77-13,675

TSOUBELIS, Dimitri George, 1948-
HOMOGENEOUS RELATIVISTIC COSMOLOGICAL
MODELS OF BIANCHI TYPE IV.

City University of New York, Ph.D., 1977
Physics, general

HOMOGENEOUS RELATIVISTIC COSMOLOGICAL MODELS OF BIANCHI TYPE IV

by

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A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1976

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

HOMOGENEOUS RELATIVISTIC COSMOLOGICAL MODELS OF BIANCHI TYPE IV

by

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Adviser: Professor Alex Harvey

The fundamental results of Differential Topology are reviewed to facilitate the presentation of the basic tenets of Relativistic Cosmology, as well as the Bianchi classifcation of spatially homogeneous space-time manifolds. A general discussion of the properties of homogeneous models is given, and a comparison with available observational data follows. General properties of Bianchi Type IV models are presented and exact solutions are offered for special cases.

Acknowledgement

The laboring classes and the scientists of past and present who have created the presuppositions for this small contribution to scientific inquiry to materialize, I thank.

As directly connected with the production of the present work, I would like to express my gratitude to Professor Alexander Harvey, for the continuous guidance and encouragement. My deep appreciation goes to Professor Jeffrey M. Cohen for giving me the benefit of consultation and enlightening discussions. I would also like to thank my wife, Evdokia, for her unebbing patience and support throughout a long period of time that closes with the completion of the present work.

TO

GIORGOS AND SOPHIA, MY PARENTS

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NOTATIONAL AND OTHER CONVENTIONS

The Einstein summation convention is used throughout. In spacetime, Greek indices run as 0,1,2,3 and Latin as 1,2,3. Equations are enumerated as in the following example: (3.14), means the 14th equation of chapter 3.References are specified in the text by the name(s) of the author(s) or editor(s), followed by the year of the publication.

1. INTRODUCTION

Cosmological Models are mathematical constructs on the basis of which we try to understand the geometry of the universe as a whole, i.e. its structure and the dynamics of its evolution. These models are constructed on the basis of general conceptions about the nature of space and time and physical theories, which have been successfully tested over scales and regimes accessible to man in the laboratory or via teleobservation.

The theory of space and time which is philosophically the most satisfactory and has received the most positive support from laboratory tests and astronomical observations is the so called "Einstein's General Theory of Relativity."The fact that models constructed on the basis of this theory can adequately account for the currently available data from cosmological observations, has rendered modern cosmology almost synonymous with General Relativity.

Of course, the complexity of our universe makes a faithful representation of its dynamical structure in principle impossible. Direct observation shows that even over the scale of galaxies the universe is inhomogeneous, anisotropic, and contains such a multitude of mass-energy fields that no simple, i.e. amenable to calculations , mathematical model can adequately describe it. Relatively simple models have been constructed, however, which seem to give a good approximation to the structure of the universe over scales of linear extent one order of magnitude larger than that of clusters of galaxies. These models, called "Spatially Homogeneous and Isotropic Models," are so highly symmetric that they show a very limited range of behavior. To obtain a wider spectrum of bahavior, one drops the isotropy restriction, whereby "Spatially Homogeneous but Anisotropic" models are obtained. In the present work we study a particular class of such models, those of "Bianchi Type IV."

The discussion of the Einstein theory of spacetime is taken up again in Chapter 3.In Chapter 2 we present some basic results from topology and differential geometry , in order to make the discussion of Einstein's theory rigorous and deeper. In Chapter 4 we delineate the geometry of spatially homogeneous spacetimes, and present the Bianchi classification scheme of their subclasses. Basic cosmological data are presented in Chapter 5, and they are linked to the geometry of isotropic models, as well as to the genesis of the study of nonisotropic ones.(For details on the material presented in the above Chapters, see KOBAYASHI & NOMIZU (1963), HAWKING & ELLIS(1973), MISNER, THORNE & WHEELER (1973), ADLER, BAZIN, & SCHIFFER (1975), EINSTEIN (1915), EINSTEIN (1956), PEEBLES (1971), RYAN & SHEPLEY (1975), KING & ELLIS (1973).)

Finally, in Chapter 6 we present three exact Bianchi IV spatially homogeneous models. The first is an empty model, and its solution is given in closed form. The second, whose solution is given in terms of a second order nonlinear differential equation, is a tilted model with "Zeldovich's stiff matter" filling. The last model represents an electromagnetic universe, and its solution is also given in closed form.

2. DIFFERENTIAL GEOMETRY

2.A. TOPOLOGICAL SPACES

A pair $(S,(U_\kappa))$, consisting of a set S and a collection of subsets U_κ of S, which satisfies the following conditions,

(i) Both the empty set $\emptyset$ and S are elements of (U_κ).

(ii) If $V_\kappa \in (U_\kappa)$, then the union $\cup_\kappa V_\kappa \in (U_\kappa)$, where $\kappa \in I$, an index set.

(iii) If $V_1, V_2, \ldots, V_n \in (U_\kappa)$, then $\cap_i V_i \in (U_\kappa)$; is called a <u>topological</u> <u>space</u>.

The collection (U_κ) is called the topology of S and the elements of (U_κ) are called <u>open sets</u> of S.

If $p \in U_\kappa$, the open set U_κ is called a <u>neighborhood</u> of p.

A topological space with the property that for any pair p, q of distinct elements of the space there exist disjoint open sets U_p, U_q such that $p \in U_p, q \in U_q$, is called <u>Housdorff</u>.

Let ϕ be a map of the topological space S into the topological space S'. We say that ϕ is <u>continuous</u> <u>at</u> <u>$p \in S$</u>, if for every neighborhood U' of the image point $\phi(p) \in S'$ there is a neighborhood U of p, such that $q \in U$ implies $\phi(q) \in U'$. If ϕ is one-one and onto, and both ϕ and its inverse ϕ^{-1} are continuous over the whole of S, then ϕ is called a <u>homeomorphism</u> of S onto S', and then S and S' are topologically equivalent or <u>homeomorphic</u>.

2.B. MANIFOLDS

By a <u>C^r</u> <u>n-dimensional</u> <u>manifold</u> <u>without</u> <u>boundary</u> we will mean a set M together with a collection of pairs $\{(U_\alpha, \phi_\alpha)\}$ of subsets U_α of M and one-one maps ϕ_α of the corresponding U_α to open sets in R^n, such that

(i) $M = \cup_\alpha U_\alpha$

(ii) If $U_\alpha \cap U_\beta$ is nonempty, then the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta (U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha (U_\alpha \cap U_\beta) \quad ,$$

is a C^r map of an open subset of R^n to an open subset of R^n, (see Fig. 1).

When the coordinates of $\phi_\alpha(p)$ in R^n are $(x^1_\alpha(p), x^2_\alpha(p), x^3_\alpha(p), \ldots, x^n_\alpha(p))$ for a point p in M, we will also call them the coordinates of p, and write $\phi_\alpha(p) = (x^1_\alpha(p), x^2_\alpha(p), \ldots, x^n_\alpha(p))$.

Each pair (U_α, ϕ_α) is called <u>a</u> <u>chart</u> and the collection $\{(U_\alpha, \phi_\alpha)\}$ <u>an</u> <u>atlas</u>. Otherwise, an atlas is a coordinate system that covers the manifold. The set of all possible atlases of M is called the <u>complete</u> <u>atlas</u> of the manifold.

We define the topology of M by taking its open sets to be unions of sets of the form U_α belonging to the complete atlas. This makes each map ϕ_α a homeomorphism, so that M is locally topologically equivalent to R^n.

A <u>function</u> f on a C^r manifold is a map from M to R^1. It is said to be of class $C^k (k \leq r)$ at a point p of M, if the expression $f \circ \phi_\alpha^{-1}$ of f on any local coordinate neighborhood U_α is a C^k function of the local coordinates at p.

A <u>C^k curve</u> <u>$\partial(t)$</u> in M is a C^k map of an interval of the real line R^1 into M.

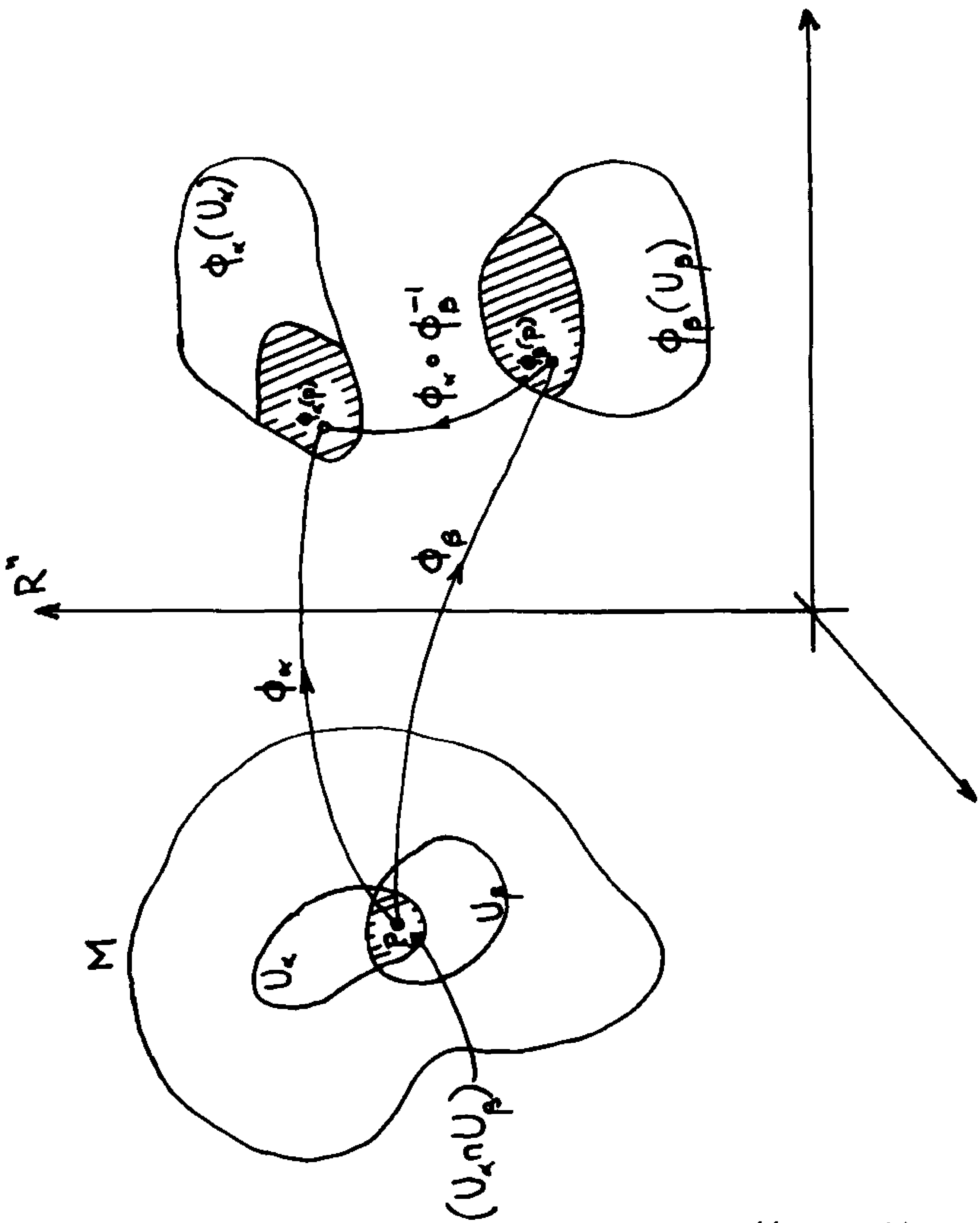


FIGURE 1: A manifold, M, and two open subsets U_α and U_β of M which overlap in the neighborhood of the point p. $\phi_\alpha(U_\alpha)$ and $\phi_\beta(U_\beta)$ are coordinates of U_α and U_β, respectively.

The n-dimensional Euclidean space will be denoted by R^n, and its topology will be taken to consist of the open sets, as these are usually understood, of R^n. A map ϕ of an open set O of R^n to an open set O' of R^m is said to be of <u>class</u> $\underline{C^r}$ if the coordinates $(x'^1, x'^2, \ldots, x'^m)$ of $\phi(p)$ in O' are r-times differentiable functions of the coordinates $(x^1, x^2, \ldots, x^n)$ of p in O, and their r-th derivatives are continuous.

2.C. THE TANGENT SPACE - TENSOR FIELDS ON A MANIFOLD

Any C^1 curve $\lambda(t)$ through the point $\lambda(t_0)$ of a manifold defines an operator $\left(\frac{\partial}{\partial t}\right)_{\lambda|t_0}$ which maps each C^1 function f to the real number $\left(\frac{\partial f}{\partial t}\right)_{\lambda|t_0}$ given by

$$(2.1)\quad \left(\frac{\partial f}{\partial t}\right)_{\lambda|t_0} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big(f(\lambda(t_0+\epsilon)) - f(\lambda(t_0))\Big) .$$

The operator $\left(\frac{\partial}{\partial t}\right)_{\lambda|t_0}$ is called the <u>tangent vector of the curve $\lambda(t)$ at $\lambda(t_0)$</u>. The tangent vectors of all the C^1 curves through a point p form a linear vector space, called the <u>tangent space of the manifold at p</u>, to be denoted by T_p. The demension of the tangent space is the same as that of the manifold. This means that one can always find a set of vectors (e_α) ($\alpha = 1, \ldots, n$, where n the dimension of the manifold), such that each element V of T_p can be written as

$$V = V^\alpha e_\alpha$$

The set (e_α) is called a <u>basis</u> of T_p and the numbers V^α <u>components</u> of V in terms of the basis (e_α). The result of operating on a function f with the basis vector e_α will be denoted by $e_\alpha(f)$ or $f_{,\alpha}$. Thus, for a vector V we have

$$(2.2)\qquad V(f) = V^\alpha e_\alpha(f) = V^\alpha f_{,\alpha}$$

If $\lambda(t)$ is the curve whose tangent vector at $\lambda(t_0)$ is V, then by definition we have

$$(2.3)\qquad V(f) = \left(\frac{\partial f}{\partial t}\right)_{\lambda|t_0} = \frac{dx^\alpha(\lambda(t))}{dt}\Big|_{t=t_0} \cdot \frac{\partial f}{\partial x^\alpha}\Big|_{\lambda(t_0)} ,$$

in terms of some local coordinate system ($x^1, \ldots, x^n$). This means that any tangent vector at the point $\lambda(t_0)$ can be expressed in terms of $\left(\frac{\partial}{\partial x^\alpha}\right)$, or that the set $\left(\frac{\partial}{\partial x^\alpha}\right)$ is a basis of $T_{\lambda(t_0)}$. This basis is called a <u>coordinate</u> <u>basis</u>, and in this case we have, (see (2.2)),

$$f_{,\alpha} = \frac{\partial f}{\partial x^\alpha} \quad , \quad V^\alpha = \frac{dx^\alpha}{dt} \tag{2.4}$$

Given the components V^α of a vector $\mathbf{V}$ at point p in some coordinate basis $\left(\frac{\partial}{\partial x^\alpha}\right)$, we can immediately give a curve to which this vector is tangent. It is the set of points with coordinates

$$x^\alpha = x^\alpha(p) + V^\alpha t \tag{2.5}$$

for t in some interval $[-\epsilon, \epsilon]$.

A <u>one-form</u> <u>(covariant vector)</u> $\underline{\boldsymbol{\omega}}$ at p is a real valued linear function on the space T_p. The number to which a vector $\mathbf{V} \in T_p$ is mapped by $\boldsymbol{\omega}$ will be denoted by $\langle \boldsymbol{\omega}, \mathbf{V} \rangle$. It is easily shown that all one-forms at p form an n-dimensional vector space, the <u>dual</u> <u>space</u> $\underline{T_p^*}$ of T_p. Given a basis $(\mathbf{e}_\alpha)$ of T_p, one forms a basis $(\boldsymbol{\sigma}^\alpha)$ of T_p^*, called dual to $(\mathbf{e}_\alpha)$ by imposing the condition

$$\langle \boldsymbol{\sigma}^\alpha, \mathbf{e}_\beta \rangle = \delta^\alpha_\beta \tag{2.6}$$

Thus, we have

$$\langle \boldsymbol{\omega}, \mathbf{V} \rangle = \langle \omega_\alpha \boldsymbol{\sigma}^\alpha, V^\beta \mathbf{e}_\beta \rangle = \omega_\alpha V^\alpha \tag{2.7}$$

Since a vector maps functions into $\mathbb{R}^1$, while one-forms act likewise on vectors, we see that we can form a one-form

df from any function f, by the correspondence:

$$\langle df, v \rangle = v(f) \tag{2.8}$$

Then, df is called the <u>differential</u> of f. The set of differentials $(dx^1, dx^2, \ldots, dx^n)$ of a local set of coordinates at p form a basis of one-forms, which is dual to the vector basis $\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right)$ at p, since

$$\langle dx^\alpha, \frac{\partial}{\partial x^\beta} \rangle = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta \tag{2.8}$$

From T_p and T_p^* we can form the Cartesian product

$$\Pi_r^s = \underbrace{T_p^* \times T_p^* \times \cdots \times T_p^*}_{\text{r-factors}} \times \underbrace{T_p \times T_p \times \cdots \times T_p}_{\text{s-factors}}, \tag{2.9}$$

and then define as a <u>tensor of type (r, s)</u> at p a function T on Π_r^s, linear in all of its arguments, mapping the element $(\sigma^1, \ldots, \sigma^r, V_1, \ldots, V_s)$ of Π_r^s into the real number $T(\sigma^1, \ldots, \sigma^r, V_1, \ldots, V_s)$. The set of all such functions is a vector space over the reals, denoted by

$$T_s^r(p) = T_p \otimes \cdots \otimes T_p \otimes T_p^* \otimes \cdots \otimes T_p^* \tag{2.10}$$

and called the <u>tensor product</u>. If (e_α), (σ^α) are dual bases of T_p and T_p^* respectively, then the collection

$$(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r} \otimes \sigma^{\beta_1} \otimes \cdots \otimes \sigma^{\beta_s}), \quad \alpha_i, \beta_j = 1, \cdots, n,$$

will be a basis of $T_s^r(p)$. In terms of such a basis, an element T of $T_s^r(p)$ can be expressed as

$$T = T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r} \otimes \sigma^{\beta_1} \otimes \cdots \otimes \sigma^{\beta_s} = T(\sigma^{\alpha_1}, \ldots, \sigma^{\alpha_r}, e_{\beta_1}, \ldots, e_{\beta_s}) e_{\alpha_1} \otimes \cdots \otimes \sigma^{\beta_s}$$

since $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r} \otimes \sigma^{\beta_1} \otimes \cdots \otimes \sigma^{\beta_s}$ denotes the element of $T^r_s(p)$ which maps the element $(\sigma^{\alpha_1'}, \ldots, \sigma^{\alpha_r'}; e_{\beta_1'}, \ldots, e_{\beta_s'})$ of Π^s_r into the number $\langle \sigma^{\alpha_1'}, e_{\alpha_1} \rangle \cdots \langle \sigma^{\alpha_r'}, e_{\alpha_r} \rangle \langle \sigma^{\beta_1}, e_{\beta_1'} \rangle \langle \sigma^{\beta_s}, e_{\beta_s'} \rangle$. Where it is more convenient, we will express relations in the tensor algebra at p in terms of the components of tensors. For example,

$$R = T \otimes T' \longleftrightarrow R^{\alpha_1 \cdots \alpha_{r+p}}{}_{\beta_1 \cdots \beta_{s+q}} = T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} T'^{\alpha_{r+1} \cdots \alpha_{r+p}}{}_{\beta_{s+1} \cdots \beta_{s+q}}$$

Among operations which give new tensors from old, those of contraction over two indices, transposition of two indices, symmetrization as well as anti-symmetrization are well known.

A $\underline{C^k}$ <u>tensor field</u> $\underline{T}$ <u>of type</u> $\underline{(r, s)}$ (r times contra-, s times co-variant) on a set $U \subset M$ is an assignment of an element of $T^r_s(p)$ to each point $p \in U$ such that the components of T with respect to any coordinate basis defined on an opensubset of U are C^k functions.

2.D. MAPS ON MANIFOLDS

Let M be a C^r n-dimensional manifold and M' a $C^{r'}$ n'-dimensional manifold, and ϕ a C^k map from M to M'. We define a map ϕ^* which maps functions f on M' to functions $\phi^* f$ on M in a linear fashion and in correspondence with ϕ by

$$\phi^* f(p) = f(\phi(p)), \tag{2.11}$$

as shown pictorially in fig. 2.

Similarly, we define a linear map ϕ_* which maps any vector V of T_p into a vector $\phi_* V$ of $T_{\phi(p)}$, by

$$V(\phi^* f)\big|_p = \phi_* V(f)\big|_{\phi(p)} \tag{2.12}$$

The map ϕ^* can be extended to cover one forms. It maps one-forms ω of $T^*_{\phi(p)}$ to one-forms $\phi^*\omega$ of T^*_p according to

$$\langle \phi^*\omega, V\rangle\big|_p = \langle \omega, \phi_* V\rangle\big|_{\phi(p)} \tag{2.13}$$

from where it also follows that

$$\phi^*(df) = d(\phi^* f) . \tag{2.14}$$

A C^k map ϕ ($k \geq 0$) is said to be an <u>immersion</u> if for each point $p \in M$ there is a neighborhood U of p such that the inverse ϕ^{-1} restricted to $\phi(U)$ is also a C^r map. This implies that $n \leq n'$. The image $\phi(M)$, called an n-dimensional <u>immersed submanifold</u> in M', may intersect itself, i.e. ϕ may be one-one only locally. An immersion becomes an <u>imbedding</u> when it is a homeomorphism onto its image.

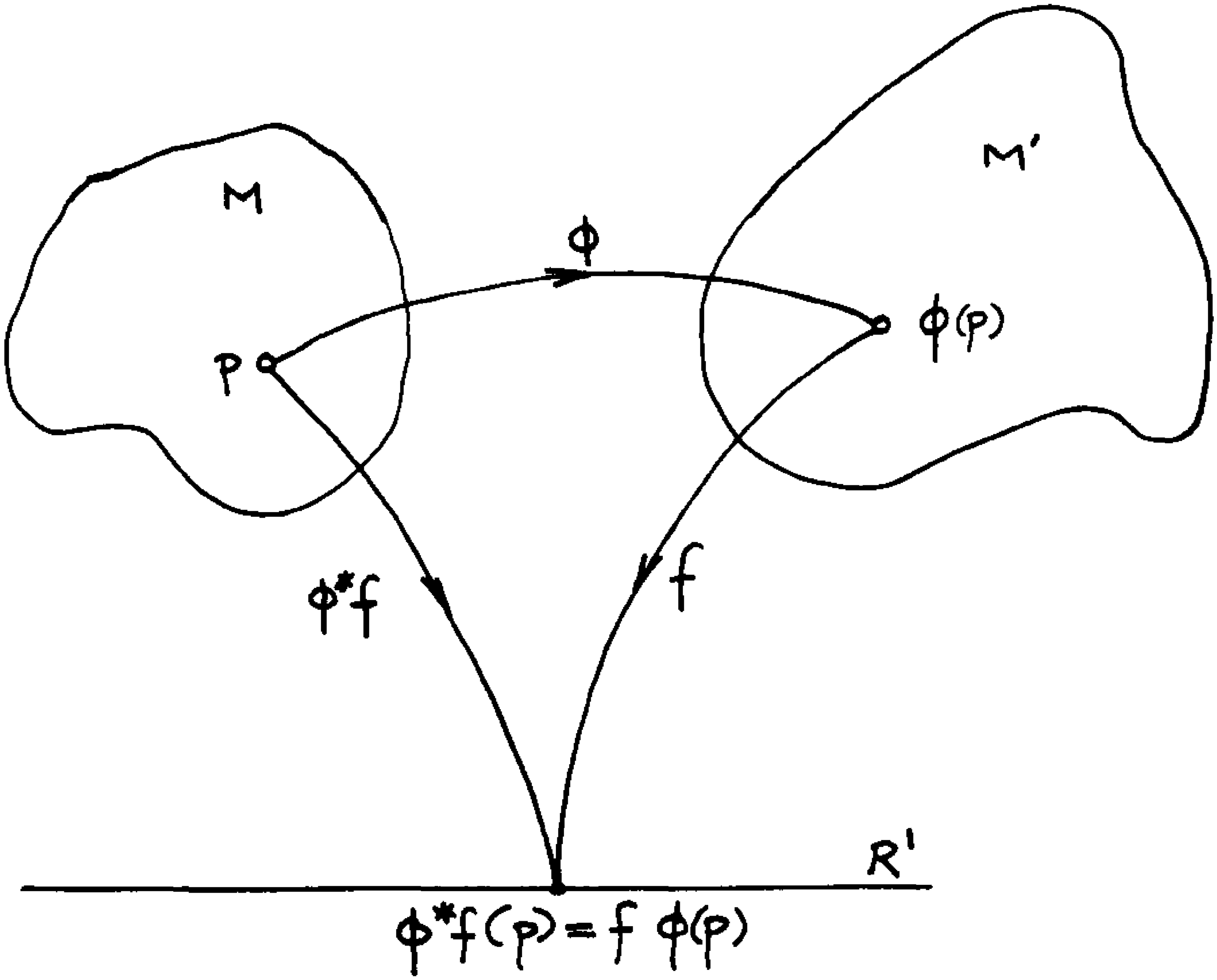


FIGURE 2: The map ϕ from the manifold M to the manifold M' induces a linear map from functions f defined on M' to functions $\phi_* f$ on M.

A map ϕ from M to M' is said to be a C^r diffeomorphism if it is a one-one C^r map and the inverse ϕ^{-1} is a C^r map from M' to M. Then $n' = n$. In this case, ϕ_* maps $T_p(M)$ to $T_{\phi(p)}(M')$ and $(\phi^{-1})^*$ maps $T^*_p(M)$ to $T^*_{\phi(p)}(M')$. Then, the map ϕ_* of $T^r_s(p)$ to $T^r_s(\phi(p))$, defined by

$$(2.15)\quad T(\omega^1, \ldots, \omega^r, V_1, \ldots V_s)\Big|_p = \phi_* T((\phi^{-1})^*\omega^1, \ldots (\phi^{-1})^*\omega^r, \phi_* V_1, \ldots \phi_* V_s)\Big|_{\phi(p)}$$

preserves symmetries and relations in the tensor algebra.

2.E. THE LIE AND EXTERIOR DERIVATIVES

On the basis of the manifold structure, as analyzed so far, we can define two types of differentiation, the Lie and the exterior ones.

Given a C^r vector field V over a manifold M, a family of local diffeomorphisms is induced on M, as follows. By the fundamental theorem for systems of ordinary diffential equations, the set of differential equations

$$\frac{dx^\kappa}{dt} = V^\kappa(x^1(t), \dots x^n(t)) ,$$

where V^κ are the components of V in terms of a set of local coordinates (x^κ), determine a unique curve $\lambda(t)$ through each point p in M, whose tangent vector at the point $\lambda(t_0)$ is $V|_{\lambda(t_0)}$, and $\lambda(0) = p$. This curve is called the <u>integral curve</u> of V starting at p. If, now, each point q in an open neighborhood U of M is moved a parameter distance t along the integral curves of V starting at q, with t less than a given $\epsilon > 0$, a map $\phi_t : U \to M$ is induced, which obviously constitutes a diffeomorphism. This diffeomorphism maps each tensor field T at q into $\phi_{t*} T|_{\phi_t(q)}$, according to (2.15).

The <u>Lie derivative</u> $L_V T$ of a tensor field T with respect to V is defined by

$$L_V T = \lim_{t \to 0} \frac{1}{t} (T|_p - \phi_{t*} T|_p) . \tag{2.16}$$

Since ϕ_* corresponds to a diffeomorphism, the Lie derivative maps tensors linearly and preserves tensor types as well as contractions. It is easily shown, on the other hand, that

(2.17)
(i) $L_V(S \otimes T) = L_V S \otimes T + S \otimes L_V T$

(ii) $L_V f = V(f)$, whose f is a function

(iii) $L_V u = [V, u] = -[u, V]$

for any vector field u, where $[V, u]$ is defined by $[V, u](f) = V(u(f)) - u(V(f))$.

If $L_V u$ vanishes, we say that u is Lie-transported along V . In this case, the vector fields form an immersed two dimensional submanifold of M. If $\lambda(t)$, $\mu(s)$ are respectively, the integral curves of V and u, then the vectors $V|_{\lambda(t_0),\mu(s_0)}$, $V|_{\lambda(t_0),\mu(s_0+\epsilon)}$, $u|_{\lambda(t_0),\mu(s_0)}$, and $u|_{\lambda(t_0+\delta),\mu(s_0)}$ form a closed "rectangle", for any point p defined by the intersection of $\lambda(t)$ and $\mu(s)$. (See fig. 3)

When (e_α) is a field of basis vectors over M then we have

$$(2.18) \qquad L_{e_\alpha} e_\beta = [e_\alpha, e_\beta] = C_{\alpha\beta}{}^{\gamma} e_\gamma = -C_{\beta\alpha}{}^{\gamma} e_\gamma$$

since the Lie derivative maps vectors into vectors in a linear fashion. The functions $C_{\alpha\beta}{}^{\gamma}$ are called the structure coefficients of the basis (e_α). In a coordinate basis these coefficients necessarily vanish, since, then, $[e_\alpha, e_\beta] = \frac{\partial^2}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2}{\partial x^\beta \partial x^\alpha}$
Finally, for any three vector fields u, V, Z on M, Jacobi's identity,

$$(2.19) \qquad [u, [V, Z]] + [V, [Z, u]] + [Z, [u, V]] = 0$$

holds.

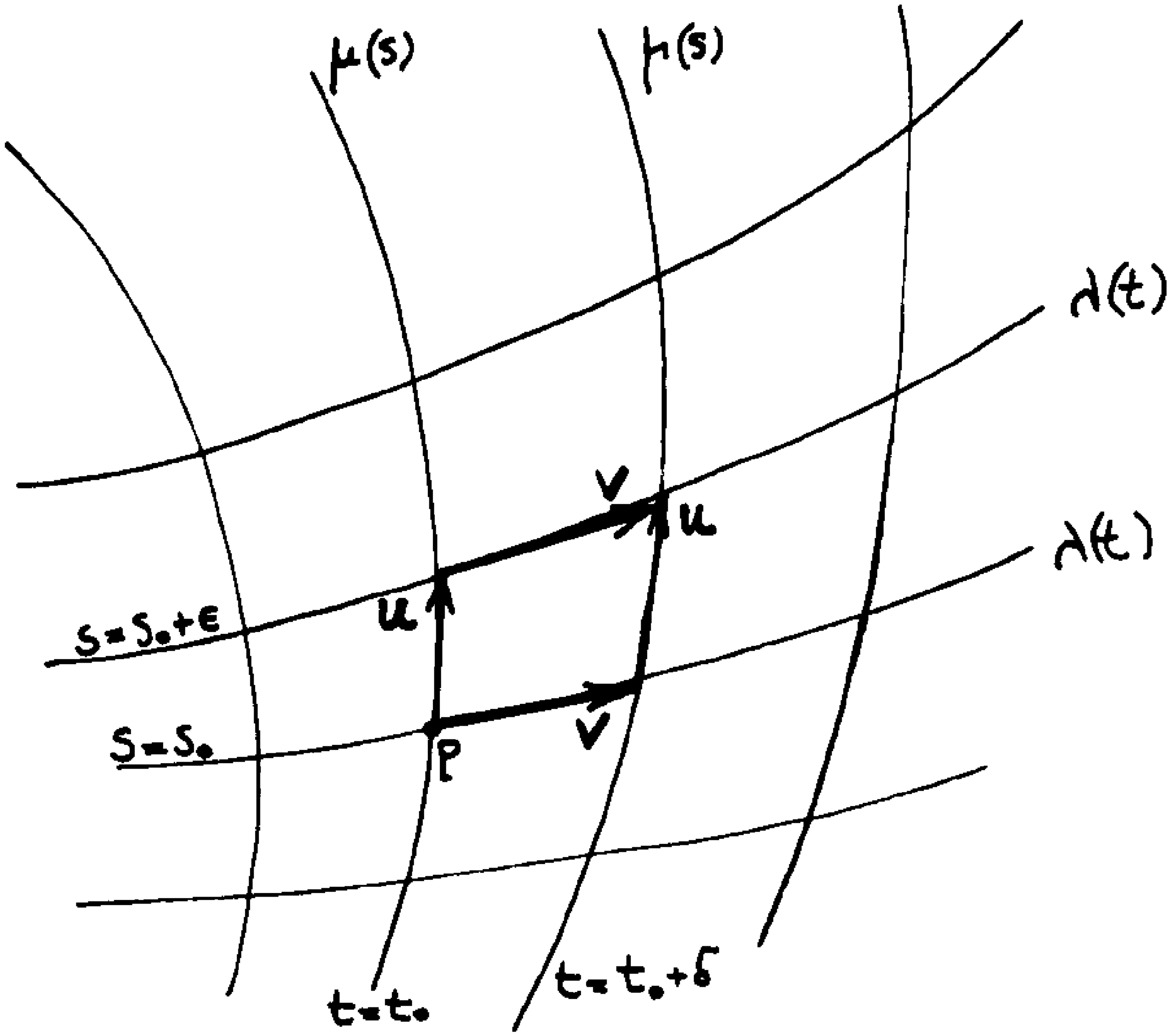


FIGURE 3: The vector field **v** is "Lie transported" along the integral curves $\mu(s)$ of the vector field **u**.

The exterior derivative is defined for a subset of tensors, the so-called <u>q-forms</u>. One defines as q-forms the set of tensors of type $(0, q)$ which are completely antisymmetric with respect to their q covariant indices, where $q \leq n$, the dimension of the manifold. If Ω and Σ are p- and q-forms respectively one forms a (p + q)-form by taking their <u>exterior or shew-symmetric tensor product</u> $\Omega \wedge \Sigma$, i.e.

$$(2.20) \qquad (\Omega \wedge \Sigma)_{\alpha \ldots \beta \gamma \ldots \delta} = \Omega_{[\alpha \ldots \beta} \Sigma_{\gamma \ldots \delta]} = \frac{1}{(p+q)!} \text{(Alternating sum over all indices of } \Omega_{\alpha \ldots \beta} \Sigma_{\gamma \ldots \delta} \text{)}$$

It follows that

$$(2.21) \qquad \Omega \wedge \Sigma = (-1)^{pq} \Sigma \wedge \Omega$$

If (σ^{α}) is a basis of one-forms, then the q-forms $(\sigma^{\alpha_1} \wedge \sigma^{\alpha_2} \wedge \ldots \sigma^{\alpha_q})$ $\alpha_i = 1, \ldots, n$, form a basis of q-forms, so that any q-form can be written as

$$(2.22) \qquad \Sigma = \Sigma_{\alpha \ldots \beta} \sigma^{\alpha} \wedge \cdots \wedge \sigma^{\beta} \quad \text{with} \quad \Sigma_{\alpha \ldots \beta} = \Sigma_{[\alpha \ldots \beta]}$$

Via the <u>exterior differentiation</u> one maps q-form fields to (q + 1)-form fields, according to the rules

(i) The exterior derivative of a zero-form field (i.e., a function) f is the one form df, as defined by (2.8) above.

(ii) The exterior derivative of a q-form field $\Sigma = \Sigma_{\alpha \ldots \beta} dx^{\alpha} \wedge \ldots \wedge dx^{\beta}$

is the (q + 1) form

$$d\Sigma = (d\Sigma_{\alpha \ldots \beta}) \wedge dx^{\alpha} \wedge \cdots \wedge dx^{\beta} \tag{2.23}$$

It is easily shown that the last operation is coordinate independent. One can also show that

$$d(\Sigma \wedge \Omega) = d\Sigma \wedge \Omega + (-1)^{p} \Sigma \wedge d\Omega \tag{2.23}$$

where Σ is any p-form, and that for any q-form field Σ

$$d(d\Sigma) = 0 \tag{2.24}$$

Since the exterior differentiation maps q-forms linearly into (q + 1)-forms, then for any basis of one-forms (σ^{α}) we must have

$$d\sigma^{\alpha} = -\omega^{\alpha}{}_{\beta} \wedge \sigma^{\beta} \tag{2.25}$$

where the $\omega^{\alpha}{}_{\beta}$'s are one-forms. They are called <u>Cartan's</u> <u>rotation</u> <u>coefficients</u> for reasons that become obvious later in our exposition. Thus, we can write

$$\omega^{\alpha}{}_{\beta} = \gamma^{\alpha}{}_{\beta\gamma} \sigma^{\gamma} \tag{2.26}$$

for some functions $\gamma^{\alpha}{}_{\beta\gamma}$. Then (2.25) reads

$$d\sigma^{\alpha} = \gamma^{\alpha}{}_{\beta\gamma} \sigma^{\beta} \wedge \sigma^{\gamma} \tag{2.27}$$

Now, for any function f, we have, according to (2.8),

$$df = e_{\alpha}(f)\sigma^{\alpha}$$

while (2.23) and (2.24) imply that

(2.28) $$d^2 f = 0$$

Then, $$(d\sigma^\alpha)e_\alpha(f) = -\sigma^\beta\wedge\sigma^\alpha e_\beta(e_\alpha(f)) =$$
$$= -\tfrac{1}{2}\sigma^\beta\wedge\sigma^\alpha\left(e_\beta(e_\alpha(f)) - e_\alpha(e_\beta(f))\right) = -\tfrac{1}{2}\sigma^\beta\wedge\sigma^\alpha C_{\beta\alpha}{}^\gamma e_\gamma(f)$$

where the structure coefficients $C_{\alpha\beta}{}^\gamma$ enter via (2.18).

Thus,

(2.29) $$d\sigma^\alpha = -\tfrac{1}{2} C_{\beta\gamma}{}^\alpha \sigma^\beta\wedge\sigma^\gamma$$

2.F. COVARIANT DERIVATIVE AND CURVATURE

The structure of a manifold is refined by demanding that it is affinely connected. A C^k affine connection ∇ (to be simply called connection hereafter) on a C^r manifold ($r \geq k+2$) is a rule by which any C^{k+1} tensor field T on M of type (p, s) is mapped into a C^k tensor field of type (p , $s+1$), called the covariant derivative of T, to be denoted by ∇T, such that

(i) ∇ is linear and commutes with contractions

(ii) $\nabla(S \otimes T) = \nabla S \otimes T + S \otimes \nabla T$ for any tensor fields S and T,

(iii) $\nabla f = df$ for any function f.

The linearity of ∇, together with the fact that it increases the covariant index by one, implies that, for a basis set (e_α) of vectors and its dual (σ^α)

$$\nabla e_\alpha = \Gamma^\gamma{}_{\alpha\beta}\, e_\gamma \otimes \sigma^\beta \tag{2.30}$$

$$\nabla \sigma^\alpha = -\Gamma^\alpha{}_{\beta\gamma}\, \sigma^\beta \otimes \sigma^\gamma$$

since $\nabla \langle \sigma^\alpha, e_\beta \rangle = \nabla(\delta^\alpha_\beta) = 0$.

The functions $\Gamma^\alpha{}_{\beta\gamma}$ are called connection coefficients, and do not transform as the components of a tensor. However, if ∇ and $\hat{\nabla}$ are two different connections, then ($\Gamma^\alpha{}_{\beta\gamma} - \hat{\Gamma}^\alpha{}_{\beta\gamma}$) are components of a tensor.

In terms of any particular basis the covariant derivative of a tensor field T of type (r, s) will be written as

(2.31) $$\nabla \mathbf{T} = T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta;\epsilon}\, \mathbf{e}_\alpha \otimes \cdots \otimes \mathbf{e}_\beta \otimes \sigma^\gamma \otimes \cdots \otimes \sigma^\delta \otimes \sigma^\epsilon$$

Next, we define the <u>covariant</u> <u>derivative</u> <u>of</u> <u>a</u> <u>tensor</u> $\underline{\mathbf{T}}$ <u>in</u> <u>the</u> <u>direction</u> <u>of</u> <u>the</u> <u>vector</u> $\underline{\mathbf{V}}$ as the tensor of the same type as $\mathbf{T}$, resulting upon acting on $\mathbf{V}$ with $\nabla \mathbf{T}$. We denote it by $\nabla_{\mathbf{V}} \mathbf{T}$, so that

(2.32) $$\nabla_{\mathbf{V}} \mathbf{T} = \nabla \mathbf{T}(\mathbf{V})$$

(2.33) $$\nabla_{\mathbf{V}} \mathbf{T} = T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta;\epsilon}\, V^\epsilon\, \mathbf{e}_\alpha \otimes \mathbf{e}_\gamma \otimes \cdots \otimes \mathbf{e}_\beta \otimes \sigma^\gamma \otimes \cdots \otimes \sigma^\delta$$

For a set of basis vectors $(\mathbf{e}_\alpha)$ and their dual one forms (σ^α) we have, in particular

$$\nabla_\alpha \mathbf{e}_\beta \equiv \nabla_{\mathbf{e}_\alpha} \mathbf{e}_\beta = \Gamma^\delta{}_{\beta\gamma}\, \mathbf{e}_\delta \otimes \omega^\gamma(\mathbf{e}_\alpha) = \Gamma^\delta{}_{\beta\gamma}\, \mathbf{e}_\delta\, \delta^\gamma_\alpha$$

or

(2.34) $$\nabla_\alpha \mathbf{e}_\beta = \Gamma^\gamma{}_{\beta\alpha}\, \mathbf{e}_\gamma \; .$$

Similarly, we obtain

(2.35) $$\nabla_\alpha \omega^\beta = -\,\Gamma^\beta{}_{\gamma\alpha}\, \omega^\gamma$$

whereby it follows that

(2.36) $$\Gamma^\alpha{}_{\beta\gamma} = \langle \omega^\alpha, \nabla_\gamma \mathbf{e}_\beta \rangle = -\langle \nabla_\gamma \omega^\alpha, \mathbf{e}_\beta \rangle$$

Knowing the results of acting with ∇ on the basis vectors and one forms allows us to find that in (2.27) the components of the covariant derivative of a tensor are given by

(2.37) $$T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta;\epsilon} = T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta,\epsilon} + \Gamma^\alpha{}_{\mu\epsilon}\, T^{\mu\cdots\beta}{}_{\gamma\cdots\delta} +$$

$+$ (similar operation in all upper indices) $+$

$- \Gamma^{j}{}_{\alpha\epsilon} T^{\alpha\cdots\beta}{}_{j\cdots\delta} \quad +$

$-$ (similar for all lower indices),

where $T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta;\epsilon} = \nabla_{\epsilon}(T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta})$,

since the coefficients of a tensor are functions.

The covariant derivative of a C^k ($k \geq 1$) tensor field $\mathbf{T}$ along a C^k curve $\lambda(t)$ is defined as $\nabla_{\partial_t}\bar{\mathbf{T}}$ where $\bar{\mathbf{T}}$ is any C^k tensor field extending $\mathbf{T}$ onto an open neighborhood of $\lambda(t)$. We denote it by $\frac{D\mathbf{T}}{\partial t}$, and for a vector $\mathbf{V}$ tangent to λ we have

$$\frac{D\mathbf{T}}{\partial \lambda} = T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta;\epsilon} V^{\epsilon} \tag{2.38}$$

In particular, we can choose local coordinates so that $\lambda(t)$ has coordinates $x(t)$, $V^{\alpha} = \frac{dx^{\alpha}}{dt}$, and then for a vector field $\mathbf{u}$

$$\frac{D\mathbf{u}}{\partial t} = \frac{\partial u^{\alpha}}{\partial t} + \Gamma^{\alpha}{}_{\beta\gamma} u^{\beta} \frac{dx^{\gamma}}{dt} \tag{2.39}$$

The tensor $\mathbf{T}$ is said to be <u>parallelly</u> <u>transported</u> along λ if $\frac{D\mathbf{T}}{\partial t} = 0$. A curve $\lambda(t)$ is said to be a <u>geodesic</u> <u>curve</u> if

$$\nabla_{\mathbf{u}}\mathbf{u} = u^{\alpha}{}_{;\beta} u^{\beta} \mathbf{e}_{\alpha} = \frac{D}{\partial t}\left(\frac{\partial}{\partial t}\right)_{\lambda}$$

is parallel to $\left(\frac{\partial}{\partial t}\right)_{\lambda}$, i.e. if there is a function f such that

$$\nabla_{\mathbf{u}}\mathbf{u} = f\mathbf{u} \quad . \tag{2.40}$$

For such a curve, one can find a new parameter $s(t)$ along the curve such that

$$\frac{D}{\partial s}\left(\frac{\partial}{\partial s}\right)_{\lambda} = 0$$

Such a parameter is called an <u>affine</u> <u>parameter</u>. The vector $V = \left(\frac{\partial}{\partial s}\right)_\lambda$ obeys, then, the equation

(2.41) $$\nabla_V V = 0\ , \text{ or } \quad \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0$$

if coordinates as in (2.34) are used.

Given a C^k connection ∇, one can define a C^{k-1} tensor field T of type (1,2) by the relation

(2.42) $$T(u,V) = \nabla_u V - \nabla_V u - [u,V]$$

where u, V are arbitrary C^k vector fields. This tensor is called the <u>torsion</u> <u>tensor</u>. In a coordinate basis its components are given by

(2.43) $$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}$$

When the torsion $T = 0$, the connection coefficients, when expressed in a coordinate basis, are symmetric in the sense that $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$. Also, in that case

(2.44) $$[u,V] = \nabla_u V - \nabla_V u$$

For a basis set of vectors (e_α) the last equation reads

(2.45) $$[e_\alpha, e_\beta] = (\Gamma^\gamma{}_{\beta\alpha} - \Gamma^\gamma{}_{\alpha\beta})\, e_\gamma$$

Then, it follows from (2.18) that

(2.46) $$C_{\alpha\beta}{}^\gamma = \Gamma^\gamma{}_{\beta\alpha} - \Gamma^\gamma{}_{\alpha\beta} = -2\,\Gamma^\gamma{}_{[\alpha\beta]}$$

From (2.44) it also follows that the Lie derivative of a tensor field $\mathbf{T}$ of type (r, s) can be expressed in terms of the covariant derivative as

$$(L_V T)^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} = T^{\alpha\ldots\beta}{}_{\gamma\ldots\delta;\epsilon} V^{\epsilon} - T^{\epsilon\ldots\beta}{}_{\gamma\ldots\delta} V^{\gamma}{}_{;\epsilon} + \tag{2.47}$$

$$- \text{(all upper indices)} \quad + T^{\alpha\ldots\beta}{}_{\epsilon\ldots\delta} V^{\epsilon}{}_{;\gamma}$$

$$+ \text{(all lower indices)}$$

The exterior derivative of a q-form $\mathbf{Z}$ can also be expressed in terms of the covariant derivative as

$$dZ = Z_{\alpha\ldots\beta;\gamma}\, dx^{\gamma} \wedge dx^{\alpha} \wedge \cdots \wedge dx^{\beta} \longleftrightarrow \tag{2.48}$$

$$(dZ)_{\alpha\ldots\beta\gamma} = (-1)^{q} Z_{[\alpha\ldots\beta;\gamma]}$$

Given a C^{k} connection ∇, C^{k+1} vector fields $\mathbf{u}, \mathbf{v}, \mathbf{z}$ a C^{k-1} vector field $R(\mathbf{u},\mathbf{v})\mathbf{z}$ is defined by

$$R(\mathbf{u},\mathbf{v})\mathbf{z} = \nabla_{\mathbf{u}}(\nabla_{\mathbf{v}}\mathbf{z}) - \nabla_{\mathbf{v}}(\nabla_{\mathbf{u}}\mathbf{z}) - \nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{z} \tag{2.49}$$

A tensor field of type (1,3) is associated with $R(\mathbf{u},\mathbf{v})\mathbf{z}$ according to

$$R(\omega, \mathbf{z}, \mathbf{u}, \mathbf{v}) = \langle \omega, R(\mathbf{u},\mathbf{v})\mathbf{z} \rangle \tag{2.50}$$

and it is called the Riemann curvature tensor. Its components are given by

$$R^{\alpha}{}_{\beta\gamma\delta} = R(\omega^{\alpha}, e_{\beta}, e_{\gamma}, e_{\delta}) = \langle \omega^{\alpha}, R(e_{\gamma}, e_{\delta}) e_{\beta} \rangle = \tag{2.51}$$

$$= \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\mu} C_{\gamma\delta}{}^{\mu}$$

The Riemann tensor measures the degree to which the covariant derivatives of a vector field $\mathbf{V}$ do not commute, as seen from

$$V^{\alpha}{}_{;\beta\gamma} - V^{\alpha}{}_{;\gamma\beta} = R^{\alpha}{}_{\delta\gamma\beta} V^{\delta} \tag{2.52}$$

The first two sets of the so-called Bianchi identities satisfied by the Riemann tensor are

$$\begin{aligned} R^{\alpha}{}_{[\beta\gamma\delta]} &= 0 \\ R^{\alpha}{}_{\beta[\gamma\delta;\epsilon]} &= 0 \end{aligned} \tag{2.53}$$

2.G. GENERALIZED EXTERIOR CALCULUS

Let, now, $\mathbf{F}$ by a tensor of type $(r, s+q)$ which is totally antisymmetric in its last q covariant indices. Then, it can be written in the form

$$\mathbf{F} = \mathbf{F}_{\kappa_1 \cdots \kappa_q}\, \sigma^{\kappa_1} \wedge \cdots \wedge \sigma^{\kappa_q} \tag{2.54}$$

where $\mathbf{F}_{\kappa_1 \cdots \kappa_q} = \mathbf{F}_{[\kappa_1 \cdots \kappa_q]}$ is a tensor of type (r,s). Then we call $\mathbf{F}$ an <u>(r,s)-tensor-valued</u> <u>q-form</u>. Thus, a function is a scalar valued 0-form, a vector is a vector-valued 0-form, an ordinary one form is a scalar valued 1-form, a tensor of type (1,1) is a vector valued 1-form and so on.

Then, we can generalize the exterior derivative as defined earlier in section 3D, as follows.

(i) $d\mathbf{F}$, for $\mathbf{F}$ as in (2.54), is an (r,s) valued (q + 1)-form.

(ii) If $\mathbf{T}$ is an (r,s) valued 0-form, then $d\mathbf{T} = \nabla \mathbf{T}$.

Consider now a vector field $\mathbf{V}$, which in terms of some basis $(\mathbf{e}_\kappa)$ is written as $\mathbf{V} = V^\kappa \mathbf{e}_\kappa$. Then

$$\begin{aligned} d\mathbf{V} &= (dV^\kappa)\mathbf{e}_\kappa + V^\kappa d\mathbf{e}_\kappa = \\ &= (dV^\kappa)\mathbf{e}_\kappa + V^\kappa \mathbf{e}_\gamma \omega^\gamma{}_\kappa \end{aligned} \tag{2.55}$$

since, according to the generalized definition of exterior exterior derivative, and (2.46)

$$d\mathbf{e}_\kappa = \nabla \mathbf{e}_\kappa = \Gamma^\gamma{}_{\kappa\beta}\, \mathbf{e}_\gamma \otimes \omega^\beta = \mathbf{e}_\gamma\, \omega^\gamma{}_\kappa .$$

Then,

$$
\begin{aligned}
d^2\mathbf{v} &= d\{e_\gamma(dv^\gamma + v^\alpha \omega^\gamma{}_\alpha)\} = \\
&= (de_\gamma)\wedge(dv^\gamma + v^\alpha \omega^\gamma{}_\alpha) + e_\gamma\, d(dv^\gamma + v^\alpha\omega^\gamma{}_\alpha) \\
&= e_\delta \omega^\delta{}_\gamma \wedge (v^\gamma{}_{,\delta}\,\omega^\delta + v^\alpha \omega^\gamma{}_\alpha) + e_\gamma(v^\alpha{}_{,\delta}\,\omega^\delta \wedge v^\alpha \omega^\gamma{}_\alpha + \\
&\quad + v^\alpha\, d\omega^\gamma{}_\alpha) = \\
&= e_\delta\,(d\omega^\delta{}_\alpha + \omega^\delta{}_\gamma \wedge \omega^\gamma{}_\alpha)\, v^\alpha\,,
\end{aligned}
$$

where (ω^α) is a 1-form basis.

or

$$d^2\mathbf{v} = e_\alpha \mathcal{R}^\alpha{}_\beta\, v^\beta \tag{2.56}$$

where the 2-forms $\mathcal{R}^\alpha{}_\beta$ are given by

$$\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\mu \wedge \omega^\mu{}_\beta\,, \tag{2.57}$$

and are called <u>curvature</u> <u>2-forms</u>.

Letting $\mathcal{R}$ be the (1,1)-tensor valued 2-form

$$\mathcal{R} = e_\alpha \otimes \omega^\beta \mathcal{R}^\alpha{}_\beta \tag{2.58}$$

we can rewrite (2.56) as

$$d^2\mathbf{v} = \mathcal{R}(\mathbf{v}) \tag{2.59}$$

Using the result (see WHEELER ET AL p. 352)

$$\langle d\mathbf{S}, \mathbf{u}\wedge\mathbf{z}\rangle = \nabla_{\mathbf{u}}\langle \mathbf{S},\mathbf{z}\rangle - \nabla_{\mathbf{z}}\langle \mathbf{S},\mathbf{u}\rangle - \langle \mathbf{S}, [\mathbf{u},\mathbf{z}]\rangle \tag{2.60}$$

valid for any tensor-valued 1-form $\mathbf{S}$, with $\mathbf{S}$ replaced by the vector-valued 1-form $d\mathbf{v}$, we find that

$$\langle d^2\mathbf{v}, \mathbf{u}\wedge\mathbf{z}\rangle = \mathcal{R}(\mathbf{u},\mathbf{v})\,\mathbf{z} \tag{2.61}$$

where $R(u, z)$ is the operator defined by (2.49) above.

Thus, from (2.59) and (2.61) we obtain

$$(2.62) \quad \langle \mathcal{R}(v), u \wedge z \rangle = R(u,z)v$$

for arbitrary C^1 vector fields v, u, z. Then taking $v = e_\alpha$, $u = e_\beta$, $z = e_\gamma$, where (e_α) a vector basis, (2.62) gives

$$e_\tau \langle \mathcal{R}^\tau{}_\alpha, e_\beta \wedge e_\gamma \rangle = e_\tau R^\tau{}_{\alpha\beta\gamma}$$

so that

$$(2.63) \quad \mathcal{R}^\tau{}_\alpha = \frac{1}{2} R^\tau{}_{\alpha\beta\gamma}\, \omega^\beta \wedge \omega^\gamma$$

2.H THE METRIC

The geometrical structure of a manifold is further restricted by demanding that the manifold "has a metric," in the sense explained presently.

A <u>C^k metric</u> g on a manifold M is a C^k symmetric tensor field of type (0,2). The scalar $|g(u,u)|^{1/2}$ will be called the magnitude of the vector u, and $g(u,v)/(|g(u,u)\cdot g(v,v)|)^{1/2}$ the cosine of the angle between u and v, provided $g(u,u)\cdot g(v,v) \neq 0$.

In terms of a vector basis (e_α) the components of g are

$$(2.64) \quad g_{\alpha\beta} = g_{\beta\alpha} = g(e_\alpha, e_\beta)$$

so that we write

$$g = g_{\alpha\beta}\, \sigma^\alpha \otimes \sigma^\beta$$

where (σ^α) is the basis of 1-forms dual to (e_α).

A metric is <u>non-degenerate</u> of a point $p \in M$ if for a given vector $u \in T_p$, $g(u,v)=0$ for any $v \in T_p$, implies $u=0$. Otherwise, the metric is non-degenerate if the matrix ($g_{\alpha\beta}$) is not singular. For a non-degenerate metric, there is a unique tensor $\tilde{g}$ of type (2,0), called the inverse of g, which is determined by

$$\tilde{g}^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma} \tag{2.65}$$

Hereafter we will identify $\tilde{g}$ with g, and take all tensors obtainable from each other by lowering or raising of indices (as in $T^{\alpha\beta} = g^{\alpha\delta} g^{\beta\epsilon} T_{\delta\epsilon}$) to represent the same geometrical object.

The <u>length</u> <u>of</u> <u>curves</u> in M will be defined as follows. The scalar

$$L = \int_{t_1}^{t_2} |g(\partial_t, \partial_t)|^{1/2} dt \tag{2.66}$$

will be called the length of the piece of the curve $\lambda(t)$ with tangent vector $\frac{\partial}{\partial t}$, with end-points $\lambda(t_1)$, $\lambda(t_2)$.

Given a metric g on a manifold, there is a unique torsion-free connection ∇ on the manifold determined by the <u>condition</u> <u>of</u> <u>compatibility</u>

$$\nabla g = 0 \longleftrightarrow dg_{\alpha\beta} = \omega_{\alpha\beta} + \omega_{\beta\alpha} \tag{2.67}$$

where

$$\omega_{\alpha\beta} \equiv g_{\alpha\gamma} \omega^{\gamma}{}_{\beta} = g_{\alpha\gamma} \Gamma^{\gamma}{}_{\beta\delta} \omega^{\delta} \equiv \Gamma_{\alpha\beta\delta} \omega^{\delta}, \tag{2.68}$$

(ω^{α}) a basis of 1-forms.

Then,

(2.69) $$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha}\right)$$

It also now follows that

(2.70) $$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$

which, together with, (2.53), implies that

(2.71) $$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

Altogether, these symmetries leave the Riemann tensor with $\frac{1}{12}n^2(n^2-1)$ algebraically independent components, $\frac{1}{2}n(n+1)$ of which can be represented by the components of the symmetric tensor

(2.72) $$R_{\alpha\beta} \equiv R^{\gamma}{}_{\alpha\gamma\beta}$$

called the Ricci tensor.

The n-form $\boldsymbol{\epsilon}$, defined in terms of its components in a basis $(\mathbf{e}_\alpha)$ by

(2.73) $$\epsilon_{\alpha_1 \ldots \alpha_n} = n!\,|g|^{1/2}\,\delta^1_{[\alpha_1} \cdots \delta^n_{\alpha_n]}\ ,$$

where g is the determinant of the matrix ($g_{\alpha\beta}$), is also called Levi-Civita tensor. We use it to define the $*$-duals of q-forms, as well as volumes of open sets of a manifold. The $*$-dual $^*\mathbf{F}$ of a q-form is the (n-q)-form

(2.74) $$^*\mathbf{F} = \frac{1}{q!}\frac{1}{(n-q)!}\,F^{\lambda_1 \ldots \lambda_q}\,\epsilon_{\lambda_1 \ldots \lambda_q \lambda_{q+1} \ldots \lambda_n}\,\omega^{\lambda_{q+1}} \wedge \cdots \wedge \omega^{\lambda_n}$$

The volume of an open set U of M is defined by

(2.75) $$V(U) = \int_U \boldsymbol{\epsilon} = \int_{\phi(U)} |g|^{1/2}\,dx^1 \cdots dx^n\ ,$$

where $\phi(U)$ is the region of R^n to which U is mapped when coordinates $(x^1, \ldots, x^n)$ are used. Similarly, we define <u>the integral of a function</u> f in an open subset U of M as

(2.76) $$\int_U f \epsilon = \int_{\phi(U)} f \, |g|^{1/2} dx^1 \ldots dx^n,$$

which is invariant under coordinate transformations.

3. THE RIEMANN-EINSTEIN SPACETIME

3.A. FOUNDATIONS

The cosmological models which we will present in later sections of this work will be based on the fundamental concepts incorporated in what we will refer to as the Riemann-Einstein theory of spacetime. This theory is commonly known as General Relativity for reasons which cannot be epistemologically justified. On the contrary, we believe that we are completely justified from the historical standpoint in associating Riemann's name with a theory which was put on a sound scientific basis by Albert Einstein. Riemann was not only responsible for the development of the mathematical strutures which make up the backbone of "General Relativity," but, as Einstein later said, "with prophetic vision he saw the physical meaning of this (Riemann's) generalization of Eucleid's geometry," EINSTEIN (1956) p. 64. (See RIEMANN (1872) for short and clear account of his vision. For a thorough analysis of the epistemological questions associated with "General Relativity," see BUNGE (1967).) In any event, we will accept the following set of axioms as the basis of our discussion of cosmological problems:

(a) Spacetime is a 4-dimensional C^{∞} connected Hausdorff manifold. We take spacetime to be a 4-dimensional Hausdorff manifold because we believe that this mathematical structure correctly represents our notion of spacetime as a continuum of events--the points of the manifold--which we can distinguish by separation and identify using four labels on each. We take it to be connected on the basis of our belief that we can obtain knowledge about our universe in its totality. This does not mean that we can obtain information about every part of the universe via direct reception of signals from everywhere. Actually there do exist horizons beyond which we cannot "see." We demand that spacetime is connected so that every part of it affects the properties of every other.

(b) The metric of the spacetime manifold is Lorentz, i.e. its signature is +2. (The signature of the metric is defined as the number of the positive eigenvalues of the matrix ($g_{\alpha\beta}$) minus the number of negative ones.) This is equivalent to the fact at any given point of the manifold we can find a basis in which $g_{\alpha\beta} = \mathrm{diag}\,(-1,1,1,1)$. When the metric is Lorentz, the vectors of the tangent space T_p at every point p of the manifold are separated into three families: <u>spacelike</u> ($g(u,u) > 0$), <u>null</u> ($g(u,u) = 0$), and <u>timelike</u> ($g(u,u) < 0$). This way we objectify our belief that space and time are fundamentally distinct although connected. Physically, it reflects the validity of Special Relativity in the small.

(c) Timelike curves in the spacetime manifold, i.e. curves whose tangent at every point is a timelike vector, are distinguished as future- and past-directed ones. This axiom attributes to spacetime our belief that past and future are absolutely distinct.

(d) Timelike curves are not closed, neither is the causal connection between two events relative. This way man's ability to intervene effectively and change his environment along predetermined lines is taken to be in accord with the laws of the universe. Futhermore, the universe as a whole and in its parts negates itself only in a future directed way, and it negates its negation both in the past and in the future. Otherwise, whenever all timelike curves focus to a point we consider this point as "singular," in the sense that it indicates inadequacy of our understanding of the universe and not the beginning or the end of the universe itself.

(e) The geometry of the spacetime manifold is locally determined by the matter fields in it, according to Einstein's field equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + g_{\alpha\beta}\Lambda = 8\pi T_{\alpha\beta} \tag{3.1}$$

Here $R \equiv R^{\alpha}{}_{\alpha}$, and is called the curvature scalar, Λ is a "cosmological" constant, and $T_{\alpha\beta}$ a symmetric tensor, called the stress-energy tensor, which depends on the matter fields and their covariant derivatives. By saying that the geometry of the manifold is locally determined by the matter fields, we do not mean that the absence of matter fields leaves the geometry indetermined or that it leads necessarily to any specific structure, Minkowski geomentry--for example. That this is not the case is proven by the existing number of "empty" exact solutions of the field equations, one of which will be explicitly obtained in a later section. We simply mean that the topological, i.e. global, properties of spacetime do not derive from the field equations.

There is no unique way for the construction of the stress-energy tensor for the various source fields. One usually generalizes local results, i.e. special relativistic results, in an invariant fashion. For details see HAWKING & ELLIS (1973) pp. 64-71, and MISNER ET AL (1973) pp. 139-140, 385-387, 568-570. The specific form of the stress-energy tensor in the cases treated later will be quoted from these references.

In the field equations (3.1) the units were chosen so that the speed of light in vacuum and the gravitational constant are both equal to unity. In the following, the magnitude $\mathbf{g}(\mathbf{u},\mathbf{u})$ of the vector $\mathbf{u}$ will be denoted by $\mathbf{u}\cdot\mathbf{u}$. Greek indices will run like 0,1,2,3 and Latin like 1,2,3, unless otherwise indicated.

3.B. KINEMATICS

The dynamic, mathematically complex structure of the Riemann-Einstein spacetime makes it very difficult to visualize the kinematic behavior of the various matter fields in it. Most observations are not invariant under a change of the position of the observer, neither is the physical significance of a multitude of geometrical quantities associated with a given field always clear. For this reason we define here a rigorous way of identifying the vantage point of an observer and the means of communication between different observers, as well as the physical significance of some tensors usually employed in describing a fluid. We concentrate on the description of a fluid because our discussion will later focus on "spatially homogeneous" cosmological models, most of which take the matter in the universe to be continuously and smoothly distributed.

A particle with rest mass m will be represented by a future-directed timelike curve in the spacetime manifold, and a vector $\mathbf{p}$ tangent to the curve and with magnitude $\mathbf{p}\cdot\mathbf{p}=-m^2$ will represent the particle's <u>momentum</u>. The <u>velocity</u> of the particle is $\mathbf{v}=\mathbf{p}/m$, so that $\mathbf{v}\cdot\mathbf{v}=-1$, always.

We define as an <u>observer</u> a particle together with a <u>frame</u>, i.e. a set of basis vectors at each point on the curve representing the particle. Obviously, any point along the observer's path can be associated with an infinity of frames. We would like, therefore, to be able to pick a one-parameter family of frames along the path of the observer in a physically realizable and useful manner. We do this by the following two-legged selection process: (a) At any one instant, out of the infinity of frames definable, we choose the <u>rest frame of the observer</u>. (b) We <u>Fermi-transport</u> this frame along the observer's path. By observer's rest frame we mean a tetrad of vectors $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, such that $\mathbf{e}_0$ is parallel to $\mathbf{u}$, the observer's velocity, and the $\mathbf{e}_i$, (i =1,2,3) are normal to $\mathbf{u}$, i.e. $\mathbf{u}\cdot\mathbf{e}_i = 0$. On the other hand, we say that a vector $\mathbf{V}$ is Fermi-transported along the curve $\lambda(t)$, whose tangent is $\mathbf{u}$, if the Fermi derivative of $\mathbf{V}$ along $\mathbf{u}$, $\nabla^F_{\mathbf{u}}\mathbf{V}$, defined by

$$\nabla^F_{\mathbf{u}} \mathbf{V} = \nabla_{\mathbf{u}} \mathbf{V} - (\mathbf{V}\cdot\mathbf{a})\mathbf{u} + (\mathbf{u}\cdot\mathbf{V})\mathbf{a} \tag{3.2}$$

where $\mathbf{a} = \nabla_{\mathbf{u}}\mathbf{u}$, vanishes. It follows from the definition that the Fermi-derivative reduces to covariant derivative when $\lambda(t)$ is a geodesic ($\mathbf{a} = 0$), and that a vector $\mathbf{V}$ normal to $\mathbf{u}$ at one instant will remain normal if $\mathbf{V}$ is Fermi-transported along $\lambda(t)$. It is also easily shown that if two vectors $\mathbf{V}, \mathbf{Z}$ are Fermi transported along $\lambda(t)$ then $\mathbf{V}\cdot\mathbf{Z}$ remains constant. It follows that if we choose an orthonormal rest frame at a point along an observer's path and Fermi transport it along his path, the frame will remain an orthonormal one. The directions of the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ can be physically realized by a set of three gyroscopes carried by the observer.

Any vector $\mathbf{V}$ along an observer's path can be separated into a part $\mathbf{V}_{\parallel}$ parallel to the observer's velocity $\mathbf{u}$ and a part normal to $\mathbf{u}$, denoted by $\mathbf{V}_\perp$. The first is given by $-\mathbf{u}\cdot\mathbf{V}$ and the second by $\langle \mathbf{P}, \mathbf{V}\rangle$, where

$$(3.3) \qquad \mathbf{P} = (\delta^\alpha{}_\beta + u^\alpha u_\beta)\, \mathbf{e}_\alpha \otimes \boldsymbol{\omega}^\beta$$

A <u>fluid</u> will be described by a dense population of non-colliding particles, i.e. a family of timelike curves one through each point of the spacetime manifold, or conguence of timelike curves. Locally, we will represent a fluid by a spacetime tube, as in fig. 4. Let t be a parameter along the curves, and $\mathbf{V}$ a vector connecting two given points one on each of two neighboring curves of the congruence. Next, move these two points the same parameter distance along the curves. The vector that connects the two points thus reached, is the same as $\mathbf{V}$ Lie-transported along the curves. Thus we have, by construction,

$$(3.4) \qquad L_{\mathbf{u}}\mathbf{V} = 0 \longleftrightarrow \nabla_{\mathbf{u}}\mathbf{V} = \nabla_{\mathbf{V}}\mathbf{u}$$

Now

$$(3.5) \qquad \mathbf{V} = \mathbf{V}_\perp + \mathbf{V}_\parallel$$

where

$$(3.6) \qquad \mathbf{V}_\perp = \langle \mathbf{P}, \mathbf{V}\rangle \quad , \quad \mathbf{V}_\parallel = -(\mathbf{V}\cdot\mathbf{u})\mathbf{u} \; .$$

Thus, equation (3.4) can be written as

$$(3.7) \qquad \nabla_{\mathbf{u}}\mathbf{V}_\perp + \nabla_{\mathbf{u}}\mathbf{V}_\parallel = \nabla_{\mathbf{V}_\perp}\mathbf{u} + \nabla_{\mathbf{V}_\parallel}\mathbf{u}$$

and, thus,

$$(3.8) \qquad \langle \mathbf{P}, \nabla_{\mathbf{u}}\mathbf{V}_\perp + \nabla_{\mathbf{u}}\mathbf{V}_\parallel\rangle = \langle \mathbf{P}, \nabla_{\mathbf{V}_\perp}\mathbf{u} + \nabla_{\mathbf{V}_\parallel}\mathbf{u}\rangle$$

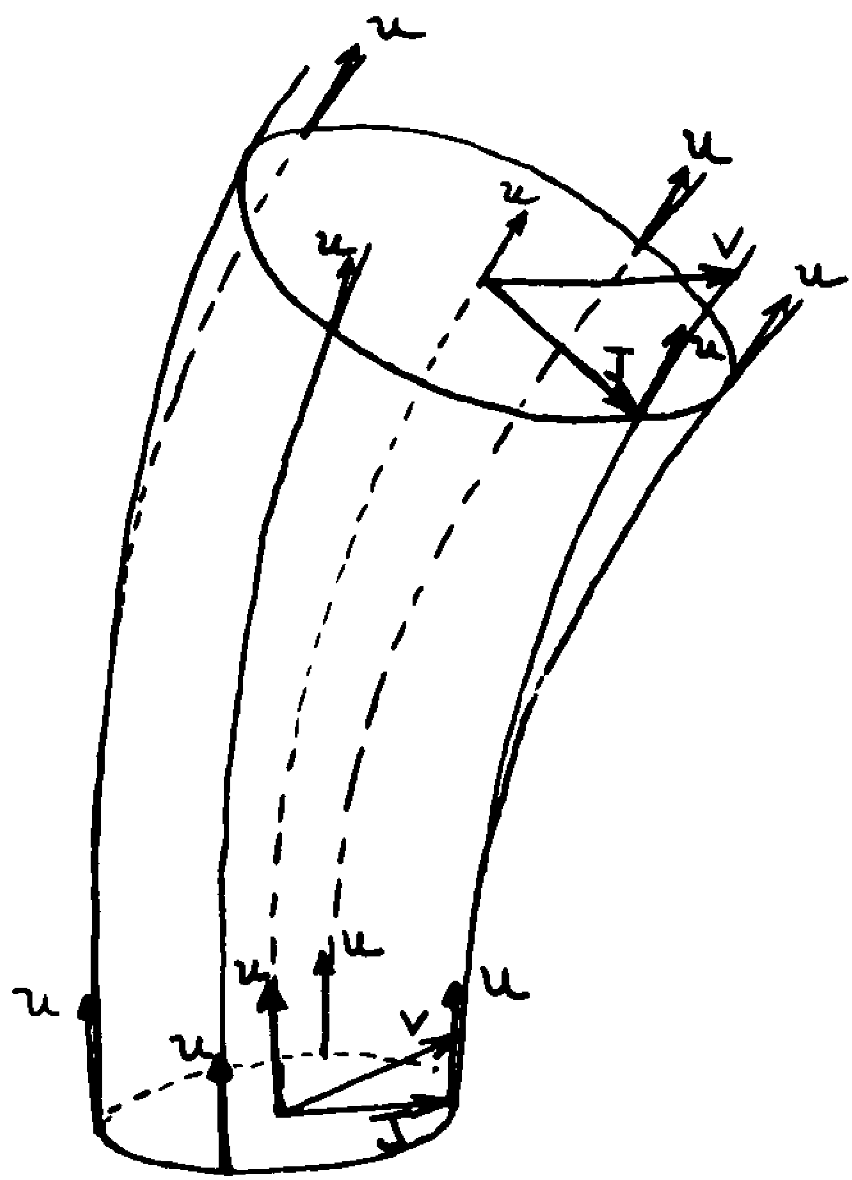


FIGURE 4: A spacetime tube constructed from a congruence of timelike curves with tangent vector $\mathbf{u}$. The vector $\mathbf{J}$ measures the seperation of nearby curves in the rest frame of an observer with velocity   .

Then

$$(3.9) \quad \langle P, \nabla_u V_\perp \rangle = \langle P, \nabla_{V_\perp} u \rangle = \nabla_{V_\perp} u ,$$

since ∇u never has components parallel to u.

Now, it is easily shown that

$$(3.10) \quad \langle P, \nabla_u V_\perp \rangle = \nabla_u^F V_\perp$$

from which, by taking $J \equiv V_\perp$, we obtain equation (3.9) in the form

$$(3.11) \quad \nabla_u^F J = \nabla_J u \quad .$$

Then

$$(3.12) \quad \nabla_u^{F2} J = \nabla_u^F (\nabla_J u) = \langle P , \nabla_u (\nabla_J u) \rangle \cdot$$

But, it follows from equation (249) that,

$$(3.13) \quad \nabla_u (\nabla_J u) = - R(J,u)u + \nabla_J (\nabla_u u) + \nabla_{[J,u]} u \, .$$

Thus, after some futher algebra, we obtain

$$(3.14) \quad \nabla_u^{F2} J = \langle P , -R(J,u)u + \nabla_J a + (J \cdot a) a \rangle ,$$

or

$$(3.15) \quad \nabla_u^{F2} J = - R(J,u)u + (J \cdot a)a + \langle P, \nabla_J a \rangle ,$$

which is the Jacobi equation for the separation vector J, expressed in terms of the Fermi derivative.

We now consider equations (3.11) and (3.15) from the standpoint of an observer moving along $\gamma(t)$ who uses an orthonormal frame with $e_0 = u$, his velocity, and Fermi transports the e_i (i =1,2,3) basis vectors. They read,

$$\frac{dJ^i}{dt} = u^i{}_{;j} J^j \tag{3.16}$$

$$\frac{d^2J^i}{dt^2} = \left(- R^i{}_{0j0} + a^i{}_{;j} + a^i a_j\right) J^j . \tag{3.17}$$

Then

$$J^i(t) = A^i{}_j(t) J^j(t_0) \tag{3.18}$$

where $A^i{}_j(t)$ is a 3 X 3 matrix, which satisfies the equation

$$\frac{d}{dt} A^i{}_j(t) = u^i{}_{;l} A^l_j(t) , \tag{3.19}$$

with the initial condition $A^i{}_j(t_0) = \delta^i_j$. This matrix can be written as

$$A^i{}_j = O^i{}_l S^l{}_j \tag{3.20}$$

where O_{ij} is an orthogonal and S_{ij} a symmetric matrix. They measure, respectively, the rotation and change in the magnitude of the vector J^i which connects the two neighboring particles. Since $\left.\frac{dO_{ij}}{dt}\right|_{t_0}$ is antisymmetric and $\left.\frac{dS_{ij}}{dt}\right|_{t_0}$ is symmetric, it follows from (3.19) that the antisymmetric part of $u^i{}_{;j}$ gives the rate of rotation and its symmetric part the rate of expansion of neighboring curves. This leads us to an invariant definition of various tensors that describe the fluid's behavior in a physically meaningful way. The tensor $\omega_{\alpha\beta}$, defined by

$$\omega_{\alpha\beta} = P_\alpha{}^\gamma P_\beta{}^\delta u_{[\gamma;\delta]} = u_{[\alpha;\beta]} + a_{[\alpha} u_{\beta]} \,, \tag{3.21}$$

is called the <u>vorticity</u> <u>tensor</u>. The <u>vorticity</u> <u>vector</u> is defined by

$$\Omega = \frac{1}{2} {}^*(u \wedge du) = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} u_\beta \omega_{\gamma\delta} e_\alpha \,, \tag{3.22}$$

and vanishes if and only if the vorticity tensor vanishes. We also define the <u>expansion</u> <u>tensor</u>

$$\Theta_{\alpha\beta} = P_\alpha{}^\gamma P_\beta{}^\delta u_{(\gamma;\delta)} = u_{(\alpha;\beta)} + a_{(\alpha} u_{\beta)} \,, \tag{3.23}$$

which can also be written in terms of its trace-free part $\sigma_{\alpha\beta}$, called the shear tensor, and the volume expansion scalar $\theta = u^\alpha{}_{;\alpha}$ as

$$\Theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3} P_{\alpha\beta} \theta \,. \tag{3.24}$$

Then the gradient of the fluid velocity can be written as

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3} \theta P_{\alpha\beta} - a_\alpha u_\beta \,. \tag{3.25}$$

In terms of the Fermi-transported orthonormal frame defined earlier and the matrix A_{ij}, the vorticity and expansion are given by

$$\begin{aligned} \omega_{ij} &= - A^{-1}{}_{\ell[i} \frac{d}{dt} A_{j]\ell} \\ \theta_{ij} &= A^{-1}{}_{\ell(i} \frac{d}{dt} A_{j)\ell} \\ \theta &= (\det A)^{-1} \frac{d}{dt} (\det A) \,. \end{aligned} \tag{3.26}$$

The deviation equation (3.17) now gives

$$\frac{d^2}{dt^2} A_{ij} = \left(- R_{i0l0} + a_{i;l} + a_i a_l \right) A_{lj} , \tag{3.27}$$

which allows for the calculation of the propogation of vorticity, shear and expansion along the congruence, once the Riemann tensor is known. For details see HAWKING & ELLIS (1973) pp. 93.

A <u>perfect</u> <u>fluid</u> is matter whose stress-energy tensor has the form

$$\mathbf{T} = (\rho + p)\, u \otimes u + p\, g \tag{3.28}$$

where the scalars ρ and p are called, respectively, the stress-energy density and pressure of the fluid. The stress-energy density ρ is given in terms of the density μ and internal energy density ϵ of the fluid, by

$$\rho = \mu (1 + \epsilon) \tag{3.29}$$

while the pressure is given by

$$p = \mu^2 \left(\frac{d\epsilon}{d\mu} \right) . \tag{3.30}$$

A <u>signal</u> <u>or</u> <u>photon</u> will be represented by a future directed null geodesic of the manifold. A photon connecting two observers reaches the receiver with a wavelength λ , frequency ω, energy E, and momentum $\mathbf{K}$, different, in general, from the ones it had when emitted. The quantity

(3.31) $$z = \frac{\lambda_2}{\lambda_1} - 1 = \frac{\omega_1}{\omega_2} - 1 = \frac{E_1}{E_2} - 1 = \frac{\mathbf{u}_1 \cdot \mathbf{K}_1}{\mathbf{u}_2 \cdot \mathbf{K}_2} - 1$$

where $\mathbf{u}_1$ is the velocity of the emitting and $\mathbf{u}_2$ that of the receiving observer, is a measure of the change in the signal's characteristics. It is called <u>red or blue shift</u>, depending on whether it is positive or negative.

The electromagnetic field will be described by a two-form field

(3.32) $$\mathbf{F} = \frac{1}{2} F_{\alpha\beta}\, \boldsymbol{\sigma}^{\alpha} \wedge \boldsymbol{\sigma}^{\beta}$$

whose F_{i0} component gives the i-th component of the electric vector field $\mathbf{E}$ and the F_{ij} component gives the k-th (i, j, k cyclically 1,2,3) component of the magnetic vector field $\mathbf{B}$.

The electromagnetic field obeys Maxwell's equations,

(3.33) $$d\mathbf{F} = 0$$

(3.34) $$d\,{}^{*}\mathbf{F} = 4\pi\, {}^{*}\mathbf{J} \quad .$$

The <u>current 1-form</u>, is given by $\mathbf{J} = \mu \mathbf{u}$, where μ is the electric charge density of the fluid to which the electromagnetic field is coupled. In this case the $\mathbf{F}$ field contributes to the stress-energy by an amount equal to

(3.35) $$T_{\alpha\beta} = -\frac{1}{4\pi}\left(F_{\alpha}{}^{\mu} F_{\mu\beta} + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\right).$$

For an expression of the stress-energy tensor of a non-perfect fluid see MISNER THORNE & WHEELER (1973) pp. 557, 568 and references cited therein. For the case of a scalar field see HAWKING & ELLIS (1973) pp. 67, 68.

4. SPATIALLY HOMOGENEOUS SPACETIMES
THE BIANCHI CLASSIFICATION

4.A. ISOMETRIES

The solution of Einsteins's equations is facilitated when the spacetime manifold is characterized by symmetric features. The particular case where spacetime is "invariant under a continuous group of isometries" will be central in our discussion of cosmological models, and for this reason we present here the basic concepts employed in the analysis of this kind of symmetry. By an <u>isometry</u> we mean a diffeomorphism $\phi : M \to M$, such that $\phi_* g$ is equal to g at every point of M. This implies that scalar products are preserved, since

$$(4.1) \qquad g(\phi_* u, \phi_* v)\big|_{\phi(p)} = \phi_* g(\phi_* u, \phi_* v)\big|_{\phi(p)} = g(u,v)\big|_p$$

for any pair of vectors u and v.

In section 2.E above, we saw that a vector field u on a manifold M induces locally a one-parameter family of diffeomorphisms. One can easily verify that this family constitutes a group. If this group is a group of isometries, then the vector field u is called a <u>Killing vector field</u>. The <u>equation of Killing</u>

$$(4.2) \qquad L_u g = 0 \quad \longleftrightarrow \quad u_{\alpha;\beta} + u_{\beta;\alpha} = 0$$

is a necessary and sufficient condition for u to be a Killing vector.

It now follows from (4.1) and (4.2) that for a Killing vector ξ, and any two vector fields $\mathbf{u}, \mathbf{v}$,

$$\xi(\mathbf{u}\cdot\mathbf{v}) = L_\xi(\mathbf{u}\cdot\mathbf{v}) = \mathbf{v}\cdot(L_\xi \mathbf{u}) + \mathbf{u}\cdot(L_\xi \mathbf{v}) \tag{4.3}$$

Suppose we choose a basis such that $\xi = \mathbf{e}_t$, one of the basis vectors, and take $\mathbf{u}, \mathbf{v}$ in (4.3) to be equal to the basis vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$, respectively. We will then obtain

$$\begin{aligned} \mathbf{e}_t(g_{\alpha\beta}) = g_{\alpha\beta,t} &= \mathbf{e}_\alpha\cdot[\mathbf{e}_t, \mathbf{e}_\beta] + \mathbf{e}_\beta\cdot[\mathbf{e}_t, \mathbf{e}_\alpha] = \\ &= C_{t\beta\alpha} + C_{t\alpha\beta} \end{aligned} \tag{4.4}$$

If we further demand that our vector basis is a coordinate one, then the structure coefficients $C_{t\beta\alpha}$ vanish, $g_{\alpha\beta,t} = \frac{\partial g_{\alpha\beta}}{\partial t}$ and (4.4) shows that, in that case, the metric is independent of the coordinate corresponding to the Killing vector.

4.B. HYPERSURFACES

Anticipating, again, application to cosmological problems we discuss here the geometric features of "hypersurfaces," as they relate to the geometry of the whole manifold. A <u>hyper-surface</u> of a manifold M is a $(n-1)$ dimensional manifold S imbedded in M. If (x^α) are coordinates of M, then we can locally find a function f, with $df \neq 0$, such that the imbedded submanifold consists of points of M for which

$$f(x^\alpha) = c \,, \tag{4.5}$$

where c is a constant.

The tangent space of S at its point p will be denoted by H_p. Obviously, H_p is a $(n-1)$ dimensional subspace of T_p. Then we can find a 1-form $n \in T_p^*$ such that $\langle n, v\rangle = 0$ for any $v \in H_p$. If f is the hypersurface forming function in the sense of (4.5), then we can take $n = df$, since

$$\langle df, v\rangle = v(f) = 0 \,, \tag{4.6}$$

as follows from the fact that f is constant along the hyper-surface.

Given a metric g on M, a metric $\bar{g}$ is induced on S, by restricting g to S. If n is spacelike $\bar{g}$ will be Lorentz and then the hypersurface S is called timelike. If n is null, $\bar{g}$ will be degenerate, and if n is timelike $\bar{g}$ is positive definite. In this case S is called spacelike.

When n is spacelike or timelike we can normalize it so that $n \cdot n = \pm 1$, respectively. Then the tensor

$$h = g \mp n \otimes n \tag{4.6}$$

is a projection operator projecting tensors from $T^s_r(p)$, into $H^s_r(p)$. In particular, it maps vectors of T_p into their parts in H_p , i.e. tangent to the hypersurface S , and similarly for one-forms. Thus any vector u can be written as

$$u = u_{\parallel} + u_{\perp} \tag{4.7}$$

where

$$u_{\parallel} = \pm (n \cdot u) n \quad , \quad u_{\perp} = \langle h, u \rangle . \tag{4.8}$$

In component form, the last expressions read

$$u_{\parallel}^{\alpha} = \pm n^{\beta} u_{\beta} n^{\alpha} , \quad u_{\perp}^{\alpha} = h^{\alpha}{}_{\beta} u^{\beta} . \tag{4.9}$$

Actually h is the induced metric on S. This is easily seen when we reduce vector fields in M along S , to their projections in the tangent space of S. As an illustration, consider the magnitude of the part of the vector V in H_p. It is given by

$$V_{\perp} \cdot V_{\perp} = g_{\alpha\beta} h^{\alpha}{}_{\gamma} h^{\beta}{}_{\delta} V^{\gamma} V^{\delta} = h_{\gamma\delta} V^{\gamma} V^{\delta} \tag{4.10}$$

The tensor

$$K_{\alpha\beta} = - h^{\gamma}{}_{\alpha} h^{\delta}{}_{\beta} n_{\gamma;\delta} \tag{4.11}$$

is called the <u>second fundamental form of S</u> for reasons to become clear presently.

Let us take a basis $(\mathbf{n}, \mathbf{e}_i)$ (= 1,2,3) such that $\mathbf{n}\cdot\mathbf{e}_i = 0$ and the $\mathbf{e}_i$ are tangent to the hypersurface. In this basis the metric of the hypersurface is given by

$$h^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} \mp (\mathbf{n}\cdot\mathbf{n})\delta^{n}{}_{\beta}\delta^{\alpha}{}_{n} = \delta^{\alpha}_{i}\delta^{j}_{\beta} \tag{4.12}$$

so that the only surviving components of $K_{\alpha\beta}$ are

$$K_{ij} = -\mathbf{e}_i\cdot\nabla_j\mathbf{n} \tag{4.13}$$

Since $\mathbf{e}_i\cdot\mathbf{n}=0$, we have

$$\mathbf{e}_i\cdot\nabla_j\mathbf{n} = -\mathbf{n}\cdot\nabla_j\mathbf{e}_i = -\mathbf{n}\cdot(\Gamma^{\alpha}{}_{ij}\mathbf{e}_{\alpha}) = -(\mathbf{n}\cdot\mathbf{n})\Gamma^{n}{}_{ij} \tag{4.14}$$

Thus (4.13) becomes

$$K_{ij} = (\mathbf{n}\cdot\mathbf{n})\Gamma^{n}{}_{ij} \quad . \tag{4.15}$$

Then

$$\nabla_j\mathbf{e}_i = \Gamma^{\alpha}{}_{ij}\mathbf{e}_{\alpha} = K_{ij}\frac{\mathbf{n}}{(\mathbf{n}\cdot\mathbf{n})} + \Gamma^{k}{}_{ij}\mathbf{e}_k \, . \tag{4.16}$$

We thus see that the covariant derivative along the hypersurface of vectors tangent to the hypersurface itself, is made up of two parts; one orthogonal and the other parallel to the hypersurface. As the parallel component, $\Gamma^{i}{}_{jk}$, is derivable from the metric h_{ij} of the hypersurface, it is an intrinsic quantity. It can be used to define parallel transport along the hypersurface and to compute its intrinsic curvature.

Equation (4.16), known as the Gauss-Weingarten equation, also delineates the significance of the second fundamental form K_{ij}. It shows that K_{ij} reflects the relation of the hypersurface to the manifold in which it is imbedded. This follows from the fact that K_{ij} measures the extent to which a vector parallelly transported along the hypersurface protrudes into the surrounding space.

With

$$ {}^{(3)}\Gamma_{ijk} \equiv \mathbf{e}_i \cdot \nabla_k \mathbf{e}_j \tag{4.17} $$

we find, using (2.51), that

$$ {}^{(3)}R^{\ell}{}_{ijk} = R^{\ell}{}_{ijk} - \frac{(K_{ij}K_k{}^{\ell} - K_{ik}K_j{}^{m})}{(n \cdot n)} \tag{4.18} $$

and

$$ R^{n}{}_{ijk} = -\frac{(K_{ij|k} - K_{ik|j})}{(n \cdot n)} \tag{4.19} $$

where $K_{ij|k}$ means covariant derivative with respect to ${}^{(3)}\Gamma_{ijk}$, and ${}^{(3)}R^{\ell}{}_{ijk}$ is the Riemann curvature tensor of the hypersurface. Equations (4.18) and (4.19) are known as the equations of Gauss and Codazzi. From (4.18) we see that

$$ {}^{(3)}R^{\ell}{}_{ijk} = R^{\ell}{}_{ijk} \tag{4.20} $$

when $K_{ij} = 0$, which justifies the second name of K_{ij} as the extrinsic curvature of the hypersurface.

4.C. SPATIALLY HOMOGENEOUS SPACETIMES

We saw in section 4.A. that a Killing vector field on a manifold generates a local one parameter group of isometries. If, now, the manifold admits r linearly independent Killing vector fields, one can show that these vector fields form an r-dimensional Lie algebra over the reals, with the algebra product being their commutation bracket. The local group of diffeomorphisms generated by these vector fields is an r-dimensional Lie group of isometries of the manifold.

Conversely, if the manifold is characterized by a continuous group of isometries we can form its Lie group. This allows us to study the effect of the group near the identity, i.e. locally. Besides a continuous group, the manifold might have a discrete group of isometries, like reflections.

The dimension r of the continuous group of isometries G_r is restricted by $r \leq \frac{n(n+1)}{2}$, where n is the dimension of the manifold. Actually, $r = \frac{n(n+1)}{2}$ if and only if the manifold has constant curvature. See EISENHART (1926) p. 239. Another restriction on G_r is that no isotropy group is allowed which is not a subgroup of the homogeneous Lorentz group. By an <u>isotropy group</u> or <u>group of stability</u> of a point $p \in M$, to be denoted by I_p, we mean a group of isometries that leaves p fixed. That I_p must be a subgroup of the homogeneous Lorentz group follows from the fact that at any point $p \in M$ we can choose coordinates such that $g_{\alpha\beta}(p) = \mathrm{diag}(-1,+1,+1,+1)$.

The points of the manifolds that can be mapped into each other by the action of G_r constitute a submanifold which we call the surface of transitivity or invariant variety of G_r. If the dimension of the group is the same as the dimension of the hypersufaces it generates, we say that G_r is simply transitive on these submanifolds.

We will call a spacetime manifold M spatially homogeneous if it is invariant under a G_r, such that the invariant varieties of G_r are spacelike hypersurfaces, on which a subgroup of G_r is simply transitive. Then, the only allowed values of r are $r = 3,6$. The invariant varieties of G_r will then be called surfaces of homogeneity or homogeneous hypersurfaces.

Given a spatially homogeneous spacetime we will take n to be the vector field normal to the hypersurfaces of homogeneity, as in section 4.B., with $n \cdot n = -1$. This field is easily shown to be both geodesic and rotation-free, i.e.

$$\nabla_n n = 0 \quad , \quad n \wedge dn = 0 \tag{4.21}$$

Then there exists a function $t(x^\alpha)$ such that

$$n = -dt \ . \tag{4.22}$$

We actually take $t = x^0$ and $e_0 = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$, $\sigma^0 = dt$, $e_0 \cdot e_0 = -1$

Let (ξ_i), (i =1,2,3) be the three Killing vectors of the 3-dimensional subgroup which is simply transitive on the hypersurface of homogeneity. Then

$$[\xi_i, \xi_j] = D_{ij}{}^k \xi_k \quad , \tag{4.23}$$

where the $D_{ij}{}^{k}$ are called the structure constants of the group. They define the type of group that generates the hypersurfaces of homogeneity.

At this point we further restrict our basis by choosing it to be invariant under the action of the group, i.e. we take

$$[\xi_i, e_\alpha] = 0 \tag{4.24}$$

Applying the Jacobi identity to ξ_i, e_α, e_β we find

$$\xi_i\left(C_{\alpha\beta}{}^{\gamma}\right) = 0 \tag{4.25}$$

which implies that

$$C_{\alpha\beta}{}^{\gamma} = C_{\alpha\beta}{}^{\gamma}(t) \, . \tag{4.26}$$

Next, we use the fact that (see 4.2)

$$L_{\xi_i} g = 0 \tag{4.27}$$

to find that

$$g_{\alpha\beta} = g_{\alpha\beta}(t) \, . \tag{4.28}$$

This implies that

$$g_{\alpha\beta,\gamma} = g_{\alpha\beta,\gamma}(t) \tag{4.29}$$

so that, using (4.26) and (2.64), we obtain

$$\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma}(t) \tag{4.30}$$

If we now choose our basis vectors (e_i) to be tangent to the surfaces of homogeneity, we can take them to be linear combinations of the ξ_i's, according to

$$\mathbf{e}_i = A_i{}^j \xi_j \quad . \tag{4.31}$$

Suppose, now, that $A_i{}^j = \delta_i^j$ at the point p of the hypersurface $t = t_0$. Then it follows from (4.24) that

$$\xi_j(A_i{}^k)\big|_p = D_{ij}{}^k \ . \tag{4.32}$$

On the other hand, it follows from

$$d\sigma^0 = -d^2t = 0 = -\tfrac{1}{2} C_{\kappa\rho}{}^0 \sigma^\kappa \wedge \sigma^\rho \ , \tag{4.33}$$

that

$$[\mathbf{e}_i, \mathbf{e}_j] = C_{ij}{}^k \mathbf{e}_k \ . \tag{4.34}$$

Using the last result and (4.32), we obtain

$$C_{ij}{}^k \big|_{t_0} = -D_{ij}{}^k \ . \tag{4.35}$$

This equation can be considered as an initial condition on the structure coefficients $C_{ij}{}^k(t)$ of the invariant basis. Our construction, however, has also imposed a development condition on the $C_{ij}{}^k$'s. To see this, we first write the $C_{ij}{}^k$'s in terms of a symmetric 3 X 3 matrix $\mathbf{m} = (m_{ij}(t))$ and a three-vector $a^i(t)$ as follows

$$C_{jk}{}^i = \epsilon_{jkl} m^{li} + \delta^i_k a_j - \delta^i_j a_k \ . \tag{4.36}$$

Then we apply the Jacobi identity on the vectors $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$, whereby we obtain

$$m_{ij}\, a^{j} = 0 \tag{4.37}$$

On the other hand, one can verify by direct differentiation that the quantity

$$h \equiv -\frac{2\, a^{i} a_{i}}{\left(\mathrm{Tr}\, m^{2} - (\mathrm{Tr}\, m)^{2}\right)} \quad , \qquad \mathrm{Tr} = \mathrm{Trace}, \tag{4.38}$$

is time-independent.

It follows from (4.35) that we can classify the groups under which the manifold is invariant by classifying the $C_{ij}{}^{k}$'s. To do it we employ m and a^{i}. First we split the groups in classes A and B according to if $a^{i}=0$ or not. Then we use the eigenvalues of m in order to distinguish the members of the two classes.

As we are free to rotate our basis vectors e_{i}, we choose them so that $a^{i}=(a,0,0)$ and $m = \mathrm{diag}(m_{1}, m_{2}, m_{3})$. Furthermore, we can rescale our basis so that on the hypersurface $a, m_{1}, m_{2}, m_{3} = \pm 1$, except when $a m_{2} m_{3} \neq 0$. In this case we can take $m_{2} m_{3} = \pm 1$, and then a is determined by invariance of h defined by (4.38). The values of the $C_{ij}{}^{k}$'s at $t=t_{0}$ are given in table II. This classification scheme is after Bianchi. The names of Behr and Schucking are also associated with it as they delineated the character of the possible groups at a deeper level than Bianchi.

TABLE 1

BEHR-SCHUCKING-BIANCHI CLASSIFICATION SCHEME

BIANCHI TYPE	GROUP TYPE	a	m_1	m_2	m_3	
I	I	0	0	0	0	
II	II	0	1	0	0	
VI	$VI_{h=0}$	0	1	-1	0	CLASS A
VII	$VII_{h=0}$	0	1	1	0	
VIII	VIII	0	1	1	-1	
IX	IX	0	1	1	1	
V	V	1	0	0	0	
IV	IV	1	0	0	1	CLASS B
VI (III if h=-1)	$VI_{h\neq 0}$	$(-h)^{\frac{1}{2}}$	0	1	-1	
VII	$VII_{h\neq 0}$	$h^{\frac{1}{2}}$	0	1	1	

At this point one is free to restrict the basis even further. If, for example, we demand that the $\mathbf{e}_i$ are Lie transported along $\mathbf{e}_0$, then

$$[\mathbf{e}_0, \mathbf{e}_i] = C_{0i}{}^{\kappa}\, \mathbf{e}_\kappa = 0 \quad , \tag{4.39}$$

and from (4.24) and (4.31) we obtain

$$C_{ij}{}^{k}(t) = C_{ij}{}^{k}\big|_{t_0} \quad . \tag{4.40}$$

In this case our metric reads

$$ds^2 = -\sigma^0 \otimes \sigma^0 + g_{ij}(t)\, \sigma^i \otimes \sigma^j \quad , \tag{4.41}$$

and the basis forms satisfy the equation

$$d\sigma^k = -\frac{1}{2} C_{ij}{}^{k}\big|_{t_0} \sigma^i \wedge \sigma^j = \frac{1}{2} D_{ij}{}^{k} \sigma^i \wedge \sigma^j \quad . \tag{4.42}$$

Another convenient basis to work with is one which is orthonormal everywhere. Let ($\mathbf{E}_\kappa$) be the vectors of such a basis, and ($\boldsymbol{\omega}^\kappa$) the one-forms which make up a basis dual to ($\mathbf{E}_\kappa$). Then the metric reads

$$ds^2 = -\boldsymbol{\omega}^0 \otimes \boldsymbol{\omega}^0 + \boldsymbol{\omega}^1 \otimes \boldsymbol{\omega}^1 + \boldsymbol{\omega}^2 \otimes \boldsymbol{\omega}^2 + \boldsymbol{\omega}^3 \otimes \boldsymbol{\omega}^3 \quad , \tag{4.43}$$

where $\boldsymbol{\omega}^0 = \sigma^0 = -dt$ and the $\mathbf{E}_i$ are tangent to the hypersurfaces of homogeneity. The usefulness of this basis is revealed in the following way. From

$$d\, g_{\kappa\beta} = 0 \quad , \tag{4.44}$$

it follows that (see (2.62), (2.63), (2.64), (2.51))

$$\Gamma_{\alpha\beta\gamma} = \Gamma_{\alpha\beta\gamma}(t) = -\Gamma_{\beta\alpha\gamma} = \frac{1}{2}(C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha}), \tag{4.45}$$

where the $C_{\alpha\beta\gamma}$'s are the structure coefficents of the $(\mathbf{E}_\alpha)$ basis. Then the Ricci tensor, given by,

$$R_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} - \Gamma^{\mu}{}_{\alpha\nu}\Gamma^{\nu}{}_{\beta\mu} + \Gamma^{\mu}{}_{\nu\mu}\Gamma^{\nu}{}_{\alpha\beta}, \tag{4.46}$$

involves only time derivatives of the $C_{\alpha\beta\gamma}$'s.

One can go back from the orthonormal basis ($\mathbf{E}_\alpha$) to the invariant basis ($\mathbf{e}_\alpha$) discussed earlier, by the transformation

$$\sigma^i = b^i{}_j(t)\omega^j \tag{4.47}$$

where (b_{ij}) is a symmetric nonsigular matrix. The substitution of (4.47) in (4.43) shows that the latter transforms into the metric (4.41) if

$$b^\ell{}_i\, g_{ij} b^j{}_k = \delta^\ell_k \quad . \tag{4.48}$$

Having determined the transformation matrix (b_{ij}) one can go back and forth from the invariant basis ($\mathbf{e}_\alpha$) to the orthonormal one ($\mathbf{E}_\alpha$) so as to take advantage of the convenient features of both. This will be exploited in our derivation of exact solutions of Einstein's equations in the case where the spacetime manifold is invariant under a Bianchi type IV group of isometries.

5. OBSERVATIONAL COSMOLOGY - FRIEDMANN MODELS

Cosmology views the universe as a historical entity, i.e. something that is always in a state of flux. Whatever features characterize the universe today are the endproducts of a long process of evolution, and only in the context of this process do such features obtain any meaning. Here we will describe, in outline form, some observations which are considered cosmological in exactly the above sense, namely, that they are indicative of the present state of the universe as a whole and the process that led to such a state. Furthermore, we will illustrate how a specific family of spacetime models based on the Riemann-Einstein theory can adequately account for the cosmological observations referred to above.

The basic cosmological observations that one tries to understand are the following.

(a) Matter in the universe is mainly agglomerated into luminous bodies with a density of the order of 10^{-31} g/cm^3 .

(b) When the observable universe is subdivided into cells of linear extent one order of magnitude greater than the linear extent of clusters of galaxies ($\sim 3\times 10^7$ light years), then matter distribution seems to be homogeneous.

(c) Galaxies recede from each other with a velocity proportional to their separation.

(d) The earth is constantly irradiated by a microwave radiation of extragalactic origin which is isotropic.

The fact that galaxies recede from each other was deduced by Edwin Hubble in 1929, on the basis of redshift data from distant galaxies. He found that the spectra of light received from such galaxies were redshifted by an amount proportional to their distance from our own galaxy, i.e.

$$z = H_0 \ell \tag{5.1}$$

where ℓ is the distance to the source galaxy. The constant H_0, called the Hubble constant, is currently estimated to be

$$H_0^{-1} = (18 \pm 2) \times 10^9 \text{ years,} \tag{5.2}$$

where the error indicated is purely statistical, as systematic errors are poorly understood today.

The microwave background discovered by Penzias and Wilson in 1965 has been found to have a thermal distribution with Planck temperature $T \sim 2.7°K$, besides being highly isotropic ($\Delta T/T \leq 0.004$). Not only is this radiation extragalactic, but it seems that it has not been scattered for about 10^9 years, neither seems it probable that it emanates from discrete sources placed at such distances from our own galaxy. (For a thorough description and analysis of the "cosmic" radiation and its implications see HAWKINS & ELLIS (1973) pp. 348-359 and references cited there.)

The observations mentioned so far can be accomodated in a non-contradictory fashion in a model which pictures the universe as having developed into its current phase via a monotonic expansion of all its parts starting with an extremely dense and hot state. The expansion of the universe was accompanied by a continuous reduction in its temperature. When the temperature dropped to a few thousand degrees Kelvin, the plasma "recombined" to hydrogen atoms, and the photon gas, which up to that point was in thermal equilibrium with matter, decoupled from matter and continued its existence in a thermalized state. Matter, on the other hand, developed local inhomogeneities from which galaxies formed. Thus, according to this model, over large enough scales matter has remained homogeneously distributed, as we observe today, and the photon gas, having been redshifted, is observed today at a temperature $\sim 2.7^\circ K$.

The sequence of phases described in the above model agrees with an exact solution of Einstein's equations obtained by Alexander Friedmann in 1922. Friedmann found that the field equations admit a solution in which the stress energy tensor has the perfect fluid form with $p = p(t)$ and $\rho = \rho(t)$, while the metric of spacetime is given by

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}\,dx^i dx^j, \tag{5.3}$$
$$\gamma_{ij}\,dx^i dx^j = d\chi^2 + \Sigma^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where

$$\Sigma = \sin\chi, \text{ or } \chi, \text{ or } \sinh\chi. \tag{5.4}$$

In this solution, the functions $a(t)$, $\rho(t)$, $p(t)$, are determined from the following equations $(\dot{a} \equiv da/dt)$

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \frac{\Lambda}{3} + \frac{8\pi}{3}\rho \tag{5.5}$$
$$\frac{\ddot{a}}{a} = -\left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} + \Lambda - 8\pi p \quad ,$$

once an equation of state is given. Equations (5.5) are the only two field equations not identically satisfied by (5.3), and the constant k takes the values 1, 0, -1 as $\Sigma = \sin\chi, \chi, \sinh\chi$ respectively. Obviously, we are dealing here with a family of models and not only one. When k = 1, 0,-1 the models are correspondingly called closed, flat, and open.

The following features make the metric (5.3) appropriate for a cosmological model. Firstly, the coordinate t defines a family of spacelike hypersurfaces, with the timelike vector $\frac{\partial}{\partial t}$ normal to these hypersurfaces. Secondly, we have a pair of scalar functions $p(t)$ and $\rho(t)$, which are defined at every point of these hypersurfaces. Thus, we can take the hypersurfaces determined by t as a one parameter family of configurations of the uneverse, with $p(t)$ and $\rho(t)$ giving the pressure and density of matter-energy in the universe at a given "cosmic instant" t = constant. The latter follows from the fact that, as already mentioned, the metric (5.3) gives a solution of Einstein's equations with a stress-energy tensor where p and ρ figure as the pressure and density of a perfect fluid.

That the model described above can also account for the observations described earlier is shown from the following considerations. First, there is a subset of the family of Friedmann models which, starting with a vanishing volume at some finite value of t, expand isotropically in all directions, until they finally either recollapse to a singularity or keep expanding forever. (For a description of the time-development of all Friedmann universes, see MISNER, THORNE & WHEELER (1973) pp. 746-747.) Since these Friedmann models are also solutions of Einstein's equations for both matter and thermal radiation "filling," (see COHEN (1967B),) they describe the type of universe mentioned earlier, where matter and radiation start from infinite temperatures and densities which keep decreasing as the universe expands.

The observed recession of galaxies from each other, as reflected in the Hubble redshift-distance relation (5.1), is incorporated in the Friedmann models in the following fashion. Consider an observer who at $t = t_0$ is at the origin $\chi = 0$ of the coordinate system used in (5.3). At that instant, he will be receiving radiation from a "ring" of radiant particles located at $\chi = \chi_e$ at an earlier time $t = t_e$ with a redshift

(5.6) $$z = \frac{a(t_0)}{a(t_e)} - 1 \quad ,$$

(see MISNER, THORNE & WHEELER (1973), p. 778)
The spacelike separation between the observer and the radiant particles is obtained directly from the line element (5.3). It reads

(5.7) $$\ell = a(t_0)\chi_e = a(t_0)\int_{t_e}^{t_0} \frac{1}{a(t)}\, dt \, .$$

Using this result for small times of travel for the radiation, so that

(5.8) $$a(t_e) \sim a(t_0) + \dot{a}(t_0)(t_e - t_0) \, ,$$

we obtain from (5.6)

$$z \sim \frac{\dot{a}(t_0)}{a(t_0)}\, \ell \,. \tag{5.9}$$

Comparing with (5.1) we conclude that the model accounts the observed red-shift distance relation, if we take

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \,. \tag{5.10}$$

From (5.6) we see that the redshift is independent of the angular position of the emitter relative to the observer at the origin, i.e. the observed redshift is isotropic. In the same fashion one can account for the isotropic nature of the observed $2.7°K$ microwave background. If we assume that at $t=t_e$ the universe was filled with a photon gas of temperature $T_e \sim 1{,}000°K$ then the currently observed temperature $T_0 \sim 2.7°K$ results from the fact that $a(t)T(t) =$ constant, as one can easily show, if we assume that the universe has expanded since t_e by a factor of $a(t_0)/a(t_e) \sim 1{,}000$.

We return now to the geometry of the Friedmann models in order to determine its connection to the "spatially homogeneous spacetimes" considered in section 4.C, as our later discussion will concentrate on a particular class of such models, the Bianchi type IV. It has been determined that the t = constant hypersurfaces of the Friedmann models are invariant under a 3-parameter group of isometries, which belongs to one of the classes shown in Table 1, according to the following scheme:

TABLE II. SPATIALLY HOMOGENEOUS MODELS CONTAINING FRIEDMANN SOLUTIONS

GROUP TYPE	FRIEDMANN MODEL
IX	$k=+1$
I	$k=0$
$VII_{h=0}$	$k=0$
V	$k=-1$
$VII_{h\neq 0}$	$k=-1$

It is the invariance under this group that makes the Friedmann models "homogeneous." This means that the observations of an observer comoving with the hypersurfaces of the model are independent of his position on a given hypersurface. On this basis, it has been accepted by most cosmologists to call a model universe spatially homogenous if its underlying geometry is that of a "spatially homogenous spacetime," in the sense of section 4.C. (For a detailed discussion of the criteria of spatial homogeneity see MACCALLUM (1972A).)

What distinguishes the Friedmann models from the spatially homogenous ones is their higher symmetry. They are invariant under a six parameter group of isometries, with a three dimensional subgroup simply transitive on spacelike hypersurfaces. The remaining three dimensions of the grOup represent the invariance of the models under rotations in the hypersurfaces of homogeneity. This subgroup is responsible for the isotropic observations of comoving observers in a Friedmann model, as described earlier. Robertson and Walker proved that the Friedmann models exhaust the whole class of spacetimes that are both spatially homogeneous and isotropic. For this reason, spacetimes which are only spatially homogeneous are also called non-isotropic.

The study of spatially homogeneous non-isotropic models was initiated for two basic reasons. The first was the singularity encountered in the past of all the Friedmann models which account for the currently available cosmological data. In all these models the density of energy content becomes infinite at some finite time in the past, approximately $H_0^{-1} \approx 18 \times 10^9$ years ago, while the spacetime manifold itself becomes singular at the same instant, i.e. all invariant quantities defined in terms of the manifold's Riemann curvature tensor blow up. The discussion of the physics of a Friedmann universe near the singularity demands a wild extrapolation of physical laws, as we know them, to regimes where energy densities exceed anything we have ever encountered before, even in the area of astrophysics. Worse than that, we have no way at all of talking physics at a point where the spacetime manifold itself has become singular. Most cosmologists have accepted this initial singularity as the beginning of the universe. The writer contends that not only is such an interpretation totally unjustifiable from the standpoint of physics, but it also runs against the foundations of the Riemann-Einstein theory. Anyhow, the hope was that non-isotropic models might escape such singularities.

The second reason for studying non-isotropic models was the need to determine the extent to which the Friedmann models were stable against non-isotropic perturbations. It was of interest to know if anisotropies existing at some phase in the models' development would damp out so as to give rise to a universe as highly isotropic and homogeneous in the large as observations indicate to be the case today.

As far as the issue of the damping of anisotropies is concerned, we want to mention here that there is no generally accepted interpretation of what isotropization means. Some cosmologists interpret it to mean that the universe approaches asymptotically a regime where expansion occurs uniformly in all directions and is described by Hubble's law. See DOROSHKEVICH, LUKASH, & NOVIKOV, I.D. (1973) and NOVIKOV, I.D. (1974). Others, like Collins and Hawking, by isotropization mean a process whereby the model approaches the Friedmann solution for large times. The latter have shown that the set of spatially homogeneous models that are characterized by such an asymptotic behavior is of measure zero. In other words, only particular initial conditions give rise to models that approach the Friedmann models asymptotically, or, equivalently, the Friedmann models are unstable against anisotropic (as well as inhomogeneous) perturbations. See COLLINS & HAWKING (1973B), and HAWKING (1974).

In terms of the initial singularity, things turned out differently. Non-isotropic models have been shown to exhibit a very wide variety of behavior, some examples of which will be discussed in terms of our later analysis of Bianchi IV models. It should be mentioned, however, that all realistic models of the universe based on the Riemann-Einstein theory are singular in the following sense. In such models some geodesics are incomplete, i.e. cannot be extended to all values of their affine parameter. As a result of this, the ways in which an anisotropic model can avoid the Friedmann-like singularities are very limited. For details on the issue of singularities see HAWKING & ELLIS (1973) pp. 256-298, ELLIS & KING (1974), ELLIS (1975), COLLINS (1974).

6. BIANCHI IV MODELS

6.A. THE GENERAL PROPERTIES OF A PERFECT FLUID MODEL

We now consider space-time models, whose spacelike surfaces of homogeneity are generated by a group of Bianchi type IV. Let $\sigma^i (i = 1,2,3)$ be the one-forms dual to the invariant basis $(\mathbf{e}_i)$ in the sense of section 4.C. Then,

$$d\sigma^i = D_{jk}{}^i \, \sigma^j \wedge \sigma^k \tag{6.1}$$

where $D_{jk}{}^i$ are the structure constants of the group. For the Bianchi IV group, the only surviving constants are $D_{12}{}^2 = -D_{21}{}^2 = D_{12}{}^3 = -D_{21}{}^3 = D_{13}{}^3 = -D_{31}{}^3 = 1$, where the invariant basis was chosen as specified in section 4.C above. Equivalently, we have

$$\begin{aligned} d\sigma^1 &= 0 \\ d\sigma^2 &= \sigma^1 \wedge \sigma^2 \\ d\sigma^3 &= \sigma^1 \wedge \sigma^2 + \sigma^1 \wedge \sigma^3 \end{aligned} \tag{6.2}$$

One can easily find a coordinate basis where the above invariant relations are satisfied. A particular choice for the Bianchi IV group is

$$\begin{aligned} \sigma^1 &= dx^1 \\ \sigma^2 &= e^{x^1} dx^2 \\ \sigma^3 &= x^1 e^{x^1} dx^2 + e^{x^1} dx^3 \end{aligned} \tag{6.3}$$

where (x^1, x^2, x^3) are Cartesian coordinates.

It now follows from section 4.C above, that the metric tensor will read

(6.4) $$ds^2 = -dt \otimes dt + g_{ij}\,\sigma^i \otimes \sigma^j$$

where $g_{ij} = g_{ij}(t) = g_{ji}$.

We choose g_{ij} as follows

(6.5) $$g_{ij} = \begin{pmatrix} a^2 + e^2b^2 + f^2c^2 & & \\ eb^2 + fgc^2 & b^2 + g^2c^2 & \\ fc^2 & gc^2 & c^2 \end{pmatrix}$$

where a,b,c,e,f,g are functions of time only.

Next, through a time-dependent transformation, we find a new set of basis one-forms, in which the metric tensor (6.4) is diagonal. The appropriate transformation is:

(6.6) $$\begin{aligned} \omega^0 &= dt \\ \omega^1 &= a\sigma^1 \\ \omega^2 &= b\,(e\sigma^1 + \sigma^2) \\ \omega^3 &= c\,(f\sigma^1 + g\sigma^2 + \sigma^3) \end{aligned}$$.

Thus, our metric will read

(6.7) $$ds^2 = -\omega^0 \otimes \omega^0 + \omega^i \otimes \omega^i$$.

The inverse of the transformation (6.6) is

(6.6I) $$\begin{aligned} \sigma^1 &= \frac{1}{a}\,\omega^1 \\ \sigma^2 &= -\frac{e}{a}\,\omega^1 + \frac{1}{b}\,\omega^2 \\ \sigma^3 &= \frac{eg-f}{a}\,\omega^1 - \frac{g}{b}\,\omega^2 + \frac{1}{c}\,\omega^3 \end{aligned}$$.

Employing equations (6.2), (6.6) and (6.6I), we find the exterior derivatives of the ω^i. They are

(6.8)
$$
\begin{aligned}
d\omega^0 &= 0 \\
d\omega^1 &= \frac{\dot a}{a}\,\omega^0\wedge\omega^1 \\
d\omega^2 &= \frac{\dot e b}{a}\,\omega^0\wedge\omega^1 + \frac{\dot b}{b}\,\omega^0\wedge\omega^2 + \frac{1}{a}\,\omega^1\wedge\omega^2 \\
d\omega^3 &= \frac{c}{a}(\dot f - e\dot g)\,\omega^0\wedge\omega^1 + \frac{\dot g c}{b}\,\omega^0\wedge\omega^2 + \frac{\dot c}{c}\,\omega^0\wedge\omega^3 \\
&\quad + \frac{c}{ab}\,\omega^1\wedge\omega^2
\end{aligned}
$$

where the dot indicates differentiation with respect to time. Comparing equations (6.8) with the relationship (see (2.29))

$$d\omega^\alpha = -\frac{1}{2} C_{\beta\gamma}{}^{\alpha}\,\omega^\beta\wedge\omega^\gamma$$

we obtain the commutation coefficients for the ω^α—basis . They read:

(6.9)
$$
\begin{aligned}
&C_{011} = -\frac{\dot a}{a} \\
&C_{012} = -\frac{\dot e b}{a}\,,\quad C_{022} = -\frac{\dot b}{b}\,,\quad C_{122} = -\frac{1}{a}\,, \\
&C_{013} = -\frac{c}{a}(\dot f - e\dot g)\,,\quad C_{023} = -\frac{\dot g c}{b}\,,\quad C_{033} = -\frac{\dot c}{c}\,, \\
&C_{123} = -\frac{c}{ab}\,,\quad C_{133} = -\frac{1}{a}\,.
\end{aligned}
$$

All others, except those obtainable from the above by the relationship $C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma}$, vanish.

Since we are using an orthonormal basis, where the metric tensor has the Minkowski form $g_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$, it follows from (4 .45) that the connection coefficients are given by

(6.10)
$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha}\right) .$$

The nonvanishing connection coefficients are:

$$\Gamma_{011} = -\frac{\dot a}{a}\,, \quad \Gamma_{012} = \Gamma_{021} = -\Gamma_{120} = -\frac{\dot e b}{2a}\,,$$

(6.11)
$$\Gamma_{013} = \Gamma_{031} = -\Gamma_{130} = -\frac{c(\dot f - e\dot g)}{2a}\,, \quad \Gamma_{022} = -\frac{\dot b}{b}\,,$$

$$\Gamma_{023} = \Gamma_{032} = -\Gamma_{230} = -\frac{\dot g c}{2b}\,, \quad \Gamma_{033} = -\frac{\dot c}{c}\,,$$

$$\Gamma_{122} = -\frac{1}{a}\,, \quad \Gamma_{123} = \Gamma_{132} = -\Gamma_{231} = -\frac{c}{2ab}\,,$$

$$\Gamma_{133} = -\frac{1}{a}\,,$$

as well as those obtainable from the above using the relationship $\Gamma_{\alpha\beta\gamma} = -\Gamma_{\beta\alpha\gamma}$.

The components of the Ricci tensor are obtained from the connection coefficients through the relation (see (4 46)).

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu}\,.$$

They are:

(6.12)
$$R_{00} = -\frac{\ddot a}{a} - \frac{\ddot b}{b} - \frac{\ddot c}{c} - \frac{\dot e^2 b^2}{2a^2} - \frac{c^2(\dot f - e\dot g)^2}{2a^2} - \frac{\dot g^2 c^2}{2b^2}$$

$$R_{01} = \frac{1}{a}\left(2\frac{\dot a}{a} - \frac{\dot b}{b} - \frac{\dot c}{c}\right) - \frac{\dot g c^2}{2ab^2}$$

$$R_{02} = \frac{3\dot e b}{2a^2} + \frac{c^2(\dot f - e\dot g)}{2a^2 b}$$

$$R_{03} = \frac{3c(\dot f - e\dot g)}{2a}$$

$$R_{11} = \frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot b}{b} + \frac{\dot c}{c}\right) - \frac{2}{a^2} - \frac{c^2}{2a^2b^2} + \frac{\dot e^2 b^2}{2a^2} + \frac{c^2(\dot f - e\dot g)^2}{2a^2}$$

$$R_{22} = \frac{\ddot b}{b} + \frac{\dot b}{b}\left(\frac{\dot a}{a} + \frac{\dot c}{c}\right) - \frac{2}{a^2} - \frac{c^2}{2a^2b^2} - \frac{\dot e^2 b^2}{2a^2} + \frac{\dot g^2 c^2}{2b^2}$$

$$R_{33} = \frac{\ddot c}{c} + \frac{\dot c}{c}\left(\frac{\dot a}{a} + \frac{\dot b}{b}\right) - \frac{2}{a^2} + \frac{c^2}{2a^2b^2} - \frac{c^2(\dot f - e\dot g)^2}{2a^2} - \frac{\dot g^2 c^2}{2b^2}$$

$$R_{12} = \left(\frac{b\dot e}{2a}\right)^{\cdot} + \frac{b\dot e}{2a}\left(2\frac{\dot b}{b} + \frac{\dot c}{c}\right) + \frac{\dot g c^2(\dot f - e\dot g)}{2ab}$$

$$R_{13} = \left(\frac{c(\dot f - e\dot g)}{2a}\right)^{\cdot} + \frac{c(\dot f - e\dot g)}{2a}\left(\frac{\dot b}{b} + 2\frac{\dot c}{c}\right)$$

$$R_{23} = \left(\frac{\dot g c}{2b}\right)^{\cdot} + \left(\frac{\dot g c}{2b}\right)\left(\frac{\dot a}{a} + 2\frac{\dot c}{c}\right) - \frac{\dot e b c(\dot f - e\dot g)}{2a^2} - \frac{c}{a^2 b}$$

We first seek solutions of Einstein's equations

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) + g_{\mu\nu}\Lambda$$

with $R_{\mu\nu}$ given by (6.12), when the source is a perfect fluid, i.e.

$$T_{\mu\nu} = (\rho+p)u_\mu u_\nu + p g_{\mu\nu} \ ,$$

where ρ is the mass-energy density, p the pressure, and u_μ the velocity of the fluid, for which we have $u_\mu u^\mu = -1$. Under these assumptions the field equation may be rewritten in the following form

$$R_{\mu\nu} = (\rho+p)u_\mu u_\nu + \frac{1}{2}(\rho-p)g_{\mu\nu} + \Lambda g_{\mu\nu} \ . \tag{6.13}$$

Since none of the Ricci tensor components vanishes identically, we must assume the fluid velocity one-form is

$$u = u_0\omega^0 + u_1\omega^1 + u_2\omega^2 + u_3\omega^3 \tag{6.14}$$

This means that, in general, the fluid is not hypersurface orthogonal, having instead velocity components along the hypersurface itself. Otherwise, we are dealing with a "tilted" cosmological model. Since a tilted matter velocity is a necessary but not sufficient condition for the fluid congruence to rotate, (see KING & ELLIS (1973)) we go on to calculate its rotation in an invariant manner. In section (3.22) above, we saw that the rotation one-form is given by

$$\Omega = \frac{1}{2} {}^*(u \wedge du)$$

Since

$$\begin{aligned} du = d(u_\alpha \omega^\alpha) &= \dot{u}_\alpha \omega^0\wedge\omega^\alpha + u_\alpha d\omega^\alpha = \\ &= \dot{u}_\alpha \omega^0\wedge\omega^\alpha - \frac{1}{2} C_{\beta\gamma}{}^\alpha u_\alpha \omega^\beta\wedge\omega^\gamma \ , \end{aligned} \tag{6.15}$$

we use equations (4.9) to obtain:

$$(6.16)\quad 2\Omega = -\frac{c}{ab}u_3^2\,\omega^0 + \left(\frac{\dot c}{c}u_2 - \frac{\dot b}{b}u_2 - \frac{\dot g c}{b}u_3\right)u_3\,\omega^1 + \left(\frac{1}{a}u_0 + \frac{\dot a}{a}u_1 - \frac{\dot c}{c}u_1 + \frac{\dot e b}{a}u_2 + \frac{c}{a}(\dot f - e\dot g)u_3\right)u_3\,\omega^2 + \left\{\left(-\frac{c}{ab}u_0 + \frac{\dot g c}{b}u_1 - \frac{c}{a}(\dot f - e\dot g)u_2\right)u_3 + \left(-\frac{1}{a}u_0 - \frac{\dot a}{a}u_1 + \frac{\dot b}{b}u_1 - \frac{\dot e b}{a}u_2\right)u_2\right\}\omega^3.$$

We notice immediately that, when $u_3 = 0$, then

$$2\Omega = \left(-\frac{1}{a}u_0 - \frac{\dot a}{a}u_1 + \frac{\dot b}{b}u_1 - \frac{\dot e b}{a}u_2\right)u_2\,\omega^3,$$

i.e. the rotation vector is parallel to m, while if we further have $u_2 = 0$, then $\Omega = 0$. This shows that our model explicitly exhibits the general property of class B space-times of having null rotation when the fluid velocity projection along the hypersurface of homogeneity (here u_1) is parallel to a^i (here (1,0,0)). See KING & ELLIS (1973). This, in other words, is an explicit case where tilt does not imply rotation.

It follows from the field equations that the stress energy tensor must satisfy the so-called "conservation equations."

$$(6.17)\qquad T^{\alpha\beta}{}_{;\beta} = 0$$

For our model, these equations read

$$(6.18)\quad T^{00}{}_{,0} + \frac{\dot a}{a}T^{11} + \frac{\dot b}{b}T^{22} + \frac{\dot c}{c}T^{33} + \frac{\dot e b}{a}T^{12} + \frac{c}{a}(\dot f - e\dot g)T^{13} + \frac{\dot g c}{b}T^{23} + \frac{2}{a}T^{01} + \left(\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c}\right)T^{00} = 0,$$

$$T^{01}{}_{;0} + \frac{\dot{e}b}{a} T^{02} + \frac{c}{a}\left(\dot{f} - e\dot{g}\right) T^{03} + \frac{2}{a} T^{11} - \frac{1}{a} T^{22}$$

$$- \frac{1}{a} T^{33} - \frac{c}{ab} T^{23} + \left(2\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) T^{01} = 0$$

$$T^{02}{}_{;0} + \frac{\dot{g}c}{b} T^{03} + \frac{3}{a} T^{12} + \frac{c}{ab} T^{13} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) T^{02} = 0$$

$$T^{03}{}_{;0} + \frac{3}{a} T^{13} + \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + 2\frac{\dot{c}}{c}\right) T^{03} = 0 \quad .$$

6.B. AN ANALYTICAL, GLOBAL SOLUTION

We now consider the tilted but nonrotating case. That is, we take $u_2 = u_3 = 0$, whereupon the field equations in the form (6.13) become, using (6.12),

$$R_{03} = \frac{3c(\dot f - e\dot g)}{2a^2} = 0 \tag{6.19}$$

$$R_{02} = \frac{3\dot e b}{2a^2} + \frac{c^2(\dot f - e\dot g)}{2a^2 b} = 0 \ ;$$

This implies that

$$\dot e = 0 = \dot f - e\dot g \tag{6.20}$$

so that

$$e = \text{Constant}, \quad f = eg + \text{Constant} \quad .$$

For convenience we let the above constants vanish since, they could be easily absorbed in the basis one forms. The rest of the field equations now read:

$$R_{00} = -\frac{\ddot a}{a} - \frac{\ddot b}{b} - \frac{\ddot c}{c} - \frac{\dot g^2 c^2}{2b^2} = (\rho + p)u_0^2 - \frac{1}{2}(\rho - p) - \Lambda \ , \tag{6.21}$$

$$R_{01} = \frac{1}{a}\left(2\frac{\dot a}{a} - \frac{\dot b}{b} - \frac{\dot c}{c} - \frac{\dot g c^2}{2b^2}\right) = (\rho + p)u_0 u_1$$

$$R_{11} = \frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot b}{b} + \frac{\dot c}{c}\right) - \frac{2}{a^2} - \frac{c^2}{2a^2b^2} = (\rho + p)u_1^2 + \frac{1}{2}(\rho - p) + \Lambda$$

$$R_{22} = \frac{\ddot b}{b} + \frac{\dot b}{b}\left(\frac{\dot a}{a} + \frac{\dot c}{c}\right) - \frac{2}{a^2} - \frac{c^2}{2a^2b^2} = \frac{1}{2}(\rho - p) + \Lambda$$

$$R_{33} = \frac{\ddot c}{c} + \frac{\dot c}{c}\left(\frac{\dot a}{a} + \frac{\dot b}{b}\right) - \frac{2}{a^2} + \frac{c^2}{2a^2b^2} + \frac{\dot g^2 c^2}{2b^2} = \frac{1}{2}(\rho - p) + \Lambda$$

$$R_{23} = \left(\frac{\dot g c}{2b}\right)^{\cdot} + \frac{\dot g c}{2b}\cdot\left(\frac{\dot a}{a} + \frac{2\dot c}{c}\right) - \frac{c}{a^2 b} = 0 \ ,$$

Correspondingly, the conservation equations (6.18) become

$$
\begin{aligned}
(6.22)\qquad & \left((\rho+p)u^{0\,2}-p\right)^{\cdot}+\frac{\dot a}{a}\left((\rho+p)u'^{2}+p\right)+\frac{\dot b}{b}p+\frac{\dot c}{c}p+ \\
& \qquad +\left(\frac{\dot a}{a}+\frac{\dot b}{b}+\frac{\dot c}{c}\right)\left((\rho+p)u^{0\,2}-p\right)+\frac{2}{a}(\rho+p)u^{0}u'=0 \\
& \left((\rho+p)u^{0}u'\right)^{\cdot}+\left(2\frac{\dot a}{a}+\frac{\dot b}{b}+\frac{\dot c}{c}\right)(\rho+p)u^{0}u'+ \\
& \qquad +\frac{2}{a}(\rho+p)u'^{2}=0 \quad .
\end{aligned}
$$

After some algebra, the last two equations give

$$
\begin{aligned}
(6.23)\qquad & \frac{\dot\rho}{\rho+p}+\frac{\dot u^{0}}{u^{0}}+\frac{\dot a}{a}+\frac{\dot b}{b}+\frac{\dot c}{c}=-\frac{2u'}{au^{0}} \\
& \frac{\dot p}{\rho+p}+\frac{\dot u'}{u'}+\frac{\dot a}{a}=0 \quad .
\end{aligned}
$$

We now define the functions w and r through

$$
(6.24)\qquad \frac{dw}{w}=\frac{d\rho}{\rho+p}\;,\quad \frac{dr}{r}=\frac{dp}{\rho+p}\;,
$$

and the hyperbolic angle of tilt through

$$
(6.25)\qquad u^{0}=\cosh\beta\;,\quad u'=\sinh\beta
$$

Then, the conservation equations in the form (6.23) can be rewritten as:

$$
\begin{aligned}
(6.26)\qquad & (\ln wabc\cosh\beta)^{\cdot}=-\frac{2\tanh\beta}{a} \\
& (\ln ra\sinh\beta)^{\cdot}=0 \quad .
\end{aligned}
$$

The form of the field equations (6.21) suggests that a considerable simplification occurs when $b=c$. We do, therefore, demand that $b=c$ at all times, and then we obtain

$$R_{22} - R_{33} = -\frac{1}{a^2} + \dot{g}^2 = 0 \ . \tag{6.27}$$

Thus,

$$\dot{g} = \pm \frac{1}{a} \ . \tag{6.28}$$

Substitution of this into the last of equations (6.21) yields

$$\frac{\dot{b}}{b} = \pm \frac{1}{a} \tag{6.29}$$

Substituting equations (6.28) and (6.29) into either the R_{22} or the R_{33} equation of the (6.21) set, we obtain

$$R_{22} = 0 = \frac{1}{2}(\rho - p) + \Lambda \ . \tag{6.30}$$

The remaining field equations are thus reduced to

$$\begin{aligned} R_{00} &= -\frac{\ddot{a}}{a} \pm 2\frac{\dot{a}}{a^2} - \frac{5}{2a^2} = (\rho+p)u_0^2 \\ R_{01} &= \mp 2\frac{\dot{a}}{a^2} + \frac{5}{2a^2} = (\rho+p)u_0 u_1 \\ R_{11} &= \frac{\ddot{a}}{a} \pm \frac{2\dot{a}}{a^2} - \frac{5}{2a^2} = (\rho+p)u_1^2 \ . \end{aligned} \tag{6.31}$$

From these equations it follows that

$$R_{00} + 2R_{01} + R_{11} = 0 = (\rho+p)(u_0^2 + 2u_0 u_1 + u_1^2) \tag{6.32}$$

which implies that either $(\rho+p)=0$ or $u_0 = -u_1$. The latter possibility is, however, unacceptable since it violates the constraint $u^\alpha u_\alpha = -1$. Thus, we must take $\rho + p = 0$, which implies that $\rho = p = 0$, since only positive values of ρ and p

are permitted. Then, equations (6.31) yield

$$\alpha = \pm \frac{5}{4} \quad . \tag{6.33}$$

Combining equations (6.28), (6.29) and (6.33), we obtain the following closed form for the metric coefficients:

$$\begin{aligned} a &= \pm \tfrac{5}{4} t + k \\ b = c &= b_0 \left(\pm \tfrac{5}{4} t + k\right)^{4/5} \\ g &= \ln g_0 \left(\pm \tfrac{5}{4} t + k\right)^{4/5} \end{aligned} \tag{6.34}$$

where k, b_0, g_0 are constants of integration. The constant k sets the time of the singularity, and, therefore, can be set equal to zero for convenience.

Thus, the metric reads

$$\begin{aligned} ds^2 = -dt^2 + \left(\tfrac{5e}{4} t\right)^2 \sigma^1 \otimes \sigma^1 + \left(\tfrac{5e}{4} t\right)^{8/5} \Big\{ \Big(1 + \\ + \ln^2 \left(\tfrac{5e}{4} t\right)^{4/5} \Big) \sigma^2 \otimes \sigma^2 + \left(\ln \tfrac{5e}{4} t\right)^{4/5} \left(\sigma^2 \otimes \sigma^3 + \sigma^3 \otimes \sigma^2\right) + \\ + \sigma^3 \otimes \sigma^3 \Big\} \quad . \end{aligned} \tag{6.35}$$

where we take $e = +1$ for $t > 0$, and $e = -1$ for $t < 0$.

Obviously, the metric is singular at $t = 0$, and we want to know if this is a true singularity, or simply a coordinate one. Also, we want to obtain information about the geometry of the hypersurfaces of homogeneity. Using equations (2.57) we find that

$$\begin{aligned} R_{1212} &= \frac{\dot a \dot b}{ab} - \frac{1}{a^2} - \frac{3c^2}{4a^2b^2} \\ R_{1313} &= \frac{\dot a \dot c}{ac} - \frac{1}{a^2} + \frac{c^2}{4a^2b^2} \\ R_{2323} &= \frac{\dot b \dot c}{bc} - \frac{1}{a^2} + \frac{c^2}{4a^2b^2} - \frac{c^2 \dot g^2}{4b^2} \quad . \end{aligned} \tag{6.36}$$

Then equations (4 .18) yield

$$(6.37)\quad \begin{aligned} {}^{(3)}R_{1212} &= -\frac{1}{a^2} - \frac{3c^2}{4a^2b^2} \xrightarrow{b=c} -\frac{7}{4a^2} \\ {}^{(3)}R_{1313} &= -\frac{1}{a^2} + \frac{c^2}{4a^2b^2} \xrightarrow{b=c} -\frac{3}{4a^2} \\ {}^{(3)}R_{2323} &= -\frac{1}{a^2} + \frac{c^2}{4a^2b^2} \xrightarrow{b=c} -\frac{3}{4a^2} \\ {}^{(3)}R &= -\frac{13}{2a^2} \end{aligned}$$

Thus, we see that the point $t=0$ corresponds to infinite values for the curvature of the homogeneous hypersurface, which shows that we are dealing with a true singularity. Equations (6.37) on the other hand show that the hypersurface of homogeneity remains anisotropic at all times, and that it flattens out in the asymptotic region of very large time.

Returning to the expression (6.35) for the metric we see that our model represents a universe which either starts out flat at $t=-\infty$ and curls up to a singularity at $t=0$, or which explodes from a singularity at $t=0$ to become asymptotically flat at $t=+\infty$.

6.C. A BIANCHI IV MODEL, WHERE THE SOURCE IS NULL OR "ZELDOVICH'S STIFF MATTER WITHOUT ROTATION"

In section 6.B. we saw that the condition $b=c$ at all times turned out to be very severe. It automatically leads to the fact that no perfect fluid could play the role of the source for our manifold. This contrasts impressively with the locally rotationally symmetric tilted universes. King and Ellis (see KING & ELLIS (1973)) have shown that any locally rotational symmetric universe has to be Bianchi type V and then there exists a coordinate system where the Ricci tensor takes the form (6.21) with $b=c$, except for the term $\frac{c^2}{2a^2b^2}$. Then Collins, on the basis of a qualitative analysis of the field equations, showed that solutions existed for a perfect fluid with barometric equation of state, i.e. with $p=(\gamma-1)\rho$ where $1 \leq \gamma$=constant≤ 2, and that these solutions exhibit a very wide range of qualitatively different behaviors. See COLLINS (1974).

These considerations led us to investigate the case where the Ricci tensor remains of the form (6.21) with $b \neq c$. Since the resulting field equations are extremely complicated for the general perfect fluid case, we restricted our attention to the case of a "Zeldovich stiff matter" source, i.e. a perfect fluid with $p=\rho$.

Returning to the field equations in the form (6.13), where $R_{\mu\nu}$ is given by (6.21), we obtain a reduction as follows.

First, we notice that if we take

$$(\ln a^2/bc)^{\cdot} = a(\rho+p)u_0u_1 + \frac{\dot{g}c^2}{2b^2} , \tag{6.38}$$

as the R_{01} equation demands, then it is easily shown that the R_{23} equation is equivalent to the equation

$$2R_{11} - R_{22} - R_{33} = 2(\rho+p)u'^2 \tag{6.39}$$

combined with the conservation equation (6.26). This implies that the R_{23} equation will be automatically satisfied provided the rest of the field equations and the conservation equations are. From the field equations (6.21) we also find

$$\begin{aligned} R_{00} + R_{11} + R_{22} + R_{33} &= \rho u_0^2 + p u_1^2 + \Lambda = \\ &= \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} - \frac{3}{a^2} - \frac{c^2}{4a^2b^2} - \frac{c^2\dot{g}^2}{4b^2} . \end{aligned} \tag{6.40}$$

Now we put $p = \rho$, whereupon the conservation equations (6.26) become

$$\begin{aligned} (\ln \rho^{1/2} abc \cosh\beta)^{\cdot} &= -\frac{2\tanh\beta}{a} \\ (\ln \rho^{1/2} a \sinh\beta)^{\cdot} &= 0 \end{aligned} \tag{6.41}$$

From the last equation we obtain

$$\rho = \frac{R^2}{a^2 \sinh^2\beta} \tag{6.42}$$

where R is a positive constant. Substitution into the first of equations (6.41) yields

$$(\ln bc \coth\beta)^{\cdot} = -\frac{2\tanh\beta}{a} . \tag{6.43}$$

The first integral (6.40) now becomes

$$(6.44)\quad \frac{\dot a\dot b}{ab}+\frac{\dot a\dot c}{ac}+\frac{\dot b\dot c}{bc}-\frac{3}{a^2}-\frac{c^2}{4b^2}\left(\frac{1}{a^2}+\dot g^2\right)=\rho(\cosh^2\beta+\sinh^2\beta)+\Lambda\,,$$

while the development equations read

$$(6.45)\quad \begin{aligned}&(\ln a)^{\cdot\cdot}+(\ln a)^{\cdot}(\ln abc)^{\cdot}-\frac{2}{a^2}-\frac{c^2}{2a^2b^2}=2\rho\sinh^2\beta+\Lambda\\ &(\ln b)^{\cdot\cdot}+(\ln b)^{\cdot}(\ln abc)^{\cdot}-\frac{2}{a^2}-\frac{c^2}{2b^2}\left(\frac{1}{a^2}-\dot g^2\right)=\Lambda\\ &(\ln c)^{\cdot\cdot}+(\ln c)^{\cdot}(\ln abc)^{\cdot}-\frac{2}{a^2}+\frac{c^2}{2b^2}\left(\frac{1}{a^2}-\dot g^2\right)=\Lambda\,.\end{aligned}$$

Letting $\Lambda=0$, we now find that adding and substracting the last two of equations (6.45) yield,

$$(6.46)\quad \begin{aligned}&(\ln bc)^{\cdot\cdot}+(\ln bc)^{\cdot}(\ln abc)^{\cdot}-\frac{4}{a^2}=0\\ &(\ln c/b)^{\cdot\cdot}+(\ln c/b)^{\cdot}(\ln abc)^{\cdot}+\frac{c^2}{b^2}\left(\frac{1}{a^2}-\dot g^2\right)=0\,,\end{aligned}$$

respectively. From the first integral (6.40) we find that

$$(6.47)\quad \frac{c^2}{b^2}\left(\frac{1}{a^2}-\dot g^2\right)=\rho(\cosh^2\beta+\sinh^2\beta)-4\frac{\dot a\dot b}{ab}-4\frac{\dot a\dot c}{ac}+-4\frac{\dot b\dot c}{bc}+\frac{12}{a^2}+\frac{2c^2}{b^2a^2}\,,$$

which, together with the definition of the new dependent variables

$$(6.48)\quad x\equiv bc\,,\quad \eta\equiv\frac{c}{b}$$

turns equations (6.46) into:

$$(6.49)\quad \begin{aligned}&(\ln x)^{\cdot\cdot}+(\ln x)^{\cdot}(\ln ax)^{\cdot}-\frac{4}{a^2}=0\\ &(\ln\eta)^{\cdot\cdot}+(\ln\eta)^{\cdot}(\ln ax)^{\cdot}-4\frac{\dot a\dot x}{ax}-\left(\frac{\dot x}{x}\right)^2+\left(\frac{\dot\eta}{\eta}\right)^2+\\ &\quad+\frac{12}{a^2}+\frac{2\eta^2}{a^2}+\rho(\cosh^2\beta+\sinh^2\beta)=0\end{aligned}$$

Next we change the time variable according to

(6.50) $$dt' = \frac{1}{a}\,dt\,,$$

and collect our euqations. They are:

(6.51) $$\begin{aligned} &\rho = \frac{R^2}{a^2\sinh^2\beta} \\ &(\ln x\coth\beta)' = -2\tanh\beta \\ &(\ln a)'' + (\ln a)'(\ln x)' - 2 - \frac{\eta^2}{2} = 2R \\ &x'' - 4x = 0 \\ &\frac{\eta''}{\eta} + (\ln x)'(\ln\eta)' - 4(\ln a)'(\ln x)' - [(\ln x)']^2 + 12 + \\ &\qquad\qquad + 2\eta^2 = -R(1+\coth^2\beta)\,. \end{aligned}$$

where $x' \equiv dx/dt'$, etc.

It will be useful for later purposes to understand the role of the new time coordinate t'. For that purpose we reproduce the solution of the case $b=c$ in the new variables. Equations (6.51) become, upon setting $\eta=1$, $R=0$,

(6.52) $$\begin{aligned} &\nu' + \mu\nu = \frac{5}{2} \\ &\mu' + \mu^2 = 4 \\ &4\mu\nu + \mu^2 = 14 \end{aligned}$$

where

(6.53) $$\nu \equiv (\ln a)'\,, \quad \mu \equiv (\ln x)'$$

These equations are easily solved, yielding

(6.54) $$\mu = \pm 2\,, \quad \nu = \pm\frac{5}{4}$$

respectively. Thus,

(6.55) $$x = bc = b^2 = x_0 \exp(\pm 2t'), \quad a = a_0 \exp(-\tfrac{5}{4} t')$$

Figure 5 gives a qualitative plot of these solutions. It is seen from this figure that in the new time coordinate the initial singularity of the expanding solution is displaced to $t' = -\infty$ while the final singularity of the contracting solution is displaced to $t' = +\infty$.

Returning, now, to equations (6.51), we see that

(6.56) $$x = A \exp(2t') + B \exp(-2t')$$

with A, B some constants, satisfies the third of these equations. On the other hand, the second of equations (6.51) can be written as

(6.57) $$(\coth\beta)' + (\ln x)'(\coth\beta) = -2$$

If we concentrate on solutions with monotonically expanding and contracting x separately, then we find from (6.56) that

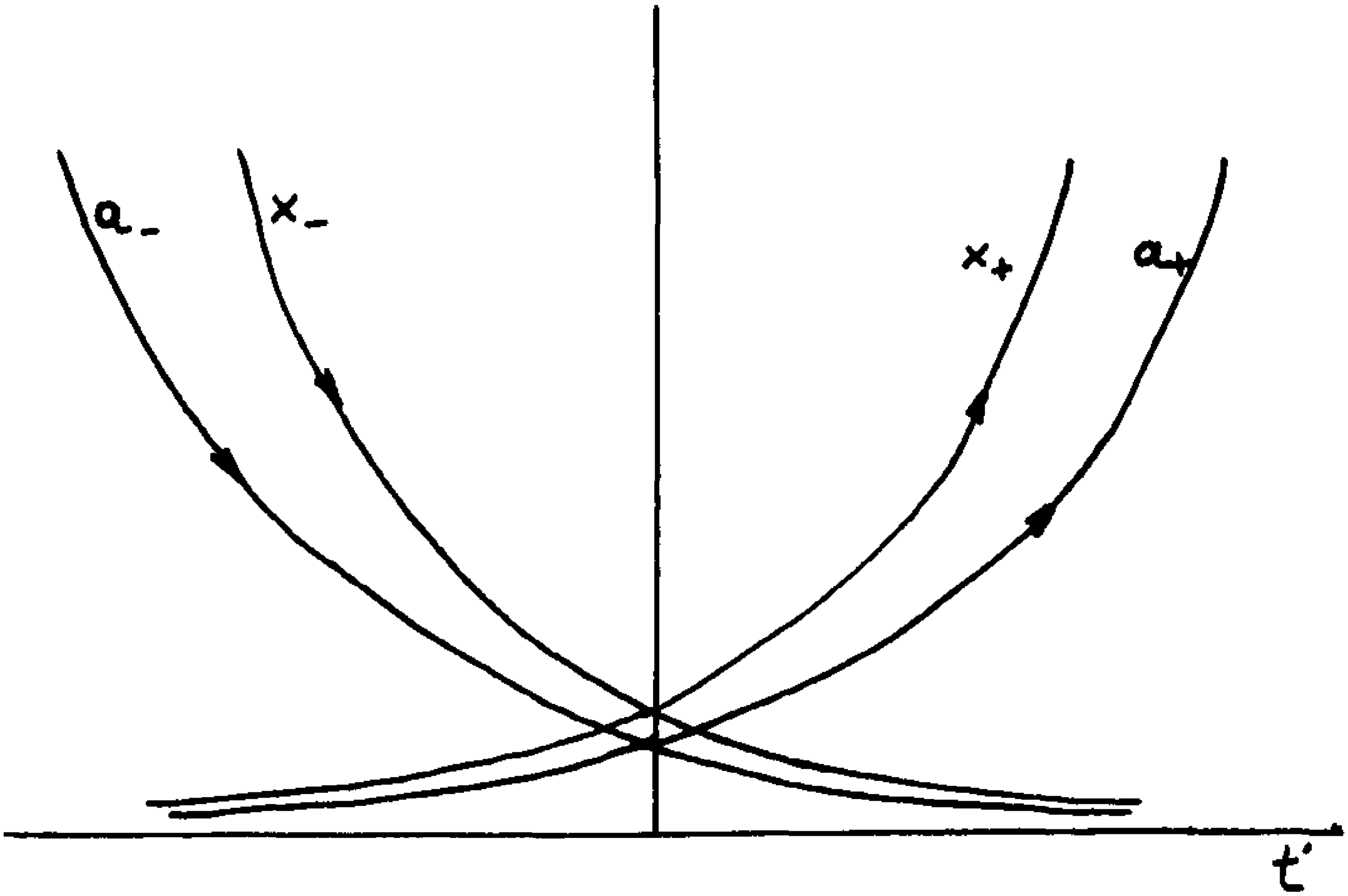
(6.58) $$(\ln x)' = +2, -2$$

respectively. Substituting these values into (6.57) we obtain

(6.59) $$\coth\beta = C \exp(\mp 2t) \mp 1$$

where C is a constant. Figure 6 gives a qualitative representation of the $\tanh\beta$ on the basis of (6.59) for the case $C = 1$. At this point, we are left with the third and last of equations (6.51) to satisfy. By the substitution

(6.60) $$\nu = (\ln a)' ,$$



$$x_+ = bc = x_0 \exp 2t' \qquad x_- = bc = x_0 \exp -2t'$$

$$a_+ = a_0 \exp 2t' \qquad a_- = a_0 \exp -2t'$$

FIGURE 5: Expanding (x_+, a_+), and contracting (x_-, a_-) Bianchi IV "empty" universes. See text.

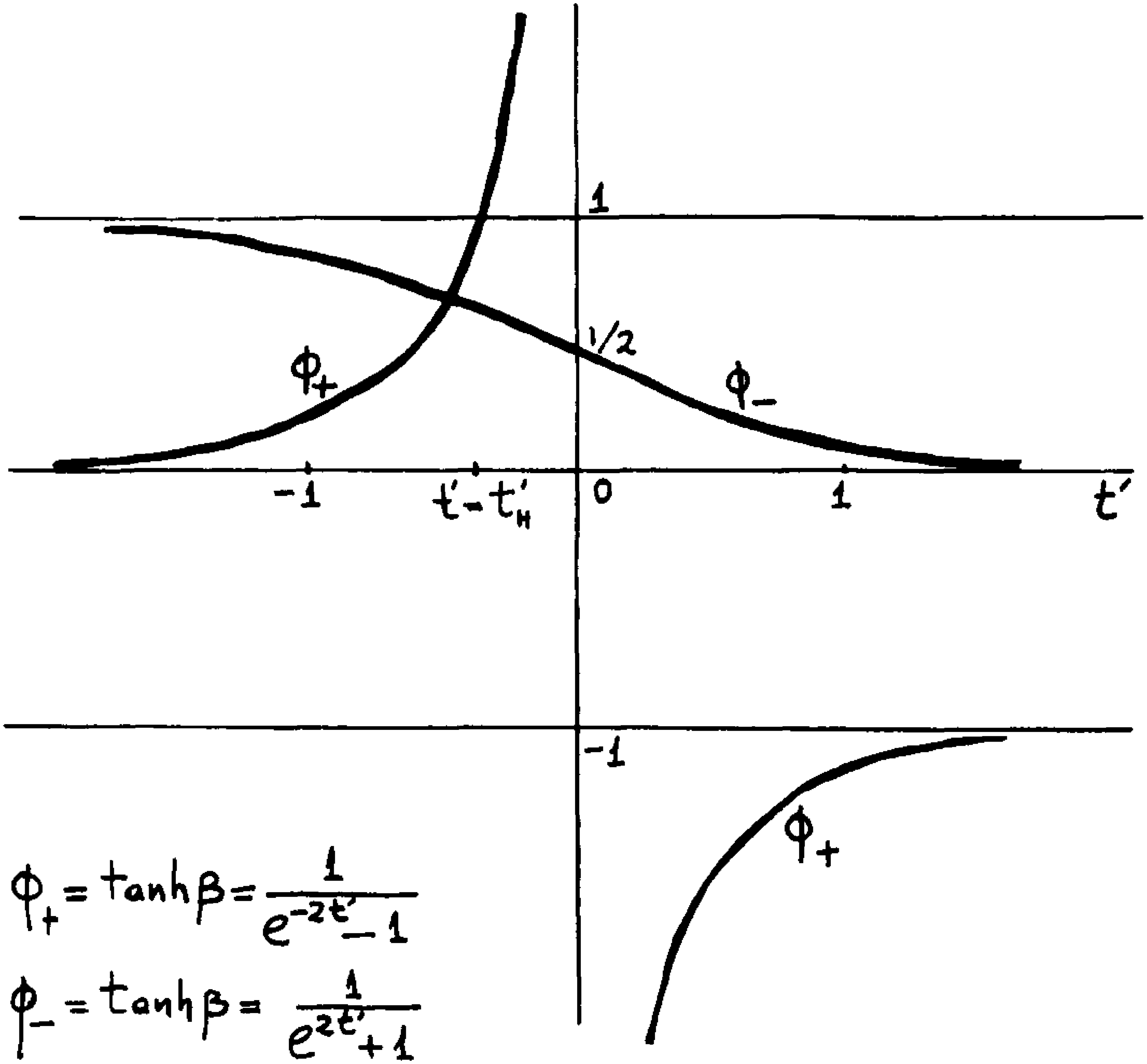


FIGURE 6: Tanhβ vs. t' for an expanding (ϕ_+) and a contracting (ϕ_-) Bianchi IV model. β is the "hyperbolic angle of tilt."

we obtain these euqations in the form

$$(6.61)\qquad \begin{aligned} &\nu' \pm 2\nu = 2(1+R) + \eta^2/2 \\ &y'' \pm 2y' + 2y^3 + \left[R(1+\coth^2 p) \mp 8\nu + 8\right] y = 0 \end{aligned}$$

where the $\pm$ corresponds to $x \sim \exp(\pm 2t')$. From the first of equations (6.61) we obtain

$$(6.62)\qquad \nu = \pm(1+R) + ke^{\mp 2t'} + \frac{1}{2} e^{\mp 2t'} \int y^2 e^{\pm 2t'} dt',$$

where k is a constant. Substitution of this and (6.59) into the second of equations (6.61) gives

$$(6.63)\qquad \begin{aligned} &y'' \pm 2y' + 2y^3 + \\ &\quad + \left\{ R\left([Ce^{\mp 2t'} \mp 1]^2 - 7\right) \mp ke^{\mp 2t'} + \right. \\ &\qquad\qquad \left. \mp 4 e^{\mp 2t'} \int y^2 e^{\pm 2t'} dt' \right\} y = 0 \end{aligned}.$$

The complexity of equation (6.63) has not allowed us to determine the exact behavior of the solution obtained.

However, an asymptotic solution for the case of an empty universe was obtained, which shows some interesting features. Letting $R=0$ in equation (6.63) we obtain, for the expanding case,

$$y'' + 2y' + 2y^3 - 4y e^{-2t'} \int y^2 e^{2t'} = 0 \tag{6.64}$$

Next, we let

$$y = 1 + \psi \tag{6.65}$$

where $\psi \ll 1$. Then, for large enough t', equation (6.64) reads

$$(\psi'' + 2\psi' + 4\psi) e^{2t'} = 8 \int \psi e^{2t'} \tag{6.66}$$

which has as a solution, (A, t'_0 are constants),

$$\psi = A e^{-2t'} \sin 2(t' - t'_0) \tag{6.67}$$

Thus, for large enough t',

$$y = 1 + A e^{-2t'} \sin 2(t' - t'_0). \tag{6.68}$$

This shows that at late times the anisotropy of the model along two principal directions damps out in an oscillatory fashion.

Moreover, from the results obtained so far we can deduce some very interesting characteristics of the general case where R=0. From fig. 6 we see that , in the expanding (ϕ_+) case, the $\tanh\beta$, starting from zero at $t'=-\infty$, monotonically increases to become unity at $t'=t'_H$, and then it approaches infinity as t' approaches zero. This behavior of the $\tanh\beta$ implies that the surfaces of homogeneity, being spacelike for $t'<t'_H$, turn into null at $t'=t'_H$. From (6.60) and equation (6.62) on the other hand, we see that the matter density vanishes at $t'=t'_H$. To the extent that the flow lines of matter can be continued into the region where $t'>t'_H$,the hypersurfaces generated by the Killing vectors will become timelike.

The region of spacetime thus entered will be inhomogeneous and stationary, since one of the Killing vectors will turn timelike.Since the density of matter in this region is finite it seems that we have a "whimper" singularity at $t'=t'_H$, of the type Collins discovered on the basis of his analysis of the axisymmetric generalization of the Friedmann open ($k=-1$) model: An empty "Cauchy horizon" (the null hypersurface) seperates a homogeneous spacelike region from an inhomogeneous stationary one, both of which are characterized by nonzero matter density. However, as our analysis of the obtained solution has not yet been completed, the conclusions discussed above must not be taken as definitive.

6.D. AN ELECTROMAGNETIC BIANCHI TYPE IV UNIVERSE

In this section we show that an exact solution of Einstein's field equations exists for the case where the Ricci tensor is given by the left-hand-side of equations (6.21) and the stress-energy tensor is that of a source-free electromagnetic field.

The spatial homogeneity of the models we are considering demands that the components of the stress-energy tensor with respect to an invariant basis be dependent only on time. Sometimes this is described by saying that the electromagnetic field must be in its "lowest mode." Here we will employ the invariant basis (6.3) again, and the corresponding orthonormal basis given by (6.6), with the restrictions imposed in section 6.B. which led to the form (6.21) of the Ricci tensor.

The electromagnetic 2-form $\mathbf{F}$, expressed in an orthonormal basis (ω^α) reads

$$\mathbf{F} = -e_\ell \omega^0 \wedge \omega^\ell + \tfrac{1}{2} \epsilon^\ell{}_{mn} h_\ell \omega^m \wedge \omega^n \tag{6.69}$$

while its dual reads

$$^*\mathbf{F} = h_\ell \omega^0 \wedge \omega^\ell + \tfrac{1}{2} \epsilon^\ell{}_{mn} e_\ell \omega^m \wedge \omega^n \tag{6.70}$$

where e_ℓ, h_ℓ (ℓ =1,2,3) are components of the electric and magnetic field respectively. Transforming into the invariant basis (σ^α) via

$$\begin{aligned} \omega^0 &= \sigma^0 \\ \omega^i &= b^i{}_j(t)\sigma^j \end{aligned} \tag{6.71}$$

with

(6.72) $$b_{11} = a\ , \quad b_{22} = b\ , \quad b_{33} = c$$
$$b_{23} = gc$$

and the rest of b_{ij} zero, we obtain (6.69) and (6.70) in the form

(6.73)
$$F = -e_\ell b^\ell{}_k\, \sigma^0 \wedge \sigma^k + \tfrac{1}{2}\epsilon^\ell{}_{mn} h_\ell b^m{}_r b^n{}_s\, \sigma^r \wedge \sigma^s$$
$${}^*F = h_\ell b^\ell{}_k\, \sigma^0 \wedge \sigma^k + \tfrac{1}{2}\epsilon^\ell{}_{mn} b^m{}_r b^n{}_s e_\ell\, \sigma^r \wedge \sigma^s$$

The components of the stress-energy tensor in the orthonormal basis are given by

(6.74)
$$8\pi T_{\alpha\beta} = \begin{pmatrix} h^2+e^2 & & & \\ 2(h_2e_3-h_3e_2) & \begin{matrix}-h_1^2+h_2^2+h_3^2+\\ -e_1^2+e_2^2+e_3^2\end{matrix} & & \\ 2(h_3e_1-h_1e_3) & -2(h_1h_2+e_1e_2) & \begin{matrix}h_1^2-h_2^2+h_3^2+\\ +e_1^2-e_2^2+e_3^2\end{matrix} & \\ 2(h_1e_2-h_2e_1) & -2(h_1h_3+e_1e_3) & -2(h_2h_3+e_2e_3) & \begin{matrix}h_1^2+h_2^2-h_3^2+\\ e_1^2+e_2^2-e_3^2\end{matrix} \end{pmatrix}$$

with $T_{\alpha\beta} = T_{\beta\alpha}$, $h^2 = \sum_\ell h_\ell^2$, and $e^2 = \sum_\ell e_\ell^2$.
Since $T \equiv T^\alpha{}_\alpha = 0$, the field equations read

(6.75) $$R_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

From (6.21) we see that $R_{02} = R_{03} = R_{12} = R_{13} = 0$, identically, so that, as follows from (6.75), we must also have $T_{02} = T_{03} = T_{12} = T_{13} = 0$. This will be automatically satisfied if we take $e_1 = h_1 = 0$, as one easily concludes by inspection of (6.74).

Summarizing, we are left with the field equations

$$R_{00} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} - \frac{\dot{g}^2c^2}{2b^2} = h_2^2 + h_3^2 + e_2^2 + e_3^2 \tag{6.76}$$

$$R_{01} = \frac{1}{a}\left(2\frac{\dot{a}}{a} - \frac{\dot{b}}{b} - \frac{\dot{c}}{c}\right) - \frac{\dot{g}c^2}{2ab^2} = 2(h_2e_3 - h_3e_2)$$

$$R_{11} = \frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) - \frac{2}{a^2} - \frac{c^2}{2a^2b^2} = h_2^2 + h_3^2 + e_2^2 + e_3^2$$

$$R_{22} = \frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right) - \frac{2}{a^2} - \frac{c^2}{2a^2b^2} = -h_2^2 + h_3^2 - e_2^2 + e_3^2$$

$$R_{33} = \frac{\ddot{c}}{c} + \frac{\dot{c}}{c}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) - \frac{2}{a^2} + \frac{c^2}{2a^2b^2} - \frac{\dot{g}^2c^2}{2b^2} =$$
$$= h_2^2 - h_3^2 + e_2^2 - e_3^2$$

$$R_{23} = \left(\frac{\dot{g}c}{2b}\right)^{\cdot} + \frac{\dot{g}c}{2b}\cdot\left(\frac{\dot{a}}{a} + \frac{2\dot{c}}{c}\right) - \frac{c}{a^2b} =$$
$$= -2(h_2h_3 + e_2e_3).$$

On the other hand, the Maxwell equations

$$dF = 0 = d^*F, \tag{6.77}$$

yield

$$(h_3ab - h_2cga)^{\cdot} = -e_2b - e_3c(g+1) \tag{6.78}$$
$$(h_2ca)^{\cdot} = e_3c$$
$$(e_3ab - e_2cga)^{\cdot} = h_2b + h_3c(g+1)$$
$$(e_2ca)^{\cdot} = -h_3c \quad .$$

At this point, we take $b=c$ and make a change of both dependent and independent variables according to

(6.79) $$dt' = \frac{1}{a}\,dt\,,\quad E_i = e_i ab\,,\quad H_i = h_i ab\,,\quad i=1,2,$$

whereupon equations (6.78) become

(6.80)
$$\begin{aligned} H_3' &= H_2 g' - E_2 - E_3 \\ H_2' &= E_3 \\ E_3' &= E_2 g' + H_2 + H_3 \\ E_2' &= -H_3 \end{aligned}$$

where $H_3' \equiv dH_3/dt'$, etc.

Now, the sum and the difference of the R_{22} and R_{33} of equations (6.76) give

(6.81)
$$\begin{aligned} &b^2(g'^2-1) = H_3^2 - H_2^2 + E_3^2 - E_2^2 \\ &\left(\frac{b'}{b}\right)' + 2\left(\frac{b'}{b}\right)^2 - 2 = 0\,, \end{aligned}$$

while the sum and the difference of the R_{00} and R_{11} equations give

(6.82)
$$\begin{aligned} &b^2\left[2\left(\frac{b'}{b}\right)' + 2\left(\frac{b'}{b}\right)^2 - 4\frac{a'b'}{ab} + \frac{g'^2}{2} + \frac{5}{2}\right] = -2\left(H_2^2 + H_3^2 + E_2^2 + E_3^2\right) \\ &2\left(\frac{b'}{b}\right)' + 2\left(\frac{b'}{b}\right)^2 + \frac{g'^2}{2} - \frac{5}{2} + 2\left(\frac{a'}{a}\right)' = 0 \end{aligned}$$

Finally, the R_{01} and R_{23} equations become

(6.83)
$$\begin{aligned} &2\left(\frac{a'}{a} - \frac{b'}{b}\right) - \frac{g'}{2} = \frac{2}{b^2}\left(H_2E_3 - H_3E_2\right) \\ &g'' + 2\frac{b'}{b}g' - 2 = -\frac{2}{b^2}\left(H_2H_3 + E_2E_3\right). \end{aligned}$$

A solution can now be obtained under the initial simplifying assumptions

$$g'^2 = 1 \;, \quad g' E_2 = -H_3 \;, \quad g' H_2 = E_3 \;. \tag{6.84}$$

Under these assumptions, it is seen that the first of equations (6.81) is automatically satisfied, while the second of (6.81) and the last of equations (6.83) are satisfied for

$$g' \cdot \frac{b'}{b} = 1 \tag{6.85}$$

Then, from the second of equations (6.82) we obtain

$$\frac{a'}{a} = \text{constant}, \tag{6.86}$$

while the first equation becomes

$$b^2 \left(5 - 4 g' \frac{a'}{a} \right) = -4 \left(H_2^2 + H_3^2 \right) \tag{6.87}$$

It is easily verified that the first of equations (6.83) is identical to (6.87), so that we are left with the Maxwell equations, which now read

$$\begin{aligned} g' H_3' &= H_3 \\ g' H_2' &= H_2 \;. \end{aligned} \tag{6.88}$$

Since g' is a constant, the last equations yield

$$\begin{aligned} H_3 &= H_{30} \exp(g' t') \\ H_2 &= H_{20} \exp(g' t') \end{aligned} \tag{6.89}$$

From (6.85), on the other hand, we have

$$b = b_0 \exp(g' t') \;, \tag{6.90}$$

while (6.87) now yields

$$a = a_0 \exp\left\{g'\left(\frac{5}{4} + \frac{H_{02}^2 + H_{03}^2}{b_0^2}\right)t'\right\} \tag{6.91}$$

In order to clarify the role of the electromagnetic field, we return to the original time variable, via (6.79). From (6.91) we find that

$$a = \left(\frac{5}{4} + H^2\right) g'(t - t_0) \tag{6.92}$$

where $H^2 = (H_{02}^2 + H_{03}^2)/b_0^2$, and t_0 are constants.

Obviously we must take $g' = \pm 1$, when $t \gtrless t_0$, respectively. Then (6.90) gives

$$b = c = \begin{cases} \text{Constant} \times (t - t_0)^{1/(\frac{5}{4} + H^2)} & \text{for } t > t_0 \\ \text{Constant} \times (t_0 - t)^{1/(\frac{5}{4} + H^2)} & \text{for } t < t_0. \end{cases} \tag{6.93}$$

From (6.92) and (6.93) and upon comparing with the empty solution obtained in section 6.B. above, we see that the electromagnetic field acts as a "positive pressure" in directions transverse to the field itself and as a "negative pressure" in the $e_2 - e_3$ "plane" along which the field lies. That is, the rate of expansion or contraction of $a(t)$ increases, while the corresponding rates for b and c decreases due to the presence of the field. This behavior is similar to that of the "Brill magnetic universe." See BRILL (1964). The remarkable feature of our solution, however, is the fact that the field is not along the distinct principal direction of expansion, here $\mathbf{e}_1$, but in the "plane" to which $\mathbf{e}_1$ is normal. A thorough survey of the existing literature, has shown that our solution is the first example of such kind of behavior.

REFERENCES

Adler, R., M. Bazin, and M. Schiffer, (1975), Introduction to General Relativity, McGraw-Hill, New York.

Batakis, N., 1974, New Solutions in Relativistic Cosmology, Ph.D. Thesis (Unpublished).

Batakis, N., and Cohen, J.M. (1972)"Closed Anisotropic Cosmological Models", Ann. Physics 73 578.

Batakis, N., and Cohen, J.M. (1975),"Cosmological Model with Expansion, Shear, and Vorticity", Phys. Rev. D, 12 1544.

Brill, D.R. (1964) "Electromagnetic Fields in a Homogeneous Nonisotropic Universe" Phys. Rev. B 133 845.

Bunge, M. (1967) "Foundations of Physics," Springer-Verlag.

Cohen, J.M. (1967A) in "Relativity Theory and Astrophysics, I. Relativity and Cosmology" ed. by J. Ehlers, A.M.S. Providence, R.I.

Cohen, J.M. (1967B) "Friedmann Cosmological Models with both Radiation and Matter" Nature 216 249.

Collins, C.B. (1964) "Tilting at Cosmological Singularities" Commun. Math. Phys. 39 131.

Collins, C.B. and Hawking, S.W. (1973A) "The Rotation and Distortion of the Universe," Mon. Not. R. Astr. Soc. 162, 307.

Collins, C.B. and Hawking, S.W. (1973B) "Why is the Universe Isotropic?" Astrophys. J. 180 317.

Demianski, M., Grischuk, L.P. (1972) Commun. Math. Phys. 25 233.

Doroshkevich, A.G., Lukash, V.N., and Novikov, I.D. (1973) Sov. Phys.-JETP 37 739.

Einstein, A. (1915) "Die Feldgleichungen der Gravitation" Preus. Akad. Wiss. Berlin, Sitzber pp. 844-847.

Einstein, A. (1956) "The Meaning of Relativity" Princeton Univ. Press, Princeton, N.J.

Eisenhart, L.P. (1926) "Riemannian Geometry" Princeton Univ. Press, Princeton, N.J.

Ellis, G.F.R. (1967) "Dynamics of Pressure-free Matter in General Relativity" J. Math. Phys. 8 1171.

Ellis, G.F.R. (1975) "Singularities and Cosmology," in Seventh Texas Symposium on Relativistic Astrophysics" Ed. by Bergmann, P.G., Fenyves, E.J., and Motz, L. Annals N.Y. Acad. Sc. 262.

Ellis, G.F.R., and MacCallum, M.A.H., (1969) "A Class of Homogeneous Cosmological Models." Commun. Math. Phys. 17 108.

Ellis, G.F.R., and King, A.R. (1974) "Was the Big Bang a Whimper?" Commun. Math. Phys. 38 119.

Farnsworth, D.L. (1967) J. Math. Phys. 8 2315.

Hawking, S.W. (1974) "The Anisotropy of the Universe at Large Times", "Confrontation of Cosmological Theories with Observational Data" ed. by M.S. Longair (see name) pp. 283-286.

Hawking, S.W. and Ellis, G.F.R. (1973) "The Large Scale Structure of Space-time" Cambridge Univ. Press, Cambridge, England.

King, A.R. and Ellis, G.F.R (1973) "Tilted Homogeneous Cosmological Models" Commun. Math. Phys. 31 209.

Kobayashi, S. and Nomizu, K. (1963) "Foundations of Differential Geometry" Interscience, New York.

Longair, M.S. (Edit.) (1974) "Confrontation of Cosmological Theories with observational Data." D. Reidel Publishing Co., Dordrecht-Holland/Boston, U.S.A.

MacCallum, M.A.H.(1972A) "On Criteria of Cosmological Spatial Homogeneity" Phys. Lett. 40A 325.

MacCallum, M.A.H. (1972B) "On 'Diagonal' Bianchi Cosmologies" Phys. Lett 40A 385.

Misner, C.W., Thorne, K.S., and Wheeler, J.A. (1973) Freeman, San Francisco.

Novikov , I.D. (1974) "Isotropization of Homogeneous Cosmological Models" in "Confrontation of Cosmological Theories with Observational Data" ed. by M.S. Long ir, pp. 273-282.

Peebles, P.J.E. (1971) "Physical Cosmology," Princeton Univ. Press, Princeton, N.J.

Riemann, B. (1872) "On the hypotheses that lie at the foundations of geometry," translated by W.K. Clifford, Nature, 8, 14.

Ryan, M.P. and Shepley, L.C. (1975) "Homogeneous Relativistic Cosmologies" Princeton Univ. Press, Princeton, N.J.

Ryan, M.P. and Shepley L.C. (1976) "Resource Letter RC-1: Cosmology," Am. J. Phys. 44 223.

Shikin, I.S. (1976) "Anisotropic Cosmological Bianchi V Model in the General (axially symmetric)case with moving matter ," Sov. Phys. JETP 41 794.

Taub, A.H. (1951) "Empty Space-times Admitting a Three Parameter Group of Motions" Ann. Math 53 472.

Tolman, C.R. (1934) "Relativity Thermodynamics and Cosmology," Oxford, Clarendon Press.