Strongly Unfoldable Cardinals Made Indestructible

by

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Abstract

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Advisor: Joel David Hamkins

I provide indestructibility results for weakly compact, indescribable and strongly unfoldable cardinals. In order to make these large cardinals indestructible, I assume the existence of a strongly unfoldable cardinal κ, which is a hypothesis consistent with $V = L$. The main result shows that any strongly unfoldable cardinal κ can be made indestructible by all $<\kappa$-closed forcing which does not collapse κ^+. As strongly unfoldable cardinals strengthen both indescribable and weakly compact cardinals, I obtain indestructibility for these cardinals also, thereby reducing the large cardinal hypothesis of previously known indestructibility results for these cardinals significantly. Finally, I use the developed methods to show the consistency of a weakening of the Proper Forcing Axiom PFA relative to the existence of a strongly unfoldable cardinal.

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Contents

Introduction

I provide in Chapter 1 a new method to obtain indestructibility for some smaller large cardinals, such as weakly compact, indescribable and strongly unfoldable cardinals. I then use the idea to prove several indestructibility results, including the construction of a forcing extension in which every strongly unfoldable cardinal becomes widely indestructible. In Chapter 2, the method is combined with another new idea, leading to significant improvements in several results of Chapter 1. The consistency result of a variant of the Proper Forcing Axiom PFA in Chapter 3 is closely related, since it combines the usual consistency proof of PFA with the ideas of Chapter 1.

Chapter 1

Indestructible Strongly Unfoldable Cardinals

Determining which cardinals can be made indestructible by which classes of forcing has been a major interest in modern set theory. Laver [Lav78] made supercompact cardinals highly indestructible, Gitik and Shelah [GS89] treated strong cardinals and Hamkins [Ham00] obtained partial indestructibility for strongly compact cardinals. I aim to extend this analysis to some *smaller* large cardinals, such as weakly compact, indescribable or strongly unfoldable cardinals. Each of these cardinals is, if consistent with ZFC, consistent with $V = L$. So is each of the large cardinal hypotheses used for the results of this and the following chapters.

The Main Theorem of this chapter makes any given strongly unfoldable cardinal κ indestructible by ${<}\kappa$-closed, κ-proper forcing. This class of posets includes all ${<}\kappa$-closed posets that are either κ^+-c.c. or ${\leq}\kappa$-strategically closed as well as finite iterations of such posets. Strongly unfoldable cardinals were introduced by Villaveces [Vil98] as a strengthening of both weakly compact cardinals and totally indescribable cardinals. The Main Theorem therefore provides similarly indestructible weakly compact and indescribable cardinals from a large cardinal hypothesis consistent with $V = L$.

The only previously known method of producing a weakly compact cardinal κ indestructible by ${<}\kappa$-closed, κ^+-c.c. forcing, was to start with a supercompact cardinal κ and apply the Laver preparation (or some alternative, such as the lottery preparation [Ham00]). Similarly, in order to obtain a totally indescribable cardinal κ indestructible by all ${\leq}\kappa$-closed forcing, one had to start with at least a strong cardinal κ and use the Gitik-Shelah method. It follows from the Main Theorem that it does, in fact, suffice to start with a strongly unfoldable cardinal κ, thereby reducing the large cardinal hypothesis significantly (see Corollary 21 and 37).

In Chapter 2, Joel Hamkins and I use the method developed here to improve the Main Theorem significantly. In fact, we show that the strongly unfoldable cardinal κ as in the Main Theorem becomes not only indestructible by ${<}\kappa$-closed, κ-proper forcing, but indestructible by all ${<}\kappa$-closed, κ^+-preserving forcing. In Chapter 3, we extend the method of Chapter 1 to prove the relative consistency of PFA ($\mathfrak{c}$-proper), a weakening of the *Proper*

Forcing Axiom PFA. While the best known upper bound for the strength of PFA is the existence of a supercompact cardinal, we obtain our result by assuming the existence of a strongly unfoldable cardinal only.

I am hoping that the theorems and ideas of this chapter will allow for similar reductions in other indestructibility results or relative consistency statements. Moreover, the described methods may help identify indestructibility for other large cardinals as well, such as for those cardinals that can be characterized by elementary embeddings which are sets. In Section 1.6, I obtain a global form of the Main Theorem: I prove that there is a class forcing extension which preserves *every* strongly unfoldable cardinal κ and makes its strong unfoldability indestructible by $<\kappa$-closed, κ-proper forcing.

Given a strongly unfoldable cardinal κ, how indestructible can we make it? Of course, if κ happens to be supercompact, then the Laver preparation of κ makes κ indestructible by all $<\kappa$-directed closed forcing. In general we cannot hope to prove such wide indestructibility for κ if we want to only rely on hypotheses consistent with $V = L$. Intuitively it seems that collapsing κ^+ to κ poses a serious problem: A strongly unfoldable cardinal κ gives for every transitive set of size κ a certain elementary embedding. If $M \in V$ is a transitive set of size κ in the forcing extension, yet M has size κ^+ in V, then there seems little reason that the strong unfoldability of κ in V provides the necessary embedding for M. Results from inner model theory confirm that this intuition is correct. For instance, if κ is weakly compact and indestructible by some $<\kappa$-closed forcing that collapses κ^+, then Jensen's Square Principle $\square_\kappa$ fails, as was pointed out to me by Grigor Sargsyan and is shown in Chapter 2. But a failure of $\square_\kappa$ for a weakly compact cardinal κ implies AD in $L(\mathbb{R})$, which has the strength of infinitely many Woodin cardinals (see [SZ01] and [Woo99]). If we want to rely on hypotheses consistent with $V = L$ only, we must therefore focus on indestructibility by posets which preserve κ^+. It is thus natural to ask for instance the following:

Question 1. *Given a strongly unfoldable cardinal κ, can we make it indestructible by all $<\kappa$-directed closed forcing that is κ^+-c.c? Or indestructible by all $\leq\kappa$-directed closed forcing?*

Already suggested in [She80] and studied intensively more recently (e.g. [RS], [Eis03]), the *κ-proper* posets have been defined for cardinals κ with $\kappa^{<\kappa} = \kappa$ as a higher cardinal analogue of *proper* posets. Similar to the proper posets, which include all forcing notions that are either c.c.c. or countably closed, the κ-proper posets include all forcing notions that are either κ^+-c.c. or $\leq\kappa$-closed. Every κ-proper poset preserves κ^+. Moreover, every finite iteration of $<\kappa$-closed, κ-proper posets is itself $<\kappa$-closed and κ-

proper (Corollary 18). Recall that proper posets can be characterized by the way in which the posets interact with *countable* elementary submodels X of H_λ for sufficiently large cardinals λ. From this characterization one obtains the definition of a κ-proper poset by generalizing "countable" to higher cardinals κ (see Section 1.3). This interaction with elementary submodels $X \prec H_\lambda$ of size κ is exactly what allowed me to handle posets of arbitrary size in the proof of the Main Theorem.

Main Theorem. *Let κ be strongly unfoldable. Then there is a set forcing extension in which the strong unfoldability of κ is indestructible by $<\kappa$-closed, κ-proper forcing of any size. This includes all $<\kappa$-closed posets that are either κ^+-c.c. or $\leq\kappa$-strategically closed as well as finite iterations of such posets.*

It follows that the existence of a strongly unfoldable cardinal κ indestructible by $<\kappa$-closed, κ-proper forcing is equiconsistent over ZFC with the existence of a strongly unfoldable cardinal. Moreover, since strongly unfoldable cardinals are totally indescribable and thus weakly compact, the theorem provides a method of making these two classic large cardinal notions indestructible by $<\kappa$-closed, κ-proper forcing.

The Main Theorem thus answers Question 1 affirmatively. At the beginning of Section 1.4, I will illustrate why the class of κ-proper posets is a natural collection of posets to consider when one tries to make strongly unfoldable cardinals indestructible. Observe that a strongly unfoldable cardinal κ is not always indestructible by $<\kappa$-closed, κ-proper forcing: If $\kappa \in V$ is strongly unfoldable, then κ is strongly unfoldable in L (see [Vil98]), but forcing over L, with for instance the poset to add a Cohen subset of κ, destroys the weak compactness of κ and thus its strong unfoldability (see Fact 26). Moreover, Hamkins showed in [Ham98] that any nontrivial small forcing over any ground model makes a weakly compact cardinal κ similarly destructible (see Theorem 27). Of course, the strong unfoldability of κ is then destroyed as well.

Note that we do not insist on $<\kappa$-*directed* closure in the statement of the Main Theorem. We insist merely on $<\kappa$-closure. This is a significant improvement since the usual indestructibility results for measurable or larger cardinals (such as [Lav78], [GS89] and [Ham00]) can never obtain indestructibility by all $<\kappa$-closed, κ-proper forcing. In fact, no ineffable cardinal κ can ever exhibit this degree of indestructibility (see Fact 30).

The proof of the Main Theorem employs the *lottery preparation*, a general tool invented by Hamkins [Ham00] to force indestructibility. The lottery preparation of a cardinal κ is defined relative to a function $f \colon \kappa \to \kappa$ and works best if f has what Hamkins calls the *Menas* property for κ. Since

Woodin's *fast function* forcing adds such a function, the lottery preparation is often assumed to be performed after some preliminary fast function forcing. For a strongly unfoldable cardinal κ though, it turns out that we do not need to do any prior forcing; a function with the Menas property for κ already exists (see Section 1.2).

The Main Theorem uses the lottery preparation of a strongly unfoldable cardinal κ to make it indestructible by all $<\kappa$-closed, κ-proper forcing. The strategy is to take the embedding characterization of strongly unfoldable cardinals and borrow lifting techniques of strong cardinals as well as those of supercompact cardinals in order to lift the ground model embeddings. I thereby follow Hamkins' strategy, who was first to use these kind of lifting arguments in the strongly unfoldable cardinal context [Ham01]. But can we obtain more indestructibility than the Main Theorem identifies? We saw the need to focus on posets which do not collapse κ^+, which therefore suggests the following question:

Question 2. *Can any given strongly unfoldable cardinal κ be made indestructible by all $<\kappa$-closed, κ^+-preserving forcing?*

In joint work with Joel Hamkins I was able to answer Question 2 affirmatively, thereby providing as much indestructibility for strongly unfoldable cardinals as we could hope for. The proof builds on the method presented in Section 1.4 and is given in Chapter 2. But how about other large cardinals? The following question remains completely open:

Question 3. *Can any given weakly compact cardinal κ be made indestructible by all ${<}\kappa$-closed, κ^+-preserving forcing? Or at least indestructible by ${<}\kappa$-closed, κ^+-c.c. forcing? And how about totally indescribable cardinals or Ramsey cardinals?*

In Section 1.6, I will apply the Main Theorem simultaneously to all strongly unfoldable cardinals and obtain the following result.

Main Theorem (Global Form). *If V satisfies* ZFC*, then there is a class forcing extension $V[G]$ satisfying* ZFC *such that*

1. *every strongly unfoldable cardinal of V remains strongly unfoldable in $V[G]$,*
2. *in $V[G]$, every strongly unfoldable cardinal κ is indestructible by ${<}\kappa$-closed, κ-proper forcing, and*
3. *no new strongly unfoldable cardinals are created.*

I review strongly unfoldable cardinals in Section 1.1 and show in Section 1.2 that there exists a class function $F : Ord \to Ord$, which exhibits the

Menas property for every strongly unfoldable cardinal simultaneously. Section 1.3 reviews κ-proper posets and in Section 1.4, I prove the Main Theorem using lifting techniques similar to those of supercompact cardinals. I mention some limitations and variations of the Main Theorem in Section 1.5 and also provide several destructibility results. The result, which makes all strongly unfoldable cardinals simultaneously indestructible, is proved in Section 1.6. In Section 1.7, I apply the Main Theorem to both totally indescribable cardinals and partially indescribable cardinals. To do so, I first prove a local analogue of the Main Theorem for a θ-strongly unfoldable cardinal with θ a successor ordinal. Section 1.8 addresses and solves the issue one faces when trying to prove the corresponding analogue for a θ-strongly unfoldable cardinal with θ a limit ordinal. Interestingly, this result provides a second and totally different alternative proof of the Main Theorem. The case when θ is a limit ordinal seems to require lifting techniques similar to those of strong cardinals. The fact that strongly unfoldable cardinals mimic both supercompact cardinals and strong cardinals allows for these two different proofs. At the end of Section 1.8, I state the local version of the Main Theorem in its strongest form.

1.1 Strongly Unfoldable Cardinals

Following [DH06], I review several characterizations of strongly unfoldable cardinals. In [Vil98] Villaveces introduced strongly unfoldable cardinals. It turns out that they are exactly what Miyamoto calls the (H_{κ^+})-*reflecting* cardinals in [Miy98]. Strongly unfoldable cardinals strengthen weakly compact cardinals similarly to how strong cardinals strengthen measurable cardinals. Their consistency strength is well below measurable cardinals, and if they exist, then they exist in the universe of constructible sets L. It was discovered independently that strongly unfoldable cardinals also exhibit some of the characteristics of supercompact cardinals (see [Miy98] and [DH06]).

While measurable cardinals are characterized by elementary embeddings whose domain is all of V, strongly unfoldable cardinals carry embeddings whose transitive domain mimics the universe V, yet is a *set* of size κ. Let ZFC^- denote the theory ZFC without the Power Set Axiom. For an inaccessible cardinal κ, we call a transitive structure of size κ a κ-*model* if $M \models \mathrm{ZFC}^-$, the cardinal $\kappa \in M$ and $M^{<\kappa} \subseteq M$.

Fix any κ-model M. Induction shows that $V_\kappa \subseteq M$ and hence the Replacement Axiom in M implies that $V_\kappa \in M$. Note that M satisfies enough of the ZFC-Axioms to allow forcing over M. Moreover, for inaccessible κ, there are plenty of κ-models. For instance, if $\lambda > \kappa$ is any regular cardinal, we may use the Skolem-Löwenheim method to build an elementary submodel X of size κ with $X \prec H_\lambda$ and $\kappa \in X$ such that $X^{<\kappa} \subseteq X$. The Mostowski collapse of X is then a κ-model. This argument also shows that any given set $A \in H_{\kappa^+}$ can be placed into a κ-model, since making sure that $\mathrm{trcl}(\{A\}) \subseteq X$ implies that A is fixed by the Mostowski collapse.

Definition 4 ([Vil98])**.** Fix any ordinal θ. A cardinal κ is *θ-strongly unfoldable* if κ is inaccessible and for any κ-model M there is an elementary embedding $j : M \to N$ with critical point κ such that $\theta < j(\kappa)$ and $V_\theta \subseteq N$. A cardinal κ is *strongly unfoldable* if κ is θ-strongly unfoldable for every ordinal θ.

One can show that a cardinal κ is *weakly compact* if and only if κ is κ-strongly unfoldable [Vil98]. Unlike Villaveces, who requires $\theta \leq j(\kappa)$, I insist in Definition 4 on strict inequality between θ and $j(\kappa)$. The two definitions are equivalent, as one can see by an argument given in the context of *unfoldable* cardinals in [Ham].

From now on, when I write $j : M \to N$, I mean implicitly that j is an elementary embedding with critical point κ and both M and N are transitive sets. I will refer to embeddings $j : M \to N$ where M is a κ-model, $\theta < j(\kappa)$

and $V_\theta \subseteq N$ as *θ-strong unfoldability* embeddings for κ. We will use the following previously known characterizations of θ-strong unfoldability:

Fact 5. *Let κ be inaccessible and $\theta \geq \kappa$ any ordinal. The following are equivalent.*

1. *κ is θ-strongly unfoldable.*

2. *(Extender embedding) For every κ-model M there is a θ-strong unfoldability embedding $j : M \to N$ such that $N = \{j(g)(s) \mid g : V_\kappa \to M$ with $g \in M$ and $s \in S^{<\omega}\}$ where $S = V_\theta \cup \{\theta\}$.*

3. *(Hauser embedding) For every κ-model M there is a θ-strong unfoldability embedding $j : M \to N$ such that $|N| = \beth_\theta$ and $j \in N$ has size κ in N.*

4. *For every $A \subseteq \kappa$ there is a κ-model M and a θ-strong unfoldability embedding $j : M \to N$ such that $A \in M$.*

5. *For every $A \subseteq \kappa$ there is a transitive set M satisfying ZFC^- of size κ containing both A and κ as elements with a corresponding elementary embedding $j : M \to N$ such that $V_\theta \subseteq N$ and $\theta < j(\kappa)$.*

Proof. The implication (1)$\to$(2) is proved the same way how one produces canonical extender embeddings for θ-strong cardinals. The proof that (2) implies (3) essentially follows from Hauser's trick of his treatment of indescribable cardinals [Hau91], for a proof see [DH06]. For the other assertions, since every subset of κ can be placed into a κ-model, it suffices to prove that (5) implies (1). Thus, suppose that M' is any κ-model. Code it by a relation A on κ via the Mostowski collapse, and fix M and $j : M \to N$ with $A \in M$ as provided by (5). Since $M \models \mathrm{ZFC}^-$, it can decode A, and thus we have $M' \in M$. As M' is closed under ${<}\kappa$-sequences and $\theta < j(\kappa)$, it follows by elementarity that N thinks that $V_\theta \subseteq j(M')$. N is correct and we see that $j \restriction M' : M' \to j(M')$ is the desired θ-strong unfoldability embedding. □

The next fact illustrates the way in which strongly unfoldable cardinals also mimic supercompact cardinals. It allows us to use lifting arguments similar to those of supercompact cardinals when proving the Main Theorem.

Fact 6 ([DH06])**.** *If κ is $(\theta+1)$-strongly unfoldable, then for every κ-model M there is a $(\theta+1)$-strong unfoldability embedding $j : M \to N$ such that $N^{\beth_\theta} \subseteq N$ and $|N| = \beth_{\theta+1}$. If κ is θ-strongly unfoldable and θ is a limit ordinal, then for every κ-model M there is a θ-strong unfoldability embedding such that $N^{<cof(\theta)} \subseteq N$ and $|N| = \beth_\theta$.*

If the GCH holds at $\delta = \beth_\theta$, we obtain in Fact 6, a $(\theta+1)$-strong unfoldability embedding $j : M \to N$ such that N has size δ^+ and $N^\delta \subseteq N$. Note that $\delta < j(\kappa)$, as $j(\kappa)$ is inaccessible in N. This special case allows for diagonalization arguments, as in Section 1.4. Moreover, by forcing if necessary, we can simply assume that the GCH holds at $\beth_\theta$ for any given $(\theta+1)$-strongly unfoldable cardinal κ:

Lemma 7. *If κ is $(\theta+1)$-strongly unfoldable for some $\theta \geq \kappa$ and $\mathbb{P}$ is any $\leq\beth_\theta$-distributive poset, then κ remains $(\theta+1)$-strongly unfoldable after forcing with $\mathbb{P}$. In particular, we can force the* GCH *to hold at $\beth_\theta$ while preserving any $(\theta+1)$-strongly unfoldable cardinal κ.*

Proof. Fix any $\leq\beth_\theta$-distributive poset $\mathbb{P}$. Let $G \subseteq \mathbb{P}$ be V-generic. Fix any κ-model $M \in V[G]$. As $\mathbb{P}$ is $\leq\kappa$-distributive, we see that $M \in V$. We may thus fix in V an embedding $j : M \to N$ with $V_{\theta+1} \subseteq N$ and $\theta < j(\kappa)$. Because the forcing is $\leq\beth_\theta$-distributive, it follows that $(V_{\theta+1})^V = (V_{\theta+1})^{V[G]}$, and j is hence the desired $(\theta+1)$-strong unfoldability embedding in $V[G]$. □

We will use the results from [Ham03] in Section 1.5 to show that after nontrivial forcing of size less than κ, a strongly unfoldable cardinal κ becomes highly destructible. All applications of the Main Theorem from [Ham03] need a cofinal elementary embedding whose target is highly closed, so let me show how this can be achieved for most θ-strongly unfoldable cardinals κ. Note first that a map $j : M \to N$ with $j \in N$ and $N \models \mathrm{ZFC}^-$ can never be cofinal: As $j \in N$ and $M \in N$, we have that $j"M$ is a set in N and therefore certainly not an unbounded class in N. It follows that θ-strong unfoldability embeddings $j : M \to N$ of κ with $N^\kappa \subseteq N$ can never be cofinal. For the same reason, Hauser embeddings as in assertion (3) of Fact 5 are not cofinal.

Lemma 8. *Let κ be a θ-strongly unfoldable cardinal for some $\theta \geq \kappa$. Suppose that θ is either a successor ordinal or cof(θ) $\geq \kappa$. Then for every κ-model M there is a θ-strong unfoldability embedding $j : M \to N$ such that j is cofinal and $N^{<\kappa} \subseteq N$.*

Proof. Fix any κ-model M. Suppose $\theta \geq \kappa$ is either a successor ordinal or $\mathrm{cof}(\theta) \geq \kappa$. By Fact 6, there is a θ-strong unfoldability embedding $j : M \to N$ such that $N^{<\kappa} \subseteq N$. As seen above, there is no reason to think that j is cofinal. Yet, by restricting the target of j to $N_0 = \bigcup j"M$, I claim that $j : M \to N_0$ is the desired θ-strong unfoldability embedding. It is crucial that $j : M \to N_0$ remains an elementary embedding. This is shown by induction on the complexity of formulas. It is then easy to see that $j : M \to N_0$ is a cofinal θ-strong unfoldability embedding. To see that N_0 is closed under $<\kappa$-sequences, note first that Ord^M is an ordinal with cofinality κ, since $M^{<\kappa} \subseteq M$. It follows that Ord^{N_0} has cofinality κ. If $s \in (N_0)^{<\kappa}$ is any sequence of less than κ many elements from N_0, then $s \in N$ by the closure of

N and rank(s) is bounded in Ord^{N_0}. This shows that $s \in N_0$ as desired. $\square$

1.2 A Menas Function for all Strongly Unfoldable Cardinals

I show in Theorem 11 that there is a function $F : Ord \to Ord$ such that for every strongly unfoldable cardinal κ, the restriction $F \upharpoonright \kappa$ is what Hamkins calls a *Menas* function for κ. This will allow us to use Hamkins' lottery preparation *directly*, without any preliminary forcing to add such a function.

For a θ-strongly unfoldable cardinal κ, I follow [Ham00] and say that a function $f : \kappa \to \kappa$ has the *(θ-strong unfoldability) Menas property for κ* if for every κ-model M with $f \in M$, there is a θ-strong unfoldability embedding $j : M \to N$ such that $j(f)(\kappa) \geq \beth_\theta^N$. Note that $\beth_\theta^N \geq \beth_\theta$ and we have equality if θ is a limit ordinal (see for instance the proof of Lemma 9). I insist that $j(f)(\kappa) \geq \beth_\theta^N$ since I want N to see that $|V_\theta| \leq j(f)(\kappa)$. This will be crucial for the lifting arguments of Theorem 43 in Section 1.8. Arguments in [Ham01] show that given a θ-strongly unfoldable cardinal κ, a function with the Menas property for κ can be added by Woodin's *fast function forcing*. But, as assertion (2) of Theorem 11 shows below, we do not have to force to have such a function. A canonical function f with the Menas property for κ already always exists.

Observe that we may assume without loss of generality that an embedding j witnessing the Menas property of f for κ is an extender embedding. In order to see this, simply follow the proof of assertion (2) of Fact 5 and use the embedding j to obtain an extender embedding $j_0 : M \to N_0$ with $j_0(f)(\kappa) \geq \beth_\theta^{N_0}$. In fact, when given a function f with the Menas property for κ, we may assume without loss of generality that an embedding j witnessing the Menas property of f satisfies *any* of the equivalent characterizations of Fact 5 or Fact 6. This follows again from the corresponding proofs of the two facts.

As expected, we say for a strongly unfoldable cardinal κ that $f \vdots \kappa \to \kappa$ has the *(strong unfoldability) Menas property* for κ, if for *every* ordinal θ, the function f has the θ-strong unfoldability Menas property for κ. Again, fast function forcing adds such a function. But, as assertion (1) of Theorem 11 shows, we do not have to force to have such a function, because it already exists.

In order to prove Theorem 11, we first need two lemmas. Let us say that a cardinal κ is *$<\theta$-strongly unfoldable* if κ is α-strongly unfoldable for every $\alpha < \theta$. Note that for $\theta \leq \kappa$, every $<\theta$-strongly unfoldable cardinal is in fact κ-strongly unfoldable and thus weakly compact.

Lemma 9. *Let κ be a θ-strongly unfoldable cardinal for some ordinal $\theta > \kappa$.*

If M is a κ-model and $j : M \to N$ is a θ-strong unfoldability embedding for κ, then κ is $<\!\theta$-strongly unfoldable in N.

Proof. Fix any θ-strong unfoldability embedding $j : M \to N$ for κ. We know by assertion (2) of Fact 5 that for ordinals $\alpha \geq \kappa$ the α-strong unfoldability of κ is characterized by the existence of extender embeddings j of transitive size $\beth_\alpha$. As $\theta > \kappa$, it thus suffices to show that for every α with $\kappa \leq \alpha < \theta$ the model N contains all these extender embeddings as elements. Fix thus any such α. I first claim that $\beth_\xi^N = \beth_\xi$ and $H_{\beth_\xi^+} \subseteq N$ for every $\xi < \theta$. As M is a κ-model, we see by elementarity that $\beth_\xi^N$ exists for every $\xi \leq j(\kappa)$. As $V_\theta \subseteq N$, it follows by induction that $\beth_\xi^N = \beth_\xi$ for each $\xi < \theta$. Thus, for each $\xi < \theta$, $P(\beth_\xi) \subseteq N$ (since for ordinals $\xi \geq \omega^2$ the power set $P(\beth_\xi)$ corresponds in N to $P(V_\xi)$ and $P(V_\xi) \subseteq V_\theta \subseteq N$). But elements of $H_{\beth_\xi^+}$ are coded via the Mostowski collapse by elements of $P(\beth_\xi)$ and the claim follows. Since $\alpha < \theta$, we see that $H_{\beth_\alpha^+} \subseteq N$. This shows that N contains all the necessary extender embeddings. □

Assertion (4) of Fact 5 allows us to switch between κ-models and subsets of κ as we desire, while assertion (5) frees us from insisting that the domain M of the embeddings has to be closed under $<\!\kappa$-sequences, a requirement that need not be upwards absolute. It follows that, if $N \subseteq V$ is a transitive class with $P(\kappa) \cup V_\theta \subseteq N$ and N thinks that κ is θ-strongly unfoldable, then κ is indeed θ-strongly unfoldable.

Lemma 10. *Suppose that κ is θ-strongly unfoldable. For every κ-model M there is a θ-strong unfoldability embedding $j : M \to N$ such that κ is not θ-strongly unfoldable in N.*

Proof. Fix any κ-model M'. Let $A \subseteq \kappa$ code M' via the Mostowski collapse. Fix an elementary embedding $j : M \to N$ as in characterization (5) of Fact 5 with $A \in M$ and $V_\theta \subseteq N$ such that N has least Levy rank. The set $A = j(A) \cap \kappa$ is an element of N. But, in N, there cannot exist a θ-strong unfoldability embedding $j_0 : M_0 \to N_0$ with $A \in M_0$: Such an embedding $j_0 \in N_0$ would by absoluteness really be an embedding as in characterization (5) of Fact 5, which would therefore contradict our choice of j since $N_0 \in N$. It follows that κ is not θ-strongly unfoldable in N. The restriction $j \restriction M' : M' \to j(M')$ is then the desired embedding. □

Lemma 9 and Lemma 10 have the following consequence.

Theorem 11. *There is a function $F : Ord \to Ord$ such that*

1. *If κ is strongly unfoldable, then $F``\kappa \subseteq \kappa$ and the restriction $F \restriction \kappa$ has the Menas property for κ. Moreover, every κ-model contains $F \restriction \kappa$ as an element.*

2. *If κ is θ-strongly unfoldable for some ordinal $\theta \geq \kappa$, then the restriction $F \cap (\kappa \times \kappa)$ has the θ-strong unfoldability Menas property for κ. Moreover, every κ-model contains $F \cap (\kappa \times \kappa)$ as an element.*

3. *The domain of F does not contain any strongly unfoldable cardinals.*

Proof. Let $F : Ord \to Ord$ be defined as follows: If ξ is a strongly unfoldable cardinal, then let $F(\xi)$ be undefined; otherwise let $F(\xi) = \beth_\eta$ where η is the least ordinal $\alpha \geq \xi$ such that ξ is not α-strongly unfoldable. Note that $F(\xi) \geq \xi$ for all $\xi \in \text{dom}(F)$. This will be used to prove assertion (2) in the case when $\theta = \kappa$.

For assertion (1), fix any strongly unfoldable cardinal κ. Let us first see that $F``\kappa \subseteq \kappa$. Suppose that $\xi < \kappa$ is $<\kappa$-strongly unfoldable. I claim that ξ is in fact strongly unfoldable and thus $\xi \notin \text{dom}(F)$. To verify the claim, fix any ordinal $\theta \geq \kappa$, any κ-model M and a corresponding θ-strong unfoldability embedding $j : M \to N$ for κ. In particular, $\text{crit}(j) = \kappa$. Since M sees that ξ is $<\kappa$-strongly unfoldable and $\theta < j(\kappa)$, it follows by elementarity that N thinks that $j(\xi)$ is θ-strongly unfoldable. As $j(\xi) = \xi$ and $V_\theta \subseteq N$, we see that N is correct. The cardinal ξ is thus θ-strongly unfoldable in V. Since θ was arbitrary, we verified the claim and thus $F``\kappa \subseteq \kappa$.

To see that every κ-model contains $F \restriction \kappa$ as an element, suppose that

$\xi < \kappa$ is α-strongly unfoldable for some $\alpha < \kappa$. Since this is witnessed by extender embeddings which are elements of V_κ, the definition of $F \upharpoonright \kappa$ is *absolute* for any κ-model. Consequently, every κ-model contains $F \upharpoonright \kappa$ as an element, as desired.

To verify the Menas property of $F \upharpoonright \kappa$ in assertion (1), fix any κ-model M. Let θ be any ordinal that is strictly bigger than κ. By Lemmas 9 and 10 there is a θ-strong unfoldability embedding $j : M \to N$ such that κ is not θ-strongly unfoldable in N, yet κ is ${<}\theta$-strongly unfoldable in N. Since the definition of $F \upharpoonright \kappa$ is absolute for M and $F \upharpoonright \kappa \in M$, it follows that $j(F \upharpoonright \kappa)(\kappa) = \beth_\theta^N$. This verifies the Menas property of $F \upharpoonright \kappa$ for κ and completes the proof of assertion (1).

For assertion (2), fix any θ-strongly unfoldable cardinal κ for some ordinal $\theta \geq \kappa$. Restricting the domain of F now to only those $\xi < \kappa$ which are not ${<}\kappa$-strongly unfoldable makes the definition of $F \cap (\kappa \times \kappa)$ absolute for κ-models. Consequently, every κ-model contains $F \cap (\kappa \times \kappa)$ as an element. The Menas property of $F \cap (\kappa \times \kappa)$ follows thus exactly as in assertion (1) as long as θ is strictly bigger than κ. But if $\theta = \kappa$, we cannot use Lemma 9. In this case, since we defined F in such a way that $F(\xi) \geq \xi$ for all $\xi \in \mathrm{dom}(F)$, it follows from Lemma 10 directly that $F \cap (\kappa \times \kappa)$ has the Menas property for κ. This completes the proof of assertion (2). Assertion (3) is clear. $\square$

Observe that in assertion (2) of Theorem 11 we cannot avoid restricting $F \upharpoonright \kappa$ to $F \cap (\kappa \times \kappa)$: If κ is not θ-strongly unfoldable for some $\theta \geq \kappa$, then any $\xi < \kappa$ which is θ-strongly unfoldable, but not strongly unfoldable, will have $F(\xi) > \theta \geq \kappa$. This shows that $F"\kappa \nsubseteq \kappa$. Consequently, $F \upharpoonright \kappa$ does not technically have the Menas property for κ even though $F \cap (\kappa \times \kappa)$ does.

1.3 κ-Proper Forcing

We review κ-proper posets as defined in [RS] and [Eis03], provide a few necessary facts about them and prove an important lemma (Lemma 17) for the Main Theorem. Since several arguments in this section are direct analogues of well known arguments for proper forcing, the reader may also compare the following material with any standard source on proper forcing (e.g. [She98], [Jec03]).

Suppose $\langle N, \in \rangle$ is a transitive model of ZFC$^-$. Let $\langle X, \in \rangle$ be an elementary substructure of $\langle N, \in \rangle$, not necessarily transitive. Assume $\mathbb{P} \in X$ is a poset and $G \subseteq \mathbb{P}$ a filter on $\mathbb{P}$. Let $X[G] = \{\tau_G \mid \tau \text{ is a } \mathbb{P}\text{-name with } \tau \in X\}$. If G is an N-generic filter, it is a well known fact that $X[G] \prec N[G]$.

The filter G is *X-generic* for $\mathbb{P}$ if for every dense set $D \in X$, we have $G \cap D \cap X \neq \emptyset$. In other words, an X-generic filter meets every dense set $D \in X$ *in* X. For transitive sets X this condition coincides with the usual requirement for a filter to be X-generic. Thus, if $\pi : \langle X, \in\rangle \to \langle M, \in\rangle$ is the Mostowski collapse of X, then G is X-generic for $\mathbb{P}$ if and only if π"G is an M-generic filter for the poset $\pi(\mathbb{P})$. It is a standard result that a V-generic filter G on $\mathbb{P}$ is X-generic if and only if $X[G] \cap V = X$. A condition $p \in \mathbb{P}$ is said to be X*-generic* (or $(X, \mathbb{P})$*-generic*) if every V-generic filter $G \subseteq \mathbb{P}$ with $p \in G$ is X-generic.

Proper posets were introduced by Shelah as a common generalization of c.c.c. posets and countably closed posets. Recall Shelah's characterization of proper posets that looks at the way in which the posets interact with elementary submodels of H_λ:

Definition 12. A poset $\mathbb{P}$ is *proper* if for all regular $\lambda > 2^{|\mathbb{P}|}$ and for all countable $X \prec H_\lambda$ with $\mathbb{P} \in X$, there exists for every $p \in \mathbb{P} \cap X$ an X-generic condition below p.

Already suggested in [She80], one obtains the definition of a κ-proper poset by essentially generalizing "countable" to higher cardinalities κ. There is a subtle difference though: It can be shown that properness can be defined equivalently by weakening the quantification "for all countable $X \prec H_\lambda \ldots$" to "for a closed unbounded set of countable $X \prec H_\lambda \ldots$". This other characterization of a proper poset shows that properness is a reasonably robust property, one that is for instance preserved by isomorphisms. In the case of κ-properness, I will prove this preservation directly in Fact 14.

Definition 13 (Shelah, [RS])**.** Assume that κ is a cardinal with $\kappa^{<\kappa} = \kappa$. A poset $\mathbb{P}$ is *κ-proper* if for all sufficiently large regular λ there is an $x \in H_\lambda$ such that for all $X \prec H_\lambda$ of size κ with $X^{<\kappa} \subseteq X$ and $\{\kappa, \mathbb{P}, x\} \in X$, there exists for every $p \in \mathbb{P} \cap X$ an X-generic condition below p.

Definition 13 is a bit subtle, as for every sufficiently large regular cardinal λ we have to consider possibly very different witnessing parameters $x \in H_\lambda$ and restrict ourselves to only those elementary substructures $X \prec H_\lambda$ which contain x as an element. Yet, it seems to me that the preservation of κ-properness by isomorphisms as in assertion (1) of Fact 14 makes essential use of this technicality. We will call any such parameter $x \in H_\lambda$ as in Definition 13 a *λ-witness* for (the κ-properness of) $\mathbb{P}$. Note that proper posets are simply $\aleph_0$-proper posets[1].

There are a few different definitions of κ-properness in the literature. Our definition is exactly the same as the one presented in [RS] and [Ros]. Moreover, the definition of a κ-proper poset as in [Eis03] is equivalent to our definition. This follows from the fact that for an uncountable cardinal κ

[1]This is not to be confused with the very different definition of an α-proper poset for a countable ordinal α (see for instance in [She98]), which we will not be concerned with.

with $\kappa^{<\kappa} = \kappa$, *every* elementary submodel $X \prec H_\lambda$ of size κ with $X^{<\kappa} \subseteq X$ has what Eisworth calls a *filtration* of X: If $X = \{x_\alpha \mid \alpha < \kappa\}$ is such an elementary submodel, then it is easy to construct a filtration $\langle X_\alpha : \alpha < \kappa \rangle$ of X inductively; simply take unions at limit steps and choose an elementary submodel $X_{\alpha+1} \prec X$ of size less than κ at successor steps in such a way that $\{x_\alpha, \langle X_\beta : \beta \leq \alpha \rangle\} \cup X_\alpha \subseteq X_{\alpha+1}$.

Definition 13 differs slightly from [She80], where the substructures X are not required to be ${<}\kappa$-closed and generic conditions are only required for a closed unbounded set of elementary substructures. Definition 13 also differs from the notion of a κ-proper poset as defined in [HR01]. There, the authors generalize Definition 12 directly and hence omit the use of λ-witnesses. They also insist that $\mathbb{P}$ is ${<}\kappa$-closed in order for $\mathbb{P}$ to be considered κ-proper. It is not clear to me whether their definition of κ-properness is preserved by isomorphisms.

Fact 14 generalizes corresponding statements about proper posets. These results show that for a cardinal κ with $\kappa^{<\kappa} = \kappa$ we have many κ-proper posets. Assertion (7) shows that κ-proper posets preserve κ^+. For the definition of ${\leq}\kappa$-strategic closure, see the remarks before Fact 23.

Fact 14. *Suppose that κ is a cardinal with $\kappa^{<\kappa} = \kappa$, and $\mathbb{P}$ and $\mathbb{Q}$ are any posets. Then:*

1. *If $\mathbb{P}$ is κ-proper and $\mathbb{Q}$ is isomorphic to $\mathbb{P}$, then $\mathbb{Q}$ is κ-proper.*

2. *If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding and $\mathbb{Q}$ is κ-proper, then $\mathbb{P}$ is κ-proper.*

3. *If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, then $\mathbb{P}$ is κ-proper if and only if $\mathbb{Q}$ is κ-proper.*

4. *If $\mathbb{P}$ is a κ^+-c.c. poset, then $\mathbb{P}$ is κ-proper.*

5. *If $\mathbb{P}$ is a $\leq\kappa$-closed poset, then $\mathbb{P}$ is κ-proper.*

6. *If $\mathbb{P}$ is a $\leq\kappa$-strategically closed poset, then $\mathbb{P}$ is κ-proper.*

7. *If $\mathbb{P}$ is a κ-proper poset, then $\mathbb{P}$ preserves κ^+.*

Proof. This is a straightforward generalization of the corresponding proofs for proper forcing. To illustrate, I prove assertion (1). Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is an isomorphism between the posets $\mathbb{P}$ and $\mathbb{Q}$. Suppose that $\mathbb{P}$ is a κ-proper poset. Then there is a cardinal $\lambda_{\mathbb{P}}$ such that all regular $\lambda \geq \lambda_{\mathbb{P}}$ are sufficiently large to witness the κ-properness of $\mathbb{P}$ as in Definition 13. Fix now any $\lambda > \mathrm{trcl}(\{\mathbb{P}, \mathbb{Q}, i, \lambda_{\mathbb{P}}\})$ and some corresponding λ-witness $x_{\mathbb{P}} \in H_\lambda$ for the κ-properness of $\mathbb{P}$. To see that $\mathbb{Q}$ is κ-proper, it suffices to show that for all $X \prec H_\lambda$ of size κ with $X^{<\kappa} \subseteq X$ and $\{\kappa, \mathbb{P}, \mathbb{Q}, i, x_{\mathbb{P}}\} \in X$, there exists for every $q \in \mathbb{Q} \cap X$ an $(X, \mathbb{Q})$-generic condition below q. Fix thus any such elementary substructure $X \prec H_\lambda$ and a condition $q \in \mathbb{Q} \cap X$. Since $\{\mathbb{P}, \mathbb{Q}, i\} \subseteq X$, it follows that $i^{-1}(q) \in \mathbb{P} \cap X$. As λ is sufficiently large, we know that there exists an $(X, \mathbb{P})$-generic condition p_0 below $i^{-1}(q)$. Since i is an isomorphism, it follows that $i(p_0)$ is the desired $(X, \mathbb{Q})$-generic condition below q. This shows that $\{\mathbb{P}, i, x_\mathbb{P}\}$ is a λ-witness for the κ-properness of $\mathbb{Q}$. As λ was chosen arbitrarily above $\mathrm{trcl}(\{\mathbb{P}, \mathbb{Q}, i, \lambda_\mathbb{P}\})$, we see that $\mathbb{Q}$ is κ-proper as desired for assertion (1). □

The following fact is well known in the specific case when X is a *transitive* set (let $X = N$) and then frequently combined with diagonalization (see Fact 19) to build generic filters. The general case is essential for us, since we will be dealing with elementary substructures $X \prec H_\lambda$ that are not necessarily transitive (e.g. in Theorem 42 as well as in Lemmas 16 and 17).

Fact 15 (Closure Fact)**.** *Let N be a transitive model of* ZFC$^-$ *and $X \prec N$ be an elementary substructure, not necessarily transitive. Suppose that $\mathbb{P} \in X$ is a poset and δ is a cardinal such that $X^{<\delta} \subseteq X$ in V. Let G denote a filter on $\mathbb{P}$. Then:*

1. *If $G \in V$ is N-generic, then $X[G]^{<\delta} \subseteq X[G]$ in V.*

2. *If $\mathbb{P}$ is $<\delta$-distributive in V and G is V-generic for $\mathbb{P}$, then $X^{<\delta} \subseteq X$ in $V[G]$ and $X[G]^{<\delta} \subseteq X[G]$ in $V[G]$.*

3. *Suppose* $\mathbb{P} \subseteq X$. *If* $\mathbb{P}$ *is* δ*-c.c. in* V *and* G *is* V*-generic for* $\mathbb{P}$*, then* $X[G]^{<\delta} \subseteq X[G]$ *in* $V[G]$.

Proof. Using $X[G] \prec N[G]$ it is easy to verify assertions (1) and (2). To see assertion (3), we follow the usual proof for the transitive case closely. Fix the cardinal δ, the structure X with $X^{<\delta} \subseteq X$ and the poset $\mathbb{P} \subseteq X$ which is δ-c.c. in V. Let $G \subseteq \mathbb{P}$ be V-generic. Observe that the closure of X shows that every antichain $A \in V$ of $\mathbb{P}$ is an element of X. Let $\dot{G}$ be the canonical $\mathbb{P}$-name for the V-generic filter on $\mathbb{P}$. I first claim that if $\tau \in V$ is a name such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\tau \in \check{X}[\dot{G}]$", then we can find a name $\sigma \in X$ such that $\mathbb{1}_{\mathbb{P}} \Vdash \tau = \sigma$. To see this, fix a name $\tau \in V$ as above. Working in V, we see that the set $D = \{\ p \in \mathbb{P} \mid \exists\sigma \in X \text{ such that } p \Vdash \sigma = \tau\}$ is dense in $\mathbb{P}$. Let $A \subseteq D$ be a maximal antichain in V and choose for each $a \in A$ a witness $\sigma_a \in X$ such that $a \Vdash \sigma_a = \tau$. By our earlier observation, we know that $A \in X$ and consequently that $\langle\sigma_a : a \in A\rangle \in X$. By mixing these names in X, we obtain a single name $\sigma \in X$ such that $\mathbb{1}_{\mathbb{P}} \Vdash \sigma = \tau$, which proves the claim. To verify that $X[G]$ is closed under ${<}\delta$-sequences in $V[G]$, fix now any $s \in X[G]^{\beta} \cap V[G]$ for some $\beta < \delta$. We may assume that s has a name $\dot{s} \in V$ such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{s}$ is a β-sequence of elements of $\check{X}[\dot{G}]$". For each $\alpha < \beta$, we may fix in V by the claim a name $\sigma_\alpha \in X$ such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{s}(\alpha) = \sigma_\alpha$". In particular, $s(\alpha) = \dot{s}_G(\alpha) = (\sigma_\alpha)_G$. The closure of X shows that $\langle\sigma_\alpha : \alpha < \beta\,\rangle \in X$. As

$G \in X[G]$, it follows that $s = \langle (\sigma_\alpha)_G : \alpha < \beta \rangle \in X$, as desired. $\square$

Note that assertion (3) of Fact 15 is false, if we omit the hypothesis $\mathbb{P} \subseteq X$. As a counterexample, suppose that δ is an uncountable cardinal with $\delta^{<\delta} = \delta$. Let $X \prec H_{\delta^{++}}$ have size δ such that $\delta \in X$ and $X^{<\delta} \subseteq X$ in V. Let $\mathbb{P} = \mathrm{Add}(\omega, \delta^+)$ be the poset which adds δ^+ many Cohen reals. The poset $\mathbb{P}$ is an element of $H_{\delta^{++}}$ and since $\mathbb{P}$ is definable there, it follows also that $\mathbb{P} \in X$. Moreover, $\mathbb{P}$ is certainly δ-c.c. and preserves δ^+. If $G \subseteq \mathbb{P}$ is V-generic, it follows that we have at least δ^+ many reals in $V[G]$, yet $X[G]$ has size δ only. This shows that $X[G]^\omega \nsubseteq X[G]$.

Fact 15 helps to establish some sufficient conditions for a finite iteration of κ-proper posets to be κ-proper.

Lemma 16. *Suppose $\mathbb{P}$ is a $<\kappa$-distributive, κ-proper poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name which necessarily yields a κ-proper poset. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-proper.*

Proof. Fix $\mathbb{P}$ and $\dot{\mathbb{Q}}$ as in the lemma. There is a cardinal $\lambda_{\mathbb{P}}$ such that all regular $\lambda \geq \lambda_{\mathbb{P}}$ are sufficiently large to witness the κ-properness of $\mathbb{P}$. Moreover, since $\mathbb{P}$ is a set, we can find in V a cardinal $\lambda_{\mathbb{Q}}$ such that $\mathbb{1}_{\mathbb{P}}$ forces that all regular $\lambda \geq \lambda_{\mathbb{Q}}$ are sufficiently large to witness the κ-properness of $\dot{\mathbb{Q}}$. Without loss of generality, assume $\mathrm{trcl}(\mathbb{P}) < \lambda_{\mathbb{P}}$ and $\mathrm{trcl}(\dot{\mathbb{Q}}) < \lambda_{\mathbb{Q}}$. To see that $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-proper, fix now any regular cardinal $\lambda \geq \max(\lambda_{\mathbb{P}}, \lambda_{\mathbb{Q}})$. As

$\lambda \geq \lambda_{\mathbb{P}}$, we may fix a λ-witness $x_{\mathbb{P}}$ for $\mathbb{P}$. Since $\mathbb{1}_{\mathbb{P}}$ forces that there exists a λ-witness for $\dot{\mathbb{Q}}$ also, we may by mixing find a $\mathbb{P}$-name $\dot{x}_{\mathbb{Q}} \in V$ that is forced by $\mathbb{1}_{\mathbb{P}}$ to be a λ-witness for $\dot{\mathbb{Q}}$. In fact, we can find such a $\mathbb{P}$-name $\dot{x}_{\mathbb{Q}}$ with $\operatorname{trcl}(\dot{x}_{\mathbb{Q}}) < \lambda$. We will show that $\{x_{\mathbb{P}}, \dot{x}_{\mathbb{Q}}\}$ serves as a λ-witness for the κ-properness of $\mathbb{P} * \dot{\mathbb{Q}}$.

Fix thus any elementary submodel $X \prec H_\lambda$ of size κ with $X^{<\kappa} \subseteq X$ such that $\{\kappa, \mathbb{P} * \dot{\mathbb{Q}}, x_{\mathbb{P}}, \dot{x}_{\mathbb{Q}}\} \in X$. Fix also any condition $r_1 \in (\mathbb{P} * \dot{\mathbb{Q}}) \cap X$. It is our goal to find an $(X, \mathbb{P} * \dot{\mathbb{Q}})$-generic condition $r \in \mathbb{P} * \dot{\mathbb{Q}}$ below r_1. Let $r_1 = \langle p_1, \dot{q}_1 \rangle$ with $p_1 \in \mathbb{P}$ and $\dot{q}_1 \in \operatorname{dom}(\dot{\mathbb{Q}})$ and $p_1 \Vdash \dot{q}_1 \in \dot{\mathbb{Q}}$. Since $\lambda \geq \lambda_{\mathbb{P}}$ and $x_{\mathbb{P}} \in X$, there exists an $(X, \mathbb{P})$-generic condition $p_0 \in \mathbb{P}$ below p_1. Let $\dot{G}$ be the canonical $\mathbb{P}$-name for the V-generic filter on $\mathbb{P}$. Note that $\mathbb{1}_{\mathbb{P}}$ forces that λ is a sufficiently large regular cardinal, that $X[\dot{G}]$ is an elementary submodel of $H_\lambda[\dot{G}]$, and that $\dot{x}_{\mathbb{Q}} \in X[\dot{G}]$ is a λ-witness for $\dot{\mathbb{Q}}$. Moreover, $\mathbb{1}_{\mathbb{P}}$ also forces that $X[\dot{G}]$ is closed under ${<}\kappa$-sequences. This follows from assertion (2) of Fact 15 and the $< \kappa$-distributivity of $\mathbb{P}$. We thus see that $p_1 \Vdash$ "$\exists x \in \dot{\mathbb{Q}}$ below $\dot{q}_1$ which is $(X[\dot{G}], \dot{\mathbb{Q}})$-generic". Let $p \leq p_1$ and $\dot{q} \in \operatorname{dom}(\dot{\mathbb{Q}})$ such that $p \Vdash$ "$\dot{q} \leq \dot{q}_1$ and $\dot{q} \in \dot{\mathbb{Q}}$ is $(X[\dot{G}], \dot{\mathbb{Q}})$-generic". Then $r = \langle p, \dot{q} \rangle$ is an element of $\mathbb{P} * \dot{\mathbb{Q}}$ below r_1.

I claim that $r \in \mathbb{P} * \dot{\mathbb{Q}}$ is the desired $(X, \mathbb{P} * \dot{\mathbb{Q}})$-generic condition below r_1. Clearly $r \leq r_1$. Thus, fix any V-generic filter $G * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ where $G \subseteq \mathbb{P}$ is

V-generic and H is $V[G]$-generic for $\mathbb{Q} = \dot{\mathbb{Q}}_G$ such that $r \in G * H$. It follows that $G \subseteq \mathbb{P}$ is X-generic since $p \in G$ and thus $X \cap Ord = X[G] \cap Ord$. Moreover, since $\dot{q}_G \in H$ it follows that $H \subseteq \mathbb{Q}$ is $X[G]$-generic and thus $X[G] \cap Ord = X[G][H] \cap Ord$. Thus $X[G * H]$ has the same ordinals as X, which implies that $G * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is an X-generic filter. This proves the claim and hence that $\{x_{\mathbb{P}}, \dot{x}_{\mathbb{Q}}\}$ is a λ-witness for the κ-properness of $\mathbb{P} * \dot{\mathbb{Q}}$. Since $\lambda \geq \max(\lambda_{\mathbb{P}}, \lambda_{\mathbb{Q}})$ was arbitrary, this concludes the proof of the fact. □

The next lemma is crucial for the proof of the Main Theorem, where I precede a κ-proper forcing $\mathbb{Q}$ with the lottery preparation $\mathbb{P}$ of κ.

Lemma 17. *Assume that κ is a cardinal with $\kappa^{<\kappa} = \kappa$. If $\mathbb{P}$ is a κ-c.c. poset of size κ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name which necessarily yields a κ-proper poset, then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-proper.*

Proof. Fix $\mathbb{P}$ and $\dot{\mathbb{Q}}$ as in the lemma. Since $\mathbb{P}$ has size κ, and κ-properness is preserved by isomorphisms (Fact 14), we may assume without loss of generality that $\mathbb{P} \subseteq \kappa$. The rest of the argument is identical to the proof of Lemma 16, except that we use now assertion (3) of Fact 15 instead of assertion (2). The hypotheses of assertion (3) hold since $X^{<\kappa} \subseteq X$ implies that $\kappa \subseteq X$. □

Corollary 18. *A finite iteration of $<\kappa$-closed, κ-proper posets is itself $<\kappa$-closed and κ-proper. A finite iteration of $<\kappa$-distributive, κ-proper posets is itself $<\kappa$-distributive and κ-proper.*

Proof. Finite forcing iterations of $<\kappa$-distributive posets are $<\kappa$-distributive. Similarly, $<\kappa$-closure is preserved by finite iterations. Apply Lemma 16 finitely often. □

Fact 19 (Diagonalization Criterion)**.** *Let δ be an ordinal. Suppose that $\langle N, \in\rangle$ is a transitive model of* ZFC$^-$*. Let $\langle X, \in\rangle$ be an elementary substructure of $\langle N, \in\rangle$, not necessarily transitive. Assume $\mathbb{P} \in X$ is a poset. If the following criteria are satisfied,*

1. *X has at most δ many dense sets for $\mathbb{P}$,*
2. *$\mathbb{P}$ is $<\delta$-closed in X and*
3. *$X^{<\delta} \subseteq X$,*

then for any $p \in \mathbb{P} \cap X$ there is an X-generic filter $G \subseteq \mathbb{P}$ with $p \in G$.

Proof. The proof is similar to the method of building generic filters for *countable transitive* models of set theory. Indeed, using conditions (2) and (3) we can meet δ many dense sets of X inside of X. This descending chain of δ many elements of X generates in V a filter $G \subseteq \mathbb{P}$ that is X-generic. □

1.4 The Main Theorem

I will now prove the Main Theorem that makes a strongly unfoldable cardinal κ indestructible by $<\kappa$-closed, κ-proper forcing. First, I will describe the basic strategy that one would like to use, illustrate some immediate problems and show how to overcome them with Lemma 20. I will also review Hamkins' *lottery preparation* [Ham00] briefly.

Suppose κ is strongly unfoldable and we want to make κ indestructible by some nontrivial forcing $\mathbb{Q}$. Let θ be an ordinal with $\text{rank}(\mathbb{Q}) < \theta$, and $G \subseteq \mathbb{Q}$ a V-generic filter. To show that κ is θ-strongly unfoldable in $V[G]$, it is our goal (by assertion (5) of Fact 5) to place any given $A \in V[G]$ with $A \subseteq \kappa$ into a transitive set M^* satisfying ZFC$^-$ of size κ containing κ as an element with a corresponding embedding $j^* : M^* \to N^*$ for which $(V_\theta \subseteq N^*)^{V[G]}$ and $\theta < j^*(\kappa)$.

To illustrate the basic method, suppose first that $\mathbb{Q}$ has size *at most* κ, say $\mathbb{Q} \in H_{\kappa^+}$. If $A \in V[G]$ with $A \subseteq \kappa$, then A has a $\mathbb{Q}$-name $\dot{A} \in H_{\kappa^+}$. In V, we can thus place both $\dot{A}$ and $\mathbb{Q}$ into a κ-model M. As κ is θ-strongly unfoldable in V, there exists in V a θ-strong unfoldability embedding $j : M \to N$. As $\mathbb{Q} \in M$, we can force with $\mathbb{Q}$ over M using the M-generic filter $G \subseteq \mathbb{Q}$. *If* the embedding j lifts to $j^* : M[G] \to N[H]$ such that $G \in N[H]$, then I claim that we have fulfilled our goal and j^* is the desired embedding. Clearly $A = \dot{A}_G \in M[G]$. To verify that "$V_\theta \subseteq N[H]$" holds in $V[G]$, let us denote the rank initial segment $(V_\theta)^{V[G]}$ by $V[G]_\theta$. It is a standard fact about forcing that for ordinals $\alpha > \text{rank}(\mathbb{Q})$ every $x \in V[G]_\alpha$ has a $\mathbb{Q}$-name $\dot{x} \in V_\alpha \times V_\alpha$. By means of a suitable pairing function, a *flat* pairing function, which does not increase rank, we may assume that $V_\alpha \times V_\alpha \subseteq V_\alpha$ for all infinite ordinals α (see for instance [Ham]). It follows that $V[G]_\alpha \subseteq V_\alpha[G]$ for all $\alpha > \text{rank}(\mathbb{Q})$. Since $\theta > \text{rank}(\mathbb{Q})$, the filter $G \in N[H]$, and $V_\theta \subseteq N$, we see that $V[G]_\theta \subseteq N[H]$. This verifies the claim. A necessary and sufficient condition for the embedding j to lift to j^*, the *lifting criterion*, is that H is an N-generic filter for $j(\mathbb{Q})$ such that j"$G \subseteq H$. We will use Silver's *master condition* argument to verify the lifting criterion when proving the Main Theorem.

Suppose now that $\mathbb{Q}$ has size *bigger than* κ. The above strategy fails completely, as we cannot place the poset $\mathbb{Q}$ into a κ-model M. Also, the $\mathbb{Q}$-name $\dot{A}$ for the subset of κ may be too big to fit into M. Yet, the next lemma provides a solution to the problem: *If* we succeed in putting $\mathbb{Q}$, $\dot{A}$ and κ into an elementary substructure $X \prec H_\lambda$ of size κ (where λ is some regular cardinal) with $X^{<\kappa} \subseteq X$ such that the filter $G \subseteq \mathbb{Q}$ is *both* X-generic and V-generic, then we can follow the above strategy with a *collapsed* version of

$\mathbb{Q}$. More specifically, if $\pi : X \to M$ is the Mostowski collapse of X, then M is a κ-model containing the collapsed poset $\pi(\mathbb{Q})$. By Lemma 20 below, the image $G_0 = \pi"G$ is an M-generic filter on $\pi(\mathbb{Q})$. We may thus force with $\pi(\mathbb{Q})$ over M using the M-generic filter G_0 and obtain the extension $M[G_0]$. Moreover, since $\kappa + 1 \subseteq X$, the lemma also shows that $A = \dot{A}_G = \pi(\dot{A})_{G_0}$ is an element of $M[G_0]$. We may therefore follow our previous strategy and try to lift any given θ-strong unfoldability embedding $j \in V$ with domain M to an embedding $j^* : M[G_0] \to N[H_0]$ in such a way that $G \in N[H_0]$.

Lemma 20. *Suppose that N is a transitive model of* ZFC$^-$. *Suppose also that $X \prec N$ is an elementary substructure of any size, $\mathbb{Q} \in X$ is a poset and $G \subseteq \mathbb{Q}$ is a filter that is both X-generic and N-generic for $\mathbb{Q}$. Let $\pi : \langle X, \in\rangle \to \langle M, \in\rangle$ be the Mostowski collapse of X and let $G_0 = \pi"G = \pi"(G \cap X)$. Then:*

1. *G_0 is M-generic for $\pi(\mathbb{Q})$ and π lifts to $\pi_1 : X[G] \to M[G_0]$, which is the Mostowski collapse of $X[G]$ in $V[G]$.*

2. *Suppose κ is a cardinal with $\kappa + 1 \subseteq X$. If $\dot{A} \in X$ is a $\mathbb{Q}$-name which necessarily yields a subset of κ, then $\dot{A}_G = \pi(\dot{A})_{G_0}$.*

Proof. To verify assertion (1), recall that we saw earlier that X-genericity of G is equivalent to G_0 being M-generic for $\pi(\mathbb{Q})$. Since every object in $X[G]$

is the interpretation of a $\mathbb{Q}$-name $\tau \in X$ by the generic filter G, we must let $\pi_1(\tau_G) = \pi(\tau)_{G_0}$. I claim that π_1 is a well-defined map. For, if $\sigma, \tau \in X$ are $\mathbb{Q}$-names with $\sigma_G = \tau_G$, then consider the Boolean value $b = [\![\sigma = \tau]\!]^{\mathbb{Q}}$. Since G is N-generic for $\mathbb{Q}$, it follows that $b \in G$. Moreover $b \in X$, as b is definable from σ, τ and $\mathbb{Q}$. Since $b \Vdash_{\mathbb{Q}}$ "$\sigma = \tau$" holds in N and hence in X, it follows that M thinks that $\pi(b) \Vdash_{\pi(\mathbb{Q})}$ "$\pi(\sigma) = \pi(\tau)$". Since $G_0 = \pi``(G \cap X)$ is M-generic and $\pi(b) \in G_0$, we see that $\pi(\sigma)_{G_0} = \pi(\tau)_{G_0}$, which shows that π_1 is well-defined. One checks similarly that π_1 preserves the membership relation, extends π and is a bijection. Since $M[G_0]$ is transitive, π_1 must be the Mostowski collapse of $X[G]$ in $V[G]$.

To see assertion (2), fix any ordinal $\alpha \in \dot{A}_G$. Since $\dot{A}$ is a name which necessarily yields a subset of κ, we have $\alpha \in X$. Consider the Boolean value $b = [\![\alpha \in \dot{A}]\!]^{\mathbb{Q}}$. It follows as in (1) that $b \in G \cap X$. Elementarity of π yields $\alpha = \pi(\alpha) \in \pi(\dot{A})_{G_0}$. This establishes $\dot{A}_G \subseteq \pi(\dot{A})_{G_0}$. The converse inclusion is similar. $\square$

The Main Theorem uses the *lottery preparation*, a general tool developed by Hamkins [Ham00] to force indestructibility for various large cardinal notions. The lottery preparation of κ is defined relative to a function $f : \kappa \to \kappa$. Usually, one assumes that f has the *Menas* property for the particular large cardinal κ (see Section 1.2). The basic building block of the lottery preparation is the *lottery sum* $\oplus\mathcal{A}$ of a collection $\mathcal{A}$ of posets. Also commonly called *side-by-side forcing*, $\oplus\mathcal{A}$ is the poset $\{\langle\mathbb{Q}, p\rangle : \mathbb{Q} \in \mathcal{A} \text{ and } p \in \mathbb{Q}\} \cup \{\mathbb{1}\}$, ordered with $\mathbb{1}$ above everything and $\langle\mathbb{Q}, p\rangle \leq \langle\mathbb{Q}', p'\rangle$ when $\mathbb{Q} = \mathbb{Q}'$ and $p \leq_{\mathbb{Q}} p'$. Because compatible conditions must have the same $\mathbb{Q}$, the forcing effectively holds a lottery among all the posets in $\mathcal{A}$, a lottery in which the generic filter selects a 'winning' poset $\mathbb{Q}$ and then forces with it. The lottery preparation $\mathbb{P}$ of κ relative to f is an Easton support κ-iteration which at stage $\gamma{<}\kappa$, if $\gamma \in \operatorname{dom}(f)$ and $f``\gamma \subseteq \gamma$, forces with the lottery sum of all ${<}\gamma$-closed posets $\mathbb{Q} \in H_{f(\gamma)^+}$ in $V^{\mathbb{P}_\gamma}$. (Note: Insisting on ${<}\gamma$-closure is slightly less general than developed in [Ham00], but sufficient for our purposes here.) Generically, if $f(\gamma)$ is large, then the stage γ forcing of $\mathbb{P}$ selects from a wide variety of posets, so that if $j : M \to N$ is an embedding such that both $\mathbb{P}$ and f are elements of M, then $j(\mathbb{P})$ selects from a wide variety of posets. It follows that the stage κ lottery of $j(\mathbb{P})$ typically includes a sufficiently rich collection of posets so that we can work below a condition $z \in j(\mathbb{P})$ that opts at stage κ for a particular desired forcing notion. For instance, suppose $\mathbb{Q}$ is any ${<}\kappa$-closed poset in the forcing extension of V after forcing with $\mathbb{P}$. Using the strong unfoldability of κ and the Menas property of f, we can fix an ordinal θ, a $\mathbb{P}$-name $\dot{\mathbb{Q}} \in V_\theta$ for $\mathbb{Q}$ and a θ-strong unfoldability embedding

$j : M \to N$ with $f \in M$ (and thus $\mathbb{P} \in M$) such that $j(f)(\kappa) \geq \beth_\theta^N$. Recall that $\beth_\theta^N \geq \beth_\theta$. It follows that the stage κ lottery of $j(\mathbb{P})$ includes the poset $\mathbb{Q}$. By simply working below a condition $z \in j(\mathbb{P})$ that opts at stage κ for $\mathbb{Q}$, we see that $j(\mathbb{P}) \upharpoonright z$ forces at stage κ with $\mathbb{Q}$ and thus factors as $j(\mathbb{P}) \upharpoonright z = \mathbb{P} * \mathbb{Q} * \mathbb{P}_{\text{tail}}$. Moreover, since $j(f)(\kappa) \geq \beth_\theta^N$, the next nontrivial forcing after stage κ occurs after stage $\beth_\theta^N$. It follows that $\mathbb{P}_{\text{tail}}$ is ${\leq}\beth_\theta^N$-closed in N. This flexibility to make $\mathbb{P}_{\text{tail}}$ highly closed is crucial for the tail forcing arguments.

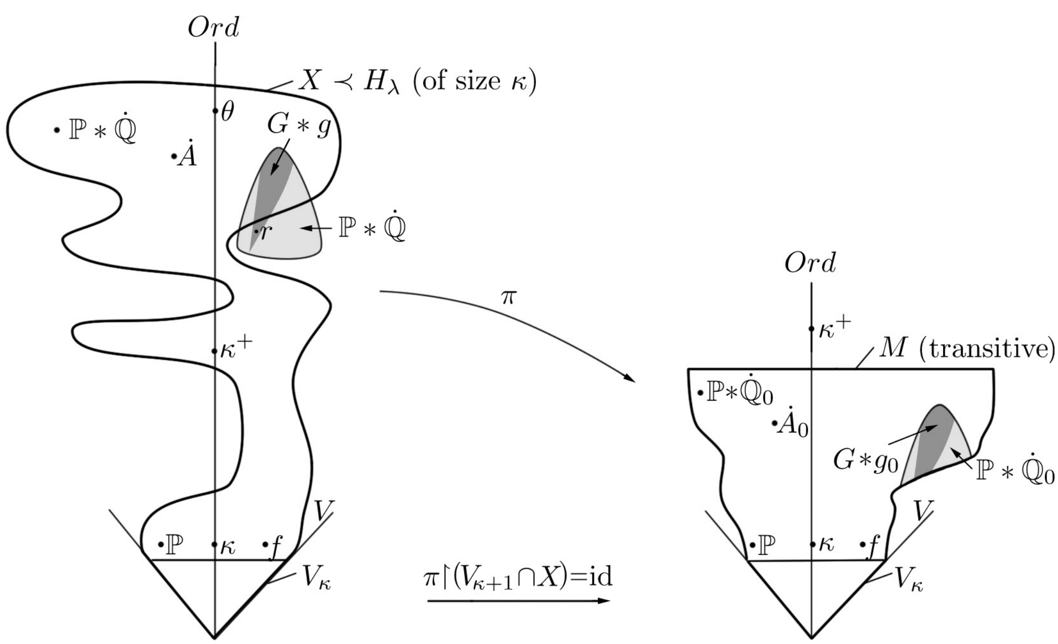
Main Theorem. *Let κ be strongly unfoldable. Then after the lottery preparation of κ relative to a function with the Menas property for κ, the strong unfoldability of κ becomes indestructible by ${<}\kappa$-closed, κ-proper forcing.*

Proof. Let κ be strongly unfoldable. By assertion (1) of Theorem 11 we know that there is a function $f \colon \kappa \to \kappa$ with the Menas property for the strongly unfoldable cardinal κ. Let $\mathbb{P}$ be the lottery preparation of κ relative to f. We will show that after forcing with $\mathbb{P}$ the strong unfoldability of κ becomes indestructible by ${<}\kappa$-closed, κ-proper forcing. The poset $\mathbb{P}$ certainly preserves the inaccessibility of κ (see [Ham00]). Fix any $\mathbb{P}$-name $\dot{\mathbb{Q}}$ which necessarily yields a ${<}\kappa$-closed, κ-proper poset. Since $\dot{\mathbb{Q}}$ is the name of a ${<}\kappa$-distributive poset, it follows that κ is inaccessible after forcing with $\mathbb{P} * \dot{\mathbb{Q}}$. It remains to show that for every ordinal θ, the poset $\mathbb{P} * \dot{\mathbb{Q}}$ preserves the embedding property of the θ-strongly unfoldable cardinal κ. Note that $\mathbb{P}$ is κ-proper, as it has size κ and $\kappa^{<\kappa} = \kappa$ (Fact 14). Moreover, since $\mathbb{P}$ is κ-c.c. and $\mathbb{P}$ has size κ, Lemma 17 shows that $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-proper. In view of characterization (4) of Fact 5, fix any ordinal $\theta \geq \kappa$ and any $\mathbb{P} * \dot{\mathbb{Q}}$ -name $\dot{A}$ which necessarily yields a subset of κ. We may assume that θ is large enough so that $\dot{\mathbb{Q}}$ and $\dot{A}$ are elements of $V_{\theta+1}$. Consider the following subset D of $\mathbb{P} * \dot{\mathbb{Q}}$,

$$D = \{r \in \mathbb{P} * \dot{\mathbb{Q}} : r \Vdash \text{``}\dot{A} \text{ can be placed into a } \kappa\text{-model } M \text{ with an embedding } j : M \to N \text{ with } \theta < j(\kappa) \text{ and } V_\theta \subseteq N\text{''}\}.$$

To prove the theorem, it suffices to show that D is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. To do so, consider any $r' \in \mathbb{P} * \dot{\mathbb{Q}}$. Note that $\operatorname{trcl}(\{\kappa, \mathbb{P}, f, \dot{\mathbb{Q}}, \dot{A}, \theta\}) \leq \beth_\theta$. Let $\lambda > \beth_\theta$ be a sufficiently large regular cardinal to witness the κ-properness of $\mathbb{P} * \dot{\mathbb{Q}}$ as in Definition 13, and let $x \in H_\lambda$ be a corresponding λ-witness for $\mathbb{P} * \dot{\mathbb{Q}}$. Since $\kappa^{<\kappa} = \kappa$, we may use the Skolem-Löwenheim method in V to build an elementary submodel $X \prec H_\lambda$ of size κ with $X^{<\kappa} \subseteq X$ such that $\{\kappa, r', \mathbb{P}, f, \dot{\mathbb{Q}}, \dot{A}, \theta, x\} \subseteq X$. Note that $V_\kappa \subseteq X$ by induction. As λ is sufficiently large and $x \in X$, we may thus fix an $(X, \mathbb{P} * \dot{\mathbb{Q}})$-generic condition $r \in \mathbb{P} * \dot{\mathbb{Q}}$ such that $r \leq r'$. The rest of the proof will show that $r \in D$, and hence that D is dense.

Let $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be any V-generic filter containing r so that $G \subseteq \mathbb{P}$ is a V-generic filter and $g \subseteq \mathbb{Q} = \dot{\mathbb{Q}}_G$ is a $V[G]$-generic filter. Since $r \in G*g$ is an $(X, \mathbb{P}*\dot{\mathbb{Q}})$-generic condition, we see that $G*g$ is an X-generic filter on $\mathbb{P}*\dot{\mathbb{Q}}$. Let $\pi : \langle X, \in \rangle \to \langle M, \in \rangle$ be the Mostowski collapse of X. The construction of X shows that M is a κ-model. Since $\pi \restriction V_\kappa = \text{id}$, we see that π also fixes κ, the poset $\mathbb{P}$, and the Menas function f. Let $\pi(\dot{\mathbb{Q}}) = \dot{\mathbb{Q}}_0$ and $\pi(\dot{A}) = \dot{A}_0$. It follows in M that $\dot{\mathbb{Q}}_0$ is a $\mathbb{P}$-name for a ${<}\kappa$-closed poset and that $\dot{A}_0$ is a $\mathbb{P} * \dot{\mathbb{Q}}_0$-name for a subset of κ. Moreover, the image $G * g_0 = \pi"(G * g)$ is an M-generic filter on $\mathbb{P} * \dot{\mathbb{Q}}_0$ by Lemma 20. Note that the poset $\mathbb{P} * \dot{\mathbb{Q}}_0$ is isomorphic to $(\mathbb{P} * \dot{\mathbb{Q}}) \cap X$. The next diagram illustrates the situation.



Let $A = \dot{A}_{G*g}$ be the subset of κ which we need to put into the domain of an elementary embedding $j^* \in V[G * g]$. We saw that $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is an X-generic filter, which implies that $X[G * g] \cap V = X$. It follows that G is X-generic for $\mathbb{P}$ and g is $X[G]$-generic for $\mathbb{Q}$. Since κ is $(\theta + 1)$-strongly unfoldable in V, fix by Fact 6, a $(\theta + 1)$-strong unfoldability embedding $j : M \to N$ with $N^{\beth_\theta} \subseteq N$ and $|N| = \beth_{\theta+1}$. Since f has the Menas property for κ, we may assume that $j(f)(\kappa) \geq \beth_\theta^N$ and $\beth_\theta^N < j(\kappa)$. Let $\delta = \beth_\theta^N$. Since $V_{\theta+1} \subseteq N$, we see that $\beth_\theta^N = \beth_\theta = \delta$. Elementarity of j and $\delta < j(\kappa)$ shows that $N \models$ "H_{δ^+} exists". It follows that $H_{\delta^+}^N = H_{\delta^+}$. Summarizing, we know that the $(\theta + 1)$-strong unfoldability embedding $j : M \to N$ with $\delta < j(\kappa)$ and $|N| = 2^\delta$ satisfies $N^\delta \subseteq N$ and $j(f)(\kappa) \geq \delta$. As indicated in the remark after Fact 6, we would like N to have size δ^+ in order to allow for diagonalization arguments over N. For simplicity, let us for the moment *assume* that $2^\delta = \delta^+$ in V. I shall show at the end of this proof how to modify the arguments in the case if $2^\delta \neq \delta^+$.

The next diagram illustrates our strategy of lifting the elementary embedding $j : M \to N$ in $V[G * g]$ in two steps. While lifting j, we will also lift the isomorphism $\pi : X \to M$ twice.

$$
\begin{array}{ccccccc}
H_\lambda[G*g] & \succ & X[G*g] & \cong & M[G*g_0] & \overset{j}{\dashrightarrow} & N[j(G)*j(g_0)] \\
\uparrow \subseteq & & \uparrow \subseteq & & \uparrow \subseteq & & \uparrow \subseteq \\
H_\lambda[G] & \succ & X[G] & \cong & M[G] & \overset{j}{\dashrightarrow} & N[j(G)] \\
\uparrow \subseteq & & \uparrow \subseteq & & \uparrow \subseteq & & \uparrow \subseteq \\
H_\lambda & \succ & X & \overset{\pi}{\cong} & M & \xrightarrow{\quad j \quad} & N
\end{array}
$$

Step 1. *In $V[G*g]$, lift the embedding $j : M \to N$ to $j : M[G] \to N[j(G)]$.*

Since $\mathbb{P} \in M$, we can certainly force with the V-generic filter $G \subseteq \mathbb{P}$ over M. By elementarity, $j(\mathbb{P})$ is the lottery preparation of $j(\kappa)$ relative to $j(f)$ computed in N. Since $\dot{\mathbb{Q}} \in V_{\theta+1} \subseteq H_{\delta^+}$ and $H_{\delta^+} \in N$, we see that $N[G] \models (\mathbb{Q} \in H_{\delta^+}$ and $\mathbb{Q}$ is ${<}\kappa$-closed$)$. As $j(f)(\kappa) \geq \delta$, we may hence opt for $\mathbb{Q}$ at the stage κ lottery of $j(\mathbb{P})$. Consequently, below the condition p which opts for $\mathbb{Q}$ at stage κ, the forcing $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$. To satisfy the lifting criterion we first need an N-generic filter $j(G)$ for the poset $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$. Since G is N-generic for $\mathbb{P}$ and g is $N[G]$-generic for $\mathbb{Q}$, it suffices to find in $V[G*g]$ a filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ that is $N[G*g]$-generic. To do so, we verify the diagonalization criterion in $V[G*g]$ for $N[G*g]$ (see Fact 19): N has size δ^+ in V, and $N[G*g]$ has thus size δ^+ in $V[G*g]$.

Since $j(f)(\kappa) \geq \delta$, the poset $\mathbb{P}_{\text{tail}}$ is $\leq\delta$-closed in $N[G * g]$ by the definition of $j(\mathbb{P})$. Lastly, since $N^\delta \subseteq N$ in V, and both $\mathbb{P}$ and $\mathbb{Q}$ are δ^+-c.c., the closure fact shows that $N[G * g]^\delta \subseteq N[G * g]$ in $V[G * g]$. So, by diagonalization in $V[G * g]$, we may construct a filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ which is $N[G * g]$-generic. If we let $j(G) = G * g * G_{\text{tail}}$, then $j``G \subseteq j(G)$, and we satisfy the lifting criterion, and j lifts. Note that $N[j(G)]^\delta \subseteq N[j(G)]$. This concludes *Step 1.*

Since $\mathbb{P}$ is κ-c.c. and $\mathbb{Q}$ is $<\kappa$-distributive, the closure fact (Fact 15) shows that $M[G]$ is still a κ-model in $V[G * g]$. Since our goal is to put $A = \dot{A}_{G*g}$ into a κ-model and the domain of the embedding so far is only $M[G]$, we need to lift the embedding again. But in general $\mathbb{Q}$ will not be an element of $M[G]$ and thus we cannot force with $\mathbb{Q}$ over $M[G]$. Instead, we shall force with the collapsed version of $\mathbb{Q}$— namely $(\dot{\mathbb{Q}}_0)_G$— over $M[G]$. That this is at all possible is a crucial step of the argument.

We had $X \prec H_\lambda$ and the Mostowski collapse $\pi : \langle X, \in\rangle \to \langle M, \in\rangle$. We also saw that G is both X-generic and V-generic for $\mathbb{P}$. Recall that $\pi \restriction V_\kappa = \text{id}$ and hence $\pi``G = G$. We can thus apply Lemma 20 and see that π lifts to $\pi_1 : X[G] \to M[G]$ by $\pi_1(\sigma_G) = \pi(\sigma)_G$, where π_1 is the Mostowski collapse of $X[G]$ in $V[G]$. Since $\dot{\mathbb{Q}} \in X$, we have $\mathbb{Q} = \dot{\mathbb{Q}}_G \in X[G]$. Let $\pi_1(\mathbb{Q}) = \mathbb{Q}_0$ be the collapsed version of $\mathbb{Q}$, which *is* an element of $M[G]$. We have $X[G] \prec H_\lambda[G]$ as usual. Recall that κ-properness of $\mathbb{Q}$ (and hence of $\mathbb{P} * \dot{\mathbb{Q}}$) enabled us to pick an X-generic condition $r \in \mathbb{P} * \dot{\mathbb{Q}}$ below r'. This showed that g is $X[G]$-generic for $\mathbb{Q}$, which now allows for the crucial application of Lemma 20 in $V[G]$ to $g \subseteq \mathbb{Q}$: It follows that $g_0 = \pi_1{}^{"}g$ is $M[G]$-generic for $\mathbb{Q}_0$ and forcing with $g_0 \subseteq \mathbb{Q}_0$ over $M[G]$ thus makes sense. Moreover, $A = \dot{A}_{G*g} = \pi(\dot{A})_{G*g_0} = (\dot{A}_0)_{G*g_0}$ is an element of $M[G * g_0]$. Since $\mathbb{Q}_0$ is ${<}\kappa$-closed in $M[G]$, Fact 15 shows that $M[G * g_0]$ is a κ-model in $V[G * g]$. Thus, to proceed showing that $r \in D$, let us lift the embedding j once more:

Step 2. *In $V[G * g]$, lift the elementary embedding $j : M[G] \to N[j(G)]$ to $j : M[G * g_0] \to N[j(G) * j(g_0)]$.*

Since $\mathbb{Q}_0$ is a ${<}\kappa$-distributive poset in $M[G]$, I claim that $M[G*g_0]$ thinks that g_0 is a ${<}\kappa$-closed subset of $\mathbb{Q}_0$. To see this, fix any $\beta < \kappa$ and any descending sequence $s = \langle s_\xi : \xi < \beta \rangle$ in $M[G * g_0]$ of elements in g_0. Then $s \in M[G]$ by the distributivity of $\mathbb{Q}_0$. Consider therefore in $M[G]$ for each $\xi < \beta$ the dense open set $D_\xi = \{q \in \mathbb{Q}_0 \mid q \leq s_\xi \text{ or } q \perp s_\xi\}$. The distributivity of $\mathbb{Q}_0$ shows that $D = \bigcap_{\xi<\beta} D_\xi$ is a dense open subset of $\mathbb{Q}_0$ in $M[G]$. As g_0 is an $M[G]$-generic filter on $\mathbb{Q}_0$, it meets the set D. Let $t \in D \cap g_0$. Since t is compatible with s_ξ and $t \in D_\xi$ for each $\xi < \beta$, it follows that t lies below all s_ξ's, which proves the claim.

As $M[G*g_0]$ is closed under $<\kappa$-sequences in $V[G*g]$, the model $M[G*g_0]$ is correct, and we have that $V[G*g] \models (g_0$ is a $<\kappa$-closed subset of $\mathbb{Q}_0)$. Moreover, g_0 is a directed set of size κ in $V[G*g]$. Consequently, we see in $V[G*g]$ that there is a *descending* chain $\langle q_\xi : \xi < \kappa\rangle$ of elements of g_0 such that every element of g_0 lies above q_ξ for some $\xi < \kappa$. It follows that every element of $j"g_0$ lies above $j(q_\xi)$ for some $\xi < \kappa$. Consider therefore in $V[G * g]$ the descending chain $\vec{c} = \langle j(q_\xi) : \xi{<}\kappa\rangle$ of elements of $j"g_0 \subseteq j(\mathbb{Q}_0)$. Since $N[j(G)]$ is closed under κ-sequences in $V[G * g]$ and $\vec{c} \in N[j(G)]^\kappa$, we have $\vec{c} \in N[j(G)]$. Moreover, $N[j(G)]$ thinks that $j(\mathbb{Q}_0)$ is $<j(\kappa)$-closed, and we can hence find a condition $q \in j(\mathbb{Q}_0)$ below all the $j(q_\xi)$'s. Since $q \leq x$ for all $x \in j"g_0$, we see that q is the desired master condition. Finally, we verify the diagonalization criterion in $V[G * g]$ easily and build an $N[j(G)]$-generic filter $j(g_0) \subseteq j(\mathbb{Q}_0)$ containing q as an element. Since $j"g_0 \subseteq j(g_0)$, we satisfy the lifting criterion, and j lifts. This concludes *Step 2*.

To see that $r \in D$, recall that we used a flat pairing function which now implies that $V[G*g]_{\theta+1} \subseteq V_{\theta+1}[G*g] \subseteq N[j(G)*j(g_0)]$. Since $A \in M[G*g_0]$ and j is elementary in $V[G*g]$ with a κ-model as its domain, we have verified that $r \in D$. Since $r \leq r'$ and r' was arbitrary in $\mathbb{P} * \dot{\mathbb{Q}}$, we established that D is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. This completes the proof in the easy case when $2^\delta = \delta^+$ holds in V. Let me point out that we did not only show that $r \in D$, but we also established that $V[G*g]_{\theta+1} \subseteq N[j(G)*j(g_0)]$ in the easy case when $2^\delta = \delta^+$. This subtle point – V_θ versus $V_{\theta+1}$ – will be useful for the proof of Theorem 38.

But what to do if GCH fails at δ? As before, we will show that the set $D \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is dense. The arguments are essentially the same as before, until we diagonalize over $N[G*g]$ in *Step 1*. It was crucial that $|N[G*g]| = \delta^+$ in $V[G*g]$. Instead, we now *force* $2^\delta = \delta^+$: In $V[G*g]$, let $\mathbb{R} = \mathrm{Add}(\delta^+, 1)$, the poset that adds a Cohen subset of δ^+ using conditions of size at most δ. If $H \subseteq \mathbb{R}$ is $V[G*g]$-generic, then $V[G*g*H] \models (2^\delta = \delta^+)$ and thus $N[G*g]$ has size δ^+ in $V[G*g*h]$ as needed. Rather than lifting in $V[G*g]$, we now lift the embedding $j : M \to N$ in $V[G*g*H]$ to $j : M[G*g_0] \to N[j(G)*j(g_0)]$. Since $\mathbb{R}$ is ${\leq}\delta$-distributive and $|V[G*g]_\theta| = \delta$ in $V[G*g]$, it then follows that j is a $(\theta+1)$-strong unfoldability embedding in $V[G*g*H]$. Of course, j need not be an element of $V[G*g]$. Yet, the embedding j naturally induces in $V[G*g*H]$ a θ-strong unfoldability embedding $j_0 : M[G*g_0] \to N_0$ such that N_0 has size δ only, by using seeds in $V_\theta[G*g*H] \cup \{\theta\}$. The embedding j_0 has thus hereditary size δ, which shows by the ${\leq}\delta$-distributivity of $\mathbb{R}$ that j_0 already exists in $V[G*g]$. Since $A \in \mathrm{dom}(j_0)$, we see that $r \in D$ and thus D is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. This completes the proof of the Main Theorem. $\square$

1.5 Consequences, Limitations and Destructibility

In this section, I discuss some corollaries and limitations of the Main Theorem. Moreover, in contrast to the indestructibility that I obtained in the Main Theorem, I also show that a strongly unfoldable cardinal κ becomes highly *destructible* after forcing with posets of size less than κ. In particular, $<\kappa$-closed, κ-proper forcing may then destroy κ.

Since strongly unfoldable cardinals strengthen weakly compact cardinals, we have the following corollary of the Main Theorem:

Corollary 21. *If κ is strongly unfoldable, then there is a set forcing extension in which the weak compactness of κ is indestructible by $<\kappa$-closed, κ-proper forcing.*

Proof. This is an immediate consequence of the Main Theorem. □

Corollary 21 suggests the following question, to which I would very much like to know the answer:

Question 22. *Can every weakly compact cardinal κ be made indestructible by all $<\kappa$-closed, κ-proper forcing? Or indestructible at least by all $<\kappa$-closed, κ^+-c.c. forcing?*

In the statement of the Main Theorem, the reader may wonder whether the $<\kappa$-closure assumption can be relaxed. Can we relax it to $<\kappa$-strategical closure? The answer is no.

Recall that a poset $\mathbb{P}$ is *$<\kappa$-strategically closed* if there is a strategy for the second player in the game of length κ allowing her to continue play, where the players alternate play to build a descending sequence $\langle p_\xi \mid \xi < \kappa\rangle$ of conditions in $\mathbb{P}$, with the second player playing at limit stages[2]. Every $<\kappa$-closed poset is of course $<\kappa$-strategically closed. A poset $\mathbb{P}$ is *$\leq\kappa$-strategically closed* if the second player has a strategy for the game of length $\kappa + 1$.

Fact 23. *A weakly compact cardinal κ can never be indestructible by all $<\kappa$-strategically closed forcing of size κ. Specifically, the Jech-Prikry-Silver poset to add a κ-Suslin tree has size κ and is $<\kappa$-strategically closed, but it destroys the weak compactness of κ.*

Proof. Suppose that κ is weakly compact and $\mathbb{P}$ is the Jech-Prikry-Silver poset for adding a κ-Suslin tree (see for instance p. 248 in [Kun99]). Since $\mathbb{P}$ adds a κ-Suslin tree, it is clear that forcing with $\mathbb{P}$ destroys the tree property of κ and thus its weak compactness. The poset $\mathbb{P}$ has size κ. Moreover, $\mathbb{P}$ is $<\kappa$-strategically closed. This can be seen essentially by the same argument

[2] Some authors call this property *κ-strategic closure*, while defining $<\kappa$-strategic closure to mean the weaker property where the second player only needs a strategy for the games of length *less* than κ.

which is given in [Kun99] and which shows that $\mathbb{P}$ is ${<}\kappa$-distributive. Note that $\mathbb{P}$ is not ${\leq}\omega_1$-closed. $\square$

Since posets of size κ with $\kappa^{<\kappa} = \kappa$ are κ-proper, we know that any attempt to generalize the Main Theorem to ${<}\kappa$-strategically closed, κ-proper posets has to fail necessarily. In fact, such an attempt fails in the second lifting argument (*Step 2* of the proof of the Main Theorem), where we built a master condition for $j\text{"}g_0$. One way to avoid the second lift is by assuming that $\mathbb{Q}$ does not add any new subsets to κ. In this case we can relax ${<}\kappa$-closure to ${<}\delta$-strategic closure for every $\delta < \kappa$. We also do not need κ-properness anymore:

Theorem 24. *Let κ be strongly unfoldable. Then after the lottery preparation of κ relative to a function with the Menas property for κ, the strong unfoldability of κ becomes indestructible by all set forcing which does not add subsets to κ and which is also ${<}\delta$-strategically closed for every $\delta < \kappa$.*

Proof. Let κ be strongly unfoldable. Recall that I defined the lottery preparation $\mathbb{P}$ of κ relative to a function $f : \kappa \to \kappa$ in such a way that the stage γ lottery in $V^{\mathbb{P}_\gamma}$ included all posets $\mathbb{Q} \in H_{f(\gamma)^+}$ that were ${<}\gamma$-closed. But the slightly more general form as presented in [Ham00] considers at stage γ the lottery sum of all those posets $\mathbb{Q} \in H_{f(\gamma)^+}$ which are ${<}\delta$-strategically closed for every $\delta < \gamma$. Let $\mathbb{P}$ be the lottery preparation of κ relative to a function f with the Menas property for κ in this more general form. Let $G \subseteq \mathbb{P}$ be a V-generic filter. Again, $\mathbb{P}$ preserves the inaccessibility of κ. Let $\mathbb{Q}$ be any poset in $V[G]$ which does not add subsets to κ and which is also ${<}\delta$-strategically closed for every $\delta < \kappa$. Let $g \subseteq \mathbb{Q}$ be a $V[G]$-generic filter for $\mathbb{Q}$. If $A \subseteq \kappa$ is a set in $V[G * g]$, then $A \in V[G]$ by the choice of $\mathbb{Q}$. The set A has thus a $\mathbb{P}$-name $\dot{A} \in V$ whose transitive closure has size at most κ. We can hence put the function f, the poset $\mathbb{P}$ *and* the $\mathbb{P}$-name $\dot{A}$ into a κ-model M in V. Let θ be sufficiently large so that $\mathbb{Q} \in V[G]_\theta$ and let $j : M \to N$ be a $(\theta + 1)$-strong unfoldability embedding for κ as in Fact 6. We may assume without loss of generality that $j(f)(\kappa) \geq \beth_\theta^N$. Since the poset $\mathbb{Q}$ enters the stage κ lottery of $j(\mathbb{P})$, we may factor $j(\mathbb{P})$ below a condition which opts for $\mathbb{Q}$ as $j(\mathbb{P}) = \mathbb{P} * \mathbb{Q} * \mathbb{P}_{\text{tail}}$. By following the arguments in *Step 1* of the proof of the Main Theorem (and the corresponding modification if the GCH fails at $\beth_\theta$), we can show that j lifts in $V[G * g]$ to a θ-strong unfoldability embedding $j : M[G] \to N[j(G)]$ where $j(G) = G * g * G_{\text{tail}}$. The point is that we avoid the need for a second lift, since $A \in M[G]$ and $V_\theta \subseteq N$ implies $V[G * g]_\theta \subseteq N[G * g] \subseteq N[j(G)]$. Since $A \subseteq \kappa$ was arbitrary in $V[G * g]$, we showed that κ is θ-strongly unfoldable in $V[G * g]$. It follows that κ is strongly unfoldable in $V[G * g]$. □

The method of proving Theorem 24 provides a local analogue even if κ is only partially strongly unfoldable:

Corollary 25. *Let κ be θ-strongly unfoldable for some ordinal $\theta \geq \kappa$. Then after the lottery preparation of κ relative to a function with the Menas property for κ, the θ-strong unfoldability of κ becomes indestructible by all set forcing of rank less than θ which does not add subsets to κ and which is also ${<}\delta$-strategically closed for every $\delta < \kappa$.*

Proof. Sketch. This is a straightforward modification of the proof of Theorem 24 if θ is a successor ordinal such that the GCH holds at $\beth_\theta$ (see also the proof of Theorem 38). Unfortunately, if the GCH fails at $\beth_\theta$ or if θ is a limit ordinal, one needs a more refined method of proof. I will give the necessary arguments in Section 1.8 when proving Theorem 44. It is then straightforward to modify the proof of Theorem 24 to prove this corollary, even if the GCH fails at $\beth_\theta$ or if θ is a limit ordinal. □

While the Main Theorem establishes that indestructibility is possible, let me in contrast now show various possibilities for *destructibility*.

Fact 26. *Suppose that κ is a weakly compact cardinal in $V = L$. Then any ${<}\kappa$-distributive forcing which adds a subset to κ will destroy the weak compactness of κ.*

Proof. Let κ be weakly compact in L. Let $\mathbb{Q}$ be a ${<}\kappa$-distributive poset and let $G \subseteq \mathbb{Q}$ be an L-generic filter. Let $A \subseteq \kappa$ be a set that is in $L[G]$ but not in L. Assume towards contradiction that κ is weakly compact in $L[G]$. Recall that this is equivalent to κ being κ-strongly unfoldable in $L[G]$. The distributivity of $\mathbb{Q}$ shows that $\mathbb{Q}$ does not add elements of rank less than κ and so $L[G]_\kappa = L_\kappa$. In $L[G]$, fix a κ-model M with $A \in M$ and a corresponding κ-strong unfoldability embedding $j : M \to N$. Note that $A = j(A) \cap \kappa$ is an element of N. Since M sees that $L[G]_\kappa = L_\kappa$, and $A \in N$ has rank less than $j(\kappa)$, it follows by elementarity that N thinks that A is constructible, a contradiction. □

Since weakly compact cardinals are downwards absolute to L and there are ${<}\kappa$-closed, κ-proper posets which add subsets to κ (the poset $\mathrm{Add}(\kappa, 1)$ to add a Cohen subset to κ for instance), we see that forcing over L with a ${<}\kappa$-closed, κ-proper poset can possibly destroy the weak compactness, and hence the strong unfoldability, of a cardinal κ.

Results from [Ham98] free us from forcing over L and show that any nontrivial small forcing over any ground model makes a weakly compact cardinal κ similarly destructible as in Fact 26. Recall that a poset $\mathbb{P}$ is *small* relative to κ, if $\mathbb{P}$ has size less than κ.

Theorem 27 ([Ham98]). *After nontrivial small forcing, any $<\kappa$-closed forcing which adds a subset to κ will destroy the weak compactness of κ.*

This result is the second Main Theorem of [Ham98]. The essential idea is that every elementary embedding witnessing the weak compactness in the forcing extension lifts a ground model embedding. This is then, similarly as in Fact 26, easily seen to be impossible. I will give the proof after Theorem 29 using methods from [Ham03].

Theorem 27 thus shows that the indestructibility obtained in the Main Theorem is necessarily destroyed by any nontrivial small forcing. But how about Theorem 24 and its Corollary 25? These results concern indestructibility of a strongly unfoldable cardinal κ by certain posets that do not add subsets to κ. Such posets preserve of course the weak compactness of κ, yet there are various possibilities that they destroy the strong unfoldability of κ:

Fact 28. *Suppose that κ is a strongly unfoldable cardinal in $V = L$. Then any $<\kappa$-distributive poset which adds a subset to θ will destroy the $(\theta + 1)$-strong unfoldability of κ.*

Proof. The proof is similar to the proof of Fact 26. Let $A \subseteq \theta$ be a set that is in $L[G]$ but not in L. In general, A will not fit into a given κ-model M, but if $j : M \to N$ is any $(\theta + 1)$-strong unfoldability embedding in $L[G]$, then $A \in N$. This suffices to obtain the same contradiction as in the proof of Fact 26. $\square$

In particular, Fact 28 shows that forcing over L with any nontrivial ${<}\kappa$-distributive poset will destroy the strong unfoldability of κ. Moreover, as before, we have again that any nontrivial small forcing over any ground model makes a strongly unfoldable cardinal κ similarly destructible:

Theorem 29. *After nontrivial small forcing, any ${<}\kappa$-closed forcing which adds a subset to θ will destroy the $(\theta+1)$ strong unfoldability of κ.*

Proof. I use the general results on the approximation and cover properties from [Ham03]. The essential idea is that every elementary embedding witnessing the $(\theta+1)$-strong unfoldability in the forcing extension lifts a ground model embedding. This is then, similar as in Fact 28, easily seen to be impossible.

Suppose that $\mathbb{P}$ is any nontrivial poset of size less than κ. We may assume that $\mathbb{P} \in V_\kappa$. Suppose that $\dot{\mathbb{Q}}$ is a name for a poset which necessarily adds a subset to θ while being ${<}\kappa$-closed. Let $g * G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be a V-generic filter. Let $A \subseteq \theta$ be a set that is in $V[g * G]$ but not in $V[g]$. The poset $\mathbb{Q} = \dot{\mathbb{Q}}_g$ is ${<}\kappa$-closed in $V[g]$, which implies $\theta \geq \kappa$. Assume towards a contradiction that κ is $(\theta+1)$-strongly unfoldable in $V[g * G]$.

The cardinal κ is inaccessible in $V[g * G]$. Let $\lambda > \kappa$ be any regular cardinal above $2^{|\mathbb{P}*\dot{\mathbb{Q}}|}$. In $V[g*G]$, use the Skolem-Löwenheim method to build an elementary substructure $\bar{X} \prec H_\lambda[g * G]$ of size κ in the language with a predicate for V, so that $\langle \bar{X}, X, \in\rangle \prec \langle H_\lambda[g * G], H_\lambda, \in\rangle$ where $X = \bar{X} \cap V$. We may assume that $\bar{X}^{<\kappa} \subseteq \bar{X}$ in $V[g * G]$ and that $\{\kappa, \mathbb{P}, \dot{\mathbb{Q}}, g, G\} \subseteq \bar{X}$. Let $\pi : \langle \bar{X}, X, \in\rangle \to \langle \bar{M}, M, \in\rangle$ be the Mostowski collapse of $\bar{X}$. It follows that $\bar{M}$ is a κ-model in $V[g * G]$. The isomorphism π fixes both $\mathbb{P}$ and the filter $g \subseteq \mathbb{P}$. Let $\mathbb{Q}_0 = \pi(\mathbb{Q})$ be the collapsed poset and let $G_0 = \pi(G)$. As π is an isomorphism, it follows that G_0 is an $M[g]$-generic filter on $\mathbb{Q}_0$. Consequently, $\bar{M}$ decomposes as $\bar{M} = M[g * G_0]$ and that $\mathbb{Q}_0$ is ${<}\kappa$-closed in $M[g]$.

Let $j : \bar{M} \to \bar{N}$ be a $(\theta + 1)$-strong unfoldability embedding for κ in $V[g * G]$. By Lemma 8, we may assume that j is cofinal and $\bar{N}^{<\kappa} \subseteq \bar{N}$ in $V[g * G]$. Since $\bar{M} = M[g * G_0]$ and j is cofinal, the model $\bar{N}$ decomposes by elementarity into $\bar{N} = N[g * j(G_0)]$, where $N = \bigcup j\text{"}M$. Since $A \subseteq \theta$ is an element of $V[g * G]_\theta \subseteq \bar{N}$, it follows that $A \in N[g]$ by the ${<}j(\kappa)$-closure of the poset $j(\mathbb{Q}_0)$ in $N[g]$. I now claim that this is impossible:

To see this claim, I use several results from [Ham03]. First observe that $\mathbb{P}*\dot{\mathbb{Q}}$ has the δ approximation and cover properties for $\delta = |\mathbb{P}|^+$ by Lemma 13 in [Ham03]. Moreover, by the proof of Lemma 15 in [Ham03], it follows from our construction of $\bar{X}$ that $M = \bar{M} \cap V$! Thus, the embedding $j : \bar{M} \to \bar{N}$ satisfies all hypotheses of the Main Theorem in [Ham03]. The elementary embedding $j \restriction M : M \to N$ exists therefore in V. In particular, $N \subseteq V$. Combined with $A \in N[g]$ this implies $A \in V[g]$, contradicting the choice of A. This verifies the claim and hence completes the proof. □

In particular, Theorem 29 shows that any nontrivial small forcing followed by nontrivial $<\kappa$-closed forcing will destroy the strong unfoldability of κ. The proof of Theorem 29 can be easily modified to also show Theorem 27:

Proof of Theorem 27. We follow the proof of Theorem 29 closely. Fix therefore the cardinal κ, the poset $\mathbb{P} * \dot{\mathbb{Q}}$ and the filter $g * G$ as before. Assume towards contradiction that κ is weakly compact in $V[g * G]$. Given a subset $A \subseteq \kappa$ which is in $V[g * G]$ but not in $V[g]$, we simply make sure that the elementary substructure $\bar{X} \prec H_\lambda[g * G]$ contains A as an element. It then follows that $\bar{M}$, the Mostowski collapse of $\bar{X}$, contains the set A also. Let $j : \bar{M} \to \bar{N}$ be a κ-strong unfoldability embedding in $V[g * G]$ witnessing the weak compactness of κ. We can assume by Lemma 8 that j is cofinal and $\bar{N}^{<\kappa} \subseteq \bar{N}$ in $V[g * G]$. Since $A = j(A) \cap \kappa$ is an element of $\bar{N}$, we now continue exactly as in the proof of Theorem 29 to reach a contradiction. It follows that κ is not weakly compact in $V[g * G]$, which completes the proof of Theorem 27. $\Box$

Lastly, I want to mention that the indestructibility which we obtained in the Main Theorem for a strongly unfoldable cardinal κ can never be achieved for a measurable or larger cardinal. In fact, it is known that no ineffable cardinal κ can be made indestructible by ${<}\kappa$-closed, κ^+-c.c. forcing (see for instance [KY] or [Ham]). Recall that an uncountable regular cardinal κ is *ineffable* if for every sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of sets with $A_\alpha \subseteq \alpha$ for each $\alpha < \kappa$, there exists a set $A \subseteq \kappa$ such that $\{\alpha \in \kappa \mid A \cap \alpha = A_\alpha\}$ is stationary. Every measurable cardinal is ineffable, and every ineffable cardinal is weakly compact. If κ is ineffable, then κ is ineffable in L. Furthermore, a tree T of height κ is a *κ-Kurepa tree* if T has at least κ^+ many paths and every level of T has size less than κ. A tree T is *slim* if for each infinite ordinal α, the α^{th} level of T has size at most $|\alpha|$. It is clear that the complete binary tree $2^{<\kappa}$ for an inaccessible cardinal κ is always a κ-Kurepa tree. In contrast, Jensen and Kunen showed that for an ineffable cardinal κ there can never exist a slim κ-Kurepa tree (see [Dev84]). As usual, I denote the set of all paths trough a tree T by $[T]$.

Fact 30. *If κ is an uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$, then there is* *a $<\kappa$-closed, κ^+-c.c. poset of size κ^+ which adds a slim κ-Kurepa tree. An ineffable cardinal κ can thus never be indestructible by all $<\kappa$-closed, κ-proper forcing.*

Proof. I will essentially follow the proof given in [Ham]. Suppose $\kappa > \omega$ is a cardinal satisfying $\kappa^{<\kappa} = \kappa$. Let $\mathbb{P}$ be the partial order with the top element $\mathbb{1}_{\mathbb{P}} = \langle \emptyset, \emptyset \rangle$ and conditions $\langle t, f \rangle$ below $\mathbb{1}_{\mathbb{P}}$, where $t \subseteq 2^{<\beta}$ is a slim tree of height β for some $\beta < \kappa$, and $f : \kappa^+ \to [t]$ is a function with $1 \leq |\mathrm{dom}(f)| \leq |\beta|$ (in particular, $[t] \neq \emptyset$). The conditions of $\mathbb{P}$ are ordered as follows: $\langle t, f \rangle \leq \langle t', f' \rangle$ iff either $t = t'$ and $f = f'$ or the tree t is a proper end-extension of t' and $f(\xi)$ extends $f'(\xi)$ for every $\xi \in \mathrm{dom}(f')$. Note that the poset $\mathbb{P}$ is atomless, since we made sure that $[t] \neq \emptyset$ for every nontrivial condition $\langle t, f \rangle \in \mathbb{P}$. While $\mathbb{P}$ is not $\leq\omega_1$-directed closed, it is easy to verify that $\mathbb{P}$ is $<\kappa$-closed and that $|\mathbb{P}| = \kappa^+$. A standard application of the Δ-system lemma shows that $\mathbb{P}$ is κ^+-c.c. It follows that $\mathbb{P}$ preserves all cardinals. If $G \subseteq \mathbb{P}$ is a V-generic filter, then the union of the first coordinates of the elements of G is easily seen to be a slim κ-tree T. Furthermore, it is clear that the pointwise union of the paths given by the second coordinates of elements in G naturally produces a partial function $F : \kappa^+ \to [T]$. Density arguments show that F is one-to-one and that $\mathrm{dom}(F) = \kappa^+$. The tree $T \in V[G]$ is thus a slim κ-Kurepa tree. The second statement of Fact 30 is immediate from the Jensen-Kunen result. □

1.6 Global Indestructibility

We will obtain in Theorem 34 a class forcing extension $V[G]$ such that *every* strongly unfoldable cardinal of V is preserved and *every* strongly unfoldable cardinal κ in $V[G]$ is indestructible by $<\kappa$-closed, κ-proper forcing. There is a subtle issue in our goal of obtaining the model $V[G]$. We need to make sure that the process of making the strongly unfoldable cardinals of V indestructible in $V[G]$ does not create any *new* strongly unfoldable cardinals. Such new large cardinals would have little reason to exhibit the desired indestructibility in $V[G]$. The question when forcing does not create any new large cardinals is the main focus of Hamkins' article [Ham03] on the *approximation* and *cover properties*. We already used these properties in Section 1.5, when we showed that small forcing makes a strongly unfoldable cardinal κ highly destructible. The following fact suffices for our purposes in this section.

Fact 31 ([Ham03])**.** *Suppose λ is any cardinal. Suppose that $\mathbb{P}$ is nontrivial forcing, $|\mathbb{P}| \leq \lambda$, and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a necessarily $\leq\lambda$-closed poset. If $G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is a V-generic filter, then every strongly unfoldable cardinal above λ in $V[G]$ is strongly unfoldable in V.*

Proof. The result follows directly from Lemma 13 and Corollary 20 in [Ham03].

□

Fact 31 suggests to precede the lottery preparation $\mathbb{P}$ by some small forcing. I will follow this idea in Theorem 34 when we add a Cohen real at stage ω of the lottery preparation.

Forcing with classes generalizes the usual set forcing, and the main ideas to do so are sketched for instance in [Kun99] or [Jec03]. A rigorous exposition of class forcing that includes class versions of the usual set forcing results can be found in [Rei06]. The main problem when forcing with a proper class $\mathbb{P}$ of forcing conditions is that there is little reason for the forcing extension to satisfy the axioms of set theory. Nevertheless, many commonly used class iterations are unproblematic. For instance, every *progressively closed* class forcing iteration [Rei06] preserves the ZFC axioms. In essence, a progressively closed class iteration $\mathbb{P}$ is an Ord-length iteration of complete subposets $\mathbb{P}_\alpha$ where $\mathbb{P} = \bigcup_{\alpha \in Ord} \mathbb{P}_\alpha$ with the additional requirement that the tail forcing becomes more and more closed as we progress through the iteration.

Theorem 34 relies on Lemma 33 below, which shows that the *indestructibility* of κ that I obtained in the Main Theorem for a strongly unfoldable cardinal κ is itself preserved by a wide variety of forcing notions, including partially ordered classes. The situation is easy for partially ordered *sets*:

Lemma 32. *Suppose that κ is any large cardinal which is indestructible by $<\kappa$-closed, κ-proper set forcing. Then any $<\kappa$-closed, κ-proper set forcing preserves κ and its indestructibility.*

Proof. This is clear by Corollary 18. □

Lemma 32 can be strengthened to apply to forcing with certain partially ordered *classes*. The reason why I used κ-proper posets in the Main Theorem is that such posets can be put into an elementary substructure $X \prec H_\lambda$ of size κ. It is clear that class forcing notions which are not posets can never technically be κ-proper. Nevertheless, the Main Theorem combined with the next lemma does in fact provide indestructibility of a strongly unfoldable cardinal κ by different types of class forcing notions also:

Lemma 33. *Suppose κ is a strongly unfoldable cardinal which is indestructible by $<\kappa$-closed, κ-proper set forcing. Let $\mathbb{P}$ be a class forcing notion that preserves* ZFC *such that for unboundedly many cardinals δ, the class $\mathbb{P}$ factors as $\mathbb{P} = \mathbb{P}_1 * \dot{\mathbb{P}}_2$ where $\mathbb{P}_1$ is a $<\kappa$-closed, κ-proper poset and $\dot{\mathbb{P}}_2$ is the name for a necessarily $<\delta$-closed class. Then:*

1. *Forcing with $\mathbb{P}$ preserves the strong unfoldability of κ.*

2. *Forcing with $\mathbb{P}$ preserves the indestructibility of κ.*

Proof. Fix the cardinal κ and the class $\mathbb{P}$ as in the theorem. Suppose that $G \subseteq \mathbb{P}$ is a V-generic class filter for the partially ordered class $\mathbb{P}$.

For assertion (1), fix any ordinal $\theta \geq \kappa$. Let $\delta > \beth_\theta^{V[G]}$ be a cardinal such that $\mathbb{P}$ factors as $\mathbb{P} = \mathbb{P}_1 * \mathbb{P}_2$ where $\mathbb{P}_1$ is a ${<}\kappa$-closed, κ-proper poset and $\mathbb{P}_2$ is a ${<}\delta$-closed class forcing notion in $V^{\mathbb{P}_1}$. Both $\mathbb{P}_1$ and $\mathbb{P}_2$ are ${<}\kappa$-distributive, which shows that $\mathbb{P} = \mathbb{P}_1 * \mathbb{P}_2$ preserves the inaccessibility of κ. Moreover, $\mathbb{P}_1$ preserves the strong unfoldability of κ by hypothesis. As $\mathbb{P}_2$ is ${<}\delta$-distributive, it does not add new elements of rank less than θ, which shows that every θ-strong unfoldability embedding in $V^{\mathbb{P}_1}$ is in fact a θ-strong unfoldability embedding in $V[G]$. The cardinal κ is thus θ-strongly unfoldable in $V[G]$. Since θ was arbitrary, we verified assertion (1).

For assertion (2), let $\mathbb{Q} \in V[G]$ be any ${<}\kappa$-closed, κ-proper poset in $V[G]$. Let $H \subseteq \mathbb{Q}$ be a $V[G]$-generic filter. As $\mathbb{Q}$ is ${<}\kappa$-distributive, it preserves the inaccessibility of κ. To verify that $\mathbb{Q}$ preserves the strong unfoldability of κ, fix thus any sufficiently large ordinal $\theta \geq \kappa$ such that $\mathbb{Q} \in V[G]_\theta$. Let $\delta > \beth_\theta^{V[G]}$ be a cardinal such that $\mathbb{P}$ factors as $\mathbb{P} = \mathbb{P}_1 * \mathbb{P}_2$ where $\mathbb{P}_1$ is a ${<}\kappa$-closed, κ-proper poset and $\mathbb{P}_2$ is a ${<}\delta$-closed class forcing notion in $V^{\mathbb{P}_1}$. If we factor the filter G correspondingly as $G = G_1 * G_2$, it then follows that $\mathbb{Q}$ is an element of $V[G_1]$. We thus see that the two-step iteration $\mathbb{P}_2 * \dot{\mathbb{Q}}$ is isomorphic to the product $\mathbb{P}_2 \times \mathbb{Q}$ in $V[G_1]$. As $\mathbb{P}_2$ is $\leq |\mathbb{Q}|$-closed, it follows that we may reverse the order of the factors of the product $\mathbb{P}_2 \times \mathbb{Q}$ and see that G_2 is in fact $V[G_1][H]$-generic for $\mathbb{P}_2$. Moreover, while $\mathbb{P}_2$ may not be ${<}\delta$-closed in $V[G_1][H]$, forcing with $\mathbb{Q}$ does preserve the ${<}\delta$-distributivity of $\mathbb{P}_2$. The class versions of these standard properties of products of posets are given in [Rei06]. We thus have that $V[G][H] = V[G_1][H][G_2]$. Lemma 32 shows that the poset $\mathbb{P}_1$ preserves the indestructibility of κ by ${<}\kappa$-closed, κ-proper posets. Moreover, we may assume without loss of generality that θ was chosen large enough, so that $V[G_1]^{<\delta} \subseteq V[G_1]$ in $V[G]$ implies that $\mathbb{Q}$ is κ-proper in $V[G_1]$. As $\mathbb{Q}$ is certainly ${<}\kappa$-closed in $V[G_1]$, it follows that κ is strongly unfoldable in $V[G_1][H]$. But as in the proof of assertion (1), this means that κ is θ-strongly unfoldable in $V[G][H]$. Since θ was arbitrary, we verified the strong unfoldability of κ in $V[G][H]$ and thus assertion (2).

Note that we could have in fact omitted the proof of assertion (1), since trivial forcing is κ-proper in $V[G]$ (the equality $\kappa^{<\kappa} = \kappa$ is preserved by $\mathbb{P}$) and assertion (2) therefore implies assertion (1). □

In particular, the Main Theorem from Section 1.4 shows that any strongly unfoldable cardinal κ becomes also indestructible by class forcing notions as described in Lemma 33. We can now combine the Main Theorem, Theorem 11 and Lemma 33 to obtain the following global indestructibility result.

Theorem 34. *If V satisfies* ZFC*, then there is a class forcing extension $V[G]$ satisfying* ZFC *such that*

1. *every strongly unfoldable cardinal of V remains strongly unfoldable in $V[G]$,*

2. *in $V[G]$, every strongly unfoldable cardinal κ is indestructible by $<\kappa$-closed, κ-proper set forcing, and*

3. *no new strongly unfoldable cardinals are created.*

Proof. We will force with the class lottery preparation relative to a suitable Menas function in order to prove the theorem. Let $F : Ord \to Ord$ be the class function as defined in Theorem 11. We saw that F has the Menas property for every strongly unfoldable cardinal $\kappa \in Ord$ and that $\mathrm{dom}(F)$ does not contain any strongly unfoldable cardinals. Let $\mathbb{P}$ be the class lottery preparation $\mathbb{P}$ relative to the function F. This is the direct limit of an Ord-stage forcing iteration with Easton support which at stage γ, if $\gamma \in \mathrm{dom}(F)$ and $F``\gamma \subseteq \gamma$, forces with the lottery sum of all $<\gamma$-closed posets $\mathbb{Q} \in H_{F(\gamma)^+}$ in $V^{\mathbb{P}_\gamma}$. Since $F(\omega) = \omega$, we know that the forcing to add a Cohen real, $\mathrm{Add}(\omega, 1)$, enters the stage ω lottery. Let $p \in \mathbb{P}$ be a condition opting for $\mathrm{Add}(\omega, 1)$. Let $G \subseteq \mathbb{P}$ be a V-generic filter containing p.

I first sketch that $V[G] \models$ ZFC. Note that for any δ which is closed under F, the lottery preparation $\mathbb{P}$ factors as $\mathbb{P}_\delta * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_\delta$ is the set lottery preparation using $F \upharpoonright \delta$ and $\mathbb{P}_{\text{tail}}$ is the class lottery preparation defined in $V^{\mathbb{P}_\delta}$ using the restriction of F to ordinals greater than or equal to δ. It follows that the tail forcing $\mathbb{P}_{\text{tail}}$ is necessarily $<\delta$-closed. Since the class of closure points of F is unbounded in Ord and $\mathbb{P}$ is the direct limit at Ord of all the previous stages, one can show that $\mathbb{P}$ is progressively closed and consequently that $V[G] \models$ ZFC (for details, see for instance [Rei06]).

We verify assertion (3) next. Observe that the class forcing iteration $\mathbb{P}$ factors below p as $\text{Add}(\omega, 1) * \dot{\mathbb{P}}_{(\omega,\infty)}$ where $\dot{\mathbb{P}}_{(\omega,\infty)}$ is a name for a necessarily $\leq\omega$-closed class iteration. Note that Fact 31 generalizes to class forcing iterations: The definition of the approximation and cover properties also applies to class forcing extensions and the proof of Lemma 13 in [Ham03] works well no matter whether the tail forcing is a set or a proper class. It follows that forcing with $\mathbb{P}$ does not create any new strongly unfoldable cardinals, which proves assertion (3).

We can now verify assertions (1) and (2) simultaneously. For assertion (2), note first that it suffices to make all cardinals κ which are strongly unfoldable in V indestructible in $V[G]$. By assertion (3), we do not have to worry about any other possibly strongly unfoldable cardinals in $V[G]$. Fix thus any strongly unfoldable cardinal $\kappa \in V$. Since $F``\kappa \subseteq \kappa$, we know that $\mathbb{P}$ factors at stage κ. Moreover, as $\kappa \notin \operatorname{dom}(F)$, we have that the stage κ forcing of $\mathbb{P}$ is trivial. This means that $\mathbb{P}$ factors as $\mathbb{P}_\kappa * \mathbb{P}_{\text{tail}}$ where $\mathbb{P}_\kappa$ is the set lottery preparation of κ relative to $F \restriction \kappa$ and $\mathbb{P}_{\text{tail}}$ is the ${\leq}\kappa$-closed class lottery preparation in $V^{\mathbb{P}_\kappa}$ relative to the restriction of F to ordinals above κ. Since $F \restriction \kappa$ has the Menas property for κ, it follows from the Main Theorem that $\mathbb{P}_\kappa$ preserves the strong unfoldability of κ and makes κ indestructible by forcing with ${<}\kappa$-closed, κ-proper posets. Note that there are unboundedly many δ such that $\mathbb{P}_{\text{tail}}$ factors as $\mathbb{P}_{\text{tail}} = \mathbb{P}_{(\kappa,\delta)} * \mathbb{P}_{[\delta,\infty)}$ where $\mathbb{P}_{(\kappa,\delta)}$ is a ${\leq}\kappa$-closed poset and $\mathbb{P}_{[\delta,\infty)}$ is a ${<}\delta$-closed class forcing iteration. Lemma 33 thus shows that $\mathbb{P}_{\text{tail}}$ preserves the strong unfoldability of κ and its indestructibility by ${<}\kappa$-closed, κ-proper set forcing. This proves assertion (1) and (2). $\square$

Theorem 34 makes every strongly unfoldable cardinal κ indestructible by ${<}\kappa$-closed, κ-proper set forcing. Combined with assertion (1) of Lemma 33 this also implies indestructibility of every strongly unfoldable cardinal κ by a wide variety of *class* forcing notions.

1.7 An Application to Indescribable Cardinals

When Villaveces introduced strongly unfoldable cardinals in [Vil98], he observed that they also strengthen *indescribable* cardinals, referring to the embedding characterization due to Hauser [Hau91]. Classically, for $m, n \in \mathbb{N}$, a Π^m_n indescribable cardinal κ is characterized by a certain reflection property of V_κ for Π^m_n formulas. A *totally indescribable* cardinal is then a cardinal κ that is Π^m_n indescribable for every $m, n \in \mathbb{N}$. Hauser's embedding characterization introduced the idea of Σ^m_n correctness at κ. Following [Ham], we say for $m \geq 1$ and $n \geq 0$ that a transitive set N is Σ^m_n *correct at* κ if $(V_{\kappa+m})^N \prec_{\Sigma_n} V_{\kappa+m}$ and $V_{\kappa+m-1} \subseteq N$. (Since V_α is Σ_1 definable in $V_{\alpha+1}$, it follows that the latter condition is redundant if $n > 0$.) Note that unlike [Hau91], we do not insist that N is closed under $(\beth_{\kappa+m-2})$-sequences. As noted in [DH06], this closure requirement can easily be dropped, as the following fact shows.

Fact 35 ([Hau91],[DH06])**.** *Let κ be an inaccessible cardinal. Let $m \geq 1$ and $n \geq 1$ be natural numbers. The following are equivalent:*

1. *κ is a Π^m_n indescribable cardinal.*

2. *For every κ-model M there is an embedding $j : M \to N$ with critical*

point κ such that N is Σ^m_{n-1} correct at κ.

3. *For every κ-model M there is an embedding $j : M \to N$ with critical point κ such that N is Σ^m_{n-1} correct at κ, the model N has size $\beth_{\kappa+m-1}$ and $N^{\beth_{\kappa+m-2}} \subseteq N$ (meaning $N^{<\kappa} \subseteq N$ when $m = 1$).*

Proof. Hauser [Hau91] provided the characterization of Π^m_n indescribable cardinals as in assertion (3). Džamonja and Hamkins observed in [DH06] that assertions (2) and (3) are equivalent. Their proof of Fact 6 of this chapter can be modified in a straightforward manner to establish the equivalence between assertion (2) and (3). □

Assertion (2) of Fact 35 implies for instance the classic result due to Hanf and Scott that a cardinal κ is weakly compact if and only if κ is Π^1_1 indescribable. More generally, we have the following.

Corollary 36 ([DH06]). *Let $m \geq 0$ be a natural number. A cardinal κ is Π^{m+1}_1 indescribable if and only if κ is $(\kappa+m)$-strongly unfoldable. A cardinal κ is totally indescribable if and only if κ is $(\kappa+m)$-strongly unfoldable for every $m \in \mathbb{N}$.*

Proof. This is immediate by characterization (2) of Fact 35. □

The Main Theorem has thus the following corollary.

Corollary 37. *If κ is strongly unfoldable, then there is a forcing extension in which the total indescribability of κ is indestructible by $<\kappa$-closed, κ-proper forcing.*

Proof. This is immediate by the Main Theorem and Corollary 36. □

If κ is a Π^m_n indescribable cardinal which is not strongly unfoldable, we need a local version of the Main Theorem for θ-strong unfoldability in order to make κ indestructible. This is fairly straightforward if θ is a successor ordinal. It is more difficult to obtain the local version if θ is a limit ordinal (see Section 1.8).

Theorem 38. *Let κ be a $(\theta+1)$-strongly unfoldable cardinal for some ordinal $\theta \geq \kappa$. Assume that the* GCH *holds at $\beth_\theta$. Then after the lottery preparation of κ relative to a function with the Menas property for κ, the $(\theta+1)$-strong unfoldability of κ becomes indestructible by $<\kappa$-closed, κ-proper forcing of size at most $\beth_\theta$.*

Proof. This is what we essentially argued when proving the Main Theorem. Let κ be $(\theta+1)$-strongly unfoldable for some ordinal $\theta \geq \kappa$. By assertion (2) of Theorem 11, we know that there is a function $f : \kappa \to \kappa$ with the Menas property for κ. Let $\mathbb{P}$ be the lottery preparation of κ relative to f and let $G \subseteq \mathbb{P}$ be V-generic. If $\mathbb{Q}$ is $<\kappa$-closed and κ-proper of size at most $\beth_\theta$ in

$V[G]$, we may assume by assertion (1) of Fact 14 that $\mathbb{Q} \in V_{\theta+1}[G]$. Following the proof of the Main Theorem, we thus *know* that θ is large enough so that in V we have names $\dot{\mathbb{Q}}$ and $\dot{A}$ that are elements of $V_{\theta+1}$. The GCH assumption brings us to the easy case when $2^\delta = \delta^+$ in V for $\delta = \beth_\theta$. In this case we were able to prove not only the set D to be dense in $\mathbb{P} * \dot{\mathbb{Q}}$, but the set

$$D^* = \{r \in \mathbb{P} * \dot{\mathbb{Q}} \ : r \Vdash \text{“}\dot{A} \text{ can be placed into a } \kappa\text{-model } M \text{ with an embedding } j : M \to N \text{ with } \theta < j(\kappa) \text{ and } V_{\theta+1} \subseteq N\text{”}\}.$$

to be dense in $\mathbb{P}*\dot{\mathbb{Q}}$. But density of D^* proves that κ remains $(\theta+1)$-strongly unfoldable after forcing with $\mathbb{P} * \dot{\mathbb{Q}}$, as desired. □

Note that the GCH assumption at $\beth_\theta$ in Theorem 38 is not too restrictive: If the assumption fails, we can simply force the GCH at $\beth_\theta$ first, which by Lemma 7 preserves the $(\theta + 1)$-strong unfoldability of κ. Moreover, we will see in Section 1.8 that the GCH assumption at $\beth_\theta$ is in fact an *unnecessary* hypothesis for the conclusion of Theorem 38 (see Theorem 43).

Using this slightly stronger result from Section 1.8, we have the following indestructibility result for indescribable cardinals.

Corollary 39. *Let κ be Π^{m+1}_1 indescribable for some natural number $m \geq 1$. Then, after the lottery preparation of κ relative to a function with the Menas* *property for* κ*, the* Π_1^{m+1}*-indescribability of* κ *becomes indestructible by* $<\kappa$*-closed,* κ*-proper forcing of size at most* $\beth_{\kappa+m-1}$*.*

Proof. This is immediate from Theorem 43 and Corollary 36. $\square$

Corollary 40. *Let* κ *be totally indescribable. Then after the lottery preparation of* κ *relative to a function with the Menas property for* κ*, the total indescribability of* κ *becomes indestructible by* $<\kappa$*-closed,* κ*-proper forcing of size less than* $\beth_{\kappa+\omega}$*.*

Proof. This is immediate from Corollaries 36 and 39. $\square$

1.8 The Limit Case

In Section 1.7, I needed and found a local analogue of the Main Theorem for $(\theta+1)$-strong unfoldability. But can we make a θ-strongly unfoldable cardinal κ for a *limit ordinal* θ also indestructible? The answer is yes (see Theorem 42). But it seems that the method of proof as in the Main Theorem does not quite work. If $j : M \to N$ with $V_\theta \subseteq N$ and $|N| = \beth_\theta$, then we can make N closed under sequences of length less than $\text{cof}(\theta)$ (see Fact 6), but not under $\text{cof}(\theta)$-sequences (since $\beth_\theta^{\text{cof}(\theta)} > \beth_\theta$). In general, this closure is not sufficient in order to apply diagonalization for N (unless $\theta = \beth_\theta = \text{cof}(\theta)$, that is unless θ is inaccessible). Instead, I will use a θ-strong unfoldability embedding $j : M \to N$ where N is generated by less than $j(\kappa)$ many seeds. Woodin [CW] was first to show how to use factor methods to lift such embeddings, and I will follow the modification of Woodin's technique by Gitik and Shelah [GS89] that they used to make strong cardinals indestructible by $\leq\kappa$-closed forcing.

There is a slight problem though that prevents us from directly using extender embeddings as in assertion (2) of Fact 5: When we built a master condition in *Step 2* of the proof of the Main Theorem, we relied on $j : M \to N$ being an embedding with $N^\kappa \subseteq N$. The closure of N under κ-sequences implied that $N[j(G)]^\kappa \subseteq N[j(G)]$ in $V[G*g]$, which in turn enabled us to find the desired master condition. But, if $j : M \to N$ is a θ-strong unfoldability embedding with $N = \{j(g)(s) \mid g : V_\kappa \to M \text{ with } g \in M \text{ and } s \in S^{<\omega}\}$ for $S = V_\theta \cup \{\theta\}$, then $j"M$ is cofinal in N and thus $j \notin N$. It follows that extender embeddings as in assertion (2) of Fact 5 can never satisfy $N^\kappa \subseteq N$. The solution to this problem is quite easy. We did not really need $N[j(G)]^\kappa \subseteq N[j(G)]$ in $V[G * g]$ in *Step 2* of the Main Theorem in order to find the desired master condition. It would have been sufficient to have $j \restriction g_0 \in N[j(G)]$. The next lemma shows how this can be achieved while still keeping the necessary properties of an extender embedding.

Lemma 41. *Assume that κ is a θ-strongly unfoldable cardinal for an ordinal*

$\theta \geq \kappa$. *Suppose that* M *is a* κ*-model,* $\alpha \in M$ *an ordinal and* $B \in M$ *a set such that* $M \models (V_\alpha$ *exists and* $B \in V_\alpha)$. *Then there is a* θ*-strong unfoldability embedding* $j : M \to N$ *such that* $N = \{j(h)(s) \mid h : D^{<\omega} \to M$ *with* $h \in M$ *and* $s \in S^{<\omega}\}$ *where* $D = V_\alpha^M$ *and* $S = V_\theta \cup trcl(B) \cup \{\theta, j \restriction B\}$. *In particular,* $j \restriction B \in N$.

Proof. The proof uses arguments from *seed* theory. Assume the ordinals κ, θ, α and the sets M, B, D are given as in the lemma. Without loss of generality assume that $\alpha \geq \kappa$. We may fix a Hauser embedding $j : M \to N$ with $\theta < j(\kappa)$ and $V_\theta \subseteq N$ such that $j \in N$. Let $b = j \restriction B$. Since $M \in N$ and $j \in N$, we have $b \in N$. As $B \in D = V_\alpha^M$, we see that both B and $j(B)$ are elements of $j(D)$. Since $b \subseteq B \times j(B)$, it follows that $b \in j(D)$. This is easy to see for a limit ordinal α. If α is a successor ordinal, one again needs to use a flat pairing function (see Section 1.4) instead of the usual von Neumann code of ordered pairs. Let $S = V_\theta \cup \mathrm{trcl}(B) \cup \{\theta, b\}$. Since $\alpha \geq \kappa$, it follows that $S \subseteq j(D)$. It hence makes sense to define the *seed hull* of S via j in N, namely the set $X_S = \{j(h)(s) \mid h : D^{<\omega} \to M$ with $h \in M$ and $s \in S^{<\omega}\}$. As usual, $X_S \prec N$ is an elementary substructure (by the Tarski-Vaught test) such that $\mathrm{ran}(j) \subseteq X_S$ and $S \subseteq X_S$. Let $\pi : X_S \to N_0$ be the Mostowski collapse of X_S. The composition map $j_0 = \pi \circ j$ is elementary with critical point κ and $\theta < j_0(\kappa)$. With $S_0 = \pi$"S, it follows by elementarity of π that j_0 :

$M \to N_0$ is an embedding with $N_0 = \{j_0(h)(t) \mid f : D^{<\omega} \to M$ with $h \in M$ and $t \in S_0^{<\omega}\}$. Since $V_\theta \cup \operatorname{trcl}(B)$ is a transitive subset of X_S, we see that $\pi \restriction V_\theta = \mathrm{id}$ and $\pi \restriction \operatorname{trcl}(B) = \mathrm{id}$. This means that π fixes each element of S except possibly b. Let $b_0 = \pi(b)$. Moreover, since $B \subseteq X_S$ and $\pi \restriction B = \mathrm{id}$, we have $b_0 = j_0 \restriction B$ as desired. It follows that $j_0 : M \to N_0$ is the desired embedding. □

As usual, one checks easily that the embedding characterization of Lemma 41 preserves the Menas property of any function $f \colon \kappa \to \kappa$ which has the Menas property for κ.

Theorem 42. *Let κ be θ-strongly unfoldable for a limit ordinal $\theta \geq \kappa$. Then after the lottery preparation of κ relative to a Menas function for κ, the θ-strong unfoldability of κ becomes indestructible by $<\kappa$-closed, κ-proper forcing of size less than $\beth_\theta$.*

Proof. Again, we follow the proof of the Main Theorem as closely as possible. Let κ be θ-strongly unfoldable for some limit ordinal $\theta \geq \kappa$. By assertion (2) of Theorem 11, we know that there is a function $f \colon \kappa \to \kappa$ with the Menas property for κ. Let $\mathbb{P}$ be the lottery preparation of κ relative to f and let $G \subseteq \mathbb{P}$ be V-generic. If $\mathbb{Q}$ is $<\kappa$-closed and κ-proper of size less than $\beth_\theta$ in $V[G]$, we may assume without loss of generality that $\mathbb{Q} \in V_\theta[G]$. We thus know that θ is large enough so that in V we have names $\dot{\mathbb{Q}}$ and $\dot{A}$ that are elements of V_θ. As before, we shall prove that the set

$$D = \{r \in \mathbb{P} * \dot{\mathbb{Q}} : r \Vdash \text{“}\dot{A} \text{ can be placed into a } \kappa\text{-model } M \text{ with an embedding } j : M \to N \text{ with } \theta < j(\kappa) \text{ and } V_\theta \subseteq N\text{”}\}$$

is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. We fix any $r' \in \mathbb{P} * \dot{\mathbb{Q}}$ and let $\lambda > \beth_\theta$ be a sufficiently large regular cardinal witnessing the κ-properness of $\mathbb{P}$. Again, we let $x \in H_\lambda$ be a λ-witness for $\mathbb{P} * \dot{\mathbb{Q}}$. We use the Skolem-Löwenheim method in V to build $X \prec H_\lambda$ of size κ with $X^{<\kappa} \subseteq X$ such that $\{\kappa, r', \mathbb{P}, f, \dot{\mathbb{Q}}, \dot{A}, \theta, x\} \subseteq X$. As λ is sufficiently large and $x \in X$, we can thus fix an $(X, \mathbb{P} * \dot{\mathbb{Q}})$-generic condition $r \in \mathbb{P} * \dot{\mathbb{Q}}$ such that $r \leq r'$. The rest of the proof will again show that $r \in D$, and hence that D is dense.

Let $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be any V-generic filter containing r as an element so that $G \subseteq \mathbb{P}$ is a V-generic filter and $g \subseteq \mathbb{Q} = \dot{\mathbb{Q}}_G$ is a $V[G]$-generic filter. Let $A = \dot{A}_{G*g}$ be the subset that has to be put into the domain of a θ-strong unfoldability embedding $j \in V[G * g]$. As before, G is X-generic for $\mathbb{P}$ and g is $X[G]$-generic for $\mathbb{Q}$. Again, let $\pi : \langle X, \in\rangle \to \langle M, \in\rangle$ be the Mostowski collapse. M is a κ-model in V. Let $\pi(\dot{\mathbb{Q}}) = \dot{\mathbb{Q}}_0$ and $\pi(\dot{A}) = \dot{A}_0$. Let θ_0 denote the ordinal $\pi(\theta)$. We want to use Lemma 41 for the κ-model M and the set $\dot{\mathbb{Q}}_0 \in M$. Since H_λ sees that $\dot{\mathbb{Q}} \in V_\theta$, it follows that $M \models (V_{\theta_0}$ exists and $\dot{\mathbb{Q}}_0 \in V_{\theta_0})$. Moreover, since $\theta_0 \leq \theta$ and $\dot{\mathbb{Q}}_0 \in V_\theta$, we have $\text{trcl}(\dot{\mathbb{Q}}_0) \subseteq V_\theta$. Since κ is θ-strongly unfoldable in V, Lemma 41 now provides a θ-strong unfoldability embedding $j : M \to N$ with $N = \{j(h)(s) \mid h : D^{<\omega} \to M \text{ with } h \in M \text{ and } s \in S^{<\omega}\}$ where $S = V_\theta \cup \{\theta, j \restriction \dot{\mathbb{Q}}_0\}$ and $D = (V_{\theta_0})^M$. Let $\delta = \beth_\theta^N$. Since f has the Menas property for κ, we may assume that $j(f)(\kappa) \geq \delta$ and $\delta < j(\kappa)$. Note that $S \subseteq N$ and thus $S \in N$ has size δ in N. As θ is a limit ordinal and $V_\theta \subseteq N$, we have that $\beth_\theta^N = \beth_\theta$. Let $b = j \restriction \dot{\mathbb{Q}}_0$. As in the proof of the Main Theorem, our strategy is to lift the embedding j in $V[G * g]$ in two steps.

Step 1. *In $V[G * g]$, lift $j : M \to N$ to $j : M[G] \to N[j(G)]$.*

We force with $G \subseteq \mathbb{P}$ over M. As before we may opt for $\mathbb{Q}$ at the stage κ lottery of $j(\mathbb{P})$. Thus $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$. Since $G * g$ is V-generic and hence N-generic for $\mathbb{P} * \dot{\mathbb{Q}}$, it suffices to find in $V[G * g]$ a filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ which is $N[G * g]$-generic. The key to solving this problem is the fact that $j : M \to N$ is an embedding where N is generated by less than $j(\kappa)$ many seeds $s \in S^{<\omega}$. We shall use the diagonalization criterion (Fact 19) in $V[G*g]$ not for $N[G*g]$, but for a suitable elementary substructure $Y[G*g] \prec N[G*g]$ of size κ. Since $\delta \leq j(f)(\kappa)$, the first nontrivial stage of forcing in $\mathbb{P}_{\text{tail}}$ is beyond δ, and so $\mathbb{P}_{\text{tail}}$ is ${\leq}\delta$-closed in $N[G * g]$. Observe that $a = \pi \restriction \dot{\mathbb{Q}}$ is an element of V_θ (since θ is a limit ordinal) and thus $a \in N$. Recall that $b = j \restriction \dot{\mathbb{Q}}_0 \in N$ by construction of j. Let $Y = \{j(h)(\langle\kappa, \theta, a, b\rangle) \mid h : D^4 \to M$ with $h \in M\}$ be the seed hull of $\langle\kappa, \theta, a, b\rangle$ in N. As usual, $\{\kappa, \theta, a, b\} \cup \operatorname{ran}(j) \subseteq Y$ and $Y \prec N$. Since $\mathbb{P}_{\text{tail}}$ is definable from $j(\mathbb{P})$ and κ, it follows that $\mathbb{P}_{\text{tail}} \in Y$. Clearly $|Y| = |M| = \kappa$. Moreover, $M^{<\kappa} \subseteq M$ in V implies $Y^{<\kappa} \subseteq Y$ in V. Consider the forcing extension $Y[G * g] = \{\tau_{G*g} : \tau \in Y$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name$\}$. Since $\mathbb{P}$ is κ-c.c. in V and $\mathbb{P} \subseteq V_\kappa \subseteq Y$, it follows by assertion (3) of Fact 15 that $Y[G]^{<\kappa} \subseteq Y[G]$ in $V[G]$. Assertion (2) of the same fact shows that $Y[G * g]^{<\kappa} \subseteq Y[G * g]$ in $V[G * g]$, since $\mathbb{Q}$ is ${<}\kappa$-distributive in $V[G]$. We have $Y[G * g] \prec N[G * g]$ as usual. It follows that $\mathbb{P}_{\text{tail}}$ is (much more than) ${<}\kappa$-closed in $Y[G * g]$. By the diagonalization criterion for $Y[G*g]$, we may construct in $V[G*g]$ a filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ which is $Y[G * g]$-generic for $\mathbb{P}_{\text{tail}}$. The crucial claim now is that G_{tail} is actually $N[G * g]$-generic. For, if $E \in N[G * g]$ is any dense open set in $\mathbb{P}_{\text{tail}}$, then $E = \dot{E}$ for some $(\mathbb{P}*\dot{\mathbb{Q}})$-name $\dot{E} \in N$. Thus, $\dot{E} = j(h_0)(s_0)$ for some function $h_0 \in M$ and some $s_0 \in S^{<\omega}$. Consider in $N[G * g]$ the subset

$$\bar{E} = \bigcap\{\tau_{G*g} \mid \tau = j(h_0)(s) \text{ for some } s \in S^{<\omega} \text{ and } \tau_{G*g} \text{ is open dense in } \mathbb{P}_{\text{tail}}\}$$

of $\mathbb{P}_{\text{tail}}$. Recall that $S \in N$ and $S^{<\omega}$ has size δ in N. As $\mathbb{P}_{\text{tail}}$ is ${\leq}\delta$-distributive in $N[G * g]$, it follows that $\bar{E}$ is dense in $\mathbb{P}_{\text{tail}}$. Note that $S \in Y$ since it is definable in N from parameters θ and b which are both elements of Y. Moreover, $\bar{E}$ is definable in $N[G * g]$ from parameters $j(h_0)$, the seed set S, the tail forcing $\mathbb{P}_{\text{tail}}$ and the filter $G * g$. As all these parameters are elements of $Y[G * g]$, it follows that $\bar{E} \in Y[G * g]$. Since G_{tail} is a $Y[G * g]$-generic filter for $\mathbb{P} * \mathbb{Q}$, we see that $G_{\text{tail}} \cap \bar{E} \neq \emptyset$. As $\bar{E} \subseteq E$, we established that G_{tail} is indeed $N[G * g]$-generic. If we let $j(G) = G * g * G_{\text{tail}}$, then $G \cong j\text{"}G \subseteq j(G)$. This satisfies the lifting criterion and j hence lifts to $j : M[G] \to N[j(G)]$. This concludes *Step 1.*

As in the proof of the Main Theorem, we will force with the collapsed version of $\mathbb{Q}$ – we called it $\mathbb{Q}_0$ – over $M[G]$. Since g was $X[G]$-generic for $\mathbb{Q}$, Lemma 20 applied and yielded the $M[G]$-generic filter $g_0 \subseteq \mathbb{Q}_0$. Recall that $g_0 = \pi_1\text{"}g$ where π_1 is the Mostowski collapse of $X[G]$ in $V[G]$, defined by $\pi_1(\tau_G) = \pi(\tau)_G$. The crucial application of Lemma 20 in $V[G * g]$ showed that g_0 is $M[G]$-generic for $\mathbb{Q}_0$ and that $A = \dot{A}_{G*g} = \pi(\dot{A})_{G*g_0} = (\dot{A}_0)_{G*g_0}$ is an element of $M[G * g_0]$. As before, $M[G * g_0]$ is a κ-model in $V[G * g]$ and $V[G * g]_\theta \subseteq V_\theta[G * g] \subseteq N[G * g]$. Thus, to finish showing that $r \in D$, it suffices to lift the embedding j once more:

Step 2. *In $V[G * g]$, lift the elementary embedding $j : M[G] \to N[j(G)]$ to $j : M[G * g_0] \to N[j(G) * j(g_0)]$.*

Again, we will verify the lifting criterion. Similar to *Step 1*, we will build a $Y[j(G)]$-generic filter $j(g_0) \subseteq j(\mathbb{Q}_0)$ and then argue that $j(g_0)$ is actually $N[j(G)]$-generic. In order to satisfy the necessary condition $j"g_0 \subseteq j(g_0)$, let us first find a master condition $q \in j(\mathbb{Q}_0)$ below $j"g_0$ so that $q \in Y[j(G)]$.

Observe that $\dot{\mathbb{Q}}_0 \in V_\theta$ and $\pi \upharpoonright \dot{\mathbb{Q}} \in V_\theta$ both have size κ in V_θ (since θ is a limit ordinal). Since $V_\theta \subseteq N$ and π_1 is definable from π and G, it follows that $g_0 = \pi_1"g \in N[G * g]$ has size κ in $N[G * g]$. Moreover, since $\mathbb{Q}_0$ is ${<}\kappa$-distributive in $M[G]$, the same density argument as before shows that g_0 is a ${<}\kappa$-closed subset of $\mathbb{Q}_0$ in $M[G*g_0]$, and also in $V[G*g]$. Absoluteness shows that $N[G * g] \models (g_0$ is a ${<}\kappa$-closed subset of $\mathbb{Q}_0 \wedge |g_0| = \kappa \wedge g_0$ is directed$)$. As before, we can thus find in $N[G * g]$ a *descending* chain $\langle q_\xi : \xi < \kappa\rangle$ of elements of g_0 such that every element of g_0 lies above q_ξ for some $\xi < \kappa$. Again, every element of $j"g_0$ lies therefore above $j(q_\xi)$ for some $\xi < \kappa$ and we aim to find a master condition below all the $j(q_\xi)$'s. Consider hence in $V[G * g]$ the descending chain $\vec{c} = \langle j(q_\xi) : \xi{<}\kappa\rangle$ of elements of $j"g_0 \subseteq j(\mathbb{Q}_0)$. Since we constructed j in such a way that $j \upharpoonright \dot{\mathbb{Q}}_0 \in N$, it follows that $j \upharpoonright \mathbb{Q}_0 \in N[j(G)]$ and therefore that $\vec{c} \in N[j(G)]$. Moreover, $N[j(G)]$ thinks that $j(\mathbb{Q}_0)$ is ${<}j(\kappa)$-closed, and we can hence find the desired master condition $q \in j(\mathbb{Q}_0)$ below all the $j(q_\xi)$'s.

In order to apply the diagonalization criterion for $Y[G*g]$, we need such a master condition q with $q \in Y[G*g]$. But I claim that we may assume without loss of generality that $q \in Y[j(G)]$. For, since $Y[j(G)] \prec N[j(G)]$ and $N[j(G)] \models (\exists q \in j(\mathbb{Q}_0)$ below $j"g_0)$, it suffices to show that the parameters $j(\mathbb{Q}_0)$ and $j"g_0$ are elements of $Y[j(G)]$. Since we put $a = \pi \upharpoonright \dot{\mathbb{Q}}$ and $b = j \upharpoonright \dot{\mathbb{Q}}_0$ into Y when defining Y, it follows that $g_0 = \pi_1"g \in Y[G * g]$ and thus $j"g_0 \in Y[j(G)]$. Also $j(\mathbb{Q}_0) \in Y[j(G)]$ since $j(\dot{\mathbb{Q}}_0) \in Y$. This proves my claim and we may hence fix a master condition $q \in j(\mathbb{Q}_0) \cap Y[G * g]$ below all of $j"g_0$.

Lastly, we want to build a $Y[j(G)]$-generic filter $j(g_0) \subseteq j(\mathbb{Q}_0)$ containing the element q. Observe that in $V[G * g]$, the structure $Y[j(G)]$ has size κ and $j(\mathbb{Q}_0) \in Y[j(G)]$. Moreover, since $G_{\text{tail}} \in V[G * g]$ is $N[G * g]$-generic for $\mathbb{P}_{\text{tail}}$, assertion (1) of Fact 15 shows that $Y[j(G)]^{<\kappa} \subseteq Y[j(G)]$ in $V[G * g]$. Since $Y[j(G)] \prec N[j(G)]$, we see that $j(\mathbb{Q}_0)$ is (much more than) ${<}\kappa$-closed in $Y[j(G)]$. By the diagonalization criterion in $V[G * g]$, we can thus construct a filter $j(g_0) \subseteq j(\mathbb{Q}_0)$ which is $Y[j(G)]$-generic for $j(\mathbb{Q}_0)$ such that $q \in j(g_0)$. Similar to *Step 1*, one verifies that this filter is actually $N[j(G)]$-generic for $j(\mathbb{Q}_0)$. Since $q \in j(g_0)$ we have $j"g_0 \subseteq j(g_0)$. The lifting criterion is satisfied and j lifts to $j : M[G * g_0] \to N[j(G) * j(g_0)]$ in $V[G * g]$ as desired. This concludes *Step 2*. We thus established that D is dense in $\mathbb{P} * \dot{\mathbb{Q}}$ and the proof is complete. □

Since the limit ordinals are unbounded in all the ordinals, it follows that Theorem 42 certainly implies the Main Theorem. We thus found a proof of the Main Theorem that is quite different from the one presented in Section 1.4. The original proof used the fact that strongly unfoldable cardinals have embeddings similar to those of supercompact cardinals, while this second proof uses the embedding characterization that mimics strong cardinals.

Interestingly, the proof of Theorem 42 can be modified in a straightforward manner to also prove the successor case (i.e. the strengthening of Theorem 38 where the GCH assumption is omitted), since we never used θ being a limit ordinal in an essential way: For instance, when we fixed $j : M \to N$ with $V_\theta \subseteq N$ and let $\delta = \beth_\theta^N$, we concluded that $\beth_\theta^N = \beth_\theta$. If θ is a successor ordinal, one can merely infer that $\beth_\theta^N \geq \beth_\theta$. But, in fact we never used the equality between δ and $\beth_\theta$; all we worked with was that $j(f)(\kappa) \geq \delta$ and that V_θ had size δ in N. This is exactly why we insisted that $j(f)(\kappa) \geq \beth_\theta^N$ when defining the Menas property for a θ-strongly unfoldable cardinal κ in Section 1.2. Another use of θ being a limit ordinal occurred when we observed that $a = \pi \upharpoonright \dot{\mathbb{Q}}$ was an element of size κ in V_θ. If θ is a successor ordinal and one uses the usual von Neumann code for ordered pairs, then a need not be an element of V_θ. But, as discussed before, if we use a flat pairing function instead, it follows that $V_\theta \times V_\theta \subseteq V_\theta$ for all infinite ordinals θ and a has thus size κ in V_θ. Lastly, note that the poset $\mathbb{Q}$ must have size at most $\beth_\theta$, in order for it to have an isomorphic copy in $V_{\theta+1}$. With these modifications, the proof of Theorem 42 yields the following:

Theorem 43. *Let κ be $(\theta+1)$-strongly unfoldable for some ordinal $\theta \geq \kappa$. Then after the lottery preparation of κ relative to a function with the Menas property for κ, the $(\theta+1)$-strong unfoldability of κ is indestructible by $<\kappa$-closed, κ-proper forcing of size at most $\beth_\theta$.*

Theorem 43 improves Theorem 38 by freeing us from any GCH assumption for $\beth_\theta$. We thus obtain the following strongest local version of the Main Theorem.

Theorem 44. *Let κ be θ-strongly unfoldable for some ordinal $\theta \geq \kappa$. Then after the lottery preparation of κ relative to a function with the Menas property for κ, the θ-strong unfoldability of κ is indestructible by $<\kappa$-closed, κ-proper forcing of rank less than θ.*

Proof. This is immediate by Theorems 42 and 43. □

Chapter 2

More Indestructibility Identified

In this chapter, I will illustrate how the Main Theorem can be improved to make any given strongly unfoldable cardinal κ indestructible by all $<\kappa$-closed, κ^+-*preserving* forcing. I thereby give an affirmative answer to Question 2 from Chapter 1. The material in this chapter was developed jointly with Joel Hamkins.

The key idea of the proof of the Main Theorem was roughly as follows: Suppose that κ is strongly unfoldable in V and we want to make κ indestructible by some nontrivial forcing $\mathbb{Q}$. Let $G \subseteq \mathbb{Q}$ be a V-generic filter for $\mathbb{Q}$. The natural strategy is to fix a κ-model M containing $\mathbb{Q}$ and try to lift a θ-strong unfoldability embedding $j : M \to N$ to the forcing extension. But, of course, if $\mathbb{Q}$ is too big to fit into a κ-model, then we cannot follow this strategy. The key observation (Lemma 20) was that *if* we succeed in putting $\mathbb{Q}$ into an elementary submodel $X \prec H_\lambda$ whose transitive collapse is a κ-model M such that the filter $G \subseteq \mathbb{Q}$ is *both* X-generic and V-generic, then we can instead force over M with the collapsed version $\pi(\mathbb{Q})$ of $\mathbb{Q}$ using the M-generic filter $\pi"G$.

A density argument in the Main Theorem then showed that *κ-properness* of the poset $\mathbb{Q}$ is a sufficient condition enabling us to restrict ourselves to forcing extensions $V[G]$ where the V-generic filter is also X-generic for a suitable elementary substructure $X \prec H_\lambda$. The goal of this chapter is to show that in fact we can restrict ourselves to this case whenever $\mathbb{Q}$ is a poset that does not collapse the cardinal κ^+. The point is that rather than building the structure $X \prec H_\lambda$ in V, we construct an elementary submodel $\bar{X} \prec H_\lambda[G]$ with $\bar{X}^{<\kappa} \subseteq \bar{X}$ in the forcing extension $V[G]$. Of course, this will only be possible if $\mathbb{Q}$ preserves the equality $\kappa^{<\kappa} = \kappa$. Letting $X = \bar{X} \cap V$, it will then follow that G is X-generic. Using results from [Ham03], we will be able, under certain conditions, to show that X is an element of V. As $\mathbb{Q}$ does not collapse κ^+, we will see that X has size κ in V also. If $\pi : \langle X, \in\rangle \to \langle M, \in\rangle$ is the Mostowski collapse of X, it will follow that M is a κ-model in V. As before, using Lemma 20, this will allow us to force over M with the collapsed version $\pi(\mathbb{Q})$ of $\mathbb{Q}$ using the M-generic filter $\pi"G$.

Suppose that $V \subseteq \bar{V}$ is an extension consisting of two transitive class models of ZFC, such as for instance when $\bar{V}$ is a forcing extension of V. Recall from [Ham03] that for a cardinal δ in $\bar{V}$, the extension $V \subseteq \bar{V}$ satisfies the δ *approximation property* if whenever $A \subseteq V$ is a set in $\bar{V}$ and $A \cap a \in V$ for any $a \in V$ of size less than δ in V, then $A \in V$. Hamkins calls the sets $A \cap a$, where a has size less than δ in V, the δ *approximations* to A over V. The pair $V \subseteq \bar{V}$ satisfies the δ *cover property* if for every set A in $\bar{V}$ with $A \subseteq V$ and $|A|^{\bar{V}} < \delta$, there is a covering set $B \in V$ with $A \subseteq B$ and $|B|^V < \delta$. The next fact from [Ham03] is crucial for the proof of Theorem 46. For convenience, I will provide its proof.

Fact 45 ([Ham03])**.** *Suppose that $V \subseteq \bar{V}$ satisfies the δ approximation and δ cover properties for some cardinal δ in $\bar{V}$. Let λ be an ordinal with cof$(\lambda) \geq \delta$ in $\bar{V}$. If $\bar{X}^{<\delta} \subseteq \bar{X}$ in $\bar{V}$ and $\bar{X} \prec \bar{V}_\lambda$ in the language with a predicate for V, so that $\langle \bar{X}, X, \in\rangle \prec \langle \bar{V}_\lambda, V_\lambda, \in\rangle$, where $X = \bar{X} \cap V$, then $X \in V$.*

Proof. Of course, $X = \bar{X} \cap V_\lambda$ and is thus an element of $\bar{V}$. We will use the δ approximation property to show that $X \in V$. Fix thus any set $a \in V$ which has size less than δ in V. We want to show that $X \cap a$ is an element of V. By the δ cover property, there is a set $B \in V$ of size less than δ in V such that $X \cap a \subseteq B$. I claim that there is such a covering set $B \in V$ with $B \subseteq X$. To see this, note first that we may assume $X \cap a$ has a covering set $B \in V_\lambda$ with $|B|^{V_\lambda} < \delta$, since $X \cap a \subseteq V_\lambda$ and $(\mathrm{cof}(\lambda) \geq \delta)^V$. As $\bar{X}$ is closed under ${<}\delta$-sequences in $\bar{V}$, we have that $X \cap a$ is an element of $\bar{X}$. By elementarity, we hence see that the set $X \cap a$ must have such a covering set B *in* X with $|B|^X < \delta$. But $|B|^X < \delta$ implies that $B \subseteq X$, as X is closed under ${<}\delta$-sequences in V, which verifies the claim. It follows that $X \cap a \subseteq B \subseteq X$ and thus $X \cap a = B \cap a$ is an element of V, as desired. Since all the δ approximations to X over V are in V, we have that $X \in V$. □

When characterizing strong unfoldability, we have been working with transitive models that satisfied ZFC$^-$. But the elementary submodel $X \prec V_\lambda$ as in Fact 45 need not satisfy Replacement, which is the reason why we will continue to build elementary substructures $X \prec H_\lambda$ for regular λ. It is easy to check that the corresponding modification of the statement of Fact 45 also holds if $(V_\lambda)^{\bar{V}}$ is replaced by $(H_\lambda)^{\bar{V}}$ for regular λ.

Theorem 46. *Let κ be a strongly unfoldable cardinal. Then after the lottery preparation of κ relative to a function with the Menas property for κ, the strong unfoldability of κ becomes indestructible by ${<}\kappa$-closed, κ^+-preserving set forcing.*

Proof. We will follow the proof of the Main Theorem in Section 1.4 closely. Let κ be strongly unfoldable and $f \colon \kappa \to \kappa$ be a function with the Menas property for κ. Let $\mathbb{P}$ be the lottery preparation of κ relative to f. We may assume without loss of generality that $f(\omega) = \omega$ and that $p \in \mathbb{P}$ is a condition that opts at stage ω for $\mathrm{Add}(\omega, 1)$, the poset to add a Cohen real. Suppose that $G \subseteq \mathbb{P}$ is a V-generic filter containing p. To show that the strong unfoldability of κ is indestructible by $<\kappa$-closed, κ^+-preserving forcing in $V[G]$, let $\mathbb{Q} \in V[G]$ be any such poset. Suppose that $g \subseteq \mathbb{Q}$ is a $V[G]$-generic filter and $\dot{\mathbb{Q}}$ is a name for $\mathbb{Q}$ which necessarily yields a $<\kappa$-closed, κ^+-preserving poset. As before, we have that κ is inaccessible in $V[G * g]$ and it remains to verify the embedding property of the strongly unfoldable cardinal κ. Fix thus any ordinal $\theta \geq \kappa$ and any subset $A \in V[G * g]$ of κ. Let $\dot{A}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$ -name for A which necessarily yields a subset of κ. We may assume that θ is large enough so that $\dot{\mathbb{Q}}$ and $\dot{A}$ are elements of $V_{\theta+1}$.

In $V[G * g]$, let $\lambda > \beth_{\theta+1}$ be any regular cardinal. In contrast to the proof in Section 1.4 (where we worked in V), we now use the Skolem-Löwenheim method in $V[G * g]$ to build an elementary submodel $\bar{X} \prec H_\lambda[G * g]$ of size κ such that $\{\kappa, \mathbb{P}, f, \dot{\mathbb{Q}}, \dot{A}, G, g\} \subseteq \bar{X}$. Using an extended language with a predicate for V, we may assume that $\langle \bar{X}, X, \in \rangle \prec \langle H_\lambda[G * g], H_\lambda, \in \rangle$ where $X = \bar{X} \cap V$. Since $\kappa^{<\kappa} = \kappa$ holds in $V[G * g]$, we may assume that $\bar{X}^{<\kappa} \subseteq \bar{X}$ in $V[G * g]$. Note that $G * g$ is X-generic for $\mathbb{P} * \dot{\mathbb{Q}}$. One way to see this uses elementarity of $\bar{X}$ in $H_\lambda[G * g]$: If $\mathcal{D}$ is the collection of all dense subsets of $\mathbb{P} * \dot{\mathbb{Q}}$ which exist in V, then $\mathcal{D} \in H_\lambda$ and thus $\mathcal{D} \in \bar{X}$. Since $H_\lambda[G * g]$ sees that $G * g$ meets every dense set $D \in \mathcal{D}$ and $G * g \in \bar{X}$, it follows from elementarity that $(G * g) \cap D \cap X \neq \emptyset$ for every $D \in \mathcal{D} \cap X$, as desired. Moreover, I make the crucial claim that X is in fact an *element* of the ground model V. Since the lottery preparation $\mathbb{P}$ factors below p as $\mathrm{Add}(\omega, 1) * \dot{\mathbb{P}}_{\mathrm{tail}}$ where $\dot{\mathbb{P}}_{\mathrm{tail}}$ is a name for a necessarily ${\leq}\omega$-closed poset and $\dot{\mathbb{Q}}$ is necessarily ${\leq}\omega$-closed, it follows that $\mathbb{P} * \dot{\mathbb{Q}}$ has what Hamkins [Ham03] calls a *closure point* at ω. This implies that $\mathbb{P} * \dot{\mathbb{Q}}$ preserves $\aleph_1$ and that the forcing extension $V \subseteq V[G * g]$ has the ω_1 approximation and ω_1 cover properties (see Lemma 13 in [Ham03]). By Fact 45 and the remark after its proof, we thus see that $X \in V$ as claimed.

We may hence consider *in* V the Mostowski collapse $\pi : \langle X, \in\rangle \to \langle M, \in\rangle$. The isomorphism π fixes every element of V_κ. Moreover, π fixes the poset $\mathbb{P}$, the Menas function f and the cardinal κ. Let $\pi(\dot{\mathbb{Q}}) = \dot{\mathbb{Q}}_0$ and $\pi(\dot{A}) = \dot{A}_0$. Since $\bar{X}$ is closed in $V[G*g]$ under ${<}\kappa$-sequences, it follows that $X^{<\kappa} \subseteq X$ in V and consequently $M^{<\kappa} \subseteq M$ in V. As $\kappa \subseteq M$, we see that M has size κ in $V[G*g]$. Since both $\mathbb{P}$ and $\mathbb{Q}$ preserve κ^+, it follows that M has size κ *in* V also. In view of Lemma 20, we summarize that the poset $\mathbb{P} * \dot{\mathbb{Q}}$, the function f, and the name $\dot{A}$ are elements of the elementary submodel $X \prec H_\lambda$ whose transitive collapse is a κ-model M in V, with the additional property that the filter $G * g$ is both X-generic and V-generic.

Let $G * g_0 = \pi$"$(G * g)$. It follows from Lemma 20 that $G * g_0$ is M-generic for $\mathbb{P} * \dot{\mathbb{Q}}_0$ and that $A = \dot{A}_{G*g} = (\dot{A}_0)_{G*g_0}$. We can hence force with $\mathbb{P} * \mathbb{Q}_0$ over M using the M-generic filter $G * g_0$. As before, we will use a $(\theta + 1)$-strong unfoldability embedding $j \in V$ with domain M and lift it in $V[G * g]$ twice so that the domain of the lifted embedding equals $M[G * g_0]$.

Since κ is $(\theta + 1)$-strongly unfoldable in V, fix by Fact 6, a $(\theta + 1)$-strong unfoldability embedding $j : M \to N$ with $N^{\beth_\theta} \subseteq N$ and $|N| = \beth_{\theta+1}$. As f has the Menas property for κ, we may assume that $j(f)(\kappa) \geq \beth_\theta^N$. The proof splits again into two cases, depending on whether the GCH holds at $\delta = \beth_\theta^N = \beth_\theta$ or not. If $2^\delta = \delta^+$ in V, then we can follow the arguments of *Steps 1* and *2* of the proof of the Main Theorem in Section 1.4 word-by-word and see that j lifts in $V[G * g]$ to a $(\theta + 1)$-strong unfoldability embedding $j : M[G * g_0] \to N[j(G) * j(g_0)]$. Since $\dot{A}$ was a name for any given subset of κ, we showed in this case that κ is $(\theta + 1)$-strongly unfoldable in $V[G * g]$. The case when $2^\delta \neq \delta^+$ is handled the same way as before and establishes that κ is θ-strongly unfoldable in $V[G * g]$. As θ was chosen arbitrarily, we see that κ is strongly unfoldable in $V[G * g]$. This completes the proof. $\square$

Theorem 46 strengthens the Main Theorem from Section 1.4 significantly.

It follows that the existence of a strongly unfoldable cardinal κ indestructible by ${<}\kappa$-closed, κ^+-preserving forcing is equiconsistent over ZFC with the existence of a strongly unfoldable cardinal. As was mentioned in Chapter 1 in the paragraph before Question 1, we could not have hoped to obtain general indestructibility by posets which *collapse* κ^+, while only relying on hypotheses consistent with $V = L$.

One way of seeing this was explained to me by Grigor Sargsyan [Sar]. The argument, which I give in Lemma 47, Theorem 48 and Corollaries 49 and 50 below, shows that if κ is weakly compact and indestructible by for instance the poset $\mathrm{Coll}(\kappa, \kappa^+)$, then Jensen's Square Principle $\square_\kappa$ fails. But the failure of $\square_\kappa$ for a weakly compact cardinal κ implies AD in $L(\mathbb{R})$, which has the strength of infinitely many Woodin cardinals (see [SZ01] and [Woo99]).

For cardinals $\kappa < \lambda$ we denote the canonical poset to collapse λ to an ordinal of size κ using conditions of size less than κ by $\mathrm{Coll}(\kappa, \lambda)$. Moreover, for a given set A of size at least λ, we denote the set of all subsets of A of size less than λ by $P_\lambda(A)$. In the proof of the next lemma, we will use Jech's generalization of closed unbounded and stationary sets to the space $P_\lambda(A)$.

Lemma 47. *Let $\lambda > \omega$ be regular and $\mathbb{P}$ a countably closed poset. If $S \subseteq \lambda$ is stationary such that all elements of S have cofinality ω, then S remains stationary in λ after forcing with $\mathbb{P}$.*

Proof. Fix the cardinal λ, the poset $\mathbb{P}$ and the stationary set S as in the lemma. Note that $\mathbb{P}$ preserves the fact that λ is an ordinal with uncountable cofinality, which allows us to consider closed unbounded sets and stationary subsets of λ after forcing with $\mathbb{P}$. To see that S remains stationary, let thus σ be any $\mathbb{P}$-name which necessarily yields a closed unbounded set in λ. It suffices to show that the set $D = \{p \in \mathbb{P} \mid p \Vdash \text{“}\sigma \text{ meets } S\text{”}\}$ is dense in $\mathbb{P}$. Fix thus any $p \in \mathbb{P}$. Let θ be a sufficiently large cardinal so that the cardinal λ, the poset $\mathbb{P}$ and the name σ are elements of H_θ. Since λ is closed unbounded in $P_\lambda(\lambda)$ and S is stationary in λ, it follows that S is stationary in $P_\lambda(\lambda)$. By a Theorem due to Menas (see for instance Theorem 8.27 in [Jec03]), we see that $\{X \in P_\lambda(H_\theta) \mid X \cap \lambda \in S\}$ is stationary in $P_\lambda(H_\theta)$. We may thus fix an elementary submodel $X \prec H_\theta$ of size less than λ such that $\{\mathbb{P}, p, S, \sigma\} \subseteq X$ and $X \cap \lambda \in S$. Let $\beta = X \cap \lambda$. Since $\beta \in S$ has cofinality ω, it is easy to construct a descending sequence $\langle p_n \mid n < \omega\rangle$ of conditions in X below p and a corresponding increasing sequence of ordinals $\langle \beta_n \mid n < \omega\rangle$ cofinal in β such that each p_n forces that β_n is an element of the club set σ. The countable closure of $\mathbb{P}$ provides a condition $q \in \mathbb{P}$ below all the p_n's. It is clear that q forces that $\sup_n \beta_n$ is an element of σ. As $\sup_n \beta_n = \beta \in S$, we see that q forces that the set S meets the club σ. This shows that $q \in D$ and thus that D is dense in $\mathbb{P}$. This completes the proof. $\square$

Jensen's Square Principle is closely related to the concept of *stationary reflection*. Recall that for an ordinal λ with uncountable cofinality and a stationary subset $S \subseteq \lambda$, the set S *reflects* if for some ordinal $\mu < \lambda$ with $\mathrm{cof}(\mu) > \omega$, the set $S \cap \mu$ is a stationary subset of μ. A classical application of $\square_\kappa$ implies the existence of many non-reflecting stationary subsets of κ^+. More specifically, if $\square_\kappa$ holds for an uncountable regular cardinal κ and S is any stationary subset of κ^+, then it is well known that not every stationary subset of S reflects (see for instance [CFM01]). In contrast, if κ is a weakly compact cardinal which is indestructible as in the theorem below for some $\lambda > \kappa$, then *every* stationary subset S of λ with $\mathrm{cof}(\alpha) = \omega$ for all $\alpha \in S$ does reflect:

Theorem 48. *Suppose that κ is weakly compact and $\lambda > \kappa$ is regular with $\lambda^{<\kappa} = \lambda$. If κ is indestructible by the forcing Coll(κ, λ) and S is a stationary subset of λ such that all elements of S have cofinality ω, then S reflects.*

Proof. Fix the cardinal λ, the weakly compact cardinal $\kappa < \lambda$ and the stationary subset $S \subseteq \lambda$ as in the theorem. Let $\mathbb{P} = \mathrm{Coll}(\kappa, \lambda)$. We will show that S reflects. Fix a regular cardinal θ sufficiently large and build an elementary submodel $X \prec H_\theta$ of size λ such that $\lambda + 1 \subseteq X$ and $S \in X$. As $\lambda^{<\kappa} = \lambda$, we may also assume that $X^{<\kappa} \subseteq X$. Let $\pi : X \to M$ be the

Mostowski collapse of X. As θ is regular, we have that M satisfies ZFC$^-$. The isomorphism π fixes all elements of λ and thus the set S.

Let $G \subseteq \mathbb{P}$ be a V-generic filter. By hypothesis, κ remains weakly compact in $V[G]$. The transitive collapse M has now size κ in $V[G]$. As $\mathbb{P}$ is ${<}\kappa$-distributive, it does not add new ${<}\kappa$-sequences of elements in V, and it follows that $M^{<\kappa} \subseteq M$ in $V[G]$. In summary, we have that M is a κ-model in $V[G]$. Since weak compactness is equivalent to κ-strong unfoldability, we may thus fix in $V[G]$ by Fact 5, a Hauser embedding $j : M \to N$ with critical point κ such that j is an element of N of size κ in N. It follows that $j"\lambda \in N$ has size κ in N. Since λ is regular in M, we see by elementarity that $j"\lambda$ is bounded in $j(\lambda)$. Let $\mu = \sup j"\lambda$. As $\mathbb{P}$ is ${<}\kappa$-distributive, we have that $\text{cof}(\mu) = \kappa$ in $V[G]$.

It suffices to show that $\mu \cap j(S)$ is a stationary subset of μ in $V[G]$. Because, if we succeed in proving this, then N will see that $\mu \cap j(S)$ is stationary, and by elementarity, M will think that $S \subseteq \lambda$ reflects. Since the isomorphism $\pi : X \to M$ fixes both S and λ, it will then follow that X thinks that S reflects. As $X \prec H_\theta$, we will see that X is correct and that the stationary subset $S \subseteq \lambda$ does indeed reflect, as desired.

Fix thus any club $C \in V[G]$ with $C \subseteq \mu$ and argue in $V[G]$. Let D denote the set $\{\alpha < \lambda \mid j(\alpha) \in C\}$. As M is closed under ${\leq}\omega$-sequences, it follows that the set D is unbounded in λ. One sees similarly that every countable increasing sequence of elements in D has its supremum in D also. Still in $V[G]$, it follows that D meets every stationary set $T \subseteq \lambda$ which satisfies $\mathrm{cof}(\alpha) = \omega$ for all $\alpha \in T$. Since S is stationary in λ in $V[G]$ by Lemma 47, we thus see that $S \cap D \neq \emptyset$. It follows that $\mu \cap j(S) \cap C \neq \emptyset$, which proves in $V[G]$ the stationarity of the subset $\mu \cap j(S)$ in μ. As discussed before, this shows in V that the set S reflects, as desired. $\Box$

Note that we did not really use in the above proof that $\mathbb{P}$ was equal to the poset $\mathrm{Coll}(\kappa, \lambda)$. We relied on countable closure of $\mathbb{P}$ in order to apply Lemma 47 and on $<\kappa$-distributivity in order to see that M remained closed under $<\kappa$-sequences in $V[G]$. We thus have the following.

Corollary 49. *Suppose that κ is weakly compact and $\lambda > \kappa$ is regular with $\lambda^{<\kappa} = \lambda$. Suppose that κ is indestructible by some countably closed, $<\kappa$-distributive poset which collapses the cardinal λ to an ordinal of size κ. If S is a stationary subset of λ such that all elements of S have cofinality ω, then S reflects.*

The next corollary shows, as mentioned before, that a weakly compact cardinal κ indestructible by the poset $\mathrm{Coll}(\kappa, \kappa^+)$ implies a failure of $\Box_\kappa$.

Corollary 50. *Suppose that κ is weakly compact and indestructible by some countably closed, ${<}\kappa$-distributive forcing which collapses the cardinal κ^+. Then $\square_\kappa$ fails. Consequently* AD *holds in $L(\mathbb{R})$, which has the strength of infinitely many Woodin cardinals.*

Proof. Fix the weakly compact cardinal κ indestructible as in the corollary. The cardinal κ^+ is regular and satisfies $(\kappa^+)^{<\kappa} = \kappa^+$. Moreover, the set $S = \{\alpha < \kappa^+ \mid \text{cof}(\alpha) = \omega\}$ is a stationary subset of κ^+. By Corollary 49, every stationary subset of S reflects. As discussed in the paragraphs before Lemma 47 and Theorem 48, this implies a failure of $\square_\kappa$ and consequently that AD holds in $L(\mathbb{R})$. □

This concludes the argument that shows that a strongly unfoldable cardinal κ indestructible by some ${<}\kappa$-closed forcing which collapses κ^+ has at least the consistency strength of infinitely many Woodin cardinals. The indestructibility that we obtained in Theorem 46 for a strongly unfoldable cardinal κ is hence in this sense best possible.

In Chapter 1, we proved several theorems and corollaries using the indestructibility we obtained in the Main Theorem. As expected, Theorem 46 allows for analogous improvements in each of these results.

Since strongly unfoldable cardinals are both weakly compact and totally indescribable, we have the following immediate corollary of Theorem 46:

Corollary 51. *If κ is strongly unfoldable, then there is a forcing extension in which the total indescribability of κ and thus the weak compactness of κ is indestructible by $<\kappa$-closed, κ^+-preserving set forcing.*

Moreover, it is straightforward to modify the proof of Theorem 44 to obtain the following local version of the Main Theorem.

Theorem 52. *Let κ be θ-strongly unfoldable for some ordinal $\theta \geq \kappa$. Then after the lottery preparation of κ relative to a function with the Menas property for κ, the θ-strong unfoldability of κ is indestructible by $<\kappa$-closed, κ^+-preserving forcing of rank less than θ.*

It is clear that Theorem 52 applies to the indestructibility results for indescribable cardinals as presented in Section 1.7:

Corollary 53. *Let κ be Π^{m+1}_1 indescribable for some natural number $m \geq 1$. Then, after the lottery preparation of κ relative to a function with the Menas property for κ, the Π^{m+1}_1-indescribability of κ becomes indestructible by $<\kappa$-closed, κ^+-preserving forcing of size at most $\beth_{\kappa+m-1}$.*

Corollary 54. *Let κ be totally indescribable. Then after the lottery preparation of κ relative to a Menas function for κ, the total indescribability of κ*

becomes indestructible by $<\kappa$*-closed,* κ^+*-preserving forcing of size less than* $\beth_{\kappa+\omega}$.

For the results in Section 1.6, it was crucial that a two-step iteration of $<\kappa$-closed, κ-proper posets was itself $<\kappa$-closed and κ-proper (see Lemmas 32 and 33). The corresponding result for posets that preserve κ^+ is trivial. This observation allows us to use the methods of Section 1.6 to obtain the following improvement of Theorem 34.

Theorem 55. *If* V *satisfies* ZFC*, then there is a class forcing extension* $V[G]$ *satisfying* ZFC *such that*

1. *every strongly unfoldable cardinal of* V *remains strongly unfoldable in* $V[G]$*,*

2. *in* $V[G]$*, every strongly unfoldable cardinal* κ *is indestructible by* $<\kappa$*-closed,* κ^+*-preserving set forcing, and*

3. *no new strongly unfoldable cardinals are created.*

Chapter 3

The Forcing Axiom PFA ($\mathfrak{c}$-proper)

We will use the ideas from Chapter 1 to prove the relative consistency of the forcing axiom PFA ($\mathfrak{c}$-proper), a weakening of the usual *Proper Forcing Axiom* PFA. The best upper bound on the consistency strength of PFA known today is the existence of a supercompact cardinal by a classic result due to Baumgartner. In contrast, we will establish the consistency of PFA($\mathfrak{c}$-proper) relative to a significantly reduced large cardinal hypothesis, namely the existence of a strongly unfoldable cardinal. Such a reduction is impossible for the forcing axiom PFA itself, since it is known that PFA implies the existence of an inner model containing a Woodin cardinal, a large cardinal hypothesis contradicting $V = L$. Nevertheless, the forcing axiom PFA ($\mathfrak{c}$-proper) has large cardinal strength also. Recent work of Neeman and Schimmerling [NS] shows that the existence of a Σ^2_1 indescribable cardinal, a weakening of strong unfoldability, is a lower bound for a weakening of PFA ($\mathfrak{c}$-proper). The material in this chapter was developed jointly with Joel Hamkins.

Recall the *Proper Forcing Axiom* PFA, which is the principle asserting that for every proper poset $\mathbb{Q}$ and every collection $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{Q}$, there exists a filter $G \subseteq \mathbb{Q}$ meeting every element of $\mathcal{D}$. In general, if $\mathcal{C}$ is any class of posets, we denote the forcing axiom that is obtained by replacing "proper poset $\mathbb{Q}$" with "proper poset $\mathbb{Q} \in \mathcal{C}$" in the above definition by PFA ($\mathcal{C}$). Similarly, we consider the following weakening of PFA, where $\mathfrak{c}$ denotes the size of the continuum $2^{\aleph_0}$.

Definition 56. The axiom PFA ($\mathfrak{c}$-proper) is the principle asserting that $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ and that for every proper poset $\mathbb{Q}$ which is also $\mathfrak{c}$-proper and every collection $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{Q}$, there exists a filter $G \subseteq \mathbb{Q}$ meeting every element of $\mathcal{D}$.

Since we defined κ-properness only for cardinals satisfying $\kappa^{<\kappa} = \kappa$, we need to include the hypothesis $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ in Definition 56. Otherwise PFA ($\mathfrak{c}$-proper) would hold vacuously whenever $\mathfrak{c}^{<\mathfrak{c}} \neq \mathfrak{c}$, which is not what we want. Every c.c.c. poset is proper and $\mathfrak{c}$-proper if $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$. It is thus clear that PFA ($\mathfrak{c}$-proper) implies Martin's Axiom $MA_{\aleph_1}$ and thus $\mathfrak{c} > \aleph_1$. A theorem of Todorcevic and Veličković [Vel92] shows that PFA ($\mathcal{C}_0$) implies

$2^{\aleph_1} = \aleph_2$, where $\mathcal{C}_0$ is the class of all posets of size at most $\mathfrak{c}$. There, the authors also noted that a weakly compact cardinal provides a lower bound on the consistency strength of PFA ($\mathcal{C}_0$). Since PFA ($\mathfrak{c}$-proper) strengthens PFA ($\mathcal{C}_0$), it follows that the forcing axiom PFA ($\mathfrak{c}$-proper) implies $\mathfrak{c} = \aleph_2$ and that it has at least the consistency strength of a weakly compact cardinal.

This lower bound is improved by recent work of Neeman and Schimmerling. Their result [NS] shows that the existence of a Σ^2_1 indescribable cardinal (see Section 1.7) is equiconsistent with PFA ($\mathcal{C}_1$), where $\mathcal{C}_1$ is the class of all $\mathfrak{c}$-*linked* posets. Since for any cardinal θ, every θ-linked poset is also θ^+-c.c., it follows that PFA ($\mathfrak{c}$-proper) strengthens PFA ($\mathcal{C}_1$). The existence of a Σ^2_1 indescribable cardinal provides hence a lower bound on the consistency strength of PFA ($\mathfrak{c}$-proper).

For an easy upper bound, as PFA ($\mathcal{C}_0$) implies $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$, one can simply use Baumgartner's result to see that PFA ($\mathfrak{c}$-proper) is consistent relative to the existence of a supercompact cardinal. We will improve this upper bound and show in Theorem 57 that a strongly unfoldable cardinal is sufficient to establish the consistency of PFA($\mathfrak{c}$-proper).

In his consistency proof of PFA, Baumgartner used a supercompact cardinal κ in V and followed the key idea of the Laver preparation of using a Laver function l in order to anticipate every possible proper forcing of the forcing extension. We will follow his proof closely, while taking advantage of the similarities between strongly unfoldable and supercompact cardinals, as well as using the method developed in Section 1.4 of lifting unfoldability embeddings through κ-proper forcing. If $G \subseteq \mathbb{Q}$ is a filter which meets every element of a collection $\mathcal{D}$ of dense subsets of $\mathbb{Q}$, then we say that G is a $\mathcal{D}$-*generic* filter on $\mathbb{Q}$.

Theorem 57. *If there exists a strongly unfoldable cardinal in V, then there is a set forcing extension of V in which* PFA ($\mathfrak{c}$-proper) *holds.*

Proof. Let κ be strongly unfoldable and let $f \colon \kappa \to \kappa$ be a function with the Menas property for κ. Let $\mathbb{P} \in V$ be a countable support iteration of length κ which at stage $\gamma < \kappa$, if $\gamma \in \mathrm{dom}(f)$ and $f``\gamma \subseteq \gamma$, forces with the lottery sum of all proper posets in $H_{f(\gamma)^+}$ in $V^{\mathbb{P}_\gamma}$.

As the lottery sum of proper posets is necessarily proper, it follows that $\mathbb{P}$ is a countable support iteration of proper posets, and thus itself proper. The poset $\mathbb{P}$ hence preserves $\aleph_1$. Note that for every uncountable cardinal $\gamma < \kappa$, the countably closed forcing to collapse γ to $\aleph_1$ is proper. A density argument thus shows that the iteration $\mathbb{P}$ collapses every cardinal between $\aleph_1$ and κ. The cardinal κ and all cardinals above κ are preserved by $\mathbb{P}$, since $\mathbb{P}$ is κ-c.c. by a standard Δ-system argument. The cardinal κ becomes thus the second uncountable cardinal after forcing with $\mathbb{P}$.

How big is the size of the continuum $\mathfrak{c}$ after forcing with $\mathbb{P}$? Counting nice names for subsets of $\aleph_1$ shows that $\mathfrak{c} \leq 2^{\aleph_1} = \aleph_2 = \kappa$ in $V^{\mathbb{P}}$. Moreover, since every V-generic filter on $\mathbb{P}$ selects unboundedly often the poset $\mathrm{Add}(\omega, 1)$ to add a Cohen real, we see that there are κ many distinct Cohen reals appearing explicitly in any such filter and consequently that $\mathfrak{c} \geq \kappa$ in $V^{\mathbb{P}}$. In summary, we have $\mathfrak{c} = 2^{\aleph_1} = \aleph_2$ in $V^{\mathbb{P}}$.

It is our goal to verify that PFA ($\mathfrak{c}$-proper) holds after forcing with $\mathbb{P}$. We saw already that $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c} = \kappa$ in $V^{\mathbb{P}}$. Let thus $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name which necessarily yields a proper and κ-proper poset. Since $\mathbb{P}$ is κ-c.c. and of size κ, it follows by Lemma 17 that $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-proper in V. Moreover, let $\dot{\mathcal{D}}$ be a $\mathbb{P}$-name which necessarily yields a collection of $\aleph_1$ many dense subsets of $\dot{\mathbb{Q}}$. Consider the following subset

$$E = \{r \in \mathbb{P} * \dot{\mathbb{Q}} \ : \ r \Vdash \text{“there exists a } \dot{\mathcal{D}}\text{-generic filter on } \dot{\mathbb{Q}}\text{”}\}$$

of $\mathbb{P} * \dot{\mathbb{Q}}$. To fulfill our goal, it suffices to show that E is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. Fix thus any $r' \in \mathbb{P} * \dot{\mathbb{Q}}$ and let $\lambda > \mathrm{trcl}(\{\kappa, 2^{2^{|\dot{\mathbb{Q}}|}}, \dot{\mathcal{D}}\})$ be a sufficiently large regular cardinal to witness the κ-properness of $\mathbb{P} * \dot{\mathbb{Q}}$ as in Definition 13. Let $x \in H_\lambda$ be a corresponding λ-witness for $\mathbb{P} * \dot{\mathbb{Q}}$. Similarly as in Section 1.4, we use the Skolem-Löwenheim method in V to build an elementary submodel

$X \prec H_\lambda$ of size κ with $X^{<\kappa} \subseteq X$ such that $\{\kappa, r', \mathbb{P}, f, \dot{\mathbb{Q}}, \dot{\mathcal{D}}, x\} \subseteq X$. Since λ is sufficiently large, $x \in X$ and $r' \in X$, we can find an $(X, \mathbb{P} * \dot{\mathbb{Q}})$ -generic condition $r \in \mathbb{P} * \dot{\mathbb{Q}}$ below r'. The rest of the proof will show that $r \in E$ and hence that E is dense.

Let $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ be any V-generic filter containing r as an element so that $G \subseteq \mathbb{P}$ is V-generic and $g \subseteq \mathbb{Q} = \dot{\mathbb{Q}}_G$ is $V[G]$-generic. Since $r \in G * g$ is an X-generic condition, it follows that $G \subseteq \mathbb{P}$ is an X-generic filter and $g \subseteq \mathbb{Q}$ is an $X[G]$-generic filter. Let $\mathcal{D} = \dot{\mathcal{D}}_G \in V[G]$ be the collection of $\aleph_1$ many dense subsets of $\mathbb{Q}$ for which we need to find a $\mathcal{D}$-generic filter in $V[G]$. Let $\pi : \langle X, \in \rangle \to \langle M, \in \rangle$ be the Mostowski collapse of X in V. The construction of X shows that M is a κ-model in V.

The isomorphism π fixes the cardinal κ and the poset $\mathbb{P}$. As $G \subseteq \mathbb{P}$ is X-generic for $\mathbb{P}$ and π fixes every element of $\mathbb{P}$, it follows from Lemma 20 that π lifts to $\pi_1 : X[G] \to M[G]$ by $\pi_1(\sigma_G) = \pi(\sigma)_G$, where π_1 is the Mostowski collapse of $X[G]$. The poset $\mathbb{Q}$ and the collection $\mathcal{D}$ of dense subsets of $\mathbb{Q}$ are elements of $X[G]$, since we made sure that $\{\dot{\mathbb{Q}}, \dot{\mathcal{D}}\} \subseteq X$. Let $\mathbb{Q}_0 = \pi_1(\mathbb{Q})$ and $\mathcal{D}_0 = \pi_1(\mathcal{D})$. It follows that $\mathcal{D}_0$ is a collection of $\aleph_1$ many dense subsets of $\mathbb{Q}_0$ in $M[G]$.

As κ is $(\lambda + 1)$-strongly unfoldable in V, fix by Fact 6, a $(\lambda + 1)$-strong unfoldability embedding $j : M \to N$ with $N^\lambda \subseteq N$ and $V_{\lambda+1} \subseteq N$. The

Menas property of the function f allows us to assume that $j(f)(\kappa) \geq \lambda$. As in Baumgartner's proof of the consistency of PFA, we will lift the embedding j in one step to $j : M[G] \to N[j(G)]$, while making sure that $N[j(G)]$ has a filter $h \subseteq j(\mathbb{Q}_0)$ meeting every element of $j(\mathcal{D}_0)$. Elementarity of the lift of j will then provide in $M[G]$ a $\mathcal{D}_0$-generic filter on $\mathbb{Q}_0$. The existence of the desired $\mathcal{D}$-generic filter on $\mathbb{Q}$ in $V[G]$ will then follow as the elementary submodel $X[G] \prec H_\lambda[G]$ is isomorphic to $M[G]$ via π_1.

Since V and M agree in their definition of $\mathbb{P}$, we see that $\mathbb{P} \in M$ and we can force with the V-generic filter $G \subseteq \mathbb{P}$ over M. The forcing $j(\mathbb{P})$ is defined from $j(f)$ in N in the same way that $\mathbb{P}$ was defined from f in M. As $N[G]^\lambda \subseteq N[G]$ in $V[G]$ and λ was chosen sufficiently large, it follows that $\mathbb{Q} \in N[G]$ is proper in $N[G]$. As $j(f)(\kappa) \geq \lambda$, we may thus opt for $\mathbb{Q}$ in the stage κ lottery and view $j(\mathbb{P})$ as $\mathbb{P} * \mathbb{Q} * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the forcing from stage $\kappa + 1$ up to $j(\kappa)$. Let $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ be $V[G * g]$-generic for this forcing, so that $G * g * G_{\text{tail}}$ is N-generic for $j(\mathbb{P})$. Since $\mathbb{P}$ uses countable support, which is bounded below κ, we see that $j"G \subseteq G * g * G_{\text{tail}}$, and we may thus lift the embedding to $j : M[G] \to N[j(G)]$ in $V[G * g * G_{\text{tail}}]$, where $j(G) = G * g * G_{\text{tail}}$.

I now claim that $N[j(G)]$ contains a $j(\mathcal{D}_0)$-generic filter on $j(\mathbb{Q}_0)$. We saw that g is $X[G]$-generic for $\mathbb{Q}$. Lemma 20 shows that $g_0 = \pi_1"g$ is hence

$M[G]$-generic for $\mathbb{Q}_0$. In particular, g_0 meets every element of the collection $\mathcal{D}_0$ of dense subsets of $\mathbb{Q}_0$. It follows that $j"g_0$ meets every element of $j"\mathcal{D}_0$. Since $\mathcal{D}$ has size $\aleph_1$ in $H_\lambda[G]$ and $\aleph_1^{V[G]} = \aleph_1^V$ is less than the critical point κ of j, it follows by elementarity that $j(\mathcal{D}_0) = j"\mathcal{D}_0$. The filter $h \subseteq j(\mathbb{Q})$ generated by $j"g_0$ is therefore $j(\mathcal{D}_0)$-generic! To verify the claim, it thus suffices to show that $j"g_0$ is an element of $N[j(G)]$. Since $X \in V$ is a subset of $H_\lambda \subseteq V_\lambda$, it follows that $X \in V_{\lambda+1} \subseteq N$. As N correctly computes the Mostowski collapse π of X, we see that the lift π_1 is an element of $N[G * g]$ and hence that $g_0 \in N[G * g]$. Since $N^\kappa \subseteq N$ in V, we have $j \restriction M \in N$ and consequently $j \restriction M[G] \in N[j(G)]$. It follows that $j"g_0 \in N[j(G)]$, which verifies the claim.

As outlined above, elementarity of j shows that $M[G]$ has a $\mathcal{D}_0$-generic filter on $\mathbb{Q}_0$. Since $X[G]$ is isomorphic to $M[G]$ via π_1, we have that $X[G]$ thinks that there is a $\mathcal{D}$-generic filter on $\mathbb{Q}$. The elementary submodel $X[G]$ is correct as $X[G] \prec H_\lambda[G]$, and the existence of the desired $\mathcal{D}$-generic filter on $\mathbb{Q}$ in $V[G]$ follows. We thus verified that $r \in E$. Since $r \leq r'$ and r' was chosen arbitrarily in $\mathbb{P} * \dot{\mathbb{Q}}$, we established that E is dense in $\mathbb{P} * \dot{\mathbb{Q}}$ and therefore that PFA ($\mathfrak{c}$-proper) holds in $V[G]$. This completes the proof. $\square$

The proof of Theorem 57 uses the method developed in Chapter 1 of lifting θ-strong unfoldability embeddings through κ-proper forcing. Motivated by the improvements of Chapter 2, where we lifted through forcing with κ^+-preserving posets, it is natural consider the following strengthening of PFA ($\mathfrak{c}$-proper):

Definition 58. The axiom PFA($\mathfrak{c}^+$-preserving) is the principle asserting that for every proper poset $\mathbb{Q}$ which preserves the cardinal $\mathfrak{c}^+$ and every collection $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{Q}$, there exists a filter $G \subseteq \mathbb{Q}$ meeting every element of $\mathcal{D}$.

In Definition 58 there is a subtle point regarding the phrase "preserves the cardinal $\mathfrak{c}^+$". All we require is that after forcing with $\mathbb{Q}$ over a ground model V, the cardinal $(\mathfrak{c}^+)^V$ remains a *cardinal.* Note that we are here not concerned what the actual size of the continuum is *after* forcing with $\mathbb{Q}$, i.e. it may possibly be bigger than $\mathfrak{c}^V$, less than $\mathfrak{c}^V$, or equal to $\mathfrak{c}^V$.

Again, the result of Todorcevic and Veličković [Vel92] on the size of the continuum shows that PFA ($\mathfrak{c}^+$-preserving) implies $\mathfrak{c} = 2^{\aleph_1} = \aleph_2$ and hence $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$. It follows that PFA($\mathfrak{c}^+$-preserving) strengthens PFA($\mathfrak{c}$-proper). The existence of a Σ^2_1 indescribable cardinal provides thus again a lower bound, while the existence of a supercompact cardinal provides an upper bound on the consistency strength of PFA ($\mathfrak{c}^+$-preserving). Even though one might expect that the methods from Chapter 2 would provide a quick answer, the following question remains open:

Question 59. *Given a strongly unfoldable cardinal κ, can we find a model of set theory satisfying the forcing axiom* PFA ($\mathfrak{c}^+$-preserving)*?*

Various problems occur when one tries to combine the new idea of Chapter 2 with the proof of Theorem 57 in order to establish the relative consistency of PFA ($\mathfrak{c}^+$-proper). The key idea in Theorem 46 was to use the Skolem-Löwenheim method in the forcing extension $V[G*g]$ to build an elementary submodel $\bar{X} \prec H_\lambda[G*g]$ with $\bar{X}^{<\kappa} \subseteq \bar{X}$ in the forcing extension $V[G*g]$, rather than in the ground model V. Using the ω_1 cover property of the forcing extension $V \subseteq V[G*g]$ and Fact 45, we then verified the crucial claim that $\bar{X} \cap V$ is an *element* of V.

A serious problem in any attempt to combine the method of Chapter 2 with the PFA argument, however, is the fact that the extension $V \subseteq V[G*g]$ that we used to prove the relative consistency of PFA ($\mathfrak{c}$-proper) in Theorem 57 need not satisfy the ω_1 approximation property. For instance, if the generic $G \subseteq \mathbb{P}$ for the countable support iteration $\mathbb{P}$ of proper posets opts at the first stage of nontrivial forcing for the countably closed poset to add a Cohen subset to ω_1, then $V[G]$ contains that new subset $A \subseteq \omega_1$, yet all the

ω_1 approximations of A are elements of V, which violates the ω_1 approximation property for $V \subseteq V[G * g]$. We thus cannot apply Fact 45 to see that $\bar{X} \cap V$ is an element of V. A solution to this problem might be to work over the ground model L, and use Gödel's Condensation Lemma to show that the transitive collapse of $\bar{X} \cap L$ is an element of L.

A related problem concerns the possible lack of closure of $\bar{X}$ under ${<}\kappa$-sequences in $V[G * g]$. It was essential in Theorem 57 that the transitive collapse M of the elementary submodel X was closed under ${<}\kappa$-sequences in V, as it allowed us to find an elementary embedding $j : M \to N$ where N was highly closed which consequently showed that $N[j(G)]$ had a $j(\mathcal{D}_0)$-generic filter for $\mathbb{Q}_0$ (see Fact 6 and its proof in [DH06]). As we built X in V and κ was inaccessible in V, it was easy to insist in the proof of Theorem 57 that $X^{<\kappa} \subseteq X$ and consequently that $M^{<\kappa} \subseteq M$ in V. In Theorem 46 we were lucky for a different reason. When we built the submodel $\bar{X}$ in $V[G * g]$, we knew that $\mathbb{Q}$ was not only κ^+-preserving, but also ${<}\kappa$-closed. It followed that both the lottery preparation $\mathbb{P}$ and the poset $\mathbb{Q}$ preserved the inaccessibility of κ, which showed that $\kappa^{<\kappa} = \kappa$ in $V[G * g]$. We were therefore able to build $\bar{X} \in V[G * g]$ with $\bar{X}^{<\kappa} \subseteq \bar{X}$ in $V[G * g]$ and saw that $\bar{X} \cap V$ inherited the ${<}\kappa$-closure in V from $\bar{X}$.

The situation is different if we try to combine the ideas of Theorems 46 and 57 in order to prove the consistency of PFA ($\mathfrak{c}^+$-preserving) relative to a strongly unfoldable cardinal. As there are many proper κ^+-preserving posets $\mathbb{Q}$ that do not preserve the equality $\kappa^{<\kappa} = \kappa$ (consider for instance the c.c.c. forcing $\mathrm{Add}(\omega, \kappa^+)$ to add κ^+ many Cohen reals), we cannot hope to build $\bar{X} \in V[G * g]$ with $\bar{X}^{<\kappa} \subseteq \bar{X}$ in $V[G * g]$. But can we maybe construct $\bar{X} \prec H_\lambda[G * g]$ in such a way that the Mostowski collapse M of $X = \bar{X} \cap V$ is $<\kappa$-closed in V? If we were able to build $\bar{X}$ in such a way that $X \in V$, then this seems possible. But, as discussed before, we do not know how to see that $X \in V$. Moreover, the idea of working over the ground model L and using Condensation to see that the transitive collapse M of X is an element of L has the disadvantage that the isomorphism between X and M need not exist in L. This seems to pose a serious problem when trying to make M closed under $<\kappa$-sequences in L.

I plan to continue the investigation of these issues and would of course be happy if Question 59 had an affirmative answer. At the same time, I am also interested in improvements on the lower bound for the consistency strength of PFA ($\mathfrak{c}$-proper) and PFA ($\mathfrak{c}^+$-preserving), with the ideal szenario being that the latter axiom turns out to be equiconsistent with the existence of a strongly unfoldable cardinal.

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