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ON EXTREMAL CIRCUITS

by

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INTRODUCTION

If a finite number of noncollinear points in the euclidean plane each lie on the boundary B of their convex hull, then the (unique) shortest circuit through the points is determined by B (cf.[7]). The general problem of finding the shortest circuits through n given points in euclidean space of dimension $N \geq 2$ has met with little success. It is understood that the intent of this problem is to find a more efficient procedure than examining all of the possible $\frac{1}{2}(n-1)!$ circuits. Note that by a circuit we explicitly mean:

Definition 0.1: A circuit on n points in euclidean space is a set of n line segments (edges) such that (i) each given point is an end point of exactly two edges and (ii) the ordering of the points induced by the set is cyclic, (cf. first paragraph §1).

Over a decade ago, Fred Supnick pointed out [9] that "it would be desirable to obtain, if possible, further results of the type exemplified by the convex case" because this poverty of examples is "undoubtedly a bottleneck to insight." In that paper he constructed what is apparently still only the second such example in the plane, consisting of sets whose points fulfill certain "hyperbolic" relations. He also gave two other examples in euclidean N space for which it was necessary that $N \geq n$, respectively $n - 1$.

The first result in this paper is the establishment (see §1) of a broad class of examples which is the first to apply without restriction on the dimension N.

Theorem 0.1: Given any necklace of n closed spheres in euclidean N space; that is, given a connected collection of spheres such that each sphere intersects precisely two others, then the circuit through the sphere centers determined by the necklace ordering is the unique shortest of all circuits on those n points.

Note if $N > 2$ the necklace may be weirdly knotted. This "necklace" theorem was first stated by Arnold Reinhold [8]; however, the proof he gave was valid only for the case that the spheres all had the same radius. We extend theorem 0.1 to a wider class of point distributions that do not necessarily have unique shortest circuits.

Definition 0.2: A sphere cluster will mean in this paper a finite collection of spheres (spheres will always be taken to be solid and closed) with at least one "consistent" circuit; that is, a circuit on the centers of these spheres satisfying (i) each edge of the circuit joins the centers of intersecting spheres and (ii) each line segment not in the circuit does not join the centers of spheres whose interiors meet. Then

Theorem 0.2: Given a sphere cluster in euclidean N -space, the minimal circuits on the sphere centers are precisely those circuits which are consistent.

The full strength of euclidean geometry is not needed to prove this theorem; in fact, the theorem remains true in any geodesic metric space (see theorem 2.1).

Perhaps the most natural question raised by theorems 0.1 and 0.2 is the question of whether n arbitrarily given points will always form the centers of some necklace or sphere cluster. This question is answered in the negative by an example of six points in the plane (see example 3.1) and, in addition, it is shown that no smaller number of points will furnish such a counterexample (see remark 3.1). Another example on eight points (example 3.4) is also given to show that it is not always possible to find a sphere cluster for those point distributions in which the minimal circuit has length no longer than the length of any "multi-circuit" each of whose "subcircuits" is on at least three of the given points (cf. the paragraph preceding example 3.4) where we mean by this terminology:

<u>Definition 0.3</u>: A multi-circuit is the generalization of a circuit obtained by omitting (ii) from definition 0.1. Thus, a multi-circuit on a set S of n given points can be thought of as a union of circuits (called subcircuits) on disjoint subsets of S whose union is S.

The question of finding maximal circuits through a given finite number of points has proved even more intractible than finding minimal circuits. For instance, it was not until 1965 that the maximal circuits were described for the convex case in the plane (see [10],[11]). And, in fact, the only other configurations with identified maximal circuits are the two examples in euclidean N space where $N \geq n, n-1$ by Supnick [9] referred to earlier, because their minimal circuits are known. The following dual to the sphere cluster theorem gives another class of examples.

<u>Theorem 0.3</u>: Given a collection of spheres in euclidean N space with at least one circuit on the sphere centers satisfying (i) the

interiors of two spheres do not intersect if there is an edge of the circuit joining their centers and (ii) two spheres do meet if there is no edge of the circuit joining their centers, then all circuits satisfying (i) and (ii) and only these circuits are maximal on the sphere centers.

Note if (i) is weakened by omitting "the interiors of", then an analogue of theorem 0.1 is obtained assuring a unique maximal circuit. The existence of collections of spheres satisfying the hypotheses of theorem 0.3 is not nearly as evident as the existence of necklaces and sphere clusters were in the minimal case, especially for large, even numbers $n = 2m$ of spheres. However, that such do exist for arbitrarily large m has been shown by Warren Becker in [1].

We also note that although the existence of a necklace on n points, like the convex case in the plane, so naturally determines a minimal circuit that it raises the question of whether necklaces might not also determine maximal circuits in the same way as in the convex case (see [10],[11]). An example is given of seven points (example 4.1) to show this hope is unfounded.

The special case of tangent necklaces and tangent sphere clusters; i.e., necklaces and sphere clusters whose spheres intersect only in (external) tangencies, are considered because the consistent circuits of these have the property that their length is not only minimal among all circuits but also among all "nontrivial" multi-circuits (cf.(5.2)), that is among multi-circuits none of whose subcircuits consists of only a single given point (cf. definition 0.3). (Should a subcircuit pass through just two of the given points p,q its length is counted, of course, as the distance from p to q and back to p.) Thus, when asking does a finite distribution of points necessarily form the centers of a tangent sphere cluster, the answer cannot possibly be affirmative unless the minimal circuits on the points are also minimal among all nontrivial multi-circuits. However, in this case a tangent sphere cluster will indeed exist (cf. (5.3)). Further, it will be seen that this case is precisely the situation that at least one solution to the personnel assignment problem on the corresponding distance matrix (with suitably large diagonal entries inserted) of the given point distribution is also a solution to the traveling salesman problem (cf. (5.4)).

Part II of this paper deals with the possibility of the self-intersection of minimal circuits on a finite number of given points in a metric space. The question has immediate meaning only for geodesic metric spaces; however, such concepts as a self-intersecting circuit and cogeodesic points are extended naturally to make sense in any metric space (see definitions 7.1 and 7.2 and the following discussion) and the main result (see theorem 7.1) is proved in this context. For geodesic metric spaces this takes the form:

Theorem 0.4: In a geodesic metric space, given distinct non-cogeodesic points $p_1,\ldots,p_n$ such that if E is any geodesic joining p_i and p_j and E' is any geodesic joining p_j and p_k then either $E \cap E' = \{p_j\}$ or p_i, p_j, p_k are cogeodesic; then it is true that if P is any minimal circuit on the n points, it follows that P is not self-intersecting.

The precise meaning of a circuit in a geodesic metric space appears toward the beginning of §2 where it is pointed out that there may correspond many circuits to a fixed cyclic permutation of the given points, and therefore in particular to a minimizing cyclic permutation. The intent of the theorem is that any such circuit is not self-intersecting. Although the converse of theorem 0.4 is false (cf. example 7.2), the condition given to insure the nonself-intersection of minimal circuits in metric space is really the best possible (cf. theorem 7.2).

I. EXTREMAL CIRCUITS ON THE CENTERS OF CERTAIN COLLECTIONS OF SPHERES

§1. Minimal circuits in euclidean space. Theorems 0.1 and 0.2 are proved as special cases of theorem 1.1 stated below, in order to understand where the geometry of euclidean space is used. The problem of finding the shortest circuits on n given distinct points in euclidean space depends only on the distances between the $\frac{1}{2}n(n-1)$ pairs of points. That is complete information is contained in an $n \times n$ symmetric matrix whose diagonal elements are all zero, or unspecified since they will be immaterial. If the $(n-1)!$ cyclic permutations of the first n positive integers are identified in pairs under symmetric rearrangement (i.e. $(i_1 i_2 \ldots i_n) = (i_n i_{n-1} \ldots i_1) \equiv (i_1 i_n i_{n-1} \ldots i_2)$ then there is a one-one correspondence between the $\frac{1}{2}(n-1)!$ cyclic permutations and the circuits on the n points where we think of the circuit corresponding to $P = (i_1 i_2 \ldots i_n)$ as the set of n edges (line segments) corresponding to

$$\delta(P) = \{\{i_1, i_2\}, \ldots, \{i_{n-1}, i_n\}, \{i_n, i_1\}\} ,$$

(cf. definition 0.1). Thus the problem of finding shortest circuits on n points in euclidean space can be considered a special case of the symmetric matrix formulation of the traveling salesman problem: given an $n \times n$ symmetric matrix (a_{ij}) of real numbers find among all $\frac{1}{2}(n-1)!$ cyclic permutations $P = (i_1 i_2 \ldots i_n)$ of the first n positive integers those which minimize

$$L(P) = \sum_{\{i,j\} \in \delta(P)} a_{ij} = a_{i_1 i_2} + \ldots + a_{i_{n-1} i_n} + a_{i_n i_1} .$$

Note that the entries a_{kk} on the diagonal of the matrix are irrelevant since they can never enter into the sum $L(P)$.

A matrix analogue of the existence of sphere clusters is obtained through:

Definition 1.1: A cyclic permutation P is "$[r_1 \ldots r_n]$-consistent with the $n \times n$ real symmetric matrix (a_{ij})" abbreviated "$[r_k]$-consistent" if the matrix is understood, whenever (i) $a_{ij} \le r_i + r_j$ if $\{i,j\} \in \delta(P)$ and (ii) $a_{ij} \ge r_i + r_j$ if both $\{i,j\} \notin \delta(P)$ and $i \neq j$.

If $r_i > 0$, $r_j > 0$; d_{ij} is the distance between p_i and p_j ($p_i \neq p_j$); S_i and S_j are respectively the spheres about p_i and p_j with radii r_i and r_j then

(1.1) $\quad S_i \cap S_j \neq \varphi \Leftrightarrow d_{ij} \le r_i + r_j$

(1.2) $\quad \text{Int } S_i \cap \text{Int } S_j = \varphi \Leftrightarrow d_{ij} \ge r_i + r_j$

Thus given a sphere cluster on n points, letting $r_1,\ldots,r_n$ be the sphere radii, (a_{ij}) be the corresponding distance matrix of the n points and P be the cyclic permutation corresponding to the consistent circuit; then P is $[r_k]$-consistent with (a_{ij}) , and moreover a circuit will be consistent with the given sphere cluster if and only if the corresponding cyclic permutation is $[r_k]$-consistent with (a_{ij}) . Theorem 0.2 then is established as a special case of the following theorem 1.1. Note theorem 0.1 is a corollary of theorem 0.2 because a necklace is a sphere cluster with exactly one consistent circuit.

Remark 1.1: An example of eight points is given in §3 (example 3.3) which cannot be the centers of any sphere cluster unless some spheres are permitted to have radius zero. Thus we would like to prove theorem 0.2 allowing this possibility also. However the preceding argument does not hold because half of equivalence (1.2) fails. For instance if $r_i = 2$ $r_j = 0$ $d_{ij} = 1$ then $\text{Int } S_i \cap \text{Int } S_j = \text{Int } S_j = \varphi$ does not imply $d_{ij} \geq r_i + r_j$. In fact theorem 0.2 as stated fails (in essentially only the one case here) if zero radii are allowed. For consider in the plane the points $p_1 = (0,0)$ $p_2 = (3,0)$ $p_3 = (5,0)$ $p_4 = (7,0)$ and spheres centered at these points with respective radii $6,2,0,2$. Then these four spheres form a sphere cluster with consistent circuit $p_1p_2p_3p_4$ because $\text{Int } S_1 \cap \text{Int } S_3 = \text{Int } S_3 = \varphi$. However $p_1p_3p_4p_2$ is also minimal but not consistent because $\text{Int } S_1 \cap \text{Int } S_4 \neq \varphi$. Theorem 0.2 will be true even permitting zero radii if condition (ii) in definition 0.2 is changed to (ii') each line segment not in the circuit does not join the centers of spheres which intersect in more than boundary points. For spheres of positive radius (ii) and (ii') are equivalent but for instance in the preceding example circuit $p_1p_2p_3p_4$ does not satisfy (ii') because S_1 and S_3 intersect in (5,0) which is an interior point of S_1. The validity of theorem 0.2 allowing zero radii follows as a special case of theorem 3.1.

Theorem 1.1: Given a real symmetric $n \times n$ matrix (a_{ij}) perhaps with unspecified diagonal entries, and real numbers $r_1, \ldots, r_n$ such that there is at least one $[r_k]$-consistent cyclic permutation, then such and only such minimize L among all cyclic permutations.

Proof: Let P be any $[r_k]$-consistent cyclic permutation for (a_{ij}) and let Q be any cyclic permutation. We show that $L(P) < L(Q)$ if Q is not $[r_k]$-consistent and that $L(P) = L(Q)$ if Q is $[r_k]$-consistent. Let $D = \{\{i,j\} : i \neq j$ and $a_{ij} < r_i + r_j\}$ then an equivalent statement of (ii) in definition 1.1 is $D \subset \delta(P)$. Further note that a consequence of (i) and (ii) in definition 1.1 is $\{i,j\} \in \delta(P) - D$ implies $a_{ij} = r_i + r_j$. Let $B = \delta(Q) \cap D$. Then $B \subset D \subset \delta(P)$ and $\delta(P)$ is a disjoint union of B, $D - B$, and $\delta(P) - D$. We have

$$L(P) = \sum_{\delta(P)} a_{ij} = \sum_{B} a_{ij} + \sum_{D-B} a_{ij} + \sum_{\delta(P)-D} a_{ij}$$

$$= \sum_{B} a_{ij} + \sum_{D-B} a_{ij} + \sum_{\delta(P)-D} (r_i + r_j)$$

$$\leq \sum_{B} a_{ij} + \sum_{D-B} (r_i + r_j) + \sum_{\delta(P)-D} (r_i + r_j)$$

and the last $\leq$ is really $<$ unless $B = D$, which is equivalent, using the definition of B , to $D \subset \delta(Q)$, which is equivalent to Q satisfying condition (ii) in definition 1.1 for $[r_k]$-consistent permutations. Continuing:

$$\sum_{B} a_{ij} + \sum_{D-B} (r_i + r_j) + \sum_{\delta(P)-D} (r_i + r_j) = \sum_{B} a_{ij} + \sum_{\delta(P)-B} (r_i + r_j)$$

$$= \sum_{B} a_{ij} + \sum_{\delta(Q)-B} (r_i + r_j)$$

$$\leq \sum_{B} a_{ij} + \sum_{\delta(Q)-B} a_{ij}$$

the last step because of the definition of D and because $\delta(Q) - B = \delta(Q) - D$, Furthermore this $\leq$ is really $<$ unless $a_{ij} = r_i + r_j$ for all $\{i,j\} \in \delta(Q) - D$, which is equivalent to

$\{i,j\} \in \delta(Q)$ implies $a_{ij} \leq r_i + r_j$, which says Q satisfies condition (i) for $[r_k]$-consistent permutations. Completing the chain:

$$\sum_B a_{ij} + \sum_{\delta(Q)-B} a_{ij} = \sum_{\delta(Q)} = L(Q) \; .$$

Thus $L(P) < L(Q)$, unless the two conditions are fulfilled which make Q an $[r_k]$-consistent permutation, and in this case $L(P) = L(Q)$.

We remark that for a given symmetric matrix a particular $[r_k]$-consistent cyclic permutation P might not also be $[r_k']$-consistent, for it is surely possible to pick $r_1',\ldots,r_n'$ so there are no $[r_k']$-consistent cyclic permutations. However if there does exist some $[r_k']$-consistent cyclic permutation Q, then P is also $[r_k']$-consistent. Because we have $L(P) \leq L(Q)$ by theorem 1.1 from P being $[r_k]$-consistent, but if P is not also $[r_k']$-consistent then another application of the theorem yields the contradiction $L(Q) < L(P)$. Thus for a particular symmetric matrix the set of $[r_k]$-consistent cyclic permutations coincides with the set of $[r_k']$-consistent cyclic permutations unless one of these sets is empty and the other is not. For this reason we will occasionally just refer to consistent cyclic permutations without explicitly mentioning for which $r_1,\ldots,r_n$.

§2. Minimal circuits in geodesic metric spaces. We extend theorem 0.2 to geodesic metric spaces, that is to any metric space such that each pair of points is joined by an arc of length equal to the distance between the points. Examples of geodesic metric spaces can be constructed from any compact, finitely arc connected (meaning that given any two points there is an arc of finite length joining them) metric space (M,d) because as Menger has shown (pg. 492, [5]) such spaces have for each pair of points a geodesic (arc of minimal length) joining them. Defining $\lambda(p,q) =$ the length of any geodesic joining p and q where p,q are distinct points in M and $\lambda(p,q) = 0$ if $p = q$, it can be verified (cf. pg. 496,-7 [5]) that λ is a new metric on M and in fact (M,λ) is a geodesic metric space. We remark that the geodesic metric λ and the original metric d are not necessarily equivalent in the sense of producing the same topology (see pg. 497 [5] or pg. 43, 44 [2] although the related footnote pg. 44 of [2] is in error).

The meaning of a circuit in euclidean space is naturally extended to geodesic metric spaces by replacing the term line segments in definition 0.1 by geodesics. However unlike euclidean space, because there can be many geodesics joining the same two points in an arbitrary geodesic metric space, there may correspond to a particular ordering of the given points numerous circuits. But they all have the same length and we will not usually want to distinguish between them. Sphere clusters and consistent circuits in a geodesic metric space (M,λ) are also defined entirely analogously to the euclidean case. A sphere cluster is a collection of spheres, i.e. sets of the form $\{q : \lambda(p,q) \leq r\}$, with at least one consistent circuit, that is a circuit through the sphere centers satisfying (i) each geodesic in the circuit joins the centers of intersecting spheres, and (ii) each geodesic not in the circuit (and not joining a pair of points already joined by a geodesic in the circuit) does not join the centers of spheres which intersect in more than boundary points. Note condition (ii) here is the analogue of (ii') in remark (1.1) and spheres of radius zero are allowed. However $r_i < 0$, i.e. $S_i = \varphi$, cannot occur in any sphere cluster because p_i is joined by a geodesic in any consistent circuit to some other center p_k such that $S_i \cap S_k$ must be nonempty. Now

<u>Theorem 2.1</u>: Given a sphere cluster in a geodesic metric space (M,λ), then the circuits of minimal length through the sphere centers are precisely those circuits which are consistent.

<u>Proof</u>: As in the euclidean case, by taking the distance matrix $(a_{ij}) = \lambda(p_i,p_j)$ of the sphere centers $p_i,\ldots,p_n$ and letting $r_1,\ldots,r_n$ be the radii of the corresponding spheres $S_1,\ldots,S_n$ this theorem follows from theorem 1.1 once we show subject to r_i and r_j nonnegative and $i \neq j$ that

(2.1) $$\lambda(p_i,p_j) \leq r_i + r_j \iff S_i \cap S_j \neq \varphi$$

(2.2) $$\lambda(p_i,p_j) \geq r_i + r_j \iff (\text{Int } S_i \cup \text{Int } S_j) \cap (S_i \cap S_j) = \varphi$$

To prove half of the first equivalence note if $p \in S_i \cap S_j$ then $\lambda(p_i,p_j) \leq \lambda(p_i,p) + \lambda(p_j,p) \leq r_i + r_j$. Conversely we can assume in addition to $\lambda(p_i,p_j) \leq r_i + r_j$ that $\lambda(p_i,p_j) > r_i$, because otherwise $p_j \in S_i \cap S_j$. Note we have implicitly used here that $r_j \geq 0$. Now it can be seen there exists a point q a distance r_i (using $r_i \geq 0$) from p_i along any geodesic going from p_i

to p_j . Therefore $q \in S_i$ and furthermore $\lambda(p_j,q) = \lambda(p_i,p_j) - r_i \leq r_j$ so $q \in S_j$. Restating (2.2) in the form: $\lambda(p_i,p_j) < r_i + r_j$ if and only if $S_i \cap S_j$ contains a point q interior to at least one of S_i and S_j , the proof is entirely similar. First if q is interior to S_i or S_j and in the other then $\lambda(p_i,p_j) \leq \lambda(p_i,q) + \lambda(p_j,q) < r_i + r_j$. To prove the converse we can assume in addition to $\lambda(p_i,p_j) < r_i + r_j$ that $r_i \leq \lambda(p_i,p_j)$ because otherwise $r_i > \lambda(p_i,p_j)$ so p_j is interior to S_i and in S_j . Then there is a point q a distance r_i away from p_i along any geodesic going from p_i to p_j . This q is in S_i and since $\lambda(p_j,q) = \lambda(p_i,p_j) - r_i < r_j$ q is interior to S_j completing the proof. We note that together (2.1) and (2.2) yield $\lambda(p_i,p_j) = r_i + r_j$ if and only if S_i and S_j do meet but only in boundary points.

Both theorem 0.2 and its generalization theorem 2.1 were established by considering the distance matrix (a_{ij}) of the given points. All such matrices besides having zero or unspecified diagonal entries and being symmetric will have positive (off diagonal) entries and satisfy the triangle inequality (i.e. for i,j,k distinct $a_{ij} + a_{jk} \geq a_{ik}$) . However it is not true that given an $n \times n$ matrix with these properties that it must be the distance matrix of some n points in euclidean space.

<u>Example 2.1</u>: Let $(a_{ij}) = \begin{pmatrix} - & 8 & 8 & 8 \\ & - & 8 & 4 \\ & & - & 4 \\ & & & - \end{pmatrix}$. Entries a_{12}, a_{13},

and a_{23} determine the vertices p_1, p_2, p_3 of an equilateral triangle and therefore a two dimensional plane in euclidean N space. Entries a_{24} and a_{34} determine a point p_4 bisecting edge p_2p_3. Then the entry a_{14} is incompatible with the geometry of the euclidean plane.

Remark 2.1: It is true, however, that every real symmetric matrix M with positive (off diagonal) entries satisfying the triangle inequality does always represent the distance matrix of a finite number of points in a geodesic metric space M'.

Proof: Suppose the order of M is n and that $\epsilon > 0$ is a smallest (off diagonal) entry. Choose any n points in a disk in the euclidean plane of diameter $< \frac{\epsilon}{2}$. Connect these points by smooth nonintersecting arcs, extending into three space, of lengths corresponding to the matrix entries. This collection of arcs in three space with distance given by ordinary euclidean arc length is the metric space M' that is sought. We note that this remark is also established as an immediate corollary of theorem XIII.5.2, pg. 286 [3] referred to at the end of this section.

Remark 2.2: Given $p_1, \ldots, p_n$ in a geodesic metric space (perhaps in particular, euclidean space), let (a_{ij}) be the corresponding distance matrix then $p_1, \ldots, p_n$ will be the centers of some sphere cluster if and only if (a_{ij}) has a consistent cyclic permutation.

Proof: The "only if" half was established in the proof of theorem 2.1. To get the converse suppose P is a cyclic $[r_k]$-consistent permutation for (a_{ij}) and define spheres S_i with centers p_i and radii r_i for $i = 1,\ldots,n$. It will follow from (2.1) and (2.2) that the set of these S_i form a sphere cluster on $p_1,\ldots,p_n$ once we show that each $r_i \geq 0$. This need not be true if $n \leq 3$ but in these cases it is clear that there is a sphere cluster on the given points. If $n > 3$ write $(1i\ldots j)$ for the $[r_k]$-consistent cyclic permutation and suppose $r_1 < 0$. Then $r_i + r_j > (r_i + r_1) + (r_j + r_1) \geq a_{1i} + a_{1j} \geq a_{ij} \geq r_i + r_j$. Contradiction. Note that $a_{1i} + a_{1j} \geq a_{ij}$ is the only use made in the argument of the fact that (a_{ij}) was not an arbitrary symmetric matrix but arose as the distance matrix of points in a metric space. Apparently no use was made of the fact that the (off diagonal) entries of (a_{ij}) are positive, but this property is already included in the triangle inequality. Because if $a_{ij} < 0$ then taking any $a_{jk} \leq a_{ki}$ the triangle inequality is violated $a_{ij} + a_{jk} < a_{ki}$.

Remark 2.3: We observe that matrices which are the distance matrices of finite numbers of points in a nongeodesic metric space (M,d) can be considered in theorem 1.1. And if such a matrix $(a_{ij}) = d(p_i,p_j)$ has an $[r_k]$-consistent cyclic permutation P then it, among all cyclic permutations will minimize L. By the preceding remark, we know that these $r_1,\ldots,r_n$ will be nonnegative so we can again consider the corresponding spheres $S_i = \{p : d(p_i,p) \leq r_i\}$. However these S_i do not necessarily form a sphere cluster. Although $a_{ij} \geq r_i + r_j$ implies S_i, S_j meet at most in boundary points (cf. proof of (2.2)), $a_{ij} \leq r_i + r_j$ does not necessarily imply that $S_i \cap S_j \neq \varphi$.

Just take any necklace in the plane, remove the set I of intersection points of two adjacent spheres and let (M,d) be the submetric space consisting of the plane minus I. It is natural to ask whether such counterexamples always arise in this way, by the absence of points that "could" have been there. More precisely we ask if given a metric space (M,d) can it be embedded isometrically in some space (M',d') such that if the distance matrix of any n points in M (or for that matter M') has a consistent permutation then the corresponding spheres in M' form a sphere cluster, i.e. (2.1) and (2.2) hold. The answer is yes, because any metric space can be isometrically embedded in a geodesic metric space! Given (M,d) let M' = the set of bounded continuous real valued functions on M and let $d'(f,g) = \sup_{x \in M} |f(x) - g(x)|$. Then (M,d) can be isometrically embedded in (M',d'), see pg. 286 [3], and it can be verified straightforwardly that (M',d') is a geodesic metric space.

§3. Nonnecessity of sphere clusters: It is clear when asking whether any n points in euclidean space will be the centers of some necklace, that the answer must be no because the existence of a necklace guarantees a unique shortest circuit. But even among point sets with unique minimal circuits it is still not true that there must be a necklace or even a sphere cluster centered on the points.

Example 3.1: In the euclidean plane let $p_1 = (0,1)$ $p_2 = (0,0)$ $p_3 = (0,-1)$ $p_4 = (5,1)$ $p_5 = (5,0)$ $p_6 = (5,-1)$. The boundary

$p_1p_2p_3p_6p_5p_4$ of the convex hull of the six points is a unique shortest circuit and therefore by theorem 0.2 this circuit must be consistent in any sphere cluster. That means the radii of the spheres at p_1 and p_4 must be no more than 2 since their interiors cannot meet the spheres at p_3 and p_6 respectively, and yet these two spheres at p_1 and p_4 must intersect, despite the distance between p_1 and p_4 being 5 . Thus there can be no sphere cluster on the above six points.

We now show that it was not false generality in the preceding example to show the nonexistence of sphere clusters instead of just the nonexistence of necklaces, because it is possible to have a set of points which are the centers of a sphere cluster, but not of any necklace, <u>and</u> which have a unique shortest circuit.

<u>Example 3.2</u>: In the euclidean plane let $p_1 = (0,4)$ $p_2 = (0,0)$ $p_3 = (0,-4)$ $p_4 = (8,4)$ $p_5 = (8,0)$ $p_6 = (8,-4)$. Then $p_1p_2p_3p_6p_5p_4$ is the unique shortest circuit and the corresponding spheres with radii $r_1 = 5$ $r_2 = 2$ $r_3 = 3$ $r_4 = 3$ $r_5 = 2$ $r_6 = 5$ form a sphere cluster on the given points. But there can be no necklace on the points because that requires that S_1 and S_3 do not meet (i.e. $r_1 + r_3 < 8$), S_4 and S_6 do not meet (i.e. $r_4 + r_6 < 8$), S_1 and S_6 do meet (i.e. $r_1 + r_6 \geq 8$) , and S_3 and S_4 do meet (i.e. $r_3 + r_4 \geq 8$) . The first two inequalities imply $r_1 + r_3 + r_4 + r_6 < 16$ and the last two the contradictory relation $r_1 + r_3 + r_4 + r_6 \geq 16$.

Now the example promised in §2 of a set of points for which there is no sphere cluster unless zero radii are allowed is given:

<u>Example 3.3</u>: In the euclidean plane let $p_1 = (-3,0)$ $p_2 = (-3,2)$ $p_3 = (-3,3)$ $p_4 = (-3,6)$ $p_5 = (3,6)$ $p_6 = (3,3)$ $p_7 = (3,2)$

$p_8 = (3,0)$. Then $r_1 = 3$ $r_2 = 1$ $r_3 = 0$ $r_4 = 3$ $r_5 = 3$ $r_6 = 0$ $r_7 = 1$ $r_8 = 3$ gives a sphere cluster on the eight points. To see there are no sphere clusters on these points without zero radii at p_3 and p_6 observe that since $p_1p_2p_3p_4p_5p_6p_7p_8$ is the unique shortest circuit we must have S_1 and S_3 meet in no more than boundary points (i.e. $r_1 + r_3 \leq 3$), S_6 and S_8 meet in no more than boundary points (i.e. $r_6 + r_8 \leq 3$), and S_1 and S_8 meet (i.e. $r_1 + r_8 \geq 6$) . Then from $r_3 \geq 0$, $r_6 \geq 0$, and $6 \leq r_1 + r_8 \leq r_1 + r_3 + r_6 + r_8 \leq 6$ it follows that $r_3 + r_6 = 0$ or $r_3 = r_6 = 0$.

We remark that the first example given in this section is the simplest possible in the sense that there are no point sets of order ≤ 5 in euclidean space which cannot be the centers of a sphere cluster. In fact we show more generally:

Remark 3.1: Given any real symmetric matrix of order ≤ 5 there exists a consistent cyclic permutation.

Proof: First we show that if a real symmetric matrix of order ≤ 5 has a unique minimal cyclic permutation P , then P satisfies conditions (i) and (ii) in definition 1.1 with strict inequality. The first nontrivial case is $n = 4$ so assume that $P = (1234)$, that is assume

$$a_{12} + a_{23} + a_{34} + a_{41} < a_{12} + a_{24} + a_{43} + a_{31}$$

$$a_{12} + a_{23} + a_{34} + a_{41} < a_{13} + a_{32} + a_{24} + a_{41}$$

or equivalently suppose $a_{23} + a_{14} < a_{13} + a_{24}$ and $a_{12} + a_{34} < a_{13} + a_{24}$. Then we must find r_1,r_2,r_3,r_4 such that

$a_{12} < r_1 + r_2 \quad a_{23} < r_2 + r_3 \quad a_{34} < r_3 + r_4 \quad a_{14} < r_1 + r_4$

$a_{24} > r_2 + r_4 \quad a_{13} > r_1 + r_3$. If we let

$\epsilon = \min(a_{24} + a_{13} - a_{23} - a_{14}, a_{24} + a_{13} - a_{12} - a_{34})$ then $\epsilon > 0$ and it can easily be verified that the following choices for the r's will suffice:

$$r_1 = \frac{a_{12} + a_{13} - a_{23}}{2} - \frac{\epsilon}{8}$$

$$r_2 = \frac{a_{12} - a_{13} + a_{23}}{2} + \frac{3\epsilon}{8}$$

$$r_3 = \frac{-a_{12} + a_{13} + a_{23}}{2} - \frac{\epsilon}{8}$$

$$r_4 = a_{24} + \frac{-a_{12} + a_{13} - a_{23}}{2} - \frac{5\epsilon}{8}$$

The case $n = 5$ is more complex. It can be seen that assuming $P = (12345)$ is equivalent to the ten inequalities:

$$a_{12} + a_{45} < a_{14} + a_{25} \qquad a_{12} + a_{34} + a_{51} < a_{14} + a_{25} + a_{31}$$

$$a_{23} + a_{51} < a_{25} + a_{31} \qquad a_{45} + a_{12} + a_{34} < a_{42} + a_{53} + a_{14}$$

$$a_{34} + a_{12} < a_{31} + a_{42} \qquad a_{23} + a_{45} + a_{12} < a_{25} + a_{31} + a_{42}$$

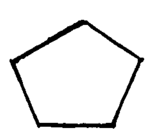
$$a_{45} + a_{23} < a_{42} + a_{53} \qquad a_{51} + a_{23} + a_{45} < a_{53} + a_{14} + a_{25}$$

$$a_{51} + a_{34} < a_{53} + a_{14} \qquad a_{34} + a_{51} + a_{23} < a_{31} + a_{42} + a_{53}$$

It is easily proved that of the following five inequalities:

$$a_{51} + a_{23} \geq a_{12} + a_{53}$$

$$a_{45} + a_{12} \geq a_{51} + a_{42} \qquad\qquad a_{12} + a_{34} \geq a_{23} + a_{14}$$



$$a_{34} + a_{51} \geq a_{45} + a_{31} \qquad\qquad a_{23} + a_{45} \geq a_{34} + a_{25}$$

no two adjacent on the pentagon can hold without contradicting one of the preceding ten inequalities. Thus there is no loss in generality to assuming in addition to the original ten inequalities that also

$$a_{12} + a_{34} < a_{23} + a_{14}$$

$$a_{34} + a_{51} < a_{45} + a_{31}$$

$$a_{45} + a_{12} < a_{51} + a_{42}$$

Letting ϵ be the minimum of the difference between the larger and smaller sides of these thirteen inequalities, so $\epsilon > 0$, and writing $\bar{a}_{12}$ for $\max(a_{12}, a_{51} + a_{23} - a_{53})$, $\bar{a}_{34} = \max(a_{34}, a_{45} + a_{23} - a_{25})$; it can be verified that P is a consistent permutation for the following choices for r_1, r_2, r_3, r_4, r_5:

$$r_1 = \frac{1}{2}(\bar{a}_{12} - a_{23} + \bar{a}_{34} - a_{45} + a_{51}) + \frac{3\epsilon}{7}$$

$$r_2 = \frac{1}{2}(a_{23} - \bar{a}_{34} + a_{45} - a_{51} + \bar{a}_{12}) + \frac{\epsilon}{7}$$

$$r_3 = \frac{1}{2}(\bar{a}_{34} - a_{45} + a_{51} - \bar{a}_{12} + a_{23}) + \frac{\epsilon}{7}$$

$$r_4 = \frac{1}{2}(a_{45} - a_{51} + \bar{a}_{12} - a_{23} + \bar{a}_{34}) + \frac{3\epsilon}{7}$$

$$r_5 = \frac{1}{2}(a_{51} - \bar{a}_{12} + a_{23} - \bar{a}_{34} + a_{45}) - \frac{2\epsilon}{7}$$

For instance: $r_1 + r_2 = \bar{a}_{12} + \frac{4\epsilon}{7} \geq a_{12} + \frac{4\epsilon}{7} > a_{12}$

$$r_2 + r_5 = a_{23} - \bar{a}_{34} + a_{45} - \frac{\epsilon}{7} = a_{23} + a_{45} - \max(a_{34}, a_{45} + a_{23} - a_{25}) - \frac{\epsilon}{7} \leq a_{25} - \frac{\epsilon}{7} < a_{25}$$

$$r_1 + r_3 = \bar{a}_{34} - a_{45} + a_{51} + \frac{4\epsilon}{7} = \max(a_{34} - a_{45} + a_{51},$$

$$a_{23} - a_{25} + a_{51}) + \frac{4\epsilon}{7} \leq a_{31} - \epsilon + \frac{4\epsilon}{7} < a_{31}$$

To complete the argument, say for the case $n = 5$, assume for the matrix (a_{ij}) that $P = (12345)$ is one of several minimal cyclic permutations. Then for each $\alpha > 0$ P will be a unique solution of the matrix with entries

$$m_{ij} = a_{ij} - \alpha \quad \text{if} \quad \{i,j\} \in \delta(P)$$

$$m_{ij} = a_{ij} \quad \text{if} \quad \{i,j\} \notin \delta(P)$$

and by the above for each such matrix we can find $r_k(\alpha)$ so that P is $[r_k(\alpha)]$-consistent. It can be checked that as $\alpha \to 0$, the $r_k(\alpha) \to$ limits, say r_k, and therefore P will be $[r_k]$-consistent for the matrix (a_{ij}).

Further understanding of when a set of points will form the centers of a sphere cluster is gained from the observation that the type of permutations theorem 1.1 really applies to are those which we will call "nondoubletons." Precisely we consider all permutations of the first n positive integers which contain no cycles of one or two elements, and identify those P, P' for which $\delta(P) = \delta(P')$; which is exactly the identification made before for the cyclic permutations. Then the same proof given for theorem 1.1 establishes:

Theorem 3.1: Given a real symmetric $n \times n$ matrix (a_{ij}), perhaps with unspecified diagonal entries, and real numbers

$r_1, \ldots, r_n$ such that there is at least one nondoubleton permutation P satisfying (i) $a_{ij} \le r_i + r_j$ if $\{i,j\} \in \delta(P)$ and (ii) $a_{ij} \ge r_i + r_j$ if both $\{i,j\} \notin \delta(P)$ and $i \neq j$ then these and only these minimize L among all nondoubleton permutations.

We note that P and Q in the proof of theorem 1.1 must not have any one element cycles because then, for instance, $L(P) = \sum_{\delta(P)} a_{ij}$ is undefined since the diagonal entries of (a_{ij}) may not even be specified. Even if they are specified the step $\sum_{\delta(Q)-D} (r_i + r_j) \le \sum_{\delta(Q)-D} a_{ij}$ uses $\{i,j\} \in \delta(Q)$ implies $i \neq j$. (The step $\sum_{\delta(P)-D} a_{ij} = \sum_{\delta(P)-D} (r_i + r_j)$ uses $\{i,j\} \in \delta(P)$ implies $i \neq j$ but this can be avoided by redefining D without the condition $i \neq j$). Dropping $i \neq j$ from (ii) in theorem 3.1 and removing the possibility of unspecified diagonal entries, would allow the inclusion of permutations with single element cycles, but that is another result. Similarly if P or Q had any two element cycles then, say $L(Q) = \sum_{\delta(Q)} a_{ij}$ would be a sum with strictly less than n elements. Reasonable conventions could be made about counting twice some elements of $\delta(P)$, $\delta(Q)$, D, $\delta(P)-D$, etc., but then the step $\sum_{\delta(Q)-D} (r_i + r_j) \le \sum_{\delta(Q)-D} a_{ij}$ could fail because $\delta(Q)-D$ and D might not be disjoint. In fact the conditions (i) $a_{ij} < r_i + r_j$ if $\{i,j\} \in \delta(P)$ and (ii) $a_{ij} > r_i + r_j$ if $\{i,j\} \notin \delta(P)$ are not sufficient to guarantee P minimal among permutations which include those with two element cycles, as the following example illustrates. Let $(a_{ij}) = \begin{pmatrix} 12 & 3 & 11 & 9 \\ & 12 & 9 & 11 \\ & & 12 & 3 \\ & & & 12 \end{pmatrix}$.

If $r_1 = r_2 = r_3 = r_4 = 5$, the cyclic permutation $P = (1234)$ satisfies (i) and (ii). Yet $L(P) = 12 < 24 = L(Q)$ where $Q = (14)(23)$.

From theorem 3.1 it follows using (2.1) and (2.2) that the existence of a sphere cluster on n given points guarantees that some circuit is minimal not only among all circuits but among all multi-circuits (see definition 0.3) which correspond to non-doubleton permutations, i.e. all multi-circuits each of whose subcircuits is on at least three of the given points. Thus there can be no sphere cluster on the points of example 3.1 because the multi-circuit $P_1P_2P_3$, $P_4P_5P_6$ is of strictly shorter length than the circuit $P_1P_2P_3P_6P_5P_4$. Similarly example 3.2 is elucidated. The natural question now is whether there must be a sphere cluster on any finite point set whose minimal circuits are also minimal among all multi-circuits with each subcircuit containing at least three of the given points. The answer is no.

Example 3.4: In the euclidean plane let $p_1 = (3,0)$ $p_2 = (6,5)$ $p_3 = (9,10)$ $p_4 = (12,15)$ $p_5 = (-12,15)$ $p_6 = (-9,10)$ $p_7 = (-6,5)$ $p_8 = (-3,10)$. Then $p_1p_2p_3p_4p_5p_6p_7p_8$ is not only the shortest circuit, it can be verified that it is shorter than any multi-circuit with each subcircuit containing three or more points. (The chief contenders are $p_1p_2p_3p_4$, $p_5p_6p_7p_8$ and $p_2p_3p_4$, $p_5p_6p_7p_8p_1$) . But there can be no sphere cluster on these eight points because S_2 and S_4 can meet only in boundary points (i.e. $r_2 + r_4 \leq 2\sqrt{34}$) , S_1 and S_4 can meet only in boundary points (i.e. $r_1 + r_4 \leq 3\sqrt{34}$) , S_1 and S_2 must meet (i.e. $r_1 + r_2 \geq \sqrt{34}$. Thus $r_4 \leq 2\sqrt{34}$. Similarly $r_5 + r_7 \leq 2\sqrt{34}$, $r_5 + r_8 \leq 3\sqrt{34}$, and $r_7 + r_8 \geq \sqrt{34}$. Thus $r_5 \leq 2\sqrt{34}$. But then $r_4 + r_5 \leq 4\sqrt{34} < 24$ contradicts that S_4 and S_5 must meet.

I believe there is no example such as this involving fewer points. However the type of argument employed in demonstrating remark 3.1 becomes too cumbersome in the cases $n = 6$, $n = 7$.

§4. Maximal circuits in euclidean space. The result on maximal circuits dual to theorem 0.2 can be proved similarly as a special case of a result on matrices.

Theorem 4.1: Given a real symmetric $n \times n$ matrix (a_{ij}) , perhaps with unspecified diagonal entries, and real numbers $r_1, \ldots, r_n$ such that there is at least one cyclic permutation P satisfying (i) $a_{ij} \geq r_i + r_j$ if $\{i,j\} \in \delta(P)$ and (ii) $a_{ij} \leq r_i + r_j$ if both $\{i,j\} \notin \delta(P)$ and $i \neq j$; then these and only these maximize L among all cyclic permutations.

Proof: Just reverse all inequalities in the proof of theorem 1.1.

Theorem 0.3 follows from theorem 4.1 by (1.1) and (1.2). Realizations of theorem 0.3 for large n are not easy to find. If n is odd an example in the plane can be given by taking points uniformly placed on the circumference of a circle and choosing the sphere radii all to be the same value of appropriate size so that each sphere meets all the others but the two most nearly diametrically opposite. A realization for any even n due to Warren Becker [1] also has the property that not all the points lie on the boundary of their convex hull.

We note that if $n = 5$ any necklace also satisfies the hypotheses of theorem 0.3 and that moreover the relationship between the maximal and minimal circuits is the same as for the convex case for odd n (cf. [10]), i.e. that each point is connected in the maximal circuit by edges to the two points furthest away, not necessarily in distance, but as determined by the ordering of the points according to the minimal circuit. The hope that this might generally be true is false.

Example 4.1: In the euclidean plane let $p_1 = (0,0)$ $p_2 = \left(\frac{3.9}{\sqrt{2}}, \frac{-3.9}{\sqrt{2}}\right)$ $p_3 = (10,-10)$ $p_4 = (9,9)$ $p_5 = (-9,9)$ $p_6 = (-10,10)$ $p_7 = \left(\frac{-3.9}{\sqrt{2}}, \frac{-3.9}{\sqrt{2}}\right)$ and let the corresponding spheres have radii $r_1 = 2.5$ $r_2 = 1.4$ $r_3 = 9$ $r_4 = 10.03$ $r_5 = 10.03$ $r_6 = 9$ $r_7 = 1.4$. It can be checked that these spheres form a sphere cluster with unique consistent circuit $p_1p_2p_3p_4p_5p_6p_7$. However $P = p_1p_5p_2p_6p_3p_7p_4$ is not maximal because $L(P) < L(Q)$ where $Q = p_1p_6p_2p_5p_4p_7p_3$.

§5. Tangent sphere clusters. Since in this section the permutations of the first n positive integers that will be considered will not be necessarily cyclic, we will use the notation $[i_1 i_2 \ldots i_n]$ for the permutation sending 1 into $i_1, \ldots, n$ into i_n. If $P = [i_1 \ldots i_n]$ then $\gamma(P)$ will denote the set of ordered pairs $\{[1i_1], \ldots, [ni_n]\}$ and given any $n \times n$ matrix (with diagonal entries specified and possibly asymmetric) $L(P)$ will be the sum $a_{1i_1} + \ldots + a_{ni_n} = \sum_{\gamma(P)} a_{ij}$. Note if P is cyclic this notation for $L(P)$ is in agreement with the previous definition used.

With these agreements we can state a result known in linear programming theory:

<u>Theorem A</u>: Given any real $n \times n$ matrix (c_{ij}) there exist real numbers $u_1,\ldots,u_n$ and $v_1,\ldots,v_n$ such that $c_{ij} \geq u_i + v_j$ for all i,j and the permutations P minimizing L are exactly those satisfying $c_{ij} = u_i + v_j$ whenever $[i,j] \in \gamma(P)$.

This theorem, or at least a constructive proof of the existence of the u_i and v_j , is a solution to the personnel assignment problem, i.e. the problem of finding a permutation P that minimizes L (among all permutations) for a given matrix (c_{ij}) . A knowledge of linear programming methods is not necessary to prove theorem A. Once the u_i and v_j are obtained it is clear that any P satisfying $c_{ij} = u_i + v_j$ whenever $[i,j] \in \gamma(P)$ is minimal because if Q is any other permutation $L(P) = \sum_{\gamma(P)} c_{ij} = \sum_{\gamma(P)} (u_i + v_j) = \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j = \sum_{\gamma(Q)} (u_i + v_j) \leq \sum_{\gamma(Q)} c_{ij} = L(Q)$. And since there is only one inequality in the preceding line it also follows that unless Q satisfies $c_{ij} = u_i + v_j$ whenever $[i,j] \in \gamma(Q)$ then Q is not minimal. The problem then is to find u_i and v_j with the desired properties, but this is essentially just the algorithm due to Kuhn (cf. [4] or [6]).

Our interest in the preceding theorem is that it can be specialized to the following, where the term "nonsingleton" permutation will be used for a permutation without any single element cycles, i.e. any $[i_1 \ldots i_n]$ satisfying $i_k \neq k$ for all $k = 1,\ldots,n$.

Theorem B: Given a real symmetric $n \times n$ matrix (a_{ij}) , perhaps with unspecified diagonal entries, there exist real numbers $r_1,\ldots,r_n$ such that $a_{ij} \geq r_i + r_j$ for all i,j , $i \neq j$ and the nonsingleton permutations P minimizing L among all nonsingleton permutations are exactly those satisfying $a_{ij} = r_i + r_j$ whenever $[i,j] \in \gamma(P)$.

Proof: Apply theorem A to the matrix (c_{ij}) where $c_{ij} = a_{ij}$ if $i \neq j$ and $c_{ij} = K$ if $i \neq j$ where K is sufficiently large, say $K = 1 + \sum_{i \neq j} |c_{ij}|$, to get $u_1,\ldots,u_n$ and $v_1,\ldots,v_n$. Let $r_i = \frac{1}{2}(u_i + v_i)$ for $i = 1,\ldots,n$. Then $a_{ij} = c_{ij} = c_{ji} = \frac{1}{2}(c_{ij} + c_{ji}) \geq \frac{1}{2}(u_i + v_j + u_j + v_i) = r_i + r_j$; for all $i \neq j$. Because $c_{ii} = K$ was chosen so large, the nonsingleton permutations minimizing L for (a_{ij}) are exactly the minimizing permutations for (c_{ij}) and therefore by theorem A exactly those P satisfying $c_{ij} = u_i + v_j$ whenever $[i,j] \in \gamma(P)$, that is those nonsingleton P satisfying $a_{ij} = r_i + r_j$ whenever $[i,j] \in \gamma(P)$. The last step because $[i,j] \in \gamma(P)$ implies $a_{ij} = c_{ij} = \frac{1}{2}(u_i + v_j + u_j + v_i)$ $= r_i + r_j$.

We note that we could in theorem B, as in theorem 3.1, identify all permutations with the same "edge" set $\delta(P) = \{\{1,i_1\},\ldots,\{n,i_n\}\}$. The more substantial differences between the two theorems are that theorem B, by restricting the inequality in condition (i) of theorem 3.1 to equality assures that any permutation satisfying (i) with equality and (ii) of theorem 3.1 will be minimal not only among the nondoubleton permutations but also among the nonsingletons. And that in the present case the existence of $r_1,\ldots,r_n$ can be asserted instead of being a hypothesis (cf. example 3.4)

If the symmetric matrix in theorem B is taken as the distance matrix of n points in euclidean space, the nonsingleton permutations correspond to the nontrivial multi-circuits on the points. Once we know that $r_1,\ldots,r_n$ in theorem B are nonnegative then by taking spheres with these radii centered at the corresponding points we have by (2.1) and (2.2) what we will call a "multiple sphere cluster," that is a collection of spheres whose only pairwise intersections are (external) tangencies and which have a "consistent" nontrivial multi-circuit, i.e. a nontrivial multi-circuit on the sphere centers each of whose edges joins the centers of tangent spheres (cf. definition 0.2). To prove $r_1,\ldots,r_n$ are nonnegative if (a_{ij}) is a distance matrix of n points, we assume some r say $r_1 < 0$. In a minimal permutation $P = [i_1\ldots i_n]$ of (a_{ij}), taking $j = i_1$ and k such that $i_k = 1$, then $r_1 + r_k = a_{1k}$, $r_1 + r_j = a_{1j}$ and $a_{jk} \geq r_j + r_k > (r_1 + r_j) + (r_1 + r_k) = a_{1j} + a_{1k} \geq a_{jk}$. Contradiction (cf. proof of remark 2.2). The only problem with this argument is that it does not cover the case $j = k$, i.e. that $(1j)$ is a two element cycle in P. (The case $1 = j = k$ does not occur since P is a nonsingleton). In fact if $(1j)$ is a two cycle in P it is possible that $r_1 < 0$, but we show under these circumstances we can pretend that we started with the positive values $r_1' = \frac{-r_1}{2}$ and $r_j' = r_j + \frac{r_1}{2}$ instead of r_1 and r_j. Note $r_j' > 0$ since $r_j + \frac{r_1}{2} > r_j + r_1 = a_{1j} > 0$. Further

$r_1' + r_j' = r_1 + r_j = a_{1j}$ and $r_j' + r_k \leq r_j + r_k \leq a_{jk}$ $(k \neq j)$. Thus it remains to show $r_1' + r_k \leq a_{1k}$ $(k \neq 1)$. But if not, i.e. if $\frac{-r_1}{2} + r_k > a_{1k}$ then adding with $r_j + \frac{r_1}{2} > a_{1j}$ gives $r_k + r_j > a_{1k} + a_{1j} \geq a_{jk}$ contradicting that $r_1, \ldots, r_n$ satisfied the conditions of theorem B.

Theorem B then translates into the following facts about euclidean space, or for that matter any geodesic metric space:

(5.1) Given any finite point distribution whatever, there is a multiple sphere cluster with these points for the sphere centers.

(5.2) The length of any circuit determined by a tangent necklace (tangent sphere cluster) is minimal among the lengths of all nontrivial multi-circuits.

(5.3) If the minimal circuits of a finite set of points are also minimal among the nontrivial multi-circuits then the points will be the centers of some tangent sphere cluster.

(5.4) Those finite point distributions whose minimal circuits are identified by their being the consistent circuits of a tangent sphere cluster on the given points are precisely those point distributions such that at least one solution to the personnel assignment problem on the corresponding distance matrix (with suitably large diagonal entries inserted) is in fact also a solution to the traveling salesman problem.

II. NONSELF-INTERSECTION OF MINIMAL CIRCUITS IN CERTAIN METRIC SPACES

§6. Discussion of previous work on the problem. Flood states (pg. 64 [4]) "in the euclidean plane the minimal tour does not intersect itself, and this intersection condition generalizes easily for arbitrary $a_{\alpha\beta}$". Apparently the intended meaning of self-intersection of tours (i.e. cyclic permutations) in this matrix context is that tour

$$[i_1 \ldots i_{p-1} i_p \ldots i_q i_{q+1} \ldots i_n]$$

is self-intersecting if tour

$$[i_1 \ldots i_{p-1} i_q i_{q-1} \ldots i_p i_{q+1} \ldots i_n]$$

is of strictly smaller length. It then follows immediately that minimal tours are not self-intersecting, however this definition does not agree with the usual meaning of a self-intersecting circuit in the plane. For instance if $p_1 = (-1,-3)$ $p_2 = (0,0)$ $p_3 = (1,-3)$ $p_4 = (1,1)$ $p_5 = (-1,1)$ then circuit $[p_1 p_2 p_3 p_4 p_5]$ does not intersect itself, yet $[p_1 p_3 p_2 p_4 p_5]$ is of strictly shorter length.

A proof that a minimal circuit on a given finite set of non-cogeodesic points in the euclidean plane or on the two dimensional euclidean sphere (nonsolid, unlike spheres in part I) could not be self-intersecting is given by Quintas and Supnick [7]. Motivated by this paper, Reinhold [8] investigated the question in (not necessarily geodesic) metric spaces, giving definitions for self-intersecting polygons (which we adopt, see definition 7.2) and for collinear points (which we do not use, cf. definition 7.1). The term polygon on the points $p_1, \ldots, p_n$ will be used for the cyclic permutations of the points identified in pairs under symmetric rearrangement (thus $[p_1p_2\cdots p_n] = [p_np_{n-1}\cdots p_1] \equiv [p_1p_n\cdots p_2]$), while the term circuit will be reserved for any of the sets of n geodesics which correspond to a given polygon in a metric space. Reinhold proved for a certain class of metric spaces that any minimal polygon on a finite set of noncollinear points was not self-intersecting. We will considerably augment this class to the largest possible (see theorem 7.2).

We first state and discuss Reinhold's theorem (but in a notation differing from his). In a metric space M, with the distance between two points p and q denoted by $\overline{pq}$, let $S(p_1p_2p_3)$ mean that p_1, p_2, p_3 are distinct and satisfy $\overline{p_1p_3} = \overline{p_1p_2} + \overline{p_2p_3}$. Reinhold defines three points p_1, p_2, p_3 as "collinear" $C(p_1, p_2, p_3)$ if there is some ordering $q_1q_2q_3$ of p_1, p_2, p_3 such that $S(q_1q_2q_3)$; and defines distinct points $p_1, p_2, \ldots, p_n$ $n > 3$ as "collinear" $C(p_1, p_2, \ldots, p_n)$ if each triple p_i, p_j, p_k of them is collinear. With the same definition of self-intersection used later here, his result is:

Theorem R: If M is a metric space satisfying for any points p,q,r,s in it: (i) $S(prs)$, $S(qrs)$ and $p \neq q \Rightarrow S(pqs)$ or $S(qps)$

(ii) $S(pqr)$ and $S(qrs) \Rightarrow S(prs)$

(iii) $C(p,q,r)$ and $C(s,q,r) \Rightarrow C(p,s,q)$

then a minimal polygon on a finite noncollinear set of points is not self-intersecting.

Reinhold remarks that the independence of conditions (i), (ii), (iii) is open. In fact the three conditions are redundant with {(ii),(iii)} being the only proper subset equivalent to the three. This follows from (iii) $\Rightarrow$ (i), (iii) $\not\Rightarrow$ (ii), (i) and (ii) $\not\Rightarrow$ (iii).

Proof (iii) $\Rightarrow$ (i): Assume (i) fails, i.e. that there exist distinct p,q,r,s satisfying S(prs), S(qrs), not-S(pqs), and not-S(qps). By (iii) S(prs) and S(qrs) $\Rightarrow$ C(p,q,s). C(p,q,s) means S(pqs) or S(qps) or S(psq). Since the first two have been excluded $\overline{ps} + \overline{sq} = \overline{pq}$, but from S(prs) and S(qrs) we know $\overline{ps} = \overline{pr} + \overline{rs}$, $\overline{qs} = \overline{qr} + \overline{rs}$ so $\overline{pq} = \overline{pr} + \overline{rs} + \overline{rs} + \overline{qr} \geq \overline{pq} + \overline{rs} + \overline{rs}$. Thus $\overline{rs} = 0$, i.e. that r = s contradicting p,q,r,s distinct.

Proof (iii) $\not\Rightarrow$ (ii): Let $M = \{p_1,p_2,p_3,p_4\}$ with $(\overline{p_i p_j}) = \begin{pmatrix} 0 & 1 & 2 & 1 \\ & 0 & 1 & 2 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

Then it is easy to check that M is a metric space and that of all possibilities $S(p_i p_j p_k)$ we have only $S(p_1 p_2 p_3)$, $S(p_2 p_3 p_4)$, $S(p_3 p_4 p_1)$, and $S(p_4 p_1 p_2)$. Therefore $C(p_1,p_2,p_3)$, $C(p_2,p_3,p_4)$, $C(p_3,p_4,p_1)$, and $C(p_1,p_2,p_4)$ so (iii) holds, but (ii) fails by $S(p_1 p_2 p_3)$ and $S(p_2 p_3 p_4)$ but not-$S(p_1 p_3 p_4)$.

Proof (i) and (ii) $\not\Rightarrow$ (iii): Let $M = \{p_1,p_2,p_3,p_4\}$ with $(\overline{p_i p_j}) = \begin{pmatrix} 0 & 1 & 1 & \sqrt{2} \\ & 0 & 2 & 1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$. Then M is a metric space and of all possible $S(p_i p_j p_k)$ we have only $S(p_2 p_1 p_3)$ and $S(p_2 p_4 p_3)$. Therefore (i) and (ii) hold vacuously and (iii) fails by $C(p_1,p_2,p_3)$ and $C(p_2,p_3,p_4)$ but not-$C(p_1,p_2,p_4)$.

We note that Reinhold's theorem does not subsume the work of Quintas and Supnick [7] referred to earlier; because the two dimensional euclidean sphere with the arc length metric fails to satisfy (ii) and (iii). In fact, as we shall see shortly, only condition

(i) is needed in the hypotheses of theorem R.

We also remark that Reinhold's definition of the collinearity of n points is a little too general. For instance taking the unit circle in the plane with arc length metric and taking four points on the circle which divide it into four equal sections, then a minimal circuit through these four points is not self-intersecting. But Reinhold's result will not apply because according to his definition these four points are collinear. (Reinhold's result is also inapplicable because this metric space does not satisfy conditions (ii) or (iii)). Reinhold's definition of collinearity further has the somewhat strange effect in this situation that if any one of the four points is displaced in either direction, the new set of four points is no longer collinear.

§7. Definitions and statement of results. We now write $(q_1 \ldots q_n)$ whenever $\overline{q_1 q_n} = \sum_{k=1}^{n-1} \overline{q_k q_{k+1}}$. In distinction with the notation $S(q_1 q_2 q_3)$ used previously we no longer demand that the q_i be distinct, only that at least two are different. It is then true that for any $p \neq q$ we have (pq); also $(q_1 \ldots q_n) \Leftrightarrow (q_n \ldots q_1)$; and if we know the q_i are distinct, for any other arrangement of the q_i besides $q_n \ldots q_1$ we have $(q_1 \ldots q_n) \Rightarrow \text{not-}(q_{i_1} \ldots q_{i_n})$. Furthermore if the q_i are distinct, $(q_1 \ldots q_n) \Rightarrow (q_{i_1} \ldots q_{i_r})$ where $r \geq 2$ and $1 \leq i_1 < \ldots < i_r \leq n$.

Definition 7.1: The points $q_1, \ldots, q_n$ will be called "cogeodesic" if there is some ordering of the q_i such that $(q_{i_1} \ldots q_{i_n})$. It follows that if $q_1, \ldots, q_n$ are cogeodesic by this definition, then they are collinear by Reinhold's definition, but not conversely, as for instance, the four point example in the last paragraph of §6 shows.

Definition 7.2: Given a polygon $[p_{i_1} \ldots p_{i_n}]$ on the vertices $p_1, \ldots, p_n$ we will refer to adjacent pairs of vertices $p_{i_k} p_{i_{k+1}}$ in the ordering (including $p_{i_n} p_{i_1}$) as "edges" of the polygon. In a polygon $[\ldots ab \ldots cd \ldots]$ in metric space M we will say edge ab "intersects" edge cd if there exists $x \in M$ so (axb) and (cxd). Note it is allowed that x might $=$ a, b, c, or d but we understand by the notation $[\ldots ab \ldots cd \ldots]$ that a, b, c, d are distinct. Adjacent edges ab, bc in the polygon $[\ldots abc \ldots]$ will be said to "intersect" (nontrivially) if there exists $x \in M$, $x \neq b$ so that (axb) and (bxc). Again x might $= a$ or c but it is understood a, b, c are distinct. A polygon in a metric space will be said to be "nonself-intersecting" in that space if no pair of its edges intersect.

The preceeding definitions of a self-intersecting polygon and of cogeodesic points are reasonable because they are consistent with the natural meanings of these terms in geodesic metric spaces. It is clear that if a circuit on n points in a geodesic metric space (M, λ) is self-intersecting then the corresponding polygon is also. Conversely, however, if a polygon in (M, λ) is self-intersecting not all of the corresponding circuits must be.

Example 7.1: Let M be the subset of the plane consisting of the points $p_1 = (-1,1)$ $p_2 = (1,1)$ $p_3 = (1,-1)$ $p_4 = (-1,-1)$

$x = (0,0)$ $y = (0,2)$ and the segments p_1p_3, p_1p_4, p_1y, p_2p_3, p_2p_4, p_2y; and let λ be the arc length metric, i.e., $\lambda(p,q)$ = length of the shortest path in M joining p and q . Then $(p_1 \times p_3)$ and $(p_2 \times p_4)$ so polygon $[p_1p_2p_3p_4]$ is self-intersecting in edges p_1p_3 and p_2p_4 . However corresponding circuit $p_1yp_2p_3xp_4$ is not self-intersecting, although corresponding circuit $p_1xp_2p_3xp_4$ is.

It is true, however, that if a polygon is self-intersecting, then at least one of the corresponding circuits must be also. This is an immediate consequence of the definition of self-intersection of a polygon and the following:

<u>Remark 7.1</u>: In a geodesic metric space, (abc) holds if and only if there is a geodesic from a to c containing b .

The "only if" part of this remark is clear, and the "if" part follows from a slight extension of the argument used to verify that $\lambda(p,q)$ = length of any geodesic joining p to q satisfies the triangle inequality (cf. pg. 496,-7; also pg. 495 [5]).

Several applications of remark 7.1 also show that the points of a finite set S are cogeodesic in a geodesic metric space according to definition 7.1 if and only if they are cogeodesic in the natural sense of there existing a geodesic containing S .

Returning to not necessarily geodesic metric spaces we state our main result:

<u>Theorem 7.1</u>: Given distinct noncogeodesic points $p_1, \ldots, p_n$ in a metric space M satisfying:

(7.1) $(p_i x p_j)$ and $(p_i x p_k) \Rightarrow (p_i p_j p_k)$ or $(p_i p_k p_j)$ where p_i, p_j, p_k, x are distinct and $x \in M$

then any minimal polygon on $p_1,\ldots,p_n$ is not self-intersecting.

The proof of this theorem is given in §8. We discuss now the strength of the theorem and its consequences. First note that the converse to theorem 7.1 as formulated is not true.

Example 7.2: Let M be the subset of the plane consisting of the four line segments forming the sides of the square on $p_1 = (1,1)$ $p_2 = (-1,1)$ $p_3 = (-1,-1)$ $p_4 = (1,-1)$ and the segment joining $p_5 = (1,0)$ to $x = (0,1)$; and let λ be the arc length metric. Then $[p_1p_2p_3p_4p_5]$ is not self-intersecting. However the hypotheses of theorem 7.1 do not hold because we have (p_1xp_2) and (p_4xp_2) but not-$(p_1p_4p_2)$ and not-$(p_4p_1p_2)$.

The reason the converse to theorem 7.1 fails is that the hypotheses of the theorem really guarantee that any minimal polygon on any subset of noncogeodesic points of $\{p_1,\ldots,p_n\}$ is not self-intersecting. In the above example, for instance, the minimal polygon on $\{p_1,p_2,p_4\}$ is self-intersecting and generally if the minimal polygons on each noncogeodesic subset of $p_1,\ldots,p_n$ are not self-intersecting, then (7.1) holds. Because if (p_ixp_j), (p_ixp_k), not-$(p_ip_jp_k)$, and not-$(p_ip_kp_j)$ for some distinct p_i,p_j,p_k,x then also not-$(p_jp_ip_k)$, for otherwise from (p_ixp_j), (p_ixp_k), and $(p_jp_ip_k)$ we get $\overline{p_jp_k} = \overline{p_jp_i} + \overline{p_ip_k} = \overline{p_jx} + \overline{xp_i} + \overline{p_ix} + \overline{xp_k} \geq 2\overline{xp_i} + \overline{p_jp_k}$ which contradicts $x \neq p_i$. However not-$(p_jp_ip_k)$, not-$(p_ip_jp_k)$ and not-$(p_ip_kp_j)$ means p_i,p_j,p_k are not cogeodesic and the minimal, in fact only polygon on these three points has an intersection of edges p_ip_j and p_ip_k .

From theorem 7.1 and the discussion in the preceeding paragraph we have:

Theorem 7.2: In a metric space M the property (abc) and (abd) imply (acd) or (adc) for any four distinct $a,b,c,d \in M$ holds if and only if any minimal polygon on a finite number of distinct noncogeodesic points is not self-intersecting.

The discussion in the paragraph preceeding theorem 7.2 also shows that condition (7.1) is equivalent to $(p_i x p_j)$ and $(p_i x p_k)$ imply p_i, p_j, p_k cogeodesic for distinct p_i, p_j, p_k, x. Thus theorem 0.4 is a corollary of theorem 7.1 by the discussion following definitions 7.1 and 7.2.

§8. Proof of theorem 7.1. We will demonstrate the contrapositive, i.e. we will assume that we have a polygon Q on the vertices $p_1, \ldots, p_n$ which is self-intersecting and then produce another polygon P on $p_1, \ldots, p_n$ which is of strictly shorter length. We consider four cases:

Case 1: Suppose $Q = [\ldots ab \ldots cd \ldots]$ has an intersection of edges ab and cd where a,b,c,d are distinct and noncogeodesic. Then we claim the polygon $P = [\ldots a(b \ldots c)d \ldots]$ is strictly shorter, where this notation is meant to stand for the polygon resulting from Q when the order of the vertices from b to c is reversed while the order of the other vertices is unaltered. To establish this claim we consider two subcases.

Case 1α: $x = a$ where x is the intersection point of ab and cd, i.e. we have (cad). Then $\overline{ab} + \overline{cd} = \overline{ab} + \overline{ca} + \overline{ad} \geq \overline{ca} + \overline{bd}$. To prove length Q > length P we need in fact $\overline{ab} + \overline{cd} > \overline{ca} + \overline{bd}$ but we get this because if equality were to hold in preceding sentence then (bad), (cad) by (7.1) implies (bcd) or (cbd) which implies (bcad) or (cbad) contradicting a,b,c,d noncogeodesic.

<u>Case 1β</u>: $x \neq a$ where again x is the intersection point of ab and cd, and we can further assume $x \neq b$, $x \neq c$, $x \neq d$ because the contrary of any one of these is really the same as case 1α by a suitable change in notation. Thus we have (axb), (cxd) and a,b,c,d,x are all distinct. Then again P = [...a(b...c)d...] is strictly shorter than Q, because $\overline{ab} + \overline{cd} = \overline{ax} + \overline{xb} + \overline{cx} + \overline{xd} \geq \overline{ac} + \overline{bd}$. If this were not really a strict inequality in the last step, then (axc) and (bxd). But (axc) and (axb) imply (axbc) or (axcb); (dxb) and (dxc) imply (dxbc) or (dxcb). Having together (axcb) and (dxbc) is impossible because (xcb) and (xbc) are incompatible with $b \neq c$. Similarly (axbc) and (dxcb) can't hold simultaneously. But (axbc) and (dxbc) imply (axc) and (dxc) imply (adxc) or (daxc) implies (adxbc) or (daxbc). Similarly (axcb) and (dxcb) lead to a,b,c,d cogeodesic, again contradicting our initial assumption for case 1.

We now consider the situation Q = [...ab...cd...] has an intersection of edges ab and cd where a,b,c,d are distinct and cogeodesic. The cogeodesic ordering of a,b,c,d can be assumed to be one of (abcd), (abdc), (acbd), (acdb), (adbc), or (adcb) by appropriately writing Q . We note that the first ordering is incompatible with the assumption that ab and cd intersect because letting x be the intersection point, whether x is distinct from a,b,c,d or not, $\overline{ad} = \overline{ab} + \overline{bc} + \overline{cd} = \overline{ax} + \overline{xb} + \overline{bc} + \overline{cx} + \overline{xd} \geq \overline{ad} + \overline{bx} + \overline{bc} + \overline{cx}$ implies $\overline{bc} = 0$ implies $b = c$. Similarly (abdc) is impossible.

Case 2: $Q = [\ldots ab \ldots cd \ldots]$ has an intersection on edges ab and cd where a,b,c,d are distinct and (acbd) or (acdb). Then $P = [\ldots a(b \ldots c)d \ldots]$ is strictly shorter than Q because if (acbd): $\overline{ab} + \overline{cd} = \overline{ac} + \overline{cb} + \overline{bd} > \overline{ac} + \overline{bd}$. And similarly if (acdb).

Case 3: $Q = [\ldots ab \ldots cd \ldots]$ has an intersection on edges ab and cd where a,b,c,d are distinct and (adbc) or (adcb). Note we can also assume in this case that Q has no intersections of type 1 or 2, i.e. of the kind in cases 1 and 2. But before continuing with this case let us first see what other possible kinds of intersection might remain.

Case 4: $Q = [\ldots abc \ldots]$ has an intersection point $x \neq b$ on edges ab and bc where a,b,c are distinct. Suppose $x \neq a$, $x \neq c$. Then (axb), (bxc) imply (acb) or (bac). Suppose $x = a$ ($x = c$ will be similar). Then (bac). Since by assumption all the vertices of the polygon are not cogeodesic, we can assert that there must be another vertix, say $Q = [\ldots abcv \ldots]$ if (acb), otherwise say $Q = [\ldots uabc \ldots]$ if (bac). In the first situation edges ab and cv intersect in c; in the second situation edges ua and bc intersect in a . If we assume, as we may, that all intersections of types 1 and 2 have been eliminated the intersection we have just produced must be of type 3. Thus all that remains is to return and argue case 3.

Suppose then that $Q = [\ldots r_m ab q_1 \ldots q_\ell cd r_1 \ldots]$, that ab and cd have an intersection of type 3 and that all intersections of types 1 and 2 have already been removed by the methods previously indicated. We will argue by example. Say (adbc). Then bq_1 and cd intersect in b and since the intersection must be of type 3, either (dbq_1c) or $(dbcq_1)$. If (dbq_1c) then q_1q_2 intersects cd at q_1 and therefore either (dq_1q_2c) or (dq_1cq_2). If $(dbcq_1)$ then bq_1 intersects $q_\ell c$ at c and either (bcq_1q_2) or $(bcq_\ell q_1)$. Continuing in this manner eventually the q's will be exhausted and an intersection of type 4 will be reached, either $(q_{\nu+1}q_{\nu-1}q_\nu)$ or $(q_{\nu-1}q_{\nu+1}q_\nu)$. The same procedure can be run through the r's also until another intersection of type 4 is reached, either $(r_\mu r_{\mu+1}r_{\mu-1})$ or $(r_\mu r_{\mu-1}r_{\mu+1})$. This argument produces a new ordering of the vertices, something perhaps like

$[r_\mu \ldots r_2 r_{m-1} r_m \; a \; r_1 \; db \; q_1q_2 \; c \; q_\ell q_3 q_{\ell-1} q_{\ell-2} \ldots q_\nu]$ which we call P.

By the construction,

$$\text{length of } P = \sum_{k=\mu}^{m-1} \overline{r_k r_{k+1}} + \overline{r_m a} + \overline{ab} + \overline{bq_1} + \sum_{k=1}^{\nu-1} \overline{q_k q_{k+1}} + \overline{q_\nu r_\mu}$$

Comparing this to length Q, we see length $P <$ length Q if and only if

$$\overline{q_\nu r_\mu} < \sum_{k=\nu}^{\ell-1} \overline{q_k q_{k+1}} + \overline{q_\ell c} + \overline{cd} + \overline{dr_1} + \sum_{k=1}^{\mu-1} \overline{r_k r_{k+1}} .$$

But again by the construction of P equality could hold between $\overline{q_\nu r_\mu}$ and $\sum_{k=\nu}^{\ell-1} \overline{q_k q_{k+1}} + \overline{q_\ell c} + \overline{cd} + \overline{dr_1} + \sum_{k=1}^{\mu-1} \overline{r_k r_{k+1}}$ only if all the vertices were cogeodesic between q_ν and r_μ. This completes the proof.

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AUTOBIOGRAPHICAL STATEMENT

David Sanders was born October 20, 1937, the first child to one of the best high school mathematics teachers around and a woman who never failed to substitute female intuition for logical thinking. Five years, five months, five days, and five hours later the third mathematician entered the family (currently completing his doctorate at Yale) and his mother with the aforementioned perception decided that was enough children.

David did his undergraduate work at Princeton University (B.A. 1959) and continued to study math at Harvard (M.A. 1960). Leaving the academic womb to learn more about the nonmathematical world he taught for a living and decided he wanted to become a teacher.

On Christmas Day 1963 he met Judy Rose. Four months later in an arboretum at the dawn of a sunny spring day, they were married. After getting his doctorate, David looks forward to a diversified career teaching mathematics, maybe even to the lot of little geniuses Judy has promised him.