partial \bar{z}_2'}} \right] \frac{\partial}{\partial \bar{z}_2'}$$

we still have

$$(8') \qquad H' \left[\frac{\dfrac{-\partial f}{\partial z_1'}}{\dfrac{\partial f}{\partial z_2'}} \right](0) \neq 0 .$$

If $z_2' = 0$ then $[f_{x_1}(0) - i\, f_{y_1}(0)]z_1 + [f_{x_2}(0) - i\, f_{y_2}(0)]z_2 = 0$ and thus $(f_{x_1}(0))x_1 + (f_{y_1}(0))y_1 + (f_{x_2}(0))x_2 + (f_{y_1}(0))y_2 = 0$. But, this means that the plane $z_2' = x_2' + i\, y_2' = 0$, considered as a real two-dimensional plane $x_2' = 0,\ y_2' = 0$, is in the real tangent space to S . After this change of variables we have $\frac{\partial f}{\partial z_2'}(0) \neq 0$, but $f_{x_1'}(0) = f_{y_1'}(0) = 0$ and, therefore, $\frac{\partial f}{\partial z_1'}(0) = \frac{\partial f}{\partial \bar{z}_1'}(0) = 0$. We now have that

$$H' \frac{-\dfrac{\partial f}{\partial z_1'}}{\dfrac{\partial f}{\partial z_2'}}(0) = \frac{\partial}{\partial \bar{z}_1'} \frac{-\dfrac{\partial f}{\partial z_1'}}{\dfrac{\partial f}{\partial z_2'}}(0) =$$

$$\frac{[f_{x_2'} + i\, f_{y_2'}][-f_{x_1'x_1'} + i\, f_{y_1'x_1'}]}{(f_{x_2'} + i\, f_{y_2'})^2}(0) + \frac{i[f_{x_2'} + i\, f_{y_2'}][-f_{y_1'x_1'} + i\, f_{y_1'y_1'}]}{(f_{x_2'} + i\, f_{y_2'})^2}(0) .$$

Therefore, (8′) reduces to the Condition

$$(9) \qquad [f_{x_1'x_1'} + f_{y_1'y_1'}](0) \neq 0 .$$

Now we shall change variables again by a rotation of the coordinate z_2'. Let $z_1'' = z_1'$, $z_2'' = \frac{[f_{y_2'}(0) - i\, f_{x_2'}(0)]}{[f_{x_2'}(0)]^2 + [f_{y_2'}(0)]^2} z_2^1$. Then $f_{x_2''}(0) = 0$ and $f_{y_2''}(0) \neq 0$. For simplicity of notation, let $z_1 = z_2''$, $z_2 = z_2''$. We now have $f_{x_1}(0) = f_{y_1}(0) = f_{x_2}(0) = 0$, $f_{y_2}(0) \neq 0$ so that the ordinary real tangent plane to S at the origin is given by $y_2 =$ constant.

We shall change variables once more, as in [5], and with these new variables $Z_1 = X_1 + i\, Y_1$, $Z_2 = X_2 + i\, Y_2$, Condition (9) will imply that the surface S is strongly pseudo-convex at 0, i.e.

$$(10) \qquad f_{X_1X_1}(0)\, f_{Y_1Y_1}(0) - (f_{X_1Y_1}(0))^2 > 0 , \quad f_{X_1}(0) = f_{Y_1}(0) = 0 .$$

Let $Z_1 = z_1$, $Z_2 = z_2 + \frac{\beta_1 + i\, \beta_2}{2}(z_1)^2$, where β_1 and β_2 are defined by $\beta_1 = \frac{f_{x_1y_1}(0)}{f_{y_2}(0)}$, $\beta_2 = \frac{f_{x_1x_1}(0) - f_{y_1y_1}(0)}{f_{y_2}(0)}$. We have

$$f_{X_1}(0) = f_{Y_1}(0) = f_{X_2}(0) = 0, \quad f_{Y_2}(0) = f_{y_2}(0) \neq 0 , \quad f_{X_1Y_1}(0) = 0,$$

$$f_{X_1X_1}(0) - f_{Y_1Y_1}(0) = 0 \quad \text{and} \quad f_{X_1X_1}(0) + f_{Y_1Y_1}(0) = f_{x_1x_1}(0) + f_{y_1y_1}(0) .$$

And thus, as in [5],

$$4((f_{X_1X_1}(0)\ f_{Y_1Y_1}(0) - (f_{X_1Y_1}(0))^2)$$
$$= (f_{X_1X_1}(0) + f_{Y_1Y_1}(0))^2 - (f_{X_1X_1}(0) - f_{Y_1Y_1}(0))^2$$
$$- 4(f_{X_1Y_1}(0))^2 = (f_{x_1x_1}(0) + f_{y_1y_1}(0))^2 ,$$

and we have that Condition (9) ==> Condition (10)

As was shown in [5], Condition (10), strong pseudo convexity of S at 0, implies that there exists U_0, a neighborhood of the origin such that on U_0, $f_{X_1X_1}\ f_{Y_1Y_1} > 0$ and the surface $S \cap U_0$ intersects the plane $Z_2 = 0$ only at the origin. This is proved by contradiction. Suppose we cannot find U_0 sufficiently small so that $S \cap U_0 \cap \{Z_2 = 0\} = 0$. Then we must have a curve through the origin which lies on S and on $Z_2 = 0$. Since $f \equiv 0$ on S, we have, for any parameter t for which $\left(\frac{dX_1}{dt}\right)^2 + \left(\frac{dY_1}{dt}\right)^2 \neq 0$ at the origin.

$$0 = \left(\frac{d^2f}{dt^2}(0)\right) = f_{X_1X_1}(0)\left(\frac{dX_1}{dt}\right)^2 + 2\ f_{X_1Y_1}(0)\left(\frac{dX_1}{dt}\right)\left(\frac{dY_1}{dt}\right)$$
$$+ f_{Y_1Y_1}(0)\left(\frac{dY_1}{dt}\right)^2 .$$

But, this happens if and only if $(f_{X_1Y_1}(0))^2 - f_{X_1X_1}(0)\ f_{Y_1Y_1}(0) > 0$. Thus (10) implies that such a curve cannot be found.

Condition (10) also implies that $f_{X_1X_1}(0)\ f_{Y_1Y_1}(0) > 0$ and thus $f_{X_1X_1}(0) + f_{Y_1Y_1}(0) \neq 0$. For simplicity, we shall assume that $f_{X_1X_1}(0) + f_{Y_1Y_1}(0) < 0$ (a symmetric argument can be used in the other case). Since f is C^∞ we can shrink U_0 sufficiently so that

(11) $\quad f_{X_1X_1} + f_{Y_1Y_1} < 0$ on U_0.

We shall assume that u_0 and v_0 are independent characteristic coordinates for P such that if $(z_1,z_2) = (u_0(\alpha_1,\alpha_2,\alpha_3),\ v_0(\alpha_1,\alpha_2,\alpha_3))$, then Conditions (10) and (11) are satisfied.

$$\mathrm{Dim}_{\mathbb{C}}\ HT_0(S) = 3 - \mathrm{rank}\begin{pmatrix} \frac{\partial \overline{u_0}}{\partial \alpha_1} & \frac{\partial \overline{u_0}}{\partial \alpha_2} & \frac{\partial \overline{u_0}}{\partial \alpha_3} \\ \frac{\partial \overline{v_0}}{\partial \alpha_1} & \frac{\partial \overline{v_0}}{\partial \alpha_2} & \frac{\partial \overline{v_0}}{\partial \alpha_3} \end{pmatrix} = 3 - 2 = 1$$

$\mathrm{Dim}_{\mathbb{C}}\ HT_p(S) = 1$ for all p in some neighborhood $\Omega' \subset \mathbb{R}^3$ of the origin. Let $U \subset U_0 \cap U$ be a neighborhood of $(0,0) \in \mathbb{C}^2$ such that for all $(z_1,z_2) \in U \cap S$, $(z_1,z_2) = (u_0(\alpha_1,\alpha_2,\alpha_3),\ v_0(\alpha_1,\alpha_2,\alpha_3))$ with $(\alpha_1,\alpha_2,\alpha_3) \in \Omega'$. As in [5], on U we can put S into the form $S = \{(z_1,z_2)\colon -y_2 + \varphi(x_1,y_1,x_2) = 0\}$ where φ is C^∞ on the set $W = \mathrm{Re}\ \pi_1 U \times \mathrm{Im}\ \pi_1 U \times \mathrm{Re}\ \pi_2 U \subset \mathbb{R}^3$, $\varphi(0,0,0) = 0$, $\varphi_{x_1}(0) = \varphi_{y_1}(0) = \varphi_{y_2}(0) = 0$, $\varphi_{x_1x_1}(0) + \varphi_{y_1y_1}(0) < 0$, and $\dfrac{\partial(\varphi_{x_1}, \varphi_{y_1}, x_2)}{\partial(x_1,y_1,x_2)} \neq 0$ on W . Using the implicite function theorem as in [5], we can express the set $\{(x_1,y_1,x_2)\colon \varphi_{x_1}(x_1,y_1,x_2) = \varphi_{y_1}(x_1,y_1,x_2) = 0\}$ on W as a curve $(x_1(x_2),y_1(x_2),x_2)$ depending on x_2 . We have $\varphi_{x_1}(x_1(x_2),y_1(x_2),x_2)$ $= \varphi_{y_1}(x_1(x_2),y_1(x_2)x_2) = 0$. Thus, for each value α_2 of x_2 , we know that φ has a maximum point at $(x_1(\alpha_2),y_1(\alpha_2),\alpha_2)$. We shall now show that we are in a position to use the results of [3] to prove Lewy's results [5].

Let S^+, $\overline{S}^+$ be defined as above, i.e.

$$S^+ = \{(z_1,z_2) \in U\colon f(z_1,z_2) > 0\}$$

$$\overline{S}^+ = \{(z_1,z_2) \in U\colon f(z_1,z_2) \geq 0\}$$

Since U is an open set in $\mathbb{C}^2$, $HT_p(U) = 2$ for all $p \in U$. Therefore, $S \cap U$ is a holomorphically transverse hypersurface in U. Let $L' = U$, $L = L' \cap \bar{S}^+ = \bar{S}^+$. We foliate L' by planes ℓ' of the form $z_2 =$ constant. Since L' is a neighborhood in $\mathbb{C}^2$, Conditions i), ii), and v) of a top hat foliation are satisfied by $(\bar{S}^+,U)$. For any constant $c = c_1 + i\,c_2$, $\ell' = \{(z_1,c)\colon (z_1,c) \in U\}$, $\ell = \{(z_1,c)\colon (z_1,c) \in U$ and $\varphi(x_1,y_1,c_1) \geq c_2\}$, we have

If $c_2 > \varphi(x_1(c_1),y_1(c_1),c_1)$, $\ell = \emptyset$. In particular, if $c_2 > 0$, $\ell = \emptyset$ and thus Condition iv) is satisfied.

If $c_2 = \varphi(x_1(c_1),y_1(c_1),c_1)$, $\ell = (x_1(c_1) + i\,y_1(c_1),c)$.

If $c_2 < \varphi(x_1(c_1),y_1(c_1)c_1)$ then the set $c_2 = \varphi(x_1,y_1,c_1)$ is a closed Jordan curve, J, in $\pi_1 U = \{z_1 \in \mathbb{C}\colon (z_1,z_2) \in U$ for any $z_2 \in \mathbb{C}\}$ about the point $x_1(c_1) + i\,y_1(c_1)$. Thus, $\ell = \{(z_1,c)\colon z_1$ is inside or on $J\}$. Thus, Condition iii) is satisfied and $(\bar{S}^+,U)$ has a top hat foliation. Therefore we can apply the results of [3] and obtain that any $u' \in CR(S)$ can be extended to $\tilde{u}' \in CR(S^+)$, i.e. any $u' \in CR(S)$ can be extended to $\tilde{u}'$ which is analytic on one side of S. This is Lewy's result [5].

VI. The 4-dimensional Case

In 4 dimensions our operator P is of the form $P = \frac{\partial}{\partial t} + i \sum_{j=1}^{3} b_j(t,x_1,x_2,x_3) \frac{\partial}{\partial x_j}$ where the b_j's are real and analytic in a neighborhood of the origin. We had hoped to be able to use the results of [1] and [3] to show that if $P, \bar{P}$, $[P,\bar{P}]$ and $[P,[P,\bar{P}]$ are linearly independent in a neighborhood of the origin, then the C^∞ initial value problem does not always have a solution for our operator P .

Let u_0, v_0, w_0 be three independent analytic characteristic coordinates for P . Let

$$S = \{(u_0(t,x_1,x_2,x_3),\ v_0(t,x_1,x_2,x_3),\ w_0(t,x_1,x_2,x_3)):\ (t,x_1,x_2,x_3) \text{ in a neighborhood of the origin}\} .$$

From [1] we have $S \subset \mathbb{C}^3$ and $\dim_{\mathbb{R}} S = 4$. We can characterize S as $S = \{(z_1,z_2,z_3):\ f^1(z_1,z_2,z_3) = f^2(z_1,z_2,z_3) = 0\}$ where f^1, f^2 are real valued analytic functions. Letting $S_j = \{(z_1,z_2,z_3):\ f^j(z_1,z_2,z_3) = 0\}$, $j=1,2$, we have two hypersurfaces S_1 and S_2 in $\mathbb{C}^3$ such that $S = S_1 \cap S_2$. We know that any solution u of $Pu = 0$ corresponds to u , a CR function on S . Let

$$S_j^+ = \{(z_1,z_2,z_3):\ f^j(z_1,z_2,z_3) > 0\}$$
$$\bar{S}_j^+ = \{(z_1,z_2,z_3):\ f^j(z_1,z_2,z_3) \geq 0\} .$$

Our idea was to find an appropriate change of coordinates in which the pairs $(\bar{S}_1^+ \cap S_2,\ U \cap S_2)$ and $(\bar{S}_1^+ \cap S_2,\ \bar{S}_1^+ \cap U)$ would have a top hat foliation, where U is a neighborhood of the origin in $\mathbb{C}^3$. Then, once we have proved that $\dim_{\mathbb{C}} HT_p(\bar{S}_1^+ \cap S_2) = 2$ for all $p \in \bar{S}_1^+ \cap S_2$, we would have, from [3], that every function u' which satisfies $\bar{\partial}_S u' = 0$ has a unique extension which is complex analytic on $S_1^+ \cap S_2^+$. We could then use this fact and the Schwarz Reflection Principle to prove that for initial values f which are real valued, C^∞ , but not analytic no C^∞ solution, u could be found.

Our assumption on P that $P, \bar{P}$, $[P, \bar{P}]$ and $[P,[P,\bar{P}]]$ are linearly independent becomes, on S , the assumption that the matrix

$$M = \begin{pmatrix} 1 & -1 & 0 & 0 \\ i & -i & 0 & 0 \\ \sigma_1 & \bar{\sigma}_1 & 2i\,\mathrm{Im}(\bar{\partial}_S \bar{\sigma}_1) & 2i\bar{\sigma}_S(\mathrm{Im}\,\bar{\partial}_S \bar{\sigma}_1) - [\bar{\partial}_S, \partial_S]\sigma_1 \\ \sigma_1 i & -\bar{\sigma}_i & -2i\,\mathrm{RE}(\bar{\partial}_S \bar{\sigma}_1) & -2i\bar{\partial}_S(\mathrm{Re}\,\bar{\partial}_S \bar{\sigma}_1) - i[\bar{\partial}_S, \partial_S]\sigma_1 \\ \sigma_2 & \bar{\sigma}_2 & 2i\,\mathrm{Im}(\bar{\partial}_S \bar{\sigma}_2) & 2i\bar{\partial}_S(\mathrm{Im}\,\bar{\partial}_S \bar{\sigma}_2) - [\bar{\partial}_S, \partial_S]\sigma_2 \\ \sigma_2 i & -\bar{\sigma}_2 i & -2i\,\mathrm{Re}(\bar{\partial}_S \bar{\sigma}_2) & -2i\bar{\partial}_S(\mathrm{Re}\,\bar{\partial}_S \bar{\sigma}_2) - i[\bar{\partial}_S, \partial_S]\sigma_2 \end{pmatrix} \mathrm{Re}$$

has rank 4 , where

$$\sigma_1 = \frac{f^2_{\bar{z}_1} f^1_{\bar{z}_3} - f^1_{\bar{z}_1} f^2_{\bar{z}_3}}{f^2_{\bar{z}_3} f^1_{\bar{z}_2} - f^1_{\bar{z}_3} f^2_{\bar{z}_2}} \qquad \sigma_2 = \frac{f^1_{\bar{z}_1} f^2_{\bar{z}_2} - f^1_{\bar{z}_1} f^1_{\bar{z}_2}}{f^2_{\bar{z}_3} f^1_{\bar{z}_2} - f^1_{\bar{z}_3} f^2_{\bar{z}_2}}$$

and

$$[\bar{\partial}_S, \partial_S] = 2i\ \mathrm{Im}(\bar{\partial}_S \sigma_1) \frac{\partial}{\partial x_2} - 2i\ \mathrm{Re}(\bar{\partial}_S \bar{\sigma}_1) \frac{\partial}{\partial y_2} + 2i\ \mathrm{Im}(\bar{\partial}_S \sigma_2) \frac{\partial}{\partial x_3} - 2i\ \mathrm{Re}(\bar{\partial}_S \sigma_2) \frac{\partial}{\partial y_3} \ .$$

We have not been able to understand all the conditions imposed on the geometry of S_1 and S_2 by rank $M = 4$. Rank $M = 3$ forces S_k to be strongly pseudo convex at the origin for one k, as in [5]. Assuming $k = 1$, we have

$$S_1 = \{(z_1,z_2,z_3):\ -y_2 + \varphi^1(x_1,y_1,x_2,x_3,y_3) = 0\}$$

$$S_2 = \{(z_1,z_2,z_3):\ -y_3 + \varphi^2(x_1,y_1,x_2,y_2,x_3) = 0\}$$

where $z_j = x_j + i\, g_j$ and φ^j are real valued and analytic and $\varphi^1_{x_1x_1}(0)\ \varphi^1_{y_1y_1}(0) - (\varphi^1_{x_1y_1}(0))^2 > 0$.

We have a 3-dimensional surface, depending on x_2, x_3, and y_3 in S_1 of the form:'

$$(x_1(x_2,x_3,y_3),\ y_1(x_2,x_3,y_3), x_2,\ \varphi^1(x_1(x_2,x_3,y_3),\ y_1(x_2,x_3,y_3),$$
$$x_2,x_3,y_3),\ x_3,y_3)$$

on which $\varphi^1_{x_1} = \varphi^1_{y_1} = 0$. But, due to the fact that φ^1 depends on y_3 and φ^2 depends on y_2 as well as on x_1,y_1,x_2 and x_3, we cannot find a suitable foliation for $(\bar{S}_1^+ \cap S_2, \cdot U \cap S_2)$. We hope to be able to return to this.

VII. Appendix

In this section we wish to construct a function $f \in C^{\infty}((-1,1))$ such that f is not analytic in any subneighborhood of $(-1,1)$.

Let $\{r_k\}$, $k=1,\ldots,\infty$ be the set of all rationals in $(-1,1)$ ordered in some fashion. Define f by

$$f(x) = \begin{cases} \displaystyle\sum_{j=1}^{\infty} \frac{1}{2^k\, 3^k\, k!}\, e^{-\frac{1}{(x-r_k)^2}} & x \neq r_\ell \;,\; \ell = 1,2,\ldots,\infty \\ \displaystyle\sum_{\substack{j=1 \\ k\neq m}}^{\infty} \frac{1}{2^k\, 3^k\, k!}\, e^{-\frac{1}{(x-r_k)^2}} & x = r_m \;,\; m \text{ any positive integer.} \end{cases}$$

$f \in C((-1,1))$ since the series converges uniformly. Assume that we can take derivatives under the summation sign; then

$$(12) \qquad f^{(n)}(x) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k\, 3^k\, k!}\, e^{-\frac{1}{(x-r_k)^2}} \sum_{j=0}^{n} c(j,n) \frac{1}{(n-r_k)^{n+2j}} \;, & x \neq r_\ell ,\; \ell = 1,\ldots,\infty \\ \displaystyle\sum_{\substack{k=1 \\ k\neq m}}^{\infty} \frac{1}{2^k\, 3^k\, k!}\, e^{-\frac{1}{(x-r_k)^2}} \sum_{j=0}^{n} c(j,n) \frac{1}{(x-r_k)^{n+2j}} \;, & x = r_m \end{cases}$$

where $c(0,0) = 1$, $c(j,0) = 0$, $j \neq 0$, and

$$c(j,n) = 2c(j-1,n-1) - [n+2j-1]\; c(j,n-1),\; j=1,2,\ldots,n$$

$$c(j,n) = 0 \;,\; j \le 0 \;,\; j > n$$

$$\sum_{j=1}^{n} |c(j,n)| \le \sum_{j=1}^{n} 2|c(j-1,n-1)| + (n+2j-1)\; |c(j,n-1)|$$

$$= 2\sum_{j=1}^{n-1} |c(j,n-1)| + (n-1) \sum_{j=1}^{n-1} |c(j,n-1)| + 2 \sum_{j=1}^{n-1} j\,|c(j,n-1)|$$

$$\le (n+1) \sum_{j=1}^{n-1} |c(j,n-1)| + (2n-2) \sum_{j=1}^{n-1} |c(j,n-1)|$$

$$\sum_{j=1}^{n} |c(j,n)| \le (3n-1) \sum_{j=1}^{n-1} |c(j,n-1)|$$

$$\sum_{j=1}^{n} |c(j,n)| \le (3n)\ (3(n-1))\ (3(n-2))\ \ldots\ 3 = 3^n n!$$

Thus the series on the right-hand side of (12) converges uniformly for each fixed n and thus $f \in C^\infty((-1,1))$, and term by term differentiation is justified. Since $\begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not analytic in any neighborhood containing a rational in $(-1,1)$. Thus f is not analytic on any subinterval of $(-1,1)$.

In general, if Ω' is any neighborhood of the origin in $\mathbb{R}^n$, $\Omega' \subset (-1,1) + \ldots \times (-1,1)$, we can define $g \in C^\infty(\Omega')$ such that g is not analytic in any variable in any subneighborhood of Ω' by

$$g(x_1,\ldots,x_n) = f(x_1) + f(x_2) + \ldots + f(x_n)$$

with f defined as above.

Bibliography

[1] Andreotti, A. and Hill, C.D., Complex characteristic coordinates and tangential Cauchy-Riemann equation, Annali della scuola Norm sup di Pisa, Series III, Vol. XXVI, Fasc II, April-June (1972), 299-324.

[2] Duff, G.F., Partial differential equations, Univ. of Toronto Press, Toronto, 1956.

[3] Hill, C.D., A kontinuitatssatz for $\bar{\partial}_M$ and Lewy extendability, Indiana Univ. Math. Journal, vol. 22, No. 4 (1972), 339-353.

[4] Hormander, L., An introduction to complex analysis in several variables, Van Nostrand, Princeton, 1966.

[5] Lewy, H., On the local character of the solutions of an atypical linear, differential equation in three variables and a related theorem for a regular function of two complex variables, Ann. of Math. 64, (1956), 514-522.

[6] Nirenberg, L., Lectures in linear partial differential equations, Regional Conference Series in Math. No. 17, 1973.

[7] Singer, I.M. and Thorpe, J.A., Lecture Notes on Elementary Topology and geometry, Scott, Foresman and Co., Glenview, Ill., 1967.

[8] Spivak, M., A comprehensive introduction to differential geometry, vol. 1, Brandeis Univ., Waltham, Mass., 1970.

Autobiography

Geraldine Taiani Plakun was born in Queens, New York on February 13,1948. She received her primary education in New York City public schools and was graduated from Andrew Jackson High School in Cambria Heights, New York in 1964. In June, 1968 she was graduated from The State University of New York at Stony Brook with a B.S. in Mathematics and began attending the Graduate School of The City University of New York the following September. In 1969 she was married to Eric Martin Plakun.