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Diamonds, triangles, and squares: A developmental study of the congruence concept

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DIAMONDS, TRIANGLES, AND SQUARES: A DEVELOPMENTAL STUDY OF THE CONGRUENCE CONCEPT

by
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Abstract

DIAMONDS, TRIANGLES AND SQUARES:

A DEVELOPMENTAL STUDY OF THE CONGRUENCE CONCEPT

by

Barbara Busse Paul

Advisor: Professor Harry Beilin

The development of the concept of geometric congruence in 3- 4- and 5-year-olds is examined. Rudimentary geometrical knowledge was assessed via pretests addressing the concepts of length, angle and the classification of size and shape. Knowledge of congruence was assessed by responses on area-matching and side-matching tasks. Childrens' inferences about geometric congruence were classified as rules. Age trends among mutually exclusive clusters were significant suggesting a developmental progression of rule use. A common factor analysis revealed four underlying dimensions in the data reflecting different modes of childrens' reasoning. The results are interpreted within the theoretical framework of Piaget and Garcia's (1991) new theory concerning the logic of meanings.

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Basic to an understanding of geometry is the concept of congruence. Two or more objects are congruent if they can be made to coincide, or as Freudenthal (1983) formulates it, they can be carried into each other by congruence mappings. As a psychological phenomenon, no one fully understands what congruence entails because the act of making things coincident, or performing congruence mappings, consists of a number of components that have not been adequately detailed.

Intuitively, congruence mapping can be decomposed into two basic components - correspondence and covering. Correspondence implies knowledge about the agreement between two things. We do not know whether this knowledge is obtained directly via perceptual processing, evolves from some sort of cognitive processing, or some interactive processing system. Geometrically, correspondence means to establish the equivalence of points on a line, the degrees in an angle, size, shape, etc.. Covering implies a motion which places one thing on top of another either mentally or physically. Together these components - correspondence and covering - fulfil the requirements of a congruence mapping by assigning to every element of one set (e.g., geometrical figure) an element of the same or another set.[1]

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[1] The covering component applies to only one or two dimensional space. It was in the context of two dimensional space in which the materials for the experimental tasks were presented and therefore intuitively, covering, can be considered a component of the concept under investigation.

The congruence concept represents a paradigm case for understanding the way in which geometrical notions develop. An examination of the concept should tell us something of the way in which congruence components relate to the attainment of the congruence concept in development.

Theoretical Framework

Two types of models compete for interpretation of the development of the congruence concept. The first is a perceptual model; the second a cognitive rule-based model.

Gibson's ecological model. J. J. Gibson's (1977) ecological model of perception maintains that perceptions are veridical; that organisms directly perceive information afforded them by objects, events and situations in the environment. Information is determined by the structure of stimulation in the world; it is this information so structured, rather than any intervening cognitive processes, that specifies what we perceive.

What is important about this perceptual model in the context of the congruence concept, is that congruences are relationships that are specified in the information provided by the world. In the best of conditions (i.e., with appropriate lighting and a functioning visual system), apprehension of physical occurrences of coincidence are direct, requiring no analysis of the coinciding objects per se. In fact, Gibson argues against the view that perceptions are mediated by any non-perceptual junctions of psychological processing (Mace, 1977). According to this model, there would be no psychological development of the congruence concept because the information contained in the coincidence or non-coincidence of one object with another is directly perceived; either the objects match or they do not.

Cognitive rule-based model. The cognitive rule-based model, under development and consideration in the present study, is derived from two sources. The first is theoretical and stems from Jean Piaget's constructivist-interactionist formulations; the second is Siegler's (1981) rule-assessment approach. What is derived from Siegler for present purposes is primarily methodological.

Knowledge about the world, according to Piaget, derives from actions taken upon objects in the world. Concepts are constructed via an interaction between information in the world obtained through the organism's actions and the organism's existing intellectual makeup. This is commonly known as Piaget's (1974; 1985) equilibration model in which the dual processes of assimilating information from our actions, and accommodating our mental schemes in order to absorb this information, accounts for the development, and thus the varying levels, of conceptual understanding. Siegler (1981) refers to the developmental process as the acquisition of increasingly powerful rules for solving problems, which apply to comparatively specific situations.

Piaget and Inhelder's (1960; 1967) writings on the child's construction of space and geometry places the pre-school child at the level of what they call a topological analysis of objects in space. For Piaget, topological analysis means that the child exhibits an understanding of neighborhood, proximity, connectivity, enclosure, openness, etc.. His more recent writings on this topic (Piaget & Garcia, 1989) have characterized the initial analysis to be intrafigural, but this new formulation shares essentially the same properties of understanding spelled out in the old theory. At this level of understanding squares, circles and triangles are all equivalent because they share the property of enclosure and connectivity. They are, however, distinguished from arc shapes, since the latter violate the properties of connectivity.

As Piaget's older views about the initial level of geometric thinking have been challenged for various reasons (e.g., Martin, 1976a, 1976b; Kapadia, 1974), the present study sought to describe an alternative model of initial types of geometric reasoning. Hypotheses were based on prior research in the area of congruence, through decomposing the congruence concept into its components and from observations during data collection. General developmental sequences were predicted according to the cognitive rule-based model. Specific hypotheses about the progressions of such sequences were not made because the sequences and the rules were the phenomena to be investigated.

Prior Research

Findings from prior research (Beilin & Klein, 1983) reported that children between the ages of four and eleven years exhibited two distinctive spontaneous strategies for solving congruence problems with triangles. In general, young children, four to six years of age, utilized a single side-matching strategy in which they brought together two geometrical figures and aligned them along one side. Older children, seven to eleven years of age, spontaneously placed one figure on top of the other (i.e., superimposed) to make congruence comparisons.

These results led Beilin & Klein to posit different reasoning processes utilized by younger and older children. It appeared as though young children were determining congruence by a rule based on the equality of one side. This was seen as corresponding with Piaget's (1979) notions of one-way mapping functions and one-one correspondence operations. On the other hand, when older children superimposed, they appeared to be using an area rule. The area rule was thought to be based on a whole-figure analysis reflecting the correspondence of shape and size. Accurate superimposing corresponds to a more general mathematical operation - that of coincidence, or congruence mapping. The development of the congruence concept was interpreted to be a result of a qualitative shift from analyzing geometric figures in terms of their component parts to an analysis of the whole figure.

The above interpretation of the side-matching strategy has not gone without criticism (Geddes et al, 1982; O'Donahue, 1983). The main criticism comes from a differentiation theory point of view. The argument is that, because of the nature of the materials used in single side-matching with two congruent triangles, the constructed shape yields a good figure, or a gestalt-like whole. This criticism cannot be taken lightly, because it is not clear from the Beilin & Klein study whether side-matching congruence judgments are based on the matching of one side or on the symmetricality of the whole figure.

In addition to the symmetricality issue, there were several other reasons why young children might have opted for single side-matching as a viable strategy for determining congruence. First, the experimental procedure in the Beilin & Klein task began with a pretest establishing an operational definition of congruence through the action of making Play-doh cookies with a cookie cutter. The experimenter followed by asking subjects why the cookies were all the same. The appropriate response was because the cookies were all cut from the same cookie cutter. The type of cookie cutter used in this task was a triangle frame, rather than a molded type of cookie cutter. Perceptually, the sides were accentuated because the triangle, which was formed by the sides, consisted of empty space. This may have led young children to focus on the sides of the figures during the tasks that followed.

Second, superimposing in the Beilin & Klein tasks required two levels of transformational ability. The first level required a motion--either physical or mental--which shifts the plane of comparison from within plane (i.e., one-dimensional, or side-matching comparison) to parallel plane (i.e., two-dimensional, or superimposing or area-matching comparison). Because the pretest may have biased young children to focus on the sides of the triangles, and because with the single side-matching strategy sameness could too easily be determined, the shift to a parallel plane comparison may never have been seen as necessary.

The Beilin & Klein tasks, in some cases, also required facility with Euclidean transformations (i.e., slides, flips, rotations). The experimental procedure required the selection of a triangle (held in a tray with other triangles) that was congruent with the model triangle. The selection triangles were displayed in random orientations, some of which were flipped and/or rotated. It has been demonstrated that young children have difficulty performing Euclidean transformations (e.g., Piaget, 1960; Perham 1978; Kosslyn, 1980). Some young children who initially tried to superimpose the figures, but were unable to perform the needed Euclidean transformations to establish coincidence, reverted to single side-matching. Because of the experimental procedure, these tasks may not have adequately tapped young children's procedural or strategic abilities. Instead, the tasks may have only highlighted the child's difficulty with specific types of transformational reasoning.

Another problem with the experimental procedure, used in the Beilin & Klein study became evident in a later study (Whitehurst & Beilin, 1984). The later study was a replication of one of the tasks in the Beilin & Klein study with two critical modifications in materials. First, in addition to triangles, knowledge of congruence was assessed with quadrilaterals. Second, the dimensions by which the selection shapes differed from the target shape were much smaller than those in the Beilin & Klein task. Children who were forced to choose only one selection shape had no problem with the Beilin & Klein task. Because differences in the selection shapes were quite apparent, accuracy of choice was very high, independent of the type of strategy used, or the age of the subject.

In the Whitehurst & Beilin study accuracy of choice was much lower. In the later study, in addition to the congruent shape, one of the other selection shapes which contained one side equal in length to one side of the target shape was also selected as congruent. With the right triangle and the square, when the shapes were side-matched along the equal side, some five-year-olds - the youngest group in the sample - maintained that two of the selection tray shapes were the same as the target shape. Because children were forced to choose only one, these children were simply coded as inaccurate. Upon reflection, these subjects could have been seen as correctly applying an inaccurate rule.

Other investigations (e.g., Silverman, York & Zuidema, 1984; Russell, 1975; Verge & Bogartz, 1978) report on young children's use of side-matching strategies in area matching tasks involving rectangles. In general, these studies replicate Piaget's (1960) findings with area conservation tasks, in which young children use one-dimensional strategies to solve problems. These findings have been contradicted by others (e.g., Anderson & Cuneo, 1978; Cuneo, 1980; Wilkening, 1980) in which it has been found that young children's judgments of rectangle area were said to obey an additive height + width model, suggesting a two-dimensional appreciation of area. Mathematically, the additive height + width model does not apply to any shape.

<u>Research Questions</u>

Several questions result from the Beilin & Klein and the Whitehurst & Beilin research. Would young children stay with a one-side equal rule when asked to superimpose geometric shapes? Would superimposing result in the use of other rules? Might these other rules suggest a concern for the total figure? Would the one-side equal rule hold when two figures are side-matched on sides of equal length when the resulting figure is highly asymmetrical? Do the rules young children use relate to the hypothesized components of the congruence concept? If so, is there a preference of one rule over another? Are there varying levels or types of consistent rule users within the pre-operational age range? Do different geometrical configurations (different shapes) affect rule use? Do boys and girls reason differently about geometric congruence? To answer these questions the research assessed young children's (3- 4- and 5-year-old) rule use on two types of congruence tasks: area-matching and side-matching tasks.

Hypothesized Rules

The area-matching task was designed to elicit spontaneous superimposing through a story scenario in which geometric figures in the form of 'characters' come from a land in which it is customary to sleep with 'blankets' that 'fit just right.' Concrete instances of fitting just right were demonstrated and tested in a pretest. Blankets were chosen because their function--'putting on top of'--is familiar to pre-school children. Two types of shapes were used--a right triangle and a square--and were of the same dimensions as the target shapes used in the Whitehurst & Beilin study. Five blankets, representing tests of the hypothesized rules, for each character were presented, and children were asked to determine for each blanket whether it fit just right.

The side-matching task presented pairs of shapes already matched on one side. Again, right triangles and squares were used for the target 'character' shapes. Children were asked to judge whether a character was lying next to a blanket that fit just right.

Three types of rule use were hypothesized--the cover-rule, one-side-equal rule, and the congruence rule. The covering rule states that in order for a surface superimposed on another to contain the possibility of congruence with the base surface it must cover it completely. Use of the cover rule reflects an implicit appreciation for parallel plane comparison, either physical or mental, in which the totality of the figure is considered. Two of the comparison shapes - one similar (i.e., equal angles) to the target but larger; the other similar but smaller - tested the extent to which the covering component of the congruence concept existed independently of the correspondence component. The larger and smaller shapes were chosen to ensure that responses were based on covering and not on similarity properties. As shown in Table 1, strict application of this rule predicted that in both tasks, in addition to the congruent blanket, the larger blanket (i.e., the geometric figure used as a cover) would also be identified as congruent (and so would not measure congruence correctly).

Table 1

Predicted Patterns of Rule Use for both Triangles and Quadrilaterals and for both Tasks by Blanket Type

Blanket	Rules		
	Cover Rule	1-side= Rule	Congruence Rule
Congruent	1	1	1
Larger	1	0	0
Smaller	0	0	0
Equal Side - Slightly Different Shape	0	1	0
Equal Side - Very Different Shape	0	1	0

Note. 1 & 0 refer to subjects judgements: 1 = fits just right, 0 = does not fit just right. Thus, for congruent geometric figures, from the childs' perspective, all rules would lead to a "correct" congruence judgment, in all other cases the judgment would be incorrect.

The one-side-equal rule represented a partial application of the correspondence component. The rule was hypothesized because of its apparent use by young children in the Beilin & Klein and the Whitehurst & Beilin studies. Two of the comparison shapes tested the extent to which the one-side-equal rule is used. It was predicted that strict application of this rule would lead in both tasks, to a judgment of congruence with the congruent blanket, but more centrally with the blankets containing one side equal in length to one side of the target shape as well (see Table 1). The slightly different shape was the same as one of the comparison shapes used in the Whitehurst & Beilin study. In that study, young children using a single side-matching strategy identified this shape as well as the congruent shape as being congruent. A very different shape in the present study was used as an additional test of the one-side-equal rule.

The congruence rule reflects the coordination of the covering component and the correspondence component, in which shape and size correspond, in addition to all sides and angles. It was predicted that a strict application of this rule would lead only to the congruent blanket to be identified as congruent in both tasks.

Additional hypothesized rules

It was apparent in the course of the study that the children, in addition to the above stated hypothesized rules, were using other rules as well. An estimation rule and a same shape rule were then hypothesized post hoc. The estimation rule allows for some slippage in accuracy. It was predicted (post hoc) that the application of this rule would lead to the slightly different shape to be identified as congruent in addition to the congruent blanket as was the case in the Whitehurst and Beilin study (see Table 2). The same shape rule states that if the blanket is the same shape or has the same number of sides it 'fits just right'. A more lenient application was excepted for this rule because of the obvious distortions of the very different blanket. Thus all of the blankets would be accepted as congruent, with the possible rejection of the very different shape.

Table 2

Modified Predicted Patterns of Rule Use for both Triangles and Quadrilaterals and for both Tasks by Blanket Type

	Modified Rules				
Blanket	Congruence Rule	Estimation Rule	1-side= Rule	Cover Rule	Same Shape
Congruent	1	1	1	1	1
Larger	0	0	0	1	1
Smaller	0	0	0	0	1
Equal Side - Slightly Different Shape	0	1	1	1/0	1
Equal Side - Very Different Shape	0	0	1	0	1/0

Note. 1 = fits just right, 0 = does not fit just right.

As shown in Table 2 a more lenient application of the cover rule was also accepted. Since the slightly different shape was seen by many children as acceptable as the congruent blanket its inclusion or rejection along with the inclusion of the larger shape was considered an application of this rule.

Predictions.

1) Children's inferences about congruence would be similar for both the area-matching and side-matching tasks.

2) There would be no difference in the use of a particular rule between the genders. No sex differences in accuracy judgments were found in the Beilin & Klein study. In the Whitehurst & Beilin study, with the younger subjects, there was no difference in strategy use or accuracy between boys and girls, and there was no overall main effect.

3) No difference in type of rules used would be found for the different shapes. In the Whitehurst & Beilin study subjects consistently used the same strategy for triangles and quadrilaterals.

4) No predictions were made concerning the order of the particular rules in development.

Method

Research Design

A mixed-model repeated measures design was used in which there were three between-subject factors--age(3 levels), sex(2), order(4) -and two within-subject factors--task(2) and shape(2). Five randomly-ordered trials were conducted for each of the two shapes within each of the two tasks. Subjects were equally distributed among the three between-subject factors resulting in a total of 120 subjects with five subjects in each cell.

Five pretests preceded the presentation of the two tasks. The first two established children's implicit knowledge of relative length and angle. The next two established classification skill regarding simple size and shape categories. The fifth pretest demonstrated and tested children's understanding of the phrase 'fits just right' (which denoted congruence and was used in the story scenarios for the tasks). The two tasks--area-matching and side-matching--involved geometrical figures in the form of 'characters' that like to sleep with 'blankets' that fit just right.

Each subject received a total of 20 trials in which they were asked to make congruence judgments (i.e., 'does the blanket fit just right?') about two geometrical figures. Ten trials--five trials with triangles and five trials with quadrilaterals--made up the area-matching task. The side-matching task consisted of five trials with triangles and five trials with quadrilaterals. Within each group of five trials, one pair of shapes was congruent, and the other four pairs were not congruent. Of the four pairs that were not congruent, one pair contained a blanket that was larger and had the potential to cover the character. One pair contained a blanket that was smaller than the character. The other two pairs contained blankets with at least one side equal in length to one side of the character figure; one was moderately different, and the other a very different, shape.

Twenty scores were calculated for each child. Each score measured adherence to one of the five hypothesized rules--the original rules plus the post hoc rules (i.e., congruence, estimation, one side equal, cover and same shape)--within each task and shape condition (see Table 2 for predictions). Initially, Analyses of variance (ANOVAs) were calculated on scores for each rule considered separately, with age, sex, task order and shape order as between-subject variables and task and shape as within-subject variables. Assessment of the main and interaction effects for each rule guided decisions concerning collapsing the scores across any one or more of the factors (see Results section).

A cluster analysis was done to see whether there were similarities in patterns of responses among individuals.

This analysis led to the analysis of age trends in the use of rules.

Variables from both the pretest and the tasks were included in an exploratory factor analysis. From this analysis an investigation of the underlying dimensions involved in the cognitive development of young children's geometric reasoning was assessed.

Subjects

One hundred and twenty preschoolers equally distributed among three age groups (i.e., 3-, 4- and 5-year-olds), participated. The three age groups consisted of 3- (Mean=3.2; Range - 2 years 7 months to 3 years 6 months), 4- (Mean=4.07; Range - 3 years 10 months to 4 years 5 months), and 5-year-olds (Mean=5.0; Range - 4 years 8 months to 5 years 4 months). Each age group consisted of 20 males and 20 females. The sample was selected from eight private preschools in heterogeneous working- and middle class neighborhoods in a large metropolitan area. All subjects came from homes where English was the dominant language spoken. Three subjects were dropped from the study. The reason for dropping two of the subjects was because of mechanical failure of the video recorder. It was discovered that the third subject had a hearing deficit.

Procedure and Materials

The order of presentation for the five pretests was the same for all subjects. The pretest order was as follows: 1) length, 2) angle, 3) size, 4) shape, and 5) puzzle board. Half of the subjects at each age, equally distributed by sex, received the area-matching task followed by the side-matching task (order 1). The other half received the side-matching task first (order 2). Order of presentation of the triangle or square was counterbalanced, with the same order retained across both tasks. The order for presenting the blankets within a shape condition (i.e., triangle or quadrilateral) was random. The puzzle board pretest, as well as the two tasks, involved a story about imaginary characters in the shape of geometrical figures that like to sleep with blankets that 'fit just right'. All task materials were displayed on a 28 cm x 36 cm white bristol board.

Procedure for length pretest. The length pretest established knowledge of relative length and an implicit knowledge of line. Because young children have difficulty understanding comparative terms such as 'longer' (Maratsos, 1973), and there is controversy over young children's understanding of unidimensional terms such as 'long' (Ravn & Gelman, 1984), the general term 'big' was used in the length and angle pretests (see Appendix A for script). Two sticks of different length were placed side by side, approximately an inch and a half apart, and perpendicular to the base of the bristol board. The bottom ends of the sticks were in alignment with one another and the base of the bristol board. Subjects were asked to choose the 'big one'. Subjects were asked to do this twice with two sets of sticks of the same dimensions, but different in color.

Materials for length pretest. The materials for the length pretest were two sets of 1 cm wooden sticks. The sticks were 10.5 cm and 11.5 cm in length. One set of sticks was green; the other red.

Procedure for angle pretest. The angle pretest established an implicit knowledge of angle. In previous research (Beilin & Klein, 1983), young children spontaneously referred to angles, in the context of geometrical shapes, as corners. Two different angles were placed before the subject. The orientation of both angles was the same; one leg of the angle was parallel to the top of the bristol board the other pointed toward the left side of the board so that the angle opened toward the subject. Subjects were asked to point to the big corner (see Appendix B for script). Subjects were asked to do this twice with two sets of angles of the same dimensions, but different in color.

Materials for angle pretest. The materials for the angle pretest consisted of two sets of two pairs of four 1 cm wooden sticks. The sticks were all 10 cm long. Within each set, one pair of sticks was joined together at one end to form a 110 degree obtuse angle. The other pair formed a 70 degree acute angle. One set of two angles was red; the other set was green.

Procedure for size sorting pretest. The size sorting pretest established the subjects' classification skill regarding simple size categories (i,e., large, small). A pile of scrambled blocks (i.e., varying on only one dimension--size) was placed before the child. Borrowing Fisher and Roberts (cited in Gelman and Baillargeon, 1983) phraseology, the subjects were asked to "put together all the blocks that belonged together" in the plastic bowls that were provided (see Appendix C for script). A separate bowl was used for each category and were arranged in a line before the subject. If the subject did not sort the blocks correctly, the experimenter demonstrated how they should be sorted. The blocks were then taken out of the bowls and scrambled. The subject was asked to sort them again.

Materials for size sorting pretest. The materials for the size sorting pretest consisted of six wooden balls of the same color and two transparent plastic containers. Three of the blocks were balls with a 2 cm diameter; the other three were balls with a 3 cm diameter. The two transparent plastic containers were 1.4 liter bowls.

Procedure for shape sorting pretest. The shape sorting pretest established the subjects' classification skill regarding simple shape categories (i,e., circle, triangle, square). A pile of scrambled blocks varying on only one dimension, shape, was placed before the child. The subjects were asked to "put together all the blocks that belonged together" in the plastic bowls that were provided (see Appendix C for script). A separate bowl was used for each shape category and were arranged in a line before the subject. If the subject did not sort the blocks correctly, the experimenter demonstrated how they should be sorted. The blocks were taken out of the bowls and scrambled. The subject was then asked to sort them again.

Materials for shape sorting pretest. The materials for the shape sorting pretest consisted of nine blocks, all the same color, and three transparent plastic containers. Three of the blocks were wooden cubes, three were wooden prisims and three were wooden balls. All of the blocks shared a common dimension: the sides of the cubes and triangles and the diameter of the balls were all 5 cm. The three plastic containers were 1.4 liter bowls.

Procedure for puzzle board pretest. The purpose of the pretest was to familiarize the subjects with the phrase 'fits just right' through concrete demonstrations of the concept. The five puzzle boards were presented in the same order to each subject.

The experimenter placed the target blanket over each character and ask the subject whether the blanket 'fits just right' (see Appendix D for script). Subjects received feedback about the correctness of their responses on each puzzle board. There were three questions per puzzle board.

Materials for puzzle board pre-test. The materials for the puzzle board pretest consisted of five puzzle boards made from two 30 cm long x 4 cm wide x 3.5 mm thick balsa wood and colored construction paper. One piece of balsa wood contained three cutout geometrical shapes. Three characters were drawn on the construction paper and mounted on the back piece of balsa wood. The pieces of wood were glued together so that the character showed through the cutout shapes. Accompanying each puzzle board was a target shape made of balsa wood with a knob glued to one of its sides. The target shape fit into at least one of the cutout shapes on the puzzle board (see Figure 1). The dimensions of the shapes on each puzzle board are specified in Table 3.

Puzzle Board
Number

1 Shape Varies

Target - Equilateral
Triangle

2 Shape Varies
Target - Ellipse

3 Same Shape
Target - Circle

4 Size Varies
Target - Circle

5 Shape Varies
Target - Ellipse

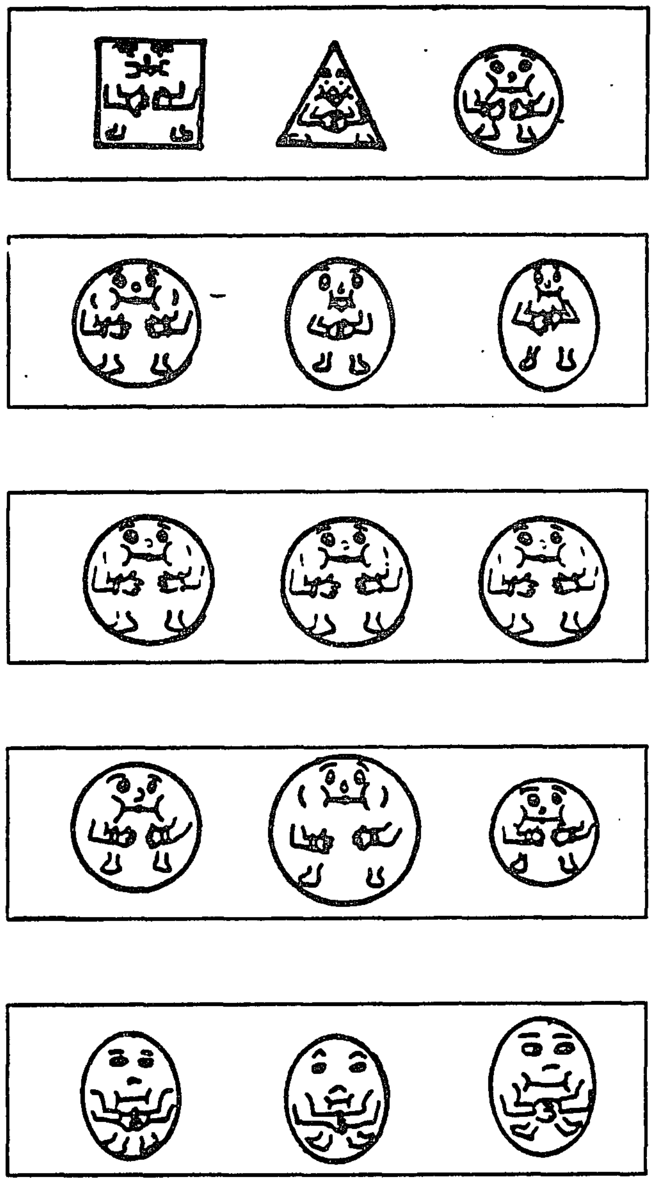

Figure 1

Target and character shapes for the puzzle board pretest.

Table 3

Dimensions for Puzzle Board Target and Character Shapes

Puzzle Board Number	Target Blanket	Left Character	Middle Character	Right Character
1	6.4cm x 6.4cm x 6.4cm (triangle)	6.4cm x 6.4cm x 6.4cm x 6.4cm (square)	6.4cm x 6.4cm x 6.4cm (triangle)	6.4 diameter (circle)
2	7cm height 50 degree arc (ellipse)	7cm diameter circle (ellipse)	7cm height 55 degree arc (ellipse)	7cm height 50 degree arc (ellipse)
3	7cm diameter (circle)	7cm diameter (circle)	7cm diameter (circle)	7cm diameter (circle)
4	7cm diameter (circle)	7cm diameter (circle)	7.6cm diameter (circle)	6.4 diameter (circle)
5	7.3cm 50 degree arc (ellipse)	7.3cm 50 degree arc (ellipse)	7.3cm 55 degree arc (ellipse)	7.6cm 50 degree arc (ellipse)

Procedure for area-matching task. The area-matching task assessed the subjects' ability to make congruence judgments with regard to several concrete instances of superimposition. The target shape (triangle or square character) was handed to the subject, and the story about the blankets was repeated (see Appendix E for script). A trial began with the random presentation of each of five triangle- or square-shaped blankets (see Materials). The order of shape presentation was counterbalanced between subjects, but remained the same order for any one subject across both tasks. When given to the subject, the blankets were in the same approximate orientation (i.e., not flipped or rotated) as the target shape. For each trial the subject was asked whether the blanket fit just right. When all five trials with one shape (e.g., triangle) were completed, the procedure was repeated with the other shape (e.g., square).

Materials for the area matching task. The materials for the area matching task consisted of two plastic shapes inscribed with characters having eyes, nose, mouth, arms, hands and feet. Corresponding to these were 10 plastic shapes (i.e., 5 triangles and 5 quadrilaterals) representing blankets for each of the characters (see Figure 2). The shape of the one character was a scalene right triangle (i.e., no sides equal), and the other was a square. The characters were made of opaque-colored plexiglass. The blankets were made of flexible, transparent yellow plastic.

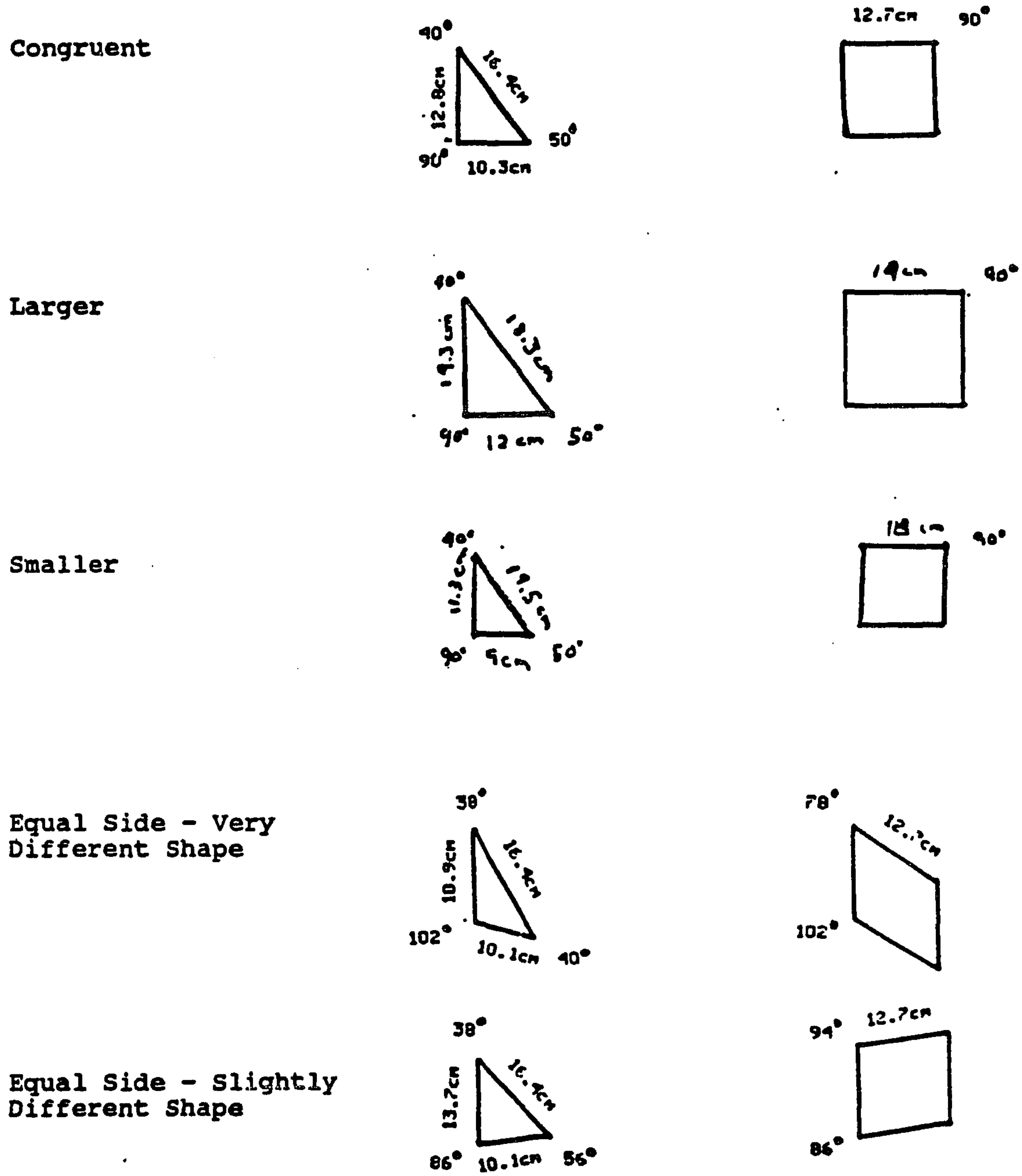

Figure 2

Dimensions of blanket shapes for the area-matching and side-matching tasks.

Since the blankets were presented one at a time (see Procedure above), and inasmuch as color could be a potentially distracting dimension for young children, all of the blankets were the same but different in color from the characters. The dimensions by which the five triangle blankets varied from the triangle character are as follows. One blanket was congruent with the character. Two were similar (having congruent angles); one larger, one smaller. The other two blankets were not similar to or congruent with the character, but contained one side which was the same length as the hypotenuse of the right triangle character. One blanket was slightly different and the other was very different in shape from its character.

The quadrilateral blankets varied from the square character in the same way as the triangle blankets varied from the triangle character except that with the two non-similar blankets, which share an equal side with the character, instead of one side being equal, all sides were equal. Of the non-similar blankets, one was a rhombus, slightly different in shape from the square character; the other was a rhombus very different in shape.

The dimensions of the characters and the selection blankets for all tasks are found in Figure 2.

Procedure for side matching task. The side matching task assessed the subject's ability to make congruence judgments based on the alignment of one side of each figure.

A trial began with the presentation of each of five triangle or quadrilateral pairs (see Materials). Within the shape condition the five cards were presented in a random order. For each trial the subject was asked whether the character was lying next to a blanket that fits just right (see Appendix F for script). After all five trials with one shape (e.g., triangles) were completed the entire procedure was repeated using the other shape (e.g., quadrilaterals).

Materials for the side-matching task. The materials for the side-matching task consisted of ten 30.5 cm long x 22.8 cm wide cards with pairs of opaque plastic shapes mounted on each card (see Figure 3). Five pairs were triangles and five pairs were quadrilaterals. One of the shapes in each pair was the character shape and the other was a blanket, or comparison, shape.

For the triangle pairs, the dimensions of the character figure were the same on all five cards, and were the same dimensions as the right scalene triangle in the area-matching tasks. Aligned along the hypotenuse of the target triangle were blankets that varied in shape and size in the same way as the blanket triangles varied in the area matching tasks.

For the quadrilateral pairs, the dimensions of the character figure were the same on all five cards, and were also the same dimensions as the square in the area-matching tasks. Aligned along one side of the target square were blankets that varied in shape and size in the same way that the blanket quadrilaterals varied in the area matching tasks.

All subjects were videotaped with a Camcord mounted on a tripod. The tapes provided data for scoring subjects' responses.

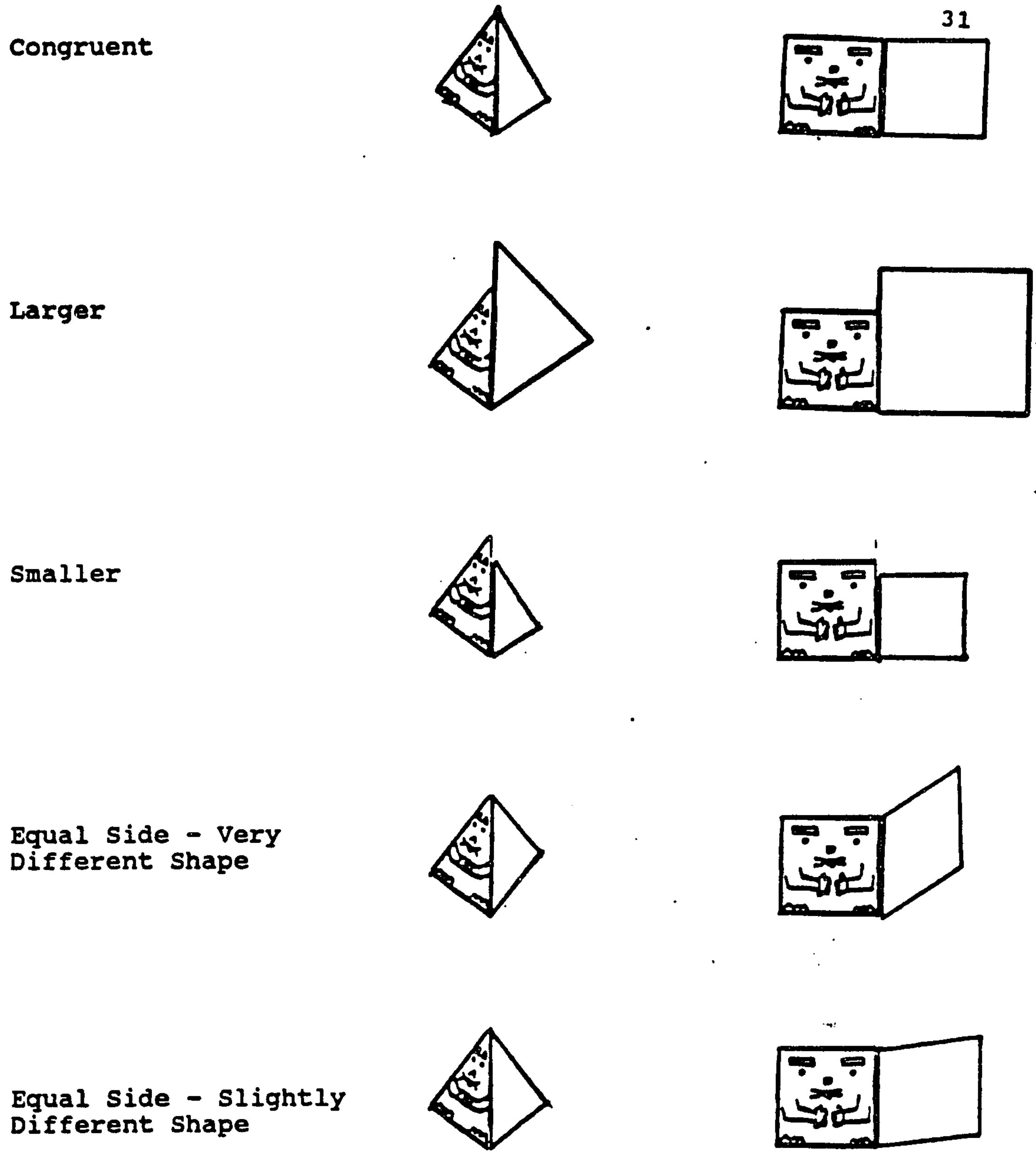

Figure 3

Orientation of character and blanket shapes for the side-matching task.

Results

Inter-rater Reliability

Twelve subjects (four 3-year-olds, four 4-year-olds and four 5-year-olds) were randomly selected to test inter-rater reliability. Out of a total of 503 instances of observed behavior there was agreement on 480 and disagreement on 23 instances resulting in an overall reliability measure of 95.4%. Appendix H shows reliability measures broken down by pretest, task and strategy. All differences were discussed and resolved.

Pretests

Length and Angle. In general, children showed a better understanding of length than angle. A 3 (age group) by 2 (length and angle) repeated measures analysis of variance was used with length and angle as a within-subject factor. A main effect was found for age group, $F(2,117) = 15.05$, $p < .001$, and for length and angle, $F(1,117) = 16.67$, $p < .001$, but there was no interaction.

For the length pretest 65% of the 3-year olds, 85% of the 4-year-olds and 95% of the 5-year-olds were correct on both trials. For the angle pretest 42.5% of the 3-year-olds, 61.5% of the 4-year-olds and 80% of the 5-year-olds were correct on both trials.

Size and Shape Pretest. A modified version of Vygotsky's (1962) method for categorizing classification skill was developed (see Appendix H for specifics of scoring). Three levels of conceptual understanding describe the sorting behavior for both the size and shape pretest.

The first level, trial and error or syncretic thinking, represented a basic inability to discriminate between different size balls or different shape blocks. Children categorized as such tended to randomly place the blocks in the bowls with no attempt to systematically group them.

The second level, thinking in complexes, represented an initial attempt to organize the blocks into systematic groups. At the lower end of the continuum children exhibited heaping behavior - piling all of the blocks into one bowl. More sophisticated collections were grouping by number or in the case of the shape pretest, one of each kind.

The third level, proto concepts, represented the highest level of classification skill possible with the materials presented. This level is labeled proto concepts because the skill required to be successful on these tasks only requires the subject to think about one dimension (i.e., size or shape).

To investigate the overall performance of subjects on both trials an average score was calculated. A 3 (age group) by 2 (size and shape) repeated measures analysis of variance was used with size and shape as a within-subject factor. A main effect was found for age group, $F(2,117) = 28.18$, $p < .001$. A marginally significant main effect was found for size and shape, $F(1,117) = 3.61$, $p < .06$, with children doing slightly better on the shape task than the size. There was no interaction between age group and the within subject factor.

Table 4 shows the distribution of subjects among the different levels of conceptual understanding. The profiles of the levels across age is similar for both the size and shape pretest. The number of children exhibiting trial and error behavior was highest for the 3-year-olds, droped considerably for the 4-year-olds and completely disappeared by 5-year-old. Level 2, thinking in complexes, was moderately high at 3-years-old, increased in the 4-year-old group and declined drastically with the 5-year-old sample. The highest level of classification skill, proto concepts, describes a few 3-year-olds, slightly more 4-year-olds and a solid majority of the 5-year-olds.

Puzzle Board Pretest

Most of the children performed very well on the puzzle board pretests. Based on their initial responses (i.e., before feedback was given), 85% of the 3-year-olds, 92.5% of the 4-year-olds and 95% of the 5-year-olds were correct on 13 out of 15 response items. The high accuracy rate on the puzzle boards demonstrated that the children had a fairly good notion of what was meant by fitting just right.

Table 4

<u>Percentage of Subjects at three size and shape categorization levels by Age on Pretest</u>

Pretest by Level	Age 3 (N=40) N	%	Age 4 (N=40) N	%	Age 5 (N=40) N	%
<u>Size</u>						
Level 1	10	25.0	2	5.0	0	0.0
Level 2	26	65.0	29	72.5	10	25.0
Level 3	4	10.0	9	22.5	30	75.0
<u>Shape</u>						
Level 1	11	27.5	5	12.5	0	0.0
Level 2	18	45.0	21	52.5	6	15.0
Level 3	11	27.5	14	35.0	34	85.0

<u>Scoring for the Tasks</u>

It was also apparent from the data collection process that the children were treating the tasks and the shapes differently regarding the blankets they chose as congruent. As a result, separate univariate F-tests were calculated for each individual hypothesized rule considering the tasks and the shapes separately. Each subject received a total of 20 trials in which they were asked to make congruence judgments (i.e., 'does the blanket fit just right?') about two geometrical figures. Ten trials--five trials with triangles and five trials with quadrilaterals--made up the area-matching task. The side-matching task consisted of five trials with triangles and five trials with quadrilaterals. Within each group of five trials, one pair of shapes was congruent, and the other four pairs were not congruent. Of the four pairs that were not congruent, one pair contained a blanket that was larger and had the potential to cover the character. One pair contained a blanket that was smaller than the character. The other two pairs contained blankets with at least one side equal in length to one side of the character figure; one was moderately different, and the other a very different, shape.

Twenty scores were calculated for each child. Each score represented the extent to which the pattern of blankets chosen matched the hypothesized pattern for each rule (5) by task (2) and shape (2) (See Appendix I for a listing of the variables). For each rule, the predicted response for each trial was assigned 1 point; an unpredicted response was assigned 0 points. Since there were five trials for each condition, five was the maximum possible score for any one rule when tasks and shapes were considered separately.

<u>Rules</u>

Initially, analyses of variance were calculated for each of the twenty types of scores to determine the main and interaction effects of the between-subject variables - age, sex, task order and shape order. The only between-subject variable that registered any meaningful explanation of variation in scores was age (See Appendix N for output). There were two significant agegroup by sex interaction effects for AMSQ and MCAMSQ but since the main effect of sex for these two variables was not significant the interaction effects were considered not to be meaningful. As shown in Table 5, 15 of the 20 variables had significant age main effects.

In general, as indicated by higher mean scores, older children were more likely to use the congruence rule or the estimation rule for the area-matching tasks and the one-side equal rule or the estimation rule for the side-matching tasks. The same shape rule tended to be a younger childs' strategy. Although the pattern of selection indicating the cover rule was evidenced, it was not age related. Because of the interdependence of these scores when considered on a continuous scale, between/within-subject comparisons across the twenty variables was not possible. The important information concluded from the analyses of variance was that age trends warranted further investigation while sex, task order, and shape order did not.

Table 5

Mean Scores of Rule-based dependent variables by Age

Rules/ Variables	AGE 3	4	5	*p*
Congruence Rule				
AMT	2.37	3.40	4.23	**
AMSQ	2.58	3.22	4.35	**
SMT	2.15	2.57	3.00	**
SMSQ	2.32	2.85	3.55	**
Estimation Rule				
ESTAMT	2.87	3.60	3.98	**
ESTAMSQ	3.38	3.98	4.25	**
ESTSMT	2.85	3.13	3.75	**
ESTSMSQ	3.07	3.35	4.30	**
One-Side = Rule				
OSAMT	2.87	3.00	3.08	**
OSAMSQ	2.68	3.18	3.30	**
OSSMT	3.35	3.58	4.50	**
OSSMSQ	3.12	3.30	4.05	**
Cover Rule				
COVAMT	3.47	3.70	3.80	
COVAMSQ	4.08	4.15	4.00	
COVSMT	3.00	3.05	3.08	
COVSMSQ	3.30	3.40	3.58	
Same Shape Rule				
SHAMT	3.68	2.80	2.38	**
SHAMSQ	4.07	3.48	2.53	**
SHSMT	3.70	3.20	2.92	**
SHSMSQ	3.95	3.43	2.88	**

** $p < .001$

Cluster Analysis

The analyses of variance highlighted the age differences among preschool children for single dependent variables. These, however, could not explain much about the patterns of rule use across the two tasks, that is, whether or not different patterns could be discerned between/within the age groups or the two different shapes. A cluster analysis was thought appropriate to distinguish groups of like performers, and could therefore inform the question of whether like-performing groups resided primarily within or across the experimental groupings.

The first step in doing the cluster analysis was to select an appropriate measure of similarity. A matching type measure was selected for two reasons: 1) the way that rule-use was defined, a subject either did or did not exhibit the use of a rule (i.e., the data were nominally scaled), and 2) the goal of the cluster analysis was to uncover similar profiles of responses among individuals or, as Dillon and Goldstein (1984) refer to the process, finding individuals that "share common attributes."

The twenty scores that were used in the analyses of variance were recalculated so that each variable took on values of either 1 or 0. A value of 1 indicated adherence to the pattern of blanket selection described for each rule as shown in Table 2. A value of 0 indicated the nonadherence to that particular pattern. Four additional scores were calculated which represented patterns of selection other than those indicated in Table 2. They were labeled no-rule and were calculated for the area-matching triangle (NORAMT), area-matching square (NORAMSQ), side-matching triangle (NORSMT) and the side-matching square (NORSMSQ) tasks. A total of 24 variables were used in the cluster analysis.

The next step in the cluster analysis was to select the appropriate clustering algorithm. The clustering algorithm defines the rules for clustering the subjects into subgroups based on inter-object similarities. A between group average method was chosen. This method "uses the average distance from individuals in one cluster to individuals in another cluster as the clustering criterion" (Hair et al., 1987). Because this method creates mutually exclusive groups with small within-group variance it was thought to be the most appropriate for revealing underlying profiles.

As shown in Table 6, five clusters with descriptive profiles were found. A Chi-square test showed that cluster membership was associated with age; $\chi^2_{(8)}=55.74$, $p < .001$. The correlation measure, phi, was .56. Twenty-one children (17.5 %) were grouped into Cluster 1. The profile for Cluster 1 showed members who were categorized as no-rule users for either task. About two thirds (14) began showing signs of partially organized strategies for making congruence judgments with squares on the area-matching task.

Table 6

Frequency Distributions of Cluster Members by Task by Cluster by Shape by Rule

Area-Matching Task	Cluster 1 (N=21)		Cluster 2 (N=36)		Cluster 3 (N=13)		Cluster 4 (N=26)		Cluster 5 (N=24)	
Rule	Triangle (N=21)	Square (N=21)	Triangle (N=36)	Square (N=36)	Triangle (N=13)	Square (N=13)	Triangle (N=26)	Square (N=26)	Triangle (N=24)	Square (N=24)
Congruence Rule	0	1	0	2	0	3	11	0	14	19
Estimation Rule	2	0	1	0	6	0	10	25	2	0
One-Side Equal Rule	1	0	0	0	1	3	1	0	0	0
Cover Rule	3	9	10	15	0	4	0	1	0	1
Same Shape Rule	1	5	13	15	0	0	0	0	1	0
No Rule	14	6	11	4	6	3	4	0	7	4

Side-Matching Task	Cluster 1 (N=21)		Cluster 2 (N=36)		Cluster 3 (N=13)		Cluster 4 (N=26)		Cluster 5 (N=24)	
Rule	Triangle (N=21)	Square (N=21)	Triangle (N=36)	Square (N=36)	Triangle (N=13)	Square (N=13)	Triangle (N=26)	Square (N=26)	Triangle (N=24)	Square (N=24)
Congruence Rule	0	0	1	2	0	0	0	1	0	3
Estimation Rule	0	0	1	4	0	0	2	9	0	15
One-Side Equal Rule	2	0	3	0	11	13	17	9	18	3
Cover Rule	0	3	1	1	0	0	1	1	0	1
Same Shape Rule	0	1	27	26	0	0	2	4	0	0
No Rule	19	17	3	3	2	0	4	2	6	2
Mean Age	3.65		3.61		4.44		4.43		4.60	

Nine of these children were categorized as using the cover-rule and five the same-shape rule. This cluster contained no five-year olds, 12 four-year-olds and nine three-year-olds.

Cluster 2 was the largest cluster with 36 members (30 %). The profile for this group was the use of either the cover-, same- shape or no-rule for the area-matching task and the same shape rule for the side-matching task. Like the members in Cluster 1 fewer children were categorized as no-rule users for the square trial on the area-matching task. For children who were coded as using the same-shape rule, nearly all selected every blanket for the area-matching triangle, side-matching triangle and side-matching square task (i.e., 11 of 13, 25 of 27, 21 of 26 respectively). The pattern was altered for the area-matching square task. Of the 15 children who were coded as using the same- shape rule on the area-matching square task, 11 rejected the very different shape as fitting just right. Many of them commented that it was a diamond. Over half (22) of the samples three-year-olds, 10 four-year-olds and only four five-year-olds were grouped into this cluster. Cluster 1 and 2 tended to include the youngest children in the sample who evidenced either no organized strategy (i.e., no rule) or a partially organized strategy (i.e., cover or same shape rule) for making congruence judgments. The use of the same-shape rule is not inconsistent with Piaget's notions of initial geometric understandings having the properties of enclosure, connectivity, etc. However, what is also evident in the performance of members of Cluster 2 is a move toward making subtle discriminations in shape. This was shown by their rejection of the very different shape during the area-matching square task as well as their use of the cover rule. When the performance on the size and shape pretest for members of these two clusters was investigated, more Cluster 2 members achieved the level of proto-concept than Cluster 1 for the shape pretest (25.0% and 14.3% respectively). Both clusters were equally inferior on the size pretest (i.e., 3 members in Cluster 2 and 2 members in Cluster 1 achieved the level of proto-concept). It appears that subtle discriminations in shape in addition to the covering component mark the initial levels of organization in the development of the congruence concept.

Thirteen children (10.8 %) were grouped into Cluster 3. The similarity between members in this group was found in their likeness of response on the side-matching tasks and their lack of preference for any particular pattern of response on the area-matching tasks. Nearly all of the children used the one-side-equal rule on the side-matching task for both the triangle (11) and the square (13). For the area-matching triangle task children tended to either use the estimation or no-rule. For the area-matching square there seemed to be a preference to use a rule rather than not use a rule but there was no particular rule favored by the group. Cluster 3 showed a shift in the age distribution from the first two clusters. Proportionately more five-year-olds were grouped into Cluster 3 than the other two clusters (54% compared to 11% and 0%). Seven five-year-olds, four four-year-olds and two three-year-olds were grouped together in this cluster.

In addition to shape discrimination and covering, the importance of the equality of corresponding sides, as highlighted by the members of Cluster 3, was revealed next in development. Although members in Cluster 1 and Cluster 2 did reasonably well on the length pretest (i.e., 81% and 67%, respectively, were correct on both trials), this recognition did not generalize to performance on the side-matching task. In contrast, all of the members of Cluster 3 were correct on both trials of the length pretest and nearly all of them used a one-side-equal rule on both shape trials of the side-matching task (i.e., Triangle-85%; Square-100%).

There was also a shift evident in Cluster 3, which reached a plateau in Clusters 4 and 5, with performance on the angle pretest (i.e., Cluster 1 - 38%; Cluster 2 - 50%; Cluster 3 - 77%; Cluster 4 - 77%; Cluster 5 - 75% on both trials). This second level in the development of the congruence concept was accompanied by attention to component parts of the figure.

Cluster 4 contained 26 members (21.7 %). In general, the members in this cluster tended to treat the triangle and the square a bit differently. When presented with the triangle, children in this group tended to use either the congruence or the estimation rule during the area-matching task. For the square trial all but one used the estimation rule. For the side-matching triangle task most used the one-side-equal rule. On the square trial just as many children used the one-side-equal rule as the estimation rule. This is the only cluster in which there appeared to be some sort of shape by task interaction evident. Four- and five-year-olds dominated membership in this cluster with 11 five-year-olds, 13 four-year-olds and only two three-year-olds making up the group.

Twenty-four (20%) of the children clustered into the fifth group. The majority of the children used the congruence rule on both the triangle (58%) and the square (79%) trials for the area-matching task. On the side-matching task the children used the one-side-equal rule on the triangle trial. For the square trial, unlike some members in the fourth cluster, most children rejected the very different shape as fitting just right and so were categorized as using the estimation rule. This cluster was dominated by the oldest group in the sample with 18 five-year-olds, one four-year-old and five three-year-olds.

Further elaboration of the discrimination process was evident with the performance of members in Clusters 4 and 5.

Cluster 4 was a transitional cluster with the use of increasingly powerful rules (i.e., Congruence and Estimation) during the area-matching tasks. Along with this was a dip, or as Strauss (1982) called it, a U-Shape function marking transition, with the square shape. As shown in Table 6, all but one member in Cluster 4 used the estimation rule with the square shape on the area-matching task while nearly half (i.e., 42%) of Cluster 4 members were more discriminating with the triangle shape. Members of Cluster 5 showed a concern for the necessity of all component parts needing to correspond with their performance on the area-matching task (i.e., greater occurrence of the Congruence Rule).

Turning to the side-matching task, we again see Cluster 5 members outperforming Cluster 4 members regarding the square shape. More members in Cluster 5 rejected the very different shape (i.e., as shown by their use of the estimation rule) than members in Cluster 4. The difference in performance between the triangle and square trials for both Cluster 4 and 5 reveals sensitivity on the part of these children when symmetry was concerned. Apprehension of asymmetry was easily extracted in the case of the square character and the very different blanket. This was not the case with the triangle character and the very different triangle blanket, hence, the tendency to accept a blanket as fitting just right on the basis of length alone for the triangle.

In sum, the cluster analysis did distinguish mutually exclusive categories of like performers sharing common attributes. The members in the clusters shared the same approximate age as well as the use of similar rules. The hierarchy of rule use began with the use of the same shape and cover rules, advancing to the one-side equal and estimation rules and finally resulted in the use of the congruence rule. That is, childrens' notions of congruence initially began by being informed by notions of shape or the capacity to cover. Congruence judgments then seem to fixate on the length of one side and under limiting conditions, as with the side-matching task, and stayed that way. Finally, notions of symmetry and correspondence of multiple component parts appeared as necessary to the child for making congruence judgments.

Factor Analysis

A common factor analysis was conducted to reveal the underlying dimensions in the data that were expected to reflect different modes of reasoning such as the rules children used.

Eight variables -- SIZE, SHAPE, AMT, AMSQ, SMT, SMSQ, AMTS, AMSQS -- were used in the factor analyses (See Table 7). The factor analysis was exploratory since there were no specific prior theoretical hypotheses concerning the underlying dimensions. A four factor maximum likelihood model with a VARIMAX (orthogonal) rotation was tested. As shown in Table 8, four factors accounted for 75.4 percent of the total variance. $^2(2) = 4.23$, p = n.s., assuming the goodness of fit of the model.

Factor interpretation. Pairs of the eight original variables held highly significant loadings on each of the four factors. The factor labels reflect what each pair of variables represents. Factor 1 was labeled Procedural Knowledge, because, as in Table 7 two variables loading highest (AMTS=.98, AMTSQS=.75) were variables representing the actual strategies (i.e., superimposing, side-matching, etc.) the children used while making their congruence judgments on the area matching task. Factor 2 was labeled Classificatory Knowledge because the variables SIZE (.97) and SHAPE (.67) represented the developmental level of the child with regard to sorting blocks by size and shape. Perceptual Acuity was the label chosen for factor 3. SMSQ (.97) and SMT (.65) represented static presentations of two shapes, aligned on one side, mounted on cardboard. Factor 4 was labeled Physical Knowledge. AMT (.82) and AMSQ (.53) represented tasks where children manipulated the materials, gaining physical information about the shapes in addition to more accurate perceptual information. Children formed judgments of congruence based on more elaborate experience with the materials.

Table 7
Summary Information: Four Dimension Common Factor Solution - Maximum Likelihood Method with VARIMAX Rotation

	Summary Information					
Variables	Factor 1	Factor 2	Factor 3	Factor 4	Communality[a]	Unique[b]
AMTS	.98	.16	-.02	.11	.99	.01
AMSQS	.75	.05	.05	.18	.60	.40
SIZE	.14	.97	.14	.13	.99	.01
SHAPE	.06	.67	.14	.28	.55	.45
SMSQ	.12	.16	.97	.14	.99	.01
SMT	-.05	.11	.65	.17	.47	.53
AMT	.19	.25	.23	.82	.83	.17
AMSQ	.32	.38	.25	.53	.59	.41
Total variance	34.6%[c]	12.7%	18.4%	9.7%	75.4%	25%
Common variance	45.9%[d]	16.8%	24.4%	12.9%		
Factor contribution of factor j (eigenvalue)[e]	2.8	1.0	1.5	.78		
$\chi^2_{(2)}=4.23, p=.121$						

[a] The communality of z_1 -- that part of the variance of z_1 accounted for by the data on the other seven variables (common factors).

[b] The uniqueness of z_1 -- that part of the variance not accounted for by the data on the other seven variables (common factors).

[c] Percentage of variance among all the variables that is accounted for by a single factor (i.e., f1)

[d] Variance among all the variables that is accounted for by a single factor (i.e., f1) as a percentage of that accounted for by all of the factors.

[e] Sum of the squared factor loadings - eigenvalue.

Age trends among factors.

A multivariate analysis of variance was calculated to determine whether the extracted underlying dimensions from the factor analysis were age related. The factor scores served as the dependent measures and age group was the independent measure. There was a significant age main effect (Approx. $F=14.5(8,226)$, $p<.001$). As shown in Table 8, Procedural Skill was the first level of organization apparent in the data (i.e., no significant age main effect).

Table 8

Mean Factor Scores by Age Group and Factor

	Age				
Factor	3	4	5	F	p
Procedural	-.18	.01	.17	.25	n.s.
Perceptual	-.33	-.03	.37	.24	.01
Physical	-.50	.05	.46	.53	.001
Classificatory	-.48	-.16	.64	.91	.001

What this means is that most of the children were coded as using a superimposition strategy to determine congruence regardless of age. Even under adverse conditions (i.e., Order 2; side-matching task first), fewer than half of the 3- and 4-year olds (i.e., 45% and 35% respectively) were influenced by seeing the character and blanket aligned on one side. The children that were influenced continued to use a side-matching strategy with the area-matching task. Only 1 of the 20 5-year-olds, in this condition, did so. In comparison, a large majority of the 3- 4- and 5-year-olds (i.e., 85%, 90% and 95% respectively) who received the area-matching task first compared the shapes with a superimposing strategy.

It appears that the other three factors, Perceptual, Physical and Classificatory appear to occur simultaneously in development. Looking at the mean factor scores (Table 8) a significant age trend is apparent for each of the three factors. Without over-interpreting the developmental nature of these underlying dimensions or latent structures, these findings suggest that these schemes are beginning to co-ordinate around five years of age. Five-year-olds are more likely to accurately implement a superimposing strategy resulting in accurate judgments of congruence, they are better able to co-ordinate length and symmetry information and they can accurately sort by shape and size. The co-ordination of these schemes may be a necessary cognitive condition requisite to the emergence of Piagetian-type concrete operations.

Discussion

The results appear to support a cognitive rule-based model of the development of the congruence concept. The Cluster analysis highlights the differerent modes of reasonining children brought to the tasks. The inferences the children made about the problems were delineated, based on the pattern of selection of the blankets,. These inferences were called rules because they represented the underlying algorithms currently operating within the child.

Theoretically, the results also suggest compatibility with Piaget's later theory (see Beilin, 1992). Beilin describes Piaget and Garcia's (1991) new theory concerning the logic of meanings. Referring to one aspect of Piaget's new theoretical model, which Beilin calls the "logical hermeneutics of action", he details what Piaget requires as a research strategy in attempting to describe preoperational thought:

> The model, as applied to the sensorimotor and preoperative periods, requires a functional analysis of how subjects approach a task. An adequate model would necessarily detail the actions of the child, the objects acted on, and above all provide an interpretation of the inferences the child is making in carrying out the task. It requires an interpretation on the part of the investigator, who is attempting to characterize the meanings inherent in the

situation and subsequently to identify the logical links in the system that make the meaning implications cohere. (Beilin, 1992, pp 6-7).

What follows is an attempt to show how the present research fits into the above stated theoretical framework. The relevant actions of the children were described in detail for the development of rudimentary classification skills for size and shape, and for the procedural strategies children employed while solving the area-matching task. (See Appendix J for details of the coding classifications.) The rules the children were coded as using were the investigator's interpretation of the inferences children were making to solve the problems. The cluster analysis showed that the initial level of the child's analysis focused on a semi-topological analysis of shape, which the children quickly adapted when confronted with a totally different shape (i.e., a diamond) or a more age appropriate task (i.e., as evident in the use of the covering-rule on the area-matching task). Further analyses showed a gradual transition toward co-ordination and cohesion during the preschool years with the additional attention to corresponding component parts and symmetry.

Finally, the underlying dimensions extracted in the factor analysis point to at least some of the logical links in the system that account for an accurate judgment of congruence. As stated in the introduction, intuitively, congruence mappings entail the co-ordination of correspondences and covering such that every element of one set is assigned an element of another. Psychologically, classification skill (i.e., skill that is developed to the point of making subtle shape discriminations) as well as developed perceptual skills also seem to play a part. Linkage among these interdependent systems, begin to appear around the age of five years and allows for the "meaning implications" of the child to begin to resemble those of the older problem solver.

Reflections on the predictions

Two catagories of findings address the research questions and the related predictions. The first concerns the between-subject factors of age and sex. The second concerns the inferences made or the rules used by the children while solving the problems presented by the tasks.

The prediction that there would be no sex differences was supported by the single dependent variable analyses of variance. As for hypothesized age differences, they were found in the repeated measure analyses of variance for the shape and size pretests, the single dependent analyses of variance, the cluster groups and the mean factor scores. The prediction that 5-year-olds would consistently use inaccurate rules more than 3- and 4-year-olds was not supported by the data. In fact, the reverse was true. The result appears to be due to the greater detail of the analyses of children's activity in this study, which was not the case in the other studies that formed the basis of this prediction. In addition, more inferior forms of reasoning were discovered, as well as their consistent employment.

The prediction of the relationship between rule use and tasks was not supported. As shown in the cluster analysis, and further supported with the factor analysis, the two tasks appeared to show a striking influence of context on performance (i.e., static presentations vs. free manipulation of materials). This finding will be readdressed in the section on educational implications.

The prediction of the relationship between rule use and shape (i.e., triangle or square) was not supported. Children at all ages treated the triangle and square very differently. Squares were easier to differentiate from one another than triangles, even when superimposing was the strategy employed. Even though the acute angle of the very different triangle was of the same degree as the very different rhombus, children did not perceive it as such. The very different triangle was still a triangle while the very different quadrilateral was a diamond not a square.

Limitations

There were two noteworthy limitations of this research. One concerns the puzzle board. The other concerns the testing of asymmetry with the very different blankets.

The puzzle board pretest was not a good test of the children's understanding of the phrase 'fits just right'. As was reported, children demonstrated a fairly good understanding of what the phrase meant, but only with immediate feedback. A better test would have included additional puzzle boards with no feedback. The justification for not including additional puzzle boards was because of the short attention span of young children, as well as the possibility of overtraining the concept. Future research of this concept should test such a trade off more carefully.

Second, the very different triangle blanket was probably not different enough. After the analyses were concluded and it was discovered that children were treating the triangle and square differently, several adults were asked to judge which were more alike -- the triangle character with its very different blanket or the square character and its very different blanket. All said the triangles were more alike. Future research could examine the integrity of the three-sided figure under extremely distorting conditions to see whether it too would be recognized as another shape by very young children. Another aspect for study could be the orientation of the triangle blanket on the side-matching task (e.g. flipped along the horizontal axis resulting in a rectangular type as opposed to a deltoid type figure).

Educational Implications

It is well known that educational practices, especially those related to early childhood curricula, are influenced by psychological research on the developing child (Bredekamp

1987). With the 1989 publication of the National Council of the Teachers of Mathematics curriculum and evaluation standards (NCTM, 1989), a new era in mathematics education in this country was marked.

One of the major issues addressed in the statement of standards concerns the way students learn mathematics. Traditional educational theory and practice holds that student learning is passive and that knowledge is accumulated through processes such as drill and practice. The new view in mathematics education, a view long held by many psychologists, certainly from Piaget onwards, is that it is through the child's activity that meaningful mathematical concepts are constructed. John Dewey's claim was that educators knew it too. Only they tended to ignore it, even in the progressive education movement.

The direct implications of the present research for educational practice are twofold. The first concerns the ever-present reliance on paper and pencil worksheets. As documented by the different levels of performance on the area-matching vs. the side-matching task, active manipulation of materials brings out the best in what children know. Children tend to use lower-level inferences when problem-types are presented statically.

Early childhood programs have adapted their curriculum to include an enormous amount of manipulatives for children to engage in discovery type activities. Many elementary school teachers look disdainfully upon manipulatives thinking active manipulation of materials is only appropriate for very young children. This most likely is not the case.

This leads to the second implication which is intimately related to the first. It concerns assessment. Fortunately, in many school districts, kindergarten and first graders are not subjected to standardized paper and pencil tests. The current trend, as recommended by the NCTM standards, as well as other educational organizations, is toward authentic or performance-based assessment at all grades. It is only through these forms of assessment that student achievement will be most reliably monitored and students will be given a better chance to express what they know.

Both curriculum and assessment development need to have what statistitions call content validity. The curriculum needs to reflect what educators believe is important for students to know as well as how students best obtain that knowledge. Assessment needs to reflect the curriculum while maintaining the integrity of the learning process. Curriculum and assessment policymakers would do well in keeping these ideas in mind while making decisions that effect so many young lives.

Appendix A

(Experimenter sits across the table from the subject. The video camera is focused on the table. The white bristol board is in position, on the table, before the subject enters the room).

E. says: "I want you to help me find out how 3-(4-,5-) year-olds think. So I want you to pay close attention and think real hard about the things I ask you."

<u>Length Pretest</u>

(Place red sticks side by side and perpendicular to the base of the bristol board so that the bottom ends of the sticks are in alignment with one another and the base of the board).

E. says: "Give me the big stick."

(Repeat procedure with green sticks then proceed to angle pretest).

Appendix B

Angle Pretest

(Place red angles apart from one another, but side by side so that one leg of each angle is parallel to the left side of the bristol board and the other leg is aligned with the top of the board so that the angles open toward the subject).

E. says: "Point to the big corner."

(Repeat procedure with green angles then proceed to block sorting pretests).

Appendix C

Block Sorting Pretests

(Place pile of scrambled size (shape) blocks in front of subject. Behind the pile, but within reach of the subject place the two (three) transparent plastic containers).

E. says: "Put together the blocks that belong together. Put them in separate boxes."

(If the subject does not sort the blocks correctly take the blocks out and demonstrate how they should be sorted).

E. says: "These (hold up one type of block) belong in one box and these (hold up the other type of block) belong in the other box."

(Take the blocks out and scramble the pile. Ask the subject to sort them again.)

E. says: "Put together the blocks that belong together. Put them in separate boxes."

(Repeat procedure with shape blocks then continue with puzzle board pretest).

Appendix D

Puzzle Board Pretest

(Place puzzle board #1 approximately 1 in. from the base of the board.)

E. says: "These are some of my friends. They come from a place where they do things kind of different. When they go to bed they like to sleep with blankets that 'fit just right'. They like to have their heads covered up and their toes covered up. Here's a blanket." (Pick up target blanket and place on top of circle character.) "Does this blanket fit just right." (Correct subject if wrong). " Let's try this one." (Place target blanket on square.) "Does this blanket fit just right." (Correct subject if wrong). "Let's try this one." (Place target blanket on triangle.) "Does this blanket fit just right". (Correct subject if wrong) "Now I'm going to get some more of my friends and I want you to tell me if the blanket fits just right.

(Place puzzle board #2 approximately 1 in. from the base of the board. Place the target blanket on each character. Ask if the blanket fits just right for each character. Do not correct subjects if they are wrong. Repeat the procedure with puzzle boards 3, and 4. After completion of puzzle boards, continue with the area-matching task.)

Appendix E

Area Matching Task

(Place the triangle (square) character in front of the subject approximately 3 in. from the base of the white bristol board.)

E. says: "This is another friend. When she/he goes to bed, she/he likes a blanket that 'fits just right.' (The choice of she/he is consistent with the sex of the child.) "Remember, they like to have their heads covered up and their toes covered up. And they don't like a blanket with any extra. Here is a blanket." (Give a blanket to the subject making sure the blanket is not flipped or rotated). "You put my friend to bed and tell me if the blanket 'fits just right.'" (If the subject appears to have made a decision but does not articulate their judgment, prompt them by asking:) "Does that blanket 'fit just right?'"

(Continue giving the subject the rest of the blankets in the assigned random order. Make sure a judgment about each blanket is elicited. Repeat procedure with the other shape. After both shape conditions have been completed continue with side-matching task).

Appendix F

Side Matching Task

(Place mounted pair of triangles (quadrilaterals) approximately 3 in. from the base of the bristol board. Present cards consecutively in the assigned random order).

E. says: "This is another one of my friends. Remember, they like blankets that are exactly the same as they are. Is my friend lying next to a blanket that 'fits just right?'"

(After all five pairs with one shape (e.g., triangles) are presented, repeat procedure with other shape (e.g., quadrilaterals)).

Appendix G

Pilot Study

Seven children, three 3-year-olds, two 4-year-olds and two 5-year-olds, participated in the pilot study. The purpose of the pilot study was to make a final determination about the coding scheme for all of the variables and to test out the procedures. After testing two 4-year-olds and one 5-year-old it was determined that several adjustments in the procedure needed to be made.

First, the testing session was much to long for one sitting. After about ten or fifteen minutes the children lost concentration and became easily distracted. Splitting the testing into two separate intervals resolved the problem. The first session consisted of the pretests, the second the tasks. It was decided that both sessions should take place on the same day.

The second was a problem involving the procedure for the puzzle board pretest. Originally, the task was designed to demonstrate and test the childs' understanding of 'fits just right'. One puzzle board with feedback on the accuracy of the child's answer was not sufficient experience with the phrase for the children. They did not know all of the parameters of what fitting meant and as a result when presented with shapes that were similar they were often wrong. Although these subjects seemed to exhibit understanding of the tasks they exhibited anxiety, not really being sure of what was expected of them. It was felt that not giving the children feedback on the puzzle board pretest could potentially backfire on the tasks especially with the youngest group. Since the purpose of the pretest was essentially to familiarize the subject with the phrase 'fits just right' and the tasks were designed to test the parameters of this understanding in a different context it was decided that feedback should be given on all of the puzzle boards. To further familiarize the subjects a fifth puzzle board was developed. The rest of the subjects tested in the pilot test were given feedback. Even though they scored similarly (with exception of the 3-year-olds) to the initial subjects they seemed more confident with their answers. Subjects participating in the pilot study were not included in the final analyses.

Appendix H

Reliability Measures by Pretests, Tasks and Strategy

Variables	Reliability Measures: Agreements	Disagreements	Total %
Pretest			
Length	23	1	
Angle	24	0	
Size	19	0	
Shape	15	1	
Puzzle Board	57	3	
Sub Total	138	5	96.5%
Area-Matching Tasks			
Triangle	58	2	
Square	59	1	
Sub Total	117	3	97.5%
Side-Matching Tasks			
Triangle	59	1	
Square	59	1	
Sub Total	118	2	98.3%
Strategy on Area-Matching Tasks			
Triangle	55	5	
Square	52	8	
Sub Total	107	13	89.2%
Grand Total	480	23	95.4%

Appendix I

Abbreviations and Descriptions of Continuous Dependent Variables

Abbreviation	Description
AMTT	Congruence rule for the area-matching triangle task.
ESTAMT	Estimation rule for the area-matching triangle task.
OSAMT	One-side equal rule for the area-matching triangle task.
COVAMT	Cover rule for the area-matching triangle task.
SHAMT	Same shape rule for the area-matching triangle task.
AMSQ	Congruence rule for the area-matching square task.
ESTAMSQ	Estimation rule for the area-matching square task.
OSAMSQ	One-side equal rule for the area-matching square task.
COVAMSQ	Cover rule for the area-matching square task.
SHAMSQ	Same shape rule for the area-matching square task.
SMT	Congruence rule for the side-matching triangle task.
ESTSMT	Estimation rule for the side-matching triangle task.
OSSMT	One-side equal rule for the side-matching triangle task.

COVSMT	Cover rule for the side-matching triangle task.
SHSMT	Same shape rule for the side-matching triangle task.
SMSQ	Congruence rule for the side-matching square task.
ESTSMSQ	Estimation rule for the side-matching square task.
OSSMSQ	One-side equal rule for the side-matching square task.
COVSMSQ	Cover rule for the side-matching square task.
SHSMSQ	Same shape rule for the side-matching square task.

Appendix J

CODING MANUAL

Variable Label	Description
RECNO	Record number - Number assigned to subject. Range is 1-125.
AGEGRP	Age Group - Approximately three months either way of 3-4-5-years old.
SEX	Gender of Subject - 0=Male 1=Female.
ORDER	Task and Shape Order - AMT: Area-Matching Triangle; AMSQ: Area-Matching Square; SMT: Side-Matching Triangle; SMSQ: Side-Matching Square. 1= AMT; AMSQ; SMT; SMSQ 2= AMSQ; AMT; SMSQ; SMT 3= SMT; SMSQ; AMT; AMSQ 4= SMSQ; SMT; AMSQ; AMT
DOBYR	Date of Birth - Year.
DOBMO	Date of Birth - Month.
DOBDA	Date of Birth - Day.
TESTYR	Date of Testing - Year.
TESTMO	Date of Testing - Month.
TESTDA	Date of Testing - Day.

CODING MANUAL CONT'D

Variable Label	Description
LENGTH	Length Pretest - For trial one the correct choice is placed to subject's left. For trial two the correct choice is placed to subject's right. 2= Subject is correct on both trials. 1= Subject is correct on one trial. 0= Subject is incorrect on both trials.
ANGLE	Angle Pretest - For trial one the correct choice is placed to subject's left. For trial two the correct choice is placed to subject's right. 2= Subject is correct on both trials. 1= Subject is correct on one trial. 0= Subject is incorrect on both trials.
SIZE	Size Pretest - Sorting blocks by size. Codes of sorting behaviors for both trial one and trial two. Levels 1 and 2 - <u>Trial and Error: Syncretic Thinking</u> Levels 1 and 2 represent little or no discrimination on the part of the subject and differ from one another by degree of execution. For level 1 not all of the blocks are put into the bowls. For level 2 all of the blocks are put into bowls but there is no systematic grouping. Level 1 Sorting Behaviors - Code=1 a) Both bowls are left empty. b) One of each size in one bowl, the other bowl left empty. c) One in each bowl of the same size. d) One in each bowl of different sizes. e) Two of the same or different size in one bowl, two of different sizes in the other bowl. Level 2 Sorting Behaviors - Code=2 a) Four blocks in one bowl, two blocks in the other bowl (the two blocks may be of the same size). b) Five blocks in one bowl, one block in the other bowl.

CODING MANUAL CONT'D

Variable Label | Description

SIZE

Level 3 and 4 - Thinking in Complexes
Levels 3 and 4 represent initial attempts at organizing the blocks into systematic groups.

Level 3 Sorting Behaviors - Code=3
a) Blocks are grouped by shape. All six balls are put into one bowl.

Level 4 Sorting Behaviors - Code=4
a) Blocks are grouped by number. Three blocks in one bowl, three blocks in the other bowl, sizes are mixed.
b) Blocks are separated by size but directions are not followed completely. Three of one shape in one bowl or distributed between two bowls, three of the other shape left in the tray.

Level 5 - No level 5 for size pretest.

Level 6 - Proto-Concepts
Proto-concepts represent the highest level of thinking prior to true conceptual thinking. Proto-concepts are in the sphere of perception or practical, action bound thinking.

Level 6 Sorting Behavior - Code=6
a) Blocks are correctly grouped by size. Three small blocks are in one bowl, three large blocks are in the other bowl.

SHAPE

Shape Pretest - Sorting blocks by shape.
Codes of sorting behaviors for both trial one and trial two.

Levels 1 and 2 - Trial and Error: Syncretic Thinking
Levels 1 and 2 represent little or no discrimination on the part of the subject and differ from one another by degree of execution. For level 1 not all of the blocks are put into the bowls. For level 2 all of the blocks are put into bowls but there is no systematic grouping. Graphic collections are also included in level 1 or 2.

CODING MANUAL CONT'D

Variable Label	Description

SHAPE

Level 1 Sorting Behaviors - Code=1
a) All three bowls are left empty.
b) Five blocks in one bowl, the other two bowls are left empty.
c) One block in one bowl, one block in another bowl, one bowl is left empty or two blocks in one bowl, two blocks in another bowl, and the third bowl is left empty.

d) One block per bowl, including one of each kind. See code 4 below for exception.

Level 2 Sorting Behaviors - Code=2
a) The following outcome patterns are coded as level 2 sorting behaviors:
1,8,0; 5,4,0; 5,2,2; 7,1,1; 4,4,1; 2,3,4 (see code 4 below for exception); 5,3,1; 5,4,0; 2,1,6.

Level 3 and 4 - Thinking in Complexes
Levels 3 and 4 represent initial attempts at organizing the blocks into systematic groups.

Level 3 Sorting Behaviors - Code=3
a) Blocks are grouped by size. All nine balls are put into one bowl.

Level 4 Sorting Behaviors - Code=4
a) Blocks are grouped by number. Three blocks in each bowl, shapes are mixed. See Code 5 for exception.
b) The balls are separated from the triangles and the squares. Particular behavioral manifestations of this are as follows:
1) Three balls in one bowl, six blocks (triangles and squares) in another bowl, the third bowl is left empty.
2) A pair of triangles and squares are in each bowl, the balls are left in the tray.
3) One ball is in each bowl, the triangles and squares are left in the tray.

Level 5 Sorting Behaviors - Code=5
a) Three blocks, one of each shape, is in each bowl.

CODING MANUAL CONT'D

Variable Label	Description
SHAPE	Level 6 - Proto-Concepts Proto-concepts represent the highest level of thinking prior to true conceptual thinking. Proto-concepts are in the sphere of perception or practical, action bound thinking. Level 6 Sorting Behavior - Code=6 a) Blocks are correctly grouped by shape. Three balls are in one bowl, three triangles are in the another bowl, and three squares are in another bowl.
PB1-PB5	Puzzle Boards 1,2,3,4,5. 3= Spontaneously correctly responds for all three characters. 2= Spontaneously correctly responds for two of the three characters. 1= Spontaneously correctly responds for one of the three characters. 0= Incorrectly responds for all three characters.
AMT1	Congruent shape for Area-Matching Triangle Trial.
AMT2	Larger shape for Area-Matching Triangle Trial.
AMT3	Smaller shape for Area-Matching Triangle Trial.
AMT4	Slightly different shape for Area-Matching Triangle Trial.
AMT5	Much different shape for Area-Matching Triangle Trial. Coding for AMT1 - AMT5 2 =Subject indicates that the blanket fits just right. 1 =Subject indicates that the blanket does not fit just right. 0 =Subject's response is ambiguous or for some other reason cannot be coded as a 1 or 2.

CODING MANUAL CONT'D

<u>Variable Label</u> <u>Description</u>

AMT1S-AMT5S - Matching strategies used by subject with each trial shape.

Coding for AMT1S - AMT5S

1 = Visual Inspection.

2 = Subject side-matches non-corresponding sides.

3 =Subject side-matches corresponding sides.

4 =Corresponding sides are not aligned but subject places the blanket on top of the character partially covering the character or places character on top of blanket.

5 =Corresponding sides aligned but incomplete superimposing. This category is used for coding the congruent shape only.

6 = Correct complete superimposing.

CODING MANUAL CONT'D

Variable Label — Description

AMSQ1 Congruent shape for Area-Matching Square Trial.

AMSQ2 Larger shape for Area-Matching Square Trial.

AMSQ3 Smaller shape for Area-Matching Square Trial.

AMSQ4 Slightly different shape for Area-Matching Square Trial.

AMSQ5 Much different shape for Area-Matching Square Trial.

Coding for AMSQ1 - AMSQ5

2 =Subject indicates that the blanket fits just right.
1 =Subject indicates that the blanket does not fit just right.
0 =Subject's response is ambiguous or for some other reason cannot be coded as a 1 or 2.

AMSQ1S-AMSQ5S - Matching strategies used by subject with each trial shape.

Coding for AMSQ1S - AMSQ5S

1 = Visual Inspection.

2 = Subject side-matches non-corresponding sides.

3 =Subject side-matches corresponding sides.

4 =Corresponding sides are not aligned but subject places the blanket on top of the character partially covering the character or places character on top of blanket.

5 =Corresponding sides aligned but incomplete superimposing. This category is used for coding the congruent shape only.

6 = Correct complete superimposing.

CODING MANUAL CONT'D

Variable Label	Description
SMT1	Congruent shape for Side-Matching Triangle Trial.
SMT2	Larger shape for Side-Matching Triangle Trial.
SMT3	Smaller shape for Side-Matching Triangle Trial.
SMT4	Slightly different shape for Side-Matching Triangle Trial.
SMT5	Much different shape for Side-Matching Triangle Trial.
	Coding for SMT1 - SMT5
	2 =Subject indicates that the blanket fits just right. 1 =Subject indicates that the blanket does not fit just right. 0 =Subject's response is ambiguous or for some other reason cannot be coded as a 1 or 2.
SMSQ1	Congruent shape for Side-Matching Square Trial.
SMSQ2	Larger shape for Side-Matching Square Trial.
SMSQ3	Smaller shape for Side-Matching Square Trial.
SMSQ4	Slightly different shape for Side-Matching Square Trial.
SMSQ5	Much different shape for Side-Matching Square Trial.
	Coding for SMSQ1 - SMSQ5
	2 =Subject indicates that the blanket fits just right. 1 =Subject indicates that the blanket does not fit just right. 0 =Subject's response is ambiguous or for some other reason cannot be coded as a 1 or 2.

CODING MANUAL CONT'D

Variable Label	Description
POSTQUEX	Post-test question responses. 1 =Subject changes mind and says blanket does/doesn't fit just right, but does not give reason. 2 =Subject changes mind and says blanket does/doesn't fit just right, and does give a reason - e.g., too big. 3 = Subject does not change mind and does not give reason. 4 =Subject does not change mind and gives a reason.

Appendix K - **MASTER ORDER LIST 3F1**

#	FNAME	AGE	SEX	TASK ORDER	SHAPE ORDER	O#
1	____________	3	F	1. AM 2. SM	1. T 2. SQ	1

Trial Order

1. AM-T __ - __ - __ - __ - __ 2. AM-SQ __ - __ - __ - __ - __

3. SM-T __ - __ - __ - __ - __ 4. SM-SQ __ - __ - __ - __ - __

Test Site ____________________ Date ______________ Tape # ___________

#	FNAME	AGE	SEX	TASK ORDER	SHAPE ORDER	O#
2	____________	3	F	1. AM 2. SM	1. T 2. SQ	1

Trial Order

1. AM-T __ - __ - __ - __ - __ 2. AM-SQ __ - __ - __ - __ - __

3. SM-T __ - __ - __ - __ - __ 4. SM-SQ __ - __ - __ - __ - __

Test Site ____________________ Date ______________ Tape # ___________

#	FNAME	AGE	SEX	TASK ORDER	SHAPE ORDER	O#
3	____________	3	F	1. AM 2. SM	1. T 2. SQ	1

Trial Order

1. AM-T __ - __ - __ - __ - __ 2. AM-SQ __ - __ - __ - __ - __

3. SM-T __ - __ - __ - __ - __ 4. SM-SQ __ - __ - __ - __ - __

Test Site ____________________ Date ______________ Tape # ___________

#	FNAME	AGE	SEX	TASK ORDER	SHAPE ORDER	O#
4	________	3	F	1. AM 2. SM	1. T 2. SQ	1

Trial Order

1. AM-T __ - __ - __ - __ - __ 2. AM-SQ __ - __ - __ - __ - __

3. SM-T __ - __ - __ - __ - __ 4. SM-SQ __ - __ - __ - __ - __

Test Site ______________ Date ____________ Tape # __________

#	FNAME	AGE	SEX	TASK ORDER	SHAPE ORDER	O#
5	________	3	F	1. AM 2. SM	1. T 2. SQ	1

Trial Order

1. AM-T __ - __ - __ - __ - __ 2. AM-SQ __ - __ - __ - __ - __

3. SM-T __ - __ - __ - __ - __ 4. SM-SQ __ - __ - __ - __ - __

Test Site ______________ Date ____________ Tape # __________

Appendix L - SUBJECT PROTOCOL

RECNO [ ][ ][ ] 1 2 3 AGEGRP [ ] 4 SEX [ ] 5 ORDER [ ] 6

DOBYR [ ][ ] 7 8 DOBMO [ ][ ] 9 10 DOBDA [ ][ ] 11 12

TESTYR [ ][ ] 13 14 TESTMO [ ][ ] 15 16 TESTDA [ ][ ] 17 18

PRETESTS

LENGTH [ ] 19 ANGLE [ ] 20 SIZE T1 T2 [ ][ ] 21 22 SHAPE T1 T2 [ ][ ] 23 24

PB1 [ ] 25 PB2 [ ] 26 PB3 [ ] 27 PB4 [ ] 28 PB5 [ ] 29

TASKS

AREA-MATCHING

AMT1 [ ] 30 AMT2 [ ] 31 AMT3 [ ] 32 AMT4 [ ] 33 AMT5 [ ] 34

AMT1S [ ] 35 AMT2S [ ] 36 AMT3S [ ] 37 AMT4S [ ] 38 AMT5S [ ] 39

AMSQ1 [ ] 40 AMSQ2 [ ] 41 AMSQ3 [ ] 42 AMSQ4 [ ] 43 AMSQ5 [ ] 44

AMSQ1S [ ] 45 AMSQ2S [ ] 46 AMSQ3S [ ] 47 AMSQ4S [ ] 48 AMSQ5S [ ] 49

SIDE-MATCHING

SMT1 [ ] 50 SMT2 [ ] 51 SMT3 [ ] 52 SMT4 [ ] 53 SMT5 [ ] 54

SMSQ1 [ ] 55 SMSQ2 [ ] 56 SMSQ3 [ ] 57 SMSQ4 [ ] 58 SMSQ5 [ ] 59

Appendix M
Letter of Consent

Dear Parent/Guardian,

I am conducting my dissertation research for the City University Graduate Center in your child's school and am asking your permission to include your child in my study. The study is about early forms of mathematical reasoning. The tasks use fun materials and last about 20 minutes. The session will be video-taped since I will be the only one administering the tasks and will be unable to code the responses at the same time. I have found in previous studies that children enjoy participating in this type of research.

A child's participation is voluntary and he or she may withdraw at any time during my work with them. No personal information about the children (name, etc.) will be used in the research. All information gathered will be treated as confidential.

I very much appreciate your cooperation with my efforts and hope you will approve of your child's participation in this study. Please fill out the information about your child in accord with your wishes, sign and return this form as soon as possible. If you have any questions about the study, please feel free to contact me. I am available every afternoon.

Thank you
Barbara Blank

Child's Name ______________________________

Date of Birth ☐☐ ☐☐ ☐☐☐☐
Month Day Year

Teacher's Name ___________________________

☐ I give permission for my child to participate.

☐ I do not wish my child to participate.

Signature __________________________ Date________________

Appendix N
Means and Initial Univariate Statistics

Cell Means and Standard Deviations

Variable .. AMT AGG AMT CORRECT RULE

FACTOR	CODE	Mean	Std. Dev.	N
AGEGRP	3			
SEX	0	2.250	1.164	20
SEX	1	2.500	1.357	20
AGEGRP	4			
SEX	0	3.200	1.056	20
SEX	1	3.600	.995	20
AGEGRP	5			
SEX	0	4.150	1.040	20
SEX	1	4.300	.923	20
For entire sample		3.333	1.324	120

Variable .. AMSQ AGG AMSQ CORRECT RULE

FACTOR	CODE	Mean	Std. Dev.	N
AGEGRP	3			
SEX	0	2.250	.851	20
SEX	1	2.900	1.021	20
AGEGRP	4			
SEX	0	3.050	.759	20
SEX	1	3.400	1.046	20
AGEGRP	5			
SEX	0	4.500	.827	20
SEX	1	4.200	.696	20
For entire sample		3.383	1.154	120

Variable .. SMT AGG SMT CORRECT RULE

FACTOR	CODE	Mean	Std. Dev.	N
AGEGRP	3			
SEX	0	1.950	1.191	20
SEX	1	2.350	1.268	20
AGEGRP	4			
SEX	0	2.550	1.099	20
SEX	1	2.600	1.095	20
AGEGRP	5			
SEX	0	3.150	.671	20
SEX	1	2.850	.587	20
For entire sample		2.575	1.066	120

Variable .. SMSQ FACTOR	CODE	AGG SMSQ CORRECT RULE Mean	Std. Dev.	N
AGEGRP	3			
SEX	0	2.150	1.387	20
SEX	1	2.500	1.100	20
AGEGRP	4			
SEX	0	2.800	1.196	20
SEX	1	2.900	1.165	20
AGEGRP	5			
SEX	0	3.700	.801	20
SEX	1	3.400	.754	20
For entire sample		2.908	1.188	120

Variable .. MCAMT FACTOR	CODE	AGG AMT MCOVER RULE 1+2+4 Mean	Std. Dev.	N
AGEGRP	3			
SEX	0	3.500	1.051	20
SEX	1	3.450	.686	20
AGEGRP	4			
SEX	0	3.650	.988	20
SEX	1	3.750	.786	20
AGEGRP	5			
SEX	0	3.800	.696	20
SEX	1	3.800	.523	20
For entire sample		3.658	.804	120

Variable .. MCAMSQ FACTOR	CODE	AGG AMSQ MCOVER RULE 1+2+4 Mean	Std. Dev.	N
AGEGRP	3			
SEX	0	4.000	.795	20
SEX	1	4.150	.933	20
AGEGRP	4			
SEX	0	4.450	.605	20
SEX	1	3.850	.587	20
AGEGRP	5			
SEX	0	4.050	.510	20
SEX	1	3.950	.394	20
For entire sample		4.075	.676	120

Variable .. OSSMT		AGG ONSIDE SM T 1+4+5		
FACTOR	CODE	Mean	Std. Dev.	N
AGEGRP	3			
SEX	0	3.150	1.040	20
SEX	1	3.550	1.050	20
AGEGRP	4			
SEX	0	3.650	1.424	20
SEX	1	3.500	1.318	20
AGEGRP	5			
SEX	0	4.350	1.040	20
SEX	1	4.650	.671	20
For entire sample		3.808	1.211	120

Variable .. MOSSMSQ		AGG MONSIDE SM SQ 1+4+/-5		
FACTOR	CODE	Mean	Std. Dev.	N
AGEGRP	3			
SEX	0	3.400	.821	20
SEX	1	3.800	.951	20
AGEGRP	4			
SEX	0	3.850	1.040	20
SEX	1	3.800	.894	20
AGEGRP	5			
SEX	0	4.600	.754	20
SEX	1	4.750	.550	20
For entire sample		4.033	.961	120

EFFECT .. AGEGRP BY SEX

Univariate F-tests with (2,114) D. F.

Variable	Hypoth. SS	Error SS	Hypoth. MS	Error MS	F	Sig. of F
AMT	.31667	137.50000	.15833	1.20614	.13127	.877
AMSQ	4.71667	87.50000	2.35833	.76754	3.07257	.050
SMT	2.45000	118.35000	1.22500	1.03816	1.17997	.311
SMSQ	2.15000	135.55000	1.07500	1.18904	.90409	.408
MCAMT	.11667	74.65000	.05833	.65482	.08908	.915
MCAMSQ	2.91667	49.95000	1.45833	.43816	3.32833	.039
OSSMT	1.71667	142.15000	.85833	1.24693	.68836	.504
MOSSMSQ	1.01667	82.30000	.50833	.72193	.70413	.497

EFFECT .. SEX

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Univariate F-tests with (1,114) D. F.

Variable	Hypoth. SS	Error SS	Hypoth. MS	Error MS	F	Sig. of F
AMT	2.13333	137.50000	2.13333	1.20614	1.76873	.186
AMSQ	1.63333	87.50000	1.63333	.76754	2.12800	.147
SMT	.07500	118.35000	.07500	1.03816	.07224	.789
SMSQ	.07500	135.55000	.07500	1.18904	.06308	.802
MCAMT	.00833	74.65000	.00833	.65482	.01273	.910
MCAMSQ	1.00833	49.95000	1.00833	.43816	2.30130	.132
OSSMT	1.00833	142.15000	1.00833	1.24693	.80865	.370
MOSSMSQ	.83333	82.30000	.83333	.72193	1.15431	.285

EFFECT .. AGEGRP

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Univariate F-tests with (2,114) D. F.

Variable	Hypoth. SS	Error SS	Hypoth. MS	Error MS	F	Sig. of F
AMT	68.71667	137.50000	34.35833	1.20614	28.48618	.000
AMSQ	64.51667	87.50000	32.25833	.76754	42.02800	.000
SMT	14.45000	118.35000	7.22500	1.03816	6.95944	.001
SMSQ	30.21667	135.55000	15.10833	1.18904	12.70638	.000
MCAMT	2.21667	74.65000	1.10833	.65482	1.69257	.189
MCAMSQ	.45000	49.95000	.22500	.43816	.51351	.600
OSSMT	29.71667	142.15000	14.85833	1.24693	11.91593	.000
MOSSMSQ	25.71667	82.30000	12.85833	.72193	17.81106	.000

EFFECT .. CONSTANT

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Univariate F-tests with (1,114) D. F.

Variable	Hypoth. SS	Error SS	Hypoth. MS	Error MS	F	Sig. of F
AMT	1333.33333	137.50000	1333.33333	1.20614	1105.45455	.000
AMSQ	1373.63333	87.50000	1373.63333	.76754	1789.64800	.000
SMT	795.67500	118.35000	795.67500	1.03816	766.42966	.000
SMSQ	1015.00833	135.55000	1015.00833	1.18904	853.64035	.000
MCAMT	1606.00833	74.65000	1606.00833	.65482	2452.57803	.000
MCAMSQ	1992.67500	49.95000	1992.67500	.43816	4547.84685	.000
OSSMT	1740.40833	142.15000	1740.40833	1.24693	1395.75484	.000
MOSSMSQ	1952.13333	82.30000	1952.13333	.72193	2704.04860	.000

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