Order Number 9224826

Polytopal graphs and arrangements of curves

Jeong, Dalyoung, Ph.D.

City University of New York, 1992

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

A

POLYTOPAL GRAPHS AND ARRANGEMENTS OF CURVES

by

DALYOUNG JEONG

A dissertation submitted to the Graduate Faculty
in Mathematics in partial fulfillment of the requirements
for the degree of Doctor of Philosophy,
The City University of New York.

1992

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

12/12/91 — Joseph Malkevitch
Date — Chair of Examining Committee

12/18/91 — [signature]
Date — Executive Officer

Stefan A. Burr
Harvey Cohn
Joseph Malkevitch
Supervisory Committee

The City University of New York

ABSTRACT

POLYTOPAL GRAPHS AND ARRANGEMENTS OF CURVES

by

Dalyoung Jeong

Advisor: Professor Joseph Malkevitch

Let P be a 4-valent 3-polytope, and let G be a plane 4-valent 3-connected graph whose vertices and edges correspond to the vertices and edges of P, respectively. Then,

$$p_3 = 8 + \sum_{k\geq 4} (k - 4)\cdot p_k, \qquad (*)$$

where p_k is the number of faces of G (or P) with k sides. The following "Eberhard-type" theorem, which extends a theorem of B. Grünbaum, is proven:

Theorem Given a collection of non-negative integers p_3^*, p_5^*, p_6^*, ... p_n^* which satisfies (*), there exists a non-negative integer p_4^* and a 4-valent 3-polytopal graph G, having an Eulerian circuit which is generated by choosing the "middle edge" to be the next edge as one approaches a vertex along an edge, and such that $p_k(G) = p_k^*$ $(3 \leq k \leq n)$. ($p_k(G)$ denotes the number of faces with k sides in G.)

Similar results for 4-valent 3-polytopal graphs having either a Hamiltonian circuit or a spanning tree without 2-valent vertices are also proven.

Additional structure theorems about plane 4-valent graphs

which can be written as the union of simple plane circuit are also proved. For example, the following theorem is proven in detail:

Theorem If $p_s^* = 1$, $p_3^* = 8 + (s-4)$, and $p_i^* = 0$ for all $i \neq 3$, 4, and k, then there is no plane 4-valent 3-connected graph G which is the graph of an arrangement of simple curves for which $p_k(G) = p_k^*$ for all $k = 3, 4, 5, \ldots, s$, and any choice of p_4^*.

ACKNOWLEDGEMENT

I cordially thank Professor Joseph Malkevitch for generously sharing his time as well as his ideas with me. He introduced me to the world of Graph Theory, especially to the subject of 3-polytopal graphs, with a number of questions which led me to the writing of this thesis.

I also wish to thank Professor Harvey Cohn and Professor Stephan Burr for agreeing to be members of my supervisory committee. Additionally, I would like to thank my friend Ron Skurnick for his careful proof-reading of this thesis.

Finally, I would like to say to my wife Youngsook: "Thank you for your understanding , patience, and continuous support from the beginning to the very end of my graduate studies."

TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION

1.1 Preliminaries

Most of the graphs discussed in this study are plane 3-connected graphs, which are interesting because they are isomorphic to the graphs of the convex 3-dimensional polytopes. There are two topics which form the main body of this thesis. These topics are discussed in detail in Chapters 2 and 3.

Chapter 2 deals with three variations on an Eberhard-type problem for 4-valent graphs. This problem concerns the existence, for a given p-vector, of a 3-polytope which has: i) a "cut-through" Eulerian circuit, ii) a Hamiltonian circuit, or iii) a proper spanning tree.

Chapter 3 talks about arrangements of simple curves, a notion developed by B. Grünbaum [8]. Some properties of arrangements of simple curves, as well as the question of the realizability of a certain kind of arrangement of simple curves, are discussed. Regarding the realizability question, an Eberhard-type problem involving the arrangement of simple curves is considered and solved.

For what follows, we define terms and notation briefly in this chapter. Basically, we follow the terminology of J. A. Bondy's *Graph Theory with Applications* [2], B. Grünbaum's *Convex Polytopes* [5], and J. Malkevitch's "Polytopal Graphs" [17].

1.2 Graphs and Subgraphs

A **graph** G is an ordered triple (V,E,ψ_G) consisting of a finite nonempty set V of **vertices**, a set E, disjoint from V, of **edges**, and an **incidence function** ψ_G that associates with each edge of G an unordered pair of vertices of G. If e is an edge of G and u,v are vertices such that $\psi_G(e) = (u,v)$, then e is said to **join** u and v, and the two vertices u and v are called the **endpoints** of e. The endpoints of an edge are said to be **incident** with the edge. Two vertices, like u and v, which are incident with the same edge, are **adjacent**. An edge with identical endpoints is called a **loop**. A graph is **simple** if it has no loops and no two of its edges join the same pair of vertices. Most of the graphs considered in this thesis are simple graphs because they are the graphs of 3-polytopes.

Two graphs G and G′ are **isomorphic** if there exists a one-to-one correspondence between their vertex sets, as well as their edge sets, which preserves adjacency. Let V(G) denote the set of vertices of G and E(G) denote the set of edges of G. A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to E(H). A subgraph H is called a **spanning subgraph** of G if V(H) = V(G). The **degree** (or the **valence**) of a vertex u, which is denoted by d(u), is the number of edges in G incident with u. The vertex u is called d(u)-valent. A graph G is **k-regular** (or **k-valent**) if d(u) = k for each vertex u in G.

1.3 Paths and connectedness

A u-v **walk** of G is a finite, alternating sequence of vertices and edges

$$u=u_0, e_1, u_1, e_2, \ \ldots \ , u_{n-1}, e_n, u_n=v$$

starting with vertex u and terminating with vertex v. A walk is **closed** if its starting and terminating vertices are the same. All the vertices except the starting and terminating vertices are called **internal** vertices. A u-v **path** is a u-v walk in which no vertex is repeated, while a u-v **trail** is a u-v walk in which no edge is repeated. A pair of u-v paths is called **internally disjoint** if the sets of internal vertices of the two u-v paths are disjoint. A closed trail is called a **circuit**. A **cycle** is a circuit whose starting and internal vertices are all distinct. A **Hamiltonian cycle** in G is a cycle whose vertex set is equal to that of G. A graph G is called **Hamiltonian** if there exists a Hamiltonian cycle in G. A graph G is **Eulerian** if there exists a circuit which traverses all the edges of G exactly once. A graph is **connected** if there exists a u-v path for each pair of vertices u and v. A **tree** is a connected graph which contains no cycles. A graph G is **trivial** if it consists of only one vertex and no edges. There is only one trivial graph up to isomorphism. A graph G is **n-connected** if and only if the removal of fewer than n vertices results in neither a disconnected graph nor the trivial graph. For example, a tree is 1-connected and a cycle is 2-connected. However, in the study of polytopes, 3-connectivity allows for the possibility of 4-connectivity or higher connectivity. H. Whitney provides another characterization of an n-connected graph.

Theorem 1.1 (H. Whitney) A non-trivial graph G is n-connected if and only if for each pair u,v of distinct vertices there are at least n internally disjoint u-v paths in G.

For the proof, see G. Chartrand [3].

1.4 Plane graphs

A graph is said to be **embeddable in the plane**, or **planar**, if it can be drawn in the plane so that its edges intersect only at their endpoints. Such a drawing of a planar graph G is called a **planar embedding** of G. We will refer to a planar embedding of a planar graph as a **plane graph** in this thesis. Figure 1.1 (a) shows an example of a planar graph and Figure 1.1 (b) shows its planar embedding. (All figures are grouped in the Appendix A). Every planar graph can be embedded on the sphere.

Theorem 1.2 (see J. A. Bondy [2]) A graph G is embeddable in the plane if and only if it is embeddable on the sphere.

A plane graph G partitions the plane into a number of connected regions. The closures of these regions are called the **faces** of G. Figure 1.1 (b) shows a plane graph with 4 faces. Each plane graph has exactly one unbounded region, called **the infinite face** (or **exterior face**). The number of edges encountered as one traverses around a face is called the number of sides of the face. Figure 1.1 (b) has two faces with 3 sides and two faces with 4 sides.

Theorem 1.3 Let G be a plane graph and let f_1 be a face of G. Then there is a plane graph G′, isomorphic to G, such that the face of G′ corresponding to f_1 is the infinite face.

Proof Since G is plane, it can be embedded on the sphere. Rotate the sphere in order to place the north pole inside the region corresponding to f_1. Now, re-embed the graph into the plane by using stereographic projection.

For a given plane graph G, one can define another planar graph G^* as follows: Corresponding to each vertex v (each face f, respectively) of G, there is a face v^* (a vertex f^*, respectively) of G^*. Two vertices f^* and g^* of G^* are joined by the edge e^* if their corresponding faces f and g of G share the edge e. The graph G^* is called the **dual** of G (Figure 1.2). Generally speaking, G^* may not be a simple graph unless G is 3-connected. For example, in Figure 1.2, there is a pair of vertices p^* and q^* joined by distinct edges in the dual G^* since G has a 2-valent vertex u.

Now, let us look at the relationship between the number of edges, and the number of vertices of degree i, of a graph G.

Since a face with i sides in G corresponds to a vertex of degree i in the dual graph G^*, we get a similar relationship between the number of edges, and the number of faces with i sides, of G.

Theorem 1.4 Let G be a graph. Let e, v_k, and p_k be the number of edges, the number of vertices of degree k, and the number of faces with k sides, respectively, of G. Then,

$$2 \cdot e = \sum_k k \cdot v_k = \sum_k k \cdot p_k.$$

For the proof, see J. A. Bondy [2] or G. Chartrand [3].

1.5 d-polytopes and Euler's formula

A set S of d-dimensional points is **convex** if, for any two points p and q in S, the line segment pq is in the set S. The **convex hull** of a set S is the intersection of all convex sets containing S. A **d-polytope** is the convex hull of some finite set of d-dimensional points. For example, 2-polytopes are the convex polygons and 3-polytopes are the convex 3-dimensional polyhedra. Actually, the main objects of our study in this thesis are the 3-polytopes. We call the 0-dimensional skeletons, 1-dimensional skeletons, and (d-1)-dimensional skeletons, respectively, of a d-polytope, the **vertices**, **edges**, and **facets** (in the case of 3-polytopes, we use the term **faces** instead of facets), respectively, of the d-polytope. The **graph of a d-polytope** P is the graph consisting of the vertices and edges of P. A **d-polytopal** graph G is a graph which is isomorphic to that of some d-polytope P. In this case, we say that **P realizes G**.

There is a simple but remarkable result discovered by Leonard Euler for any convex polyhedron.

Theorem 1.5 (Euler's Polyhedral Formula) Let v,e, and f be the numbers of vertices, edges, and faces, respectively, of a 3-polytope. Then,

$$v - e + f = 2.$$

For the proof of this theorem and for the generalization of this theorem to d-polytopes ($d \geq 3$), see B. Grünbaum [5].

As we mentioned at the beginning of this chapter, there is a nice relationship between the 3-polytopes and the planar 3-connected graphs, which is described in the "Fundamental Theorem of Polytopal Graph Theory".

Theorem 1.6 (Fundamental Theorem of Polytopal Graph Theory; also known as Steinitz' Theorem) A graph is 3-polytopal if and only if it is planar and 3-connected.

For a proof, see B. Grünbaum [5]. This theorem simplifies the combinatorial study of a 3-polytope by allowing us to look at a 3-polytopal graph in the plane which is isomorphic to the given 3-polytope, instead of the original 3-dimensional object itself. Moreover, this also allows us to apply graph theoretical techniques to the study of 3-polytopes.

Remark Let G be a plane 3-connected graph. Then, the following **Steinitz operations** (i.e., face splits) preserve the planarity and 3-connectivity of the graph G.

i) Add two new vertices u and u′ and place them on the edges e and e′, respectively, where e and e′ are sides of the same face F. Then, join the two vertices u and u′ with a new edge e_u (Figure 1.3 a).

ii) Choose a vertex v and add a new vertex v′ by placing u′ on an edge e which is not incident with v. Then, join the two vertices v and v′ with a new edge e_v (Figure 1.3 b).

iii) Choose two vertices w and w′ which are not adjacent, and join these two vertices with a new edge e_w (Figure 1.3 c).

The following is the graph-theoretical version of Euler's formula.

Theorem 1.7 (Euler's formula) Let G be a planar connected graph and let v,e, and f be the number of vertices, edges and faces, respectively, of G. Then,

$$v - e + f = 2$$

The proof of this theorem can be found easily in many books on graph theory (see [2] and [3]).

1.6 3-polytopes and Eberhard's problem.

Let G be a 3-polytopal graph and let v_k and p_k be the number of k-valent vertices and the number of faces with k sides, respectively. For example, for the graph in Figure 1.4, $v_3 = 3$, $v_4 = 4$, $v_5 = 1$ and $p_3 = 6$, $p_4 = 3$. Usually, we simply call a face with k sides a **k-gon**.

Theorem 1.8 Let G be a 3-polytopal graph and let v_k and p_k be the number of vertices of degree k and the number of faces with k sides. Then,

$$\sum_{k\geq 3}(4-k)p_k + \sum_{k\geq 3}(4-k)v_k = 8$$

For the proof of this theorem, see J. Malkevitch [15].

If G is a connected plane r-valent graph with r = 3, 4 or 5, then we have more specific and important results which form the starting point of this study.

Theorem 1.9 Let G be an r-valent 3-polytopal graph. Then,

(i) for $r = 3$, $3p_3 + 2p_4 + p_5 = \sum_{k\geq 6}(k-6)p_k$

(ii) for $r = 4$, $p_3 = 8 + \sum_{k\geq 4}(k-4)p_k$

(iii) for $r = 5$, $p_3 = 20 + \sum_{k\geq 4}(3k-10)p_k$.

Proof Let v be the number of vertices of G. Since G is r-valent, $v_r = v$ and $v_i = 0$ for all $i \neq r$. And, by Theorem 1.4, we know that $\sum_k k\cdot p_k = 2\cdot e = \sum_k k\cdot v_k$, where e is the number of edges of G. Use these facts in the equation of Theorem 1.8.

This yields the desired results (i),(ii), and (iii).

These are necessary conditions for r-valent 3-polytopal graphs, where $r = 3,4$ or 5. Note that the coefficient of p_6 in equation (i) and that of p_4 in equation (ii) are 0. Therefore, the number of 6-gons, p_6, plays no role in equation (i). Similarly, the number of 4-gons, p_4, plays no role in the equations (ii). From the first of these two facts, a question was formulated and subsequently answered by V. Eberhard [4]. This question was:

For a given number of non-hexagonal polygons whose numbers satisfy equation (i), can we construct a 3-valent 3-polytope by adding a suitable number of hexagons?

We can ask an analogous question for the 4-valent case.

To simplify further discussion, let us first define some notation and terminology. For a given 3-polytope P, let $p_k(P)$ be the number of faces with k sides (or simply p_k as before), and let s(P) be the following sequence of p_k's;

$$s(P) = (p_3, p_4, p_5, p_6, \ldots, p_n)$$

where n is the greatest integer such that $p_n > 0$.

A finite sequence $s^* = (p_3^*, p_4^*, p_5^*, p_6^*, \ldots, p_n^*)$ of non-negative integers is said to be **realizable** if there is a 3-polytope P such that $s(P) = s^*$. In this case, P is a **realization** of s^* (or P **realizes** s^*). Let $t^* = (p_3^*, p_4^*, p_5^*, p_7^*, \ldots, p_n^*)$ denote a finite sequence of non-negative integers (notice that p_6^* is missing in this sequence). We say that t^* is **augmentable** by p_6^* if there exists a 3-valent 3-

polytope P which realizes the sequence $(p_3^*, p_4^*, p_5^*, p_6^*, \ldots, p_n^*)$. Similarly, we can define the term **augmentable** by p_4^*. A sequence $(p_3, p_4, p_5, p_7, \ldots, p_n)$ of non-negative integers is called a **p-vector** if it satisfies equation (i) of Theorem 1.9. Similarly, a sequence $(p_3, p_5, p_6, p_7, \ldots, p_n)$ of non-negative integers which satisfies equation (ii) of Theorem 1.9 is also called a **p-vector**. Now, let us state Eberhard's Theorem and Grünbaum's Theorem.

Theorem 1.10 (Eberhard's Theorem) Every sequence $(p_3^*, p_4^*, p_5^*, p_7^*, \ldots, p_n^*)$ of non-negative integers which satisfy equation (i) in Theorem 1.9 is augmentable by some non-negative integer p_6^*.

Theorem 1.11 (Grünbaum's Theorem) Every sequence $(p_3^*, p_5^*, p_6^*, p_7^*, \ldots, p_n^*)$ of non-negative integers which satisfy equation (ii) in Theorem 1.9 is augmentable by some non-negative integer p_4^*.

For detailed proofs of Theorem 1.10 and Theorem 1.11, see B. Grünbaum [5] and [6]. Since Grünbaum's proof of Theorem 1.11 is quiet clever, we shall illustrate it with an example.

The procedure of Grünbaum's proof is:

(i) Construct blocks.

(ii) Place these blocks anti-diagonally.

(iii) Connect the 2- or 3-valent vertices to make them 4-valent.

For example, let $p_5^* = p_6^* = 1$, then $p_3^* = 8 + (5-4)\cdot 1 + (6 - 4)\cdot 1 = 11$. Figure 1.5 (a) shows the blocks for a 5-gon and 6-gon. Each block for each k-gon contains k-4 triangles (k = 5 or k = 6). Hence, it suffices to construct 8 triangles outside of these two blocks. Figure 1.5 (b) shows the anti-diagonal arrangement of these blocks and Figure 1.5 (c) shows how to connect the non-4-valent vertices to complete the construction. By connecting the non-4-valent vertices, we obtain another 8 triangles (each labeled by the number "3") outside of these two blocks.

Other proofs of Grünbaum's Theorem have been given by J. Malkevitch [16]. Also, in addition to Eberhard's Theorem, which is an existence theorem, there are discussions in the literature concerning the number of hexagons possible for a given p-vector (see B. Grünbaum [7] and D. Barnette [1]).

Finally, let us state a nice necessary condition for a plane graph to be Hamiltonian, due to E. Grinberg (see J. A. Bondy [2] or G. Chartrand [3]).

Theorem 1.12 (Grinberg's Theorem) Let G be a simple plane graph with a Hamiltonian cycle C and let p be the number of vertices of G. If r_i denotes the number of faces with i sides (of G) in the interior of C, and r_i' denotes the number of faces with i sides (of G) exterior to C, then,

$$\Sigma_{i\geq 3} (i - 2)(r_i - r_i') = 0.$$

CHAPTER 2 REALIZATIONS WITH A CUT-THROUGH EULERIAN CIRCUIT

2.1 Introduction

After Grünbaum extended Eberhard's ideas to construct 4-valent 3-polytopes for a given p-vector, some additional analogues of Eberhard's problem and other types of realization problems have been studied (see J. Zaks [22], [23] and [24]). In this spirit, we will discuss another type of extension of Eberhard's problem. For convenience, we will assume that all graphs in this chapter are planar, simple, and connected unless stated otherwise.

Let us define some terms. First, consider the concept of a "<u>directed trail</u>" (or "<u>directed path</u>") with certain specific properties. Note that a directed path is a trail, not a path, as defined in Chapter 1. Therefore, some vertices may be traversed several times by the same directed path. Let G be a plane 4-valent graph. Let u be a vertex in G and let e_1, e_2, e_3, and e_4 be edges which are incident with u. Suppose that the vertex u is visited by a directed path via the edge e_1 (Figure 2.1 a, All figures are grouped in Appendix A). Then, there are three possible ways to choose the next edge, that is, either e_2, e_3, or e_4. We call these three edges, e_2, e_3, and e_4 a left edge, a middle edge, and a right edge, respectively. If the edge e_3 is chosen to be the next edge of the path, then the edges e_1e_3 "cut through" the edges e_2e_4 (Figure 2.1 b). If a path in a plane 4-valent graph always takes the middle edge as the next edge, then we call this type of path a **cut-through path**. The vertex u is called **a cut-through vertex** with respect to such a path. On the other hand, a **left-right path** is an alternating sequence of left and right edge (Figure 2.1 c). However, if either e_2 or e_4 is chosen to be the next edge of the path, then we call such a path a **non-cut-through path**, and we call the vertex u a **non-cut-through vertex** (Figure 2.1 d). A circuit C is called **a cut-through circuit** (a **non-cut-through circuit**, respectively) if all the vertices in this circuit are cut-through vertices (non-cut-through vertices, respectively) with respect to the circuit C. For a more generalized definition and some properties of "coded" paths which include a cut-through path and a left-right path, see J. Malkevitch [15]. And, for a generalization of the left-right path, called a "left-most and right-most path", see H. Shank [19].

In the literature, other names are used to describe a cut-through path, such as "direct extension" (see B. Grünbaum [5]) and "straight ahead" (see S. Lins [13]). Sometimes, we will call a cut-through circuit a **cut-through component** because it can be transformed into a simple curve by using one or more of the three basic deformations (the so-called Reidemeister moves [10]) from knot theory (see Figure 2.31). A planar 4-valent graph G is **cut-through Eulerian** if there exists an Eulerian circuit C (in G) such that all the vertices of G are cut-through vertices with respect to the Eulerian circuit C (in other words, there is only one cut-through component in G).

The graph in Figure 2.2 (a) has two cut-through components, while the graph in Figure 2.2 (b) has only one cut-through component (i.e., the graph in Figure 2.2 (b) is cut-through Eulerian).

A p-vector $(p_3, p_5, p_6, \ldots, p_n)$ is said to be **realizable with the cut-through Eulerian property** if it is augmentable by some p_4, and the 4-valent 3-polytopal graph G of its realization is cut-through Eulerian. We call such a 3-polytopal graph G a **cut-through Eulerian realization** of the p-vector $(p_3, p_5, p_6, \ldots, p_n)$ (or simply, we say that such a 3-polytopal graph G is **cut-through Eulerian**). By contrast, an Eulerian circuit of a plane 4-valent graph G is called a **non-cut-through Eulerian circuit** if all of its vertices are non-cut-through vertices with respect to the given Eulerian circuit.

For a given p-vector $(p_3, p_5, p_6, \ldots, p_n)$ for which $\sum_{i \geq 5} p_i \geq 0$, B. Grünbaum's realization is not cut-through Eulerian, and J. Malkevitch's realizations are not cut-through Eulerian either, in general. For example, look at realizations of a p-vector $(p_3, p_5, p_6, \ldots, p_n)$ such that $p_3 = 9$, $p_5 = 1$ and $p_i = 0$ for all $i \geq 6$ in Figure 2.3. Each of these realizations has more than one cut-through component.

In this regard, J. Malkevitch has raised the question, "Is there a cut-through Eulerian realization for any given p-vector?" In what follows, an affirmative answer will be given.

2.2 Realizations with the cut-through Eulerian property

Let us state the main theorem in this chapter, first.

Theorem 2.1 For a given p-vector $(p_3, p_5, p_6, \ldots, p_n)$, there exists a plane 3-connected 4-valent graph G which realizes the given p-vector and which is cut-through Eulerian (simply stated, every p-vector $(p_3, p_5, p_6, \ldots, p_n)$ is realizable with the cut-through Eulerian property).

The strategy of the proof will be to show that, starting with an appropriate plane 3-polytopal cut-through Eulerian graph, we can locate configurations within the graph that can be modified in such a way that we obtain the desired p-vector while preserving the cut-through Eulerian property.

To prove this theorem, we need to discuss some operations and lemmas. Let us call the special types of face structures in Figure 2.4 (a), (b), and (c) by the names A_1-configuration, A_2-configuration, and B-configuration, respectively. The number within each region represents the number of sides of that face (or region). Now, let us define the operations α and β. The operation α adds 5 new vertices and places them one on each edge of the A_1-configuration (or A_2-configuration). Then, it joins these new vertices to make a simple circuit (see Figure 2.5 (a) and (b)). The purpose of the operation ß is to place a configuration of triangles and 4-gons (Figure 2.5 (c)) into the inside region of the B-configuration, while creating 3 new vertices on the top edge and one new vertex on each of the other edges. Clearly, these operations preserve the cut-through property. Let us denote the shapes on the right-hand side of Figure 2.5 by $A_1\alpha$, $A_2\alpha$, and $B\beta$, respectively. Let G be a plane 4-valent graph. Then $G\alpha$ is a new graph obtained from G by applying the operation α to the A_1-configuration or A_2-configuration in G. Thus, whenever we write $G\alpha$, we assume that G has an A_1-configuration or an A_2-configuration. Gß is defined in a similar manner. A string of α's and ß's denotes the consecutive application of these operations, reading from left to right, to an appropriate configuration. For example, $ß^2\alpha$ means that one applies the ß operation twice to a B-configuration and then applies the α operation to an A-configuration.

Lemma 2.2 Let G be a plane 4-valent 3-connected graph. Then $G\alpha$ and Gß are also plane 4-valent 3-connected graphs.

Proof Clearly, these operations preserve simplicity, planarity, and 4-regularity. Hence, it suffices to show that $G\alpha$ and Gß are also 3-connected. Since the operations α and ß are both finite combinations of series of Steinitz' operations, and since Steinitz' operations preserve 3-connectedness, $G\alpha$ and Gß are also 3-connected.

Remark The operations α and ß change the face structure of a graph G in the following way. Let p_k be the number of k-gons in G.

1) applying operation α to an A_1-configuration:

$$p_k \longrightarrow p_k - 1$$
$$p_{k+1} \longrightarrow p_{k+1} + 1$$
$$p_3 \longrightarrow p_3 + 1$$

2) applying operation α to an A_2-configuration:

$$p_5 \longrightarrow p_5 + 1$$
$$p_3 \longrightarrow p_3 + 1$$

3) applying operation ß to a B-configuration:

$$p_k \longrightarrow p_k - 1$$
$$p_{k+3} \longrightarrow p_{k+3} + 1$$
$$p_3 \longrightarrow p_3 + 3$$

4) Note that $A_2\alpha$ contains another A_2-configuration, so we can apply the operation α over and over again (Figure 2.5 (b)).

5) $A_1\alpha$ and Bß contain both an A_1-configuration and a B-configuration. Hence, we can apply either operation α or operation ß to $A_1\alpha$ and Bß (Figure 2.5 (a) and (c)).

Lemma 2.3 i) Let the k-gon on the left side in Figure 2.5 (c) be a triangle. Then, we can change it to a (4n+2)-gon or a (4n+3)-gon for $n \geq 1$.

ii) Let the k-gon on the left side of Figure 2.5 (a) be a 4-gon. Then, we can change it to a 4n-gon or a (4n+1)-gon for $n \geq 2$.

Proof i) Apply operations α and ß in an alternating way, starting by applying ß to a B-configuration. Then,

$$k \xrightarrow{ß} k + 3 \xrightarrow{\alpha} k + 4 \xrightarrow{ß} k + 7 \xrightarrow{\alpha} k + 8.$$

Since k = 3, we can change 3 to 6, then 7, then 10, then 11, and so on. Thus, we can create a (4n+2)-gon or a (4n+3)-gon for $n \geq 1$ (Figure 2.6 (a)).

ii) Apply operations α and β in an alternating way, starting by applying α to an A_1-configuration. Then,

$$k \xrightarrow[\alpha]{} k+1 \xrightarrow[\beta]{} k+4 \xrightarrow[\alpha]{} k+5 \xrightarrow[\beta]{} k+8.$$

Since k = 4, we can change 4 to 5, then 8, then 9, then 12, and so on. Thus, we can create a 4n-gon or a (4n+1)-gon for $n \geq 2$ (Figure 2.6 (b)).

Lemma 2.3 shows that we can obtain a k-gon ($k \geq 5$) by applying appropriate numbers of α and β operations to a triangle or 4-gon. In fact, a triangle in a B-configuration is transformed into a (4n+3)-gon by applying $(\beta\alpha)^n$ operations, or to a (4n+2)-gon by applying $(\beta\alpha)^{n-1}\beta$ operations. Similarly, a 4-gon in an A_1-configuration is changed to a 4n-gon by applying $(\alpha\beta)^n$ operations or to a (4n+1)-gon by applying $(\alpha\beta)^n\alpha$ operations.

We will use the plane 4-valent 3-connected graph G_0 in Figure 2.7 (a) as the base of our construction. Observe that G_0 is cut-through Eulerian and has 8 triangles and three 4-gons. Moreover, G_0 is symmetric with respect to the vertex u. For convenience, let us denote the left-half and right-half of G_0 by G_0' and G_0'', respectively (Figure 2.7 (b), (c)).

Lemma 2.4 Let $(p_3, p_5, p_6, p_7, \ldots, p_n)$ be a p-vector such that $p_5 > 0$ and $p_i = 0$ for all $i \geq 6$. Then, there exists a cut-through Eulerian realization G of the given p-vector.

Proof By applying the operation α to G_0' once, we get an A_2-configuration (Figure 2.8 (a)). Now, by applying operation α to this A_2-configuration, we produce one 5-gon and a new A_2-configuration (Figure 2.8 (b)). By repeatedly applying the operation α, we can construct p_5 5-gons (Figure 2.8 (c)). This completes the proof.

Note that our construction of p_5 5-gons above does not alter the right-half G_0'' at all. Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1 Let us prove this theorem by constructing a 4-valent 3-polytopal graph which realizes the given p-vector. We will introduce k-gons, one at a time, by applying a series of the operations α and β to the base graph G_0, which is cut-through Eulerian. Let $(s_3, s_5, s_6, s_7, \ldots, s_n)$ be a sequence which is a copy of the given p-vector, i.e., $s_k = p_k$ for all $k = 3$ and $k \geq 5$ (s_k is the number of k-gons that we should construct).

<u>Step 1</u> First, construct p_5 5-gons by following the procedure found in the proof of Lemma 2.4 in the left-half G_0' of the base graph G_0 (if $p_5 = 0$, then skip this process). After this construction, replace s_5 by 0. We can locate an A_1-configuration and a B-configuration in the right-half G_0'' of G_0.

<u>Step 2</u> Let k be the largest integer such that $s_k \neq 0$. If

k = 3, then go to Step 5. Otherwise, go to Step 3 to construct a k-gon.

<u>Step 3</u> Case (i) $k \equiv 0$ or $k \equiv 1 \pmod 4$:

Apply the operation α to the A_1-configuration located in Step 1 (or at the end of Step 3, see below) to modify a 4-gon to a k-gon. For example, Figure 2.9 (a) shows the first application of the operation α to a 4-gon in the right-half G_0". Apply the operations α and ß in the way described in Lemma 2.3 (cf. Figure 2.6 (b)) until the original 4-gon becomes a k-gon.

Case (ii) $k \equiv 2$ or $k \equiv 3 \pmod 4$:

Similarly, apply the operation ß to the B-configuration located in Step 1 (or at the end of Step 3, see below) to modify a triangle to a k-gon. For example, Figure 2.9 (b) shows the first application of the operation ß to a triangle in the right-half G_0". Apply the operations α and ß in the way described in Lemma 2.3 (cf. Figure 2.6 (a)) until the original triangle becomes a k-gon.

After constructing a k-gon, we can locate an A_1-configuration and a B-configuration once again, to which we can apply the operations α and ß, respectively, to construct another k-gon in the final stage of both cases (Figure 2.9 (c) - (f). Note the use of Remark 5) in page 19).

<u>Step 4</u> Since we now have a k-gon, reduce the value of s_k by 1. If the new value of s_k is 0, then we have constructed p_k-gons already. So, go to Step 2. Otherwise, return to Step 3 to construct another k-gon.

Step 5 The final graph is a cut-through Eulerian realization of the given p-vector P.

For example, Figure 2.10 illustrates the above method for a p-vector $P = (p_3, p_5, p_6, p_7, \ldots, p_n)$ such that $p_5 = 3$, $p_7 = 2$, $p_8 = 1$, $p_3 = 8 + 3\cdot(5-4) + 2\cdot(7-4) + 1\cdot(8-4) = 21$, and $p_i = 0$ for all i which are not equal to 3,5,7 or 8.

Now we know that there exists a cut-through Eulerian realization for any p-vector of the form $(p_3, p_5, p_6, p_7, \ldots, p_n)$. But we do not have any information about the number of 4-gons, p_4, yet. For the classical 3-valent Eberhard Theorem, B. Grünbaum and D. Barnette obtained some results about those values of p_6 that could occur for a fixed p-vector (see [1] and [6]). Similarly, we can establish some results about the number of 4-gons that can occur in the cut-through Eulerian realization. For this purpose, let us define another operation called "**parallel drawing**".

The operation of "**Parallel Drawing**".

Let G be a plane 4-valent graph with a cut-through Eulerian circuit. For any vertex v, we can find a cut-through circuit which returns to the vertex v for the first time. Let v be a vertex of G and ω be such a cut-through circuit which starts from v and returns to v for the first time. Let n be the length of ω (Figure 2.11 (a)). Label all the edges of ω from e_1 to e_n. Of course, all the edge of ω are distinct. So,

ω can be written as a sequence of edges, that is,

$$\omega = e_1 e_2 e_3 \ldots\ldots e_n.$$

Let v_n and $v_1 = v$ be the endpoints of the edge e_n and let v_i and v_{i+1} be the endpoints of the edge e_i for all i, where $1 \le i \le n-1$ (we can also write this circuit as a sequence of vertices, i.e., $\omega = v_1 v_2 v_3 \ldots\ldots v_n v_1$). Let F be the face surrounded by the edges e_n and e_1 in Figure 2.11 (a). Now, create a vertex v_1' on the edge e_1. This vertex v_1' splits the edge e_1 into two edges e_1' and e_1'' (Figure 2.11 (b)). There are three edges other than e_1'' that are incident with v_2 but not e_1''. Let us choose an edge which is a side of the face F among these three edges and denote it by e. Now, create another vertex v_2' on the edge e and join the two vertices v_1' and v_2'. Starting from the vertex v_2', draw a curve which is parallel to the circuit ω such that all the faces between the new curve and ω are 4-gons, until the new curve reaches the edge e' which is incident with v_n and is a side of the face F. Let $\{v_1', v_2', \ldots, v_m'\}$ be the vertex set which is created by this process (m is equal to n if the circuit ω is a simple circuit). Finally, join the two vertices v_m' and v_1' (Figure 2.11 (b)). Then, we have a new circuit ω' such that

$$\omega' = v_1 v_1' v_2' v_3' \ldots v_m' v_1' v_2 v_3 \ldots\ldots v_n v_1.$$

Note that this process preserves the cut-through property. Suppose that all the vertices $v_1, v_2, \ldots, v_n$ are distinct (i.e., the circuit ω is simple) and the k-gon in Figure 2.11 (a) is a triangle. Then, this operation produces the triangle $\Delta v_1'v_2v_2'$ and (n-1) 4-gons between ω and the new curve, and changes the triangle Δv_1uv_2 into a 4-gon (Figure 2.11 (c), (d)). Hence, this operation simply increases the number p_4 to $p_4 + n$, while fixing the other p_i's. Moreover, this operation is repeatable (Figure 2.11 (d)).

Lemma 2.5 Let G be a plane 4-valent graph with a cut-through Eulerian circuit and $p = (p_3, p_4, p_5, p_6, \ldots, p_n)$ be its face vector. From G, one can construct a sequence of plane 4-valent graphs which have the same face vector p, except for the value of p_4.

Proof Choose a triangle in G and an edge of that triangle. Apply the operation of parallel drawing. We now have another plane 4-valent graph G' with a cut-through Eulerian circuit, and all the p_k's are the same as before, except p_4. By repeating this process, we generate infinitely many different plane 4-valent graphs, each with a cut-through Eulerian circuit, and each having the same face vector, except for the value of p_4. Figure 2.12 shows the result of the application of the parallel drawing technique three times. (Thick edges are those of the original graph.)

For convenience in what follows, let us name the configurations of the simple circuits in Figure 2.13 as follows: T_1-, T_2-, T_3-, T_4-, and T_5-configuration.

Remark Now, we list the following facts.

1) $A_1\alpha$, $A_2\alpha$, and $B\beta$ contain the T_1-configuration (Figure 2.14 (a), (b) and (c)).

2) $A_1\alpha\alpha$ contains the T_2-configuration (Figure 2.14 (d)).

3) $B\beta\alpha$ contains the T_3-configuration (Figure 2.14 (e)).

4) $A_1\alpha\beta$ contains the T_4-configuration (Figure 2.14 (f)).

5) $B\beta\beta$ contains the T_5-configuration (Figure 2.14 (g)).

Theorem 2.6 (Frobenius problem, see A. Schrijver [20]) Let p and q be two relatively prime positive integers. Then, there is a positive integer m such that every integer n which is greater than m can be written as a linear combination of p and q with non-negative coefficients. In fact,

$$m = pq - p - q.$$

Theorem 2.7 For a given p-vector $P = (p_3, p_5, p_6, p_7, \ldots, p_s)$ such that $p_i > 0$ for some $i \geq 5$, there exists an integer n such that, for any $p_4 > n$, $P^* = (p_3, p_4, p_5, p_6, p_7, \ldots, p_s)$ is realizable with the cut-through Eulerian property (Note that the number of 4-gons is specified in the sequence p^*).

Proof By Theorem 2.1, we can construct a 3-polytopal graph which realizes the given sequence for some choice of p_4. Let p_4^* be the number of 4-gons in such a 3-polytopal graph. It suffices to show that there are two disjoint cut-through simple circuits whose lengths are relatively prime and whose starting edges are one of the edges of a triangle. Suppose that we can find two simple cut-through circuits ω and ω' described as above, where p and q are the lengths of ω and ω', respectively. Since p and q are relatively prime, we can find an integer m such that every integer greater than m can be written as a linear combination with non-negative coefficients, by Theorem 2.6. Then, let $n = p_4^* + m$. For any $p_4 > n$, $p_4 - p_4^* > m$. Thus, there are non-negative integers a and b such that $p_4 - p_4^* = ap + bq$. Now, apply the operation of parallel drawing a times (b times, respectively) to ω (ω', respectively). Then, the final graph has p_4 4-gons.

Now, let us find two simple cut-through circuits described as above, case by case.

<u>Case 1</u> $p_5 \geq 2$;

If $p_5 = 2$ and $p_i = 0$ for all $i \geq 6$, then we have a cut-through Eulerian realization G as in Figure 2.15 (a). In G, there are both a T_1-configuration and a T_2-configuration.

If $p_5 > 2$ and $p_i = 0$ for all $i \geq 6$, then apply a method, similar to that used in the proof of Lemma 2.4, to G to construct p_5 5-gons (Figure 2.15 (b)). This graph also has both a T_1-configuration and a T_2-configuration.

In case $p_i > 0$ for some $i \geq 6$, we can use the operations α and β to construct the given number of i-gons for $i \geq 6$ (Figure 2.15 (c) and (d)). As described in the previous remark, there is a T_1-configuration present after the application of the operations α and β. Thus,

$$n = p_4^* + 5 \cdot 7 - 5 - 7 = p_4^* + 23$$

in this case.

<u>Case 2</u> $\Sigma_{i \geq 6}\, p_i \geq 1$

Construct p_5 5-gons in the left-half of the base graph G_0. Then, there is a T_1-configuration in the left-half (Figure 2.16 (a)). Suppose that $p_i > 0$ for some $i > 6$ and that we have constructed all the faces, except for one i-gon, to realize the given p-vector. Now, let us apply an alternating sequence of the operations α and β to generate the required i-gon. After this construction, we can find a T_4-configuration or a T_3-configuration in the result of the last $\alpha\beta$ or $\beta\alpha$ operation, respectively (Figure 2.14 (e) and (f)). Thus,

$$n = p_4^* + 5\cdot 9 - 5 - 9 = p_4^* + 31$$

or

$$n = p_4^* + 5\cdot 7 - 5 - 7 = p_4^* + 23.$$

If $p_i = 0$ for all $i > 6$ but $p_6 > 1$, then we have to apply the operation β over and over to get the desired p_6 6-gons. Then, there is a T_5-configuration in the result of the last β operation (Figure 2.14 (g)). In this case,

$$n = p_4^* + 5\cdot 9 - 5 - 9 = p_4^* + 31.$$

If $p_i = 0$ for all $i > 6$ but $p_6 = 1$, then the graph in Figure 2.16 (b) is a cut-through Eulerian realization. Moreover, it contains a cut-through simple circuit of length 7. Thus,

$$n = p_4^* + 5\cdot 7 - 5 - 7 = p_4^* + 23.$$

<u>Case 3</u> $p_5 = 1$ and $p_i = 0$ for all $i > 5$.

The graph in Figure 2.17 is a cut-through realization of the given p-vector. It contains two disjoint cut-through simple circuits whose lengths are 5 and 7, respectively. Thus,

$$n = p_4^* + 5\cdot 7 - 5 - 7 = p_4^* + 23.$$

This completes the proof.

Unfortunately, there is no known sufficient condition for a 3-polytopal graph to be cut-through Eulerian. However, we do have a sufficient condition which assures that a specific plane 4-valent graph is not cut-through Eulerian. Let G be a plane graph. If the number of side of every face of G is multiple of 3, then we call G a **multi 3-gon graph**.

Theorem 2.8 (Malkevitch [15]) A plane 4-valent multi 3-gon graph G is not cut-through Eulerian.

Actually, Malkevitch showed that the edges of a plane multi 3-gon graphs can be written as the union of cut-through cycles, which we will call **simple curves**. We will discuss this topic further in Chapter 3.

2.3 Realizations having other properties

In Section 2.2, we considered the realization of a given p-vector such that this realization is cut-through Eulerian. In this section, we shall consider the following realization problems:

i) For a given p-vector, is there a 4-valent 3-polytopal realization which has a Hamiltonian circuit?

ii) For a given p-vector, is there a 4-valent 3-polytopal realization which has a HIST (Homeomorphically Irreducible Spanning Tree)? (we will see the definition of HIST later.)

To solve these problems we will use the same operations that we defined in Section 2.2. Let's prepare ourselves to handle these problems by introducing the following lemmas.

Lemma 2.9 Let G be a plane 4-valent graph having a Hamiltonian circuit, and let Gα and Gβ be the new graphs obtained from G by applying the operations α and β, respectively. Then

i) Gα is also Hamiltonian.

ii) Gβ is also Hamiltonian if at least one edge of the face F of the B-configuration of G (Figure 2.18) is contained in the Hamiltonian circuit of G. (In what follows, F will be called the inside face of the B-configuration).

Proof We can prove this lemma by checking all possible cases. Figure 2.19 (a) and (b) shows how to transform the given Hamiltonian circuit to a new graph via application of the operation α. The small graph of each case shows a part of the given Hamiltonian circuit, and the graph following each arrow is an extension of the HC upon application of the operation α. Similarly, Figure 2.20 shows how to transform the given Hamiltonian circuit to a new graph via application of the operation ß. The small graph in each diagram shows a part of the Hamiltonian circuit, and the larger graph following each arrow shows an extension of the HC in the transformed graph, via the operation ß. (Note that it is sufficient to check only 8 cases because it is not important whether the 4th edge of the face F in Figure 2.18, which is shown as a dotted line in the diagram, is contained in the

Hamiltonian circuit or not.) Both of the operations α and β preserve Hamiltonicity.

Lemma 2.10 For a p-vector $(p_3, p_5, p_6, p_7, \ldots, p_n)$ such that $p_i = 0$ for all $i \geq 6$ but $p_5 > 0$, there exists a 4-valent 3-polytopal realization G which has a Hamiltonian circuit.

Proof It is easily checked that the base graph G_0 used in the proof of Theorem 2.1 is Hamiltonian. And, there exists a 3-polytopal graph which realizes the given p-vector by Lemma 2.4. Since we apply only the operation α to construct such a graph, it is Hamiltonian, by Lemma 2.9. In fact, we have two different types of Hamiltonian circuits, depending on the number of 5-gons (Figure 2.21).

Theorem 2.11 For any given p-vector, there is a 4-valent 3-polytopal realization G which has a Hamiltonian circuit.

Proof By Lemma 2.10, we can construct a 3-polytopal Hamiltonian graph G′ which has p_5 5-gons (Figure 2.21). Now, from G′ we can construct a 3-polytopal graph G, by applying the same method as that used in Theorem 2.1, which realizes the given p-vector. Since at least one edge of each face of $A_1\alpha$ and $B\beta$, in Figure 2.19 and Figure 2.20, is contained in the Hamiltonian circuit, the operation ß preserves the Hamiltonicity. Thus, G is also Hamiltonian, by Lemma 2.9.

As a result, we see that the realization having a cut-through Eulerian circuit, constructed in Section 2.2, also has a Hamiltonian circuit.

Conjecture Every cut-through Eulerian 4-valent 3-polytopal graph is Hamiltonian.

In order to address problem ii), posed at the beginning of this section (2.3), let us first define some terminology. A tree T is called **a proper tree** if it has no 2-valent vertex. A graph G is called a **proper graph** if it is spanned by a proper tree T. A proper spanning tree T of G is called a **HIST** (Homeomorphically Irreducible Spanning Tree). Now, we are ready for the following facts.

Lemma 2.12 Let G be a 4-valent 3-polytopal graph with a HIST. Then, so is Gβ if the operation ß is applied to a B-configuration in G and no more than two edges of the inside face of a B-configuration are included in the given HIST.

Proof As in the proof of Lemma 2.8, we can check every possible case involving the operation ß. Let T be a proper spanning tree of G and let F be the inside face of a B-configuration (Figure 2.18). Then, we must consider the following cases: i) no edge of F is contained in T (Figure 2.22 (h)), ii) only one edge of F is contained in T (Figure 2.22 (a), (b), (c) and (d)), iii) exactly two edges of F are contained in T (Figure 2.22 (e), (f) and (g)), and iv) more than two edges of F are contained in T. Every transformation of a B-configuration in G, resulting from an application of the operation ß, preserves the HIST property (Figure 2.22), except in case iv). This proves the lemma.

While the checking of all the cases involving the operation ß is somewhat simple, the same cannot be said for the operation α because many more cases must be checked involving the operation α. Hence, it is better to consider a few fixed configurations which preserve the HIST property, for the sake of the simplicity.

Remark 1) Suppose that a proper tree is given in an A_1-configuration, as in Figure 2.24 (a). Then, the operation α, applied to this A_1-configuration, extends that proper tree, to another proper tree as in Figure 2.24 (b).

2) Suppose that a proper tree is given in a B-configuration as in Figure 2.24 (c). Then, the operation ß, applied to this B-configuration, extends that proper tree to another proper tree as in Figure 2.24 (d).

3) When we apply the operation α to the A_1- configuration, we obtain the same type of proper tree as that found in the B-configuration (compare the shaded part in Figure 2.24 (b) and that in Figure 2.24 (c)).

4) When we apply the operation ß to the B-configuration, we obtain the same type of proper tree as that found in A_1-configuration (compare the shaded part in Figure 2.24 (d) and that in Figure 2.24 (a)).

5) By remarks 3) and 4) above, we can apply the operations α and ß in an alternating manner to extend a given proper tree to another proper tree.

Theorem 2.13 Given any p-vector $(p_3, p_5, p_6, p_7, \ldots, p_n)$, there is a 4-valent 3-polytopal realization which has a HIST.

Proof By Theorem 2.1, we can construct a 4-valent 3-polytopal graph which realizes the given p-vector. Hence, it suffices to show that such a construction preserves the HIST property. Let us start our construction from the graph G in Figure 2.23. Clearly, G has a HIST (thick line indicates a HIST. In fact, this tree has only 1- and 3-valent vertices.). First, construct p_5 5-gons first. Then, we can find a proper spanning tree, as in Figure 2.25 (a) and (b), according as the number of 5-gons is odd or even. However, proper trees in the right-half of two different HIST's will have the same form. If there is an integer $k \geq 6$ such that $p_k > 0$, construct a k-gon by applying the same method used in the proof of Theorem 2.1.

If $k \equiv 0$ or $k \equiv 1 \pmod 4$, then apply the operation α, as in Figure 2.26 (a), and we obtain an extension of the proper tree. If $k \equiv 2$ or $k \equiv 3 \pmod 4$, then apply the operation ß, as in Figure 2.26 (b), and we obtain an extension of the proper tree. In both cases, the proper trees in the A_1- and B-configurations are the same as those in Figure 2.24 (a) and (c). According to remark 5) on the previous page, we can apply the operations α and ß to extend our proper tree to another proper tree. Thus, we can construct a k-gon while preserving the HIST property. Now, to construct another k-gon or a face with a different number of sides, we apply the operations α and ß to the last configuration. Suppose that our construction of a k-gon ends with the operation ß. Then, there is a proper tree, as in Figure 2.27 (a). And Figure 2.27 (b) and (c) show how to apply the operation α or ß to start the construction of a new k-gon, while preserving the HIST property. Similarly, if our construction of a k-gon ends with the operation α, then there is again a proper tree, as in Figure 2.28 (a). And Figure 2.28 (b) and (c) show how to apply the operation α or ß to start the construction of a new k-gon, while preserving the HIST property. Clearly, these applications preserve the HIST property. Therefore, we can construct a 4-valent 3-polytopal graph which has a HIST and which realizes the given p-vector.

Now, we have solutions to the major problems in this chapter. For the remainder of this chapter, we will consider some topics related to the concept of a "cut-through path". First, let us recall the definition of a non-cut-through path. A **non-cut-through path** is a path, all of whose vertices are not cut-through vertices. A plane 4-valent graph is called **non-cut-through Eulerian** if it has a non-cut-through Eulerian circuit. W. T. Tutte proved the following theorem (see W. T. Tutte and C. A. B. Smith [21]), but we will prove it in a different way. Aside from these proofs, there is also a knot theoritical version of the proof of this theorem (see L. H. Kauffman [10]).

Theorem 2.14 (W. T. Tutte) Every plane 4-valent graph G is non-cut-through Eulerian.

Proof Let V be the set of all the vertices of the graph G. Since G is 4-valent, we can find an Eulerian circuit C in G. If all the vertices of G are non-cut-through vertices with respect to the Eulerian circuit C, then G is a non-cut-through Eulerian graph. Otherwise, we can change every cut-through vertex in G to a non-cut-through vertex by changing the Eulerian circuit locally (by using Kotzig's χ transformation, see A. Kotzig[11]). Let us assign a label to each vertex in G. Starting from any vertex u in G, traverse the Eulerian circuit C (in one direction). Now, starting with u, list the vertices consecutively as you walk along C, until you return to the vertex u at the end of the Eulerian circuit. Every vertex of G appears exactly twice in such a list. Suppose v is a cut-through vertex and

$$v, u_1, u_2, \ \dots \ u_k, v$$

is a portion of our list (Figure 2.29). By changing this above partial list to $v, u_k, \ \dots \ u_2, u_1, v$, the vertex v becomes a non-cut-through vertex and we have a new Eulerian circuit C'. Clearly, this transformation does not affect the cut-through (or non-cut-through) status of any vertex other than v. Hence, the number of cut-through vertices in C' is exactly one less than the number in C. Now, apply this transformation to each remaining cut-through vertex. The resulting Eulerian circuit will have no cut-through vertices. This proves the theorem.

Theorem 2.15 For any given p-vector, there is a non-cut-through Eulerian 4-valent 3-polytopal realization.

Proof For a given p-vector, there is a 4-valent 3-polytopal realization (for example, by Theorem 1.11). Since this graph is 4-valent, it is Eulerian. Then, it is non-cut-through Eulerian, by Theorem 2.14.

Let G be a plane 4-valent graph with a cut-through Eulerian circuit (G may not be a 3-polytopal graph) and let V be the set of vertices of G. Suppose that $|V| = n$ and we assign the numbers from 1 to n to the vertices of G. Then, we can construct a sequence of length 2n, consisting of the integers 1,2,...,n, which is constructed by listing the number corresponding to each vertex as it is traversed by the cut-through Eulerian circuit (in one direction). This sequence is called a **Gauss code**. For example,

1 2 9 6 4 1 2 3 6 7 8 9 3 4 5 8 7 5

is the Gauss code of the graph in Figure 2.30. Gauss discovered the following simple fact.

Remark (i) Every symbol appears exactly twice in a Gauss code.

(ii) The number of symbols occurring between the first and the second appearances of the same symbol in a Gauss code is even.

There is a well-known problem, posed by Gauss, concerning the characterization of the Gauss code. And, there are some solutions for this problem. These results include forbidden structures for a Gauss code (see L. Lovasz [12]), an algorithm which checks whether a given code is a Gauss code or not (see P. Rosenstiel [18]), and a generalization of the Gauss code problem for some surfaces other than a sphere (see S. Lin [13]).

By using the simple facts from the previous remark, we obtain another result about cut-through Eulerian graphs.

Theorem 2.16 Let G be a cut-through Eulerian 4-valent graph. Then, there is a finite family of simple circuits $\{C_i\}$ in G such that:

i) C_i and C_j are vertex-disjoint if $i \neq j$.

ii) each circuit in the family is a non-cut-through circuit.

iii) every vertex of G is contained in a circuit.

Proof Let k be the number of vertices of G. To each vertex of G, assign an integer from the set $\{1,2,\ldots,k\}$ such that no two vertices are assigned the same number. And, let the finite sequence

$$n_0 \; n_1 \; \ldots \; n_{2k-2} \; n_{2k-1}$$

be the Gauss code formed by following a cut-through Eulerian circuit in G. We can locate simple circuits in G in the following way. Choose the edge n_0n_1 as the first edge and delete n_0 and n_1 from the above sequence. Since each symbol appears exactly twice in a Gauss code, we can find the same symbol in two different places. So, there exists an $i \neq 1$ such that $n_i = n_1$. Then, choose the edge n_i and n_{i+1} as the next edge and delete n_i and n_{i+1} from the Gauss code. We repeat this process until we finally select an edge with an endpoint n_j such that $n_j = n_0$. This procedure locates a simple circuit which is non-cut-through because of the special selection process employed in choosing each edge. Now, start again with the "reduced" sequence (that is, the sequence that remains after all the aforementioned deletions are made), and repeat the same process to find another non-cut-through simple circuit in G. After a finite number of repetitions of this procedure, the length of the original sequence will be reduced to 0. When this occurs, the set of simple circuits found along the way constitutes the finite family of simple circuits $\{C_i\}$ required in the theorem.

For example. in Figure 2.30, we have two disjoint simple non-cut-through circuits, namely, (1 2 3 4) and (9 6 7 5 8).

Remark If G is a graph of order of *n*, then a family of pairwise disjoint non-cut-through circuits can be found in G in O(n) time of operation.

A plane 4-valent cut-through Eulerian graph can be understood as the result of a sequence of applications of some basic deformations of knot theory, which are called the

Reidemeister moves (see L. H. Kauffman [10], Figure 2.31), to one simple curve. In other words, a cut-through Eulerian graph is, in some sense, just a projection of a circular piece of string. However, there are many plane 4-valent graphs which are the union of several cut-through components. We are going to study graphs of this type in the next chapter.

CHAPTER 3 ARRANGEMENT OF SIMPLE CURVES

3.1 Introduction

In Chapter 2, we studied plane 4-valent graphs whose edges lie on one cut-through curve which intersects itself. In this chapter, we will consider some graphs (polytopal or not polytopal) whose edges lie in the union of a family of cut-through simple curves.

B. Grünbaum generalized the notion of "an arrangement of lines" to "an arrangement of curves" in his book [8], and got many results. According to his definition, **an arrangement of simple curves** is a family of simple curves such that every two simple curves meet at precisely two points (this definition allows that three or more curves may intersect at the same point). We can generalize this notion by allowing that two simple curves meet at n points, where $n \geq 2$ and n can be odd or even. So, in this thesis, **an arrangement of simple curves** $C = \{ C_1, \ldots, C_n \}$ in the Euclidean plane E^2 is a finite family of simple closed curves with the properties:

(i) Every pair of curves has n ($n \geq 2$) points in common, at which they meet each other.

(ii) Exactly two curves meet at each point (or vertex).

As we know, there are exactly three different paths that can be traversed at a 4-valent vertex (see Chapter 2.1). In fact, there is one cut-through path and two non-cut-through paths. Therefore, a vertex in a plane 4-valent graph is either a cut-through vertex or a non-cut-through vertex. We call a cut-through vertex in a plane 4-valent graph an **intersection point**. A non-cut-through vertex in such a graph is called a **kissing point** (or an **osculation point**). For example, the arrangement in Figure 3.1 has two intersection points u and v, and one kissing point w. (All figures are grouped in Appendix A.)

The graph of an arrangement of simple curves is a plane graph, each of whose vertices is either an intersection point or a kissing point, and whose edges are the segments of curves between each pair of adjacent points. Such a graph is a plane 4-valent graph (perhaps, with loops and multiple edges).

In this chapter, we will see some relationships among the number of curves, the number of vertices, and the number of partitions in a curve of an arrangement of simple curves. Moreover, we will consider a special type of arrangement of simple curves and a kind of Eberhard-type problem.

For convenience, we shall assume that all curves in this chapter are simple curves, and that all arrangements in this chapter are those of simple curves. Additionally, we will use the terms points and vertices interchangeably throughout this chapter.

3.2 A realization of an a-vector

Let n be the number of simple curves in an arrangement. Let n_i be the number of pairs of curves that have i intersection points and m_i be the number of curves that have i edges (in other words, the number of curves that have i points). For example, the arrangement in Figure 3.2 of four simple curves has

$$n_0 = 2,\ n_2 = 4,\ m_2 = 1,\ m_4 = 2,\ \text{and}\ m_6 = 1.$$

By using a simple counting argument, we have the following.

Theorem 3.1 (B. Grünbaum [11]) Let G be a graph of an arrangement of n simple curves. Let v(G) and e(G) be the number of vertices and the number of edges of G, respectively. Then,

$$i)\quad \sum_{i\ge 0} n_i = \binom{n}{2}$$

$$ii)\quad \sum_{i\ge 0} i n_i = 2v(G)$$

$$iii)\quad \sum_{i\ge 0} m_i = n$$

$$iv)\quad \sum_{i\ge 0} i m_i = e(G)$$

$$v)\quad 2v(G) = e(G)$$

There are many different kinds of arrangements of simple curves. Among these, a special type of arrangement will now be examined in detail. This arrangement of n simple curves is such that every pair of curves in the arrangement has t intersection points (note that t must be even) and k kissing points. Then, every curve in such an arrangement has the same number of points. Let r be the number of points in a curve (note that every curve also has r edges because it is a simple curve).

The following theorem, due to J. Malkevitch, shows that a relationship exists amongst the numbers n, t, k, and r.

Theorem 3.2 (J. Malkevitch) Suppose that G is a graph of an arrangement of n simple curves such that every pair of curves has exactly t intersection points and k kissing points. If each curve has exactly r vertices, then:

$$n = \frac{r}{t+k} + 1 \qquad (*)$$

Proof From the given condition, we know the following:

$$\begin{aligned} m_r &= n, \\ m_i &= 0 \quad \textit{for all } i \neq r, \\ n_{t+k} &= \frac{n(n-1)}{2}, \\ n_i &= 0 \quad \textit{for all } i \neq t+k. \end{aligned}$$

Combining the equalities ii), iv) and v) in Theorem 3.1, we have the equation (*).

Now, let us define some terminology. An **a-vector** is a 4-tuple (n,r,t,k) of non-negative integers that satisfies the equation (*). An **(n,r,t,k)-arrangement** is a arrangement such that every curve in the arrangement has exactly r edges (or r points) and every pair of curves has exactly t intersection points (t is even), and k kissing points. We call such an arrangement an **equally distributed** arrangement. For example, the arrangement in Figure 3.3 is a (3,6,2,1)-arrangement. In some sense, an (n,r,t,k)-arrangement is a **realization** of an a-vector (n,r,t,k). Let G be the graph of an (n,r,t,k)-

arrangement. Then G is a plane 4-valent graph. If G does not have any faces which are digons, then such an arrangement is called **a digon-free (n,r,t,k)-arrangement**.

Since all the curves in an arrangement are simple, there is no (1,r,t,k)-arrangement such that $t + k > 0$. Similarly, there is no digon-free (2,r,t,k)-arrangement where $t+k > 0$. Thus, whenever we mention a digon-free (n,r,t,k)-arrangement, we assume that $n \geq 3$, unless stated otherwise.

Lemma 3.3 We have the following simple facts:

i) There exists a (2,r,t,k)-arrangement.

ii) There exists an (n,r,2,0)-arrangement for all $n \geq 2$.

iii) There exists a digon-free (n,r,2,0)-arrangement.

Proof For an a-vector (2,r,t,k), arrange two curves in the manner shown in Figure 3.4 (a). Then, we have a (2,r,t,k)-arrangement. If we arrange n curves as shown in Figure 3.4 (b), then we have an (n,r,2,0)-arrangement. For a digon-free (n,r,2,0)-arrangement, arrange (n-1) curves as before and draw one curve that cuts through the 3 digons. Then, we obtain a digon-free arrangement (Figure 3.4 (c)). This completes the proof.

For the sake of convenience, let us define the operations O_1, O_2, O_3, and O_4 (Figure 3.5 (a), (b), (c), and (d)). The operations O_1 and O_3 add two more intersection points, while operation O_4 adds four more intersection points. Operation O_2 adds one more kissing point.

Theorem 3.4 If there is an (n,r,t,k)-arrangement with $t + k > 0$, then there exists an $(n,r',t+2m,k+k')$-arrangement where m and k' are positive integers and $r' = r + (n - 1)\cdot(2m + k')$.

Proof Let G be the graph of an (n,r,t,k)-arrangement such that $t + k > 0$. Since $t + k > 0$, each pair of curves in G has at least one intersection point or kissing point. Let C_i and C_j be two different curves in G and let v be an intersection point or a kissing point (Figure 3.6 (a), (b)). By applying the operation O_1 m times to the neighbor of the point v, we can increase the number of intersection points of the two curves to $t + 2m$. Similarly, by applying the operation O_2 k' times to the neighbor of the point v, we can increase the number of kissing points of the two curves to $k + k'$ that we would like (Figure 3.6). Now, apply this method to every pair of curves, one by one. Then, every curve has r' vertices. In fact, the new graph is an $(n,r',t+2m,k+k')$-arrangement.

Lemma 3.5 There is no $(n,r,0,k)$-arrangement for $n \geq 5$.

Proof We will proceed by induction on k. Suppose that $k = 1$. Then, Figure 3.7 shows the existence of an $(n,r,0,1)$-arrangement, where $n = 2,3$, and 4. Now, suppose that there is an $(n,r,0,1)$-arrangement for some $n \geq 5$. Since $t = 0$ and $k = 1$, each curve has only $(n - 1)$ kissing points. Let C be a curve in this arrangement. If a curve resides inside the curve C (or outside the curve C, respectively), then all the other curves must reside inside C (or outside C, respectively). Otherwise, the curves inside C can not kiss the curves that are outside of C without creating an intersection point, and this violates the condition $t = 0$.

Now, let us construct a new graph G' from G. Since every curve in an (n,r,0,1)-arrangement is simple, each curve separates the plane into two regions. Assign a vertex to each curve by adding and placing a vertex in the region that does not contain any other curves. And, join two vertices if the corresponding curves meet at a kissing point (Figure 3.8). Then, the new graph G' is isomorphic to K_n, and we know that K_n is not planar if $n \geq 5$. Therefore, there is no (n,r,0,1)-arrangement for all $n \geq 5$.

Suppose that our assertion is true for all $k < m$ and that there is an (n,r,0,m)-arrangement. Then, we can construct an (n,r,0,m-1)-arrangement by changing two adjacent kissing points of each pair of curves to one kissing point. This contradicts our assumption. Therefore, there is no (n,r,0,k)-arrangement if $n \geq 5$.

Theorem 3.6 For any a-vector (n,r,t,k), there is an (n,r,t,k)-arrangement, except for the case $n \geq 5$ and $t = 0$.

Proof If $t = 0$, the result is clear by Lemma 3.5. Otherwise, there is an (n,r',2,0)-arrangement by Lemma 3.3. Apply the operation O_1 $(t - 2)/2$ times and the operation O_2 k times to every pair of curves in the manner described in Theorem 3.4. The final arrangement is an (n,r,t,k)-arrangement.

The arrangements described in Theorem 3.6 may not be digon-free arrangements. However, a digon-free arrangement is more desirable because it can be a 3-polytopal graph. Hence, let us consider the construction of digon-free arrangements.

Figure 3.9 (a) shows a (2,r',t,0)-arrangement. If a pair of curves is arranged in this way, then we say that this pair of curves forms **a cycle of lenses**.

Theorem 3.7 There exists a digon-free (3,r,t,0)-arrangement.

Proof A pair of curves that forms a cycle of lenses is a (2,r',t,0)-arrangement. Now, draw the third curve by alternately cutting the digons in the (2,r',t,0)-arrangement to change them to triangles (Figure 3.9 (b)). The final arrangement is a digon-free (3,r,t,0)-arrangement.

Lemma 3.8 Suppose that there exists a digon-free (n,r,t,0)-arrangement. Then, there is a digon-free (n+1,r',t,0)-arrangement, where r' = r + t.

Proof Let G be the graph of a digon-free (n,r,t,0)-arrangement. Choose a simple curve C_n in G. Then, we can draw a new simple curve C_{n+1} such that C_{n+1} is parallel to the curve C_n and close enough not to have any points between C_n and C_{n+1} (Figure 3.10 (a)). Since C_n intersects every curve C_i ($i \leq n - 1$) t times, C_{n+1} also intersects all of these curves t times (except for the curve C_n). Now, apply the operation O_3 along the other edges that cut through the curve C_n t/2 times. The final graph is an (n+1,r',t,0)-arrangement.

Figure 3.10 (b) shows an example with t = 2.

Lemma 3.9 Let G be the graph of an (n,r,t,k)-arrangement. If G contains the shape shown in Figure 3.11 (a), then we can transform G into the graph of an (n,r',t+2m,k)-arrangement.

Proof By transforming the shape in Figure 3.11 (a) to the configuration in Figure 3.11 (b) by using O_4 operation, we can change an (n,r,t,k)-arrangement to an (n,r',t+2,k)-arrangement. Since the new shape in Figure 3.11 (b) contains the shape in Figure 3.11 (a) again, we can apply the same transformation over and over. Therefore, we obtain an (n,r',t+2m,k)-arrangement by repeating this process m times.

Remark Lemma 3.8 provides a method for increasing the number of curves, n, while keeping the value of t fixed. On the other hand, Lemma 3.9 shows how to increase the value of t while keeping the number of curves, n, fixed.

Since the (n,r,2,0)-arrangement shown in Figure 3.4 (c) contains the special structure of Figure 3.11 (a), we have the following.

Theorem 3.10 There exists a digon-free (n,r,t,0)-arrangement for $n \geq 3$.

Proof Since the digon-free (n,r,2,0)-arrangement in Figure 3.4 (c) contains the configuration of Figure 3.11 (a), we can construct a digon-free (n,r,t,0)-arrangement, by Lemma 3.8.

To generalize this theorem for an arbitrary number of kissing points k, we need to develop some methods for altering the number of kissing points in an arrangement. In Figure 3.12 indicates some methods for increasing the value of k. Recall that k is the number of kissing points of every pair of curves. Thus, when we change k, we have to change the number of kissing points for all pairs of curves. Combining Lemma 3.5 and Theorem 3.10, we have the following result.

Theorem 3.11 There exists a digon-free (n,r,t,k)-arrangement except for the case $n \geq 5$ and $t = 0$.

Proof We can exclude the case $n \geq 5$ and $t = 0$, by Lemma 3.5. Furthermore, Theorem 3.10 provides for the existence of an (n,r,t,0)-arrangement. By applying the methods illustrated in Figure 3.12 properly, we can increase the value of k as much as we'd like. Thus, we obtain a digon-free (n,r,t,k)-arrangement.

For example, the graph in Figure 3.13 is a digon-free (5,20,4,1)-arrangement.

Now, let us consider a kind of converse of the above fact: for a given number of points (vertices), how many different (n,r,t,k)-arrangements (equally distributed arrangements) are possible?

Let G be the graph of an (n,r,t,k)-arrangement. Then, the number of edges of G, denoted by e(G), is rn, since each curve has r edges. Moreover, we know by Theorem 1.4 that $2 \cdot e(G) = 4 \cdot v(G)$, where $v(G)$ is the number of vertices of G. Therefore, $v(G) = rn/2$.

Let us consider the following relationship between r and n: From Theorem 3.2,

$$r = (n - 1)(t + k).$$

Hence,

$$\begin{aligned} r - n &= (n - 1)(t + k) - n \\ &= n(t + k) - (t + k) - n \\ &= (n - 1)(t + k - 1) - 1. \end{aligned}$$

Since $n \geq 2$ and $t + k \geq 2$, $r - n \geq 0$.

Theorem 3.12 Let v be a given number of points, where v is greater than two. Then, we have the following:

i) If v is a prime number, then a $(2,v,t,k)$-arrangement is the only arrangement possible having v points.

ii) An $(n,2v/n,t,k)$-arrangement having v points is possible if n is divisor of 2v such that

$$n \leq \sqrt{2v} \quad \textit{and} \quad (n-1) \mid \frac{2v}{n}$$

iii) Otherwise, there is no equally distributed arrangement with v points.

Proof We know that $rn = 2v$. If v is a prime number, then the only possible values for r and n are $r = v$, $n = 2$ because $r - n \geq 0$. This proves fact i).

Suppose that v is not a prime and d is a divisor of 2v such that $(d - 1)$ divides $2v/d$. Let $n = d$, $r = 2v/n$ and $t + k =$

$2v/(n(n-1))$. Then (n,r,t,k) is an a-vector and there exists an (n,r,t,k)-arrangement, by Theorem 3.10. This proves fact ii).

Fact iii) is obvious.

For example, if $v = 24$, then $2v = 48 = 2(24) = 3(16) = 4(12) = 6(8)$. In case $n = 2,3,$ or 4, $(n - 1)$ divides $48/n$, but, if $n = 6$, then $(n - 1) = 5$ does not divide $48/n$. Hence, there exist $(2,24,t_1,k_1)$-, $(3,16,t_2,k_2)$-, and $(4,12,t_3,k_3)$-arrangements, where $t_1 + k_1 = 24$, $t_2 + k_2 = 8$, and $t_3 + k_3 = 4$. On the otherhand, there is no $(6,r,t,k)$-arrangement with 24 points.

The three arrangements shown in Figure 3.13 are examples of a $(2,24,12,12)$-arrangement, a digon-free $(3,16,6,2)$-arrangement, and a digon-free $(4,12,4,0)$-arrangement.

3.3 A p-vector and a digon-free (n,r,t,k)-arrangement

Since the graph of a digon-free (n,r,t,k)-arrangement is a plane 4-valent simple graph, we may think about the following questions:

i) For a given p-vector $(p_3,p_5,p_6,p_7,\ldots,)$, is there a digon-free (n,r,t,k)-arrangement that realizes the given p-vector?

ii) What are the possible face structures of a digon-free (n,r,t,k)-arrangement?

To answer these questions, let's begin by considering a relationship between a p-vector $(p_3,p_5,p_6,\ldots)$ and an a-vector (n,r,t,k).

Theorem 3.13 Let $(p_3, p_4, p_5, p_6, \ldots)$ be a face vector of a digon-free (n,r,t,k)-arrangement. Then,

$$\sum_{i\geq 4} (i-3)p_i = \frac{n(n-1)(t+k)}{2} - 6$$

Proof Let v(G) and f(G) denote the number of vertices in G and the number of faces in G, respectively. We know that $f(G) = v(G) + 2$ where $v(G) = nr/2$, and $f(G) = \sum_i p_i$. Combining these equalities with ii) from Theorem 1.9, we obtain the required equality.

A plane 4-valent graph G is called a **k-antiprism** if it has only 2 k-gons and 2k triangles. In fact, there is only one k-antiprism for each integer k up to isomorphism. The graph in Figure 3.14 is the 6-antiprism.

Remark If we allow for a certain number of 4-gons, then there exists an (n,r,2,0)-arrangement which has a face vector such that $p_3 = 2n$, $p_n = 2$, $p_4 > 0$ and $p_i = 0$ for all $i \neq 3,4$, and n. Moreover, p_4 is always even, by Theorem 3.13. For example, Figure 3.15 shows a digon-free (5,8,2,0)-arrangement such that $p_3 = 10$, $p_5 = 2$, $p_4 = 10 > 0$ and $p_i = 0$ for all $i \neq$ 3,4, and 5. But, if there are no 4-gons (i.e., $p_4 = 0$), then we have the following interesting fact.

Theorem 3.14 Let $(p_3, p_5, p_6, p_7, \ldots)$ be a p-vector and let m be a positive integer such that $m \geq 4$, $p_m = 2$, $p_3 = 2m$, and p_i

= 0 for all $i \neq m$. Then, there exists a digon-free (3,r,t,0)-arrangement that realizes the given p-vector only when $m \equiv 0 \pmod 3$. In this case, $r = 4m/3$ and $t = 2m/3$.

Proof The given p-vector is the p-vector of the m-antiprism. Suppose that there is a digon-free (n,r,t,k)-arrangement whose graph is isomorphic to the m-antiprism. Then $n \geq 3$ because there is no digon-free (2,r,t,k)-arrangement.

<u>Claim</u>: There is at least one simple curve that contains one m-gon inside and another one outside of that curve. Otherwise, every curve must be of the form shown in Figure 3.16 by the thick line. Thus, the edges inside this curve cannot be covered by any other curve. This implies our claim. From relation v) of Theorem 3.1 and Euler's formula, we know that $f = (nr/2) + 2$. On the other hand, $f = p_3 + p_m = 2m + 2$. Therefore, we have the equality $nr = 4m$. Since there is a simple curve that separates the two m-gons, the number of edges r in this curve is greater than m. This means that $n < 4$ because this arrangement is equally distributed. Therefore, $n = 3$. From this fact, we know that only a (3,r,t,k)-arrangement can be the same as the m-antiprism. Moreover, we have the equality $3r = 4m$. Since 3 and 4 are relatively prime, 3 must divide m. Thus, $m \equiv 0 \pmod 3$. In fact, such an m-antiprism is a (3,r,t,0)-arrangement. From this fact, we see that $r = 4m/3$ and $t = 2m/3$, $r \equiv 0 \pmod 4$.

Remark For an m-antiprism such that $m \equiv 1$ or $m \equiv 2 \pmod 3$, there is no (n,r,t,k)-arrangement that is the same as m-antiprism, by Theorem 3.14. However, these m-antiprisms can be a union of 3 curves that have almost the same number of edges. If $m \equiv 1 \pmod 3$, then an m-antiprism can be covered by 3 simple curves such that two of them are of length $[4m/3]$ and one of them is of length $[4m/3]+1$. If $m \equiv 2 \pmod 3$, then an m-antiprism can be covered by 3 simple curves such that two of them are of length $[4m/3]+1$ and one of them is of length $[4m/3]$, but the curves involved have different t values and k values. For example, the 7-antiprism is covered by two simple circuits with 8 intersection points and 1 kissing point, and by one simple circuit with 8 intersection points and two kissing points (Figure 3.17 (a)). The graph in Figure 3.17 (b) shows the 8-antiprism.

Theorem 2.7 states that every multi-3-gon graph can be covered by a number of simple curves and Theorem 3.14 says that all the m-antiprisms such that $m \equiv 0 \pmod 3$ can be thought of as (3,2t,t,0)-arrangements. However, all (3,r,t,0)-arrangements are not multi-3-gon graphs. Figure 3.18 shows a (3,2t,t,0)-arrangement that is not a multi-3-gon graph.

Now, let us think about a kind of converse of Theorem 2.7, that is, is there any (n,r,t,k)-arrangement which produces a multi-3-gon graph? For (3,2t,t,0)-arrangements, we have the following sufficient conditions.

Theorem 3.15 Let G be the graph of a digon-free (3,2t,t,0)-arrangement. Then, the following are true:

i) If every curve in the graph G cuts through the others in an alternating manner, then G is a multi-3-gon graph.

ii) If G contains a cycle of lenses, then G is a multi-3-gon graph.

Proof i) Let F be a face (in the graph G) with k sides and let $e_1, e_2, \ldots, e_k$ be the sides of the face F (Figure 3.19). Let us call the three curves in the figure C_1, C_2, and C_3, respectively. Suppose that e_1 is a portion of the curve C_1. Since every curve cuts through the other curves alternately, the two adjacent sides, e_2 and e_k, are not contained in the same curve. Without loss of generality, let the side e_2 be a portion of the curve C_2. Then, e_k is a portion of the curve C_3. Similarly, we know that the side e_3 is a portion of the curve C_3, and the side e_4 is a portion of the curve C_1, etc. This implies that two sides e_i and e_j are on the same curve if $i \equiv j \pmod 3$. Thus, $k \equiv 3 \pmod 3$; that is, F is a multi-3-gon. Hence, G is a multi-3-gon graph.

ii) A cycle of lenses in G is a (2,t,t,0)-arrangement, and the number of digons is t. To make the cycle of lenses a digon-free (3,2t,t,0)-arrangement, we have to draw the third curves to cut-through each digon just once because every pair of curves has t intersection points and we have only t digons (if we cut through a digon more than once, some digons remain digons). Moreover, there are no more vertices other than those on the cycle of lenses. This means that we have to connect a cut-through edge of one digon to that of another digon that is separated from the first digon by an even number of digons (Figure 3.20). Then every curve cuts through the other curves in an alternating manner. Therefore, G is a multi-3-gon graph by part i).

Remark In Theorem 3.15, we saw that if a (3,2t,t,0)-arrangement contains a cycle of lenses, then every curve cuts through the other curves in an alternating way. However, we do not know whether the converse is true.

It can be easily checked that a (3,2t,t,0)-arrangement that has a cycle of digons is Hamiltonian. The thick line in Figure 3.21 shows a Hamiltonian circuit in a (3,24,12,0)-arrangement. We will use this fact in the proof of the following theorem.

Theorem 3.16 Let G be the graph of a digon-free (3,2t,t,0)-arrangement, where t is a positive even integer. Suppose that two curves form a cycle of lenses and that G has only k-gons and triangles. Then, the number of k-gons is even. Moreover, the k-gons are divided equally into two parts, those inside and those outside the cycle of lenses.

Proof Since two curves form a cycle of lenses, G has a subgraph as displayed in Figure 3.21. Since G is Hamiltonian, there is a Hamiltonian circuit. Let the thick line circuit in Figure 3.21 be the Hamiltonian circuit. Let

F^i and F^o denote the inside region and outside region of the given Hamiltonian circuit, respectively. Let Δ^i and Δ^o be the number of triangles in the regions F^i and F^o, respectively. Similarly, let us denote the number of k-gons in the regions F^i and F^o by $p_k^{\,i}$ and $p_k^{\,o}$, respectively. By the previous theorem, we know that $k \equiv 0 \pmod 3$. By Grinberg's Theorem (Theorem 1.11), we have

$$(3 - 2)(\Delta^i - \Delta^o) + (k - 2)(p_k^{\,i} - p_k^{\,o}) = 0 \qquad (1)$$

Let G^i be the subgraph whose edges are inside the cycle of lenses including the inside edges of the cycle of lenses, (denoted by thick line in Figure 3.22). Similarly, we can define the subgraph G^o (denoted by the dotted line in Figure 3.22). For the subgraph G^i, the following are true where d_i is the number of vertices of degree i of G^i (cf. Theorem 1.4):

$$2\cdot e(G^i) = \textstyle\sum_i i\cdot d_i = 2\cdot t + 3\cdot t = 5\cdot t \qquad (2)$$

$$2\cdot e(G^i) = \textstyle\sum_i i\cdot p_i = 3\cdot(\Delta^i - t) + k\cdot p_k^{\,i} + 2t \qquad (3)$$

Thus, we have the following equality from (2) and (3),

$$3\cdot\Delta^i + k\cdot p_k^{\,i} = 6t. \qquad (4)$$

Since k is a multiple of 3, there is an integer m such that k = 3m ($m \geq 2$). So, equation (4) can be changed as following:

$$\Delta^i + m\cdot p_{3m}^{\,i} = 2t \qquad (5)$$

By applying same argument to the subgraph G^o, we can get

$$\Delta^o + m\cdot p_{3m}^{\,o} = 2t \qquad (6)$$

Subtract (6) from (5), then we have the following:

$$(\Delta^i - \Delta^o) + m\cdot(p_{3m}^{\,i} - p_{3m}^{\,o}) = 0 \qquad (7)$$

Comparing (1) and (7), we obtain

$$(3m - 2)(p_{3m}^{\,i} - p_{3m}^{\,o}) - m\cdot(p_{3m}^{\,i} - p_{3m}^{\,o}) = 0.$$

$$2(m - 1)(p_{3m}^{i} - p_{3m}^{o}) = 0.$$

Since $m - 1 > 0$, we now have

$$p_{3m}^{i} = p_{3m}^{o}.$$

This proves the theorem.

We have seen some relationships between a p-vector and an (n,r,t,k)-arrangement, especially an arrangement of 3 curves. Now, let us consider an Eberhard-type question for an arrangement of simple curves. That is, for a given p-vector $(p_3, p_5, p_6, p_7, \ldots)$, is there a digon-free (n,r,t,k)-arrangement that realizes the given p-vector? And, if not, under what conditions is the given p-vector realizable or not realizable?

There are many theorems that assure the non-existence of certain plane graphs that have a uniform face and vertex structure (see J. Malkevitch[15]). Since the graphs of digon-free arrangements are plane graphs with a uniform vertex structure, these theorems are also valid for the graphs of suitable arrangements of curves. In addition to these results, we will see more facts about arrangements of curves.

Theorem 3.17 (W. Meyer [14]) Let G be the graph of a digon-free (n,r,2,0)-arrangement. Let P(F) be the number of sides of a face F of G. Then, $P(F) \leq 2n - 4$.

As a corollary, we have the following fact.

Corollary 3.18 A face of the graph of a digon-free

(4,6,2,0)-arrangement is either a triangle or a 4-gon.

Proof By Theorem 3.17, the largest face of the graph of any digon-free (4,6,2,0)-arrangement is a 4-gon. In fact, there are 8 triangles and six 4-gons in such an arrangement.

Lemma 3.19 For a given p-vector $(p_3, p_5, p_6, p_7, \ldots)$ such that $p_3 = 8$ and $p_i = 0$ for all $i \neq 3$, there is only one digon-free (3,4,2,0)-arrangement (up to isomorphism) that realizes the given p-vector if we do not allow the addition of 4-gons.

Proof Let G be the graph of a digon-free (n,r,t,k)-arrangement. Clearly, G is a plane 4-valent graph. Thus,

$$2 \cdot e = 4 \cdot v, \text{ so, } e = 2 \cdot v,$$

where e and v are the number of edges and the number of vertices of G, respectively. Moreover, $e = n \cdot r$ because every curve has n edges. Combining these equations with Euler's formula, we have that the number of faces f(G) in the graph G is $2 + nr/2$. Since we have only 8 triangles, $nr = 12$. Hence, $n = 3$, $r = 4$, and $t + k = 2$ by Theorem 3.12. So, only two a-vectors, (3,4,2,0) and (3,4,0,2), are possible. But, any (3,4,0,2)-arrangement is not digon-free because the third curve must be drawn in the region A or B, as illustrated in Figure 3.23. Thus, one of the regions remains a digon. Therefore, the (3,4,2,0)-arrangement is the only digon-free arrangement that has only 8 triangles.

Now, we are ready to state a theorem which shows that an Eberhard-type theorem is not valid in the arrangements of simple curves.

Theorem 3.20 For a given p-vector $(p_3, p_5, p_6, p_7, \ldots)$ such that $p_k = 1$ for an integer $k \geq 4$, $p_i = 0$ for all $i \neq 3, k$,

and $p_3 = 8 + (k - 4) = k + 4$, there is no digon-free $(n,r,2,0)$-arrangement that realizes the given p-vector.

Proof Use induction on n. Let G be the graph of a digon-free $(n,r,2,0)$-arrangement. If $n = 3$, then all the faces of G are triangles, by Lemma 3.19. If $n = 4$, then G has eight triangles and six 4-gons, by Corollary 3.18. Thus, our assertion is true in case $n = 3$ or $n = 4$.

Assume that our theorem is true for all digon-free $(n,r,2,0)$-arrangements where $n < m$. That is, no digon-free $(n,r,2,0)$-arrangement has only one k-gon $(k > 4)$, with some triangles and possibly some 4-gons for all $n < m$.

Now, suppose that there exists a graph of a digon-free $(m,r,2,0)$-arrangement that has exactly one k-gon $(k \geq 5)$, some triangles, and some 4-gons as its faces. By Theorem 1.3, there exists a graph G′, isomorphic to G, whose infinite face is the k-gon and all of whose other faces are triangles and 4-gons (Figure 3.24 (a)).

Claim 1) No two sides of the infinite face are on the same curve.

Otherwise, there is a curve that contains at least two edges of the infinite face. Suppose that C_1 is such a curve that contains at least two sides of the infinite face. Let e_1 be one of the edges of the infinite face contained on the curve C_1. Assign the labels $e_2, e_3, \ldots, e_k$ to all the other edges of the infinite face (Figure 3.24 (b)). Let e_i ($i \neq 2$) and e_j ($j \neq k$) be two edges of the curve C_1 such that no edges between e_1 and e_i, and no edges between e_1 and e_j can be an edge of the curve C_1 (note that i and j may be equal). Since $t = 2$, two edges e_p and e_q where $2 \leq p < i$, $j < q \leq k$, cannot lie on the same curve (otherwise, such a curve would intersect C_1 at least 4 times). Let C_2 (C_3, respectively) be a curve that contains the edge e_2 (e_k, respectively). Then, the two curves C_2 and C_3 must meet each other inside the curve C_1 (Figure 3.24 (b)) because $t = 2$. Since the two curves C_2 and C_3 create a digon, there must be a curve that cuts through such a digon to make this arrangement digon-free, and this curve must cut through the edges v_1v_2 and v_3v_4 to avoid making p-gons, where $p > 4$ (dotted line in Figure 3.24 (b)). Then, the curve C_1 and this curve have at least 4 intersection points since it enters the regions to the left and right of v_1v_2 and v_3v_4, respectively. However, this violates the condition that $t = 2$. In other words, every edge of the infinite face must be contained on a different curve; that is, $m \geq k \geq 5$.

Claim 2) There is no configuration of three triangles like that in Figure 3.24 (c).

Assume the contrary, i.e., assume that there is such a configuration of three triangles as in Figure 3.24 (c). When $m \geq 4$, r cannot be less than 6 because $r = (m-1)t$. Thus, the face F that is adjacent to two triangles cannot be a triangle. In fact, if the face F were a triangle, then $r = 4 < 8$ (Figure 3.24 (d)). Moreover, the face F cannot be a 4-gon because every curve is a simple curve (Figure 3.24 (e)). Hence, the face F must be the infinite face of G. This implies that the infinite face contains two edges e and e′ that are on the same curve (Figure 3.24 (c)), and this contradicts Claim 1). Thus, Claim 2) is true.

Claim 3) There is at least one curve C such that a digon-free arrangement can be preserved as a digon-free arrangement after removing such a curve C from the given arrangement. Suppose that removing any curve C in this arrangement creates a digon. This means that every curve cuts through at least one digon (Figure 3.24 (e)). Let us call the configuration in Figure 3.24 (d) a simple lens of the curve C. By Claim 2), we can find at least k simple lenses, one for each curve, such that they are edge-disjoint from each other. Therefore, we can assign two triangles to each curve, so the number of triangles in this digon-free (m,r,2,0)-arrangement is not less than 2k. On the other hand, according to equation ii) from Theorem 1.8, the number of triangles in this arrangement is $8 + (k - 4) = k + 4$. Therefore, $k + 4 \geq 2k$. But, this is absurd because $k \geq 5$. Thus, Claim 3 is true.

Now, we are ready to prove the theorem. By Claim 3), we can find a curve C such that the (m-1,r′,2,0)-arrangement, obtained from the (m,r,2,0)-arrangement by removing the curve C, is still a digon-free arrangement. And, the infinite face becomes a k-gon or a (k-1)-gon, depending on whether the face F' in Figure 3.24 (f) and (g) is a 3-gon or a 4-gon. Hence, we have a digon-free (m-1,r',2,0)-arrangement with only one k-gon or one (k-1)-gon, as well as some triangles and some 4-gons as its faces. This violates our induction hypothesis. Therefore, there is no digon-free (n,r,2,0)-arrangement that realizes the given p-vector.

It seems likely that the above theorem can be extended to (n,r,t,k)-arrangements. This leads us to the following conjecture.

Conjecture For a given p-vector $(p_3,p_5,p_6,p_7,\ldots)$ such that $p_k = 1$ for an integer $k \geq 4$, $p_i = 0$ for all $i \neq k$ and 3, and $p_3 = 8 + (k - 4) = k + 4$, there is no digon-free (n,r,t,k)-arrangement with $t \geq 2$, which realizes the given p-vector.

We have already seen one type of p-vector which is not be realizable by a digon-free (n,r,2,0)-arrangement.

Now, let us study some p-vectors which are realizable as face structures of digon-free (n,r,t,k)-arrangements. For this discussion, let us recall the operation O_2 in 3.2, which allows us to create one kissing point for each pair of curves $\{C_1,C_2\}$ and $\{C_1,C_3\}$ (Figure 3.12 (a)). Let us call such an operation a Z_1 operation. In general, let Z_i be the repetition of such an operation i times to create i kissing points for each pair of curves $\{C_1,C_2\}$ and $\{C_1,C_3\}$. Among these, the Z_3 operation is the most interesting since it can be interpreted in different ways. That is, the Z_3 operation can be thought of as a procedure for increasing the number of intersection points of each pair of curves by 2, i.e., from t to t + 2. Figure 3.25 (a) shows the Z_3 operation. The numbers of sides of the faces f_2 and f_5 are increased by 3, while the numbers of sides of the other faces f_1, f_3, f_4, and f_6 are unchanged. Similarly, a Z_{3m} operation increases the number of intersection points of each pair of curves by 2m and the number of sides of the faces f_2 and f_5 by 3m. Thus, we have the following result.

Theorem 3.22 Let $(p_3, p_5, p_6, \ldots)$ be a p-vector such that

$$p_i \equiv 0 \pmod 2 \quad \text{if } i \equiv 0 \pmod 3,$$
$$p_j = 0 \quad \text{otherwise.}$$

Then, there exists a digon-free (3,r,t,0)-arrangement such that

$$t = 2 + \Sigma_{i\geq 2} (i - 1) p_{3i}$$

Proof By Lemma 3.19, there is only one digon-free (3,4,2,0)-arrangement whose faces are all triangles. Let m be the greatest integer such that $p_{3m} \neq 0$. Then, apply a $Z_{3(m-1)}$ operation to change two triangles to two 3m-gons. Clearly, t is increased to t + 2(m-1). Since this operation creates a band of triangles, we can apply the same operation to another two triangles to produce two 3m-gons. Figure 3.25 (b) shows how to construct four 6-gons. Repeat the same operation until we have p_{3m} 3m-gons. This repetition increases the t value to $t + (m - 1)p_{3m}$. Find another integer m' such that $m' < m$ and $p_{3m'} \neq 0$, and follow the same procedure to produce $p_{3m'}$ $3m'$-gons. This proves the theorem.

Theorem 3.22 says that if the number of each type of multi 3-gon is even within a graph G, then G is the union of 3 simple curves. But, applying a Z_i operation in a different manner, we can create a multi-3-gon graph that has an odd number of 3m-gons, for some m (cf. Figure 3.25 (c)). However, no characterization of the associated p-vector has been obtainable.

An (n,r,t,k)-arrangement of n curves is, in some sense, a restricted arrangement because every pair of curves has exactly t intersection points and k kissing points. Similar to B. Grünbaum's definition of a "weak arrangement of curves", we may generalize the notion of an "equally distributed arrangement of curves" as follows: every pair of curves has at most t intersection points and at most k kissing points. Then, each curve may have a different number of edges (or vertices), i.e., each curve may have associated with it a different r value. From this generalization, we obtain a new type of arrangement.

An (n,-,t,k)-arrangement is an arrangement of n simple curves such that every pair of curves has at most t intersection points and at most k kissing points. This definition allows that some pair or pairs of curves do not meet each other. **A digon-free (n,-,t,k)-arrangement** is an (n,-,t,k)-arrangement that has no digon in its face structure. By using these new definitions, we can obtain further results. The tools we will make use of will again be the Z_i operations mentioned previously.

Remark Figure 3.26 shows the outcomes of applications of Z_i, (i = 2, 3, and 4) operations. Clearly, we can apply another Z_i operation to the resulting structure (for example, to the shaded area).

Lemma 3.23 Let $(p_3, p_4, p_5, p_6, p_7, \ldots)$ be a p-vector such that all the p_i are even integers (note that the value of p_4 is specified). Then, there exists a digon-free (3,-,t,k)-arrangement that realizes the given p-vector.

Proof Start from the digon-free (3,4,2,0)-arrangement (i.e., the octahedron). Let m be the greatest integer such that $p_m \neq 0$. By applying a Z_{m-3} operation to any edge, we obtain two m-gons and a new arrangement of three curves that has another band of triangles. We can apply a Z_{m-3} operation to one of the triangles. Repeat the same operation until we have p_m m-gons. Now, find another integer m′ such that $p_{m'}$ is not zero, and follow the same procedure. By repeating this process, we can construct a (3,-,t,k)-arrangement that realizes the given p-vector.

For example, if $p_5 = 4$ and $p_7 = 2$, then we can produce four 5-gons by applying a Z_2 operation twice, and two 7-gons by applying a Z_4 operation twice (Figure 3.27 (a)). Figure 3.27 (b) shows a transformation of the octahedron that has four 5-gons and two 7-gons.

Before stating another lemma, let us look at some variations of a Z_{3m} operation for $m \geq 2$. Suppose that we apply a Z_{3m} operation to an edge which connects two triangles. Then, we obtain two (3m+3)-gons as a result. Since a Z_{3m} operation does not create any kissing points, we can draw a curve C that is parallel to the curve C_1 almost everywhere, except where C_1 has been modified by the application of a Z_{3m} operation (see Figure 3.28 (a)). A small displacement of the curve C changes two (3m+3)-gons to: i) one (3m+3)-gon and one (3m+4)-gon, ii) two (3m+4)-gons, iii) one (3m+4)-gon and one (3m+5)-gon, iv) two (3m+5)-gons, or v) one (3m+5)-gon and one (3m+6)-gon. All of these changes are displayed in Figure 3.28. Moreover, each of these changes contains a portion which is suitable another good place for further applications of a Z_k operation for all $k \geq 2$ (see the shaded area in Figure 3.28). From this observation, we have the following lemma.

Lemma 3.24 Let $(p_3, p_5, p_6, p_7, \ldots)$ be a p-vector such that $p_3 > 0$, $p_k \geq 0$ and $p_{k+1} \geq 0$ for an integer $k \geq 9$, $p_j = 0$ for all $j \neq 3$, k, and k+1, and $(p_k + p_{k+1})$ is even. Then there is a digon-free (n,-,t,0)-arrangement that realizes the given p-vector.

Proof We know that there exists a digon-free (3,4,2,0)-arrangement whose faces are all triangles (i.e., the octahedron). Let $k = 3m + i$, where $i = 0, 1,$ and 2.

Case 1) Both p_k and p_{k+1} are even.

Apply a $Z_{3(m-1)}$ operation to one edge in the graph of the digon-free (3,4,2,0)-arrangement. Then we obtain two 3m-gons. We can change these two 3m-gons to two (3m+i)-gons by using the method described in Figure 3.28. That is, we can construct two k-gons or two (k+1)-gons. By repeating this process, we produce p_k k-gons and p_{k+1} (k+1)-gons.

Case 2) Both p_k and p_{k+1} are odd.

It suffices to show that we can construct one k-gon and one (k+1)-gon. Apply a $Z_{3(m-1)}$ operation to one edge in the graph of the digon-free (3,4,2,0)-arrangement. Then, we obtain two 3m-gons. By using the variation i), iii), or v) described above, we can construct one k-gon and one (k+1)-gon. This completes the proof.

Let $S = (s_1, s_2, s_3, s_4, s_5, \ldots, s_n)$ be a finite sequence. A **positive section** of the sequence S is a portion $[s_i, s_{i+1}, \ldots, s_{i+j}]$ of S such that $s_{i-1} = s_{i+j+1} = 0$ and $s_k > 0$ for all $k = i, i+1, \ldots, i+j$. If $i = 1$ or $i+j = n$, then we need only one of the conditions $s_{i+j+1} = 0$ or $s_{i-1} = 0$. A positive section is **even** if the sum of the elements in that section is even. For example, the sequence

$$(2,4,3,0,0,1,5,6,4,8,4,0,0,0,0,1,2,3)$$

has the three positive sections [2,4,3], [1,5,6,4,8,4], and [1,2,3]. Among these, the two positive sections, [1,5,6,4,8,4] and [1,2,3] are even.

Theorem 3.25 Let $(p_3, p_5, p_6, p_7, \ldots)$ be a p-vector such that

i) p_i is even for $i = 5,6,7$, and 8;

ii) every positive section of the subsequence $(p_9, p_{10}, p_{11}, \ldots)$ is even.

Then, there is a digon-free (n,-,t,k)-arrangement that realizes the given p-vector.

Proof First, let us look at the subsequence $(p_9, p_{10}, p_{11}, p_{12}, \ldots)$. Suppose that we have a digon-free (n,-,t,k)-arrangement which realizes this subsequence with a proper number of triangles. Then, we can realize the given p-vector, by Lemma 3.23. Thus, it suffices to show that there exists a digon-free (n,-,t,k)-arrangement that realizes the above-mentioned subsequence. Since, by hypothesis, every positive section of this subsequence is even, Lemma 3.24 indicates the required method of construction. Let us demonstrate this by way of an example. Let $p_9 = 2$, $p_{10} = 1$, $p_{11} = 4$, $p_{12} = 2$, $p_{13} = 1$ and $p_{14} = 0$. Then, $[p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}]$ is an even positive section. Now, we can construct the corresponding faces, step by step. i) Construct two 9-gons, ii) construct one 10-gon and one 11-gon, iii) construct two 11-gons, iv) construct one 11-gon and one 12-gon, v) construct one 12-gon and one 13-gon. Apply the same method to all the even positive sections to construct a digon-free (n,-,t,0)-arrangement that realizes $(p_9, p_{10}, p_{11}, p_{12}, \ldots)$. Finally, construct the required 5-,6-,7- and 8-gons. Therefore, there exists a digon-free (n,-,t,k)-arrangement which realizes the given p-vector.

The graph in Figure 3.29 is a weak arrangement which realizes the positive section $[p_{10}, p_{11}, p_{12}]$ with $p_{10} = 1$, $p_{11} = 2$, and $p_{12} = 1$.

Although there are many realizable p-vectors that are not contained in the category described in Theorem 3.25, it is not easy to classify them. So, the classification of realizable p-vectors remains as an open problem. Let us consider some questions related to the above theorems.

1) Can we extend Theorem 3.20 to a digon-free (n,-,t,k)-arrangement?

2) Can any p-gon, $p \geq 3$, be a face of a digon-free (3,r,t,0)-arrangement? If the answer is no, then which p-gons are not achieved by a digon-free (3,2t,t,0)-arrangement?

3) List all forms of a realizable p-vector as a digon-free (n,r,t,k)-arrangement or a digon-free (n,-,t,k)-arrangement (preferably, with k = 0).

4) Find the precise relationships between cut-through paths and left-right paths.

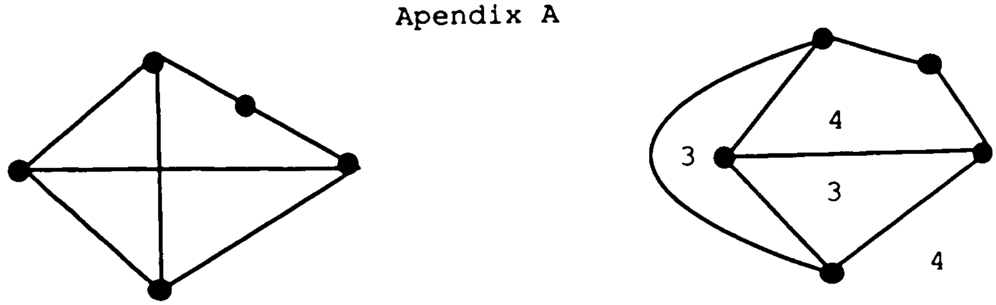
5) Is every digon-free cut-through Eulerian circuit Hamiltonian?

6) Are digon-free (n,r,t,k)-arrangements Hamiltonian?

7) Does a digon-free (n,r,t,k)-arrangement have a HIST?

In addition to these questions, there are many conjectures and questions about arrangements of simple curves which remain unanswered (see B. Grünbaum[10]).

Apendix A



(a) (b)

Figure 1.1

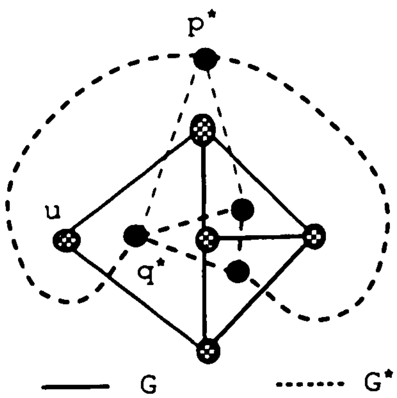


Figure 1.2

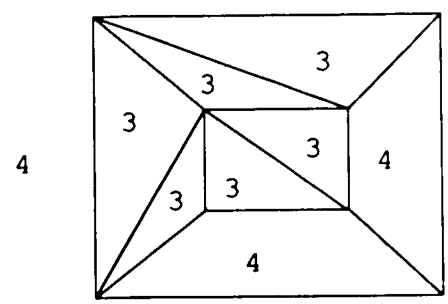
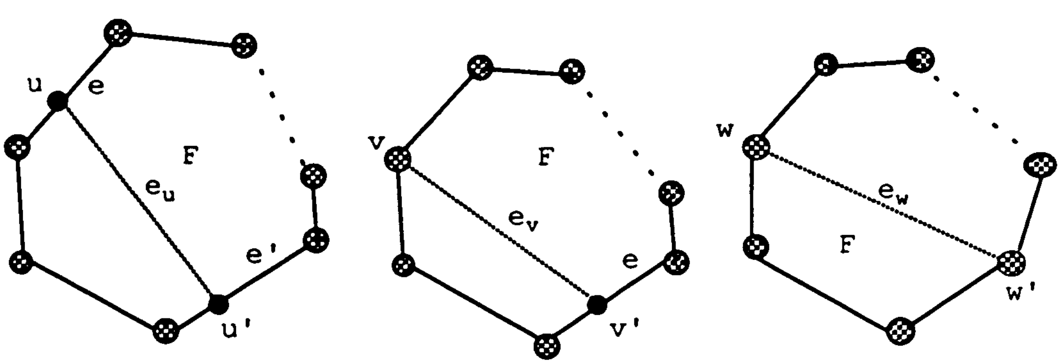


Figure 1.4



(a) (b) (c)

Figure 1.3

(a)
(b)
(c)

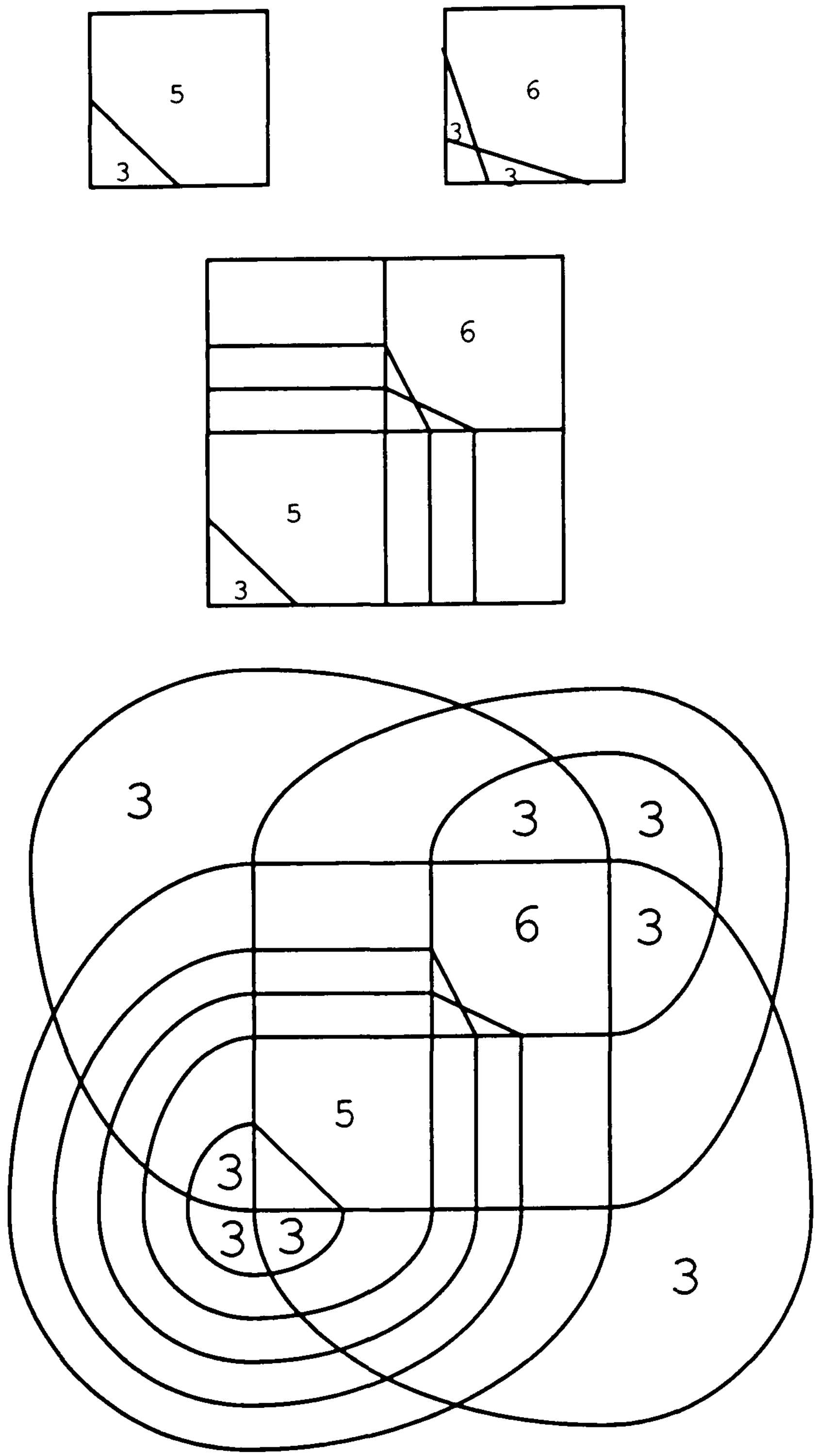


Figure 1.5

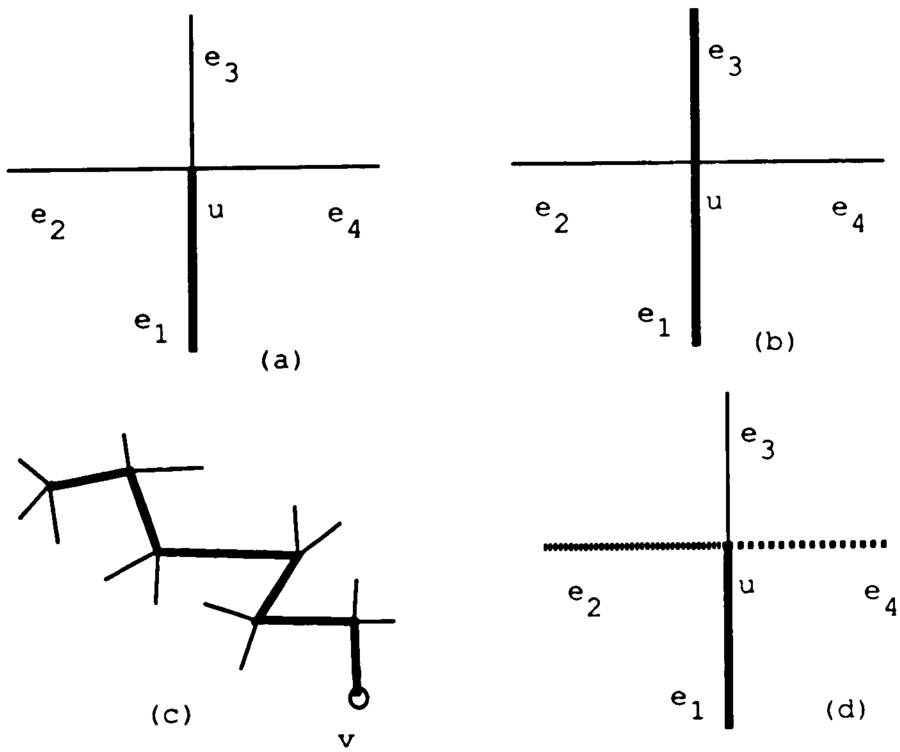


Figure 2.1

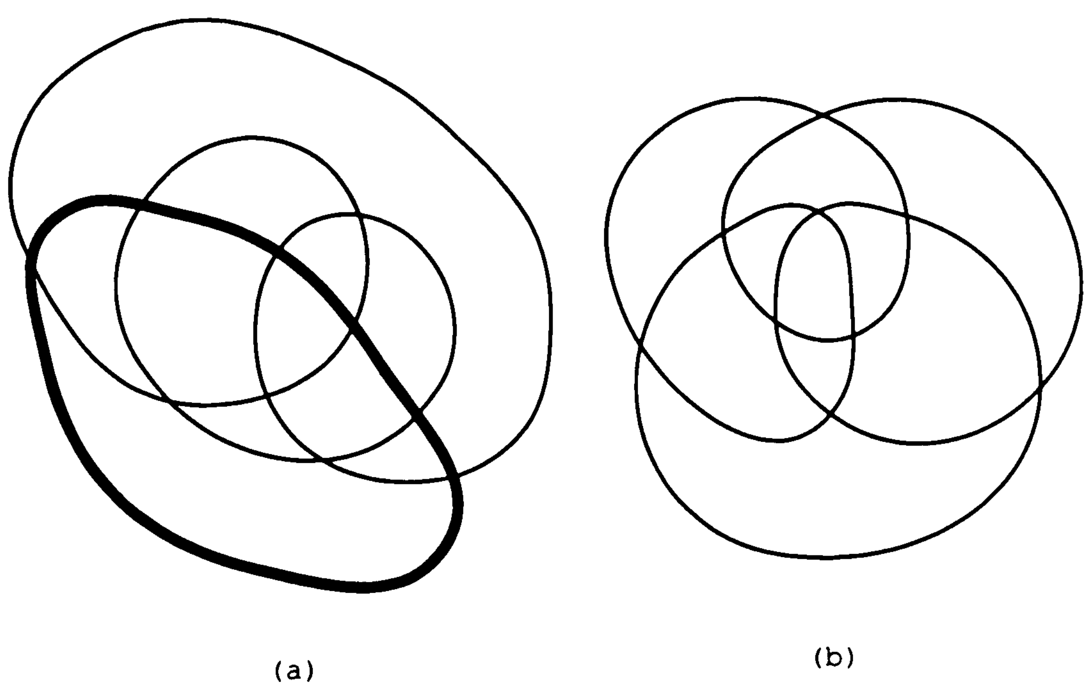
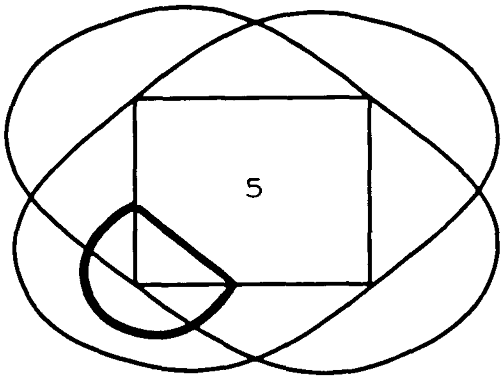
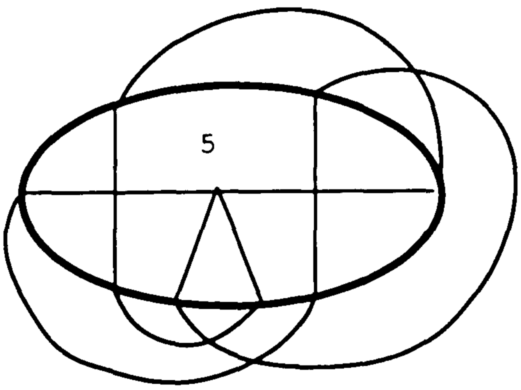


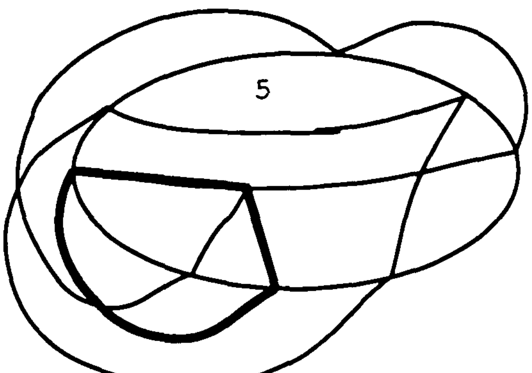
Figure 2.2



(a)

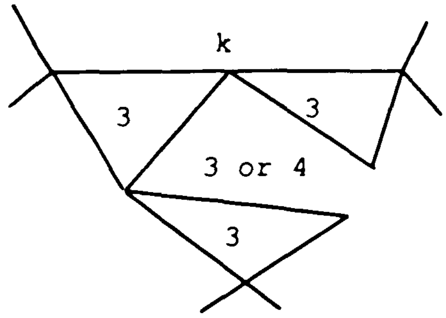


(b)

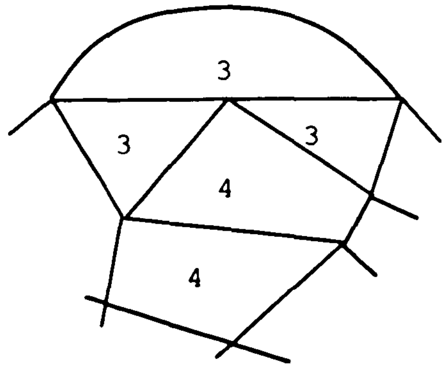


(c)

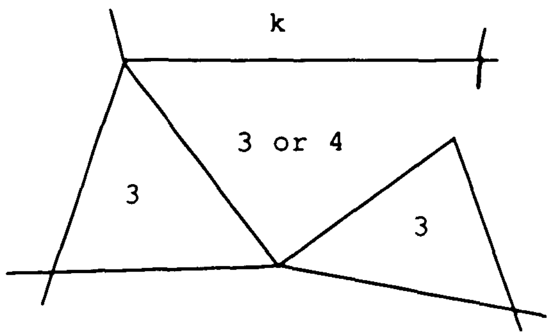
Figure 2.3



(a) A_1 - configuration

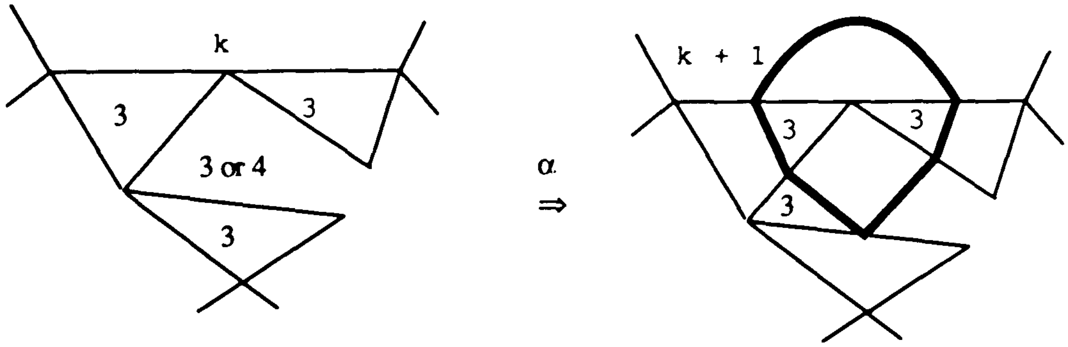


(b) A_2 - configuration

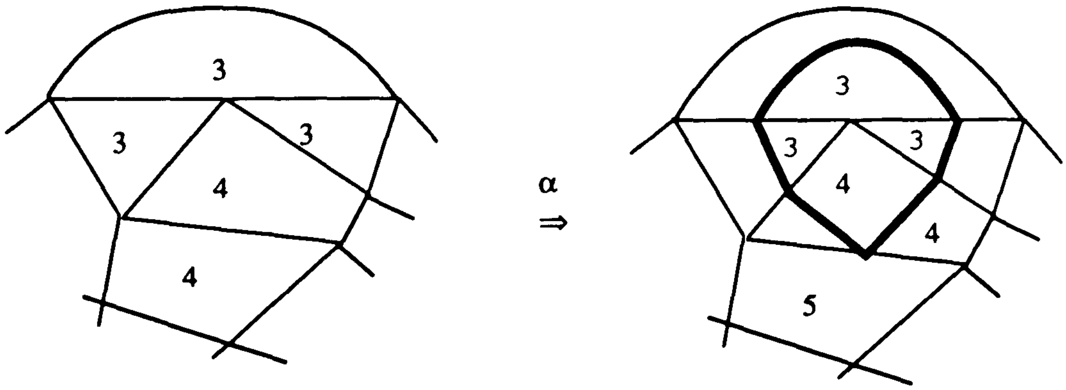


(c) B - configuration

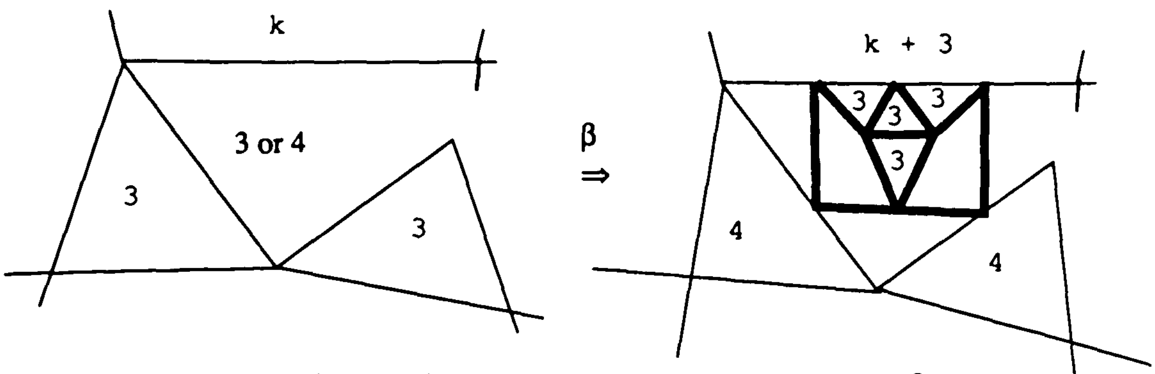
Figure 2.4



(a) A_1 - configuration (a') $A_1\alpha$



(b) A_2 - configuration (b') $A_2\alpha$



(c) B - configuration (c') $B\beta$

Figure 2.5

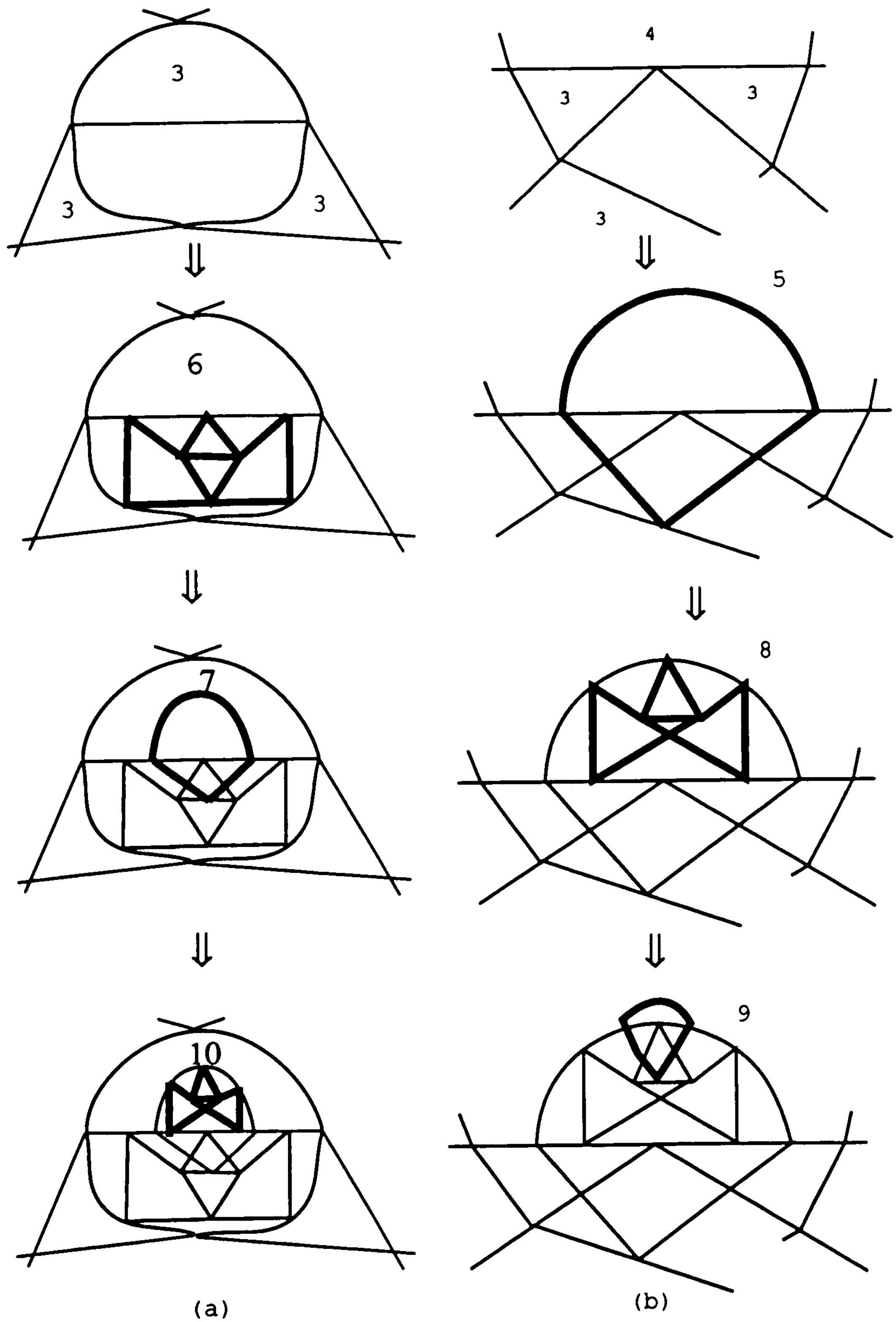
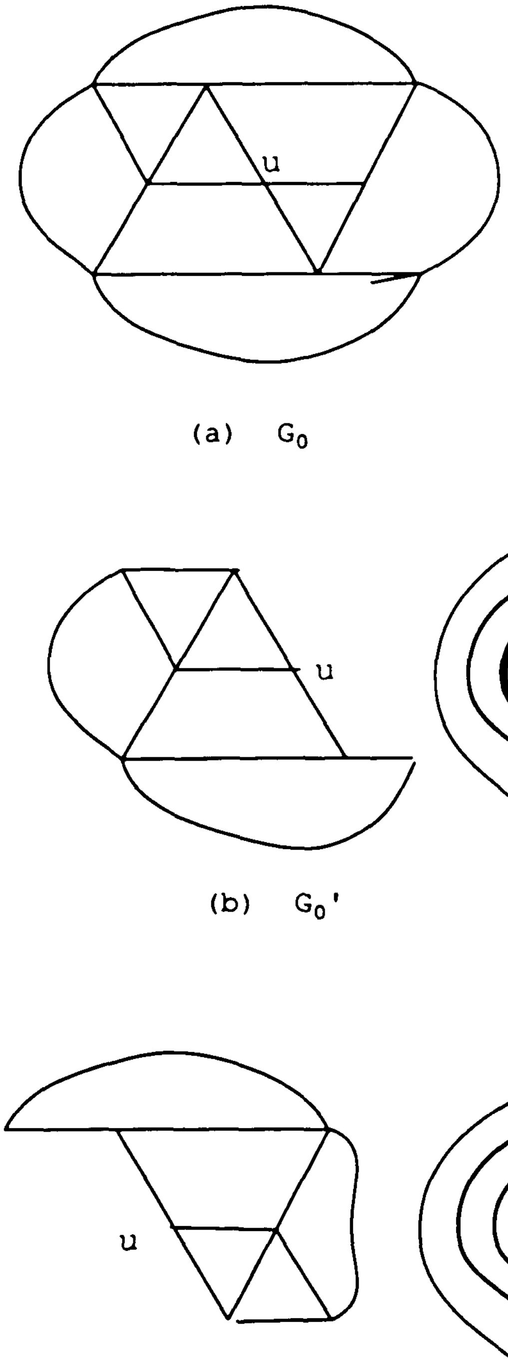


Figure 2.6



(a) G_0

(b) G_0'

(c) G_0''

Figure 2.7

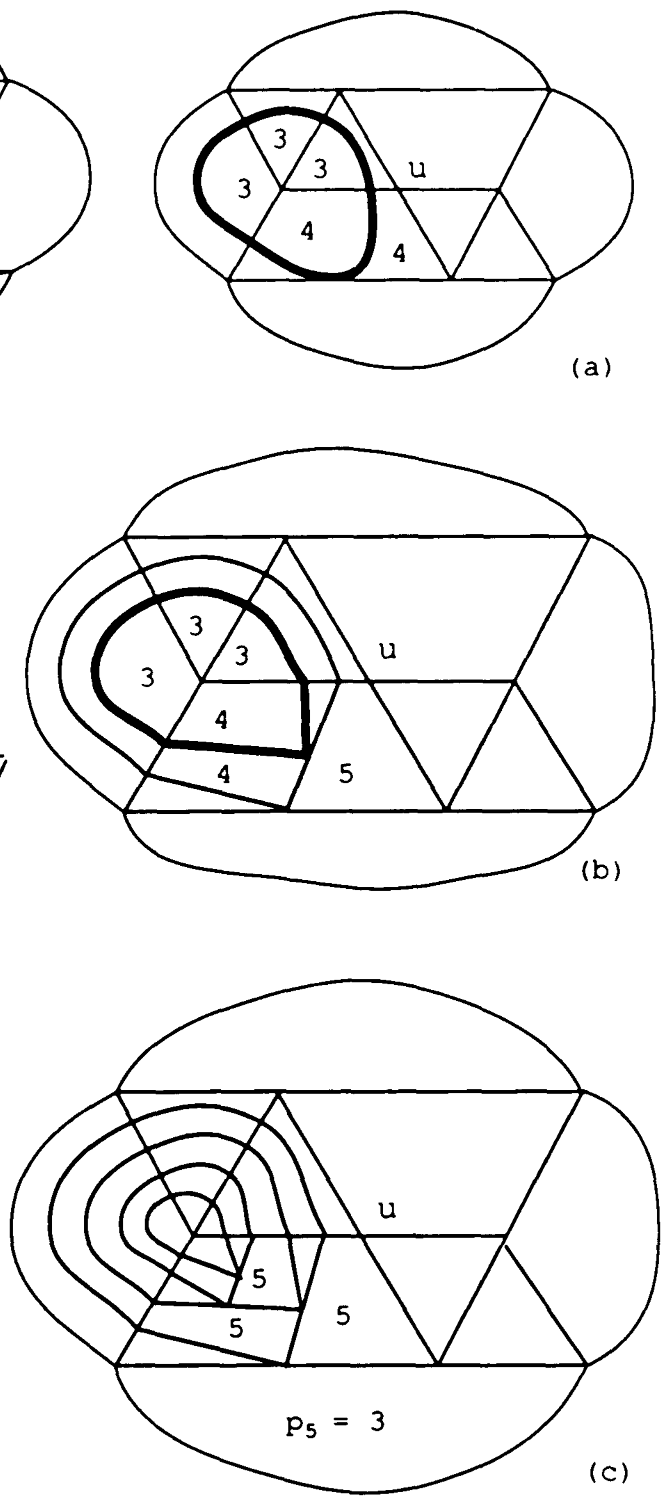
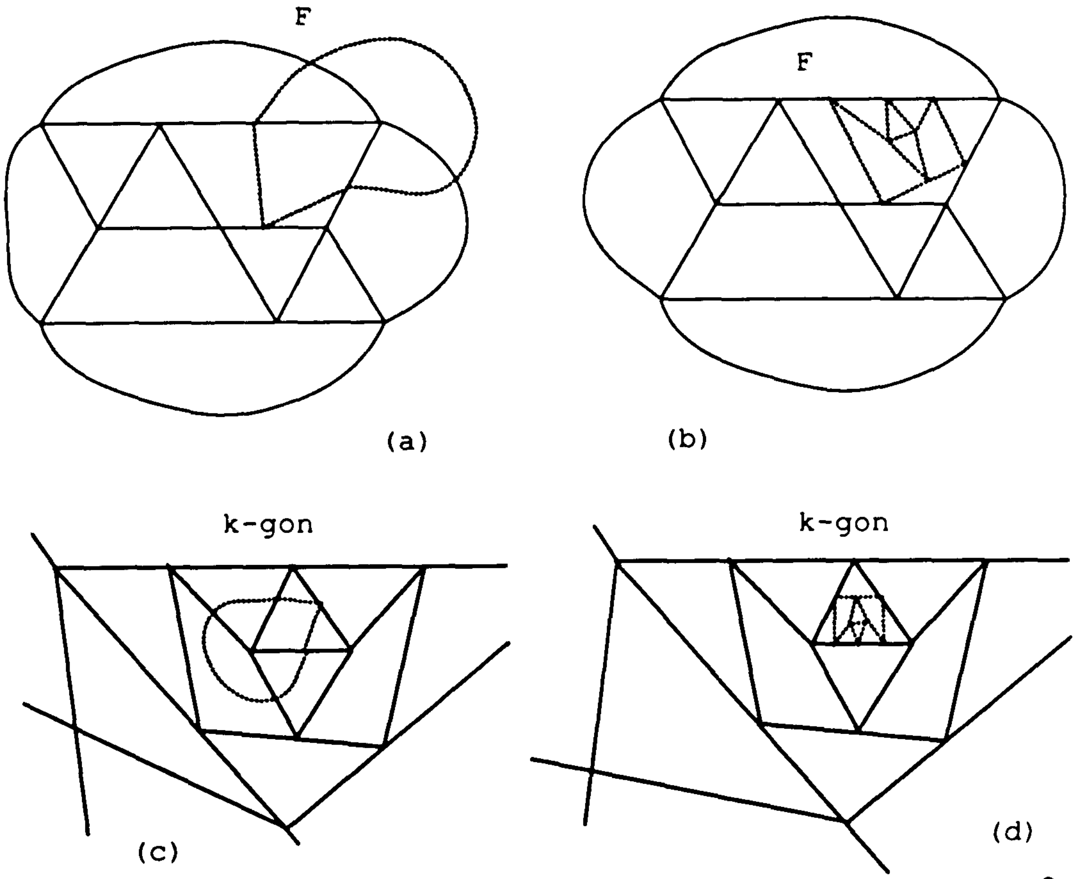
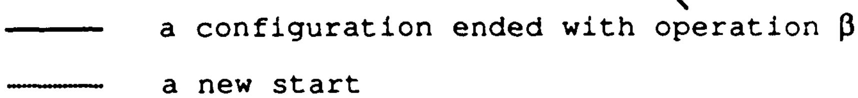
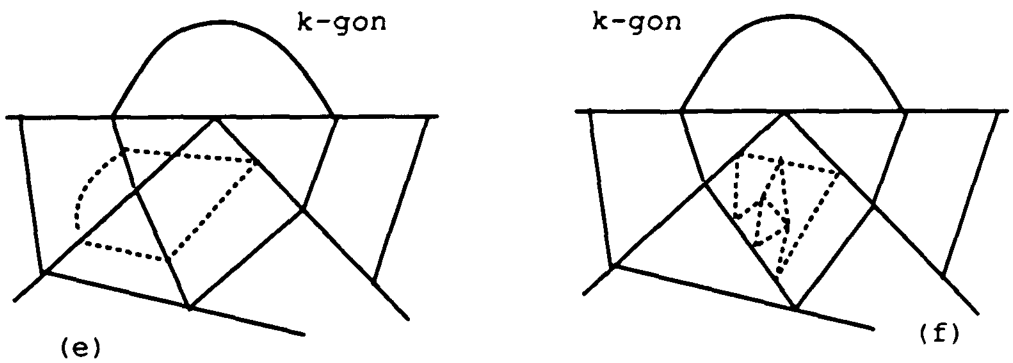


Figure 2.8


F
F
(a)
(b)
k-gon
k-gon
(c)
(d)


a configuration ended with operation β
a new start


k-gon
k-gon
(e)
(f)

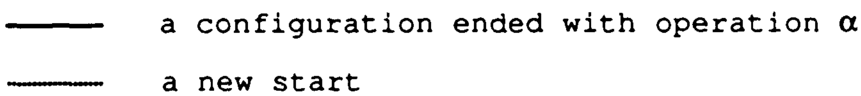
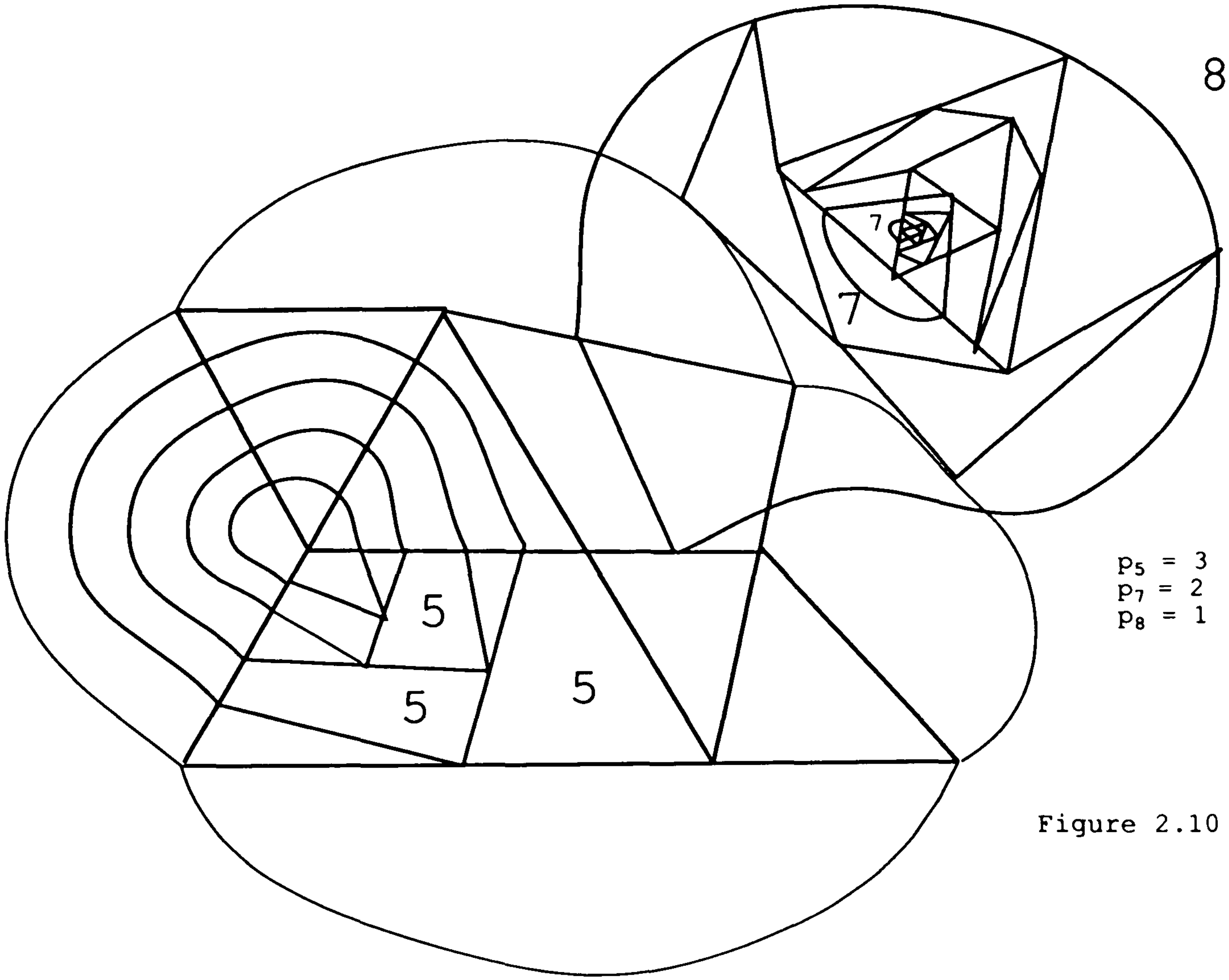
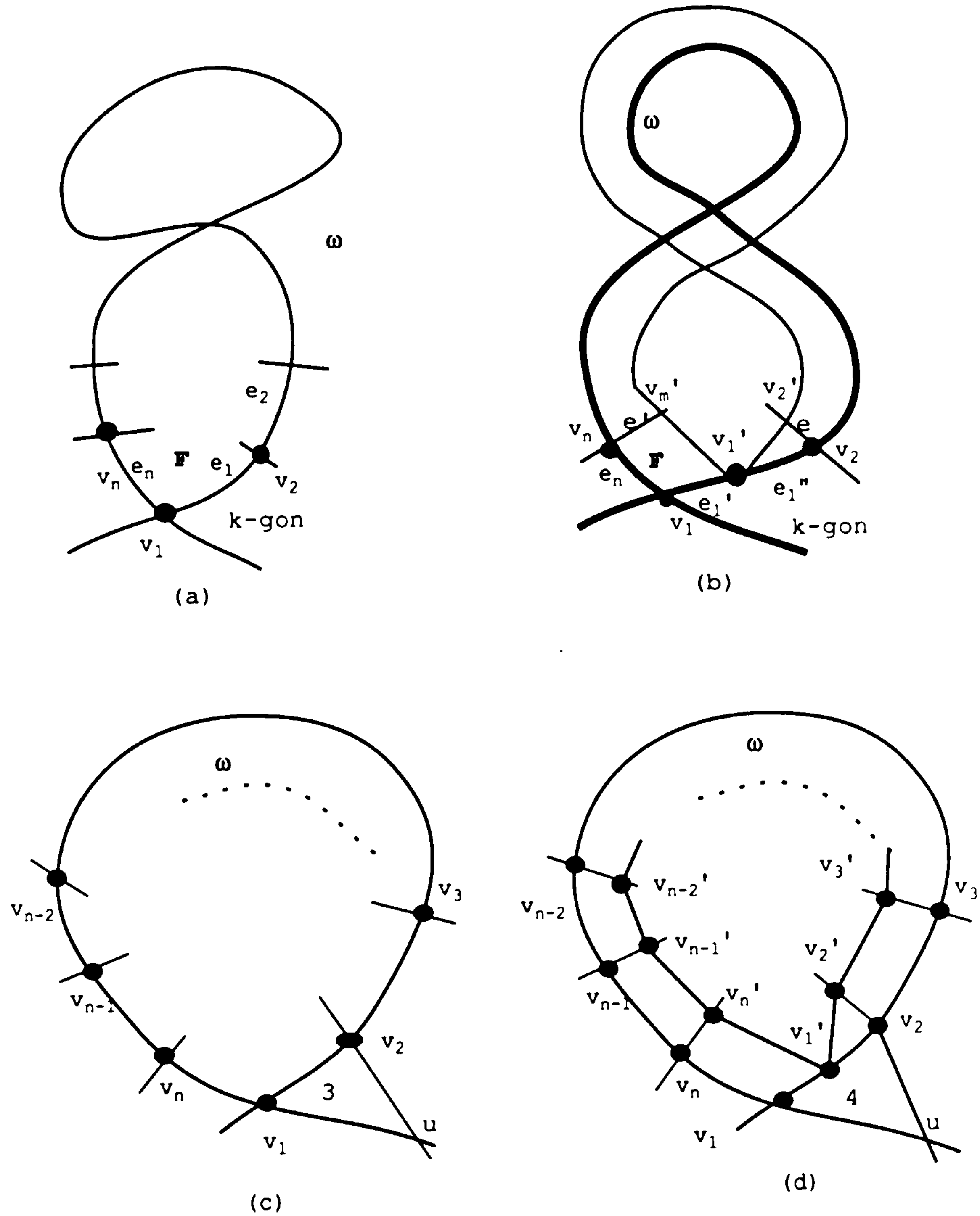

a configuration ended with operation α
a new start

Figure 2.9



$p_5 = 3$
$p_7 = 2$
$p_8 = 1$

Figure 2.10



ω
e2
vn
en
F
e1
v2
v1
k-gon
(a)
ω
vm'
v2'
vn
e'
e
v1'
v2
en
F
e1"
e1'
v1
k-gon
(b)
ω
v3
vn-2
vn-1
v2
vn
3
u
v1
(c)
ω
vn-2'
v3'
v3
vn-2
vn-1'
v2'
vn-1
vn'
v2
v1'
vn
4
u
v1
(d)

Figure 2.11

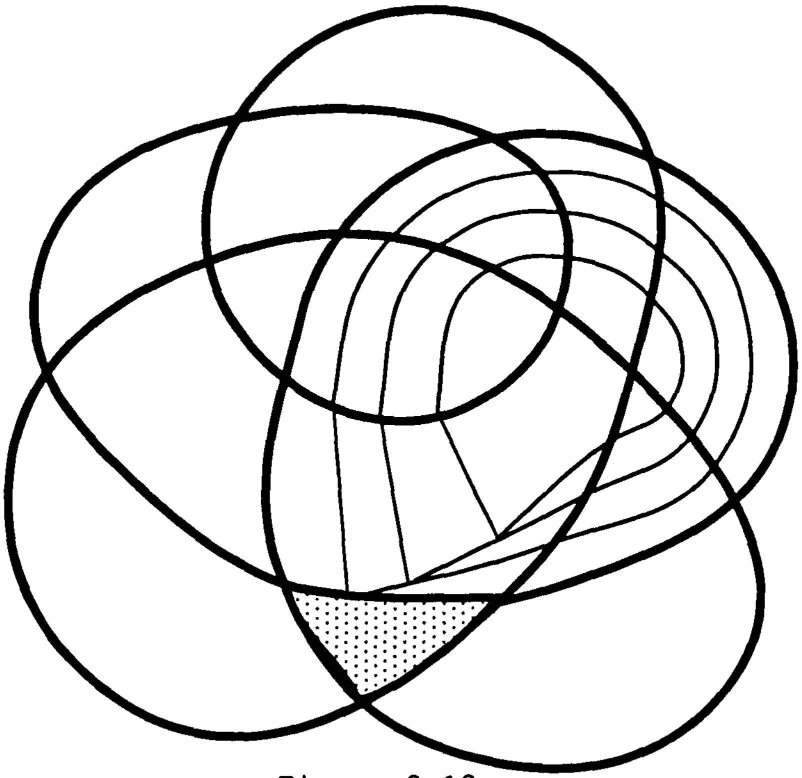
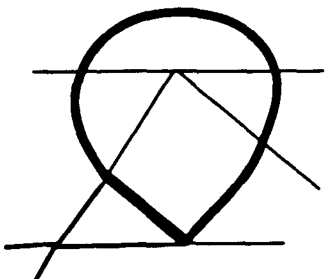
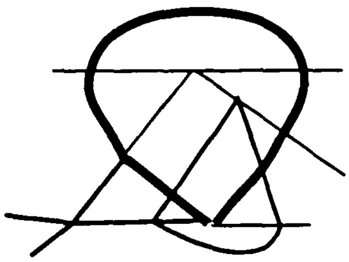


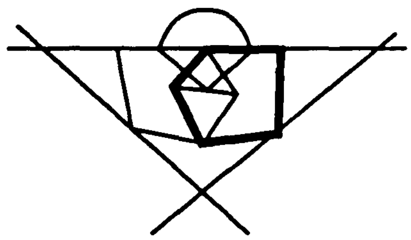
Figure 2.12



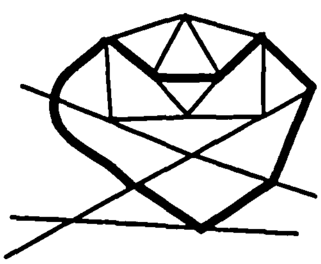
(a) T_1 configuration



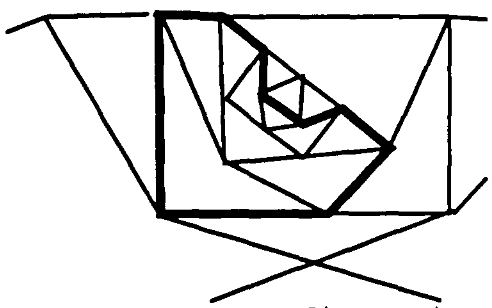
(b) T_2 configuration



(c) T_3 configuration

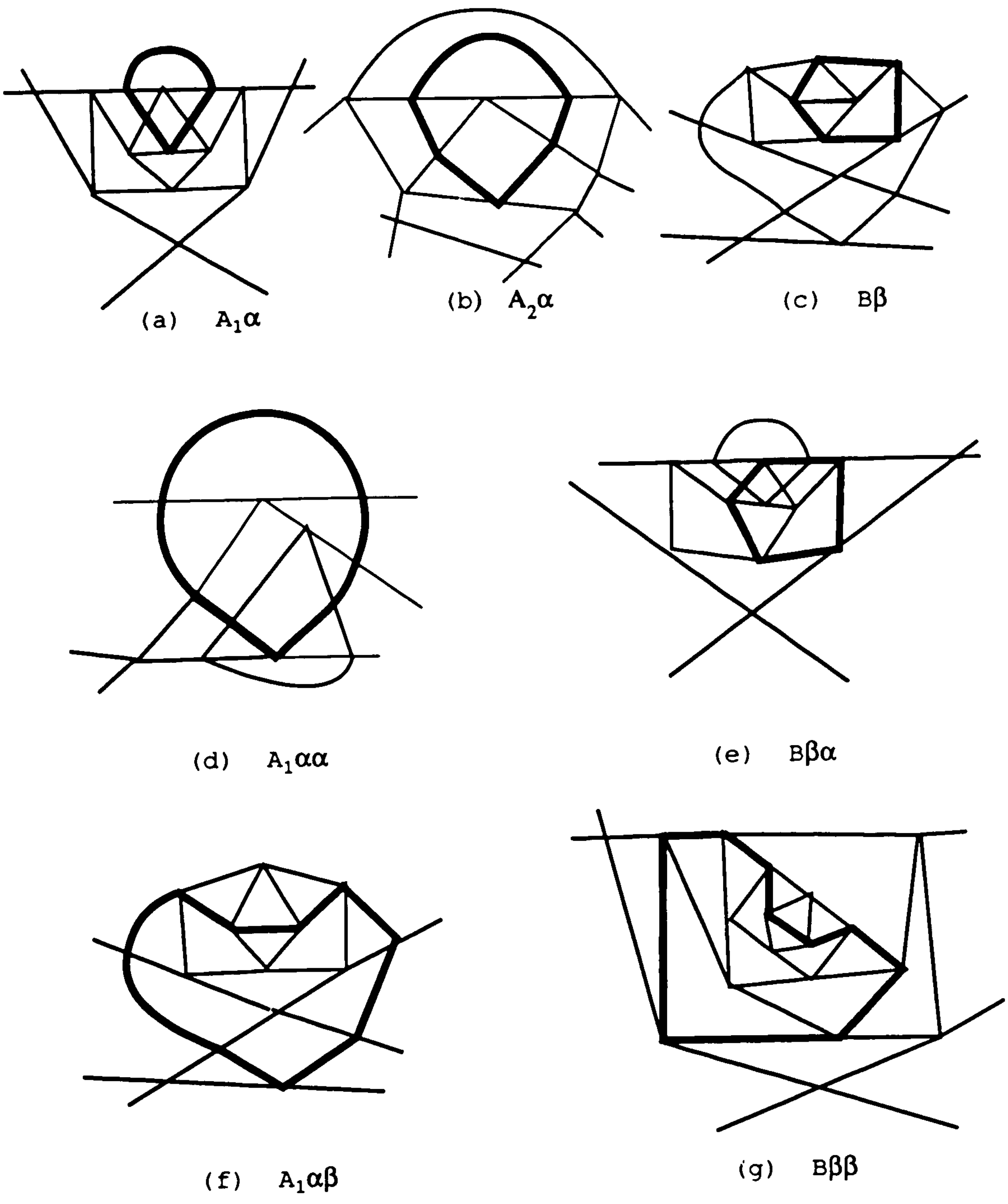


(d) T_4 configuration



(e) T_5 configuration

Figure 2.13



(a) $A_1\alpha$ (b) $A_2\alpha$ (c) $B\beta$

(d) $A_1\alpha\alpha$ (e) $B\beta\alpha$

(f) $A_1\alpha\beta$ (g) $B\beta\beta$

Figure 2.14

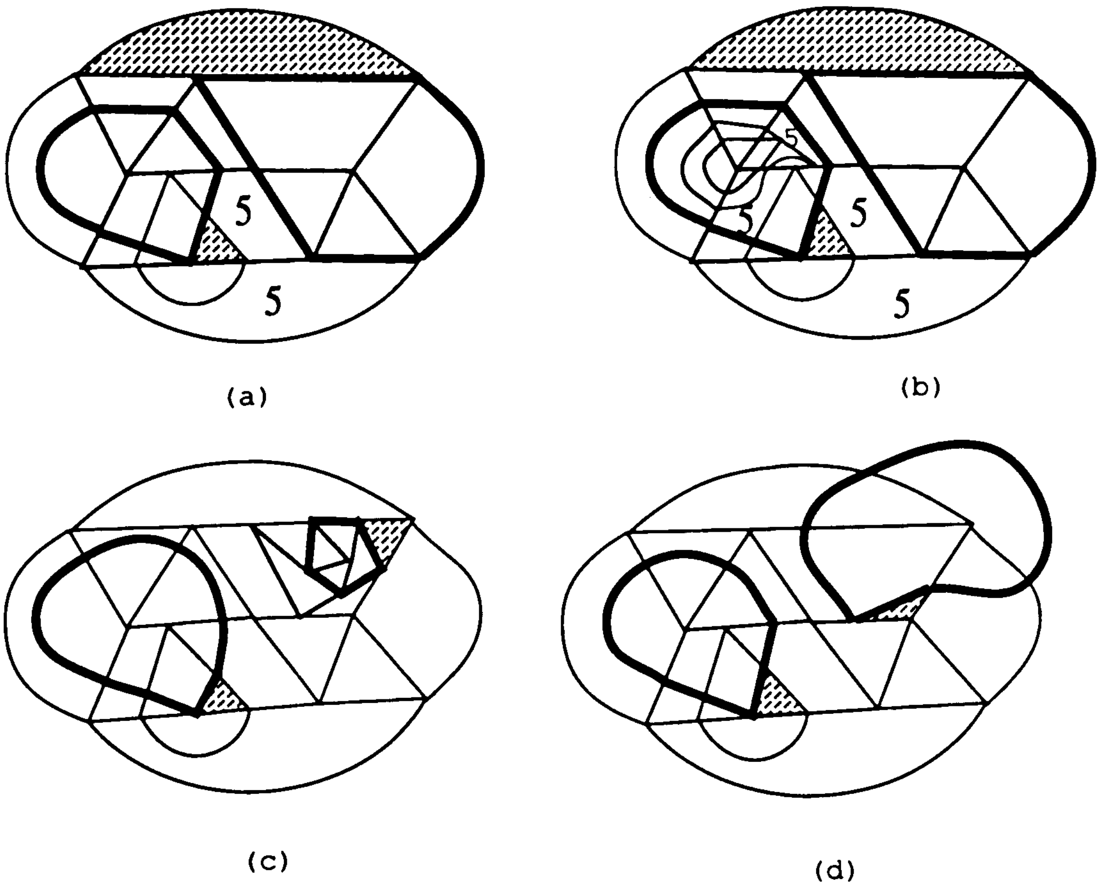


Figure 2.15

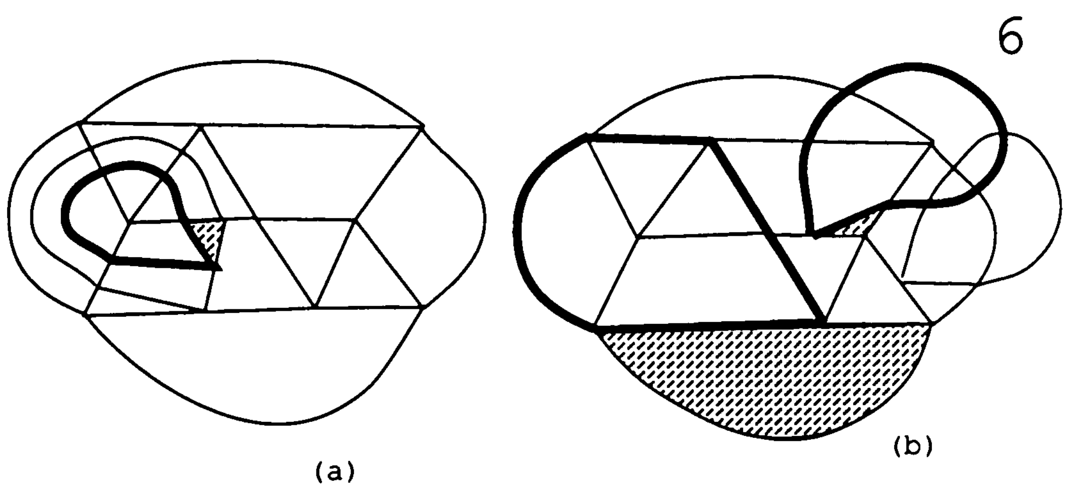
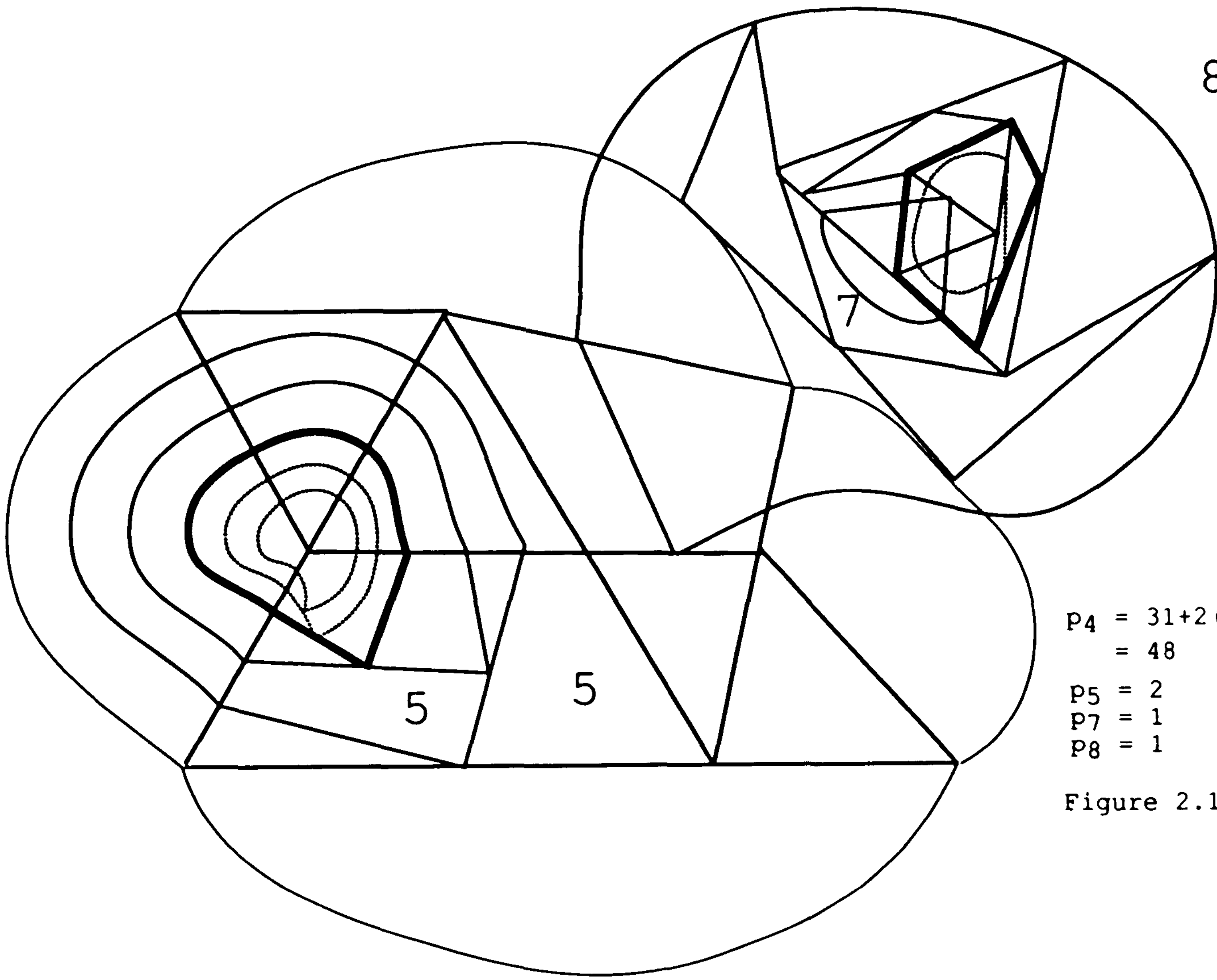


Figure 2.16



$p_4 = 31+2(5)+1(7)$

$= 48$

$p_5 = 2$

$p_7 = 1$

$p_8 = 1$

Figure 2.17 (b)

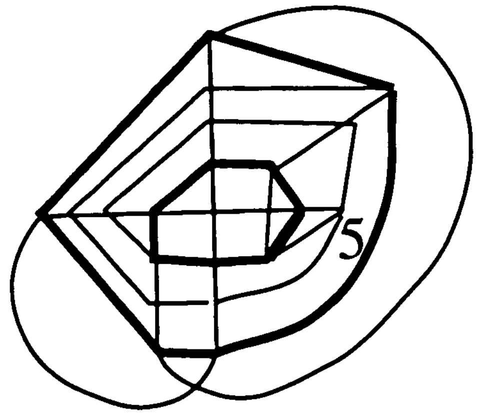


Figure 2.17

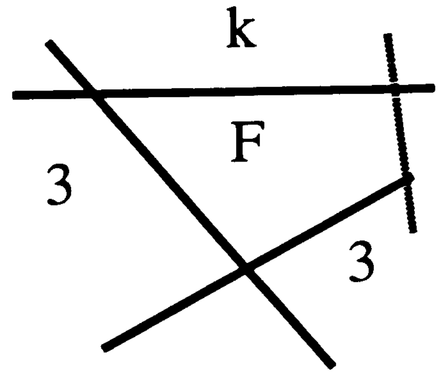


Figure 2.18

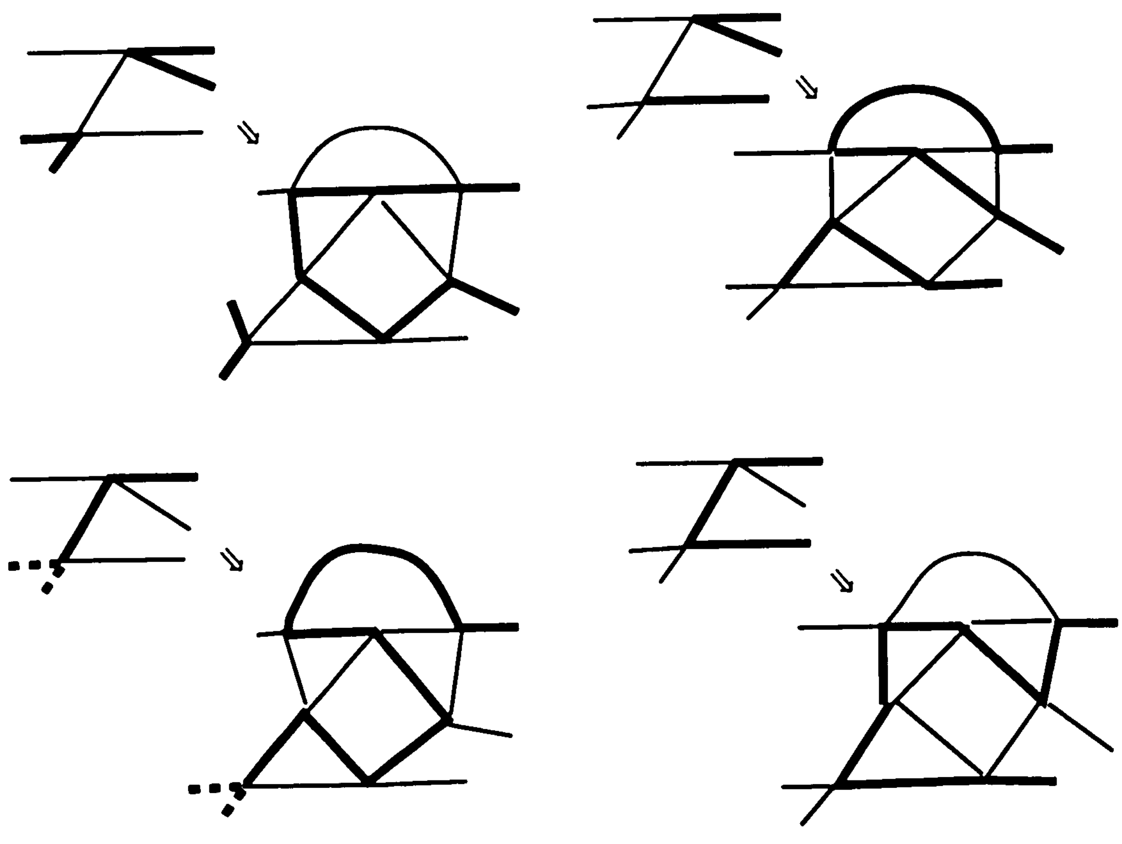


Figure 2.19 (a)

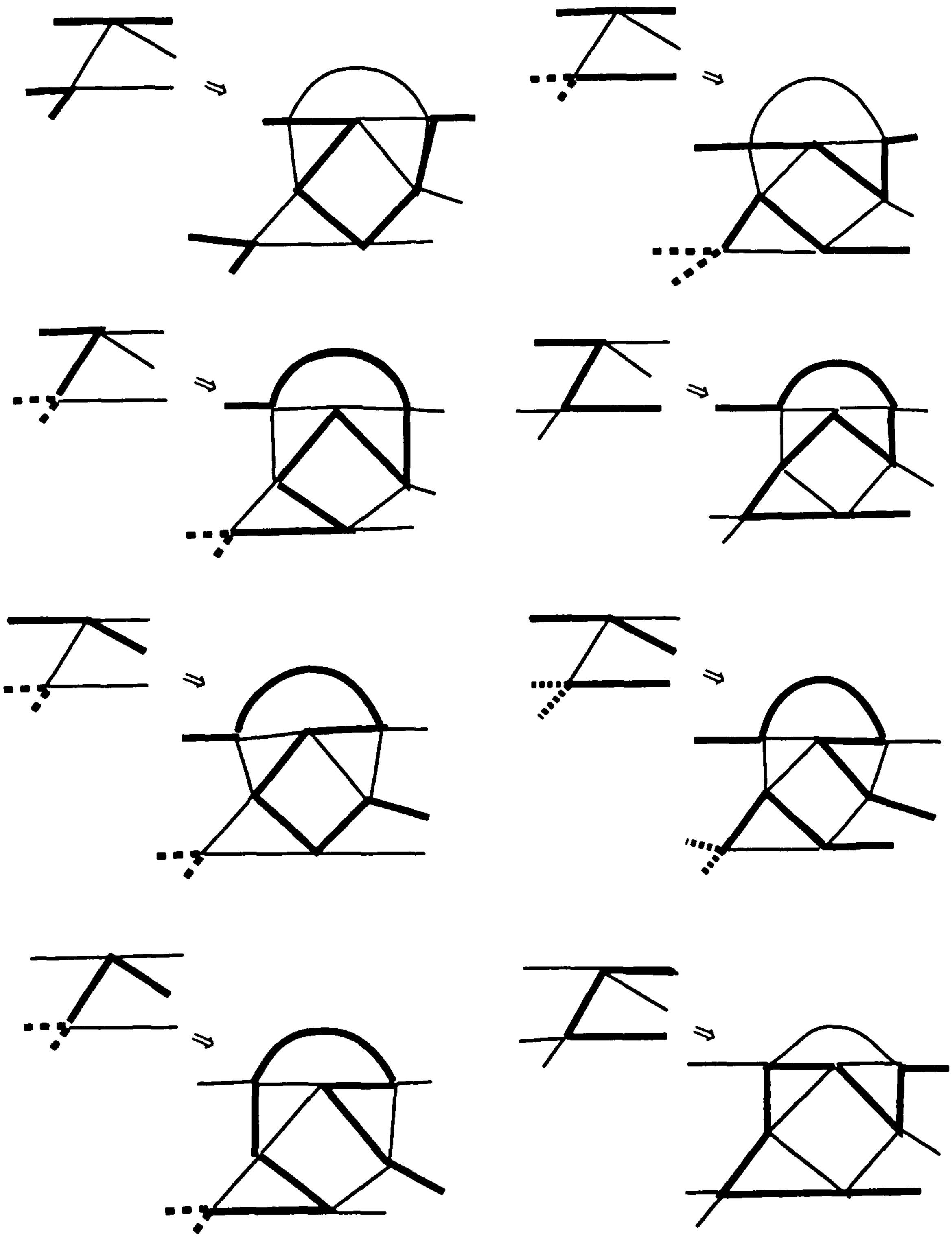


Figure 2.19 (b)

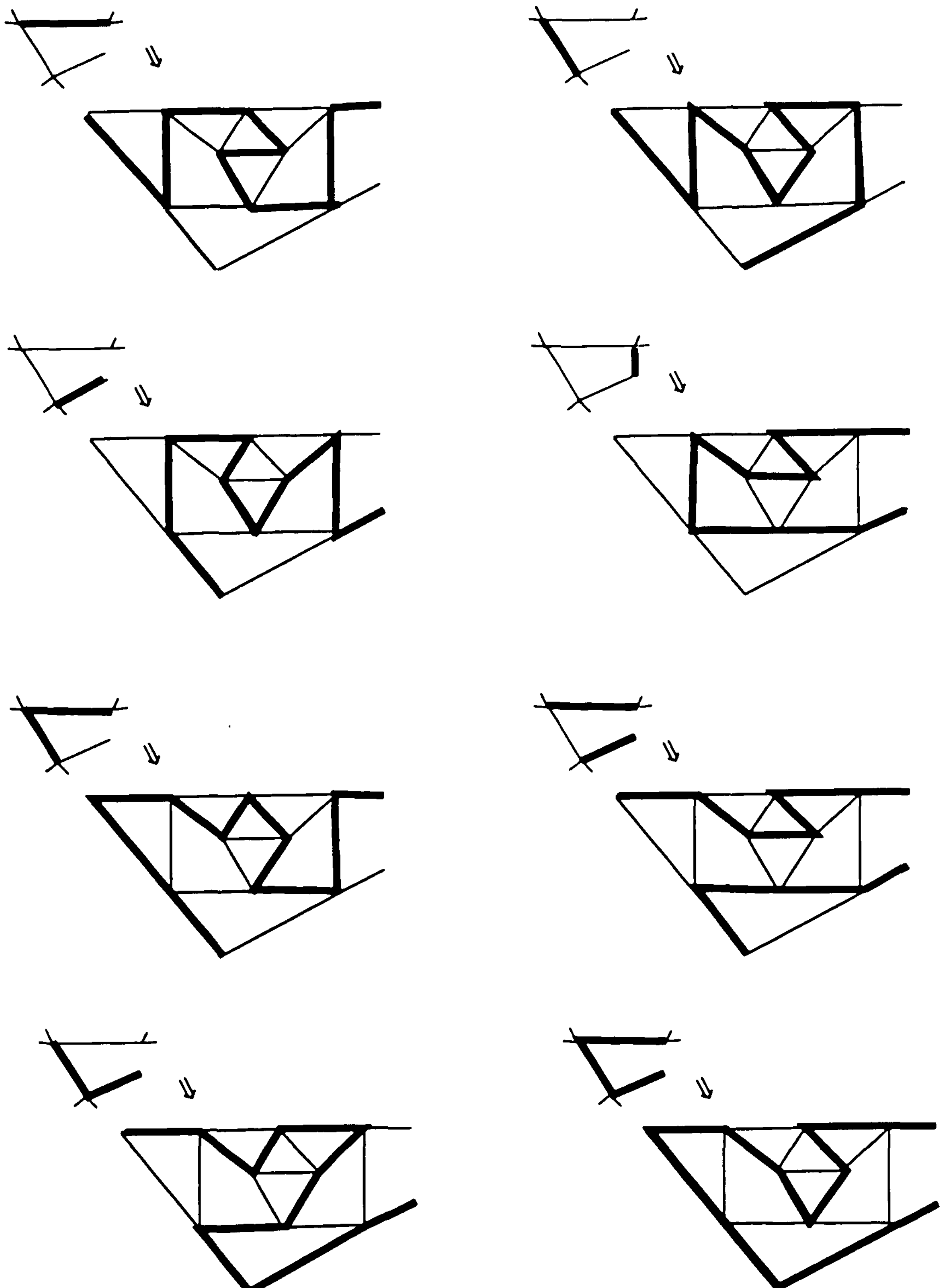
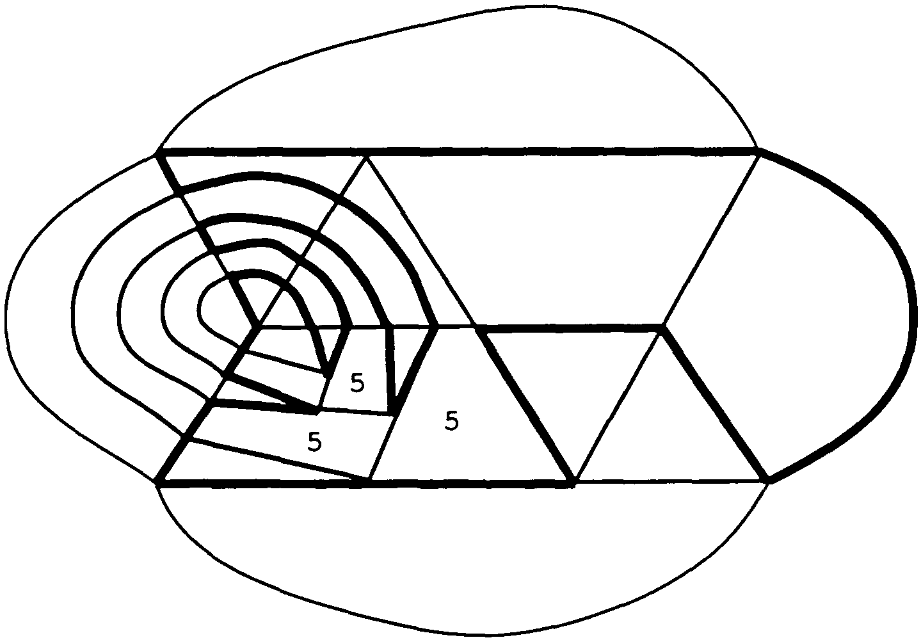
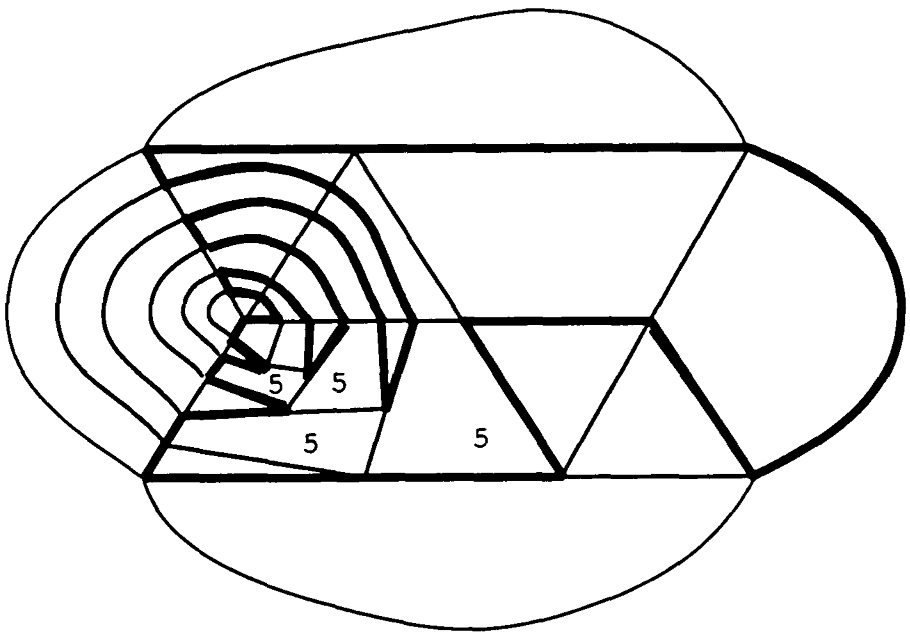


Figure 2.20



(a) odd number of 5-gons



(b) even number of 5-gons

Figure 2.21

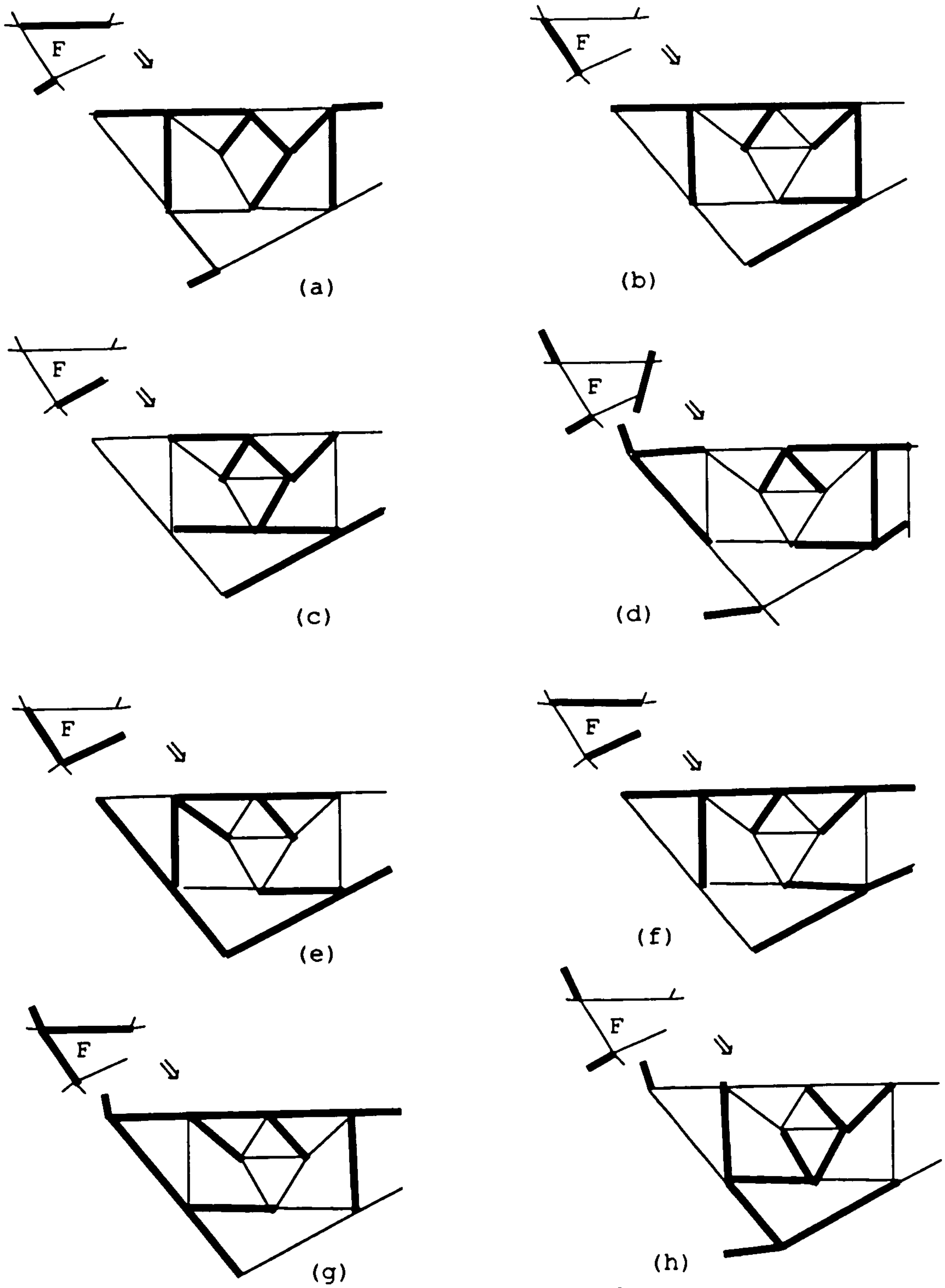


Figure 2.22

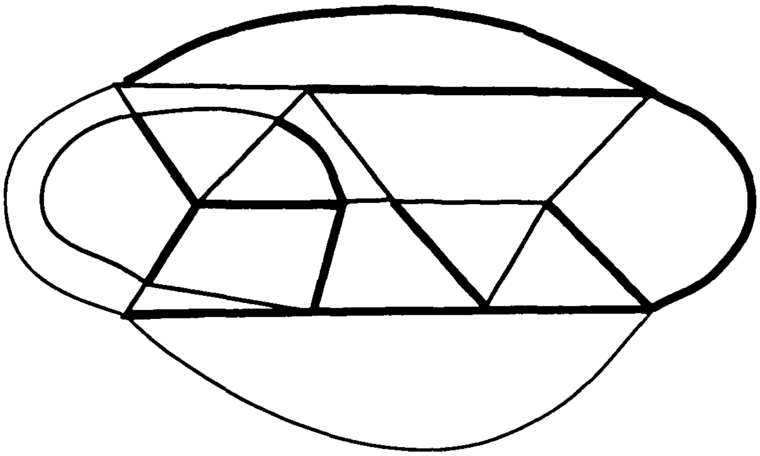


Figure 2.23

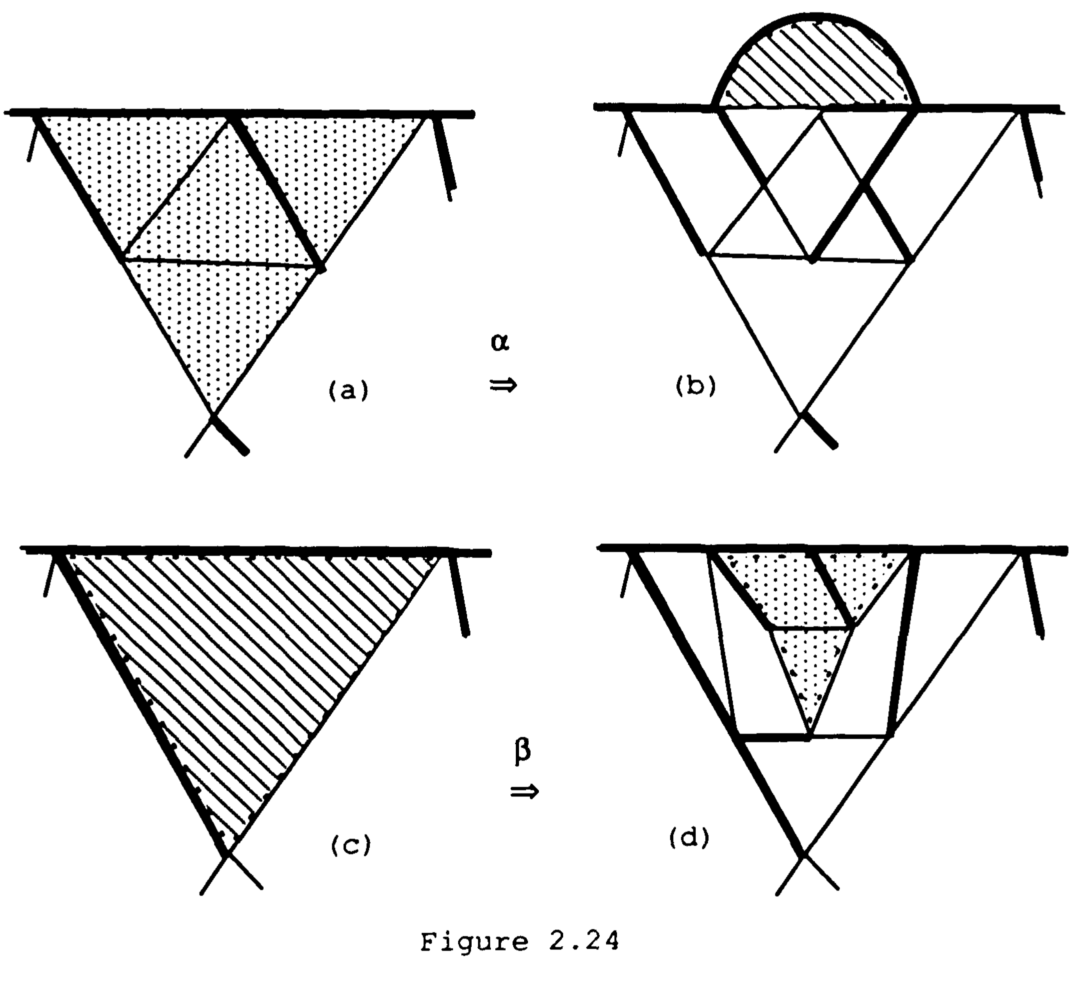
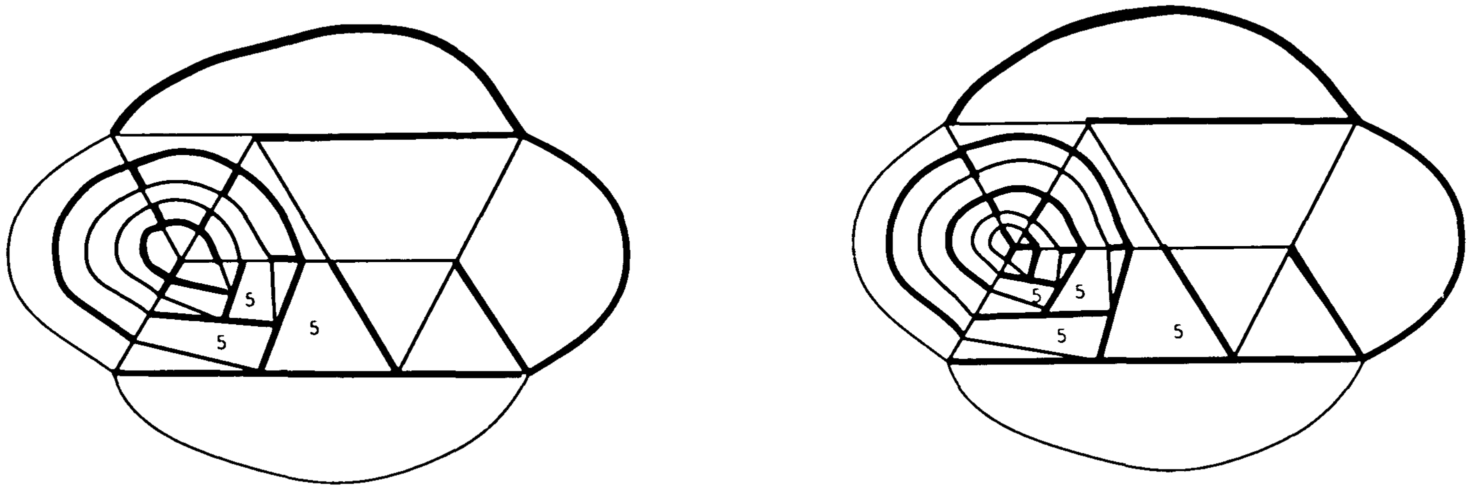
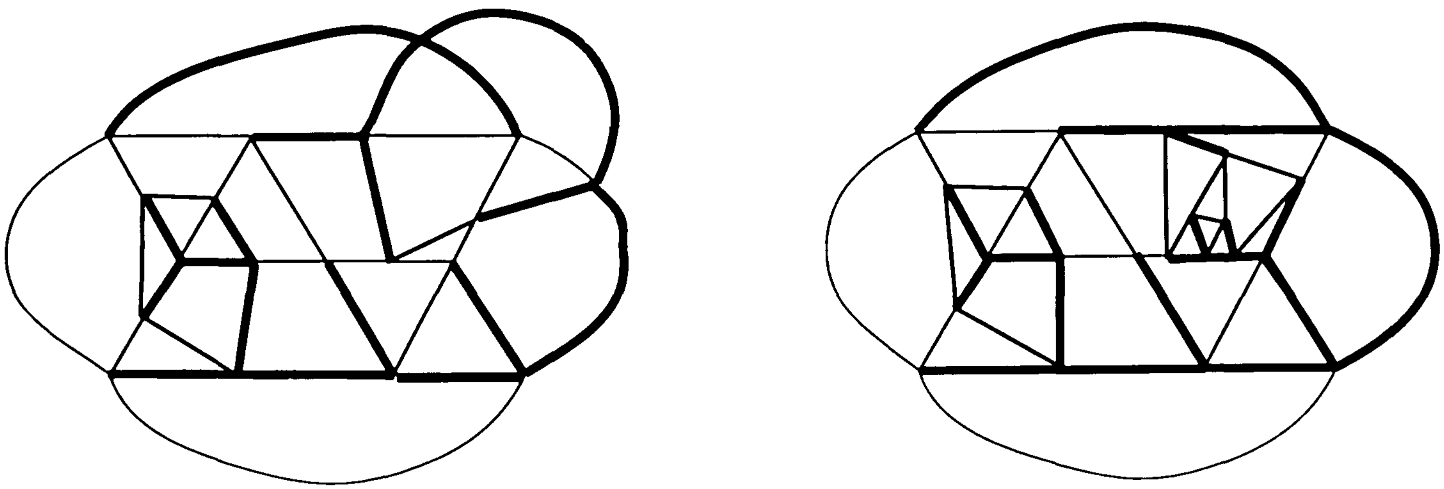


Figure 2.24



(a) odd number of 5-gons   Figure 2.25   (b) even number of 5-gons



(a)   Figure 2.26   (b)

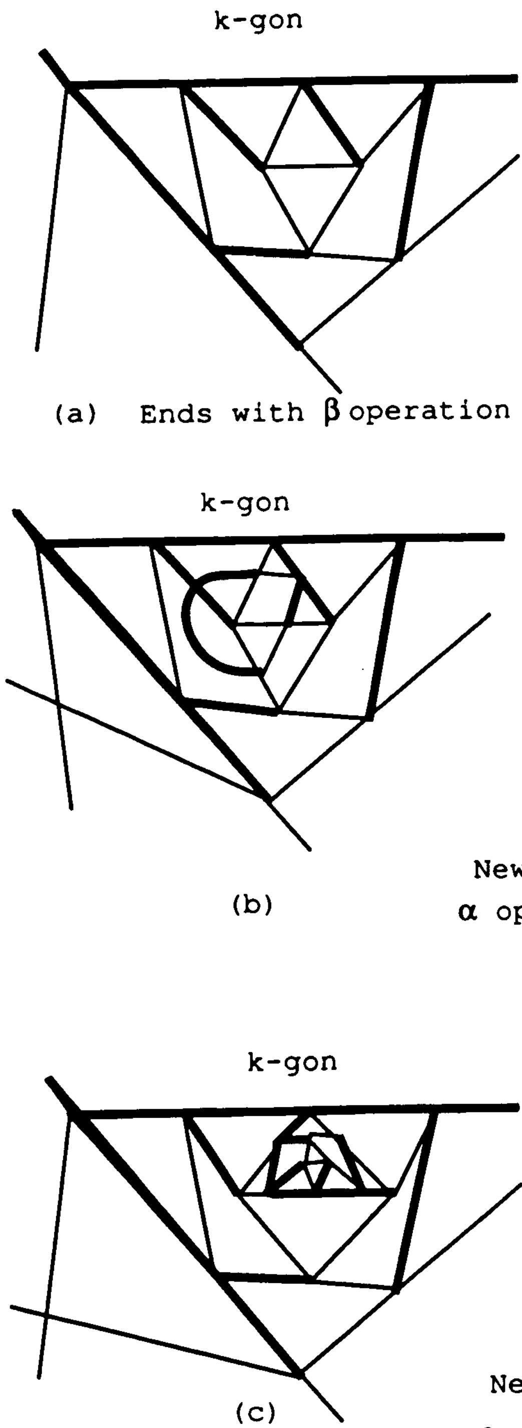


Figure 2.27

New start
α operation

New start
β operation

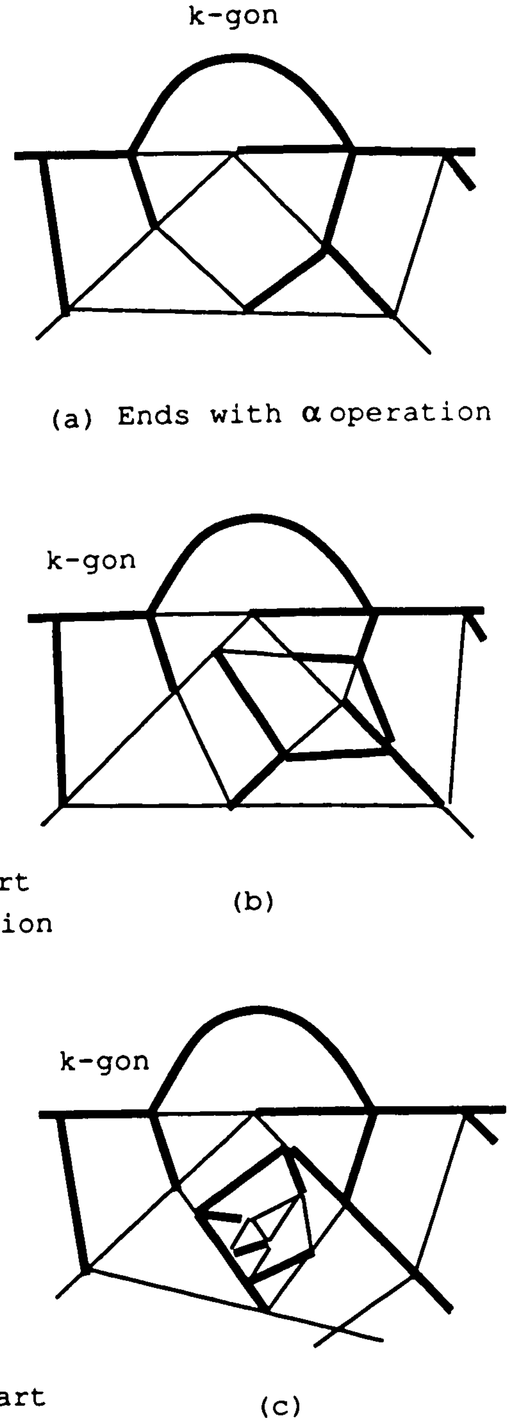


Figure 2.28

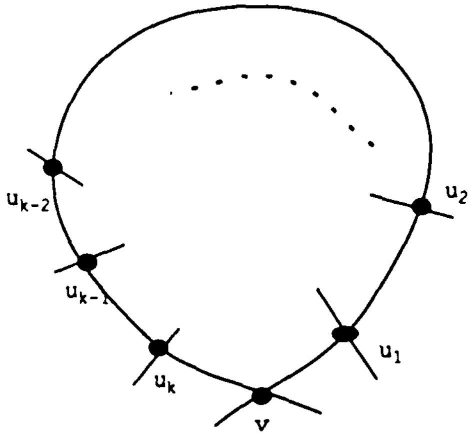


Figure 2.29

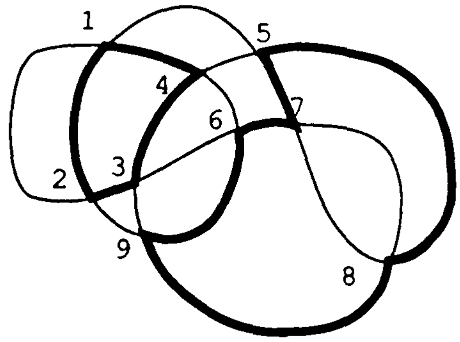


Figure 2.30

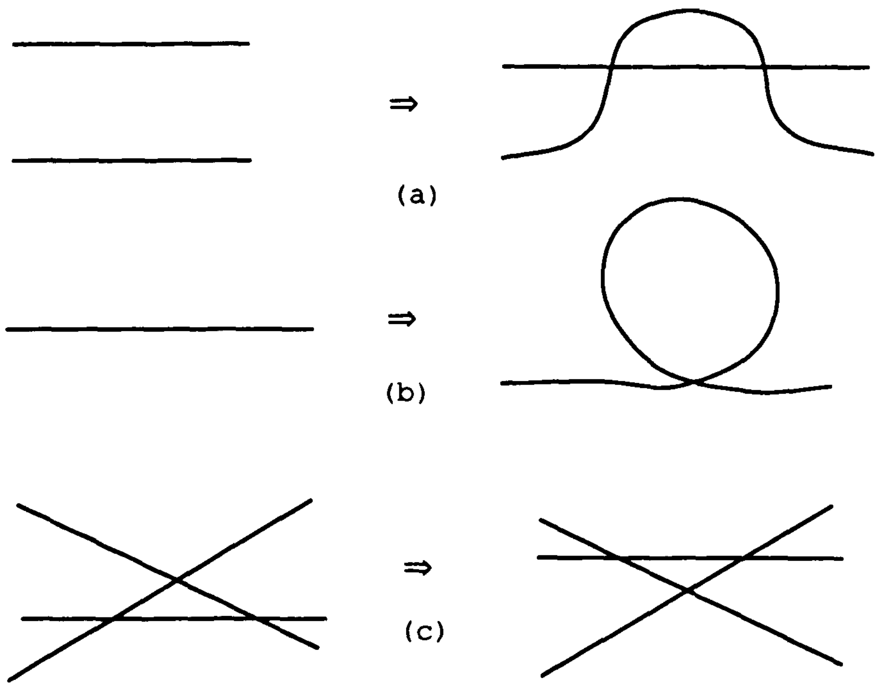


Figure 2.31

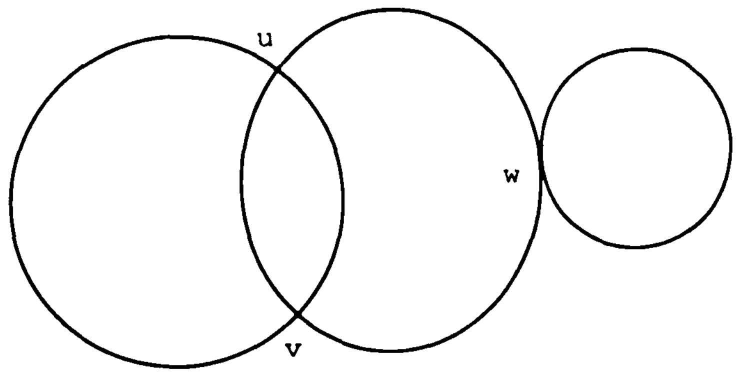


Figure 3.1

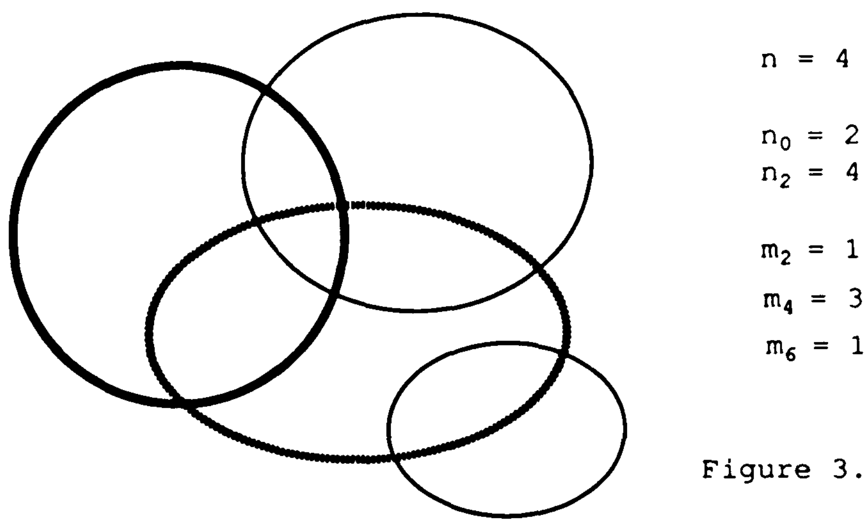


Figure 3.2

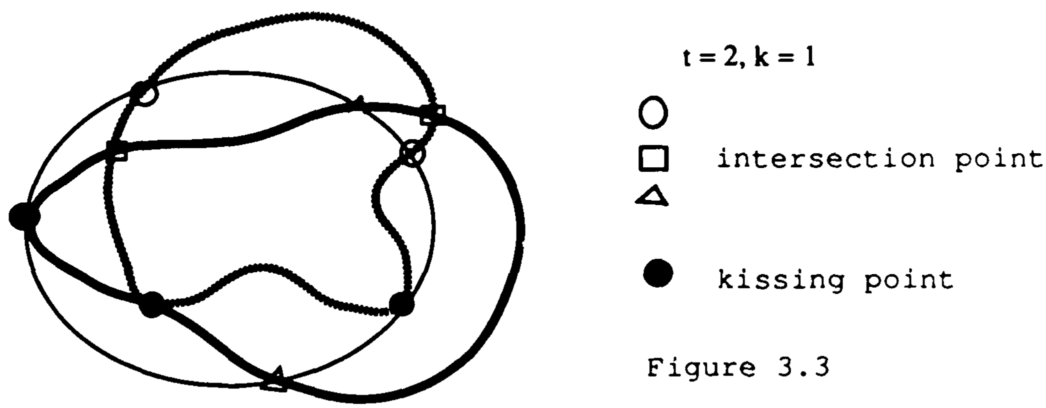
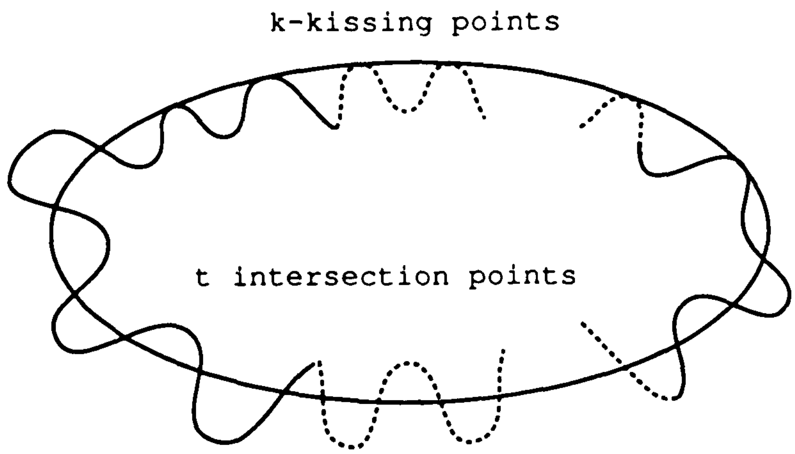
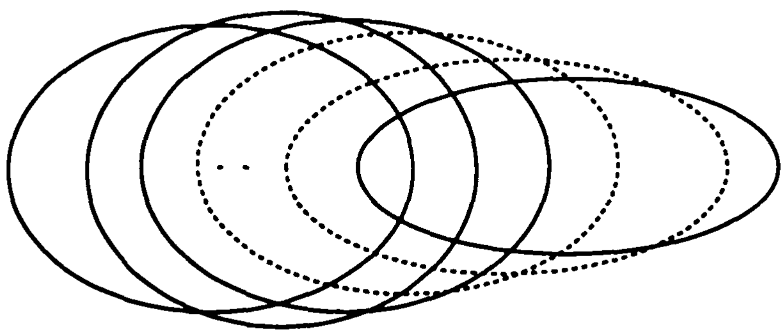


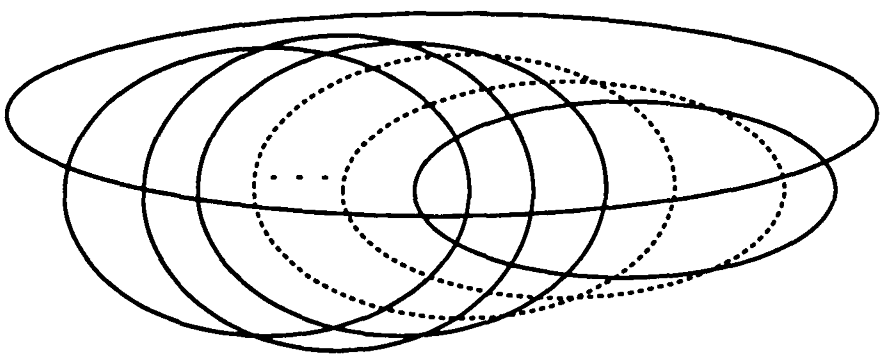
Figure 3.3



(a)



(b) a (n,r,2,0)-arrangememt

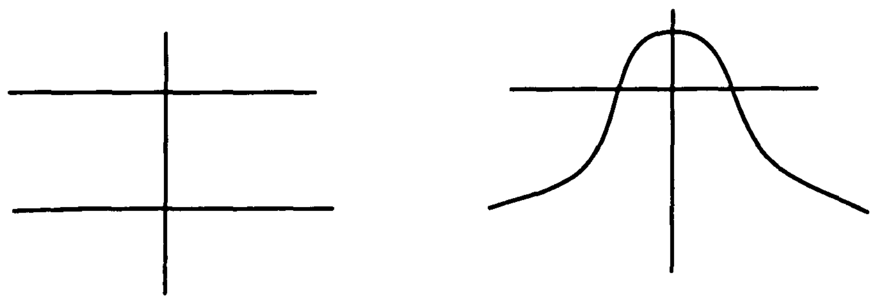


(c) a digon-free (n,r,2,0)-arrangement

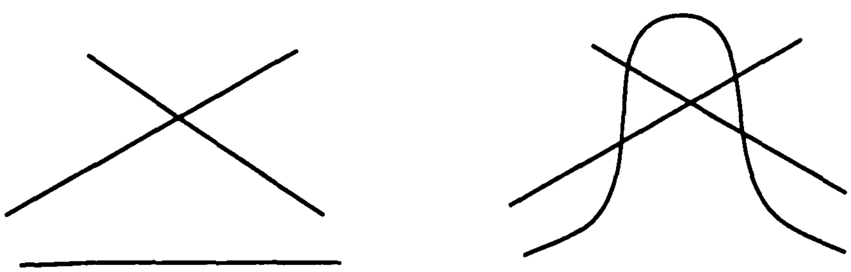
Figure 3.4

(a) O_1 operation

(b) O_2 operation

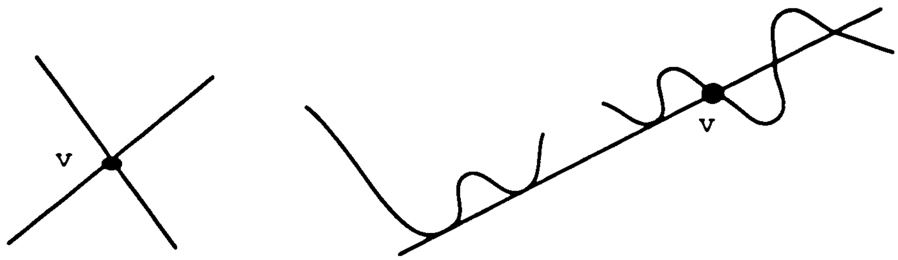


(c) O_3 operation



(d) O_4 operation

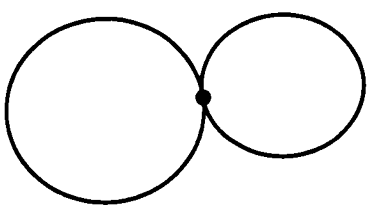
Figure 3.5



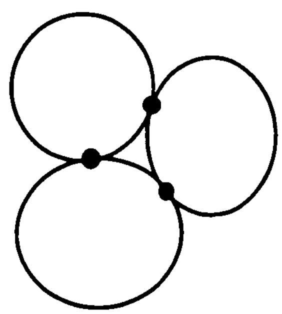
(a)

(b)

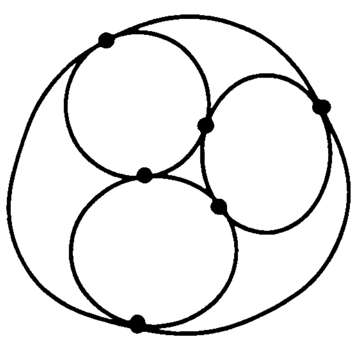
Figure 3.6



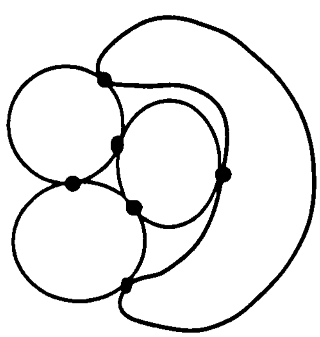
(2,1,0,1)



(3,2,0,1



or



(c) (4,3,0,1) (d) (4,3,0,1)

Figure 3.7

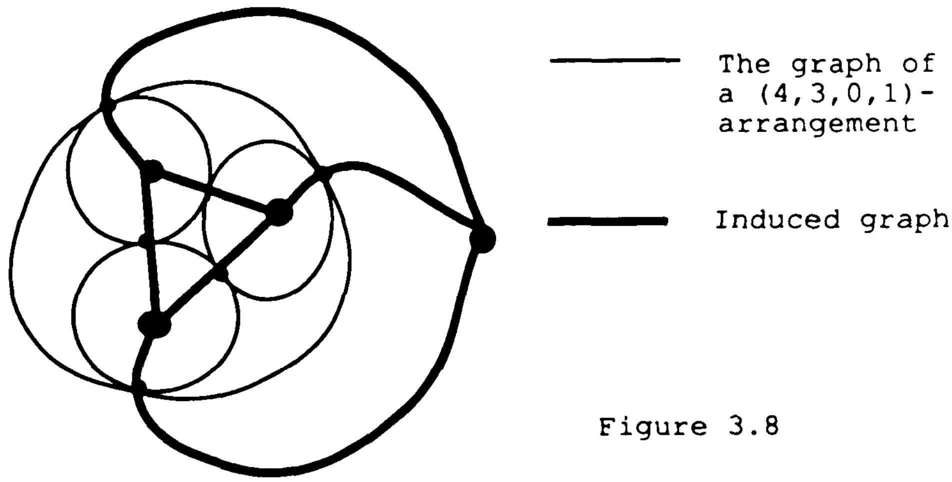


Figure 3.8

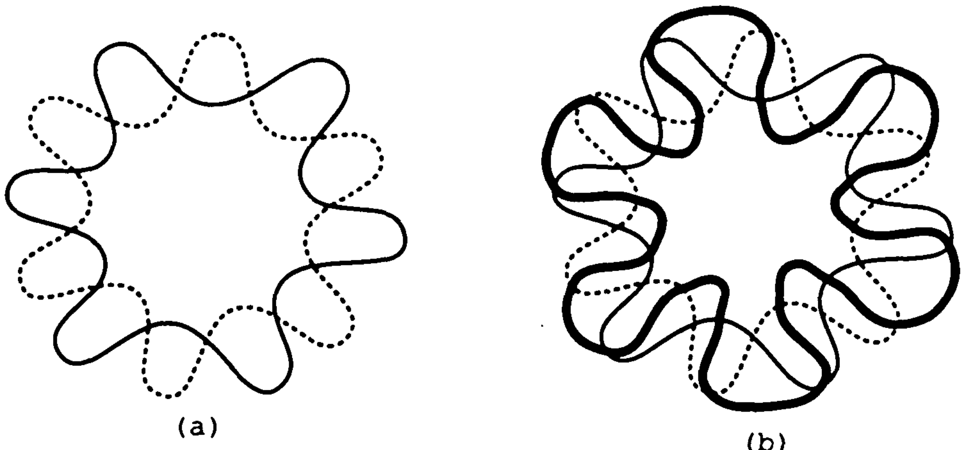


Figure 3.9

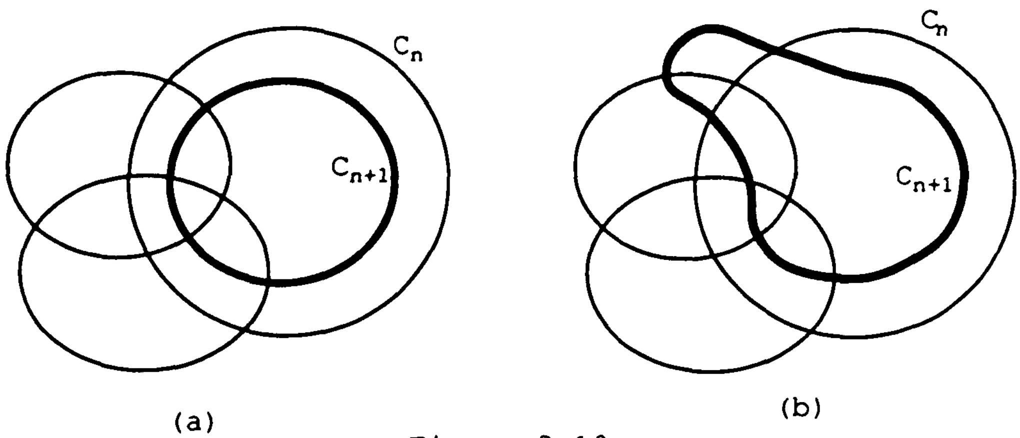


Figure 3.10

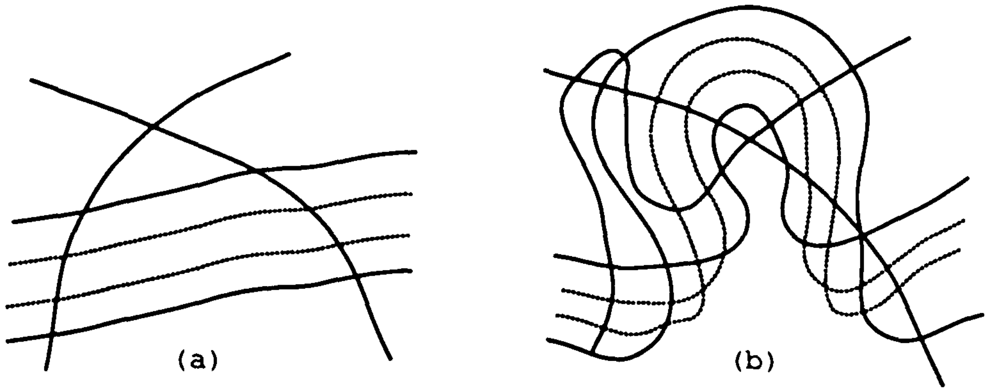


Figure 3.11

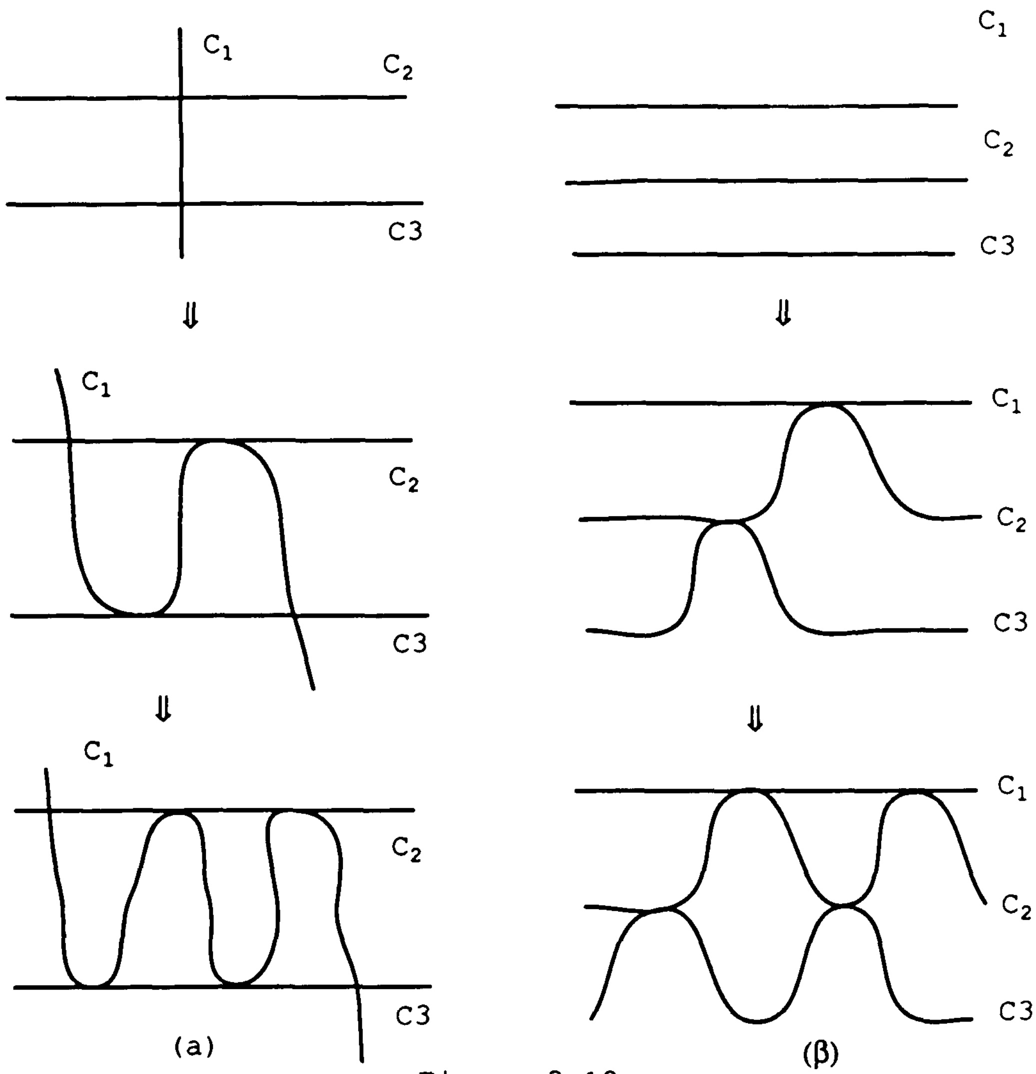
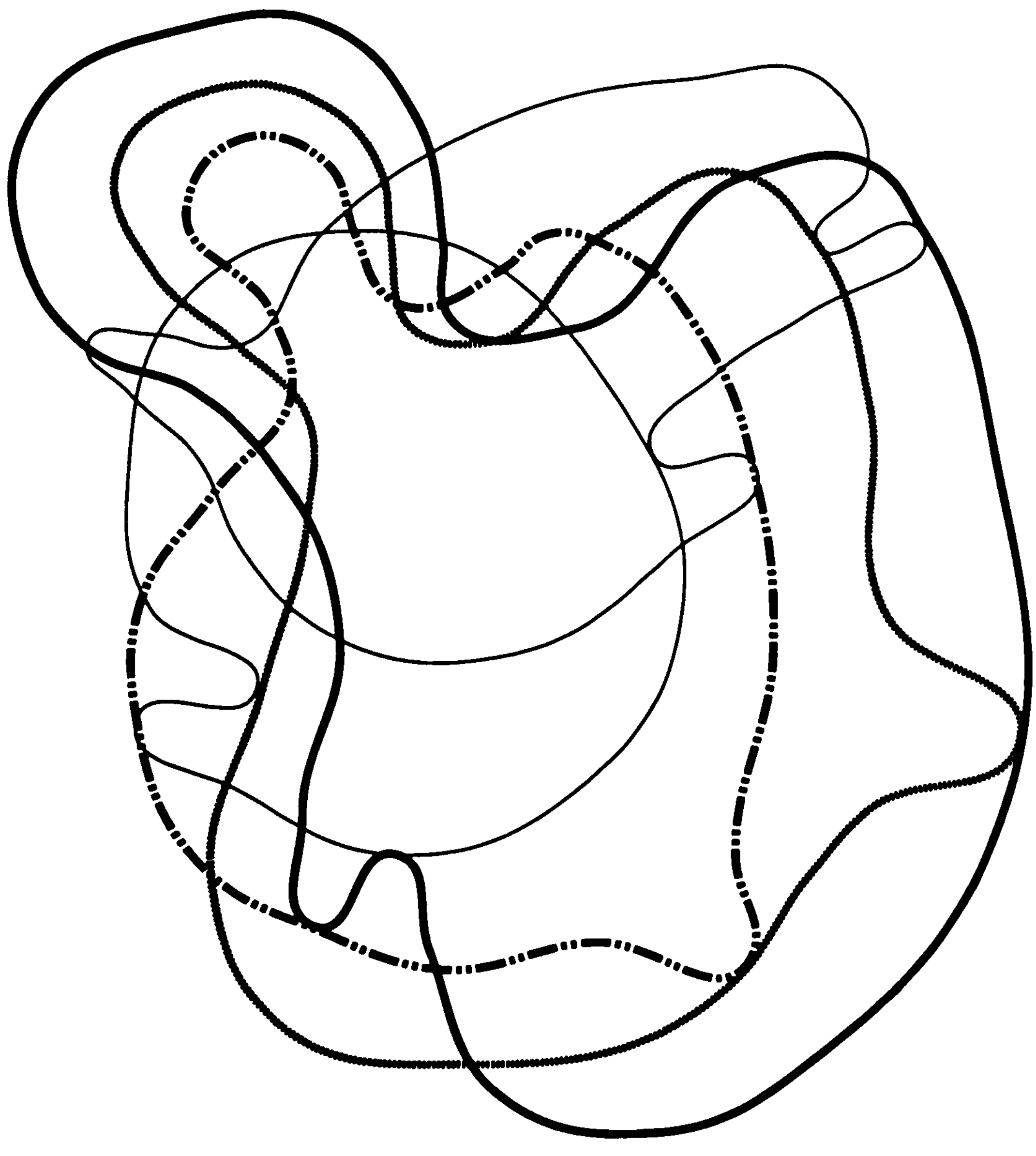
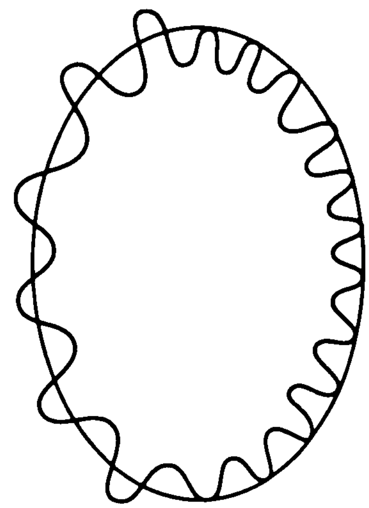


Figure 3.12

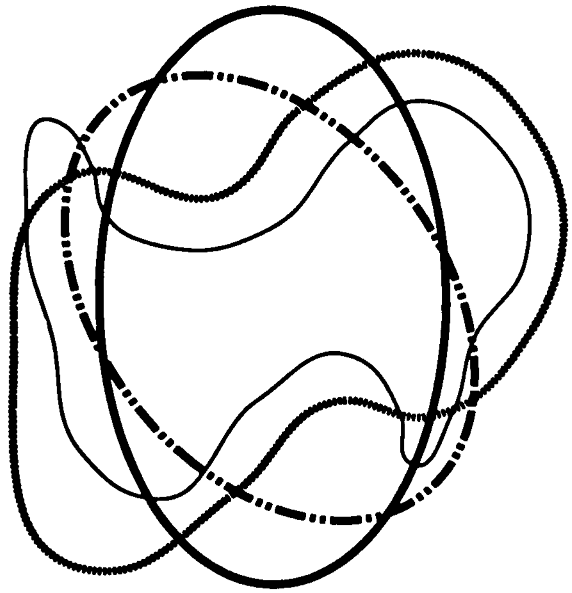


a digon-free (5,20,4,1)-arramgement

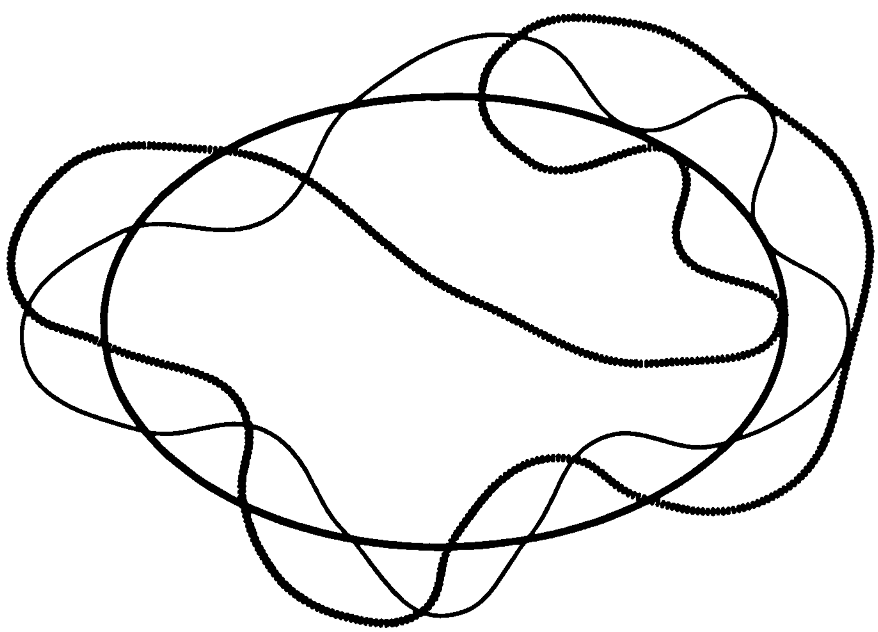
Figure 3.13



a (2,24,12,12)-arrangement

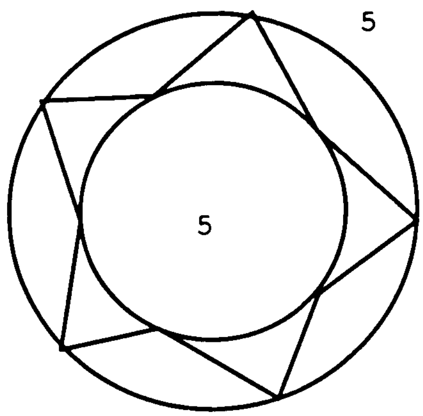


a digon-free (4,12,4,0)-arrangement

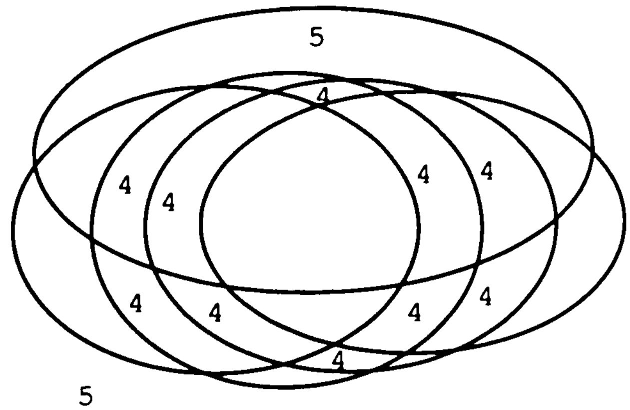


a digon-free (3,16,6,2)-arrangement

Figure 3.14

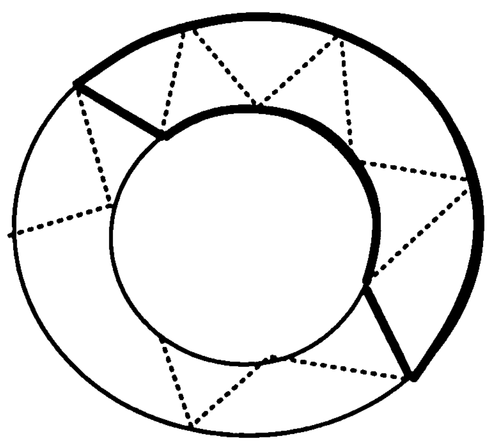


(a) $p_4 = 0$



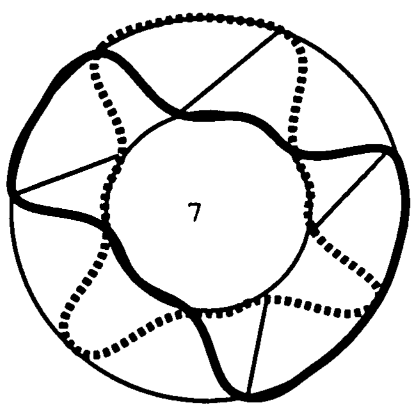
(b) $p_4 = 10 > 0$

Figure 3.15

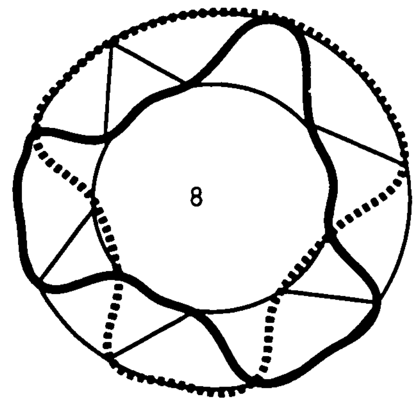


k-antiprism

Figure 3.16



(a) 7-antiprism



(b) 8-antiprism

Figure 3.17

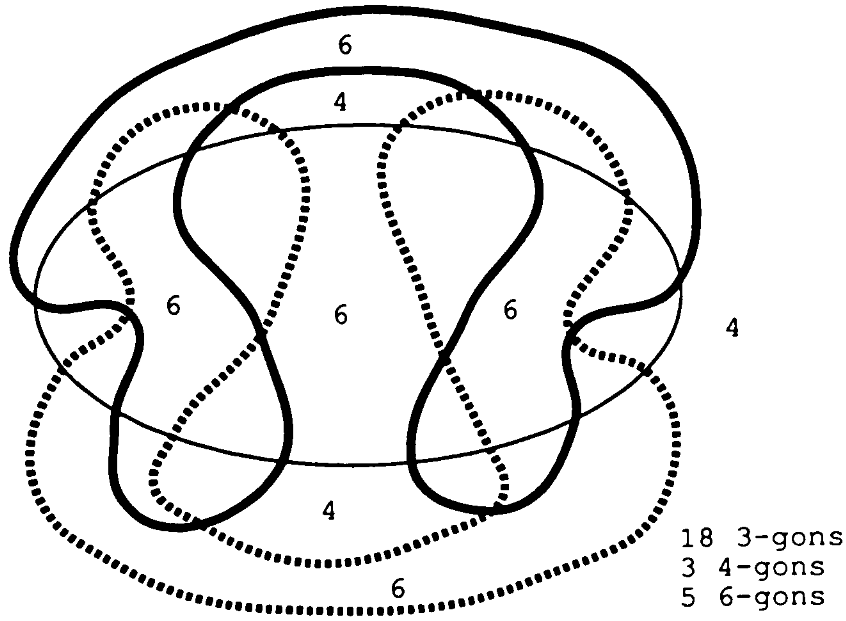


Figure 3.18 a digon-free (3,16,8,0)-arrangement

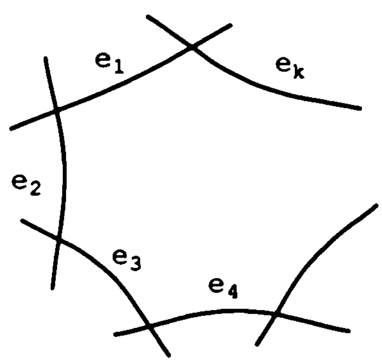


Figure 3.19

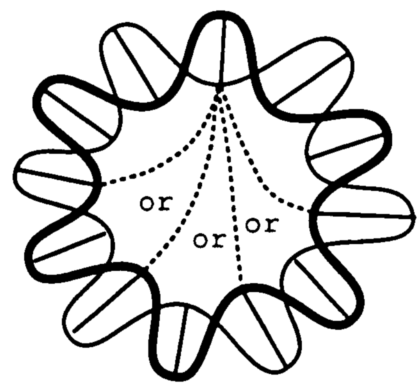


Figure 3.20

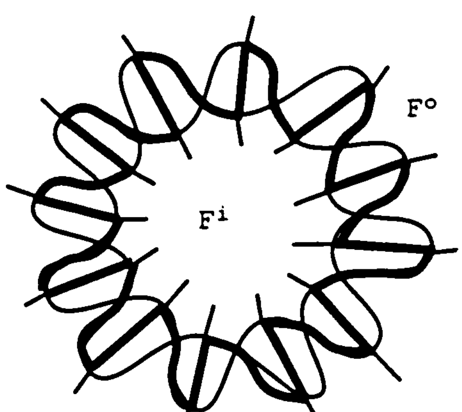


Figure 3.21

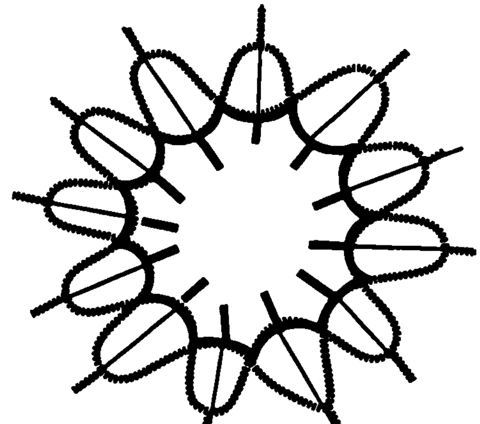
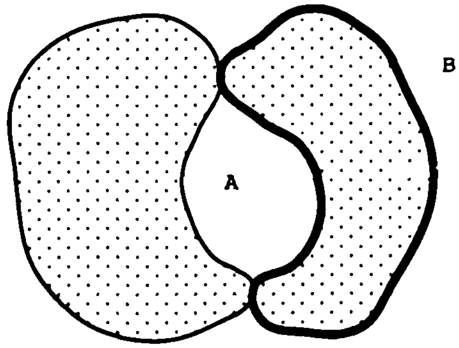


Figure 3.22



a (2,2,0,2)-arrangement

Figure 3.23

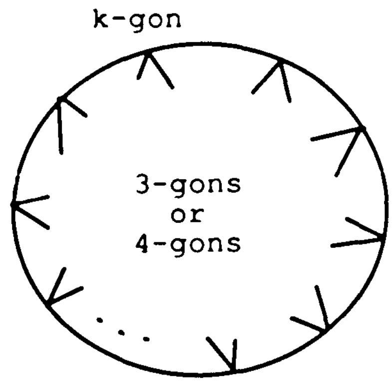
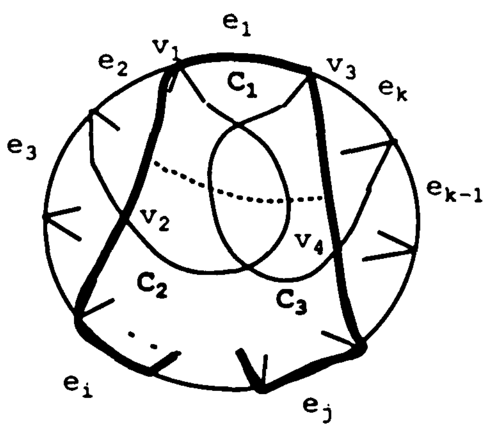
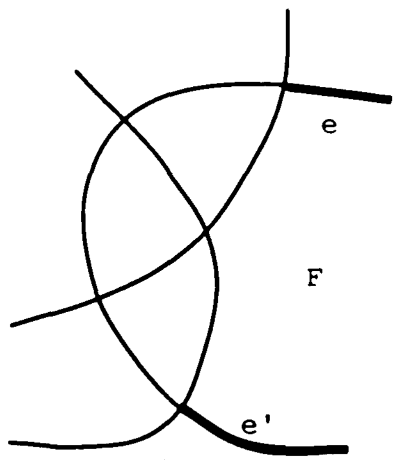


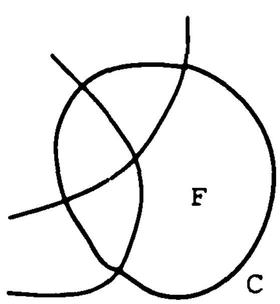
Figure 3.24 (a)



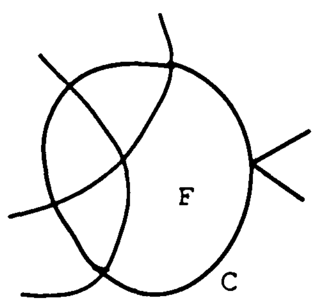
(b)



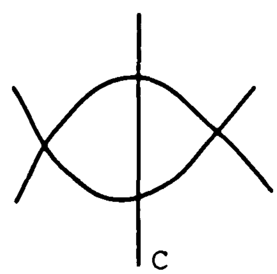
(c)



(d)



(e)



(f)

Figure 3.24

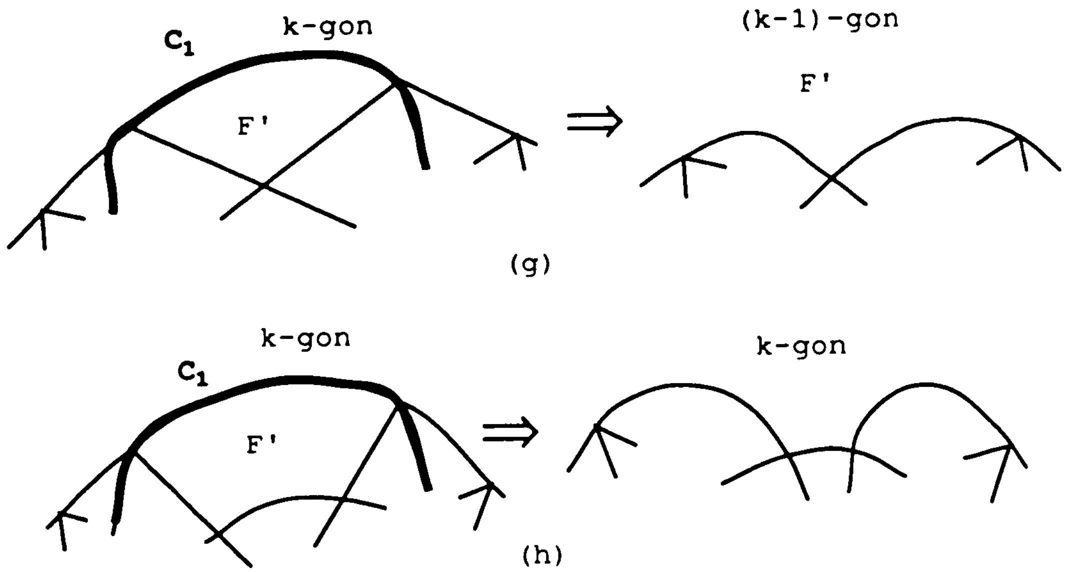


Figure 3.24

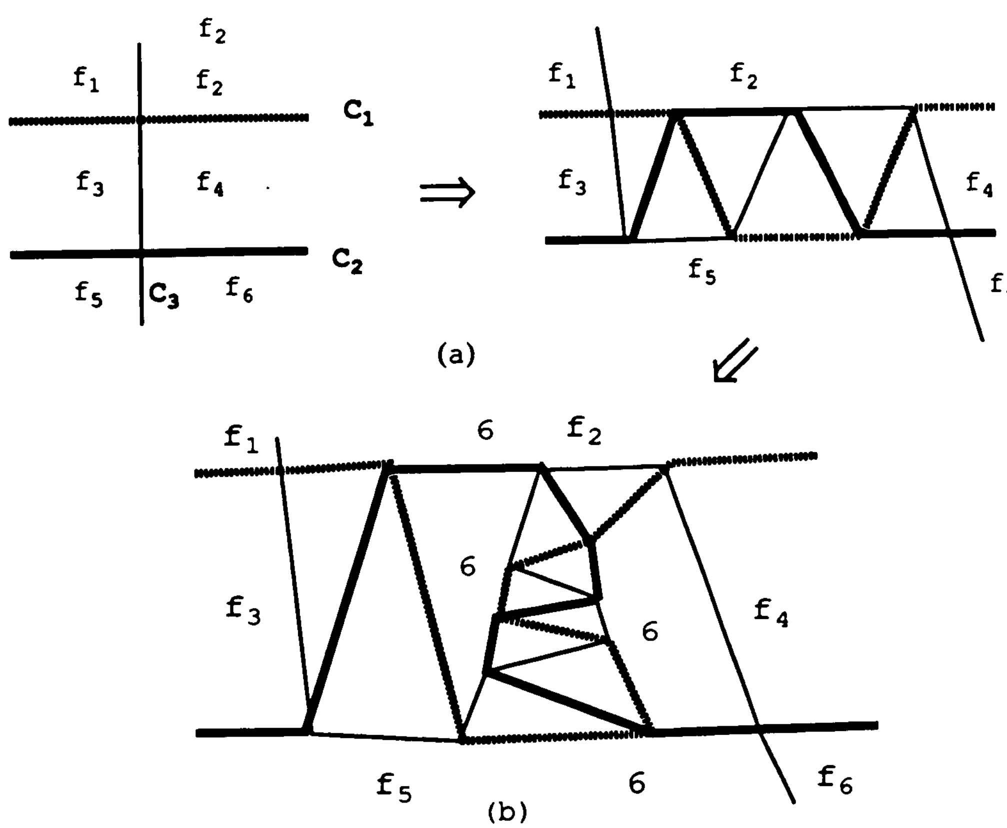
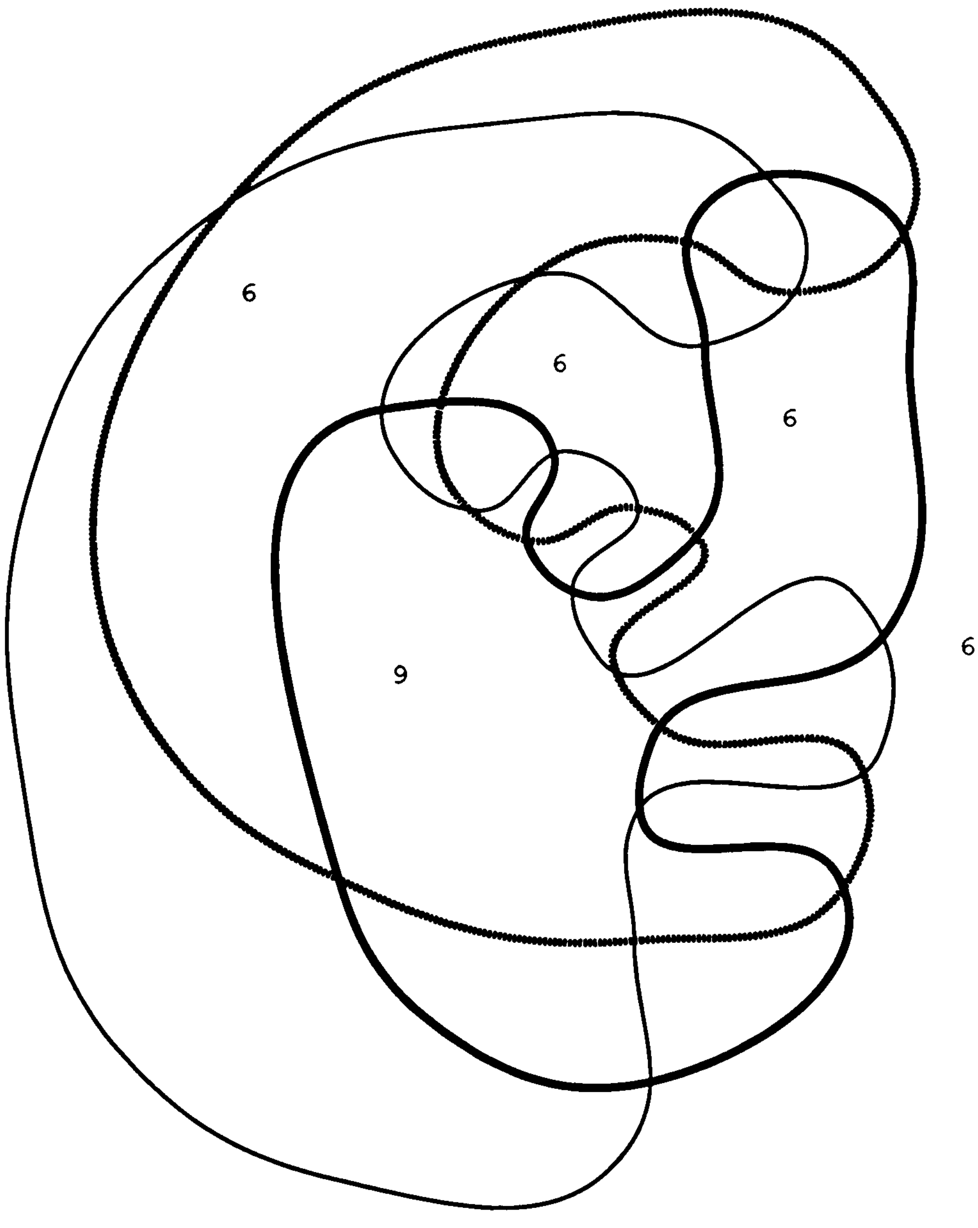


Figure 3.25



(c) $p_6 = 4$, $p_9 = 1$

Figure 3.25

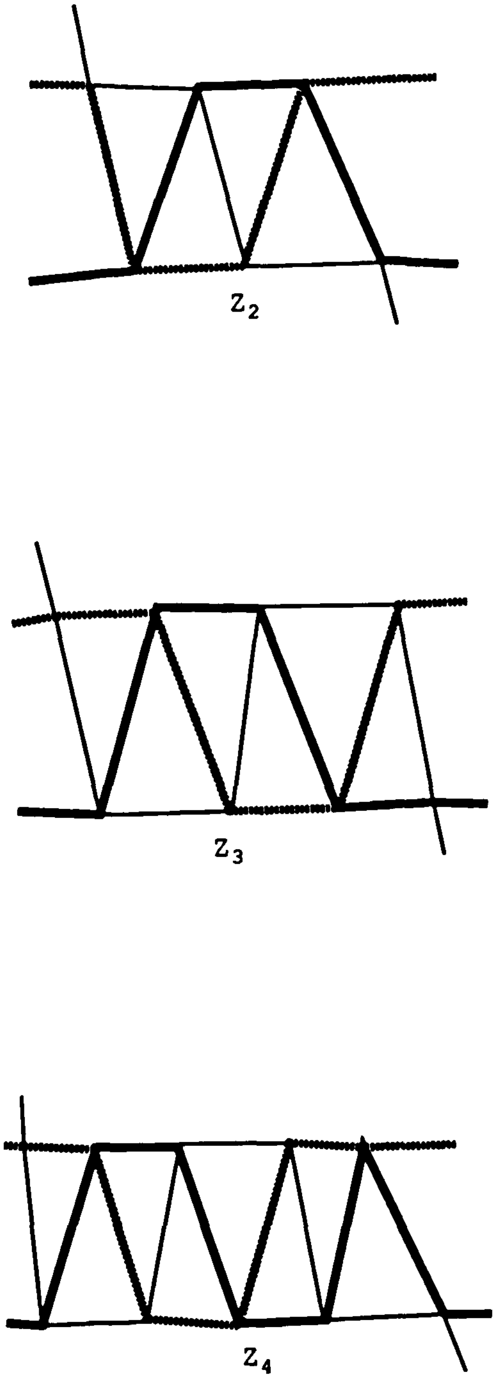


Figure 3.26

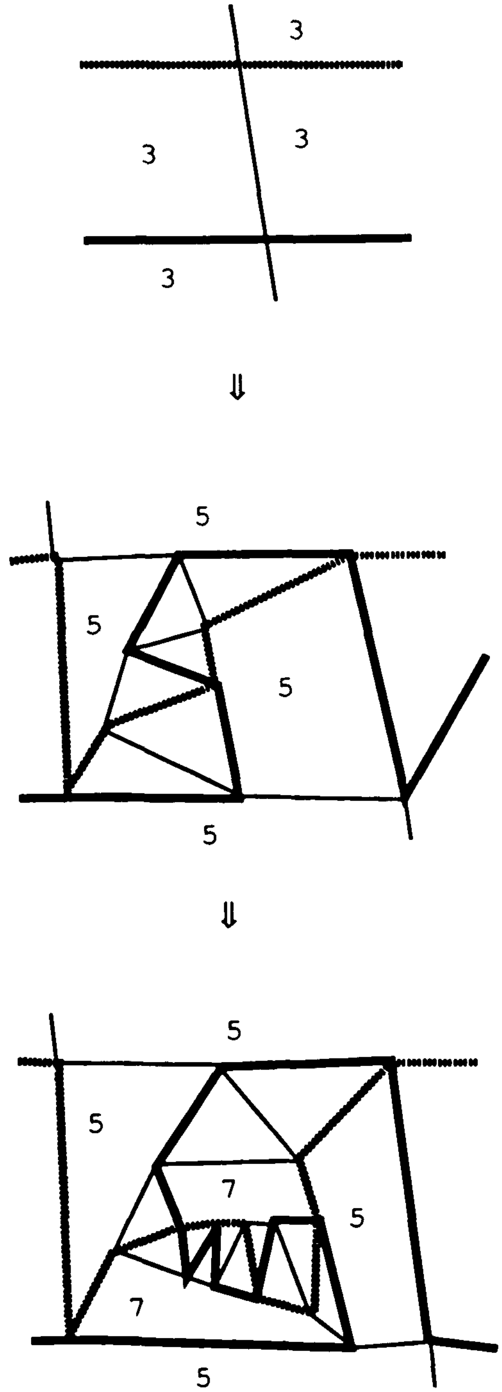
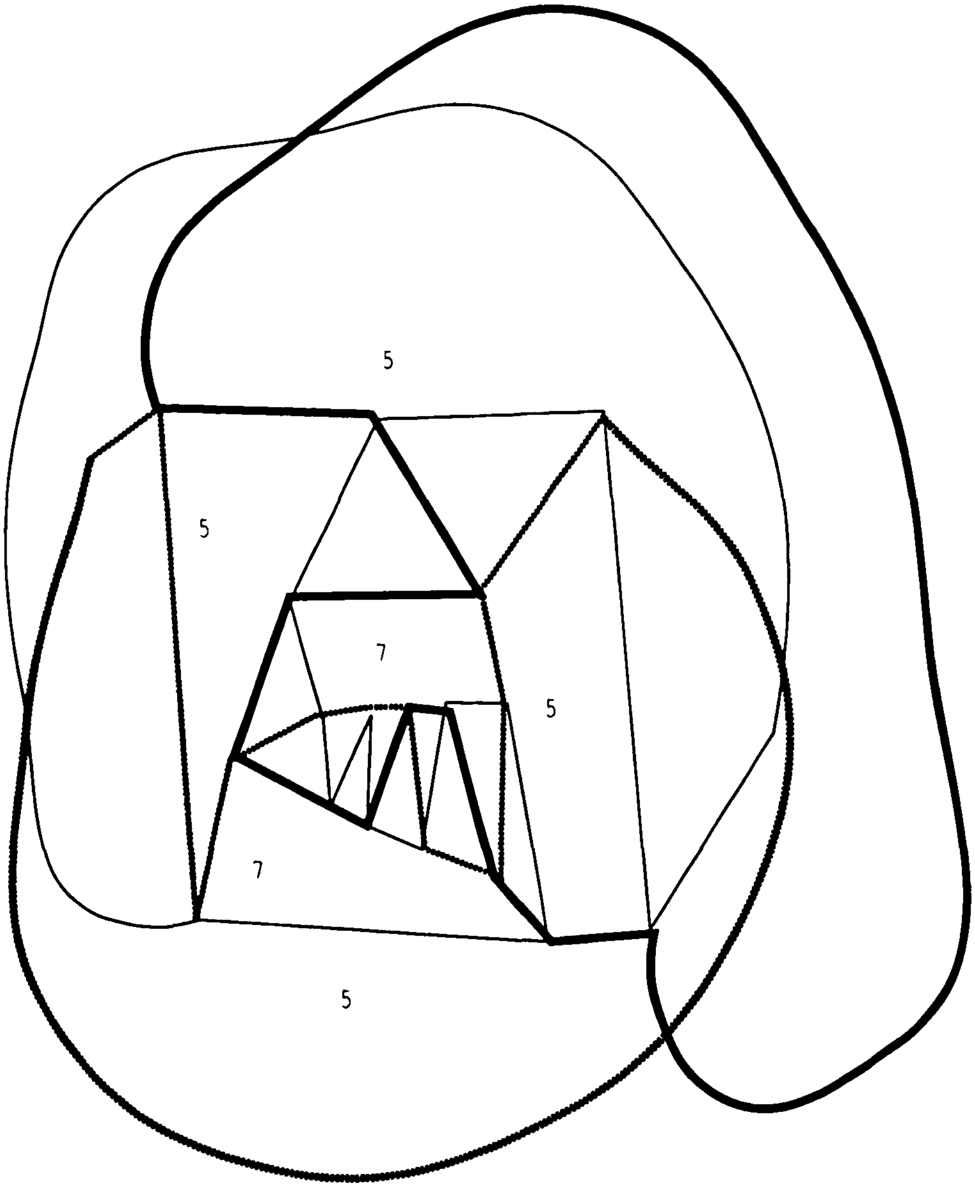


Figure 3.27 (a)



a digon-free (3,-,t,k)-arrangement

$p_3 = 16$, $p_5 = 4$ and $p_7 = 2$

Figure 3.27 (b)

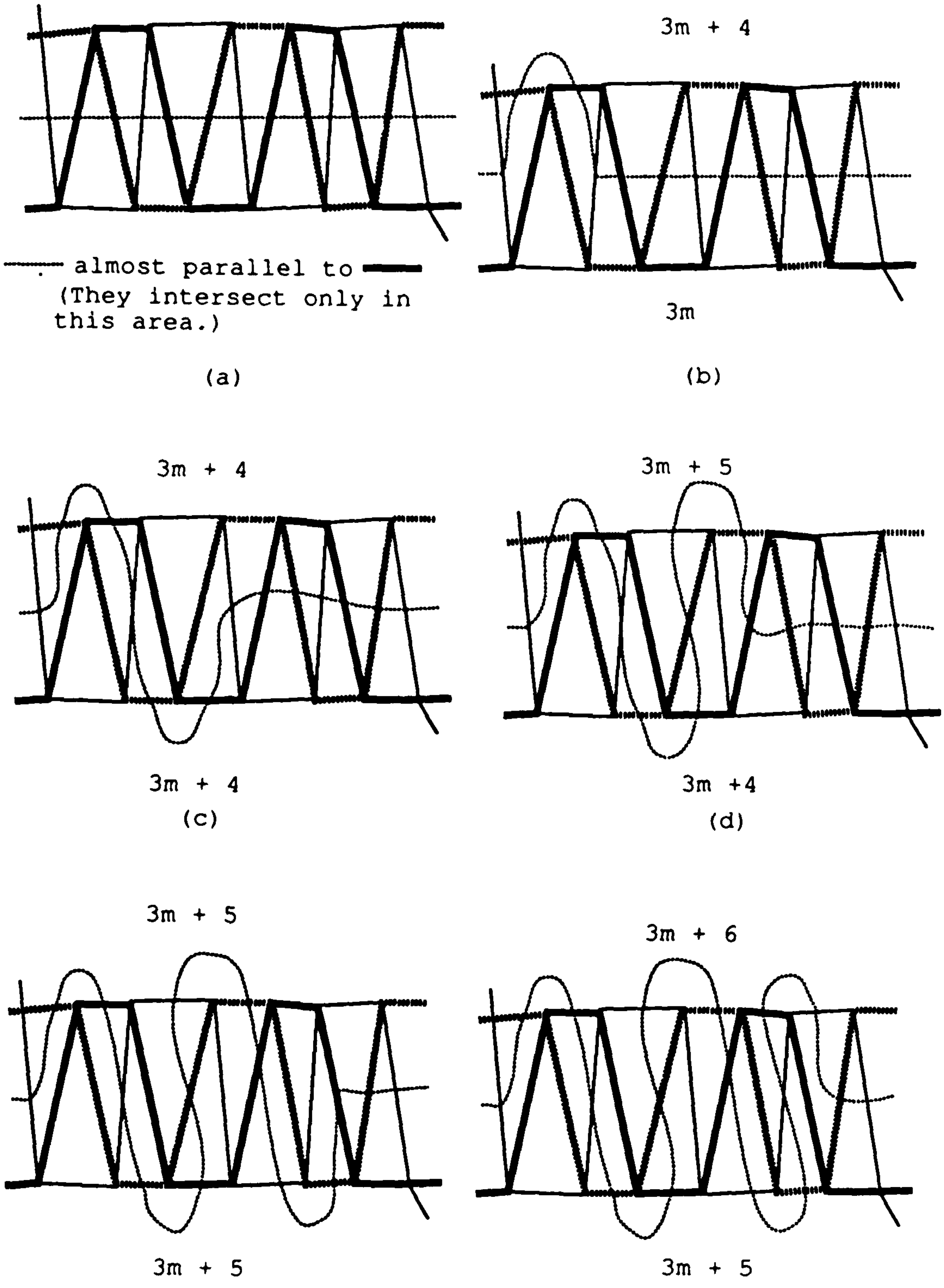

almost parallel to
(They intersect only in this area.)
(a)
3m + 4
3m
(b)
3m + 4
3m + 4
(c)
3m + 5
3m +4
(d)
3m + 5
3m + 5
(e)
3m + 6
3m + 5
(f)

Figure 3.28

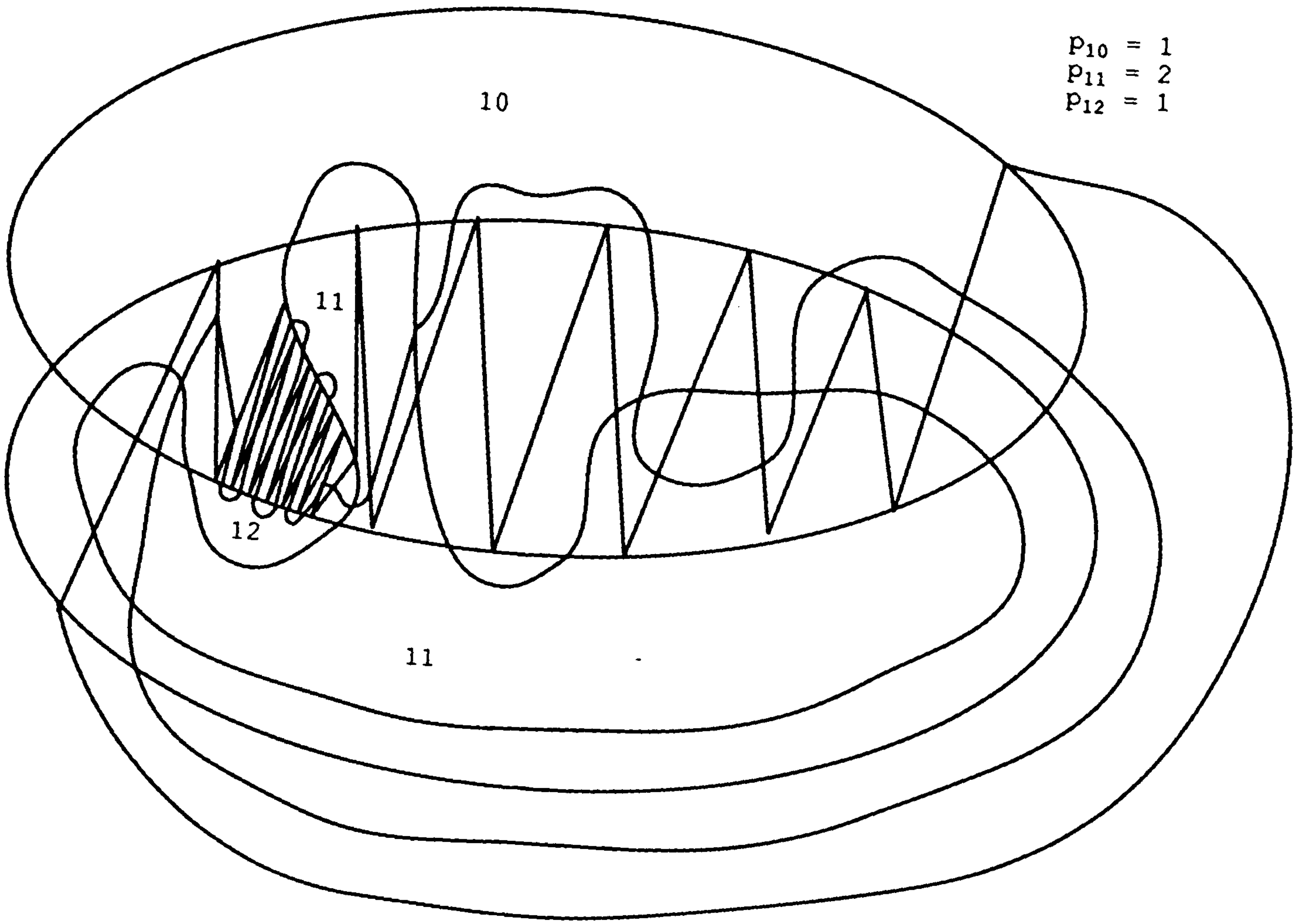


Figure 3.29

Appendix B.

Suppose that p and q are relatively prime positive integers, then every integer can be written as a linear combination of p and q. But, if we restrict the coefficients to non-negative integers, then all negative integers and some positive integers can not be written as a linear combination of p and q.

In this appendix, we will prove that the largest positive integer that cannot be written as a linear combination of p and q with non-negative coefficients is $p \cdot q - p - q$. For convenience, let us assume that p and q are two relatively prime positive integers such that $p = k \cdot q + r$ for some positive inters k and r $(0 < r < q)$.

Now, let us define some terms for the proof. Let $\mathbf{N}_0$ denote the set of all non-negative integers. Let $L(p,q)$ denote the set of all positive integers which can be written as a linear combination of p and q with non-negative coefficients, that is,

$$L(p,q) = \{\, ap + bq \mid a \in \mathbf{N}_0 \text{ and } b \in \mathbf{N}_0 \,\}.$$

And, let $L^c(p,q)$ be the set of all positive integers which are not contained in the set $L(p,q)$.

Proof of Theorem 2.6 First, let us claim several facts.

<u>Claim 1</u> Let n and b′ be two non-negative integers. If $n \in L(p,q)$, then so is $n + b'q$. If $n \in L^c(p,q)$, then so is $n - b'q$.

proof If $n \in L(p,q)$, then there exist non-negative integers a and b such that $n = a \cdot p + b \cdot q$. Hence,

$$n + b' \cdot q = (a \cdot p + b \cdot q) + b' \cdot q = a \cdot p + (b + b') \cdot q.$$

The latter part is the contraposition of the first assertion.

Claim 2 Let $r' \in \mathbf{Z}_q$, then there exists an integer $i \in \mathbf{Z}_q$ such that $r' \equiv i \cdot r \pmod q$.

proof We know that $p = k \cdot q + r$. Since p and q are relatively prime, there exist two integers α and β such that $\alpha \cdot p + \beta \cdot q = 1$. Hence,

$$\alpha(k \cdot q + r) + \beta \cdot q = 1$$

$$\Rightarrow \quad (\alpha \cdot k + \beta)q + \alpha \cdot r = 1$$

Therefore $r' = r' \cdot 1 \equiv r' \cdot \alpha \cdot r \pmod q$. If we let i be the residue of $\alpha \cdot r'$ modulo q, then we have done it.

Claim 3 Every integer can be written as a form of

$$ip + bq, \text{ where } i \in \mathbf{Z}_q \text{ and } b \in \mathbf{Z}.$$

proof Let m be an integer. Then, there are integers k' and r' such that

$$m = k' \cdot q + r', \quad r' \in \mathbf{Z}_q.$$

By Claim 2, there is an integer $i \in \mathbf{Z}_q$ such that $r' \equiv i \cdot r \pmod q$. That is, for some integer j, $i \cdot r = j \cdot q + r'$. Thus,

$$\begin{aligned} m &= k' \cdot q + r' \\ &= k' \cdot q + i \cdot r - j \cdot q \\ &= k' \cdot q + i \cdot r - j \cdot q + (i \cdot k \cdot q - i \cdot k \cdot q) \\ &= (i \cdot r + i \cdot k \cdot q) + (k' \cdot q - j \cdot q - i \cdot k \cdot q) \\ &= i \cdot (r + k \cdot q) + (k' - j - i \cdot k) \cdot q \\ &= i \cdot p + (k' - j - i \cdot k) \cdot q \end{aligned}$$

This completes the proof.

<u>Claim 4</u> Let us define two sets L^+ and L^- as following;

$$L^+ = \{ i \cdot p + b \cdot q \mid i \in \mathbf{Z}_q \text{ and } b \in \mathbf{N}_0 \},$$

$$L^- = \{ i \cdot p + b \cdot q \mid i \in \mathbf{Z}_q \text{ and } b \in \mathbf{Z} - \mathbf{N}_0 \}.$$

Then, $L^+ \cap L^- = \emptyset$, $L^+ = L(p,q)$ and $L^- = L^c(p,q)$.

<u>proof</u> Clearly, $L^+ \cap L^- = \emptyset$. If $m \in L^+$, then $m \in L(p,q)$. Suppose that $m \in L(p,q)$, then there exist two non-negative integers a and b such that $m = a \cdot p + b \cdot q$. By the division algorithm, there are two non-negative integers k' and r' ($r' \in \mathbf{Z}_q$) such that $a = k' \cdot q + r'$. Thus,

$$\begin{aligned} m &= a \cdot p + b \cdot q \\ &= (k' \cdot q + r')p + b \cdot q \\ &= k' \cdot q \cdot p + r' \cdot p + b \cdot q \\ &= r' \cdot p + (k' \cdot p + b) \cdot q. \end{aligned}$$

From this, we know that $L^+ = L(p,q)$. Therefore, $L^- = L^c(p,q)$.

Now, let us prove the main theorem. Let $A = pq - p - q$. Then, $A \in L^-$. To show that A is the greatest element in L^-, it suffices to show that all the integers between A and $A + q$ are the elements of $L(p,q)$. If it is true, then every integer larger than $A + q$ is an element of $L(p,q)$ by Claim 1.

Let $B = A + n$ where $1 \le n \le q$.

<u>Case 1</u> $n = q$. Then,

$$B = p \cdot q - p - q + q = (q-1)p + 0 \cdot q.$$

Thus, $B \in L(p,q)$.

<u>Case 2</u> $1 \le n < q$. Since $q - n \in \mathbf{Z}_q$, there is an integer m in $\mathbf{Z}_q$ such that $m \cdot r \equiv q - n \pmod{q}$ by Claim 2 (recall $p = k \cdot q + r$). Let $m \cdot r = l \cdot q + (q - n)$ for some non-negative integer l. Thus,

$$
\begin{aligned}
B &= p\cdot q - p - q + n \\
&= (q - 1)\cdot p - q + n \\
&= (q - 1)\cdot p - m\cdot p + m\cdot p - q + n \\
&= (q - 1 - m)\cdot p + m\cdot (k\cdot q + r) - (q - n) \\
&= (q - 1 - m)\cdot p + m\cdot k\cdot q + m\cdot r - (m\cdot r - 1\cdot q) \\
&= (q - 1 - m)\cdot p + m\cdot k\cdot q + 1\cdot q \\
&= (q - 1 - m)\cdot p + (m\cdot k + 1)\cdot q.
\end{aligned}
$$

Of course, $q - 1 - m$ and $m\cdot k + 1$ are non-negative integers. Therefore, $B \in L(p,q)$.

BIBLIOGRAPHY

[1] D. W. Barnette, On p-vectors of 3-polytopes, *J. Combinatorial Theory* 7 (1969), 99-103.

[2] J. A. Bondy and U. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., New York, 1976.

[3] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Wadsworth and Brooks, California, 1979.

[4] V. Eberhard, *Zur Morpologie der Polyeder*, Teubner, Leipzig, 1891.

[5] B. Grünbaum, *Convex Polytopes*, J. Wiley and sons, New York, 1967.

[6] B. Grünbaum, Some analogues of Eberhard's Theorem on convex polytopes, *Israel J. of Math.*, vol 6 (1968), 398-411.

[7] B. Grünbaum, Planar maps with prescribed types of vertices and faces, *Mathematica* 16 (1969), 28-36.

[8] B. Grünbaum, *Arrangements and Spreads*, Regional Conference Series in Mathematics No. 10, American Mathematical Society, 1972.

[9] P. Joffe, *Some Properties of Polytopal Graphs*, Ph.D. Thesis, CUNY Graduate Center, New York, 1983.

[10] L. H. Kauffman, *Formal Knot Theory*, Mathematical notes 30, Princeton university press, Princeton, 1983.

[11] A. Kotzig, Eulerian lines in finite 4-valent graphs and their transformations, *Theory of Graphs, Proceeding of a colloquium held at Tihany, Hungary, September 1966*, P.

Erdős and G. Katona, eds., Academic press, New York, 1969, 219-230.

[12] L. Lovász and M. L. Marx, A forbidden substructure characterization of Gauss Codes, *Acta Sci. Math.*, 38 (1976), 115-119.

[13] S. Lins, B. Richter and H. Shank, The Gauss Code problem off the plane, *Aequations Mathematicae* 33 (1987), 81-95.

[14] W. Meyer, Maximum face size in an arrangement of curves, *Israel J. of Math.* 14 (1973), 88-91.

[15] J. Malkevitch, *Proeprties of Planar Graphs with Uniform Vertex and Face Structure*, Memoirs Amer. Math. Soc. 99, Province, Rhode Island, 1970.

[16] J. Malkevitch, Eberhard's theorem for convex 3-polytopes, *Convexity and Related Combinatorial Geometry* (ed. D. Kay and M. Breen), Marcel Decker, New York, 1982, 209-214.

[17] J. Malkevitch, Polytopal Graphs, *Selected Topics in Graph Theory 3* (ed. L. W. Beineke and R. J. Wilson), Academic Press, 1988, 169-188

[18] P. Rosenstiehl and R. E. Tarjan, Gauss Codes, planar Hamiltonian graphs, and stack-sortable permutations, *J. of Algorithms* 5 (1984), 375-390.

[19] A. Schrijver, *Theory of linear and integer programming*, Wiley, New York, 1986.

[20] H. Shank, The theory of left-right paths. in *Combinatorial mathematics III*. Lecture notes in Math. 452. Springer, Berlin, 1975, 42-54.

[21] W. T. Tutte and C. A. B. Smith, Unicursal paths in a network of degree 4, *American Mathematical Monthly*, V. 48, 1941, 233-237.

[22] J. Zaks, The analogue of Eberhard's theorem for 4-valent graphs on the torus, *Israel J. of Math.* 9 (1971), 299-305.

[23] J. Zaks, On realizing symmetric 3-polytopes, *Israel J. of Math.*, vol 10 (1971), 244-251.

[24] J. Zaks, 6-valent analogues of Eberhard's theorem, *Israel J. of Math.* 18 (1974), 19-29.