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A time domain approach to robust control of continuous time and delta operator systems

Piou, Jean Eugène, Ph.D.

City University of New York, 1993

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A TIME DOMAIN APPROACH TO ROBUST CONTROL OF CONTINUOUS TIME AND DELTA OPERATOR SYSTEMS

BY

JEAN EUGENE PIOU

A dissertation submitted to the Graduate Faculty in Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1993

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

A TIME DOMAIN APPROACH TO ROBUST CONTROL FOR CONTINUOUS TIME AND DELTA OPERATOR SYSTEMS

By

Jean Eugène Piou

Adviser: Professor Kenneth M. Sobel

Eigenstructure assignment is a method which allows the incorporation of classical specifications on damping, settling time, and mode decoupling into a modern multivariable control framework. However, this design approach does not guarantee stability when the plant is subject to structured state space uncertainty. In systems such as missiles and aircraft, this uncertainty may arise from errors in the identification method which is used to obtain the linearized models of the plant.

A new robust stability condition based on bounds of the integral norm of the state and output is derived. This new sufficient condition is valid for linear time invariant continuous time systems which are subject to linear time varying structured state space uncertainty and unmodelled dynamics.

In collaboration with a colleague, a new sufficient condition for robust stability of a linear time invariant continuous time system which is subject to time varying structured state space uncertainty is obtained. This new result uses the so-called modal decomposition method. A new robust eigenstructure assignment design method is developed in which the desired closed loop eigenvalues are constrained to lie within chosen regions.

Eigenstructure assignment is extended to linear time invariant sampled data systems which are represented by the so-called unified delta model. The unified delta model is valid for both continuous time and sampled data operation of the plant. Sufficient conditions for robust stability based on modal decomposition and Lyapunov methods are extended to the delta model. Robust sampled data controllers are designed for the Extended Medium Range Air to Air Technology missile and the Flight Propulsion Control Coupling aircraft.

The robust stability conditions are extended to obtain performance robustness conditions. The closed loop eigenvalues of the unified delta system are guaranteed to lie within chosen damping-settling time regions for all specified time invariant structured state space uncertainty. Finally, the robust stability conditions are extended to multiple input rate fixed output rate sampled data systems.

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There are no words to express my sincere gratitude to Prof. K.M. Sobel for his advice, guidance and friendship during this intense and rewarding period of research. His patience and encouragement during hard times are much appreciated.

I would like to thank Profs. Thau, Kranc, and Combettes for the advice and training which they provided during my graduate studies. I also thank Dr. W. Yu for his collaboration and his suggestions which illuminated this thesis.

My aspiration to engage in research was fostered early on by my father Eugène Piou and my brother Jean Enard Piou who had reputations among their closest associates as being a natural philosopher and a scientist, respectively.

I have had much help in this effort, not the least of which has come from my wife Emose whose efforts have kept our home life on track.

I am grateful to the Patricia Harris Fellowship Committee and the Magnet Dissertation Year Fellowship Committee for sponsoring my research.

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1. INTRODUCTION

1.1. BACKGROUND

Eigenstructure assignment is a method which allows the incorporation of classical specifications on damping, settling time, and mode decoupling into a modern multivariable control framework. However, this design approach does not guarantee stability when the plant is subject to structured state space uncertainty. In systems such as missiles and aircraft, this uncertainty may arise from errors in the identification method which is used to obtain the linearized models of the plant.

The first result of this research is an extension of the sufficient condition for robust stability proposed by Sobel et. al.[1] to include simultaneous time varying structured state uncertainty and unmodelled dynamics. A new robustness result based on bounds of the integral norm of the state and output is presented.

In collaboration with Yu [2], we developed a new robust stability condition for a linear time invariant (LTI) continuous time system which is subject to linear time varying structured state space uncertainty. This sufficient condition is a sum of terms each of which involves the i-th right closed loop eigenvector, the i-th left closed loop eigenvector, and the real part of the i-th closed loop eigenvalue. This robustness result is less conservative than the earlier result proposed by Sobel et. al. [1] which is based upon the Gronwall lemma.

We have extended eigenstructure assignment to linear time invariant plants which are represented by Middleton and Goodwin's [3] unified delta model which is valid for both continuous time and sampled data operation of the plant. An important property of the delta model is that the discrete time eigenvalues approach the continuous time eigenvalues as the sampling period approaches zero. We show that the eigenvectors of the delta model are identical to the eigenvectors of the continuous time plant and an expression for the eigenstructure assignment feedback gain matrix is derived. A sufficient condition for the robust stability of a linear time invariant unified delta plant subject to linear time invariant structured state space uncertainty is obtained. This yields a new unified robustness condition which is applicable to both continuous time and sampled data operation. A robust sampled data controller is designed for the Extended Medium Range Air to Air Technology (EMRAAT) missile.

We have extended the robust stability sufficient condition of Yedavalli [4], which uses a Lyapunov approach, to a unified delta model which is valid for both continuous time and sampled data operation of the plant. A robust sampled data eigenstructure assignment flight control law is designed for the yaw pointing/lateral translation maneuver of the Flight Propulsion Control Coupling (FPCC) aircraft.

We present three sufficient conditions for robust performance of a linear time invariant unified delta plant. The closed loop eigenvalues are required to lie within chosen performance regions when the plant is subject to time invariant structured state space uncertainty. The first condition uses a modal decomposition approach, the second is derived from the Lyapunov stability method and the third uses the concept of matrix measure.

We have extended eigenstructure assignment to the class of multi-rate sampled data systems with multiple input rate and fixed output rate (MIFO) proposed by Araki and Hagiwara [5]. We have obtained new robustness conditions using both modal decomposition and Lyapunov approaches.

1.2 **HISTORICAL REVIEW**

1.2.1 EIGENSTRUCTURE ASSIGNMENT FOR LINEAR SYSTEMS

The important results and techniques of eigenstructure assignment were presented by Andry et. al.[6]. The problem described in Ref.6 can be summarized as follows:

Consider an LTI system described by the following equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where

(1) $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^r$;

(2) A,B, C are real constant matrices;

(3) $\mathrm{rank}(B) = m \not\equiv 0,\ \mathrm{rank}(C) = r \not\equiv 0$;

The following theorem has been proposed by Srinathkumar [7]:

Theorem [7]:

Given the controllable and observable system described by Eqs.(1) and (2), max(m,r) closed loop eigenvalues can be assigned and max(m,r) eigenvectors (or reciprocal vectors by duality) can be partially assigned with min(m,r) entries in each vector arbitrarily chosen using constant gain output feedback.

Andry et. al. [6] concluded that the eigenvector v_i must be in the subspace spanned by the columns of $(\lambda_i I-A)^{-1}B$ and this subspace is of dimension m which is equal to the number of independent control variables. Therefore, if we choose an eigenvector v_i which lies precisely in the subspace spanned by the columns of $(\lambda_i I-A)^{-1}B$, it will be achieved exactly.

In general, however a desired eigenvector v_i^d will not reside in the prescribed subspace and hence cannot be achieved. Ref.6 gives a way to find the "best possible" choice for an achievable eigenvector. This best possible eigenvector is the projection of v_i^d onto the subspace spanned by the columns of $(\lambda_i I-A)^{-1}B$ (in a least square sense).

In many practical situation, complete specification of v_i^d

is neither required nor known but rather the designer is interested only in certain elements of the eigenvectors. Ref.6 discusses how to choose desired eigenvectors with an emphasis on mode decoupling. However, this approach does not guarantee stability when the plant is subject to structured state space uncertainty. This area has been studied in our research work and is discussed in chapter 2.

Recently, Sobel and Cloutier [8] applied eigenstructure assignment to the design of an autopilot for the EMRAAT missile. An important difference between this application and other eigenstructure assignment applications which have appeared in the literature is that the lateral dynamics of the EMRAAT missile does not have a well defined dutch roll mode. Therefore, eigenstructure assignment is utilized not only for mode decoupling, but also to create distinctly separate dutch roll and roll modes. Sobel and Cloutier [8] use the approach suggested by Andry et. al.[6] in which the i-th desired eigenvector v_i^d is chosen for mode decoupling. Then, the i-th achievable eigenvector v_i^a is chosen as the projection of v_i^d onto the so-called achievability subspace. Sobel and Cloutier [8] show that their design achieves improved decoupling between an initial sideslip angle and the integrated roll rate (which is approximately equal to the bank angle) when compared to a linear quadratic regulator design proposed by Bossi and Langehough [9].

However, the design of Sobel and Cloutier [8] does not consider that the missile's aerodynamic parameters are uncertain.

1.2.2 FEEDFORWARD CONTROL

O'Brien and Broussard [10] developed a feedforward control technique which enables a plant to perfectly track a model. Sobel and Shapiro [11] developed a methodology for pitch pointing flight control systems which uses eigenstructure assignment and command generator tracking by applying the results of Ref.10 to the special case in which the command is restricted to be a step. In Ref.11, the feedback gains are computed by using eigenstructure assignment. The desired eigenvalues are chosen to obtain the desired settling time and the desired eigenvectors are chosen to decouple the pitch attitude and flight path angle of an aircraft. Feedforward gains are computed which ensure steady-state tracking of a pilot's command.

1.2.3 PSEUDO CONTROL

Sobel and Lallman [12] proposed a pseudo control strategy for reducing the dimension of the control space by using the singular value decomposition. After the design is complete, the controller is mapped back into the original control space. Later, Sobel and Shapiro [13] extended the pseudo control strategy to allow the designer more freedom by adjusting a two dimensional parameter denoted by α. Sobel and Shapiro [13] applied the extended pseudo control strategy to design a yaw pointing/lateral translation control law for the FPCC aircraft.

1.2.4 MULTI-RATE SYSTEMS

Multi-rate sampled-data control systems have been studied by Kranc [14] in 1957. The design freedom offered by the periodic operation of multi-rate systems led Araki and Hagiwara [5] to a class of multi-rate sampled data systems. This class consists of systems with multiple input rate and fixed output rate (MIFO) sampling. Later, Patel et. al.[15] used the result of Ref.5 to design an insensitive multi-rate aircraft controller. In our research we extend the result of Araki and Hagiwara [5] to the delta model with time invariant uncertainty.

1.2.5 ROBUST CONTROL FOR LINEAR SYSTEMS WITH STATE SPACE UNCERTAINTY

Chen and Wong [16] introduced robust stability sufficient conditions in the time domain for linear time invariant systems with linear or nonlinear time-varying uncertainties. In Ref.16, the Gronwall lemma is used to derive robust stability conditions which are based on the upper norm bounds of the uncertainties. The parameters of a dynamic controller are selected to satisfy the requirements of robust stability under plant uncertainties.

Consider the system described by

$$\dot{x} = Ax + Bu + \Delta A(x) + \Delta B(u) \tag{3a}$$

$$y = Cx + Du + \Delta C(x) + \Delta D(u) \tag{3b}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^r$ is the output vector,and A, B, C and D are constant matrices. $\Delta A(x)$, $\Delta B(u)$, $\Delta C(x)$, and $\Delta D(u)$ are nonlinear time-varying parametric uncertainties with the following known upper norm-bounds.

$$\|\Delta A(x)\|_1 \le \beta_1 \|x\|_1, \quad \|\Delta B(u)\|_1 \le \beta_2 \|u\|_1 \tag{3c,3d}$$

$$\|\Delta C(x)\|_1 \le \beta_3\|x\|_1, \quad \|\Delta D(u)\|_1 \le \beta_4\|u\|_1 \qquad (3e,3f)$$

where the 1-norm of a real vector $x \in \mathbb{R}^n$ is

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

and the 1-norm of a matrix A is

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |A_{ij}|$$

and the dynamic controller has the following structure:

$$\dot{x}_r = A_r x_r + B_r y \qquad (4a)$$

$$u = K_1 x_r + K_2 x \qquad (4b)$$

where $x_r \in \mathbb{R}^{n_1}$. Notice that this design method requires both state feedback and dynamic feedback. From Eqs. (3a), (3b), (4a) and (4b):

$$\begin{bmatrix} \dot{x} \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A+BK_2 & BK_1 \\ B_rC+B_rDK_2 & A_r+B_rDK_1 \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + \begin{bmatrix} \Delta A(x)+\Delta B(u) \\ B_r[\Delta C(x)+\Delta D(u)] \end{bmatrix} \qquad (5a)$$

and

$$y = [C+DK_2 \quad DK_1]\begin{bmatrix} x \\ x_r \end{bmatrix} + [\Delta C(x)+\Delta D(u)] \tag{5b}$$

In Ref.16, the following definitions are used:

$$\bar{x} = \begin{bmatrix} x \\ x_r \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A+BK_2 & BK_1 \\ B_rC+B_rDK_2 & A_r+B_rDK \end{bmatrix}, \quad \bar{C} = [C+DK_2 \quad DK_1],$$

$$\Delta\bar{A}(\bar{x}) = \begin{bmatrix} \Delta A(x)+\Delta B(u) \\ B_r[\Delta C(x)+\Delta D(u)] \end{bmatrix}, \quad \Delta\bar{C}(\bar{x}) = [\Delta C(x)+\Delta D(u)]$$

Then, the closed loop system with parametric uncertainties can be represented as

$$\dot{\bar{x}} = \bar{A}\bar{x} + \Delta\bar{A}(\bar{x}) \tag{6a}$$

$$\bar{x}(0) = \begin{bmatrix} x_o \\ x_{ro} \end{bmatrix} \tag{6b}$$

$$y = \bar{C}\bar{x} + \Delta\bar{C}(\bar{x}) \tag{6c}$$

and the closed loop feedback system is given by

$$\dot{\bar{x}} = \bar{A}\bar{x} \tag{7a}$$

$$\bar{x}(0) = \begin{bmatrix} x_o \\ x_{ro} \end{bmatrix} \tag{7b}$$

$$y = \bar{C}\bar{x} \tag{7c}$$

Define the state transition matrix $\Phi(t)$ of Eq.(7a) as

$$\Phi(t) = \exp(\bar{A}t) \tag{8a}$$

and suppose

$$\|\Phi(t)\|_1 \le m \exp(-\alpha t) \tag{8b}$$

for some constants $m > 0$, $\alpha > 0$.

Theorem[16]

For the system with nonlinear parametric uncertainty described by Eqs.(3c)-(3f). If the control parameters of Eqs.(4a) and (4b) are chosen such that the nominal closed loop system described by Eqs.(7a)-(7c) is asymptotically stable and the following inequality is satisfied:

$$\alpha > m \{ \beta_1 + (\beta_2 + \beta_4 \|B_r\|_1)(\| [K_2 \; K_1] \|_1) + \beta_3 \|B_r\|_1 \} \tag{9}$$

where $\alpha = - \max_i \mathrm{Re}[\lambda_i(\bar{A})]$

Then, the nonlinear parametric perturbed closed loop system defined by Eqs.(6a)-(6c) is also asymptotically stable.

Next, consider the system described by

$$\dot{x} = Ax + Bu \tag{10a}$$

$$y = Cx + Du + \Delta h * u \tag{10b}$$

where the operator * denotes convolution. The structural model uncertainty Δh is linear time-varying but bounded by the following inequality.

$$\|\Delta h(t)\|_1 \leq \gamma \exp(-\beta t).$$

A theorem in Ref.16 states that if the nominal closed system (which is not shown here) is asymptotically stable and if

$$1 - \frac{1}{\alpha \beta} \|B_r\|_1 \ \| [K_2 \ K_1] \|_1 > 0 \tag{11}$$

then, the system described by Eqs.(10a) and (10b) is also asymptotically stable.

The sufficient condition of Eq.(9) for the stability robustness of linear systems with time varying norm bounded state space uncertainty was extended by Sobel et.al.[1] to include the structure of the uncertainty. The result of Ref.1 requires that the nominal eigenvalues lie to the left of a vertical line in the complex plane which is determined by a norm involving the structure of the uncertainty and the nominal closed loop eigenvector matrix. Therefore, this robustness result is especially well suited to the design of control systems using eigenstructure assignment. Ref.1 also considers the use of Perron weightings to reduce conservatism. The main result of Ref.1 can be summarized as follows:

Consider a nominal linear time invariant multi-input multi-output system described by Eqs.(1) and (2). Suppose that the nominal system is subject to linear time-varying uncertainty in the entries of A and B described by $\Delta A(t)$ and $\Delta B(t)$ respectively. Then, the system with uncertainty is given by

$$\dot{x} = Ax + Bu + \Delta A(t)x + \Delta B(t)u \tag{12a}$$

$$y = Cx \tag{12b}$$

Further, suppose that bounds are available on the absolute values of the maximum variations in the elements of $\Delta A(t)$ and $\Delta B(t)$. That is

$$|\Delta a_{ij}(t)| \le (a_{ij})_{max};\ i=1,\dots,n;\ j=1,\dots,n \tag{13a}$$

$$|\Delta b_{ij}(t)| \le (b_{ij})_{max};\ i=1,\dots,n;\ j=1,\dots,m \tag{13b}$$

Define $\Delta A^{+}(t)$ and $\Delta B^{+}(t)$ as the matrices obtained by replacing the entries of $\Delta A(t)$ and $\Delta B(t)$ by their absolute values. Also, define A_{max} and B_{max} as the matrices with entries $(a_{ij})_{max}$ and $(b_{ij})_{max}$, respectively. Then

$$\{\ \Delta A(t):\ \Delta A^{+} \le A_{max}\ \} \tag{14a}$$

$$\{\ \Delta B(t):\ \Delta B^{+} \le B_{max}\ \} \tag{14b}$$

where "≤" is applied element by element to matrices and $A_{max} \in \mathbb{R}_{+}^{n\times n}$ and $B_{max} \in \mathbb{R}^{n\times m}$ where $\mathbb{R}_{+}$ is the set of non-negative numbers.

Consider the constant gain output feedback control law described by

$$u(t) = Fy(t) \tag{15}$$

Then, the nominal closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) \tag{16}$$

and the uncertain closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) + [\Delta A(t) + \Delta B(t)FC]x(t) \tag{17}$$

Problem [1]:

Given a feedback gain matrix $F \in \mathbb{R}^{m\times r}$ such that the nominal closed loop system exhibits desirable dynamic performance, determine if the uncertain closed loop system is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by Eqs.(14a) and (14b).

Theorem [1]:

Suppose that F is such that the nominal closed loop system described by Eq.(16) is asymptotically stable with distinct eigenvalues. Then, the closed loop system with uncertainty given by Eq.(17) is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by Eqs.(14a) and (14b) if

$$\alpha > \| (M^{-1})^{+}[A_{max} + B_{max}(FC)^{+}]M^{+} \|_2 \tag{18}$$

where

$$\alpha = -\max_{i} \mathrm{Re}[\lambda_i(A + BFC)]$$

and M is a modal matrix of (A + BFC).

This theorem presents a sufficient condition for the robust stability in terms of the eigenstructure of the nominal closed loop system. Robust stability is ensured provided that the nominal closed loop eigenvalues lie to the left of a vertical line in the complex plane which is determined by the nominal closed loop modal matrix and the structure of the uncertainty.

The conservatism inherent in the sufficient condition has been reduced by the use of a diagonal weighting. Let D be a diagonal matrix with real non-negative entries. Then, given a nominal closed loop modal matrix M, another valid modal matrix is MD. This is because the matrix D just serves to scale the nominal closed loop eigenvectors. Therefore, Eq.(18) may be replaced by the sufficient condition given by

$$\alpha > \| D^{-1}(M^{-1})^{+}[A_{max} + B_{max}(FC)^{+}]M^{+}D \|_{2} \tag{19}$$

or equivalently by the D-weighted 2-norm described by

$$\alpha > \| (M^{-1})^{+}[A_{max} + B_{max}(FC)^{+}]M^{+}\|_{2D} \tag{20}$$

Ref.1 uses the following lemma which gives the infimum of the right hand side of Eq.(20). This lemma is attributed to Stoer and Witzgall [17].

Lemma [17]:

Define the Perron eigenvalue of the non-negative matrix A^{+}, denoted by $\pi(A^{+})$, to be the real non-negative eigenvalue

$\lambda_{max} \geq 0$ such that $\lambda_{max} \geq |\lambda_i|$ for all eigenvalues $(i=1,\ldots,n)$ of A^+. Then,

$$\inf_D \| A^+ \|_{2D} = \pi(A^+) \tag{21}$$

Since the matrix inside the norm on the right hand side of Eq.(20) is a non-negative matrix, the sufficient condition becomes

$$\alpha > \inf_D \|(M^{-1})^+[A_{max} + B_{max}(FC)^+]M^+\|_{2D} \tag{22}$$

or by using the lemma

$$\alpha > \pi[(M^{-1})^+[A_{max} + B_{max}(FC)^+]M^+] \tag{23}$$

By taking the infimum on the right hand side of Eq.(20), the vertical line in the complex plane has been placed as close to the imaginary axis as possible with a real positive diagonal weighting. Recall that the sufficient condition for robust stability requires that the nominal eigenvalues lie to the left of this vertical line. In this sense, Ref.1 has reduced the conservatism of the sufficient condition.

1.2.4 PERFORMANCE ROBUSTNESS

An important criterion for the design of a controller for an LTI system with uncertainty is performance robustness. Assigning eigenvalues of systems under perturbations in a specified region will not only ensure stability robustness but also achieve performance robustness when the uncertainty is time invariant. Juang, Hong and Wang [18] presented a method to calculate allowable element bounds for highly structured perturbations to achieve performance robustness. Based on a Lyapunov approach, the upper bounds on perturbations are obtained to retain system eigenvalues within an arbitrarily chosen region in the complex plane. However, this result requires the solution of a Lyapunov equation which contains complex matrices. In our research, we extend the robustness conditions of Yu [2], and Wang and Lin [19], to delta operator systems. Our results consist of three new theorems where modal decomposition, Lyapunov and matrix measure approaches are considered.

1.3 CONTRIBUTIONS OF THE THESIS

1. Proposed a sufficient condition based on bounds of the integral norm of the state and output which is valid for linear time invariant continuous time systems subject to

time varying structured state space uncertainty and unmodelled dynamics.

2. In collaboration with Yu [2], we derived a new robustness result based on a modal decomposition approach. This result is applicable to LTI continuous time systems with linear time varying uncertainty. We developed a new design method which minimizes an objective function with constraints on the robustness result and on selected damping ratios and time constants. The design method is applied to the design of a robust controller for the lateral dynamics of the EMRAAT missile.

3. Extended the sufficient condition for robust stability developed in collaboration with Yu [2] to sampled data systems using the delta operator representation presented by Middleton and Goodwin [3]. Applied the robustness result to design a robust sampled data controller for the EMRAAT missile.

4. Derived a new robustness result for delta operator systems subject to time invariant uncertainty using a Lyapunov approach. Designed a feedback controller for the FPCC aircraft by minimizing the integrated roll rate with the robustness condition as one of the constraints.

5. Developed a robust performance condition for unified delta systems with time invariant structured state space uncertainty. In our research, we obtained results based on a modal decomposition approach, a Lyapunov approach, and a matrix measure approach.

6. Derived new robustness results for multi-rate sampled data systems using modal decomposition and Lyapunov approaches. We show that when the sampling periods go to zero, the multi-rate robustness conditions approach the continuous time results.

1.4 OUTLINE

Chapter 2 extends the sufficient condition for robust stability of Ref.1 to include simultaneous time varying structured state space uncertainty and unmodelled dynamics.

Chapter 3 presents a robust eigenstructure assignment design method, which we developed in collaboration with Yu [2], for LTI systems with linear time varying uncertainty by using a new robustness sufficient condition. The method is applied to the design of a robust controller for the lateral dynamics of the EMRAAT missile.

Chapter 4 extends eigenstructure assignment for a linear time invariant system to the unified delta model proposed by Middleton and Goodwin [3]. A robust sampled data controller is computed for the EMRAAT Missile. We achieve a significant improvement in the sideslip gust response as compared to an orthogonal projection eigenstructure assignment design.

Chapter 5 presents a new sufficient condition for robust stability using a Lyapunov approach. A robust sampled data design involving a pseudo control strategy is presented.

Chapter 6 considers performance robustness for an LTI system with structured state space uncertainty. We present three new robustness results for a delta model which include a modal decomposition approach, a Lyapunov approach and a matrix measure approach.

Chapter 7 presents new robustness results for multi-input fixed output rate delta operator systems based on modal decomposition and Lyapunov methods. When the sampling period goes to zero, the multi-rate conditions approach the continuous time conditions.

2. PERFORMANCE ROBUSTNESS FOR LTI SYSTEMS WITH STRUCTURED STATE SPACE UNCERTAINTY AND UNMODELLED DYNAMICS

2.1 OVERVIEW

In this chapter, we obtain a sufficient condition for the stability robustness of a linear time invariant plant which is subject to simultaneous time varying structured state space uncertainty and unmodelled dynamics by using bounds on the integral of the state and output norms.

2.2 PROBLEM FORMULATION

Consider a nominal linear time-invariant multi-input multi-output plant described by

$$\dot{x} = Ax + Bu \tag{29a}$$

$$y = Cx + Du \tag{29b}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^r$ is the output vector, and A, B, C, and D are constant matrices. We shall refer to Eqs.(29a) and (29b) as either the nominal plant or the design model.

Suppose that the nominal system is subject to simultaneous linear time-varying structured state space uncertainty and unmodelled dynamics. The linear time-varying state space uncertainty is described by uncertainty in the entries of A, B, C, and D and is denoted by $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$, respectively. The unmodelled dynamics $\Delta H(t)$ satisfy the model described by Maciejowski [20].

$$\Delta H(t) = C_{\Delta} e^{A_{\Delta} t} B_{\Delta} + D_{\Delta}\delta(t) + E_{\Delta}\dot{\delta}(t) \tag{30}$$

which affects the output via a convolution with the input. Then, the plant with uncertainty is given by

$$\dot{x} = Ax + Bu + \Delta A(t)x + \Delta B(t)u \tag{31a}$$

$$y = Cx + Du + \Delta C(t)x + \Delta D(t)u + \Delta H(t)*u(t) \tag{31b}$$

We shall refer to Eqs.(31a) and (31b) as the plant with uncertainty or the true model. Further, suppose that bounds are available on the absolute values of the maximum variations in the elements of $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$. That is,

$$|\Delta a_{ij}(t)| \leq (a_{ij})_{max};\ i=1,\dots,n;\ j=1,\dots,n \tag{32a}$$

$$|\Delta b_{ij}(t)| \leq (b_{ij})_{max};\ i=1,\dots,n;\ j=1,\dots,m \tag{32b}$$

$$|\Delta c_{ij}(t)| \le (c_{ij})_{max};\ i=1,\ldots,r;\ j=1,\ldots,n \qquad (32c)$$

$$|\Delta d_{ij}(t)| \le (d_{ij})_{max};\ i=1,\ldots,r;\ j=1,\ldots,m \qquad (32d)$$

Define $\Delta A^{+}(t)$, $\Delta B^{+}(t)$, $\Delta C^{+}(t)$, and $\Delta D^{+}(t)$ as the matrices obtained by replacing the entries of $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$ by their absolute values. Also, define A_{max}, B_{max}, C_{max}, and D_{max} as the matrices with entries $(a_{ij})_{max}$, $(b_{ij})_{max}$, $(c_{ij})_{max}$, and $(d_{ij})_{max}$, respectively. Then,

$$\{\Delta A(t):\ \Delta A^{+}(t) \le A_{max}\} \qquad (33a)$$

$$\{\Delta B(t):\ \Delta B^{+}(t) \le B_{max}\} \qquad (33b)$$

$$\{\Delta C(t):\ \Delta C^{+}(t) \le C_{max}\} \qquad (33c)$$

$$\{\Delta D(t):\ \Delta D^{+}(t) \le D_{max}\} \qquad (33d)$$

where "≤" is applied element by element to matrices and $A_{max} \in \mathbb{R}_{+}^{n\times n}$, $B_{max} \in \mathbb{R}_{+}^{n\times m}$, $C_{max} \in \mathbb{R}_{+}^{r\times n}$, $D_{max} \in \mathbb{R}_{+}^{r\times m}$ where $\mathbb{R}_{+}$ is the set of non-negative real numbers.

Further, suppose that bounds are available on the individual terms in $\Delta H(t)$. That is,

$$\bar{\sigma}(C_{\Delta} e^{A_{\Delta} t} B_{\Delta}) \le r_2 e^{-\beta t} \qquad (34a)$$

$$\bar{\sigma}(D_\Delta) \le r_0 \tag{34b}$$

Consider the output feedback dynamic compensator described by

$$\dot{x}_c = A_c x_c + B_c y \tag{35a}$$

$$u = C_c x_c + D_c y \tag{35b}$$

2.3 STABILITY ROBUSTNESS MAIN RESULT

Given a dynamic compensator (A_c, B_c, C_c, D_c) such that the nominal closed loop system exhibits desirable dynamic performance, determine if the uncertain closed loop plant is asymptotically stable for all $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$ described by Eqs.(33a)-(33d) and unmodelled dynamics described by Eq.(30) which satisfy the assumptions shown in Eqs.(34a) and (34b).

2.3.1 SUFFICIENT CONDITIONS FOR ROBUST STABILITY

Theorem 1:

Consider the linear time-invariant plant with time-varying structured uncertainty and unmodelled dynamics described by

$$\dot{x}(t) = (A+\Delta A(t))x(t) + (B+\Delta B(t))u(t) \tag{36a}$$

$$y(t) = (C+\Delta C(t))x(t) + (D+\Delta D(t))u(t) + \Delta H(t)*u(t) \tag{36b}$$

with dynamic compensator described by

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{37a}$$

$$u(t) = C_c x_c(t) + D_c y(t) \tag{37b}$$

Consider the unmodelled dynamics $\Delta H(t)$ given by

$$\Delta H(t) = C_\Delta e^{A_\Delta t} B_\Delta + D_\Delta \delta(t) + E_\Delta \dot{\delta}(t) \tag{38}$$

and let the time-varying structured state space uncertainty $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$ satisfy

$$\{\Delta A(t)\colon \Delta A^+(t) \le A_{max}\} \tag{39a}$$

$$\{\Delta B(t)\colon \Delta B^+(t) \le B_{max}\} \tag{39b}$$

$$\{\Delta C(t)\colon \Delta C^+(t) \le C_{max}\} \tag{39c}$$

$$\{\Delta D(t)\colon \Delta D^+(t) \le D_{max}\} \tag{39d}$$

where $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$ are continuous functions.

The plant and unmodelled dynamics satisfy the following assumptions:

(i): $$\bar{\sigma}(C_\Delta e^{A_\Delta t} B_\Delta) \le r_2 e^{-\beta t} \tag{40a}$$

(ii): $$\bar{\sigma}(D_\Delta) \le r_0 \tag{40b}$$

(iii): $$\bar{\sigma}(E_\Delta) \le r_1 \tag{40c}$$

(iv): $x(t)$ and $y(t)$ are continuous

(v): $\bar{\sigma}(\dot{u}(t)) \le \gamma \cdot \bar{\sigma}(u(t))$

Suppose (A_c, B_c, C_c, D_c) is such that the nominal closed loop system is asymptotically stable with a non-defective closed loop matrix. Then, the uncertain closed loop system described by Eqs.(36a) and (36b) is asymptotically stable for all $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta D(t)$ which satisfy Eqs.(39a)-(39d) and $\Delta H(t)$ which satisfies assumptions (i)-(v) if

$$\left(\frac{1}{\alpha\cdot\underline{\sigma}(Q)}\right)\left\{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\right.\right.$$

$$\left.\cdot\begin{pmatrix} A_{max}+B_{max}(D_cC)^{+}+(B^{+}+B_{max})D_c^{+}C_{max} & B_{max}C_c^{+} \\ B_c^{+}C_{max} & 0 \end{pmatrix}(MD_1Q)^{+}\right]+$$

$$\frac{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\begin{pmatrix}(B^{+}+B_{max})D_c^{+}(D^{+}+D_{max}) \\ B_c^{+}(D^{+}+D_{max})\end{pmatrix}\right]\bar{\sigma}\left[[D_c^{+}(C^{+}+C_{max}),C_c^{+}](MD_1Q)^{+}\right]}{1-\bar{\sigma}[D_c^{+}(D^{+}+D_{max})]-\bar{\sigma}(D_c)\cdot\vartheta}$$

$$\left.+\frac{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\begin{pmatrix}(B\ +B_{max})D_c^{+} \\ B_c^{+}\end{pmatrix}\right]\cdot\bar{\sigma}\left[[D_c^{+}(C^{+}+C_{max}),C_c^{+}](MD_1Q)^{+}\right]\cdot\vartheta}{1-\bar{\sigma}[D_c^{+}(D^{+}+D_{max})]-\bar{\sigma}(D_c)\cdot\vartheta}\right\}$$

$$<1 \qquad (41a)$$

and if

$$\bar{\sigma}[D_c^{+}(D^{+}+D_{max})]+\bar{\sigma}(D_c)\cdot\vartheta<1 \qquad (41b)$$

where

$$\alpha=-\max_i \operatorname{Re}[\lambda_i(\tilde{A}_{cl})]$$

$$\vartheta=r_2/\beta+r_0+\gamma\cdot r_1$$

and

$$\tilde{A}_{cl} = \begin{bmatrix} A+BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$$

and M is the modal matrix of $\tilde{A}_{cl}$.

Proof: see Appendix I

Remark 1:

We shall comment on the continuity assumption on x(t) and y(t) which is stated as assumption (iv). The fact that the closed loop system is linear together with the continuity of $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, and $\Delta D(t)$ and the continuity of $\Delta H(t)$ is sufficient for satisfying the assumption that x(t) and y(t) are continuous. This continuity result together with $\int_0^\infty \|x(t)\|_2 \, dt < \infty$ is then sufficient to guarantee that $\lim_{t\to\infty} \|x(t)\|_2 = 0$.

Remark 2:

The question of whether there exists a compensator which satisfies Eqs.(41a) and (41b) for a specified A_{max} and B_{max} is a difficult one. However, suppose that we let $A_{max} = \varepsilon A^o_{max}$ and $B_{max} = \varepsilon B^o_{max}$ with ε sufficiently small. The continuity of y(t) implies the continuity of $x_c(t)$ and u(t). This continuity result together with the linearity of the plant and compensator and assumption (v) should be sufficient to guarantee that there exists ε sufficiently small such that the uncertain closed system is asymptotically stable. Therefore, if the sufficient condition for robust stability is not satisfied for a particular A_{max} and B_{max}, then the designer can reduce the uncertainty bounds until the compensator satisfies Eqs.(41a) and (41b). This will then yield the maximum uncertainty for which the plant and compensator can satisfy the sufficient condition for robust stability.

Corollary 1: (Bounded Unmodelled Dynamics)

If $E_\Delta = 0$, then assumption (iii) is not required.

Corollary 2: (Constant Gain Output Feedback)

For constant gain output feedback $u = D_c y$, the uncertain closed loop system is asymptotically stable if

$$\left(\frac{1}{\alpha\cdot\underline{\sigma}(Q)}\right)\Bigg\{\bar{\sigma}\Big[(D_1^{-1}M^{-1})^+[A_{max}+B_{max}(D_cC)^+$$

$$+\ (B^++B_{max})\ D_c^+\ C_{max}](MD_1Q)^+\Big]$$

$$+\ \frac{\bar{\sigma}\Big[(D_1^{-1}M^{-1})^+(B^++B_{max})D_c^+(D^++D_{max})\Big]}{1-\ \bar{\sigma}[D_c^+(D^++D_{max})]\ -\ \bar{\sigma}(D_c)\cdot\vartheta}\cdot\bar{\sigma}\Big[D_c^+(C^+\ +\ C_{max})(MD_1Q)^+\Big]$$

$$+\ \frac{\bar{\sigma}\Big[(D_1^{-1}M^{-1})^+(B^++B_{max})D_c^+\Big]\cdot\bar{\sigma}\Big[D_c^+(C^++C_{max})(MD_1Q)^+\Big]\cdot\vartheta}{1-\ \bar{\sigma}[D_c^+(D^++D_{max})]\ -\ \bar{\sigma}(D_c)\cdot\vartheta}\Bigg\}<1 \tag{42a}$$

and if

$$\bar{\sigma}[D_c^+(D^++D_{max})]\ +\ \bar{\sigma}(D_c)\cdot\vartheta\ <\ 1 \tag{42b}$$

where

$$\vartheta\ =\ r_2/\beta\ +\ r_0\ +\ \gamma\cdot r_1 \tag{43}$$

$$\alpha\ =\ -\max_i\ \mathrm{Re}[\lambda_i(A+BD_cC)]$$

and M is the modal matrix of $(A+BD_cC)$.

Remark 3:

For constant gain output feedback, Eq.(42b) is an explicit constraint on the norm of the feedback gain matrix D_c. The reason for this constraint is that $\Delta D(t)$ and $\Delta H(t)$ represent high frequency paths which can destabilize the plant unless the feedback gain is sufficiently small. This observation is consistent with the results of O'Reilly [22] who indicates the robustness problem which may result when closing high frequency plant loops with constant gains. Thus, a robust constant gain controller must be a low authority controller such that the gain is small enough at high frequencies so that closing the uncertain high frequency paths does not destabilize the plant. An alternative is to use a proper, but not strictly proper, compensator described by the quadruple (A_c, B_c, C_c, D_c). In this case, Eq.(41b) is an explicit constraint on D_c which represents a high frequency path. Thus, although the proper compensator need not be a low authority controller, we require that the gain D_c be small. Another alternative is to use a strictly proper compensator described by the triple (A_c, B_c, C_c).

Corollary 3: (Strictly Proper Plant with a Strictly Proper Compensator)

Let $D = \Delta D = D_c = 0$. The uncertain plant in Eqs.(36a) and (36b) is asymptotically stable if

$$\left(\frac{1}{\alpha \cdot \underline{\sigma}(Q)}\right)\left\{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\begin{pmatrix} A_{max} & B_{max}C_c^{+} \\ B_c^{+}C_{max} & 0 \end{pmatrix}(MD_1Q)^{+}\right]\right.$$

$$+ \bar{\sigma}\left[(D_1^{-1} M^{-1})^+ \begin{bmatrix} 0 \\ B_c^+ \end{bmatrix} \right] \bar{\sigma}\left[[0, C_c^+] (MD_1Q)^+ \right] \cdot \vartheta \Bigg\} < 1 \tag{44}$$

where

$$\alpha = -\max_i \operatorname{Re}[\lambda_i(\hat{A}_{cl})]$$

and

$$\hat{A}_{cl} = \begin{pmatrix} A & BC_c \\ B_cC & A_c \end{pmatrix}$$

and M is the modal matrix of $\hat{A}_{cl}$.

Remark 4:

For the strictly proper compensator described by the triple (A_c, B_c, C_c), there are no explicit constraints on the norms of the gains. However, B_c and C_c are implicitly constrained via the singular values in Eq.(44) and A_c is implicitly constrained via the eigenvalues of $\hat{A}_{cl}$ which must be sufficiently far into the left half of the complex plane. Note that the compensator (A_c, B_c, C_c) has no high frequency paths due to its low pass characteristic. Our results are consistent with O'Reilly [22] who suggests that the designer should avoid closing high frequency plant loops. However, we observe that we can close the high frequency loops and still guarantee robustness if the high frequency gain D_c is sufficiently small as shown by either Eq.(41b) or Eq.(42b).

Remark 5:

When using the compensator represented by the quadruple (A_c, B_c, C_c, D_c), we can use eigenstructure assignment to assign ($r + n_c$) eigenvalues. Here r is the number of plant outputs and n_c is the number of compensator states. However, when D_c is constrained to be zero we lose some of the degrees of freedom in eigenstructure assignment. Thus, there may be some advantages with regard to eigenvalue and eigenvector assignment in using a non-strictly proper compensator. There appears to be a tradeoff between robustness to high frequency uncertainty and the ability to exactly assign certain eigenvalues and eigenvectors.

Corollary 4: (Norm Bounded Uncertainty; Constant Gain Output Feedback).

Let $\bar{\sigma}(\Delta A(t))\le\gamma_1$, $\bar{\sigma}(\Delta B(t))\le\gamma_2$, $\bar{\sigma}(\Delta C(t))\le\gamma_3$, $\bar{\sigma}(\Delta D(t))\le\gamma_4$ and let $u(t)=D_c y(t)$. Then, the uncertain closed loop plant is asymptotically stable if

$$\left(\frac{\kappa_2(M)}{\alpha}\right)\Bigg[(\gamma_1+\gamma_2\bar{\sigma}(D_c)\bar{\sigma}(C)+[\sigma(\bar{B})+\gamma_2]\sigma(\bar{D}_c)\gamma_3)$$

$$+\frac{[\bar{\sigma}(D_c)]^2[\bar{\sigma}(B)+\gamma_2][\bar{\sigma}(C)+\gamma_3][\bar{\sigma}(D)+\gamma_4]}{1-\bar{\sigma}(D_c)\{[\bar{\sigma}(D)+\gamma_4]+\vartheta\}}$$

$$+\frac{[\bar{\sigma}(D_c)]^2[\bar{\sigma}(B)+\gamma_2][\bar{\sigma}(C)+\gamma_3]\cdot\vartheta}{1-\bar{\sigma}(D_c)\{[\bar{\sigma}(D)+\gamma_4]+\vartheta\}}\Bigg] < 1 \qquad (45)$$

and if

$$\bar{\sigma}(D_c)\cdot[\bar{\sigma}(D)+\gamma_4+\vartheta] < 1 \qquad (46)$$

where

$$\alpha = -\max_i \operatorname{Re}[\lambda_i(A+BD_cC)]$$

and M is a modal matrix of $(A+BD_cC)$ and $\kappa_2(M)$ is the 2-norm condition number of the matrix M.

3. ROBUST EIGENSTRUCTURE ASSIGNMENT DESIGN METHOD

3.1 INTRODUCTION

In collaboration with Wangling Yu [2], we developed a new sufficient condition for the robust stability of a linear time invariant system which is subject to linear time varying structured state space uncertainty. This new sufficient condition is a sum of terms each of which involves the i-th right eigenvector, the i-th left eigenvector, and the real part of the i-th eigenvalue. This new robustness result is less conservative than an earlier result which involves the modal matrix and the real part of the eigenvalue which is closest to the imaginary axis in the complex plane.

A robust design is computed for the Extended Medium Range Air to Air Technology Missile and the new design is compared with an earlier orthogonal projection eigenstructure assignment design. The new design method uses the MATLAB™ Optimization Toolbox [23] to minimize the integral of the roll rate with constraints on the real part of the dutch roll and roll mode eigenvalues, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the sufficient condition for robust stability.

The new design satisfies the robustness condition while also yielding an improved transient response.

3.2 PROBLEM FORMULATION

Consider a nominal linear time-invariant multi-input multi-output system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{47a}$$

$$y(t) = Cx(t) \tag{47b}$$

Suppose that the nominal system is subject to linear time-varying uncertainty in the entries of A, B described by $\Delta A(t)$ and $\Delta B(t)$, respectively. We shall assume that the entries of $\Delta A(t)$ and $\Delta B(t)$ are continuous. Then, the system with uncertainty is given by

$$\dot{x}(t) = Ax(t) + Bu(t) + \Delta A(t)x(t) + \Delta B(t)u(t) \tag{48a}$$

$$y(t) = Cx(t) \tag{48b}$$

Further, suppose that bounds are available on the absolute values of the maximum variations in the elements of $\Delta A(t)$ and $\Delta B(t)$. That is,

$$|\Delta a_{ij}(t)| \le (a_{ij})_{max}; \quad i=1,\ldots,n; \quad j=1,\ldots,n \tag{49a}$$

$$|\Delta b_{ij}(t)| \le (b_{ij})_{max}; \quad i=1,\ldots,n; \quad j=1,\ldots,m \tag{49b}$$

Define $\Delta A^{+}(t)$ and $\Delta B^{+}(t)$ as the matrices obtained by replacing the entries of $\Delta A(t)$ and $\Delta B(t)$ by their absolute values. Also, define A_{max} and B_{max} as the matrices with entries $(a_{ij})_{max}$ and $(b_{ij})_{max}$, respectively. Then,

$$\{\Delta A(t): \ \Delta A^{+}(t) \le A_{max}\} \tag{50a}$$

and

$$\{\Delta B(t): \ \Delta B^{+}(t) \le B_{max}\} \tag{50b}$$

where "≤" is applied element by element to matrices and $A_{max} \in \mathbb{R}_{+}^{n\times n}$, $B_{max} \in \mathbb{R}_{+}^{n\times m}$ where $\mathbb{R}_{+}$ is the set of non-negative numbers.

Consider the constant gain output feedback control law described by

$$u(t) = Fy(t) \tag{51}$$

Then, the nominal closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) \tag{52}$$

and the uncertain closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) + [\Delta A(t) + \Delta B(t)FC]x(t) \tag{53}$$

Finally, the stability robustness problem can be stated as follows: Given a feedback gain matrix $F \in \mathbb{R}^{m\times r}$ such that the nominal closed loop system exhibits desirable dynamic performance, determine if the uncertain closed loop system is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by Eqs.(50a) and (50b).

3.3 ROBUSTNESS RESULT

The solution proposed by Sobel et.al.[1] for the stability robustness problem is described by the following theorem.

Theorem 2:

Suppose that F is such that the nominal closed loop system described by Eq.(52) is asymptotically stable with (A+BFC) non-defective. Then, the uncertain closed loop system given by Eq.(53) is asymptotically stable for all $\Delta A(t)$ and

$\Delta B(t)$ described by Eqs.(50a) and (50b) if

$$\alpha > \lambda_{\max} \, [(M^{-1})^{+}[A_{\max} + B_{\max}(FC)^{+}]M^{+}] \tag{54}$$

where

$$\alpha = -\max_{i} \operatorname{Re}[\lambda_i(A+BFC)]$$

and where M is a modal matrix of (A+BFC) and $\lambda_{\max}(\cdot)$ of a non-negative matrix denotes the real non-negative eigenvalue $\lambda_{\max} \geq 0$ such that $\lambda_{\max} \geq |\lambda_i|$ for all eigenvalues λ_i. The above theorem describes a sufficient condition for the robust stability in terms of the eigenstructure of the nominal closed loop system. Robust stability is ensured provided that the nominal closed loop eigenvalues lie to the left of a vertical line in the complex plane which is determined by a norm involving the structure of the uncertainty and the nominal closed loop modal matrix. We remark that the sufficient condition given by Eq.(54) can be rewritten as shown below.

$$\left(\frac{1}{\alpha}\right) \cdot \lambda_{\max} \, [(M^{-1})^{+}[A_{\max} + B_{\max}(FC)^{+}]M^{+}] < 1 \tag{55}$$

This alternate form of the stability robustness condition will be useful later when we compare it with the new robustness condition which is derived in this section. We now present a new sufficient condition for robust stability of a linear time invariant system subject to linear time varying structured state space uncertainty which is described by the following theorem.

Theorem 3:

Suppose that F is such that the nominal closed loop system described by Eq.(52) is asymptotically stable with (A+BFC) non-defective. Then, the uncertain closed loop system given by Eq.(53) is asymptotically stable for all $\Delta A(t)$ and $\Delta B(t)$ described by Eqs.(50a) and (50b) if

$$\lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} [A_{max} + B_{max}(FC)^+] \right\} < 1 \qquad (56)$$

where

$$\alpha_i = -\mathrm{Re}[\lambda_i(A+BFC)]$$

and where λ_i is the i-th eigenvalue of (A+BFC) with v_i and w_i^* the corresponding right and left eigenvectors, respectively; and where $(\cdot)^*$ denotes the complex conjugate transpose.

Proof: see Appendix II

We remark that the quantity $|w_i^* v_i|$ is defined by Golub and Van Loan [24] to be the condition number of the i-th eigenvalue. This condition number is a measure of the sensitivity of the i-th eigenvalue to incremental perturbations in the matrix A+BFC. The quantity $v_i w_i^*$, which appears in Eq.(56), is weakly related to this eigenvalue condition number. Thus, the new stability robustness condition of Eq.(56) seems to have the heuristic interpretation that robustness can be achieved by having the sensitive eigenvalues far into the left half complex plane and by having the eigenvalues which are near the imaginary axis to be insensitive. Such an interpretation was not possible for the result of Sobel et.al.[1] because their stability robustness condition, which is given by Eq.(55) is in terms of the spectrum as a whole rather than the individual eigenvalues.

The new stability robustness condition of Eq.(56) is less conservative than the earlier condition of Sobel et.al.[1]. This property is described by the following theorem.

Theorem 4:

The left hand side of Eq.(56) is less than or equal to the the left hand side of Eq.(55). Mathematically,

$$\lambda_{\max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} [A_{\max} + B_{\max}(FC)^+] \right\}$$

$$\leq \left(\frac{1}{\alpha}\right)\lambda_{\max}\left\{(M^{-1})^+[A_{\max}+B_{\max}(FC)^+]M^+\right\} \tag{57}$$

Proof: see Appendix III

3.4 ROBUST DESIGN FOR THE EMRAAT MISSILE

Consider the Extended Medium Range Air to Air Technology (EMRAAT) bank-to-turn missile which is described by Bossi and Langehough [9]. A sixth order model of the yaw/roll dynamics at a 10 degree angle of attack is considered with state vector, control vector, and measurement vector given by $x = [\beta, r, p, p_I, \delta_r, \delta_a]^T$, $u = [\delta_{rc}, \delta_{ac}]^T$, and $y = [\beta, r, p, p_I]^T$, respectively. Here β is the sideslip angle (deg), r is yaw rate (deg/s), p is roll rate (deg/s), p_I is integrated roll rate (deg), δ_r is rudder deflection (deg), and δ_a is aileron deflection (deg). The state space matrices A, B, and C are shown below:

EMRAAT STATE SPACE MATRICES

$$A = \begin{bmatrix} -.5007 & -.9945 & .1736 & 0 & .109 & .00691 \\ 16.83 & -.5748 & .01233 & 0 & -132.8 & 27.19 \\ -3227 & .3208 & -2.099 & 0 & -1620.0 & -1240.0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -179 & 0 \\ 0 & 0 & 0 & 0 & 0 & -179 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 179 & 0 \\ 0 & 0 & 0 & 0 & 0 & 179 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Sobel and Cloutier [8] use eigenstructure assignment with an orthogonal projection. The desired dutch roll and roll mode eigenvalues are achieved exactly because four measurements are available for feedback. The desired dutch roll eigenvectors are chosen to yield a complex mode which is composed of sideslip angle and yaw rate with no coupling to roll rate and integrated roll rate. The desired roll mode eigenvectors are chosen to yield a complex mode which is composed of roll rate and integrated roll rate with no coupling to sideslip angle and yaw rate. Then, the achievable eigenvectors are computed by using the orthogonal projection of the i-th desired eigenvector v_i^d onto the columns of $(\lambda_i I-A)^{-1}B$. The closed loop eigenvalues and the feedback gain matrix are shown in Table 1.

TABLE 1. COMPARISON OF EMRAAT DESIGNS

	Closed Loop Eigenvalues*	Feedback Gain Matrix
Eigen- structure Assignment Design of Sobel and Cloutier	$\lambda_{dr}=-23.98\pm j17.99$ $\lambda_{roll}=-10.01\pm j9.98$ $\lambda_{act} = -132.1$ $\lambda_{act} = -161.1$	$\begin{array}{ccccc} \beta & r & p & p_I & \\ -4.19 & .233 & .00374 & .731 & \delta_{rc} \\ 2.89 & -.290 & .00631 & -.812 & \delta_{ac} \end{array}$
Robust Design	$\lambda_{dr}=-14.94\pm j16.68$ $\lambda_{roll}=-40.55\pm j92.90$ $\lambda_{act} = -150.3$ $\lambda_{act} = -99.3$	$\begin{array}{ccccc} \beta & r & p & p_I & \\ -2.82 & .162 & .0127 & 1.18 & \delta_{rc} \\ .770 & -.180 & .062 & 3.14 & \delta_{ac} \end{array}$

*Eigenvalues are computed by using feedback gains which are rounded to three significant digits.

In collaboration with Yu [2], we propose a new robust design which minimizes the integrated roll rate (which is approximately the bank angle) subject to constraints on the real part of the dutch roll and roll modes, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. Mathematically, the objective function to be minimized is given by

$$J = \sum_{k=1}^{30} [p_I(kT_1)]^2 \tag{58}$$

where T_1=0.01s

The values of T_1 and k are chosen to include the time interval [0,0.3] during which most of the transient response occurs. Of course, computation of Eq.(58) requires that a linear simulation be performed during each function evaluation of the optimization. The constraints are shown below where ζ is the damping ratio.

$$\text{Re } \lambda_{dr} \in [-50,-6] \tag{59}$$

$$\text{Re } \lambda_{roll} \in [-50,-6] \tag{60}$$

$$\text{Re } \lambda_{rudder} < -100 \tag{61}$$

$$\text{Re } \lambda_{aileron} < -100 \tag{62}$$

$$\zeta_{dr} \in [0.4,0.8] \tag{63}$$

$$\zeta_{roll} \in [0.4,0.8] \tag{64}$$

$$|\dot{\delta}_a| < 275 \text{ deg/s} \tag{65}$$

$$|\dot{\delta}_r| < 275 \text{ deg/s} \tag{66}$$

$$\lambda_{max}\left\{ \sum_{i=1}^{6} \frac{(v_i w_i^*)^+}{\alpha_i} [A_{max} + B_{max}(FC)^+] \right\} < 0.999 \tag{67}$$

where for illustrative purposes we have chosen $A_{max} = 0.1 \cdot A^+$ and $B_{max} = 0$.

The actuator deflection rates are computed from the slopes of the time responses of the deflections during the time interval [0,0.03]. This interval is chosen because the slopes of the deflections are largest during this time interval. Mathematically,

$$|\dot{\delta}_r| = \frac{\max|\delta_r(mT_2)|}{mT_2} \tag{68}$$

$$|\dot{\delta}_a| = \frac{\max|\delta_a(mT_2)|}{mT_2} \tag{69}$$

where T_2=0.001s and m=0,1,...,30. The maximum deflection rates chosen for the constraints are well within the expected 400 deg/s limit for the advanced state of the art electromechanical actuator described by Langehough and

Simons [25].

The parameter vector contains the quantities which may be varied by the optimization. This twelve dimensional vector includes Re λ_{dr}, Im λ_{dr}, Re λ_{roll}, Im λ_{roll}, Re $z_1(1)$, Re $z_1(2)$, Im $z_1(1)$, Im $z_1(2)$, Re $z_3(1)$, Re $z_3(2)$, Im $z_3(1)$, Im $z_3(2)$. Here, the two dimensional complex vectors z_i contain the free eigenvector parameters. That is, the i-th eigenvector v_i may be written as

$$v_i = L_i z_i \tag{70}$$

where the columns of $L_i=(\lambda_i I-A)^{-1}B$ are a basis for the subspace in which the i-th eigenvector must reside. Thus, the free parameters are the vectors z_i rather than the eigenvectors v_i.

The optimization is performed by using subroutine *constr* from the MATLAB™ Optimization Toolbox [23] on a 486™ 25MHz personal computer. The optimization is initialized with the design proposed by Sobel and Cloutier [8] which yields an initial value of 5.0214 for the objective function of Eq.(58), a value of 4.0472 for the left hand side (LHS) of the earlier robustness condition of Eq.(55), and a value of 2.0686 for the LHS of the new robustness condition of Eq.(56). The optimization is complete after 2698 function evaluations which requires approximately one hour of computation and yields an optimal objective function of 0.0284, a value of 2.4432 for the LHS of the earlier robustness condition of Eq.(55), and a value of 0.999 for the LHS of the new robustness condition of Eq.(56). We observe that the dutch roll mode is dominant in the robust design whereas the roll mode was chosen to be dominant in the design of Sobel and Cloutier [8]. Furthermore, the optimization has assigned the roll mode damping to be the smallest value which is allowed by the constraint of Eq.(64).

The time histories for initial, robust, initial 1.1A, robust 1.1A of sideslip angle, yaw rate, roll rate, integrated roll rate, rudder deflection, and aileron deflection to a one degree initial sideslip are shown in Figures 1, 2, 3, 4, 5, and 6, respectively. We observe a significant improvement in the integrated roll rate response (which is desired to be zero) when compared to the earlier design of Sobel and Cloutier [8]. The earlier design of Sobel and Cloutier [8] has a minimum $p_I(t)$ of -.646 deg but the new design of this thesis has a minimum $p_I(t)$ of -.112 deg which is an improvement of approximately 80%. We note that this improved response is obtained with both smaller aileron and rudder deflections. Furthermore, it is interesting that the aileron in the design of Sobel and

Cloutier [8] exhibits an initial positive deflection of approximately two degrees before becoming negative, whereas this large positive initial aileron deflection does not appear in the new robust design. Next, the response to a roll rate step of 5 deg/s is considered. The roll rate, rudder deflection, and aileron deflection are shown in Figures 7, 8, and 9, respectively. The new robust design exhibits a roll rate response with a significantly smaller settling time which is achieved by using larger aileron deflection rates. Nevertheless, these deflection rates of approximately 25 deg/s are well within the allowable limits.

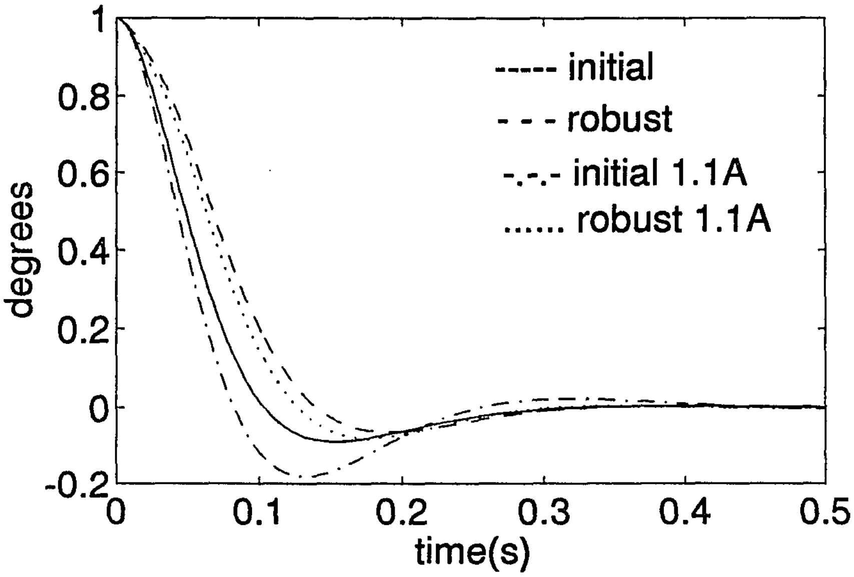

Figure 1. EMRAAT Missile Sideslip Angle.

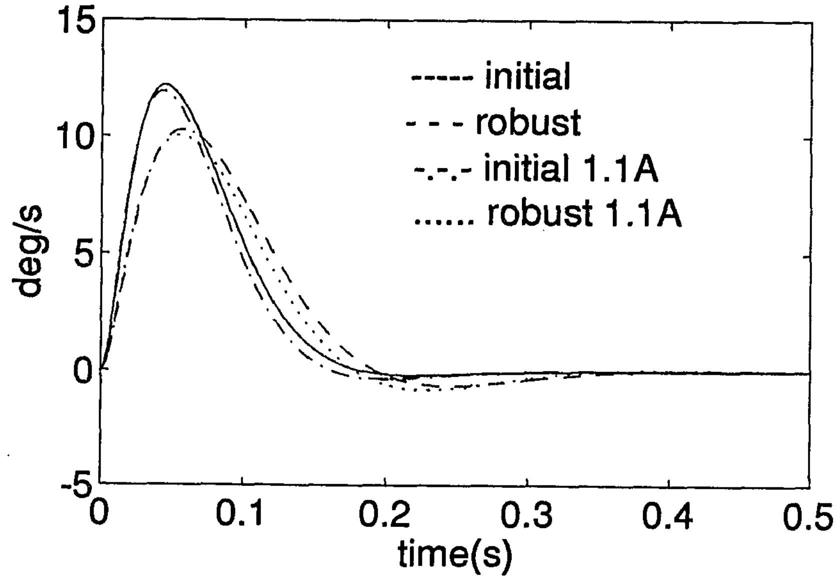

Figure 2. EMRAAT Missile Yaw Rate.

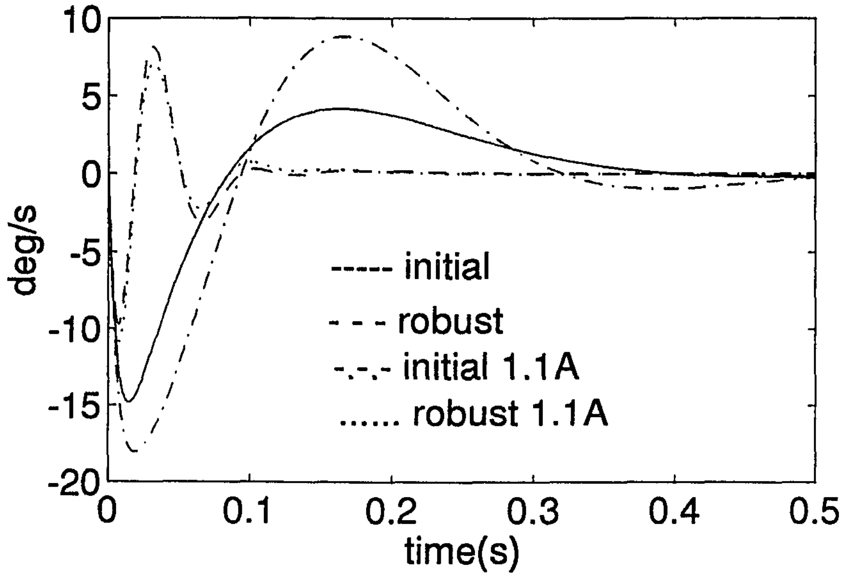

Figure 3. EMRAAT Missile Roll Rate.

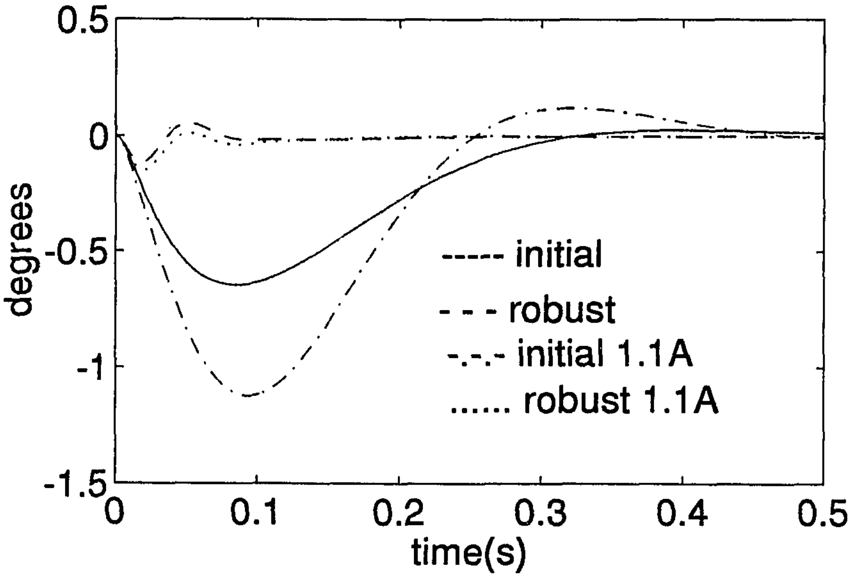

Fig.4 EMRAAT Missile Integrated Roll Rate.

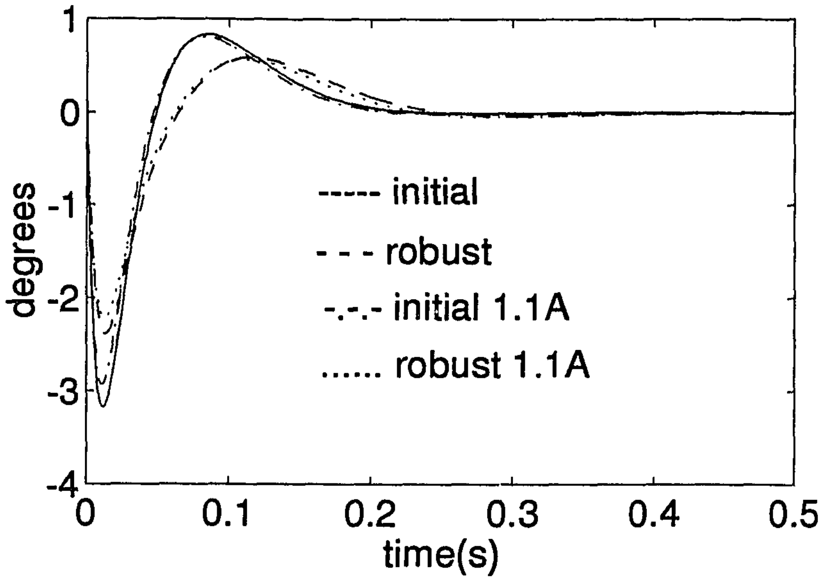

Figure 5. EMRAAT Missile Rudder Deflection.

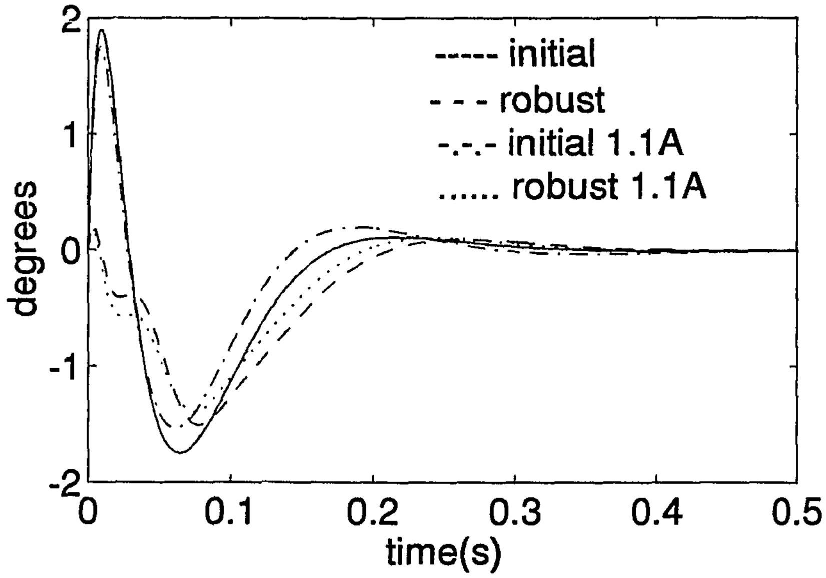

Figure 6. EMRAAT Missile Aileron Deflection

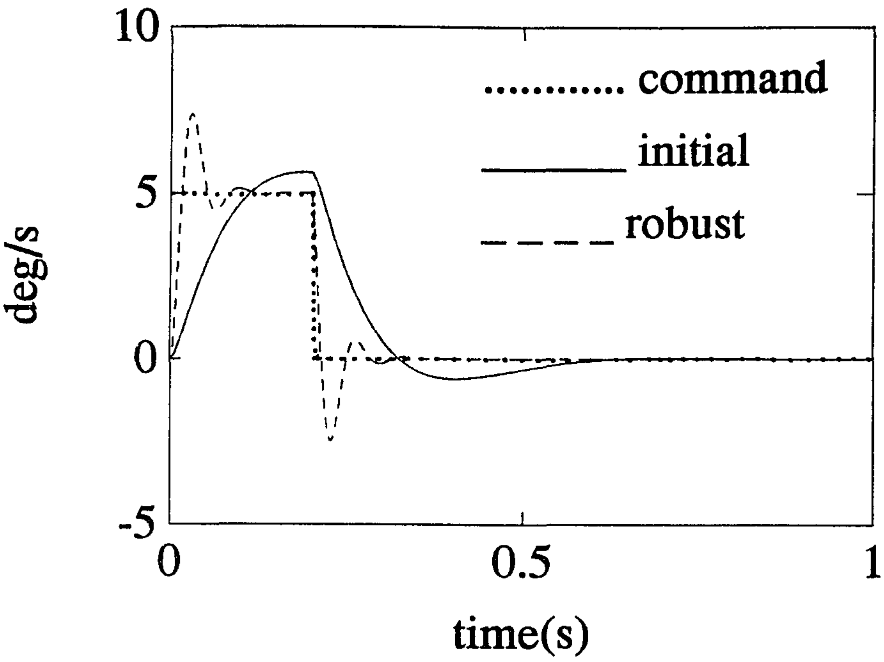

Figure 7. EMRAAT Missile Roll Rate for Step Response

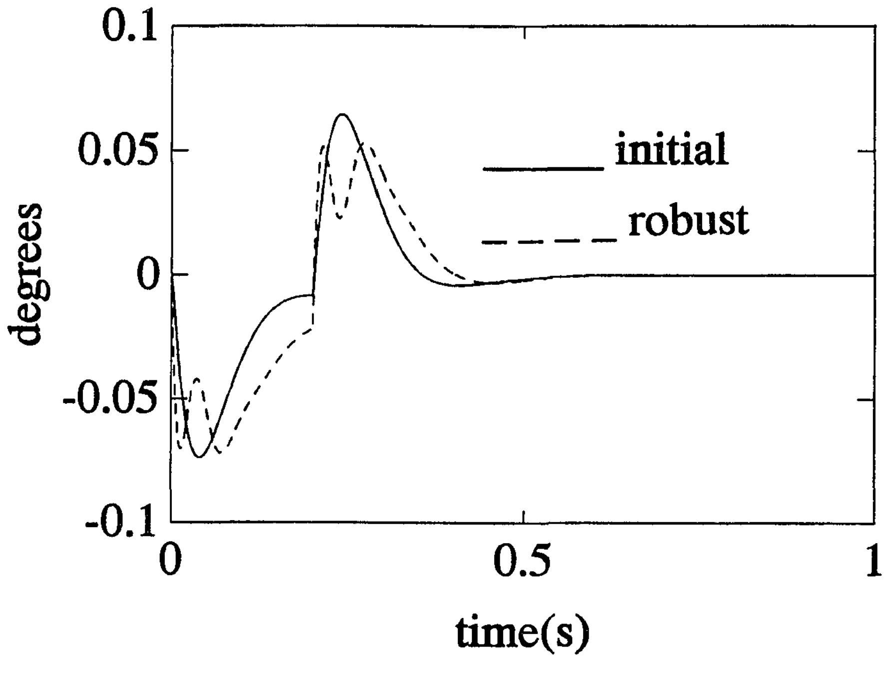

Figure 8. EMRAAT Missile Rudder Deflection for Step Response

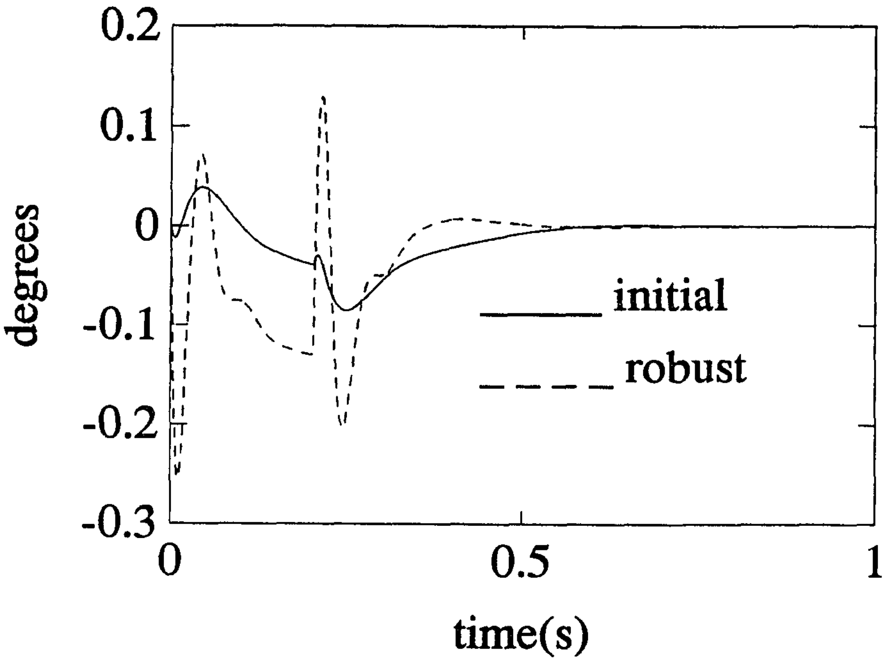

Figure 9. EMRAAT Missile Aileron Deflection for Step Response

4. ROBUST SAMPLED DATA EIGENSTRUCTURE ASSIGNMENT USING THE DELTA OPERATOR

4.1 OVERVIEW

In this section, we extend eigenstructure assignment to linear time invariant plants which are represented by Middleton and Goodwin's [3] unified delta model which is valid both for continuous time and sampled data operation of the plant. An important property of the delta model is that the discrete time eigenvalues approach the continuous time eigenvalues as the sampling period approaches zero. We show that the eigenvectors of the delta model are identical to the eigenvectors of the continuous time plant and an expression is derived for the eigenstructure assignment feedback gain matrix for the delta model. We show that in the limit as the sampling period delta goes to zero, the delta feedback gain approaches the continuous time feedback gain. We propose a sufficient condition for the robust stability of a linear time invariant unified delta plant subject to linear time invariant structured state space uncertainty. This yields a new unified robustness condition which is applicable to both continuous time and sampled data operation.

We design a robust sampled data controller for the lateral dynamics of the EMRAAT missile and this new design is compared to an orthogonal projection eigenstructure assignment design. The new design method proposed in this chapter uses the MATLAB™ Optimization Toolbox [23] and the MATLAB™ Delta Toolbox [26] to minimize the integrated roll rate with constraints on the time constants of the dutch roll and roll modes, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. This design satisfies the new robustness condition while also yielding an improved transient response as compared to the orthogonal projection design.

4.2 PROBLEM FORMULATION

Consider a nominal linear time-invariant multi-input multi-output system described by Eqs. (47a) and (47b). The corresponding sampled data system, which is obtained by using Middleton and Goodwin's [3] delta operator, is shown below.

$$\delta x = A_\delta x + B_\delta u \tag{71}$$

$$y = Cx \tag{72}$$

where

$$A_\delta = \Omega A \tag{73}$$

$$B_\delta = \Omega A \tag{74}$$

$$\Omega = \frac{1}{\Delta}\int_0^\Delta e^{A\tau}d\tau \tag{75}$$

$$\delta = \frac{q-1}{\Delta} \tag{76}$$

and where the shift operator q is defined by

$$qx_k = x_{k+1} \tag{77}$$

and where Δ is the sampling period.

The unified state space model proposed by Middleton and Goodwin [3] is valid for both the continuous and discrete time cases simultaneously. This unified model is described by

$$\rho x(t) = A_\rho x(t) + B_\rho u(t) \tag{78}$$

$$y(t) = Cx(t) \tag{79}$$

where

$$A_\rho = \begin{cases} A & \text{in continuous time} \\ A_\delta & \text{in discrete time} \end{cases} \tag{80}$$

$$B_\rho = \begin{cases} B & \text{in continuous time} \\ B_\delta & \text{in discrete time} \end{cases} \tag{81}$$

and where

$$\rho = \begin{cases} \dfrac{d}{dt} & \text{in continuous time} \\ \delta & \text{in discrete time} \end{cases} \tag{82}$$

Suppose that the nominal delta system is subject to linear time-invariant uncertainty in the entries of A_ρ, B_ρ described by dA_ρ and dB_ρ, respectively, where

$$dA_\rho = \begin{cases} dA & \text{in continuous time} \\ dA_\delta & \text{in discrete time} \end{cases} \tag{83}$$

$$dB_\rho = \begin{cases} dB & \text{in continuous time} \\ dB_\delta & \text{in discrete time} \end{cases} \tag{84}$$

and where

$$dA_\delta = \frac{1}{\Delta}\left[e^{(A+dA)\Delta} - e^{A\Delta}\right] \tag{85}$$

$$dB_\delta = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A+dA)\tau}\,d\tau\,(B + dB) - \int_0^\Delta e^{A\tau}\,d\tau\,B\right] \tag{86}$$

Then, the delta system with uncertainty is given by

$$\rho x(t) = A_\rho x(t) + B_\rho u(t) + dA_\rho x(t) + dB_\rho u(t) \qquad (87)$$

$$y(t) = Cx(t) \qquad (88)$$

Further, suppose that bounds are available on the maximum absolute values of the elements of dA and dB. That is,

$$|da_{ij}| \le (a_{ij})_{max}; \ i=1,\ldots,n; \ j=1,\ldots,n \qquad (89)$$

$$|db_{ij}| \le (b_{ij})_{max}; \ i=1,\ldots,n; \ j=1,\ldots,m \qquad (90)$$

Then, the corresponding bounds on the δ system, through Eqs. (85) and (86), are

$$|da_\delta(i,j)| \le [da_\delta(i,j)]_{max}; \ i=1,\ldots,n; \ j=1,\ldots,n \qquad (91)$$

$$|db_\delta(i,j)| \le [db_\delta(i,j)]_{max}; \ i=1,\ldots,n; \ j=1,\ldots,m \qquad (92)$$

Define dA_ρ^+ and dB_ρ^+ as the matrices obtained by replacing the entries of dA_ρ and dB_ρ by their absolute values. Also, define $A_{\rho max}$ and $B_{\rho max}$ as the matrices with entries $(a_{ij})_{max}$ and $(b_{ij})_{max}$, respectively in continuous time or with entries $[da_\delta(i,j)]_{max}$ and $[db_\delta(i,j)]_{max}$, respectively in discrete time. Then,

$$\{dA_\rho : dA_\rho^+ \le A_{\rho\max}\} \tag{93}$$

and where

$$\{dB_\rho : dB_\rho^+ \le B_{\rho\max}\} \tag{94}$$

$$A_{\rho\max} = \begin{cases} A_{\max} & \text{in continuous time} \\ A_{\delta\max} & \text{in discrete time} \end{cases} \tag{95}$$

$$B_{\rho\max} = \begin{cases} B_{\max} & \text{in continuous time} \\ B_{\delta\max} & \text{in discrete time} \end{cases} \tag{96}$$

and where

$$A_{\delta\max} = \frac{1}{\Delta}\left[e^{(A^+ + A_{\max})\Delta} - e^{A^+\Delta}\right] \tag{97}$$

$$B_{\delta\max} = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A^+ + A_{\max})\tau} d\tau \, (B^+ + B_{\max}) - \int_0^\Delta e^{A^+\tau} d\tau \, B^+\right] \tag{98}$$

and where "$\le$" is applied element by element to matrices and $A_{\max} \in \mathbb{R}_+^{n\times n}$, $B_{\max} \in \mathbb{R}_+^{n\times m}$ where $\mathbb{R}_+$ is the set of non-negative numbers.

Consider the constant gain output feedback control law described by

$$u(t) = F_\rho y(t) \tag{99}$$

where

$$F_\rho = \begin{cases} F & \text{in continuous time} \\ F_\delta & \text{in discrete time} \end{cases} \qquad (100)$$

Then, the nominal closed loop unified delta system is given by

$$\rho x(t) = A_{\rho c} x(t) \qquad (101)$$

where

$$A_{\rho c} = \begin{cases} A + BFC & \text{in continuous time} \\ A_\delta + B_\delta F_\delta C & \text{in discrete time} \end{cases} \qquad (102)$$

and the uncertain closed loop unified delta system is given by

$$\rho x(t) = A_{\rho c} x(t) + dA_{\rho c} x(t) \qquad (103)$$

where

$$dA_{\rho c} = \begin{cases} dA + dB(FC) & \text{in continuous time} \\ dA_\delta + dB_\delta (F_\delta C) & \text{in discrete time} \end{cases} \qquad (104)$$

Finally, the stability robustness problem can be stated as follows: Given a feedback gain matrix $F_\rho \in \mathbb{R}^{m \times r}$ such that the nominal closed loop unified delta system exhibits desirable dynamic performance, determine if the uncertain closed loop unified delta system is asymptotically stable for all time-invariant dA_ρ and dB_ρ described by Eqs.(93) and (94), respectively.

4.3 DELTA EIGENSTRUCTURE ASSIGNMENT AND ROBUSTNESS RESULTS

This section presents four theorems describing eigenstructure assignment and robust control for the unified delta model. Theorem 5 shows that a matrix M is a modal matrix for the delta plant if and only if it is a modal matrix for the continuous time plant. Theorem 6 describes the settling time and damping ratio regions for the delta plant in the γ-plane. Theorem 7 describes the eigenstructure assignment output feedback gain matrix for the delta plant and shows that the delta feedback gain matrix approaches the continuous time feedback gain matrix as the sampling period Δ approaches zero. Theorem 8 presents a sufficient condition for robust stability of the delta plant under time invariant structured state space uncertainty. This new robustness condition is valid simultaneously for both continuous time and discrete time operation of the plant.

Theorem 5: (Delta Eigenvectors)

Consider the continuous time plant given by $\dot{x}=Ax+Bu$ and the sampled data plant given by $\delta x=A_\delta x+B_\delta u$. The i-th eigenvalues of A and A_δ are λ_i and $\gamma_i=\frac{\exp(\lambda_i\Delta)-1}{\Delta}$, respectively. Let M be a modal matrix and let Λ be a diagonal matrix with the $\lambda_i(i=1,...,n)$ as entries. Then, $M^{-1}A_\delta M = \gamma_i I = \frac{1}{\Delta}(e^{\Lambda\Delta}-I)$ if and only if

$$M^{-1}AM = \lambda_i I = \Lambda \qquad (105)$$

Proof: see Appendix IV.

Theorem 6: (Delta Settling Time and Damping Regions)

Consider the plant described in Theorem 5 with eigenvalues λ_i in continuous time and eigenvalues γ_i for sampled data operation. The s-plane settling time region described by

$$\mathrm{Re}\ \lambda < \sigma \qquad (106)$$

maps into the γ-plane region described by

$$|1+\Delta\gamma_i| < e^{\sigma\Delta} \qquad (107)$$

and the s-plane damping region described by

$$\cos(\tan^{-1}(\mathrm{Im}\ \lambda_i/(-\mathrm{Re}\ \lambda_i))) > \zeta \qquad (108)$$

maps into the γ-plane region described by

$$|1+\Delta\gamma_i| < \exp(-\zeta\phi/(1-\zeta^2)^{1/2}) \qquad (109)$$

where

$$\phi = \arg(1+\Delta\gamma) \qquad (110)$$

Proof:see Appendix V.

Theorem 7: (Delta Eigenstructure Assignment Feedback Gain Matrix)

Suppose $u=F_\delta y$ with $F_\delta \in \mathbb{R}^{m\times r}$ such that the nominal closed loop system $A_\delta+B_\delta F_\delta C$ is asymptotically stable and non-defective. Let $\Lambda_{\delta r}$ be the r×r diagonal matrix whose entries are the assignable closed loop eigenvalues and let M_r be the n×r matrix whose columns are the corresponding achievable eigenvectors. If a solution exists for

$$(A_\delta + B_\delta F_\delta C)\, M_r = M_r \Lambda_{\delta r} \tag{111}$$

then

$$F_\delta = V_\delta \Sigma_\delta^{-1} U_{\delta 0}^T (M_r \Lambda_{\delta r} - A_\delta M_r) V_r \Sigma_r^{-1} U_{r0}^T \tag{112}$$

$$B_\delta = [U_{\delta 0} \quad U_{\delta 1}] \begin{bmatrix} \Sigma_\delta V_\delta^T \\ 0 \end{bmatrix} \tag{113}$$

and

$$CM_r = [U_{r0} \; U_{r1}] \begin{bmatrix} \Sigma_r V_r^T \\ 0 \end{bmatrix} \tag{114}$$

are the singular value decompositions of B_δ and CM_r, respectively. Furthermore, when $\Delta\to 0$, $F_\delta\to F$ where

$$F = V_B\Sigma_B^{-1}U_{B0}^T(M_r\Lambda_r - AM_r)V_r\Sigma_r^{-1}U_{r0}^T \qquad (115)$$

is the feedback gain for the continuous time plant which satisfies

$$(A + BFC)M_r = M_r\Lambda_r \qquad (116)$$

and where

$$B = [\ U_{B0} \quad U_{B1}\ ]\begin{bmatrix} \Sigma_B V_B^T \\ 0 \end{bmatrix} \qquad (117)$$

is a singular value decomposition of B and Λ_r is a diagonal rxr matrix containing the assignable eigenvalues $\lambda_i(i=1,2,\ldots,r)$.

Proof: see Appendix VI.

Theorem 8: (Unified Robustness Sufficient Condition)

Suppose that F_ρ is such that the nominal closed loop system described by Eq.(101) is asymptotically stable with $(A_\rho+B_\rho F_\rho C)$ non-defective. Then, the uncertain closed loop system given by Eq.(103) is asymptotically stable for dA_δ and dB_δ described by Eqs.(93) and (94), respectively if

$$\lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\gamma_i)} A_{\rho cmax} \right\} < 1 \qquad (118)$$

where

$$f(\gamma_i) = \begin{cases} -\mathrm{Re}(\gamma_i) & \text{continuous time} \\ \frac{1}{\Delta}\,[1-(1+\Delta\gamma_i)^+] & \text{discrete time} \end{cases} \tag{119}$$

$$A_{\rho c\max} = \begin{cases} A_{\max} + B_{\max}(FC)^+ & \text{continuous time} \\ A_{\delta\max} + B_{\delta\max}(F_\delta C)^+ & \text{discrete time} \end{cases} \tag{120}$$

$$A_{\delta\max} = \frac{1}{\Delta}\left[e^{(A^+ + A_{\max})\Delta} - e^{A^+\Delta}\right] \tag{121}$$

$$B_{\delta\max} = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A^+ + A_{\max})\tau}\,d\tau\,(B^+ + B_{\max}) - \int_0^\Delta e^{A^+\tau}\,d\tau\,B^+\right] \tag{122}$$

and where γ_i is the i-th eigenvalue of $(A_\rho + B_\rho F_\rho C)$ with v_i and w_i^* the corresponding right and left eigenvectors, respectively; and where $(\cdot)^*$ denotes the complex conjugate transpose.

Proof: see Appendix VII

4.4 ROBUST CONTROL DESIGN FOR THE EMRAAT MISSILE

We consider the Extended Medium Range Air to Air Technology (EMRAAT) bank-to-turn missile which was discussed in section 3.4. A seventh state x_7 representing a yaw rate washout filter with pole at -5 is added. The washed out yaw rate r_{wo} is fed back instead of yaw rate.

First, we design an eigenstructure assignment control law by using an orthogonal projection. The delta state space matrices A_δ and B_δ are computed by using the MATLAB™ Delta Toolbox [26]. The sampling period Δ is chosen to be 0.0025 seconds for illustrative purposes. The desired dutch roll and roll mode eigenvalues are achieved exactly because four measurements are available for feedback. The desired dutch roll eigenvectors are chosen to yield a complex mode which is composed of sideslip angle and yaw rate with no coupling to roll rate and integrated roll rate. The desired roll mode eigenvectors are chosen to yield a complex mode which is composed of roll rate and integrated roll rate (which is approximately equal to the bank angle) with no coupling to sideslip angle and yaw rate. Then, the achievable eigenvectors are computed by using the orthogonal projection of the i-th desired eigenvector v_i^d onto the subspace which is spanned by the columns of $(\gamma_i I-A_\delta)^{-1}B_\delta$. The closed loop delta eigenvalues γ_i; i=1,...,n and the feedback gain matrix F_δ are shown in Table 2. The desired and achievable closed loop eigenvectors are shown in Table 3.

TABLE 2. COMPARISON OF EMRAAT DESIGNS (Δ=0.0025 s)

	Closed Loop Eigenvalues[a]	Feedbackgain Matrix
Orthogonal Projection Design	$\gamma_{dr}=-23.65\pm j16.97$ $\gamma_{roll}=-9.98\pm j10.16$ $\gamma_{act}=-130.3$ $\gamma_{act}=-103.8$ $\gamma_{filter}=-6.97$	$\begin{array}{c} \begin{array}{cccc} \beta & r_{wo} & p & p_I \end{array} \\ \left[\begin{array}{cccc} -5.43 & .231 & .0043 & .959 \\ 4.42 & -.285 & .0050 & -1.09 \end{array}\right]\begin{array}{c} \delta_{rc} \\ \delta_{ac} \end{array} \end{array}$
Robust Design	$\gamma_{dr}=-18.26\pm j14.15$ $\gamma_{roll}=-61.39\pm j99.70$ $\gamma_{act}=-112.8$ $\gamma_{act}=-48.61$ $\gamma_{filter}=-7.42$	$\begin{array}{c} \begin{array}{cccc} \beta & r_{wo} & p & p_I \end{array} \\ \left[\begin{array}{cccc} -4.16 & .196 & .0154 & 1.27 \\ 2.36 & -.215 & .0757 & 2.16 \end{array}\right]\begin{array}{c} \delta_{rc} \\ \delta_{ac} \end{array} \end{array}$

[a] Eigenvalues are computed by using feedback gains which are rounded to three significant digits

TABLE 3 EIGENVECTORS FOR THE EMRAAT DESIGN[a]

Desired Closed Loop Eigenvectors:

$$\begin{array}{ccccccc} \begin{bmatrix}1\\x\\0\\0\\x\\x\\x\end{bmatrix} & \pm j & \begin{bmatrix}x\\1\\0\\0\\x\\x\\x\end{bmatrix} & & \begin{bmatrix}0\\0\\1\\x\\x\\x\\0\end{bmatrix} & \pm j & \begin{bmatrix}0\\0\\x\\1\\x\\x\\0\end{bmatrix} \begin{matrix}\beta\\r\\p\\p_I\\\delta_r\\\delta_a\\x_7\end{matrix} \end{array}$$

Dutch Roll Mode Roll Mode

Orthogonal Projection Design:

0.0258	0.0203	-.0083	-.0092	-.0012	0.0074	0.0327	β
1.0000	0.0000	0.0000	-.0003	-.0197	1.0000	0.3931	r
-.0006	0.0001	1.0000	0.0000	1.0000	0.1780	1.0000	p
0.0000	0.0000	-.0480	-.0501	-.0063	-.0015	-.1423	p_I
0.1309	-.1140	0.0037	0.0016	0.0024	0.7135	0.0053	δ_r
-.2382	0.0960	0.0231	0.0134	0.1258	-.9342	-.0880	δ_a
-.1385	-.1316	-.0001	0.0000	0.0006	-.0434	-.9699	x_7
Re v_{dr}	Im v_{dr}	Re v_{roll}	Im v_{roll}	actuator	actuator	filter	

Robust Design:

0.0312	0.0280	-.0006	-.0013	-.0034	-.0070	-.0693	β
1.0000	0.0000	-.0003	-.0002	-.0005	-.9184	-.4987	r
0.0038	-.1691	1.0000	0.0000	1.0000	0.4932	-.0468	p
-.0046	0.0055	-.0032	-.0073	-.0193	-.0037	0.0062	p_I
0.0958	-.0977	0.0063	-.0145	0.0075	-.7119	0.0016	δ_r
-.2083	0.0524	0.0319	-.0711	0.0393	1.0000	0.1779	δ_a
-.1678	-.1854	0.0000	0.0000	0.0001	0.0360	1.0000	x_7
Re v_{dr}	Im v_{dr}	Re v_{roll}	Im v_{roll}	actuator	actuator	filter	

---

[a] Eigenvectors are computed by using feedback gains which are rounded to three significant digits.

Next, we propose a new robust design which minimizes the integrated roll rate due to a one degree initial sideslip angle subject to constraints on the time constants of the dutch roll and roll modes, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. Mathematically, the objective function to be minimized is given by

$$J = \sum_{k=1}^{120} [p_I(k\Delta)]^2 \tag{123}$$

The upper limit on the index k is chosen to include the time interval $k\Delta \in [0,0.3]$ during which most of the transient response occurs. Of course, computation of Eq.(123) requires that a linear simulation be performed during each function evaluation of the optimization. The constraints for continuous time and the corresponding constraints for discrete time are shown in Table 4 where ζ is the damping ratio.

Table 4. Constraints for the EMRAAT Designs[a]

Continuous Time	Discrete Time
$\mathrm{Re}\ \lambda_{dr} \in [-50,-6]$	$\lvert 1+\Delta\gamma_{dr}\rvert \in [e^{-50\Delta}, e^{-6\Delta}]$
$\mathrm{Re}\ \lambda_{roll} \in [-50,-6]$	$\lvert 1+\Delta\gamma_{roll}\rvert \in [e^{-50\Delta}, e^{-6\Delta}]$
$\lambda_{rudder} < -50$	$\lvert 1+\Delta\gamma_{rudder}\rvert < e^{-50\Delta}$
$\lambda_{aileron} < -50$	$\lvert 1+\Delta\gamma_{aileron}\rvert < e^{-50\Delta}$
$\zeta_{dr} \in [0.4,0.8]$	$\lvert 1+\Delta\gamma_{dr}\rvert \in [\xi_{dr1},\xi_{dr2}]$
$\zeta_{roll} \in [0.4,0.8]$	$\lvert 1+\Delta\gamma_{roll}\rvert \in [\xi_{roll1},\xi_{roll2}]$
$\lvert\dot{\delta}_a\rvert < 275$ deg/s	$\dfrac{\lvert\delta_a[(k+1)\Delta]-\delta_a(k\Delta)\rvert}{\Delta} < 275$ deg/s
$\lvert\dot{\delta}_r\rvert < 275$ deg/s	$\dfrac{\lvert\delta_r[(k+1)\Delta]-\delta_r(k\Delta)\rvert}{\Delta} < 275$ deg/s
$\eta_c < 0.999$	$\eta_d < 0.999$

[a] Definitions of ξ_{dr1}, ξ_{dr2}, ξ_{roll1}, ξ_{roll2}, η_c and η_d are shown on page 74.

$$\xi_{dr1} = \exp\left(\frac{-0.8\phi_{dr}}{[1-(0.8)^2]^{1/2}}\right); \xi_{dr2} = \exp\left(\frac{-0.4\phi_{dr}}{[1-(0.4)^2]^{1/2}}\right)$$

$$\phi_{dr} = \arg(1+\Delta\gamma_{dr})$$

$$\xi_{roll1} = \exp\left(\frac{-0.8\phi_{roll}}{[1-(0.8)^2]^{1/2}}\right); \ \xi_{roll2} = \exp\left(\frac{-0.4\phi_{roll}}{[1-(0.4)^2]^{1/2}}\right)$$

$$\phi_{roll} = \arg(1+\Delta\gamma_{roll})$$

$$\eta_c = \lambda_{max}\left\{ \sum_{i=1}^{7} \frac{(v_i w_i^*)^+}{\alpha_i} \ [A_{max} + B_{max}(FC)^+] \right\}$$

$$\eta_d = \lambda_{max}\left\{ \sum_{i=1}^{7} \frac{(v_i w_i^*)^+}{f(\gamma_i)} \ [A_{\delta max} + B_{\delta max}(F_\delta C)^+] \right\}$$

For illustrative purposes we have chosen $A_{max} = 0.04 \cdot A^+$ and $B_{max} = 0$ with the exception that the elements of A_{max} which correspond to actuator or washout filter time constants have been set to zero. The matrices $A_{\delta max}$ and $B_{\delta max}$ are computed from Eqs.(121) and (122), respectively.

The actuator deflection rates are computed from the slopes of the time responses of the deflections during the time interval $k\Delta \in [0,0.03]$. This interval is chosen because the slopes of the deflections are largest during this time interval. Mathematically,

$$\frac{|\delta_a[(k+1)\Delta]-\delta_a(k\Delta)|}{\Delta} < \frac{\max|\delta_a(m\Delta)|}{m\Delta} \tag{124}$$

$$\frac{|\delta_r[(k+1)\Delta]-\delta_r(k\Delta)|}{\Delta} < \frac{\max|\delta_r(m\Delta)|}{m\Delta} \tag{125}$$

where $m=0,1,\ldots,12$. The maximum deflection rates chosen for the constraints are well within the expected 400 deg/s limit for the advanced state of the art electromechanical actuator described by Langehough and Simons [25].

The parameter vector contains the quantities which may be varied by the optimization. This twelve dimensional vector includes Re γ_{dr}, Im γ_{dr}, Re γ_{roll}, Im γ_{roll}, Re $z_1(1)$, Re $z_1(2)$, Im $z_1(1)$, Im $z_1(2)$, Re $z_3(1)$, Re $z_3(2)$, Im $z_3(1)$, Im $z_3(2)$. Here, the two dimensional complex vectors z_i contain the free eigenvector parameters. That is, the i-th eigenvector v_i may be written as

$$v_i = L_i z_i \tag{126}$$

where the columns of $L_i=(\gamma_i I-A_\delta)^{-1}B_\delta$ are a basis for the subspace in which the i-th eigenvector must reside. Thus, the free parameters are the vectors z_i rather than the eigenvectors v_i.

The optimization is performed by using subroutine *constr* from the MATLAB™ Optimization Toolbox [23] and subroutine *delsim* from the MATLAB™ Delta Toolbox [26] on a 486™ 25MHz personal computer. The optimization is initialized with the orthogonal projection design which yields an initial value of 9.1360 for the objective function of Eq.(123) and a value of 2.2014 for the left hand side (LHS) of the robustness condition of Eq.(118). The optimization is complete after 3640 function evaluations and yields an optimal objective function of 0.0895 and a value of 0.999 for the LHS of the robustness condition. We observe from Table 2 that the dutch roll mode is dominant in the robust design whereas the roll mode was chosen to be dominant in the orthogonal projection design. Furthermore, the optimization moves the roll mode eigenvalues to the boundary of the feasible set which corresponds to the smallest time constant and the smallest damping ratio allowed by the constraints. We observe from Table 3 that the yaw rate washout filter eigenvector for the robust design is characterized by a significant reduction in the roll rate and integrated roll rate entries as compared with the orthogonal projection design. However, the optimized dutch roll eigenvectors exhibit an increase in the roll rate and integrated roll rate entries which were desired to be zero in the orthogonal projection design. We conjecture that the optimization alters the filter eigenvector in order to improve mode decoupling whereas the dutch roll mode eigenvector is altered in order to satisfy the robustness constraint.

The time histories of integrated roll rate, rudder deflection, and aileron deflection to a one degree initial sideslip are shown in Figures 10, 11 and 12, respectively. We observe a significant improvement in the integrated roll rate response (which is desired to be zero) when compared to the initial orthogonal projection eigenstructure assignment design. The initial design has a minimum $p_I(t)$ of -.464 deg but the new design of this chapter has a minimum $p_I(t)$ of -.103 deg which is an improvement of approximately 78%. We note that this improved response is obtained with both smaller aileron and rudder deflections. Furthermore, it is interesting that the aileron in the initial design exhibits an initial positive deflection of approximately three degrees before becoming negative, whereas this positive initial aileron deflection is only approximately one degree in the robust design.

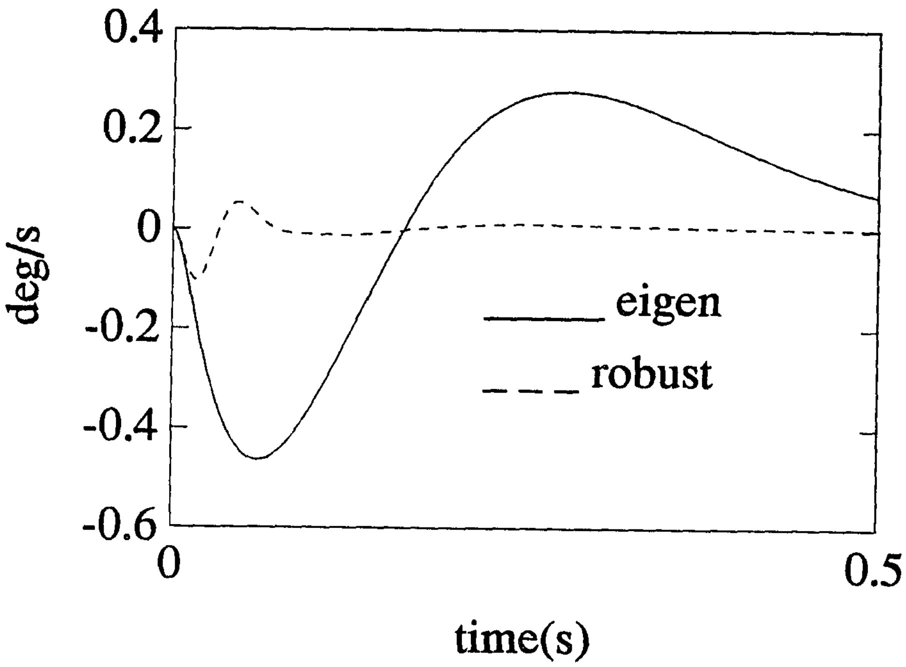

Figure. 10. Integrated Roll Rate $\beta(0)=1$ degree.

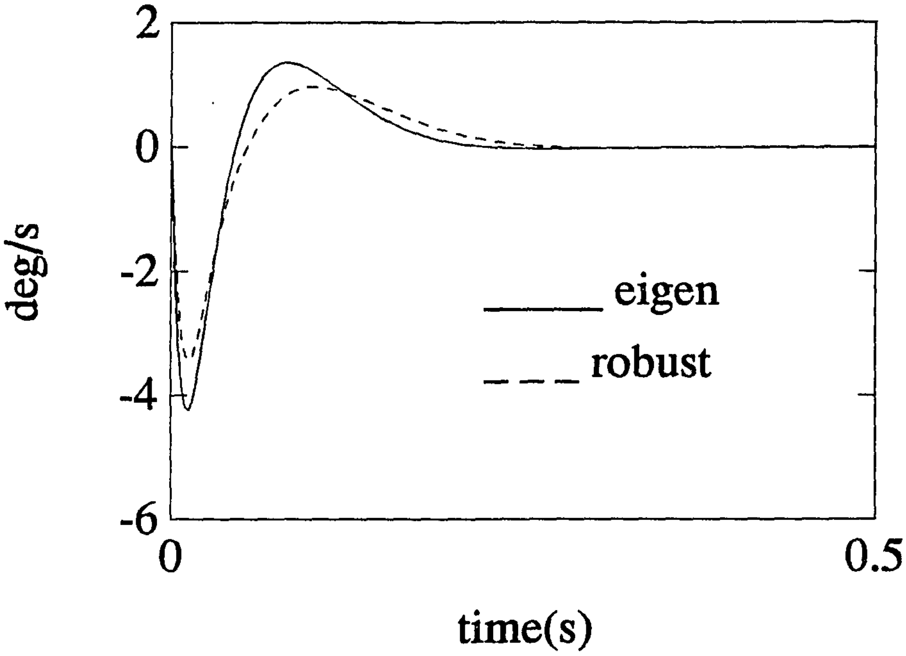

Figure 11. Rudder Deflection $\beta(0)=1$ degree.

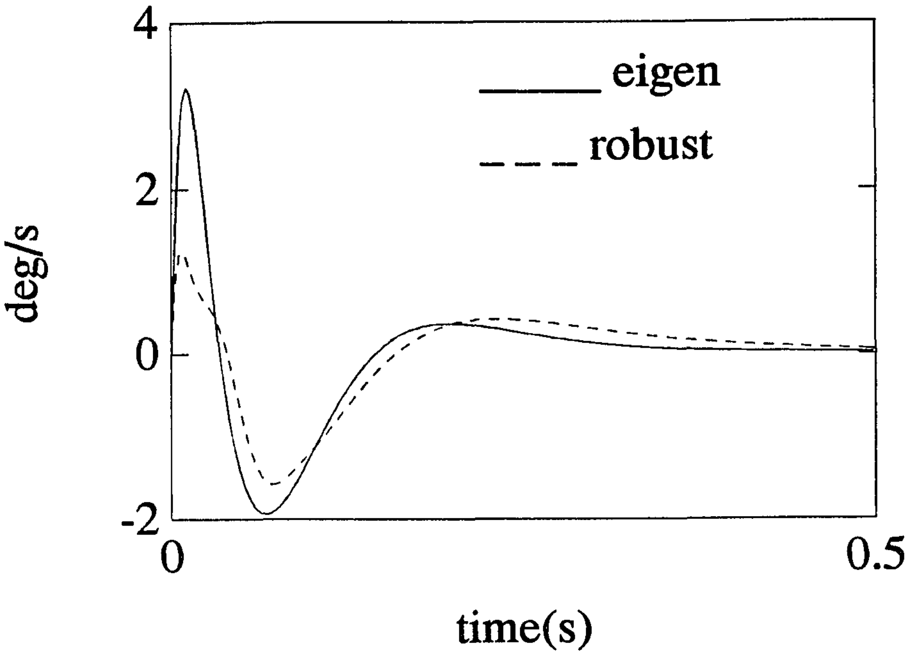

Figure. 12. Aileron Deflection $\beta(0)=1$ degree.

Next, we compute the time responses due to a "1-cosine" sideslip gust as described in MIL-F-8785-C [27]. The state equations for the lateral dynamics are shown by McRuer et. al.[28] to be given by

$$\dot{\beta} = Y_v\beta + (g/U_0)\phi - r + (Y_{\delta_a}/U_0)\delta_a + (Y_{\delta_r}/U_0)\delta_r - Y_v\beta_g \qquad (127)$$

$$\dot{p} = L'_\beta\beta + L'_p p + L'_r r + L'_{\delta_a}\delta_a + L'_{\delta_r}\delta_r - L'_\beta\beta_g - (L'_r)_g\,\dot{\beta}_g \qquad (128)$$

$$\dot{r} = N'_\beta\beta + N'_p p + N'_r r + N'_{\delta_a}\delta_a + N'_{\delta_r}\delta_r - N'_\beta\beta_g - (N'_r)_g\,\dot{\beta}_g \qquad (129)$$

where

Y_v, Y_v/U_0, Y_{δ_a}/U_0, Y_{δ_r}/U_0, L'_β, L'_p, L'_r, L'_{δ_a}, L'_{δ_r}, N'_β, N'_p, N'_r, N'_{δ_a}, and N'_{δ_r} can be obtained from the state space matrix A and where $(N'_r)_g$ and $(L'_r)_g$ are defined in Ref.28 to be

$$(N'_r)_g = N_r\left[1 - \frac{I_{xz}^2}{I_x I_z}\right]^{-1} \tag{130}$$

$$(L'_r)_g = \left(\frac{I_{xz}}{I_x}\right) N_r\left[1 - \frac{I_{xz}^2}{I_x I_z}\right]^{-1} \tag{131}$$

$$N_r = N'_r - \frac{I_{xz}}{I_z} L'_r \tag{132}$$

For the flight condition of the EMRAAT missile considered in this chapter, the parameters N'_r and L'_r are -0.5748 and 0.3208, respectively. The inertias corresponding to a full fuel condition are $I_x = 11451$, $I_z = 456282$, and $I_{xz} = -1189$. Using these values, the gust derivative coefficients are computed to be $(N'_r)_g = -0.5742$ and $(L'_r)_g = 0.05962$. The gust is defined as shown in MIL-F-8785-C [27] and is described by

$$\beta_g = \begin{cases} 0 & t<0 \\ 0.5(1-\cos 24\pi t) & 0 \le t \le 1/24 \\ 1 & t > 1/24 \end{cases}$$

where the natural frequency of the open loop complex eigenvalue pair is 24.04 rad/s.

The time histories of integrated roll rate, rudder deflection, and aileron deflection to the "1-cosine" sideslip gust are shown in Figures 13, 14 and 15, respectively. We observe a significant improvement in the integrated roll rate response (which is desired to be zero) when compared to the initial orthogonal projection eigenstructure assignment design. The initial design has a maximum $p_I(t)$ of 18.87 deg but our new design has a maximum $p_I(t)$ of 0.6990 deg which is an improvement of approximately 96%. We note that this improved response is obtained with both smaller aileron and rudder deflections.

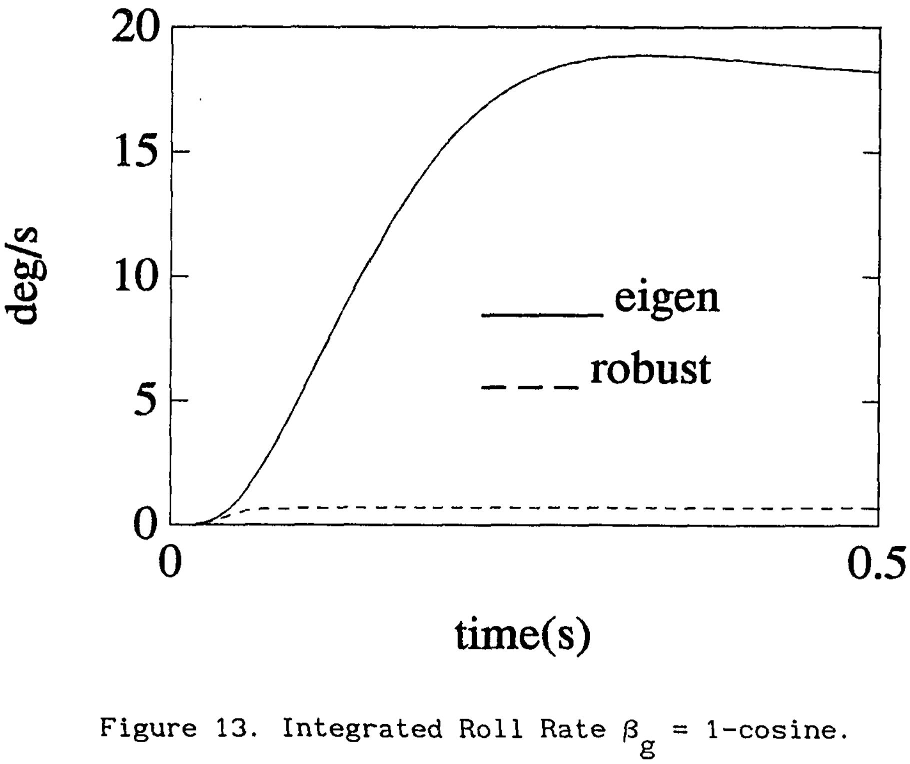

Figure 13. Integrated Roll Rate β_g = 1-cosine.

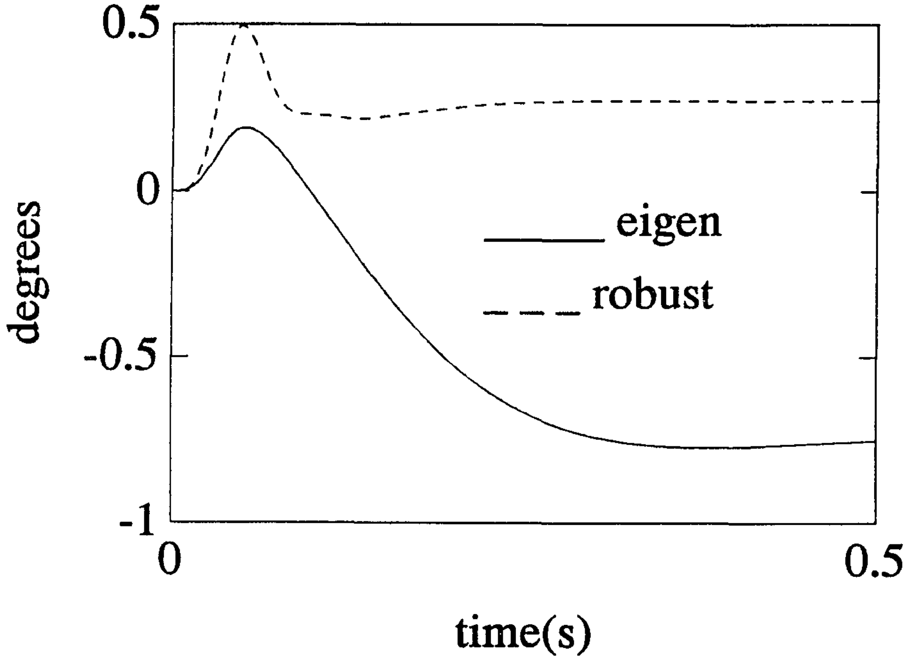

Figure 14. Rudder Deflection β_σ = 1-cosine.

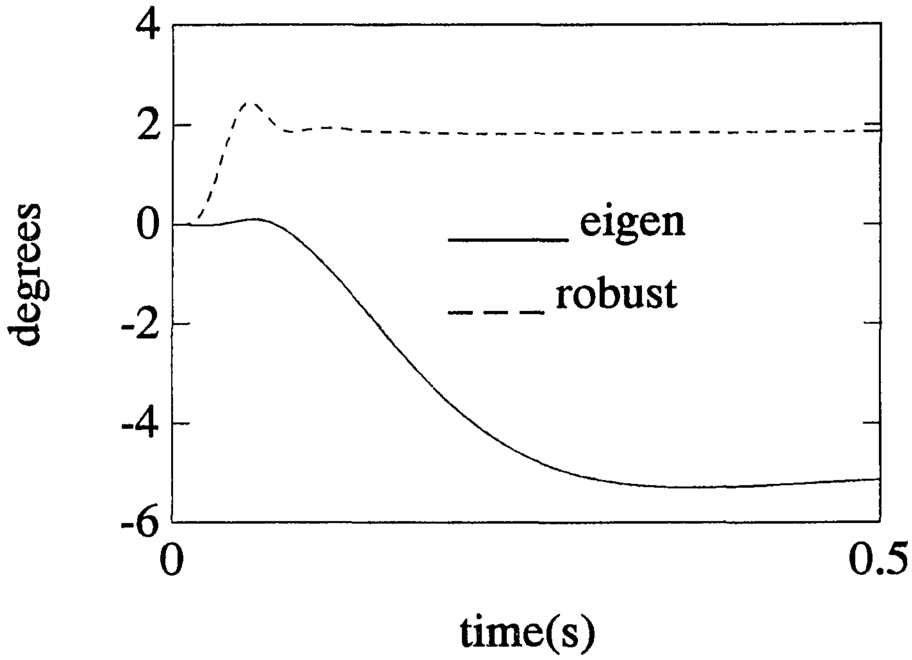

Figure 15. Aileron Deflection β_g = 1-cosine.

5. ROBUSTNESS OF DELTA OPERATOR SYSTEMS USING A LYAPUNOV APPROACH

5.1 INTRODUCTION

We extend a robust stability result which uses a Lyapunov approach to delta operator systems. A robust sampled data eigenstructure assignment flight control law is designed for the yaw pointing/lateral translation maneuver of the Flight Propulsion Control Coupling aircraft. The flight control design uses the so-called unified delta model which is valid for both continuous time and sampled data operation of the aircraft. We choose an objective function which weights both the heading angle due to a lateral flight path command and the lateral flight path angle due to a heading command. The new design method minimizes the objective function with constraints on the time constants of the dutch roll, roll and lateral flight path modes, the damping ratios of the dutch roll and roll modes, the new Lyapunov sufficient condition for robust stability, and the minimum of the smallest singular value of the return difference matrix at the aircraft inputs. This design satisfies the new robustness condition, yields a minimum of the smallest singular value of the return difference matrix at the aircraft inputs of 0.55 and yields an improved transient response as compared to the earlier design of Ref.13.

5.2 EXTENSION TO THE PSEUDO CONTROL STRATEGY

To explain the extension to the pseudo control strategy, consider the singular value decomposition of the matrix B_ρ which is given by

$$B_\rho = U\Sigma V^T = [U_3 \; U_0]\begin{bmatrix} \Sigma_3 & \\ & 0 \end{bmatrix}\begin{bmatrix} V_3^T \\ V_0^T \end{bmatrix} \tag{133}$$

where U is the matrix of left singular vectors, V is the matrix of right singular vectors, and Σ is a diagonal matrix containing the singular values in the order of descending magnitude. Suppose we partition Σ_3 as follows:

$$\Sigma_3 = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \tag{134}$$

where $\Sigma_1 = \text{diag}\,[\sigma_1,\ldots,\sigma_a]$ and $\Sigma_2 = \text{diag}\,[\sigma_{a+1},\ldots,\sigma_b]$ and where $\sigma_b \le \sigma_{b-1} \le \ldots \le \sigma_{a+1} \le \varepsilon$ with ε not necessarily close to zero. If we partition U_3 and V_3 conformally with Σ_3, then we rewrite Eq.(133) as follows:

$$B_\rho = [U_1\ U_2\ U_0]\begin{bmatrix} \Sigma_1 & & \\ & \Sigma_2 & \\ & & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \\ V_0^T \end{bmatrix} \tag{135}$$

Lemma 1:

Let the system with the pseudo control $\delta(t)$ be described by

$$\rho x(t) = A_\rho x(t) + \tilde{B}_\rho \delta(t) \tag{136}$$

$$y(t) = Cx(t) \tag{137}$$

$$\tilde{B}_\rho = U_1 + U_2[\alpha_1,\ \alpha_2] \tag{138}$$

We design a feedback pseudo control for the system described Eqs.(136)-(138). Then, the true control u(t) for the system described by Eq.(78) is given by

$$u(t) = [V_1\Sigma_1^{-1} + V_2\Sigma_2^{-1}\alpha]\delta(t) \tag{139}$$

Proof: see Appendix VIII

Remark 6:

When $\alpha=[0,\ 0]$, the control law u(t) given by Eqs.(139) reduces to the control law given by Eq.(20) in Ref.12.

5.3 STABILITY ROBUSTNESS FOR DELTA SYSTEMS USING A LYAPUNOV APPROACH

In this section, a new stability robustness sufficient condition is derived which extends the work of Yedavalli [4] to delta operator systems.

Theorem 9:

The system matrix $(A_{\rho c} + dA_{\rho c})$ of Eq.(103) is stable if

$$\sigma_{max}(E^{T}_{\rho max}\ P^{+}_{\rho}\ A_{\rho cmax})_{s} < 1 \tag{140}$$

where

$$A_{\rho cmax} = A_{\rho max} + B_{\rho max}(FC)^{+} \tag{141}$$

$$E_{\rho max} = \{I_{n} + \Delta[A_{\rho} + B_{\rho}(F_{\rho}C)]\}^{+} + (\Delta/2)A_{\rho cmax} \tag{142}$$

and where P_{ρ} satisfies the Lyapunov equation given by

$$A^{T}_{\rho c}P_{\rho} + P_{\rho}A_{\rho c} + \Delta A_{\rho c}P_{\rho}A^{T}_{\rho c} = -2I_{n}$$

and where P^{+}_{ρ} is the matrix formed by the modulus of the entries of the matrix P_{ρ}.

Proof: see Appendix IX

5.4 YAW POINTING/LATERAL TRANSLATION CONTROL LAW DESIGN

We consider the FPCC aircraft linearized lateral dynamics described by

$$\frac{d}{dt}\begin{bmatrix}\beta\\ \phi\\ p\\ r\\ \gamma\end{bmatrix} = \begin{bmatrix} -0.340 & 0.0517 & 0.001 & -0.997 & 0\\ 0 & 0 & 1 & 0 & 0\\ -2.69 & 0 & -1.15 & 0.738 & 0\\ 5.91 & 0 & 0.138 & -0.506 & 0\\ -0.340 & 0.0517 & 0.001 & 0.0031 & 0\end{bmatrix}\begin{bmatrix}\beta\\ \phi\\ p\\ r\\ \gamma\end{bmatrix}$$

$$+\begin{bmatrix} 0.0755 & 0 & 0.0246\\ 0 & 0 & 0\\ 4.48 & 5.22 & -0.742\\ -5.03 & 0.0998 & 0.984\\ 0.0755 & 0 & 0.0246\end{bmatrix}\begin{bmatrix}\delta_r\\ \delta_a\\ \delta_c\end{bmatrix}$$

The state variables are sideslip angle β, bank angle ϕ, roll rate p, and lateral directional flight path angle ($\gamma=\psi+\beta$), where ψ is the heading angle. The control variables are rudder δ_r, ailerons δ_a, and vertical canard δ_c. The angles and surface deflections are in degrees, and the angular rates are in degrees per second. The five measurements are β, ϕ, p, r, γ. First, we design an eigenstructure assignment control law by using an orthogonal projection. The delta state space matrices A_δ and B_δ are computed by using the MATLAB™ Delta Toolbox [26]. The sampling period Δ is chosen to be 0.02 seconds for illustrative purposes. The desired dutch roll, roll mode, and flight path mode eigenvalues are achieved exactly because five measurements are available for feedback. The achievable eigenvectors are computed by using the orthogonal projection of the i-th desired eigenvector v_i^d onto the subspace which is spanned by the columns of $(\gamma_i I-A_\delta)^{-1}B_\delta$. The closed loop delta eigenvalues γ_i; $i=1,\ldots,n$ and the feedback gain matrix F_δ are shown in Table 5. The desired and achievable closed loop eigenvectors are shown in Table 6.

TABLE 5. COMPARISON OF FPCC DESIGNS (Δ=0.02 s)

Closed Loop Eigenvalues[a]	Feedback Gain Matrix
Orthogonal Projection Design $\gamma_{dr}=-1.999\pm j1.921$ $\gamma_{roll}=-2.95\pm j1.883$ $\gamma_{fp}=-0.4975$	$\begin{array}{ccccc c} \beta & \phi & p & r & \gamma & \\ 1.4688 & -.2866 & -.0022 & .3799 & -1.5332 & \delta_r \\ .5652 & -2.299 & -.9044 & -.6691 & -.8890 & \delta_a \\ 9.2964 & -1.2143 & -.0514 & -1.4219 & -15.57 & \delta_c \end{array}$
Orthogonal Projection Design with Pseudo Control $\gamma_{dr}=-1.999\pm j1.921$ $\gamma_{roll}=-2.95\pm j1.883$ $\gamma_{fp}=-0.4975$	$\begin{array}{ccccc c} \beta & \phi & p & r & \gamma & \\ -.4929 & -.0332 & .0076 & .6883 & 2.7584 & \delta_r \\ .9517 & -2.353 & -.9093 & -.7565 & -2.5147 & \delta_a \\ .1203 & -.0530 & -.0245 & -.1535 & -.6023 & \delta_c \end{array}$
Robust Pseudo Control Design $\gamma_{1,2}=-3.37\pm j1.56$ $\gamma_{3,4}=-3.1 6\pm j1.473$ $\gamma_{fp}=-0.4 310$	$\begin{array}{ccccc c} \beta & \phi & p & r & \gamma & \\ -0.2833 & -.0340 & -.0062 & .2377 & .5336 & \delta_r \\ 1.5501 & -2.476 & -1.037 & -.9812 & -1.7788 & \delta_a \\ 5.6151 & -.0012 & -.1486 & -4.6301 & -10.285 & \delta_c \end{array}$
Robust Pseudo Control Design with Singular value Constraint $\gamma_{1,2}=-2.55\pm j1.19$ $\gamma_{3,4}=-2.60\pm j1.22$ $\gamma_{fp}=-0.3248$	$\begin{array}{ccccc c} \beta & \phi & p & r & \gamma & \\ -0.1658 & -.1967 & -.0808 & .4895 & .8616 & \delta_r \\ 0.8390 & -1.5479 & -.7663 & -.8565 & -1.2679 & \delta_a \\ 1.1525 & -.8794 & -.4764 & -1.9712 & -3.2578 & \delta_c \end{array}$

[a] Eigenvalues are computed by using feedback gains with significant digits to machine precision.

TABLE 6 EIGENVECTORS FOR THE FPCC DESIGNS[a]

Desired Closed Loop Eigenvectors For Orthogonal Projection Design:

$$\begin{bmatrix} x \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \pm j \begin{bmatrix} 1 \\ 0 \\ 0 \\ x \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ x \\ 1 \\ 0 \\ 0 \end{bmatrix} \pm j \begin{bmatrix} 0 \\ 1 \\ x \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \beta \\ \phi \\ p \\ r \\ \gamma \end{matrix}$$

Dutch Roll Mode   Roll Mode   Flight Path Mode

Orthogonal Projection Design:

Re v_{dr}	Im v_{dr}	Re v_{roll}	Im v_{roll}	v_{fp}	
0.2501	0.2500	0.0000	0.0000	1.0000	β
0.0000	0.0000	-.2308	-.1538	0.0000	ϕ
0.0000	0.0000	1.0000	0.0000	0.0000	p
1.0000	0.0000	0.0000	0.0000	0.0000	r
0.0000	0.0000	0.0000	0.0000	0.9998	γ

Orthogonal Projection Design with Pseudo Control:

Re v_{dr}	Im v_{dr}	Re v_{roll}	Im v_{roll}	v_{fp}	
0.2347	0.2849	0.0010	0.0032	1.0000	β
0.0000	0.0001	-.2308	-.1538	-.0184	ϕ
-.0002	-.0001	1.0000	0.0000	0.0092	p
1.0000	0.0000	-.0008	-.0005	0.2385	r
-.0155	0.0349	0.0011	0.0034	0.5228	γ

Robust Pseudo Control Design:

Re v_1	Im v_1	Re v_3	Im v_3	v_{fp}	
-.0360	0.2094	0.0944	-.2017	0.9995	β
-.2342	-.1133	-.2501	-.1211	0.0683	ϕ
1.0000	0.0000	1.0000	0.0000	-.0295	p
0.2008	0.6631	-.0465	-.6912	-.0003	r
-.0079	0.0313	0.0223	-.0231	1.0000	γ

Robust Pseudo Control Design with Singular Value Constraint:

Re v_1	Im v_1	Re v_3	Im v_3	v_{fp}	
-.2194	0.2878	0.0876	0.0016	0.9567	β
-.3117	0.1509	-.3050	-.1477	-.0363	ϕ
1.0000	0.0000	1.0000	0.0000	0.0118	p
-.1569	0.9201	0.1828	-.1099	-.0142	r
-.0316	0.0246	0.0155	0.0082	1.0000	γ

---

[a] Eigenvectors are computed by using feedback gains with significant digits to machine precision.

The yaw pointing and lateral translation time responses for the orthogonal projection design are shown in Figures 16 and 17. The orthogonal projection solution is characterized by excellent decoupling with the minimum of the smallest singular value of (I-FG) equal to 0.18. Here the transfer function matrix of the delta plant is given by $G([e^{j\omega\Delta}-1]/\Delta)$ where $0<\omega<\pi/T$. Furthermore, the Lyapunov robust stability condition of Eq.(140) is not satisfied.

In order to improve the minimum singular value of (I-FG) we design a controller by using an orthogonal projection with the pseudo control of Ref.12. This pseudo control mapping is given by Eq.(138) with $\alpha=[0,0]$. The time responses for yaw pointing and lateral translation are shown in Figures 18 and 19. This design is characterized by a lateral translation response with significant coupling between γ and ψ with the minimum of the smallest singular value of (I-FG) equal to 0.9835. The Lyapunov sufficient robust stability condition of Eq.(140) is not satisfied. We note that the yaw pointing responses exhibit excellent decoupling for both of the orthogonal projection designs.

Next, in an attempt to obtain a robust design with excellent decoupling, we propose a new robust pseudo control design. This new design method minimizes an objective function which weights the heading angle due to a lateral

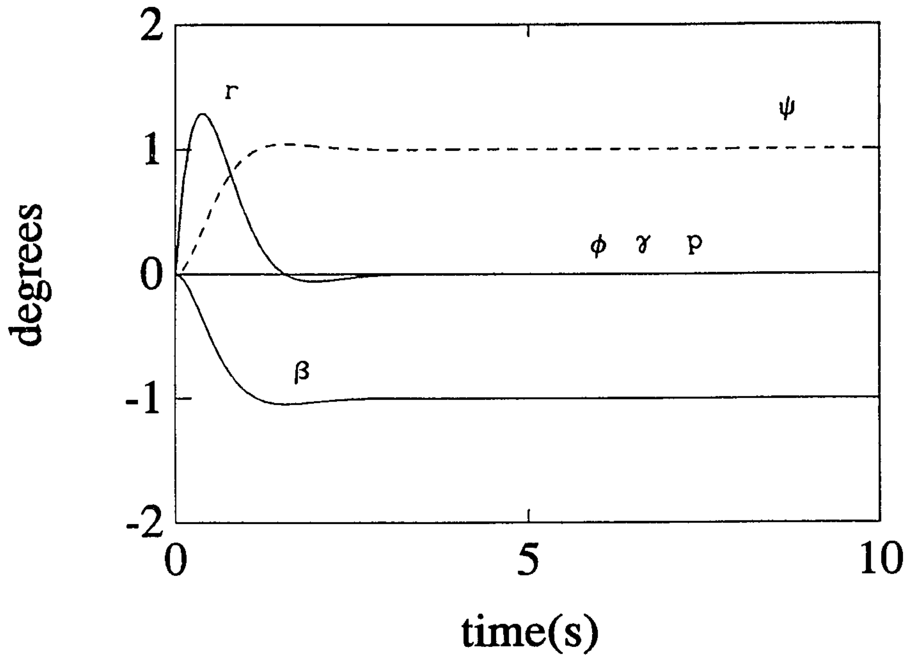

Figure.16. Yaw Pointing-Orthogonal Projection

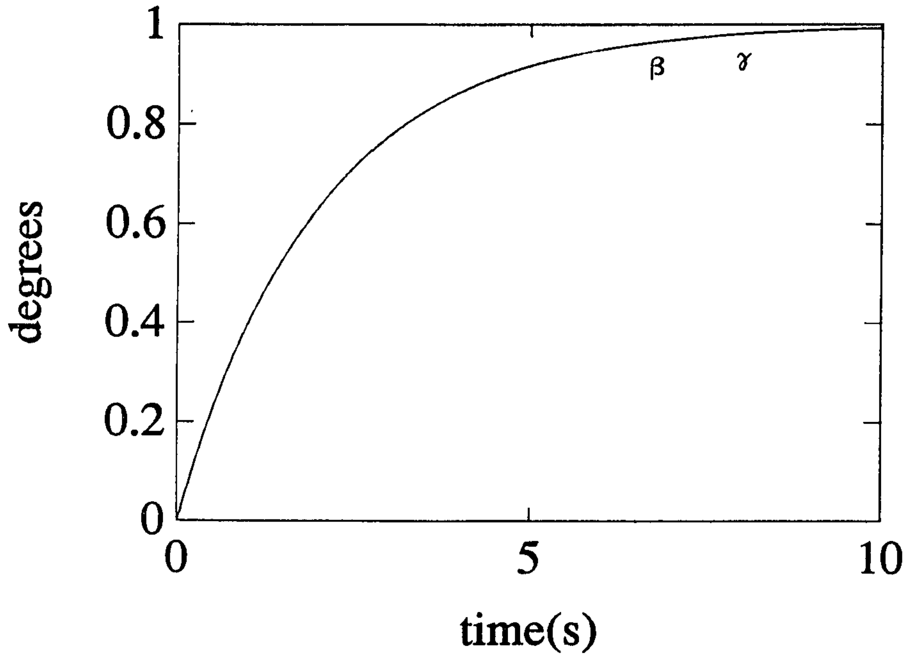

Figure 17. Lateral Translation-Orthogonal Projection.

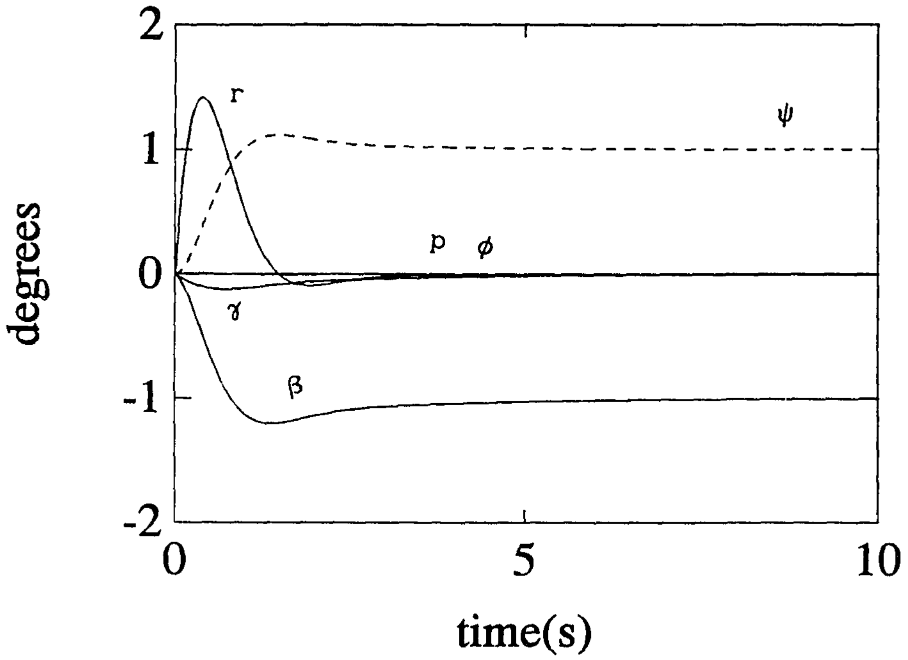

Figure 18. Yaw Pointing-Orthogonal Projection with Pseudo

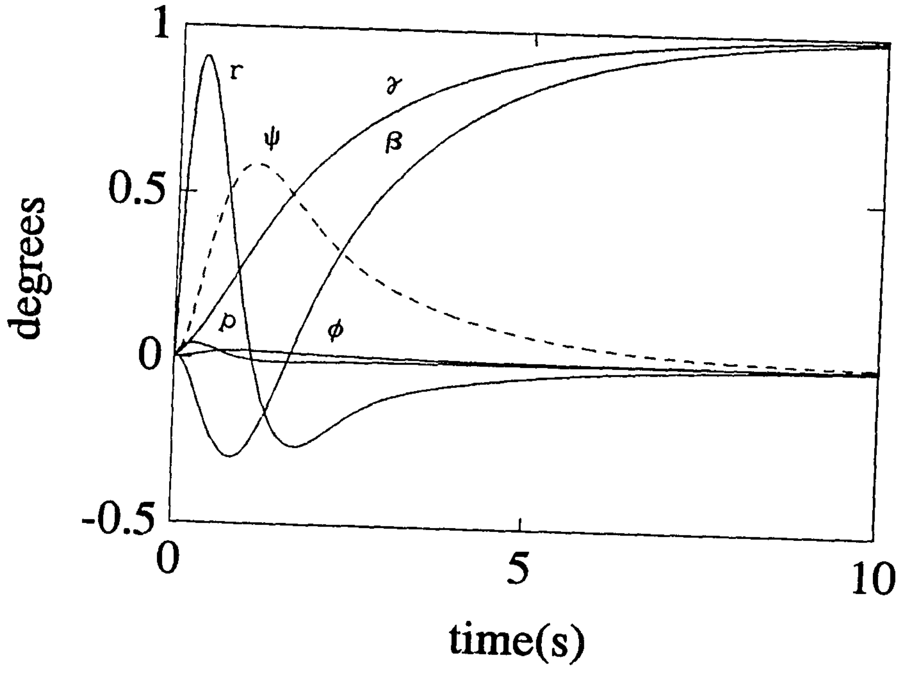

Figure 19. Lateral Translation-Orthogonal Projection with Pseudo Control.

flight path angle command and the lateral flight path angle due to a heading command. Constraints are placed on the time constants of the dutch roll, roll, and flight path modes, the damping ratios of the dutch roll and roll modes, and the new sufficient condition for robust stability. Mathematically, the objective function to be minimized is given by

$$J = \sum_{k=1}^{100} [(1-a)(\psi_k^2)_{\gamma_c} + a(\gamma_k^2)_{\psi_c}] \tag{143}$$

The upper limit on the index k is chosen to include the time interval $k\Delta \in [0,2]$ during which most of the transient response occurs. Of course, computation of Eq.(143) requires that two linear simulations be performed during each function evaluation of the optimization. The constraints for continuous time and the corresponding constraints for discrete time are shown in Table 7 where ζ is the damping ratio.

TABLE 7 Constraints for the FPCC Designs

Continuous Time	Discrete Time
For complex eigenvalues:	
Re $\lambda \in [-4,-1.5]$	$\lvert 1+\Delta\gamma \rvert \in [e^{-4\Delta}, e^{-1.5\Delta}]$
$\zeta \in [0.4,0.9]$	$\lvert 1+\Delta\gamma \rvert \in [\xi_1,\xi_2]$
For the real eigenvalue:	
$\lambda \in [-1,-0.05]$	$\lvert 1+\Delta\gamma \rvert \in [e^{-\Delta}, e^{-0.05\Delta}]$
For Lyapunov robustness:	
	$\sigma_{max}(E^T_{\rho max} P^+_{\rho} A_{\rho cmax}) < 0.999$
For multivariable stability margins (final design only):	
	$\min_{\omega} \sigma_{min}(I-FG) \geq 0.55;\ 0<\omega<\pi/T$

where

$$\xi_1 = \exp\left(\frac{-0.9\phi}{[1-(0.9)^2]^{1/2}}\right); \qquad \xi_2 = \exp\left(\frac{-0.4\phi}{[1-(0.4)^2]^{1/2}}\right)$$

$$\phi = \arg(1+\Delta\gamma)$$

For illustrative purposes we have chosen $A_{max} = 0.085 \cdot A^{+}$ and $B_{max} = 0$. After many trials, we found that a good value for the weight a in Eq.(143) is a=0.0075.

The parameter vector contains the quantities which may be varied by the optimization. This seventeen dimensional vector includes Re γ_{dr}, Im γ_{dr}, Re γ_{roll}, Im γ_{roll}, γ_{fp}, Re $z_1(1)$, Re $z_1(2)$, Im $z_1(1)$, Im $z_1(2)$, Re $z_3(1)$, Re $z_3(2)$, Im $z_3(1)$, Im $z_3(2)$, $z_5(1)$, $z_5(2)$, and the two dimensional pseudo control vector α of Eq.(138). Here, the two dimensional complex vectors z_i contain the free eigenvector parameters. That is, the i-th eigenvector v_i may be written as

$$v_i = L_i z_i \tag{144}$$

where the columns of $L_i = (\gamma_i I - A_\delta)^{-1} \tilde{B}_\delta$ are a basis for the subspace in which the i-th eigenvector must reside. Thus, the free parameters are the vectors z_i rather than the eigenvectors v_i. The vectors z_i are two dimensional because the optimization is performed in the two dimensional pseudo control space.

The optimization uses subroutine *constr* from the MATLAB™ Optimization Toolbox [23] and subroutine *delsim* from the MATLAB™ Delta Toolbox [26] on a 486™ 25MHz personal computer. The optimization is initialized with the orthogonal projection pseudo control design which yields an initial value of 20.6 for the objective function of Eq.(143) and a value of 1.66 for the right hand side (RHS) of the robustness condition of Eq.(140). The optimization yields an optimal objective function of 0.0107 and a value of 0.999 for the RHS of the robustness condition. We observe from Table 6 that both orthogonal projection designs have clearly distinct and decoupled dutch roll and roll modes whereas the robust design has two complex modes which are both coupled to roll rate. The time responses for yaw pointing and lateral translation are shown in Figures 20 and 21 from which we observe the excellent decoupling which has been achieved. Unfortunately, the minimum of the smallest singular value of (I-FG) is only 0.2607 which is less than desired.

In an attempt to achieve a design with excellent time responses, Lyapunov robustness, and an acceptable minimum of the smallest singular value of (I-FG), we repeat the optimization with the additional constraint that $\sigma_{min}(I-FG) \geq 0.55$. Once again we initialize the optimization at the orthogonal projection pseudo control design. The optimization yields an optimal objective function of 0.0556 and a value of 0.999 for the RHS of the robustness condition. The time responses for yaw pointing and lateral translation are shown in Figures 22 and 23. The lateral translation response is deemed to be excellent even though it has some small increase in coupling as compared to the design without the additional singular value constraint. Thus, we have obtained a controller which simultaneously achieves excellent time responses, Lyapunov robustness, and an acceptable minimum of the smallest singular value of the return difference matrix at the aircraft inputs.

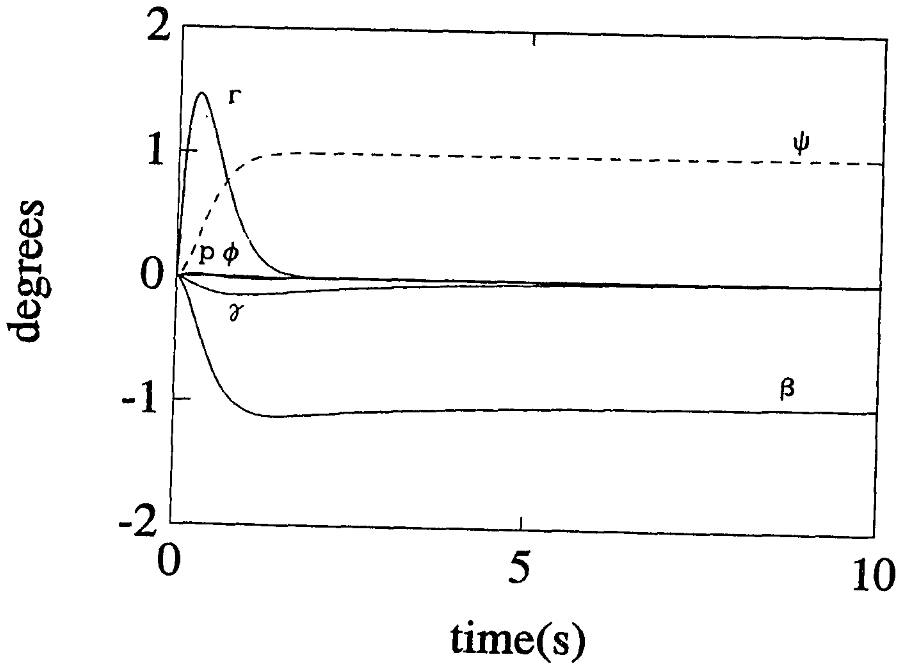

Figure 20. Yaw Pointing-Robust Pseudo Control.

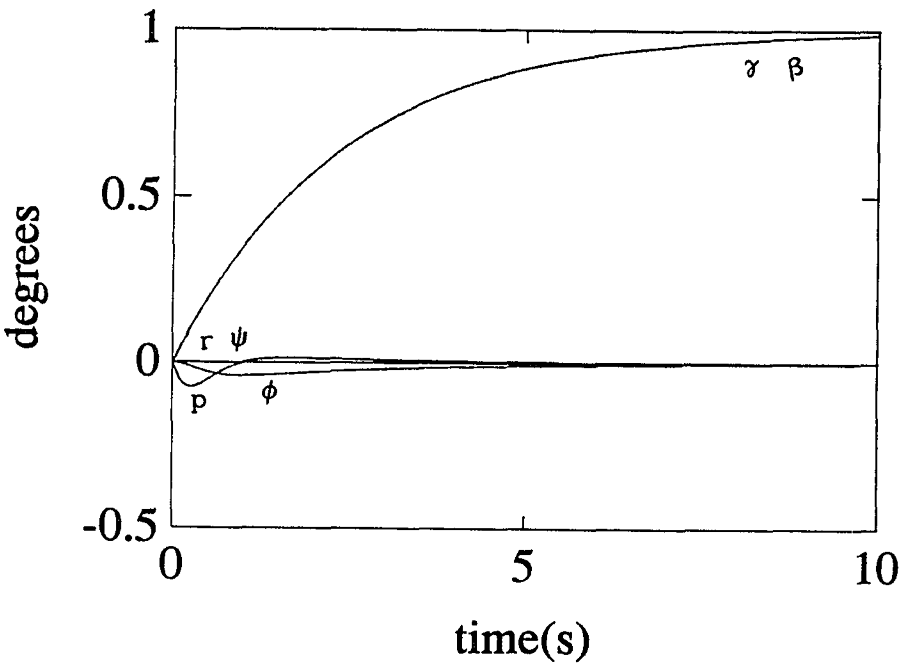

Figure 21. Lateral Translation-Robust Pseudo Control.

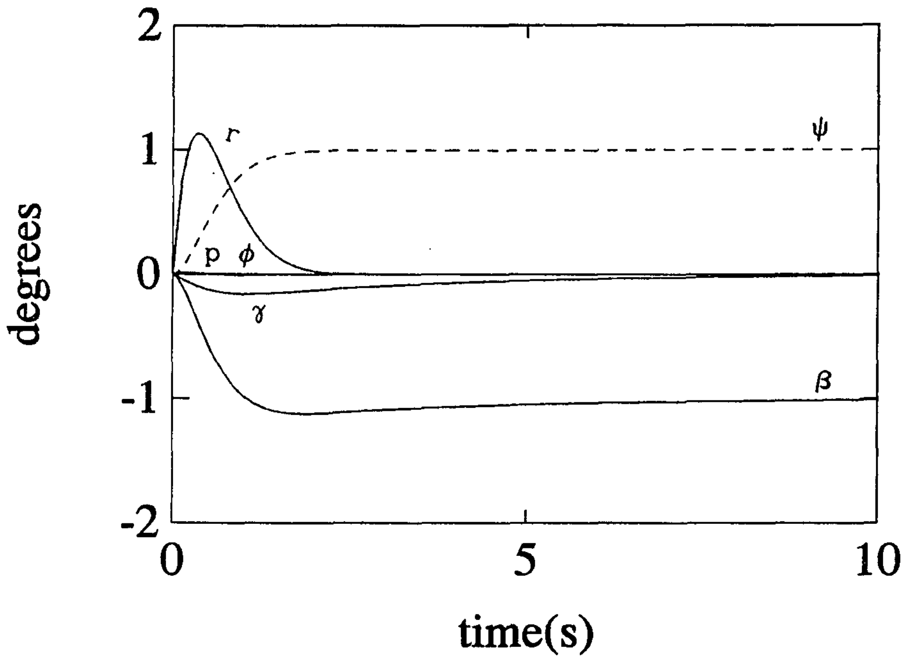

Figure 22. Yaw Pointing-Robust Pseudo Control;

Singular Value Constraint.

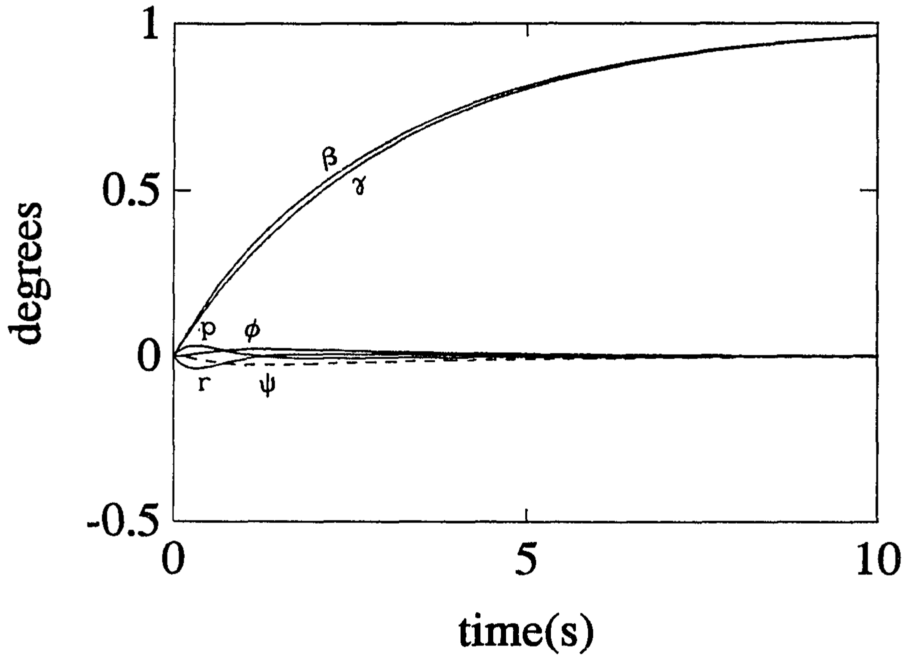

Figure 23. Lateral Translation-Robust Pseudo Control; Singular Value Constraint.

6. PERFORMANCE ROBUSTNESS OF SAMPLED DATA SYSTEMS USING THE DELTA OPERATOR.

6.1 INTRODUCTION

In this chapter, we consider the performance robustness of a unified delta system which is subject to linear time invariant structured state space uncertainty. We extend the stability robustness results of chapters 4 and 5 to obtain new sufficient conditions for performance robustness. We derive new results based upon the modal decomposition discussed in chapter 4, the Lyapunov approach obtained in chapter 5 and another result based upon a matrix measure.

A feedback gain matrix F_ρ is chosen such that all of eigenvalues of the nominal closed loop system are inside the damping ratio/settling time region R of Figure 24 for continuous time or inside the corresponding region R^d of Figure 25 for discrete time. The performance robustness problem is to obtain sufficient conditions for the eigenvalues of the uncertain closed loop system to be inside the region R in continuous time or the region R^d in discrete time for all time invariant dA_ρ and dB_ρ described by Eqs.(93) and (94).

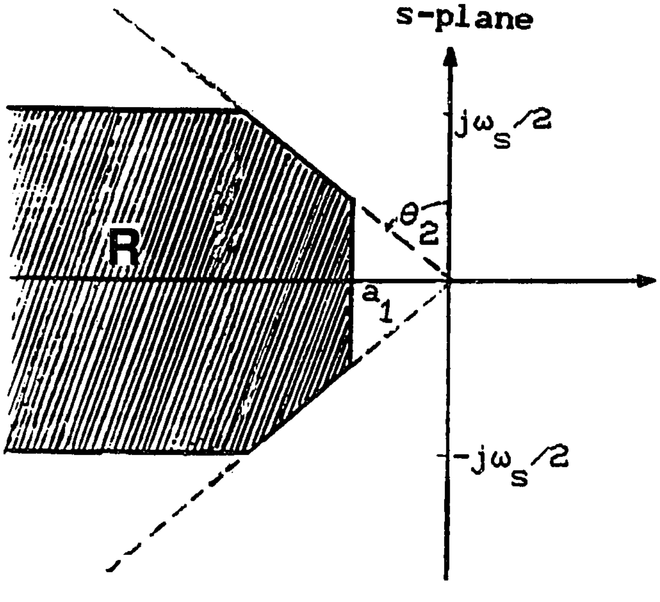

Figure 24 Robustness Region in Continuous time :R

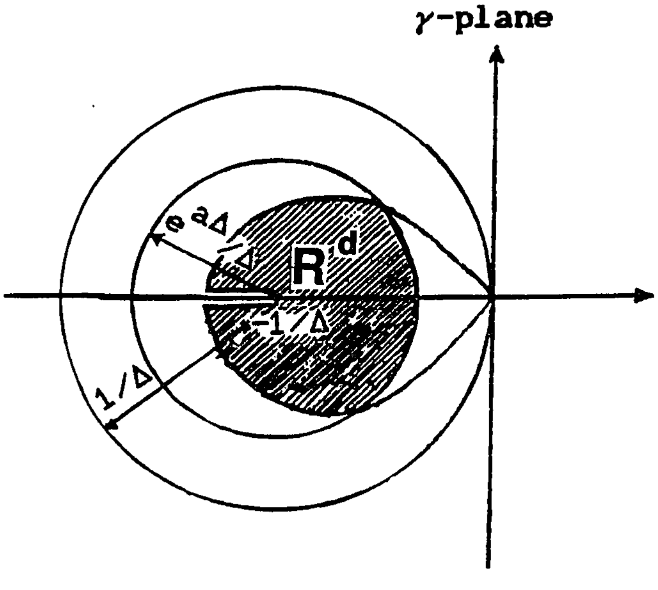

Figure 25 Corresponding Robustness Region in delta: R^d

6.2 PROBLEM FORMULATION

Consider a nominal linear time-invariant multi-input multi-output system described by Eqs.(47a) and (47b).

The corresponding sampled data system, which is obtained by using Middleton and Goodwin's [3] delta operator, is given by Eqs.(71) and (72).

The unified state space model proposed by Middleton and Goodwin [3] is valid for both the discrete time and continuous time cases simultaneously. This unified model is described by Eqs.(78) and (79).

Suppose that the nominal delta system given by Eq.(78) is subject to linear time-invariant uncertainty in the entries of A_ρ, B_ρ described by dA_ρ and dB_ρ, respectively. Then, the delta system with uncertainty is given by Eqs.(87) and (88).

Define $A_{\rho\max}$ and $B_{\rho\max}$ as the maximum bounds on dA_ρ and dB_ρ, respectively and described through Eqs.(95)-(98).

Consider the constant gain output feedback control law described by Eq.(99). Then, the nominal closed loop unified delta system is given by Eq.(101) and the uncertain closed loop unified delta system is described by Eq.(103).

6.3 PRELIMINARY RESULTS

First we review two results in continuous time attributed to Juang, Hong and Wang [18], and Wang and Lin [19]. Then, we extend the results to the unified delta model which is valid for both continuous time and discrete time.

Consider a line L , limited by $j\omega/\Delta$ and $-j\omega/\Delta$, which separates the s-plane into two regions H and $\bar{H}$ as shown in Figure 26. The L line intersects the real axis at point "a" and makes an angle "θ" with respect to the positive imaginary axis, where "θ" is assumed positive in a counterclockwise sense and $-\pi < \theta \leq \pi$. The corresponding regions in the γ-plane are represented by H^d and $\bar{H}^d$ as shown in Figure 27. Denote a specific region H by $H(a,\theta)$ in continuous time and H^d by $H^d(a,\theta)$ in discrete time.

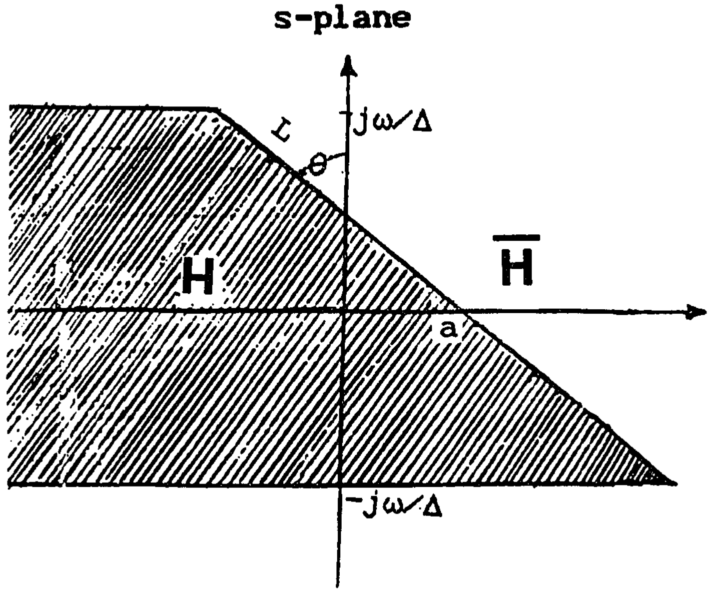

Figure 26. Description of H Region

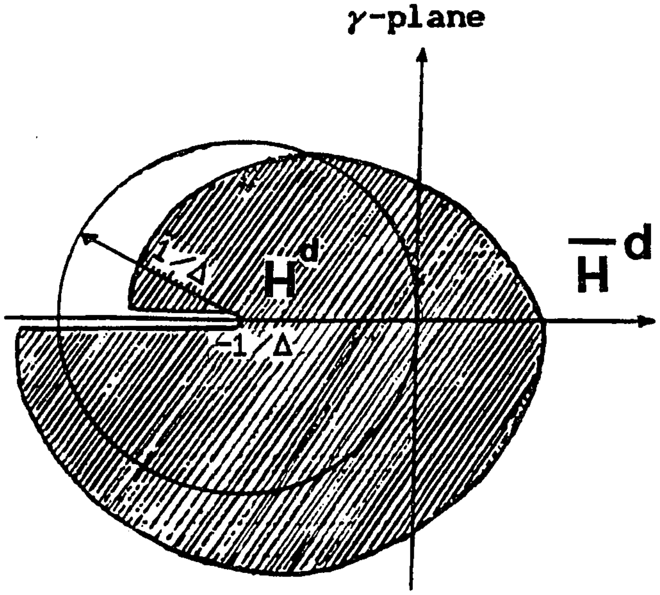

Figure 27. Description of H^d region.

Lemma 2:

All the eigenvalues of the constant matrix A_ρ of Eq.(80) lie in the H region of Figure 26 in continuous time or in the H^d region of Figure 27 in discrete time if and only if the eigenvalues of the matrix $\tilde{A}_\rho$ lie in in the respective region H or H^d.

where

$$\tilde{A}_\rho = \begin{cases} e^{-j\theta}(A-aI) & \text{in continuous time} \\ \tilde{\Omega}e^{-j\theta}(A-aI) & \text{in discrete time} \end{cases} \tag{145}$$

and where

$$\tilde{\Omega} = \frac{1}{\Delta}\int_0^\Delta \exp\{e^{-j\theta}(A-aI)\tau\}\, d\tau \tag{146}$$

Proof: see Appendix X

Next, we extend the definitions which relate the so-called transformed system with uncertainty presented by Yu [2] to the unified delta model.

Definition 1:

The nominal unified closed loop system matrix $A_{\rho c}$, the uncertainty matrix $dA_{\rho c}$, the unified transformed closed loop system matrix $\tilde{A}_{\rho c}$ and the transformed uncertainty matrix

$d\tilde{A}_{\rho c}$ are defined by

$$A_{\rho c} = A_{\rho} + B_{\rho}F_{\rho}C \tag{147}$$

$$dA_{\rho c} = dA_{\rho} + dB_{\rho}F_{\rho}C \tag{148}$$

$$\tilde{A}_{\rho c} = \tilde{A}_{\rho} + \tilde{B}_{\rho}F_{\rho}C \tag{149}$$

$$d\tilde{A}_{\rho c} = d\tilde{A}_{\rho} + d\tilde{B}_{\rho}F_{\rho}C \tag{150}$$

Then, the unified closed system with uncertainty is given by

$$\rho x(t) = A_{\rho c}x(t) + dA_{\rho c}x(t) \tag{151}$$

and the unified transformed closed loop system with uncertainty is defined by

$$\rho\tilde{x}(t) = \tilde{A}_{\rho c}\tilde{x}(t) + d\tilde{A}_{\rho c}\tilde{x}(t) \tag{152}$$

where

$$\tilde{A}_{\rho} = \begin{cases} e^{-j\theta}\ (A-aI) & \text{in continuous tme} \\ \tilde{\Omega}e^{-j\theta}\ (A-aI) & \text{in discrete time} \end{cases} \tag{153}$$

$$\tilde{B}_{\rho} = \begin{cases} e^{-j\theta}\ B & \text{in continuous time} \\ \tilde{\Omega}e^{-j\theta}\ B & \text{in discrete time} \end{cases} \tag{154}$$

$$d\tilde{A}_\rho = \begin{cases} e^{-j\theta}\, dA & \text{continuous} \\ \tilde{\Gamma} e^{-j\theta}(A+dA-aI) - \tilde{\Omega} e^{-j\theta}(A-aI) & \text{discrete} \end{cases} \tag{155}$$

$$d\tilde{B}_\rho = \begin{cases} e^{-j\theta}\, dB & \text{continuous} \\ \tilde{\Gamma} e^{-j\theta}\,(B + dB) - \tilde{\Omega} e^{-j\theta}\, B & \text{discrete} \end{cases} \tag{156}$$

and where

$$\tilde{\Gamma} = \frac{1}{\Delta}\int_0^\Delta e^{[e^{-j\theta}(A + dA - aI)]\tau}\, d\tau \tag{157}$$

The following lemma shows that the eigenvectors of the transformed continuous time plant are identical to the eigenvectors of the transformed sampled data plant.

Lemma 3:

Suppose that the continuous time plant is given by $\dot{x}=Ax+Bu$ and the sampled data plant is $\delta x = A_\delta x + B_\delta u$. Suppose that the transformed continuous time plant is described by $\dot{\tilde{x}} = \tilde{A}\,\tilde{x} + \tilde{B}u$ and the sampled data transformed plant is given by $\delta\tilde{x} = \tilde{A}_\delta \tilde{x} + \tilde{B}_\delta u$. The i-th eigenvalues of A, A_δ, $\tilde{A}$ and $\tilde{A}_\delta$ are λ_i, $\gamma_i = \dfrac{\exp(\lambda_i\Delta)-1}{\Delta}$, $\tilde{\lambda}_i$, and $\tilde{\gamma}_i = \dfrac{\exp(\tilde{\lambda}_i\Delta)-1}{\Delta}$, respectively. Let M be a modal matrix and let Λ be a diagonal matrix with the λ_i's (i=1,...,n) on the main diagonal. Then,

$$M^{-1}\tilde{A}M = \tilde{\Lambda} = \tilde{\lambda}_i I = e^{-j\theta}(\Lambda - I) \tag{158}$$

$$M^{-1}\tilde{A}_\delta M = \tilde{\Lambda}_\delta = \tilde{\gamma}_i I = \frac{1}{\Delta}(e^{\tilde{\Lambda}\Delta} - I) \tag{159}$$

where

$$\tilde{A} = e^{-j\theta}(A - aI) \text{ and } \tilde{A}_\delta = \frac{1}{\Delta}[e^{\tilde{A}\Delta} - I]$$

Proof: see Appendix XI

6.4 PERFORMANCE ROBUSTNESS SUFFICIENT CONDITIONS

We present theorems to show that the unified closed loop system with uncertainty is asymptotically stable if the eigenvalues of the unified transformed plant under time invariant uncertainty are in the H region or H^d region in continuous time and discrete time, respectively.

6.4.1 MODAL DECOMPOSITION APPROACH

Theorem 10: (Preliminary Result)

Suppose that the nominal closed loop system described by Eq.(101) has its eigenvalues in the H region of Figure 26 in continuous time or in the H^d region of Figure 27 in discrete time. Further, suppose that the matrix $A_{\rho c}$ in Eq.(147) is non-defective. The eigenvalues of the closed loop system with uncertainty described by Eq.(151) will be in the respective H or H^d region for all uncertainty described by Eqs.(93) and (94) if

$$\lambda_{max}\left\{\sum_{i=1}^{n}\frac{(v_i w_i^*)^+}{f(\tilde{\gamma}_i)}\tilde{A}_{\rho cmax}\right\} < 1 \tag{160}$$

where

$$\tilde{A}_{\rho cmax} = \begin{cases} A_{max} + B_{max}(FC)^+ & \text{continuous time} \\ \tilde{A}_{\delta max} + \tilde{B}_{\delta max}(F_\delta C)^+ & \text{discrete time} \end{cases} \tag{161}$$

$$\tilde{A}_{\delta max} = \frac{1}{\Delta}\left[e^{[(A-aI)^+ + A_{max}]\Delta} - e^{[(A-aI)^+]\Delta}\right] \tag{162}$$

$$\tilde{B}_{\delta\max} = \frac{1}{\Delta}\left[\int_0^{\Delta} e^{[(A-aI)^{+}+A_{\max}]\tau} d\tau \ (B^{+} + B_{\max}) - \int_0^{\Delta} e^{[(A-aI)^{+}]\tau} d\tau \ B^{+}\right] \tag{163}$$

$$f(\tilde{\gamma}_i) = \begin{cases} -\mathrm{Re}(\tilde{\gamma}_i) & \text{continuous time} \\ \frac{1}{\Delta}\,[1-(1+\Delta\tilde{\gamma}_i)^{+}] & \text{discrete time} \end{cases} \tag{164}$$

$$\tilde{\gamma}_i = \begin{cases} e^{-j\theta}(\lambda_i - a) & \text{continuous} \\ \frac{1}{\Delta}\left[e^{[e^{-j\theta}(\lambda_i - a)]\Delta} - 1\right] & \text{discrete} \end{cases} \tag{165}$$

and where

$$\mathrm{Re}(\tilde{\gamma}_i) = [\mathrm{Re}(\lambda_i) - a]\cos\theta + \mathrm{Im}\lambda_i \sin\theta$$

$$(1 + \Delta\tilde{\gamma}_i)^{+} = e^{\mathrm{Re}(\tilde{\gamma}_i)\Delta}$$

and where $\tilde{\gamma}_i$ is the i-th eigenvalue of $(\tilde{A}_\rho+\tilde{B}_\rho F_\rho C)$ with v_i and w_i^* the corresponding right and left eigenvectors; $(\cdot)^*$ denotes the complex conjugate transpose.

Proof: see Appendix XII

The desired performance region given by the s-plane region R in Figure 24 is the intersection of $H_1(a_1,0)$, $H_2(0,\theta_2)$ and $H_3(0,-\theta_2)$. Furthermore, the region R maps into the γ-plane region R^d which is shown in Figure 25. We now present a corollary to Theorem 10 which describes a unified sufficient condition for the eigenvalues of the system with uncertainty to be in the region R in continuous time or in the region R^d in discrete time.

Corollary 5: (Main Result)

Suppose that the nominal closed loop system described by Eq.(101) has its eigenvalues in the R region of Figure 24 in continuous time or in the R^d region of Figure 25 in discrete time. Further, suppose that the matrix $A_{\rho c}$ in Eq.(147) is non-defective. The eigenvalues of the uncertain closed loop system described by Eq.(151) will be in the respective R or R^d region for all uncertainty described by Eqs.(93) and (94) if

$$\max[\ \lambda_{\max_1}, \lambda_{\max_2}] < 1 \tag{166}$$

where

$$\lambda_{\text{maxk}} = \lambda_{\max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i)^*}{[f(\tilde{\gamma}_i)]_k} \tilde{A}_{\rho\text{cmaxk}} \right\}$$

$$\tilde{A}_{\rho cmaxk} = \begin{cases} \tilde{A}_{maxk} + \tilde{B}_{maxk}(FC)^{+} & \text{continuous time} \\ \tilde{A}_{\delta maxk} + \tilde{B}_{\delta maxk}(F_{\delta}C)^{+} & \text{discrete time} \end{cases} \tag{167}$$

where $\tilde{A}_{maxk} = A_{max}$; $\tilde{B}_{maxk} = B_{max}$

and where

$$\tilde{A}_{\delta maxk} = \frac{1}{\Delta}\left[e^{[(A-a_k I)^{+} + A_{max}]\Delta} - e^{[(A-a_k I)^{+}]\Delta}\right] \tag{168}$$

$$\tilde{B}_{\delta maxk} = \frac{1}{\Delta}\left[\int_0^{\Delta} e^{[(A-a_k I)^{+}+A_{max}]\tau}\, d\tau\, (B^{+} + B_{max}) - \int_0^{\Delta} e^{[(A-a_k I)^{+}]\tau}\, d\tau\, B^{+}\right] \tag{169}$$

$$[f(\tilde{\gamma}_i)]_k = \begin{cases} -[\mathrm{Re}(\tilde{\gamma}_i)]_k & \text{cont . time} \\ \frac{1}{\Delta}\,\{1-[(1+\Delta\tilde{\gamma}_i)^{+}]_k\} & \text{disc. time} \end{cases} \tag{170}$$

$$[(\tilde{\gamma}_i)]_k = \begin{cases} \exp(-j\theta_k)\cdot(\lambda_i - a_k) & \text{cont.} \\ \frac{1}{\Delta}\left[\exp\{e^{-j\theta_k}\cdot(\lambda_i - a_k)\Delta\} - 1\right] & \text{disc.} \end{cases} \tag{171}$$

$$[\mathrm{Re}(\tilde{\gamma}_i)]_k = [\mathrm{Re}\lambda_i - a_k]\cos\theta_k + \mathrm{Im}\lambda_i \sin\theta_k \tag{172}$$

$$[(1 + \Delta\tilde{\gamma}_i)^{+}]_k = e^{[\mathrm{Re}(\tilde{\gamma}_i)]_k \Delta} \tag{173}$$

$$k = 1: \theta_1 = 0, \; a_1 = a_1$$

$$k = 2: \theta_2 = \theta_2, \; a_2 = 0$$

and where $\tilde{\gamma}_i$ is the i-th eigenvalue of $(\tilde{A}_\rho+\tilde{B}_\rho F_\rho C)$ with v_i and w_i^* the corresponding right and left eigenvectors; $(\cdot)^*$ denotes the complex conjugate.

Proof: see Appendix XIII

6.4.2 LYAPUNOV APPROACH

Theorem 11: (Preliminary Result)

Suppose that the unified closed loop system described by Eq.(101) has its eigenvalues in the H region of Figure 26 in continuous time or in the H^d region of Figure 27 in discrete time. The eigenvalues of the closed loop system with uncertainty described by Eq.(151) will be respectively in H or H^d region for all uncertainty described by Eqs.(93) and (94) if

$$\bar{\sigma}\left[(\tilde{E}^T_{\rho\max} \tilde{P}^+_\rho \tilde{A}_{\rho c\max})_s \right] < 1 \tag{174}$$

where

$$\tilde{E}_{\rho max} = (I + \Delta \cdot \tilde{A}_{\rho c})^{+} + (\Delta/2) \cdot \tilde{A}_{\rho cmax} \qquad (175)$$

where $\tilde{A}_{\rho c}$, $\tilde{A}_{\rho cmax}$ are given by Eqs.(149) and (161) and $\tilde{P}_{\rho}$ satisfies the Lyapunov equation

$$\tilde{A}^{T}_{\rho c} \tilde{P}_{\rho} + \tilde{P}_{\rho} \tilde{A}_{\rho c} + \Delta \cdot \tilde{A}_{\rho c} \tilde{P}_{\rho} \tilde{A}^{T}_{\rho c} = - 2I$$

$$\tilde{P}_{\rho} = \begin{cases} \tilde{P} & \text{in continuous time} \\ \tilde{P}_{\delta} & \text{in discrete time} \end{cases} \qquad (176)$$

Proof: see Appendix XIV

We now present a corollary to Theorem 11 which describes a unified sufficient condition for the eigenvalues of the system with uncertainty to be in the region R in continuous time or in the region R^{d} in discrete time.

Corollary 6: (Main Result)

Suppose that the nominal closed loop system described by Eq.(101) has its eigenvalues in the R region of Figure 24 in continuous time or in the R^{d} region of Figure 25 in discrete time. The eigenvalues of the closed loop system with uncertainty described by Eq.(151) will be in the respective R or R^{d} region for all uncertainty described by Eqs.(93)

and (94) if

$$\max[\bar{\sigma}_1,\ \bar{\sigma}_2] < 1 \qquad (177)$$

where

$$\bar{\sigma}_k = \bar{\sigma}[(\tilde{E}^T_{\rho \mathrm{maxk}}\ \tilde{P}^{+}_{\rho k}\ \tilde{A}_{\rho \mathrm{cmaxk}})_s] \qquad (178)$$

$$\tilde{E}_{\rho \mathrm{maxk}} = (I + \Delta \cdot \tilde{A}_{\rho ck})^{+} + (\Delta/2) \cdot \tilde{A}_{\rho \mathrm{cmaxk}} \qquad (179)$$

where

$$\tilde{A}_{\rho ck} = \tilde{A}_{\rho k} + \tilde{B}_{\rho k} F_{\rho} C \qquad (180)$$

$$\tilde{A}_{\rho k} = \begin{cases} e^{-j\theta_k}(A - a_k I) & \text{in continuous tme} \\ \tilde{\Omega}_k e^{-j\theta_k}(A - a_k I) & \text{in discrete time} \end{cases} \qquad (181)$$

$$\tilde{B}_{\rho k} = \begin{cases} e^{-j\theta_k} B & \text{in continuous tme} \\ \tilde{\Omega}_k e^{-j\theta_k} B & \text{in discrete time} \end{cases} \qquad (182)$$

$$\tilde{\Omega}_k = \frac{1}{\Delta} \int_0^{\Delta} \exp\{e^{-j\theta_k}(A - aI)\tau\}\, d\tau \qquad (183)$$

where $\tilde{A}_{\rho \mathrm{cmaxk}}$ is given by Eq.(167) and $\tilde{P}_{\rho k}$ satisfies the Lyapunov equation

$$\tilde{A}^{T}_{\rho ck}\,\tilde{P}_{\rho k} + \tilde{P}_{\rho k}\,\tilde{A}_{\rho ck} + \Delta\cdot\tilde{A}_{\rho ck}\,\tilde{P}_{\rho k}\,\tilde{A}^{T}_{\rho ck} = -\,2I \tag{184}$$

$$\tilde{P}_{\rho k} = \begin{cases} \tilde{P}_{k} & \text{in continuos time} \\ \tilde{P}_{\delta k} & \text{in discrete time} \end{cases} \tag{185}$$

and where

$$k{=}\ 1\ :\ \theta_1 = 0,\quad a_1 = a_1$$

$$k{=}\ 2\ :\ \theta_2 = \theta_2,\quad a_2 = 0$$

Proof: see Appendix XV

6.4.3 MATRIX MEASURE APPROACH

Definition 2:

A vector norm on $\mathbb{C}^n$ is the function $\|\cdot\|_p$: $\mathbb{C}^n \rightarrow \mathbb{R}$ defined by

$$\|\ x\ \|_p \equiv \left(\sum |x_i|^p \right)^{1/p} \tag{186}$$

where $1 \le p \le \infty$, $x \in \mathbb{C}^n$.

Let $W \in \mathbb{C}^{n\times n}$. The induced norm of W corresponding to the vector norm $\|\cdot\|_p$, is defined by

$$\| W \|_{ip} \equiv \sup_{x \neq 0} \frac{\| WX \|_p}{\| X \|_p} \tag{187}$$

where $X \in \mathbb{C}^n$, $1 \leq p \leq \infty$.

The induced matrix measure is defined by

$$U_{ip}(W) \equiv \lim_{\xi \to 0^+} \frac{\| I + \xi W \|_{ip} - 1}{\xi} \tag{188}$$

where $W \in \mathbb{C}^{n\times n}$, I is the identity matrix, $1 \leq p \leq \infty$.

The following result will establish an upper bound on the matrix measure $U_{ip}(W)$ which will be required in the proof of the performance robustness condition which is based upon the matrix measure.

Lemma 4:

For each matrix $W \in \mathbb{C}^{n\times n}$, the induced matrix measure $U_{ip}(W)$ has the following property

$$U_{ip}(W) \leq U_{ip}(W^+) \tag{189}$$

where $U_{ip}(W)$ is defined by Eq.(188), $1 \leq p \leq \infty$ and W^+ denotes the matrix obtained by taking the absolute value of each element of W.

Proof: see Appendix XVI

The following lemma presents a sufficient condition for a constant matrix to have eigenvalues located in the H region in continuous time or in the H^d region in discrete time.

Lemma 5:

All the eigenvalues of a constant matrix A_ρ given by Eq.(80) are located in the H region in continuous time or in the H^d region in discrete time if

$$\begin{cases} U_{ip}(e^{-j\theta}A) < a\cos\theta & \text{cont. time} \\ (\Delta/2)\| \tilde{\Omega}e^{-j\theta}(A-aI)\|_{ip}^{2} + U_{ip}(\tilde{\Omega}e^{-j\theta}A) & \\ \qquad < U_{ip}(-\tilde{\Omega}ae^{-j\theta}) & \text{disc. time} \end{cases} \tag{190}$$

Proof: see Appendix XVII

The following corollary will be used to prove the robustness condition for the matrix measure approach. It is an expanded form of Lemma 5 where the property of the matrix measure defined by

$$U_{ip}(A + B) \le U_{ip}(A) + U_{ip}(B)$$

is used.

Corollary 7:

All the eigenvalues of a constant matrix A_ρ given by Eq.(80) are located in the H region in continuous time or in the H^d region in discrete time if

$$\begin{cases} U_{ip}(A\cos\theta) + U_{ip}(-Aj\sin\theta) < a\cos\theta & \text{cont. time} \\ \frac{\Delta}{2}\|\tilde{\Omega}e^{-j\theta}(A-aI)\|_{ip}^{2} + U_{ip}(\tilde{\Omega}A\cos\theta) + U_{ip}(-\tilde{\Omega}Aj\sin\theta) < -U_{ip}(-\tilde{\Omega}ae^{-j\theta}) & \text{disc. time} \end{cases} \qquad (191)$$

and where $\tilde{\Omega}$ is given by Eq.(146).

Proof: see Appendix XVIII

The theorem below presents a preliminary unified robustness result for the H region for continuous time and the H^d region for sampled data operation of the plant.

Theorem 12: (Preliminary Result)

Suppose that the nominal closed loop system described by Eq.(101) has its eigenvalues in the H region of Figure 26 in continuous time or in the H^d region of Figure 27 in discrete time. Further, suppose that the matrix $A_{\rho c}$ in Eq.(147) is non-defective. The eigenvalues of the closed loop system with uncertainty described by Eq.(151) will be in the respective H or H^d region for all uncertainty described by Eqs.(93) and (94) if

$$\mu_\rho < 1 \tag{192}$$

where

$$\mu_\rho = \begin{cases} \dfrac{U_{ip}(\tilde{A}_{cmax})}{a\cos\theta - U_{ip}(A_c\cos\theta) - U_{ip}(-A_c j\sin\theta)} & \text{continuous} \\ \\ \dfrac{U_{ip}(\tilde{A}_{\delta cmax}) + (\Delta/2)\| \tilde{A}^+_{\delta c} + \tilde{A}_{\delta cmax}\|^2_{ip}}{-U_{ip}(-\tilde{\Omega}ae^{-j\theta}) - U_{ip}(\Xi\cos\theta) - U_{ip}(-\Xi j\sin\theta)} & \text{discrete} \end{cases} \tag{193}$$

$$A_c = A + B(FC);\ \tilde{A}_{\delta c} = \tilde{A}_\delta + \tilde{B}_\delta(FC)$$

$$\tilde{A}_{cmax} = A_{max} + B_{max}(FC)^+;\ \tilde{A}_{max} = A_{max};\ \tilde{B}_{max} = B_{max}$$

$$\tilde{A}_{\delta cmax} = \tilde{A}_{\delta max} + \tilde{B}_{\delta max}(F_\delta C)^+$$

$$\Xi = e^{j\theta}[\ \tilde{A}_\delta + \tilde{B}_\delta F_\delta C] + \tilde{\Omega}a$$

Proof: see Appendix XIX

We now present a corollary to Theorem 12 which describes a unified sufficient condition for the eigenvalues of the system with uncertainty to be in the region R in continuous time or in the region R^d in discrete time.

Corollary 8: (Main Result)

Suppose that the nominal closed loop system described by Eq.(101) has its eigenvalues in the R region of Figure 24 in continuous time or in the R^d region of Figure 25 in discrete time. Further, suppose that the matrix $A_{\rho c}$ in Eq.(147) is non-defective. The eigenvalues of the closed loop system with uncertainty described by Eq.(151) will be in the respective R or R^d region for all uncertainty described by Eqs.(93) and (94) if

$$\max[\mu_{\rho 1}, \mu_{\rho 2}] < 1 \tag{194}$$

where

$$\mu_{\rho k} = \begin{cases} \dfrac{U_{ip}(\tilde{A}_{cmaxk})}{a_k\cos\theta_k - U_{ip}(A_c\cos\theta_k) - U_{ip}(-A_c j\sin\theta_k)} & \text{cont.} \\[2ex] \dfrac{U_{ip}(\tilde{A}_{\delta cmaxk}) + (\Delta/2)\| \tilde{A}^{+}_{\delta ck} + \tilde{A}_{\delta cmaxk}\|^2_{ip}}{-U_{ip}(-\tilde{\Omega}_k a_k e^{-j\theta_k}) - U_{ip}(\Xi_k\cos\theta_k) - U_{ip}(-\Xi_k j\sin\theta_k)} & \text{disc.} \end{cases} \tag{195}$$

where

$$k = 1 : \theta_1 = 0, \quad a_1 = a_1$$

$$k = 2: \quad \theta_2 = \theta_2, \quad a_2 = 0$$

and where

$$A_c = A + B(FC); \; \tilde{A}_{\delta ck} = \tilde{A}_{\delta k} + \tilde{B}_{\delta k}(F_\delta C)$$

$$\tilde{A}_{cmaxk} = A_{max} + B_{max}(FC)^+; \; \tilde{A}_{\delta cmaxk} = \tilde{A}_{\delta maxk} + \tilde{B}_{\delta maxk}(F_\delta C)^+$$

$$\Xi_k = e^{j\theta_k}[\, \tilde{A}_{\delta k} + \tilde{B}_{\delta k} F_\delta C] + \tilde{\Omega}_k a_k \tag{196}$$

and $\tilde{\Omega}_k$ is given by Eq.(183).

Proof: see Appendix XX

7. ROBUST MULTI-RATE DELTA OPERATOR SYSTEMS

7.1 OVERVIEW

In chapter 4 we extended the modal decomposition robustness result obtained in chapter 3 to systems represented by Middleton and Goodwin's [3] unified delta model. An alternative approach to stability robustness of a linear time invariant system was proposed by Yedavalli [4] who uses a Lyapunov approach. In chapter 5 we extended Yedavalli's [4] result to the unified delta model.

Araki and Hagiwara [5] have proposed a model for a class of multi-rate sampled data system. This class consists of systems with multiple input rate sampling and fixed output rate sampling (MIFO). Patel et. al.[15] extend the work in Ref.5 to eigenstructure assignment. However, Ref.15 does not consider the plant to be subject to uncertainty. In this chapter, we extend the results of Araki and Hagiwara [5] to obtain a delta operator representation with time invariant uncertainty. Then, we extend our robustness results of chapters 4 and 5 to MIFO delta operator systems.

7.2 PROBLEM FORMULATION

Consider a nominal linear time-invariant multi-input multi-output systems described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{197}$$

$$y(t) = Cx(t) \tag{198}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^r$ is the output vector, and A, B, C, are constant matrices.

Consider the constant gain output feedback control law is described by

$$u(t) = Fy(t) \tag{199}$$

The nominal closed loop system is given by

$$\dot{x}(t) = (A + BFC)x(t) \tag{200}$$

The continuous time plant is subject to m inputs that change from one constant value to another at n_i successive uniformly spaced time instants $t=k\Delta_i$ where the sampling periods Δ_i's have rational ratio

$$\Delta_i = (\Delta_o/n_i) \; ; \; i = 1,\ 2, \ldots, m \tag{201}$$

where the n_i's are positive integers

$$n = \mathrm{LCM}\ (n_1, n_2, \ldots, n_m) \tag{202}$$

where LCM is defined as the least common multiple.

$$\bar{n} = n_1 + n_2 + \ldots + n_m \tag{203}$$

$$\tau_o = (\Delta_o/n) \tag{204}$$

$$l_i = n/n_i; \qquad i = 1,\ 2, \ldots, m \tag{205}$$

Insert Eq.(204) into Eq.(201) to obtain

$$\Delta_i = \tau_o n/n_i \tag{206}$$

$$\Delta_i = l_i \tau_o \tag{207}$$

where Δ_o is the least common multiple (LCM) of the sampling periods referred to as the main sampling interval, and τ_o defines the base sample period.

The corresponding closed loop multi-rate sampled data system, which is obtained by using Middleton and Goodwin's [3] delta operator, is shown below:

$$\bar{\delta}x = (\bar{A}_\delta + \bar{B}_\delta \bar{F}_\delta C)x \tag{208}$$

$$y = Cx \tag{209}$$

where

$$\bar{A}_\delta = \bar{\Omega}A \tag{210}$$

$$\bar{\Omega} = \frac{1}{\Delta_o} \int_0^{\Delta_o} e^{A\tau} d\tau \tag{211}$$

$$\bar{B}_\delta = [\ \bar{B}_1\ \bar{B}_2\ \ldots\ \bar{B}_m\ ] \tag{212}$$

Each $\bar{B}_i$ is an $n \times n_i$ matrix defined by

$$\bar{B}_i = [\ b_{\delta i}\ \exp(A\Delta_i)\cdot b_{\delta i}\ \ldots\ \exp(A(n_i-1)\Delta_i)\cdot b_{\delta i}\ ] \tag{213}$$

where

$$b_{\delta i} = \frac{1}{\Delta_o} \int_0^{\Delta_i} e^{A\tau} d\tau\ b_i \tag{214}$$

with b_i the i-th column of the matrix B.

The feedback gain matrix is given by

$$\bar{F}_\delta = [\ \bar{F}_1\ \ \bar{F}_2\ \ \cdots\ \ \bar{F}_r\ ] \tag{215}$$

where each $\bar{F}_i$ is an $\bar{n}$ dimensional column vector.

The operator $\bar{\delta}$ is defined by

$$\bar{\delta} = \frac{q - 1}{\Delta_o} \tag{216}$$

which has the following property

$$(\cdot)_{\bar{\delta}} = (\bar{\cdot})_{\delta}$$

and the shift operator q is defined by

$$qx_k = x_{k+1} \tag{217}$$

The nominal unified closed loop multi-rate state space model can be described by

$$\bar{\rho}x(t) = (\bar{A}_\rho + \bar{B}_\rho \bar{F}_\rho C)x(t) \tag{218}$$

where

$$\bar{A}_\rho = \begin{cases} A & \text{in continuous time} \\ \bar{A}_\delta & \text{in discrete time} \end{cases} \tag{219}$$

$$\bar{B}_\rho = \begin{cases} B & \text{in continuous time} \\ \bar{B}_\delta & \text{in discrete time} \end{cases} \tag{220}$$

$$\bar{F}_\rho = \begin{cases} F & \text{in continuous time} \\ \bar{F}_\delta & \text{in discrete time} \end{cases} \tag{221}$$

and where

$$\bar{\rho} = \begin{cases} \frac{d}{dt} & \text{in continuous time} \\ \bar{\delta} & \text{in discrete time} \end{cases} \tag{222}$$

which has the following property

$$(\cdot)_{\bar{\rho}} = (\bar{\cdot})_{\rho}$$

Suppose that the nominal delta system is subject to linear time-invariant uncertainty in the entries of $\bar{A}_{\rho}$, $\bar{B}_{\rho}$ described by $d\bar{A}_{\rho}$ and $d\bar{B}_{\rho}$, respectively, where

$$d\bar{A}_{\rho} = \begin{cases} dA & \text{in continuous time} \\ d\bar{A}_{\delta} & \text{in discrete time} \end{cases} \tag{223}$$

$$d\bar{B}_{\rho} = \begin{cases} dB & \text{in continuous time} \\ d\bar{B}_{\delta} & \text{in discrete time} \end{cases} \tag{224}$$

and where

$$d\bar{A}_{\delta} = \frac{1}{\Delta_o} \left[\exp\{(A+dA)\Delta_o\} - \exp(A\Delta_o) \right] \tag{225}$$

$$d\bar{B}_{\delta} = [\ (d\bar{B}_1 - \bar{B}_1)\ (d\bar{B}_2 - \bar{B}_2)\ \ldots\ (d\bar{B}_m - \bar{B}_m)\ ] \tag{226}$$

Each $d\bar{B}_i$ is an $n\times n_i$ matrix defined by

$$d\bar{B}_i = [db_{\delta i}\ \ \exp\{(A+dA)\Delta_i\}\cdot db_{\delta i}\ \ldots\ \exp\{(A+dA)(n_i-1)\}\cdot db_{\delta i}\ ] \tag{227}$$

where

$$db_{\delta i} = \frac{1}{\Delta_o} \int_0^{\Delta_i} e^{(A+dA)\tau}\, d\tau\ (b_i + db_i) \tag{228}$$

with db_i the i-th column of the dB matrix.

Then, the unified closed loop system with uncertainty is given by

$$\bar{\rho}x(t) = (\bar{A}_\rho + \bar{B}_\rho\bar{F}_\rho C)x(t) + (d\bar{A}_\rho + d\bar{B}_\rho\bar{F}_\rho C)x(t) \tag{229}$$

Further, suppose that bounds are available on the maximum absolute values of the elements of dA and dB. That is

$$|da_{ij}| \leq (a_{ij})_{max};\ i=1,\ldots,n;\ j=1,\ldots,n \tag{230}$$

$$|db_{ij}| \leq (b_{ij})_{max};\ i=1,\ldots,n;\ j=1,\ldots,m \tag{231}$$

Then, the corresponding bounds on the $\bar{\delta}$ system, through Eqs.(225) and (226), are

$$|d\bar{a}_\delta(i,j)| \leq [d\bar{a}_\delta(i,j)]_{max};\ i=1,\ldots,n;\ j=1,\ldots,n \tag{232}$$

$$|d\bar{b}_\delta(i,j)| \leq [d\bar{b}_\delta(i,j)]_{max};\ i=1,\ldots,n;\ j=1,\ldots,m \tag{233}$$

Define $d\bar{A}^+_\rho$ and $d\bar{B}^+_\rho$ as the matrices obtained by replacing the entries of $d\bar{A}_\rho$ and $d\bar{B}_\rho$ by their absolute values. Also, define $\bar{A}_{\rho max}$ and $\bar{B}_{\rho max}$ as the matrices with entries $(a_{ij})_{max}$ and $(b_{ij})_{max}$, respectively in continuous time or with entries $[d\bar{a}_\delta(i,j)]_{max}$ and $[d\bar{b}_\delta(i,j)]_{max}$, respectively in discrete time. Then,

$$\{d\bar{A}_\rho : \; d\bar{A}_\rho^+ \le \bar{A}_{\rho\max} \; \} \tag{234}$$

and

$$\{d\bar{B}_\rho : \; d\bar{B}_\rho^+ \le \bar{B}_{\rho\max} \; \} \tag{235}$$

where

$$\bar{A}_{\rho\max} = \begin{cases} A_{\max} & \text{in continuous time} \\ \bar{A}_{\delta\max} & \text{in discrete time} \end{cases} \tag{236}$$

$$\bar{B}_{\rho\max} = \begin{cases} B_{\max} & \text{in continuous time} \\ \bar{B}_{\delta\max} & \text{in discrete time} \end{cases} \tag{237}$$

and where

$$\bar{A}_{\delta\max} = \frac{1}{\Delta_o} \left[\exp\{(A^+ + A_{\max})\Delta_o\} - \exp(A^+\Delta_o) \right] \tag{238}$$

$$\bar{B}_{\delta\max} = [(\bar{B}_{1\max} - \bar{B}_1^+) \;\; (\bar{B}_{2\max} - \bar{B}_2^+) \;\; \dots \;\; (\bar{B}_{m\max} - \bar{B}_m^+)] \tag{239}$$

Each $\bar{B}_i^+$ is an $n \times n_i$ matrix defined by

$$\bar{B}_i^+ = [\; b_{\delta i}^+ \;\; \exp\{A^+\Delta_i\} \cdot b_{\delta i}^+ \;\; \dots \;\; \exp\{A^+(n_i - 1)\Delta_i\} \cdot b_{\delta i}^+ \;] \tag{240}$$

where

$$b_{\delta i}^+ = \frac{1}{\Delta_o} \int_0^{\Delta_i} e^{A^+\tau} d\tau \; b_i^+ \tag{241}$$

and where b_i^+ is the i-th column of the matrix B^+. Each $\bar{B}_{i\max}$ is an $n \times n_i$ matrix defined by

$$\bar{B}_{imax}=[b_{\delta imax}\ \exp\{(A^{+}+A_{max})\Delta_i\}\cdot b_{\delta imax}\ \cdots$$

$$\cdots\ \exp\{(A^{+}+A_{max})(n_i-1)\Delta_i\}\cdot b_{\delta imax}] \qquad (242)$$

$$b_{\delta imax} = \frac{1}{\Delta_o}\int_0^{\Delta_i} e^{(A^{+}+A_{max})\tau}\, d\tau (b_i^{+} + b_{imax}) \qquad (243)$$

where b_{imax} is the i-th column of the matrix B_{max}.

Finally, the stability robustness problem can be stated as follows: Given a block feedback gain matrix $\bar{F}_\rho \in \mathbb{R}^{m\times r}$ such that the nominal closed loop multi-rate delta system exhibits desirable performance determine if the uncertain closed loop multi-rate delta system is asymptotically stable for all time-invariant $d\bar{A}_\rho$ and $d\bar{B}_\rho$ described by Eqs.(234) and (235), respectively.

7.3 ROBUSTNESS RESULTS

Theorem 13: (Modal Decomposition Approach)

Suppose that $\bar{F}_\rho$ is such that the nominal closed loop multi-rate system described by Eq.(218) is asymptotically stable with $(\bar{A}_\rho + \bar{B}_\rho\bar{F}_\rho C)$ non-defective. Then, the closed loop multi-rate system with uncertainty given by Eq.(229) is asymptotically stable for $d\bar{A}_\rho$ and $d\bar{B}_\rho$ described by Eqs.(234)

and (235), respectively if

$$\lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\bar{\gamma}_i)} \bar{A}_{\rho cmax} \right\} < 1 \qquad (244)$$

where

$$f(\bar{\gamma}_i) = \begin{cases} -Re(\bar{\gamma}_i) & \text{continuous time} \\ \frac{1}{\Delta_o}\,[1-(1+\Delta_o\bar{\gamma}_i)^+] & \text{discrete time} \end{cases} \qquad (245)$$

$$\bar{A}_{\rho cmax} = \begin{cases} A_{max} + B_{max}(FC)^+ & \text{continuous time} \\ \bar{A}_{\delta max} + \bar{B}_{\delta max}(\bar{F}_\delta C)^+ & \text{discrete time} \end{cases} \qquad (246)$$

and where $\bar{F}_\delta$, $\bar{A}_{\delta max}$ and $\bar{B}_{\delta max}$ are given by Eqs.(215), (238) and (239), respectively; $\bar{\gamma}_i$ is the i-th eigenvalue of $(\bar{A}_\rho + \bar{B}_\rho \bar{F}_\rho C)$ with v_i and w_i^* the corresponding right and left eigenvectors, respectively; $(\cdot)^*$ denotes the complex conjugate transpose.

Proof: see Appendix XXI

Theorem 14: (Lyapunov Approach)

Suppose that $\bar{F}_\rho$ is such that the nominal closed loop multi-rate system described by Eq.(218) is asymptotically stable. Then, the closed loop system with uncertainty given by Eq.(229) is asymptotically stable for all $d\bar{A}_\rho$ and $d\bar{B}_\rho$

described by Eqs.(234) and (235) if

$$\sigma_{\max}\{(\bar{E}^{T}_{\rho\max}\ \bar{P}^{+}_{\rho}\ \bar{A}_{\rho c\max})_{s}\} < 1 \tag{247}$$

where

$$\bar{A}_{\rho c\max} = \bar{A}_{\rho\max} + \bar{B}_{\rho\max}(\bar{F}_{\rho}C)^{+} \tag{248}$$

$$\bar{E}_{\rho\max} = (I_{n} + \Delta_{o}\bar{A}_{\rho c})^{+} + (\Delta_{o}/2)\bar{A}_{\rho c\max} \tag{249}$$

and where $\bar{P}_{\rho}$ satisfies the Lyapunov equation given by

$$\bar{A}^{T}_{\rho c}\bar{P}_{\rho} + \bar{P}_{\rho}\bar{A}_{\rho c} + \Delta_{o}\bar{A}_{\rho c}\bar{P}_{\rho}\bar{A}^{T}_{\rho c} = -2I \tag{250}$$

$$\bar{A}_{\rho c} = \begin{cases} A + BFC & \text{in continuous time} \\ \bar{A}_{\delta} + \bar{B}_{\delta}\bar{F}_{\delta}C & \text{in discrete time} \end{cases} \tag{251}$$

and where $\bar{P}^{+}_{\rho}$ is the matrix formed by the modulus of the entries of the matrix $\bar{P}_{\rho}$. and $(\cdot)_{s}$ denotes the symmetric part of a matrix.

Proof: see Appendix XXII

8. CONCLUDING REMARKS

8.1 CONCLUSIONS

This research has proposed new results and algorithms for robust eigenstructure assignment design of controllers for multi-input multi-output linear systems with structured state space uncertainty.

Robust stability theorems were obtained for linear time invariant systems with uncertainty. Theorem 1 presents a robustness sufficient condition for continuous time plants subject to time varying uncertainty and unmodelled dynamics. Theorem 3, which was obtained in collaboration with Yu [2], presents a new robust stability condition for linear time invariant systems which are subject to linear time varying uncertainty. Theorem 8 provides a unified performance robustness sufficient condition which is valid for both continuous time and sampled data operation of the plant. Theorem 9 extends Yedavalli's [4] Lyapunov approach for stability robustness of a linear time invariant system to the unified delta system. Corollaries 5, 6 and 8 present unified performance robustness sufficient conditions for a linear delta operator system with structured state space uncertainty. Theorems 13 and 14 extend the robustness results to MIFO delta operator systems.

Three algorithms were developed and applied to a missile and an aircraft:

- In Chapter 3, we presented a new robust design method which was obtained in collaboration with Yu [2]. This method is applied to the design of a robust controller for the lateral dynamics of the Extended Medium Range Air to Air Technology missile. The design is compared to an earlier orthogonal projection eigenstructure assignment design which was proposed by Sobel and Cloutier [8]. In the new method, we minimize the integrated roll rate with constraints on the real part of the dutch roll and roll modes, the damping ratios of the dutch roll and roll modes, the aileron and rudder deflection rates, and the new sufficient condition for robust stability. This design satisfies the new robustness condition while also yielding an improved transient response as compared to the design of Sobel and Cloutier [8].

- In chapter 4, we extended robust eigenstructure design method to the delta model. This algorithm utilizes a new robustness condition which is based on a modal decomposition. The new algorithm is applied to the design of a robust sampled data controller for the lateral dynamics of the EMRAAT missile. This approach is different from the more conventional approach of Andry et. al.[6] who suggest that the i-th eigenvector v_i should be chosen as the orthogonal projection of a desired eigenvector v_i^d onto the subspace spanned by the column of of $(\lambda_i I-A)^{-1}B$. We have obtained a robust design which satisfies the robustness condition of Theorem 8 while also yielding an improved transient response as compared to an orthogonal projection design.

- In chapter 5, we designed a robust sampled data extended pseudo control eigenstructure assignment flight control law for the yaw pointing/lateral translation maneuver of the FPCC aircraft. We chose an objective function which weights both the heading angle due to a lateral flight path command and the lateral flight path angle due to a heading command. The new design algorithm minimizes the objective function given by Eq.(143) with constraints on the time constants of the dutch roll, roll, and flight path modes, the damping ratios of the dutch roll and roll modes, the robustness condition of Theorem 9 and the minimum of the smallest singular of the return difference matrix at the aircraft inputs. This design yields an improved transient response as compared to earlier pseudo control designs.

The work of this thesis is documented in the following papers:

K.M. Sobel, J.E. Piou, W. Yu, E.Y. Shapiro, and R. Wilson, " A Time Domain Approach to Robustness of LTI Systems with Structured Uncertainty and Unmodelled Dynamics", Proceedings of the 29th IEEE Conference on Decision and Control, Honolulu, HI, December 1990, pp. 432-433.

K.M. Sobel, W. Yu, J.E. Piou, J.R. Cloutier and R. Wilson, " Robust Eigenstructure Assignment with Structured State Space Uncertainty and Unmodelled Dynamics", ACC, Boston, MA, June 1991, pp. 3137-3141.

W. Yu, J.E. Piou, and K.M. Sobel "Robust Eigenstructure Assignment for the Extended Medium Range Air to Air Technology Missile", Proceedings of the 30th IEEE CDC, U.K. December 1991, pp 2976-2981.

J.E. Piou and K. M. Sobel, "Robust Sampled Data Eigenstructure Assignment Using the Delta Operator", Journal of Guidance, Control, and Dynamics, to appear.

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J.E. Piou and K.M. Sobel, "Robust Sampled Data Eigenstructure Assignment Using the Delta Operator", AIAA, Guidance, Navigation and Control Conference Hilton Head Island, SC August 10-12, 1992, pp. 486-496.

J.E. Piou and K.M. Sobel, and E.Y. Shapiro, "Robust Lyapunov Constrained Sampled Data Eigenstructure Assignment Using the Delta Operator with Application to Missile Flight Control", IEEE Conference on Control Applications, Dayton, Ohio, September 1992, pp. 1000-1005.

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W. Yu, J.E. Piou and K.M. Sobel, "Robust Eigenstructure Assignment for the Extended Medium Air to Air Missile", Automatica, to appear.

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8.2 PROBLEMS AND RECOMMENDATION

Sufficient robustness conditions for continuous time systems with time varying structured state uncertainties have been considered. Unmodelled dynamics with output nonlinearities have been analyzed. Robustness sufficient conditions for unified delta operator systems with time invariant structured state space uncertainty have been studied. However, nonlinear state space uncertainty has not been mentioned. A possible extension to the robustness sufficient conditions will be to consider linear time varying delta operator systems with linear and nonlinear state space uncertainties.

We have obtained a unified performance robustness sufficient condition for linear time invariant delta operator systems with time invariant structured state space uncertainties. We have obtained new robustness results for multi-input fixed output multi-rate systems with time invariant uncertainty.

Recently Zhu and Johnson [29] presented new results using a spectral canonical realization for linear time varying systems. They showed that their new spectral canonical realization is the natural time varying counterpart of spectral canonical realizations for time invariant linear systems. Further research should extend the unified performance robustness sufficient conditions to include time varying uncertainties. Our results should be extended to multi input and multi output rate sampled systems with time varying uncertainty.

In this research eigenstructure assignment has been used to design controllers for linearized plants. However, the linearization of a plant is valid only at a specific operating point. A general approach to the design of robust controllers should include a wide range of operating points and the transitions between these operating points.

APPENDIX I

Proof of Theorem 1.:

Combine the uncertain plant described by Eqs.(36a) and (36b) with the dynamic compensator described by Eqs.(37a) and (37b) to obtain

$$\dot{x}(t)=(A+\Delta A)x(t)+(B+\Delta B)[C_c x_c(t)+D_c\{(C+\Delta C)x(t)+(D+\Delta D)u(t) + H(t)*u(t)\}] \tag{A1}$$

$$\dot{x}_c(t) = A_c x_c(t)+B_c[(C+\Delta C)x(t)+(D+\Delta D)u(t)+\Delta H(t)*u(t)] \tag{A2}$$

Define

$$\tilde{A}_{cl}=\begin{pmatrix} A+BD_c & CBC_c \\ B_c C & A_c \end{pmatrix}$$

and

$$\Delta\tilde{A}_{cl}=\begin{pmatrix} \Delta A+BD_c\Delta C+\Delta BD_c(C+\Delta C) & \Delta BC_c \\ B_c\Delta C & 0 \end{pmatrix}$$

Let

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}$$

Then,

$$\dot{\tilde{x}}(t)= \tilde{A}_{cl}\tilde{x}(t)+\Delta\tilde{A}_{cl}\tilde{x}(t)+\begin{bmatrix}(B+\Delta B)D_c(D+\Delta D)\\ B_c(D+\Delta D)\end{bmatrix}u(t)$$

$$+ \begin{bmatrix}(B+\Delta B)D_c\\ B_c\end{bmatrix}\Delta H(t)*u(t) \tag{A3}$$

Let $\tilde{x}(t) = MD_1Q\tilde{z}(t)$ where M is a modal matrix for $\tilde{A}_{cl}$. Then, Eq.(A3) becomes

$$\dot{\tilde{z}}(t) = Q^{-1} D_1^{-1} M^{-1} \tilde{A}_{cl} MD_1Q\tilde{z}(t) + Q^{-1}D_1^{-1} M^{-1} \Delta\tilde{A}_{cl} MD_1 Q \tilde{z}(t)$$

$$+ Q^{-1}D_1^{-1}M^{-1}\begin{bmatrix}(B+\Delta B)D_c(D+\Delta D)\\ B_c(D+\Delta D)\end{bmatrix}u(t)$$

$$+ \qquad Q^{-1}D_1{}^{-1}M^{-1}\begin{bmatrix}(B+\Delta B)D_c\\ B_c\end{bmatrix}\cdot\Delta H(t)*u(t) \tag{A4}$$

which has a solution given by

$$\tilde{z}(t) = \exp(Q^{-1}\Lambda Qt)\tilde{z}(0) + \exp(Q^{-1}\Lambda Qt)*\Big\{Q^{-1}D_1^{-1}M^{-1}\Delta\tilde{A}_{cl}MD_1Q\tilde{z}(t)$$

$$+ Q^{-1}D_1^{-1}M^{-1}\begin{bmatrix}(B+\Delta B)D_c(D+\Delta D)\\ B_c(D+\Delta D)\end{bmatrix}u(t)$$

$$+ Q^{-1}D_1^{-1}M^{-1}\begin{bmatrix}(B+\Delta B)D_c\\ B_c\end{bmatrix}\cdot[\Delta H(t)*u(t)]\Big\} \tag{A5}$$

Take 2-norms and integrate both sides of Eq.(A5) and use

$$\exp(Q^{-1}\Lambda Qt)=Q^{-1}\exp(\Lambda t)Q \tag{A6}$$

$$\begin{aligned}\int_0^\infty \|\tilde{z}(t)\|_2\, dt \le\ & \kappa_2(Q)\cdot\|\tilde{z}(0)\|_2\cdot\int_0^\infty \|\exp(\Lambda t)\|_2\, dt \\ & + \int_0^\infty \Bigg\| Q^{-1}\exp(\Lambda t)*\Bigg\{D_1^{-1}M^{-1}\Delta\tilde{A}_{cl}(t)MD_1Q\tilde{z}(t) \\ & + D_1^{-1}M^{-1}\cdot\begin{bmatrix}(B+\Delta B(t))D_c(D+\Delta D(t)) \\ B_c(D+\Delta D(t))\end{bmatrix}u(t) \\ & + D_1^{-1}M^{-1}\begin{bmatrix}(B+\Delta B(t))D_c \\ B_c\end{bmatrix}[\Delta H(t)*u(t)]\Bigg\}\Bigg\|_2\, dt\end{aligned} \tag{A7}$$

We now require a new lemma.

Lemma A.:

If a(t) and b(t) are continuous then,

$$\int_{t=0}^{\infty} \|a(t)*b(t)\|_2\, dt \le \int_{t=0}^{\infty} \|a(t)\|_2\, dt \int_{t=0}^{\infty} \|b(t)\|_2\, dt \tag{A8}$$

<u>Proof</u> <u>of</u> <u>Lemma A.:</u>

$$\int_{t=0}^{\infty} \|a(t)*b(t)\|_2\, dt = \int_{t=0}^{\infty} \Big\| \int_{\tau=0}^{t} a(\tau)\ b(t-\tau)\ d\tau\Big\|_2\, dt \tag{A9}$$

$$\int_{t=0}^{\infty} \|a(t)*b(t)\|_2 \, dt \le \int_{t=0}^{\infty} \int_{\tau=0}^{t} \|a(\tau)\, b(t-\tau)\|_2 \, d\tau dt \qquad (A10)$$

$$\le \int_{t=0}^{\infty} \int_{\tau=0}^{t} \|a(\tau)\|_2 \, \|b(t-\tau)\|_2 \, d\tau \, dt \qquad (A11)$$

The right-hand side of Eq.(A11) may be written as

$$\lim_{R\to\infty} \left[\int_{t=0}^{R} \int_{\tau=0}^{t} \|a(\tau)\|_2 \, \|b(t-\tau)\|_2 \, d\tau \, dt \right] \qquad (A12)$$

Since a(t) and b(t) are continuous i.e. all of the functions inside the integrals of Eq.(A12) are continuous functions of t and τ. Then, the order of integration may be interchanged (Churchill [31]). Then, Eq.(A12) can be written as

$$\lim_{R\to\infty} \left[\int_{\tau=0}^{R} \int_{t=\tau}^{R} \|a(\tau)\|_2 \, \|b(t-\tau)\|_2 \, dt \, d\tau \right] \qquad (A13)$$

Now use a change of variables $\gamma=t-\tau$, $d\gamma=dt$, $\gamma>0$.

$$\lim_{R\to\infty} \left[\int_{\tau=0}^{R} \int_{t=\tau}^{R} \|a(\tau)\|_2 \, \|b(t-\tau)\|_2 \, d\tau \, dt \right]$$

$$= \lim_{R\to\infty} \left[\int_{\tau=0}^{R} \int_{\gamma=0}^{R-\tau} \|a(\tau)\|_2 \, \|b(\gamma)\|_2 \, d\gamma \, d\tau \right]$$

$$\le \lim_{R\to\infty} \left[\int_{\tau=0}^{R} \int_{\gamma=0}^{R} \|a(\tau)\|_2 \, \|b(\gamma)\|_2 \, d\gamma \, d\tau \right]$$

$$= \lim_{R\to\infty} \left[\int_{\tau=0}^{R} \int_{\gamma=0}^{R-\tau} \|a(\tau)\|_2 \, \|b(\gamma)\|_2 \, d\gamma \, d\tau \right]$$

Thus,

$$\int_{t=0}^{\infty} \|a(t)*b(t)\|_2 \, dt \le \int_{t=0}^{\infty} \|a(t)\|_2 \, dt \int_{t=0}^{\infty} \|b(t)\|_2 \, dt \qquad \blacksquare$$

The proof of Theorem 1 continues. Then, Eq.(A7) becomes

$$\begin{aligned}
\int_0^{\infty} \|\tilde{z}(t)\|_2 \, dt \le\ & \kappa_2(Q)\cdot\|\tilde{z}(0)\|_2\cdot\int_0^{\infty} \|\exp(\Lambda t)\|_2 \, dt \\
&+ \|Q^{-1}\|_2 \int_0^{\infty} \|\exp(\Lambda t)\|_2 \, dt \cdot \int_0^{\infty} \Bigg\{ \|D_1^{-1}M^{-1}\Delta\tilde{A}_{cl}MD_1Q\|_2 \cdot \|\tilde{z}(t)\|_2 \\
&+ \left\| D_1^{-1}M^{-1} \begin{bmatrix} (B+\Delta B(t))D_c(D+\Delta D(t)) \\ B_c(D+\Delta D(t)) \end{bmatrix} \right\|_2 \cdot \|u(t)\|_2 \\
&+ \left\| D_1^{-1}M^{-1} \begin{bmatrix} (B+\Delta B(t))D_c \\ B_c \end{bmatrix} [\Delta H(t)*u(t)] \right\|_2 \Bigg\} \, dt
\end{aligned} \qquad \text{(A14)}$$

Use Eqs.(33a)-(33d) together with

$$\|A \ B \ C\|_2 \le \|A^+ \ B^+ \ C^+\|_2 \le \|A_{max} \ B_{max} \ C_{max}\|_2$$

and

$$\left\| \begin{pmatrix} A \\ B \end{pmatrix} \right\|_2 \le \left\| \begin{pmatrix} A^+ \\ B^+ \end{pmatrix} \right\|_2 \le \left\| \begin{matrix} A^+ \\ B^+ \end{matrix} \right\|_2$$

to obtain

$$\int_0^\infty \|\tilde{z}(t)\|_2 \, dt \le \kappa_2(Q) \cdot \|\tilde{z}(0)\|_2 \cdot \int_0^\infty \|\exp(\Lambda t)\|_2 \, dt$$

$$+ \|Q^{-1}\|_2 \cdot \int_0^\infty \|\exp(\Lambda t)\|_2 \, dt \cdot \left\{ \left\| (D_1^{-1} M^{-1})^+ \right. \right.$$

$$\left. \cdot \begin{bmatrix} A_{max} + B_{max}(D_c C)^+ + (B^+ + B_{max}) D_c^+ C_{max} & B_{max} C_c^+ \\ B_c^+ C_{max} & 0 \end{bmatrix} (MD_1 Q)^+ \right\|_2$$

$$\cdot \int_0^\infty \|\tilde{z}(t)\|_2 dt + \left\| (D_1^{-1} M^{-1})^+ \begin{bmatrix} (B^+ + B_{max}) D_c^+ (D^+ + D_{max}) \\ B_c^+ (D^+ + D_{max}) \end{bmatrix} \right\|_2 \cdot \int_0^\infty \|u(t)\|_2 \, dt$$

$$\left. + \left\| (D_1^{-1} M^{-1})^+ \begin{bmatrix} (B^+ + B_{max}) D_c^+ \\ B_c^+ \end{bmatrix} \right\|_2 \cdot \int_0^\infty \|\Delta H(t) * u(t)\|_2 \, dt \right\} \qquad \text{(A15)}$$

Next, we obtain a bound for $\int_0^\infty \|\Delta H(t) * u(t)\|_2 \, dt$

$$\int_0^\infty \|\Delta H(t) * u(t)\|_2 \, dt = \int_0^\infty \|[C_\Delta e^{A_\Delta t} B_\Delta + D_\Delta \delta(t) + E_\Delta \dot{\delta}(t)] * u(t)\|_2 \, dt$$

$$\le \int_0^\infty \|C_\Delta e^{A_\Delta t} B_\Delta\|_2 \, dt \int_0^\infty \|u(t)\|_2 \, dt + \|D_\Delta\|_2 \int_0^\infty \|\delta(t) * u(t)\|_2 \, dt$$

$$+ \|E_\Delta\|_2 \int_0^\infty \|\dot{\delta}(t) * u(t)\|_2 \, dt \qquad \text{(A16)}$$

Use assumptions (i)-(iv) and $\dot{\delta}(t) * u(t) = \dot{u}(t)$ (Kailath [30])

$$\int_0^\infty \|\Delta H(t)*u(t)\|_2 \, dt \le r_2 \int_0^\infty e^{-\beta t} \, dt \int_0^\infty \|u(t)\|_2 \, dt$$

$$+ r_0 \int_0^\infty \|u(t)\|_2 \, dt + r_1 \int_0^\infty \|\dot{u}(t)\|_2 \, dt \qquad \text{(A17)}$$

Use assumption (v)

$$\int_0^\infty \|\Delta H(t)*u(t)\|_2 \, dt \le \left(r_2/\beta + r_0 + \gamma r_1\right) \int_0^\infty \|u(t)\|_2 \, dt \qquad \text{(A18)}$$

Now we obtain a bound on $\int_0^\infty \|u(t)\|_2 \, dt$

From Eq.(37b), $u(t) = C_c x_c(t) + D_c y(t)$

Substitute Eq.(31b) to obtain

$$u(t) = C_c x_c(t) + D_c(C+\Delta C)x(t) + D_c(D+\Delta D)u + D_c \cdot \Delta H(t)*u(t)$$

$$u(t) = [D_c(C+\Delta C), C_c]\tilde{x}(t) + D_c(D+\Delta D)u(t) + D_c \cdot \Delta H(t)*u(t)$$

$$u(t) = [D_c(C+\Delta C), C_c]MD_1Q\tilde{z}(t) + D_c(D+\Delta D)u(t) + D_c \cdot \Delta H(t)*u(t) \qquad \text{(A19)}$$

Integrate both sides of Eq.(A14) to obtain

$$\int_0^\infty \|u(t)\|_2 \, dt \le \| [D_c(C+\Delta C), C_c]MD_1Q\|_2 \int_0^\infty \|\tilde{z}(t)\|_2 \, dt$$

$$+ \|D_c(D+\Delta D)\|_2 \cdot \int_0^\infty \|u(t)\|_2 \, dt + \|D_c\|_2 \cdot \int_0^\infty \|\Delta H(t) * u(t)\|_2 \, dt \tag{A20a}$$

$$\int_0^\infty \|u(t)\|_2 \, dt \le \|[D_c^+(C^+ + C_{max}), C_c^+](MD_1Q)^+\|_2 \int_0^\infty \|\tilde{z}(t)\|_2 \, dt$$

$$+ \|D_c^+(D^+ + D_{max})\|_2 \cdot \int_0^\infty \|u(t)\|_2 \, dt + \|D_c\|_2 \cdot \int_0^\infty \|\Delta H(t) * u(t)\|_2 \, dt$$

$$+ \|D_c\|_2 \cdot \int_0^\infty \|\Delta H(t) * u(t)\|_2 \, dt \tag{A20b}$$

Substitute Eq.(A18) into Eq.(A20b) and rearrange to obtain

$$\int_0^\infty \|u(t)\|_2 \, dt \le \frac{\|[\, D_c^+(C^+ + C_{max}),\; C_c^+](MD_1Q)^+\|_2}{1 - \|D_c^+(D^+ + D_{max})\|_2 + \|D_c\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right)}$$

$$\cdot \int^\infty \|\tilde{z}(t)\|_2 \, dt \tag{A21}$$

where

$$\|D_c^+(D^+ + D_{max})\|_2 + \|D_c\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right) < 1 \tag{A22}$$

Substitute Eq.(A21) into Eq.(A20b) to obtain

$$\int_0^\infty \|\Delta H(t) * u(t)\|_2 \, dt$$

$$\le \frac{\|[D_c^+(C^+ + C_{max}),\ C_c^+](MD_1Q)^+\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right)}{1 - \|D_c^+(D^+ + D_{max})\|_2 + \|D_c\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right)}$$

$$\cdot \int_0^\infty \|\tilde{z}(t)\|_2 \, dt \tag{A23}$$

Substitute Eqs.(A21) and (A23) into Eq.(A15) and use

$\|\exp(\Lambda t)\|_2 \le \exp(-\alpha t)$; $\alpha = -\max_i \mathrm{Re}[\lambda_i(\tilde{A}_{cl})]$ to obtain

$$\int_0^\infty \|z(\tilde{t})\|_2 \, dt \le$$

$$\kappa(Q_2) \cdot \|z(\tilde{0})\|_2 \cdot \int_0^\infty \|\exp(-\alpha t)\|_2 \, dt + \|Q^{-1}\|_2 \cdot \int_0^\infty \exp(-\alpha t) dt$$

$$\cdot \left\{ \left\| (D_1^{-1}M^{-1})^+ \begin{bmatrix} A_{max} + B_{max}(D_cC)^+ + (B^+ + B_{max})D_c^+C_{max} & B_{max}C_c^+ \\ B_c^+C_{max} & 0 \end{bmatrix} \right.\right.$$

$$\left. \cdot (MD_1Q)^+ \right\|_2$$

$$+ \frac{\left\| (D_1^{-1}M^{-1})^+ \begin{bmatrix} (B^+ + B_{max})D_c^+(D^+ + D_{max}) \\ B_c^+(D^+ + D_{max}) \end{bmatrix} \right\|_2}{1 - \|D_c^+(D^+ + D_{max})\|_2 + \|D_c\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right)}$$

$$\bullet\|[D_c^{+}(C^{+}+C_{max}),C^{+}](MD_1Q)^{+}\|_2$$

$$+\frac{\left\|(D_1^{-1}M^{-1})^{+}\begin{bmatrix}(B^{+}+B_{max})D_c^{+}\\ B_c^{+}\end{bmatrix}\right\|_2}{1-\|D_c^{+}(D^{+}+D_{max})\|_2+\|D_c\|_2\cdot\left(r_2/\beta+r_0+\gamma r_1\right)}$$

$$\left.\bullet\|[D_c^{+}(C^{+}+C_{max}),C_c^{+}](MD_1Q)^{+}\|_2\cdot\left(r_2/\beta+r_0+\gamma r_1\right)\right\}\cdot\int_0^{\infty}\|\tilde{z}(t)\|_2dt$$

(A24)

Let

$$\Psi=\frac{\|[D_c^{+}(C^{+}+C_{max}),C_c^{+}](MD_1Q)^{+}\|_2}{1-\|D_c^{+}(D^{+}+D_{max})\|_2+\|D_c\|_2\cdot\left(r_2/\beta+r_0+\gamma r_1\right)}$$

and note that $\int_0^{\infty}\exp(-\alpha t)=1/\alpha$; substitute into Eq.(A24) and rearrange

$$\int_0^{\infty}\|\tilde{z}(t)\|_2\,dt\left\{1-\left(\frac{\|Q^{-1}\|_2}{\alpha}\right)\right.$$

$$\cdot\left\{\left\|(D_1^{-1}M^{-1})^{+}\cdot\begin{bmatrix} A_{max} + B_{max}(D_cC)^{+}+(B^{+}+B_{max})D_c^{+}C_{max} & B_{max}C_c^{+} \\ B_c^{+}C_{max} & 0 \end{bmatrix}\right.\right.$$

$$\left.\cdot(MD_1Q)^{+}\right\|_2 + \Psi\left[\left\|(D_1^{-1}M^{-1})^{+}\begin{bmatrix}(B^{+}+B_{max})D_c^{+}(D^{+}+D_{max}) \\ B_c^{+}(D_c^{+}+D_{max})\end{bmatrix}\right\|_2\right.$$

$$\left.\left.+\left\|(D_1^{-1}M^{-1})^{+}\begin{bmatrix}(B^{+}+B_{max})D_c^{+} \\ B_c^{+}\end{bmatrix}\right\|_2\cdot\left(r_2/\beta + r_0 + \gamma r_1\right)\right]\right\}\Bigg\}$$

$$\le \kappa_2(Q)\cdot\|\tilde{z}(0)\|_2/\alpha \qquad \text{(A25)}$$

Solve Eq.(A25) to obtain

$$\int_0^{\infty}\|\tilde{z}(t)\|_2\,dt \le \frac{\kappa_2(Q)\cdot\|\tilde{z}(0)\|_2}{\xi}$$

where

$$\xi=\alpha-\|Q^{-1}\|_2\cdot\left\{\left\|(D_1^{-1}M^{-1})^{+}\right.\right.$$

$$\cdot\begin{bmatrix} A_{max}+B_{max}(D_cC)^{+}+(B^{+}+B_{max})D_c^{+}C_{max} & B_{max}C_c^{+} \\ B_c^{+}C_{max} & 0\end{bmatrix}\cdot(MD_1Q)^{+}\Big\|_2$$

$$+ \Psi\left[\left\| (D_1^{-1}M^{-1})^+ \begin{bmatrix} (B^+ + B_{max})D_c^+(D^+ + D_{max}) \\ B_c^+(D^+ + D_{max}) \end{bmatrix} \right\|_2\right.$$

$$\left.\left.+ \left\| (D_1^{-1}M^{-1})^+ \begin{bmatrix} (B^+ + B_{max})D_c^+ \\ B_c^+ \end{bmatrix} \right\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right)\right]\right\} \qquad \text{(A26)}$$

Thus,

$$\int_0^\infty \|\tilde{z}(t)\|_2 \, dt < \infty$$

if

$$\left(\frac{1}{\alpha\, \underline{\sigma}(Q)}\right) \left\{ \bar{\sigma}\left[(D_1^{-1}M^{-1})^+ \right.\right.$$

$$\left.\cdot \begin{bmatrix} A_{max} + B_{max}(D_cC)^+ + (B^+ + B_{max})D_c^+C_{max} & B_{max}C_c^+ \\ B_c^+C_{max} & 0 \end{bmatrix} (MD_1Q)^+ \right]$$

$$+ \frac{\bar{\sigma}\left[(D_1^{-1}M^{-1})^+ \begin{bmatrix} (B^+ + B_{max})D_c^+(D^+ + D_{max}) \\ B_c^+(D^+ + D_{max}) \end{bmatrix} \right]}{1 - [\bar{\sigma}(D_c^+(D^+ + D_{max}))] + \bar{\sigma}(D_c)\cdot\left(r_2/\beta + r_0 + \gamma r_1\right)}$$

$$\cdot \bar{\sigma}\left[[D_c^+(C^+ + C_{max}), C_c^+](MD_1Q)^+ \right]$$

$$+ \frac{\bar{\sigma}\left[(D_1^{-1}M^{-1})^{+}\begin{bmatrix}(B^{+}+B_{max})D_c^{+}\\ B_c^{+}\end{bmatrix}\right]\cdot\bar{\sigma}\left[[D_c^{+}(C^{+}+C_{max}),C_c^{+}\ ](MD_1Q)^{+}\right]}{1-\bar{\sigma}[D_c^{+}(D^{+}+D_{max})]-\bar{\sigma}(D_c)\cdot(\ r_2/\beta+r_0+\gamma r_1\ )}$$

$$\left.\cdot(\ r_2/\beta+r_0+\gamma r_1\ )\right\} < 1 \qquad \text{(A27)}$$

and if

$$\bar{\sigma}[D_c^{+}(D^{+}+D_{max})] + \bar{\sigma}(D_c)\cdot(\ r_2/\beta+r_0+\gamma r_1\ ) < 1 \qquad \text{(A28)}$$

Then, since $x(t)$ and $x_c(t)$ are continuous $\Rightarrow$ $\tilde{z}(t)$ is continuous which together with $\int_0^{\infty}\|\tilde{z}(t)\|_2\,dt < \infty$

$\Rightarrow \|\tilde{z}(t)\|_2 \to 0$ at $t \to \infty$

Since $\|\tilde{x}(t)\|_2 \le \|MD_1Q\|_2\cdot\|\tilde{z}(t)\|_2$ and $\|MD_1Q\|_2 < \infty$

$\Rightarrow \|\tilde{x}(t)\|_2 \to\ 0$ at $t \to \infty$

$\Rightarrow \|x(t)\|_2 \to 0$ as $t \to \infty$

Now consider $\|y(t)\|_2$

$$y(t) = (C+\Delta C)x(t) + (D+\Delta D)u(t) + \Delta H(t)*u(t)$$

$$= [C+\Delta C,\ 0]\tilde{x}(t) + (D+\Delta D)u(t) + \Delta H(t)*u(t)$$

$$= [C+\Delta C,\ 0]MD_1Q\tilde{z}(t) + (D+\Delta D)u(t) + \Delta H(t)*u(t) \qquad \text{(A29)}$$

$$\leq \|(C^+ + C_{max})(MD_1Q)^+\|_2 \cdot \|\tilde{z}(t)\|_2 + \|D^+ + D_{max}\|_2 \cdot \|u(t)\|_2$$

$$+ \|\Delta H(t)*u(t)\|_2 \qquad \text{(A30)}$$

$$\int_0^{\infty} \|y(t)\|_2 \, dt \leq \|(C^+ + C_{max})(MD_1Q)^+\|_2 \cdot \int_0^{\infty} \|\tilde{z}(t)\|_2 \, dt$$

$$+ \|D^+ + D_{max}\|_2 \cdot \int_0^{\infty} \|u(t)\|_2 \, dt + \int_0^{\infty} \|\Delta H(t)*u(t)\|_2 \, dt \qquad \text{(A31)}$$

$$\int_0^{\infty} \|y(t)\|_2 \, dt \leq \|(C^+ + C_{max})(MD_1Q)^+\|_2 \cdot \int_0^{\infty} \|\tilde{z}(t)\|_2 \, dt$$

$$+ \frac{\|D^+ + D_{max}\|_2 \cdot \|[\,D_c^+(C^+ + C_{max}),\ C_c^+](MD_1Q)^+\|_2}{1 - \|D_c^+(D_c^+ + D_{max})\|_2 + \|D_c\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right)}$$

$$+ \frac{\|[D_c^+(C^+ + C_{max}),\ C_c^+](MD_1Q)^+\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right)}{1 - \|D_c^+(D^+ + D_{max})\|_2 + \|D_c\|_2 \cdot \left(r_2/\beta + r_0 + \gamma r_1\right)}$$

$$\cdot \int_0^{\infty} \|\tilde{z}(t)\|_2 \, dt \qquad \text{(A32)}$$

Hence, if Eqs.(41a) and (41b) are satisfied it follows that

$$\int_0^\infty \|y(t)\|_2 \, dt < \infty.$$

Since y(t) is continuous and $\int_0^\infty \|y(t)\|_2 \, dt < \infty$, it follows that $\|y(t)\|_2 \to 0$ as $t \to \infty$. Hence $\|x(t)\|_2$ and $\|y(t)\|_2$ both approach zero as $t \to \infty$ which implies that the uncertain plant is asymptotically stable. ▨

APPENDIX II

Proof of Theorem 3.:

The uncertain closed loop plant may be written as

$$\dot{x}(t) = A_c x(t) + \Delta A_c(t)x(t) \tag{B1}$$

where

$$A_c = A + BFC$$

and

$$\Delta A_c(t) = \Delta A(t) + \Delta B(t)FC$$

which has a solution given by

$$x(t) = \exp(A_c t)x(0) + \int_0^t \exp[A_c(t-\tau)]\Delta A_c(\tau)x(\tau)d\tau \tag{B2}$$

Next, use the real, positive, diagonal transformation

$$x(t) = D^{-1}z(t) \tag{B3}$$

and the property that

$$\exp(DA_c D^{-1}t) = D\exp(A_c t)D^{-1} \tag{B4}$$

and

$$\exp(A_c t) = M\exp(\Lambda t)M^{-1} = \sum_{i=1}^{n} v_i w_i^* e^{\lambda_i t} \tag{B5}$$

to obtain

$$z(t)=D\sum_{i=1}^{n} v_i w_i^* e^{\lambda_i t} D^{-1} z(0)+\int_0^t D \sum_{i=1}^{n} v_i w_i^* e^{\lambda_i (t-\tau)} \Delta A_c(\tau) D^{-1} z(\tau) d\tau \tag{B6}$$

where M is a modal matrix of A_c; λ_i is the i-th eigenvalue of A_c with v_i and w_i^* the corresponding right and left eigenvectors, respectively; Λ is a diagonal matrix with the λ_i on the diagonal; and $(\cdot)^*$ denotes complex conjugate transpose.

Note that

$$\|z(t)\|_p \to 0 \text{ implies that } \|x(t)\|_p \to 0 \tag{B7}$$

Next, apply the absolute value operator, denoted by $(\cdot)^+$, to both sides of Eq.(B6) where "+" and "≤ " are applied element by element to vectors and matrices.

$$z^+(t) \le \left[D \sum_{i=1}^{n} v_i w_i^* \, e^{\lambda_i t} \, D^{-1} \, z(0) \right]^+$$

$$+ \left[\int_0^t D \sum_{i=1}^{n} v_i w_i^* \, e^{\lambda_i (t-\tau\;)} \, \Delta A_c(\tau) \, D^{-1} \, z(\tau) \, d\tau \right]^+ \qquad \text{(B8)}$$

$$\le D \sum_{i=1}^{n} (v_i w_i^*)^+ \, e^{-\alpha_i t} \, D^{-1} \, z^+(0)$$

$$+ \int_0^t \left[D \sum_{i=1}^{n} v_i w_i^* \, e^{\lambda_i (t-\tau)} \, \Delta A_c(\tau) \, D^{-1} \, z(\tau) \right]^+ d\tau \qquad \text{(B9)}$$

$$\le D \sum_{i=1}^{n} (v_i w_i^*)^+ \, e^{-\alpha_i t} \, D^{-1} \, z^+(0)$$

$$+ \int_0^t D \sum_{i=1}^{n} (v_i w_i^*)^+ \, e^{-\alpha_i (t-\tau)} \, A_{cmax} \, D^{-1} \, z^+(\tau) d\tau \qquad \text{(B10)}$$

where $\alpha_i = -\text{Re}(\lambda_i)$, $A_{cmax} = A_{max} + B_{max}(FC)^+$ and where we have used the property that $[\exp(\lambda_i t)]^+ = \exp(-\alpha_i t)$. Next, integrate both sides of Eq.(B10) to obtain

$$\int_0^\infty z^+(t)dt \le D \sum_{i=1}^{n} (v_i w_i^*)^+ \int_0^\infty e^{-\alpha_i t} dt \, D^{-1} \, z^+(0)$$

$$+ \int_{t=0}^\infty \int_{\tau=0}^t D \sum_{i=1}^{n} (v_i w_i^*)^+ \, e^{-\alpha_i (t-\tau)} \, A_{cmax} D^{-1} z^+(\tau) d\tau dt \qquad \text{(B11)}$$

Consider the double integral in Eq.(B11) which may be written as

$$\lim_{R\to\infty}\left[\int_{t=0}^{R}\int_{\tau=0}^{t} D\sum_{i=1}^{n}(v_i w_i^{*})^{+}\, e^{-\alpha_i(t-\tau)} A_{cmax} D^{-1} z^{+}(\tau)\,d\tau\,dt\right];0\le\tau\le t \tag{B12}$$

The order of integration in Eq.(B12) may be interchanged because all of the functions inside the integrals are continuous functions of t and τ (Churchill [31]). Thus, Eq.(B12) is equal to

$$\lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{t=\tau}^{R} D\sum_{i=1}^{n}(v_i w_i^{*})^{+}\, e^{-\alpha_i(t-\tau)} A_{cmax} D^{-1} z^{+}(\tau)\,dt\,d\tau\right];0\le\tau<t \tag{B13}$$

Now use the change of variables given by $\gamma=t-\tau$, $d\gamma=dt$, $\gamma\ge0$. Then, Eq.(B13) is equal to

$$\lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{\gamma=0}^{R-\tau} D\sum_{i=1}^{n}(v_i w_i^{*})^{+}\, e^{-\alpha_i\gamma} A_{cmax} D^{-1} z^{+}(\tau)\,d\gamma\,d\tau\right]; \quad 0\le\tau\le t,\quad \gamma\ge0, \tag{B14}$$

$$\le\lim_{R\to\infty}\left[\int_{\tau=0}^{R}\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_i w_i^{*})^{+}\, e^{-\alpha_i\gamma} A_{cmax} D^{-1} z^{+}(\tau)\,d\gamma\,d\tau\right];\ \alpha_i>0 \tag{B15}$$

$$=\lim_{R\to\infty}\left[\int_{\gamma=0}^{R} D\sum_{i=1}^{n}(v_i w_i^{*})^{+}\, e^{-\alpha_i\gamma} A_{cmax} D^{-1}\,d\gamma \int_{\tau=0}^{R} z^{+}(\tau)\,d\tau\right];\ \alpha_i>0 \tag{B16}$$

$$\le \lim_{R\to\infty} \left[D\sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{-\alpha_i} (e^{-\alpha_i R} - 1)\, A_{cmax} D^{-1} \int_0^\infty z^+(\tau)d\tau \right];\ \alpha_i > 0 \tag{B17}$$

Evaluating the limit of the term outside the integral in Eq.(B17), note that τ is now a dummy variable of integration, recall that the α_i's are positive, and substitute the result into Eq.(B11) to obtain

$$\int_0^\infty z^+(t)dt \le D\sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} D^{-1} z^+(0)$$

$$+ D\sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} A_{cmax}\, D^{-1} \cdot \int_0^\infty z^+(t)dt \tag{B18}$$

Take norms in Eq.(B18) and rearrange to obtain

$$\left\| \int_0^\infty z^+(t)dt \right\|_p \le \frac{\left\| D\sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} D^{-1} z^+(0) \right\|_p}{1 - \left\| D\sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} A_{cmax} D^{-1} \right\|_p} \tag{B19}$$

Thus, $\left\| \int_0^\infty z^+(t)\, dt \right\|_p < \infty$ if

$$\left\| D\sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} [A_{max} + B_{max}\, (FC)^+]\, D^{-1} \right\|_p < 1 \tag{B20}$$

We now require an additional lemma.

Lemma B:

If $\left\| \int_0^\infty z^+(t)\, dt \right\|_p < \infty$, then $\int_0^\infty \left\| z^+(t) \right\|_1 dt < \infty$.

<u>Proof of Lemma B.:</u>

Lemma B is a special case of Lemma G2 (see Appendix VII). ⁂

Returning to the proof of Theorem 3, we have from Lemma B and Eq.(B20) that $\int_0^\infty \|z^+(t)\|_1\, dt < \infty$ if

$$\left\| D \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} [A_{max} + B_{max}(FC)^+]\, D^{-1} \right\|_p < 1;\ p \geq 1 \qquad \text{(B21)}$$

Note that x(t) and z(t) are continuous because of the linearity of the uncertain closed loop plant which together with Eq.(B21) implies that $\|z^+(t)\|_1 \to 0$ as $t \to \infty$. This implies that $\|x(t)\|_1 \to 0$ as $t \to \infty$ which proves that the linear uncertain closed loop plant is asymptotically stable. We remark that although not needed for the proof of Theorem 3, it follows from Eq.(B21) that $\|z^+(t)\|_p$ and $\|x(t)\|_p$ are bounded and asymptotically vanishing for all $p \geq 1$.

Finally, Perron weightings may be used for the matrix D in

Eq.(B24) in the same manner as shown by Sobel et. al. [1] for an earlier robustness result. Thus, to reduce conservatism, Eq.(B21) may be replaced by

$$\lambda_{\max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^{*})^{+}}{\alpha_i} \left[A_{\max} + B_{\max}(FC)^{+}\right] \right\} < 1 \qquad \text{(B22)}$$

where $\lambda_{\max}(\cdot)$ of a non-negative matrix denotes the real non-negative eigenvalue $\lambda_{\max} \geq 0$ such that $\lambda_{\max} \geq |\lambda_i|$ for all eigenvalues λ_i. ∎

APPENDIX III

Proof of Theorem 4.:

We "element by element bound" the matrix on the left hand side of Eq.(57) and recall the definition $A_{cmax} = A_{max} + B_{max}(FC)^{+}$ to obtain

$$\sum_{i=1}^{n} \frac{(v_i w_i^{*})^{+}}{\alpha_i} A_{cmax} \le \sum_{i=1}^{n} \frac{v_i^{+} (w_i^{*})^{+}}{\alpha} A_{cmax} \tag{C1}$$

$$\le \sum_{i=1}^{n} \frac{v_i^{+} (w_i^{*})^{+}}{\alpha} A_{cmax} \tag{C2}$$

$$\le \left(\frac{1}{\alpha}\right) \cdot \left[v_1^{+}, v_2^{+}, \dots, v_n^{+} \right] \begin{bmatrix} (w_1^{*})^{+} \\ (w_2^{*})^{+} \\ \cdot \\ \cdot \\ \cdot \\ (w_n^{*})^{+} \end{bmatrix} \cdot A_{cmax} \tag{C3}$$

$$\le \left(\frac{1}{\alpha}\right) \cdot M^{+} (M^{-1})^{+} A_{cmax} \tag{C4}$$

Now, use the result that if $B^{+} \le A^{+}$, then

$$\lambda_{max}(B^+) = \|B^+\|_{2D_B^+} \leq \|B^+\|_{2D_A^+} \leq \|A^+\|_{2D_A^+} = \lambda_{max}(A^+) \qquad \text{(C5)}$$

where $\|\cdot\|_{2_{D_B^+}}$ and $\|\cdot\|_{2_{D_A^+}}$ are D-weighted two norms with Perron weights for B^+ and A^+, respectively. Then, applying Eq. (C5) to Eq. (C4) yields

$$\lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} A_{cmax} \right\} \leq \lambda_{max}\left\{ \left(\frac{1}{\alpha}\right) \cdot M^+ (M^{-1})^+ A_{cmax} \right\}$$

(C6)

Finally, use the result (Ogata [32]) that if A and B are matrices, then

$$\lambda_i(AB) = \lambda_i(BA) \qquad \text{(C7)}$$

to obtain

$$\lambda_{max}\left\{ \sum_{i=1} \frac{(v_i w_i^*)^+}{\alpha_i} A_{cmax} \right\} \leq \lambda_{max} \left\{ \left(\frac{1}{\alpha}\right) \cdot (M^{-1})^+ A_{cmax} M^+ \right\}$$

(C8)

APPENDIX IV

Proof of Theorem 5.:

Sufficiency:

Use the definition of A_δ to obtain

$$M^{-1}A_\delta M = \frac{1}{\Delta} M^{-1}(e^{A\Delta}-I)M \tag{D1}$$

Substitute the infinite series for exp(AΔ) into Eq.(D1) to obtain

$$M^{-1}A_\delta M = \frac{1}{\Delta} M^{-1}[I + A\Delta + \frac{A^2\Delta^2}{2!} + \ldots - I]M \tag{D2}$$

$$= \frac{1}{\Delta} [I + M^{-1}AM\Delta + \frac{M^{-1}A^2M\Delta^2}{2!} + \ldots - I] \tag{D3}$$

$$= \frac{1}{\Delta} [I + M^{-1}AM\Delta + (M^{-1}AM)^2 \frac{\Delta^2}{2!} + (M^{-1}AM)^3 \frac{\Delta^3}{3!} + \ldots - I] \tag{D4}$$

Substitute $M^{-1}AM = \Lambda$ into Eq.(D4) to obtain

$$M^{-1}A_{\delta}M = \frac{1}{\Delta}\left[I + \Lambda\Delta + \Lambda^{2}\frac{\Delta^{2}}{2!} + \Lambda^{3}\frac{\Delta^{3}}{3!} + \ldots - I\right] \tag{D5}$$

$$= \frac{1}{\Delta}\left(e^{\Lambda\Delta} - I\right) \tag{D6}$$

Necessity:

$$M^{-1}A_{\delta}M = \frac{1}{\Delta} M^{-1}\left(e^{A\Delta} - I\right)M = \frac{1}{\Delta}\left(e^{\Lambda\Delta} - I\right) \tag{D7}$$

$$\Rightarrow M^{-1}e^{A\Delta}M = e^{\Lambda\Delta} \tag{D8}$$

$$\Rightarrow e^{M^{-1}AM\Delta} = e^{\Lambda\Delta} \tag{D9}$$

$$\Rightarrow M^{-1}AM\Delta = \Lambda\Delta \tag{D10}$$

$$\Rightarrow M^{-1}AM = \Lambda \tag{D11}$$

▨

APPENDIX V

Proof of Theorem 6.:

Settling time Region:

Use

$$1+\Delta\gamma_i = \exp(\lambda_i \Delta)$$

to obtain

$$|1+\Delta\gamma_i| = \exp(\Delta \cdot \mathrm{Re}\lambda_i)$$

Then,

$$\mathrm{Re}\ \lambda_i < \alpha \Rightarrow |1+\Delta\gamma_i| < \exp(\alpha\Delta)$$

Damping Ratio Region:

$$\lambda_i = -\zeta_i \omega_{ni} + j\omega_{di} \qquad \text{(E1)}$$

$$1+\Delta\gamma_i = \exp(\lambda_i \Delta) = \exp(-\zeta_i \omega_{ni} \Delta + j\omega_{di} \Delta) \qquad \text{(E2)}$$

Use $\omega_s \Delta = 2\pi$ and $\omega_{di} = \omega_{ni}(1-\zeta_i^2)^{1/2}$ to obtain

$$1+\Delta\gamma_i = \exp\left[\frac{-\zeta_i 2\pi}{(1-\zeta_i^2)^{1/2}} \frac{\omega_{di}}{\omega_s} + j2\pi \frac{\omega_{di}}{\omega_s}\right] \qquad \text{(E3)}$$

which has a magnitude given by

$$|1+\Delta\gamma_i| = \exp\left[\frac{-\zeta_i 2\pi}{(1-\zeta_i^2)^{1/2}} \frac{\omega_{di}}{\omega_s}\right] = \exp\left[\frac{-\zeta_i \phi_i}{(1-\zeta_i^2)^{1/2}}\right] \qquad \text{(E4)}$$

and a phase angle given by

$$\arg(1+\Delta\gamma_i) = \frac{2\pi\omega_{di}}{\omega_s} \tag{E5}$$

So $\zeta > \zeta_i$

$$\Rightarrow |1+\Delta\gamma_i| < \exp\left[\frac{-\zeta\ \phi}{(1-\zeta^2)^{1/2}}\right] \tag{E7}$$

where $\phi = \arg(1+\Delta\gamma)$. ▨

APPENDIX VI

Proof of Theorem 7.:

The r assignable eigenvalues and their corresponding eigenvectors for the sampled data system satisfy the equations described by

$$(A_\delta+B_\delta F_\delta C)v_i = \gamma_i v_i; \quad i = 1,2,\ldots,r \tag{F1}$$

Combining the r equations from Eq.(F1) yields

$$(A_\delta+B_\delta F_\delta C)M_r = M_r\Lambda_{\delta r} \tag{F2}$$

Rearrange to obtain

$$B_\delta F_\delta CM_r = M_r\Lambda_{\delta r}-A_\delta M_r \tag{F3}$$

Substitute the singular value decompositions of B_δ and CM_r to obtain

$$[U_{\delta 0} \quad U_{\delta 1}]\begin{bmatrix}\Sigma_\delta V_\delta^T \\ 0\end{bmatrix} F_\delta \, [U_{r0} \quad U_{r1}]\begin{bmatrix}\Sigma_r V_r^T \\ 0\end{bmatrix} = M_r\Lambda_{\delta r}-A_\delta M_r \tag{F4}$$

$$U_{\delta 0}\Sigma_\delta V_\delta^T F_\delta U_{r0}\Sigma_r V_r^T = M_r\Lambda_{\delta r}-A_\delta M_r \tag{F5}$$

Taking the required inverses and recalling that $U_{\delta 0}$, U_{r0}, V_δ, and V_r are unitary, we obtain

$$F_\delta = V_\delta \Sigma_\delta^{-1} U_{\delta 0}^T (M_r \Lambda_{\delta r} - A_\delta M_r) V_r \Sigma_r^{-1} U_{r0}^T \qquad \text{(F6)}$$

Next, we consider the limiting behavior of F_δ as $\Delta \to 0$. Middleton and Goodwin [3] show that as $\Delta \to 0$, $A_\delta \to A$, $B_\delta \to B$, and $\Lambda_{\delta r} \to \Lambda_r$. Thus, $V_\delta \Sigma_\delta^{-1} U_{\delta 0}^T \to V_B \Sigma_B^{-1} U_{B0}^T$, and $F_\delta \to F$. ▨

APPENDIX VII

Proof of Theorem 8.:

The proof starts from showing the following bounds.

Lemma G1: (Delta Uncertainty Bounds)

$$(dA_\delta)^+ \le A_{\delta\max} = \frac{1}{\Delta}\left[e^{(A^+ + A_{\max})\Delta} - e^{A^+\Delta}\right] \qquad \text{(G1)}$$

$$(dB_\delta)^+ \le B_{\delta\max} = \frac{1}{\Delta}\left[\int_0^\Delta e^{(A^+ + A_{\max})\tau} d\tau (B^+ + B_{\max}) - \int_0^\Delta e^{A^+\tau} d\tau B^+\right] \qquad \text{(G2)}$$

Proof of Lemma G1.:

$$(dA_\delta)^+ = \frac{1}{\Delta}\left[e^{(A+dA)\Delta} - e^{A\Delta}\right]^+ = \frac{1}{\Delta}\Bigl\{\Bigl[I + (A+dA)\Delta + (A+dA)^2\frac{\Delta^2}{2!}$$

$$+ (A+dA)^3\frac{\Delta^3}{3!} + \dots \Bigr] - \Bigl[I + A\Delta + A^2\frac{\Delta^2}{2!} + A^3\frac{\Delta^3}{3!} + \dots \Bigr]\Bigr\}^+$$

$$= \frac{1}{\Delta}\Bigl\{dA\cdot\Delta + \Bigl[A\cdot dA + dA\cdot A + (dA)^2\Bigr]\frac{\Delta^2}{2!} + \Bigl[A\cdot dA\cdot A + dA\cdot A\cdot dA$$

$$+ dA\cdot A^2 + A^2\cdot dA + (dA)^2\cdot A + A\cdot(dA)^2 + (dA)^3\Bigr]\frac{\Delta^3}{3!} + \dots\Bigr\}^+$$

$$\le \frac{1}{\Delta}\Bigl\{A_{\max}\cdot\Delta + \Bigl[A^+\cdot A_{\max} + A_{\max}\cdot A^+ + A_{\max}^2\Bigr]\frac{\Delta^2}{2!} + \Bigl[A^+\cdot A_{\max}\cdot A^+$$

$$+ A_{max}\cdot A^{+}\cdot A_{max} + A_{max}\cdot (A^{+})^{2} + (A^{+})^{2}\cdot A_{max} + (A_{max})^{2}\cdot A^{+}$$

$$+ A^{+}\cdot (A_{max})^{2} + (A_{max})^{3} \Big]\frac{\Delta^{3}}{3!} + \dots \Big\}$$

$$= \frac{1}{\Delta} \Big\{ \Big[I+(A^{+}+A_{max})\Delta + (A^{+}+A_{max})^{2}\frac{\Delta^{2}}{2!} + (A^{+}+A_{max})^{3}\frac{\Delta^{3}}{3!} + \dots \Big]$$

$$- \Big[I+ A^{+}\Delta + (A^{+})^{2}\frac{\Delta^{2}}{2!} + (A^{+})^{3}\frac{\Delta^{3}}{3!} + \dots \Big] \Big\}$$

$$= \frac{1}{\Delta} \Big[e^{(A^{+}+A_{max})\Delta} - e^{A^{+}\Delta} \Big]$$

$$(dB_{\delta})^{+} = \Big\{ \frac{1}{\Delta} \Big[\int_{0}^{\Delta} e^{(A+dA)\tau}\, d\tau\, (B + dB) - \int_{0}^{\Delta} e^{A\tau}\, d\tau\, B \Big] \Big\}^{+}$$

$$= \frac{1}{\Delta} \Big\{ \Big[I\Delta + (A+dA)\frac{\Delta^{2}}{2!} + (A+dA)^{2}\frac{\Delta^{3}}{3!} + (A+dA)^{3}\frac{\Delta^{4}}{4!} + \dots \Big]$$

$$\cdot (B + dB) - \Big[I\Delta + A\frac{\Delta^{2}}{2!} + A^{2}\frac{\Delta^{3}}{3!} + A^{3}\frac{\Delta^{4}}{4!} + \dots \Big] B \Big\}^{+}$$

$$= \frac{1}{\Delta} \Big\{ \Big[I\Delta + A\frac{\Delta^{2}}{2!} + A^{2}\frac{\Delta^{3}}{3!} + A^{3}\frac{\Delta^{4}}{4!} + \dots \Big] dB + \Big[dA\cdot\frac{\Delta^{2}}{2!}$$

$$+ \Big(A\cdot dA + dA\cdot A + (dA)^{2} \Big)\frac{\Delta^{3}}{3!} + \Big(A\cdot dA\cdot A + dA\cdot A\cdot dA + dA\cdot A^{2}$$

$$+ A^{2}\cdot dA + (dA)^{2}\cdot A + A\cdot (dA)^{2} + (dA)^{3} \Big)\frac{\Delta^{4}}{4!} + \dots \Big] (B + dB) \Big\}^{+}$$

$$\le \frac{1}{\Delta} \Big\{ \Big[I\Delta + A^{+}\frac{\Delta^{2}}{2!} + (A^{+})^{2}\frac{\Delta^{3}}{3!} + (A^{+})^{3}\frac{\Delta^{4}}{4!} + \dots \Big] B_{max}$$

$$+ \Bigg[A_{max} \cdot \frac{\Delta^2}{2!} + \Big(A^+ \cdot A_{max} + A_{max} \cdot A^+ + (A_{max})^2 \Big) \frac{\Delta^3}{3!}$$

$$+ \Big(A^+ \cdot A_{max} \cdot A^+ + A_{max} \cdot A^+ \cdot A_{max} + A_{max} \cdot (A^+)^2 + (A^+)^2 \cdot A_{max}$$

$$+ (A_{max})^2 \cdot A^+ + A^+ \cdot (A_{max})^2 + (A_{max})^3 \Big) \frac{\Delta^4}{4!} + \ldots \Bigg] (B^+ + B_{max}) \Bigg\}$$

$$= \frac{1}{\Delta} \Bigg\{ \Bigg[I\Delta + (A^+ + A_{max}) \frac{\Delta^2}{2!} + (A^+ + A_{max})^2 \frac{\Delta^3}{3!} + (A^+ + A_{max})^3 \frac{\Delta^4}{4!}$$

$$+ \cdots \Bigg] (B^+ + B_{max}) - \Bigg[I\Delta + A^+ \frac{\Delta^2}{2!} + (A^+)^2 \frac{\Delta^3}{3!} + (A^+)^3 \frac{\Delta^4}{4!} + \ldots \Bigg] B^+ \Bigg\}$$

$$= \frac{1}{\Delta} \Bigg[\int_0^\Delta e^{(A^+ + A_{max})\tau} \, d\tau \, (B^+ + B_{max}) - \int_0^\Delta e^{A^+ \tau} \, d\tau \, B^+ \Bigg]$$

∎

The proof of theorem 8 now continues by observing that the uncertain closed loop system may be written as

$$\rho x(t) = A_{\rho c} x(t) + dA_{\rho c} x(t) \qquad \text{(G3)}$$

where

$$A_{\rho c} = \begin{cases} A + BFC & \text{continuous time} \\ A_\delta + B_\delta F_\delta C & \text{discrete time} \end{cases}$$

and

$$dA_{\rho c} = \begin{cases} dA + dB(FC) & \text{continuous time} \\ dA_\delta + dB_\delta (F_\delta C) & \text{discrete time} \end{cases}$$

which has a solution given by Middleton and Goodwin [3]

$$x(t) = E(A_{\rho c}, t)x(0) + \int_{0}^{t} E(A_{\rho c}, t-\tau-\Delta)dA_{\rho c}x(\tau)d\tau \tag{G4}$$

where

$$E(A_{\rho c}, t) = \begin{cases} e^{A_{\rho c}t} & \text{continuous time} \\ (I+A_{\rho c}\Delta)^{t/\Delta} & \text{discrete time} \end{cases} \tag{G5}$$

and

$$\int_{t_1}^{t_2} f(\tau)\, d\tau = \begin{cases} \displaystyle\int_{\tau=t_1}^{\tau=t_2} f(\tau)d\tau & \text{continuous time} \\ \displaystyle\Delta \sum_{k=t_1/\Delta}^{k=(t_2/\Delta)-1} f(k\Delta) & \text{discete time} \end{cases} \tag{G6}$$

Next, use the real, positive, diagonal transformation

$$x(t) = D^{-1}z(t) \tag{G7}$$

and the properties shown in Ref.3 that

$$E(DA_{\rho c}D^{-1}t) = DE(A_{\rho c}, t)D^{-1} \tag{G8}$$

and

$$E(A_{\rho c},t) = ME(\Lambda_\rho,t)M^{-1} = \sum_{i=1}^{n} v_i w_i^* \, E(\gamma_i,t) \tag{G9}$$

to obtain

$$z(t) = D\sum_{i=1}^{n} v_i w_i^* \, E(\gamma_i,t)\, D^{-1}\, z(0)$$

$$+ \int_0^t D \sum_{i=1}^{n} v_i w_i^* \, E(\gamma_i,t-\tau-\Delta)\, dA_{\rho c}(\tau)\, D^{-1}\, z(\tau)d\tau \tag{G10}$$

where M is a modal matrix of $A_{\rho c}$; γ_i is the i-th eigenvalue of $A_{\rho c}$ given by

$$\gamma_i = \begin{cases} \lambda_i & \text{continuous time} \\ \frac{1}{\Delta}(e^{\lambda_i \Delta} - 1) & \text{discrete time} \end{cases} \tag{G11}$$

with v_i and w_i^* the corresponding right and left eigenvectors, respectively; Λ_ρ is a diagonal matrix with the γ_i on the diagonal; and $(\cdot)^*$ denotes complex conjugate transpose.

Note that $\|z(t)\|_p \to 0$ implies that $\|x(t)\|_p \to 0$. Next, we apply the absolute value operator, denoted by $(\cdot)^+$, to both sides of Eq.(G10) where "+" and "≤" are applied element by element to vectors and matrices.

$$z^{+}(t) \le \left[D \sum_{i=1}^{n} v_i w_i^{*} \, E(\gamma_i, t) \, D^{-1} \, z(0) \right]^{+}$$

$$+ \left[\int_0^t D \sum_{i=1}^{n} v_i w_i^{*} \, E(\gamma_i, t-\tau-\Delta) dA_{\rho c}(\tau) D^{-1} z(\tau) d\tau \right]^{+} \qquad \text{(G12)}$$

$$\le D \sum_{i=1}^{n} (v_i w_i^{*})^{+} \, E^{+}(\gamma_i, t) \, D^{-1} \, z^{+}(0)$$

$$+ \int_0^t D \sum_{i=1}^{n} (v_i w_i^{*})^{+} \, E^{+}(\gamma_i, t-\tau-\Delta) \, A_{\rho c\max} \, D^{-1} \, z^{+}(\tau) \, d\tau \qquad \text{(G13)}$$

where

$$A_{\rho c\max} = \begin{cases} A_{\max} + B_{\max} \, (FC)^{+} & \text{continuous time} \\ A_{\delta\max} + B_{\delta\max} \, (F_{\delta}C)^{+} & \text{discrete time} \end{cases} \qquad \text{(G14)}$$

with $A_{\delta\max}$ and $B_{\delta\max}$ given by Lemma G1; and where

$$E\,(\gamma_i, t) = \begin{cases} e^{\lambda_i t} & \text{in continuous time} \\ (1+\Delta\gamma_i)^{t/\Delta} & \text{in discrete time} \end{cases} \qquad \text{(G15)}$$

$$E^{+}(\gamma_i, t) = \begin{cases} e^{\mathrm{Re}(\gamma_i)\cdot t} = e^{-\alpha_i t} & \text{continuous time} \\ [(1+\Delta\gamma_i)^{+}]^{t/\Delta} = e^{-\alpha_i k\Delta} & \text{discrete time} \end{cases} \qquad \text{(G16)}$$

and $\alpha_i = -\mathrm{Re}(\lambda_i)$.

Next, apply the $\mathcal{S}$ operator to both sides of Eq.(G13) to obtain

$$\mathcal{S}_0^t z^+(t)\,dt \le D \sum_{i=1}^{n} (v_i w_i^*)^+ \mathcal{S}_0^\infty E^+(\gamma_i, t)\,dt\, D^{-1} z^+(0)$$

$$+ \mathcal{S}_{t=0}^{\infty} \mathcal{S}_{\tau=0}^{t} D \sum_{i=1}^{n} (v_i w_i^*)^+ E^+(\gamma_i, t-\tau-\Delta)\, A_{\rho c\max} D^{-1} z^+(\tau)\,d\tau\,dt \tag{G17}$$

where

$$\mathcal{S}_0^t E^+(\gamma_i, t)dt = \begin{cases} \displaystyle\int_0^\infty e^{\mathrm{Re}(\gamma_i)t}\,dt = \int_0^\infty e^{-\alpha_i t}\,dt & \text{cont.time} \\ \displaystyle\Delta \sum_{k=0}^{\infty} [(1+\Delta\gamma_i)^+]^k = \Delta \sum_{k=0}^{\infty} e^{-\alpha_i k\Delta} & \text{disc.time} \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^\infty e^{-\alpha_i t}\,dt = \frac{1}{\alpha_i} & \text{continuous time} \\ \displaystyle\frac{\Delta}{1-\exp(-\alpha_i\Delta)} & \text{discrete time} \end{cases} \tag{G18}$$

Consider the double $\mathcal{S}$ term in Eq.(G17) which may be written as

$$\lim_{R\to\infty}\left[\mathop{S}_{t=0}^{R}\ \mathop{S}_{\tau=0}^{t} D\sum_{i=1}^{n}(v_iw_i^*)^+\ E^+(\gamma_i,t-\tau-\Delta)\ A_{\rho c\max}\ D^{-1}\ z^+(\tau)d\tau dt\right] \tag{G19}$$

In continuous time, Eq.(G19) becomes

$$\lim_{R\to\infty}\left[\int_{t=0}^{R}\int_{\tau=0}^{t} D\sum_{i=1}^{n}(v_iw_i^*)^+e^{-\alpha_i(t-\tau)}\ [A_{\max}+B_{\max}(FC)^+]D^{-1}z^+(\tau)d\tau dt\right] \tag{G20}$$

$$\leq D\sum_{i=1}^{n}\frac{(v_iw_i^*)^+}{\alpha_i}\ [A_{\max}+B_{\max}(FC)^+]D^{-1}\int_{0}^{\infty}z^+(t)dt \tag{G21}$$

where Eq.(G21) is obtained by the same derivation which was used to obtain Eq.(B17).

In discrete time, let t=kΔ and τ=jΔ. Then, Eq.(G19) becomes

$$\lim_{R\to\infty}\left[\Delta\sum_{k=0}^{(R/\Delta)-1}\ \Delta\sum_{j=0}^{k-1} D\sum_{i=1}^{n}(v_iw_i^*)^+[(1+\Delta\gamma_i)^+]^{k-j-1}\right.$$

$$\left.\cdot\ A_{\delta c\max}\ D^{-1}\ z^+(j)\right] \tag{G22}$$

where

$$A_{\delta c\max}=A_{\delta\max}+B_{\delta\max}(F_\delta C)^+$$

Upon changing the order of the summations Eq.(G22) becomes

$$\lim_{R\to\infty}\left[\Delta^2 \sum_{j=0}^{(R/\Delta\ -2)} \sum_{k=j+1}^{(R/\Delta\ -1)} D \sum_{i=1}^{n} (v_i w_i^*)^+ [(1+\Delta\gamma_i)^+]^{k-j-1} \right.$$

$$\left. \cdot A_{\delta cmax}\ D^{-1} z^+(j) \right]$$

Then, let q=k-j-1 to obtain

$$\lim_{R\to\infty}\left[\Delta^2 \sum_{j=0}^{(R/\Delta-2)} \sum_{q=0}^{(R/\Delta-j-2)} D \sum_{i=1}^{n} (v_i w_i^*)^+ [(1+\Delta\gamma_i)^+]^q A_{\delta cmax} D^{-1} z^+(j)\right] \tag{G23}$$

$$\le\lim_{R\to\infty}\left[\Delta^2 \sum_{j=0}^{R/\Delta} \sum_{q=0}^{R/\Delta} D \sum_{i=1}^{n} (v_i w_i^*)^+ \ [(1+\Delta\gamma_i)^+]^q \ A_{\delta cmax}\ D^{-1}\ z^+(j)\right] \tag{G24}$$

$$\le\lim_{R\to\infty}\left[\Delta^2\ D \sum_{i=1}^{n} (v_i w_i^*)^+ \sum_{q=0}^{R/\Delta} [(1+\Delta\gamma_i)^+]^q \ A_{\delta cmax}\ D^{-1} \sum_{j=0}^{R/\Delta} z^+(j)\right] \tag{G25}$$

$$\le\lim_{R\to\infty}\left[\Delta^2\ D \sum_{i=1}^{n} (v_i w_i^*)^+ \left(\frac{1-[(1+\Delta\gamma_i)^+]^{(R/\Delta\ +1)}}{1-(1+\Delta\gamma_i)^+}\right)\right.$$

$$\left. \cdot A_{\delta cmax}\ D^{-1} \sum_{j=0}^{R/\Delta} z^+(j)\right] \tag{G26}$$

Use Eq.(G11) to obtain

$$(1+\Delta\gamma_i)^+ \ = \ \exp(-\alpha_i \Delta)$$

and

$$\lim_{R\to\infty}\left[1 - [(1+\Delta\gamma_i)^+]^{(R/\Delta)+1} \right] = \lim_{R\to\infty}\left[1 - \exp[-\alpha_i\Delta(R+1)]\right] = 1 \qquad \text{(G27)}$$

because $\alpha_i > 0$. Substitute this result into Eq.(G26) to obtain

$$\lim_{R\to\infty}\left[\Delta^2 \sum_{k=0}^{(R/\Delta-1)} \sum_{j=0}^{k-1} D\sum_{i=1}^{n} (v_i w_i^*)^+ [(1+\Delta\gamma_i)^+]^{k-j-1} A_{\delta c\max} D^{-1} z^+(j)\right]$$

$$\leq \Delta^2\, D \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{1-(1+\Delta\gamma_i)^+} A_{\delta c\max}\, D^{-1} \sum_{j=0}^{\infty} z^+(j) \qquad \text{(G28)}$$

Combine Eqs.(G21) and (G28) and use Eq.(G6) to obtain

$$\int_{t=0}^{\infty} \int_{\tau=0}^{t} D \sum_{i=1}^{n} (v_i w_i^*)^+\, E^+(\gamma_i, t-\tau-\Delta)\, A_{\rho c\max}\, D^{-1}\, z^+(\tau)\, d\tau\, dt$$

$$\leq D \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\gamma_i)} A_{\rho c\max}\, D^{-1} \int_{0}^{\infty} z^+(\tau)\, d\tau \qquad \text{(G29)}$$

where

$$f(\gamma_i) = \begin{cases} -\mathrm{Re}(\gamma_i) = -\mathrm{Re}(\lambda_i) = \alpha_i & \text{cont.time} \\ \dfrac{1}{\Delta}\,[1-(1+\Delta\gamma_i)^+] = \dfrac{1}{\Delta}[1-e^{-\alpha_i\Delta}] & \text{disc.time} \end{cases} \qquad \text{(G30)}$$

Substitute Eq.(G29) into Eq.(G17) and use Eq.(G18) to obtain

$$\int_0^\infty z^+(t)\, dt \le D \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\gamma_i)} D^{-1} z^+(0)$$

$$+ D \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\gamma_i)} A_{\rho c\max} D^{-1} \int_0^\infty z^+(t)\, dt \qquad \text{(G31)}$$

Take norms in Eq.(G31) and rearrange to obtain

$$\left\| \int_0^\infty z^+(t)dt \right\|_p \le \frac{\left\| D \sum\limits_{i=1} \frac{(v_i w_i^*)^+}{f(\gamma_i)} D^{-1} z^+(0) \right\|_p}{1 - \left\| D \sum\limits_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\gamma_i)} A_{\rho c\max} D^{-1} \right\|_p} \qquad \text{(G32a)}$$

Thus, $\left\| \int_0^\infty z^+(t)\, dt \right\|_p < \infty$ if

$$\left\| D \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\gamma_i)} A_{\rho c\max} D^{-1} \right\|_p < 1; \quad p \ge 1 \qquad \text{(G32b)}$$

We now require an additional lemma.

Lemma G2:

If $\left\| \int_0^\infty z^+(t)\, dt \right\|_p < \infty$, then $\int_0^\infty \left\| z^+(t) \right\|_1 dt < \infty$.

Proof of Lemma G2.:

Use the equivalence of p-norms on $\mathbb{R}^n$. That is,

$$c_1 \cdot \|z\|_p \le \|z\|_1 \le c_2 \cdot \|z\|_p \tag{G33}$$

where $p \ge 1$; c_1, c_2 are constants

to obtain

$$\left\| \int_0^\infty z^+(t)\, dt \right\|_p < \infty \Rightarrow \left\| \int_0^\infty z^+(t)\, dt \right\|_1 < \infty \tag{G34}$$

It is easily shown that since $\dim[z(t)] < \infty$, it follows that

$$\left\| \int_0^\infty z^+(t)\, dt \right\|_1 = \int_0^\infty \|z^+(t)\|_1\, dt \tag{G35}$$

which implies that

$$\int_0^\infty \|z^+(t)\|_1\, dt < \infty$$

Returning now to the proof of Theorem 8, we have from Lemma G2 and Eq. (G32b) that $\int_0^\infty \|z^+(t)\|_1\, dt < \infty$ if

$$\left\| D \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\gamma_i)} A_{\rho c\max}\, D^{-1} \right\|_p < 1; \quad p \ge 1 \tag{G36}$$

Note that in continuous time, x(t) and z(t) are continuous because of the linearity of the uncertain closed loop plant which together with Eq.(G36) and a theorem in Ref.21 implies that $\|z^{+}(t)\|_1 \to 0$ as $t \to \infty$. This implies that $\|x(t)\|_1 \to 0$ as $t \to \infty$ which proves that the linear uncertain closed loop plant is asymptotically stable. We remark that although not needed for the proof of Theorem 8, it follows from Eq.(G33) that $\|z^{+}(t)\|_p$ and $\|x(t)\|_p$ are bounded and asymptotically vanishing for all $p \geq 1$.

Finally, Perron weightings may be used for the matrix D in Eq.(G36) in the same manner as shown by Sobel et.al.[1] for an earlier robustness result. Thus, to reduce conservatism, Eq.(G36) may be replaced by

$$\lambda_{\max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^{*})^{+}}{f(\gamma_i)} A_{\rho c\max} \right\} < 1 \qquad \text{(G37)}$$

where $\lambda_{\max}(\cdot)$ of a non-negative matrix denotes the real non-negative eigenvalue $\lambda_{\max} \geq 0$ such that $\lambda_{\max} \geq |\lambda_i|$ for all eigenvalues λ_i.

APPENDIX VIII

Proof of Lemma 1.:

Upon comparing Eqs.(78) and (136) we require

$$B_\rho u(t) = \tilde{B}_\rho \delta(t) \tag{H1}$$

Solving for the true control u(t), we obtain

$$u(t) = B_\rho^\dagger \tilde{B}_\rho \delta(t) \tag{H2}$$

where the pseudo inverse $B_\rho^\dagger$ may be written as

$$B_\rho^\dagger = V_3 \Sigma_3^{-1} U_3^T \tag{H3}$$

Upon combining Eqs.(138), (H2), and (H3) we obtain

$$u(t) = V_3 \Sigma_3^{-1} U_3^T \Big(U_1 + U_2 [\alpha_1,\ \alpha_2] \Big) \delta(t) \tag{H4}$$

Using Eqs.(133)-(135) yields

$$u(t) = [V_1\ \ V_2] \begin{bmatrix} \Sigma_1^{-1} & \\ & \Sigma_2^{-1} \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \Big(U_1 + U_2 [\alpha_1,\ \alpha_2] \Big) \delta(t) \tag{H5}$$

Expand Eq.(H5) to obtain

$$u(t) = (V_1\Sigma_1^{-1}U_1^T + V_2\Sigma_2^{-1}U_2^T)(U_1 + U_2[\alpha_1, \alpha_2])\delta(t) \tag{H6}$$

$$= (V_1\Sigma_1^{-1}U_1^TU_1 + V_1\Sigma_1^{-1}U_1^TU_2[\alpha_1, \alpha_2] + V_2\Sigma_2^{-1}U_2^TU_1$$

$$+ V_2\Sigma_2^{-1}U_2^TU_2[\alpha_1, \alpha_2])\delta(t) \tag{H7}$$

But from the property of the singular value decomposition, we have that U and V are unitary. Hence,

$$U_1^TU_1 = U_2^TU_2 = I \tag{H8}$$

and

$$U_1^TU_2 = U_2^TU_1 = 0 \tag{H9}$$

Thus, we obtain the desired result that

$$u(t) = (V_1\Sigma_1^{-1} + V_2\Sigma_2^{-1}[\alpha_1, \alpha_2])\delta(t) \tag{H10}$$

▨

Remark:

When $\alpha=[0, 0]$, the control law u(t) given by Eq.(H10) reduces to the control law given by Eq.(20) in Ref.12.

APPENDIX IX

Proof of Theorem 9.:

Consider

$$\rho x(t) = (A_{\rho c} + dA_{\rho c})x(t) \tag{I1}$$

Let

$$V(x) = x^T P_\rho x > 0 \tag{I2}$$

where P_ρ is the positive definite symmetric solution of

$$A^T_{\rho c} P_\rho + P_\rho A_{\rho c} + \Delta A_{\rho c} P_\rho A^T_{\rho c} = -2I \tag{I3}$$

Let

$$E_\rho = (I + \Delta\ A_{\rho c}) + (\Delta/2)dA_{\rho c} \tag{I4}$$

Then,

$$\rho V(x) = \rho x^T P_\rho x + x^T P_\rho \rho x + \Delta \rho x^T P_\rho \rho x \tag{I5}$$

Substitute for ρx from Eq.(I1), rearrange, and use Eq.(I3)

$$\rho V(x) = -x^T 2Ix + x^T(dA_{\rho c}^T P_\rho E_\rho + E_\rho^T P_\rho dA_{\rho c})x \tag{I6}$$

Then, the proof follows by using the same arguments as shown by Yedavalli [4].

Let

$$\sigma_{max}(E_{max}^T \, P_\rho^+ \, A_{\rho cmax})_s < 1$$

where $(\cdot)_s$ is the symmetric part of a matrix.

$$\sigma_{max}[\{(E_\rho^T)^+ P_\rho^+ dA_{\rho c}^+\}_s] < 1$$

$$\Rightarrow \sigma_{max}[\{(E_\rho^T P_\rho dA_{\rho c})^+\}_s] < 1$$

$$\Rightarrow \sigma_{max}[(E_\rho^T P_\rho dA_{\rho c})_s] < 1$$

$$\Rightarrow |\lambda(E_\rho^T P_\rho dA_{\rho c})_s|_{max} < 1$$

$$\Rightarrow \lambda_i[(E_\rho^T P_\rho dA_{\rho c})_s - I] < 0$$

$\Rightarrow -I_n + (E_\rho^T P_\rho dA_{\rho c})_s$ is negative definite

$\Rightarrow -2I + dA_{\rho c}^T P_\rho E_\rho + E_\rho^T P_\rho dA_{\rho c}$ is negative definite

$\Rightarrow \rho V(x)$ of Eq.(I6) is < 0 for all x

$\Rightarrow (A_{\rho c} + dA_{\rho c})$ of Eq.(I1) is asymptotically stable. ▨

APPENDIX X

Proof of Lemma 2:

Continuous time:

See Ref.18

Discrete time:

Since after rotation and translation the eigenvalues of $e^{-j\theta}(A-aI)$ are inside the region of Figure 26, then, the eigenvalues of $\frac{1}{\Delta}[\exp\{e^{-j\theta}(A-aI)\Delta\}-I]$ are inside the region of Fig.27.

▨

APPENDIX XI

Proof of Lemma 3.:

Continuous time :

See Ref.2.

Discrete time :

Let $\tilde{A} = e^{-j\theta}(A-aI)$. Use the definition of $\tilde{A}_\delta$ to obtain

$$M^{-1}\tilde{A}_\delta M = \frac{1}{\Delta} M^{-1}(e^{\tilde{A}\Delta}-I)M \tag{J1}$$

Substitute the infinite series for $\exp(\tilde{A}\Delta)$ into Eq.(J1) to obtain

$$M^{-1}\tilde{A}_\delta M = \frac{1}{\Delta} M^{-1}[I + \tilde{A}\Delta + \frac{\tilde{A}^2\Delta^2}{2!} + \dots - I]M \tag{J2}$$

$$= \frac{1}{\Delta} [I + M^{-1}\tilde{A}M\Delta + \frac{M^{-1}\tilde{A}^2M\Delta^2}{2!} + \dots - I] \tag{J3}$$

$$= \frac{1}{\Delta} [I + M^{-1}\tilde{A}M\Delta + (M^{-1}\tilde{A}M)^2 \frac{\Delta^2}{2!} + (M^{-1}\tilde{A}M)^3 \frac{\Delta^3}{3!} + \dots - I] \tag{J4}$$

Substitute $M^{-1}\tilde{A}M = \tilde{\Lambda}$ from Ref.2 into Eq.(J4) to obtain

$$M^{-1}\tilde{A}_{\delta}M = \frac{1}{\Delta}\left[I + \tilde{\Lambda}\Delta + \tilde{\Lambda}^{2}\frac{\Delta^{2}}{2!} + \tilde{\Lambda}\frac{\Delta^{3}}{3!} + \dots - I\right] \qquad \text{(J5)}$$

$$= \frac{1}{\Delta}\left(e^{\tilde{\Lambda}\Delta} - I\right) \qquad \text{(J6)}$$

∎

Remark:

From the above lemma, $\tilde{\Lambda}_{\rho} = \frac{1}{\Delta}(e^{\tilde{\Lambda}\Delta} - I)$. That is, the i-th eigenvalue is given by $\tilde{\gamma}_{i} = \frac{1}{\Delta}\{\exp(\tilde{\lambda}_{i}\Delta) - I\}$.

From a result attributed to Ref.2 $\tilde{\lambda}_{i} = e^{-j\theta}(\lambda_{i} - a)$. Then, $\mathrm{Re}(\tilde{\lambda}_{i}) = \{\mathrm{Re}(\lambda_{i}) - a\}\cos\theta + \mathrm{Im}(\lambda_{i})\sin\theta$.

APPENDIX XII

Proof of Theorem 10.:

The proof starts with the following Lemma:

Lemma K: (Delta Uncertainty Bounds)

$$(d\tilde{A}_\delta)^+ \le \tilde{A}_{\delta\max} = \frac{1}{\Delta}\left[e^{[(A-aI)^+ + A_{\max}]\Delta} - e^{(A-aI)^+\Delta}\right] \qquad \text{(K1)}$$

$$(d\tilde{B}_\delta)^+ \le \tilde{B}_{\delta\max} = \frac{1}{\Delta}\left[\int_0^\Delta e^{[(A-aI)^+ + A_{\max}]\tau} d\tau (B^+ + B_{\max}) - \int_0^\Delta e^{(A-aI)^+\tau} d\tau\, B^+\right] \qquad \text{(K2)}$$

Proof of Lemma K.:

Let $\tilde{A} = e^{-j\theta}(A-aI)$; $d\tilde{A} = e^{-j\theta}dA$; $\tilde{B} = e^{-j\theta}B$ and $d\tilde{B} = e^{-j\theta}dB$.

The following relations hold

$$\tilde{A}^+ \le (A-aI)^+$$

$$d\tilde{A}^+ \le dA^+ \Rightarrow \tilde{A}_{\max} \le A_{\max}$$

$$\tilde{B}^+ \le B^+$$

$$d\tilde{B}^+ \le dB^+ \Rightarrow \tilde{B}_{\max} \le B_{\max}$$

$$(d\tilde{A}_{\delta})^{+} = \frac{1}{\Delta}\left[e^{(\tilde{A}+d\tilde{A})\Delta} - e^{\tilde{A}\Delta}\right]^{+} = \frac{1}{\Delta}\left\{\left[\,I + (\tilde{A}+d\tilde{A})\Delta + (\tilde{A}+d\tilde{A})^{2}\frac{\Delta^{2}}{2!}\right.\right.$$

$$\left.\left. + (\tilde{A}+d\tilde{A})^{3}\frac{\Delta^{3}}{3!} + \dots\,\right] - \left[I + \tilde{A}\Delta + \tilde{A}^{2}\frac{\Delta^{2}}{2!} + \tilde{A}^{3}\frac{\Delta^{3}}{3!} + \dots\,\right]\right\}^{+}$$

$$= \frac{1}{\Delta}\left\{d\tilde{A}\cdot\Delta + \left[\tilde{A}\cdot d\tilde{A} + d\tilde{A}\cdot\tilde{A} + (d\tilde{A})^{2}\right]\frac{\Delta^{2}}{2!} + \left[\tilde{A}\cdot d\tilde{A}\cdot\tilde{A} + d\tilde{A}\cdot\tilde{A}\cdot d\tilde{A}\right.\right.$$

$$\left.\left. + d\tilde{A}\cdot\tilde{A}^{2} + \tilde{A}^{2}\cdot d\tilde{A} + (d\tilde{A})^{2}\cdot\tilde{A} + \tilde{A}\cdot(d\tilde{A})^{2} + (d\tilde{A})^{3}\right]\frac{\Delta^{3}}{3!} + \dots\right\}^{+}$$

$$\leq \frac{1}{\Delta}\left\{A_{max}\cdot\Delta + \left[\tilde{A}^{+}\cdot A_{max} + A_{max}\cdot\tilde{A}^{+} + A_{max}^{2}\right]\frac{\Delta^{2}}{2!} + \left[\tilde{A}^{+}\cdot A_{max}\cdot\tilde{A}^{+}\right.\right.$$

$$+ A_{max}\cdot\tilde{A}^{+}\cdot A_{max} + A_{max}\cdot(\tilde{A}^{+})^{2} + (\tilde{A}^{+})^{2}\cdot A_{max} + (A_{max})^{2}\cdot\tilde{A}^{+}$$

$$\left.\left. + \tilde{A}^{+}\cdot(A_{max})^{2} + (A_{max})^{3}\right]\frac{\Delta^{3}}{3!} + \dots\right\}$$

$$= \frac{1}{\Delta}\left\{\left[\,I+(\tilde{A}^{+}+A_{max})\Delta + (\tilde{A}^{+}+A_{max})^{2}\frac{\Delta^{2}}{2!} + (\tilde{A}^{+}+A_{max})^{3}\frac{\Delta^{3}}{3!} + \dots\,\right]\right.$$

$$\left. - \left[\,I+\tilde{A}^{+}\Delta + (\tilde{A}^{+})^{2}\frac{\Delta^{2}}{2!} + (\tilde{A}^{+})^{3}\frac{\Delta^{3}}{3!} + \dots\right]\right\}$$

$$= \frac{1}{\Delta}\left[\,e^{(\tilde{A}^{+}+A_{max})\Delta} - e^{\tilde{A}^{+}\Delta}\,\right]$$

$$= \frac{1}{\Delta}\left[\,e^{[(A-aI)^{+}+A_{max}]\Delta} - e^{(A-aI)^{+}\Delta}\,\right]$$

$$(d\tilde{B}_\delta)^+ = \Bigg\{ \frac{1}{\Delta} \Bigg[\int_0^\Delta e^{(\tilde{A}+d\tilde{A})\tau}\, d\tau \, (\tilde{B} + d\tilde{B}) - \int_0^\Delta e^{\tilde{A}\tau}\, d\tau \, \tilde{B} \Bigg] \Bigg\}^+$$

$$= \frac{1}{\Delta} \Bigg\{ \Bigg[I\Delta + (\tilde{A}+d\tilde{A}) \frac{\Delta^2}{2!} + (\tilde{A}+d\tilde{A})^2 \frac{\Delta^3}{3!} + (\tilde{A}+d\tilde{A})^3 \frac{\Delta^4}{4!} + \dots \Bigg]$$

$$\bullet (\tilde{B} + d\tilde{B}) - \Bigg[I\Delta + \tilde{A} \frac{\Delta^2}{2!} + \tilde{A}^2 \frac{\Delta^3}{3!} + \tilde{A}^3 \frac{\Delta^4}{4!} + \dots \Bigg] \tilde{B} \Bigg\}^+$$

$$= \frac{1}{\Delta} \Bigg\{ \Bigg[I\Delta + \tilde{A} \frac{\Delta^2}{2!} + \tilde{A}^2 \frac{\Delta^3}{3!} + \tilde{A}^3 \frac{\Delta^4}{4!} + \dots \Bigg] d\tilde{B} + \Bigg[d\tilde{A} \bullet \frac{\Delta^2}{2!}$$

$$+ \Bigg(\tilde{A} \bullet d\tilde{A} + d\tilde{A} \bullet \tilde{A} + (d\tilde{A})^2 \Bigg) \frac{\Delta^3}{3!} + \Bigg(\tilde{A} \bullet d\tilde{A} \bullet \tilde{A} + d\tilde{A} \bullet \tilde{A} \bullet d\tilde{A} + d\tilde{A} \bullet \tilde{A}^2$$

$$+ \tilde{A}^2 \bullet d\tilde{A} + (d\tilde{A})^2 \bullet \tilde{A} + \tilde{A} \bullet (d\tilde{A})^2 + (d\tilde{A})^3 \Bigg) \frac{\Delta^4}{4!} + \dots \Bigg] (\tilde{B} + d\tilde{B}) \Bigg\}^+$$

$$\leq \frac{1}{\Delta} \Bigg\{ \Bigg[I\Delta + \tilde{A}^+ \frac{\Delta^2}{2!} + (\tilde{A}^+)^2 \frac{\Delta^3}{3!} + (\tilde{A}^+)^3 \frac{\Delta^4}{4!} + \dots \Bigg] B_{max}$$

$$+ \Bigg[A_{max} \bullet \frac{\Delta^2}{2!} + \Bigg(\tilde{A}^+ \bullet A_{max} + A_{max} \bullet \tilde{A}^+ + (A_{max})^2 \Bigg) \frac{\Delta^3}{3!} +$$

$$+ \Bigg(\tilde{A}^+ \bullet A_{max} \bullet \tilde{A}^+ + A_{max} \bullet \tilde{A}^+ \bullet A_{max} + A_{max} \bullet (\tilde{A}^+)^2 + (\tilde{A}^+)^2 \bullet A_{max}$$

$$+ (A_{max})^2 \bullet \tilde{A}^+ + \tilde{A}^+ \bullet (A_{max})^2 + (A_{max})^3 \Bigg) \frac{\Delta^4}{4!} + \dots \Bigg] (B^+ + B_{max}) \Bigg\}$$

$$= \frac{1}{\Delta} \Bigg\{ \Bigg[I\Delta + (\tilde{A}^+ + A_{max}) \frac{\Delta^2}{2!} + (\tilde{A}^+ + A_{max})^2 \frac{\Delta^3}{3!} + (\tilde{A}^+ + A_{max})^3 \frac{\Delta^4}{4!}$$

$$+ \cdots \Big] (B^{+} + B_{max}) - \Big[I\Delta + \tilde{A}^{+}\frac{\Delta^{2}}{2!} + (\tilde{A}^{+})^{2}\frac{\Delta^{3}}{3!} + (\tilde{A}^{+})^{3}\frac{\Delta^{4}}{4!} + \ldots \Big] B^{+} \Big\}$$

$$= \frac{1}{\Delta} \left[\int_{0}^{\Delta} e^{(\tilde{A}^{+} + A_{max})\tau} \, d\tau \; (B^{+} + B_{max}) - \int_{0}^{\Delta} e^{\tilde{A}^{+}\tau} \, d\tau \; B^{+} \right]$$

$$= \frac{1}{\Delta} \left[\int_{0}^{\Delta} e^{[(A-aI)^{+} + A_{max}]\tau} \, d\tau \; (B^{+} + B_{max}) - \int_{0}^{\Delta} e^{(A-aI)^{+}\tau} \, d\tau \; B^{+} \right]$$ ▨

The proof of theorem 10 now continues by observing that the transformed unified closed loop system with uncertainty may be written as

$$\rho\tilde{x}(t) = \tilde{A}_{\rho c}\tilde{x}(t) + d\tilde{A}_{\rho c}\tilde{x}(t) \qquad \text{(K3)}$$

where

$$\tilde{A}_{\rho c} = \begin{cases} \tilde{A} + \tilde{B}FC & \text{continuous time} \\ \tilde{A}_{\delta} + \tilde{B}_{\delta}F_{\delta}C & \text{discrete time} \end{cases} \qquad \text{(K4)}$$

and

$$d\tilde{A}_{\rho c} = \begin{cases} d\tilde{A} + d\tilde{B}(FC) & \text{continuous time} \\ d\tilde{A}_{\delta} + d\tilde{B}_{\delta}(F_{\delta}C) & \text{discrete time} \end{cases} \qquad \text{(K5)}$$

The proof follows by in a similar manner to the proof of Theorem 8. ▨

APPENDIX XIII

Proof of Corollary 5.:

Using Theorem 10 it follows that the eigenvalues of the uncertain closed loop system described by Eq.(151) are in H_k in continuous time or H_k^d in discrete time if

$$\lambda_{maxk} = \lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{[f(\tilde{\gamma}_i)]_k} \tilde{A}_{\rho cmaxk} \right\} < 1 \tag{L1}$$

The eigenvalues of the uncertain closed loop system described by Eq.(151) are in the R region in continuous time or R^d in discrete time if they are simultaneously in each H_k in continuous time or H_k^d in discrete time. This will be true if

$$\lambda_{maxk} < 1$$

From the above comment, we have

$$\max[\lambda_{max1}, \lambda_{max2}] < 1 \tag{L2}$$

where

$$\lambda_{\max 1} = \lambda_{\max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^{*})^{+}}{[f(\tilde{\gamma}_i)]_1} \tilde{A}_{\rho c\max 1} \right\} \tag{L3}$$

$$\lambda_{\max 2} = \lambda_{\max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^{*})^{+}}{[f(\tilde{\gamma}_i)]_2} \tilde{A}_{\rho c\max 2} \right\} \tag{L4}$$

$$\tilde{A}_{\rho c\max 1} = \begin{cases} A_{\max} + B_{\max}\,(FC)^{+} & \text{continuous time} \\ \tilde{A}_{\delta\max 1} + \tilde{B}_{\delta\max 1}(F_{\delta}C)^{+} & \text{discrete time} \end{cases} \tag{L5}$$

$$\tilde{A}_{\delta\max 1} = \frac{1}{\Delta}\left[e^{[(A-a_1 I)^{+}+A_{\max}]\Delta} - e^{[A-a_1 I]^{+}\Delta} \right] \tag{L6}$$

$$\tilde{B}_{\delta\max 1} = \frac{1}{\Delta}\left[\int_0^{\Delta} e^{[(A-a_1 I)^{+}+ A_{\max}]\tau} d\tau (B^{+}+ B_{\max}) - \int_0^{\Delta} e^{(A-a_1 I)^{+}\tau} d\tau\ B^{+} \right] \tag{L7}$$

$$\tilde{A}_{\rho c\max 2} = \begin{cases} A_{\max} + B_{\max}\,(FC)^{+} & \text{continuous time} \\ \tilde{A}_{\delta\max 2} + \tilde{B}_{\delta\max 2}(F_{\delta}C)^{+} & \text{discrete time} \end{cases} \tag{L8}$$

$$\tilde{A}_{\delta\max 2} = \frac{1}{\Delta}\left[e^{[A^{+}+A_{\max}]\Delta} - e^{A^{+}\Delta} \right] \tag{L9}$$

$$\tilde{B}_{\delta max2} = \frac{1}{\Delta}\left[\int_0^{\Delta} e^{[A^+ + A_{max}]\tau} d\tau (B^+ + B_{max}) - \int_0^{\Delta} e^{A^+\tau} d\tau\, B^+\right] \tag{L10}$$

$$[f(\tilde{\gamma}_i)]_1 = \begin{cases} -Re(\tilde{\gamma}_i) = -Re(\tilde{\lambda}_i) = \tilde{\alpha}_i & \text{cont. time} \\ \frac{1}{\Delta}\,[1-(1+\Delta\tilde{\gamma}_i)^+\,] = \frac{1}{\Delta}[1-e^{-\tilde{\alpha}_i\Delta}\,] & \text{disc. time} \end{cases} \tag{L11}$$

and where

$$\tilde{\lambda}_i = (\lambda_i - a_1) \quad ; \quad \tilde{\alpha}_i = -[Re(\lambda_i) - a_1]$$

$$[f(\tilde{\gamma}_i)]_2 = \begin{cases} -Re(\tilde{\gamma}_i) = -Re(\tilde{\lambda}_i) = \tilde{\alpha}_i & \text{cont. time} \\ \frac{1}{\Delta}\,[1-(1+\Delta\tilde{\gamma}_i)^+\,] = \frac{1}{\Delta}[1-e^{-\tilde{\alpha}_i\Delta}\,] & \text{disc. time} \end{cases} \tag{L12}$$

and where

$$\tilde{\lambda}_i = e^{-j\theta_2}\lambda_i \quad ; \quad \tilde{\alpha}_i = -(Re(\lambda_i))\cos\theta_2 + Im(\lambda_i)\sin\theta_2$$

Remark:

$$k = 3: \qquad \theta_3 = -\theta_2; \quad a_3 = a_2 = 0$$

$$\tilde{A}_{\rho cmax3} = \tilde{A}_{\rho cmax2} \tag{L13}$$

$$[f(\tilde{\gamma}_i)]_3 = \begin{cases} -\mathrm{Re}(\tilde{\gamma}_i) = -\mathrm{Re}(\tilde{\lambda}_i) = \tilde{\alpha}_i & \text{cont. time} \\ \frac{1}{\Delta}\,[1-(1+\Delta\tilde{\gamma}_i)^+\,] = \frac{1}{\Delta}[1-e^{-\tilde{\alpha}_i\Delta}\,] & \text{disc. time} \end{cases} \qquad (L14)$$

and where

$$\tilde{\lambda}_i = e^{+j\theta_2}\,\lambda_i \;;\; \tilde{\alpha}_i = -(\mathrm{Re}(\lambda_i))\cos\theta_2 - \mathrm{Im}(\lambda_i)\sin\theta_2$$

Since λ_i are the eigenvalues of the real matrix (A+BFC) then, the λ_i are either real or they occur in complex conjugate pairs. If the λ_i are real, then $\mathrm{Im}(\lambda_i) = 0$ and

$$f[\tilde{\gamma}_i]_3 = f[\tilde{\gamma}_i]_2 \,.$$

Consider the complex pair

$\lambda_i = a_i + jb_i$ and $\lambda_{i+1} = \lambda_i^* = a_i - jb_i$

In this case

$$[f(\tilde{\gamma}_i)]_2 = \begin{cases} -\mathrm{Re}(\tilde{\gamma}_i) = -\mathrm{Re}(\tilde{\lambda}_i) = \tilde{\alpha}_i & \text{cont. time} \\ \frac{1}{\Delta}\,[1-(1+\Delta\tilde{\gamma}_i)^+\,] = \frac{1}{\Delta}[1-e^{-\tilde{\alpha}_i\Delta}\,] & \text{disc. time} \end{cases} \qquad (L15)$$

and where

$$\tilde{\lambda}_i = e^{-j\theta_2}\,\lambda_i \;;\; \tilde{\alpha}_i = -a_i\cos\theta_2 + b_i\sin\theta_2 \text{ or } \tilde{\alpha}_i = -a_i\cos\theta_2 - b_i\sin\theta_2$$

$$[f(\tilde{\gamma}_i)]_3 = \begin{cases} -\mathrm{Re}(\tilde{\gamma}_i) = -\mathrm{Re}(\tilde{\lambda}_i) = \tilde{\alpha}_i & \text{cont. time} \\ \frac{1}{\Delta}\,[1-(1+\Delta\tilde{\gamma}_i)^{+}\ ] = \frac{1}{\Delta}[1-e^{-\tilde{\alpha}_i\Delta}] & \text{disc. time} \end{cases} \qquad \text{(L16)}$$

and where

$$\tilde{\lambda}_i = e^{+j\theta_2}\,\lambda_i \ ; \ \tilde{\alpha}_i = -a_i\cos\theta_2 - b_i\sin\theta_2 \ \text{or} \ \tilde{\alpha}_i = -a_i\cos\theta_2 + b_i\sin\theta_2$$

Therefore, it follows that

$$[f(\tilde{\gamma}_i)]_3 = [f(\tilde{\gamma}_i)]_2 \qquad ∎$$

APPENDIX XIV

Proof of Theorem 11.:

Consider

$$\rho\tilde{x}(t) = (\tilde{A}_{\rho c} + d\tilde{A}_{\rho c})\tilde{x}(t) \tag{M1}$$

Let

$$V(\tilde{x}) = \tilde{x}^T \tilde{P}_{\rho} \tilde{x} > 0 \tag{M2}$$

where $\tilde{P}_{\rho}$ is the positive definite symmetric solution of

$$\tilde{A}^T_{\rho c}\tilde{P}_{\rho} + \tilde{P}_{\rho}\tilde{A}_{\rho c} + \Delta\tilde{A}_{\rho c}\tilde{P}_{\rho}\tilde{A}^T_{\rho c} = -2I_n \tag{M3}$$

Let

$$\tilde{E}_{\rho} = (I_n + \Delta\tilde{A}_{\rho c}) + (\Delta/2)d\tilde{A}_{\rho c}$$

Then,

$$\rho V(\tilde{x}) = \rho\tilde{x}^T \tilde{P} \tilde{x} + \tilde{x}^T \tilde{P}_{\rho} \rho\tilde{x} + \Delta\rho\tilde{x}^T \tilde{P}_{\rho} \rho\tilde{x} \tag{M4}$$

Substitute for $\rho\tilde{x}$ from Eq.(M1), rearrange, and use Eq.(M3)

$$\rho V(\tilde{x}) = -\tilde{x}^T 2I_n \tilde{x} + \tilde{x}^T(d\tilde{A}^T_{\rho c} \tilde{P}_{\rho} \tilde{E}_{\rho} + \tilde{E}^T_{\rho} \tilde{P}_{\rho} d\tilde{A}_{\rho c})\tilde{x} \tag{M5}$$

Then, the proof follows by using the same arguments as shown by Yedavalli [4].

Let

$$\sigma_{\max}[(\tilde{E}^T_{\rho\max}\ \tilde{P}^+_\rho\ \tilde{A}_{\rho c\max})_s] < 1$$

where $(\bullet)_s$ is the symmetric part of a matrix.

$$\Rightarrow \sigma_{\max}[\{(\tilde{E}^T_\rho)^+\ \tilde{P}^+_\rho\ d\tilde{A}^+_{\rho c}\}_s] < 1 \tag{M6}$$

$$\Rightarrow \sigma_{\max}[\{(\tilde{E}^T_\rho\ \tilde{P}_\rho\ d\tilde{A}_{\rho c})^+\}_s] < 1 \tag{M7}$$

$$\Rightarrow \sigma_{\max}[(\tilde{E}^T_\rho\ \tilde{P}_\rho\ d\tilde{A}_{\rho c})_s] < 1 \tag{M8}$$

$$\Rightarrow |\lambda(\tilde{E}^T_\rho\ \tilde{P}_\rho\ d\tilde{A}_{\rho c})_s|_{\max} < 1 \tag{M9}$$

$$\Rightarrow \lambda_i[(\tilde{E}^T_\rho\ \tilde{P}_\rho\ d\tilde{A}_{\rho c})_s - I_n] < 0 \tag{M10}$$

$$\Rightarrow -I_n + (\tilde{E}^T_\rho\ \tilde{P}_\rho\ d\tilde{A}_{\rho c})_s \quad \text{is negative definite} \tag{M11}$$

$$\Rightarrow -2I_n + d\tilde{A}^T_{\rho c}\ \tilde{P}_\rho\ \tilde{E}_\rho + \tilde{E}^T_\rho\ \tilde{P}_\rho\ d\tilde{A}_{\rho c} \quad \text{is negative definite} \tag{M12}$$

$\Rightarrow \rho(\tilde{x})$ of Eq.(M5) < 0 for all $\tilde{x}$

$\Rightarrow$ $(\tilde{A}_{\rho c} + d\tilde{A}_{\rho c})$ of Eq.(152) is asymptotically stable. (M13) ▨

APPENDIX XV

Proof of Corollary 6.:

Using Theorem 11, it follows that the eigenvalues of the uncertain closed loop system described by Eq.(151) are in H_k in continuous time or H_k^d in discrete time if

$$\bar{\sigma}_k = \bar{\sigma}[(\tilde{E}^T_{\rho \max k} \; \tilde{P}^+_{\rho k} \; \tilde{A}_{\rho c \max k})_s] < 1 \tag{N1}$$

The eigenvalues of the uncertain closed loop system described by Eq.(151) are in the R region in continuous time or R^d in discrete time if they are simultaneously in each H_k in continuous time or H_k^d in discrete time. This will be true if

$$\bar{\sigma}_k < 1$$

From the above comment, we have

$$\max[\bar{\sigma}_1, \bar{\sigma}_2] < 1$$

where

$$\bar{\sigma}_1 = \bar{\sigma}[(\tilde{E}^T_{\rho \max 1} \; \tilde{P}^+_{\rho 1} \; \tilde{A}_{\rho c \max 1})_s] \tag{N2}$$

$$\bar{\sigma}_2 = \bar{\sigma}[(\tilde{E}^T_{\rho max2}\ \tilde{P}^+_{\rho 2}\ \tilde{A}_{\rho cmax2})_s] \tag{N3}$$

$$\tilde{E}^T_{\rho max1} = (I + \Delta\tilde{A}_{\rho c1}) + (\Delta/2)\tilde{A}_{\rho cmax1} \tag{N4}$$

$$\tilde{E}^T_{\rho max2} = (I + \Delta\tilde{A}_{\rho c2}) + (\Delta/2)\tilde{A}_{\rho cmax2} \tag{N5}$$

$\tilde{P}_{\rho 1}$ and $\tilde{P}_{\rho 2}$ satisfy the following Lyapunov equations

$$\tilde{A}^T_{\rho c1}\tilde{P}_{\rho 1} + \tilde{P}_{\rho 1}\tilde{A}_{\rho c1} + \Delta\tilde{A}_{\rho c1}\tilde{P}_{\rho 1}\tilde{A}^T_{\rho c1} = -2I_n \tag{N6}$$

$$\tilde{A}^T_{\rho c2}\tilde{P}_{\rho 2} + \tilde{P}_{\rho 2}\tilde{A}_{\rho c2} + \Delta\tilde{A}_{\rho c2}\tilde{P}_{\rho 2}\tilde{A}^T_{\rho c2} = -2I_n \tag{N7}$$

where $\tilde{A}_{\rho c1}$ and $\tilde{A}_{\rho c2}$ are given by

$$\tilde{A}_{\rho c1} = \begin{cases} \tilde{A}_1 + \tilde{B}_1 FC & \text{continuous time} \\ \tilde{A}_{\delta 1} + \tilde{B}_{\delta 1} F_\delta C & \text{discrete time} \end{cases} \tag{N8}$$

$$\tilde{A}_{\rho c2} = \begin{cases} \tilde{A}_2 + \tilde{B}_2 FC & \text{continuous time} \\ \tilde{A}_{\delta 2} + \tilde{B}_{\delta 2} F_\delta C & \text{discrete time} \end{cases} \tag{N9}$$

where

$$\tilde{A}_1 = (A-a_1 I),\ \tilde{B}_1 = B,\ \tilde{A}_{\delta 1} = \tilde{\Omega}_1(A-a_1 I),\ \tilde{B}_{\delta 1} = \tilde{\Omega}_1 B \tag{N10}$$

$$\tilde{A}_2 = Ae^{-j\theta_2},\ \tilde{B}_2 = Be^{-j\theta_2},\ \tilde{A}_{\delta 2} = \tilde{\Omega}_2 Ae^{-j\theta_2},\quad \tilde{B}_{\delta 2} = \tilde{\Omega}_2 Be^{-j\theta_2} \tag{N11}$$

$$\tilde{\Omega}_1 = \frac{1}{\Delta}\int_0^{\Delta} e^{(A-a_1 I)\tau} d\tau,\quad \tilde{\Omega}_2 = \frac{1}{\Delta}\int_0^{\Delta} \exp\{(Ae^{-j\theta_2})\tau\} d\tau \tag{N12}$$

Remark:

$$k = 3:\quad \theta_3 = -\theta_2;\qquad a_3 = a_2 = 0$$

$$\tilde{A}_{\rho c\max 3} = \tilde{A}_{\rho c\max 2}\quad ;\quad \tilde{A}_{\rho c3} = \tilde{A}^{*}_{\rho c2}\ ;\quad \tilde{P}_{\rho c3} = \tilde{P}^{*}_{\rho c2}$$

Therefore, it follows that

$$\bar{\sigma}_3 = \bar{\sigma}_2$$ ▨

APPENDIX XVI

Proof of Lemma 4.:

Recall the following property from matrix norm.

For any matrix $W \in \mathbb{C}^{nxn}$, $\|W\|_{ip} \le \|W^{+}\|_{ip}$ (M1)

$$\Rightarrow \|I+\xi W\|_{ip} \le \|I+\xi W^{+}\|_{ip} \text{ where } \xi > 0 \qquad \text{(M2)}$$

$$\Rightarrow \lim_{\xi \to o} \frac{\|I+\xi W\|_{ip}-1}{\xi} \le \lim_{\xi \to o} \frac{\|I+\xi W^{+}\|_{ip}-1}{\xi} \qquad \text{(M3)}$$

Therefore,

$$U_{ip}(W) \le U_{ip}(W^{+}) \qquad \text{(M4)}$$

▨

APPENDIX XVII

Proof of Lemma 5.:

It is seen that all the eigenvalues of a constant matrix A_ρ are located in the H region in continuous time or in the H^d region in discrete time if and only if the eigenvalues of the corresponding state space system

$$\rho x(t) = \tilde{A}_\rho x(t) \tag{Q1}$$

lie in the region of Figure 26 in continuous time or in the region of Figure 27 in discrete time

$$\tilde{A}_\rho = \begin{cases} e^{-j\theta}(A-aI) & \text{in continuous time} \\ \tilde{\Omega}e^{-j\theta}(A-aI) & \text{in discrete time} \end{cases} \tag{Q2}$$

Continuous time:

See Ref.19.

Discrete time:

If Eq.(190) holds, then it follows that

$$\frac{\Delta}{2}\|\tilde{\Omega}e^{-j\theta}(A-aI)\|_{ip}^2 + U_{ip}(\tilde{\Omega}e^{-j\theta}A) + U_{ip}(-\tilde{\Omega}ae^{-j\theta}) < 0 \tag{Q3}$$

Use The property of the matrix measure that

$$U_{ip}(A+B) \le U_{ip}(A) + U_{ip}(B)$$

to obtain

$$\frac{\Delta}{2}\|\tilde{\Omega}e^{-j\theta}(A-aI)\|_{ip}^{2} + U_{ip}[\tilde{\Omega}e^{-j\theta}(A - aI)] < 0 \qquad (Q4)$$

We now require a new lemma

Lemma Q.:

Let λ be an eigenvalue of A, and let v be an associated eigenvector such that $\|v\|_{ip}=1$. Then,

$$|\lambda(A)| \le \| A \|_{ip}$$

Proof of Lemma Q.:

The eigenvalue λ of the matrix A and its corresponding eigenvector v satisfy the following equation

$$\lambda(A)v = Av$$

$$\|\lambda(A)v\|_{ip} = \| Av \|_{ip} \qquad (Q5)$$

$$|\lambda(A)| = \| Av \|_{ip} \qquad \text{(because v is a specific vector)} \qquad (Q6)$$

$$|\lambda(A)| \le \| A \|_{ip} \| v \|_{ip} \qquad (Q7)$$

Thus,

$|\lambda(A)| \le \| A \|_{ip}$ ▨

Use the result of Lemma Q to obtain

$$\Rightarrow \frac{\Delta}{2}|\lambda[\tilde{\Omega}e^{-j\theta}(A-aI)]|^2 + U_{ip}[\tilde{\Omega}e^{-j\theta}(A - aI)] < 0 \qquad (Q8)$$

From Ref.19 we have

$$\text{Re}\ \lambda(A) < U_{ip}(A) \qquad (Q9)$$

Use Eq.(Q9) to obtain

$$\Rightarrow \frac{\Delta}{2}|\lambda[\tilde{\Omega}e^{-j\theta}(A-aI)]|^2 + \text{Re}\ \lambda[\tilde{\Omega}e^{-j\theta}(A - aI)] < 0 \qquad (Q10)$$

which from Ref. 3 is the stability condition in the γ-plane for $\tilde{\Omega}e^{-j\theta}(A-aI)$. ▨

APPENDIX XVIII

Proof of Corollary 7.:

Continuous time:

Use the property of the matrix measure that

$$U_{ip}(A+B) \le U_{ip}(A) + U_{ip}(B) \tag{R1}$$

to obtain

$$U_{ip}(e^{-j\theta}\ A) = U_{ip}[A(\cos\theta - j\sin\theta)] \tag{R2}$$

$$\le U_{ip}(A\cos\theta) + U_{ip}(-Aj\sin\theta) < a\cos\theta \tag{R3}$$

where Eq.(B3) from the hypothesis of Corollary 7.

Thus,

$$U_{ip}(e^{-j\theta}\ A) < a\cos\theta$$

and from Lemma 5 the eigenvalues of A are in the H region in continuous time.

Discrete time:

Use the property of the matrix measure that

$$U_{ip}(A+B) \le U_{ip}(A) + U_{ip}(B)$$

to obtain

$$U_{ip}(\tilde{\Omega}Ae^{-j\theta}) = U_{ip}[\tilde{\Omega}A(\cos\theta - j\sin\theta)] \qquad (R4)$$

$$\le U_{ip}(\tilde{\Omega}A\cos\theta) + U_{ip}(-\tilde{\Omega}Aj\sin\theta) \qquad (R5)$$

$$\le -U_{ip}(\tilde{\Omega}ae^{-j\theta}) - \frac{\Delta}{2}\|\tilde{\Omega}e^{-j\theta}(A-aI)\|_{ip}^{2} \qquad (R6)$$

where Eq.(R6) follows from the hypothesis of Corollary 7. Thus,

$$U_{ip}(\tilde{\Omega}Ae^{-j\theta}) \le -U_{ip}(\tilde{\Omega}ae^{-j\theta}) - \frac{\Delta}{2}\|\tilde{\Omega}e^{-j\theta}(A-aI)\|_{ip}^{2} \qquad (R7)$$

and from Lemma 5 the eigenvalues of A_δ are in the H^d region in discrete time. ////

APPENDIX XIX

Proof of Theorem 12.:

Continuous time:

Suppose Eq.(192) is satisfied for $p=1,2,\infty$. Then, Eq.(192) can be written as

$$U_{ip}(\tilde{A}_{cmax}) + U_{ip}(A_c\cos\theta) + U_{ip}(-A_cj\sin\theta) - a\cos\theta < 0 \qquad \text{(S1)}$$

where

$$A_c = A + BFC; \; \tilde{A}_{cmax} = A_{max} + B_{max}(FC)^+$$

$$\Rightarrow U_{ip}(\tilde{A}_{cmax}) + U_{ip}(A_ce^{-j\theta}) - a\cos\theta < 0 \qquad \text{(S2)}$$

$$\Rightarrow U_{ip}(A_{cmax}$$

$$U_{ip}(\tilde{A}_{cmax}) \geq U_{ip}(\{e^{-j\theta}[dA+dB(FC)]\}^+) \geq U_{ip}(\{e^{-j\theta}[dA+dB(FC)]\}) \qquad \text{(S3)}$$

Use the definition of the matrix measure [19] to obtain

$$U_{ip}(-aIe^{-j\theta}) = -a\cos\theta$$

Then, Eqs.(S2) and (S3) yield

$$U_{ip}(e^{-j\theta}[dA+dB(FC)]) + U_{ip}(e^{-j\theta}[A+BFC])+U_{ip}(-aIe^{-j\theta}) < 0 \tag{S4}$$

$$\Rightarrow U_{ip}(e^{-j\theta}[A+BFC-aI] + e^{-j\theta}[dA+dB(FC)]) < 0 \tag{S5}$$

From Ref.19 we have

$$\mathrm{Re}\ \lambda(A) < U_{ip}(A) \tag{S6}$$

Thus,

$$\mathrm{Re}\ \lambda\{\tilde{A}_c + d\tilde{A}_c\} < 0$$

where

$\tilde{A}_c = e^{-j\theta}(A+BFC-aI)$ and $d\tilde{A}_c = e^{-j\theta}\ [dA+dB(FC)]$ ▨

Discrete time:

First we note that Ξ can be written as

$$\Xi = e^{j\theta}(\ \tilde{A}_\delta + \tilde{B}_\delta F_\delta C) + \tilde{\Omega}a = \tilde{\Omega}[\ A+ BF_\delta C\ ]$$

Suppose Eq.(192) is satisfied for $p=1,2,\infty$. Then, Eq.(192) becomes

$$U_{ip}(\tilde{A}_{\delta cmax}) + \frac{\Delta}{2}\|\tilde{A}^{+}_{\delta c}+\tilde{A}_{\delta cmax}\|^2_{ip} + U_{ip}(\tilde{\Omega}[A+BF_\delta C]\cos\theta)$$

$$+ U_{ip}(-\tilde{\Omega}[A+BF_\delta C]j\sin\theta) + U_{ip}(-\tilde{\Omega}ae^{-j\theta}) < 0 \tag{S7}$$

where

$$\tilde{A}_{\delta c} = \tilde{A}_{\delta} + \tilde{B}_{\delta}F_{\delta}C;\ \tilde{A}_{\delta cmax} = \tilde{A}_{\delta max} + \tilde{B}_{\delta max}(F_{\delta}C)^{+}$$

$$\Rightarrow U_{ip}(\tilde{A}_{\delta cmax}) + \frac{\Delta}{2}\|\tilde{A}^{+}_{\delta c}+\tilde{A}_{\delta cmax}\|^{2}_{ip} + U_{ip}(\tilde{\Omega}e^{-j\theta}[A+BF_{\delta}C-aI]) < 0 \tag{S8}$$

$$U_{ip}(\tilde{A}_{\delta cmax}) \geq U_{ip}([d\tilde{A}_{\delta}+ d\tilde{B}_{\delta}(F_{\delta}C)]) \tag{S9}$$

$$\|\tilde{A}^{+}_{\delta c} + \tilde{A}_{\delta cmax}\|_{ip} \geq \|(\tilde{A}_{\delta c}+d\tilde{A}_{\delta c})^{+}\|_{ip} \geq \|(\tilde{A}_{\delta c}+d\tilde{A}_{\delta c})\|_{ip} \tag{S10}$$

Use the result of Lemma Q.: $\|(\cdot)\|_{ip} > |\lambda(\cdot)|$ to obtain

$$\|\tilde{A}^{+}_{\delta c} + \tilde{A}_{\delta cmax}\|_{ip} > |\lambda(\tilde{A}_{\delta c} + d\tilde{A}_{\delta c})| \tag{S11}$$

Then, Eq.(S8) becomes

$$U_{ip}(d\tilde{A}_{\delta c}) + \frac{\Delta}{2}|\lambda(\tilde{A}_{\delta c}+d\tilde{A}_{\delta c})|^{2} + U_{ip}(\tilde{\Omega}e^{-j\theta}[A-aI+BF_{\delta}C]) < 0$$

$$\Rightarrow \frac{\Delta}{2}|\lambda(\tilde{A}_{\delta c} + d\tilde{A}_{\delta c})|^{2} + U_{ip}(\tilde{A}_{\delta c}+ d\tilde{A}_{\delta c}) < 0 \tag{S12}$$

Thus

$$\Rightarrow \frac{\Delta}{2}|\lambda(\tilde{A}_{\delta}+ d\tilde{A}_{\delta})|^{2} + \mathrm{Re}\lambda(\tilde{A}_{\delta}+ d\tilde{A}_{\delta}) < 0 \tag{S13}$$

where

$$\tilde{A}_{\delta c} = (\tilde{A}_{\delta} + \tilde{B}_{\delta}F_{\delta}C) \text{ and } d\tilde{A}_{\delta c} = (d\tilde{A}_{\delta} + d\tilde{B}_{\delta}\ F_{\delta}C)$$

From Ref.3, Eq.(S13) is the stability condition in the γ-plane for $\tilde{A}_{\delta} + d\tilde{A}_{\delta}$. ▨

APPENDIX XX

Proof of Corollary 8.:

Using Theorem 12 , it follows that the eigenvalues of the uncertain closed loop system described by Eq.(151) are in H_k in continuous time or H_k^d in discrete time if the following conditions hold.

Continuous time :

$$U_{ip}(\tilde{A}_{cmaxk})+U_{ip}(A_c\cos\theta_k)+U_{ip}(-A_c j\sin\theta_k)-a_k\cos\theta_k < 0 \qquad \text{(U1)}$$

The eigenvalues of the uncertain closed loop system described by Eq.(151) are in the R region if they are simultaneously in each H_k. This will be true if Eq.(U1) holds.

From the above comment, it follows that

$$U_{ip}(\tilde{A}_{cmax1}) + U_{ip}(-A_{c1}) - a_1 < 0 \qquad \text{(U3)}$$

and

$$U_{ip}(\tilde{A}_{cmax2}) + U_{ip}(A_c\cos\theta_2) + U_{ip}(-A_c j\sin\theta_2) < 0 \qquad \text{(U4)}$$

where

$$A_c = A + BFC$$

$$A_{cmax1} = A_{cmax2} = A_{max} + B_{max}(FC)^+$$

and where

$\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given by Eqs.(N9) and (N12), respectively.

Discrete time:

First of all we note that Ξ_k can be written as

$$\Xi_k = e^{j\theta_k} (\tilde{A}_{\delta k} + \tilde{B}_{\delta k} F_\delta C) + \tilde{\Omega}_k a_k = \tilde{\Omega}_k [A + BF_\delta C]$$

Suppose Eq.(195) is satisfied for p=1,2,∞. Then, Eq.(195) becomes

$$U_{ip}(\tilde{A}_{\delta cmaxk}) + \frac{\Delta}{2} \| \tilde{A}^+_{\delta ck} + \tilde{A}_{\delta cmaxk} \|^2_{ip} + U_{ip}(\tilde{\Omega}_k [A+BF_\delta C] \cos\theta_k)$$

$$+ U_{ip}(-\tilde{\Omega}_k [A+BF_\delta C] j\sin\theta_k) + U_{ip}(-\tilde{\Omega}_k a_k e^{-j\theta_k}) < 0 \qquad \text{(U5)}$$

The eigenvalues of the uncertain closed loop system described by Eq.(151) are in the R^d region if they are simultaneously in each H^d_k. This will be true if Eq.(U5) holds.

From the above comment, it follows that

$$U_{ip}(\tilde{A}_{\delta cmax1}) + \frac{\Delta}{2}\|\tilde{A}^{+}_{\delta c1}+\tilde{A}_{\delta cmax1}\|^{2}_{ip} + U_{ip}(\tilde{\Omega}_1[A+BF_{\delta}C])$$

$$+ U_{ip}(-\tilde{\Omega}_1 a_1) < 0 \qquad \text{(U6)}$$

and

$$U_{ip}(\tilde{A}_{\delta cmax2}) + \frac{\Delta}{2}\|\tilde{A}^{+}_{\delta c2}+\tilde{A}_{\delta cmax2}\|^{2}_{ip} + U_{ip}(\tilde{\Omega}_2[A+BF_{\delta}C]\cos\theta_2)$$

$$+ U_{ip}(-\tilde{\Omega}_2[A+BF_{\delta}C]j\sin\theta_2) < 0 \qquad \text{(U7)}$$

where

$$\tilde{A}_{\delta c1} = \tilde{A}_{\delta 1} + \tilde{B}_{\delta 1}F_{\delta}C;\ \tilde{A}_{\delta c2} = \tilde{A}_{\delta 2} + \tilde{B}_{\delta 2}F_{\delta}C$$

$$\tilde{A}_{\delta cmax1} = \tilde{A}_{\delta max1} + \tilde{B}_{\delta max1}F_{\delta}C;\ \tilde{A}_{\delta cmax2} = \tilde{A}_{\delta max2} + \tilde{B}_{\delta max2}F_{\delta}C$$

and where $\tilde{A}_{\delta max1}$, $\tilde{B}_{\delta max1}$, $\tilde{A}_{\delta max2}$, $\tilde{B}_{\delta max2}$, are given by Eqs.(L6), (L7), (L9) and (L10), respectively.

$(\tilde{A}_{\delta 1};\ \tilde{B}_{\delta 1})$ and $(\tilde{A}_{\delta 2};\ \tilde{B}_{\delta 2})$ are given by Eqs.(N10) and (N11), respectively.

Remark:

$$k=3:\quad \theta_3=-\theta_2;\qquad a_3=a_2=0$$

$$\tilde{A}_{\rho cmax3}=\tilde{A}_{\rho cmax2};\qquad \tilde{\Omega}_3=\tilde{\Omega}_2^{*}$$

It is easy to show that for a matrix $A \in \mathbb{C}^{n\times n}$

$$U_{ip}([A]) = U_{ip}([A^{*}])$$

Therefore, it follows that

$$\mu_{\delta 3} = \mu_{\delta 2}$$

where $\mu_{\delta k}$; k=2,3 is the discrete part of $\mu_{\rho k}$ given by Eq.(195).

APPENDIX XXI

Proof of Theorem 13.:

The control vector for the MIFO multi-rate system equation is defined for intersample points. An expanded vector

$$\bar{u} = \begin{bmatrix} \bar{u}_1(k\Delta_o) \\ \bar{u}_2(k\Delta_o) \\ \cdot \\ \cdot \\ \bar{u}_m(k\Delta_o) \end{bmatrix} ; \qquad \bar{u}_i(k\Delta_o) = \begin{bmatrix} u_i[k\Delta_o] \\ u_i[k\Delta_o+\Delta_i] \\ \cdot \\ \cdot \\ u_i[k\Delta_o+(n_i-1)\Delta_i] \end{bmatrix} \qquad \text{(V1)}$$

is defined to accommodate this. The unified multivariable, multi-rate state equations for the MIFO model are

$$\bar{\rho}x(t)=\bar{A}_\rho x(t) + \bar{B}_\rho \bar{u}(t) \qquad \text{(V2)}$$

where

$$\bar{A}_\rho = \begin{cases} A & \text{continuous time} \\ \bar{\Omega}A & \text{discrete time} \end{cases} \qquad \text{(V3)}$$

where

$$\bar{\Omega} = \frac{1}{\Delta_o}\int_0^{\Delta_o} e^{A\tau}\, d\tau$$

and where

$$\bar{B}_\rho = \begin{cases} B & \text{continuous time} \\ \bar{B}_\delta & \text{discrete time} \end{cases} \tag{V3}$$

A result from Ref.5 gives

$$\bar{B}_\delta = \Big[b_{\delta 1} \;\; e^{A\Delta_1} b_{\delta 1} \; \ldots \; e^{A(n_1-1)\Delta_1} b_{\delta 1} \; \ldots, \; b_{\delta m} \;\; e^{A\Delta_m} b_{\delta m} \; \ldots \; e^{A(n_m-1)\Delta_m} b_{\delta m} \Big] \tag{V4}$$

where

$$b_{\delta i} = \frac{1}{\Delta_o} \int_o^{\Delta_i} e^{A\tau} \, d\tau \; b_i ; \;\; i=1,\ldots,m \tag{V5}$$

with b_i the i-th column of the matrix B.

The unified multivariable multi-rate system with uncertainty for the MIFO model is

$$\bar{\rho}x(t) = \bar{A}_\rho x(t) + d\bar{A}_\rho x(t) + \bar{B}_\rho \bar{u}(t) + d\bar{B}_\rho \bar{u}(t) \tag{V6}$$

From the results of Ref.5 and the unified model described in chapter 4 the uncertainty in the B matrix in delta can be written by

$$d\bar{B}_\delta = [\ (d\bar{B}_1-\bar{B}_1)\ (d\bar{B}_2-\bar{B}_2)\ \ldots\ (d\bar{B}_m-\bar{B}_m)\ ] \qquad \text{(V7)}$$

Each $d\bar{B}_i$ is an $n \times n_i$ matrix defined by

$$d\bar{B}_i = [db_{\delta i}\quad \exp((A+dA)\Delta_i)\cdot db_{\delta i}\ \ldots\ \exp\{(A+dA)(n_i-1)\}\cdot db_{\delta i}\ ] \qquad \text{(V8)}$$

where

$$db_{\delta i} = \frac{1}{\Delta_o}\int_0^{\Delta_i} e^{(A+dA)\tau}\, d\tau\ (b_i + db_i) \qquad \text{(V9)}$$

with db_i the i-th column of the dB matrix.

The multivariable multi-rate control law can be described by described by

$$\bar{u} = \bar{F}_\rho y \qquad \text{(V10)}$$

$$\bar{F}_\rho = \begin{cases} F & \text{continuous time} \\ \bar{F}_\delta & \text{discrete time} \end{cases} \qquad \text{(V11)}$$

$$\bar{F}_\rho = [\bar{F}_1\ \bar{F}_2\ \ldots\ \bar{F}_r] \qquad \text{(V12)}$$

where $\bar{F}_\rho$ is the $n \times r$ block matrix and each block $\bar{F}_{\delta i}$ is an $\bar{n}$ column vector.

We now require a new lemma to establish a bound on $d\bar{B}_\delta$.

Lemma V.: (Multi-rate delta uncertainty bound)

$$d\bar{B}_{\delta}^{+} \le \bar{B}_{\delta max} = [(\bar{B}_{1max} - \bar{B}_{1}^{+})\ (\bar{B}_{2max} - \bar{B}_{2}^{+}\ )\ \ldots\ (\bar{B}_{mmax} - \bar{B}_{m}^{+})] \qquad \text{(V13)}$$

Each $\bar{B}_i^+$ is an $n \times n_i$ matrix defined by

$$\bar{B}_{i}^{+} = [\ b_{\delta i}^{+}\ \exp\{A^{+}\Delta_{i}\}\cdot b_{\delta i}^{+}\ \ldots\ \exp\{A^{+}(n_{i}-1)\Delta_{i}\}\cdot b_{\delta i}^{+}\ ] \qquad \text{(V14)}$$

where

$$b_{\delta i}^{+} = \frac{1}{\Delta_{o}} \int_{0}^{\Delta_{i}} e^{A^{+}\tau} d\tau\ b_{i}^{+} \qquad \text{(V15)}$$

and where b_i^+ is the i-th column of the matrix B^+. Each $\bar{B}_{imax}$ is an $n \times n_i$ matrix is defined by

$$\bar{B}_{imax} = [b_{\delta imax}\ \exp\{(A^{+} + A_{max})\Delta_{i}\}\cdot b_{\delta imax}$$

$$b_{\delta imax} = \frac{1}{\Delta_{o}} \int_{0}^{\Delta_{i}} e^{(A^{+} + A_{max})\tau} d\tau (b_{i}^{+} + b_{imax}) \qquad \text{(V16)}$$

where b_{imax} is the i-th column of the matrix B_{max}.

<u>Proof of Lemma V</u>

$$d\bar{B}_{\delta}^{+} \le [\ (d\bar{B}_{1} - \bar{B}_{1})^{+}\ (d\bar{B}_{2} - \bar{B}_{2})^{+}\ \ldots\ (d\bar{B}_{m} - \bar{B}_{m})^{+}\ ] \qquad \text{(V17)}$$

where

$$(d\bar{B}_i-\bar{B}_i)^+ \leq \Big[[db_{\delta i}-b_{\delta i}]^+ \quad [\{\exp(A+dA)\Delta_i\}\cdot db_{\delta i}-\exp\{A\Delta_i\}\cdot b_{\delta i}]^+$$

$$\ldots \quad [\exp\{A+dA)(n_i-1)\Delta_i\}\cdot db_{\delta i}-\exp\{A(n_i-1)\Delta_i\}\cdot b_{\delta i}]^+ \Big]$$

Consider a particular element of $(d\bar{B}_i-\bar{B}_i)^+$ and replace $db_{\delta i}$, $b_{\delta i}$ by their equivalents to obtain

$$[\{\exp(A+dA)n_i\Delta_i\}\cdot db_{\delta i}-\exp(An_i\Delta_i)\cdot b_{\delta i}]^+$$

$$=\frac{1}{\Delta_o}\Bigg\{\{\exp(A+dA)n_i\Delta_i\}\int_0^{\Delta_i} e^{(A+dA)\tau}\,d\tau\,(b_i+db_i)$$

$$-\exp(An_i\Delta_i)\int_0^{\Delta_i} e^{A\tau}\,d\tau\,b_i\Bigg\}^+$$

$$=\frac{1}{\Delta_o}\Bigg\{\{\exp(A+dA)n_i\Delta_i\}\int_0^{\Delta_i} e^{(A+dA)\tau}\,d\tau\,db_i$$

$$+\Bigg[\{\exp(A+dA)n_i\Delta_i\}\int_0^{\Delta_i} e^{(A+dA)\tau}\,d\tau\,b_i$$

$$-\exp(An_i\Delta_i)\int_0^{\Delta_i} e^{A\tau}\,d\tau\,b_i\Bigg]\Bigg\}^+$$

expand the first coefficient of b_i

$$\{\exp(A+dA)n_i\Delta_i\}\int_0^{\Delta_i} e^{(A+dA)\tau}\,d\tau\,b_i$$

$$= \Big\{ \exp(An_i\Delta_i) + dAn_i\frac{\Delta_i}{1!} + \Big[A\cdot dA + dA\cdot A + (dA)^2\Big] n_i^2\,\frac{\Delta_i^2}{2} + \ldots \Big\}$$

$$\cdot\Big\{ \int_0^{\Delta_i} e^{A\tau}d\tau + dA_i\frac{\Delta_i^2}{2!} + \Big[A\cdot dA + dA\cdot A + (dA)^2\Big]\,\frac{\Delta_i^3}{3!} + \ldots \Big\} b_i$$

$$= \exp(An_i\Delta_i) \int_0^{\Delta_i} e^{A\tau}d\tau\ b_i + \exp(An_i\Delta_i)\Big\{ dA\frac{\Delta_i^2}{2!} + \Big[A\cdot dA + dA\cdot A$$

$$+ (dA)^2\Big]\,\frac{\Delta_i^3}{3!} + \ldots \Big\} b_i + \Big\{ dAn_i\frac{\Delta_i}{1!} + \Big[A\cdot dA + dA\cdot A$$

$$+ (dA)^2\Big] n_i^2\,\frac{\Delta_i^2}{2} + \ldots \Big\}\cdot\Big\{ \int_0^{\Delta_i} e^{A\tau}d\tau + dA_i\frac{\Delta_i^2}{2!} + \Big[A\cdot dA + dA\cdot A$$

$$+ (dA)^2\Big]\,\frac{\Delta_i^3}{3!} + \ldots \Big\} b_i$$

Then, the particular element becomes

$$\frac{1}{\Delta_o}\Big\{ \{\exp(A+dA)n_i\Delta_i\} \int_0^{\Delta_i} e^{(A+dA)\tau}d\tau\ db_i + \Big\{ \exp(An_i\Delta_i)\Big(dA_i\frac{\Delta_i^2}{2!}$$

$$+ \Big[A\cdot dA + dA\cdot A + (dA)^2\Big]\,\frac{\Delta_i^3}{3!} + \ldots \Big) + \Big(dAn_i\frac{\Delta_i}{1!} + \Big[A\cdot dA + dA\cdot A$$

$$+ (dA)^2\Big]\, n_i^2\,\frac{\Delta_i^2}{2!} + \ldots \Big)\Big(\int_0^{\Delta_i} e^{\,A\tau}d\tau + \Big(dA_i\frac{\Delta_i^2}{2!} + \Big[A\cdot dA + dA\cdot A$$

$$+ (dA)^2\Big] \frac{\Delta_i^2}{3!} + \ldots \Big) \Big\} \Big\} \Big\}^+ b_i$$

Apply $(\cdot)^+$ on each element , use $dA^+ \le A_{max}$ and rearrange to obtain

$$[\{\exp(A+dA)n_i\Delta_i\}\cdot db_{\delta i} - \exp\{An_i\Delta_i\}\cdot b_{\delta i}]^+$$

$$\le [\{\exp(A^+ + A_{max})n\ \Delta_i\}\cdot b_{\delta imax} - \exp\{A^+n_i\Delta_i\}\cdot b^+{}_{\delta i}]$$

Then, it follows that

$$\bar{B}_{\delta max} = [(\bar{B}_{1max} - \bar{B}_1^+)\ (\bar{B}_{2max} - \bar{B}_2^+\ )\ \ldots\ (\bar{B}_{mmax} - \bar{B}_m^+)]$$

Each $\bar{B}_i^+$ is an $n \times n_i$ matrix defined by

$$\bar{B}_i^+ = [\ b_{\delta i}^+\ \exp\{A^+\Delta_i\}\cdot b_{\delta i}^+\ \ldots\ \exp\{A^+(n_i-1)\Delta_i\}\cdot b_{\delta i}^+\ ]$$

where

$$b_{\delta i}^+ = \frac{1}{\Delta_o}\int_0^{\Delta_i} e^{A^+\tau} d\tau\ b_i^+$$

and where b_i^+ is the i-th column of the matrix B^+. Each $\bar{B}_{imax}$ is an $n \times n_i$ matrix defined by

$$\bar{B}_{imax} = [b_{\delta imax}\ \exp\{(A^+ + A_{max})\Delta_i\}\cdot b_{\delta imax}\ \ldots$$

$$\ldots \exp\{(A^{+}+A_{max})(n_{i}-1)\Delta_{i}\}\cdot b_{\delta imax}]$$

$$b_{\delta imax} = \frac{1}{\Delta_{o}} \int_{0}^{\Delta_{i}} e^{(A^{+}+A_{max})\tau} d\tau (b_{i}^{+} + b_{imax})$$

where b_{imax} is the i-th column of the matrix B_{max}.

The proof of Theorem 13 now continues by observing that the transformed unified closed loop system given by

$$\bar{\rho}x(t) = \bar{A}_{\rho c}x(t) + d\bar{A}_{\rho c}x(t) \qquad \text{(V18)}$$

has a solution given by

$$x(t) = E(\bar{A}_{\rho c},t)x(0) + \int_{o}^{t} E(\bar{A}_{\rho c},t-\tau-\Delta_{o})d\bar{A}_{\rho c}x(\tau)d\tau \qquad \text{(V19)}$$

The result follows by applying Theorem 8.

$$\lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_{i}w_{i}^{*})^{+}}{f(\bar{\gamma}_{i})} \bar{A}_{\rho cmax} \right\} < 1 \qquad \text{(V20)}$$

where

$$f(\bar{\gamma}_{i}) = \begin{cases} -\mathrm{Re}(\bar{\gamma}_{i}) = -\mathrm{Re}(\bar{\lambda}_{i}) = \bar{\alpha}_{i} & \text{cont. time} \\ \frac{1}{\Delta_{o}} [1-(1+\Delta_{o}\bar{\gamma}_{i})^{+}] = \frac{1}{\Delta_{o}} [1-e^{-\bar{\alpha}_{i}\Delta_{o}}] & \text{disc. time} \end{cases}$$ ▨

Next, we consider the limiting behavior of $\bar{A}_{\delta}$, $\bar{B}_{\delta}$, $d\bar{A}_{\delta}$,

$d\bar{B}_\delta$ and $\bar{F}_\delta$ as $\Delta_o \to 0$. Using Middleton and Goodwin's [3] result, it follows that

as $\Delta_o \to 0$, $\bar{A}_\delta \to A$, $\bar{B}_\delta \to B$, $\bar{F}_\delta \to F$

$\Rightarrow (\bar{A}_\delta + \bar{B}_\delta\bar{F}_\delta C) \to (A + BFC)$

The uncertainty bounds $\bar{A}_{\delta max} \to A_{max}$, $\bar{B}_{\delta max} \to B_{max}$

$\Rightarrow (\bar{A}_{\delta max} + \bar{B}_{\delta max}\bar{F}_\delta C) \to (A_{max} + B_{max}FC)$

Thus,

$$\lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{f(\bar{\gamma}_i)} \bar{A}_{\rho cmax} \right\} \to \lambda_{max}\left\{ \sum_{i=1}^{n} \frac{(v_i w_i^*)^+}{\alpha_i} A_{cmax} \right\}$$

APPENDIX XXII

Proof of Theorem 14.:

Let

$$\bar{E}_\rho = (I_n + \Delta_o \bar{A}_{\rho c})^+ + (\Delta_o/2) d\bar{A}_{\rho c}$$

Then,

$$\bar{\rho}V(x) = \bar{\rho}x^T\bar{P}_\rho x + x^T\bar{P}_\rho\bar{\rho}x + \Delta_o\bar{\rho}x^T\bar{P}_\rho\bar{\rho}x \qquad (W1)$$

$$\bar{A}^T_{\rho c}\bar{P}_\rho + \bar{P}_\rho\bar{A}_{\rho c} + \Delta\bar{A}_{\rho c}\bar{P}_\rho\bar{A}^T_{\rho c} = -2I_n \qquad (W2)$$

Then

$$\bar{\rho}V(x) = -x^T 2I_n x + x^T(d\bar{A}^T_{\rho c}\bar{P}_\rho\bar{E}_\rho + \bar{E}^T_\rho\bar{P}_\rho d\bar{A}_{\rho c})x \qquad (W3)$$

Then, the proof follows by using the result of Theorem 9. ▨

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