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Pleating Varieties in the Maskit Embeddings of Teichmüller Spaces of Punctured Spheres

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

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Chapter 1

Maskit Embeddings

1.1 Basic Definitions

Let $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ be the extended complex plane. Every conformal homeomorphism of $\hat{\mathbf{C}}$ onto itself is a Möbius transformation , i.e., a transformation of the form

$$z \longmapsto \frac{az+b}{cz+d}, \quad ad-bc=1, \quad a,b,c,d \in \mathbf{C}.$$

Let $PSL(2,\mathbf{C})$ be the projectivized group of 2×2 matrices with determinants 1. We shall always identify the matrix representing an element of $PSL(2,\mathbf{C})$ with the corresponding Möbius transformation.

The *regular set* $\Omega(G)$ of a discrete subgroup G of $PSL(2,\mathbf{C})$ is a subset of $\hat{\mathbf{C}}$ on which G acts properly discontinuously, and $\Lambda(G) = \hat{\mathbf{C}} - \Omega(G)$ is called the *limit set* of G. If $\Lambda(G)$ contains more than two points, we call G *non-elementary.* G is called a *Kleinian group* if $\Omega(G) \neq \emptyset$.

For any subset X of $\hat{\mathbf{C}}$, $Stab_G(X) = \{g \in G : g(X) = X\}$ is a subgroup of G, which is called the *stabilizer* of X in G. If $Stab_G(X) = G, X$ is said to be *G-invariant.* If $Stab_G(X) = H$ and $g(X) \cap X = \emptyset$ for all $g \in G - H$, we say that X is *precisely invariant* under H in G.

Every connected component Ω' of $\Omega(G)$ is called a *component* of G, and $\Omega'/Stab_G(\Omega')$ is a Riemann Surface. Then the quotient space $\Omega(G)/G$ is a disjoint union of Riemann surfaces. Two components Ω_1 and Ω_2 of G are *conjugate* under G if there is a $g \in G$ suth that $g(\Omega_1) = \Omega_2$. If G is finitely generated, then, by Ahlfors' finiteness theorem, the collection of conjugacy classes of components of G is finite, and each component of $\Omega(G)/G$ is a compact Riemann surface with possibly finitely many points removed.

Every Möbius transformation can also be extended to the hyperbolic 3-space $\mathbf{H}^3$ as an isometry. A subgroup G of $PSL(2, \mathbf{C})$ is discrete if and only if G acts discontinuously on $\mathbf{H}^3$,(see [1], P.95). Therefore, a Kleinian group G determines a 3-manifold $\mathbf{H}^3/G$ with boundary if $\Omega(G) \neq \emptyset$. This 3-manifold inherits a hyperbolic structure from $\mathbf{H}^3$ and its boundary inherits a conformal structure from $\hat{\mathbf{C}}$.

An *open half space* in $\mathbf{H}^3$ is one of the components of the complement of a hyperplane in $\mathbf{H}^3$. A *polyhedron* P in $\mathbf{H}^3$ is the intersection of countably many open half spaces, where only finitely many of the hyperplanes, defining these open half spaces, meet any compact subset of $\mathbf{H}^3$. Each of the defining hyperplanes is called a *side* of P.

Let G be a Kleinian group. A *fundamental polyhedron* D for G acting on $\mathbf{H}^3$ is a polyhedron in $\mathbf{H}^3$ satisfying the following four properties:

(1) For every non-trivial element g of G, $g(D) \cap D = \emptyset$, i.e., D is precisely invariant under $\{1\}$ in G.

(2) $\cup_{g \in G} g(\bar{D}) = \mathbf{H}^3$, where $\bar{D}$ is the closure of D in $\mathbf{H}^3$.

(3) Any compact subset of $\mathbf{H}^3$ meets only finitely many G-translates of D.

(4) The sides of D are paired by elements of G.

If G has a finite sided fundamental polyhedron, G is said to be *geometrically finite*.

Let G be a Kleinian group with $\Omega(G) \neq \emptyset$. Let S be a component of $\Omega(G)/G$. A *marking* for S is a basis for $\pi_1(S)$ defined up to change of base point. If all components of $\Omega(G)/G$ are marked, the marking curves determine a distinguished set of generators for G.

Let S be a given Riemann surface which is marked, and let $Teich(S)$ be the Teichmüller space of S. $Teich(S)$ consists of isotopy classes (rel. ∂S) of quasiconformal mappings of S; the images of S under these quasiconformal mappings are again marked Riemann surfaces homeomorphic to S with different conformal structures.

Let Σ_n be an n-times punctured sphere, where $n \geq 4$, and let $\alpha_j, 1 \leq j \leq n$ be simple closed curves on Σ_n whose homotopy classes generate $\pi_1(\Sigma_n)$ such that each of the curves α_j's separates exactly one of the punctures of Σ_n from the others, and such that $\alpha_1\alpha_2 \cdots \alpha_n$ is homotopically trivial. Σ_n together with the homotopy classes represented by α_j's defined up to change of base point is called a *marked n-times punctured sphere.*

In the following sections of this chapter, We want to represent $Teich(\Sigma_n)$ as a space of Kleinian groups with a distinguished set of generators having certain special properties.

Let $\tilde{\mathcal{M}}_n, n \geq 4$, be the space of the Kleinian groups G which satisfy the following properties:

(M.1) G is a free group generated by $n-1$ parabolic Möbius transformations $X_j, j = 1, ..., n-1$.

(M.2) $\Omega(G)$ has a simply connected G-invariant component Ω_0 for which

$$\Omega_0/Stab_G(\Omega_0) = \Omega_0/G$$

is an n-times punctured sphere.

(M.3) The other components $\Omega_i, i \geq 1$, of G are all non-invariant under G and they fall into $n-2$ distinct conjugacy classes, and each $\Omega_i/Stab_G(\Omega_i)$ is a thrice punctured sphere.

1.2 Maskit Embedding of Teich(Σ_4)

In this section, we consider the case where $n = 4$. First, we are going to show that $\tilde{\mathcal{M}}_4$ is non-empty.

To this end, we need Maskit's first combination theorem. Let G_1 and G_2 be two Kleinian groups with a common subgroup J and $J \neq G_i$. A pair (B_1, B_2) of two non-empty disjoint sets is called an *interactive pair* if

(1) B_1 and B_2 are J-invariant,

(2) every element of $G_1 - J$ maps B_1 into B_2, and

(3) every element of $G_2 - J$ maps B_2 into B_1.

An interactive pair (B_1, B_2) is called *proper* if either there is a point of B_1 that is not G_2-equivalent to any point of B_2, or there is a point of B_2 that is not G_1-equivalent to any point of B_1.

Let G be a Kleinian group, and J be a subgroup of G. A closed J-invariant set B is called a *(J, G)-block* if

(1) $B \cap \Omega(G) = B \cap \Omega(J)$, and $\Omega(J)$ is precisely invariant under J in G, and

(2) for every puncture on $\Omega(J)/J$, there is a neighborhood U of the puncture (i.e. U is a punctured disc), such that

$$U \subset (B \cap \Omega(J))/J \quad \text{or} \quad U \cap [(B \cap \Omega(J))/J] = \emptyset.$$

Theorem 1.1 (Maskit's First Combination Theorem) ([14], P.149)

Let J be a geometrically finite subgroup of the discrete groups G_1 and G_2. Assume that $J \neq G_i, i = 1, 2$, and there is a simple closed curve γ dividing $\hat{\mathbf{C}}$ into two closed topological discs B_1 and B_2, where B_i is a (J, G_i)-block, and $(Int\ B_1,\ Int\ B_2)$ is a proper interactive pair. Let D_i be a fundamental set for G_i such that

(1) *$D_i \cap B_i$ is a fundamental set for J acting on B_i,*

(2) *$D_i \cap B_{3-i} = \emptyset$ or $Int(D_i \cap B_{3-i}) \neq \emptyset$, and*

(3) *$D_1 \cap \gamma = D_2 \cap \gamma$.*

Let

$$D = (D_1 \cap B_2) \cup (D_2 \cap B_1) \quad \text{and} \quad G = < G_1, G_2 > .$$

Then the following statements hold:

(i) *G is the free product of G_1 and G_2 with amalgamated subgroup J, i.e., $G = G_1 *_J G_2$.*

(ii) *G is a Kleinian group, and D is a fundamental set for G.*

(iii) *If γ is precisely invariant under J in G_1 or G_2, except perhaps for conjugates of elements of G_1 and G_2, every element of G is loxodromic.*

(iv) *G is geometrically finite if and only if G_1 and G_2 are both geometrically finite.*

Let $G_* \subset PSL(2,\mathbf{C})$ be the group generated by the following transformations:

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1+4i & 16 \\ 1 & 1-4i \end{pmatrix}.$$

Now, we prove that G_* is a Kleinian group by using Maskit's first combination theorem. Let $H_1 =< S,T >, H_2 =< X,T >$, $J =< T >$,

$$\begin{aligned} F_1 &= \{z : -2 \leq Re\, z < 2, |z+1| > 1 \text{ and} |z-1| \geq 1\}, \text{ and} \\ F_2 &= \{z : -2 \leq Re\, z < 2, |z-1-4i| \geq 1 \text{ and } |z+1-4i| > 1\}. \end{aligned}$$

F_i is a fundamental set for H_i acting on $\hat{\mathbf{C}}$ for each $i = 1,2$.

Let $C = \{z : Im\, z = 2\} \cup \{\infty\}, B_1 = \{z : Im\, z \geq 2\} \cup \{\infty\}$ and $B_2 = \{z : Im\, z \leq 2\} \cup \{\infty\}$. Then we have:

(1) $F_1 \cap C = F_2 \cap C$, and $Int(F_i \cap B_{3-i}) \neq \emptyset$ for $i = 1,2$.

(2) $F_i \cap B_i$ is a fundamental set for J acting on $B_i, i = 1,2$.

(3) B_i is a closed J-invariant set, $B_i \cap \Omega(H_i) = B_i \cap \Omega(J)$ is J-invariant and $g(B_i \cap \Omega(J)) \cap (B_i \cap \Omega(J)) = \emptyset$ for all $g \in H_i - J, i = 1,2$. It is clear that every puncture on $\Omega(J)/J$ has a neighborhood contained in the projection of B_i or disjoint from the projection of B_i. These imply that B_i is a (J,H_i)-*block*,$i = 1,2$.

(4) $Int\ B_1$ and $Int\ B_2$ are nonempty disjoint J-invariant sets and

$$h(Int\ B_1) \subset Int\ B_2, \forall h \in H_1 - J, \text{ and } h(Int\ B_2) \subset Int\ B_1, \forall h \in H_2 - J.$$

Moreover,

$$Int\ B_1 - \cup_{h \in H_2} h(Int\ B_2) \neq \emptyset$$

since it contains the set $\{z : Im\, z > 6\}$. These conditions imply that $(Int\ B_1, Int\ B_2)$ is a *proper interactive pair.*

We have proved that all the hypotheses of the first combination theorem are satisfied, and thus we have:

(1) $G_* = H_1 *_J H_2 =< H_1, H_2 >$ is discrete, and geometrically finite,

(2) $F_* = (F_1 \cap B_2) \cup (F_2 \cap B_1)$ is a fundamental set for G_* acting on $\hat{\mathbf{C}}$, and

(3) except for conjugates of elements of H_1 and H_2, every element of G_* is loxodromic since C is precisely invariant under J in H_i.

From (1) and (3), we know that G_* is torsion free, and G_* satisfies **(M.1)**.

Next, prove that G_* has a simply connected G_*-invariant component Ω_* such that Ω_*/G_* is a four times punctured sphere.

Let

$$\begin{aligned} D_1 &= \{z \in F_* : Im\, z < 0\}, \\ D_2 &= \{z \in F_* : Im\, z > 4\}, \text{ and} \\ D_0 &= \{z \in F_* : 0 < Im\, z < 4\}, \end{aligned}$$

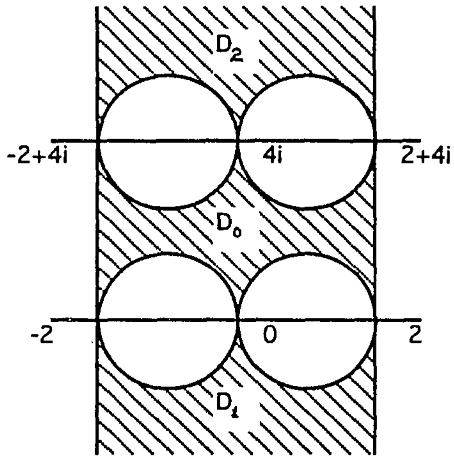


Figure 1.1: Shaded region is F_*

(see Figure 1). Note that H_1 is a Fuchsian group and D_1 is a fundamental set for H_1 acting on the lower half plane Ω_1, then

$$\Omega_1 = \cup_{h\in H_1} \ h(D_1)$$

is a component of G_* since $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is contained in $\Lambda(G_*)$. H_2 is conjugate to H_1 in $PSL(2, \mathrm{C})$, then

$$\Omega_2 = \cup_{h\in H_2} \ h(D_2) = \{z : Im\ z > 4\}$$

is a component of G_* and D_2 is a fundamental set for H_2 acting on Ω_2. Moreover, since $Stab_{G_*}(\Omega_i) = H_i$, then Ω_i/H_i is a thrice punctured sphere. Ω_i is precisely invariant under H_i in G_*, then Ω_i is a non-invariant component of $G_*, i = 1, 2$.

Let

$$\Omega_* = \cup_{g\in G_*}\ g(D_0).$$

It is clear that Ω_* is G_*-invariant. We shall prove that Ω_* is path-connected as follows:

Fix arbitrarily a point x_0 of D_0, and let x be any point of Ω_*. There is a $g \in G_*$ and an $x_1 \in D_0$ such that $g(x_1) = x$. Since D_0 is path-connected, we may assume that $g \neq 1$, and we want to find a path in Ω_* joining x_1 to $x = g(x_1)$. This proves that Ω_* is path-conneted.

There exist $g_1, ..., g_n \in \{S, S^{-1}, T, T^{-1}, X, X^{-1}\}$ such that

$$g = g_1 \circ \cdots \circ g_n.$$

For each $i = 1, .., n$, Let $x_{i+1} = g_1 \circ \cdots \circ g_i(x_1)$. Since each g_i is a side pairing of D_0, then for every i there is a path ℓ_i in Ω_* joining x_i to x_{i+1}, and thus these paths $\ell_1, .., \ell_n$ give a path in Ω_* joining x_1 to $x_{n+1} = x$.

Note that

$$D_0 \cup S(D_0) \cup S^{-1}(D_0) \cup T(D_0) \cup T^{-1}(D_0) \cup X(D_0) \cup X^{-1}(D_0)$$

is a neighborhood of $\bar{D}_0 \cap \Omega(G_*)$, then Ω_* is open.

To show that Ω_* is a component of G_*, it suffices to show that every boundary point of Ω_* in $\hat{\mathbf{C}}$ is in $\Lambda(G_*)$.

Let y be any boundary point of Ω_*. There is a sequence x_n of distinct points of Ω_* converging to y. Let $g_n \in G_*$ such that $g_n^{-1}(x_n) \in D_0$.

If there is a subsequence $\{g_{n,k}\}$ of $\{g_n\}$ with $g_{n,k} = g_{n,1}, \forall k$, let $g = g_{n,1}$. Then $g^{-1}(x_{n,k})$ are distinct points of D_0 and $g^{-1}(x_{n,k}) \to g^{-1}(y)$. This proves that

$$g^{-1}(y) \in \bar{D}_0 - \Omega_* = \{-2, 0, 2, 4i, 2+4i, -2+4i\} \subset \Lambda(G_*).$$

Then $y \in \Lambda(G_*)$.

Assume that $g_n \neq g_m$ whenever $n \neq m$. Since G_* is geometrically finite,

$$\text{(the spherical diameter of } g_n(D_0)) \longrightarrow 0 \text{ as } n \to \infty,$$

and thus $y \in \Lambda(G_*)$.

From the above arguments, Ω_* is a G_*-invariant component of G_*, and Ω_*/G_* is a four times punctured sphere. Let α, β, γ and δ be a marking of Ω_*/G_* with $\alpha\beta\gamma\delta = 1$. Then

$$\alpha \longmapsto X, \ \ \beta \longmapsto S^{-1} \text{ and } \gamma \longmapsto ST^{-1}$$

defines an isomorphism of $\pi_1(\Omega_*/G_*)$ onto G_*.

Let $p : \Omega_* \longrightarrow \Omega_*/G_* = \Sigma_4$ be the canonical projection, and $\tilde{p}$ be the homomorphism between $\pi_1(\Omega_*)$ and $\pi_1(\Sigma_4) \cong G_*$ induced by p. Since G_* acts properly discontinuously on Ω_*, p is a regular covering with deck transformation group G_*,

and

$$\begin{aligned} G_* &= \text{the deck transformation group of } p \\ &\cong \pi_1(\Sigma_4)/\tilde{p}(\pi_1(\Omega_*)). \end{aligned}$$

Thus $\tilde{p}(\pi_1(\Omega_*))$ is trivial, and Ω_* is simply connected since $\tilde{p}$ is a monomorphism. Therefore, G_* satisfies **(M.2)**.

Finally, we want to show that G_* satisfies **(M.3)**, and thus $G_* \in \tilde{\mathcal{M}}_4$. Let Ω' be any component of G_* with $\Omega' \cap \Omega_* = \emptyset$, and let x be any point of Ω'. There is a $g \in G_*$ such that

$$g(x) \in F_* = D_0 \cup D_1 \cup D_2.$$

Ω_* is G_*-invariant and $\Omega' \cap \Omega_* = \emptyset$, then $g(x)$ is not in D_0, and $g(x) \in D_1$ or $g(x) \in D_2$.

Assume that $g(x) \in D_1$. $g(\Omega')$ is also a component of G_*, and

$$g(\Omega') \cap \Omega_1 \supset g(\Omega') \cap D_1 \neq \emptyset,$$

then $g(\Omega') = \Omega_1$. Similarly, $g(\Omega') = \Omega_2$ if $g(x) \in D_2$. We conclude that Ω' is a simply connected component of G_* conjugate to Ω_1 or Ω_2, and thus $\Omega'/Stab_{G_*}(\Omega')$ is a thrice punctured sphere. Since Ω_1 and Ω_2 are not G_*-invariant, nor is Ω'.

To complete the proof, we have to show that Ω_1 and Ω_2 are not conjugate in G_*, or equivalently, $g(\Omega_1) \cap \Omega_2 = \emptyset$ for all $g \in G_*$.

If $g \in H_1 =< T, S >$, then

$$g(\Omega_1) \cap \Omega_2 = \Omega_1 \cap \Omega_2 = \emptyset.$$

Let $g \in H_2 =< T, X >$. Since $\Omega_1 \subset \{z : Im\ z < 4\} = U$ and U is invariant under H_2, then

$$g(\Omega_1) \cap \Omega_2 \subset U \cap \Omega_2 = \emptyset.$$

Since $G_* = H_1 *_J H_2$, then, for any $g \in G_* - J$, g can be written as

$$g = g_1 \circ \cdots \circ g_n, \text{ for some } n > 0,$$

where $g_i \in H_1 \cup H_2 - J$. Since $g_i(\Omega_1) \subset U, \forall i$, then $g(\Omega_1) \subset U$, and thus $g(\Omega_1) \cap \Omega_2 = \emptyset$.

Further more, we have

$$\Lambda(G_*) = \overline{\cup_{g \in G_*}\ g(W)},$$

where

$$W = \mathbf{R} \cup \{z : Im\ z = 4\} \cup \{\infty\} = \Lambda(H_1) \cup \Lambda(H_2).$$

Since $\Lambda(G_*)$ is G_*-invariant and $W \subset \Lambda(G_*)$, $\Lambda(G_*)$ contains

$$\overline{\cup_{g \in G_*}\ g(W)} = K.$$

On the other hand, G_* is non-elementary, and K is G_*-invariant, then

$$K \supset \Lambda(G_*)$$

since $\Lambda(G_*)$ is the smallest non-empty G_*-invariant closed subset of $\hat{\mathbf{C}}$.

Now, we are going to establish a bijective correspondence between conjugacy classes of groups in $\tilde{\mathcal{M}}_4$ and points in $Teich(\Sigma_4)$ as follows. Since the Teichmüller spaces of two quasiconformally equivalent Riemann surfaces are isometrically equivalent, we may set $\Sigma_4 = \Omega_*/G_*$.

Let $f : \Sigma_4 \longrightarrow f(\Sigma_4)$ represent an isotopy class in $Teich(\Sigma_4)$, and let

$$\mu \, \frac{d\bar{z}}{dz}$$

be the Beltrami differential of f. By definition, $\mu \frac{d\bar{z}}{dz}$ is invariant under local coordinate changes, then

$$\mu(g(z)) \, \frac{\overline{g'(z)}}{g'(z)} \;\; = \;\; \mu(z), \;\; \forall \, z \in \Omega_* \;\text{ and }\; \forall \, g \in G_*. \tag{1.1}$$

This gives a Beltrami differential on Ω_*, also denoted by μ, satisfying (1.1). Extend μ to $\hat{\mathbf{C}}$ by setting $\mu \equiv 0$ outside Ω_*. There is a quasiconformal homeomorphism f^μ of $\hat{\mathbf{C}}$ onto itself such that f^μ fixes 0, 1 and ∞, and

$$f^\mu_{\bar{z}} = \mu \, f^\mu_z.$$

Let g be any element of G_*. $w = f^\mu \circ g$ is again a quasiconformal automorphism of $\hat{\mathbf{C}}$ with

$$w_{\bar{z}} = \mu \, w_z,$$

then there is a Möbius transformation g_μ such that

$$f^\mu \circ g = g_\mu \circ f^\mu, \quad \text{i.e.} \quad g_\mu = f^\mu \circ g \circ (f^\mu)^{-1}.$$

It is clear that T_μ, X_μ and S_μ are parabolic since each of them has only one fixed point in $\hat{\mathbf{C}}$. Then

$$G_\mu = f^\mu\, G_*\, (f^\mu)^{-1} = < X_\mu, T_\mu, S_\mu >$$

is an element of $\tilde{\mathcal{M}_4}$ and Ω_μ / G_μ is a four times punctured sphere, where $\Omega_\mu = f^\mu(\Omega_*)$.

Since f^μ fixes 0 and ∞, T_μ fixes ∞ and S_μ fixes 0. We write

$$S_\mu = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix}, \quad T_\mu = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \quad X_\mu = \begin{pmatrix} 1+\lambda\zeta & -\lambda\zeta^2 \\ \lambda & 1-\lambda\zeta \end{pmatrix},$$

where $\xi, \eta, \lambda \in \mathbf{C} - \{0\}$, and $\zeta = f^\mu(4i)$ is the fixed point of X_μ.

$$S_\mu(\infty) = f^\mu \circ S \circ (f^\mu)^{-1}(\infty) = f^\mu \circ S(\infty) = f^\mu(1) = 1,$$

then $\eta = 1$.

$$T^{-1} \circ S = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}, \quad T^{-1} \circ X = \begin{pmatrix} -3+4i & 12+16i \\ 1 & 1-4i \end{pmatrix}$$

are parabolic, then so are

$$T_\mu^{-1} \circ S_\mu = \begin{pmatrix} 1-\xi & -\xi \\ 1 & 1 \end{pmatrix}, \quad T_\mu^{-1} \circ X_\mu = \begin{pmatrix} 1+\lambda(\zeta-\xi) & -\lambda\zeta^2 - \lambda\zeta\xi - \xi \\ \lambda & 1-\lambda\zeta \end{pmatrix}.$$

The discriminant of the equation $z^2 + \xi z + \xi = 0$ is zero, then $\xi = 4$. Similarly, we have $\lambda = 1$. Hence

$$G_\mu = < \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1+\zeta & -\zeta^2 \\ 1 & 1-\zeta \end{pmatrix} > .$$

Since

$$< \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1+\zeta & -\zeta^2 \\ 1 & 1-\zeta \end{pmatrix} >$$

is a non-elementary discrete subgroup of G_μ, $|\zeta| \geq 1$ by Jorgensen's inequality.

Since $f^\mu \circ T \circ (f^\mu)^{-1} = T_\mu = T$ and $f^\mu \circ S \circ (f^\mu)^{-1} = S_\mu = S$, then

$$f^\mu\, H_1\, (f^\mu)^{-1} = H_1 \text{ and } f^\mu\, J\, (f^\mu)^{-1} = J.$$

Let

$$H_\mu = f^\mu\, H_2\, (f^\mu)^{-1} = < T, X_\mu > .$$

Consider the sets $f^\mu(F_i), f^\mu(B_i), i = 1, 2$, and $f^\mu(C)$, where F_i, B_i and C are defined in the construction of G_*. Then these sets and groups H_1, H_μ and J satisfy all the hypotheses of Maskit's first combination theorem, and thus

$$G_\mu = H_1 *_J H_\mu,$$

except for conjugates of elements of H_1 and H_2 every element of G_μ is loxodromic, and G_μ is geometrically finite since H_1 and H_μ are geometrically finite.

Conversely, let G be any group in $\tilde{\mathcal{M}}_4$ and Ω_0 be the simply connected G-invariant component of G so that Ω_0/G is a four times punctured sphere. We want to find a quasiconformal automorphism of $\hat{\mathbf{C}}$ which conjugates G_* to G, and this map gives a quasiconformal homeomorphism of Ω_*/G_* onto Ω_0/G. By a similar argument as before, it suffices to find a quasiconformal homeomorphism of Ω_* onto Ω_0 whose Beltrami coefficient satisfies (1.1).

Let φ be a conformal homeomorphism of Ω_* onto the upper half plane $\mathbf{H}^2$, and ψ be a conformal homeomorphism of Ω_0 onto $\mathbf{H}^2$.

Let g be any element of G_*. $\varphi \circ g \circ \varphi^{-1}$ is a conformal homeomorphism of $\mathbf{H}^2$ onto itself, then $\varphi \circ g \circ \varphi^{-1}$ is a Möbius transformation by Schwarz lemma. This proves that $\varphi\, G_*\, \varphi^{-1}$ is a Fuchsian group and $\mathbf{H}^2/\varphi\, G_*\, \varphi^{-1}$ is a four times punctured sphere. Similarly, $\psi\, G_*\, \psi^{-1}$ is a Fuchsian group and $\mathbf{H}^2/\psi\, G_*\, \psi^{-1}$ is a four times punctured sphere. There are standard arguments to show that there is a quasiconformal homeomorphism f of $\mathbf{H}^2$ onto itself whose Beltrami coefficient μ_f satisfies

$$\mu_f(g(z))\overline{g'(z)} = \mu_f(z)g'(z), \ \ \forall\, z \in \mathbf{H}^2 \text{ and } \forall\, g \in \varphi\, G_*\, \varphi^{-1},$$

(see, for example, [8], §8). Let $\hat{f} = \psi^{-1} \circ f \circ \varphi$, and let $\mu_{\hat{f}}$ be the Beltrami coefficient of $\hat{f}$. It is clear that $\hat{f}$ is a quasiconformal homeomorphism of Ω_* onto Ω_0. By Marden's isomorphism theorem [13], one can extend $\hat{f}$ to a quasiconformal homeomorphism of $\hat{\mathbf{C}}$ onto itself so that the extension of $\hat{f}$ conjugates G_* to G.

As a consequence of the above argument, we have the following proposition.

Proposition 1.2 (Normalization) *Every group in $\tilde{\mathcal{M}}_4$ is geometrically finite, and it is conjugate to a group generated by*

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \textit{and} \quad \begin{pmatrix} 1+\zeta & -\zeta^2 \\ 1 & 1-\zeta \end{pmatrix},$$

where $|\zeta| \geq 1$.

We have shown that there is a function which maps the set of conjugacy classes of groups in $\tilde{\mathcal{M}}_4$ onto $Teich(\Sigma_4)$. Next, we want to check the injectivity of this function. Suppose that G and G' are groups in $\tilde{\mathcal{M}}_4$ which represent the same point in $Teich(\Sigma_4)$, i.e., $\Omega_0(G)/G$ and $\Omega_0(G')/G'$ are conformally equivalent. Then there is conformal homeomorphism f of $\Omega_0(G)$ onto $\Omega_0(G')$ which induces an isomorphism of G onto G'. Then, by Marden's isomorphism theorem again, there is a conformal automorphism of $\hat{\mathbf{C}}$ which is an extension of f and also denoted by f such that $f\, G\, f^{-1} = G'$. f is a Möbius transformation , then G and G' are conjugate.

Let

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix},$$

$$X_\zeta = \begin{pmatrix} 1+\zeta & -\zeta^2 \\ 1 & 1-\zeta \end{pmatrix}, \text{ and } \quad A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

$$ATA^{-1} = T^{-1}, \ ASA^{-1} = S^{-1}, \text{ and } AX_\zeta A^{-1} = X_{-\zeta}^{-1},$$

then $< T, S, X_\zeta >$ and $< T, S, X_{-\zeta} >$ are conjugate. This proves that every group in $\tilde{\mathcal{M}}_4$ is conjugate to $< T, S, X_\zeta >$ for some ζ with $|\zeta| \geq 1$ and $Im\,\zeta \geq 0$. On the other hand, by a similar argument as that of proving $G_* \in \tilde{\mathcal{M}}_4$, we can prove that $< T, S, X_\zeta > \in \tilde{\mathcal{M}}_4$ when $\zeta = it$ with $t > 2$. Therefore, we may identify $Teich(\Sigma_4)$ with $\mathcal{M}_4$, where $\mathcal{M}_4$ is a connected component of

$$\{\mu \in \mathbf{C} : < T, S, X_\mu > \in \tilde{\mathcal{M}}_4, |\mu| \geq 1, \text{ and } Im\,\mu \geq 0\}$$

containing $\mu = it$ with $t > 2$. We refer to $\mathcal{M}_4$ as the *Maskit embedding* of $Teich(\Sigma_4)$, and we write $G_\mu =< T, S, X_\mu >$.

Proposition 1.3 (Rough shape of $\mathcal{M}_4$)

(1) $\mu \in \mathcal{M}_4$ *if and only if* $\mu + 2 \in \mathcal{M}_4$.

(2) $\mu \in \mathcal{M}_4$ *if and only if* $-\overline{\mu} \in \mathcal{M}_4$.

(3) $\{\mu : Im\,\mu > 2\} \subset \mathcal{M}_4 \subset \{\mu : Im\,\mu > 0\}$.

Proof: $X_{\mu+2} = TX_\mu^{-1}$ and $< T, S, X_\mu > = < T, S, TX_\mu^{-1} >$, then (1) follows.

For the proof of (2), we only have to show that $-\bar{\mu} \in \mathcal{M}_4$ if $\mu \in \mathcal{M}_4$. Write the elements of G_μ and $G_{-\bar{\mu}}$ as matrices. It is easy to see that the matrix of every element of $G_{-\bar{\mu}}$ is obatined from the corresponding matrix in G_μ by replacing μ by $-\bar{\mu}$, and the fixed points of elements of $G_{-\bar{\mu}}$ and G_μ are functions of the entries of the corresponding matrices. Therefore, this map $\mu \longmapsto -\bar{\mu}$ maps $\Lambda(G_\mu)$ onto $\Lambda(G_{-\bar{\mu}})$, and hence $-\bar{\mu} \in \mathcal{M}_4$ if $\mu \in \mathcal{M}_4$.

Next, we prove that $\mathbf{R} \cap \mathcal{M}_4 = \emptyset$. From (1), we only have to show that

$$(-1, 1] \cap \mathcal{M}_4 = \emptyset.$$

By the definition of $\mathcal{M}_4$, $(-1, 1) \cap \mathcal{M}_4 = \emptyset$. If $\mu = 1$,

$$X_\mu S(z) = z - 1, \quad X_\mu S^2 = \frac{-1}{z},$$

and thus

$$< T, S, X_\mu > = < T, S, X_\mu S > = < S, X_\mu S > = < X_\mu S^2, X_\mu S >,$$

i.e., G_μ is the modular group. Hence, 1 is not in $\mathcal{M}_4$.

Finally, we prove that $\{\mu : Im\,\mu > 2\} \subset \mathcal{M}_4$. From (1), it suffices to show that $\mu \in \mathcal{M}_4$ if $Im\,\mu > 2$ and $0 \leq Re\,\mu < 2$. Consider the following groups and sets, (see Figure 1.2):

$$\begin{aligned} H_1 &= < T, S >, \; H_2 = < T, X_\mu >, \; J = < T >, \\ B_1 &= \{z : Im\,z \geq Im\,\mu/2\} \cup \{\infty\}, \\ B_2 &= \{z : Im\,z \leq Im\,\mu/2\} \cup \{\infty\}, \end{aligned}$$

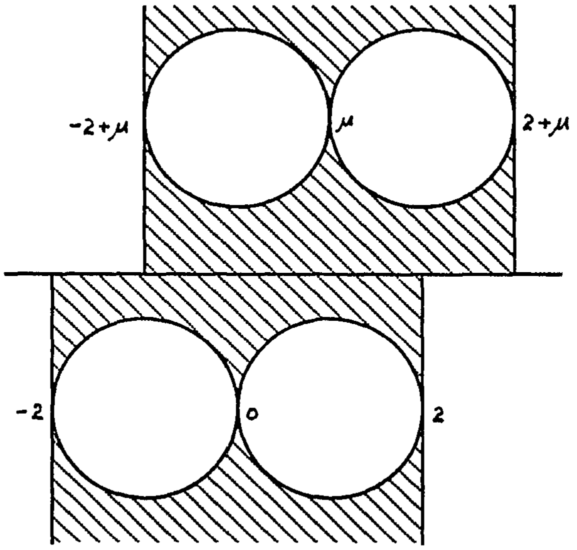


Figure 1.2:

$$\begin{aligned}
\gamma &= \{z : Im\ z = Im\ \mu/2\} \cup \{\infty\},\\
D_1 &= \{z : |z+1| > 1, \text{ and } |z-1| \geq 1\},\\
D_2 &= \{z : |z+1-\mu| > 1, \text{ and } |z-1-\mu| \geq 1\},\\
E_1 &= \{z : Im\ z \leq Im\ \mu/2 \text{ and } -2 \leq Re\ z < 2\},\\
E_2 &= \{z : Im\ z > Im\ \mu/2 \text{ and } -2 + Re\ \mu \leq Re\ z < 2 + Re\ \mu\},\\
D_0 &= E_1 \cap E_2,\ F_1 = D_0 \cap D_1 \text{ and } F_2 = D_0 \cap D_2.
\end{aligned}$$

F_j is a fundamental set for H_j acting on $\hat{\mathbf{C}}, j = 1, 2$, and

(1) B_j is a (J, H_j)-block,

(2) ($Int\ B_1$, $Int\ B_2$) is a proper interactive pair,

(3) $F_j \cap B_{3-j}$ has non-empty interior,

(4) $F_j \cap B_j$ is a fundamental set for J acting on B_j, and

(5) $F_1 \cap \gamma = F_2 \cap \gamma$.

By the first combination theorem, $G_\mu = H_1 *_J H_\mu$. By a similar argument as before.

$$\Lambda(G_\mu) = \overline{\cup_{g \in G_\mu}\ g(W)},$$

where $W = \{z : Im\ z = Im\ \mu\} \cup \mathbf{R} \cup \{\infty\}$,

$$\Omega_0(G_\mu) = \cup_{g \in G_\mu}\ g(D)$$

is the only simply connected G_μ-invariant component of G_μ with $\Omega_0(G_\mu)/G_\mu$ a four times punctured sphere, and the other components are conjugate to $\{z : Im\ z > Im\ \mu\}$ or $\{z : Im\ z < 0\}$ and are not G_μ-invariant, where

$$D = \{z \in F_1 \cap F_2 : 0 < Im\ z < Im\ \mu\}.$$

Therefore, $\mu \in \mathcal{M}_4$ if $Im\ \mu > 2$. **Q.E.D.**

1.3 Maskit Embedding of Teich(Σ_5)

Let $G_* \subset PSL(2, \mathbf{C})$ be the group genenrated by

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} 1+4i & 16 \\ 1 & 1-4i \end{pmatrix}, \text{ and } \quad Y = \begin{pmatrix} 1+4i & 4 \\ 4 & 1-4i \end{pmatrix}.$$

Let $H_1 =< T, X >, H_2 =< S, T, Y >$ and $J =< T >$, and let

$$\begin{aligned} F_0 &= \{z : -2 \leq Re\ z < 2\}, \\ F_1 &= \{z \in F_0 : |z+1-4i| > 1,\ \ |z-1-4i| \geq 1\}, \\ F_2 &= \{z \in F_0 : |z+1| > 1, |z-1| \geq 1, |z+\frac{1}{4}-i| \geq \frac{1}{4}, |z-\frac{1}{4}-i| > \frac{1}{4}\}. \end{aligned}$$

Then F_i is a fundamental set for H_i acting on $\hat{\mathbf{C}}$, $i = 1,2$.

Let

$$\begin{aligned} \gamma &= \{z : Im\ z = 2\} \cup \{\infty\}, \\ B_1 &= \{z : Im\ z \leq 2\} \cup \{\infty\}, \quad \text{and} \\ B_2 &= \{z : Im\ z \geq 2\} \cup \{\infty\}. \end{aligned}$$

Then we have:

(1) $F_1 \cap \gamma = F_2 \cap \gamma$, and $Int(F_i \cap B_{3-i}) \neq \emptyset$ for i=1,2,

(2) $F_i \cap B_i$ is a fundamental set for J acting on B_i, $i = 1,2$,

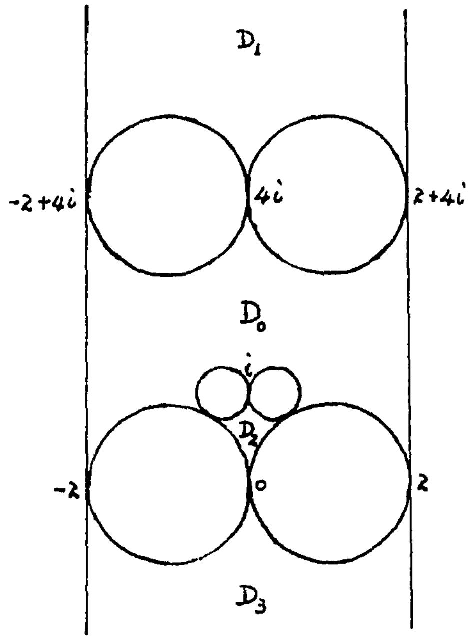
(3) γ is precisely invariant under J in H_i, $i = 1,2$,,

(4) B_i is closed and J-invariant, $B_i \cap \Omega(H_i) = B_i \cap \Omega(J)$, $B_i \cap \Omega(J)$ is precisely invariant under J in H_i, and every puncture of $\Omega(J)/J$ has a neighborhood contained in the projection of B_i or disjoint from the projection of B_i, and

(5) $Int\ B_1$ and $Int\ B_2$ are non-empty disjoint J-invariant sets and

$$h(Int\ B_i) \subset Int\ B_{3-i}, \quad \text{for all } h \in H_i - J, i = 1,2, \text{ and}$$

$$Int\ B_1 - \cup_{h \in H_2} h(Int\ B_2) \neq \emptyset.$$



The first combination theorem yields:

(1) $G_* = H_1 *_J H_2 = < H_1, H_2 >$ is discrete, and geometrically finite.

(2) $F = (F_1 \cap B_2) \cup (F_2 \cap B_1)$ is a fundamental set for G_* acting on $\hat{\mathbf{C}}$.

(3) G_* is torsion free, and except for conjugates of elements of H_1 and H_2, every element of G_* is loxodromic. Thus every parabolic element of G_* is conjugate in G_*

to one of the following:

$$S^n, T^n, X^n, (S^{-1}T)^n, (Y^{-1}S)^n, (X^{-1}T)^n,\ n \in \mathbf{Z} - \{0\}.$$

We have shown that G_* satisfies **(M.1)**. Next, we are going to prove that G_* satisfies **(M.2)** and **(M.3)**. Let

$$\begin{aligned} D_1 &= \{z : \operatorname{Im} z > 4\}, \\ D_2 &= \{z \in F : -\frac{2}{5} < \operatorname{Re} z < \frac{2}{5},\ 0 < \operatorname{Im} z < 1\}, \\ D_3 &= \{z \in F : \operatorname{Im} z < 0\}, \quad \text{and} \\ D_0 &= F - (D_1 \cup D_2 \cup D_3). \end{aligned}$$

H_1 is a Fuchsian group, D_1 is a fundamental set for H_1 acting on $\{z : \operatorname{Im} z > 4\}$, and $\Lambda(H_1) = \{z : \operatorname{Im} z = 4\} \cup \{\infty\}$, then

$$\Omega_1 = \{z : \operatorname{Im} z > 4\} = \cup_{h \in H_1} h(D_1)$$

is a component of G_*, Ω_1/H_1 is a thrice punctured sphere, and Ω_1 is not G_*-invariant since $Stab_{G_*}(\Omega_1) = H_1 \neq G_*$. Similarly, $\Omega_3 = \{z : \operatorname{Im} z < 0\}$ is a component of G_*, $Stab_{G_*}(\Omega_3) =< T, S >= H_3$, and Ω_3/H_3 is a thrice punctured sphere.

Let $\Omega_2 = \{z : |z - \frac{i}{2}| < \frac{1}{2}\}$ and $\tilde{H}_2 =< S, Y >$. $\tilde{H}_2$ is a Fuchsian group, D_2 is a fundamental set for $\tilde{H}_2$ acting on Ω_2, $\Lambda(\tilde{H}_2) = \partial\Omega_2$, $Stab_{G_*}(\Omega_2) = \tilde{H}_2$, then Ω_2 is a component of G_*, $\Omega_2/\tilde{H}_2$ is a thrice punctured sphere, and Ω_2 is not G_*-invariant.

It is clear that $S(\Omega_1) \cup Y(\Omega_1) \subset Ext(\Omega_2) \cap Ext(\Omega_3)$, then Ω_1 is not conjugate to Ω_2, and is not conjugate to Ω_3 under G_*. Similarly, Ω_2 is not conjugate to Ω_3. This proves that G_* satisfies **(M.3)**.

By a similar argument as in §**1.2**, one can prove that G_* satisfies **(M.2)**, and

$$\Omega_0 = \cup_{g \in G_*} g(D_0)$$

is the unique G_*-invariant simply connected component of G_* with $\Omega_0/G_* \cong \Sigma_5$. Moreover,

$$\Lambda(G_*) = \overline{\cup_{g \in G_*} g(W)},$$

where

$$\begin{aligned} W &= \mathbf{R} \cup \{z : \operatorname{Im} z = 4\} \cup \{z : |z - \frac{i}{2}| = \frac{1}{2}\} \cup \{\infty\} \\ &= \Lambda(H_1) \cup \Lambda(\tilde{H}_2) \cup \Lambda(H_3). \end{aligned}$$

By the same reasoning as in §**1.2**, there is a bijective correspondence between conjugacy classes of groups in $\tilde{\mathcal{M}}_5$ and points in $Teich(\Sigma_5)$, and every element of $\tilde{\mathcal{M}}_5$ is of the form fG_*f^{-1}, where f is a quasiconformal homeomorphism of $\hat{\mathbf{C}}$ onto itself and is confromal outside Ω_0. Let f be such a quasiconfromal homeomorphism fixing $0, 1$ and ∞, and let

$$\hat{S} = fSf^{-1}, \;\; \hat{T} = fTf^{-1}, \;\; \hat{X} = fXf^{-1}, \;\; \hat{Y} = fYf^{-1},$$

$$G = fG_*f^{-1} = < \hat{S}, \hat{T}, \hat{X}, \hat{Y} > .$$

Since f fixes 0 and ∞, then $\hat{T}$ fixes ∞, and $\hat{S}$ fixes 0. $\hat{S}(\infty) = 1$, then

$$\hat{S} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Write

$$\hat{T} = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} 1+\lambda\mu & -\lambda\mu^2 \\ \lambda & 1-\lambda\mu \end{pmatrix}, \quad \hat{Y} = \begin{pmatrix} 1+\rho\eta & -\rho\eta^2 \\ \rho & 1-\rho\eta \end{pmatrix},$$

where $\mu = f(4i), \eta = f(i)$ and $\lambda, \xi, \rho \in \mathbf{C} - \{0\}$.

$$\hat{T}^{-1}\hat{S} = \begin{pmatrix} 1-\xi & -\xi \\ 1 & 1 \end{pmatrix}, \quad \hat{T}^{-1}\hat{X} = \begin{pmatrix} 1+\lambda\mu-\lambda\xi & -\lambda\mu^2-\xi+\lambda\xi\mu \\ \lambda & 1-\lambda\mu \end{pmatrix}$$

are parabolic, then $\xi = 4$ and $\lambda = 1$.

$$\hat{S}^{-1}\hat{Y} = \begin{pmatrix} 1+\rho\eta & -\rho\eta^2 \\ -1+\rho-\rho\eta & 1-\rho\eta+\rho\eta^2 \end{pmatrix}$$

is paraboic, then $\rho\eta^2 = -4$, or equivalently, $\rho = -4/\eta^2$.

By considering the subgroups $<\hat{S}, \hat{X}>, <\hat{T}, \hat{Y}>, <\hat{X}, \hat{Y}>$ of G, and applying Jorgensen's inequality, we have

$$|\mu| \geq 1, |\eta| \leq 4, \text{ and } |\frac{\mu}{\eta} - 1| \geq \frac{1}{2},$$

and then

$$G =< \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1+\mu & -\mu^2 \\ 1 & 1-\mu \end{pmatrix}, \begin{pmatrix} 1-4/\eta & 4 \\ -4/\eta^2 & 1+4/\eta \end{pmatrix} > .$$

Proposition 1.4 (Normalization) *Every group in $\tilde{\mathcal{M}}_5$ is geometrically finite and is conjugate to a group $G(\mu,\nu)$ generated by*

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad X_\mu = \begin{pmatrix} 1+\mu & -\mu^2 \\ 1 & 1-\mu \end{pmatrix}, \quad Y_\nu = \begin{pmatrix} 1+2\nu & 4 \\ -\nu^2 & 1-2\nu \end{pmatrix},$$

for some $\mu, \nu \in \mathbf{C} - \{0\}$ with

$$|\mu| \geq 1, |\nu| \geq \frac{1}{2}, |\mu\nu + 2| \geq 1, \ \mathit{Im}\,\mu \geq 0, \ \text{and} \ \mathit{Im}\,\nu \geq 0.$$

Proof: Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$ATA^{-1} = T^{-1}, ASA^{-1} = S^{-1}, AX_\mu A^{-1} = X_{-\mu}^{-1}, \quad \text{and} \quad AY_\eta A^{-1} = Y_{-\eta}^{-1},$$

then $< S, T, X_\mu, Y_\eta >$ and $< S, T, X_{-\mu}, Y_{-\eta} >$ are conjugate. Then the proof is complete by setting $\nu = -2/\eta$. **Q.E.D.**

Let $\mathcal{M}_5$ be the connected component of

$$\{(\mu,\nu) \in \mathbf{C}^2 : G(\mu,\nu) \in \tilde{\mathcal{M}}_5, |\mu| \geq 1, |\nu| \geq \frac{1}{2}, |\mu\nu+2| \geq 1, \ \text{Im}\,\mu \geq 0, \ \text{and} \ \text{Im}\,\nu \geq 0\}$$

containing $(4i, 2i)$. We refer to $\mathcal{M}_5$ as the Maskit embedding of $Teich(\Sigma_5)$.

Proposition 1.5 (Rough Shape of $\mathcal{M}_5$)

(1) $(\mu,\nu) \in \mathcal{M}_5$ *iff* $(-\bar{\mu},\nu) \in \mathcal{M}_5$ *and* $(\mu,-\bar{\nu}) \in \mathcal{M}_5$.

(2) $(\mu,\nu) \in \mathcal{M}_5$ *iff* $(\mu+2,\nu) \in \mathcal{M}_5$ *and* $(\mu,\nu+1) \in \mathcal{M}_5$.

(3)

$$\{(\mu,\nu) \in \mathbf{C}^2 : Im\, \mu > 4, Im\, \nu > 1\} \subset \mathcal{M}_5 \subset \{(\mu,\nu) \in \mathbf{C}^2 : Im\, \mu > 0, Im\, \nu > 0\}.$$

Proof: By a similar argument as in Proposition 1.3 (2), one can prove (1).

(2):

$$TX_\mu^{-1} = X_{\mu+2}, \quad Y_\nu^{-1}S = Y_{\nu+1},$$

and

$$< S, T, X_\mu, Y_\nu^{-1}S > = < S, T, X_\mu, Y_\nu > = < S, T, TX_\mu^{-1}, Y_\nu >,$$

then the proof of (2) is complete.

(3): By the definition of $\mathcal{M}_5$,

$$([0,1) \times \mathbf{C}) \cap \mathcal{M}_5 = \emptyset = (\mathbf{C} \times [0,1/2)) \cap \mathcal{M}_5.$$

If $\mu = 1$, then $X_\mu S^2 = (z) = -1/z$. If $\nu = 1/2$, then $TY_\nu(z) = -4(z+4)/z$. Since

$$z \longmapsto -\frac{1}{z} \quad \text{and} \quad z \longmapsto -\frac{4(z+4)}{z}$$

are elliptic transformations, and since $G(\mu,\nu)$ are torsion free for all $(\mu,\nu) \in \mathcal{M}_5$, then

$$([0,1] \times \mathbf{C}) \cap \mathcal{M}_5 = \emptyset = (\mathbf{C} \times [0,1/2]) \cap \mathcal{M}_5,$$

and thus, from (1) and (2),

$$(\mathbf{R}\times\mathbf{C})\cap\mathcal{M}_5=\emptyset=(\mathbf{C}\times\mathbf{R})\cap\mathcal{M}_5.$$

Hence $\mathcal{M}_5\subset\{(\mu,\nu)\in\mathbf{C}^2 : Im\,\mu>0,\ Im\,\nu>0\}$.

Assume that $0\le Re\,\mu<1$ and $0\le Re\,\nu<1/2$, and let $A(z)=4/z$.

$$ATA^{-1}=\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad ASA^{-1}=\begin{pmatrix}1&4\\0&1\end{pmatrix},\quad AY_\nu A^{-1}=\begin{pmatrix}1-2\nu&-4\nu^2\\1&1+2\nu\end{pmatrix}.$$

Let

$$\begin{aligned}E&=\{z:|z+1|>1,|z-1|\ge 1,|z+1+2\nu|>1\text{ and }|z-1+2\nu|\ge 1\},\\F_1&=\{z:Im\,z\ge -Im\,\nu,-2\le Re\,z<2\},\\F_2&=\{z:Im\,z<-Im\,\nu,-2-Re\,\nu\le Re\,z<2-2Re\,\nu\}.\end{aligned}$$

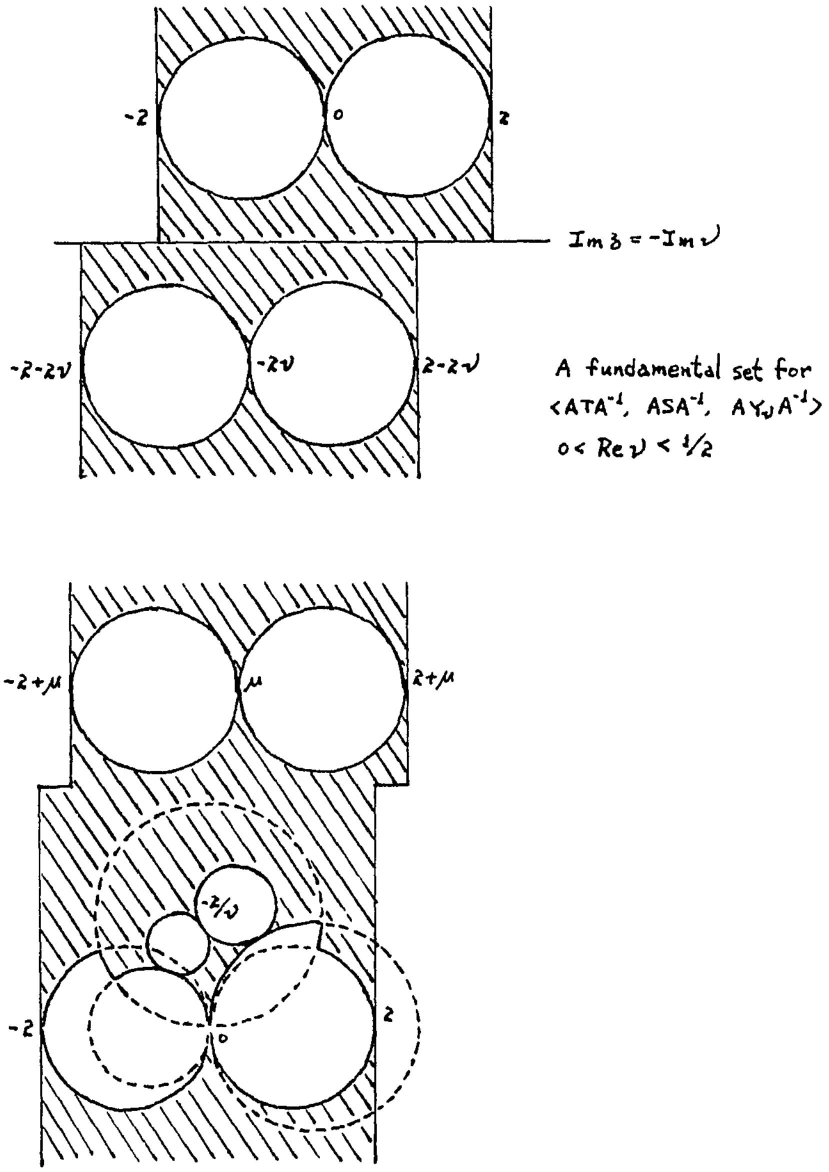
Then $F=(F_1\cup F_2)\cap E$ is a fundamental set for $<ATA^{-1},ASA^{-1},AYA^{-1}>$ acting on $\hat{\mathbf{C}}$, and $A(F)$ is a fundamental set for $<S,T,Y_\nu>$ acting on $\hat{\mathbf{C}}$, (see Figure 1.3).

Let

$$\begin{aligned}C_1&=A(\{z:|z-1+2\nu|=1\}),\\C_2&=A(\{z:|z+1+2\nu|=1\}),\ \text{ and}\\C&=A(\{z:Im\,z=Im\,(1+i-2\nu)\}\cup\{\infty\}).\end{aligned}$$

Let w be any point of C, and write $w=4/z$, where $Im\,z=1-2Im\,\nu$. Then

$$Im\,w=\frac{-4Im\,z}{|z^2|}=\frac{4(2Im\,\nu-1)}{|z|^2}\le\frac{4}{2Im\,\nu-1}<4,$$



A fundamental set for
$\langle ATA^{-1}, ASA^{-1}, AY_{\nu}A^{-1}\rangle$
$0 < \mathrm{Re}\,\nu < 1/2$

Figure 1.3:

and thus $Im\,\zeta < 4$ for all $\zeta \in C_1 \cup C_2$. By a similar argument as in the construction of G_*, we can prove that $(\mu, \nu) \in \mathcal{M}_5$ whenever $Im\,\mu > 4, Im\,\nu > 1, 0 \leq Re\,\mu < 1$ and $0 \leq Re\,\nu < 1/2$. The proof of (3) is complete by (1) and (2).

1.4 Cusps

Let $\zeta \in \partial\mathcal{M}_n, n = 4, 5$. If there is a $\omega \in \mathcal{M}_n$, a loxodromic transformation $g \in G(\omega)$, and an isomorphism $\Phi : G(\omega) \longrightarrow G(\zeta)$ taking parabolic elements of $G(\omega)$ into parabolic elements of $G(\zeta)$ such that $\Phi(g)$ is parabolic, then we say that ζ is a *cusp associated to* g, or $G(\zeta)$ is *cusp group associated to* g.

Note that $G(\tau)$ is quasiconfromally equivalent to $G(\omega)$ for every $\tau \in \mathcal{M}_n$; there is quasiconformal homeomorphism f_τ of $\hat{\mathbf{C}}$ onto itself such that $f_\tau G(\omega) f_\tau^{-1} = G(\tau)$. Therefore, ζ is also a cusp associated to $f_\tau g f_\tau^{-1} \in G(\tau)$ by considering the isomorphism

$$h \longmapsto \Phi(f_\tau^{-1} h f_\tau), \quad h \in G(\tau).$$

Example 1.1 In this example we consider the space $\mathcal{M}_4$, and we want to prove that $2i$ is a boundary point of $\mathcal{M}_4$, and it is a cusp in $\partial\mathcal{M}_4$.

Consider the polyhedron in $\mathbf{H}^3 = \{(x, y, t) \in \mathbf{R}^3 : t > 0\}$ which meets $\mathbf{C}$ in the set

$$D = \{z : -2 < Re\,z < 2, |z+1| > 1, |z+1-2i| > 1, |z-1| > 1 \text{ and } |z-1-2i| > 1\}.$$

Since $T, T^{-1}, S, S^{-1}, X_{2i}, X_{2i}^{-1}$ are side pairings of this polyhedron, then, by Poincaré theorem, this polyhedron is a fundamental polyhedron for $G(2i) = G_{2i}$ acting on $\mathbf{H}^3$, and thus D is a fundamental domain for G_{2i} acting on $\hat{\mathbf{C}}$ ([14], VI.A.3). Therefroe, G_{2i} is a Kleinian group and $\Omega(G_{2i})/G_{2i}$ is a disjoint union of four thrice punctured spheres. Hence $2i$ is not in $\mathcal{M}_4$, and $2i \in \partial\mathcal{M}_4$ since $\mu \in \mathcal{M}_4$ for $Im\ \mu > 2$.

For every $t > 2$, let $\mu = it$, and $\Phi_t : G_\mu \longrightarrow G_{2i}$ be the isomorphism with

$$\Phi_t(T) = T, \Phi_t(S) = S, \text{ and } \Phi_t(X_\mu) = X_{2i}.$$

Since every parabolic element of G_μ is a conjugate of a parabolic element of $< T, S >$ or $< T, X_\mu >$, and the parabolic elements of $< T, S >$ and $< T, X_\mu >$ are conjugates of

$$T^k, S^k, X_\mu^k, (X_\mu T^{-1})^k, (ST^{-1})^k, \quad k \in \mathbf{Z} - \{0\},$$

then Φ_t takes parabolic elements of G_μ into parabolic elements of G_{2i}.

$X_\mu^{-1}S$ is loxodromic, $X_{2i}^{-1}S$ is parabolic, and $\Phi_t(X_\mu^{-1}S) = X_{2i}^{-1}S$, then $2i$ is a cusp associated to $X_\mu^{-1}S$.

By a similar argument, $(2\sqrt{2}i, \sqrt{2}i) \in \partial\mathcal{M}_5$ is a cusp associated to $X_\mu^{-1}Y_\nu$.

McMullen proves that cusps are dense in the Bers embedding ([17]). As we remarked, every cusp in $\partial\mathcal{M}_n$ is a cusp associated to some loxodromic element of every fixed group corresponding to an element of $\mathcal{M}_n$. Since $G(\tau)$ is geometrically finite for every $\tau \in \mathcal{M}_n$, by a result of Maskit [16], for any given loxodromic element

g of $G(\tau)$ which represents a simple closed geodesic on $\Omega_0(G(\tau))/G(\tau)$ there is a cusp in $\partial\mathcal{M}_n$ associated to g, and this cusp is unique [12]. Therefore, to understand $\partial\mathcal{M}_n$ well, we have to find all loxodromic elements of $G(\tau)$ which represent simple closed geodesics on $\Omega_0(G(\tau))/G(\tau)$.

If $\zeta \in \partial\mathcal{M}_n$ is a cusp associated to $g \in G(\tau)$, and $h \in G(\tau)$ is any element, then ζ is also a cusp associated to hgh^{-1} since $\Phi(hgh^{-1}) = \Phi(h)\Phi(g)\Phi(h)^{-1}$. Thus, it suffices to find all non-conjugate loxodromic elements of $G(\tau)$ which represent simple closed geodesics on $\Omega_0(G(\tau))/G(\tau)$, or equivalentlly, to find all free homotopy classes of simple closed curves on $\Omega_0(G(\tau))/G(\tau)$ which are not homotopically equivalent to the simple closed curves which separate some puncture of the punctures on $\Omega_0(G(\tau))/G(\tau)$ from the others.

We call a simple closed curve on $\Omega_0(G(\tau))/G(\tau)$ an *essential simple closed curve* if this curve is not homotopically trivial and is not homotopically equivalent to a simple closed curve which separates some puncture of the punctures on $\Omega_0(G(\tau))/G(\tau)$ from the others. In next chapter, we are going to enumerate free homotopy classes of essential simple closed curves.

Chapter 2

Enumerating Loops and Traces

2.1 The General Setup

In this section, we are going to set up some notations and definitions that we will need later. To enumerate free homotopy classes of essential simple closed curves is a topological problem; we simply write $G = G(\tau)$, $\Sigma_n = \Omega_0(G(\tau))/G(\tau)$, $X = X_\mu$, and $Y = Y_\nu$. Let $\mathcal{G}_n$ be the set of all free homotopy classes of essential simple closed curves on Σ_n. It is a fact that every homotopy class in $\mathcal{G}_n$ contains a unique geodesic, say γ. By abuse of notation, we shall also use γ for the free homotopy class represented by γ.

We choose a simply connected fundamental domain D for G acting on $\Omega_0(G)$ with side pairings $S^\pm$, $T^\pm$, $X^\pm$,and $Y^\pm$. If s and s' are sides of D, and $E \in$

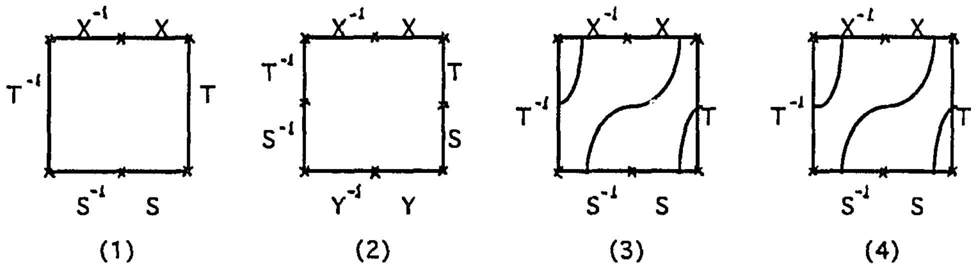


Figure 2.1: A simply connected fundamental domain D: (1) $n = 4$, (2) $n = 5$. Simple closed curves on Σ_4: (3) γ, (4) γ'.

$\{S^{\pm}, T^{\pm}, X^{\pm}, Y^{\pm}\}$ with $E(s) = s'$, then we label E^{-1} beside s, and label E beside s', see Figure 2.1. A side s of D is called the *E-side* of D if s is labelled by E.

With this labeling, we can easily write words in G to represent curves in $\mathcal{G}_n$. For example, we may write

$$W(\gamma) = T^{-1}SX \quad and \quad W(\gamma') = X^{-1}S^{-1}T$$

to represent the curves γ and γ' given as (3) and (4) in Figure 2.1 with orientations indicated by the arrows.

It is easy to see that if γ and γ' are two essential simple closed curves on Σ_n with the same underlying set but opposite orientations, and if W is a word in G corresponding to γ then W^{-1} corresponds to γ'. Because we are only interested in non-oreinted essential simple closed curves, we are going to use both W and W^{-1} to represent an essential simple closed curve on Σ_n.

Because we are only interested in elements of $\mathcal{G}_n$, we shall not distinguish a word in G from its conjugates. In particular, a cyclic permutation of a word W in G is conjugate to W in G, and we shall identify these words and call any one of them a *cyclic word.*

A word $W = E_1 \cdots E_m$ in G is called a *reduced word* if for each j,

$$E_j \in \{S^{\pm}, T^{\pm}, X^{\pm}, Y^{\pm}\}, \text{ and } E_j E_{j+1} \neq \mathbf{1}, \text{ for } j = 1, ..., m-1,$$

where $\mathbf{1}$ is the identity of G. When W represents a curve in $\mathcal{G}_n$, we also require that $E_m E_1 \neq \mathbf{1}$, and for every $j = 1, ..., m$, and every positive integer $k < m$, we call $E_{j+1} \cdots E_{j+k}$ a *subword* of W, where $E_{m+1} = E_1$.

Let $\pi : D \longrightarrow \Sigma_n$ be the canonical projection. For $E \in \{S^{\pm}, T^{\pm}, X^{\pm}, Y^{\pm}\}$, let γ_E denote the image of the E-side under π. γ_E is a simple arc on Σ_n joining two punctures of Σ_n.

For every $\gamma \in \mathcal{G}_n$, $\pi^{-1}(\gamma)$ is a disjoint union a finite number simple arcs in D with endpoints on the sides of D. We call each of these simple arcs a *strand* of γ. γ has exactly $k \geq 0$ strands with endpoints on the E-side if and only if it has exactly $k \geq 0$ strands with endpoints on the E^{-1}-side, and k is the *geometric intersection number* of γ with γ_E, which is denoted by $I_E(\gamma)$. Thus

$$\begin{aligned} I_E(\gamma) \quad = \quad & \text{the number of points which are endpoints of} \\ & \text{strands of } \gamma \text{ on the } E\text{-side (or the } E^{-1}\text{-side).} \end{aligned}$$

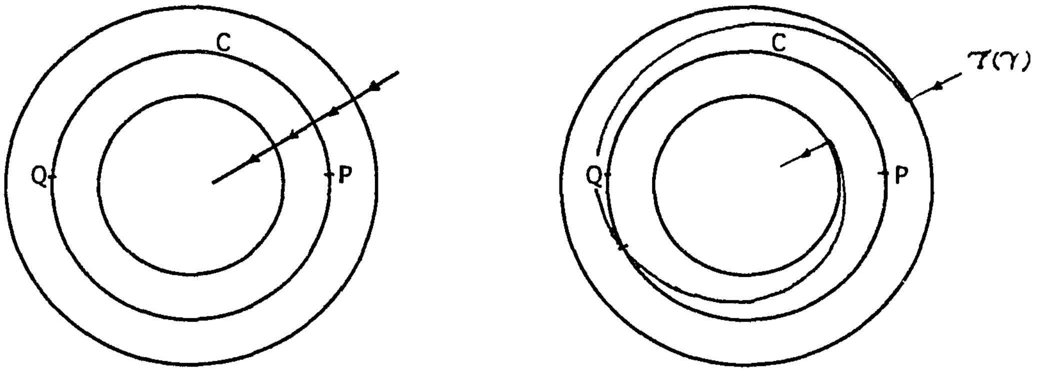


Figure 2.2: A Dehn twist $\mathcal{T}$ associtaed with the puncutes P and Q

At the end of this section, we are going to define a special kind of Dehn twist given by J. Birman ([3] P.165 - 166) , which plays an important role in the rest of our discussions. Let Σ_0 be the 2-sphere, and let P and Q be any two distinct punctures of Σ_n. As points of Σ_0, we choose a simple closed curve C on Σ_0 passing through P and Q so that one of the connected components of $\Sigma_0 - C$, say C^+, is contained in Σ_n and all punctures of Σ_n other than P and Q are contained in the other connected component, say C^-. We may orient C so that C^- lies to the right of C. Choose an annular neighborhood $\mathcal{A}$ of C on Σ_0 so that $\mathcal{A}$ does not contain any puncture of Σ_n other than P and Q. By using polar coordinates of the plane, we parametrize

$$\mathcal{A} = \{re^{i\theta} : 0 < r_1 \leq r \leq r_2 < \infty, 0 \leq \theta \leq 2\pi\}$$

so that

$$C = \{re^{i\theta} \in \mathcal{A} : r = \frac{r_1 + r_2}{2}\},$$

and P and Q correspond to

$$\frac{r_1 + r_2}{2} \quad \text{and} \quad -\frac{r_1 + r_2}{2},$$

respectively, (see Figure 2.2). For simplicity, we set $r_1 = 1$ and $r_2 = 3$. Now, we are going to define a Dehn twist $\mathcal{T}$ about C on Σ_n as follows. First, we define $\mathcal{T}$ on $\mathcal{A}$ as

$$\mathcal{T}(re^{i\theta}) = re^{i(\theta+\pi(r-1))}.$$

Extend $\mathcal{T}$ to Σ_0 so that $\mathcal{T}$ is the identity map outside $\mathcal{A}$. It is easy to see that $\mathcal{T}$ fixes all punctures of Σ_n other than P and Q, and

$$\mathcal{T}(P) = Q, \quad \text{and} \quad \mathcal{T}(Q) = P.$$

Thus the restriction of $\mathcal{T}$ to Σ_n is a homeomorphism of Σ_n onto itself. We call $\mathcal{T}$ a *Dehn twist associated with the punctures* P and Q. $\mathcal{T}$ induces an isomorphism of G which is also denoted by $\mathcal{T}$.

2.2 Words for Simple Closed Geodesics on Σ_4 and Traces

Let $G =< S, T, X >$, let P be the puncture on $\Sigma_4 = \Omega_0/G$ corresponding to the fixed point of S, let Q be the puncture on Σ_4 corresponding to the fixed point of

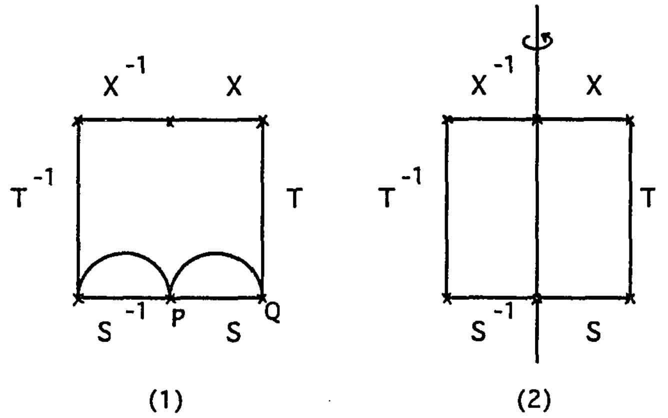


Figure 2.3: (1) Dehn twist : $\mathcal{T}$; (2) symmetry: Θ.

$T^{-1}S$, let $\mathcal{T}$ be a Dehn twist associated with the punctures P and Q, (see Figure 2.3), and let Θ be the automorphism of G defined by

$$\Theta: \ S \longmapsto S^{-1}, \ T \longmapsto T^{-1}, \ X \longmapsto X^{-1}.$$

Θ induces an (orientation reversing) homeomorphism of Σ_4, which is also denoted by Θ, (see Figure 2.3). The action of $\mathcal{T}$ on G is defined by

$$S \longrightarrow S^{-1}T, \ T \longrightarrow T, \ X \longrightarrow X.$$

Let $\mathcal{G}_4$ be the set of all free homotopy classes of essential simple closed curves on Σ_4. For $x \in \{\pm1, \pm2, \infty\}$, let γ_x be the simple closed geodesic in $\mathcal{G}_4$ represented

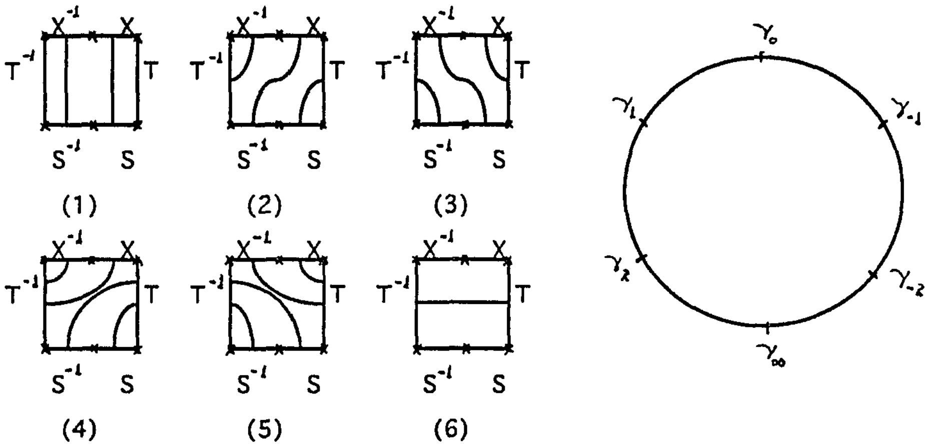


Figure 2.4: (1) γ_0, (2) γ_1, (3) γ_{-1}, (4) γ_2, (5)γ_{-2}, (6) γ_∞.

by $W_x \in G$, where

$$\begin{aligned} W_0 &= X^{-1}S,\ W_1 = XT^{-1}S,\ W_2 = TXT^{-1}S^{-1}, \\ W_{-1} &= X^{-1}TS^{-1},\ W_{-2} = T^{-1}X^{-1}TS,\ W_\infty = T. \end{aligned}$$

see Figure 2.4. By using a technique developed by Birman and Series [4], every free homotopy class in $\mathcal{G}_4$ can be written as an integral combination of two classes in

$$\mathcal{B}_4 = \{\gamma_0, \gamma_1, \gamma_2, \gamma_{-1}, \gamma_{-2}, \gamma_\infty\}$$

with relatively prime non-negative coefficients. More precisely, they are

$$a\gamma_0 + b\gamma_1,\ a\gamma_1 + b\gamma_2,\ a\gamma_2 + b\gamma_\infty,$$

$$a\gamma_0 + b\gamma_{-1},\ a\gamma_{-1} + b\gamma_{-2},\ a\gamma_{-2} + b\gamma_\infty,$$

where $a, b \geq 0$ with $\gcd(a,b) = 1$. Let

$$\begin{aligned}
\sigma_1 &= \{a\gamma_{-2} + b\gamma_\infty : a,\ b \text{ are relatively prime non-negative integers}\},\\
\sigma_2 &= \{a\gamma_{-1} + b\gamma_{-2} : a,\ b \text{ are relatively prime non-negative integers}\},\\
\sigma_3 &= \{a\gamma_0 + b\gamma_{-1} : a,\ b \text{ are relatively prime non-negative integers}\},\\
\sigma_4 &= \{a\gamma_0 + b\gamma_1 : a,\ b \text{ are relatively prime non-negative integers}\},\\
\sigma_5 &= \{a\gamma_1 + b\gamma_2 : a,\ b \text{ are relatively prime non-negative integers}\},\\
\sigma_6 &= \{a\gamma_2 + b\gamma_\infty : a,\ b \text{ are relatively prime non-negative integers}\}.
\end{aligned}$$

A result of Birman and Series [4] says that Θ and $\mathcal{T}$ act on $\mathcal{G}_4$ "linearly". More precisely, if $\gamma, \gamma' \in \sigma_i$ for some i, and if $\varphi(\gamma), \varphi(\gamma') \in \sigma_j$ for some j, where $\varphi = \Theta$ or $\mathcal{T}$, then

$$\varphi(a\gamma + b\gamma') = a\varphi(\gamma) + b\varphi(\gamma'),$$

for any two non-negative relatively prime integers a and b.

Lemma 2.1 *Let $\gamma \in \mathcal{G}_4$. Then the followings hold:*

(1) $I_X(\gamma) = I_S(\gamma)$.

(2) *There is an integer k depending on γ such that $\mathcal{T}^k(\gamma) \in \sigma_4$ whenever $\gamma \neq \gamma_\infty$.*

Proof: (1) follows immediately from previous discussions and the definitions of $I_X(\gamma)$ and $I_S(\gamma)$.

It is easy to see that (2) holds for $\gamma \in \cup_{j=2}^{5}\sigma_j$. Let $\gamma \in \sigma_6 - \{\gamma_\infty, \gamma_2\}$. Write

$$\gamma = a\gamma_2 + b\gamma_\infty, \quad a, b > 0, \gcd(a, b) = 1.$$

There exist non-negative integers k and $c < a$ such that $b = ak + c$. Since

$$\mathcal{T}^{-1}(\gamma_2 + \gamma_\infty) = \gamma_2 \text{ and } \mathcal{T}(\gamma_\infty) = \gamma_\infty,$$

then

$$\begin{aligned} \mathcal{T}^{-k}(\gamma) &= a\gamma_2 + c\gamma_\infty = (a-c)\gamma_2 + c(\gamma_2 + \gamma_\infty) \\ \mathcal{T}^{-k-2}(\gamma) &= \mathcal{T}^{-1}((a-c)\gamma_1 + c\gamma_2) = (a-c)\gamma_0 + c\gamma_1 \end{aligned}$$

By a similar argument, we can prove that (2) holds for $\gamma \in \sigma_1 - \{\gamma_\infty, \gamma_{-2}\}$. **Q.E.D.**

Now, we restrict attention to $\gamma \in \sigma_4$, and let

$$W = W(\gamma) = E_1 E_2 \cdots E_k, \quad E_j \in \{S^{\pm}, T^{\pm}, X^{\pm}\}, \tag{2.1}$$

be a cyclic reduced word in G representing γ. Note that W is a loxodromic transformation.

(1) If XX, $X^{-1}X^{-1}$, SS or $S^{-1}S^{-1}$ is a subword of W, then γ is not essential or is a curve spiraling around a puncture on Σ_n, (see (1) in Figure 2.5). This is not allowed.

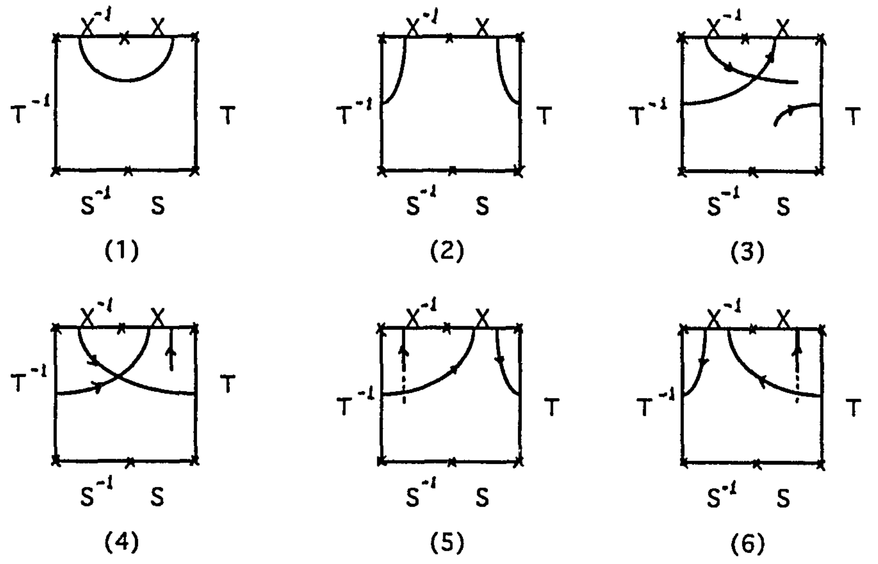


Figure 2.5: (1) XX or $X^{-1}X^{-1}$, (2) $T^{-1}XT^{-1}$ or $X^{-1}TX^{-1}$, (3) TXT, (4) XTX, (5) $X^{-1}TX$, (6) $XT^{-1}X^{-1}$.

(2) By the same reasoning as above, $T^{\varepsilon}E^{-\varepsilon}T^{\varepsilon}$ and $E^{\varepsilon}T^{-\varepsilon}E^{\varepsilon}$ cannot be a subword of W, whenever $E \in \{S, X\}$, and $\varepsilon = \pm 1$, (see (2) in Figure 2.5).

(3) If $T^{\varepsilon}E^{\varepsilon}T^{\varepsilon}$ or $E^{\varepsilon}T^{\varepsilon}E^{\varepsilon}$ is a subword of W, $E \in \{S, X\}$, and $\varepsilon = \pm 1$, then γ is not simple, (see (3) and (4) in Figure 2.5).

(4) If $E^{-\delta}T^{\varepsilon}E^{\delta}$ is a subword of W, $E \in \{S, X\}$ and $\varepsilon, \delta \in \{\pm 1\}$, then γ is not simple, (see (5) and (6) in Figure 2.5).

(5) If $E^{\delta'}T^{p}E^{\delta}$ is a subword of W, $E \in \{S, X\}$ $\delta', \delta \in \{\pm 1\}$ and $p \neq 0$ is an integer, then γ is not simple.

From the above discussion, we have:

(1) For every fixed i, if j is the smallest positive integer such that $E_i, E_{i+j} \in \{X^{\pm}\}$, then $j \geq 2$, $E_{i+1}, ..., E_{i+j-1} \in \{T^{\pm}, S^{\pm}\}$, and there is a unique $p \in \{i + 1, ..., i + j - 1\}$ such that $E_p \in \{S^{\pm}\}$.

(2) For every fixed i, if j is the smallest positive integer such that $E_i, E_{i+j} \in \{S^{\pm}\}$, then $j \geq 2$, $E_{i+1}, ..., E_{i+j-1} \in \{T^{\pm}, X^{\pm}\}$, and there is a unique $p \in \{i + 1, ..., i + j - 1\}$ such that $E_p \in \{X^{\pm}\}$.

(3) $I_E(\gamma) = \#\{j : E_j = E^{\varepsilon}, 1 \leq j \leq k, \ \varepsilon = \pm 1\}$.

For uniformity, if $X^{\delta}S^{\varepsilon}$ or $S^{\varepsilon}X^{\delta}$ is a subword of W, we write

$$X^{\delta}S^{\varepsilon} = X^{\delta}T^{0}S^{\varepsilon}, \text{ and } S^{\varepsilon}X^{\delta} = S^{\varepsilon}T^{0}X^{\delta},$$

so that W is of the form

$$\prod_{i=1}^{m} T^{\alpha_i} S^{\varepsilon_i} T^{\beta_i} X^{\delta_i}, \tag{2.2}$$

where $\alpha_i, \beta_i \in \mathbf{Z}$, $\varepsilon_i, \delta_i \in \{1, -1\}$, and $m = I_X(\gamma) = I_S(\gamma)$. We call a word in G of the form given in (2) a *cyclic semi-reduced word.* Note that the length of a word in formula (2.1) is equal to the length of a word in formula (2.2) if they correspond to the same element in G.

Next we are going to find the conditions for $\alpha_i, \beta_i, \varepsilon_i$ and δ_i.

(1) If T^k is a subword of W with $|k| \geq 2$, then γ contains a strand joining the T-side to the T^{-1}-side. Since every curve in σ_4 contains no such strands, then $|\alpha_i| \leq 1$ and $|\beta_i| \leq 1$ for every i.

(2) If $T^{-1}S^{\varepsilon}T$, $\varepsilon = \pm 1$, is a subword of W, then γ contains a strand joining the T-side to the S-side, and contains a strand joining the T-side to the S^{-1}-side, which is impossible for curves in σ_4. Similarly, $TS^{\varepsilon}T^{-1}$,$T^{-1}X^{\varepsilon}T$ and $TX^{\varepsilon}T^{-1}$ cannot be subwords of W. Thus $\alpha_i\beta_i = 0$ and $\alpha_{i+1}\beta_i = 0$ for $i = 1, ..., m$, where $\alpha_{m+1} = \alpha_1$.

(3) Since γ contains no strands joining the T-side to the S^{-1}-side, and contains no strands joining the T^{-1}-side to the S^{ε}-side, $\varepsilon = \pm 1$, then $T^{-1}S^{-1}$, ST, TS^{ε} and $S^{\varepsilon}T^{-1}$ cannot be subwords of W. Similarly, $X^{-1}T^{-1}$, TX, $X^{\varepsilon}T$ and $T^{-1}X^{\varepsilon}$ cannot be subwords of W. Then

$$\alpha_i \leq 0, \;\; \beta_i \geq 0, \;\; (\alpha_i + \beta_i)\varepsilon_i \leq 0, \;\text{ and }\; (\alpha_{i+1} + \beta_i)\delta_i \leq 0.$$

Proposition 2.2 *If* $\gamma = a\gamma_0 + b\gamma_1$ *with* $\gcd(a,b) = 1$, $a, b \geq 0$, *and if* $W(\gamma) \in G$ *is a cyclic semi-reduced word, then* $W(\gamma)$ *is of the form*

$$\prod_{i=1}^{m} T^{-\alpha_i} S^{\varepsilon_i} T^{\beta_i} X^{\delta_i},$$

where $m = I_X(\gamma) = I_S(\gamma)$, $\varepsilon_i, \delta_i \in \{1, -1\}$, *and* α_i *and* β_i *are non-negative integers satisfying the following conditions:*

(1) $\alpha_i \leq 1, \beta_i \leq 1, \alpha_i\beta_i = 0, (\beta_i - \alpha_i)\varepsilon_i \leq 0$, *for* $i = 1, ..., m$.

(2) $\alpha_{i+1}\beta_i = 0, (\beta_i - \alpha_{i+1})\delta_i \leq 0$, *for* $i = 1, ..., m$, *where* $\alpha_{m+1} = \alpha_1$.

(3) $\sum_{i=1}^{m}(\alpha_i + \beta_i) = b$.

Proof: It remains to prove (3). From (1), $\alpha_i + \beta_i \leq 1$. Let

$$k = \#\{i : \alpha_i + \beta_i = 1\}.$$

Assume that $\alpha_i + \beta_i = 1$.

$$\begin{aligned} \varepsilon_i = -1 \quad &\Rightarrow \quad \alpha_i = 0, \beta_i = 1 \Rightarrow T^{-\alpha_i} S^{\varepsilon_i} T^{\beta_i} = S^{-1}T \\ \varepsilon_i = 1 \quad &\Rightarrow \quad \alpha_i = 1, \beta_i = 0 \Rightarrow T^{-\alpha_i} S^{\varepsilon_i} T^{\beta_i} = T^{-1}S. \end{aligned}$$

Since $S^{-1}T$ or $T^{-1}S$ is a subword of W if and only if γ contains a strand joining the T-side to the S-side, then

$$\begin{aligned} k &= \#(\text{strands of } \gamma \text{ joining the } T\text{-side to the } S\text{-dise}) \\ &= \#(\text{strands of } \gamma \text{ joining the } T^{-1}\text{-side to the } X^{-1}\text{-dise}) \\ &= b \end{aligned}$$

This completes the proof. **Q.E.D.**

Let $\hat{\mathcal{G}}_4 = \mathcal{G}_4 - \{\gamma_\infty\}$. We associate to each $\gamma \in \hat{\mathcal{G}}_4$ the integer $N(\gamma)$ given as follows:

$$\begin{aligned} N(\gamma) &\geq 0 \text{ when } \gamma \in \sigma_4 \cup \sigma_5 \cup \sigma_6, \\ N(\gamma) &\leq 0 \text{ when } \gamma \in \sigma_1 \cup \sigma_2 \cup \sigma_3, \text{ and} \\ |N(\gamma)| &= \#(\text{strands of } \gamma \text{ joining the } T^{-1}\text{-side and the } T\text{-side}) \\ &\quad +\#(\text{strands of } \gamma \text{ joining the } T^{\varepsilon}\text{-side and the } E^{\delta}\text{-side}), \end{aligned}$$

where $\varepsilon, \delta \in \{1, -1\}$ and $E \in \{X, S\}$. Set

$$\mathcal{G}_4^+ = \sigma_4 \cup \sigma_5 \cup \sigma_6 - \{\gamma_\infty\}, \text{ and } \mathcal{G}_4^- = \sigma_1 \cup \sigma_2 \cup \sigma_3 - \{\gamma_\infty\}.$$

Theorem 2.3 *If $\gamma \in \hat{\mathcal{G}}_4$ and $W(\gamma) \in G$ is a cyclic semi-reduced word representing γ, then $W(\gamma)$ is of the form*

$$\prod_{i=1}^{m} T^{\alpha_i} S^{\varepsilon_i} T^{\beta_i} X^{\delta_i},$$

where $m = I_X(\gamma) = I_S(\gamma)$, $\varepsilon_i, \delta_i \in \{1, -1\}$, and α_i and β_i are integers satisfying the following conditions:

(1) $-1 \leq (\alpha_i + \beta_i)\varepsilon_i \leq 0$ *and* $-1 \leq (\alpha_{i+1} + \beta_i)\delta_i \leq 0$, *where* $\alpha_{m+1} = \alpha_1$.

(2) $|\alpha_i|, |\beta_i| \in \{\alpha, \alpha + 1\}$, *where* $\alpha = \min\{|\alpha_i|, |\beta_i| : i = 1, .., m\}$.

(3) $\sum_{i=1}^{m}(|\alpha_i| + |\beta_i|) = |N(\gamma)|$.

(4)$\alpha_i \leq 0$ *and* $\beta_i \geq 0$ *when* $\gamma \in \mathcal{G}_4^+$*, and* $\alpha_i \geq 0$ *and* $\beta_i \leq 0$ *when* $\gamma \in \mathcal{G}_4^-$.

Lemma 2.4 *If* $\gamma \in \hat{\mathcal{G}}_4$*, then*

(1) $I_X(\Theta(\gamma)) = I_X(\gamma)$, $N(\Theta(\gamma)) = -N(\gamma)$, *and*

(2) $I_X(\mathcal{T}(\gamma)) = I_X(\gamma)$, $N(\mathcal{T}(\gamma)) = N(\gamma) + I_X(\gamma)$.

Proof: (1) follows immediately from the definition of $N(\gamma)$ and that of $I_X(\gamma)$.

To prove (2), we first assume that $\gamma \in \mathcal{G}_4^+$.

Case 1: If $\gamma = a\gamma_0 + b\gamma_1$, then $N(\gamma) = b$ and $I_X(\gamma) = a + b$.

$$\begin{aligned} \mathcal{T}(\gamma) &= a\gamma_1 + b\gamma_2, \quad I_X(\mathcal{T}(\gamma)) = a + b = I_x(\gamma), \\ N(\mathcal{T}(\gamma)) &= a + 2b = N(\gamma) + I_X(\gamma). \end{aligned}$$

Case 2: If $\gamma = a\gamma_1 + b\gamma_2$, then $N(\gamma) = a + 2b$ and $I_X(\gamma) = a + b$.

$$\begin{aligned} \mathcal{T}(\gamma) &= (a+b)\gamma_2 + b\gamma_\infty, \quad I_X(\mathcal{T}(\gamma)) = a + b = I_x(\gamma), \\ N(\mathcal{T}(\gamma)) &= 2(a+b) + b = N(\gamma) + I_X(\gamma). \end{aligned}$$

Case 3: If $\gamma = a\gamma_2 + b\gamma_\infty$, then $N(\gamma) = 2a + b$ and $I_X(\gamma) = a$.

$$\begin{aligned} \mathcal{T}(\gamma) &= a\gamma_2 + (a+b)\gamma_\infty, \quad I_X(\mathcal{T}(\gamma)) = a = I_x(\gamma), \\ N(\mathcal{T}(\gamma)) &= 3a + b = N(\gamma) + I_X(\gamma). \end{aligned}$$

Now, let $\gamma \in \mathcal{G}_4^-$. Since $\Theta\mathcal{T} = \mathcal{T}^{-1}\Theta$, then

$$\mathcal{T}(\gamma) = \Theta\mathcal{T}^{-1}\Theta(\gamma), \quad I_X(\mathcal{T}(\gamma)) = I_X(\gamma), \text{ and}$$

$$N(\Theta\mathcal{T}^{-1}\Theta(\gamma)) = -N(\mathcal{T}^{-1}\Theta(\gamma)) = -N(\Theta(\gamma)) + I_X(\Theta(\gamma)) = N(\gamma) + I_X(\gamma).$$

Corollary 2.5 *If $\gamma \in \hat{\mathcal{G}}_4$ and $k \in \mathbf{Z}$, then*

$$I_X(\mathcal{T}^k(\gamma)) = I_X(\gamma), \quad \textit{and} \quad N(\mathcal{T}^k(\gamma)) = N(\gamma) + kI_X(\gamma).$$

Proof of Theorem 2.3: First, we assume that $\gamma \in \mathcal{G}_4^+ - \sigma_4$. From the proof of Lemma 2.1, there is an integer $k > 0$ such that $\gamma' = \mathcal{T}^{-k}(\gamma) \in \sigma_4$. From Corollary 2.5,

$$I_X(\gamma) = I_X(\gamma') \text{ and } N(\gamma) = N(\gamma') + kI_X(\gamma).$$

By Proposition 2.2, γ' is represented by a cyclic semi-reduced word W' of the form

$$\prod_{i=1}^{m} T^{-\alpha_i'} S^{\varepsilon_i'} T^{\beta_i'} X^{\delta_i'},$$

where $m = I_X(\gamma) = I_X(\gamma')$, $\varepsilon_i', \delta_i' \in \{1, -1\}$, and $\alpha_i' \geq 0, \beta_i' \geq 0$ are integers satisfying the conditions given in Proposition 2.2.

Let $\varepsilon', \delta' \in \{1, -1\}$ and $\alpha' \geq 0, \beta' \geq 0$ be integers with $-1 \leq (\beta' - \alpha')\varepsilon' \leq 0$.

$$\mathcal{T}(T^{-\alpha'} S^{\varepsilon'} T^{\beta'} X^{\delta'}) = T^{-\alpha} S^{\varepsilon} T^{\beta} X^{\delta},$$

where $\varepsilon = \varepsilon'$, $\delta = \delta'$, $\alpha = \alpha' + \frac{1}{2}(1 - \varepsilon')$ and $\beta = \beta' + \frac{1}{2}(1 + \varepsilon')$. It is easy to see that

$$\alpha \geq 0, \beta \geq 0, \text{ and } \alpha + \beta = \alpha' + \beta' + 1.$$

Since

$$(\beta - \alpha)\varepsilon = -(\beta' - \alpha')\varepsilon' - 1 \text{ and } -1 \leq (\beta' - \alpha')\varepsilon' \leq 0,$$

then $-1 \leq (\beta - \alpha)\varepsilon \leq 0$. Therefore, $W(\gamma) = \mathcal{T}(W')$ is of the form

$$\prod_{i=1}^{m} T^{-\alpha_i} S^{\varepsilon_i} T^{\beta_i} X^{\delta_i},$$

with $\varepsilon_i = (-1)^k \varepsilon_i'$, $\delta_i = \delta_i'$, $\alpha_i \geq 0$, $\beta_i \geq 0$,

$$\alpha_i + \beta_i = \alpha_i' + \beta_i' + k, \text{ and } -1 \leq (\beta_i - \alpha_i)\varepsilon_i \leq 0.$$

From Corollary 2.5,

$$\sum_{i=1}^{m}(\alpha_i + \beta_i) = \sum_{i=1}^{m}(\alpha_i' + \beta_i' + k) = N(\gamma') + km = N(\gamma).$$

Next, we prove that $-1 \leq (\beta_i - \alpha_{i+1})\delta_i \leq 0$ for every i. We write

$$W(\gamma) = \prod_{i=1}^{m} T^{\beta_i} X^{\delta_i} T^{-\alpha_i} S^{\varepsilon_i},$$

and prove that $-1 \leq (\beta_i - \alpha_i)\delta_i \leq 0$. Let Θ' be the automorphism of G defined by

$$S \longrightarrow X, \; T \longrightarrow T, \; X \longrightarrow S.$$

Θ' induces a homeomorphism of Σ_4 onto itself, which is also denoted by Θ'. It is easy to see that $\Theta'(\mathcal{G}_4^+) = \mathcal{G}_4^-$. Then $\Theta\Theta'(\gamma) \in \mathcal{G}_4^+$, and

$$\Theta\Theta'(W(\gamma)) = \prod_{i=1}^{m} T^{-\beta_i} S^{-\delta_i} T^{\alpha_i} X^{-\varepsilon_i}.$$

Thus

$$-1 \leq (\alpha_i - \beta_i)(-\delta_i) \leq 0, \text{ or equivalently } -1 \leq (\beta_i - \alpha_i)\delta_i \leq 0.$$

Prove that $\alpha_i, \beta_i \in \{\alpha, \alpha+1\}$. Without loss of generality, we assume that $\alpha = \min\{\alpha_1, \beta_1\}$, and assume that $\varepsilon_1 = 1$. Then $\alpha = \beta_1$. Since

$$\mathcal{T}^{-2}(S^{\varepsilon}) = TS^{\varepsilon}T^{-1}, \ \varepsilon = \pm 1,$$

then

$$\mathcal{T}^{-2}(T^{-\alpha_i}S^{\varepsilon_i}T^{\beta_i}) = T^{-(\alpha_i - 1)}S^{\varepsilon_i}T^{(\beta_i - 1)},$$

and

$$\mathcal{T}^{-2\alpha}(W(\gamma)) = \prod_{i=1}^{m} T^{-\alpha_i'}S^{\varepsilon_i}T^{\beta_i'}X^{\delta_i},$$

where $\alpha_i' = \alpha_i - \alpha, \beta_i' = \beta_i - \alpha$. Suppose that $\alpha_i > \alpha + 1$ or $\beta_i > \alpha + 1$ for some $i > 1$. Since $\beta_1' = 0$ and $\varepsilon_1 = 1$, then $\mathcal{T}^{-2\alpha}(\gamma)$ contains a strand joining the S^{-1}-side to the X^{δ_1}-side. On the other hand, $\alpha_i' > 1$ or $\beta_i' > 1$ implies that $\mathcal{T}^{-2\alpha}(\gamma)$ contains a strand joining the T-side to the T^{-1}-side, which is impossible since γ is simple.

If $\gamma \in \mathcal{G}_4^-$, we consider the curve $\mathrm{C}(\gamma)$, and then the proof is complete. **Q.E.D.**

Theorem 2.6 (Enumeration Theorem)

Let $\mathcal{L} : \hat{\mathcal{G}}_4 \longrightarrow \mathbf{Q}$ be defined by

$$\mathcal{L}(\gamma) = \frac{N(\gamma)}{I_X(\gamma)}, \ \gamma \in \hat{\mathcal{G}}_4.$$

Then $\mathcal{L}\mathcal{T}(\gamma) = \mathcal{L}(\gamma) + 1$, $\mathcal{L}\Theta(\gamma) = -\mathcal{L}(\gamma)$, and $\mathcal{L}$ is bijective.

Proof:

$$\begin{aligned}\mathcal{L}\mathcal{T}(\gamma) &= \frac{N(\mathcal{T}(\gamma))}{I_X(\mathcal{T}(\gamma))} = \frac{N(\gamma)+I_X(\gamma)}{I_X(\gamma)} = \mathcal{L}(\gamma)+1,\\ \mathcal{L}\Theta(\gamma) &= \frac{N(\Theta(\gamma))}{I_X(\Theta(\gamma))} = \frac{-N(\gamma)}{I_X(\gamma)} = -\mathcal{L}(\gamma).\end{aligned}$$

Let q be any non-negative rational number, and let $n \geq 0$ be the integer such that $0 \leq q-n < 1$. There exist integers $a > 0, b \geq 0$ with $\gcd(a,b) = 1$ such that $q - n = \frac{b}{a+b}$. Then

$$\begin{aligned}\mathcal{L}(a\gamma_0 + b\gamma_1) &= \frac{b}{a+b} = q-n,\\ \mathcal{L}\mathcal{T}^n(a\gamma_0 + b\gamma_1) &= \mathcal{L}(a\gamma_0+b\gamma_1)+n = q, \text{ and}\\ \mathcal{L}\Theta\mathcal{T}^n(a\gamma_0+b\gamma_1) &= -\mathcal{L}\mathcal{T}^n(a\gamma_0+b\gamma_1) = -q.\end{aligned}$$

This proves that $\mathcal{L}$ is surjective.

Assume that $\gamma, \gamma' \in \hat{\mathcal{G}}_4$ satisfy $\mathcal{L}(\gamma) = \mathcal{L}(\gamma')$. We want to show that $\gamma = \gamma'$. Since $\gamma = \gamma'$ iff $\mathcal{T}^k(\gamma) = \mathcal{T}^k(\gamma')$ for any fixed integer k, then, by Lemma 2.1, we may assume that $\gamma = a\gamma_0 + b\gamma_1$.

If γ' is not in σ_4, then, by Lemma 2.1 again, there is an integer $k \neq 0$ such that $\mathcal{T}^k(\gamma') \in \sigma_4$.

$$\mathcal{L}\mathcal{T}^k(\gamma') = \mathcal{L}(\gamma') + k \text{ and } 0 \leq \mathcal{L}\mathcal{T}^k(\gamma') \leq 1,$$

then $\mathcal{L}(\gamma')$ is not in the interval $[0,1]$. This is a contradiction since $\mathcal{L}(\gamma) = \mathcal{L}(\gamma')$.

Write $\gamma' = c\gamma_0 + d\gamma_1$. By assumption, $\frac{d}{c+d} = \frac{b}{a+b}$. Since $\gcd(a,b) = \gcd(c,d) = 1$, then $a = c$ and $b = d$, and thus $\gamma = \gamma'$. **Q.E.D.**

For every $\mu \in \mathcal{M}_4$, let $W(\gamma;\mu) \in G(\mu)$ be the cyclic semi-reduced word given in Theorem 2.3 representing γ. It is clear that $tr\, W(\gamma;\mu)$ is a polynomial in μ. We will write

$$tr\, W(\gamma;\mu) = a_0\mu^d + a_1\mu^{d-1} + O(\mu^{d-2}),$$

if the degree of $tr\, W(\gamma;\mu)$ is d, where $O(\mu^{d-2})$ is a polynomial in μ of degree $\leq d-2$.

Lemma 2.7 *If* $\gamma \in \hat{\mathcal{G}}_4$, *then*

$$tr\, W(\Theta(\gamma);\mu) = tr\, W(\gamma;-\mu).$$

Proof: Let

$$C = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and let $\Phi : PSL(2,\mathbf{C}) \longrightarrow PSL(2,\mathbf{C})$ be defined by

$$\Phi(A) = CAC^{-1}, \text{ for } A \in PSL(2,\mathbf{C}).$$

Set $\Psi = \Phi\Theta$. By a direct computation,

$$\Psi(S) = S,\ \Psi(T) = T,\ \Psi(X_\mu) = X_{-\mu}.$$

Then

$$\begin{aligned} tr\, W(\Theta(\gamma);\mu) &= tr\ \Theta(W(\gamma;\mu)) \\ &= tr\ \Phi\Theta(W(\gamma;\mu)) \\ &= tr\ \Psi(W(\gamma;\mu)) \\ &= tr\ W(\gamma;-\mu). \end{aligned}$$

Theorem 2.8 (Trace Formula)

If $\gamma \in \hat{\mathcal{G}}_4$, then

$$tr\ W(\gamma;\mu) = \pm(\mu^{2m} + 4N(\gamma)\mu^{2m-1}) + O(\mu^{2m-2}),$$

where $m = I_X(\gamma)$.

Proof: If $\varepsilon, \delta \in \{1,-1\}$ and α is an integer, then

$$S^\varepsilon = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}, \quad T^\alpha = \begin{pmatrix} 1 & 4\alpha \\ 0 & 1 \end{pmatrix}, \quad X^\delta = X_\mu^\delta = \begin{pmatrix} 1+\delta\mu & -\delta\mu^2 \\ \delta & 1-\delta\mu \end{pmatrix}.$$

By Lemma 2.7, we assume that $\gamma \in \mathcal{G}_4^+$. For $i = 1, ..., m$, let

$$T^{-\alpha_i} S^{\varepsilon_i} T^{\beta_i} X^{\delta_i} = \begin{pmatrix} a_i(\mu) & b_i(\mu) \\ c_i(\mu) & d_i(\mu) \end{pmatrix}, \text{ and}$$

$$\prod_{i=1}^{m} T^{-\alpha_i} S^{\varepsilon_i} T^{\beta_i} X^{\delta_i} = \begin{pmatrix} A_m(\mu) & B_m(\mu) \\ C_m(\mu) & D_m(\mu) \end{pmatrix},$$

where $\alpha_i \geq 0, \beta_i \geq 0$ and $\sum_{i=1}^{m}(\alpha_i + \beta_i) = N(\gamma)$. A direct computation yields:

$$\begin{aligned} a_i(\mu) &= (1 - 4\alpha_i\varepsilon_i)\delta_i\mu + \text{ const}, \\ b_i(\mu) &= -(1 - 4\alpha_i\varepsilon_i)\delta_i\mu^2 - 4\delta_i(-\alpha_i + \beta_i - 4\alpha_i\beta_i\varepsilon_i)\mu + \text{ const}, \\ c_i(\mu) &= \varepsilon_i\delta_i\mu + \text{ const}, \\ d_i(\mu) &= -\varepsilon_i\delta_i\mu^2 - (1 + 4\varepsilon_i\beta_i)\delta_i\mu + \text{ const}. \end{aligned}$$

If $m = 1$,

$$\begin{aligned} tr\, W(\gamma;\mu) &= -\varepsilon_1\delta_1\mu^2 + 4\varepsilon_1\delta_1(-\alpha_1 - \beta_1)\mu + \text{ const} \\ &= -\varepsilon_1\delta_1(\mu^2 + 4(\alpha_1 + \beta_1)\mu) + \text{ const} \\ &= -\varepsilon_1\delta_1(\mu^2 + 4N(\gamma)\mu) + \text{ const}. \end{aligned}$$

Applying induction on m, one can show that if $m \geq 2$, then

$$\begin{aligned} A_m &= (-1)^{m-1}(1 - 4\alpha_1\varepsilon_1)K_m\mu^{2m-1} + O(\mu^{2m-2}) \\ B_m &= (-1)^{m}(1 - 4\alpha_1\varepsilon_1)K_m\mu^{2m} + O(\mu^{2m-1}) \\ C_m &= (-1)^{m-1}\varepsilon_1 K_m\mu^{2m-1} + O(\mu^{2m-2}) \\ D_m &= (-1)^{m-1}\varepsilon_1 K_m\mu^{2m} + (-1)^m K_m[1 + 4\varepsilon_1(\sum_{i=2}^{m}\alpha_i + \sum_{i=1}^{m}\beta_i)]\mu^{2m-1} + O(\mu^{2m-2}) \end{aligned}$$

where

$$K_m = (\prod_{i=2}^{m} \varepsilon_i)(\prod_{i=1}^{m} \delta_i).$$

Then

$$tr\ W(\gamma;\mu) = A_m(\mu) + D_m(\mu) = (-1)^m(\prod_{i=1}^{m} \varepsilon_i\delta_i)(\mu^{2m} + 4N(\gamma)\mu^{2m-1}) + O(\mu^{2m-2}).$$

If $\gamma \in \mathcal{G}_4^-$, then $\Theta(\gamma) \in \mathcal{G}_4^+$, and, by Lemma 2.4 and Lemma 2.7, we have

$$\begin{aligned} tr\ W(\gamma;\mu) &= tr\ W(\Theta(\gamma);-\mu) \\ &= \pm((-\mu)^{2m} + 4N(\Theta(\gamma))(-\mu)^{2m-1}) + O(\mu^{2m-2}) \\ &= \pm(\mu^{2m} + 4N(\gamma)\mu^{2m-1}) + O(\mu^{2m-2}). \end{aligned}$$

The proof is complete. **Q.E.D.**

2.3 Words for Simple Closed Geodesics on Σ_5 and Traces

Let $G =< S,T,X,Y >$, let P and Q be the punctures on $\Sigma_5 = \Omega_0/G$ corresponding to the fixed point of Y and the fixed point of $S^{-1}Y$, respectively, and let $\mathcal{T}_1$ be a Dehn twist associated with P and Q, Θ_1 and Θ_2 be the automorphisms of G defined by

$$\Theta_1 \ : \ S \longrightarrow S^{-1},\ \ T \longrightarrow T^{-1},\ \ X \longrightarrow X^{-1},\ \ Y \longrightarrow Y^{-1},$$

$$\Theta_2 \ : \ S \longrightarrow T,\ \ T \longrightarrow S,\ \ X \longrightarrow Y,\ \ Y \longrightarrow X,$$

and $\mathcal{T}_2 = \Theta_2 \mathcal{T}_1 \Theta_2$. $\mathcal{T}_2$ is a Dehn twist associtaed with the punctures corresponding to the fixed point of X and the fixed point of $T^{-1}X$. For each $j = 1, 2$, the action of $\mathcal{T}_j$ on G is given as follows:

$$\begin{aligned} \mathcal{T}_1 &: S \longrightarrow S, \;\; T \longrightarrow T, \;\; X \longrightarrow X, \;\; Y \longrightarrow Y^{-1}S, \\ \mathcal{T}_2 &: S \longrightarrow S, \;\; T \longrightarrow T, \;\; X \longrightarrow X^{-1}T, \;\; Y \longrightarrow Y. \end{aligned}$$

Let $\gamma_\infty \in \mathcal{G}_5$ be the curve represented by T, and $\gamma'_\infty \in \mathcal{G}_5$ be the curve represented by S, and let $\hat{\mathcal{G}}_5 - \{\gamma_\infty, \gamma'_\infty\}$. Let $\mathcal{G}_5^+(T)$ be the set of curves in $\mathcal{G}_5$ which contain no strands joining the T-side the X^ε-side, $\varepsilon = \pm 1$, and let

$$\mathcal{G}_5^-(T) = \Theta_1(\mathcal{G}_5^+(T)), \;\; \mathcal{G}_5^+(S) = \Theta_2(\mathcal{G}_5^-(T)), \;\; \mathcal{G}_5^-(S) = \Theta_1(\mathcal{G}_5^+(S)).$$

As in §2.3, we associate to each $\gamma \in \hat{\mathcal{G}}_5$ two integers $N_T(\gamma)$ and $N_S(\gamma)$ given as follows: Let $E \in \{S, T\}$, and $\varepsilon, \delta \in \{1, -1\}$.

$$\begin{aligned} N_E(\gamma) &\geq 0 \quad \text{when } \gamma \in \mathcal{G}_5^+(E), \\ N_E(\gamma) &\leq 0 \quad \text{when } \gamma \in \mathcal{G}_5^-(E), \\ |N_T(\gamma)| &= \#(\text{strands of } \gamma \text{ joining the } T^{-1}\text{-side and the } T\text{-side}) \\ &\quad + \#(\text{strands of } \gamma \text{ joining the } T^\varepsilon\text{-side and the } X^\delta\text{-side}), \text{ and} \\ |N_S(\gamma)| &= \#(\text{strands of } \gamma \text{ joining the } S^{-1}\text{-side and the } S\text{-side}) \\ &\quad + \#(\text{strands of } \gamma \text{ joining the } S^\varepsilon\text{-side and the } Y^\delta\text{-side}). \end{aligned}$$

The following lemma follows immediately from definitions.

Lemma 2.9 *If* $\gamma \in \hat{\mathcal{G}}_5$*, then*

(1) $I_E(\Theta_1(\gamma)) = I_E(\gamma)$ *for* $E \in \{S, T, X, Y\}$,

(2) $I_X(\Theta_2(\gamma)) = I_Y(\gamma)$, $I_T(\Theta_2(\gamma)) = I_S(\gamma)$,

(3) $N_T(\Theta_1(\gamma)) = -N_T(\gamma)$, $N_S(\Theta_1(\gamma)) = -N_S(\gamma)$,

(4) $N_T(\Theta_2(\gamma)) = -N_S(\gamma)$, *and* $N_S(\Theta_2(\gamma)) = -N_T(\gamma)$.

Let $\gamma \in \hat{\mathcal{G}}_5$ and $W = W(\gamma)$ be a cyclic reduced word in G representing γ. By the same reasoning as in §2.3, neither one of the following is a subword of W:

$$X^\varepsilon X^\varepsilon,\ \ Y^\varepsilon Y^\varepsilon,\ \ T^\delta X^\varepsilon T^\delta,\ \ S^\delta Y^\varepsilon S^\delta,\ \ X^\varepsilon T^k X^\delta,\ \ Y^\varepsilon S^k Y^\delta,\ \ T^\varepsilon S^\delta T^\varepsilon,\ \ S^\delta T^\varepsilon S^\delta$$

where $\varepsilon, \delta \in \{1, -1\}$, and $k \neq 0$ is an integer.

If $E \in \{X^\pm, T^\pm\}$ and EY^ε (or $Y^\varepsilon E$ resp.) is a subword of W, we write $EY^\varepsilon = ES^0Y^\varepsilon$ (or $Y^\varepsilon E = Y^\varepsilon S^0 E$ resp.). Similarly, if $E \in \{Y^\pm, S^\pm\}$, we will write EX^ε and $X^\varepsilon E$ as ET^0X^ε and $X^\varepsilon T^0 E$, respectively. We call such a word a *semi-reduced word.*

Lemma 2.10 *Let* $\varepsilon = \pm 1$*, and let* α *and* β *be integers, and let* $\gamma \in \hat{\mathcal{G}}_5$ *and* $W(\gamma) = W$ *be a cyclic semi-reduced word in* G *representing* γ.

(1) *If* $W' = E' S^\alpha Y^\varepsilon S^\beta E$ *is a subword of* W *and* $E, E' \in \{X^\pm, T^\pm\}$*, then*

$$-1 \le (\alpha+\beta)\varepsilon \le 0,\ \ \alpha \le 0\ \ \beta \ge 0\ \ \textit{when } \gamma \in \mathcal{G}_5^+(S),\ \ \textit{and}\ \ \alpha \ge 0\ \ \beta \le 0\ \ \textit{when } \gamma \in \mathcal{G}_5^-(S).$$

(2) *If* $W' = E'T^{\alpha}X^{\epsilon}T^{\beta}E$ *is a subword of* W *and* $E, E' \in \{Y^{\pm}, S^{\pm}\}$*, then*

$-1 \leq (\alpha+\beta)\varepsilon \leq 0,\ \ \alpha \geq 0\ \ \beta \leq 0$ *when* $\gamma \in \mathcal{G}_5^+(T)$*, and* $\alpha \leq 0\ \ \beta \geq 0$ *when* $\gamma \in \mathcal{G}_5^-(T)$.

Proof: We will prove (1). Considering $\mathcal{T}_2$ and applying a similar argument, (2) will follow.

Without loss of generality, we may assume that $\varepsilon = 1$ and $\gamma \in \mathcal{G}_5^+(S)$. Since γ contains no strands joining the S^{-1}-side to the Y^{ϵ}-side, then $\alpha \leq 0$ and $\beta \geq 0$. We rewrite W' as

$$W' = E'S^{-\alpha}Y^{\epsilon}S^{\beta}E = E'S^{-\alpha}YS^{\beta}E,$$

where $\alpha \geq 0$ and $\beta \geq 0$. Suppose that $\beta > \alpha$. If

$$\mathcal{T}_1^{-2\alpha}(W') = E'YS^{\beta-\alpha}E$$

is a subword of $\mathcal{T}_1^{-2\alpha}(W)$, then $\mathcal{T}_1^{-2\alpha}(\gamma)$ is not simple. Contradiction!.

Suppose that $\alpha > \beta + 1$. Since

$$\mathcal{T}_1^{-2\beta}(W') = E'S^{\alpha-\beta}YE,$$

then $\mathcal{T}_1^{-2\beta}(\gamma)$ contains a strand joining the S-side to the S^{-1}-side, and contains a strand joining the Y^{-1}-side to the E-side with $E \in \{T^{\pm}, X^{\pm}\}$, which is impossible since $\mathcal{T}_1^{-2\beta}(\gamma)$ is simple. Therefore, $\beta \leq \alpha \leq \beta + 1$. **Q.E.D.**

Remark 2.1. If $E, E' \in \{S^{\pm}, Y^{\pm}\}$ and α and β are integers such that $ET^{\alpha}X^{\varepsilon}T^{\beta}E'$ is a subword of $W(\gamma)$, then γ contains $|\alpha| + |\beta| - 2$ strands joining the T^{-1}-side to the T-side whenever $|\alpha| + |\beta| > 2$, and contains no strands joining the T^{-1}-side to the T-side whenever $|\alpha| + |\beta| \leq 2$. If $|\alpha| + |\beta| \leq 2$, then γ contains $|\alpha| + |\beta|$ strands joining the T^{δ}-side to the X^{ε}-side, where $\delta, \varepsilon \in \{1, -1\}$.

If $E, E' \in \{T^{\pm}, X^{\pm}\}$ and α and β are integers such that $ES^{\alpha}Y^{\varepsilon}S^{\beta}E'$ is a subword of $W(\gamma)$, then γ contains $|\alpha| + |\beta| - 2$ strands joining the S^{-1}-side to the S-side whenever $|\alpha| + |\beta| > 2$, and contains no strands joining the S^{-1}-side to the S-side whenever $|\alpha| + |\beta| \leq 2$. If $|\alpha| + |\beta| \leq 2$, then γ contains $|\alpha| + |\beta|$ strands joining the S^{δ}-side to the Y^{ε}-side, where $\delta, \varepsilon \in \{1, -1\}$.

Remark 2.2. If $I_Y(\gamma) = 0$, then γ does not contain any strand joining the S^{-1}-side to the S-side, and thus $N_S(\gamma) = 0$. Similarly, $N_T(\gamma) = 0$ if $I_X(\gamma) = 0$.

If $I_E(\gamma) = 0$ for some $E \in \{S, T, X, Y\}$, then, by Theorem 2.3, Lemma 2.10 and the above remarks, we easily get a cyclic semi-reduced word $W = W(\gamma) \in G$ representing γ. For instance, if $I_Y(\gamma) = 0$, then W is of the form

$$\prod_{i=1}^{m} T^{t_i} X^{\omega_i} T^{r_i} S^{\delta_i},$$

where $\delta_i, \omega_i \in \{1, -1\}$, $m = I_X(\gamma) = I_S(\gamma)$, and t_i and r_i are integers satisfying the following conditions:

(1) $-1 \leq (t_i + r_i)\omega_i \leq 0$ and $-1 \leq (t_{i+1} + r_i)\delta_i \leq 0$.

(2) $|t_i|, \ |r_i| \in \{t, t+1\}$, where $t = \min\{|t_i|, \ |r_i| : i = 1, ..., m\}$.

(3) $\sum_{i=1}^{m}(|t_i| + |r_i|) = |N_T(\gamma)|$.

(4) $t_i \geq 0 \ \ r_i \leq 0$ when $\gamma \in \mathcal{G}_5^+(T)$, and $t_i \leq 0 \ \ r_i \geq 0$ when $\gamma \in \mathcal{G}_5^-(T)$.

In what follows, unless stated otherwise, we assume that $I_X(\gamma)I_Y(\gamma) \neq 0$. Since $\mathcal{G}_5^-(S) = \Theta_1(\mathcal{G}_5^+(S))$, and since $I_X(\gamma) \leq I_Y(\gamma)$ if and only if $I_Y(\Theta_2(\gamma)) \leq I_X(\Theta_2(\gamma))$, we restrict our attention to the curves $\gamma \in \mathcal{G}_5^+(S)$ with $I_X(\gamma) \geq I_Y(\gamma)$.

First, we are going to find a cyclic semi-reduced word for $\gamma \in \mathcal{G}_5(\gamma'_\infty) \cap \mathcal{G}_5^+(S)$, where $\mathcal{G}_5(\gamma'_\infty)$ is the set of curves in $\hat{\mathcal{G}}_5$ which contain no strands joining the Y^ε-side to the E-side for $\varepsilon = \pm 1$ and $E \in \{T^\pm, X^\pm\}$.

Let $m = I_X(\gamma)$ and $n = I_Y(\gamma)$. By assumption, $m \geq n > 0$, and there exist exactly n strands $\ell_1, ..., \ell_n$ of γ joining the S-side to the Y-side, there exist exactly n strands $\ell'_1, ..., \ell'_n$ of γ joining the S-side to the Y^{-1}-side, and the endpoint of ℓ_i on the Y-side and the endpoint of ℓ'_i on the Y^{-1}-side are identified by Y. Then there are integers $\alpha_i > 0$, $\beta_i > 0$ and $\varepsilon_i \in \{1, -1\}$, and there exist E_i, $E'_i \in \{X^\pm, T^\pm\}$

such that for every $i = 1, ..., n$

$$E_i S^{-\alpha_i} Y^{\varepsilon_i} S^{\beta_i} E_i'$$

is a subword of W. Thus, W must be of the form

$$W = \prod_{i=1}^{n} S^{-\alpha_i} Y^{\varepsilon_i} S^{\beta_i} W_i, \qquad W_i = \prod_{j=1}^{m_i} E_{ij},$$

$$E_{i1},\ E_{im_i} \in \{T^{\pm}, X^{\pm}\}, \quad \text{for all } i = 1, ..., n, \text{ and}$$

$$E_{ij} \in \{S^{\pm}, T^{\pm}, X^{\pm}\}, \quad \text{for all } 1 < j < m_i \text{ if } m_i > 2.$$

Also, we require W_i to be a semi-reduced word for each i. Since $m \geq n$, then, for every $i = 1, ..., n$, there exists a $j \in \{1, ..., m_i\}$ such that $E_{ij} \in \{X^{\pm}\}$.

To find W_i, we define *reduced words* representing (simple) subarcs of γ as follows. Let $W = E_1 \cdots E_k$ be a cyclic reduced word in G representing γ. Note that $k > 1$. Let p and q be any two integers with $1 \leq p \leq k$ and $1 \leq q < k$. For each $j \in \{1, ..., q\}$, there is a strand ℓ_j of γ joining the E_{p+j-1}^{-1}-side to the E_{p+j}-side. The union of these strands $\ell_1, ..., \ell_q$ projects to a subarc γ' of γ. We will use

$$W' = \check{E}_p E_{p+1} \cdots E_{p+q}$$

to represent γ', where $E_{p+j} = E_{p+j-k}$ if $p + j > k$.

Now, we first assume that W_i is a reduced word for each $i = 1, ..., n$. Note that W_i is always followed by S^{-1} since $\alpha_{i+1} > 0$ for each i. Consider the subarc γ_i

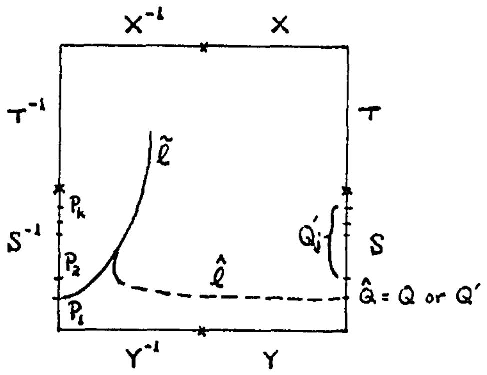


Figure 2.6:

represented by the reduced word

$$\tilde{W}_i = \check{S} W_i S^{-1}.$$

We may define $I_X(\gamma_i)$ and $I_Y(\gamma_i)$ as before. It is clear that $I_X(\gamma_i) > 0$ and $I_Y(\gamma_i) = 0$ for every i.

To simply notations, we write for every fixed i

$$\tilde{W}_i = \check{S} E_1 \cdots E_p S^{-1}.$$

Let ℓ be the strand of γ_i joining the S^{-1}-side to the E_1-side, ℓ' be the strand of γ' joining the E_p^{-1}-side to the S^{-1}-side, P and P' be the endpoints of ℓ and ℓ' on the S^{-1}-side respectively, and Q and Q' be the points on the S-side identified with P and P' by S respectively.

Assume that γ_i meets the S^{-1}-side at k points, and we label these points $P_1, ..., P_k$ so that P_j lies "below" P_{j+1}, $j = 1, ..., k-1$, see Figure 2.6. Then γ_i

meets the S-side at $k-2$ points.

Claim : $\{P, P'\} = \{P_1, P_2\}$.

Proof of the Claim: The claim is clear if $k = 2$. Assume that $k > 2$, and let $Q'_1, ..., Q'_{k-2}$ be the points on the S-side where γ_i meets. Since the arc represented by $\check{E}S^{\alpha}E'$ with $|\alpha| > 1$ and $E, E' \in \{X^{\pm}, T^{\pm}\}$ is not simple , then γ_i contains no strands joining the S-side to the S^{-1}-side. Since Q is the endpoint of some strand of γ joining the E-side to the S-side with $E \in \{S^{-1}, Y^{\pm}\}$, then Q'_j must lie above Q, and then $P \in \{P_1, P_2\}$.

If Q'_j lies below Q' for some j, then γ is not simple since $\tilde{W}_i$ is followed by S^{-1} or Y^{ϵ}, and since Q'_j is the endpoint of some strand of γ_i joining the E-side to the S-side with $E \in \{X^{\pm}, T^{\pm}\}$. Thus $P' \in \{P_1, P_2\}$. This completes the proof of the claim.

Let

$$\hat{Q} = \begin{cases} Q' & \text{when } P_1 = P, \\ Q & \text{when } P_1 = P'. \end{cases}$$

Let $\tilde{\ell}$ be the strand of γ_i with endpoint P_1, and $\tilde{Q}$ be the other endpoint of $\tilde{\ell}$ on the E-side for some $E \in \{X^{\pm}, T^{\pm}\}$. Since γ_i contains no strands joining the E^{δ}-side to the Y^{ϵ}-side, where $\delta, \varepsilon \in \{1, -1\}$, then there is a strand $\hat{\ell}$ joining $\hat{Q}$ to $\tilde{Q}$ which is disjoint from all the strands of γ_i perhaps except $\tilde{\ell}$. Let $\hat{\gamma}_i$ be the curve on Σ_5 obtained from γ_i by replacing $\tilde{\ell}$ by $\hat{\ell}$. Then $\hat{\gamma}_i$ is a simple closed curve in $\hat{\mathcal{G}}_5$ with $I_Y(\hat{\gamma}_i) = 0$ and $I_X(\hat{\gamma}_i) = I_X(\gamma_i)$. $\hat{\gamma}_i$ is represented by a cyclic semi-reduced word $\hat{W}_i$ of the form

$$\hat{W}_i = \prod_{j=1}^{m'_i} T^{t_{ij}} X^{\omega_{ij}} T^{r_{ij}} S^{\delta_{ij}},$$

where $m'_i = I_X(\hat{\gamma}_i) = I_X(\gamma_i)$, and $t_{ij}, r_{ij}, \omega_{ij}$ and δ_{ij} are integers satisfying the conditions in Theorem 2.3. Let $\hat{\gamma}_i$ be oriented so that $\pi(\hat{Q})$ is the endpoint and the orientation of $\hat{\gamma}_i$ is determined by the oriented arc with starting point $\pi(\tilde{Q})$ and endpoint $\pi(\hat{Q})$, where π is the canonical projection of the fundamental domain of G onto Σ_5. We may write $\hat{W}_i$ so that $\hat{W}_i$ represents the oriented closed curve $\hat{\gamma}_i$. Then $\delta_{im'_i} = 1$, and

$$\tilde{W}_i = \check{S}(\prod_{j=1}^{m_i} T^{t_{ij}} X^{\omega_{ij}} T^{r_{ij}} S^{\delta_{ij}})S^{-1},$$

where $\delta_{im_i} = 0$ and $\delta_{ij} \in \{1, -1\}, 1 \leq j < m_i$, and thus

$$W = \prod_{i=1}^{n} S^{-\alpha_i} Y^{\varepsilon_i} S^{\beta_i} (\prod_{j=1}^{m_i} T^{t_{ij}} X^{\omega_{ij}} T^{r_{ij}} S^{\delta_{ij}})$$

Theorem 2.11 *Let $\gamma \in \hat{\mathcal{G}}_5$, and assume that $I_X(\gamma) I_Y(\gamma) \neq 0$.*

(I) *If $I_X(\gamma) \geq I_Y(\gamma) = n$, then γ is represented by a cyclic semi-reduced word $W(\gamma)$ of the form*

$$W(\gamma) = \prod_{i=1}^{n} S^{\alpha_i} Y^{\varepsilon_i} S^{\beta_i} (\prod_{j=1}^{m_i} T^{t_{ij}} X^{\omega_{ij}} T^{r_{ij}} S^{\delta_{ij}}),$$

where $\varepsilon_i, \omega_{ij} \in \{1, -1\}$, $m_i > 0$, *and* $\alpha_i, \beta_i, t_{ij}, r_{ij}$ *and* δ_{ij} *are integers satisfying the following conditions:*

(1) $\sum_{i=1}^{n} m_i = I_X(\gamma)$, $\delta_{im_i} = 0$. *If* $m_i > 1$, *then* $\delta_{ij} = \pm 1$ *for* $1 \leq j < m_i$.

(2) $-1 \leq (\alpha_i + \beta_i)\varepsilon_i \leq 0$; $|\alpha_i|, |\beta_i| \in \{\alpha, \alpha + 1\}$, *where* $\alpha = \min\{|\alpha_i|, |\beta_i| : 1 \leq i \leq n\}$; $\sum_{i=1}^{n}(|\alpha_i| + |\beta_i|) = |N_S(\gamma)|$;

$$\begin{aligned} \alpha_i &\leq 0, \ \beta_i \geq 0 \ \textit{for all } i \textit{ when } \gamma \in \mathcal{G}_5^+(S), \textit{ and} \\ \alpha_i &\geq 0, \ \beta_i \leq 0 \ \textit{for all } i \textit{ when } \gamma \in \mathcal{G}_5^-(S). \end{aligned}$$

(3) $-1 \leq (t_{ij} + r_{ij})\omega_{ij} \leq 0$; $|t_{ij}|, |r_{ij}| \in \{t, t + 1\}$, *where* $t = \min\{|t_{ij}|, |r_{ij}| : 1 \leq i \leq n, 1 \leq j \leq m_i\}$; $\sum_{i=1}^{n} \sum_{j=1}^{m_i}(|t_{ij}| + |r_{ij}|) = |N_T(\gamma)|$;

$$\begin{aligned} t_{ij} &\leq 0, \ r_{ij} \geq 0 \ \textit{for all } i,j \textit{ when } \gamma \in \mathcal{G}_5^-(T), \textit{ and} \\ t_{ij} &\geq 0, \ r_{ij} \leq 0 \ \textit{for all } i,j \textit{ when } \gamma \in \mathcal{G}_5^+(T). \end{aligned}$$

(II) *If* $I_Y(\gamma) \geq I_X(\gamma) = m$, *then* γ *is represented by a cyclic semi-reduced word* $W(\gamma)$ *of the form*

$$W(\gamma) = \prod_{i=1}^{m} T^{t_i} X^{\omega_i} T^{r_i} \Big(\prod_{j=1}^{n_i} S^{\alpha_{ij}} Y^{\varepsilon_{ij}} S^{\beta_{ij}} T^{\delta_{ij}}\Big),$$

where $\varepsilon_{ij}, \omega_i \in \{1, -1\}$, $n_i > 0$, *and* $t_i, r_i, \alpha_{ij}, \beta_{ij}$ *and* δ_{ij} *are integers satisfying the following conditions:*

(1) $\sum_{i=1}^{m} n_i = I_Y(\gamma)$, $\delta_{in_i} = 0$. *If* $n_i > 1$, *then* $\delta_{ij} = \pm 1$ *for* $1 \leq j < n_i$.

(2) $-1 \leq (t_i + r_i)\omega_i \leq 0$; $|t_i|, |r_i| \in \{t, t+1\}$, *where* $t = \min\{|t_i|, |r_i| : 1 \leq i \leq m\}$; $\sum_{i=1}^{m}(|t_i| + |r_i|) = |N_T(\gamma)|$;

$$\begin{aligned} t_i &\leq 0, \; r_i \geq 0 \text{ for all } i \text{ when } \gamma \in \mathcal{G}_5^-(T), \text{ and} \\ t_i &\geq 0, \; r_i \leq 0 \text{ for all } i \text{ when } \gamma \in \mathcal{G}_5^+(T). \end{aligned}$$

(3) $-1 \leq (\alpha_{ij} + \beta_{ij})\varepsilon_{ij} \leq 0$; $|\alpha_{ij}|, |\beta_{ij}| \in \{\alpha, \alpha+1\}$, *where* $\alpha = \min\{|\alpha_{ij}|, |\beta_{ij}| : 1 \leq i \leq n\}$; $\sum_{i=1}^{m} \sum_{j=1}^{n_i}(|\alpha_{ij}| + |\beta_{ij}|) = |N_S(\gamma)|$;

$$\begin{aligned} \alpha_{ij} &\leq 0, \; \beta_{ij} \geq 0 \text{ for all } i,j \text{ when } \gamma \in \mathcal{G}_5^+(S), \text{ and} \\ \alpha_{ij} &\geq 0, \; \beta_{ij} \leq 0 \text{ for all } i,j \text{ when } \gamma \in \mathcal{G}_5^-(S). \end{aligned}$$

Remark 2.3: If $I_X(\gamma) = I_Y(\gamma) = n$, then

$$W(\gamma) = \prod_{i=1}^{n} S^{\alpha_i} Y^{\varepsilon_i} S^{\beta_i} T^{t_i} X^{\omega_i} T^{r_i}.$$

Proof of Theorem 2.11: Let $\gamma \in \mathcal{G}_5^+(S)$ with $I_X(\gamma) \geq I_Y(\gamma) = n$, and W be a cyclic reduced word in G representing γ. There are exactly n integers $\varepsilon_i = \pm 1$, and there are $E_i, E_i' \in \{S^{\pm}, X^{\pm}, T^{\pm}\}$ such that $E_i \neq S$, $E_i' \neq S^{-1}$ and $E_i Y^{\varepsilon_i} E_i'$ is a subword of W. Consider the image of $E_i Y^{\varepsilon_i} E_i'$ under $\mathcal{T}_1^2$.

$$\mathcal{T}_1^2(E_i Y^{\varepsilon_i} E_i') = E_i S^{-1} Y^{\varepsilon_i} S E_i',$$

then $\mathcal{T}_1^2(\gamma) \in \mathcal{G}_5(\gamma_\infty')$, and thus

$$\mathcal{T}_1^2(W) = \prod_{i=1}^{n} S^{-\alpha_i} Y^{\varepsilon_i} S^{\beta_i} (\prod_{j=1}^{m_i} T^{t_{ij}} X^{\omega_{ij}} T^{r_{ij}} S^{\delta_{ij}}),$$

and

$$W = \prod_{i=1}^{n} S^{-\alpha_i'} Y^{\varepsilon_i} S^{\beta_i'} (\prod_{j=1}^{m_i} T^{t_{ij}} X^{\omega_{ij}} T^{r_{ij}} S^{\delta_{ij}}),$$

where $\alpha_i' = \alpha_i - 1$ and $\beta_i' = \beta_i - 1$ for $i = 1, ..., n$. Since $\alpha_i > 0$, $\beta_i > 0$ for all i, then $\alpha_i' \geq 0$, $\beta_i' \geq 0$ for all i.

If $\gamma \in \mathcal{G}_5^-(S)$ and $I_X(\gamma) \geq I_Y(\gamma)$, then $\Theta_1(\gamma) \in \mathcal{G}_5^+(S)$ and $I_E(\Theta_1(\gamma)) = I_E(\gamma)$ for $E \in \{X, Y\}$. If $I_X(\gamma) \leq I_Y(\gamma)$, then $I_X(\Theta_2(\gamma)) = I_Y(\gamma) \geq I_X(\gamma) = I_Y(\Theta_2(\gamma))$. Therefore, every curve γ in $\hat{\mathcal{G}}_5$ with $I_X(\gamma)I_Y(\gamma) \neq 0$ is represented by a cyclic semi-reduced word in G of the form given in (I) or (II).

To complete the proof, we have to show that $|\alpha_i|, |\beta_i| \in \{\alpha, \alpha + 1\}$ if γ is represented a word given in (I), where $\alpha = \min\{|\alpha_i|, |\beta_i| : 1 \leq i \leq n\}$. Note that the other conditions for α's, β's, t's, r's, δ's, ε's and ω's follow from Lemma 2.10 and Remark 2.1. Without loss of generality, we assume that $\gamma \in \mathcal{G}_5^+(S)$, and assume that $\alpha = \min\{|\alpha_1|, |\beta_1|\}$ and $\varepsilon_1 = 1$. We write

$$W = \prod_{i=1}^{n} S^{-\alpha_i} Y^{\varepsilon_i} S^{\beta_i} (\prod_{j=1}^{m_i} T^{t_{ij}} X^{\omega_{ij}} T^{r_{ij}} S^{\delta_{ij}}), \quad \alpha_i, \ \beta_i \geq 0 \text{ for all } i.$$

If $n = 1$, we are done. Assume that $n > 1$. Since $\varepsilon = 1$ and $(\beta_1 - \alpha_1)\varepsilon_1 \leq 0$, then $\alpha = \beta_1$. Suppose that there is an $i_0 > 1$ such that $\max\{\alpha_{i_0}, \beta_{i_0}\} > \alpha + 1$.

$$\mathcal{T}_1^{-2\alpha}(W) = \prod_{i=1}^{n} S^{-\alpha_i'} Y^{\varepsilon_i} S^{\beta_i'} (\prod_{j=1}^{m_i} T^{t_{ij}} X^{\omega_{ij}} T^{r_{ij}} S^{\delta_{ij}}),$$

where $\alpha_i' = \alpha_i - \alpha$ and $\beta_i' = \beta_i - \alpha$, for all i. Let γ' be the curve in $\hat{\mathcal{G}}_5$ represented by $\mathcal{T}_1^{-2\alpha}(W)$, or equivalently, $\gamma' = \mathcal{T}_1^{-2\alpha}(\gamma)$. Since $\beta_1' = \beta_1 - \alpha = 0$, then γ' has a strand joining the Y^{-1}-side to the E-side for some $E \in \{X^{\pm}, T^{\pm}\}$. On the other hand, $\max\{\alpha'_{i_0}, \beta'_{i_0}\} > 1$, then γ' has a strand joining the S-side to the S^{-1}-side. Contradiction! **Q.E.D.**

As a consequence of the proof of Theorem 2.11, we have the following corollary.

Corollary 2.12 *Let $Homeo(\Sigma_5)$ be the group of all homeomorphisms of Σ_5 onto itself, and let $\Gamma = < \mathcal{T}_1, \mathcal{T}_2 >$ be the subgroup of $Homeo(\Sigma_5)$ generated by $\mathcal{T}_1$ and $\mathcal{T}_2$. If $\gamma \in \hat{\mathcal{G}}_5$, then there is a $\varphi \in \Gamma$ depending on γ such that*

$$\begin{aligned} |N_T(\varphi(\gamma))| &\geq 2 \quad \textit{whenever } I_X(\gamma) > 0, \textit{ and} \\ |N_S(\varphi(\gamma))| &\geq 2 \quad \textit{whenever } I_Y(\gamma) > 0. \end{aligned}$$

Now, we are going to compute the high order terms of $tr\ W(\gamma;\mu,\nu)$, where (μ,ν) is in $\mathcal{M}_5$ and $W(\gamma;\mu,\nu)$ is the cyclic semi-reduced word in $G(\mu,\nu)$ given in Theorem 11 representing $\gamma \in \hat{\mathcal{G}}_5$. For simplicity, we write

$$F(\gamma;\mu,\nu) = tr\ W(\gamma;\mu,\nu) = a_1\mu^p\nu^q + a_2\mu^{p-1}\nu^q + a_3\mu^p\nu^{q-1} + O(p+q-2),$$

where $a_1 \neq 0$, a_2 and a_3 are integers, and $O(p+q-2)$ is a polynomial in μ and ν with degree $\leq p+q-2$. We call $a_1\mu^p\nu^q + a_2\mu^{p-1}\nu^q + a_3\mu^p\nu^{q-1}$ the high order terms of $F(\gamma;\mu,\nu)$.

Lemma 2.13 *If $\gamma \in \hat{\mathcal{G}}_5$ with $I_X(\gamma) = m$ and $I_Y(\gamma) = n$, then*

(1) $F(\Theta_1(\gamma);\mu,\nu) = F(\gamma;-\mu,-\nu)$,

(2) $F(\Theta_2(\gamma);\mu,\nu) = F(\gamma;-2\nu,-\mu/2)$,

(3) $F(\mathcal{T}_1(\gamma);\mu,\nu) = (-1)^n F(\gamma;\mu,\nu+1)$,

(4) $F(\mathcal{T}_1^{-1}(\gamma);\mu,\nu) = (-1)^n F(\gamma;\mu,\nu-1)$,

(5) $F(\mathcal{T}_2(\gamma);\mu,\nu) = (-1)^m F(\gamma;\mu-2,\nu)$, *and*

(6) $F(\mathcal{T}_2^{-1}(\gamma);\mu,\nu) = (-1)^m F(\gamma;\mu+2,\nu)$.

Proof: Let

$$C_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & -2i \\ \frac{1}{2i} & 0 \end{pmatrix},$$

and let $\Phi_j : PSL(2,\mathbf{C}) \longrightarrow PSL(2,\mathbf{C})$ be defined by

$$\Phi_j(A) = C_j A C_j^{-1}, \quad \text{for all } A \in PSL(2,\mathbf{C}).$$

Set $\Psi_j = \Phi_j \Theta_j$. By a direct computation, we have:

$$\begin{aligned} \Psi_j(S) &= S, \;\; \Psi_j(T) = T, \;\; \Psi_1(X_\mu) = X_{-\mu}, \\ \Psi_1(Y_\nu) &= Y_{-\nu}, \;\; \Psi_2(X_\mu) = X_{-2\nu}, \;\; \Psi_2(Y_\nu) = Y_{-\mu/2}. \end{aligned}$$

By a similarly argument as in the proof of Lemma 2.7, (1) and (2) follow.

Since

$$\begin{aligned} \mathcal{T}_1 &: S \longrightarrow S, \;\; T \longrightarrow T, \;\; X_\mu \longrightarrow X_\mu, \;\; Y_\nu \longrightarrow Y_\nu^{-1} S, \\ \mathcal{T}_1^{-1} &: S \longrightarrow S, \;\; T \longrightarrow T, \;\; X_\mu \longrightarrow X_\mu, \;\; Y_\nu \longrightarrow S Y_\nu^{-1}, \end{aligned}$$

and

$$SY_\nu^{-1} = -Y_{\nu-1}, \quad Y_\nu^{-1}S = -Y_{\nu+1},$$

then (3) and (4) hold. From (2) and (3),

$$\begin{aligned} F(\mathcal{T}_2(\gamma);\mu,\nu) &= F(\mathcal{T}_1\Theta_2(\gamma);-2\nu,\frac{-\mu}{2}) \\ &= (-1)^{I_Y(\Theta_2(\gamma))}F(\Theta_2(\gamma);-2\nu,\frac{-\mu}{2}+1) \\ &= (-1)^m F(\gamma;\mu-2,\nu). \end{aligned}$$

The following proposition is an immediate consequence of Theorem 2.8, Lemma 2.9 and Lemma 2.13.

Proposition 2.14 *Let* $\gamma \in \hat{\mathcal{G}}_5$, $I_X(\gamma) = m$ *and* $I_Y(\gamma) = n$.

(1) *If* $n = 0$, *then* $F(\gamma;\mu,\nu) = \pm(\mu^{2m} + 4N_T(\gamma)\mu^{2m-1}) + O(\mu^{2m-2})$.

(2) *If* $m = 0$, *then* $F(\gamma;\mu,\nu) = \pm 4^n(\nu^{2n} + 2N_S(\gamma)\nu^{2n-1}) + O(\nu^{2n-2})$.

Now, we consider the case where $I_X(\gamma)I_Y(\gamma) \neq 0$. Let $I_X(\gamma) = m$ and $I_Y(\gamma) = n$. Assume that $m \geq n$ and $\gamma \in \mathcal{G}_5^-(T)$. Write

$$W = W(\gamma) = \prod_{i=1}^{n} S^{\alpha_i}Y^{\varepsilon_i}S^{\beta_i}(\prod_{j=1}^{m_i} T^{-t_{ij}}X^{\omega_{ij}}T^{r_{ij}}S^{\delta_{ij}}),$$

where $t_{ij},\ r_{ij} \geq 0$. Note that

$$N_T(\gamma) = -\sum_{i=1}^{n}\sum_{j=1}^{m_i}(t_{ij}+r_{ij}) \quad N_S(\gamma) = \sum_{i=1}^{n}(\beta_i - \alpha_i).$$

For $t, r \geq 0$, $\omega, \delta, \varepsilon \in \{1, -1\}$ and integers α, β, we have:

$$T^{-t}X^{\omega}T^{r} = \begin{pmatrix} \omega\mu + 1 - 4t\omega & -\omega\mu^2 + 4(t+r)\omega\mu + \text{ const.} \\ \omega & -\omega\mu + 1 + 4r\omega \end{pmatrix}$$

$$S^{\alpha}Y^{\varepsilon}S^{\beta} = \begin{pmatrix} 2\varepsilon\nu + 1 + 4\varepsilon\beta & 4\varepsilon \\ -\varepsilon\nu^2 + 2\varepsilon(\alpha-\beta)\nu + \text{ const.} & -2\varepsilon\nu + 1 + 4\varepsilon\alpha \end{pmatrix}$$

$$T^{-t}X^{\omega}T^{r}S^{\delta} = \begin{pmatrix} -\omega\delta\mu^2 + (1 + 4(t+r)\delta)\omega\mu + \text{ const.} & -\omega\mu^2 + 4(t+r)\omega\mu + \text{ const.} \\ -\omega\delta\mu + \text{ const.} & -\omega\mu + 1 + 4r\omega \end{pmatrix}$$

For $i = 1, .., n$, let $\xi = \omega_{i1}$ when $m_i = 1$, and

$$\xi_i = (\prod_{j=1}^{m_i}\omega_{ij})(\prod_{j=1}^{m_i-1}\delta_{ij}) \quad \text{when } m_i > 1,$$

$$\lambda_i = 4\sum_{j=1}^{m_i}(t_{ij}+r_{ij}), \text{ and}$$

$$W_i = \prod_{j=1}^{m_i} T^{-t_{ij}}X^{\omega_{ij}}T^{r_{ij}}S^{\delta_{ij}} = \begin{pmatrix} a_i(\mu) & b_i(\mu) \\ c_i(\mu) & d_i(\mu) \end{pmatrix}$$

If $m_i = 1$, then

$$\begin{aligned} a_i(\mu) &= \xi_i(\mu + \text{ const.}) = \xi_i(\mu^{2m_i-1} + \cdots) \\ b_i(\mu) &= -\xi_i(\mu^2 - \lambda_i\mu + \text{ const.}) = -\xi_i(\mu^{2m_i} - \lambda_i\mu^{2m_i-1} + \cdots) \end{aligned}$$

$$\begin{aligned} c_i(\mu) &= \xi_i = \xi_i(\mu^{2m_i-2} + \cdots) \\ d_i(\mu) &= -\xi_i(\mu + \text{ const.}) = -\xi_i(\mu^{2m_i-1} + \cdots). \end{aligned}$$

By induction, one can show that for $m_i \geq 1$

$$\begin{aligned} a_i(\mu) &= (-1)^{m_i}\xi_i(-\mu^{2m_i-1} + \cdots) \\ b_i(\mu) &= (-1)^{m_i}\xi_i(\mu^{2m_i} - \lambda_i\mu^{2m_i-1} + \cdots) \\ c_i(\mu) &= (-1)^{m_i}\xi_i(-\mu^{2m_i-2} + \cdots) \\ d_i(\mu) &= (-1)^{m_i}\xi_i(\mu^{2m_i-1} + \cdots). \end{aligned}$$

For every $i = 1, ..., n$, let

$$S^{\alpha_i}Y^{\varepsilon_i}S^{\beta_i}W_i = \begin{pmatrix} \hat{a}_i(\mu,\nu) & \hat{b}_i(\mu,\nu) \\ \hat{c}_i(\mu,\nu) & \hat{d}_i(\mu,\nu) \end{pmatrix},$$

and for every n let

$$\prod_{i=1}^{n} S^{\alpha_i}Y^{\varepsilon_i}S^{\beta_i}W_i = \begin{pmatrix} A_n(\mu,\nu) & B_n(\mu,\nu) \\ C_n(\mu,\nu) & D_n(\mu,\nu) \end{pmatrix}.$$

Then

$$\deg \hat{a}_i = 2m_i, \quad \deg \hat{b}_i = 2m_i + 1 = \deg \hat{c}_i, \quad \hat{d}_i = 2m_i + 2, \text{ and}$$

$$\hat{d}_i(\mu,\nu) = (-1)^{m_i-1}\xi_i\varepsilon_i(\nu^2\mu^{2m_i} - \lambda_i\nu^2\mu^{2m_i-1} + 2(\beta_i - \alpha_i)\nu\mu^{2m_i} + \cdots),$$

and, by applying induction again, we have

$$\deg A_n(\mu,\nu) = 2(n-1) + 2\sum_{i=1}^{n} m_i,$$

$$\begin{aligned}\deg B_n(\mu,\nu) &= 2n-1+2\sum_{i=1}^{n} m_i = \deg C_n(\mu,\nu),\\ \deg D_n(\mu,\nu) &= 2n+2\sum_{i=1}^{n} m_i,\end{aligned}$$

and the high order terms of $D_n(\mu,\nu)$ are determined by

$$\prod_{i=1}^{n}\hat{d}_i(\mu,\nu)=\prod_{i=1}^{n}(-1)^{m_i-1}\xi_i\varepsilon_i(\nu^2\mu^{2m_i}-\lambda_i\nu^2\mu^{2m_i-1}+2(\beta_i-\alpha_i)\nu\mu^{2m_i}+\cdots).$$

Since $F(\gamma;\mu,\nu)=A_n(\mu,\nu)+D_n(\mu,\nu)$ and $\deg A_n(\mu,\nu)<\deg D_n(\mu,\nu)-1$, then the high order terms of $F(\gamma;\mu,\nu)$ are determined by $D_n(\mu,\nu)$.

For any two polynomials

$$\begin{aligned} f(\mu,\nu) &= a_1\mu^p\nu^q+a_2\mu^{p-1}\nu^q+a_3\mu^p\nu^{q-1}+\cdots,\\ g(\mu,\nu) &= b_1\mu^{p'}\nu^{q'}+b_2\mu^{p'-1}\nu^{q'}+b_3\mu^{p'}\nu^{q'-1}+\cdots,\end{aligned}$$

the high order terms of $f(\mu,\nu)g(\mu,\nu)$ are

$$a_1b_1\mu^{p+p'}\nu^{q+q'}+(a_1b_2+a_2b_1)\mu^{p+p'-1}\nu^{q+q'}+(a_1b_3+a_3b_1)\mu^{p+p'}\nu^{q+q'-1},$$

then we have

$$\begin{aligned} F(\gamma;\mu,\nu) &= \pm(\nu^{2n}\mu^{2m}-(\sum_{i=1}^{n}\lambda_i)\nu^{2n}\mu^{2m-1}+2(\sum_{i=1}^{n}(\beta_i-\alpha_i))\mu^{2m}\nu^{2n-1}+\cdots)\\ &= \pm(\mu^{2m}\nu^{2n}+4N_T(\gamma)\mu^{2m-1}\nu^{2n}+2N_S(\gamma)\mu^{2m}\nu^{2n-1}+\cdots)\end{aligned}$$

By Lemma 2.9 and Lemma 2.13, the above trace formula also holds for $\gamma\in\mathcal{G}_5^+(T)$ with $I_X(\gamma)\geq I_Y(\gamma)$. Assume that $n=I_Y(\gamma)\geq I_X(\gamma)=m$. Then, by using Lemma

2.9 and Lemma 2.13 again,

$$\begin{aligned} F(\gamma;\mu,\nu) &= F(\Theta_2(\gamma); -2\nu, \frac{-\mu}{2}) \\ &= \pm 4^{n-m}(\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots) \end{aligned}$$

Summarizing the above discussions together with Proposition 2.14, we have the following theorem.

Theorem 2.15 (Trace Formulae)

Let $\gamma \in \hat{\mathcal{G}}_5$, $I_X(\gamma) = m$, and $I_Y(\gamma) = n$. For $(\mu,\nu) \in \mathcal{M}_5$ let $W(\gamma;\mu,\nu)$ be the cyclic semi-reduced word in $G(\mu,\nu)$ given in Theorem 2.11 representing γ.

(1) *If $m \geq n$, then*

$$tr\ W(\gamma;\mu,\nu) = \pm(\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots).$$

(2) *If $m \leq n$, then*

$$tr\ W(\gamma;\mu,\nu) = \pm 4^{n-m}(\mu^{2m}\nu^{2n} + 4N_T(\gamma)\mu^{2m-1}\nu^{2n} + 2N_S(\gamma)\mu^{2m}\nu^{2n-1} + \cdots).$$

Chapter 3

Pleating Varieties

3.1 F-Peripheral Subgroups

For $\tau \in \mathcal{M}_n$, let $\Omega_0(\tau) = \Omega_0(G(\tau))$ be the unique invariant simply connected component of $G(\tau)$, and let $\Omega(\tau) = \Omega(G(\tau))$ and $\Lambda(\tau)$ be the limit set of $G(\tau)$. Then $\Lambda(\tau) = \partial\Omega_0(\tau)$.

Let $\mathcal{C}(\tau)$ be the convex hull of $\Lambda(\tau)$ in $\mathbf{H}^3 \cup \hat{\mathbf{C}}$, and $\partial\mathcal{C}(\tau)$ be the boundary of $\mathcal{C}(\tau)$ in $\mathbf{H}^3$. Since $G(\tau)$ is not Fuchsian, then, by using the nearset point retraction with respect to $\mathcal{C}(\tau)$, one can prove that there is a one-to-one correspondence between the connected components of $\partial\mathcal{C}(\tau)$ and the components of $G(\tau)$ [11]. If $\partial\mathcal{C}'$ is a connected component of $\partial\mathcal{C}(\tau)$ corresponding to a component Ω' of $G(\tau)$, then the stabilizer of $\partial\mathcal{C}'$ in $G(\tau)$ is exactly the stabilizer $Stab(\Omega')$ of Ω' in $G(\tau)$, and

$\partial C'/Stab(\Omega')$ is homeomorphic to $\Omega'/Stab(\Omega')$ [11].

Let $\partial\mathcal{C}_0(\tau)$ be the connected component of $\partial\mathcal{C}(\tau)$ corresponding to $\Omega_0(\tau)$, and $\Sigma(\tau) = \partial\mathcal{C}_0(\tau)/G(\tau)$. $\Sigma(\tau)$ is homeomorphic to Σ_n. $\partial\mathcal{C}_0(\tau)$ is a disjoint union of 2-dimensional submanifolds of $\mathbf{H}^3$ and geodesics, and $\partial\mathcal{C}_0(\tau)$ carries an intrinsic metric with respect to which $\partial\mathcal{C}_0(\tau)$ is a complete hyperbolic surface, (see [6] : 1.5.1, 1.6.3 and 1.12.1). Thus, by [5] Lemma 5.1.4, $\Sigma(\tau)$ is a pleated surface and the pleating locus $pl(\tau)$ of $\Sigma(\tau)$ is a geodesic lamination λ. We call

$$\mathcal{P}(\lambda) = \{\tau \in \mathcal{M}_n : pl(\tau) = \lambda\}$$

the *pleating variety associated with* λ.

If $\tau \in \mathcal{M}_4$, then $pl(\tau)$ is a simple geodesic. If $\tau \in \mathcal{M}_5$ and if every leaf of $pl(\tau)$ is a simple closed geodesic, then $pl(\tau)$ is a simple closed geodesic or is a disjoint union of two simple closed geodesics.

Let γ be a simple closed geodesic on $\Sigma(\tau)$. Assume that $\Sigma(\tau)$ is only pleated along γ, i.e. $pl(\tau) = \gamma$. γ divides $\Sigma(\tau)$ into two connected components $\Sigma_1(\tau)$ and $\Sigma_2(\tau)$ with γ as their common boundary. For each $j = 1, 2$, let $\tilde{\Sigma}_j(\tau)$ be a connected component of the lift of $\Sigma_j(\tau)$ to $\mathbf{H}^3$, $H_j(\tau)$ be the stabilizer of $\tilde{\Sigma}_j(\tau)$ in $G(\tau)$, and Π_j be the hyperplane in $\mathbf{H}^3$ containing $\tilde{\Sigma}_j(\tau)$. Since $\tilde{\Sigma}_j(\tau)$ is invariant under $H_j(\tau)$, then Π_j is invariant under $H_j(\tau)$, and $H_j(\tau)$ is Fuchsian. By convexity, $\mathcal{C}(\tau)$ lies entirely in one of the closed half spaces in $\mathbf{H}^3 \cup \hat{\mathbf{C}}$ bounded by Π_j, so does $\Lambda(\tau)$.

Thus one of the two open disks, $\Delta(H_j(\tau))$, bounded by the circle where Π_j meets $\hat{\mathbf{C}}$ is disjoint from $\Lambda(\tau)$. $H_j(\tau)$ is called an *F-peripheral subgroup* of $G(\tau)$.

In general, if G is a Kleinian group but not Fuchsian, if H is a Fuchsian subgroup of G, if C is a circle containing the limit set of H, and if one of the two open disks, say Δ, bounded by C is disjoint from the limit set of G, then H is called an *F-peripheral subgorup* of G, and Δ is called a *peripheral disk* of H.

Proposition 3.1 *Let G be a Kleinian group but not Fuchsian. Then every Fuchsian subgroup of G has at most one peripheral disk.*

Proof: Let H be a Fuchsian subgroup of G, and let Δ_1 and Δ_2 be two distinct peripheral disks of H. Let C be the circle containing $\Lambda(H)$. By the definition of peripheral disks, $\partial\Delta_j = C$, and $\Delta_j \subset \Omega(G)$, then $\Lambda(G) \subset C$, which is impossible since G is not Fuchsian. **Q.E.D.**

Let G be a Kleinian group but not Fuchsian. If H is an F-peripheral subgroup of G, then, by Prposition 3.1, the peripheral disk of H is unique, which is denoted by $\Delta(H)$. It is clear that H is an F-peripheral subgroup of G if and only if AHA^{-1} is an F-peripheral subgroup of AGA^{-1} for each Möbius transformation A. Moreover, if H is an F-peripheral subgroup of G and A is a Möbius transfromation, then $\Delta(AHA^{-1}) = A(\Delta(H))$.

Let $\tau \in \mathcal{M}_5$, and assume that $pl(\tau)$ is a disjoint union of two simple closed geodesics γ_1 and γ_2. γ_1 and γ_2 divide $\Sigma(\tau)$ into three connected components $\Sigma_1(\tau)$, $\Sigma_2(\tau)$ and $\Sigma_3(\tau)$, where $\Sigma_j(\tau)$ is a sphere with two punctures and one hole for $j = 1$ or 2, $\Sigma_3(\tau)$ is a sphere with one puncture and two holes, and γ_j is the common boundary of $\Sigma_j(\tau)$ and $\Sigma_3(\tau)$. For each $j \in \{1, 2, 3\}$, let $\tilde{\Sigma}_j(\tau)$ be a connected component of the lift of $\Sigma_j(\tau)$ to $\mathbf{H}^3$, and let $H_j(\tau)$ be the stabilizer of $\tilde{\Sigma}_j(\tau)$ in $G(\tau)$. Then, by the same reasoning as above, $H_j(\tau)$ is an F-peripheral subgroup of $G(\tau)$ for each j.

Proposition 3.2 *Let $\tau \in \mathcal{M}_n$, and let $\gamma \in \mathcal{G}_n$ represented by a loxodromic transfromation $W \in G(\tau)$. If $pl(\tau) = \gamma$, then there exist non-accidental maximal parabolic transformations P_j, $j = 1, ..., n$ such that:*

(1) *The cyclic subgroups $< P_j >$ and $< P_k >$ are not conjugate in $G(\tau)$ whenever j and k are distinct intergers in $\{1, ..., n\}$.*

(2) *$P_1 P_2$ and $P_3 \cdots P_n$ are conjugate to W in $G(\tau)$.*

(3) *The two subgroups $H_1 =< P_1, P_2 >$ and $H_2 =< P_3, ..., P_n >$ of $G(\tau)$ are F-peripheral with peripheral disks Δ_1 and Δ_2, respectively. Δ_1/H_1 is a sphere with two punctures and one hole, Δ_2/H_2 is a sphere with $n-2$ punctures and one hole, and the boundary curves of the holes are γ.*

Proof: Let $\Sigma_1(\tau)$ and $\Sigma_2(\tau)$ be the two connected components of $\Sigma(\tau)-\gamma$. Then one of them is a sphere with two punctures and one hole, say $\Sigma_1(\tau)$, and the boundary curve of the hole is γ, and thus $\Sigma_2(\tau)$ is a sphere with $n-2$ punctures and one hole, and the boundary curve of the hole is γ.

On $\Sigma_1(\tau)$, we can find simple loops ω_1 and ω_2 based on the same point on $\Sigma_1(\tau)$ such that each of them encloses exactly one puncture of $\Sigma_1(\tau)$, and such that $\omega_1 \cdot \omega_2$ is freely homotopic to γ. Let P_j be the non-accidental maximal parabolic transformation in $G(\tau)$ corresponding to the homotopic class in $\pi_1(\Sigma(\tau))$ represented by ω_j, $j = 1, 2$. Then $P_1 P_2$ is conjugate to W. Similarly, there exist non-accidental maximal parabolic trasformations $P_3, ..., P_n$ in $G(\tau)$ such that each of them corresponding to one of the puncture of $\Sigma_2(\tau)$, and no two of them are conjugate in $G(\tau)$, and such that $P_3 \cdots P_n$ is conjugate in $G(\tau)$ to W. We have shown properties (1) and (2).

Let $\tilde{\Sigma}_j(\tau)$ be the connected component of the lift of $\Sigma_j(\tau)$ in $\mathbf{H}^3$ with stabilizer H_j in $G(\tau)$ for $j = 1$ or 2. From previous arguments, we know that H_j is an F-peripheral subgroup of $G(\tau)$. **Q.E.D.**

Proposition 3.3 *Let* $(\mu, \nu) \in \mathcal{M}_5$, *and let* γ_1, $\gamma_2 \in \mathcal{G}_5$ *be represented by loxodromic transfromations* W_1, $W_2 \in G(\mu, \nu)$. *If* γ_1 *and* γ_2 *are disjoint, and if* $pl(\mu, \nu) = \gamma_1 \cup \gamma_2$, *then there exist non-accidental maximal parabolic transformations* P_j, $j = 1, ..., 5$

such that:

(1) *The cyclic subgroups* $< P_j >$ *and* $< P_k >$ *are not conjugate in* $G(\mu, \nu)$ *whenever* j *and* k *are distinct intergers in* $\{1, 2, 3, 4, 5\}$.

(2) P_1P_2 *is conjugate to* W_1 *in* $G(\tau)$, P_4P_5 *is conjugate to* W_2 *in* $G(\tau)$, *and* $P_3P_4P_5$ *is conjugate to* W_1 *in* $G(\tau)$.

(3) *The subgroups* $H_1 =< P_1, P_2 >$, $H_2 =< P_4, P_5 >$ *and* $H_3 =< P_3, W_1 >$ *of* $G(\tau)$ *are F-peripheral with peripheral disks* Δ_1, Δ_2 *and* Δ_3, *respectively.* Δ_1/H_1 *is a sphere with two punctures and one hole,and the boundary curve of the hole is* γ_1. Δ_2/H_2 *is a sphere with two punctures and one hole, and the boundary curve of the hole is* γ_2. Δ_3/H_3 *is a sphere with a puncture and two holes, and the boundary curves of the holes are* γ_1 *and* γ_2.

Proof: γ_1 and γ_2 divide $\Sigma(\tau)$ into three connected components, two of them, say $\Sigma_1(\tau)$ and $\Sigma_3(\tau)$, are spheres with two punctures and one hole, and the boundary curves of the holes are γ_1 and γ_2, repectively, and the other connected component, say $\Sigma_2(\tau)$, is a sphere with one puncture and two holes, and the boundary curves of the two holes on $\Sigma_2(\tau)$ are γ_1 and γ_2.

By a similar argument as in the proof of Proposition 3.2, there exist non-accidental maximal parabolic transformations P_j, $j = 1, 2, 4, 5$ in $G(\tau)$ such that:

(1) P_1 and P_2 correspond to the two punctures of $\Sigma_1(\tau)$, respectively, and P_1P_2

is conjugate to W_1, and

(2) P_4 and P_5 correspond to the two punctures of $\Sigma_3(\tau)$, respectively, and P_4P_5 is conjugate to W_2.

There is a non-accidental maximal parabolic trasformation P_3 in $G(\tau)$ to the puncture of $\Sigma_2(\tau)$, and there is a loxodromic transformation W' in $G(\tau)$ conjugate to W_2 such that P_3W' is conjugate to W_1. After conjugating W' in $G(\tau)$, we may assume that $W' = P_4P_5$. **Q.E.D.**

Let $\tau \in \mathcal{M}_n$, and let $\gamma \in \mathcal{G}_n$ be represented by a loxodromic transformation W in $G(\tau)$. Then there exist non-accidental maximal parabolic trasformations P_j, $j = 1, ..., n$ such that these transformations satisfy the properties (1) and (2) in Proposition 3.2. After conjugations in $G(\tau)$, we may assume that

$$P_1P_2 = W = P_3 \cdots P_n.$$

Let $H_1(\tau) =< P_1, P_2 >$ and $H_2(\tau) =< P_3, ..., P_n >$. It is easy to see that $G(\tau) =< H_1(\tau), H_2(\tau) >$. We call $[H_1(\tau), H_2(\tau)]$ a *decomposition* of $G(\tau)$ associated with γ. If $H_1(\tau)$ and $H_2(\tau)$ are F-peripheral, then we call $[H_1(\tau), H_2(\tau)]$ an *F-peripheral decomposition* of $G(\tau)$ associated with γ.

Let $(\mu, \nu) \in \mathcal{M}_5$, and let γ_1 and γ_2 be two disjoint simple closed geodesics on $\Sigma(\mu, \nu)$ represented by loxodromic transformations W_1 and W_2 in $G(\mu, \nu)$, respecively. Then there exist non-accidental maximal parabolic trasformations P_j, $j = 1, ..., 5$ such that these trasnformations satisfy the properties (1) and (2) in Proposition 3.3. After conjugations in $G(\mu, \nu)$, we may assume that

$$P_1 P_2 = W_1 = P_3 P_4 P_5, \text{ and } P_4 P_5 = W_2.$$

Let $H_1(\mu, \nu) =< P_1, P_2 >$, $H_2(\mu, \nu) =< P_4, P_5 >$, and $H_3(\mu, \nu) =< P_3, W_1 >$. It is easy to see that $G(\mu, \nu) =< H_1(\mu, \nu), H_2(\mu, \nu), H_3(\mu, \nu) >$. We call

$$[H_1(\mu, \nu), H_2(\mu, \nu), H_3(\mu, \nu)]$$

a *decomposition* of $G(\mu, \nu)$ associated with γ_1, γ_2. If $H_1(\mu, \nu)$, $H_2(\mu, \nu)$ and $H_3(\mu, \nu)$ are F-peripheral, then we call

$$[H_1(\mu, \nu), H_2(\mu, \nu), H_3(\mu, \nu)]$$

an *F-peripheral decomposition* of $G(\tau)$ associated with γ_1, γ_2.

Now, we may restate Proposition 3.2 and Proposition 3.3 as follows:

(1) *If* $pl(\tau) = \gamma$, *then there is an F-peripheral decomposition of* $G(\tau)$ *associated with* γ.

(2) *If* $pl(\mu, \nu) = \gamma_1 \cup \gamma_2$, *then there is an F-peripheral decomposition of* $G(\mu, \nu)$ *associated with* γ_1, γ_2.

Modifying the proof of Proposition 3.6 of [10] we obtain:

Proposition 3.4 (A Chacterization of Pleating Varieties)

(I) *Let* $\tau \in \mathcal{M}_n$, *and* γ *be a simple closed geodesic on* $\Sigma(\tau)$ *represented by a loxodromic transformation* $W \in G(\tau)$. *Then* $pl(\tau) = \gamma$ *if and only if there is an F-peripheral decomposition of* $G(\tau)$ *associated with* γ.

(II) *Let* $(\mu, \nu) \in \mathcal{M}_5$, *and* γ_1 *and* γ_2 *be two disjoint simple closed geodesics on* $\Sigma(\mu, \nu)$ *represented by loxodromic transformations* W_1, $W_2 \in G(\mu, \nu)$, *respectively. Then* $pl(\mu, \nu) = \gamma_1 \cup \gamma_2$ *if and only if there is an F-peripheral decomposition of* $G(\mu, \nu)$ *associated with* γ_1, γ_2.

By the same arguments as in [10] Lemma 3.3 and Lemma 3.4, we have:

Lemma 3.5 (I) *Let* $\tau \in \mathcal{M}_n$, *and* γ *be a simple closed geodesic on* $\Sigma(\tau)$ *represented by a loxodromic transformation in* $G(\tau)$. *Let* $[H_1, H_2]$ *be an F-peripheral decomposition of* $G(\tau)$ *associtaed with* γ, *let* Π_j *be the hyperplane in* $\mathbf{H}^3$ *that meets* $\hat{\mathbf{C}}$ *in* $\partial\Delta(H_j)$, *and let* $N(H_j)$ *be the Nielsen region for* H_j *acting on* Π_j. *Then* $\overline{\Delta(H_j)} \cap \Lambda(\tau) = \Lambda(H_j)$, *and* $N(H_j)$ *is precisely invariant under* H_j *in* $G(\tau)$ *for* $j = 1, 2$.

(II) *Let* $(\mu, \nu) \in \mathcal{M}_5$, *and* γ_1 *and* γ_2 *be two disjoint simple closed geodesics on* $\Sigma(\mu, \nu)$ *represented by loxodromic transformations in* $G(\mu, \nu)$. *Let* $[H_1, H_2, H_3]$ *be an F-peripheral decomposition of* $G(\mu, \nu)$ *associtaed with* γ_1, γ_2, *let* Π_j *be the hyperplane in* $\mathbf{H}^3$ *that meets* $\hat{\mathbf{C}}$ *in* $\partial\Delta(H_j)$, *and let* $N(H_j)$ *be the Nielsen region for*

H_j acting on Π_j. Then $\overline{\Delta(H_j)} \cap \Lambda(\tau) = \Lambda(H_j)$, and $N(H_j)$ is precisely invariant under H_j in $G(\tau)$ for $j = 1, 2, 3$.

Lemma 3.6 *Let $\tau \in \mathcal{M}_n$, and let H be a non-elementary F-peripheral subgroup of $G(\tau)$ containing a loxodromic transformation $W \in G(\tau)$. If W represents a simple closed geodesic γ on $\Sigma(\tau)$, then $\Delta(H) \subset \Omega_0(\tau)$.*

Proof: Since $\Delta(H) \cap \Lambda(\tau) = \emptyset$, then $\Delta(H)$ is contained in a component of $G(\tau)$. Suppose that $\Delta(H)$ is not contained in $\Omega_0(\tau)$.

There exist components $\Omega_1, ..., \Omega_{n-2}$ of $G(\tau)$ such that every non-invariant component of $G(\tau)$ is conjugate to exactly one of the components $\Omega_1, ..., \Omega_{n-2}$, and each one of them is precisely invariant under its stabilizer in $G(\tau)$, (cf. Chapter 1). Let G_j be the stabilizer of Ω_j in $G(\tau)$.

Without loss of generality, we assume that $\Delta(H) \subset \Omega_1$. Since

$$W(\Omega_1) \cap \Omega_1 \supset W(\Delta(H)) \cap \Omega_1 = \Delta(H) \cap \Omega_1 \neq \emptyset,$$

then $W \in G_1$, which is impossible since $I_X(\gamma) \neq 0$ or $I_Y(\gamma) \neq 0$, see §2.3. **Q.E.D.**

Proof of Proposition 3.4: We will prove statement (I) of Proposition 3.4. By a similar argument, statement (II) of Proposition 3.4 will follow.

Using the notations given in Lemma 3.5, $\partial\Delta(H_1)$ and $\partial\Delta(H_2)$ meet at the fixed points of W, then Π_1 and Π_2 intersect in the geodesic in $\mathbf{H}^3$ joining the fixed points of W. $\Delta(H_j) \cap \Lambda(\tau) = \emptyset$, then, by convexity, $\mathcal{C}(\tau)$ is contained in one of the two closed half spaces bounded by Π_j. This proves that Π_j is a support plane for $\mathcal{C}(\tau)$, and thus $\gamma \subset pl(\tau)$.

From Lemma 3.5, $N(H_j)/G(\tau) = N(H_j)/H_j$. $N(H_1)/G(\tau)$ is a sphere with two punctures and one hole, $N(H_2)/G(\tau)$ is a sphere with $n-2$ punctures and one hole, and the boundary curves of the holes are γ. From [1] Theorem 10.4.3,

$$Area(N(H_1)/G(\tau)) = 2\pi, \text{ and } Area(N(H_2)/G(\tau)) = 2(n-3)\pi.$$

From Lemma 3.6, $N(H_j)/G(\tau) \subset \Sigma(\tau)$, then $\Sigma(\tau)$ has area at least $2(n-2)\pi$.

Bers' first area inequality [7] implies that

$$Area(\Omega(\tau)/G(\tau)) \leq 4(n-2)\pi.$$

$\Omega(\tau)/G(\tau)$ contains $n-2$ thrice punctured spheres, and the area of a thrice punctured sphere is 2π, then $\Sigma(\tau)$ has area at most $2(n-2)\pi$, and thus $Area(\Sigma(\tau)) = 2(n-2)\pi$. Therefore, $\Sigma(\tau)$ is obtained from gluing $N(H_1)/G(\tau)$ and $N(H_2)/G(\tau)$ along γ. Hence, $pl(\tau) = \gamma$. **Q.E.D.**

At the end of this section, we state a proposition that we will need later; for the proof see [10] Propsition 3.1.

Proposition 3.7 *Let* $\tau \in \mathcal{M}_n$, *and let* $H(\tau_0)$ *be a finitely generated F-peripheral* *subgroup of* $G(\tau_0)$ *with peripheral disk* Δ_0. *Suppose that*

$$\overline{\Delta_0} \cap \Lambda(\tau_0) = \Lambda(H(\tau_0)).$$

Let $\tau(t)$, $0 \leq t \leq \varepsilon_0$, *be a path in* $\mathcal{M}_n$ *with* $\tau(0) = \tau_0$, *and such that* $H_t = H(\tau(t))$ *is a Fuchsian subgroup of* $G_t = G(\tau(t))$. *Then there is an* $\varepsilon \leq \varepsilon_0$ *such that* H_t *is an F-peripheral subgroup of* G_t *for every* t *with* $0 \leq t \leq \varepsilon$.

3.2 Pleating Varieties in $\mathcal{M}_4$

In this section, we shall restrict our attentions to the pleating varieties $\mathcal{P}(\gamma)$ with $\gamma \in \mathcal{G}_4 - \{\gamma_\infty\} = \hat{\mathcal{G}}_4$. From Theorem 2.6, we may identify geodesics in $\hat{\mathcal{G}}_4$ with rational numbers, and we shall write $\mathcal{P}(q/p) = \mathcal{P}(\gamma)$, where $N(\gamma) = q$, $I_X(\gamma) = p > o$, and $\gcd(p,q) = 1$. For $\mu \in \mathcal{M}_4$, let $W(\gamma;\mu) = W(q/p;\mu) \in G(\mu)$ be the semi-reduced word given in Theorem 2.3 representing γ.

Proposition 3.8 *Let* $\gamma \in \mathcal{G}_4$, *and let* $[H_1(\mu), H_2(\mu)]$ *be a decomposition of* $G(\mu)$ *associated with* γ. *Then* $H_j(\mu)$ *is Fuchsian if and only if* $W(\gamma;\mu)$ *is a hyperbolic transformation, and thus*

$$\mathcal{P}(\gamma) \subset \{\mu \in \mathcal{M}_4 : \text{tr } W(\gamma;\mu) \text{ is real and } |\text{tr } W(\gamma;\mu)| > 2\}.$$

Proof: There exist non-accidental maximal parabolic transformations $P_j(\mu)$, $j = 1, 2, 3, 4$ such that

$$H_1(\mu) = < P_1(\mu), P_2(\mu) >, \text{ and } H_2(\mu) = < P_3(\mu), P_4(\mu) >,$$

and such that

$$P_1(\mu)P_2(\mu) = gW(\gamma;\mu)g^{-1} = P_3(\mu)P_4(\mu) \quad \text{for some } g \in G(\mu).$$

Since $W(\gamma;\mu)$ is loxodromic, then $W(\gamma;\mu)$ is hyperbolic if $H_j(\mu)$ is Fuchsian. Conversely, if $W(\gamma;\mu)$ is hyperbolic, then, by Lemma 3.9, $H_j(\mu)$ is Fuchsian. **Q.E.D.**

Lemma 3.9 *Let H be a Kleinian group generated by two parabolic transformations A and B. Then H is Fuchsian if and only if $tr\ AB$ is real.*

Proof: Let g be the Möbius transformation such that

$$\hat{A}(z) = gAg^{-1}(z) = z + 1, \text{ and } \hat{B}(z) = gBg^{-1}(z) = \frac{z}{bz+1}, \ b \in \mathbf{C} - \{0\}.$$

It is clear that H is Fuchsian if and only if $\hat{H} = gHg^{-1}$ is Fuchsian. A direct computation gives $tr\ \hat{A}\hat{B} = 2 + b$. If $tr\ AB = tr\ \hat{A}\hat{B}$ is real, then $b \in \mathbf{R} - \{0\}$, and thus $\hat{H} \in PSL(2, \mathbf{R})$. **Q.E.D.**

Example 3.1. Consider $\mathcal{P}(\gamma_0) = \mathcal{P}(0)$, where γ_0 is represented by $X_\mu^{-1}S = W(\mu)$. Since $tr\ W(\mu) = 2 + \mu^2$, then $W(\mu)$ is hyperbolic if and only if $\mu = iy$ with $y > 2$.

For every $y > 2$, let $H_1(iy) = < S, X_{iy}^{-1} >$ and $H_2(iy) = < T^{-1}S, X_{iy}^{-1}T >$. Then $H_1(iy)$ and $H_2(iy)$ are Fuchsian subgroups of $G(iy)$, and $[H_1(iy), H_2(iy)]$ is a decomposition of $G(iy)$ associated with γ_0. Next, we will prove that $H_1(iy)$ and $H_2(iy)$ are F-peripheral. Hence,

$$\mathcal{P}(0) = \{iy \ : \ y > 2\},$$

by Proposition 3.4.

Let $C_j(iy)$ be the circle containing $\Lambda(H_j(iy))$. We will prove that $C_j(iy)$ is a round circle. Let $\Delta_j(iy)$ be the open disk bounded by $C_j(iy)$ which is a bounded subset of **C**. We are going to prove that $H_j(iy)$ is F-peripheral by proving that there is fundamental set $D_j(iy)$ for $H_j(iy)$ acting on $\Delta_j(iy)$ so that $D_j(iy) \subset \Omega_0(iy)$.

Since $C_1(iy)$ passes through the fixed points of S and X_{iy}, and is orthogonal to the isometric circles of S, S^{-1}, X_{iy} and X_{iy}^{-1}, then

$$C_1(iy) = \{z : |z - \frac{iy}{2}| = \frac{y}{2}\}, \text{ and } \Delta_1(iy) = \{z : |z - \frac{iy}{2}| < \frac{y}{2}\}.$$

Similarly,

$$C_2(iy) = \{z : |z - (2 + \frac{iy}{2})| = \frac{y}{2}\}, \text{ and } \Delta_2(iy) = \{z : |z - (2 + \frac{iy}{2})| < \frac{y}{2}\}.$$

It is easy to see that $C_1(iy)$ and the isometric circles of S, S^{-1}, X_{iy} and X_{iy}^{-1} bounded a fundamental set $D_1(iy)$ for $H_1(iy)$ acting on $\Delta_1(iy)$, and $D_1(iy) \subset \Omega_0(iy)$. Hence, $H_1(iy)$ is F-peripheral. Similarly, $H_2(iy)$ is F-peripheral.

Proposition 3.10 *Let n be any integer. Then $\mathcal{P}(n) = \{-2n + iy : \ y > 2\}$.*

Proof: $tr\, W(1;\mu) = tr\, T^{-1}SX_\mu = -(\mu + 2)^2 - 2$, then $W(1;\mu)$ is hyperbolic if and only if $\mu = -2 + iy$ and $y > 2$. For $\mu = -2 + iy$ with $y > 2$, let $H_1(-2 + iy) =< T^{-1}S, X_{-2+iy} >$ and $H_2(-2+iy) =< T^{-1}ST, T^{-1}X_{-2+iy} >$. By a similar argument as in Example 3.1, we can prove that $H_1(-2+iy)$ and $H_2(-2+iy)$ are F-peripheral subgroups of $G(-2 + iy)$, and thus

$$\mathcal{P}(1) = \{-2 + iy : \ y > 2\}.$$

Let n be any integer ≥ 2. By applying induction on n, we have

$$W(n;\mu) = \begin{cases} T^{-k-1}ST^kX_\mu & \text{when } n = 2k+1,\ k \geq 1, \\ T^{-k}S^{-1}T^kX_\mu & \text{when } n = 2k,\ k \geq 1, \end{cases}$$

and $tr\, W(n;\mu) = (-1)^n[(\mu + 2n)^2 + 2]$. Then $W(n;\mu)$ is hyperbolic if and only if $\mu = -2n + iy$ and $y > 2$.

On the other hand, for $\mu = -2n + iy$, we have:

$$X_\mu = \begin{cases} T^{-k}X_{iy}T^k & \text{when } n = 2k,\ k \geq 1, \\ T^{-k}X_{iy}^{-1}T^{k+1} & \text{when } n = 2k+1,\ k \geq 1, \end{cases}$$

and

$$W(n;\mu) = \begin{cases} T^{-k}S^{-1}X_{iy}T^k & \text{when } n = 2k,\ k \geq 1, \\ T^{-k-1}SX_{iy}^{-1}T^{k+1} & \text{when } n = 2k+1,\ k \geq 1. \end{cases}$$

For $\mu = -2n + iy$ with $y > 2$, let

$$H_1(-2n+iy) = \begin{cases} < T^{-k}S^{-1}T^k, T^{-k}X_{iy}T^k > & \text{when } n = 2k,\ k \geq 1, \\ < T^{-k-1}ST^{k+1}, T^{-k-1}X_{iy}^{-1}T^{k+1} > & \text{when } n = 2k+1,\ k \geq 1, \end{cases}$$

and

$$H_2(-2n+iy) = \begin{cases} < T^{-k}S^{-1}T^{k+1}, T^{-k-1}X_{iy}T^k > & \text{when } n = 2k,\ k \geq 1, \\ < T^{-k-1}ST^k, T^{-k}X_{iy}^{-1}T^{k+1} > & \text{when } n = 2k+1,\ k \geq 1. \end{cases}$$

Then

$$H_1(-2n+iy) = \begin{cases} T^{-k}H_1(iy)T^k & \text{when } n = 2k,\ k \geq 1, \\ T^{-k-1}H_1(iy)T^{k+1} & \text{when } n = 2k+1,\ k \geq 1, \end{cases}$$

$$H_2(-2n+iy) = \begin{cases} T^{-k}H_2(iy)T^k & \text{when } n = 2k,\ k \geq 1, \\ T^{-k}H_2(iy)T^k & \text{when } n = 2k+1,\ k \geq 1, \end{cases}$$

where $H_j(iy)$ are the F-peripheral subgroups of $G(iy)$ given in Example 3.1. Since

$$G(-2n+iy) = < S, T, X_{-2n+iy} > = < S, T, X_{iy} > = G(iy),$$

then $H_j(-2n+iy)$ are F-peripheral subgroups of $G(-2n+iy)$, and thus

$$\mathcal{P}(n) = \{-2n + iy \ : \ y > 2\}, \quad \text{for any integer } n \geq 0.$$

Similarly, the proposition holds for $n < 0$. **Q.E.D.**

Let

$$\tilde{\mathcal{H}}(q/p) = \{\mu \in \mathbf{C} : tr\ W(q/p;\mu) \text{ is real and } |tr\ W(q/p;\mu)| > 2\}.$$

$\tilde{\mathcal{H}}(q/p)$ is called the *hyperbolic locus* of $tr\ W(q/p;\mu)$. Write $\mu = x + iy$, $x, y \in \mathbf{R}$. From Theorem 2.8,

$$tr\ W(q/p;\mu) = \pm(\mu^{2p} + 4q\mu^{2p-1} + O(\mu^{2p-2})).$$

Let $Tr(q/p;\mu) = \mu^{2p} + 4q\mu^{2p-1} + O(\mu^{2p-2})$. Then

$$\tilde{\mathcal{H}}(q/p) = \{\mu \in \mathbf{C} : Tr(q/p;\mu) \text{ is real and } |Tr(q/p;\mu)| > 2\}.$$

By a direct computation,

$$Im\, Tr(q/p;\mu) = (2px + 4q)y^{2p-1} + a_{2p-2}(x)y^{2p-2} + \cdots + a_1(x)y + a_0(x),$$

where $a_j(x)$ is a polynomial in x with integer coefficients for each $j = 0, ..., 2p-2$. Then $Im\, Tr(q/p;\mu) = 0$ if and only if

$$2px + 4q = \frac{-1}{y^{2p-1}}\{a_{2p-2}(x)y^{2p-2} + \cdots + a_1(x)y + a_0(x)\}.$$

On the other hand, $Tr(q/p;\mu)$ is holomorphically conjugate to μ^{2p} in a neighborhood of ∞, then $\tilde{\mathcal{H}}(q/p)$ has $2p$ branches that are asymptotic to $2p$ rays of the form $\exp(\frac{k\pi i}{p})$, $k = 0, 1, ..., 2p-1$. Therefore, there is a unique branch of $\tilde{\mathcal{H}}(q/p)$ in the strip

$$\{\mu \in \mathbf{C} : 2([-\frac{q}{p}] + 1) \geq Re\,\mu \geq 2[-\frac{q}{p}]\}$$

for $Im\ \mu$ large enough, where $[-\frac{q}{p}]$ is the greatest integer $\leq -\frac{q}{p}$. Let $\mathcal{H}(q/p)$ denote the connected component of $\tilde{\mathcal{H}}(q/p)$ containing this branch. We call $\mathcal{H}(q/p)$ the *vertical (q/p)-component* of $\tilde{\mathcal{H}}(q/p)$.

For $\mu = x + iy \in \mathcal{H}(q/p)$,

$$\frac{-1}{y^{2p-1}}\{a_{2p-2}(x)y^{2p-2} + \cdots + a_1(x)y + a_0(x)\} \longrightarrow 0 \quad \text{as } y \to \infty,$$

then $x \to \frac{-2q}{p}$. Hence, $\mathcal{H}(q/p)$ is asympototic to

$$\{-\frac{2q}{p} + iy : y > 0\}, \quad \text{as } y \to \infty.$$

From Proposition 3.10, we have

$$\mathcal{H}(n) = \mathcal{P}(n) = \{-2n + iy \ : \ y > 2\}, \quad \text{for any integer } n.$$

Using the methods of §5 of [9], we obtain:

Theorem 3.11 (Rational Pleating Varieties)

For every rational number q/p, $\mathcal{P}(q/p)$ coincides with $\mathcal{H}(q/p)$, $\mathcal{P}(q/p)$ contains no critical point of $tr\, W(q/p;\mu)$, and $\overline{\mathcal{H}(q/p)} - \mathcal{H}(q/p)$ consists of a single point of $\partial\mathcal{M}_4$ at which $|tr\, W(q/p;\mu)| = 2$, i.e., this point is a cusp of $\partial\mathcal{M}_4$.

3.3 Pleating Varieties in $\mathcal{M}_5$

Recall that $\hat{\mathcal{G}}_5 = \mathcal{G}_5 - \{\gamma_\infty, \gamma'_\infty\}$, where γ_∞ is represented by T, and γ'_∞ is represented by S. In this section, we deal with the pleating varieties

$$\mathcal{P}(\gamma) = \{(\mu, \nu) \in \mathcal{M}_5 : pl(\mu, \nu) = \gamma\},$$

and

$$\mathcal{P}(\gamma_1, \gamma_2) = \{(\mu, \nu) \in \mathcal{M}_5 : pl(\mu, \nu) = \gamma_1 \cup \gamma_2\},$$

with $\gamma, \gamma_1, \gamma_2 \in \hat{\mathcal{G}}_5$.

Lemma 3.12 *Let H be a Kleinian group generated by a hyperbolic transformation A and a parabolic transformation P. Then H is Fuchsian if and only if $tr\ AP$ is real.*

Proof: It is clear that $tr\ AP$ is real if H is Fuchsian. Assume that $tr\ AP$ is real. Conjugate H in $PSL(2, \mathbf{C})$ so that the fixed points of A are 0 and ∞, and the fixed point of P is 1. We write

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1+p & -p \\ p & 1-p \end{pmatrix},$$

where $\lambda > 0, \lambda \neq 1$ and $p \in \mathbf{C} - \{0\}$. Then

$$tr\ AP = (\lambda + \frac{1}{\lambda}) + p(\lambda - \frac{1}{\lambda}),$$

and thus $p \in \mathbf{R}$. **Q.E.D.**

Lemma 3.13 *Let H be a Kleinian group generated by three parabolic transformations A_1, A_2 and A_3. Then H is Fuchsian if and only if $tr\ A_1A_2$, $tr\ A_2A_3, tr\ A_3A_1$ and $tr\ A_1A_2A_3$ are real.*

Proof: Let $A = A_1A_2A_3$. There is a $B \in PSL(2, \mathbf{C})$ such that

$$BA_1B^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad BA_2B^{-1} = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}, \text{ and } \quad BA_3B^{-1} = \begin{pmatrix} 1+\lambda\delta & -\lambda\delta^2 \\ \lambda & 1-\lambda\delta \end{pmatrix}.$$

where $\lambda, \rho, \delta \in \mathbf{C} - \{0\}$. Then

$$\begin{aligned}
BA_1A_2B^{-1} &= \begin{pmatrix} 1+\rho & 1 \\ \rho & 1 \end{pmatrix}, \\
BA_2A_3B^{-1} &= \begin{pmatrix} 1+\lambda\delta & -\lambda\delta^2 \\ \lambda+\rho+\lambda\rho\delta & 1-\lambda\delta-\lambda\rho\delta^2 \end{pmatrix}, \\
BA_3A_1B^{-1} &= \begin{pmatrix} 1+\lambda\delta & 1+\lambda\delta-\lambda\delta^2 \\ \lambda & 1-\lambda-\lambda\delta \end{pmatrix}, \text{ and} \\
BAB^{-1} &= \begin{pmatrix} 1+\lambda+\rho+\lambda\delta+\lambda\rho\delta & \\ & 1-\lambda\delta-\lambda\rho\delta^2 \end{pmatrix}.
\end{aligned}$$

If $tr\ A_1A_2$, $tr\ A_2A_3, tr\ A_3A_1$ and $tr\ A$ are real, then $\lambda, \rho, \delta \in \mathbf{R} - \{0\}$, and thus BHB^{-1} is Fuchsian. **Q.E.D.**

Let P be a parabolic transformation with fixed point ζ, and let A be a Möbius trasformation such that $APA^{-1}(z) = z + 1$. For $t \in \mathbf{R}$, let

$$U_t = \{z \in \mathbf{C} : Im\, z > t\}.$$

We call $A^{-1}(U_t)$ a *horoball centered at* ζ. Remark that $\partial A^{-1}(U_t)$ is orthogonal to the isometric circle of P if $\zeta \in \mathbf{C}$.

Lemma 3.14 *Let* $(\mu, \nu) \in \mathcal{M}_5$*, and let* P *be a non-accidental maximal parabolic transformation in* $G(\mu, \nu)$ *with fixed point* ζ*. Then there is a horoball* U *centered at* ζ *such that* $\overline{U} - \{\zeta\} \subset \Omega_0(\mu, \nu)$*.*

Proof: Since P is not accidental, then it is conjugate in $G(\mu, \nu)$ to one of the following:

$$X_\mu, X_\mu^{-1}, Y_\nu, Y_\nu^{-1}, ST^{-1}, S^{-1}T, T^{-1}X_\mu, TX_\mu^{-1}, S^{-1}Y_\nu, SY_\nu^{-1}.$$

Thus, we only have to prove the assertion for $X_\mu, Y_\nu, ST^{-1}, T^{-1}X_\mu$ and $S^{-1}Y_\nu$.

Let $D_0(\mu, \nu)$ be the fundamental set for $G(\mu, \nu)$ acting on $\Omega_0(\mu, \nu)$ given in §1.3. For

$$P \in \{X_\mu, Y_\nu, ST^{-1}, T^{-1}X_\mu, S^{-1}Y_\nu\},$$

there is a horoball U centered at ζ scuh that $(\overline{U} - \{\zeta\}) \cap D_0(\mu, \nu)$ is a fundamental set for $< P >$ acting on $\overline{U} - \{\zeta\}$. Then U is a desired horoball for P. **Q.E.D.**

For $(\mu,\nu) \in \mathcal{M}_5$, and for $\gamma \in \hat{\mathcal{G}}_5$, let $W = W(\gamma;\mu,\nu) \in G(\mu,\nu)$ be the semi-reduced word given in Theorem 2.11 representing γ. There exist non-accidental maximal parabolic transformations $P_j = P_j(\mu,\nu)\, j = 1,...,5$ such that the subgroups of $G(\mu,\nu)$

$$\begin{aligned} H_1 &= H_1(\mu,\nu) = < P_1, P_2 >, \\ H_2 &= H_2(\mu,\nu) = < P_4, P_5 >, \\ H_3 &= H_3(\mu,\nu) = < W, P_3 > \end{aligned}$$

form a decomposition of $G(\mu,\nu)$ associated with γ, γ_1, where $\gamma_1 \in \hat{\mathcal{G}}_5$ is represented by $W_1 = W(\gamma_1;\mu,\nu) = P_4P_5 = P_3^{-1}W$.

Lemma 3.15 *Let H_j, $j = 1,2,3$ be given as above. Then:*

(1) *H_1 is Fuchsian if and only if W is hyperbolic. H_2 is Fuchsian if and only if W_1 is hyperbolic. H_3 is Fuchsian if H_1 and H_2 are.*

(2) *For $j = 1$ or 2, $< H_j, H_3 >$ is F-peripheral if H_j is F-peripheral and $< H_j, H_3 >$ is Fuchsian.*

(3) *$pl(\mu,\nu) = \gamma$ if H_1 and H_2 are F-peripheral, and if $< H_2, H_3 >$ is Fuchsian. $pl(\mu,\nu) = \gamma_1$ if H_1 and H_2 are F-peripheral, and if $< H_1, H_3 >$ is Fuchsian.*

(4) *$pl(\mu,\nu) = \gamma \cup \gamma_1$ if H_1, H_2 and H_3 are F-peripheral, and if neither $< H_1, H_3 >$ nor $< H_2, H_3 >$ is Fuchsian.*

(5) *H_3 is F-peripheral if H_1 and H_2 are F-peripheral with peripheral disks Δ_1 and Δ_2, respectively, and if $\overline{\Delta}_j \cap \Lambda(\mu,\nu) = \Lambda(H_j)$ for $j = 1,2$.*

Proof of (1): From Lemma 3.9, H_1 is a Fuchsian subgroup of $G(\mu,\nu)$ if and only if W is hyperbolic, and H_2 is a Fuchsian subgroup of $G(\mu,\nu)$ if and only if W_1 is hyperbolic. By Lemma 3.12, H_3 is a Fuchsian subgroup of $G(\mu,\nu)$ if H_1 and H_2 are.

Proof of (2): Assume that H_2 is a F-peripheral subgroup of $G(\mu,\nu)$, and assume that $< H_2, H_3 >$ is Fuchsian. Let C_2 be the circle containing $\Lambda(H_2)$, and let C be the circle containing the limit set of $< H_2, H_3 >$. Since both C and C_2 contain the fixed points of P_4, P_5 and W_1, then C and C_2 coincide, and thus Δ_2 is the peripheral disk of $< H_2, H_3 >$. Hence $< H_2, H_3 >$ is F-peripheral.

Similarly, $< H_1, H_3 >$ is F-peripheral if H_1 is F-peripheral and $< H_1, H_3 >$ is Fuchsian.

Proof of (3) and (4): These follow from (2) and Proposition 3.4.

Proof of (5): If $< H_1, H_3 >$ or $< H_2, H_3 >$ is Fuchsian, then H_3 is F-peripheral by (2).

Assume that neither $< H_1, H_3 >$ nor $< H_2, H_3 >$ is Fuchsian. Note that H_3 is Fuchsian by (1). For each j, let C_j be the circle containing $\Lambda(H_j)$. C_3 meets C_1 at the fixed points of W, and C_3 meets C_2 at the fixed pints of W_1.

For $j = 1$ or 2, let Π_j be the hyperplane in $\mathbf{H}^3$ that meets $\hat{\mathbf{C}}$ in $\partial\Delta_j$, and let N_j be the Nielsen region for H_j acting on Π_j. Since $\Delta_j \cap \Lambda(\mu, \nu) = \Lambda(H_j)$, then $N_j/H_j = N_j/G(\mu, \nu)$ is a sphere with two punctures and one hole contained in $\Sigma(\mu, \nu)$, (see [10], Lemma 3.3 and Lemma 3.4). The boundary curve of the hole of $N_1/G(\mu, \nu)$ is γ, and the boundary curve of the hole of $N_2/G(\mu, \nu)$ is γ_1. Since $< H_1, H_3 >$ and $< H_2, H_3 >$ are not Fuchsian, then γ and γ_1 are contained in the pleating locus of $\Sigma(\mu, \nu)$. A *dissection* (see [14], pp.28) of a five -times punctured sphere consists of at most two disjoint simple closed curves, thus the pleating locus of $\Sigma(\mu, \nu)$ is exactly $\gamma \cup \gamma_1$. Thus the lifts of

$$\Sigma(\mu, \nu) - (N_1/G(\mu, \nu) \cup N_2/G(\mu, \nu))$$

to $\partial\mathcal{C}_0(\mu, \nu)$ are flat pieces on $\partial\mathcal{C}_0(\mu, \nu)$.

Let $\tilde{\gamma}$ be the geodesic in $\mathbf{H}^3$ with the fixed points of W as its endpoints, and let $\tilde{\gamma}_1$ be the geodesic in $\mathbf{H}^3$ with the fixed points of W_1 as its endpoints. $\tilde{\gamma}$ is a lift of γ to $\partial\mathcal{C}_0(\mu, \nu)$, and $\tilde{\gamma}_1$ is a lift of γ_1 to $\partial\mathcal{C}_0(\mu, \nu)$. Let Π_3 be the hyperplane in $\mathbf{H}^3$ containing the lift of

$$\Sigma(\mu, \nu) - (N_1/G(\mu, \nu) \cup N_2/G(\mu, \nu))$$

to $\partial\mathcal{C}_0(\mu,\nu)$ with boundary geodesics $\tilde{\gamma}$ and $\tilde{\gamma}_1$. Then Π_3 meets $\hat{\mathbf{C}}$ in C_3. Since Π_3 is a support plane of $\mathcal{C}(\mu,\nu)$, then C_3 bounds a disk Δ_3 disjoint from $\Lambda(\mu,\nu)$, and thus H_3 is F-peripheral. **Q.E.D.**

As a consequence of Lemma 3.13 and Lemma 3.15, we have following proposition.

Proposition 3.16 *Let H_j, $j = 1,2,3$ be given as above. Then $\mathcal{P}(\gamma)$ is a subset of*

$$\tilde{\mathcal{H}}(\gamma) = \{(\mu,\nu) \in \mathcal{M})_5 : W,\ P_3P_4,\ P_4P_5,\ P_5P_1 \text{ are hyperbolic}\},$$

and $\mathcal{P}(\gamma,\gamma_1)$ is a subset of

$$\tilde{\mathcal{H}}(\gamma,\gamma_1) = \{(\mu,\nu) \in \mathcal{M}_5 : W \text{ and } P_4P_5 \text{ are hyperbolic, and tr } P_3P_4 \text{ or tr } P_5P_1 \text{ is not real}\}.$$

Proposition 3.17 *Let $\gamma \in \hat{\mathcal{G}}_5$. If $I_X(\gamma)I_Y(\gamma) = 0$, then $\mathcal{P}(\gamma) = \emptyset = \tilde{\mathcal{H}}(\gamma)$.*

Proof: Assume that $I_X(\gamma) = 0$. Then there is a simple closed curve on $\Sigma(\mu,\nu)$ isotopic to γ_∞ which divides $\Sigma(\mu,\nu)$ into two connected components, say $\Sigma'(\mu,\nu)$ and $\Sigma''(\mu,\nu)$, such that $\gamma \subset \Sigma'(\mu,\nu)$. Note that $\Sigma'(\mu,\nu)$ is a sphere with three punctures and one hole, and the punctures $\Sigma'(\mu,\nu)$ correspond to the fixed points of $Y_\nu, S^{-1}Y_\nu, T^{-1}S$, and note that $\Sigma''(\mu,\nu)$ is a sphere with two punctures and one hole, and the punctures of $\Sigma''(\mu,\nu)$ correspond to the fixed points of $X_\mu, T^{-1}X_\mu$.

Since $\gamma \in \hat{\mathcal{G}}_5$, then γ is not isotopic to γ_∞, and thus γ separates one of the punctures on $\Sigma'(\mu,\nu)$ from the other two punctures of $\Sigma'(\mu,\nu)$. Let $\Sigma'_\gamma(\mu,\nu)$ and

$\Sigma''_\gamma(\mu,\nu)$ be the two connected components of $\Sigma(\mu,\nu)-\gamma$, where $\Sigma'_\gamma(\mu,\nu)$ is a sphere with two punctures and one hole, and $\Sigma''_\gamma(\mu,\nu)$ is a sphere with three punctures and one hole, and two of the punctures of $\Sigma''_\gamma(\mu,\nu)$ correspond to the fixed points of $X_\mu, T^{-1}X_\mu$, and the other puncture of $\Sigma''_\gamma(\mu,\nu)$ corresponds to the fixed point of a non-accidental maximal parabolic P conjugate in $G(\mu,\nu)$ to a parabolic transformation in $\{Y_\nu, Y_\nu^{-1}, SY_\nu^{-1}, S^{-1}Y_\nu, T^{-1}S, TS^{-1}\}$. Let $H =< P,\ X_\mu,\ T^{-1}X_\mu >$.

Suppose that $\tilde{\mathcal{H}}(\gamma) \neq \emptyset$. Then H is Fuchsian. Let C be the circle containing $\Lambda(H)$. Since $< X_\mu, T^{-1}X_\mu >$ is a subgroup of H, then

$$\{z \in \mathbf{C} : \ Im\, z = \ Im\, \mu\} \cup \{\infty\} \subset C,$$

and thus $C = \{z \in \mathbf{C} : \ Im\, z = \ Im\, \mu\} \cup \{\infty\}$. This is a contradiction since C is not invariant under P. Similarly, $\tilde{\mathcal{H}}(\gamma) = \emptyset$ if $I_Y(\gamma) = 0$. **Q.E.D.**

Proposition 3.18 *Let $\gamma_1,\ \gamma_2 \in \hat{\mathcal{G}}_5$ be two geodesics. If $I_X(\gamma_1) = I_X(\gamma_2) = 0$, or if $I_Y(\gamma_1) = I_Y(\gamma_2) = 0$, or if $I_X(\gamma_1) = 0 = I_Y(\gamma_2)$, then γ_1 and γ_2 intersect transversely, and thus $\mathcal{P}(\gamma_1,\gamma_2) = \emptyset$.*

Proof: If $I_X(\gamma) = 0$, then there are two free homotopy classes $v_1, v_2 \in \mathcal{G}_5$ with $I_X(v_1) = I_X(v_2) = 0$, which are two adjacent curves in (1) of Figure 3.1, such that the free homototy class represented by γ can be written as $av_1 + bv_2$, where

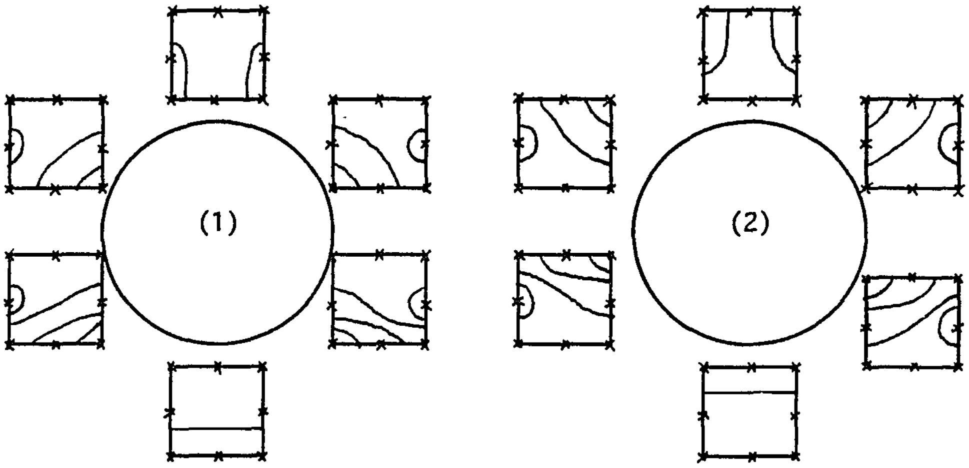


Figure 3.1: (1) Curves $\gamma \in \mathcal{G}_5$ with $I_X(\gamma) = 0$. (2) Curves $\gamma \in \mathcal{G}_5$ with $I_Y(\gamma) = 0$.

a and b are relatively prime non-negative integers, (Cf. §2.2). If $I_Y(\gamma) = 0$, then there are two free homotopy classes $v_1, v_2 \in \mathcal{G}_5$ with $I_Y(v_1) = I_Y(v_2) = 0$, which are two adjacent curves in (2) of Figure 3.1, such that the free homotopy class represented by γ can be written as $av_1 + bv_2$, where a and b are relatively prime non-negative integers. Now, we can easily see γ_1 and γ_2 intersect transversely if $I_E(\gamma_1) = I_P(\gamma_2) = 0$ and $E, P \in \{X, Y\}$. **Q.E.D.**

Example 3.2. Let γ be represented by $W = W(\gamma; \mu, \nu) = Y_\nu X_\mu^{-1}$. Note that

$$Y_\nu X_\mu^{-1} = W = (Y_\nu S^{-1})(ST^{-1})(TX_\mu^{-1}).$$

Let $W_1 = W_1(\mu, \nu) = (ST^{-1})(TX_\mu^{-1}) = SX_\mu^{-1}$, γ_1 be the geodesic represented by W_1, and let

$$\begin{aligned} H_1 &= H_1(\mu,\nu) =< Y_\nu, X_\mu^{-1} >, \\ H_2 &= H_2(\mu,\nu) =< ST^{-1}, TX_\mu^{-1} >, \text{ and} \\ H_3 &= H_3(\mu,\nu) =< W, Y_\nu S^{-1} > . \end{aligned}$$

From Proposition 3.17, $\mathcal{P}(\gamma_1) = \emptyset$. Next, we investigate $\mathcal{P}(\gamma)$ and $\mathcal{P}(\gamma, \gamma_1)$. A direct computation yields:

$$\begin{aligned} tr\, W_1 &= 2 + \mu^2, \\ tr\, W &= 2 - (\mu\nu + 2)^2, \\ tr\, Y_\nu T^{-1} &= 2 + 4\nu^2, \text{ and} \\ tr\, TX_\mu^{-1} Y_\nu S^{-1} &= 2 - ((\mu + 2)(\nu - 1) + 2)^2. \end{aligned}$$

Set $\mu = x + iy$ and $\nu = \xi + i\eta$, where $x, y, \xi, \eta \in \mathbf{R}$ with $y, \eta > 0$. Then:

$$W_1 \text{ is hyperbolic} \iff \mu = iy \text{ and } y > 2, \textit{ and}$$

$$Y_\nu T^{-1} \text{ is hyperbolic} \iff \nu = i\eta \text{ and } y > 1.$$

Claim 1. $\mathcal{P}(\gamma) = \{(2i\eta, i\eta) : \eta > \sqrt{2}\}$.

Assume that H_1 and $< H_2, H_3 >$ are Fuchsian. $W, W_1, Y_\nu T^{-1}$ and $TX_\mu^{-1} Y_\nu S^{-1}$ are hyperbolic, then $\mu = iy$, $\nu = i\eta$, $y > 2$, $\eta > 1$, and

$$Im\, tr\, W = -2(2 - y\eta)\xi y = 0, \quad |tr\, W| = |2 - (2 - y\eta)^2| > 2, \text{ and}$$

$$Im\, tr\, TX_{\mu}^{-1}Y_{\nu}S^{-1} \;=\; -2y\eta(y-2\eta)=0,\ |tr\ TX_{\mu}^{-1}Y_{\nu}S^{-1}| = |2-y^2\eta^2| > 2.$$

Thus

$$\mu = iy,\ \nu = i\eta,\ y > 2,\ \eta > 1,\ y\eta > 4, \text{ and } y = 2\eta.$$

Conversely, it is easy to see that H_1 and $< H_2, H_3 >$ are Fuchsian if μ and ν satisfy the above conditions. We have shown that

$$\mathcal{P}(\gamma) \subset \{(iy, i\eta)\ :\ y > 2,\ \eta > 1,\ y\eta > 4, \text{ and } y = 2\eta\} = \{(2i\eta, i\eta)\ :\ \eta > \sqrt{2}\}.$$

Conversely, assume that $\mu = 2i\eta$, $\nu = i\eta$ with $\eta > \sqrt{2}$. For $j = 1$ or 2, let C_j be the circle containing $\Lambda(H_j)$. Then

$$C_1 = \{z : |z - i(\eta + \frac{1}{\eta})| = \eta - \frac{1}{\eta}\}, \text{ and } C_2 = \{z : |z - (2 + i\eta)| = \eta\}.$$

Let

$$\Delta_1 = \{z : |z - i(\eta + \frac{1}{\eta})| < \eta - \frac{1}{\eta}\}, \text{ and } \Delta_2 = \{z : |z - (2 + i\eta)| < \eta\},$$

and let $D_0(2i\eta, i\eta)$ be the fundamental set for $G(2i\eta, i\eta)$ acting on $\Omega_0(2i\eta, i\eta)$ given in §1.3. $\Delta_1 \cap D_0(2i\eta, i\eta)$ is a fundamental set for H_1 acting on Δ_1, and $\Delta_2 \cap [D_0(2i\eta, i\eta) \cup T(D_0(2i\eta, i\eta))]$ is a fundamental set for H_2 acting on Δ_2. Then $\Delta_j \subset \Omega_0(2i\eta, i\eta)$, and H_j is F-peripheral for $j = 1, 2$. By (2) of Lemma 3.15, $< H_2, H_3 >$ is F-peripheral. Hence

$$\mathcal{P}(\gamma) \supset \{(2i\eta, i\eta)\ :\ \eta > \sqrt{2}\}.$$

Claim 2. $\mathcal{P}(\gamma,\gamma_1) = \{(iy, i\eta) : 2\eta > y > 2 \text{ and } y\eta > 4\}$.

Assume that H_1 and H_2 are F-peripheral. Then W and W_1 are hyperbolic, and

$$\mu = iy,\ \nu = \xi + i\eta,\ (2 - y\eta)\xi = 0,\ y > 2, \text{ and } |\xi^2 y^2 - (2 - y\eta)^2 + 2| > 2.$$

Let Δ_j be the peripheral disk of H_j for $j = 1, 2$. Then

$$\Delta_2 = \{z : |z - (2 + i\frac{y}{2})| < \frac{y}{2}\},$$

and $\partial\Delta_1 = C_1$ passes through the fixed point iy of X_μ and the fixed point $-2/\nu$ of Y_ν, and C_1 is orthogonal to the isometric circle of X_μ and is orthogonal to the isometric circle of Y_ν.

We claim that $\xi = 0$. Let $L_1 = \{z : Re\ z = 0\}$, and let L_2 be the straight line passing through $-2/\nu$ tangent to the isometric circle of Y_ν. Suppose that $\xi \neq 0$. Then $y\eta = 2$ and the center of C_1 is the intersection point of L_1 and L_2. A direct computation shows that the center of C_1 is $i/\eta = iy/2$, and the radius of C_1 is $|y - 1/\eta| = y/2$. Thus C_1 is tangent to the real axis at 0, and thus $C_1 = C_0$, where C_0 is the circle containing the limit set of $< S, Y_\nu >$. This is a contradiction. Hence $\xi = 0$.

Since $\xi = 0$ and $|\xi^2 y^2 - (2 - y\eta)^2 + 2| > 2$ then $y\eta > 4$. Since $Y_\nu T^{-1}$ is a loxodromic transformation in $G(iy, i\eta)$, then $Y_\nu T^{-1}$ is hyperbolic and $\eta > 1$. From Proposition 3.16, $tr\, T X_\mu^{-1} Y_\nu S^{-1}$ is not real, so $y \neq 2\eta$. We have shown that

$$\mathcal{P}(\gamma,\gamma_1) \subset \{(iy, i\eta) : 2\eta \neq y,\ y > 2,\ \eta > 1, \text{ and } y\eta > 4\},$$

and

$$\Delta_1 = \{z : |z - \frac{i}{2}(y + \frac{2}{\eta})| < \frac{1}{2}(y - \frac{2}{\eta})\}.$$

Since $\frac{2}{1-i\eta} = \frac{2(1+i\eta)}{1+\eta^2}$ is the fixed point of SY_ν^{-1}, and since Δ_2 is the peripheral disk of H_2, then

$$|\frac{2}{1-i\eta} - (2 + i\frac{y}{2})| \geq \frac{y}{2}, \text{ or equivalently, } 2\eta \geq y.$$

Hence

$$\mathcal{P}(\gamma, \gamma_1) \subset \{(iy, i\eta) : 2\eta > y > 2, \text{ and } y\eta > 4\}.$$

To prove that $\mathcal{P}(\gamma, \gamma_1) = \{(iy, i\eta) : 2\eta > y > 2, \text{ and } y\eta > 4\}$, it remains to show that $H_3(\mu, \nu)$ is F-peripheral for all $\mu = iy, \nu = i\eta$ with $2\eta > y > 2$ and $y\eta > 4$. Let C_3 be the circle containing $\Lambda(H_3)$. Then C_3 meets $C_1 = \partial\Delta_1$ at the fixed points of W, C_3 meets $C_2 = \partial\Delta_2$ at the fixed points of W_1, C_3 passes through the fixed point $-2/(\nu - 1)$ of $Y_\nu S^{-1}$, and is orthogonal to the isometric circle of $Y_\nu S^{-1}$. Thus C_3 is tangent to C_0 at $-2/(\nu - 1)$.

The fixed points of W_1 are

$$1 + i\frac{y \pm \sqrt{y^2 - 4}}{2}.$$

Then the center of C_3 is the intersection point of the horizontal line $Im\ z = \frac{y}{2}$ and L, where L is the straight line passing through $-2/(\nu - 1)$ tangent to the isometric circle of $Y_\nu S^{-1}$. L is orthogonal to C_0, then L passes through the center i/η of C_0, and thus the center of C_3 is

$$\frac{2\eta}{\eta^2-1}(\frac{y}{2}-\frac{1}{\eta})+i\frac{y}{2}=\frac{y\eta-2}{\eta^2-1}+i\frac{y}{2}$$

and the radius of C_3 is

$$\begin{aligned} r &= \text{the distance between the center of } C_3 \text{ and } \frac{-2}{\nu-1}=\frac{2(1+i\eta)}{1+\eta^2} \\ &= \sqrt{(\frac{y\eta-2}{\eta^2-1}-\frac{2}{1+\eta^2})^2+(\frac{y}{2}-\frac{2\eta}{1+\eta^2})^2}. \end{aligned}$$

$$\begin{aligned} r<\frac{y}{2} &\iff (\frac{y\eta-2}{\eta^2-1})^2-\frac{4(y\eta-2)}{\eta^4-1}+\frac{4}{(1+\eta^2)^2}-\frac{2y\eta}{1+\eta^2}+\frac{4\eta^2}{(1+\eta^2)^2}<0 \\ &\iff (\frac{y\eta-2}{\eta^2-1})^2-\frac{4(y\eta-2)}{\eta^4-1}-\frac{2(y\eta-2)}{\eta^2+1}<0 \\ &\iff \frac{y\eta-2}{\eta^2-1}-2<0 \\ &\iff y<2\eta. \end{aligned}$$

Then $C_3 \subset \{z : 0 < Im\, z < y\}$. Let Δ_3 be the disk bounded by C_3 disjoint form the line $Im\, z = 0$. Now, we are going to prove that $\Delta_3 \subset \Omega(iy, i\eta)$. This implies that H_3 is F-peripheral.

Note that the fundamental set $D_0(iy, i\eta)$ is bounded by the isometric circles of $X_\mu, X_\mu^{-1}, Y_\nu, Y_\nu^{-1}, S, S^{-1}$ and the vertical lines $Re\ z = -2$ and $Re\ z = 2$. Also note that the isometric circle of Y_ν^{-1} coincides with the isometric circle of SY_ν^{-1}.

Let ℓ_1 be the intersection of the isometric circle of Y_ν^{-1} with $\overline{\Delta}_3$, and let ℓ_2 be the intersection of the isometric circle of S^{-1} with $\overline{\Delta}_3$. It is clear that

$$\overline{\ell}_k - \{-2/(\nu - 1)\} \subset \Omega_0(iy, i\eta), k = 1, 2.$$

Let E_X and E_Y be the reflections in the isometric circles of X_μ^{-1} and Y_ν^{-1}, respectively, and let E be the reflection in the imaginary axis. Then $X_\mu^{-1} = EE_X$, $Y_\nu = E_Y E$, and $W = Y_\nu X_\mu^{-1} = E_Y E_X$. Thus the axis of W in Δ_3 is orthogonal to ℓ_1, and the attracting fixed point of W lies on C_3 between the endpoints of ℓ_1.

Similarly, $W_1 = SX_\mu^{-1} = E_S E_X$, the axis of W_1 in Δ_3 is orthogonal to ℓ_2, and the attracting fixed point of W_2 lies on C_3 between the endpoints of ℓ_2.

Let $\zeta = -2/(\nu - 1)$, $P = Y_\nu S^{-1}$, $\ell_1' = W^{-1}(\ell_1)$, and $\ell_2' = W_1^{-1}(\ell_2)$. Since $W = PW_1$, then $P' = W^{-1}PW = W_1^{-1}PW_1$, and $\zeta' = W^{-1}(\zeta) = W_1^{-1}(\zeta)$ is the fixed point of P'.

Let ℓ_3 be the subarc of C_3 contained in Δ_1 between the fixed points of W, and let ℓ_4 be the subarc of C_3 contained in Δ_2 between the fixed points of W_1. Then $\ell_1, \ell_2, \ell_1', \ell_2', \ell_3$ and ℓ_4 bound a fundamental set D_3 for H_3 acting on Δ_3. To prove that Δ_3 is contained in $\Omega(iy, i\eta)$, it suffices to show that D_3 is disjoint from $\Lambda(iy, i\eta)$.

Since $\Delta_1 \subset \Omega_0(iy, i\eta)$ and $\Delta_2 \subset \Omega_0(iy, i\eta)$, then ℓ_3, ℓ_4 are contained in $\Omega_0(iy, i\eta)$, and thus

$$\partial\overline{D}_3 - \{\zeta, \zeta'\} \subset \Omega_0(iy, i\eta).$$

From Lemma 3.14, there is a horoball U centered at ζ, and there is a horoball V centered at ζ' such that

$$\overline{U} - \{\zeta\} \subset \Omega_0(iy, i\eta) \text{ and } \overline{V} - \{\zeta'\} \subset \Omega_0(iy, i\eta).$$

We may choose U and V small enough so that $\ell_5 = \partial\overline{U} \cap \overline{D}_3$ and $\ell_6 = \partial\overline{U} \cap \overline{D}_3$ are connected circular arcs, and so that $\ell_5 \cap \ell_6 = \emptyset$. Let $D_3' = \overline{D_3 - (U \cup V)}$. Then $\partial D_3'$ is a simple closed curve contained in $\Omega_0(iy, i\eta)$. Since $\Lambda(iy, i\eta) = \partial\Omega_0(iy, i\eta)$ is connected, then $D_3' \cap \Lambda(iy, i\eta) = \emptyset$, and thus D_3 is disjoint from $\Lambda(iy, i\eta)$. This completes the proof.

Proposition 3.19 *Let* $\gamma \in \hat{\mathcal{G}}_5$, *and* n *be any integer. Then*

$$\tilde{\mathcal{H}}(\mathcal{T}_1^n(\gamma)) = \{(\mu, -n + \nu) \ : \ (\mu, \nu) \in \tilde{\mathcal{H}}(\gamma)\},$$

and

$$\mathcal{P}(\mathcal{T}_1^n(\gamma)) = \{(\mu, -n + \nu) \ : \ (\mu, \nu) \in \mathcal{P}(\gamma)\}.$$

Proof: Let $P_j(\gamma;\mu,\nu) \in G(\mu,\nu)$, $j = 1,\ldots,5$, be non-accidental maximal parabolic transformations such that

$$P_1(\gamma;\mu,\nu)P_2(\gamma;\mu,\nu) = P_3(\gamma;\mu,\nu)P_4(\gamma;\mu,\nu)P_5(\gamma;\mu,\nu)$$

is a loxodromic transformation representing γ, and such that

$$\begin{aligned} H_1(\gamma;\mu,\nu) &= \ < P_1(\gamma;\mu,\nu), P_2(\gamma;\mu,\nu) >, \text{ and} \\ H_2(\gamma;\mu,\nu) &= \ < P_3(\gamma;\mu,\nu), P_4(\gamma;\mu,\nu), P_5(\gamma;\mu,\nu) > \end{aligned}$$

form a decomposition of $G(\mu,\nu)$ associated with γ. Since

$$\mathcal{T}_1 \ : \ S \mapsto S, \ \ T \mapsto T, \ \ X_\mu \mapsto X_\mu, \ \ Y_\nu \mapsto Y_\nu^{-1}S = Y_{\nu+1},$$

then $\mathcal{T}_1$ maps non-accidental maximal parabolic transformations into non-accidental maximal parabolic transformations, and thus

$$\begin{aligned} H_1(\tilde{\gamma};\mu,\nu) &= \mathcal{T}_1(H_1(\gamma;\mu,\nu)) =< P_1(\tilde{\gamma};\mu,\nu), P_2(\tilde{\gamma};\mu,\nu) >, \text{ and} \\ H_2(\tilde{\gamma};\mu,\nu) &= \mathcal{T}_1(H_2(\gamma;\mu,\nu)) =< P_3(\tilde{\gamma};\mu,\nu), P_4(\tilde{\gamma};\mu,\nu), P_5(\tilde{\gamma};\mu,\nu) > \end{aligned}$$

form a decomposition of $G(\mu,\nu)$ associated with $\tilde{\gamma} = \mathcal{T}_1(\gamma)$, where

$$P_j(\tilde{\gamma};\mu,\nu) = \mathcal{T}_1(P_j(\gamma;\mu,\nu)), \ \ j = 1,\ldots,5.$$

Let

$$F_1(\gamma;\mu,\nu) \ = \ tr \ P_1(\gamma;\mu,\nu)P_2(\gamma;\mu,\nu),$$

$$\begin{aligned}
F_2(\gamma;\mu,\nu) &= tr\ P_3(\gamma;\mu,\nu)P_4(\gamma;\mu,\nu),\\
F_3(\gamma;\mu,\nu) &= tr\ P_4(\gamma;\mu,\nu)P_5(\gamma;\mu,\nu),\\
F_4(\gamma;\mu,\nu) &= tr\ P_5(\gamma;\mu,\nu)P_1(\gamma;\mu,\nu),\\
F_1(\tilde{\gamma};\mu,\nu) &= tr\ P_1(\tilde{\gamma};\mu,\nu)P_2(\tilde{\gamma};\mu,\nu),\\
F_2(\tilde{\gamma};\mu,\nu) &= tr\ P_3(\tilde{\gamma};\mu,\nu)P_4(\tilde{\gamma};\mu,\nu),\\
F_3(\tilde{\gamma};\mu,\nu) &= tr\ P_4(\tilde{\gamma};\mu,\nu)P_5(\tilde{\gamma};\mu,\nu),\\
F_4(\tilde{\gamma};\mu,\nu) &= tr\ P_5(\tilde{\gamma};\mu,\nu)P_1(\tilde{\gamma};\mu,\nu).
\end{aligned}$$

From Lemma 2.13,

$$F_j(\tilde{\gamma};\mu,\nu) = -F_j(\gamma;\mu,\nu+1),\ \ j = 1,2,3,4.$$

Then $(\mu,\nu) \in \tilde{\mathcal{H}}(\tilde{\gamma})$ if and only if $(\mu,\nu+1) \in \tilde{\mathcal{H}}(\gamma)$ by Proposition 3.16, and thus

$$\tilde{\mathcal{H}}(\mathcal{T}_1(\gamma)) = \{(\mu,-1+\nu)\ :\ (\mu,\nu) \in \tilde{\mathcal{H}}(\gamma)\}.$$

Similarly, $\tilde{\mathcal{H}}(\mathcal{T}_1^{-1}(\gamma)) = \{(\mu,1+\nu)\ :\ (\mu,\nu) \in \tilde{\mathcal{H}}(\gamma)\}$. By induction,

$$\tilde{\mathcal{H}}(\mathcal{T}_1^n(\gamma)) = \{(\mu,-n+\nu)\ :\ (\mu,\nu) \in \tilde{\mathcal{H}}(\gamma)\}.$$

Let $(\mu',\nu') = (\mu,-1+\nu)$, let $C_k(\gamma;\mu,\nu)$ be the circle containing the limit set of $H_k(\gamma;\mu,\nu)$, and let $C_k(\tilde{\gamma}:\mu',\nu')$ be the circle containing the limit set of $H_k(\tilde{\gamma};\mu',\nu')$, where $\tilde{\gamma} = \mathcal{T}_1(\gamma)$.

Since $\mathcal{T}_1(Y_{\nu'}) = Y_{\nu'+1}$, then, for $j = 1, ..., 5$,

$$P_j(\tilde{\gamma}; \mu', \nu') = \mathcal{T}_1 P_j(\gamma; \mu', \nu') = P_j(\gamma; \mu', \nu' + 1) = P_j(\gamma; \mu, \nu),$$

and thus $C_k(\gamma; \mu, \nu) = C_k(\tilde{\gamma}; \mu', \nu')$ since $C_k(\gamma; \mu, \nu)$ and $C_k(\tilde{\gamma}; \mu', \nu')$ are uniquely determined by the fixed points of $P_j(\gamma; \mu, \nu)$ and $P_j(\tilde{\gamma}; \mu', \nu')$, respectively.

On the other hand, $G(\mu', \nu') = G(\mu, \nu)$ since

$$< S, T, X_{\mu'}, Y_{\nu'} >=< S, T, X_{\mu}, Y_{-1+\nu} >=< S, T, X_{\mu}, S^{-1}Y_{\nu} >,$$

then $\Lambda(\mu', \nu') = \Lambda(\mu, \nu)$. Hence, $H_k(\gamma; \mu, \nu)$ is an F-peripheral subgroup of $G(\mu, \nu)$ if and only if $H_k(\tilde{\gamma}; \mu', \nu')$ is an F-peripheral subgroup of $G(\mu', \nu')$. Moreover, their peripheral disks coincide. This proves that

$$\mathcal{P}(\mathcal{T}_1(\gamma)) = \{(\mu, -1 + \nu) : (\mu, \nu) \in \mathcal{P}(\gamma)\}.$$

Similarly,

$$\mathcal{P}(\mathcal{T}_1^{-1}(\gamma)) = \{(\mu, 1 + \nu) : (\mu, \nu) \in \mathcal{P}(\gamma)\}.$$

By induction again, the proof is complete. Q.E.D.

By a similar argument, we obtain the following proposition.

Proposition 3.20 *Let $\gamma \in \hat{\mathcal{G}}_5$ be a geodesic, let $\gamma_1, \gamma_2 \in \hat{\mathcal{G}}_5$ be two disjoint geodesics, and let n be any integer. Then:*

(1) $\tilde{\mathcal{H}}(\mathcal{T}_2^n(\gamma)) = \{(2n+\mu,\nu) : (\mu,\nu) \in \tilde{\mathcal{H}}(\gamma)\}$.

(2) $\mathcal{P}(\mathcal{T}_2^n(\gamma)) = \{(2n+\mu,\nu) : (\mu,\nu) \in \mathcal{P}(\gamma)\}$.

(3) $\tilde{\mathcal{H}}(\mathcal{T}_1^n(\gamma_1), \mathcal{T}_1^n(\gamma_2)) = \{(\mu, -n+\nu) : (\mu,\nu) \in \tilde{\mathcal{H}}(\gamma)\}$.

(4) $\tilde{\mathcal{H}}(\mathcal{T}_2^n(\gamma_1), \mathcal{T}_2^n(\gamma_2)) = \{(2n+\mu,\nu) : (\mu,\nu) \in \tilde{\mathcal{H}}(\gamma)\}$.

(5) $\mathcal{P}(\mathcal{T}_1^n(\gamma_1), \mathcal{T}_1^n(\gamma_2)) = \{(\mu, -n+\nu) : (\mu,\nu) \in \mathcal{P}(\gamma)\}$.

(6) $\mathcal{P}(\mathcal{T}_2^n(\gamma_1), \mathcal{T}_2^n(\gamma_2)) = \{(2n+\mu,\nu) : (\mu,\nu) \in \mathcal{P}(\gamma)\}$.

Recall that Θ is the automorphism of $G(\mu,\nu)$ defined by

$$\Theta : S \mapsto T^{-1},\ T \mapsto S^{-1},\ X_\mu \mapsto Y_\nu^{-1},\ Y_\nu \mapsto X_\mu^{-1}.$$

Proposition 3.21 *Let $\gamma \in \hat{\mathcal{G}}_5$ be a geodesic, and let γ_1, γ_2 be two disjoint geodesics. Then:*

(1) $\tilde{\mathcal{H}}(\Theta(\gamma)) = \{(\mu,\nu) \ : \ (2\nu, \mu/2) \in \tilde{\mathcal{H}}(\gamma)\}$.

(2) $\mathcal{P}(\Theta(\gamma)) = \{(\mu,\nu) \ : \ (2\nu, \mu/2) \in \mathcal{P}(\gamma)\}$.

(3) $\tilde{\mathcal{H}}(\Theta(\gamma_1), \Theta(\gamma_2)) = \{(\mu,\nu) \ : \ (2\nu, \mu/2) \in \tilde{\mathcal{H}}(\gamma_1,\gamma_2)\}$.

(4) $\mathcal{P}(\Theta(\gamma_1), \Theta(\gamma_2)) = \{(\mu,\nu) \ : \ (2\nu, \mu/2) \in \mathcal{P}(\gamma_1,\gamma_2)\}$.

Proof: Let $P_j(\gamma;\mu,\nu) \in G(\mu,\nu)$, $j = 1, ..., 5$, be non-accidental maximal parabolic transformations such that

$$P_1(\gamma;\mu,\nu)P_2(\gamma;\mu,\nu) = P_3(\gamma;\mu,\nu)P_4(\gamma;\mu,\nu)P_5(\gamma;\mu,\nu)$$

is a loxodromic transformation representing γ, and such that

$$\begin{aligned} H_1(\gamma;\mu,\nu) &= < P_1(\gamma;\mu,\nu), P_2(\gamma;\mu,\nu) >, \text{ and} \\ H_2(\gamma;\mu,\nu) &= < P_3(\gamma;\mu,\nu), P_4(\gamma;\mu,\nu), P_5(\gamma;\mu,\nu) > \end{aligned}$$

form a decomposition of $G(\mu,\nu)$ associated with γ. Let $\tilde{\gamma} = \Theta(\gamma)$, $P_j(\tilde{\gamma};\mu,\nu) = \Theta(P_j(\gamma;\mu,\nu))$, and $H_k(\tilde{\gamma};\mu,\nu) = \Theta(H_k(\gamma;\mu,\nu))$.

For $\lambda \in \mathbf{C} - \{0\}$, let

$$A_\lambda = \begin{pmatrix} \lambda i & 2i \\ \frac{i}{2} & 0 \end{pmatrix} \quad \text{and} \quad B_\lambda = \begin{pmatrix} i & -2\lambda i \\ 0 & -i \end{pmatrix}.$$

By a direct computation, if $\omega \in \mathbf{C} - \{0\}$, then

$$\begin{aligned} B_\lambda A_\lambda S A_\lambda^{-1} B_\lambda^{-1} &= T^{-1}, \quad B_\lambda A_\lambda T A_\lambda^{-1} B_\lambda^{-1} = S^{-1}, \\ B_\lambda A_\lambda X_\omega A_\lambda^{-1} B_\lambda^{-1} &= Y_{\omega/2}^{-1}, \quad B_\lambda A_\lambda Y_\lambda A_\lambda^{-1} B_\lambda^{-1} = X_{2\lambda}^{-1}. \end{aligned}$$

Setting $\lambda = \mu/2$ and $\omega = 2\nu$, we have

$$\begin{aligned} B_{\mu/2} A_{\mu/2} S A_{\mu/2}^{-1} B_{\mu/2}^{-1} &= T^{-1}, \quad B_{\mu/2} A_{\mu/2} T A_{\mu/2}^{-1} B_{\mu/2}^{-1} = S^{-1}, \\ B_{\mu/2} A_{\mu/2} X_{2\nu} A_{\mu/2}^{-1} B_{\mu/2}^{-1} &= Y_\nu^{-1}, \quad B_{\mu/2} A_{\mu/2} Y_{\mu/2} A_{\mu/2}^{-1} B_{\mu/2}^{-1} = X_\mu^{-1}. \end{aligned}$$

Then

$$\begin{aligned} P_j(\tilde{\gamma};\mu,\nu) &= \Theta(P_j(\tilde{\gamma};\mu,\nu)) = B_{\mu/2} A_{\mu/2} P_j(\gamma; 2\nu, \mu/2) A_{\mu/2}^{-1} B_{\mu/2}^{-1}, \\ H_k(\tilde{\gamma};\mu,\nu) &= \Theta(H_k(\gamma;\mu,\nu)) = B_{\mu/2} A_{\mu/2} H_k(\gamma; 2\nu, \mu/2) A_{\mu/2}^{-1} B_{\mu/2}^{-1}, \text{ and} \\ G(\mu,\nu) &= \Theta(G(\mu,\nu)) = B_{\mu/2} A_{\mu/2} G(2\nu, \mu/2) A_{\mu/2}^{-1} B_{\mu/2}^{-1}. \end{aligned}$$

This proves (1) and (2). By a similar argument, (3) and (4) will follow. **Q.E.D.**

Example 3.3. Let γ and γ_1 be given as Example 3.2. It is clear that $\Theta(\gamma) = \gamma$. Let $\Theta(\gamma_1) = \gamma_2$. Then by Proposition 3.21,

$$\tilde{\mathcal{H}}(\gamma,\gamma_2) = \mathcal{P}(\gamma,\gamma_2) = \{(iy, i\eta) \;:\; y > 2\eta > 2 \text{ and } y\eta > 4\}.$$

By Proposition 3.20 and Proposition 3.19, we have

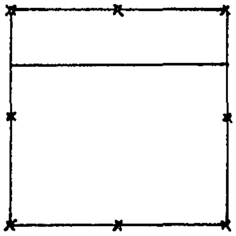
$$\begin{aligned}
\mathcal{P}(\mathcal{T}_1^n\mathcal{T}_2^m(\gamma)) &= \{(2m + 2i\eta, -n + i\eta) : \eta > \sqrt{2}\},\\
\mathcal{P}(\mathcal{T}_1^n\mathcal{T}_2^m(\gamma), \mathcal{T}_1^n\mathcal{T}_2^m(\gamma_1)) &= \{(2m + iy, -n + i\eta) : 2\eta > y > 2 \text{ and } y\eta > 4\},\\
\mathcal{P}(\mathcal{T}_1^n\mathcal{T}_2^m(\gamma), \mathcal{T}_1^n\mathcal{T}_2^m(\gamma_2)) &= \{(2m + iy, -n + i\eta) : y > 2\eta > 2 \text{ and } y\eta > 4\},
\end{aligned}$$

where m and n are any two integers.

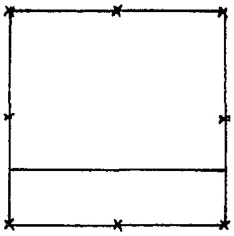
APPENDIX

Elementary Curves in $\mathcal{G}_5$: In what follows, if $\gamma \in \mathcal{G}_5$ is represented by $W \in G$, then we write $\gamma = W$.

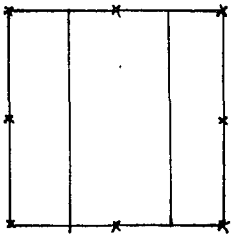
$\gamma_1 = \gamma_\infty = T$



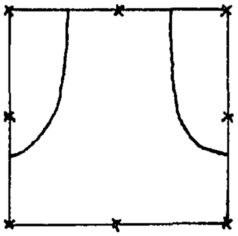
$\gamma_2 = \gamma_{\infty'} = S$



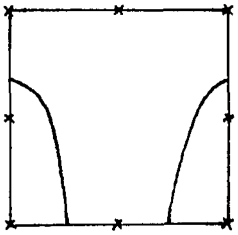
$\gamma_3 = Y^{-1}X$



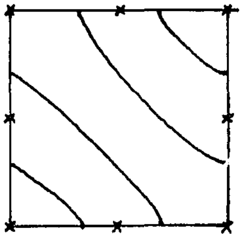
$\gamma_4 = S^{-1}X$



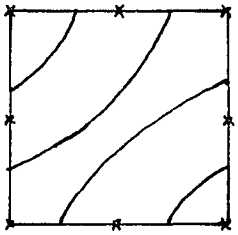
$\gamma_5 = Y^{-1}T$



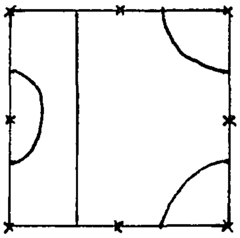
$\gamma_6 = TYS^{-1}X^{-1}$



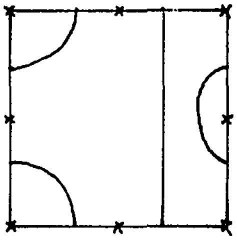
$\gamma_7 = T^{-1}Y^{-1}SX$



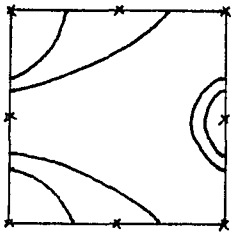
$\gamma_8 = TS^{-1}YX^{-1}$



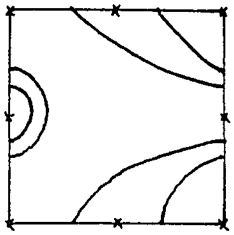
$\gamma_9 = T^{-1}SY^{-1}X$



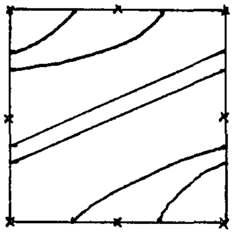
$\gamma_{10} = T^{-1}SYS^{-1}TX^{-1}$



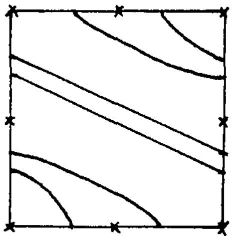
$\gamma_{11} = TS^{-1}Y^{-1}ST^{-1}X$



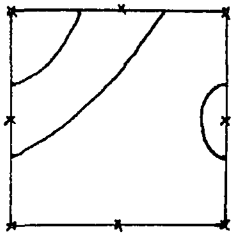
$\gamma_{12} = T^{-1}S^{-1}Y^{-1}STX$



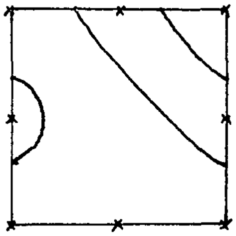
$\gamma_{13} = TSYS^{-1}T^{-1}X^{-1}$



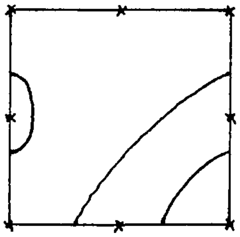
$\gamma_{14} = T^{-1}SX$



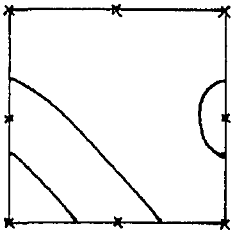
$\gamma_{15} = TS^{-1}X^{-1}$



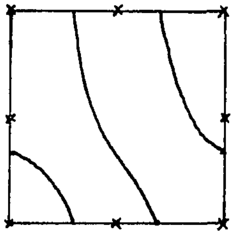
$\gamma_{16} = S^{-1}YT$



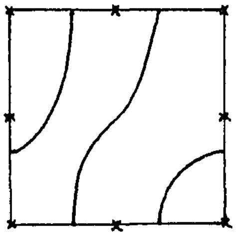
$\gamma_{17} = SY^{-1}T^{-1}$



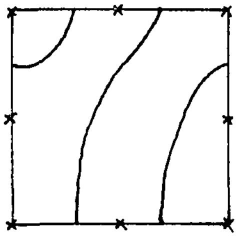
$\gamma_{18} = SY^{-1}X^{-1}$



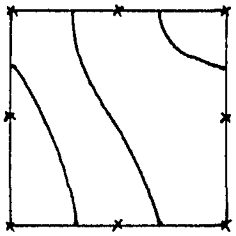
$\gamma_{19} = S^{-1}YX$



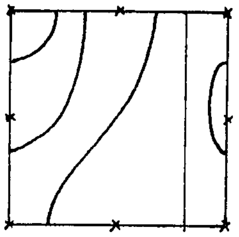
$\gamma_{20} = T^{-1}YX$



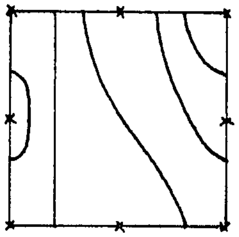
$\gamma_{21} = TY^{-1}X^{-1}$



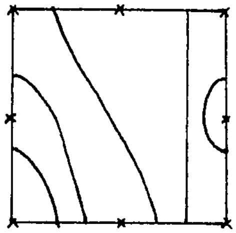
$\gamma_{22} = T^{-1}SX^{-1}Y^{-1}X$



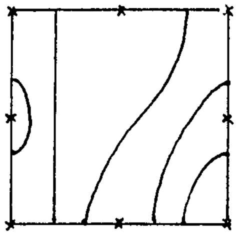
$\gamma_{23} = TS^{-1}XYX^{-1}$



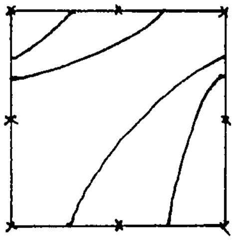
$\gamma_{24} = SY^{-1}XYT^{-1}$



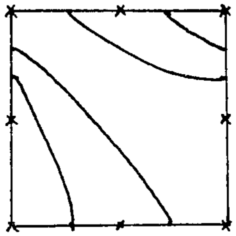
$\gamma_{25} = S^{-1}YX^{-1}Y^{-1}T$



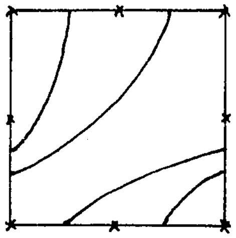
$\gamma_{26} = T^{-1}Y^{-1}TX$



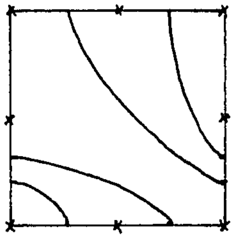
$\gamma_{27} = TYT^{-1}X^{-1}$



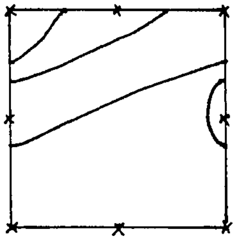
$\gamma_{28} = S^{-1}Y^{-1}SX$



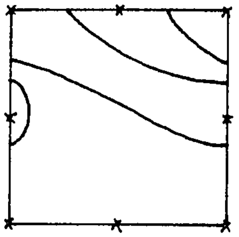
$\gamma_{29} = SYS^{-1}X^{-1}$



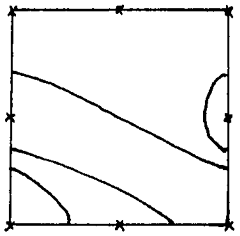
$\gamma_{30} = T^{-1}S^{-1}TX$



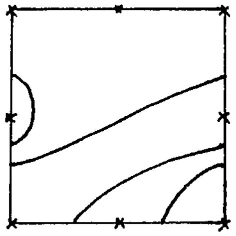
$\gamma_{31} = TST^{-1}X^{-1}$



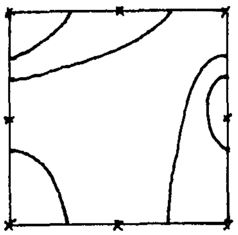
$\gamma_{32} = SYS^{-1}T^{-1}$



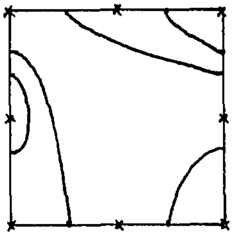
$\gamma_{33} = S^{-1}Y^{-1}ST$



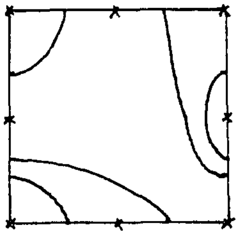
$\gamma_{34} = T^{-1}YS^{-1}TX$



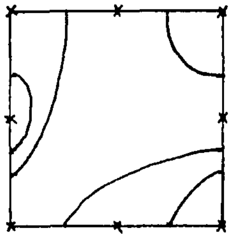
$\gamma_{35} = TY^{-1}ST^{-1}X^{-1}$



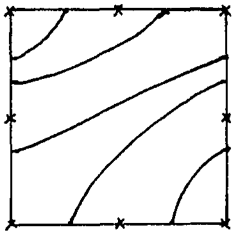
$\gamma_{36} = T^{-1}SYS^{-1}X$



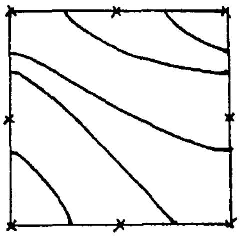
$\gamma_{37} = TS^{-1}Y^{-1}SX^{-1}$



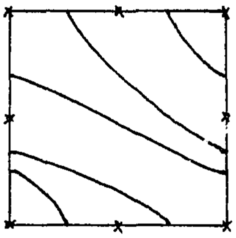
$\gamma_{38} = T^{-1}S^{-1}YTX$



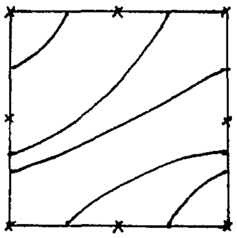
$\gamma_{39} = TSY^{-1}T^{-1}X^{-1}$



$\gamma_{40} = TSY^{-1}S^{-1}X^{-1}$



$\gamma_{41} = T^{-1}S^{-1}YSX$



By using the elementary curves given above and using the technique developed in [4], one can form a cell decomposition of $\overline{\mathcal{G}}_5$ with 168 tetrahedra given as follows.

Notation: For i, j, k, $\ell \in \{1,2,3,\ldots,41\}$, we denote by

$$\sigma = [i,\ j,\ k,\ \ell]$$

the tetrahedron with vertices γ_i, γ_j, γ_k, γ_ℓ. Then every homotopy class in σ is of the form $a\gamma_i + b\gamma_j + c\gamma_k + d\gamma_\ell$, where a, b, c and d are relatively prime non-negative integers.

σ	$\Theta_1(\sigma)$	$\Theta_2(\sigma)$	$\Theta(\sigma)$
$\sigma_1 = [3,4,9,18]$	$\sigma_2 = [3,4,8,19]$	$\sigma_3 = [3,5,9,20]$	$\sigma_4 = [3,5,8,21]$
$\sigma_5 = [3,4,9,22]$	$\sigma_6 = [3,4,8,23]$	$\sigma_7 = [3,5,9,24]$	$\sigma_8 = [3,5,8,25]$
$\sigma_9 = [3,4,19,22]$	$\sigma_{10} = [3,4,18,23]$	$\sigma_{11} = [3,5,21,24]$	$\sigma_{12} = [3,5,20,25]$
$\sigma_{13} = [3,9,18,24]$	$\sigma_{14} = [3,8,19,25]$	$\sigma_{15} = [3,9,20,22]$	$\sigma_{16} = [3,8,21,23]$
$\sigma_{17} = [3,19,20,22]$	$\sigma_{18} = [3,18,21,23]$	$\sigma_{19} = [3,18,21,24]$	$\sigma_{20} = [3,19,20,25]$
$\sigma_{21} = [1,26,30,34]$	$\sigma_{22} = [1,27,31,35]$	$\sigma_{23} = [2,29,32,36]$	$\sigma_{24} = [2,28,33,37]$
$\sigma_{25} = [1,16,26,38]$	$\sigma_{26} = [1,17,27,39]$	$\sigma_{27} = [2,15,29,40]$	$\sigma_{28} = [2,14,28,41]$
$\sigma_{29} = [1,26,30,38]$	$\sigma_{30} = [1,27,31,39]$	$\sigma_{31} = [2,29,32,40]$	$\sigma_{32} = [2,28,33,41]$
$\sigma_{33} = [1,2,10,30]$	$\sigma_{34} = [1,2,11,31]$	$\sigma_{35} = [2,1,10,32]$	$\sigma_{36} = [2,1,11,33]$
$\sigma_{37} = [1,2,12,30]$	$\sigma_{38} = [1,2,13,31]$	$\sigma_{39} = [2,1,13,32]$	$\sigma_{40} = [2,1,12,33]$
$\sigma_{41} = [10,17,32,36]$	$\sigma_{42} = [11,16,33,37]$	$\sigma_{43} = [10,14,30,34]$	$\sigma_{44} = [11,15,31,35]$
$\sigma_{45} = [1,10,17,32]$	$\sigma_{46} = [1,11,16,33]$	$\sigma_{47} = [2,10,14,30]$	$\sigma_{48} = [2,11,15,31]$
$\sigma_{49} = [1,13,31,39]$	$\sigma_{50} = [1,12,30,38]$	$\sigma_{51} = [2,12,33,41]$	$\sigma_{52} = [2,13,32,40]$
$\sigma_{53} = [4,9,14,22]$	$\sigma_{54} = [4,8,15,23]$	$\sigma_{55} = [5,9,17,24]$	$\sigma_{56} = [5,8,16,25]$
$\sigma_{57} = [14,19,20,22]$	$\sigma_{58} = [15,18,21,23]$	$\sigma_{59} = [17,21,18,24]$	$\sigma_{60} = [16,20,19,25]$

σ	$\Theta_1(\sigma)$	$\Theta_2(\sigma)$	$\Theta(\sigma)$
$\sigma_{61} = [9,14,20,22]$	$\sigma_{62} = [8,15,21,23]$	$\sigma_{63} = [9,17,18,24]$	$\sigma_{64} = [8,16,19,25]$
$\sigma_{65} = [4,14,19,22]$	$\sigma_{66} = [4,15,18,23]$	$\sigma_{67} = [5,17,21,24]$	$\sigma_{68} = [5,16,20,25]$
$\sigma_{69} = [5,9,20,34]$	$\sigma_{70} = [5,8,21,35]$	$\sigma_{71} = [4,9,18,36]$	$\sigma_{72} = [4,8,19,37]$
$\sigma_{73} = [4,8,15,37]$	$\sigma_{74} = [4,9,14,36]$	$\sigma_{75} = [5,8,16,35]$	$\sigma_{76} = [5,9,17,34]$
$\sigma_{77} = [4,14,19,28]$	$\sigma_{78} = [4,15,18,29]$	$\sigma_{79} = [5,17,21,27]$	$\sigma_{80} = [5,16,20,26]$
$\sigma_{81} = [4,19,28,37]$	$\sigma_{82} = [4,18,29,36]$	$\sigma_{83} = [5,21,27,35]$	$\sigma_{84} = [5,20,26,34]$
$\sigma_{85} = [6,15,29,40]$	$\sigma_{86} = [7,14,28,41]$	$\sigma_{87} = [7,16,26,38]$	$\sigma_{88} = [6,17,27,39]$
$\sigma_{89} = [17,29,32,36]$	$\sigma_{90} = [16,28,33,37]$	$\sigma_{91} = [14,26,30,34]$	$\sigma_{92} = [15,27,31,35]$
$\sigma_{93} = [7,14,19,20]$	$\sigma_{94} = [6,15,18,21]$	$\sigma_{95} = [6,17,21,18]$	$\sigma_{96} = [7,16,20,19]$
$\sigma_{97} = [7,14,19,28]$	$\sigma_{98} = [6,15,18,29]$	$\sigma_{99} = [6,17,21,27]$	$\sigma_{100} = [7,16,20,26]$
$\sigma_{101} = [7,16,19,28]$	$\sigma_{102} = [6,17,18,29]$	$\sigma_{103} = [6,15,21,27]$	$\sigma_{104} = [7,14,20,26]$
$\sigma_{105} = [8,11,15,37]$	$\sigma_{106} = [9,10,14,36]$	$\sigma_{107} = [8,11,16,35]$	$\sigma_{108} = [9,10,17,34]$
$\sigma_{109} = [8,11,15,35]$	$\sigma_{110} = [9,10,14,34]$	$\sigma_{111} = [8,11,16,37]$	$\sigma_{112} = [9,10,17,36]$
$\sigma_{113} = [8,15,21,35]$	$\sigma_{114} = [9,14,20,34]$	$\sigma_{115} = [8,16,19,37]$	$\sigma_{116} = [9,17,18,36]$
$\sigma_{117} = [2,4,15,37]$	$\sigma_{118} = [2,4,14,36]$	$\sigma_{119} = [1,5,16,35]$	$\sigma_{120} = [1,5,17,34]$
$\sigma_{121} = [2,4,14,28]$	$\sigma_{122} = [2,4,15,29]$	$\sigma_{123} = [1,5,17,27]$	$\sigma_{124} = [1,5,16,26]$
$\sigma_{125} = [2,4,28,37]$	$\sigma_{126} = [2,4,29,36]$	$\sigma_{127} = [1,5,27,35]$	$\sigma_{128} = [1,5,26,34]$
$\sigma_{129} = [16,19,28,37]$	$\sigma_{130} = [17,18,29,36]$	$\sigma_{131} = [15,21,27,35]$	$\sigma_{132} = [14,20,26,34]$

σ	$\Theta_1(\sigma)$	$\Theta_2(\sigma)$	$\Theta(\sigma)$
$\sigma_{133} = [2, 11, 15, 37]$	$\sigma_{134} = [2, 10, 14, 36]$	$\sigma_{135} = [1, 11, 16, 35]$	$\sigma_{136} = [1, 10, 17, 34]$
$\sigma_{137} = [2, 11, 33, 37]$	$\sigma_{138} = [2, 10, 32, 36]$	$\sigma_{139} = [1, 11, 31, 35]$	$\sigma_{140} = [1, 10, 30, 34]$
$\sigma_{141} = [7, 26, 30, 38]$	$\sigma_{142} = [6, 27, 31, 39]$	$\sigma_{143} = [6, 29, 32, 40]$	$\sigma_{144} = [7, 28, 33, 41]$
$\sigma_{145} = [7, 14, 26, 30]$	$\sigma_{146} = [6, 15, 27, 31]$	$\sigma_{147} = [6, 17, 29, 32]$	$\sigma_{148} = [7, 16, 28, 33]$
$\sigma_{149} = [1, 13, 17, 39]$	$\sigma_{150} = [1, 12, 16, 38]$	$\sigma_{151} = [2, 12, 14, 41]$	$\sigma_{152} = [2, 13, 15, 40]$
$\sigma_{153} = [1, 13, 17, 32]$	$\sigma_{154} = [1, 12, 16, 33]$	$\sigma_{155} = [2, 12, 14, 30]$	$\sigma_{156} = [2, 13, 15, 31]$
$\sigma_{157} = [6, 13, 31, 39]$	$\sigma_{158} = [7, 12, 30, 38]$	$\sigma_{159} = [7, 12, 33, 41]$	$\sigma_{160} = [6, 13, 32, 40]$
$\sigma_{161} = [6, 13, 15, 31]$	$\sigma_{162} = [7, 12, 14, 30]$	$\sigma_{163} = [7, 12, 16, 33]$	$\sigma_{164} = [6, 13, 17, 32]$
$\sigma_{165} = [6, 13, 15, 40]$	$\sigma_{166} = [7, 12, 14, 41]$	$\sigma_{167} = [7, 12, 16, 38]$	$\sigma_{168} = [6, 13, 17, 39]$

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