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TORSION-LAYERS AND GENERALIZED HERMITIAN FORMS

by

BARRY KOLB

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy , The City University of New York.

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date

CHAIRMAN OF EXAMINING COMMITTEE
Professor Alex Heller

May 14, 1974
date

Richard Sacksteder
EXECUTIVE OFFICER
Professor Richard Sacksteder

Professor Edgar A. Feldman

Professor Alphonse T. Vasquez

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THE CITY UNIVERSITY OF NEW YORK

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INTRODUCTION

Given categories with product $\mathcal{C}$ and $\mathcal{C}'$, and a product-preserving cofinal functor $F\colon \mathcal{C} \to \mathcal{C}'$, there is, according to Heller [3], a five-term exact sequence

$$K_1(\mathcal{C}) \longrightarrow K_1(\mathcal{C}') \longrightarrow K_{\#}(\Phi F) \longrightarrow K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{C}')$$

where ΦF is a new category constructed from F called the fibre of F.

For Λ a commutative ring, we denote by $\mathcal{P}_{\Lambda}^{f}$ the category whose objects are finitely generated projective Λ-modules and whose maps are isomorphisms of these modules. If Λ is an integral domain with quotient field k, and $M_{\Lambda}\colon \mathcal{P}_{\Lambda}^{f} \to \mathcal{P}_{k}^{f}$ the functor defined by tensor product over Λ with k, the fibre category ΦM_{Λ} has as objects triples (P,x,Q) where $P,Q \in \mathrm{ob}(\mathcal{P}_{\Lambda}^{f})$ and $x\colon k \otimes P \to k \otimes Q$ is an isomorphism in $\mathcal{P}_{k}^{f}$.

Identifying P with $1 \otimes P$ and Q with $1 \otimes Q$, for some $\lambda \neq 0$, $\lambda \cdot x(P) \subset Q$. In this manner one constructs out of the triple (P,x,Q) a torsion module -- indeed a collection of torsion modules, one for each element of the ideal $I_x = \{\lambda \mid \lambda \cdot x(P) \subset Q\}$. In some sense we splinter the k-equivalence x into this collection of torsion modules, and there are of course relations among these which derive from the situation in which they originate. Such a collection of torsion-modules together with additional structure given in the text proper will be called a torsion-layer. Once a splintering of an object of ΦM_{Λ} into a torsion-layer occurs, it is not possible, in general, to recover the original (P,x,Q). Indeed, Λ a complete discrete valuation ring seems to be the only non-humpty-dumpty case.

Our original desire was to probe the extent to which certain fibre categories could be replaced with categories involving torsion objects, at least insofar as the Grothendieck group is concerned. Specifically, we were interested in the fibre of $S_\Lambda: S(\mathcal{P}^f_\Lambda) \to S(\mathcal{P}^f_k)$ where $S(\mathcal{P}^f_\Lambda)$ is the category whose objects are non-degenerate symmetric bilinear forms on the objects of $\mathcal{P}^f_\Lambda$ and whose morphisms are isometries of these forms. It should be mentioned that Karoubi [6] has dealt with this question in a satisfactory fashion.

Now it proves that the fibre category ΦS_Λ can be regarded as what we will call the category of generalized hermitian forms on ΦM_Λ . The necessary ingredients for a generalized hermitian situation are a pair of involutions on a category, one covariant and the other contravariant. This is developed in Section IV.

Let $\mathfrak{J}^\Lambda$ denote the category of torsion-layers, and $\Sigma_\Lambda: \Phi M_\Lambda \to \mathfrak{J}^\Lambda$ the functor which associates with each (P,x,Q) its splintering into a torsion-layer. ΦM_Λ admits a covariant involution, denoted j , defined by $j(P,x,Q) = (Q,x^{-1},P)$. We construct an involution J on $\mathfrak{J}^\Lambda$ with the property that Σ_Λ is equivariant with respect to these involutions. The definition of J and the verification of its involutory nature is rather intricate and will occupy us throughout Section II.

The category ΦM_Λ has a multiplicative structure defined by $(P,x,Q) \circ (Q,y,R) = (P,y\circ x,R)$. The analogue of this multiplicative structure proves to be an extensional structure on $\mathfrak{J}^\Lambda$. This extensional structure admits a twisted involution compatible with the functor J. These topics are treated in Section III.

If A is a finite dimensional algebra over Λ, Λ a complete discrete valuation ring, we denote by $\mathfrak{M}_{A/\Lambda}$ the category of A-modules which when considered as Λ-modules are finitely generated and projective.

We then have a functor ${}_{\Lambda}\mathfrak{M}_{A} : \mathfrak{M}_{A/\Lambda} \to \mathfrak{M}_{k\otimes A/k}$ and from this a functor $\Phi_{\Lambda}\mathfrak{M}_{A} \to \mathfrak{J}^{A/\Lambda}$ where $\mathfrak{J}^{A/\Lambda}$ is the full subcategory of $\mathfrak{J}^{\Lambda^{\times} \to A}$ consisting of finite torsion-layers. In Section V we show that this is an equivalence of categories.

Finally, we provide assorted general results on equivalences of certain torsion-layer categories. Alas, the core of the text, which is the construction of the torsion-layer category together with its involution and extensional structure, remains an apparatus in search of wider application.

§I. Orientational Remarks.

For Λ an integral domain, we will denote by $\Lambda^\times$ the commutative monoid of non-zero elements of Λ. A submonoid I of $\Lambda^\times$ will be called an ideal of $\Lambda^\times$ provided that $\gamma \in \Lambda^\times$ and $\lambda \in I$ implies that $\gamma\lambda \in \Lambda^\times$. Note that no mention is made of the additive structure on Λ.

We observe that $\Lambda^\times$ may be regarded as a category, also to be denoted $\Lambda^\times$, whose objects are the elements of $\Lambda^\times$. The set of morphisms from α to β, $\Lambda^\times(\alpha,\beta)$, is empty provided that α does not divide β, and consists of a single element if α does divide β. In the latter case $\beta = \alpha\cdot\gamma$, where γ is unique since $\Lambda^\times$ is a cancellation monoid, and we write

$$\alpha \xrightarrow{\gamma} \beta$$

for this single element of $\Lambda^\times(\alpha,\beta)$.

We recall the definition of the category ΦM_Λ. The objects of ΦM_Λ are triples (P,x,Q) where $P,Q \in \mathrm{ob}(\mathcal{P}^f_\Lambda)$, and $x\colon k\otimes_\Lambda P \to k\otimes_\Lambda Q$ is an isomorphism. A morphism with source (P,x,Q) and target (P',x',Q') is a pair (f,g) where $f\colon P \to P'$, $g\colon Q \to Q'$ are morphisms in $\mathcal{P}^f_\Lambda$, with the property that $(1_k \otimes g)\circ x = x'\circ(1_k \otimes f)$.

Given an object (P,x,Q) of ΦM_Λ we identify P with the Λ-submodule of $k \otimes_\Lambda P$ generated by $1 \otimes P$ and Q with that generated by $1 \otimes Q$ in $k \otimes_\Lambda Q$. Let $I_x = \{\lambda \in \Lambda^\times \mid \lambda\cdot x(P) \subset Q\}$. Since P and Q are finitely generated, I_x is non-empty and it is clearly an ideal of $\Lambda^\times$. We note that I_x can be considered a full subcategory of $\Lambda^\times$.

Denote by g_λ the morphism

$$P \xrightarrow{\lambda\cdot x} Q\ , \quad \lambda \in I_x$$

so that, with this notation, $x = \lambda^{-1} \otimes g_\lambda$, and $g_{\alpha\lambda} = \alpha\cdot g_\lambda$, $\alpha \in \Lambda^\times$, since $\alpha^{-1}\lambda^{-1} \otimes g_{\alpha\lambda} = \lambda^{-1} \otimes g_\lambda = \alpha^{-1}\lambda^{-1} \otimes \alpha g_\lambda$. We now define a functor T_x, depending on x, with domain the category I_x and taking values in Tors_Λ , the category of torsion Λ-modules. For $\lambda \in I_x$, $T_x(\lambda) = Q/g_\lambda P$, and for $\lambda \overset{\alpha}{\rightarrow} \alpha\cdot\lambda$, $T_x(\lambda \overset{\alpha}{\rightarrow} \alpha\lambda)$ is defined by

1.1

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "g_\lambda"] \arrow[d, equal] & Q \arrow[r] \arrow[d, "\alpha"] & Q/g_\lambda P \arrow[r] \arrow[d, "T(\alpha)"] & 0 \\
0 \arrow[r] & P \arrow[r, "g_{\alpha\lambda}"'] & Q \arrow[r] & Q/g_{\alpha\lambda} P \arrow[r] & 0
\end{tikzcd}$$

In what follows we will suppress the dependence upon x and write $T = T_x$. Also we will write $T(\alpha)$ for $T_x(\lambda \rightarrow \alpha\lambda)$ when no confusion is possible.

Similarly, we define a functor $T^*_x = T^*: I_x^{op} \longrightarrow \mathrm{Tors}_\Lambda$ where $T^*(\lambda) = T(\lambda)$ and $T^*(\lambda \rightarrow \alpha\lambda)$ is defined by

1.2

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "g_{\alpha\lambda}"] \arrow[d, "\alpha"'] & Q \arrow[r] \arrow[d, equal] & Q/g_{\alpha\lambda} P \arrow[r] \arrow[d, "T^*(\alpha)"] & 0 \\
0 \arrow[r] & P \arrow[r, "g_\lambda"] & Q \arrow[r] & Q/g_\lambda P \arrow[r] & 0
\end{tikzcd}$$

where as before we abbreviate $T^*(\lambda \rightarrow \alpha\lambda)$ to $T^*(\alpha)$, when there is no danger of confusion. We write T_λ for $T(\lambda)$ in the sequel.

Thus, associated with each (P,x,Q) of ΦM_Λ is a pair (T,T^*) of functors, T covariant, T^* contravariant, where $T_\lambda = T^*_\lambda$ and for $\lambda \rightarrow \alpha\cdot\lambda$, $T(\alpha)$ is monic and $T^*(\alpha)$ is epic. We will refer to this pair as the <u>torsion-layer</u> associated with x, and denote it $T = T_x$, suppressing the T^* .

As mentioned in the introduction, ΦM_Λ admits the covariant involution j, where $j(P,x,Q) = (Q,x^{-1},P)$ and $j(f,g) = (g,f)$. As above there is the torsion-layer associated with x^{-1}, defined on the category $I_{x^{-1}}$. We write $x^{-1} = \mu^{-1} \otimes h_\mu$ where $\mu \in I_{x^{-1}}$ and $h_\mu = \mu \cdot x^{-1}$ considered as a morphism from Q to P. Note that since $x \cdot x^{-1} = 1$, for $\lambda \in I_x$, $\mu \in I_{x^{-1}}$, $h_\mu g_\lambda = \lambda\mu 1_P$ and $g_\lambda h_\mu = \lambda\mu 1_Q$.

In addition to the covariant involution j, ΦM_Λ admits a contravariant involution or a duality, as we shall call it. Since the category $\mathcal{P}^f_\Lambda$ has a duality, namely that given by $P \to \mathrm{Hom}_\Lambda(P,\Lambda) = P^*$ -- see, for example, Jans [4], we have immediately the duality $\mathcal{D}: \Phi M_\Lambda \to \Phi M_\Lambda$ defined by $\mathcal{D}(P,x,Q) = (Q^*,x^*,P^*)$ and $\mathcal{D}(f,g) = (g^*,f^*)$. As before x^* has associated with it a torsion-layer which we denote by $\mathcal{D}(T)$. The associated ideal is $I_{x^*} = I_x$ and since $x^* = \lambda^{-1} \otimes g_\lambda^*$, $\mathcal{D}(T)_\lambda = P^*/g_\lambda^* Q^* = \mathrm{Ext}^1_\Lambda(T_\lambda,\Lambda)$.

Let us examine this more closely. Beginning with the short exact sequence

$$0 \longrightarrow P \xrightarrow{g_\lambda} Q \longrightarrow T_\lambda \longrightarrow 0$$

and applying the functor $\mathrm{Hom}_\Lambda(-,\Lambda)$, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(T_\lambda,\Lambda) \longrightarrow Q^* \xrightarrow{g_\lambda^*} P^* \longrightarrow \mathrm{Ext}^1_\Lambda(T_\lambda,\Lambda) \longrightarrow 0$$

where $\mathrm{Hom}_\Lambda(T_\lambda,\Lambda) = 0$ since T_λ is torsion and Λ a domain. Indeed, writing down the diagrams defining $T(\alpha)$ and $T^*(\alpha)$ given above, and applying $\mathcal{D}$ to these diagrams we see that $(\mathcal{D}T)(\alpha) = \mathcal{D}(T^*(\alpha))$ and $(\mathcal{D}T)^*(\alpha) = \mathcal{D}(T(\alpha))$.

Before considering the category ΦS_Λ we need a definition. For P an object of $\mathcal{P}^f_\Lambda$, a <u>non-degenerate</u> <u>symmetric</u> <u>bilinear</u> <u>form</u> on P is an isomorphism $\xi: P \to P^*$ such that $\xi = \xi^*$, where P is identified with P^{**} in the usual manner. In general, we'll refer to such a ξ as a Λ-form, all the additional adjectives being understood.

An object of the fibre category ΦS_Λ is a triple $(P \xrightarrow{\xi} P^*, x, Q \xrightarrow{\eta} Q^*)$ where ξ and η are Λ-forms, $x\colon k \otimes P \to k \otimes Q$ is an isomorphism, and the following diagram commutes

$$\begin{tikzcd}
k \otimes P \arrow[r, "x"] \arrow[d, "1 \otimes \xi"'] & k \otimes Q \arrow[d, "1 \otimes \eta"] \\
k \otimes P^* & k \otimes Q^* \arrow[l, "x^*"]
\end{tikzcd}$$

That is, it is a pair of Λ-forms together with a k-equivalence of their k-form images.

We now associate with such an object an isomorphism of the torsion layer of x^{-1} and that of x^*. For $\lambda \in I_x \cap I_{x^{-1}}$, we can write $x^{-1} = \lambda^{-1} \otimes h_\lambda$ and $x^* = \lambda^{-1} \otimes g_\lambda^*$. From the above commutative diagram, $\xi \circ h_\lambda = g_\lambda^* \circ \eta$, and we have the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "h_\lambda"] \arrow[d, "\eta"'] & P \arrow[r] \arrow[d, "\xi"] & T_{x^{-1}}(\lambda) \arrow[r] \arrow[d, "\Gamma_\lambda"] & 0 \\
0 \arrow[r] & Q^* \arrow[r, "g_\lambda^*"'] & P^* \arrow[r] & (\mathcal{D}T_x)_\lambda \arrow[r] & 0
\end{tikzcd}$$

Denoting the torsion-layer of x^{-1} by JT, we have a family of isomorphisms

$$\{JT_\lambda \xrightarrow{\Gamma_\lambda} \mathcal{D}T_\lambda\}_{\lambda \in I_x \cap I_{x^{-1}}} .$$

It is a trivial calculation that the diagrams

$$\begin{tikzcd} JT_\lambda \arrow[r, "\Gamma_\lambda"] \arrow[d, "JT(\alpha)"'] & \mathcal{D}T_\lambda \arrow[d, "(\mathcal{D}T)(\alpha)"] \\ JT_{\alpha\lambda} \arrow[r, "\Gamma_{\alpha\lambda}"'] & \mathcal{D}T_{\alpha\lambda} \end{tikzcd} \qquad \begin{tikzcd} JT_\lambda \arrow[r, "\Gamma_\lambda"] & \mathcal{D}T_\lambda \\ JT_{\alpha\lambda} \arrow[r, "\Gamma_{\alpha\lambda}"'] \arrow[u, "JT^*(\alpha)"] & \mathcal{D}T_{\alpha\lambda} \arrow[u, "\mathcal{D}T^*(\alpha)"'] \end{tikzcd}$$

are commutative. If we consider the restrictions of JT and $\mathcal{D}T$ to $I_x \cap I_{x^{-1}}$, then the collection $\{\Gamma_\lambda\}$ provides us with what we would think of as an isomorphism of torsion-layers. In fact, we shall consider "germs" of torsion-layers rather than the torsion-layers as we have constructed them, and $\Gamma = \{\Gamma_\lambda\}$ will be an isomorphism of such germs once we have written down the definitions.

In summary, arising out of the situation defined by an object (ξ,x,η) of ΦS_Λ is an isomorphism of the torsion-layer of x^{-1} with that of x^*. Now, if we had an isomorphism of T_x with T_{x^*} it would be natural to regard this as a bilinear form. But instead we have here an isomorphism of the image of T_x under a covariant involution with the dual of T_x. Such an object we will call a generalized hermitian form.

Our goal now will be to cut the moorings of a torsion-layer from its origins in the fibre category situation in which it arose, and to construct from a torsion-layer T another torsion-layer, denoted $J(T)$, which will play the role of JT above. This we do in the following section.

§II. Torsion-Layers and Their Involution.

2.1. Definition of torsion-layers.

We will make our definitions and constructions in a more general context than that indicated by what has preceded. In the sequel, Λ will denote a commutative cancellation monoid, $\mathfrak{A}$ an abelian category, and $c(\mathfrak{A})$ the center of $\mathfrak{A}$, i.e., the commutative ring of endomorphisms of the identity functor. Let $m: \Lambda \to c(\mathfrak{A})^{x}$ be a morphism of Λ into the multiplicative monoid of $c(\mathfrak{A})$; m will remain fixed throught this section. It now makes sense to speak of $A \xrightarrow{\lambda} A$ for $A \in \operatorname{ob} \mathfrak{A}$, where we mean $A \xrightarrow{m(\lambda)A} A$. The m will be suppressed in order to alleviate the notation.

We will say that A is λ-torsion if $A \xrightarrow{\lambda} A$ is the zero-morphism. A is said to be m-torsion if it is λ-torsion for some $\lambda \in \Lambda$. Tors_m will denote the full subcategory of $\mathfrak{A}$ consisting of m-torsion modules. As in Section I, Λ and its ideals will be regarded as categories.

Definition: By a pre-torsion-layer we will mean a pair of functors $T: I \to \text{Tors}_m \mathfrak{A}$, $T^*: I^{op} \to \text{Tors}_m \mathfrak{A}$, where I is an ideal of Λ, satisfying the following properties:

(0) $T(\lambda) = T^*(\lambda)$ for $\lambda \gg 1$. The notation $\lambda \gg 1$, read 'λ large', means 'for λ in a sufficiently small ideal of Λ'. From this point on we will write T_λ for $T(\lambda)$ and $T^*(\lambda)$.

(1) For $\lambda \xrightarrow{\alpha} \alpha\lambda$, $T(\lambda \xrightarrow{\alpha} \alpha\lambda)$ is monic, $T^*(\lambda \xrightarrow{\alpha} \alpha\lambda)$ is epic and

$$\begin{tikzcd}
T_\lambda \arrow[r, "T(\alpha)"] \arrow[dr, "\alpha"'] & T_{\alpha\lambda} \arrow[d, "T^*(\alpha)"] \arrow[dr, "\alpha"] & \\
 & T_\lambda \arrow[r, "T(\alpha)"'] & T_{\alpha\lambda}
\end{tikzcd}$$

commutes for $\lambda \gg 1$. We note that the notation $T(\alpha)$ is potentially ambiguous, but the domain and range will be clear whenever this abbreviated notation is used.

In the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_{\beta\lambda} \arrow[r, "T(\alpha)"] \arrow[d, "T^*(\beta)"'] & T_{\alpha\beta\lambda} \arrow[d, "T^*(\beta)"] \\
0 \arrow[r] & T_{\lambda} \arrow[r, "T(\alpha)"'] \arrow[d] & T_{\alpha\lambda} \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

the induced morphisms of the kernels and cokernels are isomorphism for $\lambda \gg 1$. Indeed, either of these morphisms being an isomorphism implies, by the nine-lemma, that the other is also.

We will usually relax our language and say 'the pre-torsion-layer T' and suppress the T^*. Two pre-torsion-layers $T: I \to \mathrm{Tors}_m \mathfrak{A}$ and $T': I' \to \mathrm{Tors}_m \mathfrak{A}$ are said to be equivalent if there is some non-empty ideal $J \subset I \cap I'$ such that $T|J = T'|J$. An equivalence class of pre-torsion-layers will be called a <u>torsion-layer</u>. So torsion-layers are germs of pre-torsion-layers.

If S and T are torsion-layers, then a morphism $S \to T$ is a collection $\{S_\mu \xrightarrow{\varphi_\mu} T_\mu\}_{\mu \in I}$, I some ideal in Λ, such that $\{\varphi_\mu\}$ is a natural equivalence of the covariant functors S and T. More precisely, a morphism is defined when we choose pre-torsion-layer representatives of S and T with a common domain and a natural equivalence of these. We identify two such morphisms if they coincide on some smaller domain. We will later show that such a $\{\varphi_\mu\}$ is also a natural equivalence of S^* and T^*.

$\mathfrak{T}^m = \mathfrak{T}$ will denote the category whose objects are torsion-layers and whose morphisms are the morphisms of torsion-layers as defined above.

2.2. Primary example.

In the special case where $\mathfrak{A}$ is the category of Λ-modules, Λ a commutative ring, the center of $\mathfrak{A}$ can be identified with Λ. So we have the morphism $1_{\Lambda^\times}: \Lambda^\times \to c(\mathfrak{A})^\times$; we will assume that Λ is an integral domain in order that $\Lambda^\times$ be a cancellation monoid, and write $\mathfrak{T}^\Lambda$ for $\mathfrak{T}^{1_{\Lambda^\times}}$.

In Section I we associated with an object (P,x,Q) in ΦM_Λ a pair of functors (T_x, T_{x^*}).

Proposition 2.1. The pair (T_x, T_{x^*}) is a pre-torsion-layer and $(P,x,Q) \to (T_x, T_{x^*})$ defines a functor $\Sigma: \Phi M_\Lambda \to \mathfrak{T}^\Lambda$.

Proof: (0) is clear. That $T(\lambda \xrightarrow{\alpha} \alpha\lambda)$ is monic and $T^*(\lambda \xrightarrow{\alpha} \alpha\lambda)$ is epic follows easily from the defining diagrams 1.1 and 1.2. That the compositions $T^*(\alpha)T(\alpha)$ and $T(\alpha)T^*(\alpha)$ yield the appropriate x is equally clear.

Considering the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Q/\beta g_\lambda P \arrow[r, "T(\alpha)"] \arrow[d, "T^*(\beta)"'] & Q/\alpha\beta g_\lambda P \arrow[r] \arrow[d, "T^*(\beta)"] & Q/\alpha Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q/g_\lambda P \arrow[r, "T(\alpha)"'] & Q/\alpha g_\lambda P \arrow[r] & Q/\alpha Q \arrow[r] & 0
\end{tikzcd}$$

we see that (2) is satisfied. Note that $T(\alpha)[q] = [\alpha q]$ and $T^*(\beta)[q] = [q]$ where [ ] means coset class and $q \in Q$. The action of Σ on morphisms is transparent.

2.3. Derived kernel and cokernel torsion-layers.

If T is a torsion-layer, property (2) provides us with isomorphisms $\mathrm{Cok}\ T(\beta\lambda \to \alpha\beta\lambda) \xrightarrow{\sim} \mathrm{Cok}\ T(\lambda \to \alpha\lambda)$ and $\mathrm{Ker}\ T^*(\lambda \to \beta\lambda) \xrightarrow{\sim} \mathrm{Ker}\ T^*(\alpha\lambda \to \alpha\beta\lambda)$ for $\lambda \gg 1$. Fixing α and β, via these isomorphisms we identify the cokernels (kernels) and denote this common cokernel (kernel) by the symbol $C_\alpha(T)(K_\beta(T))$. We will omit the T when only one torsion-layer is under consideration.

We define morphisms $C(\alpha \xrightarrow{\beta} \alpha\beta)$ and $C^*(\alpha \xrightarrow{\beta} \alpha\beta)$ by means of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_\lambda \arrow[r, "T(\alpha)"] \arrow[d, equal] & T_{\alpha\lambda} \arrow[r] \arrow[d, "T(\beta)"] & C_\alpha \arrow[r] \arrow[d, "C(\beta)"] & 0 \\
0 \arrow[r] & T_\lambda \arrow[r, "T(\alpha\beta)"] \arrow[d, "\beta"'] & T_{\alpha\beta\lambda} \arrow[r] \arrow[d, "T^*(\beta)"] & C_{\alpha\beta} \arrow[r] \arrow[d, "C^*(\beta)"] & 0 \\
0 \arrow[r] & T_\lambda \arrow[r, "T(\alpha)"'] & T_{\alpha\lambda} \arrow[r] & C_\alpha \arrow[r] & 0
\end{tikzcd}$$

From the nature of the identifications it follows that $C(\beta)$ and $C^*(\beta)$ are independent of the particular λ, provided only that $\lambda \gg 1$. That $C(\alpha\beta) = C(\alpha)C(\beta)$ and $C^*(\alpha\beta) = C^*(\beta)C^*(\alpha)$ follows from the corresponding properties of (T,T^*). Observe that $C(\beta)$ is monic, $C^*(\beta)$ epic, and that their compositions yield the appropriate β.

This enables us to define a functor pair $C\colon \Lambda \to \mathrm{Tors}_m \mathcal{A}$, $C^*\colon \Lambda^{op} \to \mathrm{Tors}_m \mathcal{A}$ which satisfies conditions (0) and (1). This particular functor pair has an additional property from which (2) will follow.

Lemma 2.1. The sequence

$$0 \longrightarrow C_\alpha \xrightarrow{C(\beta)} C_{\alpha\beta} \xrightarrow{C^*(\alpha)} C_\beta \longrightarrow 0$$

is exact.

Proof: We have the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_\lambda \arrow[r] \arrow[d, equal] & T_{\alpha\lambda} \arrow[r] \arrow[d] & C_\alpha \arrow[r] \arrow[d, "C(\beta)"] & 0 \\
0 \arrow[r] & T_\lambda \arrow[r] & T_{\alpha\beta\lambda} \arrow[r] \arrow[d] & C_{\alpha\beta} \arrow[r] \arrow[d, "\theta"] & 0 \\
 & & C_\beta \arrow[r, equal] \arrow[d] & C_\beta \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

At this point we inaugurate the convention that the morphism $T_\lambda \to T_{\alpha\lambda}$ without any special indication above the arrow is $T(\lambda \to \alpha\lambda)$ and similarly, $T_{\alpha\lambda} \to T_\lambda$ is $T^*(\lambda \to \alpha\lambda)$.

It remains to verify that θ is $C^*(\alpha)$ and this follows from the known commutativity of four of the faces of the wedge diagram

$$\begin{tikzcd}
T_{\alpha\beta\lambda} \arrow[rr] \arrow[dr] \arrow[dd] & & C_{\alpha\beta} \arrow[rr] \arrow[dr, "C^*(\alpha)"] \arrow[dd, "\theta"'] & & 0 \\
 & T_{\beta\lambda} \arrow[rr] \arrow[dl] & & C_\beta \arrow[dl, equal] & \\
C_\beta \arrow[rr, equal] & & C_\beta & &
\end{tikzcd}$$

and the exactness of the top line.

Proposition 2.2. (C,C^*) defines a torsion-layer.

Proof: It remains only to prove (2), and we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C_{\beta\gamma} \arrow[r, "C(\alpha)"] \arrow[d, "C^*(\beta)"] & C_{\alpha\beta\gamma} \arrow[r, "C^*(\beta\gamma)"] \arrow[d, "C^*(\beta)"] & C_\alpha \arrow[r] \arrow[d, "1_{C_\alpha}"] & 0 \\
0 \arrow[r] & C_\gamma \arrow[r, "C(\alpha)"'] & C_{\alpha\gamma} \arrow[r, "C^*(\gamma)"'] & C_\alpha \arrow[r] & 0 .
\end{tikzcd}$$

where the commutativity of the left hand square follows immediately from the corresponding square for (T,T^*).

In a similar fashion we define morphisms $K(\alpha \to \alpha\beta)$ and $K^*(\alpha \to \alpha\beta)$ by means of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_\alpha \arrow[r] \arrow[d, "K(\beta)"'] & T_{\alpha\lambda} \arrow[r] \arrow[d, "T(\beta)"] & T_\lambda \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & K_{\alpha\beta} \arrow[r] \arrow[d, "K^*(\beta)"'] & T_{\alpha\beta\lambda} \arrow[r] \arrow[d, "T^*(\beta)"] & T_\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_\alpha \arrow[r] & T_{\alpha\lambda} \arrow[r] & T_\lambda \arrow[r] & 0
\end{tikzcd}$$

where $K(\beta)$ and $K^*(\beta)$ are independent of λ for $\lambda \gg 1$ and we have

Lemma 2.1'. The sequence

$$\begin{tikzcd}
0 \arrow[r] & K_\alpha \arrow[r, "K(\beta)"] & K_{\alpha\beta} \arrow[r, "K^*(\alpha)"] & K_\beta \arrow[r] & 0
\end{tikzcd}$$

is exact.

Proof: The relevant diagram is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_\alpha \arrow[r, equal] \arrow[d, "\varphi"'] & K_\alpha \arrow[d] & & \\
0 \arrow[r] & K_{\alpha\beta} \arrow[r] \arrow[d, "K^*(\alpha)"'] & T_{\alpha\beta\lambda} \arrow[r] \arrow[d] & T_\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_\beta \arrow[r] \arrow[d] & T_{\beta\lambda} \arrow[r] \arrow[d] & T_\lambda \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and that φ is $K(\beta)$ follows from a diagram similar to that in the proof of Lemma 2.1.

We merely state

Proposition 2.2$'$. The functor pair $K: \Lambda \to \mathrm{Tors}_m \mathfrak{A}$, $K^*: \Lambda^{op} \to \mathrm{Tors}_m \mathfrak{A}$ defines a torsion-layer.

2.4. Preliminaries.

We proceed with several lemmas which are needed for the construction of the involution J.

Lemma 2.2. If C is the torsion-layer cokernel of some T, C_α is α-torsion.

Proof: Referring to the diagram

$$\begin{tikzcd}
 & & T_{\alpha\lambda} \arrow[r, "e"] \arrow[dl] \arrow[d, "\alpha"] & C_\alpha \arrow[d, "\alpha"] & \\
0 \arrow[r] & T_\lambda \arrow[r] & T_{\alpha\lambda} \arrow[r, "e"'] & C_\alpha \arrow[r] & 0
\end{tikzcd}$$

we have $\alpha \circ e = e \circ \alpha = e \circ T(\alpha) \circ T^*(\alpha) = 0$, and e epic implies that α is the zero-morphism on C_α .

Lemma 2.3. For T a torsion-layer, T_λ is λ^2-torsion, $\lambda \gg 1$.

Proof: For $\mu \in I$, T_μ is torsion, so for some ν, $T_\mu \xrightarrow{\nu} T_\mu$ is the zero-morphism. We then have the short exact sequence

$$0 \longrightarrow T_\mu \longrightarrow T_{\mu\nu} \longrightarrow C_\nu \longrightarrow 0$$

where the end objects are ν-torsion, from which we can conclude that $T_{\mu\nu}$ is ν^2-torsion, hence $(\mu\nu)^2$ torsion.

For $\lambda \gg 1$, T_λ λ^2-torsion implies that the composition $T_\lambda \to T_{\lambda^3} \to T_\lambda$ is the zero-morphism and we denote by ${}_\lambda X$ the complex

$$\ldots\ 0 \longrightarrow T_\lambda \longrightarrow T_{\lambda^3} \longrightarrow T_\lambda \longrightarrow 0 \longrightarrow \ldots$$

where T_{λ^3} is taken as the degree zero term and we note that the homology of this complex is concentrated in degree zero. We will generally omit the zeros in writing such a complex.

Lemma 2.4. The chain map

$$\begin{tikzcd} T_{\alpha\lambda} \arrow[r] \arrow[d] & T_{\alpha\lambda^3} \arrow[r] \arrow[d] & T_\lambda \arrow[d, equal] & \text{written} & {}_\lambda X_\alpha \arrow[d, "i_\alpha"] \\ T_\lambda \arrow[r] & T_{\lambda^3} \arrow[r] & T_\lambda & & {}_\lambda X \end{tikzcd}$$

is a quasi-isomorphism, i.e., it induces an isomorphism on homology $H({}_\lambda X_\alpha) \xrightarrow{\sim} H({}_\lambda X)$.

Proof: Expressing the homology as the kernel of the morphism from dual cycles to boundaries, we have

$$\begin{tikzcd}
0 \arrow[r] & H({}_{\lambda}X_{\alpha}) \arrow[r] \arrow[d, "i_{(\alpha)}"] & C_{\lambda^2} \arrow[r] \arrow[d, equal] & T_{\lambda} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H({}_{\lambda}X) \arrow[r] & C_{\lambda^2} \arrow[r] & T_{\lambda} \arrow[r] & 0
\end{tikzcd}$$

and $i(\alpha)$, which we will write for $H(i_{\alpha})$ is an isomorphism by the five-lemma.

<u>Lemma 2.5.</u> The chain map

$$\begin{tikzcd}
T_{\alpha\lambda} \arrow[r] & T_{\alpha\beta\lambda^3} \arrow[r] & T_{\beta\lambda} & & {}_{\lambda}X^{\beta}_{\alpha} \\
T_{\alpha\lambda} \arrow[r] \arrow[u, equal] & T_{\alpha\lambda^3} \arrow[r] \arrow[u] & T_{\lambda} \arrow[u] & \text{written} & {}_{\lambda}X_{\alpha} \arrow[u, "j_{\beta}"']
\end{tikzcd}$$

is a quasi-isomorphism.

<u>Proof</u>: Expressing the homology as the cokernel of the morphism from the boundaries to cycles we have

$$\begin{tikzcd}
0 \arrow[r] & T_{\alpha\lambda} \arrow[r] & K_{\alpha\lambda^2} \arrow[r] & H({}_{\lambda}X^{\beta}_{\alpha}) \arrow[r] & 0 \\
0 \arrow[r] & T_{\alpha\lambda} \arrow[r] \arrow[u, equal] & K_{\alpha\lambda^2} \arrow[r] \arrow[u, equal] & H({}_{\lambda}X_{\alpha}) \arrow[r] \arrow[u, "j(\beta)"'] & 0
\end{tikzcd}$$

and $j(\beta)$, which we write for $H(j_{\beta})$, is an isomorphism by the five-lemma.

2.5. <u>Definition of the functor</u> J.

For T a torsion-layer we will define JT_{λ} as the homology of the complex ${}_{\lambda}X$, $\lambda \gg 1$. From Lemma 2.4. and 2.5. we have a canonical isomorphism $j(\beta)\circ i(\alpha)^{-1} : H({}_{\lambda}X) \rightarrow H({}_{\lambda}X^{\beta}_{\alpha})$. In the special case $\alpha = \beta$

we write $\theta_\alpha = j(\alpha)\circ i(\alpha)^{-1}$.

We have chain maps

$$\begin{tikzcd}
{}_\lambda X^\alpha_\alpha \arrow[d, "\varphi_\alpha"] \\
{}_{\alpha\lambda}X \arrow[d, "\psi_\alpha"] \\
{}_\lambda X^\alpha_\alpha
\end{tikzcd}
\qquad
\begin{tikzcd}
T_{\alpha\lambda} \arrow[r] \arrow[d, equal] & T_{\alpha^3\lambda^3} \arrow[r] \arrow[d] & T_{\alpha\lambda} \arrow[d, "\alpha"] \\
T_{\alpha\lambda} \arrow[r] \arrow[d, "\alpha"'] & T_{\alpha^3\lambda^3} \arrow[r] \arrow[d] & T_{\alpha\lambda} \arrow[d, equal] \\
T_{\alpha\lambda} \arrow[r] & T_{\alpha^3\lambda^3} \arrow[r] & T_{\alpha\lambda}
\end{tikzcd}$$

and we observe that φ_α is such that $H(\varphi_\alpha)$, which we denote $\varphi(\alpha)$, is a monomorphism and that $H(\psi_\alpha)$, denoted $\psi(\alpha)$, is an epimorphism. Their composition, in either order, yield the appropriate α . Remarking that $JT_\lambda = H({}_\lambda X)$, $JT_{\alpha\lambda} = H({}_{\alpha\lambda}X)$, we define $JT(\lambda \to \alpha\lambda)$ as $\varphi(\alpha)\circ\theta_\alpha$ and $JT^*(\lambda \to \alpha\lambda)$ as $\theta_\alpha^{-1} \circ \psi(\alpha)$.

In what follows we will be lax about distinguishing between torsion-layers and representative pre-torsion layers of these. This shouldn't lead to any confusion.

We must now verify that the JT thus defined is a torsion-layer. Once again some preliminaries are necessary. We consider the chain maps

$${}_\lambda X \xleftarrow{i_\alpha} {}_\lambda X_\alpha \text{ , } {}_\lambda X_\alpha \xrightarrow{j_\beta} {}_\lambda X^\beta_\alpha \text{ , and}$$

$$\begin{tikzcd}
T_\lambda \arrow[r] \arrow[d, equal] & T_{\lambda^3} \arrow[r] \arrow[d] & T_\lambda \arrow[d] \\
T_\lambda \arrow[r] & T_{\beta\lambda^3} \arrow[r] & T_{\beta\lambda} \arrow[d, equal] \\
T_{\alpha\lambda} \arrow[r] \arrow[u] & T_{\alpha\beta\lambda} \arrow[r] \arrow[u] & T_{\beta\lambda}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
{}_\lambda X \arrow[d, "j_\beta"] \\
{}_\lambda X^\beta \\
{}_\lambda X^\beta_\alpha \arrow[u, "i_\alpha"']
\end{tikzcd}$$

where it is hoped that the diagrams will provide some insight into the accompanying notation. We will be somewhat fuzzy notationwise and use i_α, j_β for different chain maps, where the specification of domain and range will serve to distinguish these from the other i_α, j_β. i_α and j_β are to be thought of as referring to certain operations rather than specific morphisms -- they are modes of passing from one complex to another. The same remarks apply to φ_α and ψ_α. Comparing the notations with the notated should give the reader a feeling for the inflexions of the symbols.

In the sequel we frequently deal with morphisms obtained by passing to homology and inverting certain morphisms. Given a finite connected sequence of chain maps, say $A \leftarrow B \rightarrow C$, and another such, say $A \rightarrow D \leftarrow C$ with the same right end complexes and left end complexes, we write $A \leftarrow B \rightarrow C \sim A \rightarrow D \leftarrow C$ to mean the following: if we pass to homology and reverse the left-pointing arrows, the compositions obtained coincide. It is understood that the left-pointing arrows are quasi-isomorphisms; $\sim$ will be read 'equivalent'.

Lemma 2.6. ${}_\lambda X \xleftarrow{i_\alpha} {}_\lambda X_\alpha \xrightarrow{j_\beta} {}_\lambda X_\alpha^\beta \sim {}_\lambda X \xrightarrow{j_\beta} {}_\lambda X^\beta \xleftarrow{i_\alpha} {}_\lambda X_\alpha^\beta$.

Proof: A trivial calculation shows that

$$ {}_\lambda X_\alpha \xrightarrow{j_\beta} {}_\lambda X_\alpha^\beta \xrightarrow{i_\alpha} {}_\lambda X^\beta = {}_\lambda X_\alpha \xrightarrow{i_\alpha} {}_\lambda X \xrightarrow{j_\beta} {}_\lambda X^\beta $$

and passing to homology we have $i(\alpha)j(\beta) = j(\beta)i(\alpha)$. Since these are all isomorphisms, $j(\beta)\circ i(\alpha)^{-1} = i(\alpha)^{-1}\circ j(\beta)$, which is the content of the assertion.

Lemma 2.7. ${}_\lambda X_\alpha^\alpha \xrightarrow{\varphi_\alpha} {}_{\alpha\lambda}X \xleftarrow{i_\beta} {}_{\alpha\lambda}X_\beta \sim {}_\lambda X_\alpha^\alpha \xleftarrow{i_\beta} {}_\lambda X_{\alpha\beta}^\alpha \xrightarrow{\varphi_\alpha} {}_{\alpha\lambda}X_\beta$.

Proof: One checks without difficulty that

$${}_{\lambda}X^{\alpha}_{\alpha\beta} \xrightarrow{\varphi_\alpha} {}_{\alpha\lambda}X_{\beta} \xrightarrow{i_\beta} {}_{\alpha\lambda}X = {}_{\lambda}X^{\alpha}_{\alpha\beta} \xrightarrow{i_\beta} {}_{\lambda}X^{\alpha}_{\alpha} \xrightarrow{\varphi_\alpha} {}_{\alpha\lambda}X^{\alpha}_{\alpha}$$

and, as in the previous lemma, this is equivalent to the assertion of the lemma.

Note that in the proofs it is remarked that $j_\beta \circ i_\alpha = i_\alpha \circ j_\beta$ and $i_\beta \circ \varphi_\alpha = \varphi_\alpha \circ i_\beta$. In addition, $i_{\alpha\beta} = i_\alpha \circ i_\beta = i_\beta \circ i_\alpha$ and $j_{\alpha\beta} = j_\alpha \circ j_\beta = j_\beta \circ j_\alpha$. These relations are transparent when written out. In order to demonstrate the functorial nature of JT we need $JT(\alpha\beta) = JT(\alpha) \circ JT(\beta)$. This is precisely the statement that

$${}_{\lambda}X \xleftarrow{i_\beta} {}_{\lambda}X_{\beta} \xrightarrow{j_\beta} {}_{\lambda}X^{\beta}_{\beta} \xrightarrow{\varphi_\beta} {}_{\beta\lambda}X \xleftarrow{i_\alpha} {}_{\beta\lambda}X_{\alpha} \xrightarrow{j_\alpha} {}_{\beta\lambda}X^{\alpha}_{\alpha} \xrightarrow{\varphi_\alpha} {}_{\alpha\beta\lambda}X$$

is equivalent to

$${}_{\lambda}X \xleftarrow{i_{\alpha\beta}} {}_{\lambda}X_{\alpha\beta} \xrightarrow{j_{\alpha\beta}} {}_{\lambda}X^{\alpha\beta}_{\alpha\beta} \xrightarrow{\varphi_{\alpha\beta}} {}_{\alpha\beta\lambda}X \quad .$$

This equivalence is a straight-forward application of Lemmas 2.6, 2.7, the remark following Lemma 2.7, and the observation that $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta = \varphi_\beta \circ \varphi_\alpha$.

Similarly, for the operation ψ, $\psi_{\alpha\beta} = \psi_\alpha \circ \psi_\beta = \psi_\beta \circ \psi_\alpha$, $\psi_\alpha \circ i_\beta = i_\beta \circ \psi_\alpha$ and $\psi_\alpha \circ j_\beta = j_\beta \circ \psi_\alpha$. From the last of these relations we easily conclude

<u>Lemma 2.7'</u>. ${}_{\alpha\lambda}X^{\beta}_{\beta} \xleftarrow{j_\beta} {}_{\alpha\lambda}X_{\beta} \xrightarrow{\psi_\alpha} {}_{\lambda}X^{\alpha}_{\alpha\beta} \sim {}_{\alpha\lambda}X^{\beta}_{\beta} \xrightarrow{\psi_\alpha} {}_{\lambda}X^{\alpha\beta}_{\alpha\beta} \xleftarrow{j_\beta} {}_{\lambda}X^{\alpha}_{\alpha\beta}$.

The functorial nature of JT^* , that $JT^*(\alpha\beta) = JT^*(\beta) \circ JT^*(\alpha)$ amounts to the equivalence of

$${}_{\alpha\beta\lambda}X \xrightarrow{\psi_\alpha} {}_{\beta\lambda}X^{\alpha}_{\alpha} \xleftarrow{j_\alpha} {}_{\beta\lambda}X_{\alpha} \xrightarrow{i_\alpha} {}_{\beta\lambda}X \xrightarrow{\psi_\beta} {}_{\lambda}X^{\beta}_{\beta} \xleftarrow{j_\beta} {}_{\lambda}X_{\beta} \xrightarrow{i_\beta} {}_{\lambda}X$$

and

$$ {}_{\alpha\beta\lambda}X \xrightarrow{\psi_{\alpha\beta}} {}_{\lambda}X^{\alpha\beta}_{\alpha\beta} \xleftarrow{j_{\alpha\beta}} {}_{\lambda}X_{\alpha\beta} \xrightarrow{i_{\alpha\beta}} {}_{\lambda}X \quad . $$

Again this is a follow-your-nose calculation that we omit.

That the pair (JT, JT^*) satisfies (0) and (1) is immediate. Only the proof of (2) remains. The fact that

$$\begin{tikzcd}
JT_{\beta\lambda} \arrow[r, "JT(\alpha)"] \arrow[d, "JT^*(\beta)"'] & JT_{\alpha\beta\lambda} \arrow[d, "JT^*(\beta)"] \\
JT_{\lambda} \arrow[r, "JT(\alpha)"] & JT_{\alpha\lambda}
\end{tikzcd}$$

commutes follows by a lengthy but trivial use of the arrow calculus found above. Using our canonical identifications, we write $JT_{\alpha\beta\lambda} = H({}_{\alpha\beta\lambda}X)$, $JT_{\alpha\lambda} = H({}_{\alpha\lambda}X^{\beta}_{\beta})$, $JT_{\beta\lambda} = H({}_{\beta\lambda}X^{\alpha}_{\alpha})$ and $JT_{\lambda} = H({}_{\lambda}X^{\alpha\beta}_{\alpha\beta})$. That making these identifications is harmless enough follows from the commutative nature of the arrow calculus in this case. Indeed, we might well call it a "calculus of identification justification", and we will refer to it rather than use it.

The chain map $\varphi_{\alpha}\colon {}_{\lambda}X^{\alpha\beta}_{\alpha\beta} \to {}_{\lambda}X^{\beta}_{\beta}$ induces $JT_{\lambda} \to JT_{\alpha\lambda}$ and we have the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{\alpha\beta\lambda} \arrow[r] \arrow[d, equal] & K_{\alpha\beta\lambda^2} \arrow[r] \arrow[d, "K(\alpha)"'] & JT_{\lambda} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{\alpha\beta\lambda} \arrow[r] & K_{\alpha^2\beta\lambda^2} \arrow[r] \arrow[d, "K^*(\alpha\beta\lambda^2)"'] & JT_{\alpha\lambda} \arrow[r] \arrow[d] & 0 \\
 & & K_{\alpha} \arrow[r, "\sim", "z_{\alpha}"'] \arrow[d] & \mathrm{Cok}\ JT(\lambda \to \alpha\lambda) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where the rows and columns are exact and z_α is an isomorphism by the nine-lemma. The chain map $\varphi_\alpha \colon {}_{\beta\lambda}X^{\alpha}_{\alpha} \to {}_{\alpha\beta\lambda}X$ induces $JT_{\beta\lambda} \to JT_{\alpha\beta\lambda}$ and we obtain a similar diagram of which we embed a section in

$$\begin{tikzcd}
 & K_{\alpha\beta^2\lambda^2} \arrow[rr] \arrow[dl, "K^*(\beta)"'] \arrow[dd] & & JT_{\beta\lambda} \arrow[dl] \arrow[dd] \\
K_{\alpha\beta\lambda^2} \arrow[rr] \arrow[dd, "K(\alpha)"'] & & JT_{\lambda} \arrow[dd] & \\
 & K_{\alpha^2\beta^2\lambda^2} \arrow[rr] \arrow[dl] \arrow[dd] & & JT_{\alpha\beta\lambda} \arrow[dl] \arrow[dd] \\
K_{\alpha^2\beta\lambda^2} \arrow[rr] \arrow[dd] & & JT_{\alpha\lambda} \arrow[dd] & \\
 & K_{\alpha} \arrow[rr, "\sim"] \arrow[dl, equal] & & \operatorname{Cok} JT(\beta\lambda \to \alpha\beta\lambda) \arrow[dl] \\
K_{\alpha} \arrow[rr, "\sim", "z_\alpha"'] & & \operatorname{Cok} JT(\lambda \to \alpha\lambda) &
\end{tikzcd}$$

which is commutative, and we see that we can identify the cokernels, so (2) is verified. In sum, we have

Proposition 2.3. The construction J defines an endofunctor of $\mathfrak{J}^m$.

2,6. The involutory nature of J.

In the proof of Proposition 2.3, we made use of an isomorphism $K_\alpha(T) \xrightarrow[z_\alpha]{} C_\alpha(JT)$, where we have again identified all the α-cokernels.

Proposition 2.4. $\{z_\alpha\}$ defines a morphism of torsion-layers $z \colon K(T) \to C(JT)$.

Proof: Consider

$$\begin{tikzcd}
 & K_{\alpha^2\beta^2\lambda^2} \arrow[rr] \arrow[dd] & & JT_{\alpha\beta\lambda} \arrow[dd] \\
K_{\alpha^2\beta\lambda^2} \arrow[ur, "K(\beta)"] \arrow[rr] \arrow[dd] & & JT_{\alpha\lambda} \arrow[ur] \arrow[dd] & \\
 & K_{\alpha\beta} \arrow[rr] & & C_{\alpha\beta}(JT) \\
K_{\alpha} \arrow[ur] \arrow[rr, "z_{\alpha}"'] \arrow[d] & & C_{\alpha}(JT) \arrow[ur, "C(JT)(\beta)"] & \\
0 & & &
\end{tikzcd}$$

The known commutativity of five faces and the exactness of the front-left edge gives us the commutativity of the bottom face.

Preparatory to the proof that J is an involution we require some lemmas.

Lemma 2.8. If $b_{\lambda}(T)\colon T_{\lambda} \to K_{\lambda^2}$ is induced by $T_{\lambda} \to T_{\lambda^3}$ for $\lambda >> 1$, then

$$\begin{tikzcd}
T_{\lambda} \arrow[r, "T(\alpha)"] \arrow[d, "b_{\lambda}(T)"'] & T_{\alpha\lambda} \arrow[d, "b_{\alpha\lambda}(T)"] \\
K_{\lambda^2} \arrow[r, "K(\alpha^2)"'] & K_{\alpha^2\lambda^2}
\end{tikzcd}$$

commutes.

Proof: This follows from the chain map

$$\begin{tikzcd}
T_{\lambda} \arrow[r] \arrow[d, "T(\alpha)"'] & T_{\lambda^3} \arrow[r] \arrow[d, "T(\alpha^2)"] & T_{\lambda} \arrow[d, "\alpha^2"] \\
T_{\alpha\lambda} \arrow[r] & T_{\alpha^2\lambda^3} \arrow[r] & T_{\lambda}
\end{tikzcd}$$

which induces

$$\begin{tikzcd}
T_\lambda \arrow[r] \arrow[d] & \operatorname{Ker} T^*(\lambda^3 \to \lambda) \arrow[d] \\
T_{\alpha\lambda} \arrow[r] & \operatorname{Ker} T^*(\alpha^2\lambda^3 \to \lambda)
\end{tikzcd}$$

and making our usual identifications we have

$$\operatorname{Ker} T^*(\alpha^2\lambda^3 \longrightarrow \lambda) = \operatorname{Ker} T^*(\alpha^3\lambda^3 \longrightarrow \alpha\lambda) = K_{\alpha^2\lambda^2} \quad ,$$

which yields the diagram of the lemma.

Lemma 2.9. If $d_\lambda(T)\colon JT_\lambda \to C_{\lambda^2}$ is induced by $JT_\lambda \to \operatorname{Cok}(T_{\lambda^3} \to T_\lambda)$ for $\lambda >> 1$, then

$$\begin{tikzcd}
JT_\lambda \arrow[r, "JT(\alpha)"] \arrow[d, "d_\lambda(T)"'] & JT_{\alpha\lambda} \arrow[d, "d_{\alpha\lambda}(T)"] \\
C_{\lambda^2} \arrow[r, "C(\alpha^2)"'] & C_{\alpha^2\lambda^2}
\end{tikzcd}$$

Proof: Writing $C_\lambda^{\lambda\mu}$ for the cokernel of $T_\lambda \to T_{\lambda\mu}$, we have by the definition of $JT(\alpha)$

$$\begin{tikzcd}
\cdot \arrow[ddd, "JT(\alpha)"'] & 0 \arrow[r] & JT_\lambda \arrow[r] & C_\lambda^{\lambda^3} \arrow[r] & T_\lambda \arrow[r] & 0 \\
& 0 \arrow[r] & JT'_\lambda \arrow[r] \arrow[u, "i(\alpha)"] \arrow[d, "j(\alpha)"'] & C_{\alpha\lambda}^{\alpha\lambda^3} \arrow[r] \arrow[u, "u"] \arrow[d] & T_\lambda \arrow[r] \arrow[u, equal] \arrow[d] & 0 \\
& 0 \arrow[r] & JT''_\lambda \arrow[r] \arrow[d] & C_{\alpha\lambda}^{\alpha^2\lambda^3} \arrow[r] \arrow[d, "C(\alpha)"'] & T_{\alpha\lambda} \arrow[r] \arrow[d] & 0 \\
\cdot & 0 \arrow[r] & JT_{\alpha\lambda} \arrow[r] & C_{\alpha\lambda}^{\alpha^3\lambda^3} \arrow[r] & T_{\alpha\lambda} \arrow[r] & 0
\end{tikzcd}$$

where u is an identification map, and provides us with the diagram of the lemma.

Theorem 2.1. The functor J is an involution on $\mathfrak{J}^m$, i.e., there exists a natural equivalence $\operatorname{Id}_{\mathfrak{J}^m} \overset{\omega}{\to} J^2$.

Proof: From the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & T_\lambda \arrow[r] \arrow[d, "\omega_\lambda"'] & K_{\lambda^2} \arrow[r] \arrow[d, "z_{\lambda^2}"] & JT_\lambda \arrow[r] \arrow[d, "1_{JT_\lambda}"] & 0 \\
0 \arrow[r] & J^2T_\lambda \arrow[r] & C_{\lambda^2}(JT) \arrow[r] & JT_\lambda \arrow[r] & 0
\end{tikzcd}$$

where the induced ω_λ is an isomorphism. We need only check that $\{\omega_\lambda\}$ is a morphism of torsion-layers. That follows from the diagram

$$\begin{tikzcd}
& T_{\alpha\lambda} \arrow[rr] \arrow[dd] & & J^2T_{\alpha\lambda} \arrow[dd] \\
T_\lambda \arrow[ur] \arrow[rr] \arrow[dd] & & J^2T_\lambda \arrow[ur] \arrow[dd] & \\
& K_{\alpha^2\lambda^2} \arrow[rr] & & C_{\alpha^2\lambda^2}(JT) \\
K_{\lambda^2} \arrow[ur] \arrow[rr] & & C_{\lambda^2}(JT) \arrow[ur] &
\end{tikzcd}$$

where the sides are commutative by Lemmas 2.8 and 2.9, along with the defining diagram for ω_λ. The bottom face is commutative since z is a morphism of torsion-layers, and the fact that $d_\lambda(JT)$ is monic implies that the top face of the cube is commutative, which proves our assertion.

2.7. Primary example.

In Proposition 2.1, we defined a functor $\Sigma: \Phi M_\Lambda \to \mathfrak{J}^\Lambda$. ΦM_Λ admits the involution j defined in Section I and the involution J that we have constructed is compatible with γ. We have

Proposition 2.6.

$$\begin{tikzcd}
\Phi M_\Lambda \arrow[r, "\Sigma"] \arrow[d, "j"'] & \mathfrak{J}^\Lambda \arrow[d, "J"] \\
\Phi M_\Lambda \arrow[r, "\Sigma"'] & \mathfrak{J}^\Lambda
\end{tikzcd}$$

commutes up to natural equivalence.

<u>Proof</u>: Let (P,x,Q) be an object of ΦM_Λ. We will write simply x for this object and refer to Section I for the notation. $(\Sigma\circ j)(x)_\lambda = P/h_\lambda Q$ and $(J\circ\Sigma)(x)_\lambda$ is the homology of

$$Q/g_\lambda P \xrightarrow{T_x(\lambda^2)} Q/\lambda^2 g_\lambda P \xrightarrow{T_x^*(\lambda^2)} Q/g_\lambda P$$

which is equal to $g_\lambda P/\lambda^2 Q$. Then $\hat{g}_\lambda;\ P/h_\lambda Q \to g_\lambda P/\lambda^2 Q$ is an isomorphism where $\hat{g}_\lambda$ is the morphism induced by g_λ, recalling that $g_\lambda h_\lambda = \lambda^2 \cdot 1_Q$. That $\{\hat{g}_\lambda\}$ defines a natural transformation follows from the commutativity of

$$\begin{tikzcd}
P/h_\lambda Q \arrow[r, "\hat{g}_\lambda"] \arrow[d, "T(\alpha)=\hat{\alpha}"'] & g_\lambda P/\lambda^2 Q \arrow[d, "\hat{\alpha}^2=JT(\alpha)"] \\
P/\alpha h_\lambda Q \arrow[r, "\hat{g}_{\alpha\lambda}"] & g_{\alpha\lambda}P/\alpha^2\lambda^2 Q
\end{tikzcd}$$

.

§III. The Extensional Structure.

3.1. Extensional structures

In [3] Heller introduces the notion of an extensional structure in a category with product. For the convenience of the reader we recall the definition. Let $\mathcal{K}$ be a category with product $\perp$, and $P_1, P_2, P: \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be defined by $P_1(A,B) = A$, $P_2(A,B) = B$, and $P(A,B) = A \perp B$.

<u>Definition 3.1</u>. An extensional structure in a category $\mathcal{K}$ consists of a category $\mathcal{K}_e$ together with functors $P'_1, P'_2, P': \mathcal{K}_e \to \mathcal{K}$ and a functor $\nabla: \mathcal{K} \times \mathcal{K} \to \mathcal{K}_e$ satisfying $P'_i\nabla = P_i$, $i = 1,2$ and $P'\nabla = P$.

We note that $\mathfrak{J}^m$ is a category with product given by direct sum in the category $\mathrm{Tors}_m \mathfrak{A}$. We would like to define an extensional structure in $\mathfrak{J}^m$.

<u>Definition 3.2</u>. If T' and T are torsion-layers, by a <u>left-morphism</u> $T' \to T$ is meant a collection of morphisms

$$T'_\lambda \xrightarrow{\varphi_{\lambda,\mu}} T_{\lambda\mu}$$

defined for $\lambda >> 1$, $\mu >> 1$ satisfying:

<u>L1</u>

$$\begin{tikzcd} T'_\lambda \arrow[r, "\varphi_{\lambda\mu}"] \arrow[dr, "\varphi_{\lambda,\alpha\mu}"'] & T_{\lambda\mu} \arrow[d, "T(\alpha)"] \\ & T_{\alpha\lambda\mu} \end{tikzcd}$$

and

<u>L2</u>

$$\begin{tikzcd} T'_\lambda \arrow[r, "\varphi_{\lambda,\mu}"] \arrow[d, "T'(\alpha)"'] & T_{\lambda\mu} \arrow[d, "T(\alpha)"] \\ T'_{\alpha\lambda} \arrow[r, "\varphi_{\alpha\lambda,\mu}"'] & T_{\alpha\lambda\mu} \end{tikzcd}$$

commute.

We obtain some immediate properties of left-morphisms.

__Lemma 3.1__. If $\{\varphi_{\lambda,\mu}: T'_\lambda \to T_{\lambda\mu}\}$ is a left-morphism, then

__L3__

$$\begin{tikzcd}
T'_\lambda \arrow[r, "\varphi_{\lambda,\alpha\mu}"] \arrow[d, "\alpha"'] & T_{\alpha\lambda\mu} \arrow[d, "T^*(\alpha)"] \\
T'_\lambda \arrow[r, "\varphi_{\lambda,\mu}"'] & T_{\lambda\mu}
\end{tikzcd}$$

and

__L4__

$$\begin{tikzcd}
T'_{\alpha\lambda} \arrow[r, "\varphi_{\alpha\lambda,\mu}"] \arrow[d, "T'^*(\alpha)"'] & T_{\alpha\lambda\mu} \arrow[d, "T^*(\alpha)"] \\
T'_\lambda \arrow[r, "\varphi_{\lambda,\mu}"'] & T_{\lambda\mu}
\end{tikzcd}$$

commute.

__Proof__: __L3__ ; $T(\alpha)T^*(\alpha)\varphi_{\lambda,\alpha\mu} = \alpha\varphi_{\lambda,\alpha\mu} = \varphi_{\lambda,\alpha\mu}\cdot\alpha = T(\alpha)\varphi_{\lambda,\mu}\alpha$ by __L1__ and $T(\alpha)$ monic implies that $T^*(\alpha)\varphi_{\lambda,\alpha\mu} = \varphi_{\lambda,\mu}\cdot\alpha$;

__L4__ : $T(\alpha)T^*(\alpha)\varphi_{\alpha\lambda,\mu} = \alpha\varphi_{\alpha\lambda,\mu} = \varphi_{\alpha\lambda,\mu}\cdot\alpha = \varphi_{\alpha\lambda,\mu}T'(\alpha)\cdot T'^*(\alpha) = T(\alpha)\varphi_{\lambda,\mu}T^*(\alpha)$ by __L2__ and $T(\alpha)$ monic implies that $T^*(\alpha)\varphi_{\alpha\lambda,\mu} = \varphi_{\lambda,\mu}T^*(\alpha)$.

__Definition 3.3__. If T and T'' are torsion-layers, by a __right__-__morphism__ $T \to T''$ is meant a collection of morphisms

$$T_{\lambda\mu} \xrightarrow{\psi_{\lambda,\mu}} T''_\mu$$

defined for $\lambda \gg 1$, $\mu \gg 1$, satisfying:

__R1__

$$\begin{tikzcd}
T_{\lambda\mu} \arrow[r, "\psi_{\lambda,\mu}"] \arrow[d, "T(\alpha)"'] & T''_\mu \arrow[d, "T''(\alpha)"] \\
T_{\alpha\lambda\mu} \arrow[r, "\psi_{\lambda,\alpha\mu}"'] & T''_{\alpha\mu}
\end{tikzcd}$$

and

R2

$$\begin{tikzcd}
T_{\alpha\lambda\mu} \arrow[dr, "\psi_{\alpha\lambda,\mu}"] \arrow[dd, "T^*(\alpha)"'] & \\
& T''_{\mu} \\
T_{\lambda\mu} \arrow[ur, "\psi_{\lambda,\mu}"'] &
\end{tikzcd}$$

commute.

Lemma 3.2. If $\{\psi_{\lambda,\mu} : T_{\lambda\mu} \to T''_{\mu}\}$ is a right-morphism, then

R3

$$\begin{tikzcd}
T_{\lambda\mu} \arrow[r, "\psi_{\lambda,\mu}"] \arrow[d, "T(\alpha)"'] & T''_{\mu} \arrow[d, "\alpha"] \\
T_{\alpha\lambda\mu} \arrow[r, "\psi_{\alpha\lambda,\mu}"'] & T''_{\mu}
\end{tikzcd}$$

and

R4

$$\begin{tikzcd}
T_{\alpha\lambda\mu} \arrow[r, "\psi_{\lambda,\alpha\mu}"] \arrow[d, "T^*(\alpha)"'] & T''_{\alpha\mu} \arrow[d, "T''^*(\alpha)"] \\
T_{\lambda\mu} \arrow[r, "\psi_{\lambda,\mu}"'] & T''_{\mu}
\end{tikzcd}$$

commute.

Proof: R3 : $\psi_{\alpha\lambda,\mu}T(\alpha)\circ T^*(\alpha) = \psi_{\alpha\lambda,\mu}\circ\alpha = \alpha\circ\psi_{\lambda,\mu}\circ T^*(\alpha)$ by R2 and $T^*(\alpha)$ epic implies $\psi_{\alpha\lambda,\mu}\circ T(\alpha) = \alpha\circ\psi_{\lambda,\mu}$;

R4 : identical with proof of L4.

Some examples of left-morphisms are :

(i) $K(T) \xrightarrow{k_T} T$ which is derived from the exact sequence

$$0 \longrightarrow K_\lambda \xrightarrow{k_{\lambda,\mu}} T_{\lambda\mu} \xrightarrow{T^*(\mu)} T_\mu \longrightarrow 0 \quad ;$$

(ii) $T \xrightarrow{\ell_T} T$ where $\ell_{\lambda,\mu} = T(\lambda \to \lambda\mu)$.

Corresponding to these we have the right-morphisms

(iii) $T \xrightarrow{C_T} C(T)$ which is defined by the exact sequence

$$0 \longrightarrow T_\lambda \longrightarrow T_{\lambda\mu} \xrightarrow{C_{\lambda,\mu}} C_\mu \longrightarrow 0$$

and

(iv) $T \xrightarrow{r_T} T$ where $r_{\lambda,\mu} = T^*(\lambda \to \lambda\mu)$.

Definition 3.4. By a c-extension is meant a triple (T', T, T'') of torsion-layers together with a left-morphism $\{\varphi_{\lambda,\mu} : T'_\lambda \to T_{\lambda\mu}\}$ and a right-morphism $\{\psi_{\lambda,\mu} : T_{\lambda\mu} \to T''_\mu\}$ such that for $\lambda \gg 1, \mu \gg 1$,

$$0 \longrightarrow T'_\lambda \xrightarrow{\varphi_{\lambda,\mu}} T_{\lambda\mu} \xrightarrow{\psi_{\lambda,\mu}} T''_\mu \longrightarrow 0$$

is an exact sequence in $\mathrm{Tor}_m \mathcal{A}$.

For a c-extension the following diagrams are commutative with exact rows.

(A)

$$\begin{tikzcd}
0 \arrow[r] & T'_\lambda \arrow[r] \arrow[d] & T_{\lambda\mu} \arrow[r] \arrow[d] & T''_\mu \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & T'_{\alpha\lambda} \arrow[r] & T_{\alpha\lambda\mu} \arrow[r] & T''_\mu \arrow[r] & 0
\end{tikzcd}$$

(B)

$$\begin{tikzcd}
0 \arrow[r] & T'_\lambda \arrow[r] \arrow[d, equal] & T_{\lambda\mu} \arrow[r] \arrow[d] & T''_\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T'_\lambda \arrow[r] & T_{\alpha\lambda\mu} \arrow[r] & T''_{\alpha\mu} \arrow[r] & 0
\end{tikzcd}$$

(C)

$$\begin{tikzcd}
0 \arrow[r] & T'_\lambda \arrow[r] \arrow[d] & T_{\alpha\lambda\mu} \arrow[r] \arrow[d] & T''_{\alpha\mu} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T'_\lambda \arrow[r] & T_{\lambda\mu} \arrow[r] & T''_\mu \arrow[r] & 0
\end{tikzcd}$$

(D)

$$\begin{tikzcd}
0 \arrow[r] & T'_{\alpha\lambda} \arrow[r] \arrow[d, "\alpha"'] & T_{\alpha\lambda\mu} \arrow[r] \arrow[d] & T''_\mu \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T'_\lambda \arrow[r] & T_{\lambda\mu} \arrow[r] & T''_\mu \arrow[r] & 0
\end{tikzcd}$$

This is just a restatement of the exactness condition along with an immediate application of R1-4 and L1-4. If no designation of a morphism is given in a diagram, it can be assumed to be the only one guaranteed by the context.

Associated with any torsion-layer T we have two c-extensions:

$$K \xrightarrow{k_T} T \xrightarrow{r_T} T \quad \text{and} \quad T \xrightarrow{\ell_T} T \xrightarrow{c_T} C \ .$$

Let $\mathfrak{T}^m_c$ be the category whose objects are c-extensions; a morphism $(T', T, T'') \to (S', S, S'')$ is a triple of morphisms in $\mathfrak{T}^m$, $T' \to S'$, $T \to S$, and $T'' \to S''$ such that the diagrams

$$\begin{tikzcd}
0 \arrow[r] & T'_\lambda \arrow[r] \arrow[d] & T_{\lambda\mu} \arrow[r] \arrow[d] & T''_\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S'_\lambda \arrow[r] & S_{\lambda\mu} \arrow[r] & S''_\mu \arrow[r] & 0
\end{tikzcd}$$

commute for $\lambda, \mu \gg 1$.

We have three functors $P_1, P_2, P : \mathfrak{T}^m_c \to \mathfrak{T}^m$ defined, for $X = (T', T, T'')$, by $P_1(X) = T'$, $P_2(X) = T''$ and $P(X) = T$. We will now define a functor $J_c : \mathfrak{T}^m_c \to \mathfrak{T}^m_c$ which is "twisted" with respect to J, i.e., $P_1(J_c(X)) = J(P_2(X))$, $P_2(J_c(X)) = J(P_1(X))$ and $P(J_c(X)) = J(P(X))$.

3.2. Construction of the functor J_c .

We will associate with each c-extension

$$\{T'_\lambda \xrightarrow{\varphi_{\lambda,\mu}} T_{\lambda\mu} \xrightarrow{\psi_{\lambda,\mu}} T''_\mu\}$$

a new c-extension

$$\{JT''_\mu \xrightarrow{(J\varphi)_{\mu,\lambda}} JT_{\lambda\mu} \xrightarrow{(J\psi)_{\mu,\lambda}} JT'_\lambda\} \quad .$$

We begin with a series of lemmas.

Lemma 3.3. We have the following exact sequences

(a) $$0 \longrightarrow (T''_\mu)_\alpha \longrightarrow C'_\alpha \longrightarrow C_\alpha \longrightarrow (T''_\mu)^\alpha \longrightarrow 0$$

where $(T''_\mu)_\alpha = \mathrm{Ker}(\alpha)$ and $(T''_\mu)^\alpha = \mathrm{Cok}(\alpha)$, and $C'_\alpha = C_\alpha(T')$, $C_\alpha = C_\alpha(T)$, and $C''_\alpha = C_\alpha(T'')$

(b) $$0 \longrightarrow C_\alpha \overset{\sim}{\longrightarrow} C''_\alpha \longrightarrow 0$$

(c) $$0 \longrightarrow (T'_\lambda)_\alpha \longrightarrow K_\alpha \longrightarrow K''_\alpha \longrightarrow (T'_\lambda)^\alpha \longrightarrow 0$$

(d) $$0 \longrightarrow K'_\alpha \overset{\sim}{\longrightarrow} K_\alpha \longrightarrow 0 \quad .$$

Proof: (a),(b),(c),(d) result from the application of the serpent lemma to (A),(B),(C), and (D) respectively. The morphisms are the induced morphisms.

Lemma 3.4. Given a c-extension (T', T, T''), for $\mu \gg 1$ there is an exact sequence

$$0 \longrightarrow T''_\mu \longrightarrow C'_{\mu^2} \longrightarrow JT''_\mu \longrightarrow 0 \quad .$$

Proof: Applying (a) with $\alpha = \mu^2$, while choosing $\mu \gg 1$ so that T''_μ is μ^2-torsion, we have the exact sequence

$$0 \longrightarrow T''_\mu \longrightarrow C'_{\mu^2} \longrightarrow C_{\mu^2} \longrightarrow T''_\mu \longrightarrow 0 \ .$$

From this we know that $\mathrm{Cok}[T''_\mu \to C'_{\mu^2}] \cong \mathrm{Ker}[C_{\mu^2} \to T''_\mu]$. By (b) $C_{\mu^2} \overset{\sim}{\to} C''_{\mu^2}$ and

$$\begin{tikzcd}
C_{\mu^2} \arrow[d, "\sim"'] \arrow[dr] & \\
C''_{\mu^2} \arrow[r] & T''_\mu
\end{tikzcd}$$

commutes by R2 and R4 .

Since the kernel of the horizontal arrow is JT''_μ we have the result.

Definition 3.5. If T is any torsion-layer, for $\lambda \gg 1$, $D^T_{\lambda,\mu}$ denotes the homology of the complex $0 \to T_\lambda \to T_{\lambda^3\mu^2} \to T_\lambda \to 0$. We note that this is concentrated in one degree.

Lemma 3.5. For $\lambda \gg 1$, there is an exact sequence

$$0 \longrightarrow C_{\mu^2} \longrightarrow D^T_{\lambda,\mu} \longrightarrow JT_\lambda \longrightarrow 0 .$$

Proof: We have the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & C_{\mu^2} \arrow[r, equal] \arrow[d] & C_{\mu^2} \arrow[d, "c(\lambda^2)"] & & \\
0 \arrow[r] & D^T_{\lambda,\mu} \arrow[r] \arrow[d] & C_{\lambda^2\mu^2} \arrow[r] \arrow[d, "c^*(\mu^2)"] & T_\lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & JT_\lambda \arrow[r] \arrow[d] & C_{\lambda^2} \arrow[r] \arrow[d] & T_\lambda \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

the right hand square commutes since $C_{\lambda^2\mu^2} \to T_\lambda$ is induced by $T^*(\lambda^2\mu^2)$ and $C_{\lambda^2} \to T_\lambda$ by $T^*(\lambda^2)$. The middle row is the expression of the homology as the kernel of the morphism from dual cycles to boundaries. The other morphisms are those determined by commutativity.

Lemma 3.6. For a c-extension (T', T, T''), $\lambda \gg 1$ and denoting $D^{T'}_{\lambda,\mu}$ by $D'_{\lambda,\mu}$, we have an exact sequence

$$0 \longrightarrow T''_\mu \longrightarrow D'_{\lambda,\mu} \longrightarrow JT_{\lambda\mu} \longrightarrow 0 .$$

Proof: Consider the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & T''_{\mu} \arrow[r, equal] \arrow[d] & T''_{\mu} \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D'_{\lambda,\mu} \arrow[r] \arrow[d] & C'_{\lambda^2\mu^2} \arrow[r] \arrow[d] & T'_{\lambda} \arrow[r] \arrow[d, "\varphi_{\lambda,\mu}"] & 0 \\
0 \arrow[r] & JT_{\lambda\mu} \arrow[r] \arrow[d] & C_{\lambda^2\mu^2} \arrow[r] \arrow[d] & T_{\lambda\mu} \arrow[r] \arrow[d, "\psi_{\lambda,\mu}"] & 0 \\
 & 0 & T''_{\mu} \arrow[r, equal] \arrow[d] & T''_{\mu} \arrow[d, no head] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where the middle column is provided by (a) in the case $\alpha = \lambda^2\mu^2$ and the other parts are definitions. The column on the left with the induced morphisms is the desired sequence. From these building materials we construct a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & T''_{\mu} \arrow[r, equal] \arrow[d] & T''_{\mu} \arrow[d] & & \\
0 \arrow[r] & C'_{\mu^2} \arrow[r] \arrow[d] & D'_{\lambda,\mu} \arrow[r] \arrow[d] & JT'_{\lambda} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & JT''_{\mu} \arrow[r, "(J\varphi)_{\mu,\lambda}"'] \arrow[d] & JT_{\lambda\mu} \arrow[r, "(J\psi)_{\mu,\lambda}"'] \arrow[d] & JT'_{\lambda} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

from which we will conclude

Proposition 3.2. Given a c-extension (T', T, T'') there exists an exact sequence

$$0 \longrightarrow JT''_{\mu} \xrightarrow{(J\varphi)_{\mu,\lambda}} JT_{\lambda\mu} \xrightarrow{(J\psi)_{\mu,\lambda}} JT'_{\lambda} \longrightarrow 0$$

for $\lambda, \mu \gg 1$.

Proof: We need only prove that the above diagram has commutative top square and exact columns and middle row. The middle row is given by Lemma 3.5 applied to the torsion-layer T' , the left hand column by Lemma 3.4, and the middle column by Lemma 3.6. We need only check the commutativity of the top square. $T''_{\mu} \to C'_{\mu^2}$ is induced by

$$T''_{\mu} \xleftarrow{\psi_{\lambda,\mu}} T_{\lambda\mu} \xrightarrow{T(\mu^2)} T_{\lambda\mu^3} \xleftarrow{\varphi_{\lambda\mu^2,\mu}} T'_{\lambda\mu^2} \quad ,$$

$C'_{\mu^3} \to D'_{\lambda,\mu}$ is induced by

$$T'_{\lambda\mu^2} \xrightarrow{T'(\lambda^2)} T'_{\lambda^3\mu^2} \quad ,$$

and using $\varphi_{\lambda^3\mu^2,\mu} \circ T'(\lambda^2) = T(\lambda^2)\varphi_{\lambda\mu^2,\mu}$ we have

$$\begin{tikzcd}
T''_{\mu} & T_{\lambda\mu} \arrow[l, "\psi_{\lambda,\mu}"'] \arrow[r, "T(\mu^2)"] & T_{\lambda\mu^3} \arrow[dr, "T(\lambda^2)"'] & T'_{\lambda\mu^2} \arrow[l, "\varphi_{\lambda\mu^2,\mu}"'] \arrow[r, "T'(\lambda^2)"] & T'_{\lambda^3\mu^2} \arrow[dl, "\varphi_{\lambda^3\mu^2,\mu}"] \\
 & & & T_{\lambda^3\mu^3} &
\end{tikzcd}$$

and the lower trajectory defines $T''_{\mu} \to D'_{\lambda,\mu}$.

We begin the proof of

Proposition 3.3. (JT'', JT, JT') together with $\{(J\varphi)_{\mu,\lambda}\}$ and $\{J\psi)_{\mu,\lambda}\}$ define an element of $\mathfrak{J}_c^m$.

Since the exactness has been demonstrated in Proposition 3.2, it remains only to prove that $\{(J\varphi)_{\mu,\lambda}\}$ and $\{(J\psi)_{\mu,\lambda}\}$ are left and right morphisms respectively.

We need some components in order to assemble a proof and we begin by taking a closer look at $D_{\lambda,\mu}$. First we observe that Lemmas 2.4 and 2.5 immediately generalize to ensure that the complex $T_\lambda \to T_{\lambda^3\gamma} \to T_\lambda$ has homology canonically isomorphic to that of $T_{\alpha\lambda} \to T_{\alpha\beta\lambda^3\gamma} \to T_{\beta\lambda}$ for $\lambda \gg 1$ and γ arbitrary. We will write ${}_\lambda Y$ for $T_\lambda \to T_{\lambda^3\mu^2} \to T_\lambda$ where the presence of μ is not stressed simply to unburden the notation. We pursue a path nearly identical to that of Section II and obtain a morphism

$$D_{\lambda,\mu} \xrightarrow{D_\alpha} D_{\alpha\lambda,\mu}$$

defined by

$${}_\lambda Y \xleftarrow{i_\alpha} {}_\lambda Y_\alpha \xrightarrow{j_\alpha} {}_\lambda Y_\alpha^\alpha \xrightarrow{\varphi_\alpha} {}_{\alpha\lambda} Y ,$$

where as before, the degree zero component of i_α and ψ_α is $T^*(\alpha)$, that of j_α and φ_α is $T(\alpha)$.

<u>Lemma 3.7</u>. If T is a torsion-layer

$$\begin{tikzcd}
D_{\lambda,\mu} \arrow[r, "D_\alpha"] \arrow[d] & D_{\alpha\lambda,\mu} \arrow[d] \\
JT_\lambda \arrow[r, "JT(\alpha)"'] & JT_{\alpha\lambda}
\end{tikzcd}$$

commutes where the vertical morphisms are those of Lemma 3.5.

<u>Proof</u>: $D_{\lambda,\mu} \to JT_\lambda$ is induced by the chain map

$$\begin{tikzcd}
T_{\lambda} \arrow[r] \arrow[d, "\mu^2"'] & T_{\lambda^3\mu^2} \arrow[r] \arrow[d, "T^*(\mu^2)"] & T_{\lambda} \arrow[d, equal] \\
T_{\lambda} \arrow[r] & T_{\lambda^3} \arrow[r] & T_{\lambda}
\end{tikzcd}
\qquad \text{written} \qquad
\begin{tikzcd}
{}_{\lambda}Y \arrow[d, "\nu_{\lambda}"] \\
{}_{\lambda}X
\end{tikzcd}$$

and the assertion of the lemma follows from the commutative diagram

$$\begin{tikzcd}
{}_{\lambda}Y \arrow[d, "\nu_{\lambda}"'] & {}_{\lambda}Y_{\alpha} \arrow[l, "i_{\alpha}"'] \arrow[r, "j_{\alpha}"] \arrow[d] & {}_{\lambda}Y^{\alpha}_{\alpha} \arrow[r, "\varphi_{\alpha}"] \arrow[d] & {}_{\alpha\lambda}Y \arrow[d, "\nu_{\alpha\lambda}"] \\
{}_{\lambda}X & {}_{\lambda}X_{\alpha} \arrow[l, "i_{\alpha}"] \arrow[r, "j_{\alpha}"'] & {}_{\lambda}X^{\alpha}_{\alpha} \arrow[r, "\varphi_{\alpha}"'] & {}_{\alpha\lambda}X
\end{tikzcd}$$

where the intermediate morphisms are obvious, the top row inducing D_{α}, the bottom row inducing $JT(\alpha)$.

Lemma 3.8. In our c-extension situation

$$\begin{tikzcd}
D'_{\lambda,\mu} \arrow[r, "D'_{\alpha}"] \arrow[d] & D'_{\alpha\lambda,\mu} \arrow[d] \\
JT_{\lambda\mu} \arrow[r, "JT(\alpha)"'] & JT_{\alpha\lambda\mu}
\end{tikzcd}$$

commutes where the vertical maps are those of Lemma 3.6.

Proof: $D'_{\lambda,\mu} \to JT_{\lambda\mu}$ is induced by the chain map

$$\begin{tikzcd}
T'_{\lambda} \arrow[r] \arrow[d, "\varphi_{\lambda,\mu}"'] & T'_{\lambda^3\mu^2} \arrow[r] \arrow[d, "\varphi_{\lambda^3\mu^2,\mu}"'] & T'_{\lambda} \arrow[d, "\varphi_{\lambda,\mu}"] \\
T_{\lambda\mu} \arrow[r] & T_{\lambda^3\mu^3} \arrow[r] & T_{\lambda\mu}
\end{tikzcd}$$

and the remainder of the proof consists of a mimicry of the proof of Lemma 3.7.

Lemma 3.9.

$$\begin{tikzcd}
 & C_{\mu^2} \arrow[dl, "\underline{C}(\lambda^2)"'] \arrow[dr, "\underline{C}(\alpha^2\lambda^2)"] & \\
D_{\lambda,\mu} \arrow[rr, "D_\alpha"'] & & D_{\alpha\lambda,\mu}
\end{tikzcd}$$

commutes where the downward morphisms are those of Lemma 3.5, and where $\underline{C}(\lambda^2)$ is induced by

$$C_{\mu^2} \xrightarrow{C(\lambda^2)} C_{\mu^2\lambda^2} \quad .$$

Proof: Considering D_α , the degree zero morphism of its inducing chain map is

$$T_{\lambda^3\mu^2} \xleftarrow{T^*(\alpha)} T_{\alpha\lambda^3\mu^2} \xrightarrow{T(\alpha^2)} T_{\alpha^3\lambda^3\mu^2} \quad .$$

The $T^*(\alpha)$ contribution is subsumed in the identificational structure of C_{μ^2} , and recalling the definition of $C(\alpha)$, the commutativity is clear.

We return to the proof of Proposition 3.3. Consider the diagram

(1)

$$\begin{tikzcd}
D'_{\lambda,\mu} \arrow[rr, "D'_\alpha"] \arrow[dd] & & D'_{\alpha\lambda,\mu} \arrow[dd] \\
 & C'_{\mu^2} \arrow[ul] \arrow[ur] \arrow[dd] & \\
JT_{\lambda\mu} \arrow[rr] & & JT_{\alpha\lambda\mu} \\
 & JT''_{\mu} \arrow[ul, "(J_\varphi)_{\mu,\lambda}"] \arrow[ur, "(J_\varphi)_{\mu,\alpha\lambda}"'] &
\end{tikzcd}$$

The back face commutes by Lemma 3.8, the top face by Lemma 3.9, and the remaining two faces commute by the definition of $J\varphi$. Since $C'_{\mu^2} \rightarrow JT''_{\mu}$ is epic we conclude that the bottom face commutes and this is precisely condition L1.

By Lemmas 3.7 and 3.8, two of the square faces of the diagram

(2)

$$\begin{tikzcd}
 & D'_{\lambda,\mu} \arrow[dl] \arrow[d] \arrow[dr] & \\
D'_{\alpha\lambda,\mu} \arrow[d] \arrow[drr] & JT_{\lambda\mu} \arrow[r, "(J\psi)_{\mu,\lambda}"'] \arrow[dl] & JT'_{\lambda} \arrow[d] \\
JT_{\alpha\lambda\mu} \arrow[rr, "(J\psi)_{\mu,\alpha\lambda}"'] & & JT'_{\alpha\lambda}
\end{tikzcd}$$

commute and the two triangular faces commute by the definition of $J\psi$. The fact that $D'_{\lambda,\mu} \to JT_{\lambda\mu}$ is epic permits us to conclude the commutativity of the bottom square, which is condition R1.

At this point we will sketch another route to the exact sequence

$$0 \longrightarrow JT''_{\mu} \xrightarrow{(J\varphi)_{\mu,\lambda}} JT_{\lambda,\mu} \xrightarrow{(J\psi)_{\mu,\lambda}} JT'_{\lambda} \longrightarrow 0 .$$

One objective is to indicate the duality inherent in the axiom structure. The other is to provide proofs of L2 and R2. The lemmas which follow will be given primes and the numbering refers to the numbering of the corresponding earlier lemmas. The explanations will be minimal as these primed results mirror their unprimed alter egos.

Lemma 3.4'. Given a c-extension (T',T,T''), for $\lambda \gg 1$, there is an exact sequence

$$0 \longrightarrow JT'_{\lambda} \longrightarrow K''_{\lambda^2} \longrightarrow T'_{\lambda} \longrightarrow 0 .$$

Proof: Applying (c) with $\alpha = \lambda^2$, while choosing $\lambda \gg 1$ so that T'_{λ} is λ^2-torsion, we have the exact sequence

$$0 \longrightarrow T'_{\lambda} \longrightarrow K_{\lambda^2} \longrightarrow K''_{\lambda^2} \longrightarrow T'_{\lambda} \longrightarrow 0 .$$

From $\operatorname{Cok}(T'_{\lambda} \to K_{\lambda^2}) \cong \operatorname{Ker}(K''_{\lambda^2} \to T'_{\lambda})$ and the commutativity of

$$\begin{tikzcd}
K'_{\lambda^2} \arrow[rr, "\sim"] & & K_{\lambda^2} \\
& T'_{\lambda} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

we obtain the required sequence.

<u>Lemma 3.5′</u>. For any torsion-layer T, and $\lambda \gg 1$ there is an exact sequence

$$0 \longrightarrow JT_{\lambda} \longrightarrow D_{\lambda,\mu} \longrightarrow K_{\mu^2} \longrightarrow 0 .$$

<u>Proof:</u> The relevant diagram is

$$\begin{tikzcd}
& & 0 & 0 & \\
& & K_{\mu^2} \arrow[u] \arrow[r, equal] & K_{\mu^2} \arrow[u] & \\
0 \arrow[r] & T_{\lambda} \arrow[r] & K_{\lambda^2\mu^2} \arrow[r] \arrow[u] & D_{\lambda,\mu} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & T_{\lambda} \arrow[r] \arrow[u, equal] & K_{\lambda^2} \arrow[r] \arrow[u] & JT_{\lambda} \arrow[r] \arrow[u] & 0 \\
& & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} .$$

We remark that $D_{\lambda,\mu}$ is here used in its manifestation as cokernel of the morphism from boundaries to cycles, whereas before $D_{\lambda,\mu}$ was utilized as the kernel of the morphism from dual cycles to boundaries.

<u>Lemma 3.6′</u>. For a c-extension (T',T,T'') and $\lambda \gg 1$ we have an exact sequence

$$0 \longrightarrow JT_{\lambda\mu} \longrightarrow D''_{\mu,\lambda} \longrightarrow T'_{\lambda} \longrightarrow 0 .$$

<u>Proof:</u> The sequence in question is located in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & T'_{\lambda} \arrow[r, equal] \arrow[d] & T'_{\lambda} \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{\lambda\mu} \arrow[r] \arrow[d] & K_{\lambda^2\mu^2} \arrow[r] \arrow[d] & JT_{\lambda\mu} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T''_{\mu} \arrow[r] \arrow[d] & K''_{\lambda^2\mu^2} \arrow[r] \arrow[d] & D''_{\mu,\lambda} \arrow[r] \arrow[d] & 0 \\
 & 0 & T'_{\lambda} \arrow[r, equal] \arrow[d] & T'_{\lambda} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where the middle column is (c) with $\alpha = \lambda^2\mu^2$.

Another derivation of Proposition 3.2 results when we splice together the exact sequences of these lemmas, namely that given in the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & JT''_{\mu} \arrow[r] \arrow[d, equal] & JT_{\lambda\mu} \arrow[r] \arrow[d] & JT'_{\lambda} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & JT''_{\mu} \arrow[r] & D''_{\mu,\lambda} \arrow[r] \arrow[d] & K''_{\lambda^2} \arrow[r] \arrow[d] & 0 \\
 & & T'_{\lambda} \arrow[r, equal] \arrow[d] & T'_{\lambda} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

and an easy calculation shows that it coincides with the other. We here use the kernel representation of JT qua homology whereas in the other derivation we used the cokernel representation. The commutative diagrams mirroring (1) and (2) are

(1)'

$$\begin{tikzcd}
JT_{\lambda\mu} \arrow[dr] \arrow[dd] & & JT_{\alpha\lambda\mu} \arrow[ll] \arrow[dl] \arrow[dd] \\
& JT'_{\lambda} \arrow[dd] & \\
D''_{\mu,\lambda} \arrow[dr] & & D''_{\alpha\mu,\lambda} \arrow[ll] \arrow[dl] \\
& K''_{\lambda 2} &
\end{tikzcd}$$

(2)'

$$\begin{tikzcd}
& JT_{\lambda\mu} \arrow[dl] \arrow[dd] & \\
JT_{\alpha\lambda\mu} \arrow[dd] & & \\
& D''_{\mu,\lambda} \arrow[dl] & JT''_{\mu} \arrow[l] \arrow[uul] \arrow[dl] \\
D''_{\alpha\mu,\lambda} & JT''_{\alpha\mu} \arrow[l] \arrow[uul] &
\end{tikzcd}$$

from which we conclude that R2 and L2 hold, and $J\varphi$ and $J\psi$ are morphisms of the requisite type. The proof of Proposition 3.3 is complete.

3.3. J_c is an involution.

Using the natural transformation $I \cong J^2$ we obtain

Proposition 3.4. Given a c-extension (T',T,T'') the diagram

$$\begin{tikzcd}
0 \arrow[r] & T'_{\lambda} \arrow[r, "\varphi_{\lambda,\mu}"] \arrow[d, "\omega'_{\lambda}"'] & T_{\lambda\mu} \arrow[r, "\psi_{\lambda,\mu}"] \arrow[d, "\omega_{\lambda\mu}"'] & T''_{\mu} \arrow[r] \arrow[d, "\omega''_{\mu}"] & 0 \\
0 \arrow[r] & J^2T'_{\lambda} \arrow[r] & J^2T_{\lambda\mu} \arrow[r] & J^2T''_{\mu} \arrow[r] & 0
\end{tikzcd}$$

commutes.

Proof: Consider the commutative diagram

$$\begin{tikzcd}
 & T_{\lambda\mu} \arrow[rr, "\omega_{\lambda\mu}"] \arrow[dd] & & J^2T_{\lambda\mu} \arrow[dd] \\
T'_\lambda \arrow[ur, "\varphi_{\lambda,\mu}"] \arrow[rr] \arrow[dd] & & J^2T'_\lambda \arrow[ur, "\Gamma"] \arrow[dd] & \\
 & K_{\lambda^2\mu^2} \arrow[rr] & & C_{\lambda^2\mu^2}(JT) \\
K'_{\lambda^2} \arrow[ur, "\Phi"] \arrow[rr, "Z'_\lambda"] & & C_{\lambda^2}(JT') \arrow[ur, "\Theta"] &
\end{tikzcd}$$

where the front and back faces are the defining diagrams of ω'_λ and $\omega_{\lambda\mu}$ respectively. Φ is given by

$$K'_{\lambda^2} \underset{\sim}{\xrightarrow{\nu}} K_{\lambda^2} \xrightarrow{K(\mu^2)} K_{\lambda^2\mu^2}$$

where ν is the isomorphism given by (d) and induced by $\varphi_{\lambda^3,\mu}$. Θ and Γ are defined so as to make the diagram commutative, i.e., $\Theta = Z_{\lambda^2\mu^2} \circ \Phi \circ {Z'_\lambda}^{-1}$, and threading our way through the preceding constructions we find that Θ is given by

$$C_{\lambda^2}(JT') \underset{\sim}{\xleftarrow{\nu'}} C_{\lambda^3}(JT) \xrightarrow{C(\mu^2)} C_{\lambda^2\mu^2}(JT)$$

where ν' is the isomorphism induced by $(J\psi)_{\mu,\lambda^3}$.

If we can show that $\Gamma = (J^2\varphi)_{\lambda,\mu}$, we have half the proof. Consider

$$\begin{tikzcd}
C_{\lambda^2}(JT'') \arrow[r, "\ell"] \arrow[d, "\eta_1"'] & D_{\mu,\lambda}(JT'') \arrow[d, "\eta_2"] \\
J^2T'_\lambda \arrow[r, "(J^2\varphi)_{\mu,\lambda}"] \arrow[d, "d'_\lambda"'] & J^2T_{\lambda\mu} \arrow[d, "d_{\lambda\mu}"] \\
C_{\lambda^2}(JT') \arrow[r, "\Theta"'] & C_{\lambda^2\mu^2}(JT)
\end{tikzcd}$$

where the top commutative square is the defining diagram for $(J^2\varphi)_{\mu,\lambda}$.

If we can show that the bottom square commutes we will have our conclusion, and this will be the case if the rectangle commutes, since η_1 is epic.

Examining the constructions, we see that ℓ is induced by $JT''(\mu^2)$, η_2 by $(J\varphi)_{\mu^3\lambda^2,\lambda}$, $d_{\lambda\mu}$ is an inclusion, $d'_\lambda \circ \eta_1$ is induced by $(J\psi)_{\mu,\lambda^3} \circ (J\varphi)_{\lambda^2\mu,\lambda}$. So $\Theta \circ d'_\lambda \circ \eta_1$ is induced by

$$\begin{tikzcd} JT''_{\lambda^2\mu} \arrow[r, "(J\varphi)_{\lambda^2\mu,\lambda}"] & JT_{\lambda^3\mu} \arrow[r, "(J\varphi)_{\mu,\lambda^3}"] & JT'_{\lambda^3} & JT_{\lambda^3\mu} \arrow[l, "(J\varphi)_{\mu,\lambda^3}"'] \arrow[r, "JT(\mu^2)"] & JT_{\lambda^3\mu^3} \end{tikzcd}$$

and $d_{\lambda\mu} \circ \eta_2 \circ \ell$ is induced by

$$\begin{tikzcd} JT''_{\lambda^2\mu} \arrow[r, "JT''(\mu^2)"] & JT''_{\lambda^2\mu^3} \arrow[r, "(J\varphi)_{\lambda^2\mu^3,\lambda}"] & JT_{\lambda^3\mu^3} \end{tikzcd}$$

and that these coincide is condition <u>L2</u> for $J\varphi$.

The other part of the proof, the commutativity of the right hand square, is handled in a dual manner, using kernels where there were cokernels and vice-versa. We omit the details.

3.4. The multiplicative structure of ΦM_Λ .

ΦM_Λ admits a multiplicative structure [3] provided by the relations of the form $(P,x,Q) \circ (Q,y,R) = (P,y \circ x,R)$.

<u>Proposition 3.5</u>. Given x,y as above, there is a c-extension $(\Sigma(x),\Sigma(y \circ x),\Sigma(y))$ where $\Sigma: \Phi M_\Lambda \to \mathfrak{J}^\Lambda$ is the splintering functor defined in 2.2 .

<u>Proof</u>: If $x = \lambda^{-1} \otimes g_\lambda$, $y = \mu^{-1} \otimes f_\mu$, where $\lambda \in T_x$, $\mu \in I_y$ and $y \circ x = (\lambda\mu)^{-1} \otimes f_\mu \circ g_\lambda$, then associated with

$$P \xrightarrow{g_\lambda} Q \xrightarrow{f_\mu} R$$

is the exact sequence

$$0 \longrightarrow \mathrm{Cok}\ g_\lambda \longrightarrow \mathrm{Cok}(f_\mu \circ g_\lambda) \longrightarrow \mathrm{Cok}\ f_\mu \longrightarrow 0\ .$$

This follows from the three-lemma and the fact that f_μ is monic. Written out, we have

$$0 \longrightarrow Q/g_\lambda P \xrightarrow{\varphi_{\lambda,\mu}} R/(f_\mu \circ g_\lambda)P \xrightarrow{\psi_{\lambda,\mu}} R/f_\mu Q \longrightarrow 0$$

where $\varphi_{\lambda,\mu}$ is induced by f_μ and $\psi_{\lambda,\mu}$ by 1_R. That $\{\varphi_{\lambda,\mu}\}$ is a left-morphism follows from the relations $\alpha f_\mu = f_{\alpha\mu}$ and $f_\mu \circ g_{\alpha\lambda} = f_{\alpha\mu} \circ g_\lambda$ and that $\{\varphi_{\lambda,\mu}\}$ is a right-morphism is transparent. We observe that the natural equivalence of $\Sigma \circ j$ and $J \circ \Sigma$ is compatible with the extensional structure, i.e., for x and y as before,

$$\begin{tikzcd}
0 \arrow[r] & \Sigma(j(y))_\lambda \arrow[r] \arrow[d] & \Sigma(j(y \circ x))_{\lambda\mu} \arrow[r] \arrow[d] & \Sigma(j(x))_\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J(\Sigma(y))_\lambda \arrow[r] & J(\Sigma(y \circ x))_{\lambda\mu} \arrow[r] & J(\Sigma(x))_\lambda \arrow[r] & 0
\end{tikzcd}$$

commutes. Written out we have

$$\begin{tikzcd}
0 \arrow[r] & Q/k_\mu R \arrow[r, "\hat{h}_\lambda"] \arrow[d, "\hat{f}_\mu"'] & P/h_\lambda k_\mu R \arrow[r, "\hat{1}_p"] \arrow[d, "\widehat{f_\mu \circ g_\lambda}"] & P/h_\lambda Q \arrow[r] \arrow[d, "\hat{g}_\lambda"] & 0 \\
0 \arrow[r] & f_\mu Q/\mu^2 R \arrow[r, "\widehat{\lambda^2}"'] & f_\mu \circ g_\lambda P/\lambda^2\mu^2 R \arrow[r, "\widehat{f_\mu^{-1}}"'] & g_\lambda P/\lambda^2 R \arrow[r] & 0
\end{tikzcd}$$

where we have written the inducing morphism with a '∧' over it. That the bottom sequence is the one obtained by carrying out the processes of 3.2 is a straightforward verification.

3.5. <u>The extensional structure in $\mathfrak{J}^m$</u>.

$\mathfrak{J}^m_s$ will denote the category of triples $(T_1, T_1 \oplus T_2, T_2)$ where

T_1 and T_2 are objects of $\mathcal{T}^m$ and morphisms are given by

$$T_1 \overset{\sim}{\rightarrow} T_1' \text{ , } T_2 \overset{\sim}{\rightarrow} T_2'$$

in $\mathcal{T}^m$. We let $\mathcal{T}^m_e = \mathcal{T}^m_c \coprod \mathcal{T}^m_s$ and this provides us with an extensional structure in $\mathcal{T}^m$ where $\nabla : \mathcal{T}^m \times \mathcal{T}^m \rightarrow \mathcal{T}^m_e$ is the obvious functor into $\mathcal{T}^m_s$.

$\mathcal{T}^m_e$ is a somewhat divided household consisting of the relations one would like to divide out by in the definition of the Grothendieck group $K_0(\mathcal{T}^m, \mathcal{T}^m_e)$. In working with the category ΦM_Λ, which has a multiplicative structure, the relevant functor is $K_\#$ which forms the free abelian group on the isomorphism classes of ΦM_Λ and then divides out by the relations defined by direct sum and those defined by composition. $\mathcal{T}^m_c$ plays the role of the latter, $\mathcal{T}^m_s$ that of the former.

§IV. Generalized Hermitian Forms.

4.1. Hermitian situations.

The concept of hermitian form in the classical case of a vector space over a field K provided with an involution (denoted $^{-}$) can be formulated as follows. Letting $\mathcal{V}_K$ denote the category of finite dimensional K-vector spaces, we have two involutions on $\mathcal{V}_K$. One is $\mathcal{D} = \mathrm{Hom}(-,K)$ which is contravariant and the other is $\mathcal{H}: \mathcal{V}_K \to \mathcal{V}_K$ where $\mathcal{H}(V) = V_{\mathrm{conj}}$ where V_{conj} has the same underlying abelian group as V with the action of K defined by $k \times V = \bar{k} \circ V$, where $\circ$ denotes the action on V. In this framework a hermitian form (non-degenerate) is an isomorphism $\alpha: \mathcal{H}(V) \to \mathcal{D}(V)$ such that $\mathcal{H}(\alpha) = \mathcal{D}(\alpha)$.

This formulation admits of ready generalization. Given a category $\mathfrak{A}$ with a pair of functors $F: \mathfrak{A} \to \mathfrak{A}$, $G: \mathfrak{A}^{op} \to \mathfrak{A}$ such that $F^2 \cong \mathrm{Id}$, $G^2 \cong \mathrm{Id}$, and $F \circ G \cong G \circ F$, i.e., carrying a pair of commuting involutions, one covariant and one contravariant, we will call $(\mathfrak{A}, F, G)$ a <u>hermitian situation</u>. A hermitian form with respect to this situation is given by an object A in $\mathfrak{A}$ together with an isomorphism $\alpha: F(A) \to G(A)$ such that $F(\alpha) = G(\alpha)$.

The category we are especially interested in is ΦS_Λ, an object of which is given by a pair of symmetric bilinear forms $P \xrightarrow{\xi} P^*$, $Q \xrightarrow{\eta} Q^*$ and a k-isometry α of these forms, i.e., such that the diagram

$$\begin{tikzcd}
k\otimes P \arrow[r, "\alpha"] \arrow[d, "1\otimes\xi"'] & k\otimes Q \arrow[d, "1\otimes\eta"] \\
k\otimes P^* & k\otimes Q^* \arrow[l, "\alpha^*"]
\end{tikzcd}$$

commutes.

Now, α is an object of ΦM_Λ and with a slight twist of viewpoint the above diagram can be interpreted as giving a morphism (η,ξ) from $j(\alpha)$ to (α), where $j(\alpha) = \alpha^{-1}$ and $\mathcal{D}(\alpha) = \alpha^*$. The action of j and $\mathcal{D}$ on morphisms is given by $j(\eta,\xi) = (\xi,\eta)$ and $\mathcal{D}(\eta,\xi) = (\xi^*,\eta^*)$ so the condition that $j(\eta,\xi) = \mathcal{D}(\eta,\xi)$ is equivalent to $\xi = \xi^*$ and $\eta = \eta^*$. We will identify ΦS_Λ with $(\Phi M_\Lambda, j, \mathcal{D})$, the category of hermitian forms with respect to the involutions j and $\mathcal{D}$.

4.2. Hermitian forms on torsion-Layers.

One of our goals was to be able to speak of hermitian forms on torsion-layers in order to investigate the torsion nature of ΦS_Λ. We consider a slightly more general situation which will be relevant later.

Let A be a finite dimensional algebra over a domain Λ. In the introduction we defined the category $\Phi_\Lambda \mathcal{M}_A$ whose objects are (P,x,Q) where P and Q are finitely generated A-modules which are projective as Λ-modules. We denote by Tors A/Λ the category of finitely generated A-modules which are Λ-torsion and as Λ-modules are of homological dimension ≤ 1.

If we assume in addition that A possesses an involutory anti-automorphism as part of its structure we can identify left A-modules with right A-modules. An example is $\Lambda(G)$, the group ring over Λ of the group G, with an involution θ defined by $\theta(g) = g^{-1}$. In this case $\mathrm{Hom}_\Lambda(P,\Lambda)$ is an A-module via the action $(a\cdot f)(p) = f(p\cdot a)$, and $T \to \mathrm{Ext}^1_\Lambda(T,\Lambda)$ gives us a duality on Tors A/Λ.

Letting $\mathcal{T}_1^{A/\Lambda}$ denote the category of torsion-layers with values in Tors A/Λ, we can define a duality on $\mathcal{T}_1^{A/\Lambda}$ via that given on

<u>Tors</u> A/Λ . It is also clear that $\Sigma\colon \Phi_\Lambda \mathcal{M}_A \to \mathcal{T}_1^{A/\Lambda}$ makes sense and that $\mathcal{D}\circ\Sigma \simeq \Sigma\circ\mathcal{D}$ where $\mathcal{D}$ denotes the induced duality.

<u>Proposition 4.1</u>. The restriction of J to $\mathcal{T}_1^{A/\Lambda}$, which we also denote by J , defines an involution on $\mathcal{T}_1^{A/\Lambda}$.

<u>Proof:</u> This simply amounts to showing that JT_λ is of homological dimension ≤ 1 for $\lambda \gg 1$. Resolving T_{λ^3} with Λ-projectives we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & T_{\lambda^3} \arrow[r] \arrow[d, "T^*(\lambda^2)"] & 0 \\
0 \arrow[r] & P' \arrow[r, "\alpha"] \arrow[d] & Q \arrow[r] \arrow[d, "\lambda^2"] & T_\lambda \arrow[r] \arrow[d, "T(\lambda^2)"] & 0 \\
0 \arrow[r] & P \arrow[r] & Q \arrow[r] & T_{\lambda^3} \arrow[r] & 0
\end{tikzcd}$$

where P' is also Λ-projective. From this we see that JT_λ is isomorphic to $\alpha P'/\lambda^2 Q$ for $\lambda \gg 1$ so that there is the exact

$$0 \to Q \xrightarrow{\lambda^2} \alpha P' \longrightarrow JT_\lambda \longrightarrow 0$$

and we are done.

Now, since $\mathcal{D}$ is exact and J is defined by means of homology, $J\circ\mathcal{D} \simeq \mathcal{D}\circ J$ and we can consider the hermitian situation defined by $(\mathcal{T}_1^{A/\Lambda}, J, \mathcal{D})$ and the category $\mathcal{H}(\mathcal{T}_1^{A/\Lambda}, J, \mathcal{D}) = \mathcal{H}^{A/\Lambda}$ of hermitian forms on A/Λ torsion-layers.

§V. Torsion-Layers for Complete Discrete Valuation Rings.

5.1. Special subcategories of $\mathfrak{T}^{\Lambda}$.

In the previous section we saw that Σ could be regarded as a functor taking its values in the category $\mathfrak{T}_1^{\Lambda}$ and indeed we can in general limit the range category even more.

Definition 5.1. A torsion-layer T in $\mathfrak{T}^{\Lambda}$ will be said to be bounded if there exists an integer n such that for $\lambda \gg 1$ we can find an epimorphism $\xi_{\lambda}\colon \Lambda^n \to T_{\lambda}$.

$\mathfrak{T}_B^{\Lambda}$ will denote the full subcategory of $\mathfrak{T}_1^{\Lambda}$ whose objects are bounded torsion-layers. It is clear that $\Sigma\colon \Phi M_{\Lambda} \to \mathfrak{T}_B^{\Lambda}$ makes sense.

For any torsion-layer T we can form $\varprojlim T$ where the inverse system is defined by the $T^*(\lambda \to \alpha\lambda)$.

Definition 5.2. A torsion-layer T in $\mathfrak{T}^{\Lambda}$ is finite if $\varprojlim T$ is finitely generated as a Λ-module.

$\mathfrak{T}_F^{\Lambda}$ will denote the full subcategory of $\mathfrak{T}_1^{\Lambda}$ consisting of finite torsion-layers. It is clear that any finite torsion-layer is bounded. The converse is not in general true.

Lemma 5.1. If Λ is a discrete valuation ring, any torsion-layer T in $\mathfrak{T}^{\Lambda}$ is bounded. Furthermore we can construct a family of epimorphisms $\{\xi_{\lambda}\colon \Lambda^n \to T_{\lambda}\}$ which is compatible with the inverse system defined by T^* .

Proof: For some fixed λ, $\lambda \gg 1$, choose an epimorphism $\xi_{\lambda}\colon \Lambda^n \to T_{\lambda}$. This is possible since T_{λ} is finitely generated. Let π denote a uniformizer for Λ so that (π) is the unique maximal ideal of Λ.

Since Λ^n is free and $T^*(\lambda \to \pi\lambda)$ is epic there is a morphism $\xi_{\pi\lambda}\colon \Lambda^n \to T_{\pi\lambda}$ such that $T^*(\pi)\circ\xi_{\pi\lambda} = \xi_\lambda$. We claim that $\xi_{\pi\lambda}$ is also epic. Consider the diagram

$$\begin{tikzcd}
 & T_{\pi\lambda} \arrow[r, "\eta_{\pi\lambda}"] \arrow[d, "T^*(\pi)"] & T_{\pi\lambda}/(\pi)T_{\pi\lambda} \arrow[r] \arrow[d, "\simeq"] & 0 \\
\Lambda^n \arrow[ur, "\xi_{\pi\lambda}"] \arrow[r, "\xi_\lambda"'] & T_\lambda \arrow[r, "\eta_\lambda"'] & T_\lambda/(\pi)T_\lambda \arrow[r] & 0
\end{tikzcd}$$

Since $(\pi)T_{\pi\lambda} = T(\pi)T^*(\pi)T_{\pi\lambda} = T(\pi)T_\lambda$ we have $C_\pi \simeq T_{\pi\lambda}/(\pi)T_{\pi\lambda}$ for $\lambda \gg 1$. In the above diagram η_λ and $\eta_{\pi\lambda}$ are the canonical maps onto the cokernel. The commutativity of the above diagram implies that $\eta_{\pi\lambda}\circ\xi_{\pi\lambda}$ is epic. Thus $\operatorname{im}(\xi_{\pi\lambda}) + (\pi)T_{\pi\lambda} = T_{\pi\lambda}$ and applying Nakayama's lemma we conclude that $T_{\pi\lambda} = \operatorname{im}\xi_{\pi\lambda}$, i.e., $\xi_{\pi\lambda}$ is epic. Continuing in this fashion we can produce a compatible family of epimorphisms $\{\xi_{\pi^j\lambda}\colon \Lambda^n \to T_{\pi^j\lambda}\}_{j\in\mathbf{Z}}$.

From this lemma we see that for Λ a discrete valuation ring we have a morphism $\Lambda^n \xrightarrow{\xi} \varprojlim T$ such that if $\varprojlim T \xrightarrow{\varphi_\lambda} T_\lambda$ is a canonical morphism of the inverse limit, $\varphi_\lambda\circ\xi = \xi_\lambda$, where $\{\xi_\lambda\}$ is the collection of compatible morphisms constructed in the proof of Lemma 5.1.

Lemma 5.2. If Λ is a complete discrete valuation ring, then $\mathfrak{J}^{\Lambda}_{F} = \mathfrak{J}^{\Lambda}_{B}$.

Proof: We must show that every bounded torsion-layer is finite. From the structure theorem for finitely generated torsion modules over Λ we know that T_λ is isomorphic to a coproduct $\coprod_{k=1}^{n} \Lambda/\mathfrak{m}^{i_k}$ where $\mathfrak{m} = (\pi)$ and the i_k are positive integers. Now, since

$T(\pi)T^*(\pi)T_{\pi\lambda} = \pi T_{\pi\lambda}$ and $T(\pi)$ is monic we have that $\operatorname{Ker} T^*(\pi)$ is precisely the π-torsion of $T_{\pi\lambda}$ and since $T_{\pi\lambda}/\operatorname{Ker} T^*(\pi)$ is isomorphic to T_λ we conclude that $T_{\pi\lambda}$ must be of the form

$$\coprod_{k=1}^{n} \Lambda/\mathfrak{m}^{i_k + 1} \oplus \coprod_{E} \Lambda/\mathfrak{m} ,$$

where E is some finite index set. By means of these identifications we can regard $T^*(\pi)$ as the morphism

$$(\eta,0)\colon \coprod_{k=1}^{n} \Lambda/\mathfrak{m}^{i_k + 1} \oplus \coprod_{E} \Lambda/\mathfrak{m} \longrightarrow \coprod_{k=1}^{n} \Lambda/\mathfrak{m}^{i_k}$$

where η is the canonical epimorphism induced by 1_Λ. We choose $\xi_\lambda\colon \Lambda^n \to T_\lambda$ to be the morphism $\Lambda^n \to \coprod_{k=1}^{n} \Lambda/\mathfrak{m}^{i_k}$ defined by the canonical morphism onto the quotient in each factor. By the previous lemma ξ_λ lifts to a morphism $\xi_{\pi\lambda}$ which must also be an epimorphism. Thus the indexing set E must be the empty set. From this representation of the functor T and the definition of completeness, we see that $\varprojlim T$ is isomorphic to Λ^n and we are done.

Lemma 5.3. If T is a non-trivial torsion-layer in $\mathfrak{T}^{\Lambda}$, then $\varprojlim T$ is torsion free.

Proof: First observe that since T is non-trivial and the defining $T^*(\alpha)$'s of the inverse system are epic we can conclude that $\varprojlim T$ is not the zero module. We suppose the conclusion of the lemma to be false. Then there is an $x \neq 0$ in $\varprojlim T$ and an $\alpha \neq 0$ in Λ such that $\alpha \cdot x = 0$. Suppose $x_\mu \neq 0$ is the μ-component of $x = (x_\lambda)$ in $\varprojlim T$. Then $x_{\alpha\mu} \neq 0$. But $\alpha \cdot x_{\alpha\mu} = 0$ and $\alpha \cdot x_{\alpha\mu} = T(\alpha)T^*(\alpha)x_{\alpha\mu} = T(\alpha)x_\mu$. Now, $T(\alpha)$ is a monomorphism which implies that $x_\mu = 0$ and this is a contradiction.

5.2. Algebras over complete discrete valuation rings.

We consider the case where A is a finite dimensional algebra over a complete discrete valuation ring. Referring the reader to the introduction for notation we have ${}_{\Lambda}\mathcal{M}_{A}: \mathcal{M}_{A/\Lambda} \to \mathcal{M}_{k\otimes A/k}$ and the functor $\Sigma_{A/\Lambda}: \Phi_{\Lambda}\mathcal{M}_{A} \to \mathcal{F}^{\Lambda^{X}\to A}$. As before we write $\mathcal{F}^{A/\Lambda}$ for $\mathcal{F}^{\Lambda^{X}\to A}$.

Proposition 5.1. If A is a finite dimensional algebra over a complete discrete valuation ring, $\Sigma_{A/\Lambda}$ is an equivalence of categories.

Proof: We define a functor $L: \mathcal{F}^{A/\Lambda} \to \Phi_{\Lambda}\mathcal{M}_{A}$ as follows. By Lemma 5.3 $\varprojlim T$ is torsion free as a Λ-module. Since Λ is hereditary we conclude that $\varprojlim T$ is a Λ-projective. By Lemma 5.2 it is finitely generated. Thus for $\lambda \gg 1$ we have a short exact sequence

$$0 \longrightarrow P_{\lambda} \xrightarrow{i_{\lambda}} \varprojlim T \xrightarrow{\varphi_{\lambda}} T_{\lambda} \longrightarrow 0$$

where φ_{λ} is the canonical morphism from the inverse limit. Then $\varprojlim T$ and P_{λ} are finitely generated Λ-projectives and we define $L(T)$ to be the triple $(P_{\lambda}, \lambda^{-1} \otimes i_{\lambda}, \varprojlim T)$. In order to show that this triple is essentially unique we consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P_{\lambda} \arrow[r, "q"] \arrow[d] & P_{\alpha\lambda} \arrow[d] & & \\
0 \arrow[r] & \varprojlim T \arrow[r, "\alpha"] \arrow[d, "\varphi_{\lambda}"'] & \varprojlim T \arrow[r, "\eta"] \arrow[d, "\varphi_{\alpha\lambda}"] & C_{\alpha} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_{\lambda} \arrow[r, "T(\alpha)"] \arrow[d] & T_{\alpha\lambda} \arrow[r] \arrow[d] & C_{\alpha} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The lower right hand square commutes since $\varphi_{\alpha\lambda} \circ \alpha = \alpha \circ \varphi_{\alpha\lambda} = T(\alpha)T^{*}(\alpha)\varphi_{\alpha\lambda} = T(\alpha)\varphi_{\lambda}$ and it is clear from this that $\underset{\overleftarrow{\lambda}}{\lim}\, T(\lambda \to \alpha\lambda) = \alpha 1_{\underset{\leftarrow}{\lim}\, T}$. The middle row is obtained by taking the inverse limit over exact sequences of the form of the lower row and the sequence of the middle row will be exact provided that η is epic. But this is immediate since the $T^{*}(\alpha)$ are epic so that the first derived functor of lim evaluated at T is zero. Applying the nine-lemma we conclude that q is an isomorphism and via q we can identify $P_{\alpha\lambda}$ with P_{λ} and $i_{\alpha\lambda}$ with αi_{λ}. That $L \circ \Sigma$ is isomorphic to $\mathrm{Id}_{\Phi_{\Lambda}\mathcal{M}_{A}}$ is immediate from the construction of L. The isomorphism of $\Sigma \circ L$ with $\mathrm{Id}_{\mathcal{T}_{A/\Lambda}}$ is equally transparent.

Since $\Sigma_{A/\Lambda}$ is an equivalence of categories which up to natural isomorphism commutes with the involutions involved, we have as an immediate consequence

Proposition 5.2. If Λ is a complete discrete valuation ring, then $\mathfrak{Hb}(\Phi_{\Lambda}\mathcal{M}_{A}, j, \mathcal{D})$ is equivalent to $\mathfrak{Hb}(\mathcal{T}^{A/\Lambda}, J, \mathcal{D})$.

5.3. The multiplicative and extensional structures.

We have seen that for any composable pair x, y in $\Phi_{\Lambda}\mathcal{M}_{A}$ we have a c-extension $(\Sigma(x), \Sigma(y \circ x), \Sigma(y))$. In the case where Λ is a complete discrete valuation ring the c-extensional structure provides a faithful image of the multiplicative structure on $\Phi_{\Lambda}\mathcal{M}_{A}$. More precisely, we have

Proposition 5.3. Given a c-extension (T', T, T''), where T', T, T'' are torsion-layers in $\mathcal{T}^{A/\Lambda}$, Λ a complete discrete valuation ring, it is possible to identify $L(T)$ with $L(T'') \circ L(T')$.

Proof: From condition (d) of the c-extension axioms we obtain the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P'_{\lambda} \arrow[r, "\theta", "\sim"'] \arrow[d, "i'_{\lambda}"'] & P_{\lambda\mu} \arrow[d, "i_{\lambda\mu}"] & & \\
0 \arrow[r] & \underleftarrow{\lim}\, T \arrow[r, "\underleftarrow{\lim}\, \varphi"] \arrow[d, "\gamma'_{\lambda}"'] & \underleftarrow{\lim}\, T \arrow[r] \arrow[d, "\gamma_{\lambda\mu}"] & T''_{\mu} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T'_{\lambda} \arrow[r, "\varphi_{\lambda,\mu}"'] \arrow[d] & T_{\lambda\mu} \arrow[r] \arrow[d] & T''_{\mu} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

As in the proof of Proposition 5.1 the exactness of the middle row follows from the epic nature of the structure maps of the inverse system of T'. Then $L(T) = (P_{\lambda,\mu}, (\lambda\mu)^{-1} \otimes i_{\lambda\mu}, \underleftarrow{\lim}\, T)$ and $L(T') = (P'_{\lambda}, \lambda^{-1} \otimes i'_{\lambda}, \underleftarrow{\lim}\, T')$.

For fixed λ consider the inverse system

$$t_{\lambda} = \{\ldots \xrightarrow{\pi} T_{\lambda} \xrightarrow{\pi} T_{\lambda} \xrightarrow{\pi} T_{\lambda}\} \ .$$

It is clear that $\underleftarrow{\lim}\, t_{\lambda}$ is zero since no element of T_{λ} is infinitely divisible other than zero. Furthermore since T_{λ} is finitely generated and torsion, the system t_{λ} satisfies the Mittag-Leffler condition so that $\lim^{(1)} t_{\lambda}$ is zero. Applying this information to the diagram for condition (c) of the c-extension conditions, when we pass to the limit we have that

$$\underset{\mu}{\underleftarrow{\lim}}\, \psi_{\lambda,\mu} : \underleftarrow{\lim}\, T \longrightarrow \underleftarrow{\lim}\, T''$$

is an isomorphism. We now identify $\underleftarrow{\lim}\, T$ and $\underleftarrow{\lim}\, T''$ via this isomorphism and write

$$L(T'') = (\varprojlim T', \mu^{-1} \otimes \varprojlim \varphi, \varprojlim T)$$

where $\varprojlim \varphi$ is the morphism given in the diagram above. Then

$$L(T'')\circ L(T') = (P'_\lambda, (\lambda\mu)^{-1} \otimes (\varprojlim \varphi\circ i'_\lambda), \varprojlim T)$$

which is equivalent to $L(T)$ via the isomorphisms $\theta\colon P'_\lambda \to P_{\lambda\mu}$ and $1_{\varprojlim T}$.

5.4. Certain Grothendieck groups of generalized hermitian forms.

An example of the type of situation in which we are interested is the following. Let $\hat{\mathbb{Z}}_p$ denote the ring of p-adic integers for $p \neq 2$, and let M be a finite dimensional free $\hat{\mathbb{Z}}_p$-module with an action of a finite group Π, together with a Π-equivariant non-degenerate symmetric bilinear form. Associated with the study of such objects is the five-term exact sequence of K groups which appeared on the first page of the introduction. The content of Propositions 5.2 and 5.3 is that the study of the fibre category of the sequence can be replaced with a study of the generalized hermitian forms on torsion-layers in

$$\mathcal{F}^{\hat{\mathbb{Z}}_p[\Pi]/\hat{\mathbb{Z}}_p} .$$

The sequence in question involves the functor $K_\#$ which is molded to react to the multiplicative structure of the fibre category. For the category

$$\mathcal{H}(\mathcal{F}^{\hat{\mathbb{Z}}_p(\Pi)/\hat{\mathbb{Z}}_p})$$

the appropriate functor is the Grothendieck group of a category with an extensional structure.

$K_\#$ of a category with product and a multiplicative structure is defined to be the quotient of the free abelian group generated by isomorphism classes of objects of the category modulo the relations given by the product and multiplicative structure. K_0 of a category with an extensional structure is similar but we form the quotient modulo the relations of the extensional structure. We refer the reader to [3] for details.

Denote by $\mathfrak{h}_{A/\Lambda}$ the equivalence of categories the existence of which is asserted in Proposition 5.2. Let $\mathcal{H}(\mathcal{J}_e^{A/\Lambda})$ denote the category of generalized hermitian forms constructed with respect to the hermitian situation $(\mathcal{J}_e^{A/\Lambda}, J_e, \mathcal{D}_e)$. In an obvious way this defines an extensional structure in $\mathcal{H}(\mathcal{J}^{A/\Lambda})$. In this language, Propositions 5.2 and 5.3 immediately yield

<u>Theorem 5.1</u>. If A is a finite dimensional algebra over a complete discrete valuation ring, $\mathfrak{h}_{A/\Lambda}$ induces an isomorphism

$$\Omega\colon K_\#(\mathcal{H}(\Phi_\Lambda \mathcal{M}_A)) \longrightarrow K_0(\mathcal{H}(\mathcal{J}^{A/\Lambda}), \quad (\mathcal{J}_e^{A/\Lambda}))$$

§VI. Computational Observations.

We close with two rather clear remarks, labeled lemmas, which would be of use in certain calculations. They generalize the following two suspicions. The first is that if we wanted to describe $\mathfrak{F}^{\mathbb{Z}}$ where $\mathbb{Z}$ denotes the ring of rational integers it would suffice to describe $\mathfrak{F}^{\mathbb{Z}^{\times} \to \mathbb{Z}^{\times}_{(p)}}$ for all primes p, where $Z_{(p)}$ denotes the rational integers localized at the prime p. The second is that the category $\mathfrak{F}^{\mathbb{Z}^{\times} \to \mathbb{Z}^{\times}_{(p)}}$ is equivalent to $\mathfrak{F}^{<p> \to \mathbb{Z}^{\times}_{(p)}}$, where $<p>$ denotes the monoid of non-negative powers of the prime p.

Lemma 6.1. If the abelian category $\mathfrak{A}$ is expressed as the coproduct $\coprod_i \mathfrak{A}_i$ where the $\mathfrak{A}_i$ are themselves abelian categories, then

$$\mathfrak{F}^{\Lambda \xrightarrow{m} c(\mathfrak{A})}$$

is isomorphic to

$$\coprod \mathfrak{F}^{\Lambda \xrightarrow{p_i \circ m} c(\mathfrak{A}_i)} .$$

Proof: Since $c(\mathfrak{A})$ is isomorphic to $\prod_i c(\mathfrak{A}_i)$, the monoid morphism $\Lambda \xrightarrow{m} c(\mathfrak{A})$ is equivalent to the specification of a family of morphisms $\{\Lambda \xrightarrow{m_i} c(\mathfrak{A}_i)\}$ where $m_i = p_i \circ m$, p_i being the projection onto the i^{th} factor. Then the sundry projections onto and injections into provide us with our isomorphism of categories.

For Λ a commutative cancellation monoid let $U(\Lambda)$ denote the group of units in Λ. $\Lambda/U(\Lambda)$ will be the quotient under the action of $U(\Lambda)$ which admits an evident monoid structure. We have the natural map $N_\Lambda : \Lambda \to \Lambda/U(\Lambda)$.

<u>Lemma 6.2</u>. If N_Λ has a section $S_\Lambda : \Lambda/U(\Lambda) \to \Lambda$, then S_Λ induces an equivalence of categories

$$\mathcal{S} : \mathfrak{T}^{\Lambda \overset{m}{\to} c(\mathfrak{A})} \to \mathfrak{T}^{m \circ S_\Lambda} \ .$$

<u>Proof</u>: The functor $\mathcal{S}$ is given by $\mathcal{S}(T)_\mu = T_{S_\Lambda(\mu)}$, $\mathcal{S}(T)(\mu \to \mu' \cdot \mu) = T(S_\Lambda(\mu'))$, $\mathcal{S}(T)^*(\mu \to \mu'\mu) = T^*(S_\Lambda(\mu'))$. We think of the section S_Λ as a choice of a submonoid M_Λ of Λ consisting of 1 and certain non-units of Λ so that each λ in Λ has a unique expression of the form $\lambda = u_\lambda \cdot \mu_\lambda$ where $u_\lambda \in U(\Lambda)$ and $\mu_\lambda \in M_\Lambda$. Thus $\mathcal{S}$ is essentially the functor which restricts a torsion-layer to a submonoid.

Define
$$\mathcal{S}' : \mathfrak{T}^{m \circ S_\Lambda} \longrightarrow \mathfrak{T}^m$$
by
$$\mathcal{S}'(T)_\lambda = T_{\mu_\lambda} \ , \ \mathcal{S}'(T)(\lambda \to \alpha\lambda) = T_{\mu_\lambda} \xrightarrow{u_\alpha T(\mu_\alpha)} T_{\mu_\alpha \mu_\lambda} \ ,$$
and
$$\mathcal{S}'(T)^*(\lambda \to \alpha\lambda) = T^*(\mu_\alpha) \ .$$

The verification that this defines a torsion-layer is trivial. Then $\mathcal{S} \circ \mathcal{S}' = \mathrm{Id}_{\mathfrak{T}^{m \circ S_\Lambda}}$ and $\mathcal{S}' \circ \mathcal{S} \simeq \mathrm{Id}_{\mathfrak{T}^m}$ where the natural transformation Φ is given by $T_{\mu_\lambda} \xrightarrow{T(u_\lambda)} T_\lambda$.

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