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EIGENFUNCTION EXPANSIONS IN L^p

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This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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INTRODUCTION

Let (X, μ, σ) be a measure space. We denote by $L^P(X, \mu, \sigma)$ the space of equivalence classes of complex valued functions, f , on X such that $\int_X |f|^P d\mu < \infty$, with the norm of f , $\|f\|_P \equiv (\int_X |f|^P d\mu)^{1/P}$.

Given a sequence of functions $\{f_n\}$, $n = 1, 2, \dots$ belonging to $L^P(X, \mu, \sigma)$ we say that $\{f_n\}_{n=1}^\infty$ is a basis if for every $g \in L^P(X)$, there exists a unique sequence of complex numbers $\{C_n\}_{n=1}^\infty$ such that $\lim_{N \rightarrow \infty} \sum_{n=1}^N C_n f_n = g$ in the topology of $L^P(X)$.

To be specific, let I represent an arbitrary interval of the real line. Let

$$L = a_n(x) \left(\frac{d}{dx}\right)^n + a_{n-1}(x) \left(\frac{d}{dx}\right)^{n-1} + \dots + a_0(x)$$

be a formal differential operator defined on $D_0 = \{f(x) : f(x) \in C^{n-1}(I) \text{ and } \left(\frac{d}{dx}\right)^{n-1} f \text{ is absolutely continuous on every compact subinterval of } I\}$.

In dealing with $L^2(I)$ one often considers the domain $D_1 = \{f(x) : f(x) \in D_0, f(x) \in L^2(I), \text{ and } Lf(x) \in L^2(I)\}$. Now in general the domain D_1 is too large to be useful, so one restricts the domain of L by defining boundary conditions ([5], Chapter XIII). For example, if I is the closed interval $[a, b]$ a typical boundary condition would be

$$U \equiv \sum_{j=1}^{n-1} \left(a_j \left(\frac{d}{dx}\right)^j f \Big|_{x=a} + b_j \left(\frac{d}{dx}\right)^j f \Big|_{x=b} \right) = 0.$$

In general, given a set of boundary conditions, $U_1, U_2, \dots, U_n$, one can associate with the formal differential operator L , an operator B such that the domain of $B = \{f(x) : f \in D_1 \text{ and } U_i(f) = 0, i = 1, 2, \dots, n\}$, with $Bf = Lf$ for $f \in \text{domain}(B)$. We note that domain (B) will always contain $C_0^\infty(I)$ (i.e., functions which are infinitely differentiable and vanish outside of a closed interval contained in the interior of I).

We say that $f(x)$ is a generalized eigenfunction of B if $\exists \lambda \in \mathbb{C}$ and a positive integer n such that $(\lambda I - B)^n f = 0$. If n can be taken as one, we will call $f(x)$ an eigenfunction of B .

Given $w(x) \geq 0$, $x \in I$, denote by $\|f\|_p w(x)$ the quantity $(\int_I |f(x)|^p w(x) dx)^{1/p}$. One can consider the question of whether the collection of generalized eigenfunctions, when ordered according to the magnitude of the corresponding eigenvalues, form a basis in $L^p(I)_{w(x)}$.

The classical example where $I = [0, 2\pi]$, $w(x) \equiv 1$, $L = -(\frac{d}{dx})^2$ with periodic boundary conditions $f(0) = f(2\pi)$, $f'(0) = f'(2\pi)$ gives

$\left\{ \frac{1}{2\pi} e^{inx} \right\}_{n=-\infty}^{+\infty}$ as the set of eigenfunctions. Being the eigenfunctions

of a self-adjoint differential operator this set is a complete orthogonal system and hence a basis in $L^2[0, 2\pi]$, ([3], Chapter 7). However, it is known that these functions are, in fact, a basis in $L^p[0, 2\pi]$ for $1 < p < \infty$ ([9], Chapter II).

Another example is given by the differential operator determined by $L = (\frac{d}{d\theta})^2 + 2\lambda \cot\theta (\frac{d}{d\theta})$, $\lambda > 0$, $\theta \in I = (0, \pi)$. The ultraspherical polynomials, $\{P_n^\lambda(\cos \theta)\}_{n=0}^\infty$ are eigenfunctions for L with corresponding eigenvalues $-n(n+2\lambda)$, ([11]). If we consider the weight function

$w(\theta) = (\sin \theta)^{2\lambda}$ then L becomes a formally self-adjoint formal differential operator with respect to $w(\theta)$, (i.e., if $f, g \in C_0^\infty(I)$, then

$\int_I (Lf) \bar{g} w(\theta) d\theta = \int_I f \overline{(Lg)} w(\theta) d\theta$). The sequence $\{P_n^\lambda(\cos(\theta))\}_{n=0}^\infty$ forms

a complete orthonormal system in $L^2(I)_{w(\theta)}$, and therefore a basis in

$L^2(I)_{w(\theta)}$. In addition, Pollard [15] has shown that $\{P_n^\lambda(\cos \theta)\}_{n=0}^\infty$

is a basis in $L^p(I)_{w(x)}$ for $\frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}$.

As a third example, let $I = (-\infty, +\infty)$ and $Lf(x) = (\frac{d}{dx})^2 f(x) - 2x(\frac{d}{dx})f(x)$.

If we take $w(x) = e^{-x^2}$ then L is formally self-adjoint, and the set of Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ forms a complete orthogonal system in $L^2(I)_{e^{-x^2}}$. However Pollard ([15]), has shown that for $p \neq 2$, $\{H_n(x)\}_{n=0}^{\infty}$ do not form a basis in $L^p(I)_{e^{-x^2}}$.

One way of showing that a given set of functions $\{f_n\}$ is a basis in L^p is by relating them to another set of functions $\{g_n\}$ which is known to be a basis in L^p .

In Section I we use an idea employed by Kato [8], concerning the gaps in the spectrum of an operator to establish the following perturbation result. Consider the formally self-adjoint formal differential operator $L = \left(i \frac{d}{dx}\right)^n$ defined on a finite closed interval I . Assume that with a suitable choice of boundary conditions L becomes a self-adjoint operator B ; i.e., for $f(x), g(x) \in \text{domain}(B)$, $(Bf, g) \equiv \int_I (Lf) \bar{g} dx = \int_I f \overline{(Lg)} dx \equiv (f, Bg)$. One knows that in this case there will be a discrete real spectrum, with only eigenfunctions. Now let us assume that the eigenfunctions of B when ordered according to the magnitude of the corresponding eigenvalues, form a basis in $L^p(I)$, ($p \geq 2$). Let $\tilde{B}$ be the differential operator

$$\tilde{L} = \left(i \frac{d}{dx}\right)^n + a_{n-2}(x) \left(\frac{d}{dx}\right)^{n-2} + \dots + a_0(x)$$

($a_i(x)$ bounded $i = 0, 1, \dots, n-2$) and the same boundary conditions as B . We prove that the generalized eigenfunctions of $\tilde{B}$ also form a basis in $L^p(I)$ when ordered according to the magnitude of the corresponding eigenvalues. We note that the absence of a term $a_{n-1}(x) \left(\frac{d}{dx}\right)^{n-1}$ is no great loss of generality. Indeed, let

$$\tau f(x) = \left(i \frac{d}{dx}\right)^n f(x) + a_{n-1}(x) \left(\frac{d}{dx}\right)^{n-1} f(x) + \dots + a_0(x) f(x) .$$

If $a_{n-1}(x) \in C^{n-1}$, let $Q(x) = \exp\left[-\frac{1}{n} \int_0^x a_{n-1}(x) dx\right]$, which is a solution of the equation $nQ'(x) + a_{n-1}(x)Q(x) = 0$, make the substitution $f(x) = Q(x)g(x)$. Then $\tau f(x) = Q(x)\tau'g(x)$ where τ' is a formal differential operator with no $(n-1)$ st derivative term. Now let $A: L^P(I) \rightarrow L^P(I)$ be defined by $Af = Q(x)f$. If we denote by S_N the n -th partial sum expansion in generalized eigenfunctions of τ (with given boundary conditions) and S'_N the n -th partial sum expansion corresponding to τ' , one has the following equality:

$$S_N f = (AS'_N A^{-1})f, \quad f \in L^P(I),$$

(see [1]). This shows that the L^P convergence of the generalized eigenfunction expansion for the two operators are equivalent.

We prove our main result by showing that

$$(1) \quad \lim_{n \rightarrow \infty} \|E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n)\|_{L^p \rightarrow L^p} = 0,$$

where Λ_n is the set composed of the first n eigenvalues of B , and $E(\Lambda_n)$ represents the projection of $L^P(I)$ onto the subspace spanned by the eigenfunctions associated with Λ_n , with similar definitions for $\tilde{E}$ and $\tilde{\Lambda}_n$. Thus if $\varphi_n(x)$, $n = 1, 2, \dots$, are the eigenfunctions of B , and $f(x) \sim \sum_{j=1}^{\infty} a_j \varphi_j(x)$, then $E(\Lambda_n)f(x) = \sum_{j=1}^n a_j \varphi_j(x)$. We establish (1) by analyzing the operator $R(\lambda; B) \equiv (\lambda I - B)^{-1}$, ($\lambda \in$ resolvent of B).

We show that $R(\lambda; B)$ can be considered not only as a mapping from $L^2(I)$ to $L^2(I)$ as it is defined to be, but also as a mapping from $L^P(I)$ to $L^P(I)$, $p > 2$. This is seen from the following three facts:

(a) For $p > 2$, $L^p(I) \subseteq L^2(I)$ continuously, by Holder's inequality (recall, we are dealing with a finite interval);

(b) We have that $R(\lambda; B)$ maps $L^2(I)$ into the Sobolov space $H^n(I)$;

(c) By Sobolov's theorem, $L^p(I) \subseteq H^n(I)$.

We use the gaps in the spectrum of B to obtain bounds for the norms $\|R(\lambda; B)\|_{L^2 \rightarrow L^2}$ and $\|R(\lambda; B)\|_{L^2 \rightarrow H^n}$. Then using an interpolation theorem of Calderon [2], we get bounds for $\|R(\lambda; B) - R(\lambda; \tilde{B})\|_{L^p \rightarrow L^p}$.

We then use the relations $E(\Lambda_n) = \frac{1}{2\pi i} \int_{R_n} R(\lambda; B) d\lambda$ and $\tilde{E}(\tilde{\Lambda}_n) = \frac{1}{2\pi i} \int_{\tilde{R}_n} R(\lambda; \tilde{B}) d\lambda$ where R_n and $\tilde{R}_n$ are simple closed contours that contain Λ_n and $\tilde{\Lambda}_n$ respectively. We show that for sufficiently large n we can choose $R_n = \tilde{R}_n$ and we have the estimate

$$\|E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n)\|_{L^p \rightarrow L^p} \leq \frac{1}{2\pi} \int_{R_n} \|R(\lambda; B) - R(\lambda; \tilde{B})\|_{L^p \rightarrow L^p} |d\lambda| .$$

We note that although we have treated only ordinary differential operators, the same methods apply to elliptic partial differential operators. The only problem is that except for a few special cases (such as when the domain D is an n dimensional rectangle) we have only asymptotic estimates for the number of eigenvalues whose magnitude is less than a given number, but we do not know that the gaps between two adjacent eigenvalues become arbitrarily large. However, since the gaps must "occasionally" be large, one could conclude that if the eigenfunctions of B are a basis in $L^p(D)$, then the eigenfunction expansion (relative to $\tilde{B}$) of $f(x) \in L^p(D)$ must have a fixed subsequence which converges to $f(x)$ in $L^p(D)$.

D.R. Smart [17], has also used the notion of gaps in the spectrum to obtain results similar to ours, but by using a different method of estimating the norm $\|R(\lambda; B)\|_{L^p \rightarrow L^p}$ on the contours described above. The main advantage of Smart's method over ours is that it applies to $p < 2$ as well as $p \geq 2$. The reason is that he estimates $\|R(\lambda; B)\|_{L^p \rightarrow L^p}$ directly (i.e., without using Sobolov's theorem, etc.) and therefore does not require the fact that $L^p \subseteq L^2$ continuously for $p \geq 2$. However, it appears that our method applies to a larger class of operators than his (c.f. [17], page 196).

Having established our perturbation theorem one may ask how to determine whether the eigenfunctions of our operator B are indeed a basis in $L^p(I)$. Sometimes this can be done by using the notion of equiconvergence. Two series $\sum_{n=1}^{\infty} a_n \varphi_n(x)$ and $\sum_{n=1}^{\infty} b_n \psi_n(x)$, $x \in I = [a, b]$ are said to be (uniformly) equiconvergent if

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N a_n \varphi_n(x) - b_n \psi_n(x) = 0$$

(uniformly in x). Now one can easily establish (see Section II below) that if two orthonormal sets of functions, $\{\varphi_n(x)\}_{n=1}^{\infty}$ and $\{\psi_n(x)\}_{n=1}^{\infty}$, have uniformly equiconvergent expansions for every $f(x) \in L^p(I)$, then $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a basis in $L^p(I)$ if and only if $\{\psi_n(x)\}_{n=1}^{\infty}$ is a basis. There are numerous equiconvergence theorems in the literature, (for example, see Szego, [19]), relating the eigenfunction expansion for a given differential operator with the ordinary Fourier expansion. In connection with this, we extend, in Section II, an idea of Walsh [21], to prove an equiconvergence result concerning the eigenfunction expansion with respect to the formal operator $L = -\left(\frac{d}{dx}\right)^2$ with various boundary con-

ditions, and the eigenfunction expansion for $\tilde{L} = -\left(\frac{d}{dx}\right)^2 + q(x)$ ($q(x)$ real) with the same boundary conditions as for L .

The hypothesis of Walsh's theorem is concerned with the expansion of one real complete orthonormal system with respect to another real orthonormal system. More precisely, let $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ be two complete real orthonormal systems, both systems uniformly bounded (i.e., $|u_n| \leq c$, $|v_n| \leq c$, $n = 1, 2, \dots$). Define $c_{nk} + \delta_{nk} =$

$\int_I v_n(x) u_k(x) dx$. Assume that

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |c_{nk}| \right)^2 < \infty.$$

The conclusion is that the eigenfunction expansions for $\{u_n(x)\}_{n=1}^{\infty}$ are uniformly equiconvergent; i.e., if $f \in L^2(I)$, and $f \sim \sum_{n=1}^{\infty} a_n u_n(x)$ and $f \sim \sum_{n=1}^{\infty} b_n v_n(x)$ then

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N (a_n u_n(x) - b_n v_n(x)) = 0$$

uniformly in x .

We also prove in this section a theorem which states that if we have two complete orthonormal systems $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ in $L^2(I)$, both uniformly bounded and such that $\sum_{n=1}^{\infty} \|u_n - v_n\|_2 < \infty$, then $\{u_n(x)\}$ is a basis in $L^p(I)$ if and only if $\{v_n(x)\}$ is a basis in $L^p(I)$.

The connection between this theorem and Walsh's theorem is that here we require

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |c_{nk}|^2 \right)^{\frac{1}{2}} < \infty.$$

In Section III we discuss the notion of generalized harmonic and conjugate harmonic functions. In the classical case, if we have a har-

monic functions $u(x,y)$ defined in the unit disc, its conjugate harmonic function satisfies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad v(0,0) = 0.$$

This notion of conjugacy is very important in the theory of Fourier series.

If T represents the circle group and $f \in L^1(T)$ has as its Fourier series $f(t) = \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{int}$, we consider the Poisson integral of $f(t)$ defined in the unit disc:

$$f(r,t) = p(r, \cdot) * f = \sum_{n=-\infty}^{+\infty} \hat{f}(n) r^n e^{int}.$$

Its conjugate harmonic function is

$$\tilde{f}(r,t) = \sum_{n=-\infty}^{+\infty} -i(\operatorname{sgn} n) \hat{f}(n) r^n e^{int}.$$

It is known (Katznelson [9], Chapter III) that $\lim_{r \rightarrow 1} \tilde{f}(r,t) = \tilde{f}(t)$ exists

for almost all T and that for $p > 1$, $\tilde{f}(t) \in L^p(T)$ if and only if $f(t) \in L^p(T)$. $\tilde{f}(t)$ is called the conjugate of $f(t)$, and for $1 < p < \infty$ we have $\|\tilde{f}\|_p \leq A_p \|f\|_p$. The boundedness of this conjugate mapping $f \rightarrow \tilde{f}$ is used to show that the functions $\{e^{int}\}_{n=-\infty}^{+\infty}$ are a basis in $L^p(T)$.

Following the examples of Stein and Muckenhoupt, [12], and Muckenhoupt, [11], we generalize the notion of conjugate harmonic functions for second order ordinary differential equations, which leads to the generalized conjugate mapping described by Stein, [18]. Thus, let

$$L = \left(\frac{d}{dx}\right)^2 + a(x) \frac{d}{dx}, \quad x \in I. \quad \text{Let } D = I \times [0, \infty]. \quad \text{We say that } u(x,y)$$

$((x,y) \in D)$ is harmonic for L if

$$\left(\frac{\partial}{\partial x}\right)^2 u(x,y) + a(x)\left(\frac{\partial}{\partial x}\right) u(x,y) + \left(\frac{\partial}{\partial y}\right)^2 u(x,y) = 0 .$$

We say that $v(x,y)$ is conjugate harmonic to $u(x,y)$ if v satisfies:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} ; \quad \frac{\partial u}{\partial y} = - \left(\frac{\partial v}{\partial x} + a(x)v \right)$$

and $\lim_{y \rightarrow \infty} v(x,y) = 0$.

Using ideas developed by Stein [18], we show in a special case that the boundedness of the conjugate mapping can be generalized:

$$\|v(\cdot, y)\|_p \leq A_p \|u(\cdot, y)\|_p , \quad (1 < p < \infty) .$$

It does not appear however that the boundedness of the conjugate mapping that we have described can be used in general to yield L^p convergence of eigenfunction expansions, as in the classical case, since for example, Muckenhoupt, [11], has shown that for the case $L = \left(\frac{d}{dx}\right)^2 - 2x \frac{d}{dx}$ which yields the Hermite polynomials as eigenfunctions, conjugation is a bounded mapping with respect to the weight function $w(x) = e^{-x^2}$. However, as was pointed out above, the Hermite polynomials are not a basis in $L^p(-\infty, \infty)_{w(x)}$, for $p \neq 2$.

One of the powerful theorems proven by Stein, which we use is the following abstract Littlewood-Paley inequality: Let $(M, d\mu)$ be a positive measure space and $\{T^t\}_{0 \leq t < \infty}$ a semigroup of operators mapping functions on M to functions on M . Suppose the following four conditions are satisfied.

- (I) $\|T^t f\|_p \leq \|f\|_p$.
- (II) Each T^t is a self-adjoint operator on $L^2(M)$.
- (III) $T^t f \geq 0$ if $f \geq 0$.
- (IV) $T^t 1 = 1$.

For $f(x) \in L^p(M)$, define

$$g_1(f)(x) = \left(\int_0^\infty t \left| \frac{\partial}{\partial t} T^t f(x) \right|^2 dt \right)^{\frac{1}{2}}.$$

Then

$$\|g_1(f)\|_p \leq A_p \|f\|_p.$$

As a direct application of this inequality, consider the following theorem.

Theorem: Given $f(\theta)$, $0 < \theta < \pi$, $f(\theta) \in L^p(0, \pi)$ with respect to the measure $dm_\lambda(\theta) = (\sin\theta)^{2\lambda} d\theta$. Let $\{p_n^\lambda(\cos\theta)\}_{n=0}^\infty$ be the set of ultraspherical polynomials corresponding to the index λ . Let $f(\theta) \sim \sum_{n=0}^\infty a_n p_n^\lambda(\cos\theta)$ and form the function $f(r, \theta) = \sum_{n=0}^\infty a_n r^n p_n^\lambda(\cos\theta)$, $0 \leq r \leq 1$. Let

$$g^2(\theta) = \int_0^1 (1-r) \left| \left(\frac{\partial}{\partial r} \right) f(r, \theta) \right|^2 dr.$$

Then $\|g(\theta)\|_p \leq A_p \|f(\theta)\|_p$.

Proof: It is known that $f(r, \theta) = \int_0^\pi P(r, \theta, \varphi) f(\varphi) (\sin\varphi)^{2\lambda} d\varphi$ where

$$P(r, \theta, \varphi) = \frac{\lambda}{\pi} (1-r^2) \int_0^\pi \frac{(\sin t)^{2\lambda-1} dt}{[1-2r(\cos\theta \cos\varphi + \sin\theta \sin\varphi \cos t) + r^2]^{\lambda+1}}.$$

The theorem has been proven by Stein and Muckenhoupt, [12], by obtaining complicated estimates on the kernel $P(r, \theta, \varphi)$. However, we show how the theorem follows using Stein's Littlewood-Paley theorem.

We first form the function $u(\theta, t) = f(e^{-t}, \theta)$. From the properties of the mappings $Tr: f(\theta) \rightarrow f(r, \theta)$ we know that the mappings $T^t: f(\theta) \rightarrow u(\theta, t)$ form a semigroup and satisfy the axioms (I) to (IV) stated above. We can thus conclude using the Littlewood-Paley inequality that if $(g^*(\theta))^2 \equiv \int_0^t t \left| \left(\frac{\partial}{\partial t} \right) u(\theta, t) \right|^2 dt$ then $\|g^*(\theta)\|_p \leq A_p \|f(\theta)\|_p$.

Now making the change of variables $r = e^{-t}$ we obtain

$$(g^*(\theta))^2 = \int_0^1 r |\ln r| \left| \left(\frac{\partial}{\partial r} \right) f(r, \theta) \right|^2 d\theta .$$

We have that

$$\left(\frac{\partial}{\partial r} \right) f(r, \theta) = \int_0^\pi \left(\frac{\partial}{\partial r} \right) P(r, \theta, \varphi) f(\varphi) (\sin \varphi)^{2\lambda} d\varphi$$

and so

$$\left| \left(\frac{\partial}{\partial r} \right) f(r, \theta) \right|^2 = \left| \int_0^\pi \left(\frac{\partial}{\partial r} \right) P(r, \theta, \varphi) f(\varphi) (\sin \varphi)^{2\lambda} d\varphi \right|^2 .$$

We have

$$\begin{aligned} (g(\theta))^2 &= \int_0^{\frac{1}{2}} (1-r) \left| \left(\frac{\partial}{\partial r} \right) f \right|^2 dr + \int_{\frac{1}{2}}^1 (1-r) \left| \left(\frac{\partial}{\partial r} \right) f \right|^2 dr \\ &\equiv g_1^2(\theta) + g_2^2(\theta) . \end{aligned}$$

This implies that

$$\|g(\theta)\|_p \leq \|g_1(\theta)\|_p + \|g_2(\theta)\|_p .$$

Now it is easy to verify from the expression for $P(r, \theta, \varphi)$ that

$$\left| \left(\frac{\partial}{\partial r} \right) P(r, \theta, \varphi) \right| \leq c_1 \quad \text{for } 0 \leq r \leq \frac{1}{2} . \quad \text{This implies that } g_1(\theta) \leq c_1 \|f\|_1 .$$

Thus $\|g_1(\theta)\|_p \leq c_1 \|f(\theta)\|_1 \leq c_2 \|f(\theta)\|_p$ (by Holder's inequality). Now

for $r \geq \frac{1}{2}$ we have that $1-r \leq cr |\ln r|$. Therefore,

$$\|g_2(\theta)\|_p \leq c \|g^*(\theta)\|_p \leq A_p \|f(\theta)\|_p .$$

We thus obtain

$$\|g(\theta)\|_p \leq (c_2 + A_p) \|f(\theta)\|_p .$$

SECTION I

PERTURBATION OF A SELF-ADJOINT DIFFERENTIAL OPERATOR

Let B be a Banach space and $R: B \rightarrow B$ be a densely defined unbounded closed linear operator. Let $\sigma(T)$ be the spectrum of T and $\rho(T)$ be the resolvent set of T . (For definitions of terms and proofs of assertions dealing with operator theory, see Dunford and Schwartz, [5]). For $\lambda \in \rho(T)$, let $R(\lambda; T) = (\lambda I - T)^{-1}$. Now assume that T has compact resolvent; i.e., for any $\lambda \in \rho(T)$, $R(\lambda; T)$ is a compact operator. (We recall that if there exists a $\lambda \in \rho(T)$ such that $R(\lambda; T)$ is compact, then for any other $\mu \in \rho(T)$, $R(\mu; T)$ is also compact. This follows from the formula $R(\mu; T) = R(\lambda; T) + (\lambda - \mu)R(\lambda; T)R(\mu; T)$ and the fact that the product of a compact operator and a bounded operator is a compact operator). The spectrum of T is therefore countably infinite, consisting entirely of eigenvalues λ_i , $i = 1, 2, \dots$, which are poles of $R(\lambda; T)$. Now to each eigenvalue λ_i , $i = 1, 2, \dots$, we can associate a projection P_i of B onto the T -invariant subspace generated by the generalized eigenvectors (see Introduction) corresponding to λ_i .

P_i has the form $P_i f = \sum_{\text{finite}} \psi_k(f) \phi_k$, where $\psi_k \in B^*$ (the dual of B) and $\langle \psi_k, \phi_j \rangle = \delta_{k,j}$.

Let D_{λ_i} be a circle in the complex plane with center λ_i such that all other eigenvalues lie exterior to the circle. Then it follows that $P_i f = \frac{1}{2\pi i} \int_{D_{\lambda_i}} R(\lambda; T) f d\lambda$.

Now let D_{0, r_n} be the circle centered at the origin with radius $r_n \uparrow \infty$ and such that no λ_i lies on D_{0, r_n} . It follows that

$$\sum_{|\lambda_k| \leq r_n} \langle \psi_k, f \rangle \varphi_k = \frac{1}{2\pi i} \int_{D_0, r_n} R(\lambda; T) f d\lambda.$$

We thus conclude that the sequence $\{\varphi_k\}_{k=1}^{\infty}$ is a basis in

$$B \ni \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{D_0, r_n} R(\lambda; T) f d\lambda = f \text{ in } B.$$

Let L denote the ordinary differential operator: $L(f) = (-1)^m \left(\frac{d}{dx}\right)^{2m}$, $x \in I = [0, \pi]$. (We assume L is of even order merely for convenience).

Let U_i , $i = 1, 2, \dots, n$ be n independent boundary conditions.

Thus we may write,

$$U_i(f) = \sum_{j=0}^{n-1} (a_{ij} f^{(j)}(0) + b_{ij} f^{(j)}(\pi))$$

a_{ij} , b_{ij} being constants. When the U_i collectively satisfy a certain determinant condition, the boundary conditions are called regular (see [13], page 56). For regular boundary conditions, one knows the asymptotic behavior of the eigenvalues with sufficient precision for our purpose. In fact, one has the following result (see [13], page 64).

Theorem: Let $\tilde{L}$ be a formal differential operator of the form

$$\tilde{L} = (-1)^m \left(\frac{d}{dx}\right)^{2m} + p_{2m-2} \left(\frac{d}{dx}\right)^{2m-2} + p_{2m-3} \left(\frac{d}{dx}\right)^{2m-3} + \dots + p_0$$

where $p_j(x)$ is bounded on I , $j = 0, 1, 2, \dots, 2m-2$, and let U_i , $i = 1, 2, \dots, n$ be a set of regular boundary conditions. Then the eigenvalues, μ_n , $n = 1, 2, \dots$, for $\tilde{L}$, with the given boundary conditions, satisfy the following asymptotic relationship:

$$(1) \quad \mu_n = n^{2m} \left\{ 1 + \frac{A}{n} + O\left(n^{-\frac{3}{2}}\right) \right\}; \quad A \text{ is a constant.}$$

Let $\{\varphi_j(x)\}_{j=1}^{\infty}$ be the generalized eigenfunctions for the formal

operator L with a set of regular boundary conditions, ordered according to increasing magnitude of the corresponding eigenvalues. Let $\{\psi_j(\lambda)\}_{j=1}^{\infty}$ be the corresponding generalized eigenfunctions for $\tilde{L}$ with the same boundary conditions as for L . Given a positive integer n define,

$$H^n(I) = \left\{ f: f \in C^{n-1}(I) \text{ and } \left(\frac{d}{dx} \right)^n f \in L^2(I) \right\}.$$

Let B be the operator with Domain $(B) = \{f: f \in H^{2m} \text{ and } U_i f = 0, i = 1, 2, \dots, n\}$, with $Bf = Lf$. Let K be the operator with the same domain as B and $Kf = p_{2m-2} \left(\frac{d}{dx} \right)^{2m-2} f + \dots + p_0 f$. Thus for $f \in \text{Domain}(B+K)$, $(B+K)f = \tilde{L}f$.

We will eventually prove the following:

Theorem 1: Let B be self-adjoint. If $\{\varphi_j(x)\}_{j=1}^{\infty}$ is a basis in $L^p[I]$ for $2 \leq p$, then $\{\psi_j\}_{j=1}^{\infty}$ is a basis in $L^p[0, \pi]$ and conversely.

Definition 1: For $f \in H^n$, n a non-negative integer, one defines a norm

$$\|f\|_{H^n} = \left(\sum_{j=1}^n \left\| \left(\frac{d}{dx} \right)^j f \right\|^2 \right)^{\frac{1}{2}}.$$

If s is an arbitrary positive number, we define the space H^s as an interpolation space (see Calderon [2]).

Lemma 1: Consider the vertical line

$$\mathcal{L}_n = \left\{ \lambda : \text{Re} \lambda = \frac{\mu_n + \mu_{n+1}}{2} \right\},$$

where μ_n is the n th eigenvalue of B . Then for sufficiently large n , $\mathcal{L}_n \subseteq \rho(B+K)$.

Proof: It suffices to show that for λ satisfying the given conditions,

$R(\lambda; B)$ exists and $\|K \circ R(\lambda; B)\|_{H^0 \rightarrow H^0} < 1$, for then we would have that $[I - K \circ R(\lambda; B)]^{-1}$ exists and could invoke the identity:

$$R(\lambda; B+K) = R(\lambda; B)[I - K \circ R(\lambda; B)]^{-1}.$$

We will obtain bounds for $\|K \circ R(\lambda; B)\|_{H^0 \rightarrow H^0}$ by means of bounds for $\|R(\lambda; B)\|_{H^0 \rightarrow H^{2m-2}}$ (i.e., the norm of the operator $R(\lambda; B)$ from the space H^0 to the space H^{2m-2}) and bounds for $\|K\|_{H^{2m-2} \rightarrow H^0}$.

First let us deal with the operator K . We observe that

$$\left\| \left(\frac{d}{dx} \right)^r f(x) \right\|_{H^s} \leq \|f(x)\|_{H^{r+s}}$$

(r, s positive integers). Since we are assuming that $p_i(x)$, $i = 0, 1, \dots, 2m-2$ is bounded (say $|p_i(x)| < M$), it follows easily that

$$(2) \quad \|K\|_{H^{2m-2} \rightarrow H^0} \leq (2m-2)M.$$

We now observe that equation (1) for the asymptotic value of the eigenvalues implies that for n sufficiently large, $\ell_n \subseteq \rho(B)$. So consider $R(\lambda; B)$ for $\lambda = \frac{\mu_{n+1} + \mu_n}{2} + iy \in \ell_n$. Since B is self-adjoint the Spectral Theorem (see [5]) tells us that $\|R(\lambda; B)\|_{H^0 \rightarrow H^0} =$ the reciprocal of the distance from λ to $\sigma(B)$. We thus have that

$$\|R(\lambda; B)\|_{H^0 \rightarrow H^0} = \left[\left(\frac{\mu_{n+1} - \mu_n}{2} \right)^2 + y^2 \right]^{-\frac{1}{2}}.$$

For n sufficiently large, we see from (1) that

$$(3) \quad \|R(\lambda; B)\|_{H^0 \rightarrow H^0} \leq c \left[(n^{2m-1})^2 + y^2 \right]^{-\frac{1}{2}}.$$

Now one knows from the theory of elliptic operators (see [5], Chapter XIV), which our operator L trivially is since the leading

coefficient of L is identically one, that $R(\lambda; B)$ is a bounded mapping from H^0 to H^{2m} . We want to estimate the norm of this mapping. Now (1) tells us that $\mu_n \rightarrow +\infty$ as $n \rightarrow \infty$. We can therefore choose λ_0 real so that $\mu_n > \lambda_0 \forall n$. We now express $R(\lambda; B)$ as

$$R(\lambda; B) = R(\lambda_0; B) \circ (\lambda_0 I - B) \circ R(\lambda; B).$$

$$\begin{aligned} \text{We have } \|R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} &= \|R(\lambda_0; B) \circ (\lambda_0 I - B) \circ R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \\ &\leq \|R(\lambda_0; B)\|_{H^0 \rightarrow H^{2m}} \cdot \|(\lambda_0 I - B) \circ R(\lambda; B)\|_{H^0 \rightarrow H^0} \\ &= c \|(\lambda_0 I - B) \circ R(\lambda; B)\|_{H^0 \rightarrow H^0}. \end{aligned}$$

Spectral theory tells us that

$$\|(\lambda_0 I - B) \circ R(\lambda; B)\|_{H^0 \rightarrow H^0} = \sup_{Z \in \sigma(B)} \left| \frac{\lambda_0 - Z}{\lambda - Z} \right|.$$

We claim that for $\lambda \in \ell_n$,

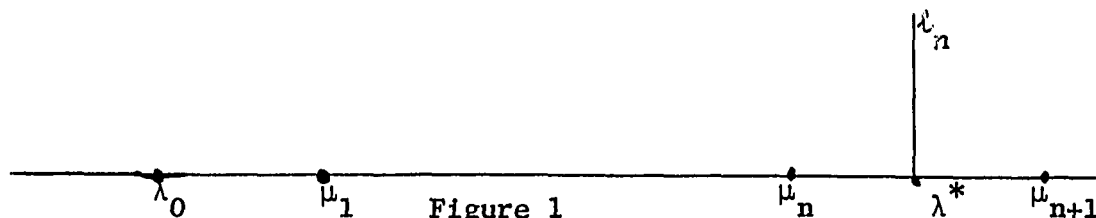
$$\sup_{Z \in \sigma(B)} \left| \frac{\lambda_0 - Z}{\lambda - Z} \right| \leq cn.$$

To see this we note that $\sigma(B)$ is real since B is self-adjoint, and therefore for $Z \in \sigma(B)$,

$$\left| \frac{\lambda_0 - Z}{\lambda - Z} \right| \leq \left| \frac{\lambda_0 - Z}{\operatorname{Re} \lambda - Z} \right|.$$

So we can confine our attention to the point $\lambda^* = \frac{\mu_{n+1} + \mu_n}{2}$ (see Figure 1).

As $z \uparrow \infty$ $\left| \frac{\lambda_0 - Z}{\lambda^* - Z} \right|$ is monotonically increasing in the interval (λ_0, λ^*) and



$\left| \frac{\lambda_0 - z}{\lambda^* - z} \right|$ is monotonically decreasing in the interval $(\lambda^*, +\infty)$. It follows easily that

$$\sup_{z \in \sigma(B)} \left| \frac{\lambda_0 - z}{\lambda^* - z} \right| = \left| \frac{\lambda_0 - \mu_{n+1}}{\lambda^* - \mu_{n+1}} \right| \sim \frac{(n+1)^{2m}}{(n+1)^{2m-1}} \leq cn.$$

(4) Thus, $\|R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \leq cn, \quad \lambda \in \ell_n.$

We now invoke the following interpolation theorem of Calderon [2]:

Theorem: If T is a bounded mapping from H^{s_1} to H^{t_1} with norm P , and also a bounded mapping from H^{s_2} to H^{t_2} with norm Q , then for $0 < \alpha < 1$, T is a bounded mapping from $H^{\alpha s_1 + (1-\alpha)s_2}$ to $H^{\alpha t_1 + (1-\alpha)t_2}$ with norm $\leq P^\alpha Q^{1-\alpha}$. Taking $s_1 = 0, s_2 = 0, t_1 = 0, t_2 = 2m, \alpha = 1/m$ and using 2) and 4) we obtain

(5) $\|R(\lambda; B)\|_{H^0 \rightarrow H^{2m-2}} \leq c \left[(n^{2m-1})^2 + y^2 \right]^{-\frac{1}{2m}} \cdot n^{\frac{m-1}{m}}.$

The right hand side clearly converges to zero as $n \uparrow \infty$. This fact and (2) show that $\|K R(\lambda; B)\|_{H^0 \rightarrow H^0}$ converges to zero as $n \uparrow \infty$. As noted at the beginning of the proof, this establishes the lemma.

Lemma 2: Let R_n denote the rectangle formed as follows:

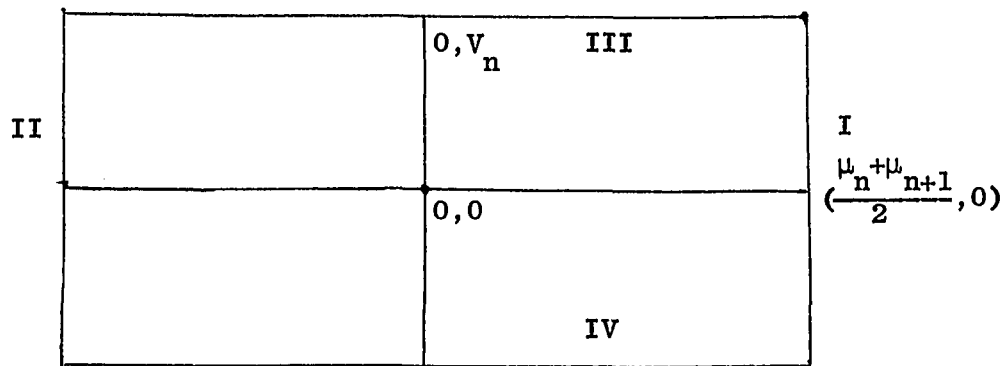


Figure 2

$$\text{Side I: } \operatorname{Re} \lambda = \frac{\mu_n + \mu_{n+1}}{2} \quad (\text{i.e., } \lambda \in \ell_n) ,$$

$$\text{Side II: } \operatorname{Re} \lambda = - \left(\frac{\mu_n + \mu_{n+1}}{2} \right) ,$$

$$\text{Side III: } \operatorname{Im} \lambda = V_n ,$$

$$\text{Side IV: } \operatorname{Im} \lambda = -V_n .$$

Then given $\epsilon > 0$ and $0 \leq t < \frac{1}{2m+1}$, we have that for sufficiently large n , $\exists V_n$ sufficiently large,

$$\int_{R_n} \|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}} |d\lambda| < \epsilon .$$

Proof: We first show that as $n \uparrow \infty$,

$$\int_{\ell_n} \|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}} \rightarrow 0 , \quad 0 \leq t < \frac{1}{2m+1} .$$

Let $\lambda \in \ell_n$. We have already shown in Lemma 1 that for n sufficiently large, $\lambda \in \rho(B+K)$. We will first estimate

$$\|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^0} .$$

We have that $R(\lambda; B+K) - R(\lambda; B) = R(\lambda; B)(I - K \circ R(\lambda; B))^{-1} - R(\lambda; B)$. We have shown that $\|K \circ R(\lambda; B)\|_{H^0 \rightarrow H^0} \rightarrow 0$ as $n \uparrow \infty$. Now when $\|K \circ R(\lambda; B)\|_{H^0 \rightarrow H^0} < 1$ we can expand $(I - K \circ R(\lambda; B))^{-1}$ in a Neumann series (see [4], page 140).

Thus

$$(I - K \circ R(\lambda; B))^{-1} = \sum_{j=0}^{\infty} (K \circ R(\lambda; B))^j ,$$

with convergence in the uniform $H^0 \rightarrow H^0$ operator topology. So we have

$$\begin{aligned} \|R(\lambda; B)(I - K \circ R(\lambda; B))^{-1} - R(\lambda; B)\|_{H^0 \rightarrow H^0} &= \|R(\lambda; B) \sum_{j=1}^{\infty} (K \circ R(\lambda; B))^j\|_{H^0 \rightarrow H^0} \\ &\leq \|R(\lambda; B)\|_{H^0 \rightarrow H^0} \left\| \sum_{j=1}^{\infty} (K \circ R(\lambda; B))^j \right\|_{H^0 \rightarrow H^0} \end{aligned}$$

$$\leq \|R(\lambda; B)\|_{H^0 \rightarrow H^0} \sum_{j=1}^{\infty} \|K^{\circ} R(\lambda; B)\|_{H^0 \rightarrow H^0}^j$$

(since for a bounded operator T , $\|T^j\| \leq \|T\|^j$)

$$\leq c \|R(\lambda; B)\|_{H^0 \rightarrow H^0} \|K^{\circ} R(\lambda; B)\|_{H^0 \rightarrow H^0}$$

for n sufficiently large since $\|K^{\circ} R(\lambda; B)\|_{H^0 \rightarrow H^0} \rightarrow 0$ as $n \uparrow \infty$.

Using (2), (3), and (5), we therefore obtain

$$(6) \quad \|R(\lambda; B + K) - R(\lambda; B)\|_{H^0 \rightarrow H^0} \leq c \left[(n^{2m-1})^2 + y^2 \right]^{-\frac{1}{2}(1+\frac{1}{m})} \cdot n^{\frac{m-1}{m}}.$$

Now let us estimate $\|R(\lambda; B)(I - K^{\circ} R(\lambda; B))^{-1} - R(\lambda; B)\|_{H^0 \rightarrow H^{2m}}$. We have

that

$$\begin{aligned} & \|R(\lambda; B)(I - K^{\circ} R(\lambda; B))^{-1} - R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \\ & \leq \|R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \cdot \|(I - K^{\circ} R(\lambda; B))^{-1}\|_{H^0 \rightarrow H^0} \\ & \quad + \|R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \\ & \leq cn \end{aligned}$$

(by (4) for sufficiently large n . We deduce,

$$(7) \quad \|R(\lambda; B + K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \leq cn.$$

Interpolating between (6) and (7) we get for $0 < t < 1$

$$(8) \quad \|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}} \leq c \left[\left[(n^{2m-1})^2 + y^2 \right]^{-\frac{1}{2}(1+\frac{1}{m})} n^{\frac{m-1}{m}} \right]^{1-t} n^t.$$

Letting $y = n^{2m-1} y'$ we have that

$$\begin{aligned}
 & n^{\left(\frac{m-1}{m}\right)(1-t)+t} \int_{-\infty}^{\infty} \left[(n^{2m-1})^2 + y^2 \right]^{-\frac{1}{2}\left(1+\frac{1}{m}\right)(1-t)} dy \\
 &= n^{\left(\frac{m-1}{m}\right)(1-t)+t-(2m-1)\left(1+\frac{1}{m}\right)(1-t)+2m-1} \int_{-\infty}^{\infty} \left[1+(y')^2 \right]^{-\frac{1}{2}\left(1+\frac{1}{m}\right)(1-t)} dy' .
 \end{aligned}$$

Now for $t < \frac{1}{2m+1}$ we have that the integral on the right converges and that the exponent of n outside the integral is negative. We thus conclude that for $t < \frac{1}{2m+1}$,

$$(9) \quad \lim_{n \rightarrow \infty} \int_{\ell_n} \|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}} = 0 .$$

We now observe that for the line $-\ell_n$ (i.e., $\operatorname{Re} \lambda = -\left(\frac{\mu_n + \mu_{n+1}}{2}\right)$) we can deduce the very same estimates as for the line ℓ_n , and therefore also conclude that:

$$(10) \quad \int_{-\ell_n} \|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (t < \frac{1}{2m+1}) .$$

As a matter of fact, since $-\ell_n$ is "further away" from $\sigma(B)$ than is ℓ_n , one can actually get better estimates for $\|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^0}$ and $\|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}}$.

We now turn our attention to sides III and IV. Let us fix n . We intend to show that given $\epsilon > 0$, we can choose V_n (see Figure 2) sufficiently large so that $\|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}} < \epsilon$, where $t < 1$ and $\lambda \in$ sides III or IV (i.e., $\lambda = x \pm iV_n$, $|x| \leq \frac{\mu_n + \mu_{n+1}}{2}$).

Now by the Spectral Theorem we have that,

$$(11) \quad \|R(\lambda; B)\|_{H^0 \rightarrow H^0} \leq \frac{1}{|y|} .$$

We claim that

$$(12) \quad \|R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \leq c \quad \text{as} \quad |y| \uparrow \infty.$$

To establish (12) we again use the inequality

$$\|R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \leq \|R(\lambda_0; B)\|_{H^0 \rightarrow H^{2m}} \|(\lambda_0 I - B)R(\lambda; B)\|_{H^0 \rightarrow H^0}$$

(where λ_0 is as in Lemma 1). Now again by the Spectral Theorem,

$$\|(\lambda_0 I - B) \circ R(\lambda; B)\|_{H^0 \rightarrow H^0} = \sup_{z \in \sigma(B)} \left| \frac{\lambda_0 - z}{\lambda - z} \right|.$$

We claim that for $|y|$ sufficiently large

$$\sup_{z \in \sigma(B)} \left| \frac{\lambda_0 - z}{\lambda - z} \right| < 2$$

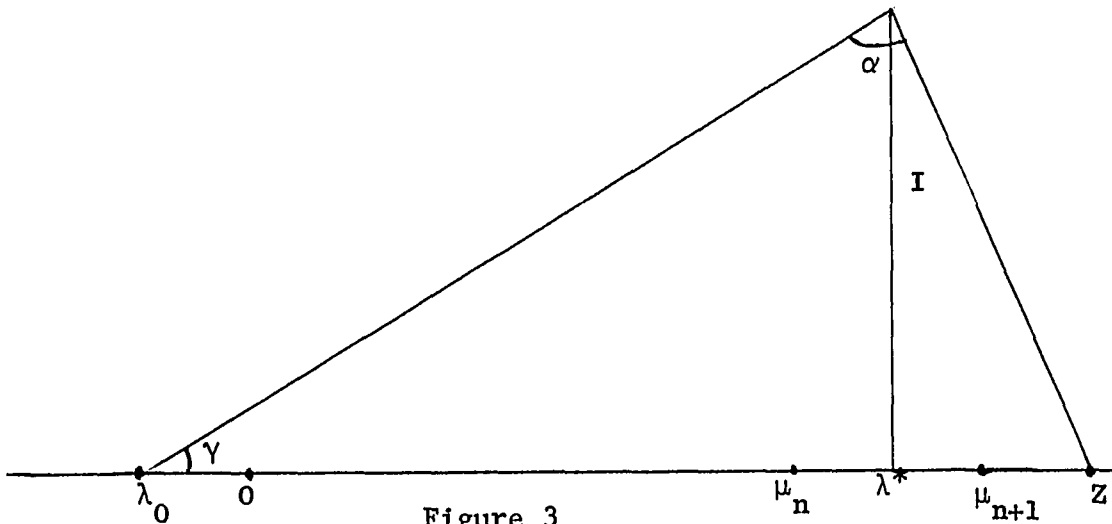


Figure 3

We have that $\sin \gamma \geq \frac{|y|}{\left[|y|^2 + \left(|\lambda_0| + \frac{\mu_n + \mu_{n+1}}{2} \right)^2 \right]^{\frac{1}{2}}}$ and this last

expression converges to one as $|y| \uparrow \infty$. It follows that

$$\left| \frac{\lambda_0 - z}{\lambda - z} \right| = \frac{\sin \alpha}{\sin \gamma} < 2$$

for $|y|$ sufficiently large. Thus (12) is established.

From (11) and (12) we conclude using the interpolation theorem stated in Lemma 1 that $\|R(\lambda; B)\|_{H^0 \rightarrow H^{2m-2}} \rightarrow 0$ as $|y| \rightarrow \infty$, and so, by (2),

$$\|K \circ R(\lambda; B)\|_{H^0 \rightarrow H^0} \rightarrow 0 \text{ as } |y| \rightarrow \infty. \text{ Thus for } |y| \text{ sufficiently large,}$$

$\lambda \in \rho(B+K)$ with $R(\lambda; B+K) = R(\lambda; B) \circ (I - K \circ R(\lambda; B))^{-1}$. So we have

$$\begin{aligned} \|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^0} &\leq \|R(\lambda; B)\|_{H^0 \rightarrow H^0} \cdot \|(I - K \circ R(\lambda; B))^{-1}\|_{H^0 \rightarrow H^0} \\ &\quad + \|R(\lambda; B)\|_{H^0 \rightarrow H^0} \end{aligned}$$

which converges to zero as $|y| \uparrow \infty$. Also,

$$\begin{aligned} \|R(\lambda; B+K)\|_{H^0 \rightarrow H^{2m}} &\leq \|R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \cdot \|(I - K \circ R(\lambda; B))^{-1}\|_{H^0 \rightarrow H^0} \\ &\quad + \|R(\lambda; B)\|_{H^0 \rightarrow H^{2m}}. \end{aligned}$$

By (12) and the fact that $\|(I - K \circ R(\lambda; B))^{-1}\|_{H^0 \rightarrow H^0} \rightarrow 1$ as $|y| \uparrow \infty$

(since $\|K \circ R(\lambda; B)\|_{H^0 \rightarrow H^0} \rightarrow 0$ as $|y| \uparrow \infty$), we conclude that

$$\|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2m}} \leq c. \text{ By the interpolation theorem again we}$$

have that if $t < 1$, then $\|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}} \rightarrow 0$ as $|y| \rightarrow \infty$,

and this convergence is uniform in x for $|x| \leq \frac{\mu + \mu_{n+1}}{2}$ since all our

estimates are uniform in this range.

It now follows that we can make the integral

$$\int_{\text{side III or IV}} \|R(\lambda; B+K) - R(\lambda; B)\|_{H^0 \rightarrow H^{2mt}} |d\lambda|, \quad (t < \frac{1}{2m+1})$$

arbitrarily small by taking $|y|$ sufficiently large. This, together with (9) and (10) prove the lemma.

Proof of Theorem 1: Suppose the eigenfunctions $\{\varphi_j(x)\}_{j=1}^{\infty}$ of B are a basis in $L^p(I)$, $p > 2$. Let $\Lambda_n = \{\lambda: \lambda \in \sigma(B) \text{ and } -(\frac{\mu + \mu_{n+1}}{2}) < \operatorname{Re} \lambda < (\frac{\mu + \mu_{n+1}}{2})\}$ and R_n be the rectangle in Lemma 2 such that $\Lambda_n \subset R_n$.

Let $E(\Lambda_n)$ be the projection given by $E(\Lambda_n)f = \frac{1}{2\pi i} \int_{R_n} (R(\lambda; B)f) d\lambda$.

Since $\{\varphi_j(x)\}_{j=1}^{\infty}$ is a basis in $L^p(I)$ and $E(\Lambda_n)f$ represents the partial sum expansion of $f(x)$ with respect to $\{\varphi_j(x)\}_{j=1}^{\infty}$, it follows that $E(\Lambda_n)f \rightarrow f$ in $L^p(I)$. If we want to show that the associated eigenfunctions of $B + K$, $\{\psi_j(x)\}_{j=1}^{\infty}$, are a basis in $L^p(I)$, we claim it suffices to show that

$$\|E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n)\|_{L^p \rightarrow L^p} \rightarrow 0$$

as $n \uparrow \infty$, where $\tilde{\Lambda}_n = \sigma(B+K) \cap \text{interior } R_n$ and

$$\tilde{E}(\tilde{\Lambda}_n)f = \frac{1}{2\pi i} \int_{R_n} R(\lambda; B+K) d\lambda,$$

n sufficiently large so that $R(\lambda; B+K)$ exists if $\lambda \in R_n$. To see this, let

$$f(x) \sim \sum_{n=1}^{\infty} a_n \varphi_n(x) \text{ and } f(x) \sim \sum_{n=1}^{\infty} b_n \psi_n(x).$$

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \|f(x) - \sum_{j=1}^n b_j \psi_j(x)\|_p &= \lim_{n \rightarrow \infty} \|f(x) - \sum_{j=1}^n a_j \varphi_j(x) \\ &+ \sum_{j=1}^n (b_j \psi_j(x) - a_j \varphi_j(x))\|_p \leq \lim_{n \rightarrow \infty} \|f - \sum_{j=1}^n a_j \varphi_j(x)\|_p \\ &+ \left\| \sum_{j=1}^n (b_j \psi_j(x) - a_j \varphi_j(x)) \right\|_p = 0. \end{aligned}$$

Now to prove that $\|E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n)\|_{L^p \rightarrow L^p} \rightarrow 0$ as $n \uparrow \infty$, we will need

the following theorem of Sobolov ([5], Chapter XIV).

Theorem: $H^s \subseteq L^p$ continuously if $\frac{1}{p} > \frac{1}{2} - s$.

We take s in this theorem to be $1/2$. Now let us define $S_{\frac{1}{2}, p}$ to be the bounded mapping from $H_{\frac{1}{2}}$ to L^p , and define $T_{p, 2}$ to be the restriction map from $L^p(I)$ to $L^2(I)$, which is well-defined and continuous by Holder's inequality since I is a finite interval.

We have that $(E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n))_{L^p \rightarrow L^p} = (S_{\frac{1}{2}, p}) \circ (E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n))_{L^2 \rightarrow H_{\frac{1}{2}}} \circ T_{p, 2}$.

Therefore

$$\|E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n)\|_{L^p \rightarrow L^p} \leq \|S_{\frac{1}{2}, p}\| \cdot \|E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n)\|_{L^2 \rightarrow H_{\frac{1}{2}}} \cdot \|T_{p, 2}\|.$$

Now Lemma 2 tells us that

$$\lim_{n \rightarrow \infty} \|E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n)\|_{L^2 \rightarrow H_{\frac{1}{2}}} = 0,$$

and therefore

$$\lim_{n \rightarrow \infty} \|E(\Lambda_n) - \tilde{E}(\tilde{\Lambda}_n)\|_{L^p \rightarrow L^p} = 0.$$

Q.E.D.

SECTION II

COMPARISON OF STURM-LIOUVILLE EXPANSIONS

In this section we shall be concerned with establishing that a given complete orthonormal set of functions $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a basis in $L^p[a,b]$, $1 < p < \infty$, by comparing it to another complete orthonormal set of functions $\{\psi_n(x)\}_{n=1}^{\infty}$ which is known to be a basis in $L^p[a,b]$.

We first make the following observation:

Lemma 1: Let $2 < p < \infty$. Assume $\{f_n(x)\}_{n=1}^{\infty}$ is an orthonormal sequence of functions on the interval $I = [a,b]$ and that $\{f_n(x)\}_{n=1}^{\infty}$ is a basis in $L^p(I)$. Then $\{f_n(x)\}_{n=1}^{\infty}$ is complete in $L^2(I)$ and is also a basis in $L^{p'}(I)$ ($1/p + 1/p' = 1$). Furthermore, if $g(x) \in L^p(I)$ (respectively $L^{p'}(I)$), the expansion of $g(x)$ in terms of $\{f_n(x)\}_{n=1}^{\infty}$ is given by

$$g(x) = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n f_n(x)$$

in $L^p(I)$ (respectively $L^{p'}(I)$), where $a_n = \int_I g(x) \overline{f_n(x)} dx$.

Proof: We first state the following known result, [16].

If $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a basis in a Banach space B , then there exist unique elements ψ_n in B^* , the dual of B , such that $\langle \psi_n, \varphi_m \rangle = \delta_{mn}$. Given $f \in B$, $f = \lim_{N \rightarrow \infty} \sum_{n=1}^N \langle \psi_n, f \rangle \varphi_n$ in B .

$\{\psi_n\}_{n=1}^{\infty}$ is called the dual basis for $\{\varphi_n\}_{n=1}^{\infty}$.

Now since we are dealing with a finite interval, it follows from Holder's inequality that $f_n(x) \in L^{p'}(I)$ and since $\langle f_m, f_n \rangle \equiv \int_I f_n(x) \overline{f_m(x)} dx = \delta_{mn}$ we must have that $\{f_n(x)\}_{n=1}^{\infty}$ is itself the dual basis in $L^{p'}(I)$. Also, since $L^p(I)$ is dense in $L^2(I)$ and $\{f_n\}_{n=1}^{\infty}$ is a basis in $L^p(I)$, it follows that finite linear combinations of the f_n are dense in $L^2(I)$,

and therefore $\{f_n\}_{n=1}^{\infty}$ is complete. The statement concerning the expansion of a function in L^P (or $L^{P'}$) in terms of $\{f_n\}_{n=1}^{\infty}$ follows from the remarks about the expansion of an element in the Banach space B in terms of the dual basis.

Q.E.D.

Definition 1: Let $\sum_{n=1}^{\infty} f_n(x)$ and $\sum_{n=1}^{\infty} g_n(x)$ be two formal series of functions defined on some interval $I = [a, b]$. We say that the two series are equiconvergent if

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N (f_n(x) - g_n(x)) = 0 \text{ on } [a, b].$$

If this limit is uniform on $[a, b]$, we say that the two series are uniformly equiconvergent.

The notion of uniform equiconvergence can be used sometimes to establish that a given sequence of functions is a basis in $L^P(I)$. In fact, we have the following:

Theorem 1: Let $1 \leq p < \infty$ and let I be a finite interval. Assume that $\{\varphi_n(x)\}_{n=1}^{\infty}$ and $\{\psi_n(x)\}_{n=1}^{\infty}$ are two sequences of functions on I that satisfy the following conditions:

- (a) $\{\varphi_n(x)\}_{n=1}^{\infty}$ and $\{\psi_n(x)\}_{n=1}^{\infty}$ are both orthonormal sequences.
- (b) Each φ_n and ψ_n belongs to $L^P(I) \cap L^{P'}(I)$.
- (c) Given any $f(x) \in L^P(I)$, the series $\sum_{n=1}^{\infty} a_n \varphi_n(x)$ and $\sum_{n=1}^{\infty} b_n \psi_n(x)$ are uniformly equiconvergent, where, as usual, $a_n = \int_I f(x) \overline{\varphi_n(x)} dx$ and $b_n = \int_I f(x) \overline{\psi_n(x)} dx$.
- (d) $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a basis in $L^P(I)$, (i.e., $\lim_{N \rightarrow \infty} \sum_{n=1}^N a_n \varphi_n(x) = f(x)$ in $L^P(I)$).

Conclusion: $\{\psi_n(x)\}_{n=1}^{\infty}$ is also a basis in $L^p(I)$.

Proof: The proof is obvious since $\lim_{N \rightarrow \infty} \|f - \sum_{n=1}^N b_n \varphi_n\|_p \leq \lim_{N \rightarrow \infty} (\|f - \sum_{n=1}^N a_n \varphi_n\|_p + \|\sum_{n=1}^N (a_n \varphi_n - b_n \psi_n)\|_p) = 0$ by assumptions (c) and (d).

We now show how to use a theorem of Walsh [21], to establish a uniform equiconvergence result between the eigenfunctions of the Sturm-Liouville operator $-(\frac{d}{dx})^2 + q(x)$, ($q(x)$ real), and the operator, $-(\frac{d}{dx})^2$, with various boundary conditions on the formal operators.

We state the following theorem of Walsh:

Theorem 2: Let $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ be two real sets of functions, orthonormal on the interval $0 \leq x \leq \pi$, the former set uniformly bounded on the interval, and such that $v_n(x) - u_n(x) = \sum_{k=1}^{\infty} c_{nk} u_k(x)$, ($n = 1, 2, \dots$) where $c_{nk} = \int_0^{\pi} [v_n(x) - u_n(x)] u_k(x) dx$, and where the series $\sum_{n=1}^{\infty} (\sum_{k=1}^{\infty} |c_{nk}|)^2$ converges. Then if $f(x)$ is any function in $L^2[0, \pi]$, the two series $f(x) \sim \sum_{k=1}^{\infty} a_k u_k(x)$ and $f(x) \sim \sum_{k=1}^{\infty} b_k v_k(x)$, where $a_k = \int_0^{\pi} f(x) u_k(x)$, $b_k = \int_0^{\pi} f(x) v_k(x)$, satisfy the condition that $\sum_{k=1}^{\infty} (a_k u_k(x) - b_k v_k(x))$ converges absolutely and converges uniformly to zero (in particular, the two series are uniformly equiconvergent).

Walsh shows the hypothesis of his theorem applies to the case $u_n(x) = \sqrt{2/\pi} \sin nx$ which is the n^{th} normalized eigenfunction (i.e., $\|u_n\|_2 = 1$) for the operator $Lu = -(\frac{d}{dx})^2 u$ with boundary conditions $u(0) = 0$, $u(\pi) = 0$, $v_n(x)$ is taken to be the n^{th} normalized eigenfunction for the Sturm-Liouville operator $\tilde{L}v = -(\frac{d}{dx})^2 v(x) + q(x)v(x)$, ($q(x)$ real and bounded on $[0, \pi]$), also with the boundary conditions

$v(0) = 0$, $v(\pi) = 0$. One can therefore conclude that for any function $f(x) \in L^2[0, \pi]$, the eigenfunction expansion in terms of

$$\{\sqrt{2/\pi} \sin nx\}_{n=1}^{\infty} \quad \text{and} \quad \{v_n(x)\}_{n=1}^{\infty}$$

are uniformly equiconvergent.

Now we have that $\{\sqrt{2/\pi} \sin nx\}$ is a basis in $L^p[0, \pi]$; this follows since any $f \in L^p[0, \pi]$, $1 < p < \infty$ may be extended to an odd function $\tilde{f}$ on $[-\pi, \pi]$. We already know (see the Introduction) that the Fourier series of $\tilde{f}$, $\sum_{n=-\infty}^{+\infty} \hat{\tilde{f}}(n) e^{inx}$, converges to $\tilde{f}$ in $L^p[-\pi, \pi]$, and it therefore follows immediately that the sine expansion of $f(x)$ converges to $f(x)$ in $L^p[0, \pi]$.

One thus concludes that for $2 \leq p < \infty$ (note: Walsh's theorem assumes $f(x) \in L^2[0, \pi]$), $\{v_n(x)\}_{n=1}^{\infty}$ is a basis in $L^p[0, \pi]$. Being the eigenfunctions of a self-adjoint differential operator, $\{v_n(x)\}_{n=1}^{\infty}$ are known to be a complete orthonormal system in $L^2[0, \pi]$, (see [3]). Therefore by Lemma 1, $\{v_n(x)\}_{n=1}^{\infty}$ is also a basis in $L^p[0, \pi]$, $1 < p < 2$.

Although Walsh treats only the boundary conditions $f(0) = 0$, $f(\pi) = 0$, we will now show how his theorem can be made to apply to other boundary conditions as well. This will include the case of periodic boundary conditions where one does not have so accurate asymptotic estimates for the eigenfunctions as in the case of separated boundary conditions.

Let us first consider as a typical example of separated boundary conditions the case: $f'(0) = 0$, $\cos \beta f(\pi) - \sin \beta f'(\pi)$, $0 < \beta < \pi$, (see [14] for a different treatment of these boundary conditions).

It is known ([20]) that with these boundary conditions:

- (1) $u_n(x) = d_n \cos \alpha_n x$ where $d_n = \sqrt{2/\pi} + O(n^{-1})$ and $\alpha_n = n + O(n^{-1})$;
- (2) $v_n(x) = \sqrt{2/\pi} \cos nx + 1/n \varphi_n(x)$, where $|\varphi_n(x)| \leq c$ and $\rho_n = n + O(n^{-1})$, where ρ_n^2 is the n^{th} eigenvalue corresponding to $v_n(x)$.

We have that $c_{nk} + \delta_{nk} = \int_0^\pi v_n(x) d_k \cos \alpha_k x dx$. Integrating by parts twice gives:

$$\begin{aligned} c_{nk} + \delta_{nk} &= \frac{1}{\alpha_k} [v_n(x) d_k \sin \alpha_k x]_0^\pi + \frac{1}{\alpha_k^2} [v_n'(x) d_k \cos \alpha_k x]_0^\pi \\ &\quad - \frac{1}{\alpha_k^2} \int_0^\pi v_n''(x) d_k \cos \alpha_k x dx. \end{aligned}$$

Since $\alpha_k = k + O(k^{-1})$ it follows that $\sin \alpha_k \pi = O(k^{-1})$. Also, since $\cos \beta v_n(\pi) + \sin \beta v_n'(\pi) = 0$ and $|v_n(x)| \leq c$, it follows that $v_n'(\pi)$ is also bounded. We therefore have:

$$(3) \quad c_{nk} + \delta_{nk} = \frac{f_{nk}}{\alpha_k^2} - \frac{1}{\alpha_k^2} \int_0^\pi v_n''(x) d_k \cos \alpha_k x dx, \quad |f_{nk}| < c.$$

Since $\alpha_k = k + O(k^{-1})$ and $\rho_n = n + O(n^{-1})$, there exists an integer $N > 0$ such that if $k > N$ and $n > N$, then $|\alpha_k - k| < \frac{1}{3}$ and $|\rho_n - n| < \frac{1}{3}$. Now for $n \leq N$ it is easy to see from equation (3) that $c_{nk} < c/k^2$, and thus $\sum_{k=1}^\infty |c_{nk}| < \infty$ for $n \leq N$.

We now consider $n > N$. For $k \leq N$ and $k = n$ we use the estimate easily derived from equations (1) and (2) and $\alpha_k = k + O(k^{-1})$, that $|c_{nk}| < c/n$.

It remains to consider $k > N$, $k \neq n$. Now in the right hand side of (3) we replace $v_n''(x)$ by $-\rho_n^2 v_n(x) + q(x)v_n(x)$. We get,

$$c_{nk} = \frac{f_{nk}}{\alpha_k^2} - \frac{1}{\alpha_k^2} \int_0^\pi q(x) v_n(x) d_k \cos \alpha_k x dx + \frac{\rho_n^2}{\alpha_k^2} c_{nk}$$

$$\text{or, } c_{nk}(\alpha_k^2 - \rho_n^2) = f_{nk} - \int_0^\pi q(x) v_n(x) d_k \cos \alpha_k x dx$$

$$\text{or, } c_{nk} = \frac{f_{nk}}{\alpha_k^2 - \rho_n^2} - \frac{1}{\alpha_k^2 - \rho_n^2} \int_0^\pi q(x) v_n(x) d_k \cos \alpha_k x dx .$$

$$\text{Since } n, k > N \text{ we can conclude that } \frac{1}{|\alpha_k^2 - \rho_n^2|} \leq \frac{c}{|k^2 - n^2|} .$$

Let us first examine the sum

$$\sum_{\substack{k=N+1 \\ k \neq n}}^{\infty} \frac{1}{|k^2 - n^2|} .$$

$$\text{We have } \sum_{\substack{k=N+1 \\ k \neq n}}^{\infty} \frac{1}{|k^2 - n^2|} < \sum_{k=1}^{n-1} \frac{1}{n^2 - k^2} + \sum_{k=n+1}^{\infty} \frac{1}{k^2 - n^2} .$$

$$\text{Consider } \sum_{k=1}^{n-1} \frac{1}{n^2 - k^2} . \text{ We have that,}$$

$$\sum_{k=1}^{n-1} \frac{1}{n^2 - k^2} \leq \int_1^{n-1} \frac{dx}{n^2 - x^2} + \frac{1}{n^2 - (n-1)^2} \leq \int_0^{n-1} \frac{dx}{n^2 - x^2} + \frac{1}{2n-1} .$$

$$\text{But } \int_0^{n-1} \frac{dx}{n^2 - x^2} dx = -\frac{1}{2n} \left[\log \frac{(n-x)}{(n+x)} \right]_0^{n-1} = -\frac{1}{2n} \log \left(\frac{1}{2n-1} \right) < c \frac{\log n}{n} ,$$

$$(n > 1). \text{ Therefore } \sum_{k=1}^n \frac{1}{n^2 - k^2} < c' \frac{\log n}{n} , (n > 1). \text{ Now consider}$$

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2 - n^2} . \text{ We have that,}$$

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2 - n^2} = \sum_{k=1}^{\infty} \frac{1}{k^2 + 2nk} < \int_1^{\infty} \frac{dx}{x^2 + 2nx} + \frac{1}{2n+1} .$$

But $\int_1^{\infty} \frac{1}{x^2+2nx} dx = \frac{1}{2n} \left[\log \frac{x}{2n+x} \right]_1^{\infty} < c'' \frac{\log n}{n}, (n > 1)$. We can

finally conclude that

$$\sum_{\substack{k=N+1 \\ k \neq n}}^{\infty} \frac{1}{|k^2-n^2|} < c \frac{\log n}{n}, (n > 1).$$

We still have to consider $\sum_{\substack{k=N+1 \\ k \neq n}}^{\infty} \left| \frac{1}{(\alpha_k^2 - \rho_n^2)} \int_I q(x) v_n(x) d_k \cos \alpha_k x dx \right|$.

We have again that $\frac{1}{|\alpha_k^2 - \rho_n^2|^2} \leq \frac{c}{|k^2 - n^2|}$. Now $\frac{1}{k^2 - n^2} = \frac{1}{2n} \frac{1}{k-n} - \frac{1}{2n} \frac{1}{k+n}$.

Using Schwartz's inequality we obtain

$$\frac{1}{(k^2 - n^2)^2} \leq \frac{1}{2n^2} \frac{1}{(k-n)^2} + \frac{1}{2n^2} \frac{1}{(k+n)^2}.$$

We have

$$\sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{(k-n)^2} = \sum_{k=1}^{n-1} \frac{1}{(n-k)^2} + \sum_{k=n+1}^{\infty} \frac{1}{(k-n)^2} < 2 \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and}$$

$$\sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{(k+n)^2} < \sum_{k=1}^{\infty} \frac{1}{k^2}. \quad \text{Therefore}$$

$$\sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{(k^2 - n^2)^2} \leq \frac{3}{2n^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{c}{n^2}.$$

Since $\{d_k \cos \alpha_k x\}$ form a complete orthonormal set of functions on $[0, \pi]$, we apply Bessel's inequality to the function $q(x)u_n(x)$ to conclude that $\sum_{k=1}^{\infty} \left| \int_0^{\pi} q(x)u_n(x) d_k \cos \alpha_k x dx \right|^2 \leq \int_0^{\pi} |q(x)v_n(x)|^2 \leq c$.

Thus from Schwartz's inequality we conclude that:

$$\sum_{\substack{k=N+1 \\ k \neq n}}^{\infty} \frac{1}{|\alpha_k^2 - \rho_n^2|} \left| \int_0^{\pi} v_n(x) d_k \cos \alpha_k x dx \right| \leq \frac{c}{n}.$$

Combining all our estimates, we get that $\sum_{k=1}^{\infty} |c_{nk}| \leq c \frac{\log n}{n}$.

Now since $\sum_{n=1}^{\infty} \left(\frac{\log n}{n} \right)^2 < \infty$ we can finally conclude that

$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |c_{nk}| \right)^2 < \infty$, which is what we wanted to prove.

Now let us consider the case of periodic boundary conditions $y(0) = y(\pi)$, $y'(0) = y'(\pi)$. However, we will now assume that $q(x)$ is of bounded variation. This assumption is made in order to have better asymptotic estimates for the eigenfunctions. So let $\{v_n(x)\}_{n=0}^{\infty}$ be the set of normalized eigenfunctions for the above equation, and let $\{\lambda_n\}_{n=0}^{\infty}$ be the corresponding eigenvalues. It is known ([3]), page 214) that the λ_n can be ordered so that

$$-\infty < \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \lambda_5 \leq \lambda_6 \dots \uparrow \infty.$$

It is further known that $(\lambda_{2n-1})^{\frac{1}{2}} = 2n + O(n^{-1})$ and $(\lambda_{2n})^{\frac{1}{2}} = 2n + O(n^{-1})$ ([10] page 39). Now using the asymptotic formula given by Tricomi [20], page 168, we have for the normalized eigenfunctions, $v_n(x)$, apart from a multiplicative constant

$$(4) \quad \tilde{v}_n(x) = \sin \alpha_n \cos(\lambda_n)^{\frac{1}{2}} x + \frac{1}{(\lambda_n)^{\frac{1}{2}}} [\cos \alpha_n + T(x) \sin \alpha_n] \sin(\lambda_n)^{\frac{1}{2}} x + O(n^{-2})$$

where $\alpha_n \in [0, \pi)$ and $T(x) = \frac{1}{2} \int_0^x q(\xi) d\xi$ and the term $O(n^{-2})$ is in-

dependent of α_n . It also follows from Tricomi that asymptotic formulas for $\left(\frac{d}{dx} \right) \tilde{v}_n(x)$, $n = 0, 1, 2, \dots$ are given by the derivatives of the asymptotic formulas, but instead of a $O(n^{-2})$ term, we now have a $O(n^{-1})$ term. We now claim the following:

Lemma 2: For $n > 0$ $\exists$ constants d_{2n-1} , e_{2n-1} , d_{2n} , e_{2n} and functions $\varphi_{2n-1}(x)$ and $\varphi_{2n}(x)$ such that

$$(5) \quad v_{2n-1}(x) = d_{2n-1}\sqrt{2/\pi} \cos 2nx + e_{2n-1}\sqrt{2/\pi} \sin 2nx + 1/n \varphi_{2n-1}(x),$$

$$(6) \quad v_{2n}(x) = d_{2n}\sqrt{2/\pi} \cos 2nx + e_{2n}\sqrt{2/\pi} \sin 2nx + 1/n \varphi_{2n}(x),$$

where $d_{2n-1}^2 + e_{2n-1}^2 = 1 = d_{2n}^2 + e_{2n}^2$, $d_{2n-1}d_{2n} + e_{2n-1}e_{2n} = 0$, and $\varphi_{2n-1}(x) = O(1) = \varphi_{2n}(x)$ and $\varphi'_{2n-1}(x) = O(n) = \varphi'_{2n}(x)$.

Proof: Consider $\tilde{v}_{2n}(x)$. We have from (4) above,

$$\begin{aligned} \tilde{v}_{2n}(x) &= \sin \alpha_{2n} \cos [2n + O(n^{-1})]x \\ &\quad + \frac{1}{2n + O(n^{-1})} [\cos \alpha_{2n} + T(x) \sin \alpha_{2n}] \sin [2n + O(n^{-1})]x \\ &\quad + O(n^{-2}). \end{aligned}$$

Simplifying we get,

$$\begin{aligned} \tilde{v}_{2n} &= \sin \alpha_{2n} \cos 2nx + \frac{\cos \alpha_{2n}}{2n} \sin 2nx + \frac{\sin \alpha_{2n}}{2n} T(x) \sin 2nx \\ &\quad + \sin \alpha_{2n} O(n^{-1}) + O(n^{-2}). \end{aligned}$$

This, in turn, gives

$$\begin{aligned} A_{2n} \tilde{v}_{2n} &= A_{2n} \sin \alpha_{2n} \cos 2nx + A_{2n} \frac{\cos \alpha_{2n}}{n} \sin 2nx \\ &\quad + A_{2n} \sin \alpha_{2n} T(x) \sin 2nx \cdot 1/n + A_{2n} \sin \alpha_{2n} O(n^{-1}) \\ &\quad + A_{2n} O(n^{-2}) \quad \text{where} \quad A_{2n} = \left[(\sin \alpha_{2n})^2 + \left(\frac{\cos \alpha_{2n}}{n} \right)^2 \right]^{-\frac{1}{2}}. \end{aligned}$$

It is easy to see that the sum of the first two terms has L^2 norm = $\sqrt{\pi/2}$, and that each of the last three terms is $O(n^{-1})$. One therefore concludes that $\exists$ constants d_{2n}, e_{2n} such that:

$$v_{2n}(x) = d_{2n}\sqrt{2/\pi} \cos 2nx + e_{2n}\sqrt{2/\pi} \sin 2nx + 1/n \varphi_{2n}(x)$$

where $\varphi_{2n}(x) = O(1)$ and $\varphi'_{2n}(x) = O(n)$ and $d_{2n}^2 + e_{2n}^2 = 1$.

Similarly, $v_{2n-1}(x) = d_{2n-1}\sqrt{2/\pi} \cos(2n-1)x + e_{2n-1}\sqrt{2/\pi} \sin(2n-1)x + 1/n \varphi_{2n-1}(x)$. Since $v_{2n-1}(x)$ and $v_{2n}(x)$ are eigenfunctions of a self-adjoint operator corresponding to different eigenvalues we have that $\int_0^\pi v_{2n-1}(x)v_{2n}(x)dx = 0$. It follows that $d_{2n-1}d_{2n} + e_{2n-1}e_{2n} = O(n^{-1})$. We can therefore assume that $d_{2n}, d_{2n-1}, e_{2n}, e_{2n-1}$ are chosen so that with the same error bounds we have that $d_{2n-1}d_{2n} + e_{2n-1}e_{2n} = 0$.

Q.E.D.

Now this time we will take for our $\{u_n\}_{n=0}^\infty$ the functions $u_0 = 1/\sqrt{\pi}$; $u_{2n-1}(x) = d_{2n-1}\sqrt{2/\pi} \cos 2nx + e_{2n-1}\sqrt{2/\pi} \sin 2nx$; $u_{2n}(x) = d_{2n}\sqrt{2/\pi} \cos 2nx + e_{2n}\sqrt{2/\pi} \sin 2nx$. We claim that the functions $\{u_n\}_{n=0}^\infty$ are a basis in $L^P[0, \pi]$. To see this we first observe that since $\{e^{inx}\}_{n=-\infty}^{+\infty}$ is a basis in $L^P[-\pi, \pi]$ and

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2} \quad \text{and} \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i},$$

it follows easily that $\{1/\sqrt{2\pi}, 1/\sqrt{\pi} \cos nx, 1/\sqrt{\pi} \sin nx\}_{n=1}^\infty$ is a basis in $L^P[-\pi, \pi]$. A change of variables ($y = x/2$) then shows that $\{1/\sqrt{\pi}, \sqrt{2/\pi} \cos 2nx, \sqrt{2/\pi} \sin 2nx\}$ is a basis in $L^P[0, \pi]$.

Since our $\{u_n(x)\}_{n=0}^\infty$ is obtained from this last set by taking linear combinations of successive pairs of $\cos 2nx$ and $\sin 2nx$, we have that $\{u_n(x)\}_{n=0}^\infty$ is also a basis in $L^P[0, \pi]$. More precisely, if

$$f \in L^P[0, \pi] \quad \text{and} \quad a_{2n-1} = \int_I f(x)u_{2n-1}(x)dx, \quad a_{2n} = \int_I f(x)u_{2n}(x)dx$$

$$b_{2n-1} = \int_I f(x)\sqrt{2/\pi} \cos 2nx, \quad b_{2n} = \int_I f(x)\sqrt{2/\pi} \sin 2nx,$$

then $a_{2n-1}u_{2n-1}(x) + a_{2n}u_{2n}(x) = b_{2n-1} \cos 2nx + b_{2n} \sin 2nx$. Having

established this we now proceed to derive (3). For example, let us evaluate $c_{2n,2k}$, $k \neq n$. We have

$$c_{2n,2k} = d_{2k} \int_0^\pi v_{2n}(x) \sqrt{2/\pi} \cos 2kx + e_{2k} \int_0^\pi v_{2n}(x) \sqrt{2/\pi} \sin 2kx .$$

Integrating by parts twice and keeping in mind that $v_{2n}(x)$ satisfies periodic boundary conditions, we obtain:

$$\begin{aligned} c_{2n,2k} &= - \frac{d_{2k}}{(2k)^2} \int_0^\pi v_{2n}''(x) \sqrt{2/\pi} \cos 2kx - \frac{e_{2k}}{(2k)^2} \int_0^\pi v_{2n}''(x) \sqrt{2/\pi} \sin 2kx dx \\ &= - \frac{1}{(2k)^2} \int_0^\pi v_{2n}''(x) u_{2k}(x) dx . \end{aligned}$$

Once equation (3) is established we then proceed exactly as before to show that $\sum_n (\sum_k |c_{nk}|)^2 < \infty$.

We have just seen how in comparing two orthonormal sequences, the inequality $\sum_{n=1}^\infty (\sum_{k=1}^\infty |c_{nk}|)^2 < \infty$ can be used to establish an equiconvergence result between the expansions of a function with respect to the two sequences. This in turn can be used to give an L^p convergence result.

We now show how the inequality $\sum_{n=1}^\infty (\sum_k |c_{nk}|^2)^{\frac{1}{2}} < \infty$ can also be used to give an L^p basis result. We first note that neither inequality implies the other. Thus if we take $c_{nk} = 1/n^2 k$ we have $\sum_{n=1}^\infty (\sum_{k=1}^\infty |c_{nk}|^2)^{\frac{1}{2}} < \infty$ and $\sum_{n=1}^\infty (\sum_{k=1}^\infty |c_{nk}|)^2 = \infty$, while if we take $c_{nk} = \delta_{nk} \cdot 1/k$, the reverse is true.

We observe that with our definition of c_{nk} ($c_{nk} + \delta_{nk} \equiv \int v_n(x) \bar{u}_k(x) dx$) we see that, the inequality $\sum_{n=1}^\infty (\sum_{k=1}^\infty |c_{nk}|^2)^{\frac{1}{2}} < \infty$ is equivalent to $\sum_{n=1}^\infty \|u_n - v_n\|_2 < \infty$. We therefore state the following:

Theorem 3: Let $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ be two complete orthonormal systems in $L^2[0,1]$ such that

$$(a) \quad \|u_n\|_{\infty} \leq c, \quad \|v_n\|_{\infty} \leq c$$

$$(b) \quad \sum_{n=1}^{\infty} \|u_n - v_n\|_2 < \infty$$

$$(c) \quad \{u_n\}_{n=1}^{\infty} \text{ is a basis in } L^p, \quad 1 < p < \infty.$$

Conclusion: $\{v_n\}_{n=1}^{\infty}$ is also a basis in $L^p[0,1]$.

Before proving this theorem we first establish the following preliminary results.

Lemma 3: Let B be a Banach space and $\{\varphi_n\}_{n=1}^{\infty}$ a basis in B . Let T be a bounded linear operator on B with a bounded inverse. Then if $\psi_n = T\varphi_n$, $n = 1, 2, \dots$, $\{\psi_n\}_{n=1}^{\infty}$ is also a basis in B .

Proof: Given $f \in B$, we must show $\exists$ a unique set of constants $\{a_i\}_{i=1}^{\infty}$ such that $\lim_{N \rightarrow \infty} \sum_{i=1}^N a_i \psi_i = f$. Let $T^{-1}f = g$. Since T and T^{-1} are both bounded, we have that $\lim_{N \rightarrow \infty} \sum_{i=1}^N a_i \psi_i = f$ if and only if $\lim_{N \rightarrow \infty} \sum_{i=1}^N a_i \varphi_i = g$. But since $\{\varphi_i\}_{i=1}^{\infty}$ is a basis, we know there is a unique set $\{a_i\}_{i=1}^{\infty}$ such that this last limit exists.

Lemma 4: Let $\{\varphi_n\}_{n=1}^{\infty}$ be a basis in a Banach space B . Let n_i , $i = 1, 2, \dots$, be an increasing sequence of positive integers. Let M be the closed subspace generated by the φ_{n_i} , $i = 1, 2, \dots$.

Conclusion: $\{\varphi_{n_i}\}_{i=1}^{\infty}$ is a basis in M . The proof can be found in Singer [16].

Lemma 5: Let B be as above. Suppose B is the direct sum of two subspaces, $B = X \oplus Y$, where X is finite dimensional and Y is closed.

Let $\{x_i\}_{i=1}^n$ be a basis for X and $\{y_i\}_{i=1}^\infty$ be a basis for Y .

Conclusion: $\{x_j\}_{j=1}^n \cup \{y_k\}_{k=1}^\infty$ is a basis for B .

Proof: The proof is trivial since given $b \in B$, we can express b uniquely as $b = x + y$, where $x \in X$ and $y \in Y$.

Proof of Theorem 3: We first note by Holder's inequality that if

$\|f\|_\infty \leq c$, then for $1 \leq r < \infty$, $\|f\|_r \leq c$ and $\|f\|_r \leq c_r \|f\|_2$. Thus if $f \in L^p[0,1]$, we have

$$(1) \quad \left| \int f(x) \bar{u}_n(x) dx \right| \leq \|f\|_p \|u_n\|_{p'} \leq c \|f\|_p, \quad (1/p + 1/p' = 1).$$

$$(2) \quad \|u_n - v_n\|_p \leq c_p \|u_n - v_n\|_2.$$

Now choose an integer m such that $\sum_{n=m+1}^\infty c \cdot c_p \|u_n - v_n\|_2 = \alpha < 1$. We define the operator on $L^p[0,1]$: $J_m(f) = \sum_{n=m+1}^\infty \langle f, u_n \rangle (v_n - u_n)$, (where $\langle f, u_n \rangle \equiv \int_0^1 f(x) \bar{u}_n(x) dx$). We have that

$$\begin{aligned} \|J_m f\|_p &\leq \sum_{n=m+1}^\infty |\langle f, u_n \rangle| \|v_n - u_n\|_p \leq \|f\|_p \sum_{n=m+1}^\infty \|u_n\|_{p'} \|v_n - u_n\|_p \\ &\leq \|f\|_p \sum_{n=m+1}^\infty c c_p \|v_n - u_n\|_2 = \alpha \|f\|_p. \end{aligned}$$

Thus J_m is an operator on $L^p[0,1]$ of norm < 1 . It follows that

$I + J_m$ is a bounded operator with a bounded inverse. Lemma 3 allows

us to conclude that $\{(I + J_m)u_n\}_{n=1}^\infty$ is a basis in L^p . But

$$(I + J_m)\{u_j\} = u_j, \quad j = 1, 2, \dots, m \quad \text{and} \quad (I + J_m)\{u_j\} = v_j,$$

$j = m+1, m+2, \dots$. We claim this implies $\{v_i\}_{i=1}^\infty$ is a basis in L^p .

To see this, let $(\bar{v}_m)^r$ denote the closure in L^r of the linear space spanned by $\{v_i\}_{i=m+1}^\infty$ and let $\{\bar{u}_i\}_{i=1}^m$ be the linear space generated by the u_i , $i = 1, 2, \dots, m$ with corresponding meaning for $\{\bar{v}_i\}_{i=1}^m$. Now since $\{u_1, u_2, \dots, u_m, v_{m+1}, v_{m+2}, \dots\}$ is a basis in

$L^p[0,1]$, we have that $L^p = \{u_k\}_{k=1}^m \oplus (\bar{v}_m)^p$. This implies by Lemma 2, that $\{v_p\}_{j=m+1}^\infty$ is a basis in $(\bar{v}_m)^p$, and we also have that dimension $L^p/(\bar{v}_n)^p = m$. Now assume for the moment that $p > 2$. We have that $(\bar{v}_m)^p \subseteq (\bar{v}_m)^2$, by Holder's inequality, and since $\{\bar{v}_i\}_{i=1}^n \cap (\bar{v}_m)^2 = \emptyset$ (since the v_i 's are mutually orthogonal) it follows that $\{\bar{v}_i\}_{i=1}^n \cap (\bar{v}_m)^p = \emptyset$. Therefore, dimension of

$$[\{\bar{v}_i\}_{i=1}^n \oplus (\bar{v}_m)^p] / (\bar{v}_m)^p = m,$$

and so we conclude that $\{\bar{v}_i\}_{i=1}^n \oplus (\bar{v}_m)^p = L^p$. Lemma 5 now tells us that $\{v_i\}_{i=1}^\infty$ is a basis in $L^p[0,1]$, ($p > 2$). For $p > 2$ one can use the fact, already established, that if an orthonormal system is a basis in $L^p[0,1]$, $p > 2$, it is a basis in $L^{p'}[0,1]$.

We note that the last theorem does not appear to have application to differential equations since if τ and τ' represent two formal differential operators of degree n , such that $\tau - \tau'$ is of degree $n - 1$ or less, and if $\varphi_n(x)$ and $\psi_n(x)$ are the n^{th} eigenfunctions for τ and τ' respectively, then, in general, $\|\varphi_n - \psi_n\|_{L^2} = O(n^{-1})$.

SECTION III

CONJUGATE HARMONIC FUNCTIONS

Consider the circle group T . Let $f(\theta) \in L^p(T)$, $1 < p < \infty$. For $\theta \in T$ and $0 \leq r < 1$ we define $f(r, \theta)$ to be the Poisson integral of $f(\theta)$: $f(r, \theta) = 1/2\pi \int_T P(r, \theta - \xi) f(\xi) d\xi$, where

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$

We have that $\|f(r, \cdot)\|_p \leq \|f\|_p$ ([9], page 16) and if we identify (r, θ) with points in the unit disc, we have that $f(r, \theta) \equiv u(x, y)$ satisfies Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Now let $v(x, y)$ be the harmonic function in the unit disc conjugate to $u(x, y)$: Thus $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Assume $v(x, y)$ is normalized so that $v(0, 0) = 0$. Then letting $\tilde{f}(r, \theta) = v(x, y)$ we have that $\|\tilde{f}(r, \theta)\|_p \leq A_p \|f(\theta)\|_p$. Also, $\tilde{f}(\theta) \equiv \lim_{r \rightarrow 1} \tilde{f}(r, \theta)$ exists for almost all θ , and therefore by Fatou's lemma the conjugate function $\tilde{f}(\theta)$ satisfies $\|\tilde{f}(\theta)\|_p \leq A_p \|f\|_p$ ([9], page 76).

We now make the following change of variables. Let $t = -\log(r)$ and $x = \theta$, $(0 \leq x \leq 2\pi)$. Defining $u(x, t) \equiv f(r, \theta)$ and $v(x, t) \equiv \tilde{f}(r, \theta)$ we have that:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0; \quad \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial x^2} = 0; \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial t}; \quad \frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x}.$$

Denote by $H^2[0, 2\pi]$, $\{f(x) : \left(\frac{d}{dx}\right)^j \in L^2[0, 2\pi], j = 0, 1, 2\}$. Let us consider the formal differential operator $L = \left(\frac{d}{dx}\right)^2$, defined on $H^2[0, 2\pi]$ with boundary conditions, for example, $f(0) = f(2\pi)$,

$f'(0) = f'(2\pi)$. L is a self-adjoint operator and has the complete set of eigenfunctions $\{e^{inx}\}_{n=-\infty}^{+\infty}$ with eigenvalues $-\lambda_n^2 = -n^2$. If $f(x) = e^{inx}$ then $\tilde{f}(x) = -i(\operatorname{sgn} n)e^{inx}$ and $u(x,t) = e^{-|n|t} e^{inx}$. We note that if $f(x) \sim \sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{inx}$, then

$$\tilde{f}(x) \sim \sum_{n=-\infty}^{+\infty} -\frac{\hat{f}(n)}{\lambda_n} \left(\frac{d}{dx} e^{inx}\right)$$

where, as above, $\lambda_n = |n|$.

Now consider the more general Sturm-Liouville operator $L = \left(\frac{d}{dx}\right)^2 + a(x)\left(\frac{d}{dx}\right)$ defined on $H^2(I)$, $I = [0,1]$. Suppose that with the imposition of two independent boundary conditions, L has a discrete set of eigenvalues $-\lambda_n^2$, ($\lambda_n > 0$) with corresponding real eigenfunctions $\varphi_n(x)$. We ask the following question: If

$$T: \varphi_n(x) \rightarrow -\frac{\left(\frac{d}{dx}\right) \varphi_n}{\lambda_n},$$

is T extendable to a bounded mapping on $L^p[0,1]$, ($1 < p < \infty$)?

We will eventually prove that this is so under the special conditions $a'(x) \leq 0$ and the boundary conditions $\varphi'(0) = \varphi'(\pi) = 0$.

It is easy to verify that with the imposition of these boundary conditions $L = \left(\frac{d}{dx}\right)^2 + a(x)\left(\frac{d}{dx}\right)$ is self-adjoint with respect to the measure $q(x) = c \exp \left[\int_0^x a(x)dx\right]$ (where c is chosen so that $\int_I q(x)dx = 1$). Thus, if $f(x), g(x) \in H^2[0,1]$ and satisfy the above boundary conditions, then we have,

$$\int_I (Lf)\bar{g} \cdot q(x)dx = \int_I f(\overline{Lg})q(x)dx.$$

It follows from a paper by McKean [23], that for a suitable class

of functions, $\mathcal{J}$, dense in all of the $L^p(q(x))$ spaces, $1 \leq p < \infty$, and for appropriate boundary conditions (including $\varphi'(0) = 0 = \varphi'(1)$), the semigroup $T^t = e^{tL}$, $(0 < t < \infty)$ can be constructed satisfying the following conditions:

For $f \in \mathcal{J}$:

- I. $\|T^t f\|_p \leq \|f\|_p$, $(1 \leq p \leq \infty)$ where now $\|f\|_p = (\int_I |f|^p q(x))^{1/p}$;
- II. Each T^t is a self-adjoint operator on $L^2(q(x))$;
- III. $T^t f(x)$ satisfies the given boundary conditions;
- IV. $T^t f \geq 0$ if $f \geq 0$;
- V. $T^t 1 = 1$ (for our boundary conditions $f'(0) = 0 = f'(1)$);
- VI. $\lim_{t \rightarrow 0} T^t f = f$ a.e.;
- VII. If we denote $T^t f(x)$ by $v(x, t)$ we have that $Lv(x, t) = \left(\frac{\partial}{\partial t}\right)v(x, t)$.

By the process of subordination we can construct the "Poisson" semigroup $P^t = e^{-t(-L)^{\frac{1}{2}}}$ from the above "Heat Diffusion" semigroup (see [18], page 46). This "Poisson" semigroup will satisfy the properties I-VI above with T^t replaced by P^t and also

VII'. Denoting $P^t f(x)$ by $u(x, t)$ we have

$$Lu(x, t) + \left(\frac{\partial}{\partial t}\right)^2 u(x, t) = 0.$$

The above properties can be deduced from the fact that

$$u(x, t) = \int_I p(t, x, y) f(y) q(y) dy \quad \text{where}$$

- (1) $p(t, \dots)$ is positive and symmetric on $I \times I$ for each $t > 0$.
- (2) $p(t, \cdot, y)$ satisfies the given boundary conditions and $p(\cdot, \cdot, y)$ satisfies the given differential equation of VII'.

$$(3) \quad \int_I p(t, x, y) q(y) dy = 1.$$

$$(4) \quad p(t_1 + t_2, x, y) = \int_0^1 p(t_1, x, \xi) p(t_2, \xi, y) q(\xi) d\xi.$$

Theorem 3.1 of McKean's paper tells us that the eigenfunctions $\{v_i\}_{i=1}^\infty$ of L with the given boundary conditions, and such that $\|v_i\|_2 = 1$, $\forall i$, form a complete orthonormal set in L^2 and can be chosen so that the corresponding eigenvalues, $-\lambda_i^2$, are such that $0 \geq -\lambda_1^2 > -\lambda_2^2, \dots \downarrow -\infty$. If $\mu > 0$ the Green's function for the operator $\mu I - L$ is given by $G_\mu(x, y) = \sum_{i=1}^\infty (\mu + \lambda_i^2)^{-1} v_i(x) v_i(y)$, the sum converging uniformly on $I \times I$. Thus if $u(x) = \int_I G_\mu(x, y) f(y) q(y) dy$, $(\mu - L)u(x) = f(x)$.

We first establish a theorem about a variant of the Hardy-Littlewood "g" function. In order to avoid technical difficulties we shall assume in what follows that $f(x)$ is a finite linear combination of the $\{v_i(x)\}_{i=1}^\infty$. There is no loss of generality in assuming this since the linear span of the $\{v_i(x)\}_{i=1}^\infty$ is dense in the space of continuous functions (see [7], page 276), and hence in $L^p(I)$. There is also no loss of generality in assuming that $f(x) \geq 0$. The reason is that any function $g(x)$ can be decomposed as $g_1^+(x) + g_2^+(x) + ig_3^+(x) + ig_4^+(x)$ where $g_i^+(x) \geq 0$, $(i = 1, 2, 3, 4)$. Now given $\epsilon > 0$ we can find continuous $h_i(x) \geq 0$ such that $\|g_i - h_i\|_p < \epsilon/2$, $(i = 1, 2, 3, 4)$. Since the $v_j(x)$ are real $\forall j$ and their linear span is dense in $C(I)$ it follows that there exists $f_i(x) = \sum_{j=1}^n c_{ij} v_j(x)$ such that $f_i(x) \geq 0$ and $\|f_i - h_i\|_p < \epsilon/2$. Thus $\|f_i - g_i^+\|_p < \epsilon$, $(i = 1, 2, 3, 4)$.

Theorem 1: Denote by Δ the formal differential operator:

$\left(\frac{\partial}{\partial x}\right)^2 + a(x)\left(\frac{\partial}{\partial x}\right) + \left(\frac{\partial}{\partial t}\right)^2$. Let $f(x) = \sum_{i=1}^n c_i v_i(x) \geq 0$ and $u(x,t)$ be the "Poisson" integral of $f(x)$. Define

$$g(f)(x) = \left(\int_0^\infty t \left(\left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right) dt \right)^{\frac{1}{2}}.$$

Conclusion: $\|g(f)\|_p \leq A_p \|f\|_p$, $1 < p < \infty$, and conversely

$\|f\|_p \leq A_p \|g(f)\|_p$ whenever f is orthogonal to the eigenfunctions of L with eigenvalue zero (i.e., the constants).

To prove this theorem we first establish the following lemmas.

Lemma 1: Let $f(x)$ be as in Theorem 1 and $u(x,t)$ be its corresponding Poisson integral. Then we have

$$\Delta(u^p) = p(p-1)u^{p-2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right].$$

Proof: This follows by straightforward differentiation using the fact that $\Delta(u) = 0$.

Lemma 2: $\int_I \sup_{t>0} |u(x,t)|^p q(x) \leq A_p \int_I |f(x)|^p q(x)$.

Proof: This follows from the general Maximal Theorem of Stein ([18], page 73).

Lemma 3: Let $\varphi(x) \geq 0$ also be a finite linear combination of the $v_i(x)$ and $w(x,t) = P^t \varphi(x)$. Then for $1 \leq r < \infty$ we have,

$$\begin{aligned} \int_I \int_0^\infty t \Delta((u(x,t))^r w(x,t)) dt q(x) dx &= \int_I (f(x))^r \varphi(x) q(x) dx \\ &\quad - \int_I (u(x,\infty))^r w(x,\infty) q(x) dx, \end{aligned}$$

where

$$u(x,\infty) = \int_I f(\xi) q(\xi) d\xi, \text{ and } w(x,\infty) = \int_I \varphi(\xi) q(\xi) d\xi.$$

Proof: We have that

$$\begin{aligned} & \int_I \int_0^\infty t \left[\left(\frac{\partial}{\partial x} \right)^2 (u^r_w) + a(x) \left(\frac{\partial}{\partial x} \right) (u^r_w) \right] dt q(x) dx \\ &= \int_0^\infty t dt \int_I \left[\left(\frac{\partial}{\partial x} \right)^2 (u^r_w) + a(x) \left(\frac{\partial}{\partial x} \right) (u^r_w) \right] q(x) dx, \end{aligned}$$

by Fubini's theorem (this is justified since $P^t(v_i)(x) = e^{-\lambda_i t} v_i(x)$, $\forall i$ (as follows from McKean's paper). However, the inner integral of the last expression

$$\begin{aligned} &= \left[\left(\frac{\partial}{\partial x} \right) (u^r_w) q(x) \right]_0^1 - \int_I \left(\frac{\partial}{\partial x} \right) (u^r_w) \left(\frac{d}{dx} \right) q(x) dx \\ &\quad + \int_I \left(\frac{\partial}{\partial x} \right) (u^r_w) a(x) q(x) dx = 0 \end{aligned}$$

since

$$\left(\frac{\partial}{\partial x} \right) (u^r_w) \Big|_{x=0} = \left(\frac{\partial}{\partial x} \right) (u^r_w) \Big|_{x=1} = 0$$

and $\frac{dq}{dx} = a(x)q(x)$. Thus we have that

$$\begin{aligned} \int_I \int_0^\infty t \Delta(u^r_w) dt q(x) dx &= \int_I q(x) \int_0^\infty t \left(\frac{\partial}{\partial t} \right)^2 (u^r_w) dt \\ &= \int_I q(x) dx \left[t \left(\frac{\partial}{\partial t} \right) (u^r_w) \right]_0^\infty \\ &\quad - \int_I q(x) dx \int_0^\infty \frac{\partial(u^r_w)}{\partial t} dt. \end{aligned}$$

Again, since $P^t v_i(x) = e^{-\lambda_i t} v_i(x)$, it is clear that $\lim_{t \rightarrow 0} t \left(\frac{\partial}{\partial t} \right) (u^r_w) = 0$

and also $\lim_{t \rightarrow \infty} t \left(\frac{\partial}{\partial t} \right) (u^r_w) \rightarrow 0$. Therefore the right hand side of the last equality

$$= \int_I (f(x))^r \varphi(x) q(x) dx - \int_I (u(x, \infty))^r \varphi(x, \infty) q(x) dx.$$

Lemma 4: $\Delta \left[\left(\frac{\partial u}{\partial t} \right)^2 \right] \geq 0$; $\Delta \left[\left(\frac{\partial u}{\partial x} \right)^2 \right] \geq 0$ where, as usual, $u(x,t) = P^t f(x)$.

Proof: Let $g(x,t) = \left(\frac{\partial}{\partial t} \right) u(x,t)$ and $h(x,t) = \left(\frac{\partial}{\partial x} \right) u(x,t)$. We have $\Delta g(x,t) = 0$ and also $\Delta h(x,t) + a'(x)h(x,t) = 0$. The latter equality follows since $\Delta h(x,t) + a'(x)h(x,t) \equiv \left(\frac{\partial}{\partial x} \right)^3 u(x,t) + a(x) \left(\frac{\partial}{\partial x} \right)^2 u(x,t) + a'(x) \left(\frac{\partial}{\partial x} \right) u(x,t) + \left(\frac{\partial}{\partial t} \right)^2 \left(\frac{\partial}{\partial x} \right) u(x,t) = \left(\frac{\partial}{\partial x} \right) [\Delta u(x,t)] = 0$.

We have upon differentiating,

$$\Delta (g^2) = 2 \left[\left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial t} \right)^2 \right] + 2g\Delta g = 2 \left[\left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial t} \right)^2 \right] .$$

Also,

$$\Delta (h^2) = 2 \left[\left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial t} \right)^2 \right] + 2h\Delta h = 2 \left[\left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial t} \right)^2 \right] - 2a'(x)h^2 \geq 0$$

since we are assuming $a'(x) \leq 0$.

Lemma 5:

$$(a) \quad \left[\frac{\partial u}{\partial x}(x, t_0) \right]^2 \leq P^{t_0/2} \left(\left[\frac{\partial u}{\partial x}(x, t_0/2) \right]^2 \right), \quad 0 < t_0 .$$

$$(b) \quad \left[\frac{\partial u}{\partial t}(x, t_0) \right]^2 \leq P^{t_0/2} \left(\left[\frac{\partial u}{\partial t}(x, t_0/2) \right]^2 \right), \quad 0 < t_0 .$$

Proof of (a): Let $F(x,t) = \left[\frac{\partial u}{\partial x}(x, t_1+t) \right]^2$, $t_1 > 0$ and let $G(x,t) = P^t F(x,0)$.

Consider $H(x,t) = F(x,t) - G(x,t)$. We want to show that

$H(x,t) \leq 0$ for $x \in I$, $0 < t < \infty$. Now consider the rectangle R_n in the (x,t) plane bounded by the sides:

- (I) $0 \leq x \leq 1$, $t = 0$,
- (II) $0 \leq x \leq 1$, $t = M$,
- (III) $x = 0$, $0 \leq t \leq M$,
- (IV) $x = 1$, $0 \leq t \leq M$.

We have that $H(x,t) = 0$ on side (I). On sides (III) and (IV) $H(x,t) \leq 0$ since on these sides $F(x,t) \equiv 0$, $\left(\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t)\right) = 0 \forall t$, and $G(x,t) \geq 0$, being the "Poisson" integral of a nonnegative function. Now since the constant functions ($f(x) = c$) are the eigenfunctions of the formal operator L with its associated boundary conditions ($f'(0) = f'(1) = 0$), with corresponding eigenvalue zero, it follows that $\frac{\partial u}{\partial x}(x,t) \rightarrow 0$ uniformly on I as $t \uparrow \infty$. Consequently, given $\epsilon > 0$ we can choose M sufficiently large so that $H(x,t) \leq \epsilon$ on the boundary of the rectangle R_m . But $\Delta H(x,t) \geq 0$ on R_m , since $\Delta F(x,t) \geq 0$ by Lemma 4 and $\Delta G(x,t) = 0$ since $G(x,t)$ is a "Poisson" integral. By the Hopf Maximal principle ([4], page 326) it follows that $H(x,t) \leq \epsilon$ in R_m . By letting $M \uparrow \infty$, one can conclude that $H(x,t) \leq 0$. Letting $t_1 = t_0/2$ and $t = t_0/2$, part (a) now follows.

Proof of (b): The proof here is simpler since the operators $\frac{\partial}{\partial t}$ and P^t commute, since $\frac{\partial}{\partial t}$ and L commute. We have

$$\begin{aligned} \left| \frac{\partial u}{\partial t}(x, t_0) \right| &= \left| P^{t_0/2} \left(\frac{\partial u}{\partial t}(x, t_0/2) \right) \right| = \left| \int_I p(t_0/2, x, y) \frac{\partial u}{\partial t}(y, t_0/2) q(y) dy \right| \\ &\leq \int_I p(t_0/2, x, y) \left| \frac{\partial u}{\partial t}(y, t_0/2) \right| q(y) dy \\ &\leq \left[\int_I p(t_0/2, x, y) \left| \frac{\partial u}{\partial t}(y, t_0/2) \right|^2 q(y) dy \right]^{1/2} \end{aligned}$$

(by the Cauchy-Schwartz inequality and the fact that

$$\int_I p(t_0/2, x, y) q(y) = 1. \text{ Thus,}$$

$$\left| \frac{\partial u}{\partial t}(x, t_0) \right|^2 \leq P^{t_0/2} \left[\frac{\partial u}{\partial t}(x, t_0/2) \right]^2.$$

Proof of Theorem 1:

Case I: $1 < p \leq 2$: Now by Lemma 1 we have

$$\begin{aligned} (g(f)(x))^2 &= \int_0^\infty t \left(\left| \frac{\partial u}{\partial t}(x, t) \right|^2 + \left| \frac{\partial u}{\partial x}(x, t) \right|^2 \right) dt \\ &= \frac{1}{p(p-1)} \int_0^\infty t u(x, t)^{2-p} \Delta(u(x, t))^p dt . \end{aligned}$$

Letting $f^*(x)$ denote $\sup_{t>0} u(x, t)$ we have that the last expression

$$\leq \frac{1}{p(p-1)} (f^*(x))^{2-p} \int_0^\infty t \Delta(u(x, t))^p dt .$$

Now define

$$J(x) = \int_0^\infty t \Delta(u(x, t))^p dt \geq 0 .$$

We have shown that $(g(f)(x))^2 \leq A_p (f^*(x))^{2-p} J(x)$. Now by Lemma 3,

with $\varphi(x) \equiv 1$ we have,

$$\begin{aligned} \int_I J(x) q(x) dx &= \int_I (f(x))^p q(x) dx - \int_I (u(x, \infty))^p q(x) dx \\ &\leq \int_I (f(x))^p q(x) dx \end{aligned}$$

(since $u(x, \infty) \geq 0$) . Therefore

$$\begin{aligned} \|g(f)\|_p^p &= \int_I (g(f))^p q(x) dx \leq A_p \int_I (f^*(x))^{(2-p)p/2} (J(x))^{p/2} q(x) dx \\ &\leq A_p \left(\int_I (f^*(x))^p q(x) dx \right)^{(2-p)/2} \left(\int_I J(x) q(x) dx \right)^{p/2} \end{aligned}$$

(by Holder's inequality $\leq A_p' \|f\|_p^{(2-p)p/2} \|f\|_p^{p \cdot p/2}$ (using Lemma 2)

$= A_p'' \|f\|_p^p$. Thus we have shown that for $1 < p \leq 2$ that

$$\|g(f)\|_p^p \leq A_p \|f\|_p^p .$$

Case II: $p \geq 4$: We note first that once we prove Case II, Theorem 1

will follow from the Marcinkiewicz interpolation theorem [22].

Now $\|g(f)\|_p^2 = \sup \int_I (g(f)(x))^2 \varphi(x) q(x) dx$, where the supremum is taken over all positive functions $\varphi(x)$, such that $\varphi(x)$ is a finite linear combination of the $v_i(x)$, $i = 1, 2, \dots$, and $\|\varphi(x)\|_r \leq 1$ where r is the exponent conjugate to $p/2$. We have

$$\begin{aligned} \int_I (g(f)(x))^2 \varphi(x) q(x) dx &= \int_I \int_0^\infty t \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] dt \varphi(x) q(x) dx \\ &\leq \int_0^\infty t \int_I p^{t/2} \left(\left[\frac{\partial u}{\partial x}(x, t/2) \right]^2 \right. \\ &\quad \left. + \left[\frac{\partial u}{\partial t}(x, t/2) \right]^2 \right) \varphi(x) q(x) dx dt, \end{aligned}$$

(by Lemma 5),

$$= \int_0^\infty t \int_I \left(\left[\frac{\partial u}{\partial x}(x, t/2) \right]^2 + \left[\frac{\partial u}{\partial t}(x, t/2) \right]^2 \right) \left[p^{t/2} \varphi(x) \right] q(x) dx dt$$

(by the self-adjointness of P^t). Now by Lemma 1,

$$\left[\frac{\partial u}{\partial x}(x, t) \right]^2 + \left[\frac{\partial u}{\partial t}(x, t) \right]^2 = \frac{1}{2} \Delta u^2(x, t).$$

Thus we deduce that

$$(a) \quad \int_I (g(f)(x))^2 \varphi(x) q(x) dx \leq \int_0^\infty \int_I t (\Delta u^2(x, t)) \varphi(x, t) q(x) dx dt$$

where $\varphi(x, t) = P^t \varphi(x)$.

Recalling that $\Delta = \left(\frac{\partial}{\partial x} \right)^2 + u(x) \left(\frac{\partial}{\partial x} \right) + \left(\frac{\partial}{\partial t} \right)^2$, it follows easily that for $F(x, t)$ and $G(x, t)$ smooth we have

$$\Delta(FG) = (\Delta F)G + F(\Delta G) + 2 \frac{\partial F}{\partial x} \frac{\partial G}{\partial x} + 2 \frac{\partial F}{\partial t} \frac{\partial G}{\partial t}.$$

If we take $F = (u(x, t))^2$ and $G = \varphi(x, t)$ and apply this identity to equation (a) we obtain (noting that $\Delta \varphi \equiv 0$)

$$(b) \quad \int_I (g(f)(x))^2 \varphi(x) q(x) \leq \int_0^\infty \int_I t \Delta [(u(x, t))^2 \varphi(x, t)] q(x) dx dt$$

$$- 2 \int_0^\infty \int_I t \left(\frac{\partial(u^2)}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial(u^2)}{\partial t} \frac{\partial \varphi}{\partial t} \right) q(x) dt .$$

Now by Lemma 3 we have

$$\begin{aligned} \int_0^\infty t \Delta(u^2(x, t) \varphi(x, t)) q(x) dx dt &= \int_I (f(x))^2 \varphi(x) q(x) dx \\ &- \int_I (u(x, \infty))^2 \varphi(x, \infty) \leq \int_I (f(x))^2 \varphi(x) q(x) dx . \end{aligned}$$

Also,

$$\begin{aligned} &\int_0^\infty \int_I t \left[\frac{\partial(u^2)}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial(u^2)}{\partial t} \frac{\partial \varphi}{\partial t} \right] q(x) dt \\ &= \int_0^\infty \int_I t \left[2u \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial x} + 2u \frac{\partial u}{\partial t} \frac{\partial \varphi}{\partial t} \right] q(x) dx dt \\ &\leq 2 \int_0^\infty \int_I t f^*(x) \left[\left| \frac{\partial u}{\partial x} \right| \left| \frac{\partial \varphi}{\partial x} \right| + \left| \frac{\partial u}{\partial t} \right| \left| \frac{\partial \varphi}{\partial t} \right| \right] q(x) dx dt , \end{aligned}$$

where $f^*(x) = \sup_{t>0} u(x, t)$,

$$\leq 2 \int_I f^*(x) q(x) dx \int_0^\infty t \left[\left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right]^{\frac{1}{2}} \cdot \left[\left| \frac{\partial \varphi}{\partial x} \right|^2 + \left| \frac{\partial \varphi}{\partial t} \right|^2 \right]^{\frac{1}{2}} dt$$

(by the Cauchy-Schwartz inequality and Fubini's theorem)

$$\leq 2 \int_I f^*(x) g(f)(x) g(\varphi)(x) q(x) dx$$

(by Schwartz's inequality). Applying the previous two inequalities in equation (6) we obtain:

$$\begin{aligned} \int_I (g(f)(x))^2 \varphi(x) q(x) dx &\leq \int_I (f(x))^2 \varphi(x) q(x) dx \\ &+ 4 \int_I f^*(x) g(f)(x) g(\varphi)(x) q(x) dx . \end{aligned}$$

Applying Holder's inequality to the two integrals on the right hand side of this inequality gives

$$\int_I (g(f)(x))^2 \varphi(x) q(x) dx \leq A_p \left[\|f\|_p^2 + \|f^*\|_p \|g(f)\|_p \|g(\varphi)\|_r \right] .$$

By Case I, $\|g(\varphi)\|_r \leq A_r$ for $\|\varphi\|_r \leq 1$ and by Lemma 2, $\|f^*\|_p \leq A_p \|f\|_p$.

Thus we have

$$\int_I (g(f)(x))^2 \varphi(x) q(x) dx \leq A_p' [\|f\|_p^2 + \|g(f)\|_p \|f\|_p] .$$

Taking the supremum over $\varphi(x)$ gives

$$\|g(f)\|_p^2 \leq A_p' [\|f\|_p^2 + \|f\|_p \|g(f)\|_p] .$$

This shows that $\|g(f)\|_p \leq A_p'' \|f\|_p$.

Q.E.D.

Now we still have to show that $\|f\|_p \leq A_p \|g(f)\|_p$ for all f such that $\int_I f(x) q(x) dx = 0$ (i.e., $f(x)$ orthogonal to the constant functions).

To do this let us define

$$g_1(f)(x) = \left(\int_0^\infty t \left| \frac{\partial u}{\partial t}(x, t) \right|^2 dt \right)^{\frac{1}{2}} .$$

Since we obviously have that $\|g_1(f)\|_p \leq \|g(f)\|_p$ it will suffice to show if $\int_I f(x) q(x) dx = 0$ then $\|f\|_p \leq A_p \|g_1(f)\|_p$.

Now it follows from Spectral Theory (see [18], page 74) that $\|f\|_2^2 = 4\|g_1(f)\|_2^2 + \|E_0(f)\|_2^2$ where $E_0 f(x) \equiv c$ for $c = \int_I f(x) q(x) dx$. It follows by polarization (see [6]) that for $f(x) \in L^2(I)$ and $\varphi(x) \in L^2(I)$ with $E_0(f) = 0$, then

$$\begin{aligned} \left| \int_I f(x) \overline{\varphi(x)} q(x) dx \right| &= \left| 4 \int_I \int_0^\infty t \frac{\partial u}{\partial t}(x, t) \overline{\frac{\partial \varphi}{\partial t}(x, t)} dt q(x) dx \right| \\ &\leq 4 \int_I g_1(f)(x) g_1(\varphi)(x) q(x) dx \end{aligned}$$

(using the Cauchy-Schwartz inequality in integrating on the t variable)
 $\leq 4\|g_1(f)\|_2 \|g_1(\varphi)\|_2$ (by the Cauchy-Schwartz inequality again). Now we have that

$$\|f\|_p = \sup_{\|\varphi\|_{p'} \leq 1} \left| \int_I f(x) \overline{\varphi(x)} q(x) dx \right|, \quad 1/p + 1/p' = 1 .$$

But for $\|\varphi\|_{p'} \leq 1$ we have $\left| \int_I f(x) \overline{\varphi(x)} q(x) dx \right| \leq 4 \int_I g_1(f)(x) g_1(\varphi)(x) q(x) dx$
 $\leq 4 \|g_1(f)\|_p \|g_1(\varphi)\|_{p'}$ (by Holder's inequality) $\leq A_p \|g_1(f)\|_p$ (by the first part of the theorem).

Q.E.D.

Lemma 6: $\exists A_p$, independent of $f(x)$ and t such that

$$\left\| \frac{\partial u}{\partial x}(x, t) \right\|_p \leq A_p \left\| \frac{\partial u}{\partial t}(x, t) \right\|_p, \quad t \geq 0, \quad 1 < p < \infty.$$

Proof: Let $\tilde{L} = \left(\frac{d}{dx}\right)^2 + a(x)\left(\frac{d}{dx}\right) + a'(x)$. We have that $\frac{d}{dx} \circ L = \tilde{L} \circ \frac{d}{dx}$

Let now P^t be the semigroup corresponding to the operator $\tilde{L}$ and the equation $\tilde{L}u(x, t) \equiv \left(\frac{\partial}{\partial x}\right)^2 u(x, t) + a(x)\left(\frac{\partial}{\partial x}\right) u(x, t) + a'(x)u(x, t) + \left(\frac{\partial}{\partial t}\right)^2 u(x, t) = 0$, together with the boundary conditions $f(0) = f(1) = 0$, which makes $\tilde{L}$ self-adjoint (with the appropriate domain) with respect to the measure $q(x)dx$. (Formally we have $\tilde{P}^t = e^{-t(-\tilde{L})^{\frac{1}{2}}}$). Now since $\frac{d}{dx} \circ L = \tilde{L} \circ \frac{d}{dx}$,

it follows that $\frac{\partial}{\partial x} P^t f(x) = \tilde{P}^t \frac{\partial}{\partial x} f(x)$ for $f(x)$ sufficiently smooth.

Thus with our old notation $u(x, t) = P^t f(x)$, we have that $\left(\frac{\partial}{\partial x}\right) u(x, t) = \tilde{P}^t \left(\frac{\partial}{\partial x}\right) f(x)$. We note that for $\{v_i(x)\}_{i=1}^\infty$, the eigenfunctions of L , $\left\{\frac{d}{dx} v_i(x)\right\}_{i=1}^\infty$ are orthogonal eigenfunctions for $\tilde{L}$, with the same eigenvalues and satisfying $f(0) = f(1) = 0$; this is easily verified. Now

$\left\{\frac{d}{dx} v_i(x)\right\}_{i=1}^\infty$ are clearly the only eigenfunctions (apart from a multiplicative factor) since if $\varphi(x)$ is an eigenfunction for $\tilde{L}$, then

$\psi(x) = \int_0^x \varphi(x) dx$ is an eigenfunction for L . Since the constants are

the eigenfunctions for L with eigenvalue zero, it follows that $\tilde{L}$, with the given boundary conditions, does not have zero as an eigenvalue.

We recall that in the proof of the second half of Theorem 1 we established that for $f(x)$ such that $E_0(f) = 0$, $\|f\|_p \leq A_p \|g_1(f)\|_p$.

We take $f(x) = \left(\frac{\partial}{\partial x}\right)u(x, t_1)$, $t_1 \geq 0$. We have just shown that with respect to the semigroup $\tilde{P}^t$, $E_0 f = 0$. Therefore we can conclude that

$$\begin{aligned} \left\| \frac{\partial u}{\partial x}(\cdot, t_1) \right\|_p &\leq A_p \left\| \left(\int_0^\infty t \left| \left(\frac{\partial}{\partial t} \right) \tilde{P}^t \frac{\partial u}{\partial x}(x, t_1) \right|^2 dt \right)^{\frac{1}{2}} \right\|_p \\ &\equiv A_p \left\| \left(\int_0^\infty t \left| \frac{\partial^2 u}{\partial x \partial t}(x, t_1 + t) \right|^2 dt \right)^{\frac{1}{2}} \right\|_p \end{aligned}$$

(since $\tilde{P}^t \frac{\partial}{\partial x} = \frac{\partial}{\partial x} P^t$). However, this last expression

$$\leq A_p \left\| g\left(\frac{\partial u}{\partial t}(x, t_1)\right) \right\|_p \leq A'_p \left\| \left(\frac{\partial}{\partial t}\right)u(\cdot, t_1) \right\|_p$$

by the first part of Theorem 1.

Q.E.D.

Theorem 2: Again let $\{v_i(x)\}_{n=1}^\infty$ be the eigenfunctions for L with $Lv_i(x) = -\lambda_i^2 v_i(x)$. Then the mapping

$$T: v_i(x) \rightarrow \frac{\left(\frac{d}{dx}\right)v_i}{\lambda_i}$$

can be extended to a bounded linear operator on $L^p(I)$, $1 < p < \infty$.

Proof: Let $f(x) = \sum_{i=1}^n a_i v_i(x)$. By Holder's inequality

$$|a_1| = \left| \int_I f(x) q(x) dx \right| \leq \|f\|_p. \text{ It follows that if } f^* = \sum_{i=2}^n a_i v_i(x),$$

$$\|f^*\|_p \leq 2\|f\|_p. \text{ Now let } f^{**}(x) = \sum_{i=2}^n -\frac{a_i}{\lambda_i} v_i(x). \text{ We have that}$$

$$P^t(f^{**})(x) = \sum_{i=2}^n -\frac{a_i}{\lambda_i} e^{-\lambda_i t} v_i(x). \text{ Applying the previous lemma with}$$

$t = 0$ we conclude that

$$\left\| \sum_{i=2}^n -\frac{a_i}{\lambda_i} \left(\frac{d}{dx}\right)v_i(x) \right\|_p \leq A_p \left\| \sum_{i=2}^n a_i v_i(x) \right\|_p \leq 2A_p \left\| \sum_{i=1}^n a_i v_i(x) \right\|_p.$$

Q.E.D.

We now deduce the following analog of a classical result for ordinary conjugate harmonic functions.

Corollary 1: Let $f(x) \in L^P(I)$ and $u(x,t) = P^t(f)(x)$. Let $u(x,t)$ and $w(x,t)$ satisfy the following pair of "Cauchy-Riemann" equations:

$$(1) \quad \left(\frac{\partial}{\partial x}\right)u(x,t) = \left(\frac{\partial}{\partial t}\right)w(x,t), \quad (2) \quad \left(\frac{\partial}{\partial t}\right)u(x,t) = -\left(\left(\frac{\partial}{\partial x}\right) + a(x)\right)w(x,t),$$

with the normalizing condition (3) $w(x,t) \rightarrow 0$ as $t \rightarrow \infty$.

Conclusion: $\|w(\cdot, t)\|_p \leq A_p \|u(\cdot, t)\|_p$, $0 < t$.

Proof: We first note that $w(x,t)$ is uniquely determined, for if there were $w_1(x,t)$ and $w_2(x,t)$ satisfying conditions (1), (2), and (3) we would have that $w_3(x,t) = w_1(x,t) - w_2(x,t)$ would satisfy the conditions

$$(1) \quad \left(\frac{\partial}{\partial t}\right)w_3(x,t) = 0, \quad (2) \quad \left(\left(\frac{\partial}{\partial x}\right) + a(x)\right)w_3(x,t) = 0 \quad \text{and}$$

(3) $w_3(x,t) \rightarrow 0$ as $t \rightarrow \infty$. Now conditions (1) and (2) imply that $w_3(x,t) = c \exp\left[\int_0^x a(x)dx\right]$. But condition (3) gives $c = 0$, or $w_3(x,t) \equiv 0$.

Now if $f(x)$ is of the form $f(x) = \sum_{i=1}^n a_i v_i(x)$, then, of course, $u(x,t) = \sum_{i=1}^n e^{-\lambda_i t} a_i v_i(x)$ and

$$w(x,t) = \sum_{i=1}^n -\frac{e^{-\lambda_i t}}{\lambda_i} \left(\frac{d}{dx}\right)v_i(x),$$

and so Theorem 2 yields the desired result. In the general case we choose a sequence of functions $f_n(x)$, $n = 1, 2, \dots$ of the above form such that $f_n(x)$ converges to $f(x)$ in $L^P(I)$. Letting $u_n(x,t) = P^t(f_n)(x)$, we have, by the elementary properties of the kernel $p(t,x,y)$ given at the beginning of the section, that $u_n(x,t)$ converges

uniformly on I to $u(x,t)$. Let T be the mapping given in Theorem 2 and $w_n(x,t)$ be the "conjugate harmonic" function for $u_n(x,t)$. Then it is clear that $w_n(x,t) = \tilde{P}^t T(f_n(x))$. But since T is a bounded mapping, the same reasoning as we have employed for $u_n(x,t)$ shows that $w_n(x,t)$ converges uniformly on I to $w(x,t)$. The corollary now follows.

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