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THE AVERAGE OF A GAUGE

by

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Introduction

The setting for our discussion is a real linear space $\mathcal{L}$ with an inner product (x,y). It is assumed that $\mathcal{L}$ is complete with respect to the norm generated by this inner product. Clearly the gauges on $\mathcal{L}$ and the bodies (closed, bounded, absolutely convex sets containing the origin Θ as an interior point) generate each other. The length (norm) $e(x) = (x,x)^{\frac{1}{2}}$ is of course a special gauge. In general, a gauge will be denoted by $\emptyset(x)$, or briefly by $\emptyset$; the symbol $\emptyset^*$ represents the conjugate of $\emptyset$.

Problem. Consider two gauges $\emptyset_1$ and $\emptyset_2$ such that $e \leq \emptyset_1 \leq \emptyset_2$. Under what conditions may one conclude that

$$(*) \qquad \frac{\emptyset_1 + \emptyset_1^*}{2} \leq \frac{\emptyset_2 + \emptyset_2^*}{2} \quad ?$$

That such an orderly relation may exist is suggested by the fact that for any gauge $\emptyset$, its "average" $\frac{\emptyset + \emptyset^*}{2}$ is well behaving with respect to e, being always $\geq$ e. Moreover, the average of the average is a better approximation to e, and so on.

Solution. With each gauge $\emptyset$ (body K) we associate its spread $\mathfrak{S}(x)$. The last concept is a very natural one and is defined as follows: For $\Theta \neq x \in \mathcal{L}$, construct the line joining Θ and x. Let $\omega(x)$ represent the (width) distance between the two support hyperplanes of K orthogonal to that line, while $\delta(x)$ represents the (diameter) length of the chord of K lying on that line. We then define

$$\mathfrak{S}_{\emptyset}(x) = \mathfrak{S}_K(x) = \omega(x) - \delta(x) .$$

Clearly, $\mathfrak{S}(x) = \mathfrak{S}(\lambda x)$ for $\lambda \neq 0$.

It turns out that for any pair of gauges $\emptyset_1$ and $\emptyset_2$ such that $e \leq \emptyset_1 \leq \emptyset_2$, the relation $\mathfrak{S}_{\emptyset_1} \leq \mathfrak{S}_{\emptyset_2}$ implies inequality (*) and also $\emptyset_1\emptyset_1{}^* \leq \emptyset_2\emptyset_2{}^*$. In particular, one may verify that for any pair of the well-known gauges $\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$, the corresponding spreads do indeed satisfy the above relation. We may express, therefore, the following:

Let $p \geq 1$. Define $q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$. For any fixed point $(x_1, x_2, \cdots, x_n)$,

$$\frac{1}{2}\left[\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |x_j|^q\right)^{\frac{1}{q}}\right]$$

and

$$\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j|^q\right)^{\frac{1}{q}}$$

are decreasing functions of p in the interval $1 \leq p \leq 2$.

Preliminaries

We begin by stating some definitions and remarks.

Definition. Let A be a bounded, absolutely convex subset of $\mathcal{L}$ containing the origin θ in its interior. The gauge of A, denoted ϕ_A, is defined as follows:

$$\phi_A(x) = \inf\left\{\lambda : \lambda > 0,\ x \in \lambda A\right\}.$$

Clearly $\phi = \phi_A$ possesses the following properties: $\phi(x) \geq 0$, $\phi(x+y) \leq \phi(x) + \phi(y)$, and $\phi(\alpha x) = |\alpha|\phi(x)$ for all scalars α. Also, $\phi(x)$ is necessarily continuous. Moreover,

$$\left\{x: \phi(x) < 1\right\} = \operatorname{int}(A)$$

$$\left\{x: \phi(x) \leq 1\right\} = \overline{A}$$

and ϕ is also the gauge of any set B such that $\operatorname{int}(A) \subseteq B \subseteq \overline{A}$.

Definition. A body is a closed, bounded, and absolutely convex set containing θ in its interior. The boundary of a body is called its surface. For $\theta \neq x \in \mathcal{L}$, the ray determined by x is $\left\{\lambda x: \lambda > 0\right\}$.

Thus if K is a body having ϕ as its gauge, then $K = \left\{x: \phi(x) \leq 1\right\}$; its surface $S = \left\{x: \phi(x) = 1\right\}$. A body and its gauge determine each other. Indeed, $\phi(x) = \frac{\|x\|}{\|S(x)\|}$, where S(x) is the point of the surface of K on the ray determined by x. Also, $K_1 \subseteq K_2$ if and only if $\phi_1(x) \geq \phi_2(x)$ for all $x \in \mathcal{L}$.

Definition. Let $\emptyset$ be the gauge corresponding to the set A. The conjugate $\emptyset^*$ of the gauge $\emptyset$ is defined as follows:

$$\emptyset^*(x) = \underset{\emptyset(y)=1}{\text{Supremum}} \; (x,y) \; .$$

It is readily seen that $\emptyset^*$ is the gauge of the set $A^* = \{w\colon \; (w,v) \leq 1 \text{ for all } v \in A\}$. The set A^* is referred to as the conjugate or polar set of A.

Definition. The average of the gauge $\emptyset$ is defined as $\frac{\emptyset + \emptyset^*}{2}$. Similarly, the average of the body K is $\frac{K + K^*}{2}$.

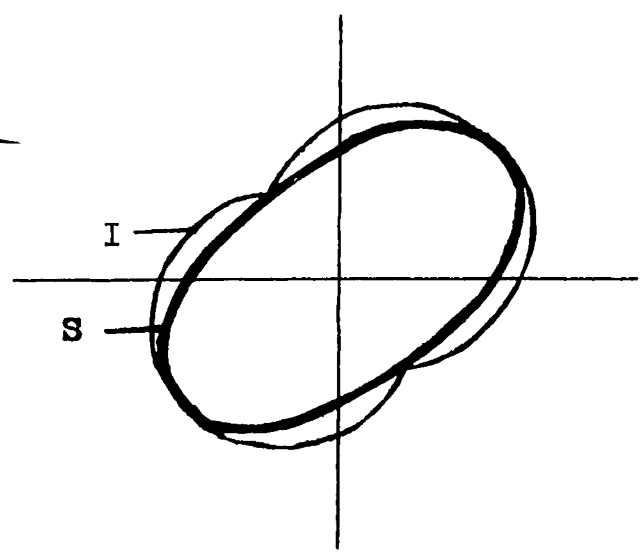
We show later that not every gauge is an average.

The Indicatrix and Conjugate

In $\mathcal{L}$, let a body K be given. Then K* is necessarily a body also; we denote its surface by *S. (Placing the asterisk on the left is unavoidable since the surface of K* does not coincide with the polar of S.) In this section, we describe a geometric method of determining K* given K, which is equivalent to deriving *S from S.

Definition. Let $x \neq \theta$ be a point of $\mathcal{L}$. The inversion of x, denoted $\frac{1}{x}$, is that point on the ray determined by x whose norm is $\frac{1}{\|x\|}$. If X is any collection of points in $\mathcal{L} - \{\theta\}$, then the inversion of X, in symbol $\frac{1}{X}$, is the set of points obtained by inverting each point of X.

We are now ready to determine *S when S is given. Consider, in $\mathcal{L}$, a fixed ray from the origin Θ. Let i be its intersection with the support hyperplane of K which is orthogonal to this ray. The collection of all such points --one for each ray-- defines the indicatrix of K. It is denoted by I_K or briefly by I. We shall presently show that the inversion of I is precisely *S.



Let ϕ be the gauge of the body K. Consider a fixed ray from the origin, and let $v \neq \Theta$ be a point on this ray. Then the distance from the origin Θ to the support hyperplane orthogonal to this ray is precisely $\phi^*\left(\frac{v}{\|v\|}\right)$. Letting i denote the point of I on this ray, we have $\phi^*\left(\frac{v}{\|v\|}\right) = \|i\|$. Thus, $\phi^*(v) = \|v\|\|i\|$. This is true for any point $v \neq \Theta$ on this ray. We, however, are interested in finding that point v for which $\phi^*(v) = \|v\|\|i\| = 1$. Clearly, the desired point is $\frac{1}{i}$. Thus, the inversion of I is, indeed, *S.

The symbols K and K_j will be reserved to represent bodies. We denote by $\hat{I}_K$ the collection of points "within and on" I_K and term it the solid indicatrix of K. Then the following statements are immediate.

1) $K_1 \subseteq K_2 \Longleftrightarrow \hat{I}_{K_1} \subseteq \hat{I}_{K_2} \Longleftrightarrow K_1^* \supseteq K_2^*$

2) $\phi_1 \geq \phi_2 \Longleftrightarrow \phi_1^* \leq \phi_2^*$

<u>Theorem 1</u>. $\phi = e \Longleftrightarrow \phi = \phi^*$

<u>Proof</u>: ($\Rightarrow$) Consider the body $U = \{x: \|x\| \leq 1\}$ corresponding to the gauge e. The Pythagorean Theorem implies that $\hat{I}_U = U$. It follows that $U = U^*$, and consequently, $e = e^*$.

($\Leftarrow$) Now suppose $\phi = \phi^*$. Clearly, for x and y in $\mathcal{L}$, $(x,y) \leq \phi(x)\phi^*(y)$. If also $x = y$, then

$$[e(x)]^2 = (x,x) \leq \phi(x)\phi^*(x) = [\phi(x)]^2 .$$

Therefore, $e \leq \phi$. It then follows that $e = e^* \geq \phi^* = \phi$. Thus, $\phi = e$. ∎

<u>Corollary</u>. $K = U \Longleftrightarrow \hat{I}_K = U \Longleftrightarrow K = K^*$

Theorem 2. $K^{**} = K$

Proof: Let $x \in K$. Then $(x,y) \leq 1$ for all $y \in K^*$. Therefore, $x \in K^{**}$ and $K \subseteq K^{**}$.

Suppose now that $x \notin K$. Then there is a hyperplane that separates x strictly from K. This hyperplane does not pass through the origin since the origin is contained in K. Therefore, there exists a point y such that

$$(z,y) < 1 \quad \text{for } z \in K \quad \text{and} \quad (x,y) > 1.$$

Thus, $y \in K^*$ and $x \notin K^{**}$. Hence, $K^{**} \subseteq K$. Consequently, we have $K^{**} = K$. ∎

Corollary. $\phi_K^{**} = \phi_K$

The Indicatrix of a Vector Sum

In expressing the theorem of this section, we need an additional piece of notation.

Definition. For two subsets X_1 and X_2 of $\mathcal{L}$, their vector sum $X_1 + X_2$ and radial sum $X_1 \oplus X_2$ are defined as follows:

$$X_1 + X_2 = \{x_1+x_2 : x_1 \in X_1 \text{ and } x_2 \in X_2\}$$

$$X_1 \oplus X_2 = \{x_1+x_2 : x_1 \in X_1 \text{ and } x_2 \in X_2, \text{ and } x_1, x_2 \text{ both on the same ray}\}.$$

The vector sum and radial sum are related as described in the theorem which follows.

Theorem 3. Let K_1 and K_2 be bodies.

Then $I_{(K_1+K_2)} = I_{K_1} \oplus I_{K_2}$.

Proof: Consider a fixed ray from the origin. Let i_1 and i_2 be the points of I_{K_1} and I_{K_2} respectively on this ray. If $\{x: f(x) = \alpha_1 > 0\}$ and $\{x: f(x) = \alpha_2 > 0\}$ are the support hyperplanes of K_1 and K_2 respectively which are orthogonal to this ray, then there exist points $v_1 \in K_1$ and $v_2 \in K_2$ for which $f(v_1) = \alpha_1$ and $f(v_2) = \alpha_2$. Consider now the point $v_1 + v_2$. It belongs to $K_1 + K_2$ and $f(v_1 + v_2) = \alpha_1 + \alpha_2$. Moreover, the hyperplane $\{x: f(x) = \alpha_1 + \alpha_2\}$ is orthogonal to our ray and intersects it in the point $i_1 + i_2$. Clearly, $i_1 + i_2$ is the point of $I_{K_1} \oplus I_{K_2}$ on this ray. To complete the proof, it remains to show that $i_1 + i_2$ is a point of $I_{(K_1+K_2)}$.

It is sufficient to show that $\{x: f(x) = \alpha_1 + \alpha_2\}$ is a support hyperplane of $K_1 + K_2$. Otherwise, there would be a point $y \in K_1 + K_2$ for which $f(y) > \alpha_1 + \alpha_2$. Then there exist $y_1 \in K_1$ and $y_2 \in K_2$ such that $y = y_1 + y_2$. Since $\{x: f(x) = \alpha_1\}$ and $\{x: f(x) = \alpha_2\}$ are support hyperplanes of K_1 and K_2 respectively, we have $f(y_1) \leq \alpha_1$ and $f(y_2) \leq \alpha_2$. Therefore, $f(y) = f(y_1 + y_2) \leq \alpha_1 + \alpha_2$. Thus, we reach a contradiction. ∎

Corollary 1. If K_1 and K_2 are bodies and α, β non-zero scalars, then $I_{(\alpha K_1 + \beta K_2)} = \alpha I_{K_1} \oplus \beta I_{K_2}$.

Corollary 2. If ϕ_1 and ϕ_2 are the gauges of K_1 and K_2 respectively, then the gauge of the set $K = \alpha K_1 + \beta K_2$ is $\phi = (\alpha \phi_1{}^* + \beta \phi_2{}^*)^*$.

Corollary 3. If K is a body having ϕ as its gauge, then $\frac{K + K^*}{2} \supseteq U = \{x: \|x\| \leq 1\}$ and $\frac{\phi + \phi^*}{2} \geq e$.

Corollary 3 is significant, but not surprising, in the light of the preceding exposition, since for any positive number α, $\frac{\alpha + \frac{1}{\alpha}}{2} \geq 1$. Indeed, equality holds in the last if and only if $\alpha = 1$. Consequently, we also have the following corollary.

Corollary 4. The average of K equals U if and only if K = U ; the average of ϕ equals e if and only if $\phi = e$.

The study of the average of a gauge is motivated by the following theorem. (Confer Schatten: [3], p.73 or [4], p.149.) "Let ϕ be an arbitrary gauge on $\mathcal{L}$. Define $\phi_1 = \frac{\phi + \phi^*}{2}$ and $\phi_n = \frac{\phi_{n-1} + \phi_{n-1}^*}{2}$ for n > 1. Then the sequence $\{\phi_n\}$ converges decreasingly to e."

Since its publication, this theorem has appeared in the literature in a variety of forms. A recent interesting version may be found in Mityagin and Shvarts: [2], p.116 . The preceding corollaries permit us to derive a different version of the above theorem, valid in Euclidean n-space E^n.

<u>Definition</u>. A sequence of bodies K_n is said to converge to K

a) in the <u>Blaschke sense</u> ($K_n \xrightarrow{\mathcal{B}} K$) if for every $\epsilon > 0$, there exists a $N(\epsilon)$ such that for all $n \geq N(\epsilon)$ we have $\Delta(K_n,K) < \epsilon$, where $\Delta(K_n,K)$ denotes the distance from K_n to K in the Blaschke sense. (Cf. Eggleston: [1], p.60)

b) <u>pointwise-radially</u> ($K_n \longrightarrow K$) if for each $x \neq \theta$ and for each $\epsilon > 0$, there exists a $N(\epsilon)$ such that $n \geq N(\epsilon)$ implies that $\| S_n(x) - S(x) \| < \epsilon$.

c) <u>uniform-radially</u> ($K_n \rightrightarrows K$) if for each $\epsilon > 0$, there exists a $N(\epsilon)$ such that $n \geq N(\epsilon)$ implies that $\| S_n(x) - S(x) \| < \epsilon$ for all $x \neq \theta$.

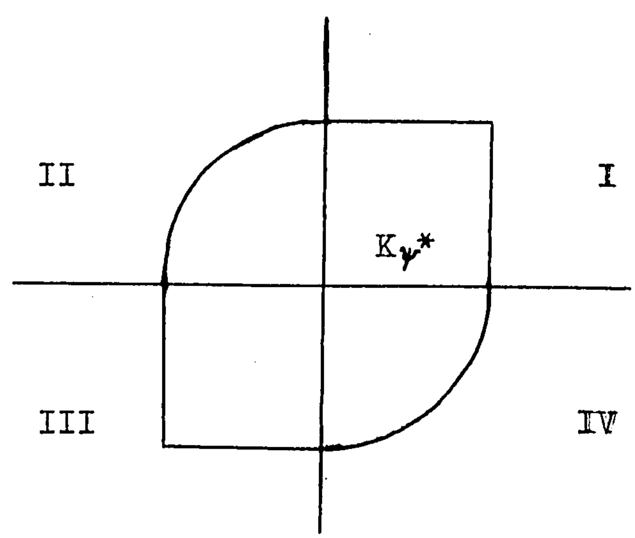
Observe that if $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ or $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$, then the three types of convergence mentioned above are equivalent. (To prove that $K_n \longrightarrow K$ implies $K_n \xrightarrow{\mathcal{B}} K$, we use the Blaschke Selection Theorem.) Thus, we may express the following.

Theorem 4. Let K be a body in E^n. Define $K_1 = \frac{K + K^*}{2}$ and $K_n = \frac{K_{n-1} + K_{n-1}^*}{2}$ for $n > 1$. Then $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ and $K_n \xrightarrow{\mathcal{B}} U$.

We conclude this section by showing that not every gauge ψ is an average, that is, of the form $\psi = \frac{\phi + \phi^*}{2}$ for some gauge ϕ. On the Euclidean plane, define:

$$\psi(x,y) = \begin{cases} |x| + |y| & \text{if } xy > 0 \\ (x^2 + y^2)^{\frac{1}{2}} & \text{if } xy \leq 0 . \end{cases}$$

ψ is not the average of any gauge. To prove this, assume to the contrary that $\psi = \frac{\phi + \phi^*}{2}$. Let K_ψ and K_ϕ be the convex sets corresponding to ψ and ϕ respectively. Corollary 2 of Theorem 3 implies that $K_\psi{}^* = \frac{K_\phi + K_\phi{}^*}{2}$ (see diagram).



Let $v \neq \Theta$ be a fixed point in either quadrant II or IV. Denoting by $I_K(v)$ and $S_K(v)$ the points of the indicatrix and surface of K respectively which lie on the ray determined by v, we obtain:

$$1 = \|I_{K_{\psi^*}}(v)\| = \tfrac{1}{2}\left(\|I_{K_\phi}(v)\| + \|I_{K_\phi^*}(v)\|\right)$$
$$\geq \tfrac{1}{2}\left(\|I_{K_\phi}(v)\| + \|S_{K_\phi^*}(v)\|\right) \geq 1.$$

Therefore, $\|I_{K_\phi}(v)\| = \|I_{K_\phi^*}(v)\| = 1$. It follows that $K_\phi = K_\phi^* = \mathbb{U}$ in quadrants II and IV. The last implies that $K_\phi \subseteq K_{\psi}^*$ and $K_\phi^* \subseteq K_{\psi}^*$. Moreover, since $\dfrac{K_\phi + K_\phi^*}{2} = K_\psi^*$, we have $K_\phi = K_\phi^* = K_\psi^*$. This of course is a contradiction.

The "Spread" Function

Definition. Let a body K be given. Then the width of K in the direction of x, denoted $\omega_K(x)$, is the distance between the two support hyperplanes of K which are orthogonal to the line determined by the origin and $x \neq \Theta$. The diameter of K in the direction of x, in symbol $\delta_K(x)$, is the length of the chord of K lying on the line determined by Θ and x.

The function $\sigma_K(x) = \omega_K(x) - \delta_K(x)$ is defined as the <u>spread of K</u>. Clearly, σ_K is defined on the whole space except at the origin. Being constant on each ray, it may also be considered as a function of direction. σ_K is completely determined by its values on the surface $\{u: \|u\| = 1\}$ of the unit ball. We reserve the letter "u" to represent an arbitrary point of this surface.

Let ϕ represent the gauge of the body K. Then $\omega_K(u) - \delta_K(u) = 2\left(\|I_K(u)\| - \|S_K(u)\|\right)$, due to the fact that K is balanced. Since $\|u\| = 1$,

$$\phi^*(u) = \frac{\|u\|}{\|S_{K^*}(u)\|} = \frac{1}{\|S_{K^*}(u)\|} = \|I_K(u)\| .$$

Moreover, $\phi(u) = \dfrac{\|u\|}{\|S_K(u)\|} = \dfrac{1}{\|S_K(u)\|}$.

Therefore, the <u>spread of the gauge ϕ</u> is given by:

$$\sigma_\phi(u) = \sigma_K(u) = 2\left(\phi^*(u) - \frac{1}{\phi(u)}\right) .$$

We stress that the last equality holds when u is any vector of norm one. For a vector x such that $0 < \|x\| \neq 1$, we define $\sigma_\phi(x) = \sigma_\phi\left(\frac{x}{\|x\|}\right)$. The following relations are immediate.

1) $\sigma_\phi \geq 0$

2) $\sigma_e \equiv 0$

3) $e \leq \phi \implies \sigma_\phi \leq \sigma_{\phi^*}$

4) If $e \leq \phi_1 \leq \phi_2$, then $\sigma_{\phi_1} \leq \sigma_{\phi_2} \implies \sigma_{\phi_1^*} \leq \sigma_{\phi_2^*}$.

Theorem 5. If ϕ_1 and ϕ_2 are gauges such that $\phi_1 \leq \phi_2$ and $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$, then $\phi_1\phi_1^* \leq \phi_2\phi_2^*$.

Proof: Let $y \neq \theta$ be arbitrarily chosen in $\mathcal{L}$. Set $y = \lambda u$, where $\lambda > 0$ and $\|u\| = 1$. Clearly, $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$ amounts to $\phi_1^*(u) - \frac{1}{\phi_1(u)} \leq \phi_2^*(u) - \frac{1}{\phi_2(u)}$. Since also $\phi_1 \leq \phi_2$, we have $\phi_1(u)\phi_1^*(u) \leq \phi_2(u)\phi_2^*(u)$. Then one obtains $\phi_1(y)\phi_1^*(y) = \lambda^2\phi_1(u)\phi_1^*(u) \leq \lambda^2\phi_2(u)\phi_2^*(u) = \phi_2(y)\phi_2^*(y)$, and consequently, $\phi_1\phi_1^* \leq \phi_2\phi_2^*$. ∎

Theorem 6. Let ϕ_1 and ϕ_2 be gauges such that $e \leq \phi_1 \leq \phi_2$. Moreover, suppose that $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$. Then

$$\frac{\phi_1 + \phi_1^*}{2} \leq \frac{\phi_2 + \phi_2^*}{2} .$$

Proof: If K_j is the body of ϕ_j and S_j is its surface (j=1,2), then $\mathfrak{S}_{K_1} \leq \mathfrak{S}_{K_2}$. Therefore, $\omega_{K_1} - \delta_{K_1} \leq \omega_{K_2} - \delta_{K_2}$. We shall denote the indicatrix of K_j by I_j and the indicatrix of K_j^* by $*I_j$. (The asterisk is placed on the left here in order to distinguish $*I_j$ from the polar of I_j.) Then, since the K_j are balanced, it follows that

$$\|I_1(x)\| - \|S_1(x)\| \leq \|I_2(x)\| - \|S_2(x)\| \quad \text{for all } x \neq \theta.$$

Observe the following.

$$\|{}^*I_2(x)\| - \|{}^*I_1(x)\| = \left\|\frac{1}{S_2(x)}\right\| - \left\|\frac{1}{S_1(x)}\right\|$$

$$= \frac{1}{\|S_2(x)\|} - \frac{1}{\|S_1(x)\|}$$

$$= \frac{\|S_1(x)\| - \|S_2(x)\|}{\|S_2(x)\|\,\|S_1(x)\|} .$$

Since $e \leq \phi_1 \leq \phi_2$, we have $K_2 \subseteq K_1 \subseteq U$. The last implies that $\|S_2(x)\|\,\|S_1(x)\| \leq 1$. Therefore,

$$\|I_1(x)\| - \|I_2(x)\| \leq \|S_1(x)\| - \|S_2(x)\| \leq \frac{\|S_1(x)\| - \|S_2(x)\|}{\|S_2(x)\|\ \|S_1(x)\|}$$

$$= \|{}^*I_2(x)\| - \|{}^*I_1(x)\| .$$

Thus,

$$\|I_1(x) + {}^*I_1(x)\| = \|I_1(x)\| + \|{}^*I_1(x)\|$$

$$\leq \|I_2(x)\| + \|{}^*I_2(x)\| = \|I_2(x) + {}^*I_2(x)\| .$$

It follows that the solid indicatrix of the average of K_1 is a subset of the solid indicatrix of the average of K_2.

Therefore, $\left(\frac{K_1 + K_1{}^*}{2}\right)^* \supseteq \left(\frac{K_2 + K_2{}^*}{2}\right)^*$ and thus,

$$\frac{\phi_1 + \phi_1{}^*}{2} \leq \frac{\phi_2 + \phi_2{}^*}{2} . \blacksquare$$

<u>Theorem 7</u>. Let ϕ_1 and ϕ_2 be gauges satisfying $\phi_2 \leq \phi_1 \leq e$, and suppose $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$. Then $\frac{\phi_1 + \phi_1{}^*}{2} \leq \frac{\phi_2 + \phi_2{}^*}{2}$.

Proof: The proof is similar to that of Theorem 6. ∎

Remark. The preceding arguments will remain valid also for bounded convex sets (not necessarily balanced) containing the origin as an interior point. This necessitates only minor changes in the text. For example, we would then define the spread as $\mathfrak{S}_K(x) = \|I_K(x)\| - \|S_K(x)\|$ and

$$\mathfrak{S}_{\emptyset}(u) = \emptyset^*(u) - \frac{1}{\emptyset(u)} .$$

The Class of Gauges $\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$ for $1 \leq p \leq \infty$

In this section $\mathcal{L}$ stands for the n-dimensional Euclidean space. There we define a class of gauges $\{\emptyset_p\}$ for $1 \leq p \leq \infty$ as follows:

$$\emptyset_p(x_1,\ldots,x_n) = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

$$\emptyset_\infty(x_1,\ldots,x_n) = \underset{1\leq j\leq n}{\text{Maximum}} |x_j| .$$

One readily verifies that for each $x = (x_1,\ldots,x_n)$, $\emptyset_p(x)$ is a decreasing function of p. Moreover, the conjugate of $\emptyset_p$ is precisely $\emptyset_q$, where $\frac{1}{p} + \frac{1}{q} = 1$ if $p > 1$, and $q = \infty$ if $p = 1$. To simplify notation, we shall write $\mathfrak{S}_p(x)$ instead of $\mathfrak{S}_{\emptyset_p}(x)$ for the spread of $\emptyset_p$.

Theorem 8. Let $x = (x_1, \cdots, x_n) \neq \theta$. Then $\mathfrak{S}_p(x)$ is a decreasing function of p for $1 \leq p \leq 2$.

Proof: Let $u = (u_1, \cdots, u_n)$ be the vector of norm one on the ray determined by x. Since the spread is constant on each ray, it is sufficient to prove the theorem for u. Without loss of generality, we may assume that $u_j > 0$ for $j = 1, 2, \cdots, n$.

To prove the theorem in case $1 < p \leq 2$, we define an auxiliary function $\gamma(t)$ for all $t \geq 1$ by:

$$\gamma(t) = \log\left(\sum_{j=1}^{n} u_j^{\,t}\right) .$$

Elementary computations show that:

$$\gamma'(t) = \frac{d}{dt}\left(\gamma(t)\right) = \frac{\sum_{j=1}^{n} u_j^{\,t}(\log u_j)}{\sum_{j=1}^{n} u_j^{\,t}}$$

and

$$\gamma''(t) = \frac{\left(\sum_{j=1}^{n} u_j^{\,t}\right)\left(\sum_{j=1}^{n} u_j^{\,t}(\log u_j)^2\right) - \left(\sum_{j=1}^{n} u_j^{\,t}(\log u_j)\right)^2}{\left(\sum_{j=1}^{n} u_j^{\,t}\right)^2} .$$

Moreover, Cauchy's Inequality implies:

$$\left(\sum_{j=1}^{n} u_j^{t}(\log u_j)\right)^2 = \left(\sum_{j=1}^{n} (u_j^{\frac{t}{2}})(u_j^{\frac{t}{2}} \log u_j)\right)^2$$

$$\leq \left(\sum_{j=1}^{n} u_j^{t}\right) \left(\sum_{j=1}^{n} u_j^{t}(\log u_j)^2\right).$$

Consequently, $\gamma''(t) \geq 0$ and therefore, $t\gamma''(t) \geq 0$ for all $t \geq 1$. The last, being the derivative of $\psi(t) = t\gamma'(t) - \gamma(t)$, implies that $\psi(t)$ is an increasing function of t.

Thus, $p \leq 2 \leq q$ implies that $\psi(q) \geq \psi(p)$.

Also,

$$\left(\sum_{j=1}^{n} u_j^{q}\right)^{\frac{1}{q}} \geq \left(\sum_{j=1}^{n} u_j^{p}\right)^{-\frac{1}{p}},$$

because

$$\left(\sum_{j=1}^{n} u_j^{q}\right)^{\frac{1}{q}} - \left(\sum_{j=1}^{n} u_j^{p}\right)^{-\frac{1}{p}} = \tfrac{1}{2}\,\mathfrak{S}_p(u) \geq 0.$$

Therefore,

$$\tfrac{1}{2}p^2 \cdot \frac{d}{dp}\left(\mathfrak{S}_p(u)\right) = \left(\sum_{j=1}^{n} u_j^{p}\right)^{-\frac{1}{p}} \cdot \psi(p) - \left(\sum_{j=1}^{n} u_j^{q}\right)^{\frac{1}{q}} \cdot \psi(q) \leq 0.$$

Thus, $\frac{d}{dp}\left(\mathfrak{S}_p(u)\right) \leq 0$ and $\mathfrak{S}_p(u)$ is a decreasing function of p for $1 < p \leq 2$.

The case $p = 1$ is settled by observing that for any p ($1 < p \leq 2$), we have $\mathfrak{S}_1(u) \geq \mathfrak{S}_p(u)$. ∎

<u>Corollary</u>. If $2 \leq q \leq q'$, then $\mathfrak{S}_q(x) \leq \mathfrak{S}_{q'}(x)$ for each $x \neq \theta$.

As a consequence of the preceding theorem, one obtains the following result.

Theorem 9. If $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for each point $(x_1, \cdots, x_n)$, both:

$$\frac{1}{2}\left[\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |x_j|^q\right)^{\frac{1}{q}}\right]$$

and

$$\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j|^q\right)^{\frac{1}{q}}$$

are decreasing functions of p.

One can derive analogous theorems for a related class of gauges; we state these here.

Theorem 10. Consider the gauge $\phi(x_1, \cdots, x_n) = \left(\sum_{j=1}^{n} \alpha_j x_j^2\right)^{\frac{1}{2}}$, where $\alpha_j > 0$ for all j. Then its conjugate is

$$\phi^*(x_1, \cdots, x_n) = \left(\sum_{j=1}^{n} \frac{1}{\alpha_j} x_j^2\right)^{\frac{1}{2}} .$$

Proof: The proof of the last is readily derived by using the Lagrange method for determining the maximum of a function over a surface. ▮

Now suppose that J is any fixed subset of the indices $(1,2,\cdots,n)$. For any number t such that $1 \leq t < \infty$, define $\phi_t(x_1,\cdots,x_n) = \left(\sum_{j=1}^{n} \alpha_j x_j^2\right)^{\frac{1}{2}}$, where $\alpha_j = t$ when $j \in J$ and $\alpha_j = 1$ when $j \notin J$. Varying t over $1 \leq t < \infty$, one obtains a family of gauges corresponding to a family of ellipsoids. Each fixed x determines an increasing function $\phi_t(x)$ of t. Moreover, $\phi_t(x) \geq e(x)$ for all x and all t $(1 \leq t < \infty)$. The analogue of Theorem 8 follows.

<u>Theorem 11</u>. Let $x \neq \theta$ be any fixed point in E^n. Then $\mathfrak{S}_t(x)$ is an increasing function of t $(1 \leq t < \infty)$.

<u>Proof</u>: Because the spread is constant on each ray, it is sufficient to consider an arbitrarily chosen point $u = (u_1,\cdots,u_n)$ of norm one. Using the fact that $\sum_{j=1}^{n} u_j^2 = 1$, one may readily verify that

$$\frac{d}{dt}\left(\mathfrak{S}_t(u)\right) = -\left(\sum_{j=1}^{n} \frac{1}{\alpha_j} u_j^2\right)^{-\frac{1}{2}} \left(\frac{1}{t^2} \sum_{j \in J} u_j^2\right)$$

$$+ \left(\sum_{j=1}^{n} \alpha_j u_j^2\right)^{-\frac{3}{2}} \left(\sum_{j \in J} u_j^2\right) \geq 0.$$

Consequently, $\mathfrak{S}_t(u)$ is an increasing function of t. ▮

The next theorem follows immediately.

<u>Theorem 12</u>. Given a subset J of the indices $(1,2,\cdots,n)$, define $\alpha_j = t$ when $j \in J$ and $\alpha_j = 1$ when $j \notin J$. Then for any fixed point $(x_1,\cdots,x_n)$, both

$$\tfrac{1}{2}\left[\left(\sum_{j=1}^{n} \alpha_j x_j^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} \frac{1}{\alpha_j} x_j^2\right)^{\frac{1}{2}}\right]$$

and

$$\left(\sum_{j=1}^{n} \alpha_j x_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \frac{1}{\alpha_j} x_j^2\right)^{\frac{1}{2}}$$

are increasing functions of t for $1 \leq t < \infty$.

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AUTOBIOGRAPHICAL STATEMENT

Charles D. Masiello was born in New York City on January 19, 1941. He received most of his early education in the New York City public schools. In 1956, he moved to New City, a small town in upstate New York, and completed his secondary education in the Clarkstown School System.

He received the Bachelor of Arts Degree from Hunter College in 1962, and the Master of Arts Degree in 1963. He then taught at that college for a brief period, before beginning his studies at the Graduate Center in 1964.