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On the Limit Points

of the Largest Eigenvalues

# of Certain Symmetric Matrices

by

Michael Gargano

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Date

Professor Alan J. Hoffman
Chairman, Examining Committee

Professor Richard Sacksteder
Executive Officer

Professor Louis Auslander

Professor H. Peyton Young

Examining Committee

The City University of New York

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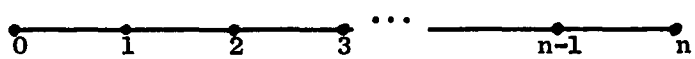
Section 1: Summary of Results

By a graph G we mean an ordered pair $(V(G),E(G))$ where $V(G)$ is the set of points and $E(G)$ is the set of edges (an edge is an unordered pair of distinct elements of $V(G)$.). We denote the cardinality of the set $V(G)$ by $|V(G)|$. We say that point i is adjacent to point j if there exists an edge in G joining i with j . The $(0,1)$ adjacency matrix $A(G) = A$ of a graph is a $|V(G)| \times |V(G)|$ symmetric matrix with zeroes on the diagonal and such that a_{ij} is 1 if point i is adjacent to point j and a_{ij} is 0 otherwise. The $(-1,1,0)$ adjacency matrix $B(G) = B$ of a graph is a $|V(G)| \times |V(G)|$ symmetric matrix with zeroes on the diagonal and such that b_{ij} is -1 if point i is adjacent to point j and b_{ij} is 1 otherwise. The $(0,1)$ eigenvalues of a graph are the eigenvalues of the associated $(0,1)$ adjacency matrix. The $(-1,1,0)$ eigenvalues of a graph are the eigenvalues of the associated $(-1,1,0)$ adjacency matrix. We note that every eigenvalue of a graph is a real number (since the adjacency matrix is symmetric with real entries). We denote the eigenvalues of a matrix in descending order as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_n$ and in ascending order as $\lambda^1 \leq \lambda^2 \leq \lambda^3 \ldots \leq \lambda^n$. Of course, $\lambda_i = \lambda^{n+1-i}$ for $1 \leq i \leq n$. The degree of a point is defined to be the number of points to which it is adjacent. A subgraph $H = (V(H),E(H))$ of a graph G has the property that $V(H)$ is a subset of $V(G)$, and $E(H)$ is a subset of $E(G)$. An induced subgraph H of a graph G is such that H is a subgraph of G and if $i,j \in V(H)$ are adjacent (not adjacent) in G then i,j are adjacent (not adjacent) in H . The complement of a graph $G = (V(G),E(G))$ is a graph $\tilde{G} = (V(\tilde{G}),E(\tilde{G}))$ with $V(\tilde{G}) = V(G)$ and i,j are adjacent in $\tilde{G}$ if and only if i,j are not adjacent in G . The clique graph K_n

has n points, any two distinct points of which are adjacent. The independent graph $\tilde{K}_n$ has n points, no two adjacent. Some graphs of special interest are:

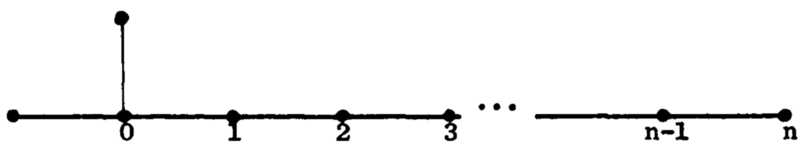
P_n



$(n \geq 0)$

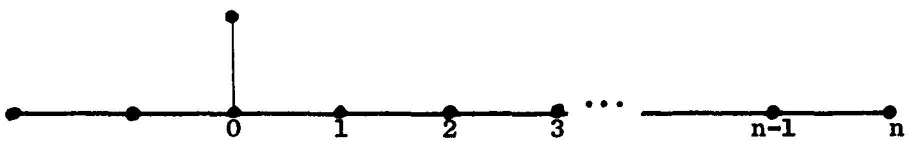
$. = P_0$ is called the trivial graph.

Y_n



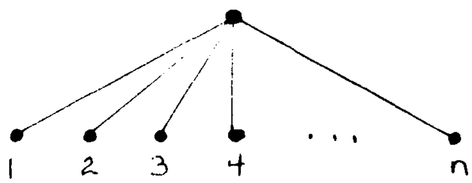
$(n \geq 0)$

Z_n



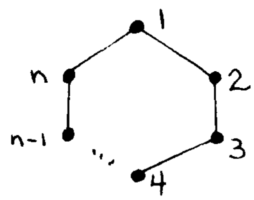
$(n \geq 0)$

$K_{1,n}$



$(n \geq 1)$

C_n



$(n \geq 3)$

We say that a graph G is connected if for any two points i and j we can find some n such that $H = P_n$ is a subgraph of G with $i=0$ and $j=n$. We define a connected component of G as a maximal connected subgraph of G.

Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers. Then b is a limit if and only if for any $\epsilon > 0$ there exists an integer N such that $|b-a_n| < \epsilon$ with $n > N$. The limit is non-trivial if there exists an integer $m > N$ such that $0 < |b-a_m|$. Let M be a set of real numbers, then b is a limit point of M if and only if there is a sequence $m_1, m_2, \ldots$ of points in M that have b as a non-trivial limit. Let C be the set of all $(0,1)$ adjacency matrices $A(G)$ where G is any graph. Let D be the set of all $(-1,1,0)$ adjacency matrices $B(G)$ where G is any graph. Let R be the set of largest eigenvalues $\lambda_1(A)$ where A is any matrix in C. Let S be the set of largest eigenvalues $\lambda_1(B)$ where B is any matrix in D. In [3] Hoffman has proved:

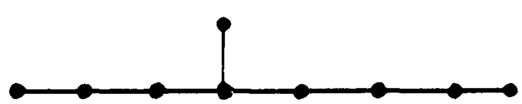
<u>Theorem</u>: Let $\tau = (\sqrt{5}+1)/2$ ("the golden ratio"). For $n = 1,2,3,\ldots$ let β_n be the positive root of $P_n(x) = x^{n+1} - (1 + x + x^2 + \ldots + x^{n-1})$. Let $\alpha_n = \sqrt{\beta_n} + \dfrac{1}{\sqrt{\beta_n}}$ then $\{\alpha_n\}$ is the set of all limit points of R smaller than $\sqrt{\tau} + \dfrac{1}{\sqrt{\tau}}$.

The main results of this investigation parallel the above theorem for $(-1,1,0)$ matrices.

<u>Theorem</u>: $\{4\cos(2\pi/2n+4) + 1 \mid n \geq 1\} \quad \cup \; \{1 + 2\sqrt{2\cos(2\pi/n) + 2} \mid n = 12, 18, 30\}$ is the set of all limit points of S smaller than 5.

<u>Theorem</u>: All the limit points of S bigger than or equal to 5 and smaller than $1 + 2\,\alpha_2$ are $\{1 + 2\lambda_1\}$ where λ_1 is the largest eigenvalue of the $(0,1)$ adjacency matrix associated with any of the following graphs:

(a) $K_{1,4}$

(b)  $= L$ or

(c) Z_n with $n \geq 6$.

Another interpretation of these results is the following. A set of $m+k$ $(k > 0)$ lines $\{\ell_1, \ell_2, \ldots, \ell_{m+k}\}$ through the origin in R^m

is said to be equiangular if $x_i \in \ell_i$ and $(x_i, x_i) = 1$ $(i = 1,2,\ldots,m+k)$ implies that $(x_i, x_j) = \pm\alpha$ and $0 < \alpha < 1$. We say that the angle of this equiangular set is $\arccos \alpha$. (See [4]).

We can ask the question: What numbers are limit points of angles of equiangular sets? We can then interpret the theorems above to yield the following information:

Let A be a $(-1,1,0)$ matrix then (1) $A - \lambda^1(A)I$ is seen to be positive semidefinite (not positive definite) since each eigenvalue is ≥ 0. Next consider $G = \left(\frac{-1}{\lambda^1(A)}\right) A + I$. G is also positive semidefinite since we have divided (1) by $-\lambda^1(A) > 0$. Therefore $G = CC^t$ where rank G = rank C and taking the rows of C to be x_i (note: m = rank G = rank C = # of non-zero eigenvalues < order G) we see that G is the Gramian $G = ((x_i, x_j))$ for some set of equiangular lines with angle $\alpha = \frac{1}{\lambda_1(-A)}$ and so $\alpha = \frac{1}{\lambda_1(B)}$ where $-A = B$ another $(-1,1,0)$ matrix. Therefore, if λ is a limit point of the largest eigenvalues of $(-1,1,0)$ matrices then $\arccos\left(\frac{1}{\lambda}\right)$ is a limit point of the set of possible angles for equiangular sets of lines.

Section 2: Preliminary Prerequisite Knowledge

Presented here is a collection of known or easily proven facts needed in the sequel. References are supplied.

<u>Theorem 2.1</u>: (Cauchy Interlacing Theorem) [5]

Let A be any real symmetric matrix of order n with eigenvalues $\lambda_n \le \dots \le \lambda_1$. Let B be a principal submatrix of order k with eigenvalues $\mu_k \le \dots \le \mu_1$, then

$$\lambda_{n-k+s} \le \mu_s \le \lambda_s \quad (s = 1,\dots,k) .$$

<u>Theorem 2.2</u>: (Courant-Weyl Inequalities) [5]

Let A and B be any real symmetric matrices of order n. Let $C = A + B$, then if $0 \le i,j,i+j+1 \le n$, we have $\lambda_{i+j+1}(C) \le \lambda_{i+1}(A) + \lambda_{j+1}(B)$ (In particular, $\lambda_1(C) \le \lambda_1(A) + \lambda_1(B)$) .

<u>Theorem 2.3</u>: (Ramsey's Theorem) [7]

Let $m,n \ge 0$ be given. Then there exists a number $N(m,n)$ such that for all graphs G with $N(m,n)$ points or more we have that G contains K_m or $\tilde{K}_n$ as an induced subgraph.

<u>Theorem 2.4</u>: (Perron-Frobenius Theorem) [5]

Let A be an $n \times n$ nonnegative indecomposable matrix, then:

(a) A has a real positive characteristic root r (a simple root) and $|\lambda_i| \le r$ for any other root λ_i of A,

(b) Corresponding to r is a positive eigenvector x,

(c) If $Ay = sy$ $(y > 0)$, then $r = s$ is the largest characteristic root of A,

(d) If $Ax \leq sx$ and $x > 0$ and $Ax \neq sx$, then $s > r$.

(e) If $a_{ij} \leq b_{ij}$ for all i,j then $\lambda_1(B) \geq \lambda_1(A)$.

Theorem 2.5: [5]

If A is a real symmetric $n \times n$ matrix then:

(a) $\lambda_1 = \max\limits_{x \neq 0} \dfrac{x^t A x}{x^t x}$

(b) $\lambda^1 = \min\limits_{x \neq 0} \dfrac{x^t A x}{x^t x}$

(c) If $x \neq 0$ then $\dfrac{x^t A x}{x^t x} = \lambda^1$ if and only if $Ax = \lambda^1 x$.

Theorem 2.6: [5]

$\lambda_1(A) \leq \lambda_1(A+J)$

Proof: $x^t(A+J)x =$

$x^t A x + x^t J x =$

$x^t A x + (x_1 + x_2 + \ldots + x_n)^2 \geq x^t A x$

and so by Theorem 2.5, $\lambda_1(A+J) \geq \lambda_1(A)$.

Theorem 2.7: [5]

Let A be any real symmetric matrix of order n then:

(a) distinct eigenvalues of A have corresponding eigenvectors which are orthogonal.

(b) let C be a principal submatrix of A and suppose $\lambda_1(C) = \lambda_1(A)$. If (y_i) is an eigenvector of C corresponding to $\lambda_1(C)$ then (x_j) is an eigenvector of A corresponding to $\lambda_1(A)$

where

$$x_j = \begin{cases} y_j & \text{if column } j \text{ is in } C \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.8: [3]

Let G be any graph and let A be the $(0,1)$ adjacency matrix of G. Then:

(a) If H is a subgraph of G then $\lambda_1(H) \le \lambda_1(G)$ (H need not be induced.)

(b) If H is a connected subgraph of G and $H \ne G$, then $\lambda_1(H) < \lambda_1(G)$,

(c) If G is connected and $x > 0$ and if $Ax \le mx$ and $Ax \ne mx$ then $m > \lambda_1(A)$.

(d) If G is connected and (aut) is an automorphism of G and $Ax = \lambda_1 x$, then if $(\text{aut})(i) = j$, then we have that $x_i = x_j$.

Proof: (a),(b) and (c) follow immediately from Theorem 2.4.

(d) Let S be the permutation matrix corresponding to (aut) and let A be the adjacency matrix of G. Then $\lambda_1 x = Ax = S^t ASx$. So we get $\lambda_1(Sx) = A(Sx)$. Therefore, by Theorem 2.4, $Sx = kx$ for some positive scalar k. But since $(Sx)^t(Sx) = (kx)(kx)$ we see $k = 1$ and so $Sx = x$. And the proof is complete.

Theorem 2.9 (Hurwitz) [9]:

Let $f_n(z)$ be a sequence of functions, each analytic in a region D bounded by a simple closed curve and let $f_n(z) \to f(z)$ uniformly in D. Also $f(z)$ is not the zero function. Let z_0 be an interior point of D; then z_0 is a zero of $f(z)$ if and only if z_0 is a limit point of the set of zeroes of the $f_n(z)$. (Here, if z_0 appears infinitely often, it is also considered a limit point.)

Section 3: Some Important Theorems

In this section we develop three important theorems which serve as major tools for proving the main theorems in Section 5. The first two theorems deal respectively with the $(-1,1,0)$ case and the $(0,1)$ case. In the first case we consider what happens to λ_1 when a large clique is adjoined to each point of a given graph. In the second case we consider what happens to λ_1 when a long path is adjoined to a single point of a given graph. Finally, we characterize those graphs whose $(0,1)$ adjacency matrices have $\lambda_1 < 2$. We begin with some lemmas:

Lemma 3.1 Let B be the $(-1,1,0)$ adjacency matrix of the clique graph K_n, then $\lambda_1 = 1$.

Proof: We see that $B = I - J$ where I is the identity matrix and J is the matrix of all ones. We see that there are $(n-1)$ eigenvectors associated with the eigenvalue 1, namely,

$$(1,-1,0,0,\ldots,0,0)$$
$$(0,1,-1,0,\ldots,0,0)$$
$$(0,0,1,-1,\ldots,0,0)$$
$$\vdots$$
$$(0,0,0,0,\ldots,1,-1)$$

and we also note that

$$(1,1,1,1,\ldots,1,1)$$

is the eigenvector associated with the eigenvalue $(1-n)$. It then immediately follows that $\lambda_1 = 1$.

Lemma 3.2 If G is such that G has at least two points, then the largest eigenvalue λ_1 of the $(-1,1,0)$ adjacency matrix B of G is bigger than or equal to 1.

Proof: If $G = K_n$, then $\lambda_1 = 1$ by Lemma 3.1. If G is not equal to a clique graph, then there exists a principal submatrix $J-I$ of order 2 of B and since this submatrix has largest eigenvalue 1, the Cauchy Interlacing Theorem implies that $\lambda_1(B) \geq 1$.

Lemma 3.3 Let B be the $(-1,1,0)$ adjacency matrix of the independent graph $\tilde{K}_n$, then $\lambda_1 = n-1$.

Proof: We note that $B = J-I$. We see that there are $(n-1)$ eigenvectors associated with the eigenvalue -1, namely,

$$
\begin{gathered}
(1,-1,0,0,\ldots,0,0) \\
(0,1,-1,0,\ldots,0,0) \\
(0,0,1,-1,\ldots,0,0) \\
\vdots \\
(0,0,0,0,\ldots,1,-1)
\end{gathered}
$$

and we note also that the vector

$$(1,1,1,1,\ldots,1,1)$$

is the eigenvector associated with the eigenvalue $(n-1)$. Therefore, $\lambda_1 = n-1$.

We next introduce the Seidel-switch [4].

Definition 3.4. Let A be the $(-1,1,0)$ adjacency matrix of some graph G, then we say A sw B (read A is switch equivalent to B) if and only if there exists a subset V_1 of points in G such that $DAD = B$ where $V_2 = V - V_1$ and D is defined by

$$
D = \begin{array}{c c}
 & \begin{array}{cc} V_1 & V_2 \end{array} \\
\begin{array}{c} V_1 \\ V_2 \end{array} & \left(\begin{array}{c|c} +I & 0 \\ \hline 0 & -I \end{array}\right)
\end{array}
$$

In other words, if $G = (V(G), E(G))$ is a graph, then B is of the form $H = (V(H), E(H))$ with $V(H) = V(G)$ and

$$E(H) = E(G) - \{ e \mid e \text{ is an edge in } E(G) \text{ and joins a point in } V_1 \text{ to a point in } V_2 \}$$
$$\cup \{(i,j) \mid i \in V_1 , j \in V_2 \text{ and } (i,j) \notin E(G)\} .$$

<u>Lemma 3.5</u> sw is an equivalence relation.

<u>Proof:</u> (1) $IAI = A$ (reflexive)

(2) if $DAD = B$ then

since $D = D^{-1}$, we also have $A = DBD$. (symmetric)

(3) if $DAD = B$ and

if $EBE = C$ then

$(ED)\, A\, (DE) = C$ and (ED)

is obviously of the required form. (transitive).

<u>Lemma 3.6</u> If A sw B then $\lambda_1(A) = \lambda_1(B)$.

<u>Proof:</u> If A sw B then in particular A is similar to B and from elementary matrix theory, we have $\lambda_1(A) = \lambda_1(B)$.

<u>Theorem 3.7</u> Let A be a fixed symmetric $m \times m$ matrix and let $\lambda_1(A) > 1$. Also let

$$B_n = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left(\begin{array}{c|c} A & -J \\ \hline -J & I-J \end{array}\right) \end{array}$$

then $\lim\limits_{n\to\infty} \lambda_1(B_n) = \lambda_1(A+J)$.

<u>Proof:</u> We first notice that $\lambda_1(B_n) \le \lambda_1(B_{n+1})$ by the Cauchy Interlacing Theorem and therefore the sequence is non-decreasing. Now consider

$$B_n = (I - J) + \left(\begin{array}{c|c} A + J - I & 0 \\ \hline 0 & 0 \end{array}\right)$$

therefore

$$\lambda_1(B_n) \le \lambda_1(I-J) + \lambda_1 \left(\begin{array}{c|c} A+J-I & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$= 1 + \lambda_1(A+J-I) \ .$$

So we see that $\{\lambda_1(B_n)\}$ is non-decreasing and bounded above by $1 + \lambda_1(A+J-I)$. It must therefore have some limit, say h . (We note here that although h is the limit of the sequence $\{\lambda_1(B_n)\}$, it may <u>not</u> be the limit point of the set $\{\lambda_1(B_n)\}$.)

Consider the $(n-1)$ eigenvectors of B_n :

$$f_1 = (0,0,\ldots,0;1,-1,0,0,\ldots,0,0)$$

$$f_2 = (0,0,\ldots,0;0,1,-1,0,\ldots,0,0)$$

$$f_3 = (0,0,\ldots,0;0,0,1,-1,\ldots,0,0)$$

$$\vdots$$

$$f_{n-1} = (0,0,\ldots,0;0,0,0,0,\ldots,1,-1)$$

associated with the eigenvale $+1$. Now let us consider the eigenvector $v = (x_1,x_2,\ldots,x_m;y_1,\ldots,y_n)$ associated with λ_1 . Since $\lambda_1 > 1$ we note that v must be orthogonal to the $(n-1)$ eigenvectors f_i $(i=1,2,\ldots,n-1)$. It follows that $v = (x_1,x_2,\ldots,x_m;y,y,\ldots,y)$. So $B_n \cdot v^t = \lambda v^t$ is equivalent to the system:

$$\text{(i)} \quad Ax^t - J(y,y,\ldots,y)^t = \lambda x^t$$

$$\text{(ii)} \quad -Jx^t + (y,y,\ldots,y)^t - J(y,y,\ldots,y)^t = \lambda(y,\ldots,y)^t$$

But the n equations of (ii) each are:

$$-(1,\ldots,1) \cdot x^t - (n-1+\lambda)y = 0 .$$

We therefore now consider the system:

$$\text{(i)}' \quad (A-\lambda I) \cdot x^t - J \cdot (y,\ldots,y)^t = 0$$

$$\text{(ii)}' \quad -(1,1,\ldots,1) \cdot x^t - (n-1+\lambda)y = 0 .$$

We are therefore interested in the largest root of:

$$\det \left(\begin{array}{c|c} & -1 \\ & -1 \\ A - \lambda I & \vdots \\ & -1 \\ \hline -1 \ -1 \ \ldots \ -1 & \frac{1}{n} - 1 - \frac{\lambda}{n} \end{array} \right) = 0 \qquad (*)$$

Since λ is bounded above, as $n \to \infty$ we get:

$$\det \left(\begin{array}{c|c} & -1 \\ & -1 \\ A - \lambda I & \vdots \\ & -1 \\ \hline -1 \ -1 \ \ldots \ -1 & -1 \end{array} \right) = 0 \qquad (**)$$

Therefore λ_1 is the largest root of (*) implies λ_1 is a root of (**) but maybe not the largest. But by setting $f_n(\lambda) = (*)$ and noting that (as $n \to \infty$) $f_n(\lambda) \to (**) = f(\lambda)$ (this is true, since $f_n(\lambda) - f(\lambda) = \left(\frac{2-\lambda}{n}\right) \cdot \det(A-\lambda I)$.) and also choosing D such that all the roots of $f(\lambda)$ are inside D, we see that λ_1 must be the largest root of (**) by Hurwitz' Theorem.

We may therefore consider the system:

$$\text{(i)}'' \quad (A-\lambda\, I)w^t - (1,1,\ldots,1)^t\, z = 0$$

$$\text{(ii)}'' \quad -(1,1,\ldots,1)w^t - z = 0\ .$$

However, (ii)$''$ implies that $z = -(1,1,\ldots,1)w^t$ and so by substitution into (i)$''$ we have that

$$(A-\lambda\, I)w^t - (1,1,\ldots,1)^t \cdot (-(1,1,\ldots,1)w^t) = 0$$

which is equivalent to

$$(A+J-\lambda\, I)w^t = 0$$

and so $\lim_{n\to\infty} \lambda_1(B_n) = \lambda_1(A+J)$. And the proof is complete.

<u>Theorem 3.8</u> Let A be the $(-1,1,0)$ adjacency matrix of a graph G and let B_n be as in Theorem 3.7. Then $\lim_{n\to\infty} \lambda_1(B_n) = \lambda_1(A+J) = \lambda_1(I+2A(G))$ where $A(G)$ is the $(0,1)$ adjacency matrix of $\tilde{G}$.

<u>Proof</u>: This is simply a restatement of Theorem 3.7 in a form for which we have a later use.

<u>Definition 3.9</u>. Let A_1 be any matrix and let A_0 be obtained from A_1 by deleting the last row and last column. Let A_i $(i=2,3,\ldots)$ be defined recursively by:

$$A_i = \left(\begin{array}{c|c} & 0 \\ & 0 \\ A_{i-1} & \vdots \\ & 0 \\ & 1 \\ \hline 00\ldots01 & 0 \end{array}\right)$$

Let $P_i(x)$ be the associated characteristic polynomial of the matrix A_i. (We shall write P_i for short.) Also $Q = P_0$ and $P = P_1$.

We also define

$$\theta = \left(x + \sqrt{x^2 - 4}\right)/2 .$$

One may easily see now that

$$\frac{1}{\theta} = \left(x - \sqrt{x^2 - 4}\right)/2$$

and also $\theta + \frac{1}{\theta} = x$

and $\theta - \frac{1}{\theta} = \sqrt{x^2 - 4}$

which follow immediately from the definition of θ. For the next theorem also see [6].

<u>Theorem 3.10</u> Let A_1 be the $(o,1)$ adjacency matrix of a graph G and let $A_0, A_1, A_2, \ldots$ be defined as in Definition 3.9. Then for $n \geq 2$ we have

$$P_n = \frac{(P-\theta\, Q)\left(\frac{1}{\theta}\right)^n - \left(P - \frac{1}{\theta}\, Q\right)\theta^n}{\frac{1}{\theta} - \theta}$$

where $\theta \neq \pm 1$.

<u>Proof</u>: We notice that

$$Q = P_0 = \det(A_0 - x\, I)$$

$$P = P_1 = \det(A_1 - x\, I)$$

and $P_{n+1} = xP_n - P_{n-1}$.

Also if $n \geq 2$ then

$$\text{(i)} \quad P_{n+1} - \theta\, P_n = (x-\theta)(P_n - \theta\, P_{n-1})$$

and

$$\text{(ii)} \quad P_{n+1} - \frac{1}{\theta} P_n = \left(x - \frac{1}{\theta}\right)\left(P_n - \frac{1}{\theta} P_{n-1}\right)$$

are easily seen to be true. Since $\theta + \frac{1}{\theta} = x$ we find equivalently that:

$$\text{(i)}' \quad P_{n+1} - \theta P_n = \frac{1}{\theta} (P_n - \theta P_{n-1})$$

and

$$\text{(ii)}' \quad P_{n+1} - \frac{1}{\theta} P_n = \theta (P_n - \frac{1}{\theta} P_{n-1}) .$$

Now using (i) the factor $\left(\frac{1}{\theta}\right)$ can be pulled out $(n-1)$ times in (i)$'$. Similarly using (ii) the factor (θ) can be pulled out $(n-1)$ times in (ii)$'$. We therefore get:

$$\text{(i)}'' \quad P_{n+1} - \theta P_n = \left(\frac{1}{\theta}\right)^n (P-\theta Q)$$

and

$$\text{(ii)}'' \quad P_{n+1} - \frac{1}{\theta} P_n = (\theta)^n \left(P - \frac{1}{\theta} Q \right) .$$

Now by subtracting (ii)$''$ from (i)$''$ we see that

$$P_n = \frac{(P-\theta Q)\left(\frac{1}{\theta}\right)^n - \left(P - \frac{1}{\theta} Q\right)(\theta)^n}{\frac{1}{\theta} - \theta}$$

where $\theta \neq \pm 1$. And we are finished.

<u>Theorem 3.11</u> Let A_i $(i=0,1,2,\ldots)$ be the $(0,1)$ adjacency matrices of a graph sequence $\{G_i\}$. Also assume for each graph $\lambda_1 \neq 2$, then $\lim_{i\to\infty} \lambda_1(A_i)$ is the largest root of $\theta P - Q = 0$.

<u>Proof</u>: Consider

$$P_i = \frac{(P-\theta Q)\left(\frac{1}{\theta}\right)^i - \left(P - \frac{1}{\theta} Q\right)(\theta)^i}{\frac{1}{\theta} - \theta} = 0$$

which implies

$$(P-\theta\, Q) \left(\frac{1}{\theta}\right)^{i} - \left(P-\frac{1}{\theta}\, Q\right) (\theta)^{i} = 0$$

and so

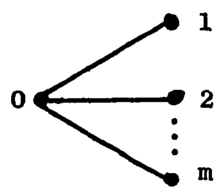
$$\left(\frac{1}{\theta}\right)^{2i} (P-\theta\, Q) - \left(P-\frac{1}{\theta}\, Q\right) = 0$$

and as $i \to \infty$ we have that the limit of the largest eigenvalues is the largest root of the polynominial equation

$$\theta\, P - Q = 0 \; .$$

<u>Lemma 3.12</u>. Consider the graph

$K_{1,m}$ =



with (0,1)

adjacency matrix A then $\lambda_1 = \sqrt{m}$.

<u>Proof</u>: We note that A is $(m+1) \times (m+1)$ and

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & & & \vdots & & & \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

The characteristic equation is easily calculated to be:

$$x^{m-1} (x^2 - m) = 0 \; .$$

Therefore, $x = \sqrt{m} = \lambda_1$.

<u>Lemma 3.13</u>. If $\{G_i\}$ is a sequence of connected graphs (with $|V_i| \to \infty$) then either (1) the size of the largest degree increases, or

(2) the size of the diameter increases

or both.

<u>Proof</u>: Let $D = \max(\deg(v))$ and let d be the diameter. We can assume that $D \geq 3$ else $d \to \infty$. Now since

$$|V| \leq 1 + D + D(D-1) + D(D-1)^2 + \dots + D(D-1)^{d-1}$$
$$= 1 + \sum_{i=0}^{d-1} D(D-1)^i \leq D^d + 1 \ .$$

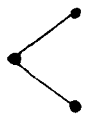
Therefore, we may conclude that $\log(|V|-1) \leq d(\log D) \leq dD$. But $d \cdot D \leq (\max\ (d,D))^2$, therefore $\sqrt{\log(|V|-1)} \leq \max\ (d,D)$. Since $|V_i| \to \infty$ we can easily see that $D \to \infty$ or $d \to \infty$ or both which was to be shown.

<u>Lemma 3.14</u>. $\lim_{n\to\infty} \lambda_1(P_n) = 2$ where $\lambda_1(P_n)$ is the largest eigenvalue of the $(0,1)$ adjacency matrix A of the graph P_n .

<u>Proof</u>: Since $\lambda_1(P_n) = \max_{x \neq 0} \frac{x^t A t}{x^t x}$ we see using $x^t = (1,1,\dots,1)$ that $\lambda_1(P_n) \geq \frac{2(n-1)}{n} = 2 - \frac{2}{n}$. We see that $\lim_{n\to\infty} \lambda_1(P_n) \geq 2$. Now by Theorem 3.11, if $\lim_{n\to\infty} \lambda_1(P_n) > 2$ then it is the largest root of $(x^3-2x)\left(\frac{x+\sqrt{x^2-4}}{2}\right) - (x^2-1) = 0$. But this equation has no largest root. Therefore, we must have that $\lim_{n\to\infty} \lambda_1(P_n) = 2$ and we are done.

<u>Lemma 3.15</u>. 1 is not a limit point of S (the set of largest eigenvalues for the $(-1,1,0)$ matrices).

<u>Proof</u>: Let $\{G_i\}$ be a sequence with limit 1 . We can assume each G_i is connected with the largest possible switch equivalent clique. Without loss of generality no G_i is a clique. Pick any point p in G_i . We can take p to be adjacent to each other vertex in G_i by using a switch. In $G_i - p$ we can find two non-adjacent vertices (else G_i is a clique).

Therefore,  $= K_{1,2}$ is an induced subgraph of G_i . But $\lambda_1(K_{1,2}) = 2$ and so $\lambda_1(G_i) \geq 2$. Note we have also shown:

<u>Lemma 3.16</u> $\lambda_1(G) = 1$ if and only if G sw K_n for some n . And if $G \not\cong K_n$ then $\lambda_1(G) \geq 2$. For the following also see [8,4].

<u>Theorem 3.17</u>. Let C be a graph of the (0,1) type. $\lambda_1(C) < 2$ if and only if C is one of the following graphs:

(1) $C = P_n$ $(n=0,1,2,\ldots)$

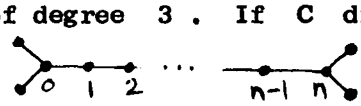
(2) $C = Y_n$ $(n=0,1,2,\ldots)$

(3) C is any of Z_2, Z_3, Z_4

<u>Proof</u>: If C is a circuit or contains a circuit, then λ_1 would be at least 2 . Therefore C must be a tree.

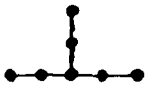
(a) Since $\lim_{n\to\infty} \lambda_1(P_n) = 2$ (by lemma 3.14), we see C may be any of $P_0, P_1, \ldots$ graphs.

(b) C cannot have a vertex of degree 4 or more. If C did have a vertex of degree 4 or more then $2 = \sqrt{4} = \lambda_1(K_{1,4}) \leq \lambda_1(C)$.

(c) C cannot contain two vertices of degree 3 . If C did contain two vertices of degree 3 then 

$= G$ $(n=0,1,2,\ldots)$ is a subgraph of C . But $\lambda_1(G) = 2$ since the associated eigenvector is $(1,1,2,2,2,\ldots,2,1,1)$.

(d) If C is a tree with a vertex of degree 3 then

$G =$  cannot be a subgraph. One can easily calculate the characteristic polynominial equation to be $(x^2-1)^2 \cdot (x^3-4x) = 0$ and so $\lambda_1(G) = 2 \le \lambda_1(C)$.

(e) Similarly, $G =$  has $\lambda_1(G) = 2$ and so G cannot be contained in C .

(f) If $G =$  then the characteristic equation is : $x^2(x^2-1)(x^2-2)(x^2-4) = 0$ and so $\lambda_1(G) = 2$ and again G cannot be a subgraph of C .

(g) Finally, we see that $\lim_{n\to\infty} \lambda_1(Y_n) = 2$. For each n , P_n is a subgraph of Y_n and therefore by Theorem 2.1 $\lim_{n\to\infty} \lambda_1(Y_n) \ge 2$.

Consider $x^t = (1,1,2,2,\ldots,2,2)$. We have then that $Ax \le 2x$ and so by Theorem 2.8 we have $\lambda_1(Y_n) < 2$ for each n . And so $\lim_{n\to\infty} \lambda_1(Y_n) = 2$.

Section 4: The Required Graphs.

In this section we find the type of graph sequence which has as a limit point of $\{\lambda_1(G_i)\}$ a number less than 5.

Let $\{G_i\}$ be a sequence of distinct graphs. Since each graph, G_i, is distinct, we see that $|V(G_i)| \to \infty$.

<u>Theorem 4.1</u>. If $\lambda_1(G)$ is a $(0,1)$ eigenvalue for some connected graph G then $1 + 2\lambda_1(G)$ is a limit poing of S. In particular, to find <u>all</u> the limit points $\{x\}$ of S such that $3 \leq x < 5$, it suffices to find all those non-trivial graphs C such that $\lambda_1(C) < 2$ (here C is considered in the $(0,1)$ sense).

Before proving Theorem 4.1, it is of interest to see what the smallest possible limit point of S can be.

<u>Theorem 4.2</u>. 3 is the smallest possible limit point of S. (Later we shall see that 3 is in fact a limit point and so 3 is the smallest limit point of S.)

<u>Proof</u>: Let $\{G_i\}$ be a sequence of distinct graphs. We know that $\lim_{i\to\infty} \lambda_1(G_i) = h > 1$. We use Ramsey's Theorem to see that as $i \to \infty$, the size of the largest clique (considered as an induced subgraph) or the size of the largest independent subgraph must increase.

Consider the following two cases:

<u>Case 1</u>. If there exists a subsequence of $\{G_i\}$ such that the size of the largest independent subgraph increases, then by Cauchy's Interlacing Theorem together with Lemma 3.3 we see that $\lambda_1(G_i) \to \infty$ as $i \to \infty$.

<u>Case 2</u>. If Casel does not occur then we are assured by Ramsey's Theorem that there exists a subsequence of $\{G_i\}$ such that the size of the largest clique subgraph increases. Without loss of generality we can assume no graph in the sequence is sw to a clique. Consider one of the graphs in the sequence and let B be its (-1,1,0) adjacency matrix. Since B is not a clique, there exists a point p in the graph which is not adjacent to a point in the largest clique subgraph. So B is of the form:

$$B = \begin{pmatrix} 0 & & & & \pm 1 \\ & \cdot & & & \\ & & \cdot & & \\ & & & 0 & \\ & \pm 1 & & & \begin{array}{c|c} C & \begin{matrix} 1 \\ 1 \\ \vdots \\ -1 \\ -1 \end{matrix} \\ \hline 1\ 1 \ldots -1 -1 & 0 \end{array} \end{pmatrix}$$

where C corresponds to the largest clique subgraph and the last row and column of B correspond to the point p . By using the Seidel Switch (sw) we may also assume without loss of generality that the number of points in C to which p is adjacent is not less than the number of points in C to which p is not adjacent.

Let $\{H_i\}$ be the graph sequence associated with the (-1,1,0) adjacency matrices

$$D_i = \begin{pmatrix} \begin{array}{c|c} C_i & \begin{matrix} 1 \\ 1 \\ \vdots \\ -1 \\ -1 \end{matrix} \\ \hline 1\ 1 \ldots -1 -1 & 0 \end{array} \end{pmatrix}$$

We note that H_i is a subgraph of G_i for each i .

Let order$(H_i) = n_i+1$, then $\text{degree}_{H_i}(p) \geq n_i/2$ (here we consider p as a point of H_i). Define $m_i = n_i - \deg(p)$. (We note $\lambda_1(G_i) \geq \lambda_1(H_i) \geq \lambda_1(H_i - p) = \lambda_1(K_j) = 1$. (These inequalities are a result of Cauchy's Interlacing Theorem.)). So $\{\lambda_1(H_i)\}$ is bounded. We now show that $\{m_i\}$ is bounded. Suppose it is not, then we have that $L_i \subset H_i$ and such that the correspoiding submatrix for L_i is the $(2m_i+1)$ by $(2m_i+1)$ matrix given by:

$$\left(\begin{array}{c|c|c} 0 & -1 \ldots -1 & 1 \ldots 1 \\ \hline \begin{matrix}-1\\ \vdots \\ -1\end{matrix} & I - J & -J \\ \hline \begin{matrix}1\\ \vdots \\ 1\end{matrix} & -J & I - J \end{array}\right).$$

We now consider the vector $v = \left(\frac{-1 + \sqrt{1+8m_i}}{2}, -1,-1,\ldots,-1,1,1,\ldots,1\right)$ whose eigenvalue is $\lambda = ((1 + \sqrt{1+8m_i})/2)$. But as $m_i \to \infty$, we have that $\lambda \to \infty$ and so $\lambda_1(G_i) \to \infty$ which is a contradiction. Therefore $\{m_i\}$ is bounded.

Since $\{m_i\}$ is bounded we pick a value, say m , that occurs infinitely often and consider the corresponding subsequence $\{H_k\}$ with associated matrix:

$$\left(\begin{array}{c|c} I - J & -J \\ \hline -J & A \end{array}\right)$$

where A is (m+1) by (m+1) and A is fixed and the last row and last column correspond to p . We can now apply Theorem 3.7 to the matrices associated with $\{H_k\}$. We find that $\lambda_1(H_k) \to \lambda_1(A+J)$ and by Theorem 3.8, $\lambda_1(H_k) \to \lambda_1(I+2K_{1,m})$. By Lemma 3.12, we finally see that $\lambda_1(H_k) \to 1 + 2\sqrt{m}$.

We see $m \neq 0$, otherwise $\{H_k\}$ would be a sequence of cliques and hence impossible. Therefore the smallest possible limit point is $3 = 1 + 2\sqrt{1}$.

We shall later see that, in fact, the sequence $\{K_w - x\}$ (w=3,4,... with K_w a w-clique and x any edge of K_w) has 3 as a limit point. This concludes the proof.

We now prove <u>Theorem 4.1</u> : If $\lambda_1(G)$ is a (0,1) eigenvalue for some connected graph G then $1 + 2\lambda_1(G)$ is a limit point of S.

<u>Proof of 4.1</u>. Consider again a sequence $\{G_i\}$ of graphs none of which is a clique. Let $\{B_i\}$ be the associated (-1,1,0) adjacency matrices and let $\lambda_1(B_i) \to h \geq 3$.

Consider a new sequence of graphs $\{H_i\}$ obtained by considering $\left\{\frac{B_i+J}{2} - \frac{I}{2}\right\} = \{A_i\}$ as the (0,1) adjacency matrices for the $\{H_i\}$ (i=1,2,...). We consider two cases:

<u>Case 1</u>. The size of the largest component is bounded for each H_i.

Since for each H_i, A_i is possibly decomposible, we have $\lambda_1(A_i) = \max\lambda_1(C)$ (we take the maximum over all the connected components C of H_i). Also by using Theorem 2.6 we have that

$$\begin{aligned}\lambda_1(B_i) &\leq \lambda_1(B_i + J)\\ &= 1 + 2\lambda_1(A_i)\\ &= 1 + 2(\max\lambda_1(C)) .\end{aligned}$$

Since the size of the largest component is bounded there exists some graph C which occurs infinitely often as a subgraph of the H_i (for each i) and such that if there is any other graph C_k with this property we have $\lambda_1(C_k) \leq \lambda_1(C)$.

We now consider the subsequence $\{H_{i_j}\}$ such that C is a component of each graph in the subsequence. We see immediately that

$$h = \lim_{i\to\infty} \lambda_1(B_i) \leq \lim_{i\to\infty} \lambda_1(B_i + J) = 1 + 2\lambda_1(C) .$$

Consider now the subsequence $\{B_{i_j}\}$ of matrices associated with $\{H_{i_j}\}$. (Recall these matrices are $(-1,1,0)$.) Each $\{B_{i_j}\}$ contains a submatrix

$$B(C,m) = \begin{array}{c} \\ m\{ \\ \\ \end{array} \begin{array}{c} \overbrace{}^{m} \\ \left(\begin{array}{c|c} I-J & -J \\ \hline -J & -J + I + 2A(C) \end{array}\right) \end{array}$$

where $m \to \infty$ as $i_j \to \infty$ and $A(C)$ is the $(0,1)$ adjacency matrix of C. Since $B(C,M)$ is a principal submatrix of B_{i_j} we see that $\lambda_1(B_{i_j}) \geq \lambda_1(B(C,m))$ and so $h = \lim_{i\to\infty} \lambda_1(B_i) \geq \lim_{m\to\infty} \lambda_1(B(C,m)) = 1 + 2\lambda_1(C)$ (by Theorem 3.7). However, even though h is a limit - is this limit non-trivial or not?

We now show that this limit is non-trivial and so $h = 1 + 2\lambda_1(C)$ is a limit point for $\{G_i\}$. This is accomplished by first defining $B(C,0) = -J + I + 2A(C)$ and so

$$B(C,m) = \begin{array}{c} \\ m\{ \\ \\ \end{array} \begin{array}{c} \overbrace{}^{m} \\ \left(\begin{array}{c|c} I - J & -J \\ \hline -J & B(C,0) \end{array}\right) \end{array} .$$

By the Cauchy Interlacing Theorem it follows that $\lambda_1(B(C,0)) \leq \lambda_1(B(C,1) \leq \ldots$

Assume $B(C,m)$ is the first of the $B(C,i)$ such that $\lambda_1(B(C,m)) = \lambda_1(B(C,m+1))$. Let $(y,y,\ldots,y;x) = \alpha$ be the associated eigenvector for $B(C,m)$ then by Theorem 2.7 $(0,y,y,\ldots;x)$ is an eigenvector for $B(C,m+1)$.

Therefore,

$$\begin{aligned} B(C,0)\cdot x - m\,y\,u &= \lambda x \\ -u^t x + (2-m)y &= \lambda y \\ -u^t x - m\,y &= 0 . \end{aligned}$$

Since $\lambda_1 \geq 2$ we see that $y = 0$ and so $\Sigma x_j = 0$. So $x \not> 0$ and $B(C,0)x = \lambda x$ and x is <u>not</u> associated with $\lambda_1(B(C,0)+J)$ since $B(C,0)+J$ is a nonnegative matrix. $\lambda_1(B(C,m)) = \lambda_1(B(C,0))$ (since $y = 0$) and this value is some eigenvalue of $(B(C,0)+J)$, that is to stay, $\lambda_1(B(C,0)) \leq \lambda_2(B(C,0)+J)$. But since

$$\begin{aligned} \lambda_2(B(C,0)+J) &\leq \lambda_2(J) + \lambda_1(B(C,0)) \\ &= 0 + \lambda_1(B(C,0)) \\ &= \lambda_1(B(C,0)) \end{aligned}$$

we have therefore, $\lambda_1(B(C,0)) = \lambda_2(B(C,0)+J)$ so that if $\lambda_1(B(C,m)) = \lambda_1(B(C,m+1))$ we see that they must both be equal to $\lambda_1(B(C,0))$ and this cannot happen forever since the limit $h = \lambda_1(B(C,0)+J)$ and $\lambda_1(B(C,0)+J) > \lambda_2(B(C,0)+J)$ (by the Perron Frobenius Theorem). Also note that this argument implies that if $\lambda_1(B(C,m)) < \lambda_1(B(C,m+1))$ then for all $k > m$ we have $\lambda_1(B(C,k)) < \lambda_1(B(C,k+1))$. And so h is a limit point. So that given any $(0,1)$ adjacency matrix of a nontrivial graph C we can find a limit point of S, $1 + 2\lambda_1(C)$, using $\lambda_1(C)$. There may, however, be other limit points not obtainable in this manner. However, all those limit points $\{x\}$ such that $3 \leq x < 5$ are obtained in this manner. We now study this:

Case 2. Size of the largest component of each H_i is not bounded. By Lemma 3.13 we must consider two subcases:

subcase 1 $(D \to \infty)$

Here we can find a subsequence $\{H_n\}$ of $\{H_i\}$ such that for each $n = 1,2,\ldots$; H_n contains a principal $(n+1)$ by $(n+1)$ submatrix of the form

$$D_n = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & & & & & \\ 1 & & & & & \\ \vdots & & & X & & \\ 1 & & & & & \\ 1 & & & & & \end{pmatrix}$$

By Theorem 2.8 and Cauchy's Interlacing Theorem we see:

$$\lambda_1(K_{1,n}) \le \lambda_1(D_n) \le \lambda_1(H_n) \ .$$

We again consider the original sequence $\{G_i\}$ with $(-1,1,0)$ matrices $\{B_i\}$. We can therefore find a subsequence $\{G_{i_j}\}$ such that each B_{i_j} has a principal submatrix of the form

$$L_j = \left(\begin{array}{c|c} I - J & -J \\ \hline -J & -J + I + 2D_j \end{array}\right)$$

and L_j is $(k+j+1)$ by $(k+j+1)$. By Cauchy's Interlacing Theorem $\lambda_1(L_j) \le \lambda_1(B_{i_j})$ so that $\lim_{i_j\to\infty} \lambda_1(L_j) \le \lim_{i_j\to\infty} \lambda_1(B_{i_j}) = h$. First fix j and we have that $\lim_{k\to\infty} \lambda_1(L_j) = 1 + 2\lambda_1(K_{1,n}) \le h$ using Theorem 3.7. However, now letting $j \to \infty$, we see that $\lim_{j\to\infty} h = \infty$ by Lemma 3.13.

Hence, h is not less than 5.

<u>subcase 2</u> $(d \to \infty)$

Here we can find a subsequence $\{H_n\}$ of $\{H_i\}$ such that for each $n = 1,2,\ldots$; H_n contains a principal $(n+1)$ by $(n+1)$ submatrix of the form

$$E_n = \begin{pmatrix} 0 & 1 & & & & & X & \\ 1 & 0 & 1 & & & & & \\ & 1 & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & X & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

By Theorem 2.8 and Cauchy's Interlacing Theorem we see that $\lambda_1(P_n) \le \lambda_1(E_n) \le \lambda_1(H_n)$. We again consider the original sequence $\{G_i\}$ with $(-1,1,0)$ adjacency matrices $\{B_i\}$. We can therefore find a subsequence $\{G_{i_j}\}$ such that each $\{B_{i_j}\}$ has a principal submatrix of the form

$$M_j = \left(\begin{array}{c|c} I - J & -J \\ \hline -J & -J + I + E_j \end{array}\right)$$

and M_j is $(k+j+1)$ by $(k+j+1)$. By the Cauchy Interlacing Theorem $\lambda_1(M_j) \le \lambda_1(B_{i_j})$ so that $\lim\limits_{i_j \to \infty} \lambda_1(M_j) \le \lim\limits_{i_j \to \infty} \lambda_1(B_{i_j}) = h$. First fix j and we see that $\lim\limits_{k \to \infty} \lambda_1(M_j) = 1 + 2\lambda_1(P_j) \le h$ using Theorem 3.7. However, letting $j \to \infty$ we see $\lim\limits_{j \to \infty} h \ge 1 + 2(2) = 5$, hence h again is not less than 5 . This completes the proof.

We must now consider those graphs C (nontrivial) such that if $\lambda_1(C)$ is the largest eigenvalue of the $(0,1)$ matrix of C then

$\lambda_1(C) < 2$. This is studied more closely in the next section.

Section 5: The Main Theorems

We are now in a position to prove the main theorems.

Theorem 5.1

$$\{4\cos(2\pi/2n+4)+1 \mid n \geq 1\} \cup \{1+2\sqrt{2\cos(2\pi/n)+2} \mid n=12,18 \text{ or } 30\}$$

are all the limit points of S smaller than 5 .

Proof: By the remarks made in Section 4, it is sufficient to find all the graphs G whose $(0,1)$ adjacency matrix has $\lambda_1 < 2$. In Theorem 3.17 we found that the required graphs are $P_n(n\geq 1)$, $Y_n(n\geq 0)$ and Z_2, Z_3 or Z_4 . We consider three cases.

Case 1. $\lambda_1(P_n) = 2\cos(2\pi/2n+4)$ with $n \geq 1$.

Proof of Case 1. We use Theorem 3.10. Here $Q = x$ and $P = x^2 - 1$. Therefore, $(P-\theta Q) - \left(P-\frac{1}{\theta}Q\right)\theta^{2n} = 0$, or

$$P(1-\theta^{2n}) + Q(\theta^{2n-1}-\theta) = 0 \quad (\text{with } \theta \neq \pm 1),$$

becomes upon substitution for Q and P : $(x^2-1)(1-\theta^{2n}) + x(\theta^{2n-1}-\theta) = 0$.

So that $\left(\theta^2+2+\frac{1}{\theta^2}-1\right)(1-\theta^{2n}) + \left(\theta+\frac{1}{\theta}\right)(\theta^{2n-1}-\theta) = 0$. This is equivalent to

$$\theta^{2n+2} - \frac{1}{\theta^2} = 0$$

or

$$\theta^{2n+4} - 1 = 0 \quad (\theta \neq \pm 1) .$$

Therefore θ is a $(2n+4)^{th}$ root of unity. Since $x = \theta + \frac{1}{\theta}$, we are looking for $\max_{\theta\neq\pm 1} \theta + \frac{1}{\theta} = \max x$. Since $(2n+4)$ is even we have that θ^* (the conjugate of θ , some $(2n+4)^{th}$ root of unity) is also a $(2n+4)^{th}$ root of unity so that

$$\theta + \frac{1}{\theta} = \theta + \frac{\theta^*}{\theta\theta^*} = \theta + \theta^* = 2\cos(2\pi k/2n+4)$$

where $k \neq 0$ or $n+2$ (since $\theta \neq +1$ or -1) and $1 \leq k \leq 2n+3$. We notice also that the maximum is achieved when $k = 1$.

So $\lambda_1(P_n) = 2\cos(2\pi/2n+4)$.

<u>Case 2</u>. $\lambda_1(Y_n) = 2\cos(2\pi/4n+8) \quad n \geq 0$

<u>Proof of Case 2</u>. Here we have that $Q = x^3 - 2x$ and $P = x^3\left(x-\frac{3}{x}\right)$.

Therefore $(x^4-3x^2)(1-\theta^{2n}) + (x^3-2x)(\theta^{2n-1}-\theta) = 0$. Recalling that $x = \theta + \frac{1}{\theta}$, we have

$$\left(\theta^4+\theta^2+\frac{1}{\theta^2}+\frac{1}{\theta^4}\right)\left(1-\theta^{2n}\right) + \left(\theta^3+\theta+\frac{1}{\theta}+\frac{1}{\theta^3}\right)\left(\theta^{2n-1}-\theta\right) = 0 .$$

So equivalently one finds that:

$$\theta^{-4} - \theta^{2n+4} + \theta^{2n} + 1 = 0$$

or

$$\theta^{2n+4}(\theta^4-1) + (\theta^4-1) = 0 .$$

Since $\theta \neq \pm 1$, we see that

$$\theta^{2n+4} + 1 = 0 \qquad n > 0$$

(if $n = 0$ then $x = \sqrt{2}$ by Case 1.) Therefore $\max x = \max\limits_{\theta \neq \pm 1} \theta + \theta^*$ as before. Using De Moivre's Theorem we see that $\theta^{2n+4} = -1 = (1)(\cos\pi + i\sin\pi)$ and the $(2n+4)^{th}$ roots of (-1) are $\cos(k\pi/2n+4) + i\sin(k\pi/2n+4)$ and so $\max x = \max 2\cos(k\pi/2n+4)$ where $k \neq 0$ or $n+2$. We note that the maximum is achieved when $k = 1$. So $\lambda_1(Y_n) = 2\cos(2\pi/4n+8)$. And so we see that $\{\lambda_1(Y_n)\}$ is a subset of $\{\lambda_1(P_n)\}$.

<u>Case 3</u>. There are three different subcases corresponding to the three proper subgraphs of Z_5 which are of interest to us.

Proof of Case 3.

Subcase 1. $\lambda_1(Z_2) = \sqrt{2\cos(2\pi/12)+2}$.

One can easily calculate the characteristic equation to be:

$$(x^2-1)^3 - 2x^2(x^2-1) = 0$$

or

$$(x^2-1)^2 - 2x^2 = 0$$

which in turn becomes $x^4 - 4x^2 + 1 = 0$ which is a quadratic in x^2 .

Therefore, the largest root is $x = \sqrt{2+\sqrt{3}} = \sqrt{2\cos(2\pi/12)+2}$.

Subcase 2. $\lambda_1(Z_3) = \sqrt{2\cos(2\pi/18)+2}$.

One can easily calculate the characteristic equation as:

$$x(x^6 - 6x^4 + 9x^2 + (-3)) = 0$$

or

$$x^6 - 6x^4 + 9x^2 - 3 = 0$$

which is a cubic in x^2 . This is the irreducible case of Cardan's Formula. Despite algebraic difficulties one may use a trigonometric approach to find the solution to: $(x^2)^3 - 6(x^2)^2 + 9(x^2) - 3 = 0$. The largest root is $x^2 = 2\cos 20^\circ + 2$ [10, page 92]. Therefore,

$$x = \sqrt{2\cos(2\pi/18)+2} .$$

Subcase 3. $\lambda_1(Z_4) = \sqrt{2\cos(2\pi/30)+2}$.

One can easily calculate the characteristic equation as:

$$x^8 - 7x^6 + 14x^4 - 8x^2 + 1 = 0$$

which is a quartic in x^2 . It is known that [10]

$$\cos(2\pi/30) = \sqrt{\frac{9+\sqrt{5}+\sqrt{30-6\sqrt{5}}}{16}}$$

and by substitution one may easily see that $\sqrt{2\cos(2\pi/30)+2} = x$ is in fact the largest root of the original equation.

These remarks complete the proof of the theorem.

Theorem 5.2. All the limit points $\{x\}$ of S such that $5 \le x < 2\alpha_2 + 1$ are of the form $x = 1 + 2\lambda_1$ with λ_1 being the largest eigenvalue of the $(0,1)$ matrix associated with the graphs:

(1) $K_{1,4}$

(2)  $= L$, and

(3) Z_n with $n \ge 6$.

Proof: Let $\{G_i\}$ be a graph sequence and let $\{B_i\}$ be the associated $(-1,1,0)$ adjacency matrices. Also suppose that $\lambda_1(B_i) \to x$ as $i \to \infty$ and $5 \le x < 2\alpha_2 + 1$. Let H_i be defined by $J + B_i = I + 2H_i$ and H_i for each is a $(0,1)$ matrix for some graphic sequence $\{H_i\}$.

Case 1. If the size of the biggest component increases in the $\{H_i\}$, then without loss of generality we can consider the sequence to be strictly increasing in size and each H_i to be a connected graph. If there is a subsequence of circuits then $\lambda_1 \to 2$. If there exists a subsequence of cyclic graphs (which are not cycles) then $\lambda_1 \to h \ge \tau > \alpha_2$. Therefore, there is a subsequence of trees. Now if there is a $\{P_n\}$ or $\{>\!\!- \cdots -\!\!<\}$ or $\{Y_n\}$ type subsequence, we have $\lambda_1 \to 2$; otherwise $\lambda_1 \to h \ge \alpha_2$. [See [3]].

Case 2. The size of the components is bounded.

As before there must be some connected graph C such that $x = 1 + 2\lambda_1(C)$. We are therefore searching for those $(0,1)$ adjacency matrices with $2 \leq \lambda_1 < \alpha_2$.

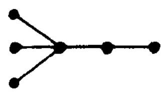
Subcase 1. If there exists a vertex of C with degree 5 or more then C must contain $K_{1,5}$ as a subgraph and so $\lambda_1(C) \geq \lambda_1(K_{1,5}) = \sqrt{5} > \alpha_2$.

Subcase 2. If there exists a vertex of C with degree 4 then

(a) if $C = K_{1,4}$ then

$\lambda_1(C) = 2$ and $x = 5$.

(b) if $C \neq K_{1,4}$ then

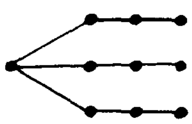
 is a subgraph of C . So we may consider

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

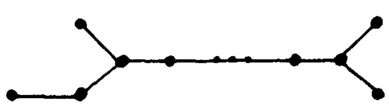
But since

$$\begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

has the same eigenvalues we can equivalently consider the graph

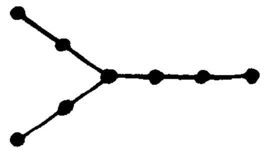
 $= C^1$. But then $\lambda_1(C^1) > \alpha_2$ since C^1 is a supergraph of the graph considered in Subcase 4.

Subcase 3. If C has 2 vertices of degree 3 and $\lambda_1(C) > 2$ then

 must be a subgraph of C .

Now as $n \to \infty$, $\lambda_1 \to h \geq \alpha_2$. [See [3]].

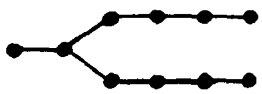
<u>Subcase 4.</u>

If $C =$ 

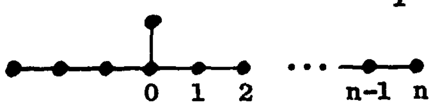
then we see

$$\lambda_1(C) = \lambda_1 \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&2&0&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0 \end{pmatrix}$$

$$= \lambda_1(C^1) = \lambda_1 \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&2&0&0&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0 \end{pmatrix}$$

where $C^1 =$  .

It may be easily calculated that the characteristic equation for C^1 is $x^6 - 6x^4 + 8x^2 + (-1) = 0$ and $2.0284 < \lambda_1 < 2.0285$ or $\lambda_1 > \alpha_2$.

<u>Subcase 5.</u> $C_n =$ 

(a) $\lambda_1(C_3) = 2$

(b) $\lambda_1(C_4)$ lies between 2.0153 and 2.0154 and so $\lambda_1(C_4) < \alpha_2$.

(c) $\lambda_1(C_5)$ lies between 2.0235 and 2.0236 and so $\lambda_1(C_n) > \alpha_2$ for all $n \geq 5$.

<u>Subcase 6</u>. If $C = Z_n$ (with $n \geq 6$) then $\lambda_1(C) < \alpha_2$. (See [3]).

This completes the proof.

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AUTOBIOGRAPHICAL STATEMENT

Michael Gargano was born in Brooklyn, New York on October 14, 1947. In 1964 he graduated from Boys' High School in Brooklyn. In 1968 he received his B.A. from Brooklyn College. In the Fall of 1968 he worked as a teacher in Boys' High School. In 1970 he received his M.A. from Brooklyn College. From 1969 to 1970 he was employed as a commercial programmer and from 1970 to 1971 he was employed as a systems programmer at Columbia University. Since 1971 he has taught mathematics and computer programming at Brooklyn College.