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A skill approach to the development of geometric thinking

Nucci, Christine, Ph.D.

City University of New York, 1992

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A SKILL APPROACH TO THE DEVELOPMENT OF GEOMETRIC THINKING

by

CHRISTINE NUCCI

A dissertation submitted to the Graduate Faculty in Educational Psychology in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1992

This manuscript has been read and accepted for the Graduate Faculty in Educational Psychology in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

A Skill Approach to the Development of Geometric Thinking

by

Christine Nucci

Advisor: Professor Nicholas J. Anastasiow

A skill theory perspective, framing a theoretical and methodological basis for understanding the acquisition of knowledge, was used in examining the extent to which children's behavior revealed a hierarchic acquisition of geometric concepts.

Sixth grade students (N=18) responded to hierarchical, geometric items, developed in a P. M. van Hiele theory domain. The responses, in the form of videotaped records, were analyzed and subjected to a scalogram method of statistical analysis using Green's (1956) Index of Consistency.

This research is designed to determine whether van Hiele's hierarchical geometry system corresponds to Fischer's Skill Theory Level 10, Representations

4/Abstractions 1. The results are interpreted as supporting Fischer's skill theory model of development. The results are discussed in terms of both their theoretical and their practical implications.

Acknowledgements

First and foremost I wish to express my sincere appreciation to my advisor Nicholas J. Anastasiow. Only with his unrelenting encouragement, support and caring was I able to complete this requirement.

Second I wish to thank my first committee member, Carol Kehr Tittle, for her time, guidance, support and gentleness throughout this research. I also wish to express my gratitude to my second committee member David Fuys. His knowledge of the geometry aspects of this study was invaluable.

I am grateful to Dorothy Geddes who generously provided me with access to data records.

I am indebted to Barbara DiBenedetto for being extremely generous with her statistical knowledge and time. Mostly, I am grateful for her friendship.

A most special thanks goes to Robert D. Palmer; without his help this would not have been possible. And, with love, thanks to my mom, Eleanor Frick Nucci, who always has had confidence in me.

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Chapter 1

Introduction

Any developmental theory has to explain why humans change from globally undifferentiated beings into ones with a complex set of skills in a variety of different but correlated domains such as cognition, motor language, social, self, music, mathematics, and spatial. Most theorists have focused on one of these areas of development. Thereby they have acknowledged that there are different sets of skills. Development among these sets of skills may proceed in a correlated fashion but not directly in concert. Freud focused on emotion-affect, Gesell on motor maturation, Piaget on cognitive development, and Lewin on the environmental forces influencing development while Vygotsky tried to account for how the environment was able to shape what was available in the genetic makeup of the individual (Miller, 1989).

Maturation has been discarded, since as a theory it describes the process of growth and does not explain development. Piagetian stages have been questioned in that the various sets of skills are not achieved in stages at the same time but vary greatly within and among individuals. Freud's developmental stages, while matching age changes of most individuals, do not necessarily explain the processes underlying the changes. Vygotsky comes closest to modern acceptance in that he proposes a means by which the environment influences the development of the individual by its context and manner of teaching. His theory does not necessarily look at different skill levels but assumes that each area would be developed under similar principles.

Fischer has brilliantly and audaciously combined elements of most available developmental theories into what he has called Skill Theory. The theory basically states that each individual has the possibility of developing a set of skills (cognition, language, mathematics, self) and does so in an environment utilizing the talents and hard wired mechanisms of the intact central nervous system. The level of skill development is a product of the maturation of the central nervous system which appears to begin to mature before birth and becomes completely operational at 33 years of age in a process he calls hierarchical tiers and levels.

He rejects the term stage of development as it implies that one develops all skills in concert, that is, one should be able to read as well as they can do mathematics. He views skill development as a product of the brain's readiness to handle the degrees of abstraction necessary for the particular skill development, and the context (people and objects) that are available to assist in the development of the skill. He has also postulated processes which account for change which he calls transformation rules. In addition, he has selected and applied a scalogram methodology which he feels is the most appropriate way to measure skill development. He has also developed a complex coding system to be able to mathematically describe the level of an individual in a particular skill area (to be called modules).

In the following sections, Fischer's skill theory is presented with its more formal terminology and each of the aspects of the theory is amplified by examples of the many research projects he has conducted with his students. Finally, a test of Fischer's theory is made by applying a hierarchical model of the acquisition of geometry, an area yet untested by Fischer to determine the fit or lack of fit between Fischer's hierarchical model and the one developed by van Hiele.

Background

Psychological theories, with differing philosophical backgrounds, have been constructed to predict and explain the nature of cognitive development and the ability to reason in contexts such as mathematics. Traditionally, these theories have been viewed as falling into one of two classes of theories - structural or functional. These two classes of theories tend to be treated as being in opposition to one another in the literature.

Skill theory (Fischer, 1980a; Fischer & Farrar, 1987) does not consider these classes as opposing. Instead, skill theory integrates principles and concepts, from both structural and functional approaches, to predict and explain the nature of development. Skill theory has been successfully applied in numerous contexts within the realms of cognitive, social and emotional development. Of particular concern to this research, skill theory has successfully predicted and explained the development of the mathematical concepts of addition, subtraction, multiplication and division (e.g. Fischer, Kenny & Rose, in press) and the development of classification skills (Roberts and Fischer, 1979). The present research examines the development of geometric concepts and thinking in terms of skill theory.

In recent years, van Hiele's model of geometry learning has been researched in terms of predicting and explaining the individual's learning of geometry and, based on this learning, how geometric concepts can best be taught. To date, van Hiele's model is the only research effort that involves constructing a hierarchy of acquisition of geometric concepts.

The purpose of this research was to validate or give support to Fischer's hierarchical model - skill theory - as a ubiquitous model of development. The van Hiele model is the one current model that serves this purpose in the context of geometry. Specifically, this research seeks to validate, confirm or lend support to Fischer's theory of development by testing the theory with a set of data collected in the context of van Hiele research. This research proposes that the van Hiele data will fit the theory of development advanced by Fischer.

In this paper, skill theory is first described with its basic concepts and tenets. Skill theory is discussed in relation to other psychological theories of development. Second, evidence for the application of skill theory to specific content areas (e.g. arithmetic) in the literature is presented. Third, an attempt is made to apply skill theory to the content area of geometry and specifically to geometric thinking - an area in relation to skill theory not yet reported in the literature. Finally, the hypotheses, procedures and results are discussed.

Chapter 2

Skill theory

Behavior varies enormously from moment to moment and across contexts. However, developmental, especially cognitive-developmental approaches and theories have traditionally characterized an individual as being at a single stage, a point on a developmental scale. This characterization has been frequently questioned in the literature (see, for example, Fischer & Lamborn, 1989) due to the demonstrated developmental unevenness within individuals, between individuals and across cultures and social groups. Skill theory (Fischer, 1980a; Fischer & Farrar, 1987) offers an alternative to the traditional predictions and explanations of the nature of this organization of behavior. Skill theory is an approach to understanding behavior that takes into account both continuities and discontinuities in development.

Skill theory both expands upon and distinguishes itself from traditional and contemporary theories of development. In particular, skill theory builds on the theories of Piaget, Vygotsky, Lewin, Werner, Freud, and also upon information-processing and social learning theorists such as Bandura (Fischer, 1980a). Skill theory integrates ideas from all of these theories to produce a tool for explaining and predicting the development of behavior and thought. As a theory of development, skill theory integrates both behavioral-developmental and cognitive-developmental concepts.

Skill theory (Fischer, 1980a), as a framework for studying development, offers an abstract representation of skill structures that emerge in development, and a set of transformation rules that relate these structures to one another. Together, the skill structures and the set of transformation rules make up the theory's tool for predicting and explaining developmental sequences and synchronies from birth through early adulthood.

According to skill theory, skills are hypothesized to develop step by step through a series of thirteen hierarchical levels divided into four tiers (Fischer & Farrar, 1987) as illustrated in Table 1. The tiers are: The reflexive, sensorimotor, representational, and abstract tiers. Each tier specifies different types of skills or abilities within a behavioral domain (for example: cognitive, social, etc.) The levels specify

Table 1

Four Tiers of Skill Development

Tier	**Level**	**Structure**
Reflex	1	Single reflex sets.
	2	Reflex mappings.
	3	Reflex systems.
	4	Systems of reflex systems or single sensorimotor sets.
Sensorimotor	5	Sensorimotor mappings.
	6	Sensorimotor systems.
	7	Systems of sensorimotor systems or single representations sets.
Representations	8	Representational mappings.
	9	Representational systems.
	10	Systems of representational systems or single abstract sets.
Abstractions	11	Abstract mappings.
	12	Abstract systems.
	13	Systems of abstract systems or single principle sets.

skills of gradually increasing complexity. A skill at the end of one level builds directly onto the skills at the next level. All new skills are built upon old skills (Piaget, 1950; 1952). Skills are always constructed by the individual acting in his or her environment.

The levels are characterized by structures that indicate the kinds of behaviors the individual controls at that level. Individuals gain proficiency in a domain as they move through levels and tiers. The level-to-level movement is specified by the five transformation rules - intercoordination, compounding, focusing, substitution and differentiation. It is by means of these transformation rules that skills develop through levels.

Skill theory posits that development is gradual, continuous, and uneven. The individual gradually constructs more complex skills - continually developing in small steps - that eventually culminate in a major reorganization at the next level. Within this systematic development, behavior varies in complexity across both context and time. And, although within the same tier different levels are achieved and demonstrated for different skills (e.g. verbal as compared to analytic skills) - a single developmental level is not consistently demonstrated. That is, the individual could be at one developmental level for verbal skills and at another developmental level for analytic skills.

Developmental variability

Variability in age of acquisition for various skills is a pervasive finding in developmental research (Bidell & Fischer, 1989). Bidell and Fischer argue that a central concern for the developmentalist is to understand this role of variability within the framework of current developmental theories. Bidell and Fischer conclude that Piaget's theory, while including the term "decalage," implies a tight time schedule for the acquisition of skills across individuals and contexts but does not offer satisfactory explanations for the wide variations observed in children and adults. They argue that developmental theories need to be expanded to include the sources of these variations. They present three ways this can be accomplished.

The first way to account for these variations is by describing skills, in terms of a more refined developmental scale. Bidell and Fischer cite studies (e.g. Gelman, 1972) where task complexity was altered to obtain results closer to Piaget's results. They consider it difficult to determine whether task simplifications made it easier for the young children to exhibit higher-level skills, or whether the task simplifications created a task requiring lower-level skills.

Research using Piagetian stages tends to force abilities into dichotomies (e.g. preoperational or operational, or the child "has" class inclusion or "does not have" class inclusion). In this sense, Bidell and Fischer argue that it is more important to interpret variation in performance in terms of a finer scale to specify the meaning of changes in the task according to specific levels of skill needed to perform each task variation. That is, for example, instead of two broad Piagetian type stages, they suggest a 12-step developmental sequence to relate specific changes in the complexity of the tasks to specific steps in the sequence of the skills' developmental pattern.

The second way suggested to incorporate variability into a developmental theory involves describing one's abilities in terms of a flexible range of skill levels, i.e. levels that can change with changes in contextual support. With different levels of contextual support, levels for any task can change within the individual. Unlike Piagetian theory that views a skill as a single point, in a fixed sequence, skill theory is similar to Vygotsky's (1962) view that skills can be advanced by an advanced peer or teacher. Vygotsky holds that the individual's ability on any task covers a range of levels. Vygotskians see differences in contextual support as helping to account for ability variation. Again, in lieu of viewing a skill like transitivity as all or none, skill theory sees it as more important to describe the individual's ability relative to specific contextual support. Thus, the researcher can systematically vary the contextual support and study how context affects the development of skills.

The third source of developmental variations that Bidell and Fischer claim needs to be incorporated into developmental theory is differences in background knowledge and experience due to social and cultural influences. That is, theories need to account for different pathways or different developmental sequences and timetables across cultural groups. An example given to illustrate the importance of cultural differences is the finding that Mexican children who participate in pottery making acquire substance conservation much earlier than their controls (Fischer & Bullock, 1981; Bidell & Fischer, 1989). Skill theory incorporates these sources of variability into its framework.

Optimal level

It is skill theory's position that developmental unevenness is the rule, not the exception. Fischer uses the term "optimal level" to characterize a true developmental level: an upper limit on the complexity of a skill that one can construct. In practice, the upper limit is only occasionally evidenced with the assistance of a teacher or guider of instruction. More typically, less advanced skills are achieved by the individual which Fischer & Pipp (1984b) call the individual's functioning level.

Optimal level is derived from the work of Vygotsky's (1962) concept of "zone of proximal development". It is the level of achievement one can achieve with the help of an adult or advanced peer who plans instruction at the current level of functioning of the student. For example, the student may be able to play basketball, but with instruction given at his or her current skill level, can advance and perform with greater precision, that is, higher skills. Whereas individuals acquire higher level skills through experience and teaching the ultimate height of a skill or the movement to higher order tiers appears to be related to the development of the CNS.

In a written, personal correspondence[1], Dr. Fischer states "...The theory as it stands specifies how the child moves across levels within domains and how optimal level shifts with development...it does not specify the "deeper" reasons for change." He continues to write, "...my best guess at what is going on is that the child grows a new type of neural network for each level. This process is induced epigenetically by a combination of biological and environmental influences that enable larger networks to function." Admitting its limitation, Dr. Fischer writes of the hope of, "more specification based on neural-network theory, evidence on brain growth, and further research on level changes across domains and children." In one article, Fischer (1987) suggests possible relationships between brain development and cognitive development based on recently reported data on cortical development in rhesus monkeys, which appeared to be similar to the cognitive capacities that develop in human infants. (Also, see Fischer, 1983a.)

Skill

For over 20 years, Fischer and his colleagues have examined the development of skills in cognitive, emotional, language, moral and social domains. These works are listed in Table 2.

At the base of skill theory is the concept of skill, a behavioral characteristic of a person-in-a-context. In skill theory, skills are always defined in terms of both the organism and the environment. Skill development is influenced by both the person's action and context (Fischer, 1989). While skills are characterized by structures with properties like those described by organismically-oriented psychologists, skills are also subject to the functional laws of environmentally-oriented psychologists. For example, in the development of conservation of length, the 5- or 6-year old who has two skills for the length of the cord (one skill for the horizontal length and one skill for the vertical length), necessary in the spring-and-cord gadget paradigm (See Figure 1), has the understanding of how the length of the

Table 2

Over 20 Years of Skill Theory Research

SKILL DOMAIN	YEAR	AUTHOR(S)	CONTEXT	SAMPLE
Cognitive	1970	Fischer	Sensory-motor	Infancy
Cognitive	1974	Fischer	Problem-solving	Early Childhood
Cognitive	1980b	Fischer	Problem-solving	Rats
Cognitive	1977	Watson & Fischer	Agent-use	Late infancy
Cognitive	1978	Bertenthal & Fischer	Self-recognition	Infancy
Cognitive	1981	Fischer & Jennings	Representations in search	Infancy
Cognitive	1982	Fischer & Canfield	Creativity	Childhood
Cognitive	1982	Fischer	Cognitive Development	First Four Years
Cognitive	1983	Bertenthal & Fischer	Search tasks	Infancy
Cognitive	1984	Fischer, Hand & Russell	Abstractions in arithmetic	Adolescence
Cognitive	1984	Fischer & Bullock	Cognitive Development	Middle Childhood
Cognitive	1984	Fischer & Pipp	Structures of unconscious thought	Early Childhood
Cognitive	1987	Pipp, Fischer & Jennings	Self and mother knowledge	Infancy
Cognitive	1988	Fischer, Kenny & Beals	Arithmetic concepts	Grade school to college
Cognitive	In Press	Fischer, Kenny & Beals	Arithmetic concepts	School years
Cognitive	In Press	Fischer & Roberts	Classification skills	Preschoolers
Cognitive	In Press	Fischer & Knight	Cognitive Development	Childhood-early adulthood
Cognitive	In Press	Kitchener & Fischer	Reflective thinking	Adolescence and early childhood
Cognitive	In Press	Plattz & Fischer	Perceptive talking	Childhood
Cognitive	In Press A and In Press B	Van Parys & Fischer	Sex, age and race concepts	Preschool years
Emotional	1981	Fischer & Watson	Resolution of the Oedipus conflice	Childhood
Emotional	1989 and In press	Fischer, Shaver & Carnochan	Emotional skills	Infancy - early adulthood
Language	1981	Fischer & Corrigan	Language skills	Infancy and early childhood
Moral	1983b	Fischer	Moral development	Childhood
Social	1980	Watson & Fischer	Social roles	Preschoolers

Table 2 (Continued)

SKILL DOMAIN	YEAR	AUTHOR(S)	CONTEXT	SAMPLE
Social	1981	Hand & Fischer	Intention and Responsibility in social interaction	Childhood
Social	1981	Hand & Fischer	Social interaction skills	Childhood
Social	1984	Fischer, Hand, Watson Van Parys & Tucker	Social categories	Preschoolers
Social	1985	Pipp, Shaver, Jennings Lamborn & Fischer	Relationships with parents	Adolescence
Social	1988	Fischer & Elmendorf	Personality & social behavior	Childhood and adolescence
Social	1989	Rose & Fischer	Social task	Childhood
Social	In Press	Hand & Fischer	Social interaction skills	Childhood
Social	In Press	Lamborn, Fischer & Pipp	Concepts of honesty and kindness	Adolescence

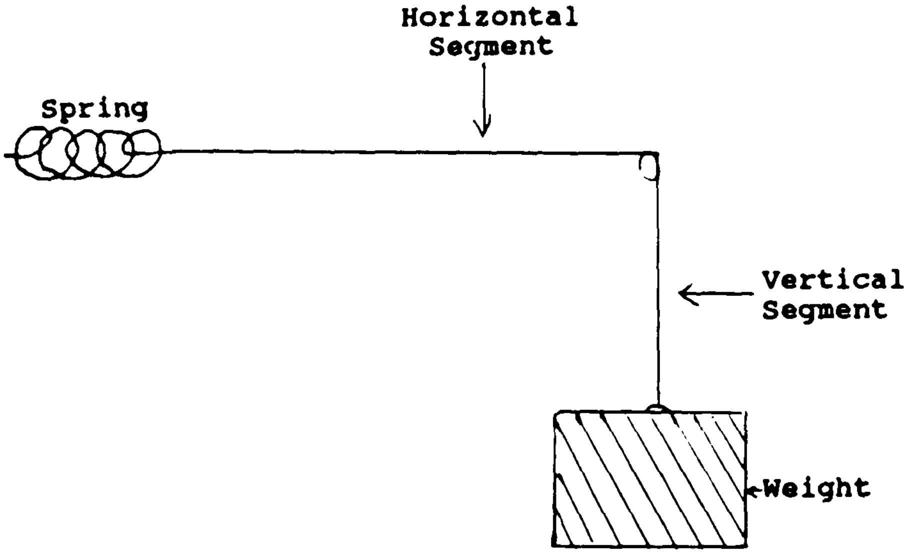


Figure 1. The spring and cord gadget.

horizontal segment relates to the length of the vertical segment, i.e. can control the relation between the vertical segment, i.e. can control the relation between the vertical and horizontal lengths and vice versa. However, the child does not yet understand that changes in length of one compensate for changes in length of the other (i.e. the child does not yet understand full conservation of the length of the cord). From the organismic perspective, the child must coordinate or integrate the two skills for predicting the length to form a more complex skill. And, from the environmental perspective, this coordination will only occur as the child plays with the gadget (i.e experience.) By repeatedly applying the two skills, the properties of the task itself induce the child to notice the connection between them. This joint action of the organism and environment in development is equally important for all the skill levels in skill theory.

In another example, the infant coordinates the skill of "reaching" with the skill of "looking" to form the skill of "reaching while looking."

Tiers and levels

The tiers are apparently genetically determined (Anastasiow, 1987; Emde, Gaensbauer & Harmon, 1976; Epstein, 1974; Fischer, 1983a; Lecours, 1975; & Luria, 1973; 1982.) There are four different types of skills: reflexive skills, sensorimotor skills, representational skills and abstract skills, proposed by Fischer. Four levels make up a tier as illustrated in Table 3. Each cycle of four levels comprising a tier is named for the type of behavior that one can acquire and perform at that time in his or her development.

At Levels 1 to 4 is the reflex set or the single set making up Tier 1; this set first emerges in infancy. At Levels 4 to 7 is the sensorimotor set or mappings that comprises Tier 2 and develops in early childhood. Levels 7 to 10 make up the representational set of the system that comprises Tier 3, which develops throughout childhood. Level 10 and beyond is the abstract set or system of systems that makes up Tier 4 and develops throughout adolescence and into adulthood. Fischer uses the point, line segment, square and cube, as shown in Figure 2, to metaphorically represent the four levels and how one acquires the skills in each tier and develops.

Table 3

Thirteen Developmental Levels Described By Skill Theory*

Tier					Level	Structure
Reflex				1	1	Single reflex sets
				2	2	Reflex mappings
				3	3	Reflex systems
Sensorimotor			1	4	4	System of reflex systems
						Single sensorimotor sets
			2		5	Sensorimotor mappings
			3		6	Sensorimotor systems
Representational		1	4		7	Systems of sensorimotor systems
						Single representational sets
		2			8	Representational mappings
		3			9	Representational systems
Abstract	1	4			10	Systems of representational systems
						Single abstract sets
	2				11	Abstract mappings
	3				12	Abstract systems
	4				13	Systems of abstract systems
						Single principle sets

* 1987 model

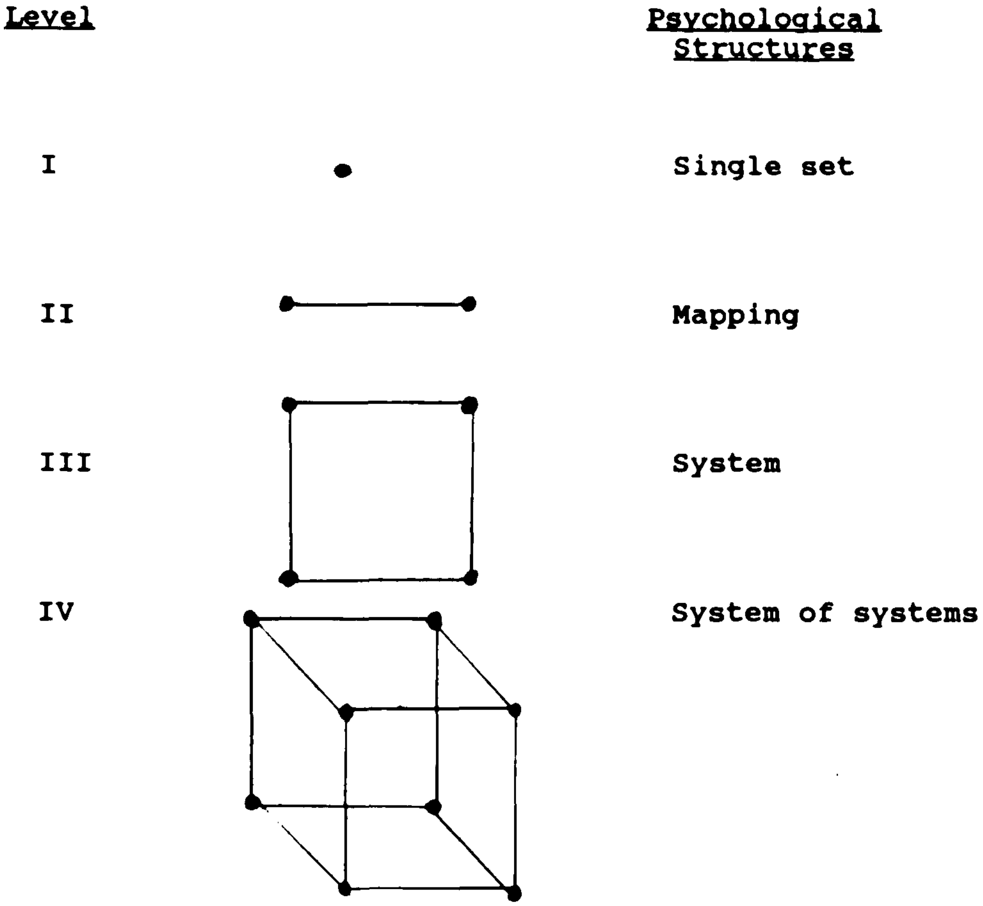


Figure 2. Fischer's geometric metaphor for the cycle of four levels.

These four tiers - the single set, mappings, the system, and the system of systems - are also recognizable within each tier. That is, within the sensorimotor tier for example, is the single sensorimotor set, sensorimotor mappings, the sensorimotor system, and the system of sensorimotor systems. Also, the fourth level of this tier (as with each of the other tiers), the system of sensorimotor systems, is considered to be the first level of the succeeding tier - the single representational set. Similarly, the fourth level of the reflex tier - the system of reflex systems - is equivalent to the first level of the sensorimotor tier for the single sensorimotor set. Level four of the representational set - the system of representational systems is equivalent to the first level of the abstract tier or the single abstract set. And, the fourth level of the abstract tier - the system of abstract systems - is equivalent to the single principle or the single principle set.

Fischer (1980a) in defining the relations between skills and levels within the skill theory framework explains that levels are used generally to characterize an individual's skills, not to characterize the individual in general. Skill theory identifies four levels within the representational tier (the last of which is the focus of this study.) The first major reorganization, around age four, is characterized by the ability to deal with simple representational relations. The three successive levels appear to correspond to Piaget's levels of pre-, concrete-, and formal-operations. This research investigates levels in the representational and abstract tiers, in particular, level 10, the system of representational systems which is equivalent to a single abstract set. Levels 9 and 11, the representational systems and abstract mappings, are considered to give a more complete picture of development. The following section explains this in more detail.

Representations are built upon sensorimotor actions, in particular, upon systems of sensorimotor systems (the last level of the sensorimotor tier is the first level of the representational tier.) However, representations are a more powerful type of skill structure than sensorimotor skill structures. Representations designate concrete characteristics of particular objects, people, or events in the world. Representation, while built on sensorimotor actions, is independent of one's own actions.

The first level of the representational tier is single representations, a relation between two sensorimotor systems. It is the combination of these sensorimotor systems that generates the single representational set. At this level which typically develops in the 20 - 24 month old, the infant can represent simple properties of objects, events, and people independently of their own immediate actions. Fischer and Roberts' (1979) study of classification skills in preschool children predicts that at about the age of one, children should be able to sort into a single simple category, that is, sort blocks into one category of shape or color as when the blocks are identical except for the variations in shape or color. For example, children can pick circles out from triangles when all the blocks are the same color and size. At about two years of age, children should be able to handle several categories simultaneously. For example, the two-year-old at this level sorts blocks into three categories - reds, yellows and blues.

The second level of this tier is representational mappings which typically first emerges around 3.5 - 4.5 years of age. At this level, one representational set is mapped onto a second representational set. Here the child can relate variations in one representation to variations in a second representation. Continuing with the example above, Fischer and Roberts predicted that at 2 1/2 years children would be able to perform multiple simple categories with variations: They sort blocks into three categories even when there are variations within each of the categories - different shades of red, yellow and blue, or different types of triangles, circles, and squares. At 3 - 3 1/2 years of age, the child can handle categories where there are variations on an interfering dimension. That is, when the blocks vary in both color and shape simultaneously, the child can still sort them into three shape categories and separately into three color categories.

Further, Fischer and Roberts predicted four-year-olds could do very simple 3-by-3 matrices. First the child, having been shown how to sort nine blocks first along one dimension and then along another to produce a matrix, sorts the blocks into three piles corresponding to the categories of one dimension and then spreads out each pile into the categories of the other dimension. Shortly after this, the child should be able to produce a simple 3-by-3 matrix which has no variations within each cell (e.g. all red triangles are identical) and two or three blocks per cell. (For a description of the complete developmental sequence of classification skills in young children see Fischer & Roberts, 1979.)

Level 3 of the representational tier is the representational system and emerges around 6 to 7 years of age. At this level, the child can relate two subsets of one representation. However, the child is still unable to relate various systems to one another and cannot control these systems in a single higher level skill. Fischer and Knight (in press-b) offer a table in their research that includes the level of Representations 3 and note in that table that this level is "also called Concrete Operations". As an example of Representations 3, they present the skill of understanding that when water is poured from one glass to another, the amount of water stays the same - the complex relations of subsets of concrete operations.

Finally, at Level 4 of this tier, the child can control the relation between two representational systems. This "system of representational systems" or the single abstract set allows the child to abstract an intangible attribute that characterizes broad categories of objects, events, or people. This level of "systems of representational systems" first emerges around 10 to 12 years of age. In their table, Fischer and Knight list Representations 4 / Abstractions 1 as "single abstractions". The skill of having the concept of the operation of addition of positive whole numbers is an example given of this coordination of concrete representational systems to produce general, intangible concepts. It is on this level that the present research focuses.

With the second level of the abstract tier, which typically emerges at 14 to 16 years of age in middle-class Americans, adolescents can relate one abstract identity concept or one abstraction with another in a simple relation. For example, in arithmetic they can relate the abstract concept of addition to the abstract concept of subtraction. That is, in explaining the relation of 5 + 7 = 12 to 12 - 5 = 7, the individual has the skill of understanding the relation of the concepts of addition and subtraction through opposition.

At the third level of the abstract tier, the individual can understand complex relations of subsets of intangible concepts with abstract systems. Abstract systems produce a much more flexible, differentiated relation between two concepts. For example, the individual understands that the operations of addition and division are related through how numbers are grouped and combined (e.g. 10 + 10 + 10 + 10 = 40 is related to 40 / 10 = 4.)

And finally, with the system of abstract systems, also considered the single principle, the individual can coordinate two or more abstract systems. An example from the arithmetic study is the principle unifying the four arithmetic operations. A response that would be considered to demonstrate thinking at this level as presented by Fischer (1980) is: "Addition, subtraction, multiplication and division are all operations, which means that they all transform numbers by either combining or separating them and doing so either in groups or one number at a time. There are relations between all possible pairs of operations. Some pairs are closely related, and others more distantly related ..." (Elaboration explaining the pairs as diagramed in the table below, and applying them to concrete arithmetic problems, using positive whole numbers, such as 5 + 7 =

12, 12 - 7 = 5, 5 x 7 = 35, and 35 / 5 = 7)" would be a level 13 response.

	Single Number	Group of Numbers
Increase	Addition	Multiplication
Decrease	Subtraction	Division

Transformation rules

The set of transformation rules, used to relate one level to the next, specifies the particular developmental steps by which a skill moves from one level to the next. Transformations produce both continuous and gradual behavioral changes across an entire profile of one's skills. The five transformation rules central to skill theory - intercoordination, compounding, focusing, substitution and differentiation - describe how a skill can be transformed in development into a new, more advanced skill. While levels describe across-level predictions, transformation rules describe within-level predictions. It is these five transformation rules that are at the heart of the mechanism for predicting specific sequences of development in skill theory. They are intended to be applied to changes in the organization of behavior during learning or problem-solving. All five transformation rules are defined structurally. That is, two or more skills with given structures are transformed into one or more skills with a new type of structure.

Two transformation rules, intercoordination and compounding, can describe how skills are combined to produce new skills. Intercoordination describes combinations that produce development from one level to the next. Compounding describes combinations that produce development within a level i.e. two skills combined to form a new skill at the same level. Focusing and substitution are smaller steps than compounding. Focusing deals with moment to moment shifts from one skill to another. It deals with shifts in attention. Substitution designates certain cases of generalization of a skill. Differentiation indicates how sets become separated into potentially distinct subsets when one of the other four transformation rules does not occur. Differentiation can also be used separately to predict a within-level step.

Substitution

Specifically, the transformation rule of substitution is the simplest: It involves putting a similar element into the same structural relation. A skill, mastered at Level L with one task, is transferred to a second similar task. Substitution will apply when all components of the tasks are similar with the exception of one. Substitution can occur only after the original skill has been mastered and before skills of greater complexity are formed in the same task domain. For example, a child holding his or her pillow pretends to go to sleep. A piece of cloth is substituted for the pillow. Thus, the child holds a piece of cloth and pretends to go to sleep.

Focusing

Focusing, also referred to in the literature as "shift-of-focus," refers to moment-to-moment shifts of attention between skills within the level or levels at which an individual is functioning in a task. Since, at any one moment, one cannot cognitively deal with all skills, focusing describes the cognitive shift from one skill to another.

An example of focusing in a classification problem, in the context of interest to this research, would be the child that sorts geometric figures by one dimension (e.g. shape or color) at a time. Although the child may be able to sort the figures along one dimension in one sort and another dimension in another sort, he or she does not sort along, for example, two dimensions in a single sort. Here the child is not able to cognitively combine the two dimensions but instead focuses on one dimension at a time.

Compounding

Skill theory considers the process of compounding the most important transformation within each of the developmental levels. In compounding, one combines two or more skills to form a new, more complex skill. Rather than isolated control of skills as in the preceding transformation rule of focusing, in a compounding transformation, two or more skills are combined to form a more complex skill at the same level. Here, for example, the 18-month-old has one Level 7 skill relating holding a pillow to placing his or her head on the pillow. The infant has a second Level 7 skill relating placing its head on the pillow to going to sleep. To control all three actions the infant needs to compound the two Level 7 skills to produce a more complex Level 7 skill.

Intercoordination

The transformation rule of intercoordination refers to the process by which two skills at one level are reciprocally coordinated to form a skill at a more developmentally advanced level. It is intercoordination that is the mechanism of transformation between developmental levels. Intercoordination specifies how the individual combines skills to develop from level to level - from Level 1 through Level 13.

An example of this rule applied to a conservation problem would be the child that has a skill to understand that the amount of liquid conserves regardless of the height and width of the container it is poured into. At Level 8, for example, the child can understand two separate representational mappings involving the height (h) and the width (w) of the cup. The child seems to understand how the height and width relate to each other yet does not recognize that the total liquid remains the same because the changes compensate for each other. This problem is resolved when the child intercoordinates the two mappings to produce a Level 9 skill for conservation of liquid.

Differentiation

Finally, the transformation rule of differentiation describes the way a single set becomes separated into distinct sets or subsets. Differentiation is a product of one of the other four transformations. Therefore, differentiation can either be a within-level or between-level change depending upon which other transformation is involved.

The development of conservation of length illustrates how differentiation occurs when a new skill is formed. The child that has not adequately differentiated the total length of the cord from the lengths of the horizontal and vertical segments confuses the total length of the cord with the segments. The lack of differentiation is resolved when the child intercoordinates the two Level 8 mappings to form the Level 9 system for conservation of the total length of the cord.

Collaboration of the organism and the environment

Transformations require both organismic and environmental factors (Fischer & Silvern, 1985). Developmentalists have recognized the limited predictive and explanatory power of both genetics and environments but also have had difficulties explaining how the environment and the organism transact to explain development. Psychologists have tended to emphasize one side or the other. For example, some have contended that the organism changes from stage to stage and the environment plays a minimal role. For Piaget, there is an introduction of the term horizontal decalage when there are instances of environmentally induced developmental unevenness. Behaviorally-oriented psychologists, in recognizing the importance of the environment and the organism, have explanatory constructs emphasizing the environment. However they tend to neglect the organism (Fischer & Silvern, 1985).

Fischer and Hogan (1989) in describing infant development discuss the need to move beyond the traditional dichotomy between organism and environment and articulate the need to build an explanatory system that subsumes both. The authors contend that a framework that integrates both organism and environment can successfully specify the processes that produce development in infancy and at other levels. Fischer explains how skill theory, as an approach to development, accomplishes this. All concepts defined in skill theory involve a transactional view.

In skill theory the collaboration of the individual and the environment results in behavior and is reflected in all concepts and constructs. This collaboration is a key principle of skill theory. According to skill theory, environmental support refers to the variation in environmental and contextual influences on one's performance on a task.

Skill theory research has assessed the influence of environmental factors by 1) varying each of the factors as an independent variable in order to measure their effects, 2) looking at specific environmental effects such as the effects of practice (e.g. Lamborn & Fischer, 1989) and the effects of the order of testing (Hand & Fischer, 1981a), and 3) by giving the same task under different conditions of environmental support (e.g. high- and low-support conditions). The factors in a typical environment that affect performance of a skill include familiarity, exposure and prior- and task-reinforcements. The factors in the testing situation that can affect the performance of a skill include practice effects, scaffolding (or the provision of task cues that guide the subject to behave in certain ways), modeling (also termed "elicited imitation"), rapport, personal and emotional support, adult (especially parental) presence and task complexity, as well as factors of the individual that may affect performance of a skill: interests, abilities, motivation and temperament; emotions and mood; health and hunger; use of strategies; mental and behavioral sets and age and developmental level. The skills in skill theory are always defined jointly by the organism and the environment in a transactional process.

Mappings and systems

In discussing relations between sets within a skill, Fischer (1980a) describes a psychological interpretation of mappings and systems. Mapping involves the individual relating two sets in one skill - a structure relating two sets. These could be two reflexes, two sensorimotor actions, two representations, or two abstractions. In each case there is a collection of ordered pairs in which the first member in each pair is from one set (A) and the second member is from another set (B). Set A is said to be mapped onto set B. The two subdivided sets are said to form a system. In a system, the individual relates two subsets of each of the two sets in one skill. Once the individual can deal with two subsets in each set, he or she can control two sources of variation in each set.

In a system of systems, there is a relation between two systems. At this point the individual can relate two systems in one skill. This allows the individual to form a new kind of set. That is the most elementary set of the next tier. The repetitive cycle continues. In all of these structures, a set is a source of variation (whether in reflexes, actions, representations or abstractions) that the individual can control.

Scalogram

The scalogram method is a multiple task method. For each step in a predicted developmental sequence there is a separate task. That is, there is independent assessment of each and every step. The scalogram method allows, to some extent, for detection of individual differences in the nature of the sequence. Typically, most studies have tested a single sequence and have precluded detection of any other sequence. In these traditional studies, individual differences, at best, have only been detected in the speed of movement through the sequence. The scalogram method is intended to actually detect and describe the nature of this sequence.

The scalogram illustrates the tasks passed and failed at particular developmental levels. If the individual's behavior depicts a pattern of increasingly more tasks passed with each increasing developmental level, the profile is said to have "a good fit". The extent to which the behavioral profile has a "good fit" can be statistically determined by Green's (1956) goodness of fit index. Typically, behavioral patterns that are highly scalable or profiles that have a good fit are indicative of particular developmental levels. Fischer has repeatedly used a scalogram method of analyses in skill theory research. Following the skill theory tradition, this research utilizes the scalogram for analyses purposes.

Chapter 3

Skill theory and content related research

The work done in the areas of classification skills (Fischer & Roberts, in press and Roberts & Fischer, 1979) and arithmetic (Fischer, Kenny, & Pipp, in press; Fischer, Kenny & Beals, 1988; Fischer, Kenny & Rose, in press; Fischer & Kenny, 1986; and Fischer, Hand & Russell, 1984) are of particular importance to this research and therefore appropriate parts are reviewed here.

Roberts & Fischer (1979) argue that a theoretical problem with interpreting Inhelder's and Piaget's (1958; 1964) proposed developmental sequence of classification skills exists. They claim that the proposed classification skills sequence proposed by Inhelder and Piaget is problematic in that they did not directly test the sequence by assessing each child on each step. Just as Kofsky (1966) tested each subject on each step of a classification task in an attempt to validate Inhelder's and Piaget's hypothesized sequence, this research tested each subject on each step of the geometric sequence and predicted that the sequence would be scalable. That is, the subjects' performance profile was predicted to fit a Guttman scale. Roberts and Fischer note that Kofsky did not find Inhelder's and Piaget's proposed sequence to be scalable; the subjects' profile did not fit a Guttman scale.

Development of classification skills

Roberts and Fischer cite findings that contradict Piaget's that preschool children demonstrate some classification skills prior to concrete operations. Roberts and Fischer predict a developmental sequence showing the nature of classification skills in one-year-olds and clearly delineate the classification skills that typically begin to emerge around six or seven years of age. They aim to clarify this nature and the sequence of youngsters' skills for understanding classes involving the dimensions of color and shape and for building matrices with these classes. They successfully used skill theory to provide a systematic approach to predicting the cognitive developmental sequences of classification skills by means of task and skill analysis and a set of rules for ordering tasks and skills into developmental sequences.

Specifically, as inherent with the skill theory method, separate tasks were used to assess each step in the proposed developmental classification skills sequence. In each task the subject sorted blocks that varied in color, shape and size. To help assure understanding of the task, a demonstration of correctly sorted blocks was shown to the subjects.

A nine-step developmental sequence was predicted by Roberts and Fischer that involved scaling each individual subject. The sequences were predicted to be evident in the performance protocols of individual children, not simply in differences between groups of children of different ages. A subject fit a sequence when he or she passed all steps up to a particular point, then failed all steps beyond that point. A scalogram analysis was used to assess the predicted scalability.

In two studies, Roberts and Fischer (1979) and Fischer and Roberts (in press) tested seventy subjects ranging from 15 months to 7 years 3 months of age and forty subjects ranging from 4 years 6 months to 6 years of age. All seventy subjects fit the predicted developmental sequence. By the scalogram analysis the items were perfectly scalable; Green's (1956) index of consistency was 1.0. Of the forty subjects 36 fit the predicted profiles; Green's index of consistency was .86.

Their findings supported the prediction that children as young as 15 months can classify or sort out one simple category from another. Two-year-olds demonstrated a sort of blocks that varied in only one dimension (e.g. color), while three-year-olds sorted blocks along a single dimension even when there were variations within the categories (e.g. shades of a color). Roberts and Fischer conclude that young children can classify when the tasks are appropriate to their ages. Also, children (4 or 5 years of age) can handle certain classification-matrix problems prior to handling those of a more complex nature used by Inhelder and Piaget.

Further, Roberts and Fischer emphasize the importance of simplifying the materials to be sorted by demonstrating to the children what they're being asked to do, and by allowing them several trials to perform correctly. In this way, they argue, the opportunity for the children to show what they are capable of doing is optimized. Ultimately, they contend that classification skills as evidenced by object sorting appear to develop in a sequential manner and, at the same time, the development of classification skills reflects a gradual process of skill construction.

Development of arithmetic skills

Fischer and Kenny (1986) present a study of the development of abstractions in an arithmetic context. To date, this is one of the most complete descriptions of the work done with skill theory in the arithmetic domain. They argue for two main points: 1) new levels of abilities emerge in late adolescence and early adulthood and 2) most skill development during this period shows slow, gradual improvement and does not reflect the cognitive advances of the newly emerging levels. These two points are not in opposition because people usually do not function at their highest developmental level. Level of performance varies systematically below the individual's highest level as a function of environmental conditions such as practice and contextual support. Under certain environmental conditions, cognitive-developmental levels are strikingly evident; while under other conditions, they are not demonstrated or observed.

In order to study the discontinuities in performance predicted to appear with a new developmental level, Fischer and Kenny hold that skills must be assessed by testing subjects 1) in familiar domains - where they have had an opportunity to construct high-level skills, 2) under environmental conditions that provide contextual support for high-level performance, and 3) with the opportunity to practice the task they are to perform.

Another aspect of development dealt with in skill theory concerns the internalization of skills. Fischer and Kenny note that, according to skill theory, skills are not present or absent but internalized to different degrees. The more a skill is internalized, the less environmental support it requires for the individual to perform it. Conversely, the less internalized a skill is, the more environmental support is necessary for performance. In familiar domains skills are internalized to the extent that short periods of practice will suffice. In unfamiliar domains, optimal performance is possible only after long periods of practice and instruction and assistance.

In their arithmetic study, Fischer and Kenny tested this concept by varying the degree of contextual support and the degree of opportunity for practice in an arithmetic context. Subjects were tested in two sessions with two conditions in each session. In the first session, subjects were given a series of arithmetic tasks with no support or practice. Later in the first session, the tasks were readministered and environmental support, consisting of being shown an acceptable answer and given time to read it over, was provided. Subjects were then told they would be retested in two weeks (i.e. the second session) and to think the tasks over. In the second session, the two conditions were repeated - assessment with no support and assessment with support (i.e. seeing an acceptable answer.) Eight subjects in each of the grades three to the sophomore year of college were tested. The arithmetic tasks in the study were designed to test the levels of abstract skills, predicted by skill theory, to develop in adolescence and early adulthood. All of the tasks involved using, defining and relating the four basic arithmetic operations dealing with positive, whole numbers.

For single abstractions all tasks required providing a general definition of one of the four basic arithmetic operations and showing how it applied to the problem the subject had calculated and worked on. To pass the level of abstract mappings, the subject had to explain in general terms how two of the four basic arithmetic operations related to each other and how that relation is evident in specific arithmetic problems (i.e. give and explain a numerical example). To demonstrate the level of abstract systems one would have to explain how two arithmetic operations are related (e.g. in opposition) in two ways and how the relation is evident in specific arithmetic problems. And, for the level of systems of abstract systems, or the single principle, the individual needed to explain a principle unifying all four of the arithmetic operations and apply it to concrete arithmetic problems.

The findings revealed spurts. This is the empirical criterion in Fischer's theory for the emergence of a new level, at the levels of the abstract set and abstract mappings. The four experimental conditions included no support or practice (nonoptimal conditions) and both support and practice (optimal conditions). The optimal conditions produced spurts in performance and the nonoptimal conditions produced slow, gradual improvement on the tasks.

The arithmetic study supports the skill theory hypothesis that development shows spurts in optimal performance with the emergence of each new cognitive level while showing gradual, continual change in performance under ordinary, nonoptimal conditions. The arithmetic study also supports the gap hypothesis. That is, individuals function at less than optimal levels under ordinary conditions. The gap between functional level (performance under ordinary conditions) and optimal levels increases as the individual's optimal level increases. At higher levels then, support and practice become more necessary for higher-level functioning. The arithmetic study revealed a smaller gap for the level of Abstractions 1 than for the level of Abstractions 2 which is consistent with the tenet that the higher the level of reasoning the greater the gap.

Research results

The research of Fischer and his colleagues has demonstrated confirmation of proposed hierarchies of skill acquisition and development numerous times. In some of Fischer's studies significant Green Indices of Consistency are reported while in other studies Fischer gives the percent of subjects demonstrating the predicted profiles. In Pipp, Fischer and Jennings' (1987) study, tasks for assessing infant's knowledge of the self yielded a Green's Index of Consistency of .78 while tasks assessing the infant's knowledge of the mother yielded a Green's Index of Consistency of .70. The authors state that these indices reveal significantly scalable sequences for the predicted order of steps since indices above .50 are considered significant by Green's method of analysis.

Fischer, Hand, Watson, Van Parys and Tucker (1984) examined the development of social categories and used skill theory to predict a developmental sequence of role understanding from infancy to adolescence. Support of the predicted sequence required performance that fit a Guttman scale. Of more than 200 subjects (Ss) tested in their studies, over 90% of them fit the predicted sequence perfectly. The writers claim that this percentage is virtually as high as could be obtained, given the degree of measurement error in the task.

Fischer and Elmendorf (1986), in researching developmental sequences for understanding "nice" and "mean" social interactions, simply reported that in the scalogram analysis, scalability was nearly perfect, with 70 of 72 middle-class children between 3 and 12 years of age exactly fitting the predicted performance profiles. They did not offer the value of the Index of Consistency.

In reporting their research on a skill approach to the development of reflective thinking, Kitchener and Fischer (in press) did not include statistics but enumerated three findings, citing that the results were too numerous to be reported in detail in the article. In this study they found that 1) Ss performed better in the high support conditions then in low, 2) an age related ceiling was evident and 3) in the high support condition there was evidence of an age-related developmental spurt with levels.

In one of two experiments investigating the development of classification skills Roberts and Fischer (1978) report that 70 out of 70 preschool children fit the predicted developmental sequence. Thus, the authors note that by scalogram analysis the items were perfectly scalable: Green's Index of Consistency was 1.0. In the second experiment 36 of the 40 Ss tested fit the predicted profiles. Green's Index of Consistency was .86.

Chapter 4

Skill theory research method

As a research method and research tool skill theory has two distinct purposes: to design research, and to make sense of research findings ex post facto or to broadly characterize developmental stages.

Skill theory provides the researcher with: 1) a structured methodology for investigating how specific behaviors develop or how tasks are performed; 2) a method for analyzing a task or skill into its constituents; 3) a method for investigating relations between skills or tasks; 4) a developmental perspective on general task or skilled behavior; and 5) an interpretive framework for understanding individual case studies of development (Rose & Fischer, 1989.) These skill theory research methods require: a high degree of control over testing conditions; a high degree of structure in assessment methods; and an assessment of separate skills to help the developmentalist know just what is going on (as compared with tasks assessing many skills at once).

The skill theory research methods generate five types of data: 1) evidence for relations between constituents underlying a skilled behavior at a specific developmental levels; 2) theory-relevant sequences of skill development of behavior through hierarchically related levels; 3) mean ages of passing each developmental level of skilled behavior; 4) effects of gender, task order, age and testing condition on performance (i.e. differences in means assessed with ANOVAs); and 5) correlation levels of skilled behavior with other tests or tasks.

In the section that follows, an alternative theory of the development of geometric skills is reviewed. In particular, van Hiele's geometric, hierarchical theory was selected as it is most similar to Fischer's theory of skill development.

Chapter 5

Van Hiele's theory

Background

The Dutch educators, P. M. van Hiele and Dina van Hiele-Geldof, developed a model of geometry learning in response to difficulties their students encountered with secondary school geometry. While teaching in Montessori secondary schools, they observed that students lacked sufficient experiences in thinking at appropriate "levels" to handle secondary school geometry. The van Hieles investigated levels of geometric thinking and the role of instruction in helping students achieve necessary levels of geometric thought in order to be successful with secondary school geometry. In particular, P. M. van Hiele formulated a model of the structure of geometric thought levels and specified principles designed to help students gain insight into geometry. Dina van Hiele-Geldof formulated an instructional approach intended to guide and assist students in moving from one level of geometric thought to another.

Together the van Hieles' work has stimulated extensive research in Russia and in the West. The

Soviets, it is reported, have substantially revised their schools' geometry curriculum on the basis of the van Hiele's work[1]. In the United States, researchers have examined van Hiele's levels of thinking in relation to 1) elementary and secondary school geometry learning and instruction (see, for example, Fuys, Geddes, & Tischler, 1988); 2) teacher education (see, for example, Fuys, et al. 1988); and 3) psychological theories such as Piaget's (see, for example, Orton, 1987). Of particular importance to the present work, Fuys, et al. (1988) hold that the van Hiele model could be appropriate for investigations in developmental and cognitive psychology research settings. As an example, they note Denis's (1987) study that investigated the extent to which there is a relationship between Piagetian stages and van Hiele levels. The present study could be viewed as an investigation of the extent to which there is a relationship between skill theory levels and van Hiele levels. Fuys et al. also suggest research on the van Hiele model from a Vygotskian perspective because of the correspondence between Vygotsky's "zone of proximal development" and their use of the term "potential level of thinking" in the van Hiele realm. The view of this study recognizes a correspondence between the concept of the "developmental range" in skill theory and Vygotsky's "zone of proximal development." As a direction for future research, Fuys et al. suggest investigating metacognition and levels of thinking, examined systematically within a theoretical framework. This research examines van Hiele levels of geometric thinking systematically within Fischer's skill theory.

Levels

Fuys, Geddes, and Tischler (1988) succinctly present the basic tenets of van Hiele's theory in their research. They present van Hiele's view that the child, assisted by appropriate instructional experiences, passes through five sequential levels of thinking. Achievement of each level is contingent upon having successfully passed through earlier levels.

Fuys, et al. (1988) operationalized the five levels of thinking (Level 0 to Level 4) as follows. At Level 0, the child identifies, names compares and operates on geometric figures such as triangles, angles and parallel lines, according to their appearance. At Level 1, the child analyzes figures in terms of their components and relationships among components and discovers properties and rules of classes of shapes empirically by, for example, measuring. At Level 2, the child logically interrelates previously discovered properties and rules by giving or following informal arguments. At Level 3, theorems are deductively proven and interrelationships among networks of theorems are established. At Level 4, theorems in different postulational systems are established and these systems are analyzed and compared.

In reference to the general nature of these five levels of thinking, van Hiele holds that what is intrinsic at one level is extrinsic at the next level (van Hiele, 1959; 1984). For example, at Level 0 where the child, in fact, determines figures by their properties, he or she is not aware of these as properties per se. Examples of structure of thinking and sample responses of those structures are included in Table 4.

Development: Van Hiele's theory in relation to skill theory

In skill theory, jumps in the learning curve are interpreted as evidence of the emergence for a new

Table 4

Van Hiele Levels, Objects of Thought and Examples

Level	Objects of thought	Structure of thinking	Response-explanation examples
0	Geometric Figures. Sensory experience with, for example, individual figures.	Names geometric figures. Sorting (based on perception) of figures.	Squares go together because "they look the same".
1	Objects - such as classes of figures - that are products of level 0 activities.	Discovers or recognizes properties as characteristics of classes.	A square has 4 sides, 4 right angles, sides of equal length, 2 sets of parallel lines.
2	Properties (of geometric figures) or necessary characteristics of shapes become the objects of thought.	Formulates logical relationship between properties of geometric figures. Explains these relationship informally.	If a quadrilateral has opposite sides parallel, then the opposite sides must be congruent.
3	Ordering relations are the objects of thought on which the individual operates.		None available
4	The foundations of the ordering relations of level 3 are the objects of thought.		None available

developmental level. For van Hiele, jumps in the learning curve, or discontinuous development, also reveal the presence of new levels. However, when jumps in the learning curve are not being observed, van Hiele would hold that no learning or development is occurring. Unlike skill theory, van Hiele does not view learning or development as having any gradual or continuous nature, as well as a stage-like nature. Van Hiele seems to view learning and development as an "all-or-none" phenomena.

A geometric hierarchy

The research and instructional model of geometric thinking put forth by P. M. van Hiele and Dina Geldof-van Hiele involves five levels of thinking in geometry. According to the van Hieles, with instruction the individual passes through these five hierarchical levels, only achieving consecutive levels of thought by having passed through any and all previous levels of thought. The five levels and their characteristics are presented in Table 4. Similar to Fischer's characterization of skill structures, van Hiele (1959, p. 6) holds that levels are "characterized by differences in objects of thought" and each object of thought is a product of cognitive activities at the previous level. See Table 5 for examples of "objects of thought" at each of van Hiele's five levels.

Table 5

Objects of Thought

Level	Objects of Thought
0	Geometric Figures.
1	The individual operates on classes of figures (from Level 0) and discovers properties for these classes.
2	The properties (discovered at Level 1) are the objects that the individual acts upon leading to logical orderings of these properties.
3	The orderings relations of Level 2 become the objects that the individual acts upon.
4	The foundations of Level 3 ordering relations are the objects of thought.

Chapter 6

The modular approach in Fischer's skill theory and van Hiele's theory

In skill theory, a single task is used to assess each developmental step. A series of tasks is termed a developmental sequence. Rose and Fischer (1989) suggest that each task, or each developmental step should contain eight modules (specified below) in order to make the task clear for the child and to help the researcher clarify what each task is asking for. For Rose and Fischer, a well defined task is likely to be one that contains all of the following modules: 1) definition of the task; 2) skill diagram; 3) integrated summary; 4) experimenter's introduction; 5) task: experimenter's modeling; 6) eliciting the child's imitation; 7) follow-up questions; and 8) criteria for passing the task. The authors present a social skills (i.e. kindness) story task as an example of these module parts and caution the reader not to preclude applying the modules to, for example, sorting and classification type tasks (as used in this research). The principles underlying the modules are applicable to both story and non-story tasks.

Based on the above, this research contains modules for 10 tasks, one for each of the ten steps in each of the developmental levels 9, 10 and 11 (Fischer's 1987 model). This series of ten tasks is the hypothesized developmental sequence for predicting and explaining the nature of geometric thinking, in skill theory terms for the late preschool through early high school-years. According to skill theory, the number of steps in the developmental levels is not constant. But for each step a separate task is used to assess it. In this research, the use of ten different tasks to assess the ten steps in each of the three levels is consistent with the research methods put forth by skill theory.

The hypothesized developmental sequence in this research is based on the developmental sequence used by Fuys, et al. (1988) in their research on van Hiele levels of geometric thinking. Fuys et al. also use a modular approach, designed to reflect van Hiele levels and phases. (Phases are discussed later in this chapter.) They present a table of modules completed or passed by their subjects that appears to correspond to an analysis similar to the scalogram analyses used in skill theory (see Table 6).

Table 6

Achievement Test Scores and Modules Completed by Subjects.

Grade Equivalency Scores			Modules		
Student	Reading	Mathematics	1	2	3
Andy	12.9	HS	X	X	X
Norma	12.9	HS	X	X	/
John	12.9	HS	X	X	/
Jeffrey	12.0	HS	X	X	X
Juan	12.0	HS	X	X	/
Luce	9.0	HS	X	X	/
David	8.8	HS	X	X	X
Murielle	7.0	HS	X	X	/
Gene	6.8	7.4	X	/	/
Frieda	6.8	7.1	X		
Arthur	6.8	6.8	X		
Bruce	6.7	8.1	X		/
Romana	5.9	7.1	X	/	/
Sherry	5.5	5.4	X	/	
Adam	5.4	4.4	X	/	
Deanna	4.5	5.3	X		
Cara	4.5	5.2	X		
Kathy	4.5	5.0	X	/	

Note: X indicates that the entire module was completed.
/ indicates that the module was partially done.
HS indicates that the student scored 9.0 or above.

Scalogram Profile Used in Skill Theory.

	Tasks					
Scale step	A	B	C	D	E	F
0	+	-	-	-	-	-
1	+	+	-	-	-	-
2	+	+	+	-	-	-
3	+	+	+	+	-	-
4	+	+	+	+	+	-
5	+	+	+	+	+	+

Note: Correct performance of a task is indicated by a "+".

Next, each of the eight parts of the modules are described with examples given for the first task (i.e. the first developmental step) in Level 9, of the developmental sequence hypothesized here. Following, in accordance with the Skill Manual (1989) and Fischer's documentation of his theory (1980a), modules for each of the remaining tasks in the hypothesized developmental sequence, were presented.

Maturation

Van Hiele speaks of a "maturing" factor or a genetic base for development through the levels of thinking, similar in some ways, to the concept of tiers in skill theory. However, in skill theory, the organism and the environment both contribute to development in a transactional sense while for van Hiele, progression from one level to the next is more dependent on the appropriate instructional experience than on the biological maturation. Also, while skill theory relies on the importance of environmental support such as "contextual support" and "elicited imitation" for inducing functional and optimal level behavior, van Hiele relies upon the importance of appropriate instructional experiences (such as discovering properties of geometric figures using concrete manipulatives) in achieving successive levels of thinking. Fuys, et al. note interesting findings concerning the interviewer's role in fostering a student's thinking in their van Hiele study and suggest that teacher's cognitive and metacognitive teaching moves be examined in relation to the students' level of thinking. An example of the teacher's cognitive teaching move could be the reviewing of prerequisite materials, including concepts and terminology, that would be used in later teaching. And, an example of the teacher's metacognitive teaching move could be the phrasing of questions that focuses the student's attention on the kind of thinking that's expected in a task. When looking at a particular figure that question could be "Are you thinking about what other shape this could be?". Hopefully, this type of question guides the student to think about their thinking.

Van Hiele is concerned with the level of material that is used in instruction. He contends that instructional that is above the student's level of thought is subject to reduction of level by the student. That is, the individual cognitively reorganizes the world (the instructional material in this case) to fit into his or her already existing cognitive structures.

Language structure

Van Hiele's theory asserts that language structure is a critical factor in determining movement through the van Hiele levels - from the global or concrete structures of Level 0, to the visual, geometric structures of Levels 1 and 2, to the abstract structures of Levels 3 and 4. Van Hiele holds that each level has its own language and system of language relations. As a result, individuals reasoning at different levels would not be capable of understanding each other. Further, what is linguistically "correct" at one level may be "incorrect" at another level. For example, a Level 0 child may not be capable of saying that a square is a ("special type of") rectangle because of its appearance - it doesn't look like a rectangle. But, at a later level, a child may say just the opposite - a square is a "special type of rectangle" because of its appearance, that is, its properties.

Phases of transition

Where skill theory offers transformation rules to predict and explain development, van Hiele offers five phases to predict and explain progress from one level to the next: information, guided orientation, explicitation, free orientation, and integration. Fuys, et al. (1988) offer the following examples of these five phases with examples for the transition from Level 0 to Level 1. First, for information, the individual gets acquainted with the working domain and examines examples and non-examples, such as quadrilaterals and non-quadrilaterals. Second, for guided orientation, the individual does tasks involving different relations of the network that is to be formed, for example, folding, measuring and looking for symmetry. Third, for explicitation, the individual becomes conscious of the relations, expresses them in words, and learns the technical language which accompanies the subject matter, for example, expresses ideas about properties of figures. Free orientation, the fourth phase, is characterized by the individual learning by doing more complex tasks in his own way in the network of relations. For example, the individual, using properties he or she knows belong to a particular shape, applies them to a new shape, such as "kites". Finally, with the fifth phase, termed integration, all that one has learned about a subject (e.g. properties of a figure) is summarized, and then the individual reflects on his or her actions and obtains an overview of the newly formed network of relations now available.

While van Hiele's theory offers five phases to predict and explain progress from level to level, it does not specify principles or mechanisms that underlie transition. That is, van hiele does not give a psychologically principled account of development as skill theory does with transformations. Skill theory incorporates mechanisms that produce change in addition to the transformation rules: task, optimal level, functional level and degree of environmental support and emotions. In skill theory, while the levels and transformation rules are used to predict developmental sequences in detail, variations are explained in addition as a function of at least four other mechanisms or sources (Fischer & Lamborn, 1989). These four mechanisms will be elaborated on here.

First, as an organismically oriented view, skill theory views the task at hand as actively helping to organize the individual's behavior; an implication of the collaboration of person and environment. For example, skill theory views a change in task as a change in skill; the reverse is also considered true.

Second, as noted earlier, optimal level limits the individual's performance even under ideal testing conditions. With instruction, for example, the individual will not learn the task if it is beyond his or her optimal level.

Third, the degree of environmental support, as a source of variation, is typically considered high or low. High-support conditions include the presence of aids or cues that minimize memory demands, the use of familiar materials, the clear definition of task demands, and the opportunity for practice (Lamborn & Fischer, 1989). High-support does not include direct intervention or performance of part of the task by an adult or advance peer, but indirect aid. When the individuals are not in high-support conditions, their behavior is below their optimal level. The highest step that the individual demonstrates consistently under low-support conditions for a task domain is his or her functional level.

And fourth, emotions, as an important part of behavior, "deserve to be included in any theory of mechanisms of development" (Fischer and Lamborn, p. 26). Further, Fischer and Elmendor (1986) claim that emotions have a "direct effect on the organization of behavior, an effect analogous in some ways to that of tasks." Organismic states, in their opinion, bias the individual toward one kind of behavioral organization or another. When the individual experiences an emotion such as anger or joy, the experience alters the organismic context. The experience directly affects the organization of the individual's behavior in terms of a script for anger or joy. And, the developmental step is produced by the interaction of the script for the emotion with other factors such as task, optimal level, and environmental support. An example in the literature involves the adolescent who becomes angry with her mother having told her a "social lie" to spare the teen's feelings. The adolescent's script for anger was predominant in organizing her behavior. This organization made it extremely difficult for her to perform at her optimal level for the concepts of honesty and kindness and understand her mother's social-lie motivation. In order for the adolescent to move up to her optimal level understanding for these concepts, she needed environmental support for integrating the two concepts. She also needed environmental support to shift away from her anger mode to an emotional state that would facilitate the components necessary to understand the mother's social lie.

Math anxiety would be one example of this idea that is relevant to this research. Clearly, skill theory could explain the cognitive block attributed to math anxiety frequently observed among certain groups. In addition to the affects of emotions on the organization of behavior, there are also developmental changes in emotions. Each developmental level brings changes in the understanding of emotions and the conditions that elicit them just as there are significant cognitive and social changes.

Research and additional levels

Two final issues of concern in relating van Hiele's theory to skill theory concerns the fact that most research on skill theory is done at the earlier levels of development (levels 1 - 6). Substantially less research is available on skill levels 7 through 13. The majority of the research done with van Hiele levels is done at Levels 0, 1 and 2. Little research is available on Levels 3 and 4. Also, Fischer (1987) has postulated the existence of levels beyond level 13 in his theory; however, at present, there is no empirical evidence available. Diana van Hiele-Geldof (1984) also has explicitly suggested that levels beyond van Hiele's Level 4 exist, however research is necessary in order to determine this.

The van Hiele hierarchical levels in relation to skill theory

To interpret the van Hiele hierarchy in terms of skill theory this study focused on the representational and abstract tiers. From Appendix B one can see Level 0 descriptors and sample student responses are based on the concrete appearance of figures characteristic of the representational tier. At Level 0, all cognitive activities are based on the figures' appearance as a whole. No activities within this level include analyzing components or properties of the figures. All Level 0 descriptors are consistent with skill theory's representational structures.

Level 1 descriptors indicate thinking and activity at levels characteristic of the abstract tier. Only beginning with Level 1 responses does the subject demonstrate thinking that is characteristic of the abstract tier.

Examples from the literature

For example, at van Hiele Level 1 the student "identifies and tests relationships among components of figures ...". It is in an abstract set of skill theory that the individual "abstracts an intangible attribute that characterizes broad categories of objects, events, or people" (Fischer, 1980a, p. 494). Specifically in skill theory, level 10, the individual can control the relation between two representational systems. Consider a 15-year-old boy who can control a system of systems for the sets of two sets of geometric figures. He can integrate several of the systems from the previous Level 9 into a single Level 10 system that controls the relations among, for example, the sides of a figure - the vertical sides, the horizontal sides (two sets of parallel sides) and the resulting angles. When he is thinking about how the sides compose a certain figure he can simultaneously consider how the angles affect the figure. It is these types of descriptors from van Hiele's Level 1 that can allow one to justify the application of the beginning of the abstract tier of skill theory.

The focus on the representational and abstract tiers gives an indication of the age level appropriate for investigation. Table 7 indicates the estimated age periods for middle class Americans at which skill theory levels first develop. The level of representational systems are shown to first develop in the grade school years, while the level of systems of representational systems (or the single abstract sets) are indicated to develop in the junior high school years. Based on these ages, the sixth grade students interviewed are appropriate age-wise.

This research uses skill theory to analyze parts of van Hiele modules 1 and 2. Parts of the two modules are being examined because of the different skill domains they represent: the first module is visual and

Table 7

Age Periods at Which Levels First Develop

	Level	Age	Periods
1	Single reflex sets.	3-4	weeks
2	Reflex mappings.	7-8	weeks
3	Reflex systems.	10-11	weeks
4	Systems of reflex systems or single sensorimotor sets.	4	months
5	Sensorimotor mappings.	7-8	months
6	Sensorimotor systems.	11-13	months
7	Systems of sensorimotor systems or single representational sets.	18-24	months
8	Representational mappings.	4-5	years
9	Representational systems.	6-7	years
10	Systems of representational systems or single abstract sets.	10-12	years
11	Abstract mappings.	14-16	years
12	Abstract systems.	18-20	years
13	Systems of abstract systems.	24-26	years

descriptive. The second module involves thinking at the first two van Hiele levels - level 0 (visual) and level 1 (analytic). Module 2, although involving some work of a visual nature as in levels 0 and 1, mainly deals with thinking at levels 2 (analytic) and 3 (informal deduction).

This research employs the van Hiele level descriptors (see Appendix B) to score all the subjects from videotaped records.

Skill theory to design research and provide a model of general behavioral development

The two uses of skill theory Rose and Fischer (1989) present are 1) "to design research" and 2) "to make sense of research findings ex post facto (only if the research was designed in a way compatible with the skill orientation to begin with), or to broadly characterize developmental stages" (p. 27). This research employs both uses of skill theory.

Under the first use of skill theory, three issues are generated:

1) predictions about development are tested through empirical research;

2) predictions arise from the developmental analysis of the behaviors of interest; and
3) specification of developmental steps are used as a metric to answer questions about how specific behaviors relate to other behaviors.

In the present study, the items or tasks involve how sixth grade children begin to use properties of geometric figures to identify and sort shapes, characterize groups of shapes, and identify and explain subclass relations. Some items look at how the sixth graders begin to use partial and appropriate information about a geometric figure to give arguments about what the figure might be. Other items look at how the sixth grade subjects deduce one property from another. Later items involve the sixth grade subjects in 1) analyzing a new set of figures (kites) in terms of properties and 2) recognizing inclusion relationships involving kites. Finally, items involving angle measurement - predictions of and actual measurements - are included.

Chapter 7

Method

Problem and Hypotheses

This research attempted to validate, confirm and lend support to Fischer's theory of development by testing the theory with the set of data collected by the Brooklyn College Project (BCP), which is the only hierarchical set of data on optimal levels of geometric thinking. This research proposes that the data will fit Fischer's theory of skill development.

The following hypotheses were tested:

1) The scoring of Ss' behaviors from the videotapes will reveal a developmental sequence as described in skill theory terms. Or, the hypothesis can be stated as: The scoring of Ss' behaviors from videotapes will give support to Fischer's skill theory model of development.

2) The sixth grade Ss will demonstrate the skill theory level of Representations 4/Abstractions 1; Level 10.

Procedures

Subjects

The performance of 18 sixth grade elementary school students was scored from videotapes in the present study. All subjects were videotaped working on all module tasks that they participated in. The Ss were selected to reflect the diverse student population of the New York City Public schools in terms of ethnicity and academic achievement level. The 18 sixth graders - 9 boys and 9 girls - were recruited from two large, inner city, K - 6 public schools in Brooklyn. Both schools serve predominantly minority populations: 17 Ss came from one school predominantly serving a variety of ethnic backgrounds and one S came from the other school serving a mainly Hispanic background. Of the 18 sixth graders, ten were African-American (5 boys and 5 girls), 5 were Caucasian (3 boys and 2 girls), and 3 were Hispanic (1 boy and 2 girls). The ages of the sixth graders ranged from 10 years 11 months to 12 years 4 months. The average age, at the time of testing, was 11 years 7 months.

As determined by scores on mathematics and reading subtests of the Metropolitan Achievement Test (Intermediate, Form L; see Table 6), sixth grade Ss with a score of 6.8 or so were considered on grade level at the time of testing. Ss one or more years above grade level were classified as high achievers and those one or more years below grade level were classified as low achievers. Two girls and five boys were classified above grade level. Three girls and three boys were classified on grade level, and four girls and one boy were classified below grade level.

Materials

Each subject was interviewed individually in six to eight 45 minute sessions. The sessions were videotaped yielding from 4 1/2 to 6 hours of videotape for each subject.

The Ss were walked from their nearby schools to the Brooklyn College campus by a Project staff member and interviewed in the mathematics education seminar room in the School of Education. Ss sat at a table across from the interviewer. The videotaping of Ss performance was done with a videocamera visible to the S and situated behind the interviewer. The interviewer was this writer or one of two professors of mathematics education at the college and a staff member of the van Hiele project. The videotapes show the Ss participating in all tasks to be scored.

In considering skill theory research methods Rose and Fischer (1989, p. 28) suggest that, "...whenever possible, design a study to use an already existing sequence of tasks... (or) consider modifying one." Based on this suggestion, this study utilizes the van Hiele model (1957) of levels of learning and geometric thinking as interpreted and applied in the research conducted by the BCP Staff. This model is hierarchical and, as applied by the BCP, involves assessment of optimal performance. (The BCP terms this "potential level" of the student. They define this as the students performance in an instructional context that has specially designed materials and an interviewer trained to encourage higher levels of thinking.)

Given the validity of Fischer's skill theory, the van Hiele data was expected to conform to the skill theory level of the system of representational systems, which are single abstract sets. The level is referred to as level 10 in the 1987 model of skill theory. To get a fuller picture of development, the adjacent levels, Representations 3 (Level 9) and Abstractions 2 (Level 11) were also examined.

The Modules and Tasks

Development

In order to characterize van Hiele's levels, the BCP reviewed all the available work done by the van Hieles (Fuys, et al., 1988); this included translating Dina van Hiele-Geldof's doctoral dissertation (van Hiele-Geldof, 1984) from Dutch to English. Since the van Hieles emphasize the role of instruction in assisting students with moving from one level of thought to the next, the BCP developed three instructional modules - each based on van Hiele's model - to be used as a research tool in a clinical one-to-one interview setting to assess children's levels of thinking in geometry.

The hierarchical nature of each of the three modules is diagrammed in Items 1, 2 and 3 in Appendix A, respectively.

Module 1 concerns classification of two-dimensional shapes. Basic geometric concepts (e.g. parallelism, angle and congruence) and properties of quadrilaterals are treated in Module 1.

Module 2 focuses on angle measurement; angle sums for triangles, quadrilaterals and pentagons; and angle relationships in triangles and parallelograms (i.e., exterior angles, opposite angles).

Module 3 utilizes the topics of area measurement; area of rectangles, triangles, parallelograms, trapezoids and figures whose vertices lie on two parallel lines.

Given the skill theory levels investigated in this study, tasks from Modules 1 and 2 were used. (See Appendix A.)

The investigative tool employed in this research is made up of 30 items; there are three subtests with ten items in each. The first three items revealed interesting characteristics of the subjects. In a game-like fashion, the geometric language, both standard and non-standard, was assessed. For each of these items, a pair of shapes was presented to the child. First, the interviewer said something that was the same about the pair and the child said something that was different about them. The roles were reversed with each pair. First, a pair of triangles, one large and one small, was presented. For item 2, the second pair of shapes was a square and a rectangle. Item 3 had a two-dimensional triangle and a three-dimensional triangle (or a pyramid. An example of non-standard language used by a child, in responding to item 1 was, "both have three points". A response, for the same item, using standard language that was given by a child was "they both have three angles".

In the next four items, the interviewer presents to the subject, sheets with geometric concepts on them. First the subject has an opportunity to identify the concepts and then communicate their familiarity with them. Respectively, the concepts included in these items were rectangles, parallel lines in figurations, right angles and right angles in figurations. Items 8, 9 and 10 involved the subject describing congruence, opposite sides or opposite angles and diagonals, respectively. The interviewer did not introduce any formal geometric vocabulary and subjects were given as much opportunity as possible to introduce their own geometric vocabulary spontaneously.

In item 11, the interviewer presented the subject with a set of card board, cut-out polygons and mats for sorting the shapes on. The interviewer said "These shapes came from several different boxes but they got all mixed up. This is how someone tried to put them back in groups which they belong." The interviewer then placed a couple of pieces on each mat, sorting by number of sides, and said "Can you guess where this will go? Why? And this? Why? Can you arrange the rest of the pieces using this idea?" A score of 1 was assigned when the child completed the sort by the number of sides. A score of 0 was assigned when the sort by number of sides was not followed. For example, where a child sorted the shapes into triangles, squares and all other shapes, a score of 0 was assigned.

A similar procedure was followed for item 12, however the child was shown a collection of quadrilaterals along with some mats. Again subjects were asked "How could we place these into groups of things that belong together?" Eventually, a sort by square, rectangle, parallelogram, trapezoid, quadrilateral was expected and if demonstrated, received a score of 1. Items 11 and 12, like the following items, assess the child's ability to think about shapes in terms of properties. Item 12 provides a richer context for sorting than does item 11. The items continued to build on each preceding one.

Items 13, 14 and 15 served to assess the child's ability to characterize the groups of shapes in terms of properties. The interviewer pointed to the group of squares and said "If you were talking with your friend over the phone and you wanted to describe these pieces, what could you say about them?" This was done for the shapes of square, rectangle, and parallelogram. If four defining properties were mentioned, a score of 1 was assigned. Subjects were encouraged to say as much as they could about the shapes.

Items 16 and 17 assessed the extent to which the subject could identify and explain subclass relations, that is, for example, all squares are rectangles and all rectangles are parallelograms. The interviewer said "When you sorted the first set of shapes, do you remember that you had a group of triangles, and one of quadrilaterals or four-sided figures, and five-sided ones, and six sided ones? Where would all the shapes on the table have gone?" When the subject had responded correctly, the interviewer picked up a square and said "So I could move this square to the quadrilateral or four-sided group - a square is a special kind of quadrilateral. What makes it special?" The interviewer then asked "Can we move this square to the rectangle group? Why? (or Why not?)"

For item 18, a score of 1 was assigned if the subject named two possible or two impossible shapes with the first uncovering. That is, the interviewer uncovered a cardboard cut-out in four stages, asking at each stage "What could this be? Could it be anything else? Why? What couldn't it be Why?" In the same way, a score of 1 was assigned for items 19 and 20, given the second and third uncoverings, respectively.

Items 21, 22 and 23 were assessments of the subject's ability to analyze a new set of figures (kites) in terms of properties, and to recognize inclusion relationships involving kites. (To include as many concepts as possible within the restraints of this study, the "Minimal Properties" category was omitted because the following tier of the hierarchy (i.e. "Kites: Sorting, ...") has "minimal properties" items included in it and follows naturally, both geometrically and psychologically, from the "Guessing Shapes" tier.) The subjects were shown a collection of cut-out shapes arranged on three cards: "These are kites, these are not kites. And which of these are kites?" The only square included is on the third card. Subjects are asked to place the shapes on the third card on the first two cards and to explain their thinking. For item 21, a score of 1 was assigned for appropriate placement. If necessary, placement was corrected so that all kites were on the first card and, if this was the case, a score of 0 was assigned. For item 22, subjects were then asked why a rectangular cut-out does not go in the kite pile, and why a square cut-out does. A response of the rectangle not having a line of symmetry received a score of 1. For item 23, the interviewer asked "How would you describe a kite?" A score of 1 was assigned if the subject listed at least three defining properties.

The next four items - 24, 25, 26 and 27 - formalized the inclusion relations. Subjects were reminded of earlier discussions of how a rectangle is a special kind of quadrilateral. "To show this we sometimes put an arrow like this between the quadrilateral and rectangle card. You could think of it as a one-way street sign." For item 24, subjects were asked if every rectangle is a quadrilateral, and for item 25, if every quadrilateral is a rectangle, and about which placement of the arrow is correct. For items 26 and 27, the subjects were then shown name cards for square, quadrilateral, and kite, and some more arrows. "Can you put arrows down between these cards to show some relationships?" A response of inclusion was score 1 in both items 26 and 27. These items, done without property cards being in sight, assessed whether it was natural for the subject to think of inclusion in terms of properties. Subjects that described or demonstrated this thinking, by placing arrows appropriately, as in the inclusion of every rectangle as a quadrilateral and the exclusion of certain quadrilaterals as rectangles, were assigned a score of 1 for items 24 and 25, respectively. For items 26 and 27, a score of 1 was assigned if the subject described or demonstrated the inclusion of another shape (for example, a parallelogram or trapezoid) as a quadrilateral and the exclusion of certain quadrilaterals as these shapes.

The final three items of the tool concerned angle measurement and angle relationships in polygons. These items assessed both understanding of and skill in angle measurement, recognition of how angle measures of adjacent angles can be added and understanding of the fact that the angle sum of a triangle is 180 degrees. If a subject experienced difficulty with angle measurement, a score of 0 was assigned. Subjects were shown a diagram of two adjacent angles. For item 28 they were asked to measure the two angles, and then to predict the outside angle (the sum). Subjects were then shown a diagram with three adjacent angles. For item 29 they were asked to predict angle measurements, after making some measurements. This led to discussion of the measure of a straight angle. Demonstration or description of the number of degrees in a straight angle was assigned a score of 1. For item 30, subjects were shown a triangle, and were asked to measure angles in it. Then a triangle was shown with only two measurements marked, and subjects were asked if they could predict the measure of the unmarked one. Those subjects that did this, received a score of 1.

In selecting various geometry topics to be dealt with the BCP took into consideration the need to assess levels of thinking across different topics and different van Hiele levels, and to guarantee the topics' relevancy to the educational experiences of the students. At the same time, the Project needed to minimize the effects of previous learning on the new learning taking place during the interviews. The primary purpose of the modules was to provide a context for clinical interviews in which students' level of thinking, cognitive processes and learning difficulties could be assessed.

As a result of analyzing van Hiele's works and discussions with Dr. van Hiele and other scholars involved in van Hiele research the BCP formed a detailed description of the van Hiele model. The BCP developed van Hiele level descriptors that were defined hierarchically, in behavioral terms, for research purposes. It was by applying these level descriptors that the subjects' videotaped behavior was assessed by the BCP for levels of thinking. Appendix B gives the complete descriptors of the levels accompanied with sample student responses.

The BCP developed and validated the three hierarchical modules in three phases over 14 consecutive months. The development of initial versions of the modules made up the first phase. Based on pilot-testing of the modules, they were revised. In particular, at first behavioral goals were clarified, the script was expanded, and manipulatives (for assessment and instruction) were added to the modules.

The second phase of the development of the modules primarily focused on assessment of level 3 thinking (informal deduction) through a fourth module. However, because of limited subject-interview time, pertinent tasks of this fourth module were added to the three previous modules and the fourth module was omitted. For example, Activity 4 (kites) was added to Module 1 as a culminating activity to assess van Hiele levels 0, 1 and especially 2 (through tasks involving subclass relationships.)

The third and final phase of the modules' development mainly involved validation of the modules. Specifically, questions for assessing levels of thinking were given to qualified outside readers. The outside readers' responses overall were positive but did include one suggestion - the interviewers were advised to be less directive.

As a result of this feedback, interviewers allowed the subjects more time to think and respond and conscientiously were less directive. At this time, the usefulness of the module-based clinical interviews for assessing levels of thinking was determined to be appropriate. Based on the mathematical knowledge of the mathematics educators involved in van Hiele research, phases of development and validation of the modules, feedback from the three outside readers familiar with van Hiele research and pilot-testing, the hierarchy appeared and was judged valid. Other researchers (e.g. Hoffer, 1983; Burger & Shaughnessy, 1986) involved in van Hiele research have also used much the same hierarchy and level descriptors in their work.

Scored Behavior

Each task was scored according to the following definitions. In accordance with developing a scalogram profile, the scorer assigned a "1" to the task behavior that demonstrated a specific level and a "0" to task behavior that did not demonstrate thinking at a specific level.

Criteria for Scoring Items

The subtest for the structure of representational systems (Fischer, Level 9) consisted of ten items. The items were administered in a consistent order and, as with each of the three subtests, each item consisted of skills necessary to successfully complete each succeeding item. The coders scored the items according to criteria like the following:

Item 1: If the subject (S) names or labels the triangles presented, using either standard or non-standard geometric terms appropriately for this, score 1.

Otherwise, score 0.

Scoring criteria for the remaining twenty nine tasks are in Appendix C.

Scoring and interrater reliability

To include at least 25% of the sample size in an interrater reliability check, five subjects were randomly selected. The videotapes of three boys and two girls were viewed by another dissertation-level graduate student, with a concentration in the measurement area, in the Department of Educational Psychology. The second scorer was trained with a set of tapes that were not part of this data set. Using the coding system, she assigned scores for each of these five Ss' responses to the given 30 items.

A level of .75 would be considered satisfactory agreement based on Cohen's Kappa (K) as described by Frick and Semmel (1978, p. 175). Cohen's Kappa is considered most appropriate for determining the interrater agreement since there are only two "categories" of responses (1, the behavior is observed or 0, the behavior is not observed) considered in this investigation, and the probability of chance agreement is relatively high. Frick and Semmel note that this agreement measure on videotaped tests representative of actual classrooms, as is the case here, should exceed .75 for acceptable observer consistency.

To calculate Cohen's Kappa a 2 x 2 contingency table was constructed to classify the nature of agreement or disagreement between two observers. A tally was made in the contingency table on the main diagonal.

$$N = \sum_{i=1}^{2} n_{i+} = \sum_{i=1}^{2} n_{+1}.$$

= total number of items.

nii = total number of agreements for the ith category (main diagonal.)

and

ni+ = marginal for the criterion coder on the ith category.

Nominal agreement for C categories, Po, is defined as

$$Po = \frac{1}{N} \sum_{o=1}^{C} nii$$

Where n = total number of items coded.

and $Pe = \sum_{i=1}^{C} Pi^2$ = chance agreement for C categories.

Pi is the proportion of tallies made by all observers for the ith category.

$$K = \frac{Po - Pe}{1 - Pe}$$

and $Pe = \frac{1}{N^2} \sum_{i=1}^{C} (ni+)(n+i)$.

The percent agreement between the two scorers was determined to be .82. See Table 8.

The writer scored all children from the videotapes. The subjects performance on each task was scored as pass (1) or fail (0). The criteria for scoring each task are outlined in this chapter. A scalogram profile was developed for each subject. (See Table 9.) In a personal communication[2], Dr. Fischer (6/29/90) said that if the subject does the step correctly he or she will be viewed as possessing the skill involved in that step. Further, Dr. Fischer recommended using an order analysis on the data - Green's (1956) Index of Consistency, which will test the hypotheses of the fit of van Hiele's and

Table 8

Interrater Reliability: Calculating Cohen's Kappa

Subject #2

Observer 1

Observer 2

	(1)	(0)
(1)	9	3
(0)	1	17

Subject #4

Observer 1

Observer 2

	(1)	(0)
(1)	15	2
(0)	3	10

Subject #9

Observer 1

Observer 2

	(1)	(0)
(1)	23	0
(0)	1	6

Subject #11

Observer 1

Observer 2

	(1)	(0)
(1)	26	1
(0)	3	0

Subject #14

Observer 1

Observer 2

	(1)	(0)
(1)	14	3
(0)	2	11

Subject Recap

Observer 1

Observer 2

	(1)	(0)	Ni±
(1)	87	9	96
(0)	10	44	54
	97	53	N=150

Table 8 (Continued)

$N = 150$

$$P_o = \frac{1}{N} \sum_{0=1}^{C} nii \quad =$$

$$= \frac{1}{150} \quad (87 + 144) \quad = \quad .87$$

$$P_e = \frac{1}{N^2} \sum_{i=1}^{C} (ni +)(n +1) =$$

$$= \quad (96)(97) \quad + \quad (54) \quad (53) \quad = \quad .27$$

$$K = \frac{P_o - P_e}{1 - P_e} \quad =$$

$$= \frac{.87 - .27}{1 - .27} \quad = \quad .82$$

$\therefore$ Since $K \geq .75$, interrater reliability is accepted.

Table 9

Scalogram Profile for Data

Apriori Tasks

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	Scale Score
1	+	-	-	-	-	-	-	-	+	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3
2	+	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	4
3	+	+	+	-	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	6
4	+	+	+	-	-	+	-	-	-	-	+	+	+	-	-	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	8
5	+	+	+	+	+	+	+	+	-	-	+	-	-	-	-	-	-	+	-	-	-	-	-	-	-	-	-	-	-	-	10
6	+	+	+	+	+	-	+	+	-	-	-	-	-	+	+	-	+	-	-	-	-	-	-	-	-	-	-	-	-	-	10
7	+	+	+	+	+	+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	+	-	-	-	-	-	-	-	-	-	14
8	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-	-	+	+	-	-	-	-	+	-	-	-	-	-	-	-	17
9	+	+	+	+	+	+	+	+	+	+	-	-	-	-	+	+	-	-	+	+	+	+	-	+	-	-	-	-	-	-	17
10	+	+	+	+	+	+	+	+	+	+	-	-	-	-	-	+	-	-	+	+	-	+	+	+	-	-	-	+	+	-	18
11	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-	+	+	-	-	-	-	-	-	+	-	-	-	-	-	18
12	+	+	+	+	+	+	+	+	+	+	+	+	+	-	-	+	+	+	+	+	+	+	-	+	+	-	+	-	-	+	23
13	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-	+	+	+	+	+	+	+	+	+	-	-	-	-	24
14	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-	-	+	+	+	+	-	+	-	+	+	-	+	+	+	25
15	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-	-	-	27
16	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-	+	+	+	+	+	+	+	+	+	+	+	+	29
17	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	30
18	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	30
1-n_g	18	17	17	15	15	15	15	14	13	12	11	10	10	10	10	10	9	9	9	9	8	8	8	7	7	6	5	5	5	5	
2-n_g	0	1	1	3	3	3	3	4	5	6	7	8	8	8	8	8	9	9	9	9	10	10	10	11	11	12	13	13	13	13	
3-n_g+1,$\bar{g}$	0	0	0	1	1	1	0	1	0	2	0	0	2	1	3	3	3	3	0	1	2	2	2	3	0	1	2	0	1	/	
4-n_g+2,$\bar{g}$	/	0	0	0	0	0	0	1	0	1	0	0	2	1	1	1	1	2	0	0	0	0	0	0	0	0	0	0	/	/	
5-n_g+2,$\underline{g}$	0	0	0	2	0	1	1	1	2	1	2	2	3	4	2	4	3	3	1	0	3	0	1	2	1	1	2	1	/	/	
6-n_g+1.$\overline{g-1}$	/	0	0	0	2	0	1	1	1	2	1	0	2	3	4	2	4	3	3	1	0	3	0	1	2	1	0	1	1	/	
7-$\underline{n_g}$+1	17	17	15	15	15	15	14	13	12	11	10	10	10	10	10	9	9	9	9	8	8	8	7	7	6	5	5	5	5	/	
8-n_g-$\bar{1}$	/	0	1	1	3	3	3	3	4	5	6	7	8	8	8	8	8	9	9	9	9	10	10	10	11	11	12	13	13	13	
9-n_g+2	17	15	15	15	15	14	13	12	11	10	10	10	10	10	9	9	9	9	8	8	8	7	7	6	5	5	5	5	/	/	

Fischer's categories. The formula for this index is described in detail in the next chapter and illustrated in Table 10.

Once the scoring of videotapes was completed, a scalogram was developed. Based on whether consistencies or inconsistencies are found, the findings and the van Hiele model were examined and interpreted in terms of skill theory. For example, should the findings indicate inconsistencies or conflicts, the extent to which they are due to the interviews or the model was addressed.

Table 10

Calculating Green's Index of Consistency

$$\text{Rep A} = 1 - \frac{1}{NK}\left(\sum_{g=1}^{K-1} ng + 1,\ \bar{g}\right) - \left(\frac{1}{NK}\sum_{g=2}^{K-2} ng+2,\ g+1,\ \bar{g},\ \overline{g-1}\right)$$

$$= 1 - \frac{1}{(18)\ (30)}\left(\text{Sum}(3)\right) - \frac{1}{(18)(30)}\left(\text{Sum}\ (4)\right)$$

$$= 1 - \frac{1}{540}\ (35) - \frac{1}{540}\ (10) = .92$$

$$\text{Rep B} = 1 - \frac{1}{NK}\left(\sum_{g=1}^{K=1} ng + 1,\bar{g} \right. - \frac{1}{N^2K}\ \mathop{}_{g=2}^{k=2}\ ng+2,\bar{g} \quad ng+1,\overline{g-1}$$

$$= 1 - \frac{1}{NK}\ \text{Sum}(3\ ) - \frac{1}{N^2K}\left(\text{Sum}\ (5) \bullet \text{Sum}\ (6)\right)$$

$$= 1 - \frac{1}{540}\ (35) - \frac{1}{18^2(30)}\left(\ 72\ \right)$$

$$= .935 - .0074$$

$$= .9276$$

$$\text{Rep I} = 1 - \frac{1}{N^2K}\sum_{g=1}^{K=1}(ng + 1)(n\ \bar{g}) - \frac{1}{N^4K}\sum_{g=2}^{K=2}(ng + 2)(ng + 1)(n\ \bar{g})(n\ \overline{g-1})$$

$$= 1 - \frac{1}{N^2K}\left(\text{Sum}\ (7) \bullet \text{Sum}\ (2)\right)_{g=1} - \frac{1}{N^4K}\left(\text{Sum}\ (9) \cdot \text{Sum}\ (7) \cdot \text{Sum}\ (2) \bullet \text{Sum}\ (8)\right)$$

$$= 1 - \frac{1}{18^2(30)}\ (1727) - \frac{1}{18^4(30)}\ (107{,}153)$$

$$= .7883$$

$$\text{I} = \frac{\text{Rep A} - \text{Rep I}}{1 - \text{Rep I}}$$

$$= \frac{.92 - .7883}{1 - .7883}$$

$$= .6221$$

$$\text{I} = \frac{\text{Rep B} - \text{Rep I}}{1 - \text{Rep 1}}$$

$$= \frac{.93 - .7883}{1 - .7883}$$

$$= .66$$

∴ Since I $\geq .50$, it is considered "scalable" or significant

Chapter 8

Results

The number of modules and tasks completed varied from subject to subject. More advanced Ss tended to complete more tasks in the allotted time. Less advanced Ss tended to spend more time on the early module tasks.

Using Green's (1956) method of scalogram analysis, a close approximation to the scale reproducibility (Rep) was computed. The concept of reproducibility refers to the reproduction of an individual's item responses from knowledge of his or her scale score. It is a measure of the success of this reproduction when the scores are so assigned that the number of errors is minimized. Dichotomous items are required. In this study the items consisted of the presence (1) or absence (0) of criterion behaviors. Using various indices of inconsistency (Numbers 1 - 9 as shown on the bottom of Table 9) two separate reproducibility estimates were computed. (See Table 8.) The formulas for these estimates of reproducibility are:

Rep A = $1 - \frac{1}{Nk}\sum_{g=1}^{k-1}(n_{g+1,\bar{g}}) - \left(\frac{1}{Nk}\sum_{g=2}^{k-2} n_{g+2,\, g+1,\, \bar{g},\, \overline{g-1}}\right).$

and

Rep B = $1-\frac{1}{Nk}\sum_{g=1}^{k-1}(n_{g+1},\bar{g})-\frac{1}{N^2k}\sum_{g=2}^{k-2}n(g+2,\bar{g})$
$(n_{g+1},\overline{g+1})$.

where N is the number of respondents and k is the number of items.

To simplify calculations, each index of inconsistency, derived from the scalogram profiles, was assigned a single number as recommended by Green (1956). That is, given k = number of items, N = number of respondents, i = subscript referring to item i and g = subscript referring to item g in rank order the following numbers were assigned:

For g = 1, 2....k-1,

(1) = ng = number of items correct in the given column.

(2) = $n\bar{g}$ = number of items incorrect in given column.

(3) = ng+1, $\bar{g}$ = number of respondents who gave the positive response to items g+1 and the negative response to item g.

(4) = ng+2, $\bar{g}$ = number of respondents who gave the positive response to item g+2 and the negative response to item g.

(5) $= ng+2$, $\overline{g}$ = number of respondents who gave the positive responses to item g+2 and the negative response to item g.

(6) $= ng+1$, $\overline{g-1}$ = number of respondents who gave the positive response to item g+1 and the negative response to item g-1.

(7) $= ng+1$ = number of respondents who gave the positive response to item g+1.

(8) $= n\overline{g-1}$ = number of respondents who gave the negative response to item g-1.

(9) $= ng+2$ number of respondents who gave the positive response to item g+2.

From this

$$\text{Rep A} = 1 - \frac{1}{NK}(\text{sum}(3)) - \frac{1}{NK}(\text{sum}(4))$$

and

$$\text{Rep B} = 1 - \frac{1}{Nk}(\text{sum}(3)) - \frac{1}{N^2k}(\text{sum}(5) \times \text{sum}(6)).$$

The two estimates of the reproducibility (Rep A and Rep B) yielded very similar estimates; Rep A yielded .92 and Rep B yielded .93. See Table 10. An additional reproducibility value, Rep I is necessary for ultimately obtaining Index of Consistency values and therefore was calculated and is included in Table 10.

Employing Green's method of scalogram analysis involved developing the respondents profiles in a Guttman-type scale and calculating Indices of Consistency. Green's Index of Consistency (I) is unity if the items are perfectly scalable and has an expected value of zero when the items are independent. Having obtained the reproducibility estimates, the value of the Index of Consistency was computed. To obtain the Indices of Consistency the following calculations were performed:

$$\text{Rep I} = 1 - \frac{1}{N^2k}\sum_{g=1}^{k=1} n_{g+1}n_{\bar{g}} - \frac{1}{N^4k}\sum_{g=2}^{k=2} n_{g+2}n_{\bar{g}+1}n_{g}n_{\bar{g}-1}$$

$$I = \frac{\text{Rep A} - \text{Rep I}}{1 - \text{Rep I}}$$ and

$$I = \frac{\text{Rep B} - \text{Rep I}}{1 - \text{Rep I}}$$

Statistical analyses produced Indices of Consistency of .62 and .66 for the reproducibility values A and B respectively. See Table 10. The items are considered "scalable" since the Indices of Consistency are greater than .50 (Green, 1956). Based on this statistical analysis the sixth grade Ss are considered to have demonstrated the skill theory level of Representations

4/Abstractions 1, Level 10, and thus lend support to Fischer's model of development.

The first ten items (the ten steps of Level 9, Representations 3) were completed without error by nearly two third's (11) of the Ss. Four additional Ss (Nos. 7, 5, 6 and 3) had one, two, three and four errors, respectively. Given the age group involved, it was expected that the Ss would demonstrate Level 9, Representations 3.

Three Ss (Nos. 15, 17 and 18) performed the second group of tasks (items 11 - 20) without error. See Table 9. An additional one third of the sample (Ss Nos.8,s 11, 12, 13, 14 and 16) demonstrated substantial thinking at the skill theory Level 10. Another one third of the sample (Ss Nos. 4, 5, 6, 7, 9 and 10) demonstrated some Level 10 thinking while three Ss (Nos. 1, 2 and 3) did not show any thinking at this level. The third group of tasks (items 21 - 30), used to assess Level 11 thinking, was completed without error by three Ss (Nos. 16, 17, and 18). Three additional Ss (Nos. 12, 14 and 15) had at least six correct response. Six Ss (Nos. 7, 8, 9, 10, 11 and 13) showed some thinking at this level (i.e. had at most five correct items) while one third of the Ss (i.e. six subjects; Nos. 1, 2, 3, 4, 5, and 6) did not demonstrate any Level 11 thinking. These figures are summarized in Table 11. Scale scores presented in the scalogram profile (See Table 9) yielded a mean value of 18.5, a median value of 17.5 and two mode values of 17 and 18. Each of these values is indicative of Level 10 thinking; responses characteristic of Representations 4/Abstractions 1.

Table 11

Percentage of Respondents Demonstrating Levels

Level	Percentage of subjects who performed tasks:				Percentage of subjects who showed:	
	Without Error	With 6 or more correct responses	With 5 or less correct responses	With no correct responses	Some thinking at this level	Substantial thinking at this level
9 (Representations 3)	61	22	17	0	100	83
10 (Representations 4 / Abstractions 1)	22	28	33	17	83	50
11 (Abstractions 2)	17	17	33	33	66	33

Chapter 9

Discussion

The results of this study support the hypothesis that the scoring of Ss' behaviors from the videotapes reveal a developmental sequence as described by skill theory. That is, the scoring of Ss' behaviors from the videotapes gives support to Fischer's skill theory model of development. The results of this study confirm the hypothesis that sixth grade Ss demonstrate the skill theory level of Representations 4/Abstractions 1; Level 10. The present study has extended prior work on the validity of Fischer's developmental model by applying it to the content area of geometry, an area previously untested.

Although the component of practice that Fischer recommends was not included in the study, the results seem to suggest that optimal performance was demonstrated and that the performance was characteristic of predicted skill theory levels. Given the age level of the subjects, the levels of thinking observed are consistent with the skill theory model of development. Nearly all Ss demonstrated strong Level 9 thinking, a majority of Ss demonstrated level 10 thinking and few Ss demonstrated

Level 11 thinking; an advanced level for the age group observed. (See Table 11.)

As a research tool and method skill theory provided this study with the following. First, it provided a structured methodology for investigating pathways specific geometric behaviors take in development. The skill theory method requires a hierarchy and Fischer suggests using an already existing one. The validity of a hierarchy is a crucial issue. The geometric hierarchy used in this study, based on van Hiele's model of geometry learning, was operationalized by mathematics educators. This fact makes this study the only skill theory research, in an area of mathematics, where the hierarchy was developed by experts in that field.

Skill theory provided a method for analyzing geometry skills or tasks into their constituents. Independent assessment of each of the thirty items was performed in this study. This provision has practical implications for education. Too frequently, in the classroom, a geometry concept is presented in one lesson e.g. the concept of rectangle. Analyzing, and thus presenting for instruction, this concept into its constituents (e.g. opposite sides, congruent angles, parallel sides, etc.) appears to be an effective instructional approach. Such an approach allows for learners' experiences with the many constituents of the concept.

Skill theory also provided an interpretive framework for understanding the individual's development. The profiles presented in the scalogram allows one to determine precisely where a student is, where he or she is having difficulties, and where to start instruction. For example, Subject 3 (S3) in this study, demonstrated some Level 9 thinking. (See Table 9.) Within that level, S3 correctly identified a rectangle (i.e. item 1 = 1) however, was unable to select two rectangles from the selection sheet (i.e. item 4 = 0.) From a skill theory perspective, S3 was unable to differentiate from among the shapes. Skill theory may further contend that S3 was unable to focus on specific properties that make up a rectangle when presented with both examples and nonexamples of the shape.

According to skill theory, abstractions develop at about tenn to twelve years of age, the period when formal operations are said to also begin. The Ss in this study, with an average age of 11 years 7 months demonstrated this emergence of abstract skills.

In the first level of abstractions (Level 10 or single abstractions,) the child controls individual, intangible categories and the relations between them. The S can coordinate two or more concrete instances to form an intangible category. This was demonstrated in the ten geometry tasks (items 11 - 20) that were used to assess this level, by the Ss. Level 10 involves only the simplest of abstract skills. The tasks for this level required providing defining properties of geometric shapes. From Table 11 one can see that most of the Ss demonstrated thinking at this level. That is, 83% showed some thinking at Level 10 and 50% showed substantial thinking at this level.

In looking at the extremes of Level 10, (Representations 3 or Level 9 and Abstractions 2 or Level 11) there were ten geometry tasks (items 1 - 10) for assessing the preceding level and ten geometry tasks (items 21 - 30) for assessing the succeeding level. The proceeding level, Level 9 or the representational systems, has an estimated age region of emergence of 6 to 7 1/2 years. At this level, also referred to by some as the level of concrete operations, the child relates two subsets of one representation. As expected, in tasks 1 - 10, the S identified geometric concepts in various contexts and positions. As predicted, the sixth grade Ss demonstrated this. Table 11 shows that 83% of the Ss demonstrated a substantial amount of thinking at this level and 100% of the Ss showed at least some thinking at this level.

The last ten items administered in this study assessed the extent to which the Ss demonstrated Level 11 (abstract mappings) thinking and skills. Typically emerging at about fourteen to sixteen years of age, adolescents relate one intangible category to another in a simple way. In these items, the S had to explain, for example, in general terms inclusion of specific shapes (e.g. squares) in general categories (e.g. kites) and the relation of angle measurements of certain geometric shapes (e.g. 180 degrees in a triangle.) Table 11 shows that only 33% of the Ss showed a substantial amount of thinking at this level while 66% showed at least some thinking at this level. Also from Table 11 one can see that the percentage of Ss demonstrating a level of thinking without error is largest for Level 9 and smallest for Level 11. The percentage of Ss demonstrating no thinking at a particular level is nonexistent for Level 9 (i.e. the percentage of Ss with no correct responses = 0) and highest for Level 11 (i.e. 33%). And the percentage of Ss showing at least some thinking is highest for Level 9 (i.e. 100%) and lowest for Level 11. Table 11 shows that the level of thinking predicted in this study has percentages the fall between the other levels in all columns. For example, the percent without error is both higher for Level 9 (61%) and lower for Level 11 (17%) than it is for Level 10 (22%).

The finding of skill theory research that is reported in the literature suggests that Ss typically operate at one developmental level when they act spontaneously, that is, for example, without contextual support, and at a higher developmental level when they are provided by contextual support, as predicted by skill theory. This study assessed Ss that were provided with contextual support (both instructional materials and guidance from the interviewer.) The findings of this study suggest that sixth grade students with a mean age of 11 years 7 months will demonstrate Level 10 thinking under optimal conditions i.e. with contextual support.

The results of this study suggest that, in practice, performance on the thinking skills of geometric concepts can best be characterized by a developmental range, both within and between individuals. Clearly the range of thinking skills varied greatly between Ss in this study. For example, Subject 1 demonstrated only some Level 9 thinking while Subject 17 demonstrated strong Level 11 thinking. Within individual variability is observed in Ss that demonstrated only some thinking at a level. For example, Subjects 9, 10 and 13 demonstrate some thinking at Level 10, then fail to show thinking at that level and then show Level 10 thinking again. The extent to which this is due to contextual support needs additional investigation.

The findings of this research have other implications for education. The skill theory method uses distinct tasks in assessing steps within a developmental level. The extent to which this was demonstrated here suggests that the geometry curriculum needs to be presented in a similar fashion; with distinct tasks being used for concepts at various developmental levels.

Geometry concepts, that involve skills and abilities of certain developmental levels, need to be dealt with in a hierarchical order. This may involve the educator "breaking down" lessons and identifying the exact skills and abilities necessary to demonstrate them. Ordering the concepts for a hierarchical presentation would be most effective. For example, as observed in the BCP, hierarchical experiences with properties of shapes require skills of increasingly advanced developmental levels. That is, for example, at an early level the student tells what shape a figure is given certain properties. Later, at a developmentally more advanced level, the student identifies a minimum set of properties that can characterize a figure.

The results of this study also suggest the type of geometric concepts that could be taught most effectively at various developmental and grade levels. Traditionally, geometry concepts taught prior to tenth grade have been fragmented and few. The concepts tend to involve skills largely of skill theory Levels 8 and 9. At the tenth grade level, the geometry concepts taught tend to involve skills characteristic of skill theory level 11. This leads to a "jump" in types of skills taught and an omission of geometric skills at Level 10. This study suggests that geometric concepts need to be examined in terms of the skills and abilities they involve. Geometric skills at each developmental level could then be dealt with for effective learning. Further, for optimal student performance, this study suggests that geometry activities, similar to those developed and implemented by the Brooklyn College staff in the van Hiele context, need to be utilized at each developmental level in the classroom setting. Features necessary for optimal performance, that correlate with skill theory, included 1) beginning activities in domains familiar to the student, 2) offering the student high support to move from level to level and 3) allowing for practice time. But, in accordance with skill theory and the results observed here, geometry skills need to be identified within levels so distinct that steps, and therefore abilities, are dealt with for effective learning.

Finally, while this study did not show spurts and plateaus in development, the findings suggest that in the classroom setting, students could benefit by being provided with reinforcement-type activities and materials that involve skills and abilities at a particular developmental level where correct performance is observed. Strengthening the skills at that level while introducing new concepts may allow for learning experiences to occur during both spurts and plateaus. With movement to a next hierarchical level, this study suggests that students would best benefit from working with more developmentally advanced geometric concepts.

Chapter 10

Directions for future research

This research, designed from a skill theory perspective, incorporated three ways Fischer et al. claim can offer satisfactory explanations for any wide variations observed. One way, describing skills in terms of a more refined developmental scale than has typically been used in the past, was to include ten step developmental sequences for each of the levels examined. With small changes from one task to the next, or small increases in levels of skills needed to perform each succeeding task, one is able to identify skills necessary to successfully perform given tasks. Using this method, for other developmental levels of geometric thinking, is a direction the literature has not yet taken but would add to a fuller understanding of developmental processes. In particular, future investigations defining the third and fourth levels of abstractions, in terms of this type of more refined developmental scale, would help explain the extent to which tenets of skill theory hold for these latter developmental levels. While little work has been done on these levels in any contexts, no work has been done on these levels in a geometry context.

A second way Fischer and his colleagues suggest to account for variations observed, is by examining different levels of contextual support. The data, in this study, in the form of videotape, includes records of the tasks examined, with different levels of contextual support. The different levels of contextual support in this data involve the modeling, questioning and guiding done by the interviewer and instructional materials. For instance, when a student was unable to identify certain geometric concepts, the interviewer presented the student with materials that showed many examples of that concept. The interviewer and student would then discuss similarities among the many examples. Then, the interviewer would ask the student to identify the concept from a sheet showing examples and non-examples of the concept. The role of contextual support (both materials and the more knowledgeable individual) needs further investigation in regards to the extent to which they facilitate and explain Fischer's transformations between skill levels. Future research investigating the extent to which the varying degrees of support situations influence the students' geometric thinking would further clarify developmental processes.

Also, at certain points in this study, the interviewer asks the student whether or not he or she had learned the geometric concept at hand in school or whether or not he or she had ever heard of the given concept. For example, before item 27, the interviewer asks the student if he or she has learned about angles in school. Future work, taking this source of variation into account, (i.e. variation in background geometry knowledge and experience) would help further explain the development of geometric thinking skills.

Further, the role of the organization of instructional materials and the effect of the way the interviewer approaches instruction needs investigation in this content area. As noted earlier in this study, van Hiele is concerned with the level of instructional material used. Van Hiele holds that materials that are above the student's thought level are cognitively "reduced" by the student or the student reorganizes the world (i.e. the materials in this case) to fit his or her existing cognitive structure. For Fischer, materials that are above the student's level of thought help guide the student to a higher level of thinking via transformation rules. For example, a student may "compound" two properties of polygons skills to form a higher skill level. The extent to which instructional materials advance the student to a higher skill level or the extent to which the student "reduces" the material to fit his or her already existing cognitive structures needs to be examined further.

The tasks included in this study allowed a central hypothesis of skill theory to be observed; that of optimal level. According to skill theory, it is the optimal level that consistently changes in a stage-like way, whereas most behavior does not show this type of change. In this way, children can show both stage-like developmental levels and wide variations in performances. Whether there are "deeper" reasons for developmental change, such as biological ones, is a question that needs further investigation.

To study discontinuities in performance that are predicted to appear with each, new developmental level, one needs to take into account the student's opportunity for practice in addition to presenting the student with a familiar domain and contextual support for high level performance. This study did not allow the students opportunities for practice. Future studies allowing for this variable (i.e. practice opportunities) would help further explain the developmental processes of geometric thinking.

This study had items corresponding to the van Hiele Module 1 and the very beginning of Module 2. Future studies applying a skill theory set up for the rest of Module 2 and for Module 3 would be informative. Modules 2 and 3 involve significantly more instruction than Module 1. Given this, future investigations of the Modules in a skill theory context may allow one to observe higher skill levels at optimal levels of performance. For example, with high contextual support in Module 2 the students may deduce the sum of the angles of a triangle from guiding experiences with angle measurement of other polygons. In Module 3, the students may derive area rules for triangles from the area rule for a rectangle and further explore a rule that unifies all area rules. As noted above, research on the later levels of skill theory is nearly non-existent. The extent to which Modules 2 and 3 could supply a geometric hierarchy for the end of the representational tier and for the entire abstract tier needs investigation.

Footnotes

1 Russian psychologists and educators, having been active for about a half of century in their study of how students learn and how to improve their curriculum (Hoffer, 1983 refers the reader to Kilpatrick & Wirszup, 1977), were immediately attracted to van Hiele's ideas. Isaak Wirszup (1976) formally introduced van Hiele ideas to American audiences in 1974 when he described breakthroughs in the teaching of geometry and reported on Russian studies at a professional conference. In the early 1980's, with little published in English that described van Hiele-related research, but with growing interest, the National Science Foundation and the National Institute of Education funded three major research studies that dealt with the van Hiele model in geometry. One of these studies was the Brooklyn College Project: "Geometric Thinking Among Adolescents in Inner City Schools"; November 1979 to January 1982. The staff consisted of Brooklyn College faculty members David Fuys, Dorothy Geddes (Director), C. James Lovett and Rosmond Tischler in the School of Education's Mathematics Education Program.

Having recently completed the master's program in mathematics education at Brooklyn College I had an opportunity to work as a research assistant on this project. I was responsible for interviewing some of the sixth grade students. My continued interest in the teaching and learning of geometry in the elementary school, educational psychology and developmental processes led to this dissertation.

2 Appendix B is used with written permission from the authors. See Fuys, et al. (1988) in references. Appendix B is from pages 58-71 of that text.

Reference Notes

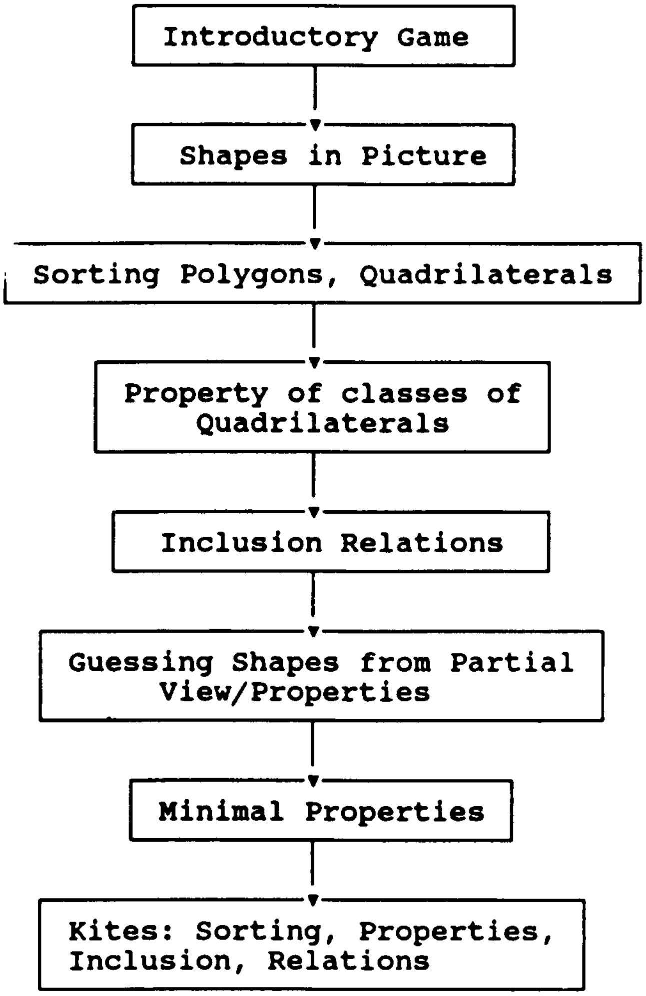
1 Fischer, K. W. Personal written communication to Dr. Nicholas J. Anastasiow, July 18, 1990.

2 Fischer, K. W. Personal communication to Christine Nucci, June 26, 1990.

Appendix A

Item 1

Module 1: Hierarchy and Corresponding Items.



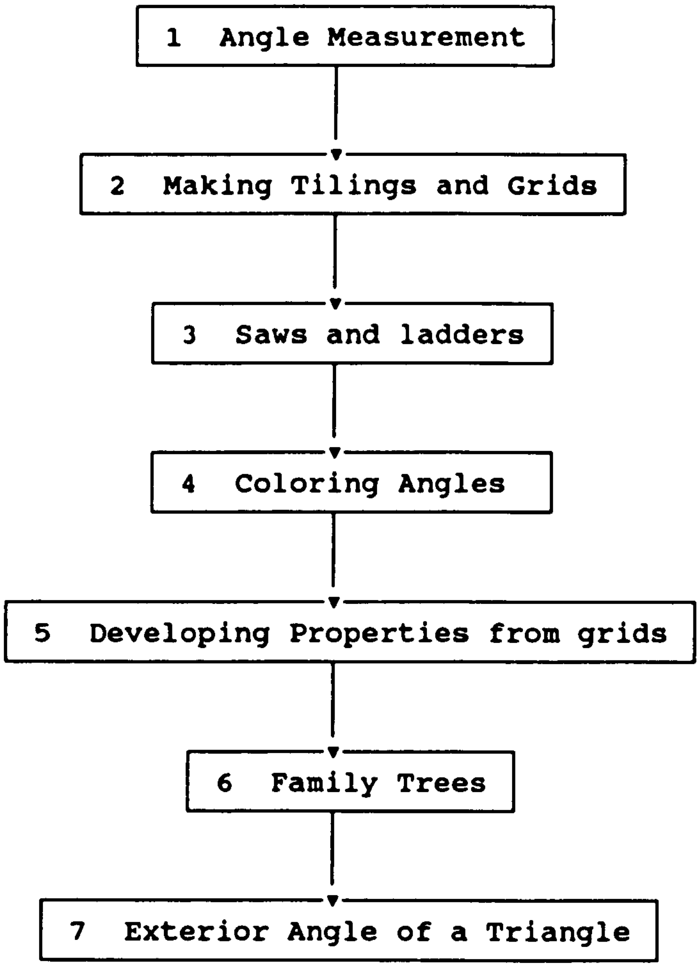
Module 1 Hierarchy	Items
Introductory Game	1, 2, 3
Shapes in Picture	4, 5, 6, 7, 8, 9, 10
Sorting Polygons, Quadrilaterals	11, 12
Property of classes of Quadrilaterals	13, 14, 15
Inclusion Relations	16, 17
Guessing Shapes from Partial View/Properties	18, 19, 20
Minimal Properties	
Kites: Sorting, Properties, Inclusion, Relations	21, 22, 23, 24, 24, 26, 27

Appendix A

Item 2

Module 2: Hierarchical listing of tasks.

Module 2 Hierarchy | Items

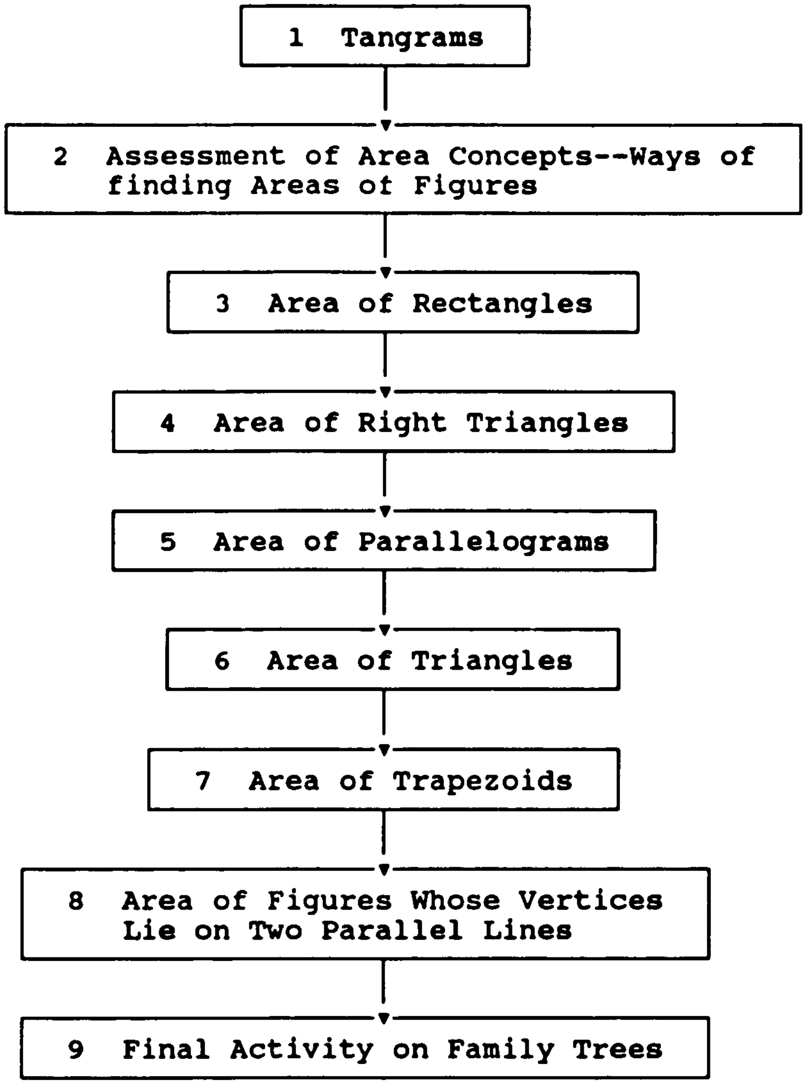


28, 29, 30

Appendix A

Item 3

Module 3: Hierarchical listing of tasks.



Appendix C

Scoring Criterion

Item 2: If the S names or labels the rectangles presented, using either standard or non-standard geometric terms appropriately for this, score 1.

Otherwise, score 0.

Item 3: If the S names or labels the two dimensional and three dimensional triangles (pyramids) presented, using standard or non-standard geometric terms appropriately for this, score 1.

Otherwise, score 0.

Item 4: If the S identifies two rectangles from the selection sheets, without error, score 1.

Otherwise, score 0.

Item 5: If the S identifies at least two sets of parallel lines in the figurations presented, without error, score 1.

Otherwise, score 0.

Item 6: If the S identifies at least two right angles from the selection sheets without error, score 1.

Otherwise, score 0.

Item 7: If the S identifies two right angles in the figurations without error, score 1.

Otherwise, score 0.

Item 8: If the S sufficiently describes congruence in two of three figures, sides or angles, score 1.

Otherwise, score 0.

Item 9: If the S sufficiently describes opposite sides or angles, without error, score 1.

Otherwise, score 0.

Item 10: If the S sufficiently describes diagonals, without error, score 1.

Otherwise, score 0.

The subtest for the structure of systems of representational systems (Fischer, Level 10), equivalent to the abstract set, consists of ten items. These items, to assess the existence of this first level of abstractions, involve only the simplest level of abstract skills. The coders scored the items according to the following criterion.

Item 11: If the S sorts the polygons presented, according to the number of sides, score 1.

Otherwise, score 0.

Item 12: If the S sorts four types of quadrilaterals without error, score 1.

Otherwise, score 0.

Item 13: If the S lists, identifies, describes or demonstrates four essential, defining properties of a square, score 1.

Otherwise, score 0.

Item 14: If the S lists, identifies, describes or demonstrates four essential, defining properties of a rectangle, score 1.

Otherwise, score 0.

Item 15: If the S lists, identifies, describes or demonstrates four essential, defining properties of a parallelogram, score 1.

Otherwise, score 0.

Item 16: If the S identifies a square as a "special type" of parallelogram, score 1.

Otherwise, score 0.

Item 17: If the S identifies a square or a rectangle as a special type of parallelogram, score 1.

Otherwise, score 0.

<u>Item 18:</u> If the S names two possible or two impossible shapes, without error, with the first uncovering clue, score 1.

Otherwise, score 0.

<u>Item 19:</u> If the S names two possible or two impossible shapes, without error, with the second uncovering, score 1.

Otherwise, score 0.

<u>Item 20:</u> If the S names a possible shape with the third uncovering clue, score 1.

Otherwise, score 0.

* "Special type" refers to the fact that a specific shape is a member of a more general class of shapes.

The subtest for the structure of abstract mappings (Fischer, Level 11), the last of the structures investigated here, also consists of ten items. The coders scored the items according to the following criterion.

<u>Item 21:</u> If the S identifies the 'kite' shape sorting rule of at least one line of symmetry through its vertices, score 1.

Otherwise, score 0.

Item 22: If the S describes why the rectangle is not a kite (as defined in item 21), score 1.

Otherwise, score 0.

Item 23: If the S lists three properties of a kite, score 1.

Otherwise, score 0.

Item 24: If the S describes or demonstrates the exclusion of particular quadrilaterals as rectangles, score 1.

Otherwise, score 0.

Item 25: If the S describes or demonstrates the exclusion of certain quadrilaterals (i.e. all but squares and rectangles) as rectangles, score 1.

Otherwise, score 0.

Item 26: If the S describes or demonstrates the inclusion of every ________ (any one specific quadrilateral is acceptable here) as a quadrilateral, score 1.

Otherwise, score 0.

Item 27: If the S describes or demonstrates the exclusion of certain quadrilaterals as ________ (any specific shape such as rectangle, parallelogram, etc., is acceptable here), score 1.

Otherwise, score 0.

Item 28: If the S accurately predicts the measure of the given angles, score 1.

Otherwise, score 0.

Item 29: If the S demonstrates or describes the measures of the angles in the straight angle, score 1.

Otherwise, score 0.

Item 30: If the S demonstrates or describes the measure of angles in the triangle, score 1.

Otherwise, score 0.

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