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SOME GENERALIZATIONS OF ONE-RELATOR GROUPS

by

KATALIN A. BENCSATH

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Chairman of Examining Committee
Prof. Gilbert Baumslag

12/16/82
date

Executive Officer
Prof. Alex Heller

L. Auslander
Prof. Louis Auslander

Prof. Alex Heller

Supervisory Committee

The City University of New York

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Abstract

SOME GENERALIZATIONS OF ONE-RELATOR GROUPS

by

Katalin A. Bencsath

Adviser: Professor Gilbert Baumslag

In this work, a parallel is drawn between certain one-relator quotients of free products arising as homomorphic images of surface groups and the surface groups themselves. After extending G. Baumslag's theorem that $\mathfrak{X}$-groups are free-by-cyclic to further classes of one-relator groups, the Freiheitssatz, residual finiteness and having locally free derived group are established for the mentioned quotients. A constructive proof is given for the solution of the word-problem in these groups.

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INTRODUCTION.

The motivation for the present investigation that turned out to be more special than one would wish for, came from the desire to at least begin an attempt to carry over some of the one-relator theory to groups with several defining relations.

The theory of one-relator groups relies greatly on the presence of many free subgroups in them as conjectured by M. Dehn and proved by W. Magnus in the Freiheitssatz [18]. The cyclically pinched one-relator groups, forming an important subclass of one-relator groups that contains the fundamental groups of 2-dimensional orientable compact manifolds, are residually finite. Lead by this, we are concerned here with manifestations of freeness, such as being residually finite; having free subgroups; obeying an analogue of the Freiheitssatz.

In [11], G. Baumslag considers $\mathcal{X}$-groups, a class of one-relator groups containing the surface groups, and proves that they all are free-by-cyclic as subgroups of cyclic extensions of finitely generated free groups. The extensions of this result, described in Section 1 of Chapter II., were obtained by employing his methods to further classes of one-relator groups containing the surface groups.

In the rest of Chapter II., motivated by E. Rapaport's work [26] on certain one-relator quotients of surface groups, we investigate a class of one-relator quotients of free products of free abelian groups, establish for them a sharper version of the analogue of the Freiheitssatz given by Lyndon [16] and Pride [2] for certain free products, solve the word problem constructively and find that their derived groups are locally free. This seems to indicate that the quotients of surface groups obtained by introducing commuting relations between certain generators inherit many of the properties of surface groups.

Chapter I. Preliminaries.

Section 0: Notational conventions.

Let G be any group, g and h elements of G. Unless expressly otherwise indicated, $g^{-1}h^{-1}gh$, the commutator of g and h is abbreviated by $[g,h]$.

If X and Y are sets, $Y\subset X$ means that Y is a (not necessarily proper) subset of X. If X is a subset of G,

gp(X) stands for the intersection of all the subgroups of G containing X. It is termed the subgroup of G generated by X;

$gp_G(X)$ stands for the intersection of all the normal subgroups of G containing X. It is termed the normal subgroup of G generated by X (as a normal subgroup) and is also referred to as the normal closure of X in G.

The symbols $H\leq G$, $H\trianglelefteq G$ mean that H is a subgroup or a normal subgroup, respectively, of G.

If $H\leq G$, $K\leq G$ then $[H,K]=gp([h,k]\,|\,h\in H, k\in K)$. If m is a positive integer, $\gamma_m G=[\gamma_{m-1}G,G]$, by agreement $\gamma_1 G=G$; $G^m=gp(g^m\,|\,g\in G)$.

If p is a fixed prime, m,n positive integers, G any group then G(m,n) is used to denote the subgroup $\gamma_m G G^{p^n}$.

If θ is a homomorphism from G into some group H then the image of $g \in G$ in H is denoted by $g\theta$.

Aut(G) stands for the group of automorphisms of G.

If $H \leq G$ then the subset $\mathcal{S}$ of G is a *right-transversal* for H in G if $1 \in \mathcal{S}$ and $\mathcal{S}$ contains exactly one element from each right-coset Hg $(g \in G)$, called the coset representative of $x \in Hg$ and denoted by $\overline{x}$ (when H is clear from the context).

If X is any set, *Wd(X)* is the set of all finite strings formed from the elements of X and their formal inverses. If Y is any set, $x \mapsto y$ $(x \in X)$ a fixed mapping, $w(\underset{\sim}{x}) \in Wd(X)$ then $w(\underset{\sim}{y})$ is the element of $Wd(Y)$ obtained by replacing every X-symbol of $w(\underset{\sim}{x})$ by its image in Y. If $x \in X$, $w \in Wd(X)$ then $\exp(x,w)$ is the sum of the exponents of all occurrences of x in w (interpreting x to have exponent 1).

Let P be any property of groups preserved under isomorphisms (e.g. freeness, finiteness etc.). The group G is said to be locally P if every finitely generated subgroup of G has P; G is residually P if for every non-trivial element $g \in G$, the image of g is non-trivial in some homomorphic image of G with property P.

The remainder of this chapter rather informally reviews some of the definitions and most important theorems used in the sequel.

Section 1: Free groups.

Let F be a group, X a set, $\eta: X \rightarrowtail F$ an injection:

Definition:

F is a free group with free basis X or shortly F is free on X if

i) $F=gp(X\eta)$

ii) for every group H and set mapping $\sigma: X \longrightarrow H$ there exists a (unique) homomorphism $\theta: F \longrightarrow H$ such that $(x\eta)\theta = x\sigma$ i.e. the diagram

$$\begin{tikzcd}
X \arrow[rr, "\eta"] \arrow[dr, "\sigma"'] & & F \arrow[dl, "\theta"] \\
& H &
\end{tikzcd}$$

is commutative.

It is customary to identify X and $X\eta$ and consider X as a subset of F. The cardinality of the basis X is an invariant of F called the free rank of F. For any set X, there exists a free group with basis X, uniquely determined up to isomorphism. By convention, the free group on the empty set is the trivial group.

Let $k \geq 1$ be an integer, X a generating set of the group F, i.e. $F=gp(X)$. A *product* of the form $x(i_1)^{\nu(1)} x(i_2)^{\nu(2)} \ldots x(i_k)^{\nu(k)}$ with $x(i_j) \in X$, $\nu(j) \in \mathbb{Z}$ $(j=1,\ldots,k)$ is (*freely*) *reduced* if $\nu(j) \neq 0$ $(j=1,\ldots,k)$ and $x(i_{j+1}) \neq x(i_j)$ $(j=1,\ldots,k-1)$.

Let $f \in F$ be given by the product

$$x(i_1)^{\mu(1)}x(i_2)^{\mu(2)}\dots x(i_m)^{\mu(m)}.$$

If we drop pairs of adjacent elements that are inverses to each other and join together the adjacent occurrences of the same element into a single power then repeat these steps until neither can be performed any more, we arrive at the empty product or else a reduced X-product. This product will still be equal to f so if the procedure ended up with the empty product then f=1.

A reduced X-product is a <u>positive word</u> if every exponent in it is positive. By agreement, 1 is considered a positive word.

A characterization of great importance is given for free groups by

<u>Theorem 1</u> (Normal Form Theorem for Free Groups):
Suppose $X \subset F$, $F=gp(X)$. Then F is free on X if and only if every reduced X-product is different from 1 in F. Equivalently, X is a free basis for the free group F if and only if every non-trivial element f is uniquely represented by a reduced X-product called the <u>reduced</u> or <u>free group normal form</u> of f (with respect to X).

If $f=x(i_1)^{\nu(1)}x(i_2)^{\nu(2)}\dots x(i_k)^{\nu(k)}$ in reduced form then <u>the length of f</u> (with respect to X) is $\ell(f)=|\nu(1)|+|\nu(2)|+\dots+|\nu(k)|$. By definition, $\ell(1)=0$.

The product $x(i_1)^{\mu(1)}x(i_2)^{\mu(2)}\dots x(i_r)^{\mu(r)}$ is an *initial segment* of the product $x(i_1)^{\nu(1)}x(i_2)^{\nu(2)}\dots x(i_k)^{\nu(k)}$ if $r \leq k$, $\mu(1)=\nu(1)$, $\mu(2)=\nu(2)$, ... , $|\mu(r)| \leq |\nu(r)|$ and $\mu(r)\cdot\nu(r) \geq 0$.

If F is free on X, $H \leq F$, then there exists a right Schreier transversal for H in F, i.e. a right transversal $\mathcal{S}$ for H in F that is closed under taking initial segments: for each $1 \neq s \in \mathcal{S}$, every initial segment of (the reduced form of) s also belongs to $\mathcal{S}$.

Subgroups of free groups are again free, in fact the Subgroup Theorem for Free Groups is:

Theorem 2 (Nielsen-Schreier):

If F is free on X, $H \leq F$ and $\mathcal{S}$ is a right Schreier-transversal of H in F then H is also a free group for which the set

$$Y = \{sx\,\overline{sx}^{-1} \mid s \in \mathcal{S},\ x \in X,\ sx\overline{sx}^{-1} \neq 1\}$$

is a free basis.

Section 2: Free products.

Definition.

Suppose that $\{G_i \mid i \in I\}$ and G are disjoint groups and $\eta_i : G_i \rightarrowtail G$ $(i \in I)$ are monomorphisms with the property that $G_i\eta_i \cap G_j\eta_j = \{1\}$ whenever $i \neq j$, $i,j \in I$. G is said to be the free product of the $\{G_i \mid i \in I\}$ - in notation

$$G = \prod_{i \in I}^{*} G_i \quad - \quad \text{if}$$

i) $G = gp(\bigcup_{i \in I} G_i)$,

ii) G has the universal mapping property:

For every group H and collection of homomorphisms $\theta_i : G_i \longrightarrow H$ $(i \in I)$ there exists a homomorphism $\theta : G \longrightarrow H$ such that $(G_i\eta_i)\theta = G_i\theta_i$ $(i \in I)$.

It is customary to identify the subgroups $G_i\eta_i$ of G with G_i and thus consider G_i as subgroups of G.

Note that for any collection $\{G_i \mid i \in I\}$ of disjoint groups $\prod_{i \in I}^{*} G_i$ exists and is uniquely determined up to isomorphism. The G_i's are termed the free (product) factors of G.

Let $k \geq 1$ be an integer, $\{G_i \mid i \in I\}$ subgroups of G with the property $G_i \cap G_j = \{1\}$ $(i \neq j)$. A product $g(i_1)g(i_2)\cdots g(i_k)$ is reduced in the free product sense if $g(i_j) \neq 1$ $(j=1,\ldots,k)$ and $i_{j+1} \neq i_j$ $(j=1,\ldots,k-1)$. For clarity, we refer to such a product as <u>*reduced product</u>.

Suppose that $g \in G$ is given by the product $g(i_1)g(i_2)\dots g(i_m)$. If we replace adjacent elements that come from the same subgroup by the element of that subgroup that is their product and drop all the identity elements occurring then repeat these steps until neither can be performed any more, we arrive at the empty product or else a *reduced product. This product, of course, is equal to g so if the procedure ended with the empty product then $g=1$.

Free Products are also characterized in terms of "normal forms":

Theorem 3 (Normal Form Theorem for Free Products): Suppose $\{G_i \mid i \in I\}$ are subgroups of G such that $G_i \cap G_j = \{1\}$ for $i \neq j$ $(i, j \in I)$ and $G = gp(\bigcup_{i \in I} G_i)$. Then $G = \prod^{*}_{i \in I} G_i$ if and only if every *reduced product is different from 1 in G. Equivalently, $G = \prod^{*}_{i \in I} G_i$ if and only if every non-trivial element g is uniquely represented by a *reduced product called the *free-product normal form* or **reduced form* of g.

free-product

If $g = g(i_1)g(i_2)\dots g(i_k)$ in *reduced form then $l^{*}(g)$ the (*free-product*) *length* of g is k and $g(i_j)$ $(1 \leq j \leq k)$ are the *syllables of g*. The *reduced form above is cyclically reduced if either $k=1$ or else $i_1 \neq i_k$.

A group G is termed *freely decomposable* if it is the free product of some of its subgroups, none of which are trivial.

A free group F of rank at least 2 is freely decomposable:

if F is free on $X=\{x(i)\mid i\in I\}$ with $|I|\geq 2$ and $\langle x(i)\rangle$ denotes the free group on $\{x(i)\}$ (infinite cyclic group generated by $x(i)$) then $F=\prod^{*}_{i\in I}\langle x(i)\rangle$.

Also, taking the free groups F_j on $\{x\mid x\in X_j\}$ $(j=1,\ldots,m)$ where $X_j\neq\emptyset$, $\bigcup_{j=1}^{m}X_j=X$ and the X_j's are pairwise disjoint, we have

$$F=\prod_{j=1}^{m}{}^{*}\;F_j\;.$$

More generally, we have

<u>Theorem 4</u> (Grushko-Neumann):

If F is a free group, $\theta:F\twoheadrightarrow\prod^{*}_{i\in I}G_i$ an epimorphism, then there exist $F_i\leq F$ $(i\in I)$ so that $F=\prod^{*}_{i\in I}F_i$ and $G_i=F_i\theta$ $(i\in I)$ (i.e. free decompositions of epimorph images of free groups can be "pulled back").

The subgroup theorem for free groups can be obtained also from

<u>Theorem 5</u> (Kurosh Subgroup Theorem):

Let $G=\prod^{*}_{i\in I}G_i$ and suppose that $H\leq G$. Then H itself is a free product:

$H=F*(\prod^{*}H_j)$ where F is a free group and each H_j is the intersection of H with a conjugate of some factor G_i.

When G is the free product of two of its subgroups G_1 and G_2, we also write $G=G_1*G_2$.

Section 3: Generalized free products.

Definition:

Let G, G_1, G_2 be groups, $U \leq G_1$, $V \leq G_2$, $\phi: U \rightarrowtail\!\!\twoheadrightarrow V$ an isomorphism, $\eta_i: G_i \rightarrowtail G$ $(i=1,2)$ monomorphisms with the property

$$Q = G_1\eta_1 \cap G_2\eta_2 = U\eta_1 = V\eta_2.$$

G is the free product of G_1 and G_2 amalgamating U and V under ϕ if

i) $G = gp(G_1\eta_1 \cup G_2\eta_2)$,

ii) for every group H and pair of homomorphisms $\theta_1: G_1 \longrightarrow H$, $\theta_2: G_2 \longrightarrow H$ with the property $U\theta_1 = V\theta_2$, there exists a (unique) homomorphism $\theta: G \longrightarrow H$ such that $(G_i\eta_i)\theta = G_i\theta_i$ $(i=1,2)$.

It is customary to identify the subgroup $G_i\eta_i$ of G with G_i and consider G_i $(i=1,2)$ as subgroups of this group G denoted by $G = \{G_1 * G_2; U\phi = V\}$, or $G = \{G_1 * G_2; Q\}$.

Note that given G_1, G_2, U, V and ϕ the free product with amalgamation exists and $G = \{G_1 * G_2; U\phi = V\}$ is unique up to isomorphism. The kernel of the "obvious" homomorphism $\theta: G_1 * G_2 \longrightarrow \{G_1 * G_2; U\phi = V\}$ induced by the identity maps on G_1 and G_2 is the normal subgroup in $G_1 * G_2$ generated by the set $\{uv^{-1} \mid u \in U, v \in V, u\phi = v\}$. Let $k \geq 1$ be an integer, G_1, G_2 subgroups of G with the property $G_1 \cap G_2 = Q$. A *reduced product $g(i_1)g(i_2)\ldots g(i_k)$ is <u>strictly alternating</u> if $g(i_j) \notin Q$ $(i_j \in \{1,2\}, j=1,\ldots,k)$.

Suppose now that in both G_1 and G_2, right transversals for Q have been chosen.

A <u>normal form</u> is a product of a single factor q ($q \in Q$) or of the form $t(1)t(2)\ldots t(k)q$ ($q \in Q, k \geq 1$) where $t(1)t(2)\ldots t(k)$ is a strictly alternating product of coset-representatives.

Let $g \in G$ be given by the *reduced product $q=p(1)p(2)\ldots p(m)$. Since the representative of Q in both G_1 and G_2 is 1, no confusion arises from simply denoting the representative of any $p \in G_1 \cup G_2$ by $\overline{p}$. Then we have $p(j) = \overline{p(j)}q(j)$ ($q(j) \in Q$) uniquely. If we replace each of the syllables $p(j)$ of g by $\overline{p(j)}q(j)$ then replace every $q(j)p(j+1)q(j+1)$ subproduct by the element equal to it in the factor containing $p(j+1)$ and *reduce the result then repeat these steps until neither of them can be performed any more, we arrive at a normal form or else an empty product. In the latter case, $g=1$ holds. The significance of this procedure is indicated in

<u>Theorem 6</u> (Normal Form Theorem for Generalized Free Products):

Suppose Q, G_1, G_2 are subgroups of G such that $G_1 \cap G_2 = Q$, $G = gp(G_1 \cup G_2)$. $G = \{G_1 * G_2; Q\}$ if and only if every strictly alternating product is different from 1 in G. Equivalently, $G = \{G_1 * G_2; Q\}$ if and only if every non-trivial element g is uniquely represented by a normal form once right-transversals for Q in both G_1 and G_2 have been chosen.

If $g=t(1)t(2)\ldots t(k)q$ in normal form then k is its <u>representative length</u> $rl(g)$. For $q \in Q$, $rl(q)=0$. If $g \in G-Q$ then the number of syllables of any strictly alternating product representing g equals $rl(g)$.

Section 4: Presentation of groups.

Let G be a group, X a basis of the free group F, $\rho: F \longrightarrow G$ a homomorphism, $R \subset F$ such that $\ker \rho = gp_F(R)$. If $gp(X\rho)=G$, $\langle X;R\rangle$ is termed a presentation for G (under the presentation map ρ).

If $\theta: G \longrightarrow G^1$ is an isomorphism then $\rho^1=\rho\theta$ is a homomorphism $F \longrightarrow G^1$ with $gp(X\rho^1)=G^1$ and $\ker \rho^1=gp_F(R)$. Thus $\langle X;R\rangle$ is also a presentation for G^1 (under the presentation-map ρ^1). With abuse of notation we write $G=\langle X;R\rangle$.

If F is free on X, $G=F/N$ where $N=gp_F(R)$ for some $R \subset F$ and $\nu: F \twoheadrightarrow G$ is the natural homomorphism then $G=gp(X\nu)$, $gp_F(R)=\ker \nu$ so $\langle X;R\rangle$ is a presentation for G (under ν). Therefore every group has a presentation and every pair of sets (X,R) with $R \subset Wd(X)$ can be viewed as a presentation $\langle X;R\rangle$ for (the groups in the isomorphism class of) the quotient of the free group F on X by the normal closure N of (the image of) R. Most of the time, the presentation map ρ is suppressed.

If $G=\langle X;R\rangle$, X is termed a set of generators for G and R a set of (defining) relators for G. Since $r\rho=1$ $(r \in R)$ in G, the alternative notation $G=\langle X;\{r=1 \mid r \in R\}\rangle$ is used as well and $r=1$ is termed a relation in G; relations of the type $rs^{-1}=1$ are also written as $r=s$ $(r,s \in R)$. Elements $w \in gp_F(R)-R$ are called consequences of R.

A presentation $\langle X;R\rangle$ is finite if both X and R are finite sets, finitely generated if X is finite, finitely related if R is finite.

A group G is <u>finitely presented</u> if some presentation for G is finite; G is <u>finitely generated</u> or <u>finitely related</u> if some presentation for G has the respective property.

If $\theta: G \longrightarrow T$ is a homomorphism, $\ker \theta = gp_G(S)$ and $G=\langle X;R\rangle$ then $T=\langle X ; R \cup S\rangle$. The very useful converse is

<u>Theorem 7</u> (von Dyck):

Suppose $G=\langle X;R\rangle$, $T=\langle \bar{X};\bar{R}\rangle$ where X and $\bar{X}$ are equally indexed sets and $w(\underset{\sim}{\bar{x}}) \in \bar{R}$ whenever $w(\underset{\sim}{x}) \in R$. Then $x \longmapsto \bar{x}$ $(x \in X)$ is a homomorphism of G onto T.

This theorem and the next one are important tools in the process of gathering information about groups given by presentations.

<u>Theorem 8</u> (Tietze):

Suppose that the presentation $\langle X;R\rangle$ turns into the presentation $\langle \bar{X};\bar{R}\rangle$ by a finite number of steps T1--T4 (called Tietze transformations) listed below. Then the groups $G=\langle X;R\rangle$ and $\bar{G}=\langle \bar{X};\bar{R}\rangle$ are isomorphic. Furthermore, two finite presentations $\langle X;R\rangle$ and $\langle \bar{X};\bar{R}\rangle$ define isomorphic groups if and only if a finite number of Tietze transformations lead from one presentation to the other.

The steps mentioned are:

T1: Replace $\langle X;R\rangle$ by $\langle X;R\cup S\rangle$ where S is a set of consequences of R.

T2: Replace
$\langle X;R\cup S\rangle$ by $\langle X;R\rangle$ if S is a set of consequences of R.

T3: Replace $\langle X;R\rangle$ by $\langle X\cup Y;R\cup\{y(j)w(j)^{-1}\mid j\in J\}\rangle$ where $Y=\{y(j)\mid j\in J\}$ is any indexed set disjoint from X and $W=\{w(j)\mid j\in J\}$ is any subset of Wd(X) indexed by the same set J.

T4: Replace $G=\langle X\cup Y;R\cup\{y(j)w(j)^{-1}\mid j\in J\}\rangle$ by $\langle X;R\rangle$ provided that none of the elements of R involve any of the Y-symbols and $\{w(j)\mid j\in J\}\subset Wd(X)$.

Since Wd(X) is countable when X is a finite set, R must be countable in every finitely generated presentation $\langle X;R\rangle$. More is true for finite presentations:

Theorem 9 (B.H. Neumann):
If G is a finitely presented group and X is finite, $\bar{R}$ is countable in $G=\langle X;\bar{R}\rangle$ then also $G=\langle X;R\rangle$ for some finite subset R of $\bar{R}$.

Thus every finite generating set of a finitely presented group G is the set of generators in some finite presentation for G.

It is often necessary to find a presentation for a subgroup H of a group G from a given presentation $G=\langle X;R\rangle$. Following the Reidemeister-Schreier rewriting procedure, this can be accomplished by first finding a Schreier transversal $\mathcal{S}$ and, as indicated by Theorem 2., a generating set for H^0, the pre-image of H in the free group F on X. Then it is possible to methodically re-express every conjugate srs^{-1} $(s \in \mathcal{S}, r \in R)$ as a product of these generators $\sigma(srs^{-1})$ and a presentation for H is given by $H = \langle Y ; \sigma(srs^{-1}) \mid s \in S, r \in R \rangle$.

Details of the method can be found in either of [17, p. 103] or [20, p. 87].

Section 5: Special presentations, further constructions.

Split extensions (or semidirect products).

Let G,B,T be groups, $\eta_1 : T \longrightarrow G$, $\eta_2 : B \longrightarrow G$ monomorphisms. G is termed the *split extension* of B by T if

i) $G = gp(T\eta_1 \cup B\eta_2)$,

ii) $B\eta_2 \trianglelefteq G$,

iii) $T\eta_1 \cap B\eta_2 = \{1\}$.

As usual, we identify T and B with their injected images in G and write $G = B \rtimes T$. Then T acts on B via conjugation. Given any homomorphism $\phi : T \longrightarrow Aut(B)$, the split extension of B by T in which the action of T on B coincides with the action of $T\phi$ on B exists, is uniquely determined up to isomorphism and is denoted by $B \rtimes_{\phi} T$. It is easy to see that every non-trivial element $1 \neq g \in G$ of the split extension of B by T has a unique expression of the form $g = tb$ $(t \in T, b \in B)$, termed its *split extension* or *semidirect normal form*.

If $B = \langle X;R \rangle$, $T = \langle Y;S \rangle$, $G = B \rtimes_{\phi} T$ and $\hat{y} = y\phi \in Aut(B)$ then

$$G = \langle X \cup Y \; ; \; R \cup S \bigcup_{y \in Y} \{y^{-1}xy = x\hat{y} \mid x \in X\} \rangle \; .$$

Direct limit of groups.

Let G, G_i $(i=1,2,...)$ be groups, $\eta_i : G_i \longrightarrow G$, $\phi_i : G_i \longrightarrow G$ $(i=1,2,...)$ monomorphisms with the property that $G_i\eta_i \leq G_{i+1}\eta_{i+1}$ and the diagrams

$$\begin{tikzcd} G_i \arrow[rr, "\phi_i"] \arrow[dr, "\eta_i"'] & & G_{i+1} \arrow[dl, "\eta_{i+1}"] \\ & G & \end{tikzcd}$$

commute for all $i=1,2,\ldots$.

Then G is the ascending union of its subgroups $G_i\eta_i$ $(i=1,2,\ldots)$. G is termed the *direct limit* of the groups $\{G_i \mid i=1,2,\ldots\}$ with connecting isomorphisms $\{\phi_i \mid i=1,2,\ldots\}$. Given $\{G_i \mid i=1,2,\ldots\}$ and $\{\phi_i \mid i=1,2,\ldots\}$ the direct limit exists and, up to isomorphism, is uniquely determined. We write $G=\bigcup_{i\in I}{\uparrow}G_i$, once $\{\phi_i \mid i=1,2,\ldots\}$ is specified.

If $G_i = \langle X_i ; R_i\rangle$ $(i=1,2,\ldots)$, $G=\bigcup_{i\in I}{\uparrow}G_i$ with connecting isomorphisms $\{\phi_i \mid i=1,2,\ldots\}$ then

$$G = \langle \bigcup_{i=1}^{\infty} X_i \; ; \; \bigcup_{i=1}^{\infty} R_i \; \bigcup_{i=1}^{\infty} \{x(i)=x(i+1) \mid x(i)\in X_i\}\rangle \; ,$$

where $x(i)\phi_i = x(i+1)\in X_{i+1}$.

If $G_i (i=1,2,\ldots)$ are free groups then the direct limit $L=\bigcup_{i\in I}{\uparrow}G_i$ is a locally free group, i.e. every finitely generated subgroup of L is free.

We now list some other "standard" presentations.

If F is the free group on X then $F=\langle X;\emptyset\rangle$.

If A is the free abelian group on X then $A=F/F'$ where F' is the commutator subgroup of the free group F on X and $A=\langle X;\{[x,y] \mid x,y\in X\}\rangle$. We also use the notation $A=\langle X;\text{abel.}\rangle$.

If $G_i=\langle X_i ; R_i\rangle$ $(i\in I)$ and $G=\prod^{*}_{i\in I} G_i$ then $G=\langle \bigcup_{i\in I} X_i ; \bigcup_{i\in I} R_i\rangle$.

If $G=\langle X;R\rangle$ is any group, Y a set disjoint from G and

$S \subset Wd(X \cup Y)$ then the abbreviation $\langle G,Y;S\rangle$ stands for $\langle X \cup Y; R \cup S\rangle$. If $H=\langle X;R\rangle$ and $w \in Wd(X)$ then the group $G=\langle H;w\rangle$ is termed a one-relator quotient of H; when $R=\emptyset$, G is called a one-relator group.

__HNN extensions__ (or Britton extensions).

Suppose that $G=\langle X;R\rangle$, $H,K \leq G$ and $\phi : H \rightarrowtail\!\!\twoheadrightarrow K$ is an isomorphism. Let $\{t\}$ be disjoint from G. The group with presentation $G^{*}=\langle G,t\ ;\ t^{-1}ht=h\phi \mid h \in H\rangle$ is called the __HNN__ (short for Higman-Neumann-Neumann) or __Britton extension__ of G with stable letter t and associated subgroups H and K under ϕ. Without going into details, we record

__Theorem 10__ (Higman, Neumann, Neumann):

The "obvious" mapping $g \mapsto g$ from G into G^{*} is an embedding.

Also, upon choosing right-transversals for H in G and K in G, normal forms can be introduced for the elements of G^{*}. One consequence of Britton's lemma is that every element in G^{*} has a unique normal form. Detailed description of HNN-extensions can be found in [17, p. 178].

Chapter II. Results.

Section 1: One-relator generalizations of surface groups.

In [11], G. Baumslag points out that it is possible to embed any countable free group into a free group of rank two such a way that positive words get mapped into positive words. Using this information, he shows that $\mathfrak{X}$-groups, i.e. 1-relator groups with the type of presentation $\langle a,b,\dots,c;[u,v]=1\rangle$ where u and v are positive words, are all cyclic extensions of free groups. As a consequence, $\mathfrak{X}$-groups turn out to be residually finite and they have free derived groups - properties they share with the fundamental groups of compact two-dimensional orientable manifolds, shortly referred to as surface groups. Although there are one-relator groups $\langle a,b,\dots,c;[u,v]\rangle$ that are not free-by-cyclic (see[11]), we can generalize the quoted result for two classes of groups both of which also contains the surface groups. We put Ψ for the class of groups with presentations of the type

$$(1)\quad \langle \{a,\dots,b\}\cup\{c,\dots,d\}\cup S\ ;\ [u,v]w=1\rangle$$

where the sets $\{a,\dots,b\}$, $\{c,\dots,d\}$ are disjoint from S and each other, $u=u(a,\dots,b)$, $v=v(c,\dots,d)$ are non-trivial positive words and $w\in Wd(S)$.

Similarly, we put Ψ^0 for the class of groups with presentation of the type

$$(2)\quad \langle \{a,\dots,b\}\cup\{c,\dots,d\}\cup S\ ;\ [u,v][q,w]=1\rangle$$

where the sets of generators, u, v and w satisfy the conditions above and $q=q(a,\dots,b)$ is a positive word.

The embeddings of the free groups F^0 on $\{a,\dots,b\}$ and F^1 on $\{c,\dots,d\}$ into the two-generator free groups on, say, $\{a',b'\}$ and $\{c',d'\}$ which map positive words into positive words give rise to an embedding of the Ψ-group

$$\langle \{a,\dots,b\} \cup \{c,\dots,d\} \cup S \; ; \; [u,v]w=1 \rangle$$

into the Ψ-group

$$\langle \{a',b',c',d'\} \cup S \; ; \; [u',v']w=1 \rangle \; .$$

Similarly, the mentioned embeddings of F^0 and F^1 give rise to an embedding of the Ψ^0-group

$$\langle \{a,\dots,b\} \cup \{c,\dots,d\} \cup S \; ; \; [u,v][q,w]=1 \rangle$$

into the Ψ^0-group

$$\langle \{a',b',c',d'\} \cup S \; ; \; [u',v'][q',w]=1 \rangle \; .$$

First we aim at proving

Theorem 11:

Let G be a Ψ-group as given by (1). Then G is free-by-cyclic i.e. an extension of a free group by a cyclic group.

Proof

Since the property of being free-by-cyclic is inherited by subgroups, we may, without loss of generality, assume that the Ψ-group G is

$$G=\langle \{a,b,c,d\} \cup S \; ; \; [u,v]w=1 \rangle \; .$$

Put $b=xa$, $c=ya$, $d=ta$ and obtain the alternative presentation

$G = \langle a,x,y,t,S \; ; \; [u(a,xa),v(ya,ta)]w=1\rangle .$

As the relator has 0 exponent sum on a, G is an extension of $N=gp_G(x,y,t,S)$ by the infinite cyclic group generated by a.

With the notation

$x(i)=a^{-i}xa^{i}$, $y(i)=a^{-i}ya^{i}$, $t(i)=a^{-i}ta^{i}$, ... , $h(i)=a^{-i}ha^{i}$,

$S(i)=\{h(i) \mid h\in S\}$ for all $i\in Z$,

$N=gp(x(i),y(i),t(i),S(i) \mid i\in Z)$.

Let r denote the relator expressed in terms of these generators for N. To prove that N is free it suffices to show (see e.g. [19]) that in r the smallest and largest labels of at least one of x,y,t are different and occur exactly once. Straightforward computation leads to the following useful results:

Lemma 12:

If $u(a,b)=b^{\beta(1)}a^{\alpha(1)}\dots b^{\beta(m)}a^{\alpha(m)}$ with $\beta(1)>0,\dots,\beta(m)>0,\alpha(1)>0,\dots,\alpha(m-1)>0,\alpha(m)\geq 0$ then u(a,b) can be put in the form

$$\begin{aligned}
&a^{\beta(1)+\alpha(1)+\dots+\beta(m)+\alpha(m)}\cdot\\
&x(\beta(1)+\alpha(1)+\dots+\beta(m)+\alpha(m)) \;\dots\; x(1+\alpha(1)+\beta(2)+\dots+\beta(m)+\alpha(m))\\
&x(\beta(2)+\alpha(2)+\dots+\beta(m)+\alpha(m)) \;\dots\; x(1+\alpha(2)+\beta(3)+\dots+\beta(m)+\alpha(m))\\
&\dots\\
&x(\beta(m)+\alpha(m))\cdot x(\beta(m)-1+\alpha(m)) \;\dots\; x(1+\alpha(m)) .
\end{aligned}$$

Lemma 13:

If $u(a,b)=a^{\alpha(1)}b^{\beta(1)}a^{\alpha(2)}\dots b^{\beta(m-1)}a^{\alpha(m)}$ with $\alpha(1)>0,\dots,\alpha(m-1)>0,\alpha(m)\geq 0,\beta(1)>0,\dots,\beta(m-1)>0$ and we interpret $m=1$ as $u(a,b)=a^{\alpha(1)}$ then $u(a,b)$ can be put in the form

$$a^{\alpha(1)+\beta(1)+\alpha(2)+\dots+\beta(m-1)+\alpha(m)}\,.$$
$$x(\beta(1)+\alpha(2)+\dots+\beta(m-1)+\alpha(m))\ \dots\ x(1+\alpha(2)+\dots+\beta(m-1)+\alpha(m))$$
$$x(\beta(2)+\alpha(3)+\dots+\beta(m-1)+\alpha(m))\ \dots\ x(1+\alpha(3)+\dots+\beta(m-1)+\alpha(m))$$
$$\cdot\ \cdot\ \cdot$$
$$x(\beta(m-1)+\alpha(m))\cdot x(\beta(m-1)-1+\alpha(m))\ \dots\ x(1+\alpha(m))\ .$$

If $u(a,b)=a^{\alpha}b^{\beta}$ with $\alpha>0$, $\beta>0$, its form is $a^{\alpha+\beta}\cdot x(\beta)\,x(\beta-1)\dots x(1)$.

Lemma 14:

If $v(c,d)=c^{\gamma(1)}d^{\delta(1)}\dots c^{\gamma(n)}d^{\delta(n)}$ with $\gamma(1)>0,\dots,\gamma(n)>0,\delta(1)>0,\dots\delta(n-1)>0,\delta(n)\geq 0$ then $v(c,d)$ can be put in the form

$$a^{\gamma(1)+\delta(1)+\dots+\gamma(n-1)+\delta(n)}\,.$$
$$y(\gamma(1)+\delta(1)+\dots+\gamma(n)+\delta(n))\ \dots .\ y(1+\delta(1)+\gamma(2)+\dots+\gamma(n)+\delta(n))$$
$$t(\delta(1)+\gamma(2)+\dots+\gamma(n)+\delta(n))\ \dots\ t(1+\gamma(2)+\dots+\gamma(n)+\delta(n))$$
$$y(\gamma(2)+\delta(2)+\dots+\gamma(n)+\delta(n))\ \dots\ y(1+\delta(2)+\gamma(3)+\dots+\gamma(n)+\delta(n))$$
$$\cdot\ \cdot\ \cdot$$
$$y(\gamma(n)+\delta(n))\ \cdot\ y(\gamma(n-1)-1+\delta(n))\ \dots\ y(1+\delta(n))$$
$$t(\delta(n))\ \cdot\ t(\delta(n)-1)\ \dots\ t(1)\ .$$

Lemma 15:

If $v(c,d) = d^{\delta(1)} c^{\gamma(1)} \dots c^{\gamma(n-1)} d^{\delta(n)}$ with $\delta(1) > 0, \dots, \delta(n-1) > 0, \delta(n) \geq 0, \gamma(1) > 0, \dots, \gamma(n-1) > 0$ and we interpret $n=1$ as $v(c,d) = d^{\delta(1)}$ then $v(c,d)$ can be put in the form

$$a^{\delta(1)+\gamma(1)+\dots+\gamma(n-1)+\delta(n)} \cdot$$

$$t(\delta(1)+\gamma(1)+\dots+\gamma(n-1)+\delta(n)) \dots t(1+\gamma(1)+\delta(2)+\dots+\delta(n))$$

$$y(\gamma(1)+\delta(2)+\dots+\gamma(n-1)+\delta(n)) \dots y(1+\delta(2)+\gamma(3)+\dots+\delta(n))$$

$$t(\delta(2)+\gamma(2)+\dots+\gamma(n-1)+\delta(n)) \dots t(1+\gamma(2)+\delta(3)+\dots+\delta(n))$$

$$\cdot \cdot \cdot$$

$$y(\gamma(n-1)+\delta(n)) \cdot y(\gamma(n-1)-1+\delta(n)) \dots y(1+\delta(n))$$

$$t(\delta(n)) \cdot t(\delta(n)-1) \dots t(1) .$$

To summarize: In every case , once all the a-occurrences have been moved to the left, the exponent of a is exactly the sum of all the exponents of the word in question, therefore it is positive. If more than one label is present, the labels form a strictly decreasing sequence of positive integers. Moreover, the expression $v(c,d)$ contains always at least one label.

So if X,Y denote the respective "a-free" segments of the re-expressed $u(a,b)$ and $v(c,d)$ and if, for every integer i $X(i)$, $Y(i)$ are the words obtained from X,Y by adding i to each label and finally if

$$k = \exp(a,u) + \exp(b,u) , \quad j = \exp(c,u) + \exp(d,u)$$

then $k \neq 0$, $j \neq 0$ and we have

$$[u(a,b),v(c,d)] = X^{-1}a^{-k}\,Y^{-1}a^{-j}\,a^{k}Xa^{j}\,Y = X^{-1}Y(k)^{-1}X(j)\;Y\quad .$$

Since none of the letters of Y are present in w, the smallest label of any letter of Y occurs exactly once in r (on account of Y) and the largest label of such letter also occurs exactly once (on account of $Y(k)^{-1}$). Hence N is free, indeed, and the theorem is true.

With some further inspection, we can read more out of this analysis and verify

Theorem 16

Let G be a Ψ^{0}-group as given by (2). Then G is free-by-cyclic.

Proof

We may again restrict our attention to $G=\langle\{a,b,c,d\}\cup S\ ;\ [u,v][q,w]=1\rangle$. Keeping the former notations we express q(a,b) as suggested by Lemmas 12 and 13, so that q(a,b) becomes $a^{e}Q$ where $e=\exp(a,q)+\exp(b,q)$ and Q is a (possibly empty) $\underset{\sim}{x}$-word. Then we obtain

$$[u(a,b),v(c,d)][q(a,b),w] =$$
$$= X^{-1}Y(k)^{-1}X(j)Y\;Q^{-1}a^{-e}\,w^{-1}a^{e}\;Q\;w =$$
$$= X^{-1}Y(k)^{-1}X(j)Y\;Q^{-1}w(e)^{-1}\;Q\;w\;.$$

The letters of Y still do not occur beyond Y so at least one letter of Y has its smallest label once only in Y, its maximum subscript once only in $Y(k)$. Thus this group is also free-by-cyclic, proving the theorem.

Residual finiteness, too, is a property inherited by subgroups. Since finitely generated cyclic extensions of free groups are residually finite by a theorem of G. Baumslag [8], we have the

Corollary 17:

If G is a $\mathcal{Y}$-group or a $\mathcal{Y}^{o}$-group then G has free derived group and G is residually finite.

Thus $\mathcal{Y}$ and $\mathcal{Y}^{o}$-groups share these properties of $\mathcal{X}$ groups. The surface group $<x,y,z,t \; ; \; [x,z][y,t]=1>$ is clearly both a $\mathcal{Y}$ and a $\mathcal{Y}^{o}$-group. So Corollary 17 is an extension of the corresponding theorems on $\mathcal{X}$-groups in [11] to the classes $\mathcal{Y}$ and $\mathcal{Y}^{o}$.

Section 2: An analogue of the Freiheitssatz.

The fundamental role that Magnus' Freiheitssatz plays in the quite expansive theory of one-relator groups has been an inspiration to establish a number of extensions and sharper versions; for an assessment, see e.g. [17]. In [16], R.C. Lyndon extends the Freiheitssatz for free products of subgroups of the additive group of real numbers by showing that all of the free factors that contribute a syllable to a given cyclically reduced element r, must also contribute a syllable to every non-trivial element in the normal closure of r. These groups include the free products of free abelian groups for which the property also follows from the extension of the Freiheitssatz proven by B. Baumslag and S. Pride [2] for the one-relator quotients of free products of locally residually free groups.

Prompted by a conjecture of Papakyriakopoulos [24]; E. Rapaport [26] investigated certain one-relator quotients of the fundamental groups of compact orientable manifolds of finite genus and found them torsion-free. This gave motivation to consider the class of ß-groups, i.e. groups with presentation of the type

$$<(a,b,\ldots,c,d;\text{abel.}),(\bar{a},\bar{b},\ldots,\bar{c},\bar{d};\text{abel.});$$
$$[a,\bar{a}][b,\bar{b}]\ldots[c,\bar{c}][d,\bar{d}]=1> ,$$

a beginning of an attempt to carry over some of the aspects of the one-relator theory. Of course, ß-groups are one-relator quotients of free products of free abelian groups. We state an appropriately sharpened version of the extensions of the Freiheitssatz mentioned before as

Theorem 18:

Let $k, m(1), n(1), m(2), n(2), \ldots, m(k), n(k)$ be positive integers satisfying $m(j) \leq n(j)$ $(j=1,\ldots,k)$. Let $X = \bigcup_{j=1}^{2k} X_j$ be a disjoint union where

$$X_1 = \{a(1),\ldots,a(m(1))\}, \quad X_2 = \{a(1),\ldots,a(n(1))\},$$
$$X_3 = \{b(1),\ldots,b(m(2))\}, \quad X_4 = \{b(1),\ldots,b(n(2))\},$$
$$\ldots$$
$$X_{2k-1} = \{d(1),\ldots,d(m(i))\}, \quad X_{2k} = \{d(1),\ldots,d(n(i))\},$$

and let Z, disjoint from X, be also a disjoint union $Z = \bigcup_{\lambda\in\Lambda} Z_\lambda$.

Suppose that Y is a subset of X and that the group G is given by the presentation

$$G = \langle X \cup Z; \bigcup_{\lambda\in\Lambda} (Z_\lambda \text{ abel.}) \bigcup_{j=1}^{2k} (X_j \text{ abel.}) \cup \{r\} \rangle$$

where

$$r = \prod_{h=1}^{m(1)} [a(h), \bar{a}(h)] \prod_{h=1}^{m(2)} [b(h), \bar{b}(h)] \ldots \prod_{h=1}^{m(k)} [d(h), \bar{d}(h)] .$$

If Y omits at least one generating symbol of X that is present in r, then the subgroup of G generated by $Y \cup Z$ is a free product of free abelian groups. Indeed, when $Y = X - \{t\}$ (t present in r) and $t \in X_{j(t)}$ $(1 \leq j(t) \leq 2k)$ then

$$\mathrm{gp}(Y) = \langle X_{j(t)} - \{t\}; \text{abel.}\rangle * \prod_{\substack{j=1 \\ j\neq j(t)}}^{2k}{}^{*} \langle X_j; \text{abel.}\rangle .$$

<u>Proof</u>

Clearly, $G=\mathrm{gp}(X) * \prod^{*}_{\lambda\in\Lambda} \langle \mathfrak{z}_\lambda; \text{abel.}\rangle$. The fact, that in a free product, subgroups of the free factors generate their own free product allows us to assume $\Lambda=\emptyset$ i.e.

$$(1) \quad G = \langle \bigcup_{j=1}^{2k} X_j ; \bigcup_{j=1}^{2k} (X_j \text{ abel.}) ; r\rangle ,$$

and $Y=X-\{t\}$ (t present in r). Furthermore, without loss of generality, we may assume that t, the omitted symbol is in $X_1 \cup X_2$. For otherwise Tietze transformations can replace r by its cyclic permutation which begins with the full $\prod_{h=1}^{m(j(t))} [x(h), \bar{x}(h)]$ subproduct of r corresponding to the set $X_{j(t)}$ whose element t originally was. Once so arranged, the appropriate relabeling of the sets X_j produces the desired type of presentation of G. The proof now consists of showing that G is an HNN extension of the "right kind of a group".

<u>Case 1</u>: $t\in X_1$, say $t=a(s)$ $1\leq s\leq m(1)$

Introduce the notation

$$u \equiv [a(1),\bar{a}(1)] \dots [a(s-1),\bar{a}(s-1)] \,,$$

$$v \equiv [a(s+1),\bar{a}(s+1)] \dots [a(m(1)),\bar{a}(m(1))] \prod_{h=1}^{m(2)} [b(h),\bar{b}(h)] \dots \prod_{h=1}^{m(k)} [d(h),\bar{d}(h)] \,.$$

Then $r=1$ is of the form $u[t,\bar{a}(s)]v=1$ or, equivalently,

$$(\bullet) \quad t^{-1}\bar{a}(s)t = \bar{a}(s)vu$$

and here the right hand side does not involve $a(s)$ $(=t)$.

After the Tietze transformation that replaces $r=1$ by $(\bullet)$, the only other relations involving $t=(a(s))$ are $[a(h),t]=1$ or, equivalently, $t^{-1}a(h)t=a(h)$ $(h\neq s)$.

This yields the presentation

$$(2) \quad G = \langle (X_1-\{a(s)\}) \cup \bigcup_{j=2}^{2k} X_j \cup \{t\} ; \bigcup_{j=2}^{2k} (X_j \text{ abel.}), (X_1-\{a(s)\} \text{ abel.}), t^{-1}a(h)t = a(h) \ (h\neq s, a(h)\in X_1),\ t^{-1}\bar{a}(s)t = \bar{a}(s)vu\rangle \,.$$

Now put $w=\bar{a}(s)vu$ and observe that in

$$P=\langle a(1),\dots,a(s-1),a(s+1),\dots,a(m(1));\text{abel.}\rangle * \prod_{j=2}^{2k}{}^{*} \langle X_j;\text{abel.}\rangle$$

$w^i \neq 1$ $(i\neq 0)$ because one of the following holds:

$$w^i = \bar{a}(1)^i \quad (\text{when } k=1, m(1)=1, s=1)$$

or

$$w^i = (\bar{a}(s)[a(s+1),\bar{a}(s+1)] \ldots [a(m(1)),\bar{a}(m(1))]$$
$$[a(1),\bar{a}(1)] \ldots [a(s-1),\bar{a}(s-1)])^i$$
$$(\text{when } k=1, m(1)\neq 1, s\neq 1)$$

or

$$w^i = (\bar{a}(1)[a(2),\bar{a}(2)] \ldots [a(m(1)),\bar{a}(m(1))]$$
$$\prod_{h=1}^{m(2)} [b(h),\bar{b}(h)] \ldots \prod_{h=1}^{m(k)} [d(h),\bar{d}(h)])^i$$
$$(\text{when } k\neq 1, m(1)\neq 1, s=1)$$

or

$$w^i = (\bar{a}(s)[a(s+1),\bar{a}(s+1)] \ldots [a(m(1)),\bar{a}(m(1))]$$
$$\prod_{h=1}^{m(2)} [b(h),\bar{b}(h)] \ldots \prod_{h=1}^{m(k)} [d(h),\bar{d}(h)]$$
$$[a(1),\bar{a}(1)] \ldots [a(s-1),\bar{a}(s-1)])^i$$
$$(\text{when } k\neq 1, m(1)\neq 1, s\neq 1) \quad .$$

Besides lack of cancellations, w^i begins and ends with free abelian basis elements in P, none of which is amongst $\{a(1),\ldots,a(s-1),a(s+1),\ldots,a(m(1))\}$. Hence G is the HNN extension of P with its subgroups

$$\langle X_1 - \{a(s)\};\text{abel.}\rangle * \langle \bar{a}(s)\rangle \quad \text{and} \quad \langle X_1 - \{a(s)\};\text{abel.}\rangle * \langle w\rangle$$

associated as in (2).

Now gp(P)=gp(Y) in G and this takes care of Case 1.

The procedure is analogous for

<u>Case</u> 2: $t \in X_2$ say $t=\bar{a}(s)$.

Then $1 \leq s \leq m(1)$ since $\bar{a}(s)$ is supposed to be present in r. With u, v as before, use that $r=1$ is equivalent to $u[a(s),t]v=1$ i.e. $t^{-1}a(s)t=a(s)(vu)^{-1}=a(s)u^{-1}v^{-1}$.

We put now $w'=a(s)u^{-1}v^{-1}$ and by a duplicate of the previous argument (with the trivial changes involved) G proves to be the HNN extension of

$$P = \langle X_2-\{\bar{a}(s)\};\text{abel.}\rangle * \prod_{\substack{j=1\\ j\neq 2}}^{2k}{}^{*} \langle X_j;\text{abel.}\rangle$$

with the subgroups

$$\langle X_2-\{\bar{a}(s)\};\text{abel.}\rangle * \langle a(s)\rangle \quad \text{and} \quad \langle X_2-\{\bar{a}(s)\};\text{abel.}\rangle * \langle w'\rangle$$

associated as in the presentation

$$(3) \quad G = \langle (X_2-\{\bar{a}(s)\}) \cup \Big(\bigcup_{\substack{j=1\\ j\neq 2}}^{2k} X_j\Big) \cup \{t\};$$

$$\bigcup_{\substack{j=1\\ j\neq 2}}^{2k} (X_j \text{ abel.}), (X_2-\{\bar{a}(s)\} \text{ abel.}),$$

$$t^{-1}\bar{a}(h)t = \bar{a}(h) \quad (h\neq s, \bar{a}(h)\in X_2), \quad t^{-1}a(s)t = a(s)(vu)^{-1}\rangle .$$

This completes the proof.

With the notation

$$H = \prod_{j=1}^{2k}{}^{*} \langle X_j;\text{abel.}\rangle * \prod_{\lambda\in\Lambda}{}^{*} \langle Z_\lambda;\text{abel.}\rangle ,$$

$G=\langle H;r\rangle$ is a one-relator quotient of a free product of free abelian groups, r is cyclically reduced when read as an element of H. Put $Y' = \bigcup_{\substack{j=1\\ j\neq j(t)}}^{2k} X_j$ when t is the only omitted symbol. According to Corollary 1 in [25],

$$gp(Y' \cup \mathfrak{Z}) = \prod_{\substack{j=1\\ j\neq j(t)}}^{2k}{}^{*} \langle X_j ;\text{abel.}\rangle * \prod_{\lambda \in \Lambda}{}^{*} \langle \mathfrak{Z}_\lambda ;\text{abel.}\rangle .$$

Note that Y' is a proper subset of Y. In particular, Theorem 18 implies for the $\mathfrak{F}$-groups G generated by $X=\{a,b,\ldots,c,d,\bar{a},\bar{b},\ldots,\bar{c},\bar{d}\}$ that the natural homomorphism from the free product of the free abelian groups A on $\{a,b,\ldots,c,d\}$ and $\bar{A}$ on $\{\bar{a},\bar{b},\ldots,\bar{c},\bar{d}\}$ into $G=\langle A*\bar{A};r\rangle$ is an injection on all subgroups of $A*\bar{A}$ generated by a proper subset of X. Consequently, $\mathfrak{F}$-groups have "many" free subgroups.

Section 3: A decomposition theorem.

In this section, we unfold a generalized free product structure of $\mathfrak{B}$-groups that parallels a similar property of surface groups. This will also pave the road to demonstrating further properties in the later sections. The notations introduced and used in this section will be carried over and employed in sections 4 and 5 as well.

Theorem 19:

The group G given by the presentation

$$G = \langle (a,b,\dots,d;\text{abel.}),(\bar{a},\bar{b},\dots,\bar{d};\text{abel.});$$
$$[a,\bar{a}][b,\bar{b}]\dots[c,\bar{c}][d,\bar{d}]=1\rangle$$

is the generalized free product of two groups, each a free product of finitely generated free abelian groups.

Proof

Put

$$A=\langle a,b,\dots,c,d;\text{abel.}\rangle \;, \qquad \bar{A}=\langle \bar{a},\bar{b},\dots,\bar{d};\text{abel.}\rangle \;,$$

so A and $\bar{A}$ are finitely generated free abelian groups of equal rank r. Let $\mathfrak{Z}$ and $\bar{\mathfrak{Z}}$ be also finitely generated free abelian groups of equal rank r-1, say

$$\mathfrak{Z}=\langle x,y,\dots,z;\text{abel.}\rangle \;, \qquad \bar{\mathfrak{Z}}=\langle \bar{x},\bar{y},\dots,\bar{z};\text{abel.}\rangle \;.$$

Now let

$$U = A * \mathcal{Z}\ , \qquad\qquad V = \bar{A} * \bar{\mathcal{Z}},$$

$$N = gp_U(b,\ldots,c,d,x,y,\ldots,z)\ , \ M = gp_V(\bar{a},\bar{b},\ldots,\bar{c},\bar{x},\bar{y},\ldots,\bar{z})\ .$$

Then $\{a^i \mid i \in \mathbb{Z}\}$, $\{\bar{d}^i \mid i \in \mathbb{Z}\}$ are respective transversals for N in U and M in V and the Reidemeister-Schreier rewriting gives that

$$N = gp(\{b,\ldots,c,d\} \cup \bigcup_{i \in \mathbb{Z}} \{x(i),y(i),\ldots,z(i)\}),$$

$$M = gp(\{\bar{a},\bar{b},\ldots,\bar{c}\} \cup \bigcup_{i \in \mathbb{Z}} \{\bar{x}(i),\bar{y}(i),\ldots\bar{z}(i)\})$$

where

$$x(i)=a^i x a^{-i}\ , \ y(i)=a^i y a^{-i}\ ,\ldots,\ z(i)=a^i z a^{-i} \qquad (i \in \mathbb{Z})\ ,$$

$$\bar{x}(i)=\bar{d}^i \bar{x}\bar{d}^{-i}\ , \ \bar{y}(i)=\bar{d}^i \bar{y}\bar{d}^{-i}\ ,\ldots,\ \bar{z}(i)=\bar{d}^i \bar{z}\bar{d}^{-i} \qquad (i \in \mathbb{Z})\ .$$

In fact, Tietze transformations yield

$$N=\langle b,\ldots,c,d;\text{abel.}\rangle * \prod_{i \in \mathbb{Z}}{}^{*} \langle x(i),\ldots,y(i),z(i);\text{abel.}\rangle\ ,$$

$$M=\langle \bar{a},\bar{b},\ldots,\bar{c};\text{abel.}\rangle * \prod_{i \in \mathbb{Z}}{}^{*} \langle \bar{x}(i),\ldots,\bar{y}(i),\bar{z}(i);\text{abel.}\rangle\ .$$

Observe that in N

$$\begin{aligned} H &= gp(b,\ldots,c,d,x(0),\ldots,y(0),z(0),x(0)x(1)^{-1}) = \\ &= gp(b,\ldots,c,d,x(0),\ldots,y(0),z(0),x(1)) = \\ &= \langle b,\ldots,c,d;\text{abel.}\rangle * \langle x(0),\ldots,y(0),z(0);\text{abel.}\rangle * \langle x(1)\rangle = \\ &= \langle b,\ldots,c,d;\text{abel.}\rangle * \langle x(0),\ldots,y(0),z(0);\text{abel.}\rangle * \langle x(0)x(1)^{-1}\rangle \end{aligned}$$

and in M

$$\begin{aligned}K &= \mathrm{gp}(\bar{a},\bar{b},\ldots,\bar{c},\bar{x}(0),\ldots,\bar{y}(0),\bar{z}(0),\\
&\qquad\qquad [\bar{c},\bar{y}(0)]\ldots[\bar{b},\bar{x}(0)]\bar{z}(0)\bar{z}(1)^{-1}) =\\
&= \mathrm{gp}(\bar{a},\bar{b},\ldots,\bar{c},\bar{x}(0),\ldots,\bar{y}(0),\bar{z}(0),\bar{z}(1)) =\\
&= \langle\bar{a},\bar{b},\ldots,\bar{c};\text{abel.}\rangle*\langle\bar{x}(0),\ldots,\bar{y}(0),\bar{z}(0);\text{abel.}\rangle*\langle\bar{z}(1)\rangle =\\
&= \langle\bar{a},\bar{b},\ldots,\bar{c};\text{abel.}\rangle*\langle\bar{x}(0),\ldots,\bar{y}(0),\bar{z}(0);\text{abel.}\rangle\\
&\qquad\qquad *\langle[\bar{x}(0),\bar{b}]\ldots[\bar{y}(0),\bar{c}]\bar{z}(0)\bar{z}(1)^{-1}\rangle\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\theta :\ & b\longmapsto\bar{x}(0),\ldots,\ c\longmapsto\bar{y}(0),\ d\longmapsto\bar{z}(0)\ ,\\
& x(0)\longmapsto\bar{a}\ ,\ldots,\ y(0)\longmapsto\bar{b},\ z(0)\longmapsto\bar{c}\ ,\\
& x(0)x(1)^{-1}\ \longmapsto[\bar{x}(0),\bar{b}]\ldots[\bar{y}(0),\bar{c}]\bar{z}(0)\bar{z}(1)^{-1}\end{aligned}$$

defines an isomorphism between H and K, so $G=\{U*V\ ;H\theta=K\}$ exists.

If we perform Tietze transformations on the ready-made presentation (from the generalized free-product structure) for G, we obtain:

$$\begin{aligned}G = \langle & a,b,\ldots,c,d,x,y,\ldots,z,\bar{a},\bar{b},\ldots,\bar{c},\bar{d},\bar{x},\bar{y},\ldots,\bar{z};\\
& (a,b,\ldots,c,d;\text{abel.}),(x,y,\ldots,z;\text{abel.}),\\
& (\bar{a},\bar{b},\ldots,\bar{c},\bar{d};\text{abel.}),(\bar{x},\bar{y},\ldots,\bar{z};\text{abel.}),\\
& \bar{x}=b,\ \bar{y}=c,\ldots,\ \bar{z}=d,\\
& x=\bar{a},\ y=\bar{b},\ldots,\ z=\bar{c},\\
& xax^{-1}a^{-1}=[\bar{x},\bar{b}]\ldots[\bar{y},\bar{c}]\bar{z}\bar{d}\bar{z}^{-1}\bar{d}^{-1}\ \rangle =\end{aligned}$$

$$
\begin{aligned}
&= \langle a,b,\dots,c,d,\bar{a},\bar{b},\dots,\bar{c},\bar{d};\ (a,b,\dots,c,d;\text{abel.}),\\
&\qquad\qquad (\bar{a},\bar{b},\dots,\bar{c},\bar{d};\text{abel.}),\\
&\qquad\qquad \bar{a}a\bar{a}^{-1}a^{-1}=[b,\bar{b}]\dots[c,\bar{c}][d,\bar{d}]\rangle =
\end{aligned}
$$

$$
\begin{aligned}
&= \langle (a,b,\dots,c,d;\text{abel.}),(\bar{a},\bar{b},\dots,\bar{c},\bar{d};\text{abel.}),\\
&\qquad [a,\bar{a}][b,\bar{b}]\dots[c,\bar{c}][d,\bar{d}]=1\rangle = G.
\end{aligned}
$$

Consequences of this decomposition will be discussed in the sections to follow.

Section 4: The residual finiteness of β-groups.

The generalized free product decomposition of β-groups arrived at in Section 3, allows us to take use of G.Baumslag's result [4; Proposition 2] that provides criteria for the residual finiteness of generalized free products. Thus residual finiteness is another property β-groups share with the surface groups. We will first prepare grounds by establishing two preliminary results.

Lemma 20:

Let p be a prime, $H=S*D$ where S and D are residually finite p-groups. Then if $h(1),\dots,h(r)\notin S$, there exists a finite p-group epimorph of H in which the images of $h(1),\dots,h(r)$ are not in the image of S.

Proof

First observe that without loss of generality, we may assume that S,D are finite p-groups. For let $s(1),\dots,s(m)$, $d(1),\dots,d(n)$ be the non-trivial syllables present in the *reduced forms of $h(1),\dots,h(r)$. Since S and D are residually finite p-groups, there exist normal subroups M of index a power of p in S and N of index a power of p in D with $s(i)\notin M$ $(i=1,\dots,m)$ and $d(j)\notin N$ $(j=1,\dots,n)$. But then the pair of natural homomorphisms $S\twoheadrightarrow S/M=\tilde{S}$, $D\twoheadrightarrow D/N=\tilde{D}$ extends to an epimorphism of H onto $\tilde{H}=\tilde{S}*\tilde{D}$ under which none of the syllables in the $\tilde{h}(i)$'s will be trivial hence $\tilde{h}(1),\dots,\tilde{h}(r)$ will still not be in $\tilde{S}$.

Now if S,D are finite p-groups then following from the fact that the free products of finite p-groups are residually finite-p [13], we can find a normal subgroup K of index a power of p in $H=S*D$ that avoids each element in the finite set $\{h(i)s^{-1} \mid 1 \neq s \in S, i=1,\ldots,r\}$. Hence, under the natural homomorphism, the image of h(i) in H/K will not belong to the image of S $(i=1,\ldots,r)$.

Proposition 21:

Let p be a prime. Suppose that $G=S*(D \times E)$ where S,D,E are residually finite p-groups. Then for any element $g \notin gp(S,D)$ there exists a finite p-group epimorph of G in which the image of g is not in the image of gp(S,D).

Proof

Let $g \notin gp(S,D)$ be given. First notice that without loss of generality, we may again assume that S,D,E are finite p-groups. For, let $s(1),s(2),\ldots,s(m)$, $d(1),d(2),\ldots d(n)$, $e(1),e(2),\ldots,e(k)$ be the non-trivial S-, D-, E-syllables present in the *reduced form of g after replacing the $(D \times E)$-syllables by their direct product normal form. To have $g \notin gp(S,D)$, $k \geq 1$ must hold. As S,D,E are residually finite p-groups, we can find subgroups M of index a power of p in S excluding each of $s(1),s(2),\ldots,s(m)$, N of index a power of p in D excluding each of $d(1),d(2),\ldots,d(n)$ and K of index a power of p in E excluding each of $e(1),e(2),\ldots,e(k)$. Then the natural homomorphisms

$S \twoheadrightarrow S/M=\tilde{S}$, $D \twoheadrightarrow D/N=\tilde{D}$, $E \twoheadrightarrow E/K=\tilde{E}$ extend to a homomorphism $\phi: G \twoheadrightarrow \tilde{S}*(\tilde{D}\times\tilde{E})=\tilde{G}$ under which $\tilde{g}=g\phi \notin gp(\tilde{S},\tilde{D})$ for $\tilde{g}$ has non-trivial syllables $\tilde{s}(1),\ldots,\tilde{s}(m)$, $\tilde{d}(1),\ldots,\tilde{d}(n)$, $\tilde{e}(1),\ldots,\tilde{e}(k)$.

So assume now that S,D,E are finite p-groups. Following from the properties of free products, in $G=S*(D\times E)$, $gp(S,D)=S*D$, $gp(D,E)=D\times E$. Put $P=S*D$, $T=D\times E$, $\bar{G}=\{P*T;D\}$ with D amalgamated under the identity. Then in $\bar{G}$, $gp(S,D)=S*D$, $gp(D,E)=D\times E$. Let α be the homomorphism from G onto $\bar{G}$ determined by the identities on S and on $D\times E$. Since the homomorphisms $\theta^1: P \longrightarrow G$ and $\theta^2: T \longrightarrow G$ defined by the identity maps on S,D,E coincide on D, they extend to a homomorphism θ of $\bar{G}$ onto G. But $\alpha\theta=1_G$ and $\theta\ =1_{\bar{G}}$ so α,θ are invertible proving $\bar{G}=G$ i.e. $G=\{P*T;D\}=\{(S*D)*(D\times E);D\}$.

Since S and D are finite, there exists, by Lemma 20, some normal subgroup M of index a power of p in P with $sM \notin DM$ $(1\neq s\in S)$. P/M is a finite p-group therefore nilpotent, say of exponent p^n and of nilpotency class m. Then $P(m,n)\subset M$ [23, Thm. 14.23, p. 12] therefore $sP(m,n)\notin DP(m,n)$ $(1\neq s\in S)$ also holds, in particular $s\neq 1 \bmod P(m,n)$ $(1\neq s\in S)$. Since S and D are both retracts of P,

$$S \cap P(m,n) = S(m,n) \ , \qquad D \cap P(m,n) = D(m,n) \ ,$$

so in $\tilde{P}=P/P(m,n)$,

$$\tilde{S}=SP(m,n)/P(m,n) \cong S/S(m,n)\, , \ \tilde{D}=DP(m,n)/P(m,n) \cong D/D(m,n)\cong D.$$

Let ϕ be the natural homomorphism $D \longrightarrow\!\!> \tilde{D}$. Since ϕ is in fact an isomorphism, $\tilde{G}=\{\tilde{P}*(D\times E);\tilde{D}=D\}$ exists and is clearly a homomorphic image of G. Since $\tilde{s} \neq 1$ $(1 \neq s \in S)$ and at least one T-syllable of the *reduced form of $g \in G-gp(S,D)$ had to have a non-trivial E-component in its direct product normal form, $\tilde{g} \notin \tilde{P}$. Now D is a retract in $D\times E$ and the finite p-group $\tilde{P}$ is a retract of $\tilde{G}$. Thus the image of $\tilde{g}$ is separate from the image of $\tilde{P}$ in some finite p-group epimorph of $\tilde{G}$; this yields the desired epimorph of G.

We are now ready to prove

<u>Theorem</u> <u>22</u>:

β-groups are residually finite.

<u>Proof</u>

We continue with the notations introduced in Section 3. Let p be any fixed prime. Since U and V are free products of free abelian groups, all their subgroups are residually finite p-groups.

For $r \geq 1$ integer, put

$$I(r)=\{-p^{r-1}+1,\ldots,-1,0,1,\ldots,p^{r}-p^{r-1}\},$$

$$N(r)=\langle b,c,\ldots,d;\text{abel.}\rangle * \prod_{i \in I(r)}^{*} \langle x(i),y(i),\ldots,z(i);\text{abel.}\rangle$$

$$M(r)=\langle \bar{a},\bar{b},\ldots,\bar{c};\text{abel.}\rangle * \prod_{i \in I(r)}^{*} \langle \bar{x}(i),\bar{y}(i),\ldots,\bar{z}(i);\text{abel.}\rangle$$

Then $N(r)$, $M(r)$ are finitely generated,

$$N=N(r)*\prod_{i\notin I(r)}^{*}\langle x(i),y(i),\ldots,z(i);\text{abel.}\rangle,$$

$$M=M(r)*\prod_{i\notin I(r)}^{*}\langle \bar{x}(i),\bar{y}(i),\ldots,\bar{z}(i);\text{abel.}\rangle.$$

Recall that

$$H=\langle b,c,\ldots,d;\text{abel.}\rangle*\langle x(0),y(0),\ldots,z(0);\text{abel.}\rangle*\langle x(1)\rangle$$

$$K=\langle \bar{a},\bar{b},\ldots,\bar{c};\text{abel.}\rangle*\langle \bar{x}(0),\bar{y}(0),\ldots,\bar{z}(0);\text{abel.}\rangle*\langle \bar{z}(1)\rangle$$

and put

$$H^{+}=\langle b,c,\ldots,d;\text{abel.}\rangle*\langle x(0),y(0),\ldots,z(0);\text{abel.}\rangle$$
$$*\langle x(1),y(1),\ldots,z(1);\text{abel.}\rangle,$$

$$K^{+}=\langle \bar{a},\bar{b},\ldots,\bar{c};\text{abel.}\rangle*\langle \bar{x}(0),\bar{y}(0),\ldots,\bar{z}(0);\text{abel.}\rangle$$
$$*\langle \bar{x}(1),\bar{y}(1),\ldots,\bar{z}(1);\text{abel.}\rangle.$$

Then for all r, H^{+}, K^{+} are free factors of $N(r)$ and $M(r)$ and $H \leq H^{+} \leq N(r)$, $K \leq K^{+} \leq M(r)$. Comparison with the splitting extension of $N(r)$ by the automorphism of order p^{r} of $N(r)$ that is the identity on $gp(b,c,\ldots,d)$ and is otherwise the "circular action" induced by the permutation $i \longmapsto i+1$ $(-p^{r-1}+1 \leq i < p^{r}-p^{r-1})$, $p^{r}-p^{r-1} \longmapsto -p^{r-1}+1$ shows that the natural homomorphism from U onto $U(r)=\langle U;a^{p^{r}}=1\rangle$ is one-to-one on $N(r)$. Similarly, the natural homomorphism from V onto $V(r)=\langle V;\bar{a}^{p^{r}}=1\rangle$ is one-to-one on $M(r)$.

In the case of $N(r)$, we simplify the notation by writing $N(r,j,t)$ instead of $N(r)(j,t)$ (see p. 3). The same will be done for $M(r)$, $U(r)$, $V(r)$. Note that $N(r,j,t)$, being fully invariant in $N(r)$, is a normal subgroup in $U(r)$. Similarly, $M(r,j,t)$ is a normal subgroup in $V(r)$.

Write now $H^{+}=S*(D\times E)$ and $K^{+}=\bar{S}*(\bar{D}\times\bar{E})$ where

$$S=\langle b,c,\ldots,d;\text{abel.}\rangle*\langle x(0),y(0),\ldots z(0);\text{abel.}\rangle,$$

$$D=\langle x(1)\rangle,\qquad E=\langle y(1),\ldots,z(1);\text{abel.}\rangle,$$

$$\bar{S}=\langle\bar{a},\bar{b},\ldots,\bar{c};\text{abel.}\rangle*\langle\bar{x}(0),\bar{y}(0),\ldots\bar{z}(0);\text{abel.}\rangle,$$

$$\bar{D}=\langle\bar{z}(1)\rangle,\qquad \bar{E}=\langle\bar{x}(1),\ldots,\bar{y}(1);\text{abel.}\rangle.$$

Then $H=gp(S,D)$ and $K=gp(\bar{S},\bar{D})$.

Proposition 21 applied to H^{+}, K^{+} yields that $H^{+}/H^{+}(j,t)$, $K^{+}/K^{+}(j,t)$ are finite p-groups,

$$\bigcap_{j,t} H^{+}(j,t)=1,\qquad \bigcap_{j,t} K^{+}(j,t)=1,$$

$$\bigcap_{j,t} HH^{+}(j,t)=H,\qquad \bigcap_{j,t} KK^{+}(j,t)=K.$$

Furthermore, since $H(j,t)\theta=K(j,t)$, $H\cap H^{+}(j,t)=H(j,t)$, $K\cap K^{+}(j,t)=K(j,t)$, the families $\{H^{+}(j,t)\}$, $\{K^{+}(j,t)\}$ are (H,K,θ)-compatible, i.e. $hH^{+}(j,t)\longmapsto(h\theta)K^{+}(j,t)$ is an isomorphism between the images of H and K in $H^{+}/H^{+}(j,t)$ and $K^{+}/K^{+}(j,t)$.

However, H^{+}, K^{+} are free product factors in $N(r)$, $M(r)$, so

$$N(r,j,t)\cap H^{+}=H^{+}(j,t),\qquad M(r,j,t)\cap K^{+}=K^{+}(j,t).$$

Hence $N(r)/N(r,j,t)$, $M(r)/M(r,j,t)$ are finite p-groups, and $N(r,j,t)$, $M(r,j,t)$ are (H,K,θ)-compatible.

Put now $U(r,j,t)=gp(a^{p^{r}},N(j,t))$, $V(r,j,t)=gp(\bar{d}^{p^{r}},M(j,t))$. Then $U/U(r,j,t)\cong U(r)/N(r,j,t)$, $V/V(r,j,t)\cong V(r)/M(r,j,t)$, therefore $\{U(r,j,t)\}$, $\{V(r,j,t)\}$ are (H,K,θ)-compatible families of the desired kind in U and V. Invoking Proposition

2 of [4] now completes the proof.

Section 5: The word problem for $\underline{B}$-groups.

Dyson and Mostowski proved (see e.g. [21]) that finitely presented residually finite groups have solvable word-problem. This yields immediately

Theorem 23:

$\underline{B}$-groups have solvable word-problem.

A more constructive proof can be obtained for this fact if we proceed along the lines of sec. 4.2 of [20] and further utilize the specific generalized free product structure of G.

Suppose an ordering is put on each of the sets $\{b,c,\dots,d\}$, $\{x,y,\dots,z\}$ and we adopt the ordering inherited under labeling by i on $\{x(i),y(i),\dots,z(i)\}$ $(i \in \mathbb{Z})$. Then every element in the free abelian groups on each of these sets has a unique normal form.

For given freely reduced $u \equiv u(a,b,\dots,c,d,x,y,\dots,z)$ perform the following procedure:

i) Attach label 0 to each occurrence of the letters $x,y,\dots,z$.

ii) Check if there are any a-powers in other than the left-most position. If so, pick one such occurrence and shift it to the left-most position while adding the exponent of the a-power on the move to each labeled letter traversed. Repeat until all occurrences of a are collected in the front. This yields the (unique) split-extension normal form of the element of G determined by u, $a^{\sigma}u'$ where $\sigma = \exp(a,u)$, $u' \in N$.

iii) Call a subword of u' homogeneous if it contains exactly one label value or else no labels at all. Then the syllables of u' are recognized as the maximal homogeneous subwords of u'. Put now each syllable of u' into its unique free abelian normal form and *reduce to obtain u''. The expression $a^{\sigma}u''$, termed the <u>proper form</u> of u, is uniquely determined by u, as a consequence of the splitting of U over the free product N of free abelian groups.

Similarly, after ordering the sets $\{\bar{a},\bar{b},\ldots,\bar{c}\}$, $\{\bar{x},\bar{y},\ldots,\bar{z}\}$ and adopting the inherited order on $\{\bar{x}(i),\bar{y}(i),\ldots,\bar{z}(i)\}$, we can follow an analogous procedure with $v \equiv v(\bar{a},\bar{b},\ldots,\bar{c},\bar{d},\bar{x},\bar{y},\ldots,\bar{z}) \in V$. Then, if $\exp(\bar{d},v)=\tau$, we end up with $\bar{d}^{\tau}v''$, the unique proper form of $v \in V$. Remembering that $H=(b,c,\ldots,d;\text{abel.})*(x(0),y(0),\ldots,z(0);\text{abel.})*(x(1))$ and $K=(\bar{a},\bar{b},\ldots,\bar{c};\text{abel.})*(\bar{x}(0),\bar{y}(0),\ldots,\bar{z}(0);\text{abel.})*(\bar{z}(1))$ we can state now

<u>Lemma 24</u>:

Given $u=u(a,b,\ldots,c,d,x,y,\ldots,z)$ we can decide in a finite number of steps whether $u \in H$ or $u \notin H$ holds. In the former case, we can also determine whether or not $u=1$ in H.
Similarly, given $v=(\bar{a},\bar{b},\ldots,\bar{c},\bar{d},\bar{x},\bar{y},\ldots,\bar{z})$ we can decide in a finite number of steps whether or not $v \in K$. If $v \in K$, we can

also determine if $v=1$ or not.

Proof

It is clearly sufficient to handle one of the statements, say for H. Now we only need to point out that since $H \leq N$, if $u=a^{\sigma}u^{*}$ in its proper form with $\sigma \neq 0$, we conclude $u \notin N$ hence $u \notin H$. When $\sigma=0$, i.e. $u=u^{*}$, the free product structure of N implies that $u \in H$ if and only if the set of integers occurring as labels in u^{*} is a subset of $\{0,1\}$ but no letter different from x has label 1. The fact now, that u^{*} is also the free product normal form of u for $u \in H$, takes care of the decision whether $u=1$ or not in H.

Repeated use of Lemma 24 accomplishes the solution of the word-problem in G as follows:

If $w=w(1)w(2)\ldots w(n)$ is an alternating product in $A*\bar{A}$, interpret it as an alternating product in $U*V$. Apply the method of Lemma 24 to each of the syllables $w(i)$ $(i=1,2,\ldots,n)$. $rl(w) \geq 1$ hence $w \neq 1$ in G if for some i $w(i) \notin H \cup K$. Otherwise, if $w(i) \in H \cup K$ for all $i=1,2,\ldots,n$, end up with each syllable expressed by its respective H- or K-normal form. Apply θ^{-1} to each K-syllable, i.e. replace every occurrence of $\bar{a}$ by $x(0)$, $\bar{b}$ by $y(0)$, $\ldots$, $\bar{c}$ by $z(0)$, $\bar{x}(0)$ by b, $\bar{y}(0)$ by c, $\ldots$, $\bar{z}(0)$ by d, $\bar{z}(1)$ by $x(1)x(0)^{-1}[b,y(0)]\ldots[c,z(0)]d$. This turns w into an H-product hence we can determine whether $w=1$ or not.

Section 6: The derived group of a β-group is locally free.

Still looking for ties between surface groups and β-groups, in this section we point out some similarities between their derived groups. We present the proof of the statements in this section for the four-generator case only; the notation for the corresponding proofs for the β-groups with more generators becomes very cumbersome.

The following two lemmas will be helpful in the sequel.

Lemma 25:

Let F be a free group on the set $\{x(j),y(j) \mid j\in Z\}$. For $j\in Z$, let $u(j)=y(j)y(j+1)^{-1}x(j)$, $v(j)=x(j)^{-1}y(j)x(j+1)$. Then the set $\{u(j),v(j) \mid j\in Z\}$ freely generates a free subgroup of F.

Proof

By an induction on ℓ, the number of (u,v)-segments in a (u,v)-product, we want to show that the spelling of every reduced (u,v)-product is a reduced (x,y)-product and therefore the set $\{u(j),v(j) \mid j\in Z\}$ is a free basis for the free subgroup it generates.

i) First we note that $u(j)$, $v(j)$ are cyclically reduced (x,y)-products therefore no x or y symbols cancel upon forming $u(j)^m$, $v(j)^n$ $(j\in Z)$, $(m\neq 0,\ n\neq 0)$, hence $u(j)^m\neq 1$, $v(j)^n\neq 1$ $(m,n\neq 0)$ so $w\neq 1$ if $\ell(w)=1$.

ii) Suppose now that $\ell(w)=2$ and

$$w = u(j)^{\pm 1}\, v(h)^{\pm 1} \qquad \text{or} \qquad w = v(h)^{\pm 1}\, u(j)^{\pm 1} .$$

Then the spelling of w is one of the expressions

$$\begin{array}{ll}
y(j)\ \ y(j+1)^{-1}x(j) \quad\ x(h)^{-1}\ \ y(h)\ \ x(h+1) & , \\
y(j)\ \ y(j+1)^{-1}x(j) \quad\ x(h+1)^{-1}y(h)^{-1}x(h) & , \\
x(j)^{-1}y(j+1)\ \ y(j)^{-1} \quad\ x(h)^{-1}\ \ y(h)\ \ x(h+1) & , \\
x(j)^{-1}y(j+1)\ \ y(j)^{-1} \quad\ x(h+1)^{-1}y(h)^{-1}x(h) & , \\
x(h)^{-1}\ \ y(h)\ \ x(h+1)\ \ y(j)\ \ y(j+1)^{-1}x(j) & , \\
x(h+1)^{-1}y(h)^{-1}x(h) \quad\ y(j)\ \ y(j+1)^{-1}x(j) & , \\
x(h)^{-1}\ \ y(h)\ \ x(h+1)\ \ x(j)^{-1}y(j+1)\ \ y(j)^{-1} & , \\
x(h+1)^{-1}y(h)^{-1}x(h) \quad\ x(j)^{-1}y(j+1)\ \ y(j)^{-1} & .
\end{array}$$

When $j=h$ then $j+1 \neq h$, therefore if there is cancellation at all in the first expression on the left-hand-side or in the last expression of the right-hand-side, it is limited to the middle ($x(j)x(h)^{-1}$, $x(h)x(j)^{-1}$, respectively). Similarly, when $j=h+1$, then $j+1 \neq h$, therefore if there is cancellation at all in the second expression on the left-hand-side or in the third expression on the right-hand-side, it is limited to the middle ($x(j)x(h+1)^{-1}$, $x(h+1)x(j)^{-1}$, respectively).

If $w=u(j)^{-1}u(h)^{-1}$ or $w=v(h)^{-1}v(j)^{-1}$ then the spelling of w is one of

$$\begin{array}{ll}
y(j)\ \ y(j+1)^{-1}x(j) \quad\ y(h)\ \ y(h+1)^{-1}x(h) & , \\
y(j)\ \ y(j+1)^{-1}x(j) \quad\ x(h)^{-1}y(h+1)\ \ y(h)^{-1} & , \\
x(j)^{-1}y(j+1)\ \ y(j)^{-1} \quad\ y(h)\ \ y(h+1)^{-1}x(h) & , \\
x(j)^{-1}y(j+1)\ \ y(j)^{-1} \quad\ x(h)^{-1}y(h+1)\ \ y(h)^{-1} & ,
\end{array}$$

$$x(h)^{-1}\ y(h)\ x(h+1)\ x(j)^{-1}y(j)\ x(j+1)\ ,$$
$$x(h)^{-1}\ y(h)\ x(h+1)\ x(j+1)^{-1}y(j)^{-1}x(j)\ ,$$
$$x(h+1)^{-1}y(h)^{-1}x(h)\quad x(j)^{-1}y(j)\ x(j+1)\ ,$$
$$x(h+1)^{-1}y(h)^{1}x(h)\quad x(j+1)^{-1}y(j)^{-1}x(j)\ .$$

We may assume $h\neq j$ since otherwise i) applies. Now if $h\neq j$ then $h+1\neq j+1$ either, therefore cancellation is possible only in the very first expression of the right-hand-side (in the case $h+1=j$) and is limited to the middle $x(h+1)x(j)^{-1}$. Thus $w\neq 1$ when $\ell(w)=2$ follows.

iii) The detailed analysis above also yields that the last two and first two letters of the considered products never cancel. Therefore, upon joining a new u- or v-segment to a reduced (u,v)-product w' to obtain the reduced product w, $\ell(w)>\ell(w')+2$ (ℓ stands for the length with respect to the basis $\{x(j),y(j)\mid j\in Z\}$). Thus the induction works and completes the proof.

<u>Lemma 26</u>: Let h be a fixed non-positive integer,

$$S(h)=<\bigcup_{i\geq h}\{x(i,j),y(i,j)\mid j\in Z\};$$
$$\bigcup_{i\geq h}\{x(i+1,j)=u(i,j),y(i+1,j)=v(i,j)\mid j\in Z\}>,\quad\text{where}$$

$u(i,j)=y(i,j)y(i,j+1)^{-1}x(i,j)$, $\quad v(i,j)=x(i,j)^{-1}y(i,j)x(i,j+1)$ $(j\in Z, i\geq h)$.

Then $S(h)$ is a free group freely generated by $\{x(h,j),j(h,j)\mid j\in Z\}$.

Proof

We will proceed by induction. For $i=h$, the relations are

$$x(h+1,j)=y(h,j)y(h,j+1)^{-1}x(h,j) \quad \text{and}$$
$$y(h+1,j)=x(h,j)^{-1}y(h,j)x(h,j+1) \quad (j\in Z) .$$

With their use, we can replace every (x,y)-product whose largest first label is in the set $\{h,h+1\}$ by an expression on $\{x(h,j),y(h,j)\mid j\in Z\}$. Let $k>h$ be an integer and assume inductively that, with the help of all relations corresponding to $h\leq i<k$, $j\in Z$, we have successively replaced every (x,y)-product whose largest first label does not exceed k by some expression involving only $\{x(h,j),y(h,j)\mid j\in Z\}$. Now the relations

$$\{x(k+1,j)=y(k,j)y(k,j+1)^{-1}x(k,j) \quad \text{and}$$
$$y(k+1,j)=x(k,j)^{-1}y(k,j)x(k,j+1) \quad (j\in Z)\}$$

allow us to express every word w whose largest first label is $k+1$, as a product of symbols with first labels not exceeding k, and thus, by the inductive assumption, also as an $\{x(h,j)y(h,j)\mid j\in Z\}$-product.

Doing this replacement procedure for $k=h+1,h+2,\ldots$ we use all the relations given for $S(h)$. Subsequently, we can apply a Tietze transformation that eliminates all the "unnecessary" generating symbols $\{x(h,j),y(h,j)\mid i>h, j\in Z\}$ and all the relations expressing them in terms of the rest of the generators. Hence $S(h) = \langle x(h,j),y(h,j)\mid j\in Z\rangle$, indeed.

We can now prove (the limited version of)

Theorem 27:

Suppose that G is a $2k$-generator $\mathfrak{B}$-group. Then G is the splitting extension of a locally free group by a free abelian group of rank k. In particular, the group

$$G = \langle a, b, \bar{a}, \bar{b} \; ; \; [a,b]=1, [\bar{a},\bar{b}]=1, [a,\bar{a}][b,\bar{b}]=1 \rangle$$

is the (splitting) extension of a locally free group by a free abelian group of rank two.

Proof

We adopt the convention $[u,v]=uvu^{-1}v^{-1}$ for calculational-notational ease. Put $x=\bar{a}a^{-1}$, $y=\bar{b}b^{-1}$, $B=gp_G(x,y)$. To prove the theorem, we show that

i) G/B is free abelian of rank two;

ii) B is the direct limit of free groups of countably infinite ranks.

To see i) observe that the mapping $a \mapsto a$, $\bar{a} \mapsto a$, $b \mapsto b$, $\bar{b} \mapsto b$ of G into itself takes each of the relators of G to the identity therefore it defines a homomorphism ϕ of G into itself. Also,

$$G\phi = \langle a, b, \bar{a}, \bar{b} \; ; \; [a,b]=1, [\bar{a},\bar{b}]=1, [a,\bar{a}][b,\bar{b}]=1, a=\bar{a}, b=\bar{b} \rangle =$$
$$= \langle a, b \; ; \; [a,b]=1 \rangle = G/B,$$

proving part i).

To see now ii), we apply first the Reidemeister-Schreier rewriting process with transversal $\{a^i b^j \mid i,j \in Z\}$ and obtain generators for B:

$$\left.\begin{aligned} x(i,j) &= a^i b^j \ \bar{a}a^{-1} \ b^{-j} a^{-i} \\ y(i,j) &= a^i b^j \ \bar{b}b^{-1} \ b^{-j} a^{-i} \end{aligned}\right. \qquad (i,j \in Z)$$

and observe that the actions of a and b amount to shifts in the labels:

$$a\, x(i,j)\, a^{-1} = x(i+1,j) , \quad b\, x(i,j)\, b^{-1} = x(i,j+1) ,$$
$$a\, y(i,j)\, a^{-1} = y(i+1,j) , \quad b\, y(i,j)\, b^{-1} = y(i,j+1) , \quad (i,j \in Z) .$$

Using this, we have

$$\begin{aligned} [\bar{a},\bar{b}] = \bar{a}\bar{b}\bar{a}^{-1}\bar{b}^{-1} &= \bar{a}(a^{-1}a)\bar{b}(b^{-1}b)(a^{-1}a)\bar{a}^{-1}(b^{-1}b)\bar{b}^{-1} = \\ &= (\bar{a}a^{-1})a(\bar{b}b^{-1})ba^{-1}(a\bar{a}^{-1})b^{-1}(b\bar{b}^{-1}) = \\ &= x(0,0)(ay(0,0)a^{-1})(bx(0,0)^{-1}b^{-1})y(0,0)^{-1} = \\ &= x(0,0)y(1,0)x(0,1)^{-1}y(0,0)^{-1} \ \text{ and} \end{aligned}$$

$$\begin{aligned} [a,\bar{a}][b,\bar{b}] = a\bar{a}a^{-1}\bar{a}^{-1}b\bar{b}b^{-1}\bar{b}^{-1} &= a(\bar{a}a^{-1})a^{-1}(a\bar{a}^{-1})b(\bar{b}b^{-1})b^{-1}(b\bar{b}^{-1}) = \\ &= x(1,0)x(0,0)^{-1}y(0,1)y(0,0)^{-1} \end{aligned}$$

leading to the presentation

$$B = \left\langle \bigcup_{i \in Z} \{x(i,j), y(i,j) \mid j \in Z\} \ ; \right.$$
$$\left. \bigcup_{i \in Z} \{x(i+1,j) = u(i,j) ,\ y(i+1,j) = v(i,j) \mid j \in Z\} \right\rangle .$$

Here, as before, for $i,j \in Z$,

$$u(i,j) = y(i,j)y(i,j+1)^{-1}x(i,j) , \quad v(i,j) = x(i,j)^{-1}y(i,j)x(i,j+1) .$$

For $h \leq 0$ integer, put

$$S(h) = \left\langle \bigcup_{i \geq h} \{x(i,j)y(i,j) \mid j \in Z\} \ ; \right.$$
$$\left. \bigcup_{i \geq h} \{x(i+1,j) = u(i,j) ,\ y(i+1,j) = v(i,j) \mid j \in Z\} \right\rangle .$$

and put S for the subgroup of $S(-1)$ generated by its subset

$\{u(-1,j), v(-1,j) \mid j \in Z\}$. By Lemma 26, $S(0)$ and $S(-1)$ are both free groups with free bases $\{x(0,j), y(0,j) \mid j \in Z\}$ and $\{x(-1,j), y(-1,j) \mid j \in Z\}$, respectively. Therefore the set mapping $x(0,j) \longmapsto u(-1,j)$, $y(0,j) \longmapsto v(-1,j)$ $(j \in Z)$ extends to a unique homomorphism $\phi_0 : S(0) \longmapsto S(-1)$ and $S(0)\phi_0 \leq S$. Lemma 25, applied to $S(-1)$, yields that $\{x(0,j)\phi_0, y(0,j)\phi_0 \mid j \in Z\}$ is a free basis for its subgroup S so the mapping $x(0,j)\phi_0 \longmapsto x(0,j)$, $y(0,j)\phi_0 \longmapsto y(0,j)$ $(j \in Z)$ from S into $S(0)$ also extends to a unique homomorphism η of S. But $\eta\phi_0$ is just the restriction of the identity of $S(-1)$ to its subgroup S and $\phi_0\eta$ is the identity on $S(0)$. Thus $S(0)$ can be identified with its injected image S in $S(-1)$ via the connecting map ϕ_0. We can now repeat the same argument for all $h<0$ and conclude that $S(h)$ can be identified with its injected image in $S(h-1)$ under the connecting map (from $S(h)$ into $S(h-1)$)

$$\phi_h : x(h,j) \longmapsto u(h-1,j), \quad y(h,j) \longmapsto v(h-1,j) .$$

This shows that the direct limit $L = \bigcup_{-h=0}^{\infty} \uparrow S(h)$ of the free groups $S(h)$ under the family of connecting maps ϕ_h $(h=0,1,\ldots)$ exists. The presentation coming from the direct limit structure of L is exactly the presentation of B:

$$
\begin{aligned}
L = {} & \langle S(0), \bigcup_{h<0} \{x(h,j), y(h,j) \mid j \in Z\} \ ; \\
& \qquad \bigcup_{h<0} \{x(h,j)=u(h-1,j), y(h,j)=v(h-1,j) \mid j \in Z\} \rangle \\
= {} & \langle \bigcup_{i\geq 0} \{x(i,j), y(i,j) \mid j \in Z\} \ \bigcup_{h<0} \{x(h,j), y(h,j) \mid j \in Z\} \ ; \\
& \bigcup_{i\geq 0} \{x(i+1,j)=u(i,j), y(i+1,j)=v(i,j) \mid j \in Z\} \\
& \bigcup_{h\leq 0} \{x(h,j)=u(h-1,j), y(h,j)=v(h-1,j) \mid j \in Z\} \rangle \\
= {} & \langle \bigcup_{i\in Z} \{x(i,j), y(i,j) \mid j \in Z\} \\
& \bigcup_{i\geq 0} \{x(i+1,j)=u(i,j), y(i+1,j)=v(i,j) \mid j \in Z\} \\
& \bigcup_{i\leq 0} \{x(i+1,j)=u(i,j), y(i+1,j)=v(i,j) \mid j \in Z\} \rangle \\
= {} & \langle \bigcup_{i\in Z} \{x(i,j), y(i,j) \mid j \in Z\} \\
& \bigcup_{i\in Z} \{x(i+1,j)=u(i,j), y(i+1,j)=v(i,j) \mid j \in Z\} \rangle \ .
\end{aligned}
$$

Thus B is locally free. As a consequence of Theorem 27, the derived group of G is locally free, indeed.

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