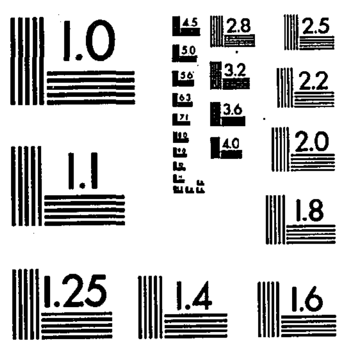

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TOPOLOGICAL EQUIVALENCE OF FLOWS ON HOMOGENEOUS SPACES, DIVERGENCE OF ONE-PARAMETER SUBGROUPS, AND ASYMPTOTIC HOMOTOPY CLASSES

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TOPOLOGICAL EQUIVALENCE OF FLOWS ON HOMOGENEOUS SPACES, DIVERGENCE OF ONE-PARAMETER SUBGROUPS, AND ASYMPTOTIC HOMOTOPY CLASSES

by

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August 15, 1985
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DEDICATION

This dissertation is dedicated to my mother, Paula Ovadia Benardete, and my father, Mair Jose Benardete, for encouraging me to pursue the life of the mind.

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Section 1 Introduction

Let G be a connected Lie group, Γ a discrete subgroup, $\phi: R \longrightarrow G$ a one-parameter subgroup, and $\phi^*: (R \times G/\Gamma) \longrightarrow G/\Gamma$, where $\phi^*: (t, x\Gamma) \longmapsto (\phi(t)x)\Gamma$, the G-induced flow. G/Γ is a homogenous space.

Since the book "Flows on Homogenous Spaces" by Auslander, Green, and Hahn [2], the ergodic theory and topological dynamics of these flows have been much studied because they provide interesting examples which can be understood in this algebraic setting. (See for example, Auslander [1], Brezin and Moore [4].) The equivalences studied have usually been parameter preserving, measure theoretic or topological maps. (See Parry [19], Walters [26], Ratner [21,22], and Witte [27].) In this paper we study topological equivalences which do not preserve the parameter. We use results concerning the divergence of one-parameter subgroups. These results may have their own geometric interest.

Definition Two flows are topologically equivalent if and only if there is a homeomorphism taking trajectories to trajectories, preserving the sense but not necessarily the parameterization.

Definition A topological conjugacy of flows is a parameter preserving topological equivalence.

Definition Let G and G' be Lie groups and Γ and Γ' discrete subgroups. An affine map from G/Γ to G/Γ' is a map of the form $\bar{a}\,\bar{A}: x\Gamma \longmapsto (a(Ax))\Gamma'$ where $a \in G'$ and A is an isomorphism of G onto G' which extends an isomorphism of Γ onto Γ'.

<u>Definition</u> A discrete subgroup Γ is a <u>lattice</u> of G if and only if there exists a finite Borel measure on G/Γ invariant under affine maps of the form $\bar{a}: x\Gamma \longmapsto (ax)\Gamma$, for all a in G.

<u>Definition</u> H is a <u>uniform</u> subgroup of G if and only if G/H is compact.

A uniform discrete subgroup is a lattice [20].

<u>Notation</u> For any Lie group G, L(G) is its Lie algebra.

<u>Theorem A</u> Let G and G' be connected Lie groups, Γ and Γ' be lattices of G and G', and ϕ^* and ϕ'^* flows on G/Γ and G'/Γ' induced by one-parameter subgroups ϕ and ϕ' of G and G'. If any of Conditions (a), (b), or (c) hold, then ϕ^* and ϕ'^* are topologically equivalent if and only if they are topologically equivalent by an affine map.

<u>Condition (a)</u> G and G' are simply connected and nilpotent.

<u>Condition (b)</u> G and G' are simply connected and solvable, and, for any X in L(G) and X' in L(G'), ad(X) and ad(X') have only real eigenvalues.

<u>Condition (c)</u> G and G' are semi-simple with trivial center and no compact direct factor. There is no direct factor G_i of G, such that G_i is isomorphic to PSL(2,R) and ΓG_i is a closed subgroup of G. There is no compact connected subgroup in the centralizer of ϕ.

Theorem A generalizes a classical result in the case where G is R^n, Γ is Z^n, and G/Γ is the n-torus [8]. When G and G' are nilpotent, Theorem A sharpens a result of L. Auslander, F. Hahn, and L. Markus. They showed that if ϕ^* and ϕ'^* are topologically equivalent, then the induced toral flows are topologically equivalent [2, p. 51]. B. Marcus showed that when G = G' = PSL(2,R) and ϕ and ϕ' are unipotent, then topologically equivalent flows are topologically equivalent by an affine map [14]. Theorem A does not imply Marcus' result, but it does imply a similar result when G = G' = SL(3,R).

The restriction on eigenvalues in Condition (b) insures that isomorphisms of the lattices extend to isomorphisms of the Lie groups, and that $\exp: L(G) \longrightarrow G$ is a diffeomorphism. The restrictions on G, G', Γ, and Γ' in Condition(c) insure that the Mostow rigidity theorem can be applied to show that isomorphisms of the lattices extend to isomorphisms of the Lie groups.

In order to prove Theorem A, in Theorem 1 we abstract and generalize a proof of the toral case. This proof depends on the fact that one-parameter subgroups of R^n diverge from each other. Therefore to apply Theorem 1, we need Theorem B on the divergence properties of one-parameter subgroups.

ϕ is <u>recurrently approached</u> by ϕ' if, for positive and negative time, ϕ' keeps returning to within a bounded distance of ϕ, where distance is induced from a left or right invariant Riemannian metric. ϕ is <u>isolated</u>, if ϕ being recurrently approached by ϕ' implies that $\phi = \phi'$ up to sense-preserving reparameterization. (See Section 2 and Section 10(i) for precise definitions and more details.)

<u>Theorem B</u> Let G be a connected Lie group and ϕ a one-parameter subgroup.

(a) If G is simply connected and nilpotent, or if G is simply connected and solvable with $\exp: L(G) \longrightarrow G$ a diffeomorphism, then every ϕ of G is isolated.

(b) Let G be semi-simple. $\phi(t) \equiv \exp(tX)$ is isolated if $[X,Y] = 0$ implies that when ad(Y) is semisimple then ad(Y) has some eigenvalue which is not pure imaginary and not 0. Let the center of G be finite. Then ϕ is isolated if there is no compact subgroup in the centralizer of ϕ.

In Section 2 isolation concepts are defined and some elementary properties of them are demonstrated in Proposition 1. In Section 3, Theorem 1 is proved. Theorem 2, which demonstrates the isolation of subgroups in the nilpotent and solvable case, is proved in Section 4. Theorem 3, which concerns the isolation of subgroups of SL(n,R), is proved in Section 5. Theorem 3 requires Proposition 2, which reduces the problem to Proposition 3, which deals with unipotent subgroups. Proposition 2 is in Section 6 and Proposition 3 is in Section 7. Theorem 4, which demonstrates isolation of subgroups in the semi-simple case, is proved in Section 8. Section 9 proves our main results, Theorems A and B. Section 10 contains examples, counterexamples, comments, and questions. Section 11 develops the rudiments of a theory of asymptotic homotopy classes, and applies it to give an alternative proof of the classification of nilflows (Theorem A(a)).

Section 2 Isolation of One-Parameter Subgroups

In this section different divergence properties of one-parameter subgroups are defined and discussed.

Let G be a connected Lie group, ϕ and ϕ' one-parameter subgroups, and $d:(G \times G) \longrightarrow [0,\infty)$ a continuous function.

Definition 1 (a) Let $\mathcal{D}_G$ be the set of continuous functions $d:(G \times G) \longrightarrow [0,\infty)$ such that for all g, h, and k in G, $d(gk,hk) = d(g,h)$, and for all closed sets X in G, if the set $\{d(x,e) \mid x \in X$ and e is the identity$\}$ is bounded then X is compact.

(b) Assume G is a Lie subgroup of GL(n,R) in some fixed representation. Let $\mathcal{N}_G$ be the set of $d:(G \times G) \longrightarrow [0,\infty)$ such that $d(g,h) \equiv \|gh^{-1}\|$ for some norm $\| \ \|$ on M(n,R), the vector space of (n,n) matrices. We write $\mathcal{D}$ and $\mathcal{N}$ if G is understood.

Example Every metric induced by a right invariant Riemannian metric is in $\mathcal{D}$, since for such a metric, closed bounded sets are compact [11, p. 56].

Definition 2 (a) ϕ is recurrently approached in d by ϕ' if and only if there exist sequences indexed by the integers

$$\{t_k \mid k \in Z,\ t_k \in R,\ \mathrm{sign}(k) = \mathrm{sign}(t_k),\ t_k \to \pm\infty \text{ as } k \to \pm\infty\}$$

and $\{s_k \mid k \in Z,\ s_k \in R,\ \mathrm{sign}(k) = \mathrm{sign}(s_k),\ s_k \to \pm\infty \text{ as } k \to \pm\infty\}$, such that $\sup\{d(\phi(t_k),\phi'(s_k)) \mid k \in Z\} < \infty$.

(b) ϕ is recurrently approached by ϕ' if and only if ϕ is recurrently approached in d by ϕ' for some $d \in \mathcal{D}$. (see Proposition 1(a), below)

(c) ϕ is <u>recurrently approached in norm</u> by ϕ' if and only if ϕ is recurrently approached in d by ϕ' for some d $\in \mathcal{N}$. (see Proposition 1(a), below)

<u>Definition 3 (a), (b), and (c)</u> ϕ is (a) <u>isolated in d</u> ((b) <u>isolated</u>, (c) <u>isolated in norm</u>) if and only if ϕ being (a) recurrently approached in d ((b) recurrently approached, (c) recurrently approached in norm) by ϕ' implies that $\phi = \phi'$ up to sense-preserving reparameterization.

<u>Proposition 1</u> (a) If d and d' are both in $\mathcal{B}$ or both in $\mathcal{N}$, then the properties of being recurrently approached in d by ϕ' and of being isolated in d are equivalent respectively to the properties of being recurrently approached in d' by ϕ' and of being isolated in d'.

(b) ϕ is recurrently approached by ϕ' if and only if ϕ' is recurrently approached by ϕ.

(c) Let $d' : (G \times G) \longrightarrow [0,\infty)$ be a continuous function such that $d'(g,h) = d'(gk,hk)$ for all g, h, and k in G. If ϕ is recurrently approached by ϕ', then ϕ is recurrently approached in d' by ϕ'. If ϕ is isolated in d', then ϕ is isolated.

(d) If ϕ is recurrently approached by ϕ', then ϕ is recurrently approached in norm by ϕ'. If ϕ is isolated in norm, then ϕ is isolated.

(e) If G is a closed subgroup of SL(n,R), then $\mathcal{N} \subset \mathcal{B}$, and the properties of being recurrently approached by ϕ' and of being isolated are equivalent respectively to the properties of being recurrently approached in norm by ϕ' and of being isolated in norm. If G is not closed in SL(n,R), the properties may not be equivalent.

(f) Let $d \in \mathcal{D}$ and let $d' : (g,h) \longmapsto d(g^{-1}, h^{-1})$, for all g and h in G. (For example, d' could be a metric induced by a left invariant Riemannian metric.) Then the properties of being recurrently approached by ϕ' and of being isolated are equivalent respectively to the properties of being recurrently approached in d' by ϕ' and of being isolated in d'.

(g) Let α and β be non-zero real numbers having the same sign. ϕ is recurrently approached or recurrently approached in norm by ϕ' if and only if the subgroup $t \longmapsto \phi(\alpha t)$ is respectively recurrently approached or recurrently approached in norm by the subgroup $t \longmapsto \phi'(\beta t)$. ϕ is isolated or isolated in norm if and only if the subgroup $t \longmapsto \phi(\alpha t)$ is respectively isolated or isolated in norm.

Proof (a) Let X be the closure in G of $\{\phi(t_k)\phi'(-s_k) \mid k \in Z\}$. For $d \in \mathcal{D}$ and e the identity of G,

$$\sup_{k \in Z} d(\phi(t_k), \phi'(s_k)) < \infty \iff \sup_{k \in Z} d(\phi(t_k)\phi'(-s_k), e) < \infty$$

$$\iff \sup_{x \in X} d(x,e) < \infty \iff X \text{ is compact.}$$

This demonstrates the result for d and d' in $\mathcal{D}$.

Let $\| \ \|$ and $\| \ \|'$ be norms on M(n,R), and for all g and h in G, let $d(g,h) \equiv \|gh^{-1}\|$ and $d'(g,h) \equiv \|gh^{-1}\|'$. Since there exist positive constants c and C such that for all y in M(n,R) $c\|y\|' \leq \|y\| \leq C\|y\|'$, Proposition 1(a) follows for d and d' in $\mathcal{N}$. (See Definitions 1(a), 1(b), 2(a), and 3(a).)

(b) For d induced by a right invariant Riemannian metric, d(g,h) = d(h,g), for all g and h in G. Since $d \in \mathcal{B}$, the result follows by Definition 2(b) and Proposition 1(a).

(c) Let X be the closure in G of $\{\phi(t_k)\phi'(-s_k) \mid k \in Z\}$. Then for some $d \in \mathcal{B}$ and e the identity of G,

$$\sup_{k \in Z} d(\phi(t_k), \phi'(s_k)) < \infty \Longrightarrow \sup_{x \in X} d(x,e) < \infty \Longrightarrow X \text{ is compact}$$
$$\Longrightarrow \sup_{x \in X} d'(x,e) < \infty \Longrightarrow \sup_{k \in Z} d'(\phi(t_k), \phi'(s_k)) < \infty .$$

(See Definitions 2(a), 2(b), 3(a), and 3(b).)

(d) Since $G \longrightarrow GL(n,R)$ is continuous, it follows that, for any norm $\| \ \|$, $d: (g,h) \longmapsto \|gh^{-1}\|$ is continuous on G x G . The result follows by Proposition 1(c).

(e) Let $d \in \mathcal{N}$ be induced by a norm $\| \ \|$. Let X be a closed set in G. Since G is closed in SL(n,R) which is closed in M(n,R), X is closed in M(n,R). Therefore

$$\sup_{x \in X} d(x,e) < \infty \text{ implies } \sup_{x \in X} \|x\| < \infty ,$$

which implies X is compact in M(n,R). Since it is a closed subgroup, G has the induced topology, and so X is compact in G [25, p. 59]. Therefore, by Definition 1(a), $d \in \mathcal{B}$.

As a counterexample, R^2 is not a closed subgroup of SL(n,R) in the representation $R^2 \longrightarrow T^4 \longrightarrow SL(8,R)$. Every subgroup of R^2 is isolated but not isolated in norm.

(f) The result follows by Definitions 2(a), 2(b), 3(a), and 3(b) from the following biconditionals.

$$\sup_{k\in Z} d'(\phi(t_k),\phi'(s_k)) < \infty \Longleftrightarrow \sup_{k\in Z} d(\phi(-t_k),\phi'(-s_k)) < \infty$$
$$\Longleftrightarrow \sup_{k\in Z} d(\phi(t_k),\phi'(s_k)) < \infty.$$

(g) The result follows immediately from Definitions 2 and 3. QED Proposition 1

Section 3   Theorem 1

Theorem 1 generalizes the following proof of the following classical theorem [8].

Classical Theorem For v and v' in R^n, let $\phi: t \longmapsto tv$ and $\phi': t \longmapsto tv'$ be one-parameter subgroups of R^n, and $\phi^*: (t, x + z^n) \longmapsto tv + x + Z^n$ and $\phi'^*: (t, x + Z^n) \longmapsto tv' + x + Z^n$ the induced flows on the torus R^n/Z^n. Then ϕ^* and ϕ'^* are topologically equivalent if and only if there exists A in GL(n,Z) and $s > 0$ such that $A(v) = sv'$.

Proof Let ϕ^* and ϕ'^* be toplogically equivalent by a homeomorphism f. After translating if necessary, we can assume that $f(0 + Z^n) = (0 + Z^n)$. Let $\tilde{f}$ be the lift of f to R^n. Let $A \equiv \tilde{f}|_{Z^n} \in GL(n,Z) \subset GL(n,R)$. $\tilde{f}$ takes ϕ to ϕ' bijectively as ordered sets. Hence, $A^{-1}\tilde{f}$ takes ϕ to $A^{-1}\phi'$ bijectively as ordered sets. $A^{-1}\tilde{f}$ moves points by a bounded amount since $A^{-1}\tilde{f}$ restricted to Z^n is the identity. Since one-parameter subgroups of R^n (i.e. straight lines) diverge, $\phi = A^{-1}\phi'$ and $A\phi = \phi'$ as ordered sets. That is there exists $s > 0$ such that $A(v) = sv'$.

The converse implication is clear. (See Lemma, below.)

QED Classical Theorem

The definitions of isolation in Section 2 and the definitions made below abstract relevant properties from the above proof.

Let G be a connected Lie group and Γ a discrete subgroup.

Definition 1 Let ϕ be a one-parameter subgroup of G. ϕ has compact recurrence in G/Γ if and only if there is a sequence indexed by the integers

$$\{r_k \mid k \in Z,\ r_k \in R,\ \text{sign}(r_k) = \text{sign}(k),\ r_k \longrightarrow \pm\infty \text{ as } k \rightarrow \pm\infty\}$$

such that there is a compact set in G/Γ containing $\{\phi(r_k)\Gamma \mid k \in Z\}$.

Examples (a) If G/Γ is compact, then every ϕ has compact recurrence in G/Γ.

(b) If G = SL(n,R), Γ = SL(n,Z), and ϕ is unipotent, then by the Margulis lemma ϕ has compact recurrence in G/Γ [7].

Definition 2 (G,Γ) has the automorphism extension property if and only if every automorphism of Γ extends to a Lie group automorphism of G.

Example $G = R^n$, $\Gamma = Z^n$, and Aut(Γ) = GL(n,Z) $\subset$ GL(n,R) = Aut(G).

Definition 3 Let e be the identity of G. Let $p: G \longrightarrow G/\Gamma$ be considered as a covering map, where p is the canonical projection. Then (G,G/Γ) has the homeomorphism lifting property if and only if every homeomorphism from (G/Γ,eΓ) to (G/Γ,eΓ) lifts to a homeomorphism from (G,e) to (G,e).

Example If G is simply connected, (G,G/Γ) has the homeomorphism lifting property.

Theorem 1 Let G be a connected Lie group, Γ a discrete subgroup, $p: G \longrightarrow G/\Gamma$ the canonical projection, ϕ and ϕ'

one-parameter subgroups, and $f: G/\Gamma \to G/\Gamma$ a topological equivalence of the flows ϕ^* and ϕ'^*.

(a) If ϕ is isolated, and ϕ has compact recurrence in G/Γ, and $f:(G/\Gamma, e\Gamma) \to (G/\Gamma, e\Gamma)$ has a lift $\tilde{f}:(G,e) \to (G,e)$ where e is the identity of G, and $\tilde{f}|_\Gamma$ is the identity on Γ, then ϕ^* and ϕ'^* are topologically equivalent by the identity.

(b) In (a) above, if $\tilde{f}|_\Gamma$ is not assumed to be the identity but (G,Γ) has the automorphism extension property, then ϕ^* and ϕ'^* are topologically equivalent by an affine map preserving $e\Gamma$.

(c) Let Γ be a lattice in G, and $(G, G/\Gamma)$ have the homeomorphism lifting property, and (G,Γ) have the automorphism extension property. Let ϕ be isolated. Then for any other ϕ' in G, the flows ϕ^* and ϕ'^* are topologically equivalent if and only if they are topologically equivalent by an affine map.

Proof We state without proof the following simple lemma.

Lemma (a) Let $\bar{a}\bar{A}$ be an affine map from G/Γ to G'/Γ'. Let ψ be a one-parameter subgroup of G. Then $\bar{a}\bar{A}$ is a topological conjugacy and hence a topological equivalence of the flows ψ^* and $(a(A\psi)a^{-1})^*$. (For notation, see below.)

(b) ψ^* on G/Γ and ψ'^* on G'/Γ' are topologically equivalent by an affine map if and only if there exist $\alpha > 0$ and an affine map $\bar{b}\bar{B}: G/\Gamma \to G'/\Gamma'$ such that $b(B\psi)b^{-1}: t \mapsto \psi'(\alpha t)$.

Notation: For any ϕ of G and $a \in G$, $a\phi a^{-1}: t \mapsto a\phi(t)a^{-1}$.

We return to the proof of Theorem 1. $\underline{e}$ is the identity.

(a) Since ϕ has compact recurrence in G/Γ, we can pick a sequence indexed by the integers

$$\{t_k \mid k \in Z,\ t_k \in R,\ \text{sign}(t_k) = \text{sign}(k),\ t_k \longrightarrow \pm\infty \text{ as } k \longrightarrow \pm\infty\}$$

such that there is a compact set D in G/Γ containing $\{\phi(t^k)\Gamma \mid k \in Z\}$. Since ϕ and ϕ' are the lifts through e of the trajectories through $e\Gamma$ of the flows ϕ^* and ϕ'^*, $\tilde{f}\phi = \phi'$ as an ordered set. The sequence $\{t_k \mid k \in Z\}$ determines a sequence indexed by the integers

$$\{s_k \mid k \in Z,\ s_k \in R,\ \text{sign}(s_k) = \text{sign}(k),\ s_k \longrightarrow \pm\infty \text{ as } k \longrightarrow \pm\infty\}$$

such that $\tilde{f}\phi(t_k) = \phi'(s_k)$.

We show that ϕ is recurrently approached in d by ϕ', where d is any function in $\mathcal{D}$. (See Section 2, Definition 1.)

$$\sup_{k\in Z} d(\phi(t_k), \phi'(s_k)) = \sup_{k\in Z} d(\phi(t_k), \tilde{f}\phi(t_k)).$$

$$\sup_{k\in Z} d(\phi(t_k), \tilde{f}\phi(t_k)) = \sup_{k\in Z} d(x_k\gamma_k, \tilde{f}(x_k\gamma_k)),$$

where for some compact set $\tilde{D} \subset G$ such that $p(\tilde{D}) = D$, we have chosen $x_k \in \tilde{D}$ and $\gamma_k \in \Gamma$ such that $\phi(t_k) = x_k\gamma_k$. Such $\tilde{D}$, x_k, and γ_k exist since p is a covering map with the deck transformations being right multiplication by elements of Γ.

$$\sup_{k\in Z} d(x_k\gamma_k, \tilde{f}(x_k\gamma_k)) = \sup_{k\in Z} d(x_k\gamma_k, \tilde{f}(x_k)\gamma_k),$$

since $\tilde{f}_{|\Gamma}$ is the identity map on Γ.

$$\sup_{k\in Z} d(x_k\gamma_k, \tilde{f}(x_k)\gamma_k) = \sup_{k\in Z} d(x_k, \tilde{f}(x_k)),$$

since d is right invariant.

$$\sup_{k\in Z} d(x_k, \tilde{f}(x_k)) \leq \sup_{x\in\tilde{D}} d(x, f(x)) < \infty,$$ since $\tilde{D}$ is compact.

Therefore, ϕ is recurrently approached in d by ϕ'.

Since ϕ is isolated, $\phi = \phi'$ up to sense-preserving reparameterization. Therefore, the identity map is a topological equivalence of ϕ^* and ϕ'^*.

(b) $\tilde{f}_{|\Gamma}$ is an automorphism of Γ. By the automorphism extension property of (G,Γ), there is an automorphism A of G extending $(\tilde{f}_{|\Gamma})^{-1}$. By the lemma, ϕ^* and $(A\phi')^*$ are topologically equivalent by the map $\bar{A}f:(G/\Gamma,e\Gamma)\longrightarrow(G/\Gamma,e\Gamma)$, which has the lift $A\tilde{f}:(G,e)\longrightarrow(G,e)$, where $(A\tilde{f})_{|\Gamma}$ is the identity on Γ. Applying Theorem 1(a) shows that ϕ^* and $(A\phi')^*$ are topologically equivalent by the identity. Therefore, ϕ^* and ϕ'^* are topologically equivalent by the affine map $\bar{A}^{-1}:(G/\Gamma,e\Gamma)\longrightarrow(G/\Gamma,e\Gamma)$.

(c) Let $f:G/\Gamma\longrightarrow G/\Gamma$ be a topological equivalence of ϕ^* and ϕ'^*. That Γ is a lattice implies that G/Γ has finite volume and that ϕ^* is a measure-preserving flow. Therefore, by the Poincaré recurrence theorem [6, p. 8], we can find a point $a\Gamma$ in some compact set D in G/Γ, such that the trajectory of $a\Gamma$, $t\longmapsto(\phi(t)a)\Gamma$, keeps returning to D in both positive and negative time.

Let $b\Gamma = f(a\Gamma)$. Then, by the lemma, $\bar{b}^{-1}f\bar{a}:(G/\Gamma,e\Gamma)\longrightarrow(G/\Gamma,e\Gamma)$ is a topological equivalence of $(a^{-1}\phi a)^*$ and $(b^{-1}\phi' b)^*$. $a^{-1}\phi a$ is isolated since ϕ is isolated. $a^{-1}\phi a$ has compact recurrence in G/Γ since $p(a^{-1}\phi a)$ is the image under the homeomorphism $\bar{a}^{-1}$ of the recurrent trajectory $t\longmapsto(\phi(t)a)\Gamma$. $\bar{b}^{-1}f\bar{a}$ lifts to a homeomorphism from (G,e) to (G,e) by the homeomorphism lifting property of $(G,G/\Gamma)$.

By hypothesis, (G,Γ) has the automorphism extension property. Therefore, by applying Theorem 1(b), we obtain that $(a^{-1}\phi a)^*$ and $(b^{-1}\phi' b)^*$ are topologically equivalent by an affine map. Therefore, ϕ^* and ϕ'^* are topologically equivalent by an affine map. QED Theorem 1

Section 4 Theorem 2

In the nilpotent case, Theorem 2(a), below, is a corolary of Proposition 3. However, the proof of Theorem 2 has greater conceptual and geometric clarity.

Theorem 2 Let G be a connected, simply connected, solvable Lie group, and let the exponential map from the Lie algebra L(G) to G be a diffeomorphism. Let

$$\{t_k \mid k \in Z,\ t_k \in R,\ \mathrm{sign}(k) = \mathrm{sign}(t_k),\ t_k \longrightarrow \pm\infty \text{ as } k \longrightarrow \pm\infty\}$$

and $\{s_k \mid k \in Z,\ s_k \in R,\ \mathrm{sign}(k) = \mathrm{sign}(s_k),\ s_k \longrightarrow \pm\infty \text{ as } k \longrightarrow \pm\infty\}$

be sequences indexed by the integers. Let ϕ and ϕ' be one-parameter subgroups of G. Let d be a metric induced by a right invariant Riemannian metric on G.

(a) If $\sup \{d(\phi(t_k), \phi'(s_k)) \mid k \in Z\} < \infty$, then $\phi = \phi'$ up to sense-preserving reparameterization. That is, every one-parameter subgroup of G is isolated. (See Section 2.)

(b) Let G be nilpotent. If

$$\sup_{k>0} d(\phi(t_k), \phi'(s_k)) < \infty \quad \text{or} \quad \sup_{k<0} d(\phi(t_k), \phi'(s_k)) < \infty ,$$

then $\phi = \phi'$ up to sense-preserving reparameterization. There are counterexamples when G is not nilpotent.

Note: Theorem 2(a) applies whenever G is connected, simply connected, and nilpotent, since then the exponential map is a diffeomorphism [11, p. 269]. Theorem 2(b) is a sharper result which applies only to the nilpotent case.

Proof Since $\exp: L(G) \longrightarrow G$ is a diffeomorphism, every connected Lie subgroup is closed.

Lemma 1 (a) Theorem 2 holds in case the dimension of G is less than or equal to 3.

(b) There is a 2-dimensional non-nilpotent counterexample to the conclusion of Theorem 2(b).

Proof We postpone the proof of Lemma 1 to the end of this section.

We proceed by induction on the dimension of G.

Assume the result if the dimension of G is $\leq n$, where $n \geq 3$.

Let the dimension of G equal $n + 1$. We choose a connected, normal, Lie subgroup H of G, having dimension ≥ 1 and codimension ≤ 2 [5, p. 46]. If G is nilpotent, let H be any one-parameter subgroup contained in the non-trivial center of G.

$H\backslash G$ is a connected, simply connected, solvable Lie group, with $\exp: L(H\backslash G) \longrightarrow H\backslash G$ a diffeomorphism and with its dimension $\leq n$. If G is nilpotent, then $H\backslash G$ is nilpotent [25, p. 238, 5, p. 400, and 23, p. 7].

Let $p: G \longrightarrow H\backslash G$ be the canonical projection, and e and $\bar{e}$ the identities of G and $H\backslash G$ respectively. We show that p is contracting.

Lemma 2 Let d be induced by any right invariant Riemannian metric $\langle\, ,\rangle$ on G. $\langle\, ,\rangle$ and d induce a certain right invariant Riemannian metric $\overline{\langle\, ,\rangle}$ and metric $\bar{d}$ on $H\backslash G$ such that for all $g \in G$ and $k \in G$, $d(g,k) \geq \bar{d}(p(g),p(k))$.

Proof Let W be the orthogonal complement to the tangent space T_eH with respect to $\langle\, ,\, \rangle$. Let $\overline{\langle\, ,\, \rangle}_{\bar{e}}$ be the inner product on $T_{\bar{e}}(H\backslash G)$ induced from W by T_ep restricted to W. Any $v \in T_eG$ can be written $v = h + w$ where $h \in T_eH$ and $w \in W$. For any $v \in T_eG$, T_ep is contracting since

$$\langle v,v\rangle \geq \langle w,w\rangle = \overline{\langle (T_ep)w,(T_ep)w\rangle}_{\bar{e}} = \overline{\langle (T_ep)v,(T_ep)v\rangle}_{\bar{e}} .$$

By right translation by elements of $H\backslash G$, $\overline{\langle\, ,\, \rangle}_{\bar{e}}$ induces $\overline{\langle\, ,\, \rangle}$ a right invariant Riemannian metric on $H\backslash G$, which induces a right invariant metric $\bar{d}$ on $H\backslash G$. Then for all $g \in G$ and $v \in T_gG$, $\langle v,v\rangle \geq \overline{\langle (T_gp)v,(T_gp)v\rangle}$. It follows that p shrinks path lengths and thus that, for all $g \in G$ and $k \in G$, $d(g,k) \geq \bar{d}(p(g),p(k))$. QED Lemma 2

Let X and Y be in L(G), and $\phi(t) \equiv \exp(tX)$ and $\phi'(t) \equiv \exp(tY)$, for all t in R. Since $\sup_{k\in Z} d(\phi(t_k),\phi'(s_k)) < \infty$, it follows by Lemma 2 that $\sup_{k\in Z} \bar{d}(p\phi(t_k),\, p\phi'(s_k)) < \infty$. Since by the inductive hypothesis $p\phi$ is isolated, it follows that $p\phi = p\phi'$ up to sense-preserving reparameterization. Therefore, there exists $\alpha \in L(H)$ such that $\alpha + X = rY$ for some $r > 0$. Since L(H) is an ideal, L(H) and X generate a subalgebra L(K) which has a dimension one greater than the dimension of L(H). L(K) is the Lie algebra of K, a connected, simply connected, solvable Lie group of dimension $\leq n$, with $\exp: L(K) \longrightarrow K$ a diffeomorphism. If G is nilpotent, K is nilpotent and isomorphic to R^2. Since X and Y are in L(K), ϕ and ϕ' are in K.

Let d_K be the restriction of d to K. Since K is closed in G, every closed set in K which is bounded in d_K is also closed in G and bounded in d, and hence is compact in G and compact in K [11, p. 56]. Therefore $d_K \in \mathcal{B}_K$. (See Section 2.)

Let d' be any metric on K induced by a right invariant Riemannian metric on K. Since by hypothesis

$\sup_{k\in Z} d_K(\phi(t_k),\phi'(s_k)) < \infty$, $(d_K = d$ on K), it follows that $\sup_{k\in Z} d'(\phi(t_k),\phi'(s_k)) < \infty$. (See Proposition 1(a).) By our inductive hypothesis, since ϕ is isolated as a subgroup of K, $\phi = \phi'$ up to sense-preserving reparameterization.

The same steps prove Theorem 2(b), since when G is nilpotent all the groups involved are nilpotent.

It only remains to give the proof of Lemma 1.

Lemma 1 Proof

Proposition 1(a,d,f) shows that it is sufficient to prove the result for some d induced by either a left or right invariant Riemannian metric, or by some norm if $G \subset GL(n,R)$.

Let n be the dimension of G.

If n = 1, the result is trivial.

We examine 3 cases: (i) n = 2, (ii) n = 3 and G contains a normal Lie subgroup of dimension one, (iii) n = 3 and G does not contain a normal Lie subgroup of dimension one.

Case (i) If n = 2, G is isomorphic to either the abelian group R^2, or to the non-nilpotent matrix group

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \; a > 0,\ b \in R\right\}$$

which we denote by S_0. $L(S_0)$, the Lie algebra of S_0, is the Lie algebra of matrices

$$\left\{\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \middle| \; c \in R,\ d \in R\right\} \quad [5,\ p.\ 44].$$

Theorem 2 is trivial for R^2, where we use the Euclidean metric.

Let us write the elements of S_0 as ordered pairs $(a,b) \in R^+ \times R$, where $(a,b)(a',b') = (aa',\ ab' + b)$. After sense-preserving reparameterization, the three types of one-parameter subgroups of S_0 are: (1) $t \longmapsto (e^t, d(e^t-1))$, (2) $t \longmapsto (e^{-t}, d(e^{-t} - 1))$, and (3) $t \longmapsto (1, td)$. d is any real number.

Let ϕ and ϕ' be one-parameter subgroups. We show that if ϕ is recurrently approached in norm by ϕ', then $\phi = \phi'$ up to sense-preserving reparameterizarion. The result is clear if ϕ is type (1) and ϕ' is type (2) or type (3), or if ϕ is type (3) and ϕ' is type (1) or type (2). By Proposition 1(g), it only remains to check the case when ϕ and ϕ' are both type (1).

So, let $\phi(t) \equiv (e^t, d(e^t - 1))$ and $\phi'(s) \equiv (e^s, d'(e^s - 1))$, for all t and s in R.

$$\phi(t)\phi'(-s) = (e^{t-s}, e^t d'(e^{-s} - 1) + d(e^t - 1)) =$$
$$(e^{t-s}, e^t(d-d') + d'(e^{t-s})).$$

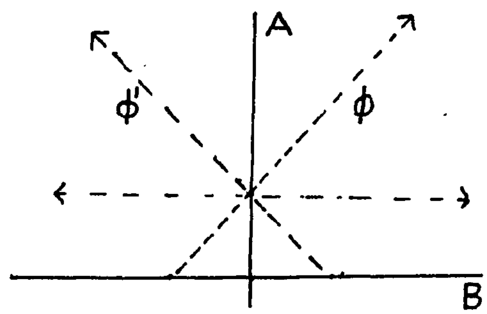
Then $\sup_{k\in Z} \|\phi(t_k)\phi'(-s_k)\| < \infty$ implies that $e^{(t_k - s_k)}$ is bounded.

Therefore $d = d'$, and thus $\phi = \phi'$.

We could also verify that the subgroups are isolated using the metric d induced by the left invariant Riemannian metric $(1/a^2)(da \otimes da + db \otimes db)$, which is the Poincaré metric on the upper half-plane. If $(a,b) \in R^+ \times R$ is considered as the complex number $z = b + ai$, it can be computed that

$$d(z_1,z_2) = \log \frac{|z_2 - \bar{z}_1| + |z_2 - z_1|}{|z_2 - \bar{z}_1| - |z_2 - z_1|} \quad [3, \text{ p. } 130].$$

Since the one-parameter subgroups are, after reparameterization, the Euclidean straight lines through (1,0), the result is intuitively clear.



To get a counterexample to the conclusion of Theorem 2(b) when G is non-nilpotent, consider the subgroups $\phi: t \longmapsto (e^t, e^t - 1)$ and $\phi': s \longmapsto (e^s, -e^s + 1)$ of S_0. (See the diagram.) Reparameterizing the positive directions of the subgroups as Euclidean half-lines in the upper half-plane, we have for $t > 0$, $\phi: t \longmapsto (t + (t+1)i)$ and $\phi': t \longmapsto (-t + (t+1)i)$.

$$\lim_{t\to\infty} d(\phi(t),\phi'(t)) = \lim_{t\to\infty} \log \frac{|2(-t + (t+1)i)| + |-2t|}{|2(-t + (t+1)i)| - |-2t|} =$$

$$\lim_{t\to\infty} \log((\sqrt{2t^2 + 2t + 1} + t)/(\sqrt{2t^2 + 2t + 1} - t)) < \infty .$$

Therefore taking $t_k = s_k = k$, we obtain that

$\sup_{k>0} d(\phi(t_k), \phi'(s_k)) < \infty$, but ϕ is not a sense-preserving reparameterization of ϕ'.

Case (ii) Let n = 3, H be a normal Lie subgroup of G of dimension one, and $p:G \longrightarrow G/H$. Using the same argument we use in proving the inductive step in Theorem 2, we show, first, that if $\sup_{k \in Z} d(\phi(t_k), \phi'(s_k)) < \infty$, then $p\phi = p\phi'$ up to sense-preserving reparameterization. This implies that ϕ and ϕ' lie in the same 2-dimensional subgroup of G and thus, by Case (i), that $\phi = \phi'$ up to sense-preserving reparameterization. If G is nilpotent, G/H is nilpotent and isomorphic to R^2. Therefore, by a similar argument, Theorem 2(b) follows in this case.

Case (iii) Let n = 3 and G not have a one-dimensional normal Lie subgroup. Since a nilpotent Lie group has a non-trivial connected center, G is not nilpotent. The commutator subgroup $[G,G]$ must be 2-dimensional, and thus isomorphic to either R^2 or S_0. Let X, Y, and Z be a basis for L(G), with X and Y in $L([G,G])$. Since $[G,G]$ is 2-dimensional, ad(z) must be invertible on $L([G,G])$, but $L(S_0)$ has no invertible derivation. Therefore $[G,G]$ must be isomorphic to R^2. So $[X,Y] = 0$. By a result of Saito, since $\exp:L(G) \longrightarrow G$ is a diffeomorphism, ad(Z) has no pure imaginary eigenvalue [23, and 5, p. 400].

It follows that L(G) is isomorphic to the Lie algebra of matrices of the form

$$\begin{pmatrix} c & -c\theta & 0 & a \\ c\theta & c & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ for some fixed } \theta \neq 0.$$

Exponentiating, we see that G is isomorphic to the group of matrices of the form

$$\begin{pmatrix} e^z\cos(z\theta) & -e^z\sin(z\theta) & 0 & x \\ e^z\sin(z\theta) & e^z\cos(z\theta) & 0 & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

That is, G is the semidirect product of R^2 by R, where $z \in R$ acts on R^2 by a rotation by $z\theta$ followed by a dilation by e^z. Writing G as ordered triples, we obtain

$$(x,y,z)(x',y',z') = ((x,y) + e^z R_{z\theta}(x',y'), z + z'),$$

where $R_{z\theta}$ is counterclockwise rotation of R^2 by $z\theta$.

After sense-preserving reparameterization, the three types of one-parameter subgroup of G are:

(1) $t \longmapsto ((e^t R_{t\theta} - I)(u,v), t)$,

(2) $t \longmapsto ((e^{-t} R_{-t\theta} - I)(u,v), -t)$, and

(3) $t \longmapsto (ta, tb, 0)$.

u, v, a, and b are any real numbers.

We show that if ϕ is recurrently approached in norm by ϕ', then $\phi = \phi'$ up to sense-preserving reparameterization. The result is clear if ϕ is of type (1) and ϕ' is of type (2) or type (3), or if ϕ is of type (3) and ϕ' is of type (3) or type (2). By Proposition 1(g), it only remains to check the case where ϕ and ϕ' are both of type (1).

So let for all t in R,

$$\phi(t) \equiv ((e^t R_{t\theta} - I)(u,v), t) \text{ and}$$

$$\phi'(t) \equiv ((e^t R_{t\theta} - I)(u',v'), t).$$

$$\phi(t)\phi'(-s) =$$

$$((e^t R_{t\theta} - I)(u,v) + e^t R_{t\theta}(e^{-s} R_{-s\theta} - I)(u',v'),\ t - s) =$$

$$(e^t R_{t\theta}(u-u',v-v') + e^{t-s} R_{(t-s)\theta}(u',v') - (u,v),\ t - s).$$

Since $\sup_{k\in Z} \|\phi(t_k)\phi'(-s_k)\| < \infty$, it follows that $\sup_{k\in Z} |t_k - s_k| < \infty$. Therefore

$\sup_{k\in Z} \|(e^{t_k}) R_{(t_k\theta)}(u-u',v-v')\| < \infty$, which implies that $u = u'$ and $v = v'$. Therefore $\phi = \phi'$.

We could also give a geometric argument, using the metric d induced by the left invariant Riemannian metric $e^{-2z}(dx \otimes dx + dy \otimes dy) + dz \otimes dz$. QED Lemma 1

QED Theorem 2

Section 5 Theorem 3

In this section we state and prove Theorem 3, which deals with the isolation of one-parameter subgroups of SL(n,R). Proposition 2 and Proposition 3, on which Theorem 3 depends, are proved in Section 6 and Section 7 respectively.

The proof goes roughly as follows. If ϕ and ψ are one-parameter subgroups on SL(n,R), they can be factored as $\phi = E_1H_1U_1$ and $\psi = E_2H_2U_2$ where, for i = 1, 2, E_i has compact closure, H_i is positive semisimple and U_i is unipotent. In this section we show that if ϕ is recurrently approached by ψ, then H_1U_1 is recurrently approached by H_2U_2. By Proposition 2, $H_1 = H_2$ up to sense-preserving reparameterization. By Proposition 3, $U_1 = U_2$ up to the same reparameterization. This proves Theorem 3.

Theorem 3 Let ϕ and ψ be one-parameter subgroups of SL(n,R). Suppose that ϕ is recurrently approached in norm by ψ. Then there is a sense-preserving reparameterization ψ' of ψ and a one-parameter subgroup χ of SL(n,R) with only positive eigenvalues, such that $\phi(t) = E_1(t)\chi(t)$ and $\psi'(t) = E_2(t)\chi(t)$, where E_1 and E_2 are one-parameter subgroups of SL(n,R). $E_1 \subset \overline{E_1} \equiv K_1$ and $E_2 \subset \overline{E_2} \equiv K_2$, where K_1 and K_2 are tori contained in the centralizer of χ. Furthermore, there are one-parameter subgroups H and U of SL(n,R) such that $\chi(t) = H(t)U(t)$. H(t) is semisimple with only positive eigenvalues, and U(t) is unipotent. E_1, H, and U are in each other's centralizer. E_2, H, and U are in each other's centralizer.

Proof See Section 2 for definitions and properties concerning being recurrently approached in norm.

Lemma 1 (a) Let $\| \ \|$ be any norm on M(n,R). Let K be a compact set in GL(n,R). Then there exist positive constants c and C such that, for all $g \in M(n,R)$ and $h \in K$,

$$c\|g\| \leq \|gh\| \leq C\|g\| \text{ and } c\|g\| \leq \|hg\| \leq C\|g\|.$$

(b) Let ϕ, ψ, E_1, E_2, $E_1\phi$, and $E_2\psi$ be one-parameter subgroups of GL(n,R), and K_1 and K_2 compact sets such that $E_1 \subset K_1 \subset GL(n,R)$ and $E_2 \subset K_2 \subset GL(n,R)$. Then ϕ is recurrently approached in norm by ψ if and only if $E_1\phi$ is recurrently approached in norm by $E_2\psi$.

Proof (a) It is enough to prove the result for some particular norm $\| \ \|$ on M(n,R). So let $\|g\| \equiv \sup_{|v|=1} |gv|$ where $| \ |$ is the Euclidean norm on R^n. Then for g and h in M(n,R), $\|gh\| \leq \|g\|\,\|h\|$. Let $C \equiv \sup_{h \in K} \|h\|$ and $c \equiv (\sup_{h \in K} \|h^{-1}\|)^{-1}$. Then

$$c\|g\| = c\|ghh^{-1}\| \leq c\|h^{-1}\|\,\|gh\| \leq \|gh\| \leq \|g\|\,\|h\| \leq C\|g\|.$$ Also

$$c\|g\| = c\|h^{-1}hg\| \leq c\|h^{-1}\|\,\|hg\| \leq \|hg\| \leq \|h\|\,\|g\| \leq C\|g\|.$$

(b) Lemma 1(b) follows directly from Lemma 1(a). (See Section 2, Definition 2) QED Lemma 1

We generalize the following decompositions to one-parameter subgroups.

Complete Multiplicative Jordan Decomposition For any element $g \in GL(n,R)$, there exist unique elements e, h, and u in GL(n,R) such that (i) g = ehu, (ii) g, h, and u commute with each other, and (iii) u is unipotent, h is semisimple with all its eigenvalues positive, and e is semisimple with all its eigenvalues of modulus 1. It follows that e is contained in a compact connected subgroup. (The letters e, h, and u are suggested by the terms elliptic, hyperbolic, and unipotent [11, p. 430].)

<u>Real Canonical Form</u> The complete multiplicative Jordan decomposition of an element $g \in GL(n,R)$ can be read off from the real canonical form of g. For an (n,n) real matrix, the real canonical form is the same as Jordan form for the real eigenvalues. For a complex eigenvalue of the form a + bi, the blocks consist of $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ on the diagonal and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ just below the diagonal [12, p. 126].

<u>Lemma 2</u> Let ϕ be a one-parameter subgroup of SL(n,R). Then there exist one-parameter subgroups E, H, U, and χ of SL(n,R), which are in each other's centralizer, and such that $\phi(t) = E(t)H(t)U(t)$ and $\chi(t) \equiv H(t)U(t)$. Furthermore, E is contained in $\bar{E} \equiv K$, a toral subgroup of the centralizer of χ, H is semisimple and has all its eigenvalues positive, and U is unipotent. E, H, and U are the unique subgroups having these properties.

<u>Proof</u> Let $\phi(t) \equiv e^{tX}$, for all t in R. Pick a basis for R^n such that X is in real canonical form. Consider the S + N decomposition of X. Then $U(t) \equiv e^{tN}$, for all t in R. $E(t)H(t) = e^{tS}$, for all t in R, is the O-P polar decomposition with respect to the basis. Looking at the matrix

representation with respect to the chosen basis, we see that the subgroups are in each other's centralizer and have the other required properties. The uniqueness follows from the uniqueness of the complete multiplicative Jordan decomposition for any element in GL(n,R). QED Lemma 2

Now let ϕ and ψ be the one-parameter subgroups of SL(n,R) referred to in the statement of Theorem 3. Choosing, as in Lemma 2, E_1, $\overline{E_1} \equiv K_1$, H_1, U_1, $\chi_1 \equiv H_1U_1$, E_2, $\overline{E_2} \equiv K_2$, H_2, U_2, and $\chi_2 \equiv H_2U_2$, we obtain the following:

(i) E_1, H_1, and U_1 are in each other's centralizer. E_2, H_2, and U_2 are in each other's centralizer.

(ii) $E_1 \subset K_1$ which is contained in the centralizer of χ_1 and $E_2 \subset K_2$ which is contained in the centralizer of χ_2. K_1 and K_2 are tori.

(iii) There exist X_1 and X_2 such that $\chi_1(t) = e^{tX_1}$ and $\chi_2(t) = e^{tX_2}$ where X_1 and X_2 have all their eigenvalues real.

(iv) $\phi(t) = E_1(t)\chi_1(t)$ and $\psi(t) = E_2(t)\chi_2(t)$.

By using Lemma 1(b), χ_1 is recurrently approached in norm by χ_2. Using Proposition 2 (Section 6) and Proposition 3 (Section 7), we can show that χ_2 is a sense-preserving reparameterization of χ_1. Thus after a sense-preserving reparameterization of ψ, we can assume that $H_1 = H_2 \equiv H$, $U_1 = U_2 \equiv U$, and $\chi_1 = \chi_2 \equiv \chi$. QED Theorem 3

Section 6   Proposition 2

The following proposition roughly shows that if ϕ is recurrently approached by ψ where ϕ and ψ have only positive eigenvalues, then up to sense-preserving reparameterization they have the same semisimple parts. This is done by showing that they have the same eigenvalues (Lemma 1) and the same generalized eigenspaces (Lemma 2). Therefore the unipotent part of ϕ is approached by the unipotent part of ψ. This result is the hypotheses used in Proposition 3.

Proposition 2 Let $\phi(t) \equiv e^{tX}$ and $\psi(t) \equiv e^{tY}$, for all t in R, be one-parameter subgroups of SL(n,R) such that all the eigenvalues of X and Y are real. Let ϕ be recurrently approached in norm by ψ.

(a) ϕ is unipotent if and only if ψ is unipotent.

(b) If ϕ and ψ are not unipotent, there is a sense-preserving reparameterization ψ' of ψ, where for all t in R $\psi'(t) \equiv e^{tY'}$, such that the largest eigenvalue of X and the largest eigenvalue of Y' are positive and equal.

(c) Let ϕ and ψ be non-unipotent such that $\lambda_1 = \mu_1 > 0$, where λ_1 is the largest eigenvalue of X and μ_1 is the largest eigenvalue of Y. Then there are one-parameter subgroups H, U_1, and U_2 with the following properties.

(i) H is semisimple with only real eigenvalues. U_1 and U_2 are unipotent. For all t and s in R, $\phi(t) = H(t)U_1(t)$, $\psi(t) = H(t)U_2(t)$, $H(t)U_1(s) = U_1(s)H(t)$, and $H(t)U_2(s) = U_2(s)H(t)$.

(ii) U_1 is recurrently approached in norm by U_2.

(iii) $\sup_{k\in Z} |t_k - s_k| < \infty$ where $\{t_k\}$ and $\{s_k\}$ are the sequences for which $\sup_{k\in Z} \|U_1(t_k)U_2(-s_k)\| < \infty$. (Section 2, Definition 2)

Proof We first make some definitions and observations which will also be used in the proof of Proposition 3.

Definitions (1) Let spec(X) and spec(Y) be the distinct eigenvalues of X and Y respectively. Let $\mathcal{V}$ and $\mathcal{W}$ be bases of R^n in which X and Y are, respectively, in Jordan form. Let, for all t in R, $H_1(t) \equiv e^{tS_1}$, $U_1(t) \equiv e^{tN_1}$,

$H_2(t) \equiv e^{tS_2}$, and $U_2(t) \equiv e^{tN_2}$ be one-parameter subgroups, where $X = S_1 + N_1$ and $Y = S_2 + N_2$ are the S + N decompositions. It follows that, for all t and s in R, $H_1(t)U_1(s) = U_1(s)H_1(t)$ and $H_2(t)U_2(s) = U_2(s)H_2(t)$. For $\lambda \in$ spec(X) and $\mu \in$ spec(Y), let

$$V_\lambda \equiv \{x \in R^n \mid S_1x = \lambda x\} \text{ and } W_\mu \equiv \{x \in R^n \mid S_2x = \mu x\}.$$

That is V and W are generalized eigenspaces of X and Y.

$$R^n = \oplus\{V_\lambda \mid \lambda \in \text{spec}(X)\} = \oplus\{W_\mu \mid \mu \in \text{spec}(Y)\}.$$

(2) For $v \in \mathcal{V}$ and $x \in R^n$, v^*x equals the v coordinate of x with respect to the basis $\mathcal{V}$.

(3) Since ϕ is recurrently approached in norm by ψ, let $\{t_k\}$ and $\{s_k\}$ be the sequences with the requisite properties (Section 2, Definition 2), such that $\sup_{k\in Z} \|\phi(t_k)\psi(-s_k)\| < \infty$.

(4) Chains in $\mathcal{V}$ and $\mathcal{W}$ are sequences of the

form $(v, N_1 v, N_1^2 v, \ldots)$ and $(w, N_2 w, N_2^2 w, \ldots)$ respectively. $\mathcal{V}$ and $\mathcal{W}$ are partitioned into maximal chains. The first and last elements in a maximal chain are called <u>initial</u> and <u>terminal</u> elements respectively. For $v \in \mathcal{V}$, <u>$\delta(v)$</u>, the <u>degree</u> of v, equals r if $N_1^r v \neq 0$ but $N_1^{(r+1)} v = 0$. <u>$\delta(w)$</u>, for $w \in \mathcal{W}$, is similarly defined. If $v \in \mathcal{V} \cap V_\lambda$ and $\delta(v) = r$, then

$$\phi(t)v = e^{\lambda t}(v_0 + tv_1 + \ldots + \frac{t^r}{r!}v_r), \text{ where } \underline{v_i} \equiv N_1^i v.$$

Similarly, if $w \in \mathcal{W} \cap W_\mu$ and $\delta(w) = r$, then

$$\psi(t)w = e^{\mu t}(w_0 + tw_1 + \ldots + \frac{t^r}{r!}w_r), \text{ where } \underline{w_i} \equiv N_2^i w.$$

The following observations are used repeatedly.

<u>Observations</u> (1) Since $\sup_{k \in Z} \|\phi(t_k)\psi(-s_k)\| < \infty$, it follows that for $v \in \mathcal{V}$ and $w \in \mathcal{W}$, $\sup_{k \in Z} |v^*\phi(t_k)\psi(-s_k)w| < \infty$.

(2) Let $v \in \mathcal{V} \cap V_\lambda$ and $w \in \mathcal{W} \cap W_\mu$. Then there are chains $(v_0, \ldots, v_p)$ and $(w_0, \ldots, w_q)$, where $v = v_p$, $w = w_0$, v_0 is an initial element, and w_q is a terminal element.

(3) $v^*\psi(t)w = v^*\psi(t)w_0 =$

$$v^*e^{\mu t}(w_0 + tw_1 + \ldots + \frac{t^q}{q!}w_q) =$$

$$e^{\mu t}((v^*w_0) + t(v^*w_1) + \ldots + \frac{t^q}{q!}(v^*w_q)).$$

(4) $v^*\phi(t)w = v_p^*\phi(t)w =$

$$v_p^*\phi(t)((v_0^*w)v_0 + (v_1^*w)v_1 + \ldots + (v_p^*w)v_p) =$$

$$e^{\lambda t}(\frac{t^p}{p!}(v_0^*w) + \frac{t^{p-1}}{(p-1)!}(v_1^*w) + \ldots + (v_p^*w)).$$

(5) In particular, if $v_i^* w = 0$ for $i < p$ (equivalently, if v is of maximal degree in the set $\{v \in V_\lambda \cap \mathcal{V} \mid v^* w \neq 0\}$), then $v^*\phi(t)w = e^{\lambda t}(v^*w)$.

(6) Symmetrically, it is also the case that Ψ is recurrently approached in norm by ϕ, where $\sup_{k \in Z} \|\Psi(s_k)\phi(-t_k)\| < \infty$. (The reason is that the closure of

$\{\Psi(s_k)\phi(-t_k) \mid k \in Z\}$ is compact because the closure of

$\{\phi(t_k)\Psi(-s_k) \mid k \in Z\}$ is compact. (See Proposition 1(b,e).)

<u>Proof of Proposition 2(a)</u> We show first that if ϕ is unipotent, then Ψ is unipotent.

Assume that ϕ is unipotent and Ψ is not unipotent. Let μ be a non-zero eigenvalue of Y, with eigenvector $w \in \mathcal{W}$. Let v be of maximal degree in the set $\{v \in V_0 \cap \mathcal{V} = \mathcal{V} \mid v^*w \neq 0\}$. Then

$$v^*\phi(t)\Psi(-s)w = v^*\phi(t)e^{-\mu s}w = e^{-\mu s}v^*\phi(t)w = e^{-\mu s}(v^*w).$$

(See Observation 5.) Since $\sup_{k \in Z} |v^*\phi(t_k)\Psi(-s_k)w| < \infty$, $v^*w = 0$, which is a contradiction.

By symmetry (Observation 6), if Ψ is unipotent then ϕ is unipotent.

<u>Proof of Proposition 2(b)</u> Since X and Y are in the Lie algebra of SL(n,R), it follows that trace(X) = trace(Y) = 0. Since spec(X) $\neq \{0\}$ and spec(Y) $\neq \{0\}$, it follows that λ_1, the largest eigenvalue of X, and μ_1, the largest eigenvalue of Y, are positive. Let $Y' \equiv (\lambda_1/\mu_1)Y$.

<u>Proof of Proposition 2(c)</u> We need the following 2 lemmas.

Lemma 1 If ϕ and ψ are non-unipotent and $\lambda_1 = \mu_1 > 0$, then spec(X) = spec(Y) and $\sup_{k \in Z} |t_k - s_k| < \infty$.

Proof The proof is postponed. (See below.)

Lemma 2 If ϕ and ψ are non-unipotent, then, for all $\lambda \in$ spec(X) = spec(Y), $V_\lambda \supset W_\lambda$.

Proof The proof is postponed. (See end of Section 6.)

Lemma 2 and the fact that $R^n = \oplus V_\lambda = \oplus W_\lambda$ imply, by dimensional considerations, that for all λ, $V_\lambda = W_\lambda$. Therefore, for all λ, S_1 and S_2 agree on V_λ. Therefore $S_1 = S_2$, and hence $H_1 = H_2$. By letting $H \equiv H_1 = H_2$, Proposition 1(c(i)) is proved.

$$\infty > \sup_{k \in Z} \|\phi(t_k)\psi(-s_k)\| = \sup_{k \in Z} \|H(t_k)U_1(t_k)U_2(-s_k)H(-s_k)\| =$$

$$\sup_{k \in Z} \|U_1(t_k)U_2(-s_k)H(t_k - s_k)\|.$$

Since, by Lemma 1, $\sup_{k \in Z} |t_k - s_k| < \infty$, it follows that $\sup_{k \in Z} \|U_1(t_k)U_2(-s_k)\| < \infty$. (See Theorem 3, Lemma 1) So U_1 is recurrently approached in norm by U_2.

To complete the proof of Proposition 2, it only remains to prove Lemma 1 and Lemma 2.

Proof of Lemma 1 (See above) Lemma 1 requires the following sublemma.

Sublemma (a) For all $\mu \in$ spec(Y) there exists $\lambda \in$ spec(X), such that $\sup_{k \in Z} (\lambda t_k - \mu s_k) < \infty$.

spec(Y) such that if $\mu = \rho(\lambda)$, then $\sup_{k \in Z} |\lambda t_k - \mu s_k| < \infty$.

spec(Y) such that if $\mu = \rho(\lambda)$, then $\sup_{k \in Z} |\lambda t_k - \mu s_k| < \infty$.

Proof of Sublemma (a) Let w be an eigenvector associated to μ, and pick v of maximal degree from $\{v \in \mathcal{V} \mid v*w \neq 0\}$. For some λ, we have $v \in V_\lambda$. Then

$$v*\phi(t)\Psi(-s)w = v*\phi(t)e^{-\mu s}w = e^{-\mu s}v*\phi(t)w = e^{-\mu s}e^{\lambda t}(v*w).$$

$\sup_{k\in Z} |v*\phi(t_k)\Psi(-s_k)w| < \infty$ implies that

$\sup_{k\in Z} (\lambda t_k - \mu s_k) < \infty$.

(b) By symmetry (Observation 6), for all $\lambda' \in \text{spec}(X)$ there exists $\mu' \in \text{spec}(Y)$ such that $\sup_{k\in Z} (\mu' s_k - \lambda' t_k) < \infty$. Start with μ, pick λ, set $\lambda' = \lambda$, and then pick μ'. By addition, $\sup_{k\in Z} (\lambda t_k - \mu s_k) < \infty$ and $\sup_{k\in Z} (\mu' s_k - \lambda' t_k) < \infty$ imply that $\mu' = \mu$. Start with λ', pick μ', set $\mu = \mu'$, and then pick λ. By adding the same formulas, we see that $\lambda = \lambda'$. This establishes the existence of a bijection ρ. If $\mu = \rho(\lambda)$, $\sup_{k\in Z} (\lambda t_k - \mu s_k) < \infty$ and $\sup_{k\in Z} (\mu s_k - \lambda t_k) < \infty$ together imply that $\sup_{k\in Z} |\lambda t_k - \mu s_k| < \infty$.

QED Sublemma

We return to the proof of Lemma 1.

Let $\mu = \rho(\lambda)$. $\sup_{k\in Z} |\lambda t_k - \mu s_k| < \infty$ implies that $\rho(\lambda) = 0$ if and only if $\lambda = 0$. While if $\lambda \neq 0$, $(\lambda/\mu) = \lim_{k\to\infty} (s_k/t_k) = \lim_{k\to-\infty} (s_k/t_k)$, where the limits, in fact, exist. By the conditions on $\{t_k\}$ and $\{s_k\}$ (Section 2, Definition 2), $\lim_{k\to\infty} (s_k/t_k) \geq 0$, and so ρ must be an order preserving bijection. So $\rho(\lambda_1)$ equals μ_1, which by assumption equals λ_1. So, for all μ such that $\mu = \rho(\lambda)$, since

$\lambda/\mu = \lim_{k\to\infty} (s_k/t_k) = \lambda_1/\mu_1 = 1$, it follows that $\lambda = \mu$.

Therefore ρ is the identity and thus $\text{spec}(X) = \text{spec}(Y)$. Picking $\lambda = \lambda_1$ and $\mu = \rho(\lambda_1) = \mu_1$,

$\sup_{k\in Z}|\lambda t_k - \mu s_k| < \infty$ implies that $\sup_{k\in Z}|t_k - s_k| < \infty$.

QED Lemma 1

Proof of Lemma 2 (See above) The proof is by contradiction.

Suppose there exists λ such that it is not the case that $V_\lambda \supset W_\lambda$.

(i) Pick w of minimal degree in the set $\{w \in W_\lambda \mid w \notin V_\lambda\}$. There is a maximal chain $(w=w_0, w_1, \ldots, w_q)$ in $\mathcal{W}$.

(ii) Pick v of maximal degree in the set

$\{v \in (\mathcal{V} - V_\lambda) \mid v^*w \neq 0\}$. $v \in V_{\lambda'}$ for some $\lambda' \neq \lambda$.

$$v^*\phi(t)\Psi(-s)w = v^*\phi(t)e^{-\lambda s}(w - sw_1 + \frac{s^2}{2!}w_2 \ldots) =$$

$e^{-\lambda s}v^*\phi(t)w$, by our choice of w. $e^{-\lambda s}v^*\phi(t)w = e^{-\lambda s}e^{\lambda' t}(v^*w)$, by our choice of v (Observation 5). Since $\sup_{k\in Z}|v^*\phi(t_k)\Psi(-s_k)w| < \infty$, it follows that $\sup_{k\in Z}(\lambda' t_k - \lambda s_k) < \infty$.

If $\lambda = 0$, then $\lambda' = 0$, which is a contradiction since $\lambda' \neq \lambda$.

If $\lambda > 0$,

$$\lim_{k\to -\infty}(s_k/t_k) \leq \lambda'/\lambda \leq \lim_{k\to\infty}(s_k/t_k).$$

Since, by Lemma 1, both limits are equal to 1, it follows that $\lambda' = \lambda$, which is a contradiction.

Similarly, if $\lambda < 0$, we get the contradiction $\lambda' = \lambda$.

Therefore, $V_\lambda \supset W_\lambda$. QED Lemma 2

QED Proposition 2

Section 7 Proposition 3

Proposition 3 Let ϕ and ψ be unipotent one-parameter subgroups of $SL(n,R)$. Let ϕ be recurrently approached in norm by ψ, where $\{t_k \mid k \in Z\}$ and $\{s_k \mid k \in Z\}$ are the sequences chosen such that $\sup_{k \in Z} \|\phi(t_k)\psi(-s_k)\| < \infty$ for any norm $\| \ \|$. Then ψ is a sense-preserving reparameterization of ϕ. Furthermore, if $\sup_{k \in Z} |t_k - s_k| < \infty$, then $\psi = \phi$.

Proof For all t in R, let $\phi(t) \equiv e^{tX}$ and $\psi(t) \equiv e^{tY}$, where X and Y are nilpotent. We use again the definitions and observations made in proving Proposition 2 (Section 6), with the difference that here X and Y have no semisimple part and $\text{spec}(X) = \text{spec}(Y) = \{0\}$. For convenience, we restate Observations 1-5 in this context.

Observations (1') For all $v \in \mathcal{V}$ and $w \in \mathcal{W}$, $\sup_{k \in Z} |v^*\phi(t_k)\psi(-s_k)w| < \infty$.

(2') Let $v \in \mathcal{V}$ and $w \in \mathcal{W}$. There exist chains $(v_0,\ldots,v_p)$ and $(w_0,\ldots,w_q)$ where $v = v_p$, $w = w_0$, v_0 is an initial element, and w_q is a terminal element.

(3') $v^*\psi(t)w = v_p^*\psi(t)w_0 =$

$$(v_p^*w_0) + t(v_p^*w_1) + \ldots + \frac{t^q}{q!}(v_p^*w_q).$$

(4') $v^*\phi(t)w = v_p^*\phi(t)w_0 =$

$$\frac{t^p}{p!}(v_0^*w_0) + \frac{t^{p-1}}{(p-1)!}(v_1^*w_0) + \ldots + t(v_{p-1}^*w_0) + (v_p^*w_0).$$

(5') In particular, $v^*\phi(t)w = v_p^*w_0$ if $v_i^*w_0 = 0$ for $i < p$, and $v^*\phi(t)w = (v_p^*w_0) + t(v_{p-1}^*w_0)$ if

$v_i^* w_0 = 0$ for $i < p-1$.

Clearly ϕ is the identity if and only if ψ is the identity, and in that case Proposition 3 trivially follows. So assume ϕ is not the identity and ψ is not the identity. The following two lemmas directly prove Proposition 3.

<u>Lemma 1</u> (a) There is a real number $\alpha > 0$ such that $\sup_{k \in Z} |\alpha t_k - s_k| < \infty$.

(b) Let $\psi'(t) \equiv \psi(\alpha t)$, for all t in R. Then ϕ is recurrently approached in norm by ψ'. The sequences $\{t'_k\}$ and $\{s'_k\}$ chosen so that $\sup_{k \in Z} \|\phi(t'_k)\psi'(-s'_k)\| < \infty$, have the property that $\sup_{k \in Z} |t'_k - s'_k| < \infty$.

<u>Lemma 2</u> If $\sup_{k \in Z} |t_k - s_k| < \infty$, then $\phi = \psi$.

It remains to prove Lemma 1 and Lemma 2.

<u>Proof of Lemma 1</u> We need the following sublemma.

<u>Sublemma</u> (a) Let $v \in \mathcal{V}$ and $w \in \mathcal{W}$, such that $\delta(w) = 0$. If $\delta(v) > \delta(w)$, then $v^* w = 0$.

(b) Let $v \in \mathcal{V}$ and $w \in \mathcal{W}$ such that $\delta(w) = 1$. If $\delta(v) > \delta(w)$, then $v^* w = 0$.

(c) There are chains (v_0, v_1) in $\mathcal{V}$ and (w_0, w_1) in $\mathcal{W}$ where v_1 and w_1 are eigenvectors (i.e. $\delta(v) = \delta(w) = 0$), such that v_1 is of maximal degree in the set $\{v \in \mathcal{V} \mid v^* w_1 \neq 0\}$ and either $v_0^* w_0 = 0$ or v_0 is of maximal degree in the set $\{v \in \mathcal{V} \mid v^* w_0 \neq 0\}$.

<u>Proof of Sublemma</u> (a) Let $\delta(w_0) = 0$. Let v_0 be of

maximal degree in $\{v \in \mathcal{V} \mid v^*w_0 \neq 0\}$. If $\delta(v_0) \neq 0$, then there is a chain (v_0, v_1) in $\mathcal{V}$. $v_1^*\phi(t)\Psi(-s)w_0 = v_1^*\phi(t)w_0 = (v_1^*w_0) + t(v_0^*w_0)$ (Observation 5'). Since $\sup_{k\in Z} |v_1^*\phi(t_k)\Psi(-s_k)w_0| < \infty$, it follows that $v_0^*w_0 = 0$, which is a contradiction. So $\delta(v_0) = 0$.

(b) Let $\delta(w_0) = 1$. Then there is a chain (w_0, w_1) in $\mathcal{W}$. Let v_0 be of maximal degree in the set $\{v \in \mathcal{V} \mid v^*w_0 \neq 0\}$. If $\delta(v_0) > 1$, there is a chain (v_0, v_1) in $\mathcal{V}$.

$$v_1^*\phi(t)\Psi(-s)w_0 = v_1^*\phi(t)(w_0 - sw_1) = v_1^*\phi(t)w_0 - sv_1^*\phi(t)w_1 =$$

$v_1^*\phi(t)w_0$, since $\delta(w_1) = 0$, $\delta(v_1) > 0$, and thus (a) applies (Observation 5').

$$v_1^*\phi(t)w_0 = v_1^*w_0 + t(v_0^*w_0) \qquad \text{(Observation 5')}.$$

Since $\sup_{k\in Z} |v_1^*\phi(t_k)\Psi(-s_k)w_0| < \infty$, $v_0^*w_0 = 0$, which is a contradiction. Therefore, $\delta(v_0) \leq 1$. (We note in passing, that continuing inductively we could show that for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$, if $\delta(v) > \delta(w)$, then $v^*w = 0$.)

(c) Since we are assuming Ψ is not the identity, there is a chain (w_0, w_1) in $\mathcal{W}$ where w_1 is an eigenvector (i.e. $\delta(w_1) = 0$). Pick $v_1 \in \mathcal{V}$ such that $v_1^*w_1 \neq 0$. By part (a) above, $\delta(v_1) = 0$ and v_1 is of maximal degree in the set $\{v \in \mathcal{V} \mid v^*w_1 \neq 0\}$.

If there is no chain of the form (v_0, v_1), then

$$v_1^*\phi(t)\Psi(-s)w_0 = v_1^*\phi(t)(w_0 - sw_1) = v_1^*w_0 - s(v_1^*w_1)$$

(See Observation 5'.)

Since $\sup_{k \in Z} |v_1^* \phi(t_k)\Psi(-s_k)w_0| < \infty$, $v_1^* w_1 = 0$, which is a contradiction.

So let (v_0, v_1) be a chain in $\mathcal{V}$. By part (b) above, either $v_0^* w_0 = 0$ or v_0 is of maximal degree in the set $\{v \in \mathcal{V} \mid v^* w_0 \neq 0\}$. QED Sublemma

We return to the proof of Lemma 1.

(a) Let (v_0, v_1) and (w_0, w_1) be the chains referred to in Sublemma (c).

$$v_1^* \phi(t)\Psi(-s)w_0 = v_1^* \phi(t)(w_0 - sw_1) =$$

$$v_1^* \phi(t)w_0 - sv_1^* \phi(t)w_1 = (v_1^* w_0 + t(v_0^* w_0)) - s(v_1^* w_1).$$

(By the choice of v_0 and v_1, and by Observation 5'.)

Let $\alpha \equiv (v_0^* w_0)/(v_1^* w_1)$. Since $\sup_{k \in Z} |v_1^* \phi(t_k)\Psi(-s_k)w_0| < \infty$, it follows that $\sup_{k \in Z} |\alpha t_k - s_k| < \infty$ and $\alpha > 0$.

(b) Let $\Psi'(t) \equiv \Psi(\alpha t)$ for all t in R, and let $t_k' \equiv t_k$, and $s_k' \equiv s_k/\alpha$.

$$\sup_{k \in Z} \|\phi(t_k')\Psi'(-s_k')\| = \sup_{k \in Z} \|\phi(t_k)\Psi(-s_k)\| < \infty .$$

So ϕ is recurrently approached in norm by Ψ', with the sequences being $\{t_k'\}$ and $\{s_k'\}$. Furthermore

$$\sup_{k \in Z} |t_k' - s_k'| = \sup_{k \in Z} |t_k - s_k/\alpha| < \infty .$$

QED Lemma 1

Proof of Lemma 2 We are done if we can show that for all maximal chains $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{W}_1 \subset \mathcal{W}$, all $v \in \mathcal{V}_1$ and $w \in \mathcal{W}_1$, and all $t \in R$, it follows that $v^* \phi(t) w = v^* \Psi(t) w$. Since then, for all $t \in R$, $\phi(t) = \Psi(t)$.

So let $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{W}_1 \subset \mathcal{W}$ be maximal chains. We proceed by induction on $\delta(w) - \delta(v)$.

Let $\delta(w) - \delta(v)$ be minimal. That is let w be the terminal element of $\mathcal{W}_1$ and v be the initial element of $\mathcal{V}_1$. Then, by Observations 3' and 5', $v^*\phi(t)w = v^*w = v^*\Psi(t)w$.

So suppose that for $v \in \mathcal{V}_1$ and $w \in \mathcal{W}_1$, $v^*\phi(t)w = v^*\Psi(t)w$ if $\delta(w) - \delta(v) \leq r-1$. Now let $\delta(w) - \delta(v) = r$. There are chains of the form $(v_0, \ldots, v_p)$ and $(w_0, \ldots, w_q)$ where, $v_p = v$, $w_0 = w$, v_0 is the initial element of $\mathcal{V}_1$, and w_q is the terminal element of $\mathcal{W}_1$ (Observation 2'). By Observation 1', $\sup_{k\in Z} |v_p^*\phi(t_k)\Psi(-s_k)w_0| < \infty$. By hypothesis, $\sup_{k\in Z} |t_k - s_k| < \infty$, which implies that

$$\sup_{k\in Z} |v_p^*\Psi(t_k)\Psi(-s_k)w_0| = \sup_{k\in Z} |v_p^*\Psi(t_k - s_k)w_0| < \infty .$$

Therefore,

$$\infty > \sup_{k\in Z} |(v_p^*\phi(t_k)\Psi(-s_k)w_0) - (v_p^*\Psi(t_k)\Psi(-s_k)w_0)| .$$

It follows that

$$\infty > \sup_{k\in Z} |(v_p^*\phi(t_k)w_0 + v_p^*\phi(t_k)(-s_k w_1 + \ldots + \frac{(-s_k)^q w_q}{q!})) - (v_p^*\Psi(t_k)w_0 + v_p^*\Psi(t_k)(-s_k w_1 + \ldots + \frac{(-s_k)^q w_q}{q!}))| =$$

$$\sup_{k\in Z} |v_p^*\phi(t_k)w_0 - v_p^*\Psi(t_k)w_0| ,$$

because for i = 1, ..., q, we have $\delta(w_i) - \delta(v_p) \leq r-1$, and the induction hypothesis holds, and thus $v_p^*\phi(t_k)w_i = v_p^*\Psi(t_k)w_i$.

By Observations 3' and 4', $v_p^* \phi(t) w_0 - v_p^* \Psi(t) w_0$ is a polynomial in t with constant term equal to 0. Since a non-constant polynomial in t converges to ∞ as t goes to ∞,

$$\infty > \sup_{k \in Z} |v_p^* \phi(t_k) w_0 - v_p^* \Psi(t_k) w_0|$$

implies that $v_p^* \phi(t) w_0 = v_p^* \Psi(t) w_0$.

QED Lemma 2

QED Proposition 3

Section 8 Theorem 4

Theorem 4 is a consequence of Theorem 3.

Theorem 4 Let $\phi(t) = \exp(tX)$, for all t in R, be a one-parameter subgroup of a connected, semisimple Lie group G. If Condition (i) or Condition (ii) holds, then ϕ is isolated. If G has finite center, Condition (i) and Condition (ii) are equivalent.

Condition (i) G has finite center, and there is no compact connected Lie subgroup in the centralizer of ϕ.

Condition (ii) For each Y in the centralizer of X (i.e. $[X,Y] = 0$), when ad(Y) is semisimple then ad(Y) has some eigenvalue which is not pure imaginary and not 0.

Proof First we prove Theorem 4 in the case where G is centerless.

Since G is centerless, the adjoint representation of G is faithful [11, p. 129]. Therefore we can consider G to be a Lie subgroup of GL(n,R), where n is the dimension of G. Since G is semisimple, the commutator subgroup $[G,G]$ equals G, which implies that G is a Lie subgroup of SL(n,R) [25, p. 243]. Also, since G is semisimple, G is a closed Lie subgroup of SL(n,R) [11, p. 152].

Let ϕ be recurrently approached in norm by ψ, a one-parameter subgroup of G. Applying Theorem 3, we see that there exists a sense-preserving reparameterization ψ' of ψ such that, for all t in R, $\phi(t) = E_1(t)H(t)U(t)$ and $\psi'(t) = E_2(t)H(t)U(t)$, where E_1, E_2, H, and U are subgroups of SL(n,R) with the properties indicated in Theorem 3. However, since G is a subgroup of SL(n,R) as the adjoint representation of a centerless semi-simple Lie group, it follows that if, for $g \in G$, $g = ehu$ is the complete multiplicative Jordan decompositon of g as an element in SL(n,R), then e, h, and u are also in G. (See Section 5 and [11, Problem 6, p. 435].) Therefore the subgroups E_1, E_2, H, and U are in G.

If Condition (i) holds, since there is no compact, connected subgroup in the centralizer of ϕ, it follows that E_1 and thus also E_2 are trivial. Therefore $\phi = \psi'$. This shows that ϕ is isolated in norm and hence isolated (Proposition 1(d)).

We next show that when G is centerless, Condition (ii) is equivalent to Condition (i), and thus that Condition (ii) also implies that ϕ is isolated. There is a compact connected Lie subgroup in the centralizer of ϕ in G if and only if there is an element Y in the centralizer of X such that the subgroup $t \mapsto \exp(tY)$ has compact closure. Since Ad(G) is closed in SL(n,R), $t \mapsto \exp(tY)$ has compact closure in G if and only if $t \mapsto \exp(t(\mathrm{ad}(Y)))$ has compact closure in SL(n,R). But for any element W in the Lie algebra of SL(n,R), by putting W in real canonical form (Section 4, [12, p. 126]), we see that $t \mapsto e^{tW}$ has compact closure in SL(n,R) if and only if every eigenvalue of W is pure imaginary or 0 and W is semisimple.

Now let G have non-trivial center.

Let Z(G) be the center of G, which in this case is discrete. Let $p: G \longrightarrow G/Z(G)$ be the canonical projection, which is thus a covering map. There exist metrics d on G and $\bar{d}$ on $G/Z(G)$ induced by right invariant Riemannian metrics on G and $G/Z(G)$ respectively, such that $d(g,h) \geq \bar{d}(p(g),p(h))$ for all g and h in G. Therefore, ϕ is isolated if $p\phi$ is isolated.

If Condition (ii) holds for ϕ, it holds for the subgroup $p\phi$ of the centerless group $G/Z(G)$ $(L(G) = L(G/Z(G)))$. Since Theorem 4 holds in the centerless case, $p\phi$ and hence ϕ are isolated.

Let $Z(G)$ be finite. Let Condition (i) hold for ϕ. There is no compact, connected, Lie subgroup in the centralizer of ϕ if and only if there is no compact, connected, Lie subgroup in the centralizer of $p\phi$, which is a one-parameter subgroup of the centerless group $G/Z(G)$. Since Theorem 4 holds in the centerless case, Condition (i) holding for $p\phi$ is equivalent to Condition (ii) holding for $p\phi$ and thus to Condition (ii) holding for ϕ. Therefore, if $Z(G)$ is finite, Condition (i) is equivalent (ii), and thus Condition (i) implies that ϕ is isolated. QED Theorem 4

Section 9 Proof of Theorem A and Theorem B

In this section, using the previous results, we prove Theorem A and Theorem B (Introduction).

Proof of Theorem A for Condition (a) on Nilpotent Groups

We can assume that $G = G'$ and $\Gamma = \Gamma'$, since if G/Γ and G'/Γ' are homeomorphic, they are homeomorphic by an affine map $\bar{A}: G/\Gamma \longrightarrow G'/\Gamma'$. Furthermore, (G,Γ) has the automorphism extension property (Section 3 (Lemma), and [13, Theorem 5, p. 292]) .

We verify that the other conditions of Theorem 1(c) hold. $(G, G/\Gamma)$ has the homeomorphism lifting property, since G is simply connected. By Theorem 2, ϕ is isolated. QED

Proof of Theorem A for Condition (b) on Solvable Groups

By the condition that the eigenvalues are real, we can assume that $G = G'$ and $\Gamma = \Gamma'$. Because, if G/Γ and G'/Γ' are homeomorphic, they are homeomorphic by an affine map. Furthermore, (G,Γ) has the automorphism extension property (Section 3 (Lemma), Gorbacevic [10], Mosak and Moskowitz [17], Saito [23, p. 166]) .

We verify that the other conditions of Theorem 1(c) hold. $(G, G/\Gamma)$ has the homeomorphism lifting property, since G is simply connected. Since for all X in L(G), no eigenvalue of ad(X) is pure imaginary, it follows that the exponential map is a diffeomorphism (Saito [23], and [5, p. 400]) . Therefore, by Theorem 2, ϕ is isolated. QED

Proof of Theorem A for Condition (c) on Semisimple Groups

Lemma Let G and G' be connected, centerless, semisimple Lie groups, with lattices Γ and Γ' and identity elements e and e' respectively. Then any homeomorphism $f:(G/\Gamma, e\Gamma) \longrightarrow (G'/\Gamma', e'\Gamma')$ lifts to a homeomorphism $f:(G,e) \longrightarrow (G',e')$.

Note To see that the conclusion of the lemma does not hold for any Lie group and lattice, consider the following example. $G = G' = R \times S^1$, $\Gamma = \Gamma' = Z \times e$, $G/\Gamma = S^1 \times S^1$, and $f: S^1 \times S^1 \longrightarrow S^1 \times S^1$ such that $f(g,h) = (h,g)$ for all g and h in S^1.

Proof of Lemma Let $p: G \longrightarrow G/\Gamma$ and $p': G' \longrightarrow G'/\Gamma'$ be the canonical projections. By covering space theory, it is enough to show that $f_* p_* \pi_1(G,e) \subset p'_* \pi_1(G',e')$ [15 , p. 156]. Therefore, it is enough to show that $p_* \pi_1(G,e)$ and $p'_* \pi_1(G',e')$ are the centers of $\pi_1(G/\Gamma, e\Gamma)$ and $\pi_1(G'/\Gamma', e'\Gamma')$ respectively.

Consider the following commutative diagram of sequences where the horizontal rows are exact. $q: \tilde{G} \longrightarrow G$ is the universal covering homomorphism, $Z(\tilde{G})$ is the center of $\tilde{G}$, and $\Delta \equiv q^{-1}(\Gamma)$ [25 , p. 62, 15 , p. 158].

$$
\begin{array}{ccccccccc}
1 \to & \pi_1(G,e) & \longrightarrow & \pi_1(\tilde{G}/\Delta, \tilde{e}\Delta) \approx \pi_1(G/\Gamma, e\Gamma) & & & & \\
 & \downarrow \approx & & \downarrow \approx & & & & \\
1 \to & Z(\tilde{G}) & \longrightarrow & \Delta & \longrightarrow & \Gamma & \to 1 \\
 & \| & & \downarrow & & \downarrow & \\
1 \to & Z(\tilde{G}) & \longrightarrow & \tilde{G} & \longrightarrow & G & \to 1
\end{array}
$$

Since G is centerless, the Borel density theorem implies that Γ is centerless [20 , pp. 84-87]. Therefore Z(G) is the center of Δ, which implies that $p_*\pi_1(G,e)$ is the center of $\pi_1(G/\Gamma, e\Gamma)$.

The same argument shows that $p'_*\pi_1(G',e')$ is the center of $\pi_1(G'/\Gamma', e'\Gamma')$. QED Lemma

Since ϕ^* and ϕ'^* are topologically equivalent, G/Γ and G'/Γ' are homeomorphic by a map f, which we can take, after translation, to be basepoint preserving. By the lemma, f lifts to $\tilde{f}:(G,e) \longrightarrow (G,e)$. $\tilde{f}$ restricted to Γ is an isomorphism from Γ to Γ'. Since Condition (c) of Theorem A is precisely the condition to which the rigidity theorem of Mostow (which uses work of Prasad and Margulis in the non-uniform lattice case) applies, $\tilde{f}|_\Gamma$ extends to an isomorphism $A: G \longrightarrow G'$ [18, p. 4].

Therefore we can consider $G = G'$ and $\Gamma = \Gamma'$ (Section 3 (Lemma)) .

We verify that the conditions of Theorem 1(c) hold. By the lemma, $(G, G/\Gamma)$ has the homeomorphism lifting property. By the Mostow rigidity theorem, (G,Γ) has the automorphism extension property. By Theorem 4, ϕ is isolated.

QED Theorem A

<u>Proof of Theorem B</u> Theorem B(a), which concerns nilpotent and solvable groups, is Theorem 2. Theorem B(b), which concerns semisimple groups, is Theorem 4.

QED Theorem B

Section 10 Examples, Comments, and Questions

In this section we comment on, ask further questions raised by, illustrate, and show the limits of our theorems.

(a) Example 1 A nilpotent example of Theorem A is the well known Heisenberg group, the group of (3,3) matrices with 0 below the diagonal, 1 on the diagonal, and any real numbers above the diagonal. Take the subgroup with integral entries as the lattice Γ.

(b) Example 2 Non-nilpotent solvable examples of Theorem A can be obtained by starting with a matrix in $SL(n,Z)$ which is semisimple with all its eigenvalues real. Then there exists an action $\sigma: R \longrightarrow SL(n,R)$ of R on R^n such that $\sigma(1)$ is the matrix we started with. Using the action σ, let $G = R^n \rtimes R$, and using the action $\sigma|_Z$, let $\Gamma = Z^n \rtimes Z$.

(c) Example 3 The condition for solvable groups (Theorem A Condition (b)) is not necessary. The following is an example of a connected, simply connected, solvable Lie group for which an eigenvalue of $ad(X)$ is pure imaginary. But still topologically equivalent flows ϕ_1^* and ϕ_2^* are topologically equivalent by an affine map.

Let $\sigma(t)$ be the counterclockwise rotation of R^2 by $2\pi t$. Let S_2 be the semidirect product $R^2 \rtimes R$, where σ is the action of R on R^2. Let Γ_2 be the subgroup $Z^2 \rtimes Z$, which is isomorphic to Z^3 since $\sigma(1)$ is the identity. Then S_2 is a connected, simply connected, solvable Lie group. S_2/Γ_2 is compact. R^2 is the commutator of S_2. There is an X in $L(S_2)$

such that ad(X) has eigenvalue $2\pi i$. ($X = 1 \in L(Z) = Z$) The exponential map is not a diffeomorphism since in fact all one-parameter subgroups not contained in R^2 intersect [2 , pp. 18-21, 31-38]. (S_2, Γ_2) does not have the automorphism extension property, since, for example, the map $(x,y,z) \longmapsto (A(x,y),z)$ is an automorphism of Γ_2 for any A in SL(2,Z), but it is an automorphism of S_2 if and only if, for all t, A commutes with $\sigma(t)$. But $A\sigma(1/4) \neq \sigma(1/4)A$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Any one-parameter subgroup contained in the subgroup R^2 is conjugate to a subgroup of the form $t \longmapsto (tx,0,0)$, $x > 0$. Any one-parameter subgroup not contained in R^2 is conjugate to a subgroup of the form $t \longmapsto (0,0,tz)$. The map $(x,y,z) \longmapsto (y,x,-z)$, which is an automorphism of S_2 and of Γ_2, takes a subgroup of the form $t \longmapsto (0,0,tz)$ where z is negative to a subgroup of the form $t \longmapsto (0,0,tz)$ where z is positive. Therefore it follows that the flows ϕ_1^* and ϕ_2^* are topologically equivalent by an affine map if both the subgroups ϕ_1 and ϕ_2 are contained in R^2 or if neither of the subgroups ϕ_1 and ϕ_2 are contained in R^2 (Section 3 (Lemma)) . However, if ϕ_1 is contained in R^2 and ϕ_2 is not contained in R^2, then the flows ϕ_1^* and ϕ_2^* are not topologically equivalent. To see this consider $\phi_1 : t \longmapsto (t,ty,0)$ where y is irrational, and $\phi_2 : t \longmapsto (0,0,t)$. All orbits in the flow ϕ_2^* are periodic, but in the flow ϕ_1^* the orbit of $(0,0,0)\Gamma_2$ is not periodic.

(d) Example 4 In the solvable case, sometimes when Condition (b) of Theorem A fails, the conclusion fails. That is there are topologically equivalent flows which are not topologically equivalent by an affine map. The following example is in [2, p. 33].

Let $G = S_2$ and $\Gamma = \Gamma_2$ from Example 3 above. Let $G' = R^3$ and $\Gamma' = Z^3$. Let $\phi_1 : t \longmapsto (0,0,-t)$ be a subgroup of S_2, and $\phi_2 : t \longmapsto (0,0,t)$ a subgroup of R^3. $f: (x,y,z) \longmapsto (-\nabla(-z)(x,y),-z)$ is a diffeomorphism from S_2 to R^3 which induces a diffeomorphism $\bar{f}$ from S_2/Γ_2 to R^3/Z^3. $\bar{f}$ is a topological equivalence, and in fact a topological conjugacy, of ϕ_1^* and ϕ_2^*, but ϕ_1^* and ϕ_2^* are not topologically equivalent by an affine map, since S_2 and R^3 are not isomorphic.

(e) Example 5 In the solvable case, sometimes when Condition (b) of Theorem A fails, the conclusion fails even when $G = G'$ and $\Gamma = \Gamma'$.

Let $G = G' = S_2 \times R^3$ and $\Gamma = \Gamma' = \Gamma_2 \times R^3$. The subgroups ϕ_1 of S_2 and ϕ_2 of R^3 referred to in Example 4 can be considered as subgroups of G. The map f of Example 2 can be considered as a diffeomorphism of R^3 with S_2 as well. $f = f^{-1}$. Then $F(g,h) \equiv (f(h),f(g))$ is a diffeomorphism of G with G which induces a diffeomorphism $\bar{F}$ of G/Γ with G/Γ. $\bar{F}$ is a topological equivalence, and in fact a topological conjugacy, of ϕ_1^* and ϕ_2^*.

Suppose ϕ_1^* and ϕ_2^* are topologically equivalent by an affine map. Then there exists $a \in G$, $s > 0$, and an automorphism A of G which extends an automorphism of Γ, such that

$(A\phi_1)(t) = (a\phi_2 a^{-1})(st)$ (Section 3 (Lemma)) . Since ϕ_2 is a subgroup of R^3, $a\phi_2 a^{-1} = \phi_2$. Since $A(\Gamma) = \Gamma$, s must equal 1. So $A\phi_1 = \phi_2$.

Let $A':R^2 \longrightarrow R^2$ be the isomorphism gotten by restricting A to the commutator $[G,G]$, which is the normal subgroup R^2 of S_2 It must be the case that

$$A:((x,y,-t),(0,0,0)) \longmapsto ((A'(x,y),0),(0,0,t)).$$

But then there exists $g_1 = ((x_1,y_1,-t_1),(0,0,0))$ and $g_2 = ((x_2,y_2,-t_2),(0,0,0))$ such that $A(g_1g_2) \neq A(g_1)A(g_2)$. This contradiction shows that the flows ϕ_1^* and ϕ_2^* are not topologically equivalent by an affine map.

(f) Automorphism Extension for Solvable Groups

In Condition (b) of Theorem A, we require that ad(X) have only real eigenvalues in order to insure that (G,Γ) has the automorphism extension property, which is required in Theorem 1(c). The following example shows that not every (G,Γ) has the automorphism extension property, where Γ is a lattice in G and G is a connected, simply connected, solvable Lie group such that $\exp:L(G) \longrightarrow G$ is a diffeomorphism (Milovanov [16]) .

Example 6 Let R_θ be clockwise rotation of R^2 by θ. Let $\sigma(t)$ be in SL(4,R) where

$$\sigma(t):(x_1,x_2,x_3,x_4) \longmapsto (e^{kt}R_{2\pi t}(x_1,x_2),e^{-kt}R_{2\pi t}(x_3,x_4)),$$

and $e^k + e^{-k} = m$ for some integer $m > 2$. Let $G \equiv R^4 \rtimes R$, where R acts on R^4 by σ. Let X_1, X_2, X_3, and X_4 be the standard basis of R^4. Let $c \equiv e^k - e^{-k}$.

Let $Y_1 \equiv -e^{-k}X_1 + e^k c^{-1}X_3$, $Y_2 \equiv -e^{-k}X_2 + e^k c^{-1}X_4$, $Y_3 \equiv -X_1 + c^{-1}X_3$, and $Y_4 \equiv -X_2 + c^{-1}X_4$. The $\{Y_i\}$ is a basis and it generates Γ_u, a uniform discrete subgroup of R^4. Since $\sigma(1)(Y_1) = Y_3$, $\sigma(1)(Y_2) = Y_4$, $\sigma(1)(Y_3) = -Y_1 + mY_3$, and $\sigma(1)(Y_4) = -Y_2 + mY_4$, it follows that $\sigma(1)$ is an automorphism of Γ_u. Therefore $\Gamma \equiv \Gamma_u \rtimes Z$ is a uniform discrete subgroup of G. For any $A \in SL(2,Z)$,

$$\alpha : (x_1,x_2,x_3,x_4) \longmapsto (A(x_1,x_2),A(x_3,x_4))$$

is an automorphism of Γ_u which commutes with $\sigma(1)$. Therefore $\Psi:(v,r) \longmapsto (\alpha v,r)$, for all (v,r) in $\Gamma_u \rtimes Z$, is an automorphism of Γ. Any extension of Ψ to G must be of the form $(v,r) \longmapsto (\alpha v,r)$, for all (v,r) in G. But if A has been chosen to be $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, then the map $(v,r) \longmapsto (\alpha v,r)$ is not an automorphism of G since $\sigma(1/4)\alpha \neq \alpha\sigma(1/4)$.

<u>Question</u> Which (G,Γ) have the automorphism extension property, where G is a connected, simply connected, solvable Lie group with $\exp:L(G) \longrightarrow G$ a diffeomorphism, and Γ is a lattice in G?

<u>(g) Semisimple Case</u> We first give examples of isolated and not isolated subgroups.

<u>Example 7</u> Let G be a semisimple subgroup of SL(n,R). Let $\phi(t) \equiv e^{tX}$, for all t in R. If X has n distinct real eigenvalues, then ϕ is isolated.

<u>Proof</u> Consider ϕ as a subgroup of SL(n,R). ϕ does not have compact closure. If $[X,Y] = 0$, then Y is a diagonal matrix.

Therefore, there is no compact, connected, subgroup in the centralizer of ϕ. By Theorem 3, ϕ is isolated as a subgroup of SL(n,R). Therefore, ϕ is isolated as a subgroup of G.

<u>Example 8</u> Every non-trivial unipotent subgroup of SL(3,R) is isolated.

<u>Proof</u> The proof is by contradiction.

Let $\phi: t \mapsto e^{tX}$, for all t in R, be a non-trivial unipotent one-parameter subgroup of SL(3,R) which is not isolated. Then, if ϕ is recurrently approached by ψ, it follows by Theorem 3 that $\psi = E\phi$, where, for all t in R, $E(t) = e^{tY}$, $E(t)\phi(t) = \phi(t)E(t)$, E has compact closure, and XY = YX. X is nilpotent, and Y must be conjugate to a matrix of the form

$$\begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus X and Y are respectively the nilpotent and semisimple parts of the linear operator X + Y. Putting X + Y in real canonical form (Section 5, [12, p. 126]), we see that X must be 0, which is a contradiction.

<u>Example 9</u> ϕ and ϕ' are subgroups of SL(3,R), where for $t \in R$,

$$\phi(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} \text{ and } \phi'(t) = \begin{pmatrix} e^t\cos t & -e^t\sin t & 0 \\ e^t\sin t & e^t\cos t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

Since ϕ is recurrently approached by ϕ', neither ϕ nor ϕ' is isolated. Observe that ϕ and ϕ' have a compact subgroup isomorphic to the unit circle in their centralizer.

Example 10 Let ϕ and ϕ' be subgroups of SL(4,R), where

$$\phi(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \phi'(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix},$$

for all t in R. ϕ is a unipotent subgroup which is recurrently approached in norm by ϕ'. Neither ϕ nor ϕ' are isolated.

The following are examples to which Condition (c) of Theorem A applies.

Example 11 Let G, G', Γ, and Γ' satisfy Condition (c) of Theorem A. In addition, let G be a subgroup of SL(n,R) and let ϕ be chosen as in Example 7. To be more specific, we could let G = PSL(m,R) and Γ = PSL(m,Z), for any m > 2 [20, chapter 10]. Let ϕ' be a subgroup of G'. Then ϕ^* and ϕ'^* are topologically equivalent if and only if they are topologically equivalent by an affine map.

Example 12 Let G = SL(3,R), and let Γ, G', and Γ' satisfy Condition (c) of Theorem A. Let ϕ and ϕ' be one-parameter subgroups of G and G' respectively, with ϕ unipotent. Then ϕ^* and ϕ'^* are topologically equivalent if and only if they are topologically equivalent by an affine map (Example 8).

Questions (a) In Example 9, are the flows ϕ^* and ϕ'^* on SL(3,R)/SL(3,Z) topologically equivalent?

(b) In general, can the analysis be extended to non-isolated subgroups?

<u>(h) Question</u> Is there an example of Lie groups and lattices G, G', Γ, and Γ' such that ϕ^* and ψ^* are topologically equivalent, but there is no sense-preserving reparameterization ψ' of ψ such that ϕ^* and ψ'^* are topologically conjugate?

<u>(i) Isolation</u> The following remarks are meant to help clarify the geometric meaning of "isolation" .

(1) Theorem 2(b) shows that for connected, simply connected, nilpotent Lie groups, if ϕ is recurrently approached by ϕ' for either positive or negative time, then $\phi = \phi'$ up to sense-preserving reparameterization. This result is not in general true for the non-nilpotent solvable groups. (See Section 4 for details.)

(2) Let d be a metric induced by a right or left invariant Riemannian metric on a connected Lie group G. ϕ is <u>asymptotically Hausdorff bounded</u> from ϕ' if and only if

$$\sup_{s>0} \inf_{t>0} d(\phi(t),\phi'(s)) < \infty \text{ , and } \sup_{s<0} \inf_{t<0} d(\phi(t),\phi'(s)) < \infty \text{ .}$$

(a) ϕ being asymptotically Hausdorff bounded from ϕ' does not imply that ϕ' is asymptotically Hausdorff bounded from ϕ. (Let ϕ' have compact closure and ϕ not have compact closure.)

(b) If ϕ' does not have compact closure, then ϕ being asymptotically Hausdorff bounded from ϕ' implies that ϕ does not have compact closure and that ϕ is recurrently approached by ϕ'.

(c) Let G be a closed Lie subgroup of SL(n,R). Then ϕ being recurrently approached by ϕ' implies that ϕ is asymptotically Hausdorff bounded from ϕ'.

($\sup_{k \in Z} d(\phi(t_k), \phi'(s_k)) < \infty$ implies that ϕ is recurrently approached in norm by ϕ' (Proposition 1(d)) . By Theorem 3, for all t in R, $\phi(t) = E_1(t)\chi(t)$ and $\phi'(t) = E_2(t)\chi(t)$, where E_1 and E_2 have compact closure, E_1 and χ are in each other's centralizer, and E_2 and χ are in each other's centralizer. Statement (c) above follows.)

Section 11 Asymptotic Homotopy Classes for Flows

In this section we develop the rudiments of a theory of asymptotic homotopy classes, which generalizes Schwartzman's theory of asymptotic homology cycles [24]. We use this theory to give an alternate proof of Theorem A(a), the classification of nilflows.

Roughly speaking, for x a recurrent or non-wandering point of any flow on a differentiable manifold M, the sequences of classes in $\pi_1(M,x)$ gotten by closing up trajectories as they return near x are algebraic topological invariants of the flow. To make computations easier by simplifying this set of invariants, we map $\pi_1(M,x)$ into a tower of nilpotent Lie groups, and take the limits of the projective classes of the induced sequences.

(a) Some Functors on Finitely Generated Groups

Let Γ be a finitely generated group with lower central sequence $\underline{\Gamma_i}$, where $\Gamma_1 = \Gamma$, and $\Gamma_{i+1} = [\Gamma, \Gamma_i]$. Up to isomorphism, there exists a unique, connected, simply connected, nilpotent Lie group $\underline{N_i}$, such that

$$\underline{\overline{\Gamma}_i} \equiv (\Gamma/\Gamma_i)/(\text{Torsion subgroup of } (\Gamma/\Gamma_i))$$

is a uniform discrete subgroup of N_i (Malcev [13, Theorem 6]).

$$\cdots \rightarrow N_{i+1} \rightarrow N_i \rightarrow \cdots \rightarrow N_1$$

is the Lie nilpotent tower of Γ.

Any sequence of elements $\{\gamma_k\}$ in Γ determines sequences

$\{\gamma_{(i,k)}\}$ in the N_i. A ray in a nilpotent Lie group is a one-parameter subgroup restricted to $[0,\infty)$. ϕ^+ denotes the ray determined by ϕ. The space of rays is topologically the disjoint union of a sphere and a point. Since $\exp : L(N_i) \longrightarrow N_i$ is a diffeomorphism, each point in N_i which is not the identity determines a unique ray. The identity determines the identity ray. The sequences $\{\gamma_{(i,k)}\}$ determine two sets of rays, ρ_i and R_i. Let ρ_i be the set of limit points in ray space of the sequence of rays containing $\{\gamma_{(i,k)}\}$. By the compactness of ray space, ρ_i is non-empty. Let R_i be the collection of rays for which there is a sequence $t_j \longrightarrow \infty$, and subsequences $\{\gamma_{(i,k_j)}\}$ of $\{\gamma_{(i,k)}\}$ such that

$$\sup_j d(\phi^+(t_j), \gamma_{(i,k_j)}) < \infty .$$

d is a metric induced by a right invariant Riemannian metric on N_i.

These constructions are functorial from the category of finitely generated groups and their isomorphisms. For f an isomorphism of groups, let f_i be the induced isomorphism of N_i. f_i is the extension of an isomorphism of $\overline{\Gamma}_i$. Let f_i also denote the induced map on the ray space of N_i.

(b) Flows

Let ϕ^* be any flow on a differentiable manifold M which has finitely generated fundamental group. For example, M is compact. Make the above constructions for the group $\pi_1(M,x)$. If x is a recurrent point for ϕ^*, there exist sequences $t_k \longrightarrow \infty$, $\phi^*(t_k,x) \longrightarrow x$, and $\{\gamma_k \in \pi_1(M,x)\}$, where γ_k is the class of the loop gotten by closing the trajectory $\phi^*(t,x)$ from 0 to t_k, by a path from $\phi^*(t_k,x)$ to x which stays near x. More precisely, there is a sequence $r_k \longrightarrow 0$ such that the path from $\phi^*(t_k,x)$ to x stays within the ball of radius r_k around x. Let $\rho_i(x,\phi^*)$ and $R_i(x,\phi^*)$ be the union over all such sequences $\{\gamma_k\}$ of the sets of rays ρ_i and R_i.

<u>Proposition 4</u> Let f be a topological equivalence of the flows ϕ^* on M and ϕ'^* on M'. It follows that if x is a recurrent point of M, then

$$(\pi_1 f)_i(\rho_i(x,\phi^*)) = \rho_i(fx,\phi'^*) \quad \text{and}$$

$$(\pi_1 f)_i(R_i(x,\phi^*)) = R_i(fx,\phi'^*)$$

for i = 1, 2, 3,... .

<u>Proof</u> The proposition follows by the functoriality of the constructions. QED

We can get a similar proposition for x a non-wandering point of ϕ^*, if we consider sequences $m_k \longrightarrow x$, $t_k \longrightarrow \infty$, $\phi^*(t_k,m_k) \longrightarrow x$, and $\{\gamma_k\}$ gotten by closing the trajectory $\phi^*(t, m_k)$ from 0 to t_k, by paths from x to m_k and $\phi^*(t_k,m_k)$ to x which stay near x (Fried [9, p. 357]) .

(c) Classification of Nilflows

<u>Theorem A(a)</u> Let N and N' be connected, simply connected, nilpotent Lie groups, Γ and Γ' lattices in N and N', and ϕ^* and ϕ'^* flows on N/Γ and N'/Γ' induced by one-parameter

subgroups ϕ and ϕ'. If ϕ^* and ϕ'^* are topologically equivalent, then they are topologically equivalent by an affine map.

Proof By applying an affine map to N', we can assume that $N = N'$, $\Gamma = \Gamma'$, and $f(e\Gamma) = e\Gamma$. (See Section 3 (Lemma) and Section 9(a).)

We can identify $\pi_1(N/\Gamma, e\Gamma)$ with the torsion free, finitely generated, nilpotent group Γ. Γ is uniform in N [20, p. 29]. We can take $N = N_n$, the top group in the Lie nilpotent tower of Γ. (Since Γ is nilpotent, for large i all the N_i are equal.) $e\Gamma$ is a recurrent point for ϕ^* and ϕ'^* [2, p. 56].

For any one-parameter subgroups ψ and ψ' and sequence $\{\gamma_k\}$ in Γ, if there are sequences $t_k \longrightarrow \infty$ and $s_k \longrightarrow \infty$ such that $\sup_{k \in Z} d(\psi(t_k), \gamma_k) < \infty$ and $\sup_{k \in Z} d(\psi'(s_k), \gamma_k) < \infty$, then, by Theorem 2(b), $\psi = \psi'$ up to sense-preserving reparameterization. Therefore, $R_n(e\Gamma, \phi^*)$ is just the ray ϕ^+, and $R_n(e\Gamma, \phi'^*)$ is the ray ϕ'^+. By Proposition 4, $(\pi_1 f)_n(\phi^+) = \phi'^+$.

Since $(\pi_1 f)_n$ is an automorphism of N extending an automorphism of Γ, it follows that there is an automorphism $A \equiv (\pi_1 f)_n$ of N extending an automorphism of Γ such that, for all t in R, $A\phi(t) = \phi'(\alpha t)$ for some $\alpha > 0$. This implies that ϕ^* and ϕ'^* are topologically equivalent by the affine map $\bar{A}$ (Section 3 (Lemma)). QED Theorem A(a)

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