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ON THE STRUCTURE OF FIBRATIONS

by

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SECTION 1

INTRODUCTION

The object of this paper is to establish extensions of given partially ordered families of fibrations to fibrations having the colimit-weak colimit property with respect to these given families.

The definitions and results contained herein are formulated within the category of quasi-topological spaces and maps Q . The basic properties of Q are described by E. Spanier in [7]. In the forthcoming book by E. Dyer and S. Eilenberg [4], it is demonstrated that this category contains all necessary objects, maps, and properties, needed in the exposition of this paper. For notational convenience we drop the prefix "quasi" with the understanding that we confine our constructions to the category Q and all categories derived from it.

We begin by constructing an extension of a partially ordered family over the same base space. We call this construction, which has the colimit property with respect to the given family, a vertical extension of this family. We note that some of these results in Section 2 were obtained independently by Mr. Lewis Berkhout [unpublished].

In Section 3 we show that, given a family of fibrations $\{p_\sigma\}_{\sigma\in\Sigma}$ over a numerable open cover of a space B which are fiber homotopically equivalent over the non-empty intersections, we are able to construct a fibration over B . This horizontal extension is fiber homotopically equivalent to each fibration in the family when restricted to the respective covering space, and it has the colimit property with respect to $\{p_\sigma\}_{\sigma\in\Sigma}$.

We define $\mathcal{J}$ as the category whose objects are fibrations over locally contractible in the large base spaces and whose morphisms are fiber homotopic classes of maps between the total spaces and cofibrations between the base spaces. We finally prove that this category contains distinguished weak colimits provided that the colimit of the base spaces is locally contractible in the large and that the inclusion maps of these bases into this colimit are cofibrations. Such are the main results in Section 4.

SECTION 2

A VERTICAL EXTENSION

A vertical extension of a given set of fibrations over a common base space:

We begin this section with a formulation of basic results and definitions to be used throughout this paper.

Lemma 2.1: Let $\mathcal{D}$:

$$\begin{array}{ccc} X & \xrightarrow{q_1} & X_1 \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & Y \end{array}$$

[2.1.1]

be a pullback diagram such that $(j_1 \sqcup j_2)\colon X_1 \sqcup X_2 \to Y$ is a projection, and if two points in either X_1 or X_2 have the same image in Y, then a point in X also has that image. Then $\mathcal{D}$ is a pushout diagram.

Proof: Let:

$$\begin{array}{ccc} X & \xrightarrow{q_1} & X_1 \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle j_1'} \\ X_2 & \xrightarrow{j_2'} & P \end{array}$$

[2.1.2]

be the pushout diagram associated with $\{X \xrightarrow{q_k} X_k,\ k = 1,2\}$. We will show that Y and P are homeomorphic by a map $F\colon P \to Y$. Since $\mathcal{D}$

commutes, there exists a unique map $F: P \to Y$ such that $F j_k' = j_k$, $(k = 1,2)$.

Define a function $F^{-1}: Y \to P$ as follows:

$$F^{-1}(j_k(x_k)) = j_k'(x_k) \ , \quad (k = 1,2) \ .$$

To show that F^{-1} is well-defined, we first show that it is defined for every $y \in Y$. Let $y \in Y$. Since $(j_1 \sqcup j_2): X_1 \sqcup X_2 \to Y$ is a projection, there exists an element $x_\ell \in X_\ell$, $(\ell = 1 \text{ or } 2)$ such that $j_\ell(x_\ell) = y$. Thus, $F^{-1}(y) = j_\ell'(x_\ell)$.

If $j_1(x_1) = j_2(x_2)$, then there exists an element $x \in X$ such that $q_k(x) = x_k$, $(k = 1,2)$. Thus $F^{-1}j_1(x_1) = j_1'(x_1) = j_1'q_1(x) = j_2'q_2(x) = j_2'(x_2) = F^{-1}j_2(x_2)$.

If $x_\ell, x_\ell' \in X_\ell$ such that $x_\ell \neq x_\ell'$ and $j_\ell(x_\ell) = j_\ell(x_\ell')$, then there exists an element $x \in X$ such that $j_\ell q_\ell(x) = j_\ell(x_\ell)$. Let us assume, without loss of generality, that $\ell = 1$. Then $j_1q_1(x) = j_1(x_1) = j_1(x_1')$. Since $j_2q_2(x) = j_1q_1(x)$ and $q_2(x) \in X_2$, we can conclude from the above that $F^{-1}j_1(x_1) = F^{-1}j_2q_2(x)$ and $F^{-1}j_1(x_1') = F^{-1}j_2q_2(x)$. Thus $F^{-1}j_1(x_1) = F^{-1}j_1(x_1')$. Therefore F^{-1} is well-defined.

To show that F^{-1} is continuous, we need only show that $F^{-1}(j_1 \sqcup j_2)$ is continuous since $(j_1 \sqcup j_2)$ is a projection.

$F^{-1}(j_1 \sqcup j_2) = (j_1' \sqcup j_2')$. $(j_1' \sqcup j_2')$ is continuous by the topology given the pushout space P. Thus F^{-1} is continuous.

$F^{-1}F\, j_k' = F^{-1}j_k = j_k' : X_k \to P$. Thus, by the uniqueness of the pushout map, $F^{-1}F = 1_P$.

$FF^{-1}j_k = F\,j'_k = j_k : X_k \to Y$. Thus $FF^{-1}(y) = FF^{-1}j_\ell(x_\ell) = j_\ell(x_\ell) = y$, implying that $FF^{-1} = 1_Y$. Therefore F is a homeomorphism implying that $\mathfrak{D}$ is a pushout diagram.

Let $\mathfrak{M}$ be the mapping category associated with Q , i.e., $p \in \mathrm{Ob}\mathfrak{M}$ provided that p is a morphism in Q , and $f \in \mathfrak{M}(p,p')$ if and only if $f = (f',f'')$, a pair of Q-morphisms such that:

$$\begin{array}{ccc} X & \xrightarrow{\quad f' \quad} & X' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\quad f'' \quad} & B' \end{array}$$

[2.2.1]

is a commuting diagram.

We let D and R: $\mathfrak{M} \to Q$ be the covariant domain and range functors respectively. Finally, we denote by $\mathfrak{M}/B$ the subcategory of $\mathfrak{M}$ whose objects are maps in Q over B and whose morphisms have as the second coordinate the identity map over B .

<u>Theorem 2.2</u>: $\mathfrak{M}$ and $\mathfrak{M}/B$ have pushouts, pullbacks, limits, and colimits.

For proof of this theorem, the reader is referred to [4].

<u>Lemma 2.3</u>: Let $\mathfrak{D}$:

$$\begin{array}{ccc} p & \xrightarrow{\quad (f_1,r_1) \quad} & p_1 \\ \downarrow{\scriptstyle (f_2,r_2)} & & \downarrow{\scriptstyle (g_1,s_1)} \\ p_2 & \xrightarrow{\quad (g_2,s_2) \quad} & q \end{array}$$

[2.3.1]

be a commuting diagram in $\mathfrak{M}$. Then $\mathfrak{D}$ is a pushout in $\mathfrak{M}$ if and only if $D(\mathfrak{D})$ and $R(\mathfrak{D})$ are pushouts in Q .

<u>Proof</u>: First we assume that $\mathfrak{D}$ is a pushout diagram in $\mathfrak{M}$ and show that $D(\mathfrak{D})$ and $R(\mathfrak{D})$ are pushout diagrams in Q . This is done by taking the following pushout diagrams in Q:

$$\begin{array}{ccc} D(p) & \xrightarrow{f_1} & D(p_1) \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1'} \\ D(p_2) & \xrightarrow{g_2'} & X \end{array}$$

[2.3.2]

$$\begin{array}{ccc} R(p) & \xrightarrow{r_1} & R(p_1) \\ \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle s_1'} \\ R(p_2) & \xrightarrow{s_2'} & B \end{array}$$

[2.3.3]

and showing that X and B are homeomorphic to $D(q)$ and $R(q)$ respectively.

Since $\mathfrak{D}$ is a commuting diagram in $\mathfrak{M}$, $D(\mathfrak{D})$ and $R(\mathfrak{D})$ are commuting diagrams in Q . Thus there exist unique maps $f: X \to D(q)$ and $r: B \to R(q)$ such that $fg_k' = g_k : D(p_k) \to D(q)$ and $rs_k' = s_k: R(p_k) \to R(q)$, $(k = 1,2)$.

We will now prove that there exists a unique map $q': X \to B$ such that $\mathfrak{B}$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1,r_1)} & p_1 \\ \Big\downarrow{\scriptstyle (f_2,r_2)} & & \Big\downarrow{\scriptstyle (g_1',s_1')} \\ p_2 & \xrightarrow{(g_2',s_2')} & q' \end{array}$$

[2.3.4]

is a commuting diagram in $\mathfrak{M}$. Since $s_1' r_1 = s_2' r_2 : R(p) \to B$, $s_1' r_1 p = s_2' r_2 p : D(p) \to B$. Thus $s_1' p_1 f_1 = s_1' r_1 p = s_2' r_2 p = s_2' p_2 f_2 : D(p) \to B$. Therefore there exists a unique map $q' : X \to B$ such that $q' g_k' = s_k' p_k$, $(k = 1,2)$. We can thus conclude that $q' \in \mathrm{Ob}\mathfrak{M}$, $(g_k', s_k') \in \mathfrak{M}(p_k, q')$, $(k = 1,2)$, and $\mathfrak{D}$ is a commuting diagram in $\mathfrak{M}$.

Since $\mathfrak{D}$ is a pushout in $\mathfrak{M}$, there exists a unique map (f^{-1}, r^{-1}): $q \to q'$ such that $(f^{-1}, r^{-1})(g_k, s_k) = (g_k', s_k') : p_k \to q'$, $(k = 1,2)$. This implies that $f^{-1} g_k = g_k' : D(p_k) \to X$ and $r^{-1} s_k = s_k' : R(p_k) \to B$, $(k = 1,2)$.

$f^{-1} f g_k' = f^{-1} g_k = g'_k : D(p_k) \to X$, and $r^{-1} r s_k' = r^{-1} s_k = s_k' : R(p_k) \to B$, $(k = 1,2)$. Thus by the uniqueness of the pushout map, $f^{-1} f = 1_X$ and $r^{-1} r = 1_B$.

$(f,r)(f^{-1}, r^{-1})(g_k, s_k) = (f,r)(g_k', s_k') = (g_k, s_k) : P_k \to q$, $(k = 1,2)$. Thus, by the uniqueness of the pushout map, $(f,r)(f^{-1}, r^{-1}) = 1_q$, implying that $f f^{-1} = 1_{D(q)}$ and $r r^{-1} = 1_{R(q)}$. Therefore $f : X \to D(q)$ and $r : B \to R(q)$ are homeomorphisms.

Now we assume that $D(\mathfrak{D})$ and $R(\mathfrak{D})$ are pushouts in Q. To show that $\mathfrak{D}$ is a pushout diagram in $\mathfrak{M}$, we let $\mathfrak{D}'$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1,r_1)} & p_1 \\ \Big\downarrow{\scriptstyle (f_2,r_2)} & & \Big\downarrow{\scriptstyle (h_1,t_1)} \\ p_2 & \xrightarrow{(h_2,t_2)} & q_0 \end{array}$$

[2.3.5]

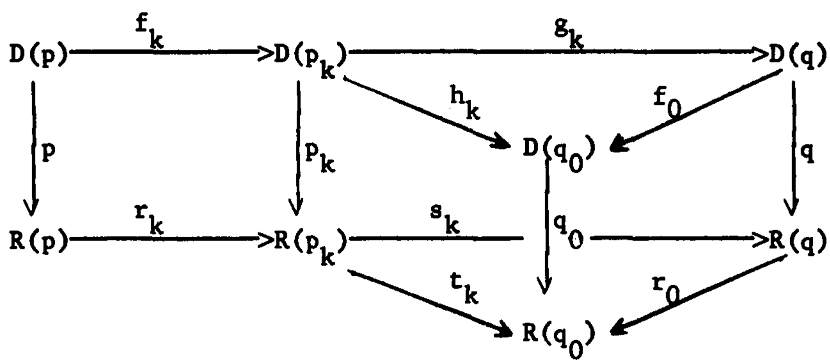
be a commuting diagram in $\mathfrak{M}$, and prove that there exists a unique map $(f_0,r_0) \in \mathfrak{M}(q,q_0)$ such that $(f_0,r_0)(g_k,s_k) = (h_k,t_k): p_k \to q_0$, $(k = 1,2)$. Since $\mathfrak{D}'$ commutes in $\mathfrak{M}$, $D(\mathfrak{D}')$ and $R(\mathfrak{D}')$ commute in Q . Therefore there exist unique maps $f_0: D(q) \to D(q_0)$ and $r_0: R(q) \to R(q_0)$ such that $f_0g_k = h_k: D(p_k) \to D(q_0)$ and $r_0s_k = t_k: R(p_k) \to R(q_0)$, $(k = 1,2)$.

Now we show that $(f_0,r_0) \in \mathfrak{M}(q,q_0)$. Since $R(\mathfrak{D}')$ commutes, $t_1r_1 = t_2r_2: R(p\ ) \to R(q_0)$. Thus $t_1r_1p = t_2r_2p: D(p) \to R(q_0)$. Since $r_kp = p_kf_k: D(p) \to R(p_k)$, $(k = 1,2)$, it follows that $t_1p_1f_1 = t_2p_2f_2: D(p) \to R(q_0)$. Therefore there exists a unique map $g: D(q) \to R(q_0)$ such that $gg_k = t_kp_k: D(p_k) \to R(q_0)$.

$r_0qg_k = r_0s_kp_k = t_kp_k$. Thus $r_0q = g$.

$q_0f_0g_k = q_0h_k = t_kp_k$. Thus $q_0f_0 = g$.

Therefore $r_0q = q_0f_0$, implying that $(f_0,r_0) \in \mathfrak{M}(q,q_0)$:



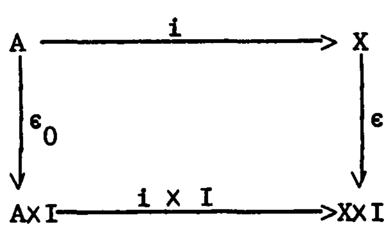
[2.3.6]

If $(f_0', r_0'): q \to q_0$ such that $(f_0', r_0')(g_k, s_k) = (h_k, t_k): p_k \to q_0$, $(k = 1,2)$, then $f_0' g_k = h_k: D(p_k) \to D(q_0)$ and $r_0' s_k = t_k: R(p_k) \to R(q_0)$. It follows then that $f_0' = f_0$ and $r_0' = r_0$. Therefore (f_0, r_0): $q \to q_0$ is unique.

Thus $\mathfrak{D}$ is a pushout in $\mathfrak{M}$.

For $X \in \text{Ob}Q$, we define the path space of X as follows: $PX = \{(\omega, r) \in (R^+, X) \times R^+ \mid \omega$ is continuous and $\omega(t) = \omega(r)$ for all $t \geq r\}$, with the topology making the inclusion map $i: PX \to (R^+, X) \times R^+$ an injection. r is called the length of the path (ω, r).

<u>Definition 2.4</u>: The map $i: A \to X$ is a cofibration provided that:



[2.4.1]

is a weak pushout diagram.

<u>Theorem 2.5</u>: Let $A \subset X$ be a closed subset and $i: A \to X$ be the injection map. Then the following conditions are equivalent:

(1) i is a cofibration.

(2) There exist maps $\varphi: X \to I$ and $H: X \times I \to X$ such that

$\varphi^{-1}(0) = A$, $H|X \times \{0\} = 1_X$, $H|A \times I = \pi_1$, and $H|\varphi^{-1}[0,1) \times \{1\}: \varphi^{-1}[0,1) \to A$.

(3) There exist maps $\varphi: X \to I$ and $H: \varphi^{-1}[0,1) \times I \to \varphi^{-1}[0,1)$ such that

$\varphi^{-1}(0) = A$, $H|\varphi^{-1}[0,1) \times \{0\} = 1_{\varphi^{-1}[0,1)}$, $H|A \times I = \pi_1$, and

$H|\varphi^{-1}[0,1) \times \{1\}: \varphi^{-1}[0,1) \to A$.

(4) There exist maps $\varphi: X \to I$ and $h: \varphi^{-1}[0,1) \to PX$ such that

$\varphi^{-1}(0) = A$, $\eta_0 h(v) = v$, $\eta_T h(v) \in A$, and $\ell h(v) = \varphi(v)$ for every $v \in \varphi^{-1}[0,1)$.

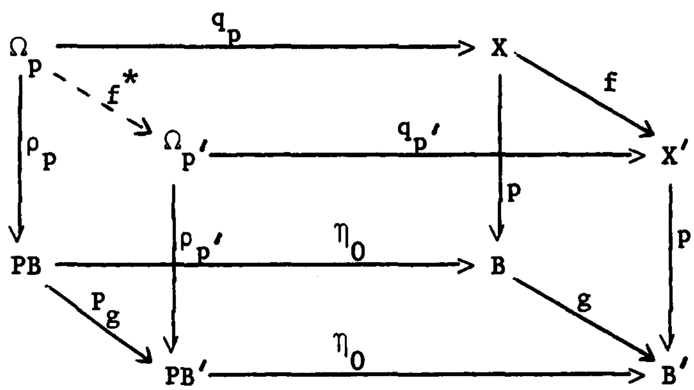
The proof of this theorem can be found in [4].

For a map in Q, $p: X \to B$, we denote as Ω_p the pullback space of:

$$\begin{array}{ccc} \Omega_p & \xrightarrow{q_p} & X \\ \downarrow{\rho_p} & & \downarrow{p} \\ PB & \xrightarrow{\eta_0} & B \end{array}$$

[2.6.1]

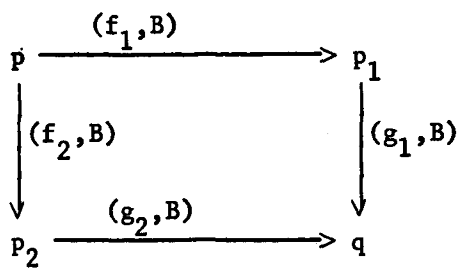
Definition 2.6: $\Gamma: \mathfrak{M} \to \mathfrak{M}$ is a covariant functor where $\Gamma(p) = pq_p: \Omega_p \to B$ and for $(f,g) \in \mathfrak{M}(p,p')$, $\Gamma(f,g) = (f^*,g)$ where $f^*: \Omega_p \to \Omega_{p'}$ is the unique map in Q induced by the pullback operation.



[2.6.2]

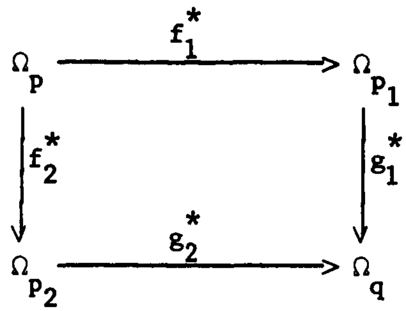
We are now in a position to prove the following proposition.

Proposition 2.7: Let $\mathfrak{D}$:



[2.7.1]

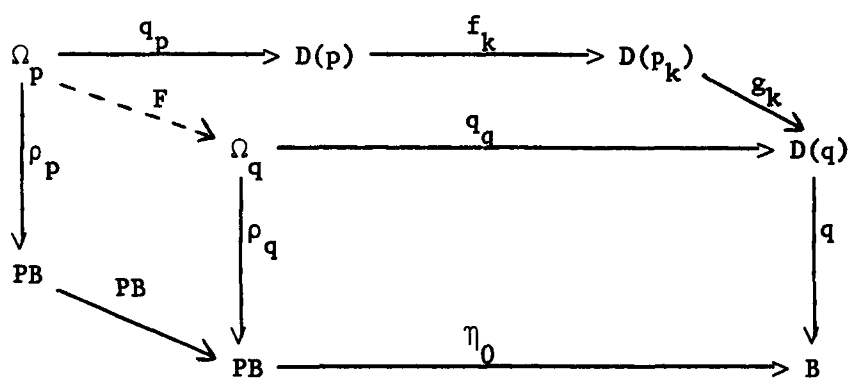
be a pushout diagram in $\mathfrak{M}/B$ such that f_1 is a cofibration. Then $D(\Gamma(\mathfrak{D}))$:



[2.7.2]

is a pushout diagram in Q .

Proof: In showing that $D(\Gamma(\mathcal{A}))$ commutes we note that $qg_kf_kq_p = \eta_0\rho_p$, $(k = 1,2)$. Thus, by the uniqueness of the induced map, $F: \Omega_p \to \Omega_q$, in the pullback operation, $F = g_1^* f_1^* = g_2^* f_2^*$

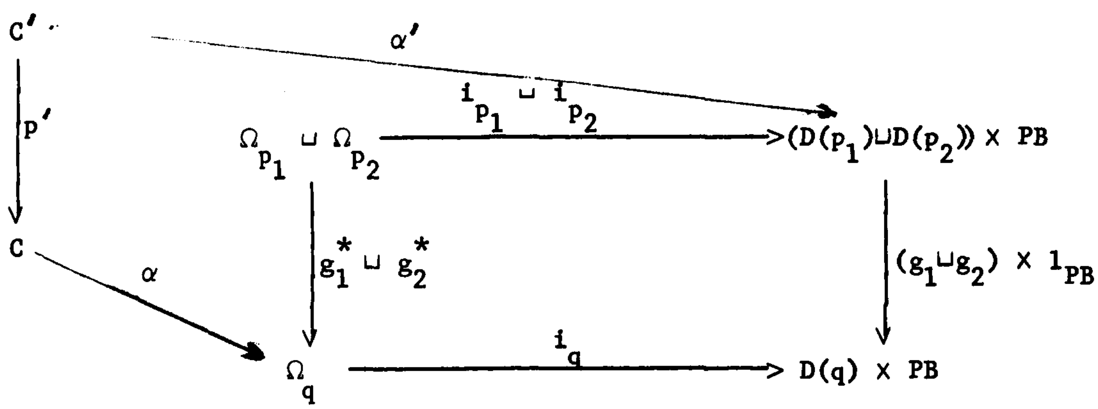


[2.7.3]

To prove that $D(\Gamma(\mathcal{A}))$ is a pushout diagram, we will show that the hypothesis of Lemma 2.1 are satisfied. To do this we first show that $(g_1^* \sqcup g_2^*): \Omega_{p_1} \sqcup \Omega_{p_2} \to \Omega_q$ is a projection.

$D(\mathcal{D})$ is a pushout diagram by Lemma 2.3. Thus $(g_1 \sqcup g_2) : D(p_1) \sqcup D(p_2) \to D(q)$ is a projection. We can thus conclude that for $(x,(\omega,r)) \in \Omega_q$, there exists an element $x_\ell \in D(p_\ell)$, $(\ell = 1 \text{ or } 2)$ such that $g_\ell(x_\ell) = x$. Since $(g_\ell, B) \in \mathfrak{M}(p_\ell q)$, $(x_\ell,(\omega,r)) \in \Omega_{p_\ell}$. Thus $g_\ell^*(x_\ell,(\omega,r)) = (g_\ell(x_\ell),(\omega,r)) = (x,(\omega,r))$, implying that $(g_1^* \sqcup g_2^*)$ is a surjection.

Now we show that Ω_q has the unique topology making this map a projection. Ω_q has the topology making the inclusion map $i_q : \Omega_q \to D(q) \times PB$ an injection. Thus $\alpha: C \to \Omega_q$ is an admissible map if and only if $i_q\alpha: C \to D(q) \times PB$ is an admissible map. Since $(g_1 \sqcup g_2)$ and 1_{PB} are projections, $(g_1 \sqcup g_2) \times 1_{PB}$ is a projection. Thus $i_q\alpha: C \to D(q) \times PB$ is an admissible map if and only if there exists a compact Hausdorff space C', an onto map $p': C' \to C$, and an admissible map $\alpha': C' \to (D(p_1) \sqcup D(p_2)) \times PB$ such that $\mathcal{D}'$:



[2.7.4]

is a commuting diagram.

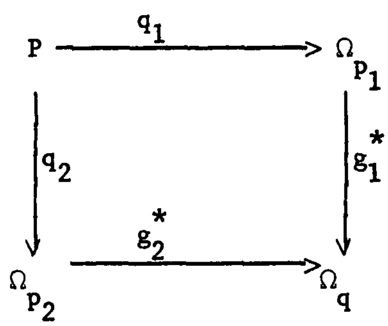
To show that $\alpha': C' \to (D(p_1) \sqcup D(p_2)) \times PB$ factors through $\Omega_{p_1} \sqcup \Omega_{p_2}$, we let $\alpha'(c') \in D(p_\ell) \times PB$, ($\ell = 1$ or 2). $[(g_1 \sqcup g_2) \times 1_{PB}]\, \alpha'(c') = (g_\ell \times 1_{PB})\, \alpha'(c') = i_q \alpha p'(c')$. Since i_q is an injection, $\alpha p'(c') \in \Omega_q$. Thus $(g_\ell \times 1_{PB})\, \alpha'(c') \in \Omega_q$ implying that $\alpha'(c') \in \Omega_{p_\ell}$. Therefore α' factors through $\Omega_{p_1} \sqcup \Omega_{p_2}$, i.e., $\alpha' = (i_{p_1} \sqcup i_{p_2})\alpha''$ where $\alpha'': C' \to \Omega_{p_1} \sqcup \Omega_{p_2}$. Since $(i_{p_1} \sqcup i_{p_2})$ is an injection, α'' is admissible if and only if α' is admissible. Since $\mathcal{D}'$ commutes, we can conclude that $i_q \alpha p' = ((g_1 \sqcup g_2) \times 1_{PB})\alpha' = ((g_1 \sqcup g_2) \times 1_{PB})(i_{p_1} \sqcup i_{p_2})\alpha'' = i_q(g_1^* \sqcup g_2^*)\alpha'': C' \to D(q) \times PB$. i_q is an injection. Thus $\alpha p' = (g_1^* \sqcup g_2^*)\alpha'': C' \to \Omega_p$. Therefore Ω_q has the unique topology making $(g_1^* \sqcup g_2^*)$ a projection.

Now we show that if two points in either Ω_{p_1} or Ω_{p_2} have the same image in Ω_q, then a point in Ω_p also has that image. f_1 is a cofibration and $D(\mathcal{D})$ is a pushout. Thus g_2 is a cofibration.

If $g_2^*(x_2, (\omega, r)) = g_2^*(x_2', (\omega', r'))$, then $q_q\, g_2^*(x_2, (\omega, r)) = q_q\, g_2^*(x_2', (\omega', r'))$ and $\rho_q\, g_2^*(x_2, (\omega, r)) = \rho_q\, g_2^*(x_2', (\omega', r'))$. Thus $x_2 = x_2'$ and $(\omega, r) = (\omega', r')$ implying that g_2^* is 1-1.

If $g_1^*(x_1, (\omega, r)) = g_1^*(x_1'(\omega', r'))$, then $q_q\, g_1^*(x_1, (\omega, r)) = q_q\, g_1^*(x_1', (\omega', r'))$ and $\rho_q\, g_1^*(x_1, (\omega, r)) = \rho_q\, g_1^*(x_1', (\omega', r'))$. Thus $g_1(x_1) = g_1(x_1')$ and $(\omega, r) = (\omega', r')$. $g_1(x_1) = g_1(x_1')$ implies that there exists an element $x_0 \in D(p)$ such that $g_1 f_1(x_0) = g_1(x_1) = g_1(x_1')$. We note that since $(f_1,B) \in \mathfrak{M}(p,p_1)$, $(x_0,(\omega,r)) \in \Omega_p$ and $g_1^* f_1^*(x_0,(\omega,r)) = g_1^*(x_1,(\omega,r)) = g_1^*(x_1',(\omega',r'))$.

We have left to show that $D\Gamma(\mathcal{D})$ is a pullback diagram in Q . Let:



[2.7.5]

be a pullback diagram in Q . Since $D\Gamma(\mathcal{D})$ commutes, there exists a unique map $H: \Omega_p \to P$ such that $q_k H = f_k^*: \Omega_p \to \Omega_{p_k}$, $(k = 1,2)$.

$$H(x_0,(\omega,r)) = (f_1(x_0),(\omega,r), f_2(x_0),(\omega,r)) = (f_1^*(x_0,(\omega,r)),f_2^*(x_0,(\omega,r))).$$

We note that since $D(\mathcal{D})$ is a pushout and f_1 is a cofibration, $g_1(x_1) = g_2(x_2)$ if and only if $x_1 \in D(p)$ and $f_2(x_1) = x_2$, i.e., $g_1(x_1) = g_2(x_2)$ if and only if there exists an element $x_0 \in D(p)$ such that $f_k(x_0) = x_k$, $(k = 1,2)$.

To show that H is a homeomorphism, we define $H^{-1}: P \to \Omega_p$ as follows:

$H^{-1}(x_1,(\omega,r),\ x_2,(\omega,r)) = (x_0,(\omega,r))$ where $f_k(x_0) = x_k$, $(k = 1,2)$. We note that since f_1 is an injection, x_0 is uniquely determined implying that H^{-1} is well defined.

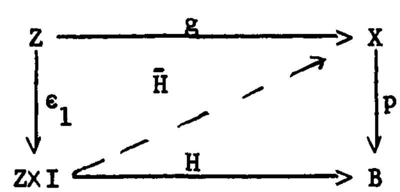
$f_1^* H^{-1}(x_1,(\omega,r),\ x_2,(\omega,r)) = f_1^*(x_0,(\omega,r)) = (f_1(x_0),(\omega,r)) = (x_1,(\omega,r))$. Thus $f_1^* H^{-1} = q_1$. Since q_1 is continuous and f_1^* is an injection, H^{-1} is continuous.

$H^{-1}H(x_0,(\omega,r)) = (x_0,(\omega,r))$. Thus $H^{-1}H = 1_{\Omega_p}$.
$HH^{-1}(x_1,(\omega,r),x_2,(\omega,r)) = H(x_0,(\omega,r)) = (f_1(x_0),(\omega,r),f_2(x_0),(\omega,r)) = (x_1,(\omega,r),x_2,(\omega,r))$. Thus $HH^{-1} = 1_P$. Therefore H is a homeomorphism implying that $D(\Gamma(\mathcal{D}))$ is a pullback.

Thus, by Lemma 2.1, $D(\Gamma(\mathcal{D}))$ is a pushout.

Definition 2.8: The Q morphism $p\colon X \to B$ is a fibration provided that there exists a map $\lambda_p\colon \Omega_p \times R^+ \to X$ such that $\lambda_p[(x,(\omega,r)),0] = x$ and $p\lambda p[(x,(\omega,r)),s] = \omega(s)$. λ_p is called the lifting function of p.

This is equivalent to saying that p has the homotopy lifting property, i.e., given maps $g\colon Z \to X$ and $H\colon Z \times I \to B$ such that $H|Z \times \{1\} = pg\colon Z \to B$, there exists a map $\bar{H}\colon Z \times I \to X$ such that $p\bar{H} = H$ and $\bar{H}|Z \times \{1\} = g$.



[2.8.1]

A proof of this equivalence statement can be found in [4].

<u>Definition 2.9</u>: $(f,g) \in \mathfrak{M}(p,p_1)$ is a relative fibration of fibrations p and p_1 provided that there exist lifting functions λ_p and λ_{p_1} such that the following is a commuting diagram:

$$\begin{array}{ccc} \Omega_p \times R^+ & \xrightarrow{f^* \times R^+} & \Omega_{p_1} \times R^+ \\ \downarrow{\scriptstyle \lambda_p} & & \downarrow{\scriptstyle \lambda_{p_1}} \\ D(p) & \xrightarrow{f} & D(p_1) \end{array}$$

[2.9.1]

We have reached the point where we can now construct the first extension of a set of fibrations over a common base space.

<u>Theorem 2.10</u>: Let $\mathcal{D}$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1,B)} & p_1 \\ \downarrow{\scriptstyle (f_2,B)} & & \downarrow{\scriptstyle (g_1,B)} \\ p_2 & \xrightarrow{(g_2,B)} & q \end{array}$$

[2.10.1]

be a pushout diagram in $\mathfrak{M}/B$ where p, p_1, and p_2 ,are fibrations, (f_1,B) is a relative fibration, and f_1 is a cofibration. Then q

is a fibration, (g_2,B) is a relative fibration, and g_2 is a cofibration.

Proof: Since (f_1,B) is a relative fibration, there exist lifting functions λ_p and λ_{p_1} of p and p_1 respectively such that

$f_1 \lambda_p = \lambda_{p_1}(f_1^* \times R^+): \Omega_p \times R^+ \to D(p_1)$. Since f_1 is a cofibration, there exist maps $\varphi: D(p_1) \to I$ and $h': \varphi^{-1}[0,1) \to PD(p_1)$ such that $\varphi^{-1}(0) = D(p)$, $\eta_0 h'(v) = v$, $\eta_T h'(v) \in D(p)$, and $\ell h'(v) = \varphi(v)$ for every $v \in \varphi^{-1}[0,1)$.

We will now construct a map $h: \varphi^{-1}[0,1) \to PD(p_1)$ with the above properties of h' such that $p_1 h(v)(s) = p_1(v)$ for every $v \in \varphi^{-1}[0,1)$ and for all $s \in [0,\varphi(v)]$.

Let $h(v)(s) = \lambda_{p_1}[(h'(v)(s),(\omega_{(v,s)},s)),s]$ where

$$\omega_{(v,s)}(t) = p_1 h'(v)(s - t) , \quad 0 \le t \le s .$$

$\omega_{(v,s)}(0) = p_1 h'(v)(s)$. Thus $(h'(v)(s),(\omega_{(v,s)},s)) \in \Omega_{p_1}$ implying that h is well defined and continuous.

We now show that h has all of the required properties.

$\eta_0 h(v) = h(v)(0) = \lambda_{p_1}[(h'(v)(0),(\omega_{(v,0)},0)),0] = h'(v)(0) = v$.

Thus $\eta_0 h(v) = v$ for every $v \in \varphi^{-1}[0,1)$.

$\eta_T h(v) = h(v)(\varphi(v)) = \lambda_{p_1}[(h'(v)(\varphi(v)),(\omega_{(v,\varphi(v))},\varphi(v))),\varphi(v)]$.

$h'(v)(\varphi(v)) \in D(p)$. Thus $h'(v)(\varphi(v)) = f_1(x_v)$. Therefore, since (f_1,B) is a relative fibration,

$$\eta_{\tau}h(v) = \lambda_{p_1}[(f_1(x_v),(\omega_{(v,\varphi(v))},\varphi(v))),\varphi(v)]$$

$$= f_1\lambda_p[(x_v,(\omega_{(v,\varphi(v))},\varphi(v))),\varphi(v)] \ .$$

Thus $\eta_{\tau}h(v) \in D(p)$ for every $v \in \varphi^{-1}[0,1)$.

$$\ell h(v) = \ell h'(v) = \varphi(v) \quad \text{for every} \quad v \in \varphi^{-1}[0,1) \ .$$

$$p_1h(v)(s) = p_1\lambda_{p_1}[(h'(v)(s),(\omega_{(v,s)},s)),s] = \omega_{(v,s)}(s)$$

$$= p_1h'(v)(s - s) = p_1h'(v)(0) = p_1(v) \ .$$

Thus $p_1h(v)(s) = p_1(v)$ for every $v \in \varphi^{-1}[0,1)$ and for all $s \in [0,\varphi(v)]$.

To prove that q is a fibration, we will exhibit a lifting function $\lambda_q: \Omega_q \times R^+ \to D(q)$. We will be able to construct such a function by defining a map $\bar{\lambda}_{p_1}: \Omega_{p_1} \times R^+ \to D(q)$ such that $\bar{\lambda}_{p_1}(f_1^* \times R^+) = g_2 \ \lambda_{p_2}(f_2^* \times R^+)$ and noting that $D(\Gamma(\mathcal{D})) \times R^+$ is a pushout diagram.

$\bar{\lambda}_{p_1}$ is defined on each of three closed sections of Ω_{p_1} .

For $(x_1,(\omega,r)) \in \Omega_{p_1}$ such that $x_1 \in \varphi^{-1}[\frac{2}{3},1]$:

$$\bar{\lambda}_{p_1}[(x_1,(\omega,r)),s] = g_1 \ \lambda_{p_1}[(x_1,(\omega,r)),s] \qquad 0 \le s \le r \ .$$

For $(x_1,(\omega,r)) \in \Omega_{p_1}$ such that $x_1 \in \varphi^{-1}[\frac{1}{3},\frac{2}{3}]$:

We let (ω,r) be the composite of two maps $(\omega_{\alpha}(x_1),2r - 3r\ \varphi(x_1))$ and $(\omega_{\beta}(x_1),3r\varphi(x_1) - r)$ where $\omega_{\alpha}(x_1)(s) = \omega(s)$ and $\omega_{\beta}(x_1)(s) = \omega(s + 2r - 3r\varphi(x_1))$.

Let $\tilde{\alpha}(x_1)(rt) = \lambda_{p_1}[(h(x_1)(t/3),(\omega_\alpha(x_1),2r - 3r\varphi(x_1))),rt]$,

$0 \le t \le 2 - 3\varphi(x_1)$.

$$\bar{\lambda}_{p_1}[(x_1,(\omega,r)),rt] = \begin{cases} g_1\tilde{\alpha}(x_1)(rt) = \hat{\alpha}(x_1)(rt), \quad 0 \le t \le 2 - 3\varphi(x_1) \\ \hat{\beta}(x_1)(rt) = g_1\lambda_{p_1}[(\eta_\tau\tilde{\alpha}(x_1),(\omega_\beta(x_1),3r\varphi(x_1)-r)), \\ \qquad\qquad rt - (2r - 3r\varphi(x_1))] , \\ 2 - 3\varphi(x_1) \le t \le 1 . \end{cases}$$

We note that since $p_1\eta_\tau\tilde{\alpha}(x_1) = \eta_\tau\omega_\alpha(x_1) = \eta_0\omega_\beta(x_1),(\eta_\tau\tilde{\alpha}(x_1),(\omega_\beta(x_1),$ $3r\varphi(x_1) - r)) \in \Omega_{p_1}$. Thus $\bar{\lambda}_{p_1}$ is well defined on this closed section of Ω_{p_1} .

For $(x_1,(\omega,r)) \in \Omega_{p_1}$ such that $x_1 \in \varphi^{-1}[0,\frac{1}{3}]$:

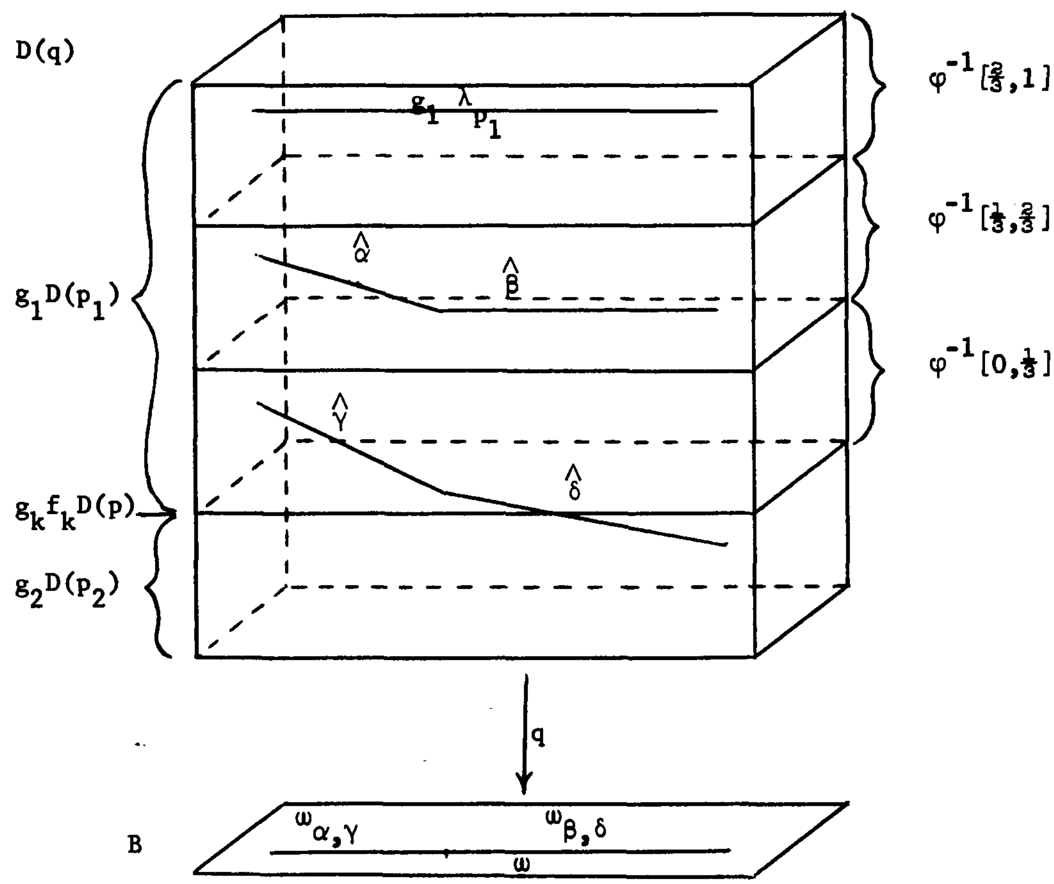
We let (ω,r) be the composite of two maps $(\omega_\gamma(x_1),3r\varphi(x_1))$ and $(\omega_\delta(x_1),r - 3r\varphi(x_1))$ where $\omega_\gamma(x_1)(s) = \omega(s)$ and $\omega_\delta(x_1)(s) =$ $\omega(s + 3r\varphi(x_1))$. Let

$$\tilde{\gamma}(x_1)(rt) = \lambda_{p_1}[(h(x_1)(t/3),(\omega_\gamma(x_1),3r\varphi(x_1))),rt], \ 0 \le t \le 3\varphi(x_1) .$$

$$\bar{\lambda}_{p_1}[(x_1,(\omega,r)),rt] = \begin{cases} \hat{\gamma}(x_1)(rt) = g_1\tilde{\gamma}(x_1)(rt) , \quad 0 \le t \le 3\varphi(x_1) \\ \hat{\delta}(x_1)(rt) = g_2\lambda_{p_2}[(f_2(\eta_\tau\tilde{\gamma}(x_1)),(\omega_\delta(x_1),r-3r\varphi(x_1))), \\ \qquad\qquad rt - 3r\varphi(x_1)] , \\ 3\varphi(x_1) \le t \le 1 . \end{cases}$$

As with the previous section, we note that since $p_1\eta_\tau\,\tilde{\gamma}(x_1) = \eta_\tau\,\omega_\gamma(x_1) = \eta_0\,\omega_\delta(x_1), (\eta_\tau\,\tilde{\gamma}(x_1), (\omega_\delta(x_1), r - 3r\varphi(x_1))) \in \Omega_{p_1}$. We further note that $\eta_\tau\,\tilde{\gamma}(x_1) \in D(p)$. Thus $(\eta_\tau\tilde{\gamma}(x_1), (\omega_\delta(x_1), r-3r\varphi(x_1))) \in \Omega_p$ implying that $(f_2(\eta_\tau\tilde{\gamma}(x_1)), (\omega_\delta(x_1), r - 3r\varphi(x_1))) \in \Omega_{p_2}$.

Therefore $\bar{\lambda}_{p_1}$ is well defined on this closed section of Ω_{p_1}.



[2.10.2]

To show that $\bar{\lambda}_{p_1}$ is well defined, we need only prove that the function, as defined, agrees at $\varphi(x_1) = \frac{1}{3}$ and $\varphi(x_1) = \frac{2}{3}$.

When $\varphi(x_1) = \frac{1}{3}$:

$$(\omega_\gamma(x_1), 3r\varphi(x_1)) = (\omega, r) = (\omega_\alpha(x_1), 2r - 3r\varphi(x_1)) \quad \text{and}$$

$$(\omega_\beta(x_1), 3r\varphi(x_1) - r) = (\Pi_T\omega, 0) = (\omega_\delta(x_1), r - 3r\varphi(x_1)) \ .$$

Thus $\bar{\lambda}_{p_1}[(x_1,(\omega,r)),rt] = g_1\lambda_{p_1}[(h(x_1)(t/3),(\omega,r)),rt]$, $0 \le t \le 1$

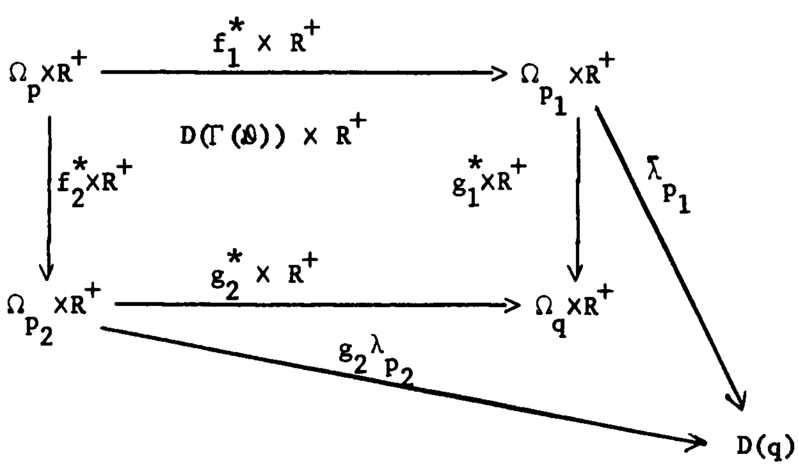
When $\varphi(x_1) = \frac{2}{3}$:

$$(\omega_\alpha(x_1), 2r - 3r\varphi(x_1)) = (p_1(x_1), 0) \quad \text{and}$$

Thus $\bar{\lambda}_{p_1}[(x_1,(\omega,r)),rt] = g_1\lambda_{p_1}[(x_1,(\omega,r)),rt]$, $0 \le t \le 1$.

Thus $\bar{\lambda}_{p_1}[(x_1,(\omega,r)),rt] = g_1\lambda_{p_1}[(x_1,(\omega,r)),rt]$, $0 \le t \le 1$.

We will now show that $\bar{\mathfrak{D}}$:



[2.10.3]

is a commuting diagram. Let $(x,(\omega,r)) \in \Omega_p$. Then

$\bar{\lambda}_{p_1}(f_1^* \times R^+)[(x,(\omega,r)),s] = \bar{\lambda}_{p_1}[(f_1(x),(\omega,r)),s]$. Since $\varphi(x) = 0$,

$(\omega_\gamma(x_1), 3r\varphi(x_1)) = (p(x), 0)$ and $(\omega_\delta(x_1), r - 3r\varphi(x_1)) = (\omega, r)$.

Thus $\bar{\lambda}_{p_1}[(f_1(x),(\omega,r)),s] = g_2\lambda_{p_2}[(f_2(x),(\omega,r)),s]$

$$= g_2\lambda_{p_2}(f_2^* \times R^+)[(x,(\omega,r)),s] .$$

Therefore $\bar{\mathcal{D}}$ commutes, and since $D(\Gamma(\mathcal{D})) \times R^+$ is a pushout diagram, there exists a unique map $\lambda_q : \Omega_q \times R^+ \to D(q)$ such that $\lambda_q(g_1^* \times R^+) = \bar{\lambda}_{p_1}$ and $\lambda_q(g_2^* \times R^+) = g_2\lambda_{p_2}$.

We now prove that λ_q is a lifting function.

$$\lambda_q[(g_1(x_1),(\omega,r)),0] = \bar{\lambda}_{p_1}[(x_1,(\omega,r)),0] = g_1(x_1) .$$

$$\lambda_q[(g_2(x_2),(\omega,r)),0] = g_2\lambda_{p_2}[(x_2,(\omega,r)),0] = g_2(x_2) .$$

Thus $\lambda_q[(z,(\omega,r)),0] = z$ for every $(z,(\omega,r)) \in \Omega_q$.

$q\lambda_q[(g_1(x_1),(\omega,r)),s] = q\bar{\lambda}_{p_1}[(x_1,(\omega,r)),s] = \omega(s)$ by the construction of $\bar{\lambda}_{p_1}$ and the fact that g_k is a fiber map , $(k = 1,2)$.

$q\lambda_q[(g_2(x_2),(\omega,r)),s] = qg_2\lambda_{p_2}[(x_2,(\omega,r)),s] = p_2\lambda_{p_2}[(x_2,(\omega,r)),s] = \omega(s)$.

Thus $q\lambda_q[(z,(\omega,r)),s] = \omega(s)$ for every $[(z,(\omega,r)),s] \in \Omega_q \times R^+$.

Therefore λ_q is a lifting function implying that q is a fibration.

$\lambda_q(g_2^* \times R^+) = g_2\lambda_{p_2} : \Omega_{p_2} \times R^+ \to D(q)$. Thus (g_2,B) is a relative fibration. Finally, since $\mathcal{D}$ is a pushout in $\mathfrak{M}/B$, $D(\mathcal{D})$ is a pushout in Q . Thus, since f_1 is a cofibration, g_2 is a cofibration.

For an immediate application of this theorem, let us consider the following: Let p and p' be two fibrations over B, and let $(f,B) \in \mathfrak{m}/B\,(p,p')$. We define $M(f)$, the mapping cylinder of f, as the pushout space of the following pushout diagram:

$$\begin{array}{ccc} D(p) & \xrightarrow{\ \epsilon_1\ } & D(p)\times I \\ \Big\downarrow f & & \Big\downarrow j_1 \\ D(p') & \xrightarrow{\ j_2\ } & M(f) \end{array}$$

[2.11.1]

We note that since $p'f = p = p\,\pi_1\,\epsilon_1 : D(p) \to B$, there exists a unique map $M(f,B) : M(f) \to B$ such that $p\,\pi_1 = M(f,B)j_1$ and $p' = M(f,B)j_2$.

<u>Corollary 2.11</u>: $M(f,B)$ is a fibration.

<u>Proof</u>: Let $\lambda_p : \Omega_p \times R^+ \to D(p)$ be a lifting function of p. Define $\lambda_{p\pi_1} : \Omega_{p\pi_1} \times R^+ \to D(p) \times I$ as follows:

$$\lambda_{p\pi_1}[((x,\epsilon),(\omega,r)),s] = (\lambda_p[(x,(\omega,r)),s],\epsilon)\ .$$

The reader can verify the fact that $\lambda_{p\pi_1}$ is a lifting function implying that $p\pi_1 : D(p) \times I \to B$ is a fibration. We note that (ϵ_1,B) is a relative fibration. We note also that ϵ_1 is a cofibration and:

$$\begin{array}{ccc} p & \xrightarrow{\ (\epsilon_1,B)\ } & p\pi_1 \\ \Big\downarrow (f,B) & & \Big\downarrow (j_1,B) \\ p' & \xrightarrow{\ (j_2,B)\ } & M(f,B) \end{array}$$

[2.11.2]

is a pushout diagram in $\mathfrak{M}/B$. Thus the hypothesis of Theorem 2.10 are satisfied implying that $M(f,B)$ is a fibration.

We end this section by proving that q and p_2 of Theorem 2.10 are fiber homotopically equivalent should such a relationship exist between p_1 and p .

Theorem 2.12: Let $f: X \to Y$ be a cofibration and a homotopy equivalence. Then X is a strong deformation retract of Y, i.e., there exist maps $f^{-1}: Y \to X$ and $H: Y \times I \to Y$ such that $f^{-1}f = 1_X$, $H|Y \times \{0\} = 1_Y$, $H|X \times I = \pi_1$, and $H|Y \times \{1\} = ff^{-1}$.

For a proof of this theorem, the reader is referred to [5].

Proposition 2.13: Let $\mathcal{D}$:

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow{g_2} & Y \end{array}$$

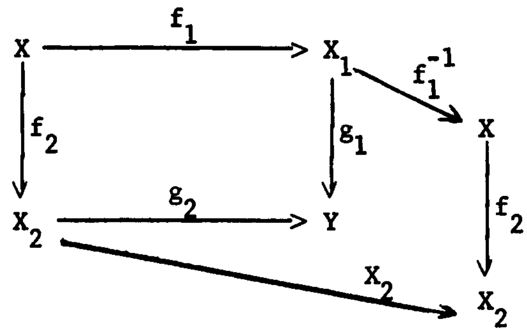
[2.13.1]

be a pushout diagram where f_1 is a cofibration and a homotopy equivalence. Then g_2 is a homotopy equivalence.

Proof: By Theorem 2.12, there exist maps $f_1^{-1}: X_1 \to X$ and $H: X_1 \times I \to X_1$ such that $f_1^{-1}f_1 = 1_X$ and $H: 1_{X_1} \simeq f_1f_1^{-1}$ rel X

$$1_{X_2} f_2 = f_2 = f_2 1_X = f_2 f_1^{-1} f_1 : X \to X_2 .$$

Therefore:

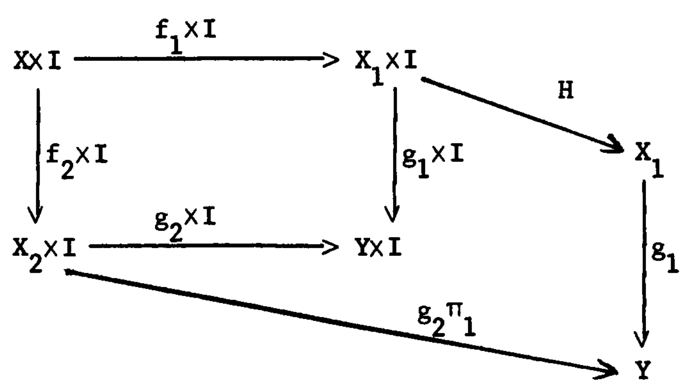


[2.13.2]

is a commuting diagram. Since $\mathcal{D}$ is a pushout diagram, there exists a unique map $g_2^{-1}: Y \to X_2$ such that $g_2^{-1}g_1 = f_2f_1^{-1}: X_1 \to X_2$ and $g_2^{-1}g_2 = 1_{X_2} : X_2 \to X_2$.

$$g_1H(f_1 \times I)(x,t) = g_1H(f_1(x),t) = g_1f_1(x) = g_2f_2(x) = g_2\pi_1(f_2(x),t)$$
$$= g_2\pi_1(f_2 \times I)(x,t) .$$

Therefore:



[2.13.3]

is a commuting diagram.

Thus, since $\mathcal{D} \times I$ is a pushout diagram, there exists a unique map $G: Y \times I \to Y$ such that $G(g_1 \times I) = g_1 H: X_1 \times I \to Y$ and $G(g_2 \times I) = g_2 \pi_1: X_2 \times I \to Y$.

By the uniqueness of the pushout map:
$G(g_1 \times I)|X_1 \times \{0\} = g_1 H|X_1 \times \{0\} = g_1$ and $G(g_2 \times I)|X_2 \times \{0\} = g_2 \pi_1 |X_2 \times \{0\} = g_2$ implies that $G|Y \times \{0\} = 1_Y$.

Also $G(g_1 \times I)|X_1 \times \{1\} = g_1 H|X_1 \times \{1\} = g_1 f_1 f_1^{-1} = g_2 f_2 f_1^{-1} = g_2 g_2^{-1} g_1$ and $G(g_2 \times I)|X_2 \times \{1\} = g_2 \pi_1 |X_2 \times \{1\} = g_2 = g_2 g_2^{-1} g_2$ implies that $G|Y \times \{1\} = g_2 g_2^{-1}$.

Therefore $G: 1_Y \simeq g_2 g_2^{-1}$.

Proposition 2.14: Let p and p' be fibrations over B, and let $(f,B) \in \mathfrak{M}/B(p,p')$ be a relative fibration where $f: D(p) \to D(p')$ is a cofibration and a homotopy equivalence. Then f is a fiber homotopy equivalence.

Proof: Since f is both a cofibration and a homotopy equivalence, there exist maps $f^{-1}: D(p') \to D(p)$ and $H: D(p') \times I \to D(p')$ such that $f^{-1}f = 1_{D(p)}$ and $H: 1_{D(p')} \simeq ff^{-1}$ rel $D(p)$. Let $\varphi: D(p') \to I$ be a haloing function of f. Let

$$\overline{D(p')} = \{(x',t) \in D(p') \times I \mid t \leq \varphi(x')\},$$

and give $\overline{D(p')}$ the topology making the inclusion map $i: \overline{D(p')} \to D(p') \times I$ an injection.

We will now show that there exists a map $\bar{H}: \overline{D(p')} \to D(p')$ such that $\bar{H}(x',0) = x'$ and $\bar{H}(x',\varphi(x')) = ff^{-1}(x')$ for every $x' \in D(p')$.

Let $h: D(p') \to PD(p')$ be the adjoint of H, $h(x')(t) = H(x',t)$. Define $\bar{h}: D(p') \to PD(p')$ as: $\bar{h}(x')(t \cdot \varphi(x')) = h(x')(t)$.

To prove that $\bar{h}$ is continuous, we make use of the following lemma found in [4]: $j: PY \to (I,Y) \times R^+$ defined by: $j(\omega,r) = [(\tilde{\omega},1),r]$ where $\tilde{\omega}(t) = \omega(rt)$, $t \in I$, is an injection.

Let $Y = D(p')$. $\pi_1 j[\bar{h}(x')(s), \varphi(x')] = \tilde{\bar{h}}(x')(s) = \bar{h}(x')(s \cdot \varphi(x')) = h(x')(s)$. Thus $\pi_1 j\bar{h} = h$. $\pi_2 j\bar{h} = \varphi$. Therefore, since the projections are continuous $j\bar{h}$ is continuous, and since j is an injection, $\bar{h}$ is continuous.

Let $\bar{H}$ be the adjoint of $\bar{h}$: $\bar{H}(x',t) = \bar{h}(x')(t)$. We note that $\bar{H}: \overline{D(p')} \to D(p')$

$\bar{H}(x',0) = \bar{h}(x')(0) = h(x')(0) = H(x',0) = x'$ for every $x' \in D(p')$.

$\bar{H}(x',\varphi(x')) = \bar{h}(x')(\varphi(x')) = h(x')(1) = H(x',1) = ff^{-1}(x')$ for every $x' \in D(p')$.

We will now show that there exist maps $\tilde{f}^{-1}: D(p') \to D(p)$ and $\tilde{H}: \overline{D(p')} \to D(p')$ such that $(\tilde{f}^{-1},B) \in \mathfrak{M}/B(p',p)$, $\tilde{f}^{-1}f = 1_{D(p)}$, $\tilde{H}(x',0) = x'$, $\tilde{H}(x',\varphi(x')) = f\tilde{f}^{-1}(x')$ and $p'\tilde{H}(x',s) = p'(x')$ for every $x' \in D(p')$. Since (f,B) is a relative fibration, there exist lifting functions λ_p and $\lambda_{p'}$ of p and p' respectively such that

$$\begin{array}{ccc} \Omega_p \times R^+ & \xrightarrow{f^* \times R^+} & \Omega_{p'} \times R^+ \\ \downarrow{\scriptstyle \lambda_p} & & \downarrow{\scriptstyle \lambda_{p'}} \\ D(p) & \xrightarrow{f} & D(p') \end{array}$$

[2.14.1]

is a commuting diagram. Let

$$\tilde{f}^{-1}(x') = \lambda_p[(f^{-1}(x'),(\omega_{(x',\varphi(x'))},\varphi(x'))),\varphi(x')]$$

and

$$\tilde{H}(x',s) = \lambda_{p'}[(\bar{H}(x',s),(\omega_{(x',s)},s)),s]$$

where $\omega_{(x',s)}(t) = p'\bar{H}(x',s-t)$, $0 \le s \le \varphi(x')$, $0 \le t \le s$.

Since $\omega_{(x',\varphi(x'))}(0) = p'\bar{H}(x',\varphi(x')) = p'ff^{-1}(x')$

$$= pf^{-1}(x'),(f^{-1}(x'),(\omega_{(x',\varphi(x'))},\varphi(x'))) \in \Omega_p .$$

Also since $\omega_{(x',s)}(0) = p'\bar{H}(x',s),(\bar{H}(x',s),(\omega_{(x',s)},s)) \in \Omega_{p'}$. Thus $\tilde{f}^{-1}$ and $\tilde{H}$ are well defined.

We will now show that these functions have the required properties.

$$p\tilde{f}^{-1}(x') = p\lambda_p[(f^{-1}(x'),(\omega_{(x',\varphi(x'))},\varphi(x'))),\varphi(x')]$$

$$= \omega_{(x',\varphi(x'))}(\varphi(x')) = p'\bar{H}(x',0) = p'(x') .$$

Thus $(\tilde{f}^{-1},B) \in \mathfrak{m}/B(p',p)$.

$$\tilde{f}^{-1}f(x) = \lambda_p[f^{-1}f(x),(\omega_{(f(x),0)},0)),0] = f^{-1}f(x) = x .$$

Thus $\tilde{f}^{-1}f = 1_{D(p)}$.

$$\tilde{H}(x',0) = \lambda_{p'}[(\bar{H}(x',0),(\omega_{(x',0)},0)),0] = \bar{H}(x',0) = x'$$

for every $x' \in D(p')$.

$$\tilde{H}(x',\varphi(x')) = \lambda_{p'}[(\bar{H}(x',\varphi(x')),(\omega_{(x',\varphi(x'))},\varphi(x'))),\varphi(x')]$$

$$= \lambda_{p'}[ff^{-1}(x'),(\omega_{(x',\varphi(x'))},\varphi(x'))),\varphi(x')]$$

$$= f\lambda_p[f^{-1}(x'),(\omega_{(x',\varphi(x'))},\varphi(x'))),\varphi(x')]$$

$$= f\tilde{f}^{-1}(x')$$

for every $x' \in D(p')$.

$$p'\tilde{H}(x',s) = p'\lambda_{p'}[(\bar{H}(x',s),(\omega_{(x',s)},s)),s]$$

$$= \omega_{(x',s)}(s) = p'\bar{H}(x',s - s)$$

$$= p'\bar{H}(x',0) = p'(x')$$

for every $x' \in D(p')$ and for all $s \le \varphi(x')$.

Define $F\colon D(p') \times I \to D(p')$ as follows:

$$F(x',t) = \tilde{H}(x',\min[t,\varphi(x')])$$

$$F(x',0) = \tilde{H}(x',0) = x' \quad \text{for every } x' \in D(p') ,$$

$$F(x',1) = \tilde{H}(x',\varphi(x')) = f\tilde{f}^{-1}(x') \quad \text{for every } x' \in D(p') ,$$

$$p'F(x',s) = p'\tilde{H}(x',\min[s,\varphi(x')]) = p'(x') \quad \text{for every } x' \in D(p')$$

and for all $s \in I$. Thus $F\colon 1_{D(p')} \simeq f\tilde{f}^{-1}$ under a vertical homotopy, and since $\tilde{f}^{-1}f = 1_{D(p)}$, we can conclude that f is a fiber homotopy equivalence.

We are now in position to prove the final theorem of this section.

<u>Theorem 2.15</u>: Let $\mathcal{D}$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1,B)} & p_1 \\ \Big\downarrow{\scriptstyle (f_2,B)} & & \Big\downarrow{\scriptstyle (g_1,B)} \\ p_2 & \xrightarrow{(g_2,B)} & q \end{array}$$

[2.15.1]

be a pushout diagram in $\mathcal{M}/B$ such that p , p_1 , and p_2 , are fibrations, (f_1,B) is a relative fibration, and f_1 is a cofibration and a fiber homotopy equivalence. Then g_2 is a fiber homotopy equivalence.

<u>Proof</u>: By Theorem 2.10, q is a fibration, (g_2,B) is a relative fibration, and g_2 is a cofibration.

Since f_1 is a cofibration and a fiber homotopy equivalence and $D(\mathcal{D})$ is a pushout diagram, we can conclude by Proposition 2.13 that g_2 is a homotopy equivalence.

The above two statements satisfying the hypothesis of Proposition 2.14, allows us to conclude that g_2 is a fiber homotopy equivalence.

SECTION 3

A HORIZONTAL EXTENSION

A horizontal extension of a given set of fibrations over a numerable open cover:

Where in the preceeding section we extended a set of fibrations over a common base space, we are by no means restricted to such extensions.

For example: Given a fibration p over B and a map $h: B' \to B$, the pullback operation induces a fibration p' over B', i.e., given the following pullback diagram:

$$\begin{array}{ccc} X' & \xrightarrow{q_1} & X \\ \Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle p} \\ B' & \xrightarrow{h} & B \end{array}$$

[3.1.1]

where p is a fibration, we can conlude that p' is a fibration.

The following theorem is a consequence of this result.

<u>Theorem 3.1</u>: Let p be a fibration over $B \subset B'$ and let $r: B' \to B$ be a retract. Then there exists a fibration p' over B' such that $p'|p'^{-1}(B) \cong p$.

We will now use several of the results of Section 2 to construct the following horizontal extension.

<u>Theorem 3.2</u>: Let $[g]: B \to I$ and let $U = [g]^{-1}(0,1]$. Let p and p_1 be fibrations over U , and let p_2 be a fibration over B .

Let $(f_1,U) \in \mathfrak{M}/U(p,p_1)$ be a relative fibration where f_1 is a cofibration and a fiber homotopy equivalence, and let $(f_2,i) \in \mathfrak{M}(p,p_2)$. Finally, let $\mathcal{D}$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1,U)} & p_1 \\ \Big\downarrow {\scriptstyle (f_2,i)} & & \Big\downarrow {\scriptstyle (g_1,i)} \\ p_2 & \xrightarrow{(g_2,B)} & q \end{array}$$

[3.2.1]

be a pushout diagram in $\mathfrak{M}$. Then q is a fibration, and (g_2,B) is a relative fibration.

<u>Proof</u>: Since (f_1,U) is a relative fibration, there exist lifting functions λ_p and λ_{p_1} of p and p_1 respectively such that:

$$\begin{array}{ccc} \Omega_p \times R^+ & \xrightarrow{f_1^* \times R^+} & \Omega_{p_1} \times R^+ \\ \Big\downarrow {\scriptstyle \lambda_p} & & \Big\downarrow {\scriptstyle \lambda_{p_1}} \\ D(p) & \xrightarrow{f_1} & D(p_1) \end{array}$$

[3.2.2]

is a commuting diagram. Since f_1 is a cofibration and a fiber homotopy equivalence as well, there exist maps $f_1^{-1}: D(p_1) \to D(p)$ and $H: D(p_1) \times I \to D(p_1)$ such that

a) $f_1^{-1} f_1 = 1_{D(p)}$,

b) $H|D(p_1) \times \{0\} = 1_{D(p_1)}$,

c) $H(f_1 \times I) = f_1\pi_1 \colon D(p) \times I \to D(p_1)$,

d) $H(x_1,\varphi(x_1)) = f_1 f_1^{-1}(x_1)$ for every $x_1 \in D(p_1)$, where

$\varphi_1 \colon D(p_1) \to I$ is a haloing function of f_1 ,

e) $p_1 H = p_1\pi_1 \colon D(p_1) \times I \to B$.

To prove that q is a fibration, we will exhibit a lifting function, λ_q . First we note that $\mathfrak{D}'$:

$$\begin{array}{ccc}
\Omega_{ip} \times R^{+} & \xrightarrow{f_1^{*} \times R^{+}} & \Omega_{ip_1} \times R^{+} \\
\Big\downarrow {\scriptstyle f_2^{*} \times R^{+}} & & \Big\downarrow {\scriptstyle g_1^{*} \times R^{+}} \\
\Omega_{p_2} \times R^{+} & \xrightarrow{g_2^{*} \times R^{+}} & \Omega_q \times R^{+}
\end{array}$$

[3.2.3]

is a pushout diagram in Q , since $\mathfrak{D}$ is a pushout diagram in $\mathfrak{M}$. For $(x_1,(\omega,r)) \in \Omega_{ip_1}$, let $t_1 = \frac{1}{2}(\varphi_1(x_1))([g]\omega(0))$, and consider (ω,r) as a composition of two paths (ω^{rt_1},rt_1) and $(\omega_{rt_1},r - rt_1)$ where $\omega^{rt_1}(s) = \omega(s)$ and $\omega_{rt_1}(s) = \omega(s + rt_1)$. Let

$$\alpha(x_1)(rt) = \lambda_{p_1}[(H(x_1,2t/[g]\omega(0)),(\omega^{rt_1},rt_1)),rt],\ 0 \le t \le t_1 .$$

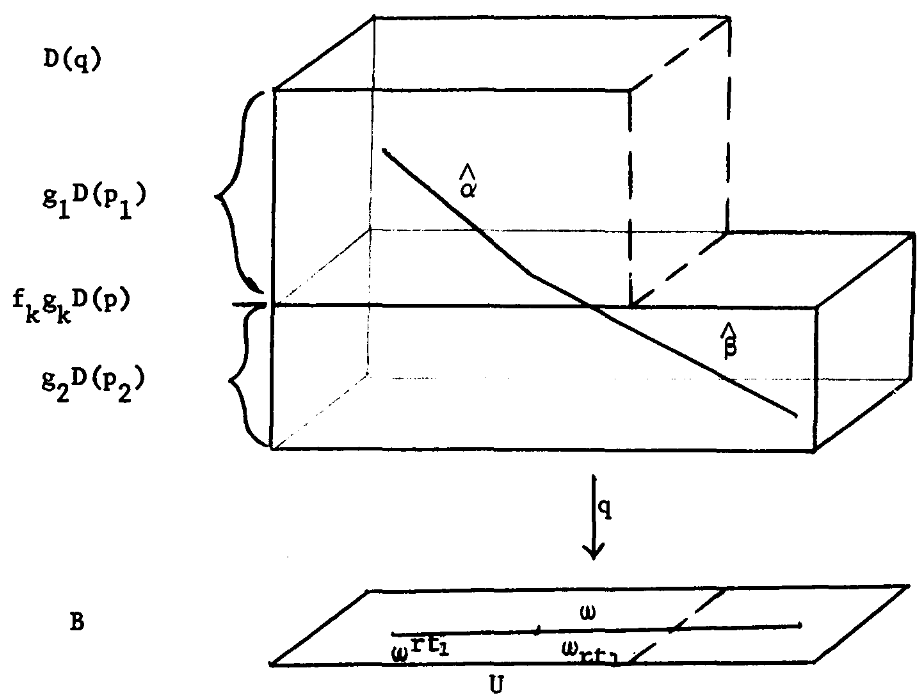
We note that since $(x_1,(\omega^{rt_1},rt_1)) \in \Omega_{p_1}$, $\alpha(x_1)$ is well defined.

Define $\bar{\lambda}_{p_1} \colon \Omega_{ip_1} \times R^{+} \to D(q)$ as follows:

$$\bar{\lambda}_{p_1}[(x_1,(\omega,r)),rt] = \begin{cases} \hat{\alpha}(x_1)(rt) = g_1\alpha(x_1)(rt)\ , & 0 \le t \le t_1 \\ \hat{\beta}(x_1)(rt) = g_2\lambda_{p_2}[(f_2(\eta_T\alpha(x_1)),(\omega_{rt_1},r-rt_1)),rt-rt_1], & t_1 \le t \le 1 \end{cases}$$

Since $\eta_T\alpha(x_1) \in D(p)$ ((f_1,U) is a relative fibration) and $p_1\eta_T\alpha(x_1) = \omega^{rt_1}(rt_1) = \omega_{rt_1}(0)$, $(\eta_T\alpha(x_1),(\omega_{rt_1},r-rt_1)) \in \Omega_{ip}$.

Therefore $(f_2(\eta_T\alpha(x_1)),(\omega_{rt_1},r-rt_1)) \in \Omega_{p_2}$. Thus $\bar{\lambda}_{p_1}$ is well defined .



[3.2.4]

$$\bar{\lambda}_{p_1}(f_1^* \times I)[(x,(\omega,r)),s] = \bar{\lambda}_{p_1}[(f_1(x),(\omega,r)),s]$$

$$= g_2\lambda_{p_2}[(f_2(x),(\omega_0,r\text{-}0)),s]$$

$$= g_2\lambda_{p_2}(f_2^* \times I)[(x,(\omega,r)),s] \ .$$

Therefore $\bar{\lambda}_{p_1}(f_1^* \times I) = g_2\lambda_{p_2}(f_2^* \times I)$. Since $\mathcal{D}'$ is a pushout diagram, we can now conclude that there exists a unique map $\lambda_q : \Omega_q \times R^+ \to D(q)$ such that $\lambda_q(g_1^* \times I) = \bar{\lambda}_{p_1} : \Omega_{ip_1} \times R^+ \to D(q)$ and $\lambda_q(g_2^* \times I) = g_2\lambda_{p_2} : \Omega_{p_2} \times R^+ \to D(q)$.

$$\lambda_q[(g_1(x_1),(\omega,r)),0] = \bar{\lambda}_{p_1}[(x_1,(\omega,r)),0] = \hat{\alpha}(x_1)(0)$$

$$= g_1\lambda_{p_1}[(H(x_1,0),(\omega^{rt_1},rt_1)),0]$$

$$= g_1H(x_1,0) = g_1(x_1) \ .$$

$$\lambda_q[(g_2(x_2),(\omega,r)),0] = g_2\lambda_{p_2}[(x_2,(\omega,r)),0] = g_2(x_2) \ .$$

Therefore $\lambda_q[(z,(\omega,r)),0] = z$ for every $(z,(\omega,r)) \in \Omega_q$.

$q\lambda_q[(g_1(x_1),(\omega,r)),s] = q\bar{\lambda}_{p_1}[(x_1,(\omega,r)),s] = \omega(s)$ by the construction of $\bar{\lambda}_{p_1}$ and the fact that g_k is a fiber map $(k = 1,2)$.

$q\lambda_q[(g_2(x_2),(\omega,r)),s] = qg_2\lambda_{p_2}[(x_2,(\omega,r)),s] = p_2\lambda_{p_2}[(x_2,(\omega,r)),s] = \omega(s).$

Thus $q\lambda_q[(z,(\omega,r)),s] = \omega(s)$ for all $[(z,(\omega,r)),s] \in \Omega_q \times R^+$.

Therefore λ_q is a lifting function of q implying that q is a fibration. Since $\lambda_q(g_2^* \times I) = g_2\lambda_{p_2} : \Omega_{p_2} \times R^+ \to D(q)$, (g_2,B) is a relative fibration.

The fact that $f_1 : D(p) \to D(p_1)$ is a fiber homotopy equivalence is crucial to the proof of this theorem, for if we assume that all of the hypothesis of Theorem 3.2 hold except this condition, we would be faced with the following counterexample:

Let $B = [-1,1]$ and define $[g]:B \to I$ as: $[g](s) = \max[0,s]$. Thus $[g]^{-1}(0,1] = U = (0,1]$. $D(p) = U \times S^0$; $p = \pi_1$; and $\lambda_p : \Omega_p \times R^+ \to D(p)$ is defined as: $\lambda_p[((x,\epsilon_i),(\omega,r)),s] = (\omega(s),\epsilon_i)$. $D(p_1) = U \times I$; $p_1 = \pi_1$; and $\lambda_{p_1} : \Omega_{p_1} \times R^+ \to D(p_1)$ is defined as:

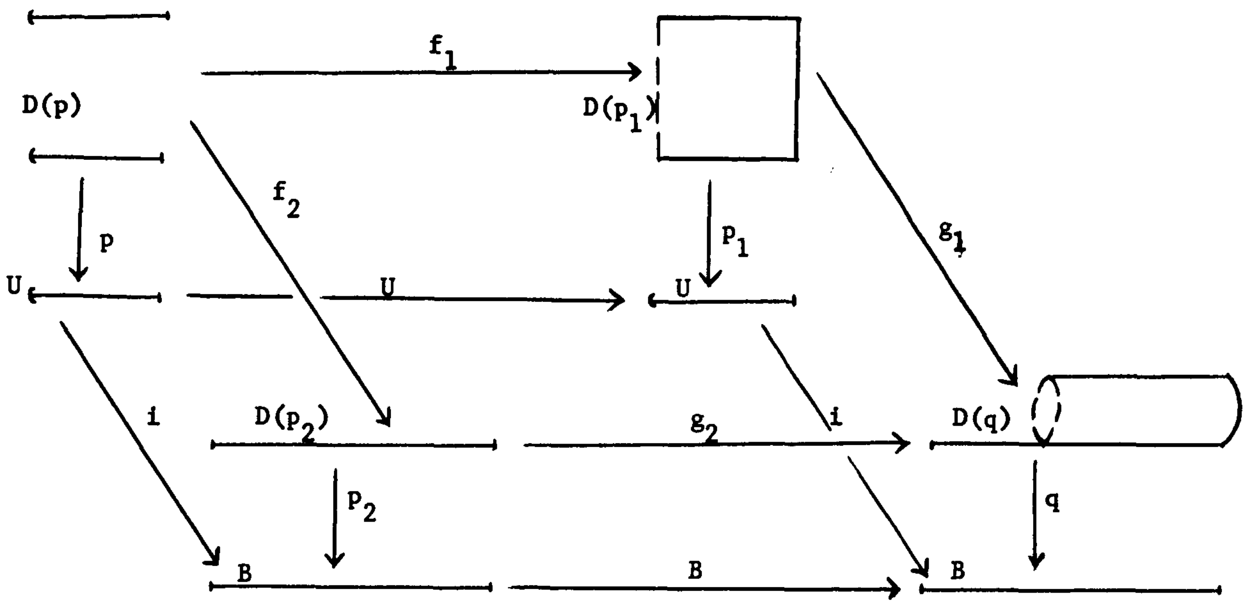
$\lambda_{p_1}[((x,t),(\omega,r)),s] = (\omega(s),t)$. $D(p_2) = B$; $p_2 = 1_B$.

Let $f_1 : D(p) \to D(p_1)$ be the inclusion map. We note that f_1 is a cofibration, and (f_1,U) is a relative fibration. Let $i: U \to B$ be the inclusion map and let $f_2 = i\pi_1 : D(p) \to D(p_2)$. Let $\mathcal{D}$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1,U)} & p_1 \\ \Big\downarrow{\scriptstyle (f_2,i)} & & \Big\downarrow{\scriptstyle (g_1,i)} \\ p_2 & \xrightarrow{(g_2,B)} & q \end{array}$$

[3.2.5]

be the associated pushout diagram. We note that $q^{-1}(1) = I/0 \sim 1 = S^1$ and $q^{-1}(0) = pt$. If q is a fibration then it has the homotopy lifting property which would imply that $S^1 \simeq pt$. But this is impossible. Thus q is not a fibration.



[3.2.6]

The following theorem lists sufficient conditions for a map over a space with a numerable open cover to be a fibration.

<u>Theorem 3.3</u>: Let $\{U_\sigma\}_{\sigma\in\Sigma}$ be a numerable open cover of a space B , and let p be a map over B such that $p|p^{-1}(U_\sigma)$ is a fibration for every $\sigma \in \Sigma$. Then p is a fibration.

The proof of this theorem can be found in [4].

The following result is a direct consequence of Theorems 3.2 and 3.3.

<u>Corollary 3.4</u>: Let B be a space with a numerable open cover $\{U_1, U_2\}$

and a partition of unity $\{[g_1],[g_2]\}$. Let p, p_1, and p_2 , be fibrations over $U_1 \cap U_2$, U_1 , and U_2 , respectively. Let $(f_k, i_k) \in \mathfrak{M}(p, p_k)$ be a relative fibration where the appropriate restriction of f_k , $f_k| : D(p) \to p_k^{-1}(U_1 \cap U_2)$, is a cofibration and a fiber homotopy equivalence and i_k is the inclusion map $(k = 1,2)$. Finally, let $\mathcal{D}$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1, i_1)} & p_1 \\ \downarrow{\scriptstyle (f_2, i_2)} & & \downarrow{\scriptstyle (h_1, j_1)} \\ p_2 & \xrightarrow{(h_2, j_2)} & q \end{array}$$

[3.4.1]

be a pushout diagram in $\mathfrak{M}$. Then q is a fibration.

<u>Proof</u>: Since $\mathcal{D}$ is a pushout, $\mathcal{D}_1$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1, i_1)} & p_1 \\ \downarrow{\scriptstyle (f_2|, U_1 \cap U_2)} & & \downarrow{\scriptstyle (h_1|, U_1)} \\ p_2|p_2^{-1}(U_1 \cap U_2) & \xrightarrow{(h_2|p_2^{-1}(U_1 \cap U_2), i_1)} & q|q^{-1}(U_1) \end{array}$$

[3.4.2]

is a pushout diagram. $(f_2|, U_1 \cap U_2)$ is a relative fibration, and $f_2|$ is a cofibration and a fiber homotopy equivalence. $[g_2]^{-1}(0,1] \cap U_1 = U_1 \cap U_2$. Thus, by Theorem 3.2, $q|q^{-1}(U_1)$ is a fibration. By an identical argument, we can conclude that $q|q^{-1}(U_2)$ is a fibration.

Therefore, by Theorem 3.3, q is a fibration.

We have seen that in order for our extensions constructed in Theorem 3.2 and Corollary 3.4 to be fibrations, it is necessary that the appropriate restrictions of our maps, $f_1|$ and $f_2|$, be fiber homotopy equivalences. We will now show that this condition is the only limitation we need place on our maps in order to produce a horizontal extension with the universal property.

<u>Theorem 3.5</u>: Let B be a space with a numerable open cover $\{U_1,U_2\}$ and a partition of unity $\{[g_1],[g_2]\}$. Let p, p_1, and p_2, be fibrations over $U_1 \cap U_2$, U_1, and U_2, respectively. Let $(f_k,i_k) \in \mathfrak{M}(p,p_k)$ where the appropriate restriction of f_k , $f_k|: D(p) \to p_k^{-1}(U_1 \cap U_2)$ is a fiber homotopy equivalence. Let $p(f_k)$ be the pushout space of $\mathcal{D}_k$:

$$\begin{array}{ccc} p & \xrightarrow{(f_k,i_k)} & p_k \\ \Big\downarrow{\scriptstyle (\epsilon_{k-1},U_1\cap U_2)} & & \Big\downarrow{\scriptstyle (\hat{g}_k,U_k)} \\ (p\pi_1)_k & \xrightarrow{(g_{1k},i_k)} & p(f_k) \end{array}$$

[3.5.1]

a pushout diagram in $\mathfrak{M}$, where $(p\pi_1)_k = \left\{D(p) \times \left[\frac{k-1}{2}, \frac{k}{2}\right] \xrightarrow{p\pi_1} U_1 \cap U_2\right\}$ and $\epsilon_{k-1}(x) = (x,k-1)$. Finally, let $\bar{\mathcal{D}}$:

$$\begin{array}{ccc}
p & \xrightarrow{(g_{11}\varepsilon_{\frac{1}{2}},\, i_1)} & p(f_1) \\
\Big\downarrow {\scriptstyle (g_{12}\varepsilon_{\frac{1}{2}},\, i_2)} & & \Big\downarrow {\scriptstyle (\bar{g}_1,\, j_1)} \\
p(f_2) & \xrightarrow{(\bar{g}_2,\, j_2)} & \bar{q}
\end{array}$$

[3.5.2]

be a pushout diagram in $\mathfrak{M}$. Then

(1) $\bar{q}$ is a fibration,

(2) Given maps $(h_k, \ell_k) \in \mathfrak{M}(p_k, \bar{p})$, $(k = 1,2)$, such that $\ell_1 i_1 = \ell_2 i_2 \colon U_1 \cap U_2 \to R(\bar{p})$ and $h_1 f_1$ is fiber homotopic to $h_2 f_2$, then there exists a map $(h, \ell) \colon \bar{q} \to \bar{p}$, such that

$$(h,\ell)(\bar{g}_k, j_k)(\hat{g}_k, U_k) = (h_k, \ell_k) \ , \ (k = 1,2) \ .$$

(3) Should (f_k, i_k) be a relative fibration and $f_k| \colon D(p) \to p_k^{-1}(U_1 \cap U_2)$ be a cofibration, $(k = 1,2)$, then the fibration q, the pushout space of $\mathcal{D}$:

$$\begin{array}{ccc}
p & \xrightarrow{(f_1,\, i_1)} & p_1 \\
\Big\downarrow {\scriptstyle (f_2,\, i_2)} & & \Big\downarrow {\scriptstyle (g_1,\, j_1)} \\
p_2 & \xrightarrow{(g_2,\, j_2)} & q
\end{array}$$

[3.5.3]

a pushout diagram in $\mathfrak{M}$, is fiber homotopically equivalent to $\bar{q}$.

<u>Proof</u>: (1) We note that $(\epsilon_{k-1}, U_1 \cap U_2) : p \to (p\pi_1)_k$ is a relative fibration, ϵ_{k-1} is a cofibration and a fiber homotopy equivalence, and $\mathcal{D}_k$ is a pushout diagram in $\mathfrak{M}$. Thus, by Theorem 3.2, $p(f_k)$ is a fibration, and $(\hat{g}_k, U_k)$ is a relative fibration $(k = 1,2)$.

We will now show that $\bar{q}|\bar{q}^{-1}(U_k)$ is a fibration. Without loss of generality, we assume that $k = 1$. Let us define

$$\lambda_{p(f_2)|} : \Omega_{p(f_2)|p(f_2)^{-1}(U_1 \cap U_2)} \times R^+ \to p(f_2)^{-1}(U_1 \cap U_2)$$

as follows:

$$\lambda_{p(f_2)|}\left[(\hat{g}_2(x_2), (\omega,r)), s\right] = \hat{g}_2 \lambda_{p_2}\left[(x_2, (\omega,r)), s\right]$$

$$\lambda_{p(f_2)|}\left[(g_{12}(x,\epsilon), (\omega,r)), rt\right] = \begin{cases} g_{12}(\lambda_p[(x,(\omega,r)),rt],\epsilon) \quad 0 \le t \le 1. \text{ For } \\ \qquad \epsilon \in [\tfrac{1}{2},\tfrac{2}{3}] \\ g_{12}(\lambda_p[(x,(\omega,r)),rt],\epsilon + t/6) \\ \qquad 0 \le t \le 6(\epsilon - \tfrac{2}{3}) \\ g_{12}(\lambda_p[(x,(\omega,r)),rt],2\epsilon - \tfrac{2}{3}) \\ \qquad 6(\epsilon - \tfrac{2}{3}) \le t \le 1 \text{ . For } \epsilon \in [\tfrac{2}{3},\tfrac{5}{6}] \\ g_{12}(\lambda_p[(x,(\omega,r)),rt],\epsilon + t/6) \\ \qquad 0 \le t \le 6(1 - \epsilon) \\ \hat{g}_2 \lambda_{p_2}[(f_2(x),(\omega_\delta, r\epsilon)), rt - 6(r - r\epsilon)] \\ \qquad 6(1-\epsilon) \le t \le 1 \text{ . For } \epsilon \in [\tfrac{5}{6},1] \\ \text{where } \omega_\delta(s) = \omega(s + 6r(1-\epsilon)) \text{ .} \end{cases}$$

We note that, as in the construction of $\bar{\lambda}_{p_1}$ in Theorem 2.10, $\lambda_{p(f_2)}$

We note that, as in the construction of $\bar{\lambda}_{p_1}$ in Theorem 2.10, $\lambda_{p(f_2)|}$

is a well defined lifting function of $p(f_2)|p(f_2)^{-1}(U_1 \cap U_2)$, and we can conclude that $(g_{12}|\epsilon_{\frac{1}{2}}, U_1 \cap U_2) : p \to p(f_2)|p(f_2)^{-1}(U_1 \cap U_2)$ is a relative fibration.

We will now prove that $g_{12}|\epsilon_{\frac{1}{2}}$ is a cofibration and a fiber homotopy equivalence. Define a haloing function $\varphi_2 : p(f_2)^{-1}(U_1 \cap U_2) \to I$ as follows:

$$\varphi_2(\hat{g}_2|(x_2)) = 1 \quad \text{for every} \quad x_2 \in D(p_2) \quad \text{and}$$

$$\varphi_2(g_{12}|(x,t)) = 2(t - \tfrac{1}{2}) \quad \text{for every} \quad (x,t) \in D(p) \times [\tfrac{1}{2},1] \;.$$

$$\varphi_2(g_{12}|\epsilon_1(x)) = \varphi_2(g_{12}|(x,1)) = 1 = \varphi_2(\hat{g}_2|f_2|(x)) \;.$$

Therefore φ_2 is well defined. Let $V_2 = \varphi_2^{-1}[0,1)$ and define $H_2 : V_2 \times I \to V_2$ as follows:

$$H_2(g_{12}|(x,t),s) = g_{12}|(x, t + s(\tfrac{1}{2} - t)) \;.$$

$\varphi_2^{-1}(0) = g_{12}|\epsilon_{\frac{1}{2}} D(p)$. $H_2|V_2 \times \{0\} = 1_{V_2}$. $H_2|V_2 \times \{1\} : V_2 \to D(p)$. $H_2(g_{12}|\epsilon_{\frac{1}{2}} \times I) = g_{12}|\epsilon_{\frac{1}{2}} \pi_1 : D(p) \times I \to p(f_2)^{-1}(U_1 \cap U_2)$. Thus, $g_{12}|\epsilon_{\frac{1}{2}}$ is a cofibration.

In order to prove that $g_{12}|\epsilon_{\frac{1}{2}}$ is a fiber homotopy equivalence, we need only show that this condition holds for $g_{12}|$, since $\epsilon_{\frac{1}{2}} : D(p) \to D(p) \times [\tfrac{1}{2},1]$ is a fiber homotopy equivalence. We note that $\mathcal{D}_2|$:

$$\begin{array}{ccc}
p & \xrightarrow{(f_2|,\,U_1\cap U_2)} & p_2|p_2^{-1}(U_1\cap U_2) \\
\downarrow{\scriptstyle (\epsilon_1,\,U_1\cap U_2)} & & \downarrow{\scriptstyle (\hat{g}_2|p_2^{-1}(U_1\cap U_2),\,U_1\cap U_2)} \\
(p\pi_1)_2 & \xrightarrow{(g_{12}|,\,U_1\cap U_2)} & p(f_2)|p(f_2)^{-1}(U_1\cap U_2)
\end{array}$$

[3.5.4]

is a pushout diagram in $\mathfrak{M}/U_1 \cap U_2$ such that ϵ_1 is a cofibration and a fiber homotopy equivalence and $(\epsilon_1, U_1 \cap U_2)$ is a relative fibration. Thus, by Theorem 2.15, we can conclude that $\hat{g}_2|p_2^{-1}(U_1 \cap U_2)$ is a fiber homotopy equivalence. Therefore $(\hat{g}_2|p_2^{-1}(U_1 \cap U_2))(f_2|) = g_{12}|\epsilon_1\colon D(p) \to p(f_2)^{-1}(U_1 \cap U_2)$ is a fiber homotopy equivalence, i.e., there exist maps $(g_{12}|\epsilon_1)^{-1}\colon p(f_2)^{-1}(U_1 \cap U_2) \to D(p)$, $\bar{H}_{21}\colon 1_{D(p)} \simeq (g_{12}|\epsilon_1)^{-1}(g_{12}|\epsilon_1)$, and $\bar{H}_{22}\colon 1_{p(f_2)^{-1}(U_1 \cap U_2)} \simeq (g_{12}|\epsilon_1)(g_{12}|\epsilon_1)^{-1}$ where $\bar{H}_{21}$ and $\bar{H}_{22}$ are vertical homotopies. Let

$$g_{12}|^{-1} = \epsilon_1(g_{12}|\epsilon_1)^{-1}\colon p(f_2)^{-1}(U_1 \cap U_2) \to D(p) \times [\tfrac{1}{2},1] = D(p\pi_1)_2 .$$

$$g_{12}|g_{12}|^{-1} = (g_{12}|)\epsilon_1(g_{12}|\epsilon_1)^{-1} = (g_{12}|\epsilon_1)(g_{12}|\epsilon_1)^{-1} \simeq 1_{p(f_2)^{-1}(U_1 \cap U_2)} .$$

$$g_{12}|^{-1}g_{12}| = \epsilon_1(g_{12}|\epsilon_1)^{-1}g_{12}| \simeq \epsilon_1(g_{12}|\epsilon_1)^{-1}g_{12}|\epsilon_1\epsilon_1^{-1} \simeq \epsilon_1 1_{D(p)}\epsilon_1^{-1}$$

$$= \epsilon_1\epsilon_1^{-1} \simeq 1_{D(p\pi_1)_2} .$$

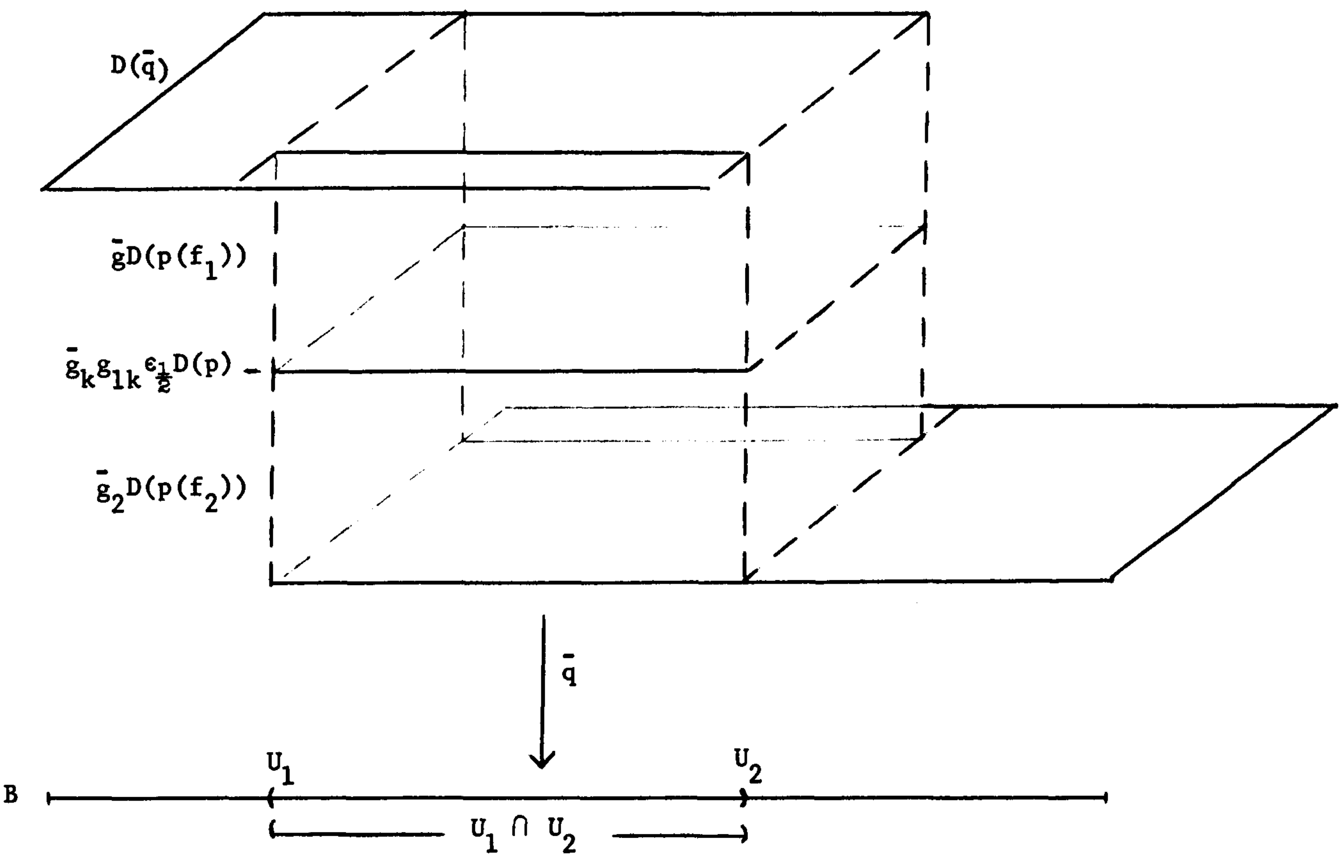
Thus $g_{12}|\colon D(p) \times [\tfrac{1}{2},1] \to p(f_2)^{-1}(U_1 \cap U_2)$ is a fiber homotopy equivalence. We now have $\bar{\mathfrak{D}}|$:

$$\begin{array}{ccc}
p & \xrightarrow{\quad (f_1, i_1) \quad} & p(f_1) \\
\Big\downarrow {\scriptstyle (g_{12}|\epsilon_{\frac{1}{2}},\, U_1 \cap U_2)} & & \Big\downarrow {\scriptstyle (\bar{g}_1|,\, U_1)} \\
p(f_2)|p(f_2)^{-1}(U_1 \cap U_2) & \xrightarrow{(\bar{g}_2|p(f_2)^{-1}(U_1 \cap U_2),\, i_1)} & \bar{q}|\bar{q}^{-1}(U_1)
\end{array}$$

[3.5.5]

a pushout diagram in $\mathfrak{M}$ satisfying the hypothesis of Theorem 3.2. Thus, $\bar{q}|\bar{q}^{-1}(U_1)$ is a fibration.

By a similar argument, we can prove that $\bar{q}|\bar{q}^{-1}(U_2)$ is a fibration. Therefore, by Theorem 3.3, $\bar{q}$ is a fibration.



[3.5.6]

(2) Let $(h_k, \ell_k) \in \mathfrak{M}(p_k, \bar{p})$, $(k = 1,2)$, be maps such that $\ell_1 i_1 = \ell_2 i_2 : U_1 \cap U_2 \to R(\bar{p})$ and $\tilde{H}: h_1 f_1 \simeq h_2 f_2 : D(p) \to D(\bar{p})$ under a vertical homotopy.

$$\mathfrak{H}|D(p) \times \left[\frac{k-1}{2}, \frac{k}{2}\right] : D(p) \times \left[\frac{k-1}{2}, \frac{k}{2}\right] \to D(\bar{p}) \ .$$

$$\left(\mathfrak{H}|D(p) \times \left[\frac{k-1}{2}, \frac{k}{2}\right], \ell_k i_k\right)\left(\epsilon_{k-1}, U_1 \cap U_2\right) = (\mathfrak{H}|D(p) \times \{k-1\}, \ell_k i_k)$$

$$= (h_k f_k, \ell_k i_k) = (h_k, \ell_k)(f_k, i_k) : p \to \bar{p} \ .$$

Thus, since $\mathfrak{D}_k$ is a pushout diagram, there exists a unique map $(\bar{H}_k, \ell_k) : p(f_k) \to \bar{p}$ such that $(\bar{H}_k, \ell_k)(\hat{g}_k, U_k) = (h_k, \ell_k)$ and

$$(\bar{H}_k, \ell_k)(g_{1k}, i_k) = \left(\mathfrak{H}|D(p) \times \left[\frac{k-1}{2}, \frac{k}{2}\right], \ell_k i_k\right), \quad (k = 1,2).$$

$$(\bar{H}_1, \ell_1)(g_{11}\epsilon_{\frac{1}{2}}, i_1) = (\bar{H}_1 g_{11}\epsilon_{\frac{1}{2}}, \ell_1 i_1) = (\mathfrak{H}|D(p) \times \{\tfrac{1}{2}\}, \ell_k i_k)$$

$$= (\bar{H}_2 g_{12}\epsilon_{\frac{1}{2}}, \ell_2 i_2) = (\bar{H}_2, \ell_2)(g_{12}\epsilon_{\frac{1}{2}}, i_2) \ .$$

Thus, since $\bar{\mathfrak{D}}$ is a pushout diagram, there exists a unique map $(h,\ell) : \bar{q} \to \bar{p}$ such that $(h,\ell)(\bar{g}_k, j_k) = (\bar{H}_k, \ell_k)$, $(k = 1,2)$.

$$(h,\ell)(\bar{g}_k, j_k)(\hat{g}_k, U_k) = (\bar{H}_k, \ell_k)(\hat{g}_k, U_k) = (h_k, \ell_k) \ .$$

(3) If $f_k| : D(p) \to p_k^{-1}(U_1 \cap U_2)$ is a cofibration as well as a fiber homotopy equivalence and (f_k, i_k) is a relative fibration, then we can conclude, by Corollary 3.4, that q , the pushout space of $\mathfrak{D}$:

$$\begin{array}{ccc} p & \xrightarrow{(f_1, i_1)} & p_1 \\ \Big\downarrow{\scriptstyle (f_2, i_2)} & & \Big\downarrow{\scriptstyle (g_1, j_1)} \\ p_2 & \xrightarrow{(g_2, j_2)} & q \end{array}$$

[3.5.7]

a pushout diagram in $\mathfrak{M}$, is a fibration.

We will now prove that q is fiber homotopically equivalent to $\bar{q}$. Define a map $r_k : D(p(f_k)) \to D(p_k)$ as follows: $r_k(\hat{g}_k(x_k)) = x_k$ and $r_k(g_{1k}(x,\epsilon)) = f_k(x)$. We note that $(r_k, U_k) \in \mathfrak{m}(p(f_k), p_k)$ and that r_k is a fiber homotopy equivalence. We note further that $r_k g_{1k} \epsilon_{\frac{1}{2}} = f_k : p \to p_k$, $(k = 1,2)$. Thus, $g_1 r_1 g_{11} \epsilon_{\frac{1}{2}} = g_1 f_1 = g_2 f_2 = g_2 r_2 g_{12} \epsilon_{\frac{1}{2}}$. Therefore, since $\bar{\mathcal{S}}$ is a pushout diagram, there exists a unique map $(f,B): \bar{q} \to q$ such that $(f,B)(\bar{g}_k, j_k) = (g_k, j_k)(r_k, U_k)$, $(k = 1,2)$.

Since $\bar{g}_k|: D(p(f_k)) \to \bar{q}^{-1}(U_k), r_k$, and $g_k|: D(p_k) \to q^{-1}(U_k)$ are fiber homotopy equivalences, $(k = 1,2)$, we can conclude that $f|\bar{q}^{-1}(U_k): \bar{q}^{-1}(U_k) \to q^{-1}(U_k)$ is a fiber homotopy equivalence. Thus, by utilizing the following theorem found in [5], we can conclude that f is a fiber homotopy equivalence.

<u>Theorem 3.6</u>: Let $\{U_\sigma\}_{\sigma\in\Sigma}$ be a numerable open cover of a space B. Let p and $\bar{p}$ be maps over B, and let $(f,B) \in \mathfrak{m}/B(p,\bar{p})$ such that $f|p^{-1}(U_\sigma): p^{-1}(U_\sigma) \to \bar{p}^{-1}(U_\sigma)$ is a fiber homotopy equivalence for every $\sigma \in \Sigma$. Then $f: D(p) \to D(\bar{p})$ is a fiber homotopy equivalence.

We will end this section by generalizing Theorem 3.5 and constructing a horizontal extension of a family of fibrations $\{p_\sigma\}_{\sigma\in\Sigma}$ over a numerable open cover of B, $\{U_\sigma\}_{\sigma\in\Sigma}$.

In order to construct this extension, we make use of the following result which stabilizes a lifting function upon extension.

<u>Theorem 3.7</u>: Let ${}_1\lambda_p$ and ${}_2\lambda_p$ be two distinct lifting functions of a fibration p. Then there exists a map $H: \Omega_p \times R^+ \times I \to D(p)$ such that

(a) $H|\Omega_p \times R^+ \times \{0\} = {}_1\lambda_p$,

(b) $H|\Omega_p \times R^+ \times \{1\} = {}_2\lambda_p$, and

(c) $H|\Omega_p \times R^+ \times \{t\}$ is a lifting functions of p for all $t \in I$.

For a proof of this theorem, the reader is referred to [4].

Proposition 3.8: Let B be a space with a numerable open cover $\{U_1, U_2\}$ and a partition of unity $\{[g_1],[g_2]\}$. Let q be a fibration over B, and let λ_1 be a lifting function of $q|q^{-1}(U_1)$. Finally, let $\epsilon \in (0,1)$. Then λ_1 restricted to $\Omega_{q|q^{-1}[g_1]^{-1}[\epsilon,1]} \times R^+$ can be extended to a lifting function of q .

Proof: Since q is a fibration, there exists a lifting function $\lambda_q\colon \Omega_q \times R^+ \to D(q)$. We note that $q|q^{-1}(U_1)$ is a fibration with a lifting function $\lambda_q|\Omega_{q|q^{-1}(U_1)} \times R^+ \to q^{-1}(U_1)$. Thus, there exists a map $H\colon \Omega_{q|q^{-1}(U_1)} \times R^+ \times I \to q^{-1}(U_1)$ such that

(a) $H|\Omega_{q|q^{-1}(U_1)} \times R^+ \times \{0\} = \lambda_q|\Omega_{q|q^{-1}(U_1)} \times R^+$,

(b) $H|\Omega_{q|q^{-1}(U_1)} \times R^+ \times \{1\} = \lambda_1$, and

(c) $H|\Omega_{q|q^{-1}(U_1)} \times R^+ \times \{t\}$ is a lifting function for all $t \in I$.

Define $\lambda\colon \Omega_q \times R^+ \to D(q)$ as follows:

$$\lambda[(x,(\omega,r)),s] = \begin{cases} \lambda_q[(x,(\omega,r)),s] & \text{for } \omega(s) \in [g_1]^{-1}[0,\epsilon/2] \\ H\left([(x,(\omega,r)),s],\dfrac{2[g_1](\omega(s))-\epsilon}{\epsilon}\right) & \text{for } \omega(s) \in [g_1]^{-1}[\epsilon/2,\epsilon] \\ \lambda_1[(x,(\omega,r)),s] & \text{for } \omega(s) \in [g_1]^{-1}[\epsilon,1] \end{cases}$$

We note that for $\omega(s) \in [g_1]^{-1}(\epsilon/2)$, $\dfrac{2[g_1](\omega(s))-\epsilon}{\epsilon} = \dfrac{2\cdot\epsilon/2-\epsilon}{\epsilon} = \dfrac{\epsilon-\epsilon}{\epsilon} = 0$, and for $\omega(s) \in [g_1]^{-1}(\epsilon)$, $\dfrac{2[g_1](\omega(s))-\epsilon}{\epsilon} = \dfrac{2\epsilon-\epsilon}{\epsilon} = \dfrac{\epsilon}{\epsilon} = 1$. Thus λ is well defined and continuous. By its construction, we conclude that λ is a lifting function of q and that

$$\lambda|\Omega_{q|q^{-1}[g_1]^{-1}[\epsilon,1]} \times R^+ = \lambda_1 .$$

Theorem 3.9: Let $\{V_k\}_{k\in Z^+}$ be a sequentially numerable open cover of a space B with a partition of unity $\{[g_k]\}_{k\in Z^+}$. Let $\{p_k\}_{k\in Z^+}$ be a family of fibrations over $\{V_k\}_{k\in Z^+}$ which are fiber homotopically equivalent over the non-empty intersections. Then there exists a fibration $\bar{q}$ over B and a family of maps $\{(\hat{\bar{g}}_k, j_k)\}_{k\in Z^+}$ from $\{p_k\}_{k\in Z^+}$ to $\bar{q}$ such that

(a) $\hat{\bar{g}}|D(p_k) \to \bar{q}^{-1}(V_k)$ is a fiber homotopy equivalence for all $k \in Z^+$, and

(b) if $\{(h_k,\ell_k)\}_{k\in Z^+}$ is a family of maps from $\{p_k\}_{k\in Z^+}$ to $\bar{p}$ which are fiber homotopic over the non-empty intersections, then there exists a map $(h,\ell) \in \mathfrak{M}(\bar{q},\bar{p})$ such that $(h,\ell)(\hat{\bar{g}}_k, j_k) = (h_k,\ell_k)$ for every $k \in Z^+$.

<u>Proof</u>: Let $W_1 = V_1$, and let $W_n = W_{n-1} \cup V_n$ for $n \geq 2$. We note that $\{W_{n-1}, V_n\}$ constitutes a numerable open cover of W_n with a partition of unity

$$\left\{\sum_{j=1}^{n-1} [g_j] \Big/ \sum_{j=1}^{n} [g_j],\ [g_n] \Big/ \sum_{j=1}^{n} [g_j]\right\} = \left\{[\bar{g}_{n-1}], [\tilde{g}_n]\right\},\ (n \geq 2).$$

Let $\bar{q}_0 = \bar{q}_1 = p_1$, and let $\lambda_{\bar{q}_0} = \lambda_{\bar{q}_1} = \lambda_{p_1}$. We note that $\bar{q}_1$ is a fibration over W_1 and $1_{p_1} \in \mathfrak{m}(p_1, \bar{q}_1)$ such that (a) and (b) are satisfied for $\{p_1\}$.

We now assume that there exist a fibration $\bar{q}_{n-1}$ over W_{n-1} and a family of maps $\{(\hat{\bar{g}}_{k(n-1)}, j_{k(n-1)})\}_{k=1,2,\ldots,n-1}$ from $\{p_k\}_{k=1,2,\ldots,n-1}$ to $\bar{q}_{n-1}$ such that (a) and (b) are satisfied for $\{p_k\}_{k=1,2,\ldots,n-1}$. We also assume that there exists a lifting function $\lambda_{\bar{q}_{n-1}}$ of $\bar{q}_{n-1}$ which is an extension of

$$\lambda_{\bar{q}_{n-2}} \Big| \Omega_{\bar{q}_{n-2}} \Big| \bar{q}_{n-2}^{-1}[\bar{g}_{n-2}]^{-1}[1/2^{n-2}, 1] \times R^+ .$$

We will now show that there exist a fibration $\bar{q}_n$ over W_n and a family of maps $\{(\hat{\bar{g}}_{k(n)}, j_{k(n)})\}_{k=1,2,\ldots,n}$ from $\{p_k\}_{k=1,2,\ldots,n}$ to $\bar{q}_n$ such that (a) and (b) are satisfied for $\{p_k\}_{k=1,2,\ldots,n}$, and we will exhibit a lifting function of $\bar{q}_n$ which is an extension of

$$\lambda_{\bar{q}_{n-1}} \Big| \Omega_{\bar{q}_{n-1}} \Big| \bar{q}_{n-1}^{-1}[\bar{g}_{n-1}]^{-1}[1/2^{n-1}, 1] \times R^+ .$$

$\bar{q}_{n-1}|\bar{q}_{n-1}^{-1}(W_{n-1} \cap V_n)$, $\bar{q}_{n-1}$, and p_n, are fibrations over $W_{n-1} \cap V_n$, W_{n-1}, and V_n, respectively. Let $(\alpha_{n-1}, r_{n-1}) \in \mathfrak{m}(\bar{q}_{n-1}|\bar{q}_{n-1}^{-1}(W_{n-1} \cap V_n), \bar{q}_{n-1})$ and $(\beta_n, s_n) \in \mathfrak{m}(p_n|p_n^{-1}(W_{n-1} \cap V_n), p_n)$ be the inclusion maps. Let $\bar{f}_{k(n)}: p_k^{-1}(V_k \cap V_n) \to p_n^{-1}(V_k \cap V_n)$ be the hypothesized fiber homotopy equivalence when $V_k \cap V_n \neq \emptyset$ and the trivial map when $V_k \cap V_n = \emptyset$, $(k = 1,2,\ldots,n-1)$. Since $\bar{q}_{n-1}$ and $\{(\hat{\bar{g}}_{k(n-1)}, j_{k(n-1)})\}_{k=1,2,\ldots,n-1}$ satisfy (a) and (b) for $\{p_k\}_{k=1,2,\ldots,n-1}, \bar{q}_{n-1}|\bar{q}_{n-1}^{-1}(W_{n-1} \cap V_n)$ and $\{(\hat{\bar{g}}_{k(n-1)}|p_k^{-1}(W_{n-1} \cap V_n), j_{k(n-1)}|V_k \cap V_n)\}_{k=1,2,\ldots,n-1}$ satisfy (a) and (b) for $\{p_k|p_k^{-1}(W_{n-1} \cap V_n)\}_{k=1,2,\ldots,n-1} = \{p_k|p_k^{-1}(V_k \cap V_n)\}_{k=1,2,\ldots,n-1}$. Thus there exists a map $(\bar{f}_n, W_{n-1} \cap V_n) \in \mathfrak{m}(\bar{q}_{n-1}|\bar{q}_{n-1}^{-1}(W_{n-1} \cap V_n), p_n|p_n^{-1}(W_{n-1} \cap V_n))$. We note that $\bar{f}_n|\bar{q}_{n-1}^{-1}(V_k \cap V_n): \bar{q}_{n-1}^{-1}(V_k \cap V_n) \to p_n^{-1}(V_k \cap V_n)$ is a fiber homotopy equivalence for $k = 1,2,\ldots,n-1$. Thus, by Theorem 3.6, we can conclude that $\bar{f}_n: \bar{q}_{n-1}^{-1}(W_{n-1} \cap V_n) \to p_n^{-1}(W_{n-1} \cap V_n)$ is a fiber homotopy equivalence.

Let $(\hat{g}_n, s_n) = (\beta_n, s_n)(\bar{f}_n, W_{n-1} \cap V_n): \bar{q}_{n-1}|\bar{q}_{n-1}^{-1}(W_n \cap V_n) \to p_n$, and note that if we consider the fibrations $\bar{q}_{n-1}|\bar{q}_{n-1}^{-1}(W_n \cap V_n)$, $\bar{q}_{n-1}$, and p_n, and the maps (α_{n-1}, r_{n-1}) and $(\hat{g}_n, s_n)$, then Theorem 3.5 allows us to conclude that there exist a fibration $\bar{q}_n$ over W_n and maps $(\bar{g}_n \hat{g}_n, s_n): \bar{q}_{n-1} \to \bar{q}_n$ and $(\bar{g}_{p_n} \hat{g}_{p_n}, i_n): p_n \to \bar{q}_n$ such that (a) and (b) are satisfied for $\{\bar{q}_{n-1}, p_n\}$. Given our assumptions regarding

$\bar{q}_{n-1}$ and $\{(\hat{\bar{g}}_{k(n-1)}, j_{k(n-1)})\}_{k=1,2,\ldots,n-1}$, we can now conclude that there exists a family of maps $\{(\hat{\bar{g}}_{k(n)}, j_{k(n)})\}_{k=1,2,\ldots,n}$ from $\{p_k\}_{k=1,2,\ldots,n}$ to $\bar{q}_n$ such that (a) and (b) are satisfied for $\{p_k\}_{k=1,2,\ldots,n}$. $(\hat{\bar{g}}_{k(n)}, j_{k(n)}) = (\bar{g}_n\hat{\bar{g}}_n, s_n)(\hat{\bar{g}}_{k(n-1)}, j_{k(n-1)})$, $(k = 1,2,\ldots,n-1)$.

Theorem 3.2 allows us to construct a lifting function λ_n of $\bar{q}_n|\bar{q}_n^{-1}(W_{n-1})$ which is an extension of $\lambda_{\bar{q}_{n-1}}$. Thus, by Proposition 3.8, there exists a lifting function $\lambda_{\bar{q}_n}$ of $\bar{q}_n$ which is an extension of $\lambda_n|\Omega_{\bar{q}_n|\bar{q}_n^{-1}[\bar{g}_{n-1}]^{-1}[1/2^{n-1},1]} \times R^+$ and is thus an extension of $\lambda_{\bar{q}_{n-1}}|\Omega_{\bar{q}_{n-1}|\bar{q}_{n-1}^{-1}[\bar{g}_{n-1}]^{-1}[1/2^{n-1},1]} \times R^+$.

Let $\bar{q}$ and $\{(\gamma_k, \mu_k)\}_{k\in Z^+}$ be the colimit of $\{\bar{q}_k\}_{k\in Z^+}$ and the inclusion maps $\{(\bar{g}_k\hat{\bar{g}}_k, s_k)\}_{k\in Z^+}$. Since $B = \text{colimit } W_n$, $\bar{q}$ is a map over B. We note that since $\lambda_{\bar{q}_n}$ is an extension of $\lambda_{\bar{q}_{n-1}}$ modulo $[\bar{g}_{n-1}]^{-1}(0,1/2^{n-1})$, there exists a map $\lambda_{\bar{q}}: \Omega_{\bar{q}} \times R^+ \to D(\bar{q})$ with the properties of a lifting function. Thus $\bar{q}$ is a fibration. Let $(\hat{\bar{g}}_k, j_k) = (\gamma_k, \mu_k)(\hat{\bar{g}}_{k(k)}, j_{k(k)}): p_k \to \bar{q}$, $(k \in Z^+)$. We note that since $\bar{q}_n$ and $\{(\hat{\bar{g}}_{k(n)}, j_{k(n)})\}_{k=1,2,\ldots,n}$ satisfy (a) and (b) for $\{p_k\}_{k=1,2,\ldots,n}$, $\bar{q}$ and $\{(\hat{\bar{g}}_k, j_k)\}_{k\in Z^+}$ satisfy (a) and (b) for $\{p_k\}_{k\in Z^+}$.

The following lemma attributed to Milnor allows us to reduce the case of a family of fibrations $\{p_\sigma\}_{\sigma\in\Sigma}$ over a numerable open cover $\{U_\sigma\}_{\sigma\in\Sigma}$ to one of a family of fibrations $\{p_k\}_{k\in Z^+}$ over a sequentially numerable open cover $\{V_k\}_{k\in Z^+}$.

Lemma 3.10: Given a numerable open cover $\{U_\sigma\}_{\sigma\in\Sigma}$ of a space B, there exists a sequentially numerable open cover $\{V_k\}_{k\in Z^+}$ of B such that each V_k is a disjoint union of open sets, each of which lies in some U_σ.

For a proof of this lemma, the reader is referred to [5].

We note that if we are given a family of fibrations $\{p_\sigma\}_{\sigma\in\Sigma}$ over $\{U_\sigma\}_{\sigma\in\Sigma}$, fiber homotopically equivalent over the non-emply intersections, the suitable restrictions of these fibrations to each V_k constructed in Lemma 3.10 yield a family of fibrations $\{p_k\}_{k\in Z^+}$ over $\{V_k\}_{k\in Z^+}$ which are themselves fiber homotopically equivalent over the non-empty intersections of this cover.

By Theorem 3.9, there exists a fibration $\bar{q}$ over B and a family of maps $\{(\hat{\bar{g}}_k, j_k)\}_{k\in Z^+}$ from $\{p_k\}_{k\in Z^+}$ to $\bar{q}$ such that (a) and (b) are satisfied for $\{p_k\}_{k\in Z^+}$, and by the way in which each p_k was constructed, we can construct a family of maps $\{(\hat{\bar{g}}_\sigma, j_\sigma)\}_{\sigma\in\Sigma}$ from $\{p_\sigma\}_{\sigma\in\Sigma}$ to $\bar{q}$ such that (a) and (b) hold for $\{p_\sigma\}_{\sigma\in\Sigma}$. We thus have the following theorem:

<u>Theorem 3.11</u>: Let Σ be an indexing set, and let $A(\Sigma)$ be the category whose objects are subsets of Σ with cardinality ≤ 2 and whose morphisms are the identities and ordered pairs $\langle\sigma,\tau\rangle: \{\sigma,\tau\} \to \{\sigma\}$. Let $p: A(\Sigma) \to \mathcal{J}$ be a covariant functor from $A(\Sigma)$ to the category of fibrations $\mathcal{J}$ such that

(a) $\{Rp(\{\sigma\})\}_{\sigma\in\Sigma}$ is a numerable open cover of a space B,

(b) $Rp(\langle\sigma,\tau\rangle): Rp(\{\sigma,\tau\}) \to Rp(\{\sigma\})$ is an injection, and

(c) $Dp(\langle\sigma,\tau\rangle)|: Dp(\{\sigma,\tau\}) \to p^{-1}(\{\sigma\})(Rp(\{\sigma,\tau\}))$ is a fiber homotopy equivalence.

Then there exist a fibration $\bar{q}$ over B and a family of maps $\{(\hat{\bar{g}}_\sigma, j_\sigma)\}_{\sigma\in\Sigma}$ from $\{p(\{\sigma\})\}_{\sigma\in\Sigma}$ to $\bar{q}$ such that

(a) $\hat{\bar{g}}_\sigma|: Dp(\{\sigma\}) \to \bar{q}^{-1}(Rp(\{\sigma\}))$ is a fiber homotopy equivalence for all $\sigma \in \Sigma$,

(b) $\hat{\bar{g}}_\sigma|: p^{-1}(\{\sigma\})(Rp(\{\sigma,\tau\}))$ is fiber homotopic to $\hat{\bar{g}}_\tau|: p^{-1}(\{\tau\})(Rp(\{\sigma,\tau\}))$, and

(c) if $\{(h_\sigma,\ell_\sigma)\}_{\sigma\in\Sigma}$ is a family of maps from $\{p(\{\sigma\})\}_{\sigma\in\Sigma}$ to $\bar{p}$ which are fiber homotopic over the non-empty intersections, then there exists a map $(h,\ell) \in \mathfrak{M}(\bar{q},\bar{p})$ such that $(h,\ell)(\hat{\bar{g}}_\sigma, j_\sigma) = (h_\sigma,\ell_\sigma)$ for every $\sigma \in \Sigma$.

By Theorem 3.6, we can conclude that if $\bar{q}$ and $\bar{q}'$ are two maps over B satisfying (a), (b), and (c), then they are fiber homotopically equivalent. Thus $\bar{q}$ is unique up to a fiber homotopy equivalence.

SECTION 4

THE CATEGORY $\mathcal{J}$

An extension of a family of fiber homotopically equivalent classes of fibrations over locally contractible in the large base spaces:

Throughout this section we assume that all base spaces considered are pathwise connected. We begin with a summary of the results found in Section 5 of a paper by E. Dyer and D. Kahn [6].

Definition 4.1: Let $\bar{p}$ be a fibration. Then $\lambda_{\bar{p}} : \Omega_{\bar{p}} \times R^+ \to D(\bar{p})$ is a transitive lifting function of $\bar{p}$ provided that for $(x,(\omega,r)) \in \Omega_{\bar{p}}$ and $(\omega',r') \in PR(\bar{p})$ such that $\omega(r) = \omega'(0)$,

$$\lambda_{\bar{p}}\left[(\lambda_{\bar{p}}[(x,(\omega,r)),r],(\omega',r')),r'\right] = \lambda_{\bar{p}}\left[(x,(\omega\cdot\omega',r+r')),r+r'\right]$$

where $\omega\cdot\omega'(s) = \begin{cases} \omega(s) & 0 \le s \le r \\ \omega'(s-r) & r \le s \le r + r' \end{cases}$.

Theorem 4.2: For every fibration p over B, there exists a fibration $\bar{p}$ over B, fiber homotopically equivalent to p, which has a transitive lifting function $\lambda_{\bar{p}} : \Omega_{\bar{p}} \times R^+ \to D(\bar{p})$.

We will now construct a fibration $\hat{p}$ over B which we call the associated fibration of p . Given a fibration p over B and a fiber homotopically equivalent fibration $\bar{p}$ with a transitive lifting function $\lambda_{\bar{p}} : \Omega_{\bar{p}} \times R^+ \to D(\bar{p})$, let $b_0 \in B$ and $F = \bar{p}^{-1}(b_0)$.

We define $\Omega_{b_0}(B)$ as the space of loops in B emanating from b_0.

$\Omega_{b_0}(B) = \{(\omega,r) \in PB \mid \omega(0) = \omega(r) = b_0\}$ with the topology making the inclusion map $i_\Omega: \Omega_{b_0}(B) \to PB$ an injection.

Define a left action of $\Omega_{b_0}(B)$ on F, $F \times \Omega_{b_0}(B) \xrightarrow{a} F$ as follows:

$x_F \cdot (\omega_L, r_L) = \lambda_{\bar{p}}\left[(x_F, (\omega_L, r_L)), r_L\right]$. $x_F \cdot (\omega_{b_0}, 0) = x_F$ and

$(x_F \cdot (\omega_{L1}, r_{L1})) \cdot (\omega_{L2}, r_{L2}) = x_F \cdot (\omega_{L1} \cdot \omega_{L2}, r_{L1} + r_{L2})$.

Let $P(b_0, B, B) = P_{b_0}$ be the space of paths in B emanating from b_0 with the topology making the inclusion map $i_p: P_{b_0} \to PB$ an injection. We note that there exists a right action of $\Omega_{b_0}(B)$ on P_{b_0} formed by juxtaposing the loop with the path.

We now define an equivalence relation on $F \times P_{b_0}$ generated in the following way: $(x_F, (\omega, r)) \sim (x'_F, (\omega', r'))$ provided that there exists a loop $(\omega_L, r_L) \in \Omega_{b_0}(B)$ such that $x'_F \cdot (\omega_L, r_L) = x_F$ and $(\omega_L \cdot \omega, r_L + r) = (\omega', r')$, i.e., for $(x_F, (\omega, r)) \in F \times P_{b_0}$ and $(\omega_L, r_L) \in \Omega_{b_0}(B)$, $(x_F \cdot (\omega_L, r_L), (\omega, r)) \sim (x_F, (\omega_L \cdot \omega, r + r_L))$. Let $D(\hat{p}) = F \underset{\Omega_{b_0}(B)}{\times} P_{b_0} = F \times P_{b_0} /\sim$ with the topology making $\beta: F \times P_{b_0} \to D(\hat{p})$ a projection, where β takes each point into its equivalence class.

Define $\hat{p}: D(\hat{p}) \to B$ as follows: $\hat{p}(\langle x_F, (\omega, r) \rangle) = \omega(r)$.

Theorem 4.3: $\hat{p}$ is a fibration with a transitive lifting function and fiber $F \underset{\Omega_{b_0}(B)}{\times} \Omega_{b_0}(B)$ homeomorphic to F. Furthermore, should B be locally contractible in the large, $\hat{p}$ would be fiber homotopically equivalent to p .

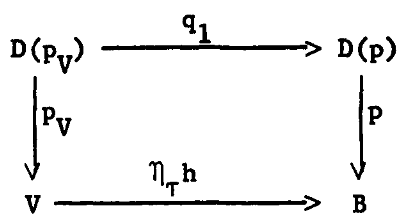
We now let $B \subset B'$, and let $D(\hat{p}_{B'}) = F \underset{\Omega_{b_0}(B)}{\times} P(b_0,B',B')$.

Define $\hat{p}_{B'} : D(\hat{p}_{B'}) \to B'$ as follows: $\hat{p}_{B'}(<x_F,(\omega,r)>) = \omega(r)$. As with the map $\hat{p}$, we can conclude that $\hat{p}_{B'}$ is a fibration with a transitive lifting function $\lambda_{\hat{p}_{B'}}$.

We will now demonstrate that if the inclusion map $i: B \to B'$ is a cofibration, then $\hat{p}_{B'}$ is an extension of $\hat{p}$ having the property that any map from $\hat{p}$ to a fibration q can be extended to a map from $\hat{p}_{B'}$ to q , up to a fiber homotopy. Thus, if B is locally contractible in the large, $\hat{p}_{B'}$ would be an extension of p, up to a fiber homotopy. First we must prove the following lemma and its corollary.

Lemma 4.4: Let $i: B \to B'$ be a cofibration, let p and p' be fibrations over B and B' respectively, and let $(g,i) \in \mathfrak{M}(p,p')$. Then there exists a fibration $\tilde{p}$ over B' and maps $(\tilde{g},i) \in \mathfrak{M}(p,\tilde{p})$ and $(\hat{g}_1^{-1},B') \in \mathfrak{M}/B'(\tilde{p},p')$ such that $\hat{g}_1^{-1}: D(\tilde{p}) \to D(p')$ is a fiber homotopy equivalence, $(\tilde{g},i)$ is a relative fibration, and $\hat{g}_1^{-1}\,\tilde{g} = g: D(p) \to D(p')$.

<u>Proof</u>: Since $i: B \to B'$ is a cofibration, there exist maps $\varphi: B' \to I$ and $h: \varphi^{-1}[0,1) \to PB'$ such that $\eta_0 h(v) = v$, $\eta_T h(v) \in B$, and $\ell h(v) = \varphi(v)$ for every $v \in \varphi^{-1}[0,1)$. Let p_V be the fibration induced by the following pullback diagram $\mathcal{D}_V$:



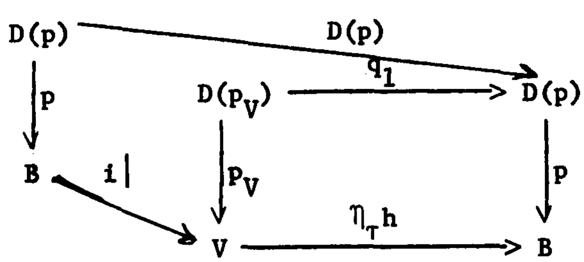
[4.4.1]

where $V = \varphi^{-1}[0,1)$. We will now show that there exists a relative fibration $(g_V, i|) \in \mathfrak{M}(p, p_V)$ where $i|: B \to V$.

$D(p_V) = \{(x,v) \in D(p) \times V \mid p(x) = \eta_T h(v)\}$ with the topology making the inclusion map $\alpha_V: D(p_V) \to D(p) \times V$ an injection. $p_V(x,v) = v$. $\lambda_{p_V}: \Omega_{p_V} \times R^+ \to D(p_V)$ is defined as:

$$\lambda_{p_V}\left[((x,v),(\omega,r)),s\right] = \left(\lambda_p\left[(x, P\eta_T h(\omega,r)),s\right], \omega(s)\right) .$$

The reader can verify that λ_{p_V} is a lifting function of p_V . We note that $\eta_T h i p(x) = p(x) = p\, 1_{D(p)}(x)$. Thus:



[4.4.2]

is a commuting diagram. Since $\mathcal{D}_V$ is a pullback diagram, we can conclude that there exists a unique map $g_V: D(p) \to D(p_V)$ such that $q_1 g_V = 1_{D(p)}$ and $p_V g_V = i|p: D(p) \to V$. (We note that this last condition implies that $(g_V, i|) \in \mathfrak{M}(p, p_V)$. We will now show that $(g_V, i|)$ is a relative fibration.

$$g_V \lambda_p[(x,(\omega,r)),s] = (\lambda_p[(x,(\omega,r)),s],\omega(s)) = \lambda_{p_V}[((x,p(x)),Pi|(\omega,r)),s]$$

$$= \lambda_{p_V}(g_V^* \times R^+)[(x,P\eta_T h\ Pi|(\omega,r)),s]$$

$$= \lambda_{p_V}(g_V^* \times R^+)[(x,(\omega,r)),s] .$$

Thus $g_V \lambda_p = \lambda_{p_V}(g_V^* \times R^+): \Omega_p \times R^+ \to D(p_V)$. Let $\hat{i}: V \to B'$ be the inclusion map and define a map $(\hat{g},\hat{i}) \in \mathfrak{M}(p_V, p')$ such that $(\hat{g},\hat{i})(g_V, i|) = (g,i): p \to p'$. Define $\hat{g}: D(p_V) \to D(p')$ as follows:

$$\hat{g}(x,v) = \lambda_{p'}[(g(x),(h^{-1}(v),\varphi(v))),\varphi(v)] \text{ where}$$

$$h^{-1}(v)(s) = h(v)(\varphi(v) - s) .$$

For $(x,v) \in D(p_V)$, $p(x) = \eta_T h(v) = \eta_0 h^{-1}(v)$. Thus $(g(x),(h^{-1}(v),\varphi(v))) \in \Omega_{p'}$ implying that $\hat{g}$ is well defined.

$$p'\hat{g}(x,v) = p'\lambda_{p'}[(g(x),(h^{-1}(v),\varphi(v))),\varphi(v)]$$

$$= h^{-1}(v)(\varphi(v)) = h(v)(0) = v = p_V(x,v) .$$

Thus $p'\hat{g} = \hat{i}\, p_V$, implying that $(\hat{g},\hat{i}) \in \mathfrak{M}(p_V, p')$. $\hat{g} g_V(x) = \hat{g}(x,p(x)) = \lambda_{p'}[(g(x),(h^{-1}(p(x)),0)),0] = g(x)$. Thus $\hat{g} g_V = g$, implying that $(\hat{g},\hat{i})(g_V, i|) = (g,i)$. We note that $\{V,B'\}$ constitutes a numerable open cover of B' with a partition of unity $\{1-\varphi/2,\ 1+\varphi/2\}$.

We can now conclude that $\tilde{p}: M(\hat{g}) \to B'$ is a fibration, where $\tilde{p}$ is the pushout space of $\mathfrak{D}$:

$$\begin{array}{ccc} p_V & \xrightarrow{(\hat{g},\hat{i})} & p' \\ \Big\downarrow{\scriptstyle (\epsilon_1, V)} & & \Big\downarrow{\scriptstyle (\hat{g}_1, B')} \\ p_V\pi_1 & \xrightarrow{(g_{11},\hat{i})} & \tilde{p} \end{array}$$

[4.4.3]

a pushout diagram in $\mathfrak{M}$. We note that $\tilde{p} = M(\hat{g},\hat{i})$. Thus there exists a unique map $(\hat{g}_1^{-1}, B') \in \mathfrak{M}(\tilde{p}, p')$ such that $(\hat{g}_1^{-1}, B')(\hat{g}_1, B') = 1_{p'}$, $(\hat{g}_1^{-1}, B')(g_{11}, \hat{i}) = (\hat{g},\hat{i})(\pi_1, V)$, and $\hat{g}_1^{-1}: D(\tilde{p}) \to D(p')$ is a fiber homotopy equivalence. Let $\tilde{g} = g_{11}\ \epsilon_0\ g_V: D(p) \to D(\tilde{p})$. $(g_{11},\hat{i})(\epsilon_0, V),(g_V, i|) = (\tilde{g}, i)$. Thus $(\tilde{g}, i) \in \mathfrak{M}(p, \tilde{p})$. $\hat{g}_1^{-1} g_{11}\epsilon_0 g_V = \hat{g}\ \pi_1\ \epsilon_0\ g_V = \hat{g}\ g_V = g$. Thus $\hat{g}_1^{-1}\ \tilde{g} = g: D(p) \to D(p')$.

We have left to prove that $(\tilde{g}, i)$ is a relative fibration. We will do this by exhibiting an appropriate lifting function of $\tilde{p}$.

For $\epsilon \in I$ and $(\omega, r) \in PB'$ such that $\omega(0) \in V$, let $\varphi_m \omega(rt) = \max\{\varphi\ \omega(s)\ ,\ 0 \le s \le rt\}$ and define $\alpha_{(\omega,\epsilon)}: I \to R^+$ as follows:

$$\alpha_{(\omega,\epsilon)}(t) = \epsilon(1 + t) + \left[\frac{2t \cdot \varphi_m\ \omega(rt)}{1 - \varphi_m\ \omega(rt)}\right] .$$

We note that $\alpha_{(\omega,\epsilon)}$ is well defined for all t such that $\omega(s) \in V$ for every $s \in [0, rt]$. Note also that $\alpha_{(\omega,\epsilon)}(0) = \epsilon$.

Define $\lambda_V: \Omega_{\hat{i}p_V\pi_1} \times R^+ \to D(\tilde{p})$ as follows:

$$\lambda_V[(((x,v),\epsilon),(\omega,r)),rt] = \begin{cases} g_{11}((\lambda_p[(x,P\eta_T h(\omega,r)),rt],\omega(rt)),\alpha_{(\omega,\epsilon)}(t)) , \\ \quad 0 \le t \le \min\{1,t_1\} \text{ where } t_1 \text{ is such} \\ \quad \text{that } \alpha(t_1) = 1 \\ \hat{g}_1\lambda_{p'}[(\hat{g}\{\lambda_p[(x,P\eta_T h(\omega,r)),rt_1],\omega(rt_1)\}, \\ \quad (\omega_{rt_1},r-rt_1)),rt-rt_1] , \; t_1 \le t \le 1 \text{ if} \\ \quad t_1 \le 1 . \end{cases}$$

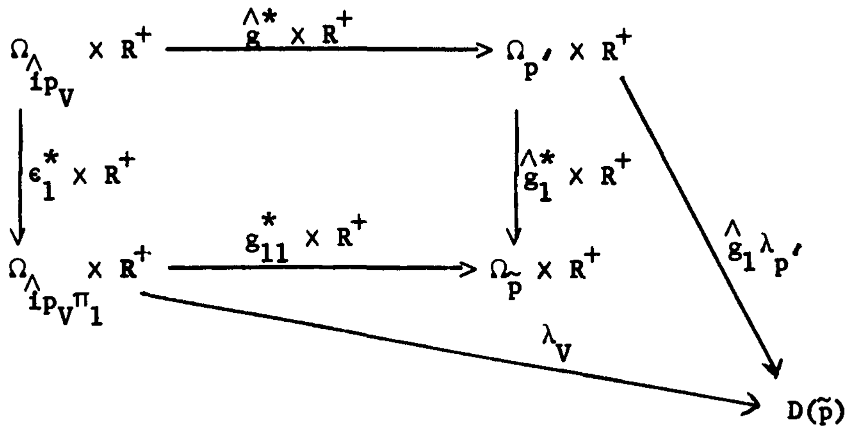
To prove that λ_V is well defined, we need only show that it is well defined at $t = t_1$ when $t_1 \le 1$:

$$\begin{aligned} & g_{11}((\lambda_p[(x,P\eta_T h(\omega,r)),rt_1],\omega(rt_1)),\alpha_{(\omega,\epsilon)}(t_1)) \\ &= g_{11}((\lambda_p[(x,P\eta_T h(\omega,r)),rt_1],\omega(rt_1)),1) \\ &= g_{11}\epsilon_1(\lambda_p[(x,P\eta_T h(\omega,r)),rt_1],\omega(rt_1)) \\ &= \hat{g}_1\hat{g}\{\lambda_p[(x,P\eta_T h(\omega,r)),rt_1],\omega(rt_1)\} \\ &= \hat{g}_1\lambda_{p'}[(\hat{g}\{\lambda_p[(x,P\eta_T h(\omega,r)),rt_1],\omega(rt_1)\},(\omega_{rt_1},r-rt_1)),0] \\ &= \hat{g}_1\lambda_{p'}[(\hat{g}\{\lambda_p[(x,P\eta_T h(\omega,r)),rt_1],\omega(rt_1)\},(\omega_{rt_1},r-rt_1),rt_1-rt_1] . \end{aligned}$$

Since $\epsilon = 1$ implies that $t_1 = 0$, we can conclude that

$$\begin{aligned} \lambda_V(\epsilon_1^* \times R^+)[((x,v),(\omega,r)),rt] &= \lambda_V[(((x,v),1),(\omega,r)),rt] \\ &= \hat{g}_1\lambda_{p'}[(\hat{g}\{(x,v)\},(\omega,r)),rt] . \end{aligned}$$

Thus:



[4.4.4]

is a commuting diagram, and since the inner square is a pushout diagram, we can conclude that there exists a unique map $\lambda_{\tilde{p}} : \Omega_{\tilde{p}} \times R^+ \rightarrow D(\tilde{p})$ such that $\lambda_{\tilde{p}}(g_{11}^* \times R^+) = \lambda_V$ and $\lambda_{\tilde{p}}(\hat{g}_1^* \times R^+) = \hat{g}_1 \lambda_{p'}$.

$$\lambda_{\tilde{p}}[(\hat{g}_1(x'),(\omega,r)),0] = \hat{g}_1 \lambda_{p'}[(x',(\omega,r)),0] = \hat{g}_1(x'), \quad \text{and}$$

$$\lambda_{\tilde{p}}[(g_{11}((x,v),\epsilon),(\omega,r)),0] = \lambda_V[((x,v),\epsilon),(\omega,r)),0] = g_{11}((x,v),\epsilon) \ .$$

Thus $\lambda_{\tilde{p}}[(\tilde{x},(\omega,r)),0] = \tilde{x}$ for all $\tilde{x} \in D(\tilde{p})$.

$$\tilde{p}\lambda_{\tilde{p}}[(\hat{g}_1(x'),(\omega,r)),s] = \tilde{p}\hat{g}_1 \lambda_{p'}[(x',(\omega,r)),s]$$

$$= p'\lambda_{p'}[(x',(\omega,r)),s] = \omega(s) \ , \quad \text{and}$$

$$\tilde{p}\lambda_{\tilde{p}}[(g_{11}((x,v),\epsilon),(\omega,r)),s] = \tilde{p}\lambda_V[(((x,v),\epsilon),(\omega,r)),s] = \omega(s)$$

by construction. Thus $\tilde{p}\lambda_{\tilde{p}}[(\tilde{x},(\omega,r)),s] = \omega(s)$ for every $\tilde{x} \in D(\tilde{p})$ and for all $s \in R^+$. Therefore $\lambda_{\tilde{p}}$ is a lifting function of $\tilde{p}$.

For $(x,(\omega,r)) \in \Omega_p$: $\lambda_{\tilde{p}}(\tilde{g}^* \times R^+)[(x,(\omega,r)),rt] = \lambda_{\tilde{p}}[(\tilde{g}(x),Pi(\omega,r)),rt]$

$$= \lambda_{\tilde{p}}[(g_{11}\epsilon_0 g_V(x),Pi(\omega,r)),rt]$$

$$= \lambda_{\tilde{p}}[(g_{11}((x,p(x)),0),Pi(\omega,r)),rt]$$

$$= \lambda_V[(((x,p(x)),0),Pi(\omega,r)),rt]$$

$$= g_{11}(\lambda_p[(x,P\eta_T hPi(\omega,r)),rt],Pi\omega(rt),\alpha_{(\omega,0)}(t))$$

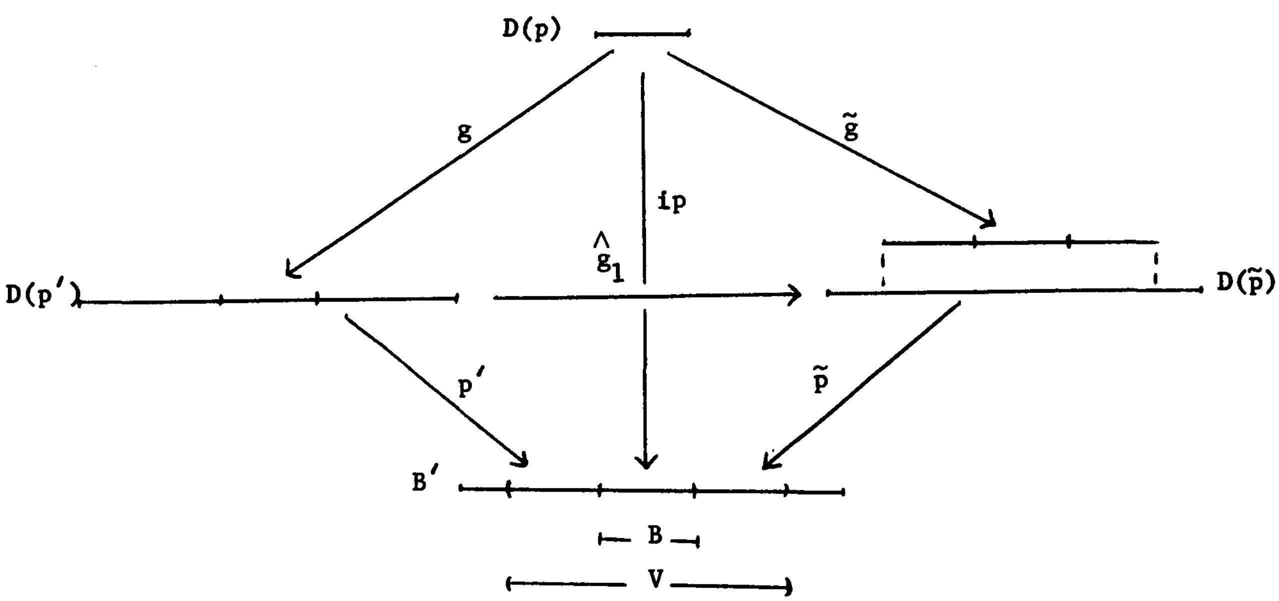
$$= g_{11}(\lambda_p[(x,(\omega,r)),rt],Pi\omega(rt),0(1+t) + \frac{2t\cdot 0}{1-0})$$

$$= g_{11}(\lambda_p[(x,(\omega,r)),rt],Pi\omega(rt),0)$$

$$= g_{11}\epsilon_0 g_V \lambda_p[(x,(\omega,r)),rt]$$

$$= \tilde{g}\lambda_p[(x,(\omega,r)),rt] .$$

Therefore $\lambda_{\tilde{p}}(\tilde{g} \times R^+) = \tilde{g}\lambda_p : \Omega_p \times R^+ \to D(\tilde{p})$ implying that $(\tilde{g},i)$ is a relative fibration.



[4.4.5]

<u>Corollary 4.5</u>: If we further assume that λ_p and $\lambda_{p'}$ are transitive lifting functions, then $\lambda_{\tilde{p}}$ is transitive when restricted to Ω_p, i.e., for $(x,(\omega,r)) \in \Omega_p$ and $(\bar{\omega},\bar{r}) \in PB'$ such that $Pi\omega(r) = \bar{\omega}(0)$, then $\lambda_{\tilde{p}}[(\lambda_{\tilde{p}}[(\tilde{g}(x),Pi(\omega,r)),r],(\bar{\omega},\bar{r})),\bar{r}] = \lambda_{\tilde{p}}[(\tilde{g}(x),(Pi\omega\cdot\bar{\omega},r+\bar{r})),r+\bar{r}]$.

<u>Proof</u>: If $1 \le t_1$:

$$\lambda_{\tilde{p}}[(\lambda_{\tilde{p}}[(\tilde{g}(x),Pi(\omega,r)),r],(\bar{\omega},\bar{r})),\bar{r}] = \lambda_{\tilde{p}}[(\tilde{g}\lambda_p[(x,(\omega,r)),r],(\bar{\omega},\bar{r})),\bar{r}]$$

$$= \lambda_{\tilde{p}}[(g_{11}((\lambda_p[(x,(\omega,r)),r],\omega(r)),0),(\bar{\omega},\bar{r})),\bar{r}]$$

$$= g_{11}(\lambda_p[(\lambda_p[(x,(\omega,r)),r],P\eta_T h(\bar{\omega},\bar{r})),\bar{r}],\bar{\omega}(\bar{r}),\alpha_{(\omega,0)}(1))$$

$$= g_{11}(\lambda_p[(x,(\omega\cdot P\eta_T h\bar{\omega},r+\bar{r})),r+\bar{r}],\bar{\omega}(\bar{r}),\alpha_{(\omega,0)}(1))$$

$$= g_{11}(\lambda_p[(x,P\eta_T h(Pi\omega\cdot\bar{\omega},r+\bar{r})),r+\bar{r}],Pi\omega\cdot\bar{\omega}(r+\bar{r}),\alpha_{(\omega,0)}(1))$$

$$= \lambda_{\tilde{p}}[(\tilde{g}(x),(Pi\omega\cdot\bar{\omega},r+\bar{r})),r+\bar{r}] .$$

If $t_1 \le 1$:

$$\lambda_{\tilde{p}}[(\lambda_{\tilde{p}}[(\tilde{g}(x),Pi(\omega,r)),r],(\bar{\omega},\bar{r})),\bar{r}] = \lambda_{\tilde{p}}[(\tilde{g}\lambda_p[(x,(\omega,r)),r],(\bar{\omega},\bar{r})),\bar{r}]$$

$$= \lambda_{\tilde{p}}[(g_{11}((\lambda_p[(x,(\omega,r)),r],\omega(r)),0),(\bar{\omega},\bar{r})),\bar{r}]$$

$$= \hat{g}_1\lambda_{p'}[(\hat{g}\{\lambda_p[(\lambda_p[(x,(\omega,r)),r],P\eta_T h(\bar{\omega},\bar{r})),\bar{r}t_1],\bar{\omega}(\bar{r}t_1)\},$$
$$(\bar{\omega}_{rt_1},\bar{r} - \bar{r}t_1)),\bar{r} - \bar{r}t_1]$$

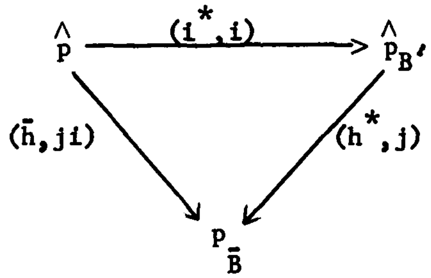
$$= \hat{g}_1\lambda_{p'}[(\hat{g}\{\lambda_p[(x,(\omega\cdot P\eta_T h\bar{\omega},r+\bar{r})),r+\bar{r}t_1],Pi\omega\cdot\bar{\omega}(r+\bar{r}t_1)\},$$
$$(Pi\omega\cdot\bar{\omega}_{rt_1},r+\bar{r}-r-\bar{r}t_1)),\bar{r}-\bar{r}t_1]$$

$$= \hat{g}_1\lambda_{p'}[(\hat{g}\{\lambda_p[(x,P\eta_T h(Pi\omega\cdot\bar{\omega},r+\bar{r})),r+\bar{r}t_1],Pi\omega\cdot\bar{\omega}(r+\bar{r}t_1)\},$$
$$(Pi\omega\cdot\bar{\omega}_{rt_1},r+\bar{r}-r-\bar{r}t_1)),\bar{r}-\bar{r}t_1]$$

$$= \lambda_{\tilde{p}}[(\tilde{g}(x),(Pi\omega\cdot\bar{\omega},r+\bar{r})),r+\bar{r}] .$$

We will now prove that $\hat{p}_{B'}$ is an extension of $\hat{p}$, up to a fiber homotopy.

<u>Theorem 4.6</u>: Let B be a locally contractible in the large pathwise connected space, B' a pathwise connected space, and $i: B \to B'$ a cofibration. $b_0 \in B$. Let p be a fibration over B, and define $\bar{p}$, $\hat{p}$, and $\hat{p}_{B'}$, as in the beginning of this section. (We note that since B is locally contractible in the large, there exists a map $(\tilde{h},B) \in \mathfrak{M}(\hat{p},p)$ such that $\tilde{h}: D(\hat{p}) \to D(p)$ is a fiber homotopy equivalence). Then for $p_{\bar{B}}$, a fibration over $\bar{B}$, and $(\bar{h},ji) \in \mathfrak{M}(p,p_{\bar{B}})$ where $j: B' \to \bar{B}$, there exists a map $(h^*,j) \in \mathfrak{M}(\hat{p}_{B'},p_{\bar{B}})$, unique up to a fiber homotopy, such that $\mathfrak{D}$:



[4.6.1]

is a commuting diagram, up to a fiber homotopy, where $i^*(<x_F,(\omega,r)>) = <x_F, Pi(\omega,r)>$.

<u>Proof</u>: This theorem reduces in the following way:

Let $\tilde{p}_{B'}$ be the fibration induced by the following pullback diagram $\bar{\mathfrak{D}}$:

$$\begin{array}{ccc} D(\tilde{p}_{B'}) & \xrightarrow{\quad q_1 \quad} & D(p_{\bar{B}}) \\ \Big\downarrow {\scriptstyle \tilde{p}_{B'}} & & \Big\downarrow {\scriptstyle p_{\bar{B}}} \\ B' & \xrightarrow{\quad j \quad} & \bar{B} \end{array}$$

[4.6.2]

We note that since $ji\hat{p} = p_{\bar{B}}\bar{h}: D(\hat{p}) \to \bar{B}$, there exists a unique map $\tilde{g}: D(\hat{p}) \to D(\tilde{p}_{B'})$ such that $q_1\tilde{g} = \bar{h}: D(\hat{p}) \to D(p_{\bar{B}})$ and $\tilde{p}_{B'}\tilde{g} = i\hat{p}: D(\hat{p}) \to B'$. This second condition is equivalent to saying that $(\tilde{g},i) \in \mathfrak{M}(\hat{p},\tilde{p}_{B'})$. Should we find a map $(\bar{h}^*,B') \in \mathfrak{M}/B'(\hat{p}_{B'},\tilde{p}_{B'})$ such that $(\bar{h}^*,B')(i^*,i) = (\tilde{g},i)$, then $(q_1,j)(\bar{h}^*,B')(i^*,i) = (q_1,j)(\tilde{g},i) = (\bar{h},ji)$. Thus $(q_1\bar{h}^*,j) = (h^*,j)$ would satisfy the result of this theorem. Therefore we need only consider extensions to fibrations over B' .

Without loss of generality, we can assume that $\tilde{p}_{B'}$ has a transitive lifting function. (If not, by Theorem 4.2, we can conclude that there exist a fibration $\bar{\tilde{p}}_{B'}$ over B' and a map $(f,B') \in \mathfrak{M}(\tilde{p}_{B'},\bar{\tilde{p}}_{B'})$ such that f is a fiber homotopy equivalence and $\bar{\tilde{p}}_{B'}$ has a transitive lifting function. We would then consider $(f\tilde{g},i) \in \mathfrak{M}(\hat{p},\bar{\tilde{p}}_{B'})$ as our new initial map).

We now begin the proof of this theorem modulo these considerations.

Since $i: B \to B'$ is a cofibration, Lemma 4.4 allows us to conclude that there exist a fibration $p_{B'}$ and maps $(g,i) \in \mathfrak{M}(\hat{p},p_{B'})$ and $(\hat{g}_1^{-1},B') \in \mathfrak{M}/B'(p_{B'},\tilde{p}_{B'})$ such that $\hat{g}_1^{-1}$ is a fiber homotopy equivalence, $\hat{g}_1^{-1}g = \tilde{g}$, and (g,i) is a relative fibration. Since (g,i) is a relative fibration and $<x_F,(\omega,r)> = \lambda_{\hat{p}}[(<x_F,(\omega_{b_0},0)>,(\omega,r)),r]$,

$g(<x_F,(\omega,r)>) = g\lambda_{\hat{p}}[(<x_F,(\omega_{b_0},0)>,(\omega,r)),r] = \lambda_{p_{B'}}[(g(<x_F,(\omega_{b_0},0)>),$

$Pi(\omega,r)),r]$. Define $\bar{h}^*: D(\hat{p}_{B'}) \to D(p_{B'})$ as follows:

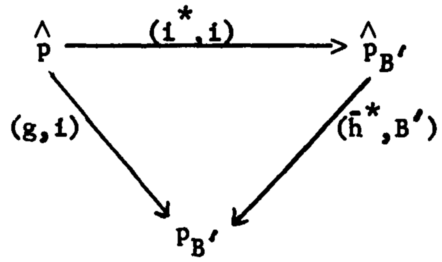
$$\bar{h}^*(<x_F,(\omega,r)>) = \lambda_{p_{B'}}[(g(<x_F,(\omega_{b_0},0)>),(\omega,r)),r] .$$

To show that $\bar{h}^*$ is well defined, let $(x_F,(\omega,r)) \sim (x_F',(\omega',r'))$. Then there exists a loop $(\omega_L,r_L) \in \Omega_{b_0}(B)$ such that $x_F'\cdot(\omega_L,r_L) = x_F$ and $(Pi\omega_L\cdot\omega,r_L + r) = (\omega',r')$.

$$\begin{aligned}
\bar{h}^*(<x_F,(\omega,r)>) &= \lambda_{p_{B'}}[(g(x_F,(\omega_{b_0},0)>),(\omega,r)),r] \\
&= \lambda_{p_{B'}}[(g(<x_F'\cdot(\omega_L,r_L),(\omega_{b_0},0)>),(\omega,r)),r] \\
&= \lambda_{p_{B'}}[(g(<x_F',(\omega_L,r_L)>),(\omega,r)),r] \\
&= \lambda_{p_{B'}}[(\lambda_{p_{B'}}[(g(<x_F',(\omega_{b_0},0)>),Pi(\omega_L,r_L)),r_L],(\omega,r)),r] \\
&= \lambda_{p_{B'}}[(g(<x_F',(\omega_{b_0},0)>),(Pi\omega_L\cdot\omega,r_L+r)),r_L+r] \\
&= \lambda_{p_{B'}}[(g(<x_F',(\omega_{b_0},0)>),(\omega',r')),r'] \\
&= \bar{h}^*(<x'_F,(\omega',r')>) .
\end{aligned}$$

Thus $\bar{h}^*$ is well defined.

For $<x_F,(\omega,r)> \in D(\hat{p})$, $\bar{h}^*i^*(<x_F,(\omega,r)>) = \bar{h}^*(<x_F,Pi(\omega,r)>) = \lambda_{p_{B'}}[(g(<x_F,(\omega_{b_0},0)>),Pi(\omega,r)),r] = g(<x_F,(\omega,r)>)$. Thus:



[4.6.3]

is a commuting diagram. Therefore $\tilde{h}^* = \hat{g}_1^{-1}\hat{h}^*: D(\hat{p}_{B'}) \to D(\tilde{p}_{B'})$ such that $(\tilde{h}^*,B')(i^*,i) = (\hat{g}_1^{-1},B')(g,i) = (\tilde{g},i)$, up to a fiber homotopy. Letting $(h^*,j) = (q_1,j)(\tilde{h}^*,B'): \hat{p}_{B'} \to p_{\bar{B}}$, we can conclude that $(h^*,j)(i^*,i) = (q_1,j)(\tilde{h}^*,B')(i^*,i) = (q_1,j)(\tilde{g},i) = (\bar{h},ji)$, up to a fiber homotopy.

Having now proven existence, we have left to show that (h^*,j) is unique up to a fiber homotopy.

Let $(h_1^*,j) \in \mathfrak{M}(\hat{p}_{B'},p_{\bar{B}})$ such that $(h_1^*,j)(i^*,i) = (\bar{h},ji)$, up to a fiber homotopy. Then $p_{\bar{B}}h_1^* = j\hat{p}_{B'} : D(\hat{p}_{B'}) \to \bar{B}$. Thus, Since $\mathfrak{D}$ is a pullback diagram, there exists a unique map $\tilde{h}_1^*: D(\hat{p}_{B'}) \to D(\tilde{p}_{B'})$ such that $q_1\tilde{h}_1^* = h_1^*$ and $\tilde{p}_{B'}h_1^* = \hat{p}_{B'}$. This second condition is equivalent to saying that $(\tilde{h}_1^*,B') \in \mathfrak{M}/B'(\hat{p}_{B'},\tilde{p}_{B'})$. Let $\bar{h}_1^* = \hat{g}_1\tilde{h}_1^* : D(\hat{p}_{B'}) \to D(p_{B'})$, and let $g_1 = \bar{h}_1^*i^* : D(\hat{p}) \to D(p_{B'})$. Since we assumed that $(h_1^*,j)(i^*,i) = (\bar{h},ji)$, up to a fiber homotopy, we can conclude that $(\bar{h}_1^*,B')(i^*,i) = (g_1,i) = (g,i)$, up to a fiber homotopy, i.e., there exists a map $G: D(\hat{p}) \times I \to D(p_{B'})$ such that

$G|D(\hat{p}) \times \{0\} = g$, $G|D(\hat{p}) \times \{1\} = g_1$, and $(G|D(\hat{p}) \times \{t\}, i) \in \mathfrak{m}(\hat{p}, p_{B'})$.

We will now define a vertical homotopy between $\bar{h}_1^*$ and $\bar{h}^*$ which will imply the existence of such a homotopy between h_1^* and h^* . Define $H: D(\hat{p}_{B'}) \times I \to D(p_{B'})$ as follows:

$$H(<x_F,(\omega,r)>,t) = \begin{cases} \lambda_{p_{B'}}[(G(<x_F,(\omega_{b_0},0)>,2t),(\omega,r)),r], \ 0 \le t \le \tfrac{1}{2} \\ \lambda_{p_{B'}}[(\bar{h}_1^*(<x_F,(\omega^{\beta(t)},\beta(t))>),(\omega_{\beta(t)},r-\beta(t))),r-\beta(t)], \\ \qquad\qquad \tfrac{1}{2} \le t \le 1 \\ \text{where } \ \beta(t) = r\cdot(2t-1), \omega^{\beta(t)}(s) = \omega(s), \ \text{and} \\ \omega_{\beta(t)}(s) = \omega(s+\beta(t)) . \end{cases}$$

To show that H is well defined, we need only show that it is well defined at $t = \frac{1}{2}$.

$$\lambda_{p_{B'}}[(G(<x_F,(\omega_{b_0},0)>,2\cdot(\tfrac{1}{2})),(\omega,r)),r] = \lambda_{p_{B'}}[(\bar{h}_1^*(<x_F,(\omega_{b_0},0)>),(\omega,r)),r]$$

$$= \lambda_{p_{B'}}[(\bar{h}_1^*(<x_F,(\omega^0,0)>),(\omega_0,r-0)),r-0] = \lambda_{p_{B'}}[(\bar{h}_1^*(<x_F,(\omega^{\beta(\frac{1}{2})},\beta(\tfrac{1}{2}))>),$$
$$(\omega_{\beta(\frac{1}{2})},r-\beta(\tfrac{1}{2}))),r-\beta(\tfrac{1}{2})] .$$

$$\begin{aligned} H(<x_F,(\omega,r)>,0) &= \lambda_{p_{B'}}[(G(<x_F,(\omega_{b_0},0)>,0),(\omega,r)),r] \\ &= \lambda_{p_{B'}}[(g(<x_F,(\omega_{b_0},0)>),(\omega,r)),r] \\ &= \bar{h}^*(<x_F,(\omega,r)>) . \end{aligned}$$

Thus $H|D(\hat{p}_{B'}) \times \{0\} = \bar{h}^*$.

$$
\begin{aligned}
H(\langle x_F,(\omega,r)\rangle,1) &= \lambda_{p_{B'}}[(\bar{h}_1^*(\langle x_F,(\omega^r,r)\rangle),(\omega_r,r-r)),r-r] \\
&= \lambda_{p_{B'}}[(\bar{h}_1^*(\langle x_F,(\omega,r)\rangle),(\omega_{b_0},0)),0] \\
&= \bar{h}_1^*(\langle x_F,(\omega,r)\rangle) \ .
\end{aligned}
$$

Thus $H|D(\hat{p}_{B'}) \times \{1\} = \bar{h}_1^*$.

$$
p_{B'}H(\langle x_F,(\omega,r)\rangle,t) = \begin{cases} \omega(r) & \text{if } 0 \le t \le \frac{1}{2} \\ \omega_{\beta(t)}(r-\beta(t)) = \omega(r) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}
$$

Thus H is vertical.

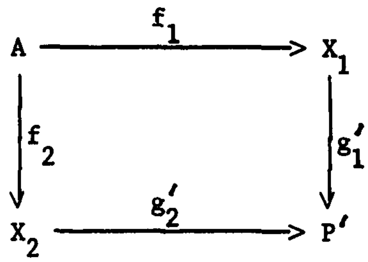
We note that since B is locally contractible in the large, there exists a fiber homotopy equivalence between p and $\hat{p}$. Thus $\hat{p}_{B'}$ can be viewed as an extension of p , modulo a fiber homotopy. We are now ready to construct an extension of a family of fibrations over locally contractible in the large base spaces included, by cofibrations, in a given space.

Definition 4.7: $\mathcal{D}$:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f_1\ } & X_1 \\
\Big\downarrow{\scriptstyle f_2} & & \Big\downarrow{\scriptstyle g_1} \\
X_2 & \xrightarrow{\ g_2\ } & P
\end{array}
$$

[4.7.1]

is a distinguished weak pushout diagram provided that it is a weak pushout diagram, and for any other weak pushout diagram $\mathcal{D}'$:



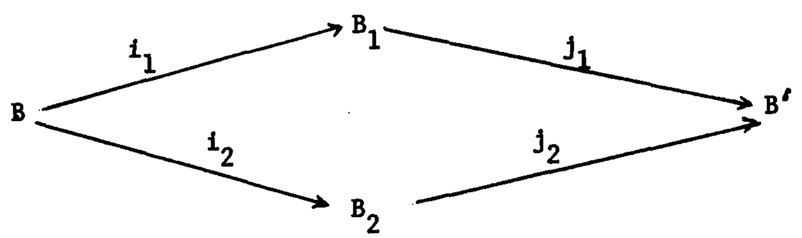
[4.7.2]

there exist pushout maps $r: P \to P'$ and $s: P' \to P$ such that $sr = 1_P$.

Definition 4.8: $\mathcal{J}$ is the category whose objects are fibrations over pathwise connected locally contractible in the large base spaces and whose morphisms are fiber homotopic classes of fiber maps between the total spaces and cofibrations between the bases.

We will now prove that distinguished weak pushouts exist in $\mathcal{J}$.

Let $\mathcal{D}$:



[4.9.1]

be a commuting diagram in $R\mathcal{J}$. $b_0 \in B$. Let p, p_1, and p_2, be objects of $\mathcal{J}$ over B, B_1, and B_2, respectively, and let $([f_k], i_k) \in \mathfrak{M}_{\mathcal{J}}(p, p_k)$, $(k = 1,2)$. We define $\hat{p}$, $\hat{p}_1$, and $\hat{p}_2$, as the associated fibrations of p, p_1, and p_2, respectively, and we let $(\alpha,B) \in \mathfrak{M}/B(p,\hat{p})$ and $(\alpha_k,B_k) \in \mathfrak{M}/B_k(p_k,\hat{p}_k)$ be the respective fiber homotopy equivalences, $(k = 1,2)$.

Let (a) $\hat{p}_{B_1}$, $\hat{p}_{B_2}$, and $\hat{p}_{B'}$, be the extensions of $\hat{p}$ to B_1, B_2, and B', respectively, and

(b) $\hat{p}_{kB'}$ be the extension of $\hat{p}_k$ to B', as defined in Theorem 4.6. We note the existence of the following morphisms in $\mathcal{J}$: $([i_k^*],i_k) \in \mathfrak{M}_{\mathcal{J}}(\hat{p},\hat{p}_{B_k}),([(j_k\ i_k)^*],j_k\ i_k) \in \mathfrak{M}_{\mathcal{J}}(\hat{p},\hat{p}_{B'})$, and $([j_k^*],j_k) \in \mathfrak{M}_{\mathcal{J}}(\hat{p}_k,\hat{p}_{kB'})$, $(k = 1,2)$.

Since $([j_k^*\alpha_k f_k],j_k\ i_k) \in \mathfrak{M}_{\mathcal{J}}(p,\hat{p}_{kB'})$, we can conclude, by Theorem 4.6, that there exists a unique morphism in $\mathcal{J}$: $([h_k^*],B') \in \mathfrak{M}_{\mathcal{J}}(\hat{p}_{B'},\hat{p}_{kB'})$ such that $([h_k^*],B')([(j_k\ i_k)^*],j_k\ i_k) = ([j_k^*\alpha_k f_k],j_k\ i_k)$, $(k = 1,2)$.

Let $\mathcal{D}_k$:

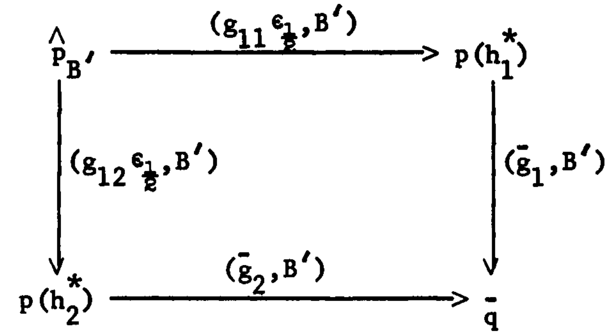
$$\begin{array}{ccc} \hat{p}_{B'} & \xrightarrow{(h_k^*,B')} & \hat{p}_{kB'} \\ \Big\downarrow{\scriptstyle(\epsilon_{k-1},B')} & & \Big\downarrow{\scriptstyle(\hat{g}_k,B')} \\ (\hat{p}_{B'},\pi_1)_k & \xrightarrow{(g_{1k},B')} & p(h_k^*) \end{array}$$

[4.9.2]

be a pushout diagram in $\mathfrak{M}$. Since (ϵ_{k-1},B') is a relative fibration and ϵ_{k-1} is a cofibration, Theorem 2.10 allows us to conclude that

$p(h_k^*)$ is a fibration, $(k = 1,2)$.

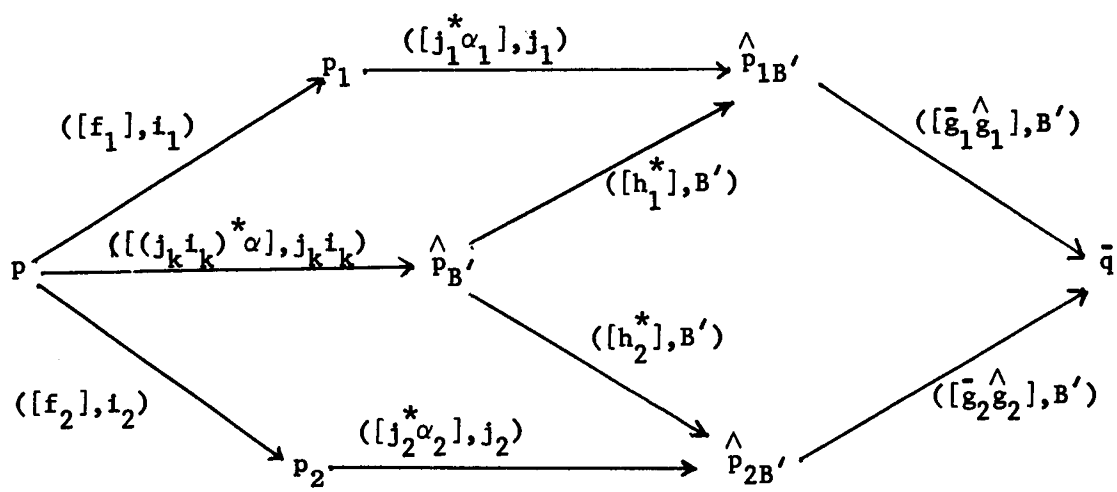
Let $\mathfrak{D}_{\bar{q}}$:



[4.9.3]

be a pushout diagram in $\mathfrak{M}$. Since $(g_{1k}\varepsilon_{\frac{1}{2}}, B')$ is a relative fibration and $g_{1k}\varepsilon_{\frac{1}{2}}$ is a cofibration, we can conclude that $\bar{q}$ is a fibration.

We will now prove that $\bar{\mathfrak{D}}_{\mathcal{J}}$:



[4.9.4]

is a commuting diagram in $\mathcal{J}$ by exhibiting a vertical homotopy $H: D(\hat{P}_{B'}) \times I \to D(\bar{q})$ such that $H: \bar{g}_1\hat{g}_1 h_1^* \simeq \bar{g}_2\hat{g}_2 h_2^* : D(\hat{P}_{B'}) \to D(\bar{q})$.

$$H(<x_F,(\omega,r)>,t) = \begin{cases} \bar{g}_1 g_{11}(<x_F,(\omega,r)>,t) & 0 \le t \le \frac{1}{2} \\ \bar{g}_2 g_{12}(<x_F,(\omega,r)>,t) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$\bar{g}_1 g_{11}(<x_F,(\omega,r)>,\tfrac{1}{2}) = \bar{g}_1 g_{11}\epsilon_{\frac{1}{2}}(<x_F,(\omega,r)>) = \bar{g}_2 g_{12}\epsilon_{\frac{1}{2}}(<x_F,(\omega,r)>)$$
$$= \bar{g}_2 g_{12}(<x_F,(\omega,r)>,\tfrac{1}{2}) .$$

Thus H is well defined.

$$H(<x_F,(\omega,r)>,0) = \bar{g}_1 g_{11}(<x_F,(\omega,r)>,0) = \bar{g}_1 g_{11}\epsilon_0(<x_F,(\omega,r)>)$$
$$= \bar{g}_1 \hat{g}_1 h_1^*(<x_F,(\omega,r)>) .$$

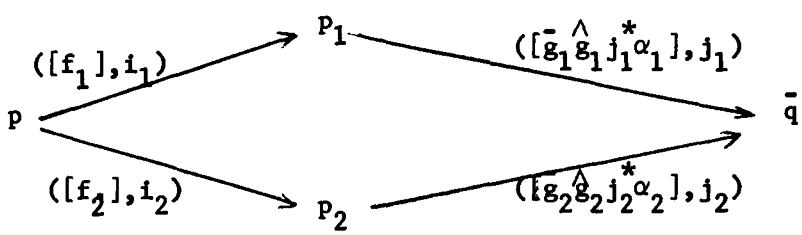
Thus $H|D(\hat{p}_{B'}) \times \{0\} = \bar{g}_1 \hat{g}_1 h_1^*$.

$$H(<x_F,(\omega,r)>,1) = \bar{g}_2 g_{12}(<x_F,(\omega,r)>,1) = \bar{g}_2 g_{12}\epsilon_1(<x_F,(\omega,r)>)$$
$$= \bar{g}_2 \hat{g}_2 h_2^*(<x_F,(\omega,r)>) .$$

Thus $H|D(\hat{p}_{B'}) \times \{1\} = \bar{g}_2 \hat{g}_2 h_2^*$.

Thus $([\bar{g}_1\hat{g}_1],B')([h_1^*],B') = ([\bar{g}_2\hat{g}_2],B')([h_2^*],B'): \hat{p}_{B'} \to \bar{q}$,
implying that $\bar{\mathcal{D}}_{\mathcal{J}}$ commutes.

<u>Theorem 4.9</u>: If $\mathcal{D}$ is a pushout diagram in Q , then $\mathcal{D}_{\mathcal{J}}$:



[4.9.5]

is a distinguished weak pushout diagram in $\mathcal{J}$.

Proof: We note that $\mathfrak{D}_{\mathcal{J}}$ is a commuting diagram, since it is an augmentation of $\bar{\mathfrak{D}}_{\mathcal{J}}$. We will now prove that $\mathfrak{D}_{\mathcal{J}}$ is a weak pushout diagram.

Let $([\tilde{f}_k],\ell_k) \in \mathfrak{M}_{\mathcal{J}}(p_k,p')$ such that $([\tilde{f}_1],\ell)([f_1],i_1) = ([\tilde{f}_2],\ell_2)([f_2],i_2)\colon p \to p'$. By Theorem 4.6, there exist unique maps in $\mathcal{J}$: $([\tilde{f}_k^*],(\ell_1,\ell_2)) \in \mathfrak{M}_{\mathcal{J}}(\hat{p}_{kB'},p')$ and $([\tilde{f}^*],(\ell_1,\ell_2)) \in \mathfrak{M}_{\mathcal{J}}(\hat{p}_{B'},p')$ such that $([\tilde{f}_k^*],(\ell_1,\ell_2))([j_k^*\alpha_k],j_k) = ([\tilde{f}_k],\ell_k)$ and $([\tilde{f}^*],(\ell_1,\ell_2))([(j_ki_k)^*\alpha],j_ki_k) = ([\tilde{f}_kf_k],\ell_ki_k)$, $(k = 1,2)$.

Since $\bar{\mathfrak{D}}_{\mathcal{J}}$ commutes, $([\tilde{f}^*],(\ell_1,\ell_2))([(j_ki_k)^*\alpha],j_ki_k) = ([\tilde{f}_k^*],(\ell_1,\ell_2))([j_k^*\alpha_k],j_k)([f_k],i_k) = ([\tilde{f}_k^*],(\ell_1,\ell_2))([h_k^*],B')([(j_ki_k)^*\alpha],j_ki_k)$. We note that by Theorem 4.6, the extension map is unique up to a fiber homotopy. Thus $([\tilde{f}_k^*],(\ell_1,\ell_2))([h_k^*],B') = ([\tilde{f}^*],(\ell_1,\ell_2))$, $(k = 1,2)$. Therefore there exists a vertical homotopy $F\colon \tilde{f}_1^*h_1^* \simeq \tilde{f}_2^*h_2^*\colon D(\hat{p}_{B'}) \to D(p')$.

We will now define a map $\tilde{f}_{\bar{q}} : D(\bar{q}) \to D(p')$ such that $([\tilde{f}_{\bar{q}}],(\ell_1,\ell_2))([\bar{g}_k\hat{g}_kj_k^*\alpha_k],j_k) = ([\tilde{f}_k],\ell_k)$, $(k = 1,2)$.

$$\tilde{f}_{\bar{q}}(\bar{g}_k\hat{g}_k(<x_{F_k},(\omega,r)>)) = \tilde{f}_k^*(<x_{F_k},(\omega,r)>) \ , \quad (k = 1,2) \ .$$

$$\tilde{f}_{\bar{q}}(\bar{g}_kg_{1k}(<x_F,(\omega,r)>,t)) = F(<x_F,(\omega,r)>,t) \ .$$

$$\tilde{f}_{\bar{q}}(\bar{g}_kg_{1k}(<x_F,(\omega,r)>,\epsilon_{k-1})) = F(<x_F,(\omega,r)>,\epsilon_{k-1}) = \tilde{f}_k^*h_k^*(<x_F,(\omega,r)>)$$

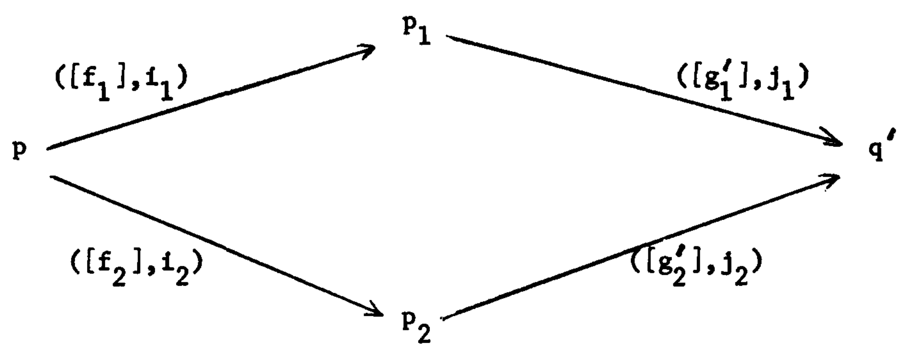
$$= \tilde{f}_{\bar{q}}(\bar{g}_k\hat{g}_kh_k^*(<x_F,(\omega,r)>)) \ .$$

Thus $\mathfrak{f}_{\bar{q}}$ is well defined, and by construction $(\mathfrak{f}_{\bar{q}},(\ell_1,\ell_2)) \in \mathfrak{m}(\bar{q},p')$.

$([\mathfrak{f}_{\bar{q}}],(\ell_1,\ell_2))([\bar{g}_k\hat{g}_k j_k^* \alpha_k],j_k) = ([\mathfrak{f}_k^*],(\ell_1,\ell_2))([j_k^*\alpha_k],j_k) = ([\mathfrak{f}_k],\ell_k)$,

$(k = 1,2)$. Thus $\mathcal{D}_{\mathcal{J}}$ is a weak pushout diagram in $\mathcal{J}$.

We will now prove that $\mathcal{D}_{\mathcal{J}}$ is distinguished. Let $\mathcal{D}_{q'}$:



[4.9.6]

be another weak pushout diagram in $\mathcal{J}$. Then there exist morphisms in $\mathcal{J}$, $([\eta],B') \in \mathfrak{m}/B'(q',\bar{q})$ and $([\mu],B') \in \mathfrak{m}/B'(\bar{q},q')$ such that $([\eta],B')([g_k'],j_k) = ([\bar{g}_k\hat{g}_k j_k^*\alpha_k],j_k)$, and $([\mu],B')([\bar{g}_k\hat{g}_k j_k^*\alpha_k],j_k) =$ $([g'_k],j_k)$, $(k = 1,2)$. Thus

$([\eta],B')([\mu],B')([\bar{g}_k\hat{g}_k j_k^*\alpha_k],j_k) = ([\bar{g}_k\hat{g}_k j_k^*\alpha_k],j_k)$, and $([\mu],B')([\eta],B')([g_k'],j_k) = ([g_k'],j_k)$, $(k = 1,2)$.

Since all maps on the base are identities, we can conclude that $([\eta],B')([\mu],B') = ([D(\bar{q})],B')$ proving our theorem. Furthermore, we can conclude that $([\mu],B')([\eta],B') = ([D(q')],B')$, implying that $\bar{q}$ and q' are fiber homotopically equivalent.

We now list conditions under which distinguished weak colimits exist in $\mathcal{J}$.

<u>Theorem 4.10</u>: Let B be a pathwise connected locally contractible in the large space, and let $\mathcal{C}$ be a subcategory of $\mathcal{J}$ indexed by Λ such that for each $\lambda \in \Lambda$, there exists a cofibration $i_\lambda^B: R(p_\lambda) \to B$. Then there exist an object q over B in $\mathcal{J}$ and a family of morphisms $\{([\hat{g}_\lambda], i_\lambda^B) \in \mathfrak{m}_{\mathcal{J}}(p_\lambda, q)\}_{\lambda \in \Lambda}$ which is a distinguished weak colimit of $\mathcal{C}$.

<u>Proof</u>; If $([f_\lambda^{\lambda'}], i_\lambda^{\lambda'}) \in \mathfrak{m}_{\mathcal{C}}(p_\lambda, p_{\lambda'})$, then $([i_{\lambda'}^{B^*} \alpha_{\lambda'} f_\lambda^{\lambda'}], i_\lambda^B) \in \mathfrak{m}_{\mathcal{J}}(p_\lambda, \hat{p}_{\lambda' B})$. Thus, by Theorem 4.6, there exists a unique map in $\mathcal{J}$, $([\bar{f}_\lambda^{\lambda'}], B) \in \mathfrak{m}(\hat{p}_{\lambda B}, \hat{p}_{\lambda' B})$ such that $([i_{\lambda'}^{B^*} \alpha_{\lambda'} f_\lambda^{\lambda'}], i_\lambda^B) = ([\bar{f}_\lambda^{\lambda'}], B)([i_\lambda^{B^*} \alpha_\lambda], i_\lambda^B)$.

For a family of fibrations over B, $\{p_\sigma\}_{\sigma \in \Sigma}$, define $\coprod_{\sigma \in \Sigma}^{T} p_\sigma$: $\coprod_{\sigma \in \Sigma} D(p_\sigma) \to B$ as follows:

$$\coprod_{\sigma \in \Sigma}^{T} p_\sigma(x_{\sigma_0}) = p_{\sigma_0}(x_{\sigma_0}) .$$

We now define $\lambda_\Sigma : \Omega_{\coprod_{\sigma \in \Sigma}^{T} p_\sigma} \times R^+ \to \coprod_{\sigma \in \Sigma} D(p_\sigma)$ as follows:

$$\lambda_\Sigma\left[(x_{\sigma_0}, (\omega, r)), s\right] = j_{\sigma_0} \lambda_{p_{\sigma_0}}\left[(x_{\sigma_0}, (\omega, r)), s\right]$$

where $j_{\sigma_0} : D(p_{\sigma_0}) \to \coprod_{\sigma \in \Sigma} D(p_\sigma)$ is the inclusion map. We note that λ_Σ is a lifting function, making $\coprod_{\sigma \in \Sigma}^{T} p_\sigma$ a fibration.

For notational convenience, let

$$\coprod^{T}_{\hat{p}_{\lambda B}\xrightarrow[([f_\lambda^{\lambda'}],\,i_\lambda^{\lambda'})\,\in\,\mathfrak{m}(\mathcal{C})]{([\bar{f}_\lambda^{\lambda'}],B)}\hat{p}_{\lambda' B}} \hat{p}_{\lambda B} = \coprod^{T}_{\lambda\to\lambda'} \hat{p}_{\lambda B} .$$

Define $([g_{1k}],B)\colon \coprod^{T}_{\lambda\in\Lambda}\left(\coprod^{T}_{\lambda\to\lambda'} \hat{p}_{\lambda B}\right) \to \coprod^{T}_{\lambda\in\Lambda} \hat{p}_{\lambda B}$ as follows:

$$([g_{11}],B) = \left(\coprod_{\lambda\in\Lambda}\left(\coprod_{\lambda\to\lambda'}[j_\lambda]\right),B\right), \text{ and } ([g_{12}],B) = \left(\coprod_{\lambda\in\Lambda}\left(\coprod_{\lambda\to\lambda'}[j_{\lambda'}\bar{f}_\lambda^{\lambda'}]\right),B\right) .$$

Let $\mathcal{D}_q$:

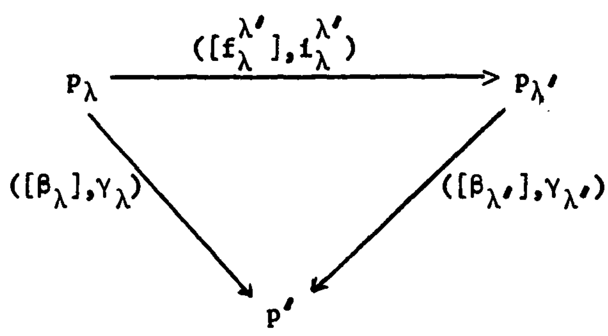
$$\begin{array}{ccc}
\coprod^{T}_{\lambda\in\Lambda}\left(\coprod^{T}_{\lambda\to\lambda'} \hat{p}_{\lambda B}\right) & \xrightarrow{([g_{11}],B)} & \coprod^{T}_{\lambda\in\Lambda} \hat{p}_{\lambda B} \\
\Big\downarrow {\scriptstyle ([g_{12}],B)} & & \Big\downarrow {\scriptstyle ([\hat{g}_1],B)} \\
\coprod^{T}_{\lambda\in\Lambda} \hat{p}_{\lambda B} & \xrightarrow{([\hat{g}_2],B)} & q
\end{array}$$

[4.10.1]

be a distinguished weak pushout diagram in $\mathcal{J}$.

We will now show that q and $\{([\hat{g}_1 j_\lambda i_\lambda^{B^*}\alpha_\lambda],i_\lambda^B)\}_{\lambda\in\Lambda}$ is a distinguished weak colimit of $\mathcal{C}$.

Let $p' \in \mathrm{Ob}\mathcal{J}$ and let $\{([\beta_\lambda],\gamma_\lambda)\}_{\lambda\in\Lambda}$ be a family of maps from $\{p_\lambda\}_{\lambda\in\Lambda}$ to p' such that for every $([f_\lambda^{\lambda'}],i_\lambda^{\lambda'}) \in \mathfrak{m}_{\mathcal{J}}(p_\lambda,p_{\lambda'})$,



[4.10.2]

is a commuting diagram.

We note that, by Theorem 4.6, there exist maps $([\beta_{\lambda B}], \gamma_B) \in \mathfrak{m}_{\mathcal{J}}(\hat{P}_{\lambda B}, p')$, such that $([\beta_{\lambda B}], \gamma_B)([i_\lambda^{B^*}\alpha_\lambda], i_\lambda^B) = ([\beta_\lambda], \gamma_\lambda) \in \mathfrak{m}(P_\lambda, p')$ for each $\lambda \in \Lambda$. Thus

$$([\beta_{\lambda' B}], \gamma_B)([\bar{f}_\lambda^{\lambda'}], B)([i_\lambda^{B^*}\alpha_\lambda], i_\lambda^B) = ([\beta_\lambda], \gamma_\lambda)$$
$$= ([\beta_{\lambda B}], \gamma_B)([i_\lambda^{B^*}\alpha_\lambda], i_\lambda^B) .$$

Since, by Theorem 4.6, the extension map is unique in $\mathcal{J}$, we can conclude that $([\beta_{\lambda' B}], \gamma_B)([\bar{f}_\lambda^{\lambda'}], B) = ([\beta_{\lambda B}], \gamma_B): \hat{P}_{\lambda B} \to p'$. Thus

$$\left(\coprod_{\lambda\in\Lambda} [\beta_{\lambda B}], \gamma_B\right)\left([g_{11}], B\right) = \left(\coprod_{\lambda\in\Lambda} [\beta_{\lambda B}], \gamma_B\right)\left([g_{12}], B\right): \coprod_{\lambda\in\Lambda}^{T} \left(\coprod_{\lambda\to\lambda'}^{T} \hat{P}_{\lambda B}\right) \to p' .$$

We can therefore conclude that, since $\mathcal{D}_q$ is a distinguished weak pushout diagram, there exists a morphism $([\beta_q], \gamma_B): q \to p'$ in $\mathcal{J}$ such that

$$([\beta_q], \gamma_B)([\hat{g}_k], B) = \left(\coprod_{\lambda\in\Lambda} [\beta_{\lambda B}], \gamma_B\right): \coprod_{\lambda\in\Lambda}^{T} \hat{P}_{\lambda B} \to p' , \ (k = 1,2) .$$

Thus

$$([\beta_q],\gamma_B)([\hat{g}_1 j_\lambda i_\lambda^{B^*}\alpha_\lambda],i_\lambda^B) = \Big(\coprod_{\lambda\in\Lambda}[\beta_{\lambda B}],\gamma_B\Big)\Big([j_\lambda i_\lambda^{B^*}\alpha_\lambda],i_\lambda^B\Big)$$

$$= ([\beta_\lambda],\gamma_\lambda) \ ,$$

which implies that q and $\{([\hat{g}_1 j_\lambda i_\lambda^{B^*}\alpha_\lambda],i_\lambda^B)\}_{\lambda\in\Lambda}$ is a weak colimit of $\mathcal{C}$. Since any other weak colimit would satisfy the condition of being a weak pushout of

$$\Big\{\coprod_{\lambda\in\Lambda}^T \Big(\coprod_{\lambda\to\lambda'}^T \hat{P}_{\lambda B}\Big) \xrightarrow{([g_{1k}],B)} \coprod_{\lambda\in\Lambda}^T \hat{P}_{\lambda B} \ , \ (k = 1,2)\Big\} \ ,$$

and since $\mathcal{D}_q$ is a distinguished weak pushout diagram, the colimit is distinguished.

It should be noted that B need not be locally contractible in the large in order for the distinguished weak colimit of $\mathcal{C}$ to exist, but this colimit would no longer be in $\mathcal{J}$.

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AUTOBIOGRAPHICAL STATEMENT

Louis J. Braun was born May 12, 1942 in Detroit, Michigan. He completed his primary education in Detroit graduating Mumford High School in 1959. Mr. Braun attended Wayne State University for two years and in September 1961, entered New York University as a Junior. He graduated in 1963 magna cum laude and did two years of graduate work at the Courant Institute. In 1965 he began his doctoral studies at The City University of New York. He is a member of Phi Beta Kappa, a former member of the executive committees of the Americans for Democratic Action and The United States Youth Council, and the Director of Development of Animal Sanctuary-Animal Care.

In 1969, he married the former Joanne Liquore. They are expecting a child in October 1971.