COUNTABLE SHORT RECURSIVELY SATURATED MODELS OF ARITHMETIC

by

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Abstract

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Advisor: Professor Roman Kossak

Short recursively saturated models of arithmetic are exactly the elementary initial segments of recursively saturated models of arithmetic. Since any countable recursively saturated model of arithmetic has continuum many elementary initial segments which are already recursively saturated, we turn our attention to the (countably many) initial segments which are not recursively saturated. We first look at properties of countable short recursively saturated models of arithmetic and show that although these models cannot be cofinally resplendent (an expandability property slightly weaker than resplendency), these models have non-definable expansions which are still short recursively saturated.

In this thesis we also investigate properties of the automorphism groups of countable short recursively saturated models of arithmetic. In particular, we show that Kaye's Theorem concerning closed normal subgroups of the automorphism groups of countable recursively saturated models of arithmetic applies to countable short recursively saturated models as well. That is, the closed normal subgroups of countable short recursively saturated models of PA are exactly the stabilizers of the invariant cuts of the models. This result, among other results in this thesis, is used to show that there are several kinds of countable short recursively saturated models of arithmetic whose automorphism groups are not isomorphic (as topological groups) to each other.

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Introduction

The study of recursively saturated models has been a major area of interest for model theorists for the past few decades. The fact that every model has a recursively saturated elementary extension of the same cardinality allows one to use the power of recursive saturation in studying any first order structure. This is more apparent when one considers countable models, and in particular, countable models of PA, as will be discussed below and in Section 1.2.

The notion of recursive saturation was introduced in the 1970's by Barwise and Schlipf [BS75], and independently Ressayre [Res77]. A model is recursively saturated if it realizes all finitely realized recursive types with a finite number of parameters (we say that a type $\{\phi_n(x,\bar{a}) : n \in \mathbb{N}\}$ is recursive if the set $\{\ulcorner\phi_n(x,\bar{y})\urcorner : n \in \mathbb{N}\}$ is recursive, where $\ulcorner\phi(x,\bar{y})\urcorner$ is the Gödel number of the formula $\phi(x,\bar{y})$). Since every model has a recursively saturated elementary extension of the same cardinality, it follows that every complete consistent theory in a finite language has countable recursively saturated models.

Barwise, Schlipf, and Ressayre showed that countable models are recursively saturated if and only if they have a stronger property called resplendency (see Section 1.2). Another important result concerning recursive saturation is that countable models of PA admit nonstandard satisfaction classes (see also Section 1.2) if and only if they are recursively saturated ([Lac81] and [KKL81]). In the past few decades there have been many results by Kaye, Kossak, Kotlarski, Schmerl, and Smoryński, to name a few, regarding the structure of recursively saturated models of arithmetic and their automorphisms.

Countable recursively saturated models of arithmetic have continuum many elementary initial segments. All but countably many are recursively saturated. The ones that are not recursively saturated are short recursively saturated. A model is short recursively saturated if it realizes all finitely realized *bounded* recursive types with finitely many parameters (a type $p(x)$ is bounded if it contains the formula $x < a$ where a is an element of the model). These models are discussed in Kossak [Kos83], Kotlarski [Kot83] [Kot84], Lesan [Les78], and Smoryński [Smo81], but other than that not much has been done with regard to short recursively saturated models of PA (Although recent work of Nurkhaidarov [Nur06] deals with initial segments of recursively saturated models of PA, and thus indirectly, with initial segments of short recursively saturated models of PA). The fact that these models do not admit a "shorter" version of resplendency (see Section 2.1), makes them not as "attractive" as their taller "cousins" (as my advisor referred to them once), the recursively saturated models. However, because these models are elementary initial segments of recursively saturated models, studying short recursively saturated models will help us understand better their recursively saturated extensions. Also, while up to isomorphism, countable recursively saturated models of arithmetic are uniquely determined by their complete theory and standard system (see Section 1.2), there are countably many non-isomorphic short recursively saturated models that have the same theory as well as the same standard system. The differences between short recursively saturated models which have the same recursively saturated elementary end extension will be explored in this thesis. In addition to that, we investigate the differences and similarities between countable recursively saturated models and their short recursively saturated elementary initial segments. A special emphasis will be given to the differences between the automorphism groups of the different models mentioned above.

In the first chapter of this thesis we give some background information regarding models of arithmetic, recursively saturated models of arithmetic, and short recursively saturated models of arithmetic. Aside from some new observations most of the material is taken from previous works.

In Chapter 2 we prove some results about short recursively saturated models of arithmetic. We present a full proof of the fact that short recursively saturated models of PA are not cofinally resplendent (Section 2.1). This fact was discovered by Solovay, but until now a proof of this fact was only outlined [Smo82a]. The second section of this chapter gives a positive result. We show that although short recursively saturated models of arithmetic are not cofinally resplendent, they have nontrivial expansions which are short recursively saturated. This can be seen as a weak instance of chronic resplendency of countable recursively saturated models.

Automorphisms of models in general have been studied extensively in the past 25 years (see [Hod93]). The automorphism group of a model gives us more insight about the structure of the model. For example, the only elements that are fixed by all automorphisms of a recursively saturated model of PA are the definable elements. In contrast, some short recursively saturated models of PA have non definable elements that are fixed by all automorphisms of the model (see Proposition 3.3.6).

Chapter 3 begins with some background material on the automorphism groups of models of PA, in particular, the automorphism groups of recursively saturated models of PA, as well as a topology on these groups. The last section of this chapter introduces new material, mainly observations about the automorphism groups of short recursively saturated saturated models of PA and their topology.

In the first section of Chapter 4 we discuss a subgroup of the automorphism group of a short recursively saturated model of PA. This subgroup arises from restricting the automorphism group of a recursively saturated elementary end extension of the model to the short recursively saturated model. In the following two sections we prove more facts concerning the automorphism groups of short recursively saturated models of arithmetic, culminating in the extension of Kaye's theorem on closed normal subgroups of automorphism groups of recursively saturated models of arithmetic to closed normal subgroups of automorphism groups of short recursively saturated models of arithmetic. This result enables us to show that there are non-isomorphic automorphism groups (as topological groups) of short recursively saturated models of arithmetic. The last section of this chapter discusses differences between two types of short recursively saturated models, those that are short arithmetically saturated (see definition in Section 1.3) and those that are not.

The thesis concludes (Chapter 5) with a list of questions and remarks.

Chapter 1

Background

I will assume that the reader is familiar with the basic facts of Model Theory, in particular compactness theorem, completeness theorem, Skolem-Löwenheim theorems, elementary chain lemma, types, and saturation. Chang and Keisler [CK77] is a good book for reference. I will also assume that the reader is familiar with the axioms of PA, Skolem terms, definability, cofinal and end extensions, cuts and overspill. The book by Kaye [Kay91] is a good reference.

The first section of this chapter is devoted to notation. The last two sections are an exposition of topics discussed in the thesis. In the second section we discuss recursively saturated models of PA and other facts related to recursive saturation. In the last section we give an introduction to short recursively saturated models of PA and list important results about these models.

1.1 Notation

In this section, let M be a model of PA.

Let I be a subset of M. If for every $x \in I$ and for every $y \in M$, $y < x \to y \in I$, we shall write $I \subseteq_{end} M$. In this case, I is said to be an *initial segment* of M, and M is an *end extension* of I. An initial segment closed under the successor function is called a *cut*.

We will use the following notation from [Kay94]. Let 2^a_n be defined inductively by $2^a_0 = a$ and $2^a_{n+1} = 2^{2^a_n}$. Let $\log^n x$ be defined inductively by $\log^0 x = x$ and $\log^{n+1} x = \lfloor \log_2(\log^n x + 1) \rfloor$. We define the following cuts:

$$2^a_{\mathbb{N}} = \{x : x < 2^a_n \text{ for some } n \in \mathbb{N}\},$$

and

$$\log^{\mathbb{N}} a = \{x : x < \log^n x \text{ for all } n \in \mathbb{N}\}.$$

Let J be a subset of M. If for every $x \in M$ there is a $y \in J$ with $y \geq x$, then we write $J \subseteq_{cof} M$ and say that J is *cofinal* in M.

Let N be a submodel of M. If N is an elementary submodel of M, we will write $N \preceq M$ (If N is a proper subset, we will write $N \prec M$).

$N \preceq_{end} M$ will denote that M is an elementary end extension of N. In this case, N is an elementary initial segment (or, equivalently, an elementary cut) of M.

$N \preceq_{cof} M$ will denote that M is an elementary cofinal extension of N.

Let Γ be a collection of formulas (for example, Σ_n or Π_n). By $N \preceq_\Gamma M$ we denote that N is a submodel of M, and that for any $\phi(\bar{x}) \in \Gamma$, and any $\bar{a} \in N$, $N \models \phi(\bar{a}) \longleftrightarrow M \models \phi(\bar{a})$.

$\mathrm{Th}(M)$ is the collection of all sentences true in M. $\mathrm{Th}_\Gamma M$ is the collection of all Γ sentences that are true in M.

A type $p(x)$ is a collection of formulas in the variable x. A type is recursive if the set of Gödel numbers of the formulas in the type is recursive. The type of an element $a \in M$, $\mathrm{tp}_M(a)$, is the collection of all formulas $\phi(x)$, such that $M \models \phi(a)$. If there is no reason for ambiguity, we shall write $\mathrm{tp}(a)$ instead of $\mathrm{tp}_M(a)$.

Let $\bar{a} \in M$. A Skolem term $t_\phi(\bar{a})$ is a term defined by the formula $\phi(\bar{y}, x)$ in the following manner:

$$t_\phi(\bar{a}) = \begin{cases} \min\{x : \phi(\bar{a}, x)\} & \text{if } \exists x \phi(\bar{a}, x), \\ 0 & \text{otherwise.} \end{cases}$$

By $\mathrm{Scl}^M(\bar{a})$ we denote the Skolem closure of $\bar{a}$ in M, that is, $\mathrm{Scl}^M(\bar{a}) = \{t(\bar{a}) : t$ is a Skolem term$\}$. If there is no reason for ambiguity, we shall write $\mathrm{Scl}(\bar{a})$.

By $\mathrm{Def}(M)$ we denote the set of all the subsets of M which are definable in M with parameters.

A subset of M is said to be M-*finite*, if it is definable and bounded in M. For every $a \in M$ there is a unique M-finite set A, such that

$$a = \sum_{x \in A} 2^x.$$

In this case we will say that a codes A. Also, for any M-finite set A, there is an $a \in M$ such that a codes A. If a codes the set A, we will write $x \in a$ if and only if $x \in A$.

Let $d \in M$. We will say that the sequence $x_0, x_1, ..., x_d$, is coded by a, if a codes the set $\{\langle 0, x_0\rangle, \langle 1, x_1\rangle, ..., \langle d, x_d\rangle\}$ where $\langle x, y\rangle = \frac{1}{2}[(x+y)^2 + 3x + y]$ is the Gödel's pairing function. The length of the sequence coded by a is denoted by $\mathrm{lh}(a)$ and by $(a)_i$ we denote the $i+1$ term in the sequence coded by a. An ω sequence $x_0, x_1, x_2, \ldots$ is said to be coded by c, if c codes a sequence of nonstandard length, and for any $i \in \mathbb{N}$, $(c)_i = x_i$.

The last two notions we introduce in this section are the theories PA^* and ACA_0.

Let $\mathcal{L}_{\mathsf{PA}}$ denote the language of arithmetic. Let $\mathcal{L}^*$ be an extension of $\mathcal{L}_{\mathsf{PA}}$ and M^* be an expansion of M to the language $\mathcal{L}^*$. Then $M^* \models \mathsf{PA}^*$ iff $M \models \mathsf{PA}$ and the induction scheme is true in M^* for all formulas of $\mathcal{L}^*$.

Let $\mathcal{X}$ be a collection of subsets of M. Then $(M, \mathcal{X}) \models \mathsf{ACA}_0$ iff $\mathcal{X}$ is closed under arithmetical definability, and $(M, X_1, ..., X_n) \models \mathsf{PA}^*$, for every $X_1, ..., X_n \in \mathcal{X}$.

1.2 Recursively saturated Models of PA

In this section we give a list of basic facts concerning recursively saturated models of PA.

We start with some definitions.

Definition 1.2.1. The *standard system* of a model M, $\mathrm{SSy}(M)$, is the family of all subsets of $\mathbb{N}$ that are coded in M. That is, $X \in \mathrm{SSy}(M)$ iff $X = \{x \in \mathbb{N} : x \in a\}$ for some $a \in M$. (Equivalently, $X \in \mathrm{SSy}(M)$ iff $X = Y \cap \mathbb{N}$ where $Y = \{x : \phi(x, a)\}$ for some $\phi \in \mathcal{L}_{\mathsf{PA}}$ and $a \in M$.)

For any nonstandard $M \models \mathsf{PA}$, if $X \in \mathrm{SSy}(M)$, then X has arbitrarily small nonstandard codes. Therefore, whenever $N \subseteq_{end} M$ and N is nonstandard, $\mathrm{SSy}(N) = \mathrm{SSy}(M)$.

Definition 1.2.2. Let M be a model of PA. We say that M is $\mathrm{SSy}(M)$-*saturated* iff

1. For every type $p(v, \bar{w})$ whose set of Gödel numbers of formulas in p is in $\mathrm{SSy}(M)$, and for every tuple $\bar{b}$ in M, if $p(v, \bar{b})$ is finitely realized in M, then $p(v, \bar{b})$ is realized in M; and

2. For every $a \in M$, $\mathrm{tp}(a) \in \mathrm{SSy}(M)$.

Recall that a model M is *recursively saturated* iff every recursive type $p(v, \bar{b})$ which is finitely realized in M, is realized in M.

Wilmers (see [Smo81]) proved that $M \models \mathsf{PA}$ is recursively saturated iff it is $\mathrm{SSy}(M)$-saturated. Using this result we can interchange these notions, and many times when proving results about recursively saturated models we will use coded types instead of recursive types.

Another important observation is that every countable recursively saturated model of PA is uniquely determined by its complete theory and its standard system. That is, if M and N are countable recursively saturated models of PA with $\mathrm{Th}(M) = \mathrm{Th}(N)$ and $\mathrm{SSy}(M) = \mathrm{SSy}(N)$, then M and N are isomorphic.

Definition 1.2.3. A family $\mathfrak{X}$ of subsets of $\mathbb{N}$ is called a Scott set if it is

a. closed under Boolean operations;

b. closed under relative recursion; and

c. whenever $T \in \mathfrak{X}$ is an infinite binary tree, then there is an infinite path $P \subseteq T$ in $\mathfrak{X}$.

For every model M of PA, $\mathrm{SSy}(M)$ is a Scott set. Also, for every countable Scott set $\mathfrak{X}$, there is a model N of PA with $\mathrm{SSy}(N) = \mathfrak{X}$ [Fri73]. Moreover, If $\mathfrak{X}$ is a countable Scott set, then for every $T \in \mathfrak{X}$ (that is, the set of Gödel numbers of formulas in T is in $\mathfrak{X}$) with T a consistent complete arithmetic theory extending PA, then there is a countable recursively saturated model M with $\mathrm{SSy}(M) = \mathfrak{X}$ and $\mathrm{Th}(M) = T$.

Countable recursively saturated models of PA have many important properties that are not shared by other countable models of PA.

The following definition and notation are taken from Kaye. The formula $\mathrm{term}(x)$ is an $\mathcal{L}_{\mathsf{PA}}$ formula representing the set of Gödel numbers of terms of PA. The formula $\mathrm{form}(x)$ is an $\mathcal{L}_{\mathsf{PA}}$ formula representing the set of Gödel numbers of formulas of PA. When $t, \phi \in M \models \mathsf{PA}$ are standard numbers, $M \models \mathrm{term}(t)$ if and only if t is the Gödel number of an $\mathcal{L}_{\mathsf{PA}}$ term, and $M \models \mathrm{form}(\phi)$ if and only if ϕ is the Gödel number of an $\mathcal{L}_{\mathsf{PA}}$ formula. By overspill, whenever M is nonstandard, there are nonstandard t and ϕ in which $M \models \mathrm{term}(t)$ and $M \models \mathrm{form}(\phi)$.

By $\mathrm{val}(t, a)$ we denote an $\mathcal{L}_{\mathsf{PA}}$ definable function whose value is b iff t is a term and $t(a) = b$. Suppose $a \in M$ codes a sequence of length $n+1$. Let $i \in M$. If $i \leq n$, then by $a[b/i]$ we denote the element c coding a sequence of length $n+1$ with $(c)_j = (a)_j$ for all $j \neq i$ and $(c)_i = b$. If $i > n$ then $a[b/i]$ is the element c coding the sequence of length $i+i$ with $(c)_j = (a)_j$ for all $j \leq n$, $(c)_i = b$, and if $n < j < i$, $(c)_j = 0$.

For the definitions below, we will identify terms and formulas with their Gödel numbers. We will also assume a Gödel numbering in which a subformula of a formula will always have a smaller Gödel number. Let $\Psi(X, \phi, a)$ be the following formula based on Tarski's definition of truth:

$$[\exists t, s(\mathrm{term}(t) \wedge \mathrm{term}(s) \wedge \phi = (t = s) \wedge \mathrm{val}(t, a) = \mathrm{val}(s, a))] \vee$$

$$[\exists t, s(\mathrm{term}(t) \wedge \mathrm{term}(s) \wedge \phi = (t < s) \wedge \mathrm{val}(t, a) < \mathrm{val}(s, a))] \vee$$

$$[\exists \psi_1, \psi_2(\mathrm{form}(\psi_1) \wedge \mathrm{form}(\psi_2) \wedge \phi = (\psi_1 \wedge \psi_2) \wedge X(\langle \psi_1, a \rangle) \wedge X(\langle \psi_2, a \rangle))] \vee$$

$$[\exists \psi_1, \psi_2(\mathrm{form}(\psi_1) \wedge \mathrm{form}(\psi_2) \wedge \phi = (\psi_1 \vee \psi_2) \wedge (X(\langle \psi_1, a \rangle) \vee X(\langle \psi_2, a \rangle)))] \vee$$

$$[\exists \psi(\mathrm{form}(\psi) \wedge \phi = \neg \psi \wedge \neg X(\langle \psi, a \rangle))] \vee$$

$$[\exists i, \psi(\mathrm{form}(\psi) \wedge \phi = \exists v_i \psi \wedge \exists b X(\langle \psi, a[b/i] \rangle))] \vee$$

$$[\exists i, \psi(\mathrm{form}(\psi) \wedge \phi = \forall v_i \psi \wedge \forall b X(\langle \psi, a[b/i] \rangle))].$$

Definition 1.2.4. A set S is a *partial nonstandard satisfaction class* if there exists a nonstandard number b such that whenever $M \models \text{form}(\phi)$ and $a \in M$,

$$\langle \phi, a \rangle \in S \text{ iff } \phi < b \text{ and } (M, S) \models \Psi(S, \phi, a).$$

If, in addition, $(M, S) \models \mathsf{PA}^*$ then we say that S is a *partial nonstandard inductive satisfaction class.*

The way that S was defined guarantees that whenever $\phi(x_0, x_1, \ldots, x_n) \in \mathcal{L}_{\mathsf{PA}}$, and $a \in M$ then $M \models \phi((a)_0, (a)_1, \ldots, (a)_n)$ iff $\langle \phi, a \rangle \in S$.

The main results relating partial satisfaction classes to recursive saturation are:

a. If $M \models \mathsf{PA}$ has a partial nonstandard satisfaction class then M is recursively saturated. [Lac81]

b. If M is a countable recursively saturated model of PA then M has a partial nonstandard inductive satisfaction class. [KKL81]

Not only can we expand countable recursively saturated models of PA to have satisfaction classes, but also we can expand them to have other relations, such as automorphisms, indiscernible sets, and elementary submodels, which are consistent with the theory of the model:

Definition 1.2.5. A Model M of PA is *resplendent* if for every Σ^1_1 sentence $\exists \bar{X} \Phi(\bar{X}, \bar{a})$, $\bar{a} \in M$, with $\text{Th}(M, \bar{a}) + \exists \bar{X} \Phi(\bar{X}, \bar{a})$ consistent, there are sets $\bar{X} \subseteq M$ such that $(M, \bar{X}) \models \Phi(\bar{X}, \bar{a})$.

Every resplendent model is recursively saturated. If M is a countable recursively saturated model then it is resplendent. This was proven by Barwise and Schlipf [BS75], and independently by Ressayre [Res77]. Actually, more is true: every countable recursively saturated model is chronically resplendent, i.e. if the hypothesis in the above definition is true and M is recursively saturated, then we can find $\bar{X} \subseteq M$ such that $(M, \bar{X}) \models \Phi(\bar{X}, \bar{a})$ and $(M, \bar{X})$ is recursively saturated, and hence again, resplendent.

The next notion that we introduce is that of arithmetic saturation.

Definition 1.2.6. A model M is *arithmetically saturated* if for all $\bar{a}, \bar{b} \in M$, every finitely realizable type $p(v, \bar{b})$, whose set of Gödel numbers is arithmetic in $\mathrm{tp}(\bar{a})$, is realized in M.

Clearly, every arithmetically saturated model is recursively saturated, but the converse is not always true. There are recursively saturated models that are not arithmetically saturated. Some differences between arithmetically saturated models and recursively saturated models that are not arithmetically saturated will be mentioned below.

Definition 1.2.7. A cut I of $M \models \mathsf{PA}$ is *strong*, if for every $e \in M$, there is a $c > I$, such that for all $i \in I$, $(e)_i > I$ iff $(e)_i > c$. Otherwise, we say that I is *weak*.

The next theorem is due to Kirby and Paris.

Theorem 1.2.8 ([KP77])**.** *Let M be a model of* PA*. Then* $(\mathbb{N}, \mathrm{SSy}(M)) \models \mathsf{ACA}_0$ *if and only if* $\mathbb{N}$ *is strong in M.*

It follows from the above theorem and Wilmers' Theorem (see page 10) that when M is a recursively saturated model of PA, M is arithmetically saturated if and only if $\mathbb{N}$ is strong in M.

The next result is by Smoryński and Stavi.

Theorem 1.2.9 ([SS80])**.** *A cofinal extension of a recursively saturated model of* PA *is recursively saturated.*

The above two theorems, together with results from Section 2.1, could be used to give a proof of the next theorem.

Theorem 1.2.10. *Every countable recursively saturated model $M \models \mathsf{PA}$ has cofinal extensions M_0 and M_1, such that M_0 is arithmetically saturated and M_1 is not.*

The last result in this introductory section is a general result about models of PA. I thought it should be mentioned here since it will used in the thesis a few times.

Theorem 1.2.11 (Friedman's Embedding Theorem [Fri73])**.** *Let M, N be countable nonstandard models of* PA*. Then there exist $I \prec_{\Sigma_n} N$ an initial segment of N and a cofinal embedding $h : M \to I$ iff* $\mathrm{SSy}(M) \subseteq \mathrm{SSy}(N)$ *and* $\mathrm{Th}_{\Sigma_{n+1}}(M) \subseteq \mathrm{Th}_{\Sigma_{n+1}}(N)$*. Moreover, there exist $I \prec_{\Sigma_n} N$ an initial segment of N and an isomorphism $h : M \to I$ iff* $\mathrm{SSy}(M) = \mathrm{SSy}(N)$ *and* $\mathrm{Th}_{\Sigma_{n+1}}(M) \subseteq \mathrm{Th}_{\Sigma_{n+1}}(N)$*.*

1.3 Short Recursively Saturated Models of PA

Again, we start with some definitions. Most definitions and results in this section are taken from Smoryński's paper [Smo81].

Definition 1.3.1. A model M is *simple* if it is generated by one element, i.e., $M = \mathrm{Scl}(a)$ for some $a \in M$.

If M is a model of PA and $a \in M$, let

$$M(a) = \{b \in M : b < t(a) \text{ for some Skolem term } t\}.$$

It follows from Tarski-Vaught test and the least number principle that M(a) is an elementary initial segment of M. In fact, it is the smallest elementary initial segment of M containing a. In many cases $M(a)$ is a proper initial segment of M, but not always. However, when M is recursively saturated then $M(a)$ is always proper, as will be shown in Proposition 1.3.4.

Definition 1.3.2. A model M is *short* if it is a cofinal elementary extension of a simple model, i.e., $M = M(a)$ for some $a \in M$. If M is not short, we say that M is *tall*.

Notice that every tall model has a short elementary end extension. This follows from the MacDowell-Specker Theorem [MDS61], which states that every model of PA has an elementary end extension. Thus, if $M \models \mathsf{PA}$, there is $N \models \mathsf{PA}$ with

$M \prec_{end} N$. Let $a \in N \backslash M$. Then $M \prec_{end} \mathrm{Scl}(M \cup \{a\})$. But $\mathrm{Scl}(a)$ is cofinal in $\mathrm{Scl}(M \cup \{a\})$ (to see this, let $t(b,a) \in \mathrm{Scl}(M \cup \{a\})$, for some $b \in M$. Then $t(b,a) < \max\{t(x,a) : x \leq a\} \in \mathrm{Scl}(a)$). Hence, $\mathrm{Scl}(M \cup \{a\})$ is short, so M has a short elementary end extension. On the other hand, it follows from Friedman's Embedding Theorem that every nonstandard short model has tall initial segments.

However, for cofinal extensions we have:

Proposition 1.3.3. *Let $M \prec_{cof} N$. Then M is short iff N is short.*

Proof. Suppose M is short. Then there is an element $a \in M$ such that $\mathrm{Scl}(a) \prec_{\mathrm{cof}} M$. But since $M \prec_{cof} N$, $\mathrm{Scl}(a) \prec_{cof} N$, so N is short.

Conversely, suppose $\mathrm{Scl}(b)$ is cofinal in N. Since M is cofinal in N, there is an $a \in M$ with $a > b$. To show that M is short, it is sufficient to show that for every $c \in M$, there is a Skolem term $t(v)$ such that $t(a) > c$. Since $c \in M$, $c \in N$, so $c < s(b)$ for some Skolem term $s(v)$. Define $t(v) = \max\{s(u) : u \leq v\}$. Notice that $t(v)$ is an increasing function and that $s(v) \leq t(v)$ for all v. Hence, $s(b) \leq t(b) \leq t(a)$, and so $c < t(a)$. □

We will now explore the connection between short models and recursive saturation. First, note the following fact:

Proposition 1.3.4. *If M is a short model, then M is not recursively saturated.*

Proof. Let $p(v, a)$ be the following type:

$$p(v, a) = \{v > t(a) : t \text{ is a Skolem term}\},$$

where a is an element whose Skolem closure is cofinal in M. This type is recursive since we can enumerate all Skolem terms recursively. It is finitely realized since for any finite collection of formulas from p, $\{v > t_i(a)\}_{i<k}$ for some $k \in \mathbb{N}$,

$$s(a) = \max\{t_i(a) : i < k\} + 1$$

is in $\mathrm{Scl}(a)$ and clearly realizes this finite collection. However, this type is not realized in M, since an element realizing this type would contradict the fact that $\mathrm{Scl}(a)$ is cofinal in M. □

The type that was constructed in the proof above was not realized because it was unbounded. However, some short models can realize all bounded finitely realized recursive types.

Definition 1.3.5. A type $p(v, a)$ is *bounded*, if it contains the formula $v < t(a)$ for some Skolem term t.

Definition 1.3.6. A model M is *short recursively saturated* if it realizes every bounded recursive type which is finitely realized.

Definition 1.3.7. A model M is *short* $\mathrm{SSy}(M)$*-saturated* if

1. Every bounded coded type which is finitely realized in M is realized in M; and
2. For any $a \in M$, $\mathrm{tp}(a) \in \mathrm{SSy}(M)$.

Proposition 1.3.8 ([Smo81]). *Let $M \models \mathsf{PA}$. Then M is short recursively saturated iff it is short* $\mathrm{SSy}(M)$*-saturated.*

Proof. Suppose that M is short $\mathrm{SSy}(M)$-saturated. Let $p(v, a)$ be a bounded recursive type finitely realized in M. Since the type is recursive, it is in $\mathrm{SSy}(M)$ (since any Scott set contains all recursive sets). Thus, this type is coded in M. Since by our assumption M is short $\mathrm{SSy}(M)$-saturated, this type is realized.

Conversely, suppose that M is short recursively saturated. Let $p(v, a)$ be a bounded coded type finitely realized in M. Since this type is coded, there is a $c \in M$ such that for any $\mathcal{L}_{\mathsf{PA}}$ formula $\phi(v, w)$,

$$\ulcorner \phi(v, w) \urcorner \in c \longleftrightarrow \phi(v, a) \in p.$$

Consider the following type:

$$q(v, a, c) = \{\ulcorner \phi(v, w) \urcorner \in c \longrightarrow \phi(v, a) : \phi \in \mathcal{L}_{\mathsf{PA}}\}.$$

This type is recursive, bounded, and finitely realized in M. Thus, it is realized. But any $x \in M$ realizing this type must realize $p(v, a)$.

Remains to show that for any $b \in M$, $\mathrm{tp}(b) \in \mathrm{SSy}(M)$. Let $b \in M$. Let $d > \mathbb{N}$. Consider the type

$$p(v, b, d) = \{v < d\} \cup \{\ulcorner \phi \urcorner \in v \longleftrightarrow \phi(b) : \phi \in \mathcal{L}_{\mathsf{PA}}\}.$$

This type is recursive and bounded. Any finite collection of $\phi \in \mathcal{L}_{\mathsf{PA}}$ can be coded by a standard number. Since d is nonstandard, this type is finitely realized, hence, it is realized. □

It is easy to see that all recursively saturated models are short recursively saturated. The question that arises is: which short recursively saturated models are recursively saturated? The next result shows that short recursively saturated models that are tall are recursively saturated.

Proposition 1.3.9 ([Smo81]). *Let M be a tall model. If M is short recursively saturated, then M is recursively saturated.*

Proof. Let $p(v, a)$ be a finitely realizable recursive type in M. Since M is tall, we can find $b \in M$ with $b > \mathrm{Scl}(a)$. Let $\Phi(v, a)$ be a finite conjunction of formulas in $p(v, a)$. Since p is finitely realizable, $M \models \exists v \Phi(v, a)$. Also, because $b > \mathrm{Scl}(a)$, $b > \min\{v \in M : \exists v \Phi(v, a)\} \in \mathrm{Scl}(a)$. Therefore, $M \models \exists v < b\ \Phi(v, a)$ for any finite conjunction Φ in p. Hence, the bounded recursive type

$$q(v, a, b) = p(v, a) \cup \{v < b\}$$

is also finitely realized, and since M is short recursively saturated, $q(v, a, b)$ is realized in M. But any element which realizes q, clearly realizes p. Hence, M is recursively saturated. □

From Proposition 1.3.4 we know that the above result is not true for short recursively saturated models that are short. But are there any short recursively saturated models of PA that are short? The answer is positive and follows from the next result.

Proposition 1.3.10. *Let M be a recursively saturated model of* PA*, and let $N \prec_{end} M$. Then N is a short recursively saturated model.*

Proof. Let $p(v)$ be a bounded recursive finitely realized type in N. Since $N \prec M$, $p(v)$ is a bounded, recursive, finitely realized type in M. Hence, it is realized in M by some element c. But since c is bounded by an element of N, and N is an initial segment of M, c is in N, so by elementarity again, c realizes $p(v)$ in N. □

Remark: This result, as most of the results in this thesis, is true for models of PA^* in a finite language. This fact will be used in Section 2.2.

It follows from the above theorem that if M is recursively saturated and $a \in M$, since $M(a) \prec M$, $M(a)$ is short recursively saturated. However, since $M(a)$ is a short model, by Proposition 1.3.4, $M(a)$ is not recursively saturated. Hence, the smallest elementary initial segment of M, $M(0)$, is short recursively saturated. In particular, if $\mathrm{Th}(M) = \mathrm{Th}(\mathbb{N})$, $M(0) = \mathbb{N}$. Therefore, the standard model is short recursively saturated (this can also be easily proved directly).

From this point on, since short recursively saturated models that are tall are recursively saturated, whenever we refer to a model as short recursively saturated we mean a short recursively saturated model which is short.

Proposition 1.3.11 ([Kot83]). *Let M be a recursively saturated model of* PA*. Then the family of short recursively saturated elementary initial segments of the model forms a dense linear order with a least element and no last element. In particular, if M is countable, then the family of short recursively saturated elementary initial segments of M has the order type $1 + \mathbb{Q}$.*

Proof. Any elementary initial segment of M must contain $M(0)$, since the definable elements are cofinal in $M(0)$. Since $M(0)$ is short recursively saturated, $M(0)$ is the least element of this family.

Suppose for a contradiction that for some $b \in M$, $M(b)$ is the greatest elementary short recursively saturated initial segment. Since M is recursively saturated, it is tall (see Proposition 1.3.4). Thus, there is $c \in M \backslash M(b)$. But then $M(c)$ is a short recursively saturated elementary initial segment of M properly containing $M(b)$, which contradicts the assumption.

To see that this family is dense, let $M(a) \subsetneq M(b)$ for some $a, b \in M$, and consider the recursive type

$$p(v, a, b) = \{t(a) < v : \ t \text{ a Skolem term }\} \cup \{t(v) < b : \ t \text{ a Skolem term }\}.$$

If c in M realizes this type then $M(a) \subsetneq M(c) \subsetneq M(b)$. To see that this type is realized by some $c \in M$, let $\Phi(v, a, b)$ be a finite conjunction of formulas from $p(v, a, b)$. Enumerate the (finite) collection of Skolem terms in Φ by $t_0, t_1, \ldots, t_k$. Let $d = \max(t_0(a), t_1(a), \ldots, t_k(a)) + 1$. Clearly, d is greater than all $t_i(a)$ with $i \leq k$. On the other hand, since d is definable from a it must be in $M(a)$. Since $M(a) \subsetneq M(b)$, for any Skolem term t, $t(d) \in M(a) < b$. Thus, d must realize $\Phi(v, a, b)$, and so $p(v, a, b)$ is finitely realized. Hence, $p(v, a, b)$ is realized by some $c \in M$.

Finally, when M is countable, since every short recursively saturated elementary initial segments of M must be of the form $M(a)$ for some $a \in M$, this family is countable. Since every countable dense linear order with no end points is isomorphic to $\mathbb{Q}$, this family (which has a least element) is isomorphic to $1 + \mathbb{Q}$. □

This result can be contrasted with another result of Kotlarski [Kot83] which says that the family of recursively saturated initial cuts of M has the ordering type of the Cantor set.

We will say that a model is *extremely short* if $M = M(0)$. Since $\mathrm{Scl}(0)$ is cofinal in $M(0)$, extremely short models have no proper elementary initial segments. On the other hand, all short recursively saturated models that are not extremely short have elementary recursively saturated initial segments. To show this we introduce the following notation. Let M be a model which is not extremely short. For any $a \in M \backslash M(0)$, let

$$M[a] = \{b \in M : t(b) < a \text{ for all Skolem terms } t\}.$$

Notice that $M[a]$ is closed under Skolem terms, hence it is an elementary initial segment of M (and of $M(a)$ as well). In fact, it is the largest elementary initial segment of M not containing a. Moreover, we have:

Proposition 1.3.12. *Let M be a recursively saturated model of* PA*, and $a \in M \backslash M(0)$. Then $M[a]$ is tall, hence it is recursively saturated.*

Proof. Suppose that $M[a]$ is short. Since $M[a]$ is the largest initial segment not containing a, there is no $b \in M$ such that $M[a] \prec M(b) \prec M(a)$, contradicting Proposition 1.3.11. Hence, $M[a]$ is tall. □

Notice that since any elementary initial segment of $M[a]$ is an initial segment of $M(a)$, any non extremely short countable recursively saturated model of PA has continuum many recursively saturated elementary initial segments and countably (and densely) many short recursively saturated elementary initial segments. However, while recursively saturated models of PA have no largest proper elementary initial segments, a short recursively saturated model of PA, $M(a)$, has a largest proper elementary initial segment, namely, $M[a]$.

Proposition 1.3.10 showed that any recursively saturated model of PA has short recursively saturated elementary initial segments. Kossak [Kos83] showed that the converse holds as well. That is, any short recursively saturated model of PA has a recursively saturated elementary end extension. Here we present a proof of this fact for countable models.

Proposition 1.3.13 ([Smo81])**.** *Let M be a nonstandard countable short recursively saturated model of* PA*. Then M has a countable recursively saturated elementary end extension.*

Proof. (Sketch) Since M is short, $M = M(a)$ for some $a \in M$. Because M is short recursively saturated, $\mathrm{Th}(M)$ is coded in M. To see this, let $\{\phi_i\}_{i\in\mathbb{N}}$ be a recursive enumeration of all $\mathcal{L}_{\mathsf{PA}}$ sentences, and consider the type

$$\{v < a\} \cup \{i \in v \longleftrightarrow \phi_i : i \in \mathbb{N}\}.$$

Since this type is bounded, recursive, and finitely realized (since any finite sequence of standard numbers is coded by a standard number), it is realized. Thus, $\mathrm{Th}(M) \in \mathrm{SSy}(M)$. Therefore, by the remarks following Definition 1.2.4, there is a countable recursively saturated model N with $\mathrm{Th}(N) = \mathrm{Th}(M)$ and $\mathrm{SSy}(N) = \mathrm{SSy}(M)$. To show that M has a recursively saturated elementary end extension it suffices to show that M is isomorphic to an elementary initial segment of N.

Since $\mathrm{tp}(a) \in \mathrm{SSy}(M)$, $\mathrm{tp}(a) \in \mathrm{SSy}(N)$, so it is coded in N. Since $\mathrm{Th}(M) = \mathrm{Th}(N)$, and $\mathrm{tp}(a)$ is finitely realized in M, $\mathrm{tp}(a)$ is finitely realized in N. Hence, $\mathrm{tp}(a)$ is realized in N by some b. Enumerate all elements of M by $a_0, a_1, \ldots$, with $a_0 = a$, and enumerate all elements of $N(b) \prec_{\mathrm{end}} N$ by $b_0, b_1, \ldots$, with $b_0 = b$. Notice that whenever $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, the map $a_i \mapsto b_i$, for all $i < \mathrm{lh}(\bar{a})$, is a partial isomorphism. Since $\mathrm{tp}(a) = \mathrm{tp}(b)$, $a_0 \mapsto b_0$ is a partial isomorphism. We will extend this mapping and make sure that after ω steps we include all elements of M and $N(b)$, using a "back and forth" construction.

Suppose that after $2k$ steps we established that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ for $\bar{a} \in M$ and $\bar{b} \in N(b)$. Step $2k + 1$("forth"): For $a_k \in M$ we will find $c \in N(b)$ such that

$\text{tp}(a_k, \bar{a}) = \text{tp}(c, \bar{b})$. Let $p(u, \bar{v}) = \text{tp}(a_k, \bar{a})$. Then p is coded in M, so it is also coded in N. Since $p(v, \bar{a})$ is finitely realized in M and $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$, it follows that $p(v, \bar{b})$ is finitely realized in N. Hence, $p(u, \bar{b})$ is realized in N by some c. Therefore, $\text{tp}(a_k, \bar{a}) = \text{tp}(c, \bar{b})$. Since $M \models a_k < t(a_0)$ for some Skolem term t, $N \models c < t(b_0)$, so $c \in N(b)$.

Step $2k+2$("back"): Now take $b_k \in N(b)$ and find $d \in M$ such that $\text{tp}(b_k, c, \bar{b}) = \text{tp}(d, a_k, \bar{a})$. This can be shown by a similar argument to the "forth" step, but now because M is short recursively saturated we have to be a bit more careful and notice that the type $q(u, w, \bar{v}) = \text{tp}(b_k, c, \bar{b})$ contains the formula $u < t((\bar{v})_0)$ for some Skolem term t (since $b_k < t((\bar{b})_0)$ for that same Skolem term t) , so $q(u, a_k, \bar{a})$ is not only finitely realized and recursive, but also bounded and thus is realized in M. □

Recall from the last section that countable recursively saturated models are determined up to isomorphism by their theory and standard system. For countable short recursively saturated models we need a third condition.

Theorem 1.3.14 ([Smo81]). *Let M and N be countable short recursively saturated models. Then $M \cong N$ if and only if $\text{Th}(M) = \text{Th}(N)$, $\text{SSy}(M) = \text{SSy}(N)$ and there are isomorphic simple models $\text{Scl}(a)$ and $\text{Scl}(b)$ with $M = M(a)$ and $N = N(b)$.*

Proof. (sketch) If $M \cong N$ it is obvious that the conditions are satisfied. Conversely, by theorem 1.3.13, there is a recursively saturated model R with the same standard system and theory as M and N. The construction in theorem 1.3.13 showed also that whenever there is an element $c \in R$ with $\mathrm{tp}(c) = \mathrm{tp}(a)$ we can construct an isomorphism from M to $R(c)$. Since $\mathrm{Scl}(a) \cong \mathrm{Scl}(b)$, there is some Skolem term t with $\mathrm{tp}(t(b)) = \mathrm{tp}(a) = \mathrm{tp}(c)$, so there is an isomorphism from N to $R(c)$ as well (sending $t(b)$ to c). Therefore, $M \cong N$. □

Another notion which will be discussed extensively throughout the thesis is that of gaps.

Definition 1.3.15. Let $a \in M \models \mathsf{PA}$. Then, the *gap* of a is defined as follows:

$$\mathrm{gap}(a) = M(a) \backslash M[a].$$

It can be shown that $\mathrm{gap}(a) = \{b \in M : a \leq f(b) \text{ and } b \leq f(a) \text{ for some } f \in \mathcal{F}\}$, where $\mathcal{F}$ is the collection of all definable increasing functions in $\mathcal{L}_{\mathsf{PA}}$.

We call $\mathrm{gap}(0)$ the *least gap* of M. If $M = M(a)$ for some $a \in M$ we call $\mathrm{gap}(a)$ the *last gap* of M. Notice that M is short iff M has a last gap.

Definition 1.3.16. A type $p(v)$ is said to be *rare* if whenever an element realizes $p(v)$, it is the only element in its gap realizing $p(v)$.

Proposition 1.3.17. *Let M be a countable recursively saturated model of* PA. *Let $a, b \in M$ be such that for each $n \in \mathbb{N}$, $a + n < b$. Then there are $c, d \in M$ such that $a < c < d < b$ and* $\mathrm{tp}(c) = \mathrm{tp}(d)$.

Proof. Consider the recursive type:

$$p(v, w, a, b) = \{a < v < w < b\} \cup \{\phi(v) \longleftrightarrow \phi(w) : \phi \in \mathcal{L}_{\mathsf{PA}}\}$$

For any finite collection of formulas, say k many, there are 2^k many possible truth values. Since 2^k is a standard number and between a and b there are $b - a - 1 > \mathbb{N}$ many elements, there must be two elements between a and b which realize the same truth values for this collection of formulas. Thus, p is finitely realized, so p is realized by some c and d. □

Corollary 1.3.18. *Let M be a countable recursively saturated model of* PA*. Then every gap in $M \backslash \mathbb{N}$ realizes non-rare types.*

Some models contain elements which do not realize rare types, yet these elements are rare in their gap. That is, there is a model K and $a \in K$, which do not realize a rare type, but for any $b \in \text{gap}(a)$, $\text{tp}(b) = \text{tp}(a) \longrightarrow b = a$. For example, let M be a recursively saturated model of PA, and $a \in M \backslash M(0)$ be an element realizing a type which is not rare (such exist by the above corollary). But then, by Ehrenfeucht-Gaifman Lemma (Lemma 4.2.2), the only element in $K = \text{Scl}(a)$ realizing $\text{tp}(a)$ is a.

When a model is recursively saturated or short recursively saturated this cannot happen:

Proposition 1.3.19. *Let M be a recursively saturated or short recursively saturated model of* PA*. Let $a \in M$. If a is the only element in its gap realizing* $\text{tp}(a)$*, then* $\text{tp}(a)$ *is rare.*

Proof. Let $p(v) = \mathrm{tp}(a)$. Suppose that there is a model N in which $p(v)$ is realized by two elements, say $a_1 < a_2 \in N$. Then $N \models a_1 < a_2 < t(a_1)$ for some Skolem term t. Then the type:

$$q(v, w) = p(v) \cup p(w) \cup \{v < w < t(v)\}$$

is finitely realized. Hence, it is realized in M by some b_1 and b_2. Since $\mathrm{tp}(b_1) = \mathrm{tp}(a)$, by Theorem 1.3.14 $M(b_1) \cong M(a)$ which implies that there is another element in $\mathrm{gap}(a)$ realizing $\mathrm{tp}(a)$. Contradiction. □

Definition 1.3.20. A gap is called a *labeled gap* if it contains an element realizing a rare type. Otherwise, we will say that the gap is a *non-labeled gap*.

In Chapters 3 and 4 we explore some of the differences between automorphism groups of short recursively saturated models of PA whose last gap is labeled and those whose last gap is non-labeled.

The last notion we define in this section is that of short arithmetic saturation.

Definition 1.3.21. Let M be a model of PA. M is *short arithmetically saturated* if for all $\bar{a}, \bar{b} \in M$, every finitely realizable bounded type $p(v, \bar{b})$, whose set of Gödel numbers is arithmetic in $\mathrm{tp}(\bar{a})$, is realized in M.

The following follows almost directly from Theorem 1.2.8 and the remarks following the theorem.

Theorem 1.3.22. *Let M be countable short recursively saturated model. Then the following statements are equivalent.*

1. *M is short arithmetically saturated.*
2. $(\mathbb{N}, \mathrm{SSy}(M)) \models \mathsf{ACA}_0$.
3. *$\mathbb{N}$ is strong in M.*

Chapter 2

Expansions

This chapter examines some properties of countable short recursively saturated models of PA. In particular, we look at expansions of such models. In Section 2.1 we show that these models are not cofinally resplendent (a notion similar to resplendency which will be defined in the following section). In Section 2.2 we give a positive result. We show that every short recursively saturated model $M \models \mathsf{PA}$ has an undefinable subset X such that $(M, X) \models \mathsf{PA}^*$, and (M, X) is short recursively saturated.

2.1 Cofinal Resplendency

Recall that every resplendent model is recursively saturated. Hence, short recursively saturated models are not resplendent. Smoryński had hoped to find a property similar to resplendency for short recursively saturated models.

Definition 2.1.1. A Model M of PA is *cofinally resplendent* if for every Σ_1^1 sentence

$\exists\bar{X}\Phi(\bar{X},\bar{a})$, $\bar{a} \in M$, such that there is a cofinal extension of M, N with $N \models \exists\bar{X}\Phi(\bar{X},\bar{a})$, then there are sets $\bar{X} \subseteq M$ such that $(M,\bar{X}) \models \Phi(\bar{X},\bar{a})$.

Since every countable recursively saturated model of PA is resplendent it is also cofinally resplendent. In 1981, Smoryński [Smo81] asked whether short recursively saturated models of PA are cofinally resplendent. He used an argument of Stavi to show that not all short recursively saturated models are cofinally resplendent. In an addendum to that paper [Smo82a], Smoryński outlined an argument of Solovay showing that no short recursively saturated model of PA is cofinally resplendent. In this section we give a detailed yet slightly different proof of this fact. But first we need some preparation.

Proposition 2.1.2 ([KS]). *Let $\mathfrak{X}$ be a countable Scott set. Then there are $\mathfrak{X}_0$ and $\mathfrak{X}_1$, countable Scott sets extending $\mathfrak{X}$ with* $(\mathbb{N},\mathfrak{X}_0) \models \mathsf{ACA}_0$ *and* $(\mathbb{N},\mathfrak{X}_1) \not\models \mathsf{ACA}_0$.

Proof. We will first construct $\mathfrak{X}_0$. Let $\mathfrak{Y}_0 = \mathfrak{X}$, and let $\mathfrak{Y}_{i+1} = \mathrm{Def}(\mathbb{N}, \{X : X \in \mathfrak{Y}_i\})$ for all $i \in \mathbb{N}$. Let $\mathfrak{X}_0 = \bigcup_{i\in\mathbb{N}} \mathfrak{Y}_i$. This set is arithmetically closed. Thus, $(\mathbb{N},\mathfrak{X}_0) \models \mathsf{ACA}_0$.

To get the set $\mathfrak{X}_1$, let $\{X_1, X_2, \ldots\}$ be an enumeration of $\mathfrak{X}$. Let

$$X = \{\langle x, n\rangle : x \in X_n\}.$$

Now consider the following theory in $\mathcal{L}_{\mathsf{PA}} \cup \{c\}$

$$T = \mathsf{PA} \cup \{p_x \mid c : x \in X\} \cup \{p_x \nmid c : x \notin X\},$$

where p_x is the x^{th} prime. Every finite fragment of T has a model of the form $(\mathbb{N}, c)$ for some $c \in \mathbb{N}$. Thus, it is consistent. Since the theory is consistent, using a Henkin construction, there is a completion of T, $\overline{T}$, which is $\Delta_2(X)$. Let K be the prime model of $\overline{T}$. Since c codes X and for each $n \in \mathbb{N}$, X_n is computable in X, $\mathfrak{X} \subseteq \mathrm{SSy}(K)$. However, since every element of K is of the form $t(c)$ for some Skolem term t, each $Y \in \mathrm{SSy}(K)$ is computable in $\bar{T}$. Hence, $\mathrm{SSy}(K) \subseteq \Delta_2(X)$. Thus, by the arithmetical hierarchy, there are $\Sigma_2(X)$ sets that are not in $\mathrm{SSy}(K)$. Let $\mathfrak{X}_1 = \mathrm{SSy}(K)$. Then $(\mathbb{N}, \mathfrak{X}_1) \not\models \mathsf{ACA}_0$. □

Proposition 2.1.3. *Let M be a nonstandard countable model. Then for every countable Scott set $\mathfrak{X}$ extending* $\mathrm{SSy}(M)$*, M has an elementary cofinal extension N with* $\mathrm{SSy}(N) = \mathfrak{X}$.

Proof. Let $T = \mathsf{PA} \cup \mathrm{Th}_{\Sigma_1}(M)$. Since $T \in \mathrm{SSy}(M)$, it has a complete consistent extension $\bar{T} \in \mathrm{SSy}(M)$. Since $\bar{T} \in \mathfrak{X}$, by the remark following Definition 1.2.3, there is $N' \models \bar{T}$ such that $\mathrm{SSy}(N') = \mathfrak{X}$. Thus, $N' \models \mathrm{Th}_{\Sigma_1}(M)$, so by Friedman's Embedding Theorem there is $K \subseteq N'$ such that $M \cong K$. Let's identify M with K, and let

$$N = \{x \in N' : \exists y \in M x \leq y\}.$$

Then $M \prec_{cof} N \subseteq_{end} N'$. Since N is a nonstandard initial segment of N', $\mathrm{SSy}(N) = \mathrm{SSy}(N') = \mathfrak{X}$ and the result follows. □

Combining the above two results and Theorem 1.3.22 gives us

Proposition 2.1.4. *Let M be a short recursively saturated model of* PA*. Then M has cofinal extensions M_0 and M_1, such that $\mathbb{N}$ is strong in M_0 and weak in M_1. That is, M_0 is short arithmetically saturated and M_1 is not.*

Recall that S is a nonstandard partial satisfaction class of M if there is a nonstandard b in M such that whenever $M \models \mathrm{form}(\phi)$ and $a \in M$, then

$$\langle \ulcorner \phi \urcorner, a \rangle \in S \text{ iff } \ulcorner \phi \urcorner < b \text{ and } M \models \Psi(S, \phi, a),$$

where $\Psi(S, \phi, a)$ is the sentence which follows Tarski's definition of truth expressing satisfaction in the sense of the model, defined in Section 1.2. Let $\Phi(b, X)$ be the following formula:

$$\forall \phi \forall a \, \mathrm{form}(\phi) \longrightarrow [X(\langle \phi, a \rangle) \longleftrightarrow \phi < b \wedge \Psi(X, \phi, a)].$$

If b is nonstandard and $M \models \exists X \Phi(b, X)$, then M has a nonstandard partial satisfaction class. Recall that when M has a nonstandard partial satisfaction class, M is a recursively saturated. Thus, for $M \models \mathsf{PA}$, if there is a nonstandard b such that $M \models \exists X \Phi(b, X)$, then M is recursively saturated. On the other hand, for any standard number n, there are definable sets Sat_{Σ_n}, such that $\mathrm{Sat}_{\Sigma_n} = \{\langle \phi, a \rangle : \phi$ is Σ_n formula and $M \models \phi(a)\}$ (see [Kay91]). Now, suppose $b \in \mathbb{N}$. Then there is $n \in \mathbb{N}$ such that for any formula ϕ, $\phi < b \rightarrow \phi \in \Sigma_n$. Thus, the set $S = \mathrm{Sat}_{\Sigma_n} \cap \{\langle \phi, a \rangle : \phi < b\}$ must realize $\Phi(b, S)$. Therefore, for any standard number b, $\exists X \Phi(b, X)$ is true in any model $M \models \mathsf{PA}$. Hence we get:

Lemma 2.1.5. *If M is a model of* PA *which is not recursively saturated then $\exists X\,\Phi(b, X)$ is true if and only if b is a standard number.*

Now we are ready to prove the main result of this section.

Proposition 2.1.6. *Let M be a short recursively saturated model of* PA. *Then M is not cofinally resplendent.*

Proof. Suppose M is cofinally resplendent. Either $\mathbb{N}$ is strong in M, or $\mathbb{N}$ is weak in M. Suppose that $\mathbb{N}$ is strong in M. Then by Proposition 2.1.4 there is a cofinal extension M_1 of M such that $\mathbb{N}$ is weak in M_1. Look at the Σ^1_1 sentence

$$\Psi = \exists X \exists Y \{0 \in X \wedge \forall y\ (y \in X \rightarrow y + 1 \in X)\} \wedge \{\forall x\ (x \in X \rightarrow \Phi(x, (Y)_x))\} \wedge$$

$$\{\exists c \forall e \notin X \exists i \in X\ (c)_i < e \wedge (c)_i \notin X\},$$

where $(Y)_x = \{y : \langle y, x\rangle \in Y\}$.

$M_1 \models \Psi$ because:

1. The first braces say that X contains zero and is closed under the successor function, hence X contains $\mathbb{N}$.

2. The second braces say that X cannot contain more than $\mathbb{N}$, since if x is nonstandard and for some set S, $M_1 \models \Phi(x, S)$, then M_1 is recursively saturated which is impossible since M_1 is short. Therefore, $X = \mathbb{N}$ (in M_1).

3. The bottom part of the sentence says that X is weak in M_1.

Hence, this sentence is true in M_1 since it just expresses the fact the $\mathbb{N}$ is weak in M_1.

Since M is cofinally resplendent (by our assumption), and since we found a cofinal extension of M, M_1 in which the above sentence is true, then M must satisfy this sentence as well. However, this is impossible since $\mathbb{N}$ is strong in M and this sentence expresses that $\mathbb{N}$ is weak in M. Contradiction. Therefore, $\mathbb{N}$ cannot be strong in M. Assume now that $\mathbb{N}$ is weak in M. Then again, by Proposition 2.1.4 there is a cofinal extension of M, M_0, such that $\mathbb{N}$ is strong in M_0. Now, consider the following Σ_1^1 sentence

$$\Theta = \exists X \exists Y \{0 \in X \wedge \forall y \ (y \in X \to y+1 \in X)\} \wedge \{\forall x \ (x \in X \to \Phi(x, (Y)_x))\} \wedge$$

$$\{\forall c \exists e \notin X \forall i \in X \ ((c)_i \in X \leftrightarrow (c)_i < e)\}.$$

$M_0 \models \Theta$ since the first line of the sentence says that $X = \mathbb{N}$ (since M_0 is not recursively saturated), and the bottom line says that $\mathbb{N}$ is strong in M_0. Again, since by our assumption M is cofinally resplendent, $M \models \Theta$. But this is impossible since $\mathbb{N}$ is weak in M by our assumption. Contradiction.

Thus, when M is short recursively saturated, M is not cofinally resplendent. □

2.2 Short Recursively Saturated Expansions

In the previous section we have shown that a short recursively saturated model of PA cannot be cofinally resplendent. In this section we show that there are non-definable inductive expansions of a countable short recursively saturated model of PA such that the expanded model is still short recursively saturated (clearly, any definable expansion is still short recursively saturated). In order to prove this result we will use forcing in arithmetic. The following definition and basic results can be found in Odifreddi's paper [Odi83] and in Kossak and Schmerl's upcoming book [KS].

Let $K \models \mathsf{PA}$ and $\mathcal{L} = \mathcal{L}_{\mathsf{PA}}(K)$. Let $\mathcal{L}^* = \mathcal{L} \cup \{U\}$, where U is a unary predicate symbol. Let $\sigma \in K$ code a binary sequence. Let $\tau \in K$. By $\sigma \trianglelefteq \tau$ and $\tau \trianglerighteq \sigma$ we denote that τ codes a binary sequence which extents σ.

Definition 2.2.1. Let $\phi \in \mathcal{L}^*$. Let $\sigma \in K$ code a binary sequence.

We define $\sigma \Vdash \phi$ (σ forces ϕ) by induction on ϕ as follows:

1. If ϕ is atomic in $\mathcal{L}$, then $\sigma \Vdash \phi$ iff $K \models \phi$.
2. If $\phi = U(c)$, then $\sigma \Vdash \phi$ iff $(\sigma)_c = 1$.
3. If $\phi = \phi_1 \vee \phi_2$, then $\sigma \Vdash \phi$ iff $\sigma \Vdash \phi_1$ or $\sigma \Vdash \phi_0$.
4. If $\phi = \neg\psi$, then $\sigma \Vdash \phi$ iff for every $\tau \trianglerighteq \sigma$, $\tau \nVdash \psi$.
5. If $\phi = \exists v \psi(v)$, then $\sigma \Vdash \phi$ iff $\sigma \Vdash \psi(c)$ for some $c \in K$.

Here are some useful facts about forcing in arithmetic.

Proposition 2.2.2. *Let $\phi \in \mathcal{L}^*$, and $\sigma \in K$.*

1. *If $\sigma \Vdash \phi$ and $\tau \trianglerighteq \sigma$, then $\tau \Vdash \phi$.*
2. *If $\sigma \Vdash \phi$, then $\sigma \nVdash \neg\phi$.*
3. *There exists $\tau \trianglerighteq \sigma$ such that $\tau \Vdash \phi$ or $\tau \Vdash \neg\phi$.*
4. *The set $\{\tau : \tau \Vdash \phi\}$ is definable in K.*

Definition 2.2.3. Let $X \subseteq K$. We will say that $X \Vdash \phi$ (X *forces* ϕ) if there exists $\sigma \in K$ coding a binary sequence such that $\forall i < \mathrm{lh}(\sigma)((\sigma)_i = 1 \longleftrightarrow i \in X)$ and $\sigma \Vdash \phi$.

Remark: We will denote the relation $\forall i < \mathrm{lh}(\sigma)((\sigma)_i = 1 \longleftrightarrow i \in X)$ by $\sigma \subseteq X$.

Definition 2.2.4. A set $X \subseteq K$ is *generic* if for all $\phi \in \mathcal{L}^*$ $X \Vdash \phi$ or $X \Vdash \neg\phi$.

We now list three important facts about generic sets:

Proposition 2.2.5. *Let $X \subseteq K$ be a generic set. Then,*

1. *X is not definable in K.* (See [Odi83])
2. *X is inductive. That is, $(K, X) \models \mathsf{PA}^*$.* (See [KS])
3. *For any $\phi \in \mathcal{L}^*$, $(K, X) \models \phi$ iff $X \Vdash \phi$.* (See [Odi83])

Now to the main result:

Theorem 2.2.6. *Let M be a countable recursively saturated model of PA, $a \in M$. Then there exists an inductive non-definable set $X \subseteq M$ such that*

$$(M(a), X \cap M(a)) \prec (M, X)$$

with (M, X) recursively saturated and $(M(a), X \cap M(a))$ short recursively saturated.

Proof. Since M is recursively saturated, $\mathrm{Th}(M, a) \in \mathrm{SSy}(M)$. Let $\mathcal{L}^* = \mathcal{L}_{\mathsf{PA}}(a) \cup \{U\}$, where U is again, a unary predicate symbol. We will construct an $\mathcal{L}^*$-theory, T_U

extending $\mathrm{Th}(M, a)$ such that whenever $N \models T_U$, the set $Y = \{x : N \models U(x)\}$ is a generic set.

Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a recursive enumeration of all sentences in $\mathcal{L}^*$. Let $\sigma_0 = 0$. For every $\phi \in \mathcal{L}^*$, there is an $\mathcal{L}_{\mathsf{PA}}(a)$ formula $\Psi_\phi(x)$ such that

$$\sigma \Vdash \phi \text{ iff } \Psi_\phi(\sigma) \in \mathrm{Th}(M, a).$$

Thus, to get σ_{i+1} we ask whether the sentence $\exists\sigma \trianglerighteq \sigma_i\ \Psi_{\phi_i}(\sigma)$ is in $\mathrm{Th}(M, a)$. If there is such σ, we let

$$\sigma_{i+1} = \min\{\sigma \trianglerighteq \sigma_i : \Psi_{\phi_i}(\sigma)\}.$$

If $\exists\sigma \trianglerighteq \sigma_i\ \Psi_{\phi_i}(\sigma) \notin \mathrm{Th}(M, a)$, we let $\sigma_{i+1} = \sigma_i$. Notice that in the latter case, $\sigma_i \Vdash \neg\phi_i$. Also notice that the construction of the sequence $\{\sigma_i\}_{i\in\mathbb{N}}$ is recursive in $\mathrm{Th}(M, a)$ and each σ_i is definable in (M, a) without parameters, hence each σ_i is in $\mathrm{Scl}(a)$.

Now, let $\psi_i = \phi_i$ if $\sigma_{i+1} \Vdash \phi_i$, and let $\psi_i = \neg\phi_i$ if $\sigma_{i+1} \Vdash \neg\phi_i$. Let $T_U = \{\psi_i\}_{i\in\mathbb{N}}$. This theory is complete by the construction and is consistent since it is finitely realized by definable sets. In every model of T_U, the set realizing U is a generic set. To see this, notice that by our construction, for any formula $\phi \in \mathcal{L}^*$,

$$T_U \vdash \forall x \exists \sigma \subseteq U(\sigma \Vdash \phi(x) \vee \sigma \Vdash \neg\phi(x)),$$

where $\sigma \subseteq U$ is defined by $\forall i < \mathrm{lh}(\sigma)\ (\sigma)_i = 0 \vee (\sigma)_i = 1 \wedge ((\sigma)_i = 1 \longleftrightarrow U(i))$.

Thus, if $N \models T_U$ and $Y = \{x \in N : N \models U(x)\}$,

$$(N, Y) \models \forall x \exists \sigma \subseteq U(\sigma \Vdash \phi(x) \vee \sigma \Vdash \neg\phi(x)),$$

which shows that Y is generic.

Since $\mathrm{SSy}(M)$ is a Scott set, and since the construction is recursive in $\mathrm{Th}(M, a)$ which is in $\mathrm{SSy}(M)$, $T_U \in \mathrm{SSy}(M)$. Thus, there is a recursively saturated model $N \models T_U$ with $\mathrm{SSy}(N) = \mathrm{SSy}(M)$. Therefore, the reduct of N to $\mathcal{L}_{\mathsf{PA}} \cup \{a\}$, $N|_{\mathcal{L}_{\mathsf{PA}} \cup \{a\}} \models \mathrm{Th}(M, a)$, and so $(N|_{\mathcal{L}_{\mathsf{PA}} \cup \{a\}}, a^N) \cong (M, a^M)$, where $a^N \in N$ is the interpretation of the constant a in N. Let X be the image of the set Y under an isomorphism from (N, a^N) to (M, a^M). Since $(N, a^N, Y) \models T_U$, $(M, a^M, X) \models T_U$. In particular, X is generic. Therefore, by Proposition 2.2.5, $(M, X) \models \mathsf{PA}^*$.

Let $X_0 = \mathrm{Scl}(a) \cap X$. We will show that $(\mathrm{Scl}(a), X_0) \prec (M, X)$. To see this, let $\phi(x) \in \mathcal{L}^*$ and $c \in \mathrm{Scl}(a)$ be such that $(M, X) \models \phi(c)$. Let t be a Skolem term such that $t(a) = c$. Since $(M, X) \models \phi(t(a))$, $\phi(t(a)) \in T_U$. Thus, by our construction, there is a $\sigma_i \in \mathrm{Scl}(a)$ such that

$$T_U \vdash \sigma_i \subseteq U \wedge \sigma_i \Vdash \phi(t(a)),$$

so $(M, X) \models \sigma_i \subseteq U \wedge \sigma_i \Vdash \phi(t(a))$. Since σ_i codes an initial segment of X in M, σ_i codes an initial segment of X_0 in $\mathrm{Scl}(a)$. Thus, $(\mathrm{Scl}(a), X_0) \models \sigma_i \subseteq U$. Also, since the relation $\sigma_i \Vdash \phi(t(a))$ is definable by an $\mathcal{L}_{\mathsf{PA}}(a)$ formula, and since $\mathrm{Scl}(a) \prec M$, $\mathrm{Scl}(a) \models \sigma_i \Vdash \phi(t(a))$. Therefore, $(\mathrm{Scl}(a), X_0) \models \sigma_i \subseteq U \wedge \sigma_i \Vdash \phi(t(a))$, which implies that $(\mathrm{Scl}(a), X_0) \models \phi(c)$.

Finally, since (M, X) and $(\mathrm{Scl}(a), X_0)$ are models of PA^*, $(M(a), X \cap M(a))$ is a short elementary initial segment of (M, X). Thus, by Proposition 1.3.10 and the remark that follows, we get that $(M(a), X \cap M(a))$ is short recursively saturated. □

An interesting fact follows from the above construction. Schlipf [Sch78] proved that whenever (M, X) is a countable recursively saturated model of PA and $X \notin \mathrm{Def}(M)$, then X has continuum many automorphic images. Schmerl (see [KS]) proved that whenever X is a non-definable class ($X \subseteq M$ is a class of M if for any $a \in M$, the set $\{x \in M : x \in X \text{ and } x < a\}$ is definable in M, in particular, any inductive set is a class) of M with M a countable recursively saturated model of PA, then X has continuum many images under the automorphisms of M. However, from the construction above, when M is short recursively saturated, we get a counterexample to both results mentioned above.

Theorem 2.2.7. *Let M be a countable recursively saturated model of* PA *with $a \in M$. Then $M(a)$ has an inductive non-definable subset Y such that $(M(a), Y)$ is short recursively saturated and Y has at most countably many automorphic images under the automorphism group of $M(a)$.*

Proof. We will use the set $Y = X \cap M(a)$, where X is the set constructed in Theorem 2.2.6. We claim that whenever f and g are automorphisms of $M(a)$ and $f(a) = g(a)$, then $f(Y) = g(Y)$. Now, since $M(a)$ is countable, a must have at most countably many images and the result follows.

To prove the claim, let $c \in M(a)$. Then there are $b \in \mathrm{Scl}(a)$ with $b > \max\{c, f^{-1}(c), g^{-1}(c)\}$, and $\sigma \in \mathrm{Scl}(a)$ such that

$$T_U \vdash \forall x < b[(\sigma)_x = 1 \longleftrightarrow U(x)],$$

where T_U is the theory constructed in Theorem 2.2.6. In particular, this sentence in true in $(M(a), Y))$. Since $\sigma \in \mathrm{Scl}(a)$, let $t(v)$ and $s(v)$ be the Skolem terms such that $t(a) = \sigma$ and $s(a) = b$. Then,

$$(M(a), Y) \models \forall x < s(a)(x \in Y \longleftrightarrow t(a)_x = 1).$$

Hence, since f and g are automorphisms of M,

$$\forall x < f(s(a))(x \in f(Y) \longleftrightarrow (f(t(a)))_x = 1 \longleftrightarrow (t(f(a)))_x = 1),$$

and also

$$\forall x < g(s(a))(x \in g(Y) \longleftrightarrow (g(t(a)))_x = 1 \longleftrightarrow (t(g(a)))_x = 1).$$

But then, whenever $f(a) = g(a)$, for all $x < f(s(a)) = g(s(a))$

$$x \in f(Y) \longleftrightarrow (t(f(a)))_x = 1 \longleftrightarrow (t(g(a)))_x = 1 \longleftrightarrow \ x \in g(Y).$$

Since, $c < f(b) = f(s(a))$, $c \in f(Y) \longleftrightarrow c \in g(Y)$. $\square$

Chapter 3

Automorphisms

In this chapter we begin to explore automorphisms of short recursively saturated models of arithmetic. The first section is an introduction to automorphisms and automorphism groups of first order structures. The second section lists results about automorphisms and automorphism groups of recursively saturated models of PA. In particular, we are interested in normal subgroups of the automorphism groups. The last section is devoted to automorphisms and automorphism groups of short recursively saturated models of PA. We will discuss the relation between automorphisms of a recursively saturated model of PA and automorphisms of a short recursively saturated initial segment of that model. This will enable us to modify and apply results concerning automorphisms of recursively saturated models of PA to automorphisms of short recursively saturated models of PA.

3.1 Automorphisms of Countable First Order Structures

Let M be a countable first order structure. Let $G = \mathrm{Aut}(M)$, that is, G is the automorphism group of M.

We will start with few definitions.

Definition 3.1.1. The *pointwise stabilizer* of a set $X \subseteq M$, denoted $G_{(X)}$ is the subgroup of G containing all automorphisms of M which fix X pointwise, that is,

$$G_{(X)} = \{g \in G : g(a) = a \text{ for all } a \in X\}.$$

The *setwise stabilizer* of a set $X \subseteq M$, denoted $G_{\{X\}}$ is the subgroup of G containing all automorphisms of M which fix X setwise, that is,

$$G_{\{X\}} = \{g \in G : g(X) = X\}.$$

Remark: If $X = \{\bar{a}\}$, we will denote $G_{(X)}$ and $G_{\{X\}}$ by $G_{(\bar{a})}$ and $G_{\{\bar{a}\}}$, respectively.

We can define a topology on G by letting the basic open subgroups of G be the pointwise stabilizers of tuples of M. That is, H is a basic open subgroup of G iff $H = G_{(\bar{a})}$ for some $\bar{a} \in M$. The basic open sets of G are cosets of the basic open subgroups. Equivalently, the basic open sets are the sets of the form $S_{\bar{a},\bar{b}} = \{g \in G : g(\bar{a}) = \bar{b}\}$. This definition makes G a topological group, i.e., a group in which the operations multiplication and inversion are continuous.

Recall that when M is a model of PA, tuples are coded by single elements. In this case, if c codes $\bar{a}$, c is definable from $\bar{a}$ and $\bar{a}$ is definable from c, so $G_{(\bar{a})} = G_{(c)}$. Therefore, when M is a model of PA, the basic open subgroups are exactly stabilizers of single elements.

Some important facts about this topology are:

1. A subgroup is open iff it contains a basic open subgroup.

2. If a subgroup is open then it is closed.

3. A subgroup H is closed iff whenever $g \in G$ has the property that for any $\bar{a} \in M$ there is an $h \in H$ such that $g(\bar{a}) = h(\bar{a})$, then $g \in H$.

For any countable first order structure M, $\text{Aut}(M)$ is metrizable. Moreover, $\text{Aut}(M)$ is a Polish group, that is, the topology is a complete, separable metric space.

For further results on automorphisms of first order structures see [Hod93] and [KM94].

3.2 Automorphisms of Countable Recursively Saturated Models of PA

Let M be a first order structure with $\bar{a}, \bar{b} \in M$. Since $\text{Aut}(M)$ preserves all first order properties of M, if there exists $f \in \text{Aut}(M)$ with $f(\bar{a}) = \bar{b}$, then $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$. When M is a countable recursively saturated model, we get the converse as well.

Proposition 3.2.1. *Let M be a countable recursively saturated structure and suppose* *that* $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$*. Then there is an automorphism* $f \in \mathrm{Aut}(M)$ *which sends* $\bar{a}$ *to* $\bar{b}$*.*

This property is called ω-homogeneity, and models which have this property are said to be ω-homogeneous.

We will now turn our attention to stabilizers of initial segments, which prove to be important in the study of normal subgroups of the automorphism groups of recursively saturated models. Let M be a model of PA and $f \in \mathrm{Aut}(M)$. Let $I_{fix}(f)$ be the largest initial segment which f fixes pointwise, i.e.,

$$I_{fix}(f) = \{x \in M : \forall y \leq x \ f(y) = y\}.$$

Now, suppose that $a \in I_{fix}(f)$. If $c < 2^a$, then c codes a unique sequence of size $b \leq a$, and all the elements coded by c are less than a. But since b is fixed by f, the length of the sequence coded by $f(c)$ is the same as the length of the sequence coded by c (Because $M \models \mathrm{lh}(c) = b$, so $M \models \mathrm{lh}(f(c)) = f(b)$, and $f(b) = b$). Since each of the elements that c codes is also fixed (because these elements are less than a), $f(c)$ codes exactly the same sequence as c. But since the code of this sequence is unique $f(c) = c$. Thus, we have proved:

Proposition 3.2.2. *Let* M *be a model of* PA *and let* $f \in \mathrm{Aut}(M)$*. Then* $I_{fix}(f)$ *is closed under exponentiation.*

When M is a countable recursively saturated model of PA, we get the converse:

Lemma 3.2.3 ([Smo82b]). *Let M be a countable recursively saturated model of* PA. *Let I be an initial segment closed under exponentiation. Then there exists an automorphism $f \in \mathrm{Aut}(M)$ with $I_{fix}(f) = I$.*

Before we explore the connection between initial segments and normal subgroups, we need few more definitions and results.

Let M be a model of PA. A set $X \subseteq M$ is *invariant* if $f(X) = X$ for all $f \in \mathrm{Aut}(M)$. Clearly, $\mathbb{N}$ and M are both invariant.

The next proposition can be found in [KKK91]

Proposition 3.2.4. *Let M be a countable recursively saturated model of* PA. *Let I be a proper initial segment of M. Then I is invariant iff $I \cap \mathrm{Scl}(0)$ is cofinal in I, or $\mathrm{Scl}(0)\backslash I$ is downward cofinal in $M\backslash I$.*

Notice that when $M \models \mathrm{Th}(\mathbb{N})$, $\mathrm{Scl}(0) = \mathbb{N}$. Hence, M has only one proper invariant initial segment closed under exponentiation, namely, $\mathbb{N}$. However,

Proposition 3.2.5 ([Kay94]). *Let M be a countable recursively saturated model of* PA *such that $M \not\models \mathrm{Th}(\mathbb{N})$. Then M has continuum many invariant initial segments closed under exponentiation.*

Proof. (Sketch) Define the relation $\sim$ on $\mathrm{Scl}(0)$ as follows:

$$a \sim b \longleftrightarrow M \models b < 2_n^a \wedge a < 2_n^b \text{ for some } n \in \mathbb{N}.$$

This relation is an equivalence relation. The set of equivalence classes of nonstandard elements, $A = \{[a] : a \in M \backslash \mathbb{N}\}$, forms a linear order defined as follows:

$$[a] <_A [b] \longleftrightarrow [a] \neq [b] \text{ and } a < b.$$

Using an overspill argument, one can show that this ordering is a dense linear order without endpoints. Since this set is countable, there is an isomorphism from $(A, <_A)$ to $(\mathbb{Q}, <)$, the set of rational numbers. Therefore, there are continuum many initial segments of Scl(0) closed under exponentiation (for a more detailed account see [Kay91] pages 76-77).

Now, let $I \subset_{end} \mathrm{Scl}(0)$ be closed under exponentiation. Then the initial segment $I' = \{x \in M : x < y \text{ for some } y \in I\}$ is closed under exponentiation and I is cofinal in I'. Thus, by Proposition 3.2.4, I' is invariant and the result follows. □

We are now ready explore the connection between initial segments and normal subgroups.

Proposition 3.2.6. *Let M be a first order structure and let $G = \mathrm{Aut}(M)$. Let I be a subset of M. Then I is invariant iff $G_{(I)}$ is normal in G.*

Proof. Let $h \in G$. Then $hG_{(I)}h^{-1} = G_{(h(I))}$. Thus, $G_{(I)}$ is normal if and only if I is invariant. □

Another fact which is true for any first order structure M and any set $I \subseteq M$ is that in $G = \mathrm{Aut}(M)$, $G_{(I)}$ is a closed subgroup. This follows from the fact that if

$f \in \overline{G_{(I)}}$, then for any $a \in I$, if $f(a) = b$, there must be a $g \in G_{(I)}$ with $g(a) = b$. But since $a \in I$, for any such $g \in G_{(I)}$, $g(a) = a$, which implies that $a = b$, i.e., $f(a) = a$. Thus, $f \in G_{(I)}$.

The next theorem is Kaye's theorem which classifies all closed normal subgroups of countable recursively saturated models of PA as stabilizers of invariant initial segments.

Theorem 3.2.7 ([Kay94])**.** *Let M be a countable recursively saturated model of* PA *and let $G = \mathrm{Aut}(M)$. Let $H \leq G$. Then H is a closed normal subgroup of G iff $H = G_{(I)}$ for some invariant initial segment $I \subseteq M$.*

The importance of this result is that it gives an example of recursively saturated models of PA with automorphism groups that are not isomorphic as topological groups. For example, when $M \models \mathrm{Th}(\mathbb{N})$ is recursively saturated it has only one proper invariant initial segment, as noted before. Hence, its automorphism group has only two closed normal subgroup, namely, the trivial ones $G_{(\mathbb{N})} = G$ and $G_{(M)} = \{id\}$. However, when $M \models$ PA but $M \not\models \mathrm{Th}(\mathbb{N})$, then by Proposition 3.2.5, M has continuum many invariant initial segments closed under exponentiation, so its automorphism group has continuum many closed normal subgroups.

What about normal subgroups of G in general?

For any initial segment I, let

$$G_{(>I)} = \{f \in G : I_{fix}(f) > I\}.$$

Proposition 3.2.8. *Let $M \models \mathsf{PA}$ and let I be an initial segment of M. Then*

$$G_{(>I)} \trianglelefteq G \longleftrightarrow I \text{ is an invariant initial segment.}$$

Proof. Similar to Proposition 3.2.6. Let $h \in G$. Notice that $hG_{(>I)}h^{-1} = G_{(>h(I))}$. Thus, $G_{(>I)}$ is normal if and only if I is invariant. □

When $I = \log^{\mathbb{N}}(a)$ for some $a \in M$, $G_{(>I)} = G_{(2^a_{\mathbb{N}})}$. Thus for such I, $G_{(>I)}$ is closed [Kay94]. On the other hand, if $I \neq \log^{\mathbb{N}}(a)$ for all a in M, $\overline{G_{(>I)}} = G_{(I)}$. Since by Smoryński's Lemma (Lemma 3.2.3), for any I closed under exponentiation there is an $f \in G$ with $I_{fix}(f) = I$, $G_{(>I)} \subsetneq G_{(I)}$. Thus, when $M \not\models \mathrm{Th}(\mathbb{N})$, since by Proposition 3.2.5 there are continuum many invariant initial segments closed under exponentiation, and since only countably many of them are of the form $I = \log^{\mathbb{N}}(a)$ for some $a \in M$, there are continuum many normal subgroups which are not closed. On the other hand, when $M \models \mathrm{Th}(\mathbb{N})$, there is only one known normal nontrivial (and non-closed) subgroup, namely, $G_{(>\mathbb{N})}$.

Kaye conjectured [Kay94] that every normal subgroup of the automorphism group of any countable recursively saturated model of PA is either of the form $G_{(I)}$ or $G_{(>I)}$ for some invariant cut I.

3.3 Automorphisms of Countable Short Recursively Saturated Models of PA

In this section we discuss automorphisms of countable short recursively saturated models of PA. We will show that unlike recursively saturated models, whenever a short recursively saturated model is not extremely short, the model is not ω-homogeneous. We will also discuss some differences between short recursively saturated models whose last gap is labeled and those whose last gap is non-labeled. Finally, we show that any automorphism of a short recursively saturated model of PA fixes elements cofinally high in the model.

For this section, fix M a countable recursively saturated model of PA. Let a be an element of M. In Section 1.3 we showed that $M(a)$ is a short recursively saturated elementary submodel of M. Let $G = \mathrm{Aut}(M)$ and $G(a) = \mathrm{Aut}(M(a))$.

We start with few observations. Notice that since automorphisms preserve all $\mathcal{L}_{\infty,\omega}$ definable sets, they map gaps to gaps and in particular, they fix the last gap setwise. Hence, we get

Lemma 3.3.1. *If $f \in G(a)$ then $f(\mathrm{gap}(a)) = \mathrm{gap}(a)$.*

This gives us a necessary condition for a restriction of an automorphism of M to be an automorphism of $M(a)$. It must fix $\mathrm{gap}(a)$ setwise. Moreover, this is a sufficient condition, since any automorphism which fixes $\mathrm{gap}(a)$ setwise must fix $M(a)$ setwise. Hence, it is a bijection which preserves all definable properties of $M(a)$. Thus,

Proposition 3.3.2. *Let $f \in G$. The restriction of f to the domain of $M(a)$, $f|_{M(a)}$, is in $G(a)$ iff $f(\mathrm{gap}(a)) = \mathrm{gap}(a)$.*

This result yields some corollaries which follow from the fact that M is ω-homogeneous.

Corollary 3.3.3. *Let $b, c \in M(a)$. If there are $d, e \in \mathrm{gap}(a)$ (not necessarily distinct) with $\mathrm{tp}(b, d) = \mathrm{tp}(c, e)$, then there is an automorphism of $M(a)$ sending b to c. In particular, if $b, c \in \mathrm{gap}(a)$ and $\mathrm{tp}(b) = \mathrm{tp}(c)$ then there is $f \in G(a)$ with $f(b) = c$.*

Proof. Suppose that there are such d and e. Since M is ω-homogeneous and $\mathrm{tp}(b, d) = \mathrm{tp}(c, e)$, there is an automorphism $f \in G$ with $f(b) = c$ and $f(d) = e$. Since d and e are in $\mathrm{gap}(a)$ and automorphisms send gaps to gaps, $f(\mathrm{gap}(a)) = \mathrm{gap}(a)$. Hence, by the previous proposition, $f|_{M(a)}$ is an automorphism of $M(a)$, and $f|_{M(a)}(b) = c$. □

This corollary give us a way of showing existence of automorphisms of $M(a)$ with a given property. Instead of constructing automorphisms, many times we will show that one exists using the corollary above.

Another way of showing existence of automorphisms of $M(a)$ is the following: We can extend the language of PA to have a constant symbol for a (or any other element in $\mathrm{gap}(a)$). Then, the expansion (M, a) is still recursively saturated. Clearly, $\mathrm{gap}(a)$ is fixed setwise by all automorphisms of (M, a). Hence, by Proposition 3.3.2, the restriction of any automorphism of (M, a) to $M(a)$ is an automorphism of $M(a)$ (which in addition fixes a).

Corollary 3.3.3 may give the impression that $M(a)$ is ω-homogeneous. When $M(a) = M(0)$, i.e., when $M(a)$ is extremely short, there is only one gap, $\mathrm{gap}(a)$. It follows from Corollary 3.3.3 that for any two elements $b, c \in M(a)$ if $\mathrm{tp}(b) = \mathrm{tp}(c)$ there is an automorphism of $M(a)$ which sends b to c. Hence, whenever $M(a)$ is extremely short, $M(a)$ is ω-homogeneous. However, when $M(a) > M(0)$, $M(a)$ is not ω-homogeneous. To prove this, we will use the following lemma.

Lemma 3.3.4. *Let $a > M(0)$. Then, for every $b \in \mathrm{gap}(a)$, there exists an element $c < \mathrm{gap}(a)$ with $\mathrm{tp}(b) = \mathrm{tp}(c)$.*

Proof. Let $p(v, b)$ be the following recursive type,

$$\{\phi(v) \leftrightarrow \phi(b) : \phi \in \mathcal{L}_{\mathsf{PA}}\} \cup \{t(v) < a : t \text{ is a Skolem term}\}.$$

An element c realizing this type, will have the same type as b, and will be less than $\mathrm{gap}(a)$. Since this type is bounded and recursive, to finish the proof, we need to show that it is finitely realized.

Suppose for a contradiction , that $p(v, b)$ is not finitely realized. Then, there must be a finite conjunction of formulas $\Phi(x)$ such that $M(a) \models \Phi(b)$, but for all elements v below $\mathrm{gap}(a)$, $M(a) \not\models \Phi(v)$. But then $x = \min\{v : \Phi(v)\}$ defines an element in $\mathrm{gap}(a)$, which is impossible since $a > M(0)$, so $\mathrm{gap}(a)$ contains no definable elements. □

Proposition 3.3.5. *Let $a > M(0)$. Then $M(a)$ is not ω-homogeneous.*

Proof. Let b and c be as in the lemma above. Since $b \in \text{gap}(a)$ and $c \notin \text{gap}(a)$, there is no automorphism of $M(a)$ which sends b to c, since by Lemma 3.3.1, $\text{gap}(a)$ is fixed setwise. But $\text{tp}(b) = \text{tp}(c)$, so $M(a)$ is not ω-homogeneous. □

So far we discussed only automorphisms of $M(a)$ which extend to automorphisms of M. In the next chapter we will prove that there are continuum many automorphisms of $M(a)$ which do not extend to M.

Recall that a type of an element is rare if whenever it is realized in a gap, it is realized by only one element in that gap. By results of Gaifman [Gai76], any recursively saturated model of PA has gaps with elements which realize rare types (labeled gaps). On the other hand, it was shown in [KKS93] that every recursively saturated model of PA has gaps in which no element realizes rare types (non-labeled gaps). The next two propositions show some of the differences between automorphisms of short recursively saturated models whose last gap is labeled and models whose last gap is non-labeled.

Since elements realizing rare types cannot be mapped to any other element in their gap and since the last gap of any short recursively saturated model is fixed setwise, we get:

Proposition 3.3.6. *Let $M(a)$ be a short recursively saturated model such that* $\text{gap}(a)$ *is labeled. Let $b \in \text{gap}(a)$ realize a rare type. Then for all $f \in G(a)$, $f(b) = b$.*

Proposition 3.3.7. *Let $M(a)$ be a short recursively saturated model with* $\mathrm{gap}(a)$ *non-labeled. Then for any $b \in \mathrm{gap}(a)$ there is an $f \in G(a)$ with $f(b) \neq b$.*

Proof. Since no $b \in \mathrm{gap}(a)$ realizes a rare type, by Proposition 1.3.19, there is an element $c \neq b$ in $\mathrm{gap}(a)$ with $\mathrm{tp}(c) = \mathrm{tp}(b)$. So by Lemma 3.3.3, there is an automorphism $f \in G(a)$ with $f(b) = c$. □

Thus, short recursively saturated models whose last gaps are labeled have a very interesting property. Unlike recursively saturated models, they have non-definable elements that are fixed by all automorphisms of the model. Therefore, elements realizing rare types in the last gap of a short recursively saturated model have properties similar to the properties of definable elements. One reason for this is the fact that although non-definable in the model, these elements are definable from any element which is in their gap.

Proposition 3.3.8. *Let $a \in M$ realize a rare type. Let $b \in \mathrm{gap}(a)$. Then $a \in \mathrm{Scl}(b)$.*

Proof. Let $b \in \mathrm{gap}(a)$, and let $t(x)$ be a Skolem term such that $t(a) > b$ and $t(b) > a$. Now, consider the recursive type

$$p(v, a, b) = \{v \neq a\} \cup \{t(v) > b\} \cup \{t(b) > v\} \cup \{\phi(a) \leftrightarrow \phi(v) : \phi \in \mathcal{L}_{\mathsf{PA}}\}.$$

Since a is rare, this type is not realizes. Thus, this type is not finitely realized. Hence, there must be a finite conjunction of formulas $\Phi(x)$ which is true only for a between

$\min(\{v : t(v) > b\})$ and $t(b)$. Thus, a is definable from b by the formula

$$\Psi(x, b) = \Phi(x) \wedge t(x) > b \wedge t(b) > x.$$

□

Although short recursively saturated models whose last gaps are non-labeled do not have elements in their last gap which are fixed by all automorphisms, every automorphism of these models fixes some elements in the last gap. To show this, I will use a lemma of Blass [Bla72] and Gaifman [Gai76].

Lemma 3.3.9 (Blass and Gaifman Lemma)**.** *Let M be a model of* PA*. Let $a < b \in M$. If $b \in \mathrm{gap}(a)$ then there is a Skolem term $t(x)$ such that $M \models a < b \leq t(a) = t(b)$.*

The above lemma implies the following proposition:

Proposition 3.3.10. *Let M be a model of* PA*. Let $f \in \mathrm{Aut}(M)$ and let $a > \mathrm{Scl}(0)$ be such that $f(a) \in \mathrm{gap}(a)$. Then there is $c \in \mathrm{gap}(a)$ such that $f(c) = c$.*

Proof. Let $b = f(a)$. Since $b \in \mathrm{gap}(a)$, by the Blass-Gaifman Lemma, there is a Skolem term t such that $t(a) = t(b)$. Let $c = t(a) = t(b)$. Then, $f(c) = f(t(a)) = t(f(a)) = t(b) = c$. Hence, c is fixed. □

Let $f \in G(a)$. By $\mathrm{fix}(f)$ we denote the set of elements in $M(a)$ fixed by f.

Corollary 3.3.11. *Let $M(a)$ be a short recursively saturated model of* PA*. Let $f \in G(a)$. Then for every $f \in G(a)$, $\mathrm{fix}(f) \prec_{cof} M(a)$.*

Proof. Since for all $f \in G(a)$, $f(a) \in \text{gap}(a)$, by the above result, there is $c \in \text{gap}(a)$ such that $f(c) = c$. But then $\text{Scl}(c) \subseteq \text{fix}(f)$. Notice that the Blass-Gaifman Lemma implies that for any $b \in \text{gap}(a)$, $\text{Scl}(b) \subset_{cof} M(a)$. Thus, $\text{Scl}(c) \subset_{cof} M(a)$, so $\text{fix}(f) \subseteq_{cof} M(a)$. Finally, for any automorphism g of any model K of PA, $\text{fix}(g) \prec K$. Thus, $\text{fix}(f) \prec_{cof} M(a)$. □

This result can be contrasted with the fact that arithmetically saturated models of PA have automorphisms which move all non-definable elements. When a model is short recursively saturated, by the corollary above, this cannot happen (although, when $M(a)$ is short arithmetically saturated, for every $b \in M(a)$ there is $f \in G(a)$ such that $\text{fix}(f) = \text{Scl}(b)$. See Proposition 4.4.4 for proof).

So far we showed differences between the action of automorphisms of short recursively saturated models whose last gap is labeled and those whose last gap is non-labeled. In the next chapter, we will exploit these results to show that automorphism groups of short recursively saturated models whose last gap is labeled and automorphism groups of short recursively saturated models whose last gap is non-labeled are not isomorphic as topological groups.

Chapter 4

Automorphisms and the Automorphism Group of $M(a)$

In this chapter we continue exploring properties of automorphisms of short recursively saturated models of PA. The first section is devoted to a subgroup of the automorphism group of a short recursively saturated model of PA. This subgroup consists of automorphisms that can be extended to automorphisms of the recursively saturated elementary end extension of the model. In Section 4.2 we discuss conjugations of automorphisms of short recursively saturated models of PA. We show that each automorphism of these models has continuum many conjugates. Section 4.3 is devoted to normal subgroups of the automorphism groups of short recursively saturated models. We prove that Kaye's characterization of closed normal subgroups of automorphism groups of recursively saturated models of PA as stabilizers of invariant cuts is true for automorphism groups of short recursively saturated models of PA as well. The final section of this chapter discusses differences between the automorphism groups of short arithmetically saturated models and the automorphism groups of short recursively saturated models that are not short arithmetically saturated.

4.1 The Subgroup $G|_{M(a)}$

As before, we fix a countable recursively saturated model M of PA and $a \in M$. Then, as proved in Section 1.3, $M(a)$ is a short recursively saturated initial segment of M. Again, we denote $\mathrm{Aut}(M)$ by G, and $\mathrm{Aut}(M(a))$ by $G(a)$. Let

$$G|_{M(a)} = \{g|_{M(a)} : g \in G_{\{\mathrm{gap}(a)\}}\},$$

i.e., $G|_{M(a)}$ is the set of restrictions to $M(a)$ of those automorphism of M which fix $\mathrm{gap}(a)$ setwise. Clearly, $G|_{M(a)}$ is a subgroup of $G(a)$. In this section we show that $G|_{M(a)}$ is in fact a proper subgroup of $G(a)$. This will be done by showing that there are automorphisms of $M(a)$ that do not extend to automorphisms of M. Later in the section, we discuss other properties of $G|_{M(a)}$.

Let K be a model, and X a subset of K. Let H be a subgroup of $\mathrm{Aut}(K)$. By $\mathcal{A}_H(X)$ we denote the set of all automorphic images of X under H. That is,

$$\mathcal{A}_H(X) = \{g(X) : g \in H\}.$$

Lemma 4.1.1 (Kueker-Reyes' Lemma [Kue75])**.** *Let K be a countable structure. Let $X \subseteq K$. Suppose that for all $\bar{a} \in K$ there are $x \in X$, $y \notin X$ such that $(K, \bar{a}, x) \cong (K, \bar{a}, y)$. Then $|\mathcal{A}_{\mathrm{Aut}(K)}(X)| = 2^{\aleph_0}$.*

Remarks: If $K \models \mathsf{PA}$ is a countable model and $X \subseteq K$, to show that X has continuum many images in $\mathrm{Aut}(K)$, since all finite sequences are coded in PA, it is enough to show that for all $a \in K$ there are $x \in X$ and $y \notin X$ such that $(K, a, x) \cong (K, a, y)$. If in addition, K is recursively saturated, then by the ω-homogeneity of K, it is enough to show that for any $a \in K$ there are $x \in X$ and $y \notin X$ such that $\mathrm{tp}(a, x) = \mathrm{tp}(a, y)$. When K is a countable short recursively saturated model of PA, it is enough to show that in the expanded model (K, b), where b is an element of the last gap of K, for any $a \in K$ there are $x \in X$ and $y \notin X$ with $\mathrm{tp}(a, x) = \mathrm{tp}(a, y)$ (this follows from Proposition 3.3.3).

Lemma 4.1.2. *Let M, $M(a)$, G, and $G(a)$ be as above. Then there exists $X \subseteq M(a)$ such that X is coded in M and $|\mathcal{A}_{G(a)}(X)| = 2^{\aleph_0}$.*

Proof. We will start by defining in $M(a)$ a cofinal sequence $\{(b)_n : n \in \mathbb{N}\}$. Let $q(w, a)$ be the type:

$$q(w, a) = \{(w)_0 = a\} \cup \{(w)_{n+1} = \max((w)_n^2, t_n(a)) : n \in \mathbb{N}\},$$

where $\langle t_n : n \in \omega \rangle$ is some recursive enumeration of all Skolem terms. Since this type is recursive, we only need to show that the type is finitely realized. Any finite collection of formulas from $q(w, a)$ involves only finitely many terms $(w)_{n_1}, (w)_{n_2}, ..., (w)_{n_k}$, with $k \in \mathbb{N}$ and $n_1 < n_2 < ... < n_k \in \mathbb{N}$. This collection can be realized by an element c coding the finite sequence $(w)_0, (w)_1, ..., (w)_{n_k}$, with $(w)_0 = a$ and for $i < n_k$ $(w)_{i+1} =$

$\max((w)_n^2, t_n(a))$ (since any finite sequence is coded in the model). Therefore, this type is realized in M. Let b realize this type. Now define the following recursive type,

$$p(v,b) = \{(b)_n < (v)_n < (b)_{n+1} : n \in \mathbb{N}\} \cup$$

$$\{\forall x < (b)_n (v)_n \neq t(x,(b)_n) : n \in \mathbb{N}, t \text{ is a Skolem term}\}.$$

Again, since the type is recursive, we only need to show that it is finitely realized. Take a finite set of formulas from $p(v,b)$. Let k be a natural number larger than the largest n used in this finite set of formulas and also larger than the number of Skolem terms used. Since for every $i < k$, between $(b)_i$ and $(b)_{i+1}$ there are at least $(b)_i^2 - (b)_i$ elements and there are no more than $(b)_i \cdot k$ many elements of the form $t(x,(b)_i)$ with $x < (b)_i$, there exists c_i between $(b)_i$ and $(b)_{i+1}$ which is not definable from this finite set of formulas. Since $c_0, ..., c_{k-1}$ is a finite sequence it can be coded by some $c \in M$ with $(c)_i = c_i$, for all $i < k$, hence the type $p(v,b)$ is finitely realized so it is realized by some c.

Let $X = \{(c)_n : n \in \mathbb{N}\}$. By our construction, $X \subseteq M(a)$ and X is coded in M by c. To finish the proof, by Kueker-Reyes Lemma (working in $(M(a), a)$), we need to show that for every $d \in M(a)$ there are $x \in X$, $y \notin X$ such that $\mathrm{tp}(d,x) = \mathrm{tp}(d,y)$. We argue by contradiction. Suppose that for some $d \in M(a)$ there are no such x and y. Because b codes a cofinal sequence in $M(a)$, we can find $i \in \mathbb{N}$ such that $(b)_i > d$. Since between $(b)_i$ and $(b)_{i+1}$ there is only one $x \in X$, by our assumption, for all

$y \neq x$ between $(b)_i$ and $(b)_{i+1}$, $\text{tp}(d,x) \neq \text{tp}(d,y)$. Thus, the recursive type

$$r(u,x,b,d) = \{(b)_i < u < (b)_{i+1}\} \cup \{u \neq x\} \cup \{\phi(d,x) \leftrightarrow \phi(d,u) : \phi \in \mathcal{L}_{\mathcal{PA}}\}$$

is not realized by any y so it is not finitely realized. Hence, there is formula Φ such that $M \models \Phi(d,x)$ and for any $y \neq x$ between $(b)_i$ and $(b)_{i+1}$ $M \models \neg\Phi(d,y)$. But then the term $t(d,(b)_i) = \min\{v : M \models \Phi(d,v) \wedge v > (b)_i\}$ defines x from $(b)_i$ and d which contradicts the fact that x cannot be defined by terms less than or equal to $(b)_i$. Hence, for any $d \in M(a)$ there are $x \in X$ and $y \notin X$ such that $\text{tp}(d,x) = \text{tp}(d,y)$. Thus, $|\mathcal{A}_{G(a)}(X)\}| = 2^{\aleph_0}$. □

Proposition 4.1.3. *There are continuum many automorphisms of $M(a)$ which are not extendible to M.*

Proof. We will use the set X constructed in the previous lemma. Since X is coded in M by an element $c \in M$ and since c has only countably many automorphic images, X has only countably many automorphic images in M. However, by the previous lemma X has continuum many automorphic images in $M(a)$. Thus, continuum many automorphisms of $M(a)$ do not extend to M. □

It follows from this corollary that $G|_{M(a)}$ is a proper subgroup of $G(a)$.

We will now investigate other properties of the subgroup $G|_{M(a)}$. We will refer to the construction of X in Lemma 4.1.2 frequently, throughout this section.

Corollary 4.1.4. *$G|_{M(a)}$ is not open in $G(a)$.*

Proof. To show that $G|_{M(a)}$ is not open in $G(a)$, we need to show that it does not contain any basic open subgroup of $G(a)$. Let $c \in M(a)$. We will show that there is an automorphism $g \in G(a)_{(c)}$, such that $g \notin G|_{M(a)}$. Since $c \in M(a)$, $(M(a), c)$ is a short recursively saturated initial segment of (M, c). Hence, we can repeat the construction of X from Lemma 4.1.2 in this expanded structure. Thus, we get an $X \subseteq M(a)$ coded in (M, c) (and hence also in M), with continuum many automorphic images in $(M(a), c)$. But since X has only countably many automorphisms in M (since it is coded), continuum many automorphisms of $(M(a), c)$ do not extend to M. Hence, $G|_{M(a)}$ does not contain any basic open subgroup, so it is not open in $G(a)$. □

Since any closed subgroup in this topology is also open, and $G|_{M(a)}$ is not open, it is not closed. We will give now a direct proof of this fact. In addition, we will show that $G|_{M(a)}$ is dense.

Proposition 4.1.5. *$G|_{M(a)}$ is dense and not closed in $G(a)$.*

Proof. Let $g \in G(a)$. We will show that g is in the closure of $G|_{M(a)}$. This shows that $G|_{M(a)}$ is dense in $G(a)$. Then, since $G|_{M(a)}$ is a proper subgroup of $G(a)$, $G|_{M(a)}$ is not closed. To show that g is in the closure of $G|_{M(a)}$, we need to show that any open set that contains g, also contains an $h \in G|_{M(a)}$. That is, we need to show that for any $c, d \in M(a)$ such that $g(c) = d$, there is $h \in G|_{M(a)}$ with $h(c) = d$.

Suppose $g(c) = d$. Let $b = g(a)$. Then $\mathrm{tp}(a, c) = \mathrm{tp}(b, d)$. Now, since M is

ω-homogeneous, there is an automorphism $f \in G$, such that $f(a) = b$ and $f(c) = d$. But since $g(a) = b$ and $g \in G(a)$, by Lemma 3.3.1, $b \in \text{gap}(a)$. Thus, f fixes $\text{gap}(a)$ setwise. Let $h = f|_{M(a)}$. Since $h \in G|_{M(a)}$ and $h(c) = d$, g is in the closure of $G|_{M(a)}$. □

The next proposition shows that the index of $G|_{M(a)}$ in $G(a)$ is not small. That is, there are uncountably many cosets of $G|_{M(a)}$.

Proposition 4.1.6. $\left[G(a) : G|_{M(a)}\right] = 2^{\aleph_0}$.

Proof. Since $|G(a)| = 2^{\aleph_0}$, $\left[G(a) : G|_{M(a)}\right] \leq 2^{\aleph_0}$. Remains to show $\left[G(a) : G|_{M(a)}\right] \geq 2^{\aleph_0}$. Suppose for a contradiction that $\left[G(a) : G|_{M(a)}\right] = \lambda < 2^{\aleph_0}$. Let X be as in Lemma 4.1.2. Recall that $|\mathcal{A}_{G|_{M(a)}}(X)| = \aleph_0$ and $|\mathcal{A}_{G(a)}(X)| = 2^{\aleph_0}$. Now, for $g \in G(a)$ and all $h_1, h_2 \in G|_{M(a)}$, $h_1(X) = h_2(X) \longleftrightarrow gh_1(X) = gh_2(X)$. Hence,

$$|\{gh(X) : h \in G|_{M(a)}\}| = |\{h(X) : h \in G|_{M(a)}\}| = |\mathcal{A}_{G|_{M(a)}}(X)| = \aleph_0,$$

that is, X has countably many automorphic images under the action of any coset of $G|_{M(a)}$. Now,

$$|\mathcal{A}_{G(a)}(X)| \leq |\mathcal{A}_{G|_{M(a)}}(X)| \cdot \left[G(a) : G|_{M(a)}\right] = \aleph_0 \cdot \lambda < 2^{\aleph_0}.$$

But this contradicts the fact that $|\mathcal{A}_{G(a)}(X)| = 2^{\aleph_0}$. Therefore, $\left[G(a) : G|_{M(a)}\right] \geq 2^{\aleph_0}$, and so $\left[G(a) : G|_{M(a)}\right] = 2^{\aleph_0}$. □

We will now proceed to show that $G|_{M(a)}$ is not a normal subgroup.

Lemma 4.1.7. *With the set X as above, there is an automorphism $f \in G(a)$ such that $|\mathcal{A}_{G|_{M(a)}}(f(X))| = 2^{\aleph_0}$.*

Proof. We will construct an automorphism f which fixes a, using "back-and-forth" inside $M(a)$. Let b, c, X be as in Lemma 4.1.2, that is, X is coded by $c \in M$ with the property that $(b)_n < (c)_n < (b)_{n+1}$ for all $n \in \mathbb{N}$, and $\{(b)_n : n \in \mathbb{N}\}$ is a cofinal sequence in $M(a)$, $(b)_0 = a$, and $(b)_{n+1} = \max((b)_n^2, t_n(a))$. Notice that each $(b)_n$ is defined from a, hence, f will fix it as well. Another important fact to recall from the construction is that $(c)_n \notin \mathrm{Scl}(x, (b)_n)$ for all $x < (b)_n$.

We are now ready to construct f. Enumerate $M(a) = \{a_0, a_1, \dots\}$, and also enumerate $M = \{m_0, m_1, \dots\}$. Let $\bar{d}_0 = a$ and $\bar{e}_0 = a$. Suppose that $2n$ steps have been done in the construction of f and that $\mathrm{tp}(\bar{d}_n) = \mathrm{tp}(\bar{e}_n)$.

Step $2n + 1$: we will do both a "forth" and a "back". For a_n find $u \in M(a)$ such that $\mathrm{tp}(\bar{d}_n, a_n) = \mathrm{tp}(\bar{e}_n, u)$ and then find $v \in M(a)$ such that $\mathrm{tp}(\bar{d}_n, a_n, v) = \mathrm{tp}(\bar{e}_n, u, a_n)$. Such u and v exist because $\mathrm{tp}(\bar{d}_n) = \mathrm{tp}(\bar{e}_n)$ implies that there is an automorphism of M sending $\bar{d}_n$ to $\bar{e}_n$, fixing a (and also $\mathrm{gap}(a)$ and $M(a)$ setwise). This automorphism will send a_n to some $u \in M(a)$ and some $v \in M(a)$ to a_n. Hence, $\mathrm{tp}(\bar{d}_n, a_n, v) = \mathrm{tp}(\bar{e}_n, u, a_n)$.

Step $2n + 2$: In this step we will only do a "forth". Let $d = \langle \bar{d}_n, a_n, v \rangle$ (that is $\langle \langle \bar{d}_n, a_n \rangle, v \rangle$) and $e = \langle \bar{e}_n, u, a_n \rangle$. Since $\{(b)_n : n \in \mathbb{N}\}$ is cofinal in $M(a)$, we can find $k \in \mathbb{N}$ such that $(b)_k > d$. For this k we will use $(c)_k$, the $k + 1$

element of X, and find an x such that $\mathrm{tp}(d,(c)_k) = \mathrm{tp}(e,x)$ (this ensures that $\mathrm{tp}(\bar{d}_n, a_n, v, (c)_k) = \mathrm{tp}(\bar{e}_n, u, a_n, x)$), with the additional condition on x that there is a $y \neq x$ with $\mathrm{tp}(m_n, a, x) = \mathrm{tp}(m_n, a, y)$. Consider the following recursive type in M,

$$p(w,z) = \{w \neq z\} \cup \{\varphi(d,(c)_k) \leftrightarrow \varphi(e,w) : \varphi \in \mathcal{L}_{\mathsf{PA}}\} \cup$$

$$\{\varphi(m_n, a, w) \leftrightarrow \varphi(m_n, a, z) : \varphi \in \mathcal{L}_{\mathsf{PA}}\}.$$

First, notice that $\mathrm{tp}(d) = \mathrm{tp}(e)$, so there exists an x such that $\mathrm{tp}(d,(c)_k) = \mathrm{tp}(e,x)$. Also, because $(c)_k$ is between $(b)_k$ and $(b)_{k+1}$, x is also between these elements. Moreover, by the way the set X was constructed, $(c)_k$ is not defined from d and $(b)_k$, so x is not defined from e and $(b)_k$. Therefore, since the interval $((b)_k, (b)_{k+1})$ is of nonstandard length, by recursive saturation there are infinitely many $w_i \in M(a)$ such that $\mathrm{tp}(e,x) = \mathrm{tp}(e,w_i)$ (since otherwise we would have been able to define x from e and $(b)_k$). So there are countably many $w_i \in M(a)$, such that $\mathrm{tp}(d,(c)_k) = \mathrm{tp}(e,w_i)$. Now, let $\Phi(w,z)$ be a finite conjunction of (say r many) formulas of the form $\varphi_j(m_n,a,w) \leftrightarrow \varphi_j(m_n,a,z)$, where $\{\varphi_j : j \in \mathbb{N}\}$ is an enumeration of all formulas of $\mathcal{L}_{\mathsf{PA}}$. Since there are only finitely (2^r) many possible truth values to r many formulas, and since we have infinitely many w_i satisfying $\mathrm{tp}(d,(c)_k) = \mathrm{tp}(e,w_i)$, we can pick from this set $w_{i_1} \neq w_{i_2}$ satisfying

$$\bigwedge_{j=1}^{r} (\varphi_j(m_n, a, w_{i_1}) \leftrightarrow \varphi_j(m_n, a, w_{i_2})).$$

Hence, $p(w, z)$ is finitely realized, so it is realized by some x and y, respectively. Therefore, we found an x such that $\text{tp}(\bar{d}_n, a_n, v, (c)_k) = \text{tp}(\bar{e}_n, u, a_n, x)$ and also a $y \neq x$ such that $\text{tp}(m_n, a, x) = \text{tp}(m_n, a, y)$. Now, set $\bar{d}_{n+1} = (\bar{d}_n, a_n, v, (c)_k)$ and $\bar{e}_{n+1} = (\bar{e}_n, u, a_n, x)$ and continue to the next odd step. After ω steps we will get the automorphism $f \in G(a)$.

To see that $|\mathcal{A}_{G|_{M(a)}}(f(X))| = 2^{\aleph_0}$ we will use Kueker-Reyes' Lemma. Consider the recursively saturated structure (M, a). By our construction, for any $m_n \in (M, a)$, there are $x \in f(X)$ and $y \notin f(X)$, such that $\text{tp}(m_n, x) = \text{tp}(m_n, y)$. Hence, by Kueker-Reyes' lemma, $f(X)$ has continuum many automorphic images in (M, a) (all in $\text{gap}(a)$). But, since every automorphism of (M, a) is also an automorphism of M which fixes a, we get $\text{Aut}(M, a)|_{(M(a),a)} \leq G|_{M(a)}$. Hence, there are continuum many images of $f(X)$ under $G|_{M(a)}$. □

Corollary 4.1.8. *$G|_{M(a)}$ is not a normal subgroup of $G(a)$.*

Proof. We will use the set X and the automorphism f as above to prove the corollary. By Lemma 4.1.2, since X is coded in M, X has only countably many images under the action of $G|_{M(a)}$. We will show that X has continuum many images under the action of $\{f^{-1}gf : g \in G|_{M(a)}\}$. Thus, showing that there are $g_i \in G|_{M(a)}$ with $f^{-1}g_if \neq h$ for any $h \in G|_{M(a)}$, which implies that $G|_{M(a)}$ is not normal in $G(a)$. Notice, that since f^{-1} is an automorphism of $M(a)$, for any $Y, Z \subseteq M(a)$,

$$Y = Z \leftrightarrow f^{-1}(Y) = f^{-1}(Z).$$

Hence, $|\{gf(X) : g \in G|_{M(a)}\}| = |\{f^{-1}gf(X) : g \in G|_{M(a)}\}|$. But, by the previous lemma, $|\{gf(X) : g \in G|_{M(a)}\}| = 2^{\aleph_0}$, thus $|\{f^{-1}gf(X) : g \in G|_{M(a)}\}| = 2^{\aleph_0}$, which completes the proof. □

4.2 Conjugates

Tzouvaras [Tzo91] proved that any nontrivial automorphism of a countable recursively saturated model of PA has continuum many conjugates. In this section, we prove that the same is true for automorphisms of countable short recursively saturated models. That is, we show that any nontrivial automorphism of $M(a)$ has continuum many conjugates. We will start with the following lemma:

Lemma 4.2.1. *Let N be a short recursively saturated model of* PA*. Then, for every $b, c \in N$, if $c \notin \mathrm{Scl}(b)$, then there is $d < c \in N$ such that $tp(b, c) = tp(b, d)$.*

Proof. Consider the type:

$$p(v, b, c) = \{v < c\} \cup \{\phi(b, c) \leftrightarrow \phi(b, v) : \phi \in \mathcal{L}_{\mathsf{PA}}\}.$$

An element $d \in N$ realizing this type will have the required properties. Since this type is bounded and recursive, we only need to show that it is finitely realized. Suppose, for a contradiction, that it is not finitely realized. Then, there exists a formula $\Phi(x, y) \in \mathcal{L}_{\mathsf{PA}}$, such that $N \models \Phi(b, c)$ and

$$N \models \forall v(\Phi(b, v) \rightarrow v \geq c).$$

But then c can be defined from b by $t(b) = \min\{v : N \models \Phi(b, v)\}$, contradicting the fact that $c \notin \mathrm{Scl}(b)$. Hence, the type must be realized by some $d \in N$. □

To prove the next proposition, we will use the above lemma, and an important result (which we present without proof) that was proved by Ehrenfeucht [Ehr73], and by Gaifman [Gai76].

Lemma 4.2.2 (Ehrenfeucht-Gaifman Lemma). *Let K be a model of* PA. *Let $a, b \in K$ and let $t(x)$ be a Skolem term such that $K \models a \neq b = t(a)$. Then,* $\mathrm{tp}(a) \neq \mathrm{tp}(b)$.

Again, for the rest of the section, we fix M, a countable recursively saturated model of PA, and $M(a)$ an elementary short recursively saturated initial segment of M. Also, we let $G(a) = \mathrm{Aut}(M(a))$.

Proposition 4.2.3. *Let $g \in G(a)$ be a nontrivial automorphism. Then, for any $b \in M(a)$ there exist $c \in M(a)$ and $h \in G(a)_{(b)}$ such that $hg(c) \neq gh(c) = g(c)$.*

Proof. Since $g \in G(a)$, by Corollary 3.3.11, there is an element $r \in \mathrm{gap}(a)$, such that $g(r) = r$. We will consider two cases.

Case 1: Suppose that $g(b) = b$. Since g is nontrivial, let $c \in M(a)$ be such that $g(c) = d \neq c$. Thus,

$$\mathrm{tp}(r, b, c) = \mathrm{tp}(g(r), g(b), g(c)) = \mathrm{tp}(r, b, d).$$

Therefore, by Ehrenfeuch-Gaifman's Lemma, $d \notin \mathrm{Scl}(r, b, c)$. So by Lemma 4.2.1 $\exists e < d \in M(a)$ such that $\mathrm{tp}(r, b, c, d) = \mathrm{tp}(r, b, c, e)$. Now, since $r \in \mathrm{gap}(a)$, by

Corollary 3.3.3, there is an automorphism $h \in G(a)$ sending (r, b, c, d) to (r, b, c, e). Thus, $h(b) = b$ and $gh(c) = g(c) = d \neq e = h(d) = hg(c)$.

Case 2: Suppose that $g(b) = d \neq b$. Then $\text{tp}(r, b) = \text{tp}(r, d)$. Then again, by Ehrenfeucht-Gaifman's Lemma, $d \notin \text{Scl}(r, b)$. Hence, by Lemma 4.2.1 $\exists e < d \in M(a)$ such that $\text{tp}(r, b, d) = \text{tp}(r, b, e)$. Let $c = b$, so again we get that $\text{tp}(r, b, c, d) = \text{tp}(r, b, c, e)$. Hence, there is an $h \in G(a)$ with the same properties as in case 1. $\square$

Since there is a definable bijection from $M(a) \times M(a)$ into $M(a)$, the graph of g, $\Gamma(g)$, can be regarded as a subset of $M(a)$. In particular, let $\Gamma(g) = \{\langle x, g(x)\rangle : x \in M(a)\}$.

Proposition 4.2.4. *Let $g \in G(a)$ be nontrivial. Then the graph of g, $\Gamma(g)$, has continuum many images under the action of $G(a)$.*

Proof. Let r be as in the proposition above, i.e., $r \in \text{gap}(a)$ and $g(r) = r$. We will use Kueker-Reyes' Lemma (Lemma 4.1.1). By the remark after the lemma, it is enough to show that for any $b \in M(a)$, there are $x \in \Gamma(g)$ and $y \notin \Gamma(g)$ such that in the expanded structure $(M(a), r)$, $\text{tp}(b, x) = \text{tp}(b, y)$. But in the proof of the proposition above, we showed that for any $b \in M(a)$, there are $c \in M(a)$, $d = g(c) \neq c$, and $e \neq d$, such that $\text{tp}(r, b, c, d) = \text{tp}(r, b, c, e)$. Thus, in $(M(a), r)$, $\text{tp}(b, c, d) = \text{tp}(b, c, e)$. In particular, $\text{tp}(b, \langle c, d\rangle) = \text{tp}(b, \langle c, e\rangle)$. But $\langle c, d\rangle \in \Gamma(g)$ and $\langle c, e\rangle \notin \Gamma(g)$. Hence, by Kueker-Reyes' Lemma, $\Gamma(g)$ has continuum many images under $G(a)$. $\square$

Now, since for any $g \in G(a)$ the image of $\Gamma(g)$ under any $f \in G(a)$ is the graph of fgf^{-1}, we get

Corollary 4.2.5. *Let $g \in G(a)$ be nontrivial. Then g has continuum many conjugates in $G(a)$.*

4.3 Normal Subgroups

In Section 3.2 we stated Kaye's Theorem (Theorem 3.2.7) which characterizes closed normal subgroups of the automorphism groups of recursively saturated models of PA as stabilizers of invariant initial segments of the model. In this section we show that the same characterization holds for the automorphism groups of short recursively saturated models of PA as well. Using this result we show that there are short recursively saturated models of PA whose automorphism groups are not isomorphic as topological groups.

In addition to the results mentioned above, we investigate other properties of automorphisms of short recursively saturated models of PA, especially those concerning initial segments of such models. We use results from Section 3.2, where automorphisms of recursively saturated models of PA were discussed, to prove similar results for automorphisms of short recursively saturated models.

We start by recalling Proposition 3.2.2 which says that for any model of PA and any automorphism f of the model, $I_{fix}(f)$, the largest initial segment of M fixed pointwise by f, is closed under exponentiation. Smoryński's Lemma (Lemma 3.2.3), says that any initial segment of a recursively saturated model of PA closed under exponentiation is of the form $I_{fix}(f)$ for some automorphism f. We will show the same for short recursively saturated models.

As before, we fix M a countable recursively saturated model of PA, $a \in M$, $G = \mathrm{Aut}(M)$, and $G(a) = \mathrm{Aut}(M(a))$.

Lemma 4.3.1. *Let I be an initial segment of $M(a)$ closed under exponentiation. Then there exists an automorphism $f \in G(a)$ with $I_{fix}(f) = I$.*

Proof. Since I is an initial segment of $M(a)$ closed under exponentiation, it is an initial segment of the recursively saturated structure (M, a) (and clearly, still closed under exponentiation). Hence, by Smoryński's Lemma, there is an automorphism h of (M, a) such that $I_{fix}(h) = I$. Let $f = h|_{M(a)}$. Now, since h fixes $M(a)$ setwise, $f \in G(a)$ (in fact, $f \in G|_{M(a)}$), and $I_{fix}(f) = I$. □

We will say (see [KKS93]) that a type $p(v)$ is *ubiquitous* if for every $a \in M$, whenever there is an element in $\mathrm{gap}(a)$ realizing $p(v)$, then the set of elements realizing $p(v)$ in $\mathrm{gap}(a)$ is both cofinal and downward cofinal in $\mathrm{gap}(a)$. We will need the following lemma from [KKS93].

Lemma 4.3.2. *For every $a \in M$ if $\mathrm{gap}(a)$ is non-labeled then the type of every element of $\mathrm{gap}(a)$ is ubiquitous.*

Proposition 4.3.3. *Suppose that* $\mathrm{gap}(a)$ *is non-labeled. Then* $M(a)$ *has a largest nontrivial invariant initial segment, namely,* $M[a]$*.*

Proof. Since $\mathrm{gap}(a)$ is fixed setwise by all automorphisms of $M(a)$, then so is $M[a] = M(a)\setminus \mathrm{gap}(a)$. To see that there are no larger nontrivial initial segments, let I be an initial segment of $M(a)$ such that $M[a] < I < M(a)$. We will show that I is not invariant.

Since $\mathrm{gap}(a)$ is non-labeled, by the previous lemma, every element of $\mathrm{gap}(a)$ is ubiquitous. Hence, for any $b \in I$ there is a $c \in M(a)\setminus I$ with $\mathrm{tp}(b) = \mathrm{tp}(c)$. Since both b and c are in $\mathrm{gap}(a)$ there is an automorphism of $M(a)$ sending b to c. Therefore, I is not invariant. □

It follows from the above proposition that countable short recursively saturated models of PA whose last gap is non-labeled have no invariant initial segments in the last gap. On the other hand, countable short recursively saturated models whose last gap is labeled have continuum many invariant initial segments in their last gap.

Proposition 4.3.4. *Suppose that* $\mathrm{gap}(a)$ *is labeled and let* I *be a proper initial segment of* $M(a)$*. Then* I *is invariant iff* $I \cap \mathrm{Scl}(b)$ *is cofinal in* I*, or* $\mathrm{Scl}(b)\setminus I$ *is downward cofinal in* $M(a)\setminus I$*, for some* b *realizing a rare type in* $\mathrm{gap}(a)$*.*

Proof. Let $b \in \mathrm{gap}(a)$ realize a rare type. Then, by Proposition 3.3.6, b is fixed by all automorphisms of $M(a)$, so $\mathrm{Scl}(b)$ is fixed pointwise by all automorphisms of $M(a)$.

Hence, if $I \cap \mathrm{Scl}(b)$ is cofinal in I, or if $\mathrm{Scl}(b) \backslash I$ is downward cofinal in $M(a) \backslash I$, then I is fixed setwise, i.e., I is invariant.

Conversely, Suppose that $I \cap \mathrm{Scl}(b)$ is not cofinal in I, and $\mathrm{Scl}(b) \backslash I$ is not downward cofinal in $M(a) \backslash I$, for any b realizing a rare type in gap(a). Again, let b be an element realizing a rare type in gap(a), and consider the structure (M, b). In this structure, since b is in the language, $\mathrm{Scl}(0) = \mathrm{Scl}(b)$. Hence, in (M, b), $I \cap \mathrm{Scl}(0)$ is not cofinal in I, and $\mathrm{Scl}(0) \backslash I$ is not downward cofinal in $(M, b) \backslash I$. Since (M, b) is recursively saturated, it follows from Proposition 3.2.4 that I is not invariant in (M, b). Let g be an automorphism of (M, b) which moves I. Then, since g fixes b and $b \in \mathrm{gap}(a)$, g fixes gap(a) setwise. Therefore, $g|_{M(a)} \in G(a)$ and $g|_{M(a)}(I) \neq I$. Hence, I is not invariant in $M(a)$. □

Recall that for any model and any subset of the model, the set of automorphisms which fix the subset pointwise is a closed subgroup of the automorphism group. Also, recall Proposition 3.2.6, which says that for any model M and any subset $I \subseteq M$, I is an invariant subset of the model iff the set of automorphisms which fix I pointwise is a normal subgroup of the automorphism group of M.

We will say that $I \subset_{end} M$ is coded by ω from below if there is an $a \in M$ with $\{(a)_i : i \in \mathbb{N}\}$ an increasing sequence which is cofinal in I. We will say that I is coded by ω from above if there is a $b \in M$ with $\{(b)_i : i \in \mathbb{N}\}$ a decreasing sequence which is downward cofinal in $M \backslash I$.

The following lemma can be proved using an easy overspill argument.

Lemma 4.3.5. *Let I be an initial segment of M. Then if I is coded by ω from below, it is not coded by ω from above.*

The following result is based on a similar result by Kossak and Kotlarski [KK88].

Proposition 4.3.6. *Let $g \in G(a)$ and $b \in M(a)$. Then there are $g' \in G$ and $I \subset_{end} M(a)$, such that $b \in I$ and $g|_I = g'|_I$.*

Proof. First, recall Proposition 3.3.11 which states that we can find $c \in M(a)$ with $c > b$ and $g(c) = c$. Let $I = 2^c_{\mathbb{N}}$. Notice that $g(I) = I$. Also, notice that the cut I is closed under exponentiation and is coded by ω from below. Therefore, by Lemma 4.3.5, it cannot be coded by ω from above. We need to show that $g|_I$ can be extended to an automorphism, g', of M. We will construct g' by "back and forth".

Let $\{a_n\}_{n\in\mathbb{N}}$ be an enumeration of M. Suppose that we already established that for all $x \in I$, $(M, \bar{a}, x) \equiv (M, \bar{b}, g(x))$, where both $\bar{a}$ and $\bar{b}$ contain $a_0, ..., a_{n-1}$. We will show only the "forth" step. The "back" is similar. We will find $b_n \in M$ such that for all $x \in I$, $(M, \bar{a}, a_n, x) \equiv (M, \bar{b}, b_n, g(x))$.

Let $d > I$ be in gap(a). Consider the following recursive type,

$$r(v) = \{\forall x < d\ (\phi(\bar{a}, a_n, x) \longleftrightarrow \langle \ulcorner \phi \urcorner, x\rangle \in v) : \phi \in \mathcal{L}_{\mathsf{PA}}\} \cup \{v < 2^{d^2}\}.$$

This type is finitely realized in M, since for any finite collection of formulas $\phi_1, ..., \phi_k$, the set $\{\langle \ulcorner \phi_i \urcorner, x\rangle : x < d, i \leq k, \phi(\bar{a}, a_n, x)\}$ is bounded and definable. Also, since the largest possible element in this set is of size $\langle r, d-1\rangle$, where r is a standard number representing the largest Gödel number among the Gödel numbers of the formulas $\phi_1, ..., \phi_k$, this set can be realized by an element of size smaller than $2^{\langle r,d-1\rangle+1} < 2^{d^2}$. Thus, there is an $\alpha > d$ in gap(a) realizing this type. Hence, for all $x \in I$ and all $\phi \in \mathcal{L}_{\mathsf{PA}}$,

$$M \models \phi(\bar{a}, a_n, x) \longleftrightarrow \langle \ulcorner\phi\urcorner, x\rangle \in \alpha.$$

For any $k \in \mathbb{N}$, let

$$\alpha_k = \min\{\gamma : \forall x < 2^c_k \forall \ulcorner\phi\urcorner < c(\langle \ulcorner\phi\urcorner, x\rangle \in \gamma \leftrightarrow (\langle \ulcorner\phi\urcorner, x\rangle \in \alpha)\}.$$

Notice that $\alpha_k \in 2^c_{\mathbb{N}}$. Let $\beta = g(\alpha)$ and for any $n \in \mathbb{N}$ let $\beta_n = g(\alpha_n)$. Since I is fixed by g, $\beta_k \in 2^c_{\mathbb{N}}$. Also, since g is an automorphism of $M(a)$, for any $x < 2^c_k$ and any $\phi \in \mathcal{L}_{\mathsf{PA}}$, $\langle \ulcorner\phi\urcorner, x\rangle \in \beta_k \longleftrightarrow \langle \ulcorner\phi\urcorner, x\rangle \in \beta$.

Let $\{\phi_j\}_{j\in\mathbb{N}}$ be an enumeration of all formulas in $\mathcal{L}_{\mathsf{PA}}$. Let $i \in \mathbb{N}$. For every $k \in \mathbb{N}$,

$$M \models \exists v \forall x < 2^c_k \bigwedge_{j \leq i} (\phi_j(\bar{a}, v, x) \longleftrightarrow \langle \ulcorner\phi_j\urcorner, x\rangle \in \alpha_k),$$

since a_n is a witness for the existence of such v. Therefore, by our inductive assumption,

$$M \models \exists v \forall x < 2^c_k \bigwedge_{j \leq i} (\phi_j(\bar{b}, v, x) \longleftrightarrow \langle \ulcorner\phi_j\urcorner, x\rangle \in \beta_k).$$

But since for any $\phi \in \mathcal{L}_{\mathsf{PA}}$, $M \models \forall x < 2^c_k(\langle \ulcorner\phi\urcorner, x\rangle \in \beta_k \longleftrightarrow \langle \ulcorner\phi\urcorner, x\rangle \in \beta)$,

$$M \models \exists v \forall x < 2^c_k \bigwedge_{j \leq i} (\phi_j(\bar{b}, v, x) \longleftrightarrow \langle \ulcorner\phi_j\urcorner, x\rangle \in \beta).$$

Hence, for all $r \in I$ and all $i \in \mathbb{N}$,

$$M \models \exists v \forall x < r \bigwedge_{j \leq i} (\phi_j(\bar{b}, v, x) \longleftrightarrow \langle \ulcorner \phi_j \urcorner, x \rangle \in \beta).$$

Now, Consider the type:

$$p(u) = \{(u)_i = \max\{r : \exists v \forall x < r \bigwedge_{j \leq i} (\phi_j(\bar{b}, v, x) \longleftrightarrow \langle \ulcorner \phi_j \urcorner, x \rangle \in \beta)\} : i \in \mathbb{N}\}.$$

Since this type is recursive and finitely realized (since any finite definable set is coded in the model), there is an e realizing it. Notice that by overspill on the cut I, for all $i \in \mathbb{N}$ $(e)_i > I$. Also, notice that $(e)_i$ codes a decreasing sequence. Since I is not coded by ω from above, there must be $s > \mathbb{N}$ with $(e)_i > (e)_s > I$ for all $i < s$. Consider the following recursive type:

$$q(v) = \{\forall x < (e)_s \bigwedge_{j \leq i} (\phi_j(\bar{b}, v, x) \longleftrightarrow \langle \ulcorner \phi_j \urcorner, x \rangle \in \beta) : i \in \mathbb{N}\}.$$

Notice that the way that $(e)_s$ was chosen guarantees that $q(v)$ is finitely realized. Let b_n realize this type. For any $x \in I$ and $\phi \in \mathcal{L}_{\mathsf{PA}}$,

$$M \models \phi(\bar{a}, a_n, x) \longleftrightarrow \langle \ulcorner \phi \urcorner, x \rangle \in \alpha \longleftrightarrow \langle \ulcorner \phi \urcorner, g(x) \rangle \in \beta \longleftrightarrow \phi(\bar{b}, b_n, g(x)).$$

Hence, $(M, \bar{a}, a_n, x) \equiv (M, \bar{b}, b_n, g(x))$. □

The next lemma was used by Schmerl to prove Kaye's theorem. After stating it, we will modify it (using the proposition above) to apply to short recursively saturated models.

Lemma 4.3.7 ([Sch01]). *Let M be a countable recursively saturated model of* PA*, $g \in G = \mathrm{Aut}(M)$ be a nontrivial automorphism of M. Let $I = I_{fix}(g)$, and suppose that there are arbitrarily small $x > I$ such that $g(x) < x$. Suppose that $a < b \in M$ and $h \in G_{(I)}$ are such that $b = h(a)$. Then there are $u, v, w \in M$ such that $g(v) = u < v$, $\mathrm{tp}(u, v) = \mathrm{tp}(u, w)$, and $\mathrm{tp}(v, w) = \mathrm{tp}(a, b)$.*

The analogous lemma is

Lemma 4.3.8. *Let $M(a)$ be a countable short recursively saturated model of* PA*, and let $g \in G(a)$ be a nontrivial automorphism of $M(a)$. Let $I = I_{fix}(g)$, and suppose that there are arbitrarily small $x > I$ such that $g(x) < x$. Suppose that $c < d \in M(a)$ and $h \in G(a)_{(I)}$ are such that $d = h(c)$. Then there are $b \in \mathrm{gap}(a)$, and $u, v, w \in M(a)$, such that $g(b) = b$, $g(v) = u < v$, $\mathrm{tp}(u, v, b) = \mathrm{tp}(u, w, b)$, and $\mathrm{tp}(v, w, b) = \mathrm{tp}(c, d, b)$.*

Proof. Let c, d, and h be given with the above properties. Since $g, h \in G(a)$, by Corollary 3.3.11, there are $x, y \in \mathrm{gap}(a)$ with $x, y > d$, and $g(x) = x$ and $h(y) = y$. By Blass-Gaifman's Lemma, there exists $b \in \mathrm{gap}(a)$ with $x, y < b$ and $b = t(x) = t(y)$ for the same Skolem term t. Therefore, $g(b) = h(b) = b$. Hence, by Proposition 4.3.6, there are $g' \in G$, $h' \in G$, and $J \subset_{end} M(a)$ with $b \in J$ such that $g|_J = g'|_J$ and $h|_J = h'_J$ (notice that we can use the same initial segment for g and h since the choice of the segment in the proof of Proposition 4.3.6 depended only on the element fixed by the automorphism, and both g and h fix the same element b). Now, both g' and h' are automorphisms of the structure (M, b). Also, $I_{fix}(g') = I$, and $h' \in G_{(I)}$. Thus, we can apply the previous lemma to g' and h', i.e., there are $u, v, w \in (M, b)$ such that $g'(v) = u < v$, $\mathrm{tp}_{(M,b)}(u, v) = \mathrm{tp}_{(M,b)}(u, w)$, and $\mathrm{tp}_{(M,b)}(v, w) = \mathrm{tp}_{(M,b)}(c, d)$. This implies that $\mathrm{tp}(u, v, b) = \mathrm{tp}(u, w, b)$, and $\mathrm{tp}(v, w, b) = \mathrm{tp}(c, d, b)$, respectively. Now, since both $d, c < b$, then $w, v < b \in J$. Therefore, $g(v) = g'(v) = u$ which completes the proof. □

Proposition 4.3.6 will be used again to modify the next result.

Lemma 4.3.9 ([Kay94]). *Let M be a countable recursively saturated model of* PA*. For each initial segment I of M closed under exponentiation, the closure of $G_{(>I)}$ is $G_{(I)}$ in all cases except when $I = \log^{\mathbb{N}}(a)$ for some $a \in M$. In that case, $G_{(>I)}$ is already closed being $G_{(J)}$ for $J = 2^a_{\mathbb{N}}$.*

Using this result we get:

Lemma 4.3.10. *Let $M(a)$ be a countable short recursively saturated model of* PA*. For each initial segment I of $M(a)$ closed under exponentiation, the closure of $G(a)_{(>I)}$ is $G(a)_{(I)}$ in all cases except when $I = \log^{\mathbb{N}}(b)$ for some $b \in M(a)$. In that case, $G(a)_{(>I)}$ is already closed being $G(a)_{(J)}$ for $J = 2^b_{\mathbb{N}}$.*

Proof. Suppose that $I = \log^{\mathbb{N}}(b)$ and $J = 2^b_{\mathbb{N}}$ for some $b \in M(a)$. Clearly, $G(a)_{(J)} \subseteq G(a)_{(>I)}$. Let $g \in G(a)_{(>I)}$. Then g fixes some $I' \supset I$ pointwise. Since $I = \log^{\mathbb{N}}(b)$, $I' \supseteq \log^n(b)$ for some $n \in \mathbb{N}$. Since g fixes I', it must fix $2^b_{\mathbb{N}} = J$ because $I_{fix}(g)$ is closed under exponentiation. Hence, $g \in G(a)_{(J)}$, and so $G(a)_{(J)} = G(a)_{(>I)}$.

Suppose now that $I \neq \log^{\mathbb{N}}(b)$ for all $b \in M(a)$. Since $G(a)_{(I)}$ is closed, $\overline{G(a)_{(>I)}} \subseteq G(a)_{(I)}$. Let $g \in G(a)_{(I)}$. We need to show $g \in \overline{G(a)_{(>I)}}$. Suppose that $g(c) = d$ for some $c, d \in M(a)$. We will show that there is $h \in G(a)_{(>I)}$ with $h(c) = d$.

Find $e \in \text{gap}(a)$ with $c, d < e$ and $g(e) = e$. By Proposition 4.3.6 there is a $g' \in G$ and J an initial segment of $M(a)$ containing e (and hence a, b, and I) such that $g'|_J = g|_J$. Now, since g' fixes e, it is an automorphism of the recursively saturated structure (M, e). Let $H = \text{Aut}((M, e))$. Then $g' \in H_{(I)}$. By Lemma 4.3.9, $H_{(I)}$ is in the closure of $H_{(>I)}$, and so there is an automorphism $h' \in H_{(>I)}$, with $h'(c) = d$. But this automorphism fixes $M(a)$ setwise, since it fixes $e \in \text{gap}(a)$. Let $h = h'|_{M(a)}$. Then $h \in G(a)_{(>I)}$ and $h(c) = d$. Hence, g is in the closure of $G(a)_{(>I)}$. □

Theorem 4.3.11. *Let $g \in G(a)$. Then either the closure of the set*

$$\{f_1^{-1} g^{-1} f_1 f_2^{-1} g f_2 : f_1, f_2 \in G(a)\},$$

or the closure of the set

$$\{f_1^{-1} g f_1 f_2^{-1} g^{-1} f_2 : f_1, f_2 \in G(a)\}$$

is a normal subgroup of $G(a)$.

Proof. The proof is almost identical to the proof of the analogous result from [Sch01].

Let $I = I_{fix}(g)$. Suppose that $g(x) < x$ for an arbitrarily small $x > I$. We will show that the first alternative works (if there is $e > I$ such that $g(x) > x$ for all $I < x < e$, replace g with g^{-1}).

Let J be the largest invariant initial segment such that $J \subseteq I$. We will show that the closure of $\{f_1^{-1}g^{-1}f_1f_2^{-1}gf_2 : f_1, f_2 \in G(a)\}$ is $G(a)_{(J)}$. Since $g \in G(a)_{(J)}$, and $G(a)_{(J)}$ is closed and normal, $\overline{\{f_1^{-1}g^{-1}f_1f_2^{-1}gf_2 : f_1, f_2 \in G(a)\}} \subseteq G(a)_{(J)}$. We need to show $\overline{\{f_1^{-1}g^{-1}f_1f_2^{-1}gf_2 : f_1, f_2 \in G(a)\}} \supseteq G(a)_{(J)}$.

Notice that if $\log^{\mathbb{N}}(b)$ is invariant for some $b \in M(a)$, then also $2^b_{\mathbb{N}}$ is invariant. Since there are no initial segments closed under exponentiation between $\log^{\mathbb{N}}(b)$ and $2^b_{\mathbb{N}}$, and since I is closed under exponentiation, if $J = \log^{\mathbb{N}}(b)$, then $I = J$ (otherwise $I \supseteq 2^b_{\mathbb{N}}$ which contradicts the fact that J is the largest invariant initial segment contained in I). It follows then from Lemma 4.3.10, that if $I \neq J$ then $\overline{G(a)_{(>J)}} = G(a)_{(J)}$. We will assume that $I \neq J$ (The case $I = J$ is simpler and we will remark on how to proceed in that case shortly).

Let $g' \in G(a)_{(J)}$. To show that g' is in the closure of $\{f_1^{-1}g^{-1}f_1f_2^{-1}gf_2 : f_1, f_2 \in G(a)\}$, we need to show that for any $p \in M(a)$ there are $f_1, f_2 \in G(a)$ with

$$f_1^{-1}g^{-1}f_1f_2^{-1}gf_2(p) = g'(p).$$

Let $p \in M(a)$ and $q = g'(p)$. If $p = q$, just take $f_1 = f_2 = id$.

Suppose $p < q$. By the assumption on J, $\overline{G(a)_{(>J)}} = G_{(J)}$. Hence, there exists $h' \in G(a)_{(>J)}$ with $h'(p) = q$. Since h' fixes an initial segment greater than J, and I is not invariant (since $I \neq J$), then using short recursive saturation, one can show that there is an $f \in G(a)$ such that $f^{-1}h'f$ fixes I. Let $h = f^{-1}h'f$. Then $h \in G(a)_{(I)}$ (remark: when $I = J$ we can take $h' = g'$, and f to be the identity). Let $c = f^{-1}(p)$, and $d = f^{-1}(q)$. So $h(c) = d > c$. Therefore, we can apply Lemma 4.3.8. That is, there are $b \in \text{gap}(a)$, and $u, v, w \in M(a)$, such that $g(b) = b$, $g(v) = u < v$, $\text{tp}(u, v, b) = \text{tp}(u, w, b)$, and $\text{tp}(v, w, b) = \text{tp}(c, d, b)$. Hence, there are automorphisms f_3 and f_4 of $M(a)$, with $f_3(u, v, b) = (u, w, b)$, and $f_4(v, w, b) = (c, d, b)$. Let $f_1 = f_3^{-1} f_4^{-1} f^{-1}$, and $f_2 = f_4^{-1} f^{-1}$. Then,

$$f_1^{-1} g^{-1} f_1 f_2^{-1} g f_2(p) = f f_4 f_3 g^{-1} f_3^{-1} g f_4^{-1} f^{-1}(p) = f f_4 f_3(v) = f(d) = q.$$

The case $p > q$ is similar, with few changes. Use the same h' and f, but let $h = f^{-1} h'^{-1} f$. Also, let $c = f^{-1}(q)$, and $d = f^{-1}(p)$. Then again $h(c) = d > c$. Hence, we can use the same lemma again and the same f_3 and f_4. Let $f_1 = f_4^{-1} f^{-1}$ and $f_2 = f_3^{-1} f_4^{-1} f^{-1}$, and we get

$$f_1^{-1} g^{-1} f_1 f_2^{-1} g f_2(p) = f f_4 g^{-1} f_3 g f_3^{-1} f_4^{-1} f^{-1}(p) = f f_4(v) = f(c) = q.$$

□

As a corollary we get:

Theorem 4.3.12. *Let $M(a)$ be a countable short recursively saturated model of* PA*. Let N be a closed normal subgroup of $G(a) = \text{Aut}(M(a))$. Then $N = G(a)_{(I)}$ for some invariant initial segment $I \subset M(a)$.*

Proof. Let

$$I = I_{fix}(N) = \bigcap_{h \in N} I_{fix}(h).$$

We will show that $N = G(a)_{(I)}$. Since all the automorphisms in N fix I pointwise, $N \subseteq G(a)_{(I)}$.

If there is a $g \in N$ such that $I_{fix}(g) = I$ (as, in particular, in the case where $I = \log^{\mathbb{N}}(b)$ for some $b \in M(a)$, since otherwise $I \geq 2^b_{\mathbb{N}} > \log^{\mathbb{N}}(b)$), then by the previous theorem and since N is normal and closed

$$N \supseteq \overline{\{f_1^{-1}g^{-1}f_1f_2^{-1}gf_2 : f_1, f_2 \in G(a)\}} \supseteq G(a)_{(I)}.$$

Suppose that there is no such g. Let $J > I$. By the way I was defined, there is a $g \in N$ with $I' = I_{fix}(g) \leq J$. Hence, again, by the previous theorem and since N is normal and closed

$$N \supseteq \overline{\{f_1^{-1}g^{-1}f_1f_2^{-1}gf_2 : f_1, f_2 \in G(a)\}} \supseteq G(a)_{(I')} \supseteq G(a)_{(J)}.$$

Therefore, $N \supseteq G(a)_{(>I)}$. But since N is closed and $I \neq \log^{\mathbb{N}}(b)$ for all $b \in M(a)$, then $N \supseteq \overline{G(a)_{(>I)}} = G(a)_{(I)}$. (This actually shows that there must be a $g \in N$ with $I_{fix}(g) = I$).

Finally, notice that by Proposition 3.2.6, since $G(a)_{(I)}$ is normal in $G(a)$, I is an invariant initial segment of $M(a)$. □

Corollary 4.3.13. *Let $N(b)$ be a short recursively saturated model of* PA *whose last gap is labeled and $K(c)$ a short recursively saturated model of* PA *whose last gap is non-labeled. Then* $\mathrm{Aut}(N(b))$ *is not isomorphic to* $\mathrm{Aut}(K(c))$ *as topological groups.*

Proof. Notice that it follows from Theorem 4.3.12 that for any countable short recursively saturated model of PA, the family of closed normal subgroups is linearly ordered by inclusion. Since the last gap in $N(b)$ is labeled, it follows from Proposition 4.3.4 that there is no largest proper invariant initial segment of $N(b)$. This implies that there is no smallest nontrivial normal subgroup.

On the other hand, since the last gap in $K(c)$ is non-labeled, it follows from Proposition 4.3.3 that there is a largest proper invariant initial segment, $K[c]$. This implies that $\mathrm{Aut}(K(c))_{(K[c])}$ is the smallest non-trivial normal subgroup of $\mathrm{Aut}(K(c))$.

Suppose for a contradiction that $\mathrm{Aut}(K(c))$ is isomorphic to $\mathrm{Aut}(N(b))$ as topological groups and let f be such isomorphism. Since f is an isomorphism of topological groups, it must map closed normal subgroups to closed normal subgroups. But then the image of $\mathrm{Aut}(K(c))_{(K[c])}$ must be the smallest nontrivial normal subgroup of $N(b)$ which gives us a contradiction. Therefore, $\mathrm{Aut}(N(b)) \ncong \mathrm{Aut}(K(c))$ as topological groups. □

4.4 Strong Standard Cuts Versus Weak Standard Cuts

In this section we show that countable short arithmetically saturated models of PA and countable short recursively saturated models of PA that are not short arithmetically saturated have automorphism groups which are not isomorphic (as topological group).

The analogous result for recursively saturated models can be found in [KKK91]. At the end of the section we prove that there are short recursively saturated models which are not short arithmetically saturated yet their automorphism groups are not topologically isomorphic (regardless of the types in their last gap).

Again, for any model M and any $a \in M$, let $G = \mathrm{Aut}(M)$ and $G(a) = \mathrm{Aut}(M(a))$. For any automorphism g of a model M, let $\mathrm{fix}(g)$ denote the set of elements in M fixed by g. For any type $p(v, u)$, let p_b^M denote the set of elements realizing the type $p(v, b)$ in M.

Recall that arithmetically saturated models are exactly the recursively saturated models whose standard cut is strong, and short arithmetically saturated models are the short recursively saturated models whose standard cut is strong.

Proposition 4.4.1 ([KKK91])**.** *Let M be a recursively saturated model of* PA *and suppose that $\mathbb{N}$ is weak in M. Let $g \in G$ and let $p(x)$ be a complete parameter free type realized in M. Then $\mathrm{fix}(g) \cap p^M \neq \emptyset$.*

Proposition 4.4.2. *Let $M(a)$ be a countable short recursively saturated model of* PA *and suppose that $\mathbb{N}$ is weak in $M(a)$. Let $g \in G(a)$ and let $p(x)$ be a complete parameter free type realized in $M(a)$. Then $\mathrm{fix}(g) \cap p^{M(a)} \neq \emptyset$.*

Proof. Let $p(v)$ be a complete type realized in $M(a)$. Let $b \in \mathrm{gap}(a)$ be such that $g(b) = b$. Let $d \in M(a)$ realize $p(v)$ and let $q(v, w) = \mathrm{tp}(d, b)$. Since $q(v, b)$ is complete and is realized in $M(a)$, and since b is in the last gap of $M(a)$, $q(v, b)$ must contain the formula $v < t(b)$ for some Skolem term t. Let M be a countable recursively saturated elementary end extension of $M(a)$. Then, by Theorem 4.3.6, there are a proper initial segment $I \subset_{end} M(a)$ containing both b and $t(b)$, and an automorphism $h \in G$, such that $h|_I = g|_I$.

Since h fixes b, it is an automorphism of the expanded recursively saturated structure (M, b). Since $(M(a), b) \prec (M, b)$, $q(v, b)$ is a complete type realized in (M, b). In the structure (M, b), q is parameter free. Furthermore, since M is an elementary end extension of $M(a)$, $\mathbb{N}$ is weak in M as well. Thus, by Proposition 4.4.1, $\text{fix}(h) \cap q^{(M,b)} \neq \emptyset$. Since all realization of q are by elements less than $t(b)$, $\text{fix}(h) \cap q^{(M,b)} = \text{fix}(g) \cap q_b^{M(a)}$, and since any element realizing $q(v, b)$ must realize $p(v)$, $\text{fix}(g) \cap p^{M(a)} \neq \emptyset$. $\square$

Remark: The above result is true in the uncountable case as well, and can be proven by mimicking the proof from [KKK91]. However, since we are interested here in the countable case, we supplied a simpler proof using results from previous sections and Proposition 4.4.1.

Proposition 4.4.3 ([KKK91]). *Let $M \models$ PA be a countable recursively saturated model and assume that $\mathbb{N}$ is strong in M. Then there is $g \in G$ which moves all non-definable elements. Moreover, for every $a \in M$ there is $g \in G$ such that* $\text{fix}(g) = \text{Scl}(a)$.

Notice that since every automorphism of a short recursively saturated model of PA fixes some elements in the last gap (see Corollary 3.3.11), the above result does not apply to short recursively saturated models that are not extremely short. However, we still get the following:

Proposition 4.4.4. *Let* $M(a) \models \mathsf{PA}$ *be a countable short recursively saturated model and assume that* $\mathbb{N}$ *is strong in* $M(a)$*. Then for every* $b \in \mathrm{gap}(a)$*, there is* $g \in G(a)$*, such that* $\mathrm{fix}(g) = \mathrm{Scl}(b)$*.*

Proof. Let $b \in M(a)$ and consider M, a countable recursively saturated elementary end extension of $M(a)$. Since M is an elementary end extension of $M(a)$, $\mathbb{N}$ is strong in M as well. Thus, by the previous result there is a $g' \in G$ with $\mathrm{fix}(g') = \mathrm{Scl}(b)$. Since g' fixes $b \in \mathrm{gap}(a)$, it fixes $\mathrm{gap}(a)$ setwise, so $g'|_{M(a)} \in G(a)$. Let $g = g'|_{M(a)}$. Then by elementarity , $\mathrm{Scl}^M(b) = \mathrm{Scl}^{M(a)}(b)$, so $\mathrm{fix}(g) = \mathrm{Scl}^{M(a)}(b)$. □

Note that when $M(a)$ is extremely short we can take $b \in \mathrm{Scl}(0)$. Thus, in this case, there is an automorphism of the model moving all non-definable elements.

Theorem 4.4.5. *Let* $M(a) \models \mathsf{PA}$ *be a countable short recursively saturated model. Then* $\mathbb{N}$ *is strong in* $M(a)$ *iff there exists* $g \in G(a)$ *and an open subgroup* $H < G(a)$ *such that for every* $f \in G(a)$*,* $f^{-1}gf \notin H$*.*

Proof. Suppose that $\mathbb{N}$ is strong in $M(a)$. We will assume that $M(a)$ is not extremely short, and later comment on the extremely short case. Let $b \in \mathrm{gap}(a)$. Let $c \in M(a) \setminus \mathrm{gap}(a)$ be such that $\mathrm{tp}(b) = \mathrm{tp}(c)$ (such c exists by Lemma 3.3.4). Since b

is in the last gap and c is not, for any automorphism $f \in G(a)$, $f(c) \neq b$. Also, by Ehrenfeucht-Gaifman's lemma (Lemma 4.2.2), if $d \in \mathrm{Scl}(b)$ and $d \neq b$, for any automorphism $f \in G(a)$, $\mathrm{tp}(d) \neq \mathrm{tp}(b) = \mathrm{tp}(f(c))$. Thus, for any automorphism $f \in G(a)$, $f(c) \notin \mathrm{Scl}(b)$. Let $H = G(a)_{(c)}$. By Proposition 4.4.4, there is $g \in G(a)$ with $\mathrm{fix}(g) = \mathrm{Scl}(b)$. But then, for all $f \in G(a)$, since $f(c) \notin \mathrm{Scl}(b)$, $g(f(c)) \neq f(c)$, so $f^{-1}gf(c) \neq c$. Therefore, $f^{-1}gf \notin H$.

Remark: in the extremely short case, simply take $b \in \mathrm{Scl}(0)$ and $c \notin \mathrm{Scl}(0)$.

To prove the converse, suppose that $\mathbb{N}$ is weak in $M(a)$. Let $g \in G(a)$ and let H be any open subgroup of $G(a)$. Since H is an open subgroup, there exists $b \in M(a)$ such that $G(a)_{(b)} \subseteq H$. Let $d \in \mathrm{gap}(a)$ and let

$$p(v, d) = \{\phi(v, u) : M(a) \models \phi(b, d)\}.$$

Since this type is complete and realized in $M(a)$ by b, then by Proposition 4.4.2, $\mathrm{fix}(g) \cap p^{M(a)} \neq \emptyset$. Let $c \in \mathrm{fix}(g) \cap p^{M(a)}$. Since $\mathrm{tp}(c, d) = \mathrm{tp}(b, d)$, and since $d \in \mathrm{gap}(a)$, by Corollary 3.3.3, there is an automorphism $f \in G(a)$, such that $f(b) = c$. But then

$$f^{-1}gf(b) = f^{-1}g(c) = f^{-1}(c) = b.$$

Therefore, $f^{-1}gf \in H$. □

Remark: This result is true for countable recursively saturated models of PA and was proved in 1991 by Kaye, Kossak, and Kotlarski [KKK91].

Since short arithmetically saturated models of PA have strong standard cuts, and since short recursively saturated models of PA which are not short arithmetically saturated have weak standard cuts, we get:

Corollary 4.4.6. *Let M_1 and M_2 be countable short recursively saturated models of* PA*, and suppose that M_1 is short arithmetically saturated and M_2 is not. Then* $\mathrm{Aut}(M_1) \ncong \mathrm{Aut}(M_2)$ *as topological groups.*

In Section 4.3 we showed that the automorphism groups of models whose last gaps are labeled are not topologically isomorphic to the automorphism groups of models whose last gaps are non-labeled. In this section we showed that that the automorphism groups of short arithmetically saturated models are not topologically isomorphic to the automorphism groups of short recursively saturated models that are not short arithmetically saturated. Since any recursively saturated model has labeled as well as non labeled gaps, regardless of its standard system, we get four topologically non-isomorphic automorphism groups.

The characterization of closed normal subgroups in Theorem 4.3.12, allows us to find other countable short recursively saturated models of PA whose automorphism groups are non-isomorphic (as topological groups).

In 1994 , Richard Kaye [Kay94] showed that there are recursively saturated models of PA (and hence short recursively saturated models), $M \nvDash \mathrm{Th}(\mathbb{N})$ which have a smallest nonstandard invariant initial segment (when $\{x : x < y \text{ for all } y \in \mathrm{Scl}(0) \backslash \mathbb{N}\} \neq$

$\mathbb{N}$). In the same paper, he showed that some recursively saturated models of PA (and thus short recursively saturated models), $N \not\models \mathrm{Th}(\mathbb{N})$, do not have a smallest nonstandard invariant initial segment (when $\{x : x < y \text{ for all } y \in \mathrm{Scl}(0)\backslash\mathbb{N}\} = \mathbb{N}$).

The first case is true for all models which code the nonstandard definable elements of the model, since by recursive saturation one can show that there are elements between the nonstandard definable elements and $\mathbb{N}$. Every arithmetically saturated model codes the set of nonstandard definable elements of the model, (see [KKK91]), but also some non arithmetically saturated models code such sets [Kay94]. Hence, by Theorem 4.3.12 the automorphism groups of such models have a largest proper normal subgroup $G(a)_{(I)}$, where $I = \{x : x < y \text{ for all } y \in \mathrm{Scl}(0)\backslash\mathbb{N}\}$.

The second case is true for some other non arithmetically saturated models of PA which were shown to have this property in [Kay94]. Thus, the automorphism groups of these models have no largest proper normal subgroup. This implies that the automorphism groups of models of the first case are not topologically isomorphic to models of the second case.

The above distinction between the non arithmetically saturated models of the first case and those of the second case, depends on the standard systems of their corresponding models. Combining this with the remarks following Corollary 4.4.6, gives us six types of short recursively saturated models $M \models \mathsf{PA}$ with $M \not\models \mathrm{Th}(\mathbb{N})$, whose automorphism groups are non isomorphic as topological groups:

1. Short arithmetically saturated with a labeled last gap.

2. Short arithmetically saturated with a non-labeled last gap.

3. Non-short arithmetically saturated which have a smallest nonstandard invariant initial segment and a labeled last gap.

4. Non-short arithmetically saturated which have a smallest nonstandard invariant initial segment and a non-labeled last gap.

5. Non-short arithmetically saturated which do not have a smallest nonstandard invariant initial segment and have a labeled last gap.

6. Non-short arithmetically saturated which do not have a smallest nonstandard invariant initial segment and have a non-labeled last gap.

When $M \models \mathrm{Th}(\mathbb{N})$ we get the four types discussed after Corollary 4.4.6. Other possible distinctions will be discussed in Chapter 5.

Chapter 5

Questions

In this chapter we will summarize some of the important results of this dissertation and list several open questions. As before, let M be a countable recursively saturated model of arithmetic and let $a \in M$. We will use the same notation from previous chapters to denote the various automorphism groups and subgroups.

In section 2.1 we discussed cofinal resplendency and showed that only recursively saturated models possess this property. The notion of cofinal resplendency sprouted from an attempt to modify the notion of resplendency to apply to short recursively saturated models.

Question 1. *Can we modify the notion of cofinal resplendency to get a similar property which is realized by short recursively saturated models of arithmetic?*

A positive answer to this question may help expand the language to languages containing automorphisms and other interesting subsets.

In section 2.2 we showed that there is a set $X \subseteq M$ such that $(M(a), X \cap M(a))$ is still short recursively saturated (see proposition 2.2.6). The set constructed in the proof had only countably many images under the action of the automorphism group of $M(a)$ (see proposition 2.2.7). An interesting question is the following:

Question 2. *Is there $X \subseteq M$ such that $(M(a), X \cap M(a))$ is short recursively saturated and $X \cap M(a)$ having continuum many images under the action of $G(a)$?*

In section 4.1 we discussed many properties of the subgroup $G|_{M(a)}$. One property that is still not resolved involves maximality.

Question 3. *Is the subgroup $G|_{M(a)}$ maximal in $G(a)$?*

In section 4.3 we proved that Kaye's theorem characterizing closed normal subgroups of the automorphism groups of countable recursively saturated models of arithmetic applies also to countable short recursively saturated models. In the end of section 3.2 we mentioned Kaye's conjecture characterizing all normal subgroups of countable recursively saturated models arithmetic. The natural question that arises is:

Question 4. *Do all normal subgroups of $G(a)$ are of the form $G(a)_{(I)}$ or $G(a)_{(>I)}$ for some invariant initial segment I?*

In this dissertation we showed that there are several countable short recursively saturated models of PA with non-isomorphic automorphism groups.

Question 5. *Can we find other types of countable short recursively saturated models of* PA *which have non-isomorphic automorphism groups as topological groups?*

I believe that the answer to this question is positive. In particular, when $M \models \mathrm{Th}(\mathbb{N})$ is countable short recursively saturated model, it may have exactly one nontrivial normal subgroup (in contrast any countable recursively saturated model of true arithmetic has no nontrivial normal subgroups). This happens when the last gap of the model is non-labeled, yet the last gap has elements realizing a $quasi-selective$ type. A type is quasi-selective if the Skolem closure of any element realizing this type, say a, contains only elements from $\mathrm{Scl}(0)$ and $\mathrm{gap}(a)$. Such gaps were proven to exist by Kossak, Kotlarski, and Schmerl in [KKS93]. If a short recursively saturated model $M(a)$ of true arithmetic has such last gap, the only invariant nonstandard initial segment of the model is $M[a]$. Thus, its only nontrivial closed normal subgroup is $G(a)_{(M[a])}$. In the same paper, they have shown that there are gaps which are non-labeled, yet any automorphism which fixes the gap setwise, must fix elements below this gap. Thus, countable short recursively saturated models with such last gaps have other invariant initial segments which implies other nontrivial closed normal subgroups.

A harder problem regarding automorphisms that one might want to investigate is the following:

Question 6. *Are there two non-isomorphic countable short recursively saturated models of* PA *whose automorphism groups are isomorphic?*

Remark: This question is also open for countable recursively saturated models.

Question 7. *Let N be a recursively saturated model of* PA *and let $b \in N$. Is it always true that* $\mathrm{Aut}(N) \not\cong \mathrm{Aut}(N(b))$ *as topological groups?*

For many models the answer is positive. For example, since every recursively saturated model of PA has a smallest nontrivial closed normal subgroup ($G_{(M(0))}$), its automorphism group is not isomorphic as a topological group to the automorphism group of any short recursively saturated model of PA whose last gap is labeled (since we have shown in the proof of Theorem 4.3.13 that these models do not have a smallest non-trivial closed normal subgroup).

One of the most important questions in model theory which arises with regard to automorphism groups involves the small index property. A countable model has the small index property if the only subgroups of its automorphism group which have a countable index are the open subgroups. If a model has this property and its automorphism group is non-isomorphic to an automorphism group of a second model as topological groups, then these automorphism groups are non-isomorphic also as abstract groups. Thus, we ask:

Question 8. *Which short recursively saturated models (if any) possess the small index property?*

Since countable arithmetically saturated models of arithmetic have this property [Las94], one might want to investigate first short arithmetically saturated models.

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