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ASPECTS OF MORSE THEORY

by

DIANA KALISH

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Diana Kalish
May 1984

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to

Gil, Robert, Jonas, and Judith

Abstract

ASPECTS OF MORSE THEORY

by

Diana Kalish

Adviser: Professor Edgar Feldman

In this paper the Morse Index Theorem is proven in the case where two submanifolds, P and Q, are at the endpoints of a geodesic. The index of the Hessian of the energy function is computed on the space of continuous piecewise smooth paths using P-focal points along the geodesic and a boundary condition at Q.

The Fundamental Theorem of Morse Theory is then generalized to the above case to obtain, under certain conditions, the homotopy type of the space of continuous paths joining P and Q as a countable CW-complex with a cell of dimension λ for each geodesic from P to Q of index λ. The description of the index of a geodesic in terms of P-focal points and a Q-boundary condition involving the second fundamental form of Q lends itself, in certain instances to a simple computation. Several such examples are worked out.

Chapter I

THE MORSE INDEX THEOREM WHERE THE ENDS ARE SUBMANIFOLDS

§1. Introduction

Let M be a complete Riemannian manifold with submanifolds P and Q. The energy function E is defined on the space $\Omega(M;P,Q)$ of piecewise C^∞ paths joining P and Q. A path $\gamma \in \Omega(M;P,Q)$ is a critical point for E when γ is a geodesic intersecting P and Q orthogonally. The tangent space, $T\Omega_\gamma$, consists of piecewise C^∞ vector fields along γ with initial and final vectors tangential to P and Q, respectively. A symmetric bilinear map, I, is defined on $T\Omega_\gamma \times T\Omega_\gamma$ to R and is called the Morse index form.

When Q is a point, the Morse index theorem yields the index of I as the sum of the P-focal points along γ counted with multiplicities. Both Ambrose [1] and Bolton [2] have proven index theorems in the general case, where Q is a submanifold. Ambrose defines a "(P,Q) conjugate point," while Bolton uses the notion of a signed (P,Q) focal point which is employed in the calculations of the index of I.

In this paper the index is found using P-focal points. This method allows for a simpler proof of the theorem, as well as easy computation of the P-focal points (as demonstrated in the examples in Chapter II).

§2. Definitions

M is a complete Riemannian manifold of dimension d.

$\gamma(t)$, $t \varepsilon [0,T]$, is a geodesic in M.

P and Q are submanifolds of M with $\gamma(0) \varepsilon P$; $\gamma'(0) \perp P_{\gamma(0)}$; $\gamma(T) \varepsilon Q$; $\gamma'(T) \perp Q_{\gamma(T)}$.

r is the dimension of $Q_{\gamma(T)}$.

H is the linear space of continuous piecewise C^∞ vector fields along γ which are orthogonal to γ and whose initial and final vectors are in $P_{\gamma(0)}$ and $Q_{\gamma(T)}$, respectively. Then

$$H = \{V(t) = \sum_{i=1}^{d-1} h_i(t)E_i(t) \quad \text{with} \quad V_{(0)} \varepsilon P_{\gamma(0)};\ V(T) \varepsilon Q_{\gamma(T)}\},$$

where $E_1, \ldots, E_{d-1}$ are orthonormal parallel vector fields along γ and orthogonal to γ, and $h_1, \ldots, h_{d-1}$ are real valued continuous piecewise C^∞ functions defined on $[0,T]$.

A Jacobi field X is a vector field along γ which satisfies the differential equation $X'' - RX = 0$, where $RX = R(\gamma',X)\gamma'$ is the curvature tensor of the Levi-Civita connection.

A P-Jacobi field is a Jacobi field which is orthogonal to γ with $J(0) \varepsilon P_{\gamma(0)}$ and $J'(0) - S_0J(0) \perp P_{\gamma(0)}$, where S_0 is the second fundamental form of P at $\gamma(0)$ with respect to $\gamma'(0)$.

$J_1, \ldots, J_{d-1}$ are $d-1$ linearly independent P-Jacobi fields which span the space of P-Jacobi fields.

B is the set of all $X \varepsilon H$ such that

$$X(t) = \sum_{i=1}^{d-1} f_i(t)J_i(t) \quad \text{with} \quad X(T) = 0,$$

and $f_1, \ldots, f_{d-1}$ are real valued continuous piecewise C^∞ functions defined on $[0,T]$.

I is a symmetric bilinear map from $H \times H$ to R defined as follows. Let $X \in H$, $Y \in H$,

$$I(X,Y) = \int_0^T \langle RX(t) - X''(t), Y(t)\rangle dt + \Sigma\langle X'(p_i^-) - X'(p_i^+), Y(p_i)\rangle + \langle X'(t) - S_t X(t), Y(t)\rangle \Big|_0^T$$

where p_i = jumps of X' in $(0,T)$; S_0 is the second fundamental form of P at $\gamma(0)$ with respect to $\gamma'(0)$; and S_T is the second fundamental form of Q at $\gamma(T)$ with respect to $\gamma'(T)$.

A P-focal point is a point $\gamma(t)$, $t \in [0,T]$, for which there exists a P-Jacobi field which vanishes at t.

The muliplicity, m, of the P-focal point $\gamma(t)$ is the dimension of the space of P-Jacobi fields which vanish at t.

A is a symmetric bilinear map defined on the space spanned by $J_1, \ldots, J_{d-1}$ for which

$$\sum_{i=1}^{d-1} \alpha_i J_i(T)$$

is contained in $Q_{\gamma(T)}$ and is defined as follows:

$$A(V,W) = \langle V'(T) - S_T V(T), W(T)\rangle$$

for

$$V = \sum_{i=1}^{d-1} \alpha_i J_i; \quad V(T) \in Q_{\gamma(T)}$$

and

$$W = \sum_{i=1}^{d-1} \beta_i J_i; \quad W(T) \in Q_{\gamma(T)}.$$

§3. The Index Theorem

Theorem (Morse Index Theorem With Variable Endpoints). The index of I is equal to the number of points $\gamma(t)$, with $0 < t < T$, such that $\gamma(t)$ is a P-focal point; each such P-focal point counted with its multiplicity plus the index of A. (Assume T is not a P-focal point.)

The above theorem states:

$$i(I) = \sum_{i=1}^{k} m_i + i(A),$$

when T is not a P-focal point, where $i(I) =$ index of I; $i(A) =$ index of A; m_i is the multiplicity of $\gamma(t_1)$; and $\gamma(t_1), \ldots, (t_k)$ are the set of P-focal points along γ; $0 < t_1 < \ldots < t_k < T$.

Our aim will be to write $H = B \oplus B^C$, where I is positive on B. We will show that B^C is a finite dimensional space and construct a subspace of B^C on which I is negative definite and whose dimension is equal to or greater than any other subspace of H on which I is negative definite. This will yield i(I). However, one of the lemmas will only apply to a Hilbert space, so that we will first complete H to a Hilbert space $\overline{H}$ and write $\overline{H} = W \oplus B^C$. We will then show $B \subset W$ such that $H = B \oplus B^C$, with B and B^C having the desired properties.

The proof of the theorem will be a consequence of the following remark and lemmas.

Remark. Introduce an inner product on the space of continuous piecewise C^∞ functions from $[0,T] \to R$ defined by

$$(f,g)_L = \int_0^T f(t)g(t) + f'(t)g'(t)dt.$$

This inner product defines an L norm. Complete this space in the L norm to a Hilbert space L. Then L is the set of absolutely continuous functions from $[0,T] \to R$ such that

$$\int_0^T |X'(t)|^2 dt < \infty.$$

Let

$$\overline{H} = \underset{d-1 \text{ copies}}{\bigoplus} L.$$

Then $\overline{H}$ is the completion of H with respect to the L norm, where

$$(\|V\|_L)^2 = \int_0^T \|V(t)\|^2 + \|V'(t)\|^2 dt.$$

By a lemma of Sobalev,

$$\|f\|_\infty = \sup |f(t)| \leqslant \text{constant } \|f\|_L, \quad f \in L, \quad t \in [0,T].$$

Since evaluation at a point is a continuous linear functional in the supremum norm, Sobolev's lemma implies that evaluation at a point is a continuous linear functional in the L norm.

<u>Definition</u>. Let $p_i^{j_i}$ $(i = 1, \ldots, k;\ j_i = 1, \ldots, m_i)$ and p^l $(l = 1, \ldots, r;\ r = \dim Q)$ be continuous linear functionals defined on $\overline{H}$ as follows.

At the P-focal point $\gamma(t_i)$ let $e_1, \ldots, e_{m_i}$ be a set of orthonormal vectors spanning the space perpendicular to the space spanned by $J_1(t_i), \ldots, J_{d-1}(t_i)$ and orthogonal to $\gamma'(t_i)$.

Let $p_i^{j_i}(V) = \langle V(t_i), e_{j_i}\rangle$, $j_i = 1, \ldots, m_i$, $V \in \overline{H}$.

At $\gamma(T)$ let $e_1, \ldots, e_r$ be an orthonormal set spanning $Q_{(T)}$ and orthogonal to $\gamma'(T)$.

Let $p^l(V) = \langle V(T), e_l\rangle$, $l = 1, \ldots, r$, $V \in \overline{H}$.

<u>Lemma 1</u>. $p_i^{j_i}$ and p^l $(j_i = 1, \ldots, m_i;\ i = 1, \ldots, k;\ l = 1, \ldots r)$ are

$$\left(\sum_{i=1}^{k} m_i + r\right)$$

linearly independent, continuous linear functionals on $\overline{H}$.

Proof. The continuity follows from the preceding remark. To show the functionals are linearly independent assume $\Sigma\alpha_i^{j_i} p_i^{j_i} + \Sigma\beta_l p^l = 0$.

Choose an element $V \in \overline{H}$ with $V(T) = e_1$, where e_1 is member of the O.N. set $\{e_1, \ldots, e_r\}$ used in the definition of p^l and such that $V(t_i)$ is in the span of $J_1(t_i), \ldots, J_{d-1}(t_i)$ for all t_i for which $\gamma(t_i)$ is a P-focal point. Then V will be contained in the kernel of all the functionals $p_i^{j_i}, p^l$ except p^1. It then follows that

$$\begin{aligned} 0 &= (\Sigma\alpha_i^{j_i} p_i^{j_i} + \Sigma\beta_l p^l)V = \beta_1 p^1(V) \\ &= \beta_1\langle V(T), e_1\rangle = \beta_1\langle e_1, e_1\rangle = \beta_1. \end{aligned}$$

$$\Rightarrow \beta_1 = 0.$$

In a similar manner, all $\alpha_i^{j_i}, \beta_l$ are equal to zero, which proves that the linear functionals $p_i^{j_i}, p^l$ are linearly independent.

This proves lemma 1.

Definition.

$$W = \left[\bigcap_{\substack{i=1,\ldots,k \\ j_i=1,\ldots,m_i}} \text{Kernel } p_i^{j_i}\right] \bigcap \left[\bigcap_{l=1,\ldots,r} \text{Kernel } p^l\right].$$

Lemma 2. $\overline{H} = W \oplus W^{\perp}$, and the dimension of $W^{\perp}$ is equal to

$$\sum_{i=1}^{k} m_i + r.$$

Proof. Lemma 2 follows from the theorem which states: If H is a Hilbert space and $f_1, \ldots, f_n$ are n linearly independent continuous functionals defined on H and if

$$M = \bigcap_{i=1}^{n} \text{Kernel } f_i,$$

then (1) $M^{\perp}$ exists; (2) $H = M \oplus M^{\perp}$; and (3) the dimension of $M^{\perp}$ is equal to n. This proves lemma 2.

The next two lemmas apply to vector spaces in general.

Lemma 3. Let X be any vector space, and suppose $W \subset X$, $V \subset X$, the dimension of V is n and $X = W \oplus V$. Then for any subspace $\overline{V} \subset X$ whose dimension is also n and such that $W \cap \overline{V} = 0$, we have

$$X = W \oplus \overline{V}.$$

Proof. We wish to show that for $x \varepsilon X$ we have $x = w + \overline{v}$; $w \varepsilon W$, $\overline{v} \varepsilon V$. Let $\overline{v}_1, \ldots, \overline{v}_n$ be a basis for $\overline{V}$; let $\overline{v}_i = w_i + v_i$, $w_i \varepsilon W$, $v_i \varepsilon V$, $i = 1, \ldots, n$; then $v_i = \overline{v}_i - w_i$, $i = 1, \ldots, n$.

The following argument shows that v_i are linearly independent and so form a basis for V.

$$0 = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i (\overline{v}_i - w_i)$$

$$= \sum_{i=1}^{n} a_i \overline{v}_i + \sum_{i=1}^{n} (-a_i) w_i .$$

Then $\Sigma a_i \overline{v}_i = 0$ follows from $\overline{V} \cap W = 0$. Since $\overline{v}_i$ are linearly independent, we have $a_i = 0$, $i = 1, \ldots, n$. Thus v_i are linearly independent.

Let $x \varepsilon X$, so that $x = w + v = w + \Sigma\alpha_i v_i = w + \Sigma\alpha_i(\overline{v}_i - w_i) = w - \Sigma\alpha_i w_i + \Sigma\alpha_i \overline{v}_i$. Letting $w - \Sigma\alpha_i w_i = \tilde{w} \varepsilon W$ and $\Sigma\alpha_i\overline{v}_i = \overline{v} \varepsilon \overline{V}$, we can write $x = \tilde{w} + \overline{v}$. This, together with $W \cap \overline{V} = 0$, yields

$$X = W \oplus \overline{V}.$$

This proves lemma 3.

Lemma 4. Let X be any vector space; V is an n-dimensional subspace of X; $W \subset X$; and $V \oplus W = X$. Let $\overline{V} \subset X$ such that $\overline{V} \cap W = 0$. Then the dimension of $\overline{V}$ cannot be greater than n.

Proof. Suppose there exists $n+1$ linearly independent vectors in $\overline{V}$; $\overline{v}_1, \ldots, \overline{v}_{n+1}$.

Let Y be the span of $\overline{v}_1, \ldots, \overline{v}_n$. By lemma 3,

$$Y \oplus W = X.$$

Therefore, $\overline{v}_{n+1} = y + w$; $y \in Y$, $w \in W$, so that $0 = y - \overline{v}_{n+1} + w$. Since $y - \overline{v}_{n+1} \in V$ and $w \in W$, we have $y - \overline{v}_{n+1} = 0$, or $\overline{v}_{n+1}$ is a linear combination of $\overline{v}_1, \ldots, \overline{v}_n$. This contradicts the assumption that $\overline{v}_1, \ldots, \overline{v}_n, \overline{v}_{n+1}$ form a linearly independent set.

This proves lemma 4.

In lemma 2 we showed $\overline{H} = W \oplus W^{\perp}$. We will next construct a finite dimensional subspace of $\overline{H}$ and will denote it by B^C. The dimension of B^C will equal the dimension of $W^{\perp}$ and $B^C \cap W = 0$, so that by lemma 3 we will have $\overline{H} = W \oplus B^C$. Finally, we will show $B \subset W$ and $H = B \oplus B^C$.

The next two definitions will yield

$$\sum_{n=1}^{k} m_1 + r$$

elements of $\overline{H}$ whose span will be denoted by B^C.

Definition. Since T is not a P-focal point, we can choose r linearly independent P-Jacobi fields $K_1, \ldots, K_r$ with the following properties: (1) $K_1(T), \ldots, K_r(T)$ span $Q_{\gamma(T)}$; (2) A is negative definite on the span of $K_1, \ldots, K_N$, where $N = \text{index } A$ and $N \leqslant r$; and (3) A is positive on the span of $K_{N+1}, \ldots, K_r$. (Recall that A was defined as a symmetric bilinear map on those P-Jacobi fields which, when evaluated at T, lie in $Q_{\gamma(T)}$. The dimension of this space is r.)

Definition. Consider the P-focal point $\gamma(t_i)$ with multiplicity m_i. Let $Y_i^1, \ldots, Y_i^{m_i}$ be m_i linearly independent P-Jacobi fields such that

$$Y_i^{j_i}(t_i) = 0 \quad \text{for} \quad j_i = 1, \ldots, m_i,$$

and such that

$$Y_i'^{1}(t_i), \ldots, Y_i'^{m_i}(t_i)$$

form an orthonormal set. Let $Z_i^{j_i}$ be parallel vector fields along γ such that

$$\tilde{Z}_i^{j_i}(t_i) = -Y_i'^{j_i}(t_i) \quad \text{for} \quad j_i = 1, \ldots, m_i.$$

Let $\phi_i:[0,T] \to R$ be a C^∞ function such that: (1) $\phi_i(t_i) = 1$; (2) $\phi_i(t_i)$ has small support about t_i; and (3) $0 \leqslant \phi_i(t) \leqslant 1$. Let

$$Z_i^{j_i}(t) = \phi_i(t)\tilde{Z}_i^{j_i}(t); \quad i = 1, \ldots, m_i.$$

Let

$$V_i^{j_i}(t) = \begin{cases} Y_i^{j_i}(t) + \lambda Z_i^{j_i}(t) & \text{for} \quad 0 \leqslant t \leqslant t_i \\ \lambda Z_i^{j_i}(t) & \text{for} \quad t_i \leqslant t \leqslant T, \end{cases}$$

where $\lambda > 0$.

Definition. Let $B^C \subset H$ denote the span of

$$\Sigma m_i + r \quad (i = 1, \ldots, k)$$

vectors $V_i^{j_i}$ $(i = 1, \ldots, k;\ j_i = 1, \ldots, m_i)$ and K_l $(l = 1, \ldots, r)$.

Claim. B^C satisfies: (1) the dimension of

$$B^C = \sum_{i=1}^{k} m_i + r;$$

and (2) $B^C \cap W = 0$; $W = (\bigcap \operatorname{Ker} p_i^{j_i}) \cap (\bigcap \operatorname{Ker} p^l)$.

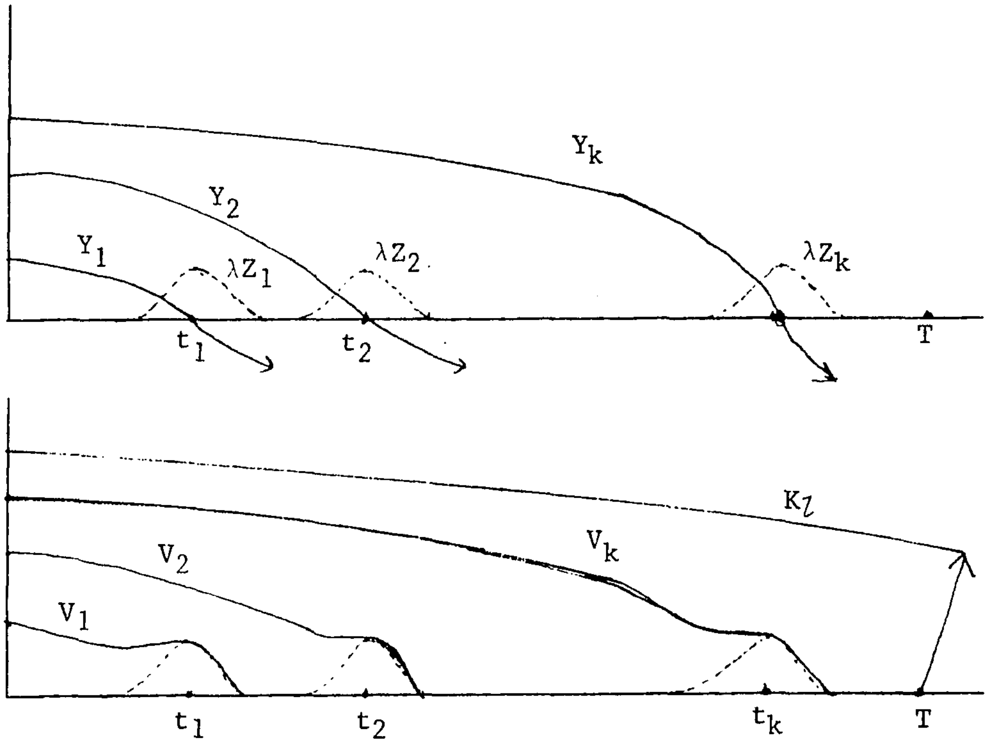
Proof. (1) We need to show that the set of

$$\sum_{i=1}^{k} m_i + r$$

vectors $\{V_i^{j_i}, K_l\}$ form a linearly independent set. Suppose

$$\sum_{\substack{i=1,\dots,k \\ j_i=1,\dots,m_i}} a_i^{j_i} V_i^{j_i} + \sum_{l=1,\dots,r} b_l K_l = 0.$$

The supports of all ϕ_i can be made small enough so that the following argument is true.



At T:

$$0 = \Sigma a_i^{j_i} V_i^{j_i}(T) + \Sigma b_l K_l(T) = \sum_{l=1}^{r} b_l K_l(T).$$

$K_1(T), \dots, K_r(T)$ being linearly independent implies $b_l = 0$ for $l = 1, \dots, r$.

At t_k:

$$0 = \sum_{\substack{i=1,\ldots,k \\ j_i=1,\ldots,m_i}} a_i^{j_i} V_i^{j_i}(t_k) = \sum_{\substack{i=k \\ j_i=1,\ldots,m_i}} a_i^{j_i} V_i^{j_i}(t_k)$$

$$= \sum_{j_k=1}^{m_k} a_k^{j_k} \lambda Z_k^{j_k}(t_k)$$

$$= \sum_{j_k=1}^{m_k} a_k^{j_k} \lambda \phi_k(t_k) \tilde{Z}_k^{j_k}(t_k)$$

$$= -\lambda \sum_{j_k=1}^{m_k} a_k^{j_k} Y_k'^{j_k}(t_k),$$

since

$$\phi_k(t_k) = 1$$

and

$$\tilde{Z}_k^{j_k}(t_k) = -Y'^{j_k}_k.$$

But $Y_k'^{1}(t_k), \ldots, Y_k'^{m_k}(t_k)$ form an orthonormal set implies

$$a_k^{j_k} = 0 \quad \text{for} \quad j_k = 1, \ldots, m_k.$$

Using the above argument for $t = t_{k-1}, \ldots, t = t_1$, we get $a_i^{j_i} = 0$ for $i = 1, \ldots, k$ and $j_i = 1, \ldots, m_i$.

Proof of (2). We need to show that if $X \varepsilon B^C$ and $X \varepsilon W$ then $X = 0$. $X \varepsilon B^C$ means X is of the form

$$X = \sum_{i,j_i} a_i^{j_i} V_i^{j_i} + \sum_{l=1}^{r} b_l K_l,$$

and $x \varepsilon W$ means

$$x \varepsilon \left[\bigcap_{i,j_i} \operatorname{Ker} p_i^{j_i}\right] \cap \left[\bigcap_{l} \operatorname{Ker} p^{l}\right].$$

At T:

$$X(T) = \sum_{l=1}^{r} b_l K_l(T),$$

so that

$$0 = p^i(X) = < \sum_{l=1}^{r} b_l K_l(T), e_i>; \quad i = 1, \ldots, r.$$

Since $e_1, \ldots, e_r$ span $Q_{\gamma(T)}$ and $K_1(T), \ldots, K_r(T)$ also span $Q_{\gamma(T)}$ and are linearly independent, we get

$$\sum_{l=1}^{r} b_l K_l(T) = 0$$

and

$$b_l = 0; \quad l = 1, \ldots, r.$$

At t_k:

$$0 = p_k^i(X) = <X(t_k), e_i> = < \sum_{j_k=1}^{m_k} a_k^{j_k} V_k^{j_k}(t_k), e_i>$$

$$= <-\lambda \sum_{j_k=1}^{m_k} a_k^{j_k} Y_k'^{j_k}(t_k), e_i>; \quad i = 1, \ldots, m_k.$$

However, the space spanned by $\{Y_k'^{j_k}(t_k)\}_{j_k = 1, \ldots, m_k}$ is orthogonal to the space spanned by $J_1(t_k), \ldots, J_{d-1}(t_k)$ and $\gamma'(t_k)$, and so is the same as the space spanned by $e_1, \ldots, e_{m_k}$ at t_k. Therefore, $X(t_k)$ is both in the space spanned by $e_1, \ldots, e_{m_k}$ and orthogonal to the space spanned by $e_1, \ldots, e_{m_k}$. This implies that $0 = X(t_k) = -\lambda \Sigma a_k^{j_k} Y_k'^{j_k}(t_k)$, and since $\{Y_k'^{j_k}(t_k)\}_{j_k = 1, \ldots, m_k}$ forms an orthonormal set, we get

$$a_k^{j_k} = 0 \quad \text{for} \quad j_k = 1, \ldots, m_k.$$

The above argument works in exactly the same way for $t_{k-1}, \ldots, t_1$, which yields $a_k^{j_k} = 0$ for all k and for all j_k. Thus $X = 0$, and the claim is proven.

Lemma 5. $\overline{H} = w \oplus B^C$.

Proof. By lemma 2, $\overline{H} = W \oplus W^{\perp}$, where the dimension of

$$W^{\perp} = \sum_{i=1}^{k} m_i + r.$$

By the claim proven above B^C has dimension

$$\sum_{i=1}^{k} m_i + r$$

and $B^C \cap W = 0$. Therefore, by lemma 3, we have $\overline{H} = W \oplus B^C$.

This proves lemma 5.

Lemma 6. $H = B \oplus B^C$.

Proof. If $X \varepsilon B$, then

$$X = \sum_{i=1}^{d-1} f_i J_i$$

such that $X(T) = 0$. Therefore, at the P-focal points X is in the span of the P-Jacobi fields at each focal point, and so X is contained in the kernels of all the linear functionals $p_i^{j_i}$ and p^l. Thus, $x \varepsilon W$. Therefore, $B \subset W$. However, $B \subset W$ and $W \cap B^C = 0$ together imply that

$$B \cap B^C = 0.$$

To complete the proof we need to show that if $x \varepsilon H$ then $x = b + c$, where $b \varepsilon B$ and $c \varepsilon B^C$. It follows from lemma 5 that $x = w + c$ for $w \varepsilon W$ and $c \varepsilon b^c$. Since $x \varepsilon H$ and $c \varepsilon H$, we have $w \varepsilon H$. Then $w \varepsilon H$ and $w \varepsilon W$ implies $w \varepsilon B$. This follows from the fact that any broken C^{∞} vector field which can be expressed as

$$\sum_{i=1}^{d-1} f_i(t) E_i(t)$$

and is in the span of $J_1(t_i), \ldots, J_{d-1}(t_i)$ at all P-focal points t_i can also be written as

$$\sum_{i=1}^{d-1} g_i(t)J_i(t)$$

for broken C^∞ functions g_i [3, p. 231].

This proves lemma 6.

Lemma 7 will show I is positive on B and lemma 8 will exhibit a subspace of B^C on which I is negative definite.

Lemma 7. $I(V,V) \geqslant 0$ for $V \varepsilon B$.

Proof. Let $V \varepsilon B$. Then

$$V = \sum_{i=1}^{d-1} f_i J_i$$

and

$$\begin{aligned} I(V,V) &= \langle \sum_{i=1}^{d-1} f_i(T)J_i'(T) - S_T[\sum_{i=1}^{d-1} f_i(T)J_i(T)], V(T)\rangle \\ &\quad + \int_0^T \langle \Sigma f_i'(t)J_i(t), \Sigma f_i'(t)J_i(t)\rangle dt \\ &= \int_0^T \langle \Sigma f_i'(t)J_i(t), \Sigma f_i'(t)J_i(t)\rangle dt \quad \text{since} \quad V(T) = 0. \\ &\geqslant 0. \end{aligned}$$

For the first equality, see [3, p. 229]. This proves lemma 7.

Lemma 8. $\operatorname{index}(I_{|B^C}) = \sum_{i=1}^{k} + \operatorname{index}(A)$.

Proof. $K_1, \ldots, K_r$ were chosen to be r linearly independent P-Jacobi fields such that A is negative definite on the span of $K_1, \ldots, K_N$ and positive on $K_{N+1}, \ldots, K_r$. (Note: $N = \operatorname{index} A$.)

We wish to show that I is negative definite on the span of

$$\{V_i^{j_i}\}_{\substack{i=1,\ldots,k \\ j_i=1,\ldots,m_k}}$$

and $K_1, \ldots, K_N$, and that I is positive on the span of $K_{N+1}, \ldots, K_r$.

$$I\Big(\sum_{i,j_i} \alpha_i^{j_i} V_i^{j_i} + \sum_{l=1}^{N} \beta_l K_l\Big) =$$

$$(1)\quad I(\Sigma \alpha_i^{j_i} V_i^{j_i})$$

$$(2)\quad +2I\Big(\sum_{i,j_i} \alpha_i^{j_i} V_i^{j_i}, \sum_{l=1}^{N} \beta_l K_l\Big)$$

$$(3)\quad +I\Big(\sum_{l=1}^{N} \beta_l K_l\Big).$$

<u>Computation of (3)</u>. Let $K = \sum_{l=1}^{N} \beta_l K_l$.

$$I(K) = \int_0^T \langle RK - K'', K\rangle dt + \sum_{\substack{\text{jumps} \\ \text{of } K'}} \langle K'(p_i^-) - K'(p_i^+), K(p_i)\rangle$$

$$+ \langle K'(t) - S_t K(t), K(t)\rangle \Big|_0^T$$

$$= \langle K'(T) - S_T K(T), K(T)\rangle.$$

This follows from the fact that K is a P-Jacobi field and is smooth and therefore satisfies $RK - K'' = 0$ and $K'(0) - S_0 K(0) \perp P_{\gamma(0)}$.

So $I(K) = \langle K'(T) - S_T K(T), K(T)\rangle = A(K) < 0$, since A is negative definite on the span of $K_1, \ldots, K_N$ and

$$K = \sum_{l=1}^{N} \beta_l K_l.$$

<u>Computation of (2)</u>. When (2) is expanded we get linear combinations of terms of the form $I(V_i^{j_i}, K_l)$.

$$I(V_i^{j_i}, K_l) = \int_0^T \langle RK_l - K_l'', V_i^{j_i}\rangle dt + \langle K_l'(t) - S_t K_l(t), V_i^{j_i}(t)\rangle \Big|_0^T$$

$$= 0,$$

since K_l is a P-Jacobi field and so satisfies $RK_l - K_l'' = 0$ and

$K'_l(0) - S_0K(0) \perp P_{\gamma(0)}$ $(V_i^{ji}(0) \in P_{\gamma(0)})$. Also, $V_i^{ji}(T) = 0$. Note that in the first equality above there are no jumps of K' since K is a P-Jacobi field and therefore smooth.

This shows that (2) is equal to zero.

Computation of (1). Let

$$\hat{Y}_i^j = \begin{cases} Y_i^j(t) & 0 \leqslant t \leqslant t_i \\ 0 & t_i \leqslant t \leqslant T. \end{cases}$$

Then $V_i^j(t) = \hat{Y}_i^j(t) + \lambda Z_i^j(t)$; $t \in [0,T]$.

$$I(V_i^j, V_h^l) = \int_0^T \langle V_i'^j, V_h'^l\rangle + \langle RV_i^j, V_h^l\rangle dt - \langle S_tV_i^j(t), V_h^l(t)\rangle\Big|_0^T \quad [3, \text{ p. } 220]$$

$$= \int_0^T \langle \hat{Y}_i'^j + \lambda Z_i'^j, \hat{Y}_h'^l + \lambda Z_h'^l\rangle + \langle R(\hat{Y}_i^j + \lambda Z_i^j), \hat{Y}_h^l + \lambda Z_h^l\rangle dt$$

$$- \langle S_tV_i^j(t), V_h^l(t)\rangle\Big|_0^T =$$

(a) $I(\hat{Y}_i^j, \hat{Y}_h^l)$

(b) $+\lambda^2 I(Z_i^j, Z_h^l)$

(c) $+\lambda I(Z_i^j, \hat{Y}_h^l) + \lambda I(\hat{Y}_i^j, Z_h^l)$.

Let $h \leqslant i$.

$$(a) = I(\hat{Y}_i^j, \hat{Y}_h^l) = \int_0^T \langle R\hat{Y}_i^j - \hat{Y}_i''^j, \hat{Y}_h^l\rangle dt + \langle \hat{Y}_i'^j(t_i^-) - \hat{Y}_i'^j(t_i^+), \hat{Y}_h^l(t_i)\rangle$$

$$+ \langle \hat{Y}_i'^j - S_t\hat{Y}_i^j, \hat{Y}_h^l\rangle\Big|_0^T$$

$$= 0.$$

This follows from $\hat{Y}_i''^j - R\hat{Y}_i^j = 0$, $\hat{Y}_h^l(t_i) = 0$, $\hat{Y}_h^l(T) = 0$, and $\hat{Y}_i'^j(0) - S_0\hat{Y}_i^j(0) \perp P_{\gamma(0)}$. When $i \leqslant h$ exactly the same argument works.

(c) = $-2\lambda\delta_{ih}\delta_{jl}$, which is shown to be true as follows. For $i \neq h$, we get

$$\begin{aligned} I(Z_i^j,\hat{Y}_h^l) &= \int_0^T \langle R\hat{Y}_h^l - \hat{Y}_h''^{l}, Z_i^j\rangle dt + \langle \hat{Y}_h'^{l}(t_h^-) - \hat{Y}_h'^{l}(t_h^+), Z_i^j(t_h)\rangle \\ &\quad + \langle \hat{Y}_h'^{l}(t) - S_t\hat{Y}_h^l(t), Z_i^j(t)\rangle\Big|_0^T \\ &= 0, \end{aligned}$$

since $\hat{Y}_h^l$ is a P-Jacobi field on $[0,t_h]$,

$$Z_i^j(0) = Z_i^j(T) = 0$$

and

$$Z_i^j(t_h) = 0$$

when the support of ϕ_i is small enough.

For $i = h$, we get the same as above except at t_i

$$\begin{aligned} I(Z_i^j,\hat{Y}_i^l) &= \langle \hat{Y}_i'^{l}(t_i^-) - \hat{Y}_i'^{l}(t_i^+), Z_i^j(t_i)\rangle \\ &= \langle \hat{Y}_i'^{l}(t_i^-), Z_i^j(t_i)\rangle \qquad \left(\text{since } \hat{Y}_i^l(t) = 0 \text{ for } t \in [t_i,T]\right) \\ &= \langle Y_i'^{l}(t_i), -Y_i'^{j}(t_i)\rangle \\ &= -\delta_{jl}\|Y_i'^{j}(t_i)\|^2 \qquad \left(\text{since } Y_i'^{1}, Y_i'^{2}, \ldots, Y_i'^{m_i} \text{ are an orthonormal set evaluated at } t_i\right) \\ &= -\delta_{jl}. \end{aligned}$$

Therefore, $\lambda I(Z_i^j,\hat{Y}_h^l) + \lambda I(\hat{Y}_i^j,Z_h^l) = -2\lambda\delta_{ih}\delta_{jl}$.

Putting together the results from (a), (b), and (c), we have

$$I(V_i^j,V_h^l) = \lambda^2 I(Z_i^j,Z_h^l) - 2\lambda\delta_{ih}\delta_{jl}.$$

Notation. Let

$$(\Sigma a_i^{j_i}V_i^{j_i}, \Sigma b_i^{j_i}V_i^{j_i}) = \Sigma a_i^{j_i}b_i^{j_i}$$

and

$$\|\Sigma a_i^{j_i}V_i^{j_i}\|^2 = \Sigma (a_i^{j_i})^2.$$

Let M be a $\left(\sum_{i=1}^{k} m_i\right)$ by $\left(\sum_{i=1}^{k} m_i\right)$ matrix, and let

$$M(\Sigma\alpha_i^{j_i} V_i^{j_i}) = (M)\begin{bmatrix} \alpha_1^1 \\ \vdots \\ \alpha_1^{m_1} \\ \vdots \\ \alpha_k^{m_k} \end{bmatrix}.$$

Then if A is the $\left(\sum_{i=1}^{k} m_i\right)$ by $\left(\sum_{i=1}^{k} m_i\right)$ matrix given by

$$A = \begin{bmatrix} I(Z_1^1, Z_1^1) & \cdots & I(Z_1^1, Z_1^{m_1}) I(Z_1^1 Z_2^1) & \cdots & I(Z_1^1, Z_k^{m_k}) \\ I(Z_1^2, Z_1^1) & & & & I(Z_1^2, Z_k^{m_k}) \\ \vdots & & & & \\ I(Z_k^{m_k}, Z_1^1) & & \cdots & & I(Z_k^{m_k}, Z_k^{m_k}) \end{bmatrix},$$

and E is the (Σm_i) by (Σm_i) identity matrix, and if

$$X = \sum_{i_i j_i} \alpha_i^{j_i} V_i^{j_i},$$

we have

$$I(X,X) = ((\lambda^2 A - 2\lambda E)X, X).$$

This will enable us to complete the computation of (1) which began on p. 16.

For

$$X = \sum_{i_i j_i} \alpha_i^{j_i} V_i^{j_i}$$

we have

$$
\begin{aligned}
I(X,X) &= ((\lambda^2 A - 2\lambda E)X,X) \\
&= ((-2\lambda E)X,X) && \text{for } A = 0 \\
&= -2\lambda(X,X) < 0 && \text{for any } \lambda < 0,\ X \neq 0.
\end{aligned}
$$

For $A \neq 0$, $\|A\| \neq 0$, so let $0 < \lambda < \frac{2}{\|A\|}$. Then

$$
\begin{aligned}
I(X,X) &= ((\lambda^2 A - 2\lambda E)X,X) \\
&= \lambda^2 (AX,X) - 2\lambda(X,X) \\
&\leqslant \lambda^2 \|AX\|\,\|X\| - 2\lambda\|X\|^2 \\
&\leqslant \lambda^2 \|A\|\,\|X\|^2 - 2\lambda\|X\|^2 \\
&< \lambda\frac{2}{\|A\|}\|A\|\,\|X\|^2 - 2\lambda\|X\|^2 \\
&= 0.
\end{aligned}
$$

This gives $I(\Sigma\alpha_i^{j_i}V_i^{j_i}, \Sigma\alpha_i^{j_i}V_i^{j_i}) < 0$ for $\Sigma\alpha_i^{j_i}V_i^{j_i} \neq 0$.

The results from (1), (2), and (3) show that I is negative definite on the span of $\{V_i^{j_i}, K_1, \ldots, K_N\}_{\substack{i=1,\ldots,k \\ j_i=1,\ldots,m_i}}$

In order to finish proving lemma 8 we need to show I is positive on the span of $K_{N+1}, \ldots, K_r$.

$$
\begin{aligned}
I\Big(\sum_{l=N+1}^{r} b_l K_l\Big) &= \int_0^T \langle R(\Sigma b_l K_l) - (\Sigma b_l K_l)'', \Sigma b_l K_l\rangle dt \\
&\quad + \langle (\Sigma b_l K_l)' - S_t(\Sigma b_l K_l), \Sigma b_l K_l\rangle \Big|_0^T \\
&= \langle (\Sigma b_l K_l)' - S_T(\Sigma b_l K_l), \Sigma b_l K_l\rangle_T \\
&= A\Big(\sum_{l=N+1}^{r} b_l K_l\Big) > 0,
\end{aligned}
$$

since A is positive on the span of $K_{N+1}, \ldots, K_r$. The above equalities result from the fact that

$$\sum_{l=N+1}^{r} b_l K_l$$

is a P-Jacobi field and so is smooth, and satisfies $RX - X'' = 0$ and $X'(0) S_0 X(0) \perp P_{\gamma(0)}$.

Lemma 9. Let $\tilde{B}^C \subset H$ such that $H = B \oplus \tilde{B}^C$. Then

$$\text{index}(I_{|\tilde{B}^C}) \leqslant \text{index}(I_{|B^C}).$$

Proof. Assume the lemma is false. Then there exists a subspace of $\tilde{B}^C$ on which I is negative definite and whose dimension is greater than

$$\sum_{i=1}^{k} m_i + N.$$

This implies that the dimension of the subspace of $\tilde{B}^C$ on which I is positive is less than $r - N$ which can be seen by considering the following:

$$\Big(\sum_{i=1}^{k} m_i + N\Big) + (r - N) = \sum_{i=1}^{k} m_i + r = \dim B^C = \dim \tilde{B}^C.$$

From lemma 8 we know that I is negative definite on the span of $K_1, \ldots, K_N$ and I is positive on the span of $K_{N+1}, \ldots, K_r$. Let

$$K_{N+1} = \tilde{V}_1 + V_1$$
$$\vdots$$
$$K_r = \tilde{V}_{r-N} + V_{r-N},$$

where $\tilde{V}_i \in \tilde{B}^C$ and $V_i \in B$ for $i = 1, \ldots, r-N$.

Our aim will be to show that $\tilde{V}_1, \ldots, \tilde{V}_{r-N}$ form a set of $r - N$ linearly independent vectors on whose span I is positive, thus yielding a contradiction.

$$I\left(\sum_{i=1}^{r-N} \alpha_i\tilde{V}_i\right) = I\left(\sum_{i=1}^{r-N} \alpha_i(K_{N+i} - V_i)\right)$$

$$= I\left(\sum_{i=1}^{r-N} \alpha_i K_{N+i} - \sum_{i=1}^{r-N} \alpha_i V_i\right)$$

$$= I(\Sigma\alpha_i K_{N+i}) + I(\Sigma\alpha_i V_i) - 2I(\Sigma\alpha_i K_{N+i}, \Sigma\alpha_i V_i).$$

$I\left(\sum_{i=1}^{r-N} \alpha_i K_{N+1}\right) \geqslant 0$ since I is positive on the span of $K_{N+1}, \ldots, K_r$

$I\left(\sum_{i=1}^{r-N} \alpha_i V_i\right) \geqslant 0$ since $\Sigma\alpha_i V_i \in B$

$$I\left(\sum_{i=1}^{r-N} \alpha_i K_{N+1}, \Sigma\alpha_i V_i\right) = 0,$$

since $\Sigma\alpha_i K_{N+1}$ is a P-Jacobi and therefore satisfies $RX - X'' = 0$, is smooth, and so has no jumps, and satisfies $X'(0) - S_0X(0) \perp P_{\gamma(0)}$. Also,

$$\sum_{i=1}^{r-N} \alpha_i V_i \in B$$

implies $\Sigma\alpha_i V_i(T) = 0$.

Therefore

$$I\left(\sum_{i=1}^{r-N} \alpha_i\tilde{V}_i\right) \geqslant 0.$$

Next show $\tilde{V}_1, \ldots, \tilde{V}_{r-N}$ are linearly independent. Suppose

$$0 = \sum_{i=1}^{r-N} \alpha_i\tilde{V}_i.$$

Then

$$0 = \Sigma\alpha_i\tilde{V}_i = \Sigma\alpha_i(K_{N+i} - V_i)$$

$$= \Sigma\alpha_i K_{N+i} - \Sigma\alpha_i V_i.$$

But $\Sigma\alpha_i K_{N+i} \varepsilon B^C$, and $\Sigma\alpha_i V_i \varepsilon B$, which implies

$$\sum_{i=1}^{r-N} \alpha_i K_{N+i} = 0,$$

so that

$$\alpha_i = 0; \quad i = 1, \ldots, r-N$$

This contradicts the assumption that there exists a subspace of $\tilde{B}^C$ on which I is negative definite and whose dimension is greater than $\Sigma m_i + N$ and so lemma 9 is true.

Proof of Index Theorem. We have proven $H = B \oplus B^C$, I is positive on B, and

$$\dim B^C = \sum_{i=1}^{k} m_i + r.$$

Suppose V is a subspace of H such that I is negative definite on V. If $v \varepsilon V$, $v \neq 0$, we know $v \notin B$ since I is positive on B. Thus $V \cap B = 0$. By lemma 4,

$$\dim V \leqslant \sum_{i=1}^{k} m_i + r.$$

Case 1. $\dim V = \Sigma m_i + r$. Then, by lemma 3, $H = B \oplus V$. It then follows by lemma 9 that

$$\text{index}(I_{|V}) \leqslant \text{index}(I_{|B^C}).$$

Case 2. $\dim V < \Sigma m_i + r$. Adjoin vectors in $H - B$ to V to form a subspace $\overline{V}$ of dimension $\Sigma m_i + r$ and such that $V \subset \overline{V}$ and $\overline{V} \cap B = 0$. Then, as in case 1,

$$\text{index}(I_{|\overline{V}}) \leqslant i(I_{|B^C}).$$

Since $V \subset \overline{V}$, we have $\text{index}(I_{|V}) \leqslant \text{index}(I_{|B^C})$.

This proves the Index Theorem.

Chapter II

THE FUNDAMENTAL THEOREM OF MORSE THEORY IN THE CASE WHERE THE ENDS ARE SUBMANIFOLDS

§1. Introduction

Let M be a complete Riemannian manifold with P and Q two closed and bounded submanifolds. Let Ω^* be the space of continuous paths joining P and Q while Ω is the space of piecewise C^∞ paths joining P and Q. The main theorem of Chapter II will be to show that the homotopy type of Ω^* or Ω is a countable CW-complex which contains a cell of dimension λ for each geodesic which is a critical point of the energy function of index λ. The proof will follow Milnor's exposition [4, pp. 88-95].

From Chapter I we can compute the index of a geodesic orthogonal to P and Q by finding the P-focal points along the geodesic with multiplicities plus a boundary term depending on Q. A few examples will be given for which the homotopy type of the path space joining two submanifolds is found.

§2. The Fundamental Theorem of Morse Theory

Let M be a complete Riemannian manifold with P and Q two submanifolds which are closed and bounded. Let

$$\Omega^* = \Omega^*(M;P,Q)$$

be the set of all continous paths

$$\omega:[0,1] \to M$$

joining P and Q in the compact open topology. This topology is induced by the metric

$$d^*(\omega_1,\omega_2) = \max_t \rho(\omega_1(t),\omega_2(t)),$$

where ρ denotes the topological metric on M coming from its Riemannian metric. Let

$$\Omega = \Omega(M;P,Q)$$

be the set of all piecewise C^∞ paths

$$\omega:[0,1] \to M$$

joining P and Q in the topology induced by the metric

$$d(\omega_1,\omega_2) = d^*(\omega_1,\omega_2) + \left[\int_0^1 \left(\frac{ds_1}{dt} - \frac{ds_2}{dt}\right)^2 dt\right]^{\frac{1}{2}},$$

where $\omega_1,\omega_2 \in \Omega$ with arc lengths $s_1(t)$ and $s_2(t)$, respectively. In this topology the energy function

$$\overset{b}{\underset{a}{E}}(\omega) = \int_a^b \left(\frac{ds}{dt}\right)^2 dt$$

will be a continuous function from Ω to the real numbers.

Definition. Two submanifolds P and Q are said to be conjugate along a geodesic γ orthogonal to P and Q if there exists a non-zero P-Jacobi field along γ which is also a Q-Jacobi field.

Theorem (Fundamental Theorem of Morse Theory extended to the case with variable endpoints). Let M be a complete Riemannian manifold with P and Q as two closed and bounded submanifolds of M which are not conjugate along any geodesic orthogonal to P and Q. Then $\Omega(M;P,Q)$ (or $\Omega^*(M;P,Q)$) has the homotopy type of a countable CW-complex which contains one cell of dimension λ for each geodesic orthogonal to P and Q of index λ.

The proof of the above main theorem will be based on the following theorem.

Theorem 1. If f is a differentiable function on a manifold M with no degenerate critical points, and if each $M^a = f^{-1}(-\infty, a]$ is compact, then M has the homotopy type of a CW-complex, with one cell of dimension λ for each critical point of index λ [4, p. 20; see proof on p. 23].

We will prove the main theorem for the path space Ω and then, since the map $i:\Omega \to \Omega^*$ is shown to be a homotopy equivalence, the theorem is true for Ω^*. The proof will closely follow Milnor's proof given in [4, pp. 88-95] with modifications for the case where the endpoints are submanifolds P and Q instead of points p and q.

For $c > 0$, let

$$\Omega^c = E^{-1}([0,c])$$

and

$$\text{Int}\,\Omega^c = E^{-1}([0,c)].$$

Choose a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$. Let $\Omega(t_0, \ldots, t_n)$ be the subspace of Ω consisting of paths $\omega:[0,1] \to M$ such that

(1) $\omega(0) \in P$ and $\omega(1) \in Q$;

(2) $\omega|[t_{i-1}, t_i]$ is a geodesic for $i = 2, \ldots, n-1$; and

(3) $\omega|[0,t_1]$ and $\omega|[t_{n-1},1]$ are geodesics orthogonal to P and Q, respectively.

Define the subspaces

$$\Omega(t_0, \ldots, t_n)^c = \Omega^c \cap \Omega(t_0, \ldots, t_n)$$

$$\text{Int}\,\Omega(t_0, \ldots, t_n)^c = \text{Int}\,\Omega^c \cap \Omega(t_0, \ldots, t_n).$$

Lemma. Let M be a complete Riemannian manifold and let c be a fixed positive number such that $\Omega^c \neq \phi$. Then for all sufficiently fine subdivisions $(t_0, \ldots, t_n)$ of $[0,1]$ the set $\mathrm{Int}\,\Omega(t_0, \ldots, t_n)^c$ can be given the structure of a smooth finite dimensional manifold.

Proof. Let S denote the set

$$\{x \in M : D(x,P) \leqslant \sqrt{c}\},$$

where D is the distance from a point x to the submanifold P. If $\omega \in \Omega^c$ then $\omega([0,1])$ lies within this subset $S \subset M$. This can be seen from

$$[D(\omega(t),P)]^2 \leqslant [\overset{t}{\underset{o}{L}}\omega]^2 \leqslant [\overset{1}{\underset{o}{L}}\omega]^2 \leqslant E(\omega) < c.$$

Since P is closed and bounded and M is complete, S is closed and bounded and therefore compact. Hence, there exists $\bar{\varepsilon} > 0$ so that whenever $x,y \in S$ and $\rho(x,y) < \bar{\varepsilon}$ there is a unique geodesic from x to y of length $<\bar{\varepsilon}$ and so that this geodesic depends differentially on x and y. Also by [3, p. 151] there exists $r_1 > 0$ such that the exponential function maps the tubular neighborhood of $\perp_{r_1}(P)$ diffeomorphically onto its image, called the tubular neighborhood of P in M, with the property that all of its points are joined to P by unique geodesics which minimize arc length to P. Similarly, there exists $r_2 > 0$ which does the same for Q. Let

$$\varepsilon = \min[\bar{\varepsilon}, r_1, r_2].$$

Choose the subdivision $(t_0, \ldots, t_n)$ of $[0,1]$ so that each difference $t_i - t_{i-1} < \varepsilon^2/c$. Then for

$$\omega \in \Omega(t_0, \ldots, t_n)^c$$

we have

$$\begin{aligned}(L_{t_{i-1}}^{t_i}\,\omega)^2 &= (t_i - t_{i-1})(E_{t_{i-1}}^{t_i}\,\omega)\\ &\leqslant (t_i - t_{i-1})(E\omega)\\ &\leqslant (t_i - t_{i-1})c < \varepsilon^2,\end{aligned}$$

while

$$\begin{aligned}[D(\omega(t_1),P)]^2 \leqslant (L\omega)_o^{t_1\,2} &= (t_1 - 0)E\omega_o^{t_1}\\ &\leqslant (t_1 - 0)E(\omega) \leqslant \frac{\varepsilon^2}{c}\cdot c = \varepsilon^2 \leqslant r_1^2.\end{aligned}$$

Similarly, $D(\omega(t_{n-1}),Q) \leqslant r_2$. Thus the geodesic $\omega|[t_{i-1},t_i]$ for $i = 2, \ldots, n-1$ is uniquely and differentially determined by the two end-points, while the geodesic $\omega|[0,t_1]$ orthogonal to P and the geodesic $\omega|[t_{n-1},1]$ orthogonal to Q is uniquely and differentially determined by $\omega(t_1)$ and $\omega(t_{n-1})$, respectively.

The broken geodesic is uniquely determined by the (n-1)-triple

$$\omega(t_1),\omega(t_2), \ldots, \omega(t_{n-1}) \in M \times \ldots \times M.$$

This correspondence

$$\omega \to (\omega(t_1), \ldots, \omega(t_{n-1}))$$

defines a homeomorphism between $\mathrm{Int}\,\Omega(t_0, \ldots, t_n)^c$ and the open subset of the (n-1) fold product $M \times \ldots \times M$ given by the set of all

$$\{p_1, \ldots, p_{n-1}\} \in S \times \ldots \times S$$

such that

$$\frac{D(p_1,P)^2}{t_1} + \sum_{i=2}^{n-1} \rho(p_{i-1},p_i)^2 \Big/ t_i - t_{i-1} + \frac{D(p_{n-1},Q)^2}{t_n - t_{n-1}} < c.$$

Taking over the differentiable structure from this product completes the proof of the lemma.

Let $\text{Int}\,\Omega(t_0, \ldots, t_n)^c$ be denoted by B, and let

$$E' : B \to R$$

denote the restriction of E to B.

Theorem 2. $E' : B \to R$ is smooth. Furthermore, for each $a < c$ the set $B^a = (E')^{-1}[0,a]$ is compact and is a deformation retract of the corresponding set Ω^a. (Similarly, B is a deformation retract of $\text{Int}\,\Omega^c$.) The critical points of E' are precisely the same as the critical points of E in $\text{Int}\,\Omega^c$; namely, the unbroken geodesics, from P to Q, which are orthogonal to P and Q, of length less than $\sqrt{c}$. The index of the Hessian E'_{**} at each such critical point γ is equal to the index of E_{**} at γ.

Proof. Since the broken geodesic $\omega \in B$ depends smoothly on the (n-1)-triple

$$\omega(t_1), \ldots, \omega(t_{n-1}) \in M \times \ldots \times M$$

the energy $E'(\omega)$ also depends smoothly on this (n-1)-tuple:

$$E'(\omega) = D(\omega(t_1),P)^2/(t_1 - 0) + \sum_{i=2}^{n-1} \rho(\omega(t_{i-1}),\omega(t_i))^2/(t_i - t_{i-1})$$
$$+ D(\omega(t_{n-1}),Q)^2/(t_n - t_{n-1}).$$

For $a < c$ the set B^a is homeomorphic to the set of all (n-1)-tuples $(p_1, \ldots, p_{n-1}) \in S \times \ldots \times S$ such that

$$D(p_1,P)^2/(t_1 - 0) + \sum_{i=2}^{n-1} \rho(p_{i-1},p_i)^2/(t_i - t_{i-1})$$
$$+ D(p_{n-1},Q)^2/(t_n - t_{n-1}) \leqslant a.$$

As a closed subset of a compact set, this set is compact.

A retraction $r:\mathrm{Int}\Omega^c \to B$ is defined as follows. Let $r(\omega)$ denote the unique broken geodesic in B such that $r(\omega)$ defined on $[0,t_1]$ and $[t_{n-1},1]$ is the unique shortest geodesic of length $<\varepsilon$ joining P to $\omega(t_1)$ and $\omega(t_{n-1})$ to Q, respectively, while each $r(\omega)_{|[t_{i-1},t_i]}$ is the unique geodesic of length $<\varepsilon$ from $\omega(t_{i-1})$ to $\omega(t_i)$ for $i = 2, \ldots, n-1$. The inequality

$$D(P,\omega(t))^2 \leqslant (L\omega)^2 \leqslant E\omega < c$$

implies that $\omega([0,1]) \subset S$. Hence the inequality

$$\rho(\omega(t_{i-1}),\omega(t_i))^2 \leqslant (t_i - t_{i-1})(\overset{t_i}{\underset{t_{i-1}}{E}}\ \omega) \leqslant \frac{\varepsilon^2}{c}\cdot c = \varepsilon^2$$

and

$$D(\omega(t_1),P)^2 \leqslant (\overset{t_1}{\underset{0}{L}}\omega)^2 \leqslant (t_1 - 0)(\overset{1}{\underset{0}{E}}\omega) \leqslant \frac{\varepsilon^2}{c}\cdot c = \varepsilon^2 < r_1^2,$$

and similarly

$$D(\omega(t_{n-1}),Q) < r_2$$

implies that $r(\omega)$ can be so defined. Then $r(\omega) \in B$, since

$$E(r(\omega)) \leqslant E(\omega) < c.$$

This retraction r fits into a 1-parameter family of maps

$$r_u:\mathrm{Int}\,\Omega^c \to \mathrm{Int}\,\Omega^c$$

as follows. For $0 \leqslant u \leqslant t_1$ let

$r_u(\omega)_{|[0,u]} =$ the unique geodesic which minimizes arc length from $\omega(u)$ to P.

$r_u(\omega)_{|[u,1]} = \omega_{|[u,1]}$

For $t_{i-1} \leqslant u \leqslant t_i$; $i = 2, \ldots, n-1$, let

$$r_u(\omega)|_{[0,t_{i-1}]} = r(\omega)|_{[0,t_{i-1}]}$$

$$r_u(\omega)|_{[t_{i-1},u]} = \text{minimal geodesic from } \omega(t_{i-1}) \text{ to } \omega(u)$$

$$r_u(\omega)|_{[u,1]} = \omega|_{[u,1]}$$

For $t_{n-1} \leqslant u \leqslant 1$ $(=t_n)$ let

$$r_u(\omega)|_{[0,t_{n-1}]} = r(\omega)|_{[0,t_{n-1}]}$$

$$r_u(\omega)|_{[t_{n-1},t_n-u+t_{n-1}]} = \omega|_{[t_{n-1},t_n-u+t_{n-1}]}$$

$$r_u(\omega)|_{[t_n-u+t_{n-1},t_n]} = \begin{array}{l}\text{the unique geodesic which minimizes}\\ \text{arc-length from } \omega(t_n-u+t_{n-1}) \text{ to } Q.\end{array}$$

Then r_0 is the identity map of $\operatorname{Int}\Omega^c$ and $r_1 = r$. Also, $r_u(\omega)$ is continuous as a function of both variables. This proves B is a deformation retract of $\operatorname{Int}\Omega^c$.

Since $E(r_u(\omega)) \leqslant E(\omega)$ we also have each B^a is a deformation retract of Ω^a.

The critical points of E are the unbroken geodesics which are orthogonal to P and Q and so lie in the submanifold B. Using the first variation formula, the critical points of E' are precisely the unbroken geodesics orthogonal to P and Q [3, p. 216].

Consider the tangent space TB_γ to the manifold B at a geodesic γ orthogonal to P and Q. This will be identified with the space K, defined as follows.

Definition. Let K be the space of vector fields Y along γ such that $Y|_{[0,t_1]}$ is a P-Jacobi field, $Y|_{[t_{i-1},t_i]}$ is a Jacobi field for $i = 2, \ldots, n-1$, and $Y|_{[t_{n-1},1]}$ is a Q-Jacobi field. This identification can be justified as follows. Let

$\overline{\alpha}(-\varepsilon,\varepsilon) \to B$

be any variation of γ through broken geodesics in B. Then the corresponding variation vector field

$\frac{\partial\alpha}{\partial u}(0,t)$

along γ is a broken Jacobi field in K [3, pp. 174, 222]. Then the index of E_{**} at γ is equal to the index of $E_{**}|_{K} = E'_{**}$ [3, p. 232]. This completes the proof of theorem 2.

Theorem 3. Let M be a complete Riemannian manifold and let P and Q be two closed and bounded submanifolds which are not conjugate along any geodesic orthogonal to P and Q of length $\leqslant\sqrt{a}$. Then Ω^a has the homotopy type of a finite CW-complex with one cell of dimension λ for each geodesic orthogonal to P and Q in Ω^a at which E_{**} has index λ.

Proof. Theorem 3 follows from theorem 2 together with theorem 1 and from the fact that the null space of E_{**} consists of the intersection of the spaces of P-Jacobi fields and Q-Jacobi fields [3, p. 221].

Theorem 4. The map $i:\Omega \to \Omega^*$ is a homotopy equivalence.

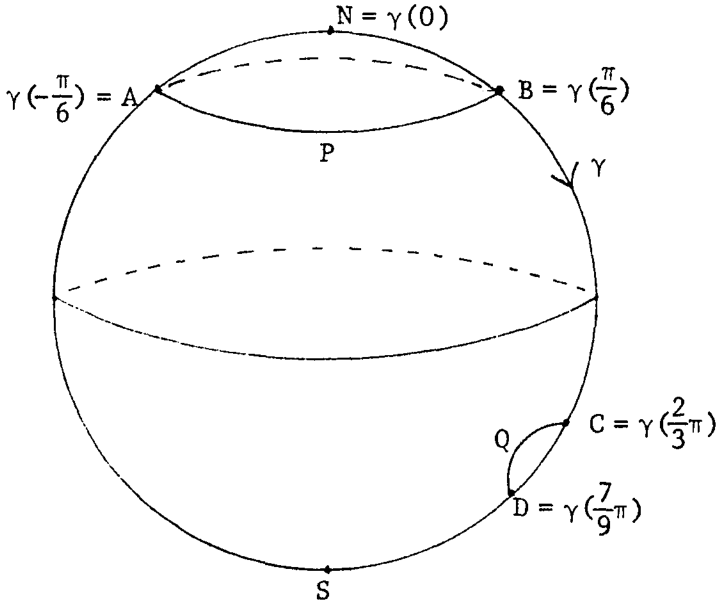
Proof. The proof is exactly the same as the proof of theorem 17.1 in [2, p. 93], substituting $\Omega(M;P,Q)$ for $\Omega(M;p,q)$, $\Omega^*(M;P,Q)$ for $\Omega(M;q,q)$, P for p, and Q for q.

The proof of the main theorem on page 24 above is again exactly the same as Milner's proof of theorem 17.3 [4, p. 95].

§3. Examples

1. The homotopy type of the space of paths joining two nonintersecting circles on the 2-sphere.

Let M be the unit 2-sphere, S^2; $P = S^1$; $Q = S^1$. Consider the circles placed on S^2 as illustrated.



Let $\gamma(t)$ be the geodesic whose points lie on the great circle through N and B and whose direction is clockwise, so that $\gamma(0) = N$; $\gamma(\frac{\pi}{6}) = B$, etc.

Let E(t) be a unit parallel vector field along the geodesic $\gamma(t)$ such that $E(t) \in M_{\gamma(t)}$ and $E(t) \perp \gamma'(t)$.

Let $J(t) = (\sin t)E(t)$ be a Jacobi field along $\gamma(t)$. The first four geodesics from P to Q which are orthogonal to P and Q are listed in order of increasing length.

γ_1; $\gamma(t)$ for $\frac{\pi}{6} \leqslant t \leqslant \frac{2}{3}\pi$ along BC

γ_2; $\gamma(t)$ for $\frac{\pi}{6} \leqslant t \leqslant \frac{7}{9}\pi$ along BD

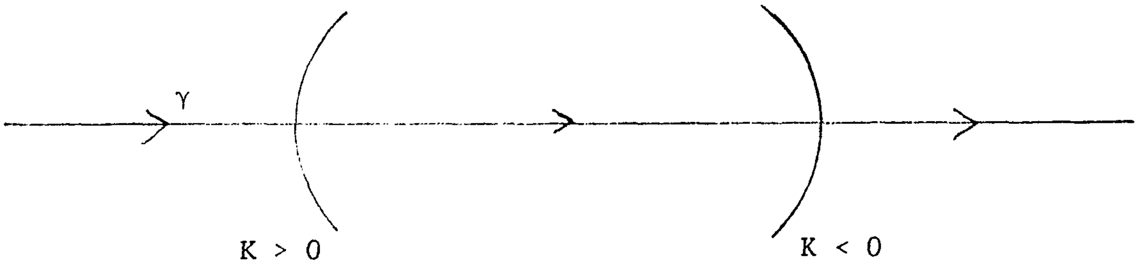
γ_3; $\gamma(t)$ for $-\frac{\pi}{6} \leqslant t \leqslant \frac{2}{3}\pi$ along AC

γ_4; $\gamma(t)$ for $-\frac{\pi}{6} \leqslant t \leqslant \frac{7}{9}\pi$ along AD.

The curvature, K, of a circle on S^2 is given by

$$K = \cot r,$$

where r is the length of arc from the center of the circle on the sphere to the circumference. K is positive when the circle is bending away from the direction of the geodesic and negative when the circle is bending towards the direction of the geodesic.



The computation of $A(J(t_0))$ is found to be

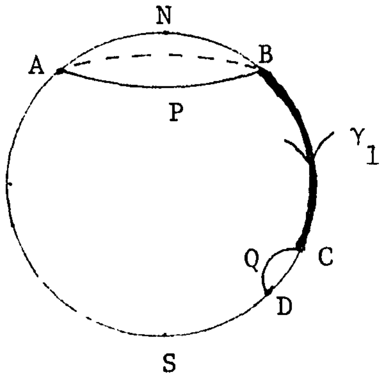
$$\begin{aligned} A(J(t_0)) &= \langle J' - S_Q J, J\rangle\big|_{t_o} = \langle J' - (-K_Q)J, J\rangle\big|_{t_o} \\ &= (\cos t + K_Q \sin t)(\sin t)\langle E(t), E(t)\rangle\big|_{t_o} \\ &= (\cos t_0 + K_Q \sin t_0)(\sin t_0), \end{aligned}$$

where S_Q is the second fundamental form of Q with respect to $\gamma'(t_0)$ at $Q_\gamma(t_0)$.

Let $i(\gamma_j)$ denote the index of I for the geodesic γ_j and let $i(A)$ denote the index of A for γ_j.

The computation of the indices of the geodesics are as follows.

$\underline{\gamma_1}$; BC



The space of P-Jacobi fields is spanned by J(t) for $\frac{\pi}{6} \leqslant t \leqslant \frac{2}{3}\pi$. That J is a Jacobi field follows from

$$\begin{aligned} \langle J' - S_pJ, J\rangle\big|_{\pi/6} &= (\cos\tfrac{\pi}{6} + K_p\sin\tfrac{\pi}{6})(\sin\tfrac{\pi}{6}) \\ &= (\cos\tfrac{\pi}{6} + (-\cot\tfrac{\pi}{6})(\sin\tfrac{\pi}{6}))(\sin\tfrac{\pi}{6}) \\ &= 0. \end{aligned}$$

This shows $(J' - S_pJ)\big|_{\pi/6} \perp P_{\gamma(\pi/6)}$.

$$\begin{aligned} A(J(\tfrac{2}{3}\pi)) &= (\cos\tfrac{2}{3}\pi + K_Q\sin\tfrac{2}{3}\pi)(\sin\tfrac{2}{3}\pi) \\ &\cong (\cos\tfrac{2}{3}\pi + (5.7)\sin\tfrac{2}{3}\pi)(\sin\tfrac{2}{3}\pi) \quad \left(\cot\tfrac{\pi}{18} \cong 5.7\right) \\ &\cong (-.5 + (5.7)(.8))(.8) > 0. \end{aligned}$$

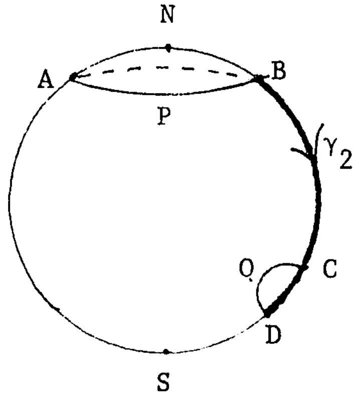
Since A is positive on the span of the P-Jacobi field (there is only one in this case) we have i(A) = 0. The P-focal points occur where J(t) vanishes, which will be at $n\pi$; $n = 0, 1, 2, \ldots$ The multiplicity at each P-focal point is one. Therefore, J has no P-focal points along γ_1. This yields

$$\begin{aligned} i(\gamma_1) &= (\text{number of P-focal points with multiplicity}) + i(A) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Each time we extent γ_1 a complete revolution we obtain another geodesic joining P and Q which is orthogonal to P and Q, and for each complete revolution we pick up two P-focal points. We will denote the geodesics by $\gamma_1 + n(2\pi)$ for $n = 0, 1, 2, \ldots$ The boundary term at Q remains the same for all these geodesics. Therefore, we have

$$\begin{aligned} i(\gamma_1 + n(2\pi)) &= 2n + i(A) \\ &= 2n + 0 \\ &= 2n. \end{aligned}$$

$\underline{\gamma_2}$; BD



A P-Jacobi field along γ_2 is given by

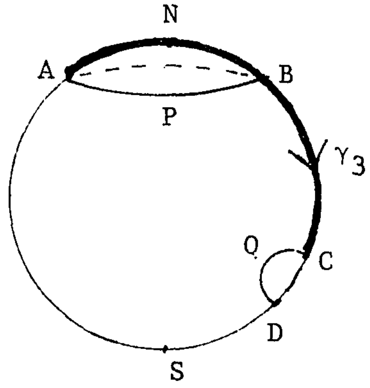
$$J(t) \quad \text{for} \quad \frac{\pi}{6} \leqslant t \leqslant \frac{7}{9}\pi .$$

$$A(J(\frac{7}{9}\pi) = (\cos\frac{7}{9}\pi + K_Q \sin\frac{7}{9}\pi)(\sin\frac{7}{9}\pi)$$
$$\cong (-.8 + (-5.7)(.6))(.6) < 0.$$

So $i(A) = 1$ and there are no P-focal points. We have

$$i(\gamma_2) = 0 + 1 = 1$$
$$i(\gamma_2 + n(2\pi)) = 2n + 1.$$

$\underline{\gamma_3}$; AC



A P-Jacobi field along γ_3 is given by

$$J(t) \quad \text{for} \quad -\frac{\pi}{6} \leqslant t \leqslant \frac{2}{3}\pi .$$

That J is a P-Jacobi field on this interval follows from

$$\langle J' - S_P J, J\rangle\big|_{-\pi/6} = (\cos(-\tfrac{\pi}{6}) + \cot(\tfrac{\pi}{6})\sin(-\tfrac{\pi}{6}))(\sin-\tfrac{\pi}{6})$$
$$= 0.$$

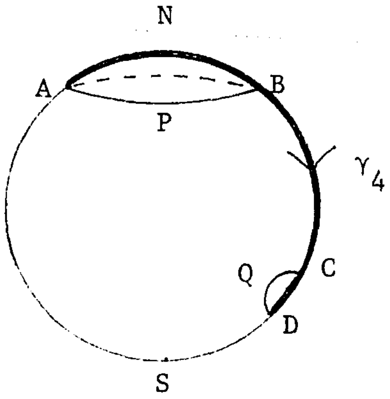
$A(J(\frac{2}{3}\pi)) > 0$ from the calculation for γ_1, so we get $i(A) = 0$. There is one P-focal point which occurs at the north pole. We have

$$i(\gamma_3) = 1+0 = 1$$

and

$$i(\gamma_3 + n(2\pi)) = 2n+1.$$

$\underline{\gamma_4}$; AD



A P-Jacobi field is given by

$$J(t) \quad \text{for} \quad -\frac{\pi}{6} \leqslant t \leqslant \frac{7}{9}\pi.$$

$A(J(\frac{7}{9}\pi)) < 0$ from the calculation for γ_2, so we get $i(A) = 1$. There is one P-focal point at the north pole. So we have

$$i(\gamma_4) = 1+1 = 2$$

and

$$i(\gamma_4 + n(2\pi) = 2n + 2.$$

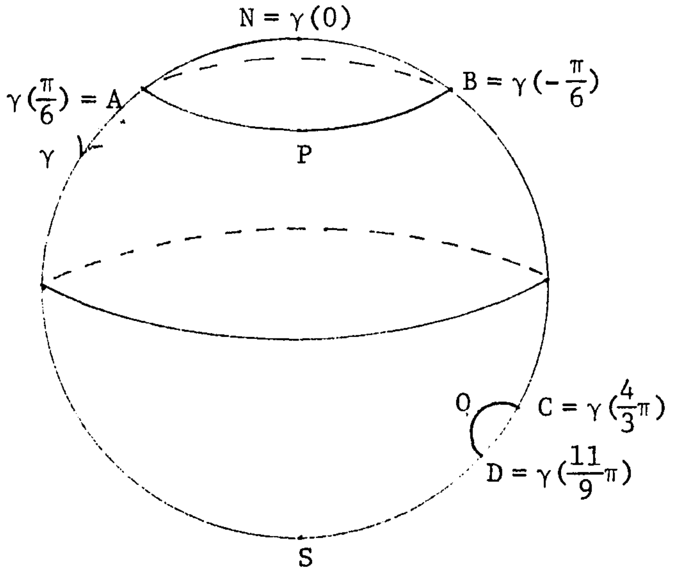
For the calculation of the next four orthogonal geodesics we redefine $\gamma(t)$, $E(t)$, and $J(t)$. Let $\gamma(t)$ be the geodesic whose points lie on the great circle through N and A and whose direction is counter-clockwise, so that $\gamma(0) = N$; $\gamma(\frac{\pi}{6}) = A$, etc. Let $E(t)$ be a unit parallel vector field along $\gamma(t)$ such that $E(t) \in M_{\gamma(t)}$ and $E(t) \perp \gamma'(t)$. Let $J(t) = (\sin t)E(t)$. The next four orthogonal geodesics from P to Q, listed in order of increasing length, are

γ_5; $\gamma(t)$ for $\frac{\pi}{6} \leqslant t \leqslant \frac{11}{9}\pi$ along $-AD$

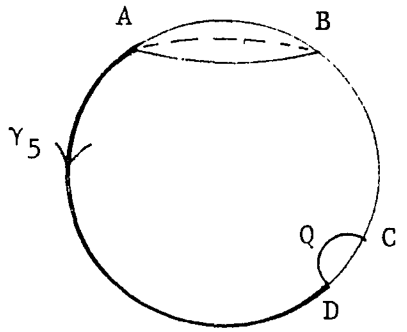
γ_6; $\gamma(t)$ for $\frac{\pi}{6} \leqslant t \leqslant \frac{4}{3}\pi$ along $-AC$

γ_7; $\gamma(t)$ for $-\frac{\pi}{6} \leqslant t \leqslant \frac{11}{9}\pi$ along $-BD$

γ_8; $\gamma(t)$ for $-\frac{\pi}{6} \leqslant t \leqslant \frac{4}{3}\pi$ along $-BC$.



$\underline{\gamma_5}$; $-AD$



A P-Jacobi field is given by

$J(t)$ for $\frac{\pi}{6} \leqslant t \leqslant \frac{11}{9}\pi$.

$$A(J(\tfrac{11}{9}\pi)) \cong (\cos\tfrac{11}{9}\pi + (5.7)(\sin\tfrac{11}{9}\pi))(\sin\tfrac{11}{9}\pi)$$

$$\cong (-.8 + (5.7)(-.6))(-.6)$$

$$> 0.$$

Hence, $i(A) = 0$, and there is one P-focal point at the south pole.

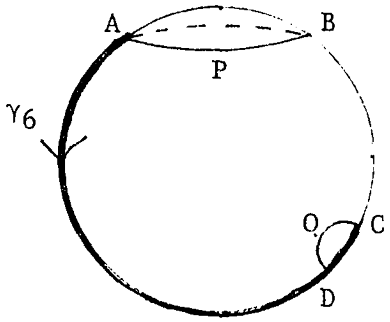
Therefore, we have

$$i(\gamma_5) = 1+0 = 1$$

and

$$i(\gamma_5 + n(2\pi)) = 2n + 1.$$

$\underline{\gamma_6}$; $-AC$



A P-Jacobi field is given by

$$J(t) \quad \text{for} \quad \frac{\pi}{6} \leqslant t \leqslant \frac{4}{3}\pi.$$

$$A(J(\tfrac{4}{3}\pi)) \cong (\cos(\tfrac{4}{3}\pi) + (-5.7)\sin\tfrac{4}{3}\pi)(\sin\tfrac{4}{3}\pi)$$

$$\cong (-.5 + (-5.7)(-.8)(-.8)$$

$$< 0.$$

So $i(A) = 1$, and there is one P-focal point at the south pole.

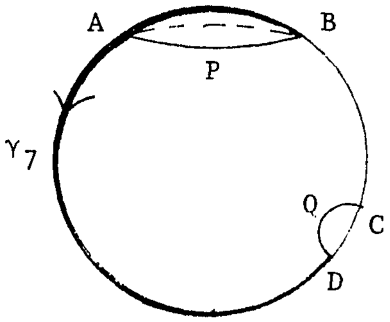
Therefore, we have

$$i(\gamma_6) = 1+1 = 2$$

and

$$i(\gamma_6 + n(2\pi)) = 2n + 2.$$

$\underline{\gamma_7}$; $-BD$



A P-Jacobi field is given by

$$J(t) \quad \text{for} \quad -\frac{\pi}{6} \leqslant t \leqslant \frac{11}{9}\pi.$$

$$A(J(\frac{11}{9}\pi) > 0$$

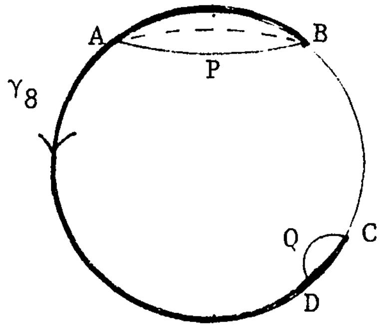
from the calculation for γ_5. So $i(A) = 0$, and there are two P-focal points--one at the north pole and one at the south pole. Therefore, we have

$$i(\gamma_7) = 2 + 0 = 2$$

and

$$i(\gamma_7 + n(2\pi)) = 2n + 2.$$

$\underline{\gamma_8}$; $-BC$



A P-Jacobi field is given by

$$J(t) \quad \text{for} \quad -\frac{\pi}{6} \leqslant t \leqslant \frac{4}{3}\pi.$$

$$A(J(\tfrac{4}{3}\pi)) < 0$$

from the calculation for γ_6. So $i(a) = 1$, and there are two P-focal points--one at the north pole and one at the south pole. Therefore, we have

$$i(\gamma_8) = 2+1 = 3$$

and

$$i(\gamma_5 + n(2\pi)) = 2n + 3.$$

Putting together all of the previous computations, we have:

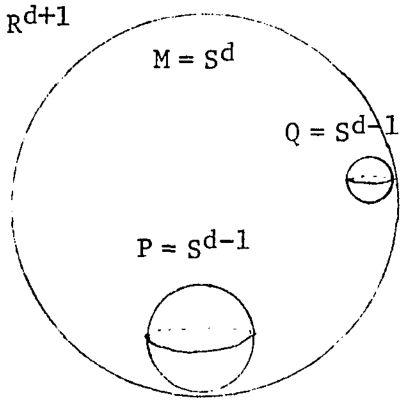
	n = 0	n = 1	n = 2	...	etc.
$i(\gamma_1 + n(2\pi)) = 2n$	0	2	4		
$i(\gamma_2 + n(2\pi)) = 2n+1$	1	3	5		
$i(\gamma_3 + n(2\pi)) = 2n+1$	1	3	5		
$i(\gamma_4 + n(2\pi)) = 2n+2$	2	4	6		
$i(\gamma_5 + n(2\pi)) = 2n+1$	1	3	5		
$i(\gamma_6 + n(2\pi)) = 2n+2$	2	4	6		
$i(\gamma_7 + n(2\pi)) = 2n+2$	2	4	6		
$i(\gamma_8 + n(2\pi)) = 2n+3$	3	5	7		

Therefore, the homotopy type of the space of paths joining two non-intersecting circles on S^2 is the countable CW-complex given by:

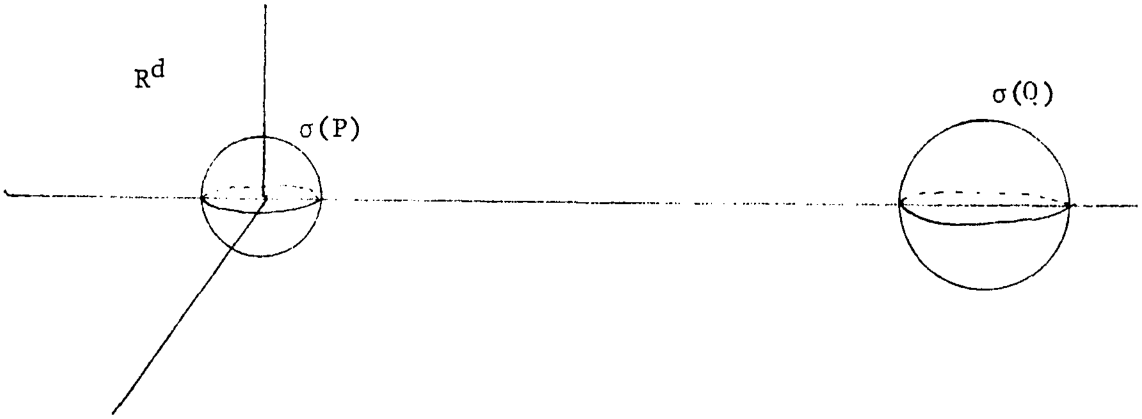
$$\begin{aligned} & e^0 \cup e^1 \cup e^1 \cup e^2 \cup e^1 \cup e^2 \cup e^2 \cup e^3 \\ & \cup e^2 \cup e^3 \cup e^3 \cup e^4 \cup e^3 \cup e^4 \cup e^4 \cup e^5 \\ & \cup e^4 \cup e^5 \cup \ldots \text{etc.} \end{aligned}$$

Remark. This computation is consistent with a similar computation done by Professor Vasquez using methods of algebraic topology.

2. The homotopy type of the space of paths joining two non-intersecting (d-1)-spheres on a d-sphere. $M = S^d$; $P = S^{d-1}$; $Q = S^{d-1}$.



$\sigma \downarrow$ stereographic projection



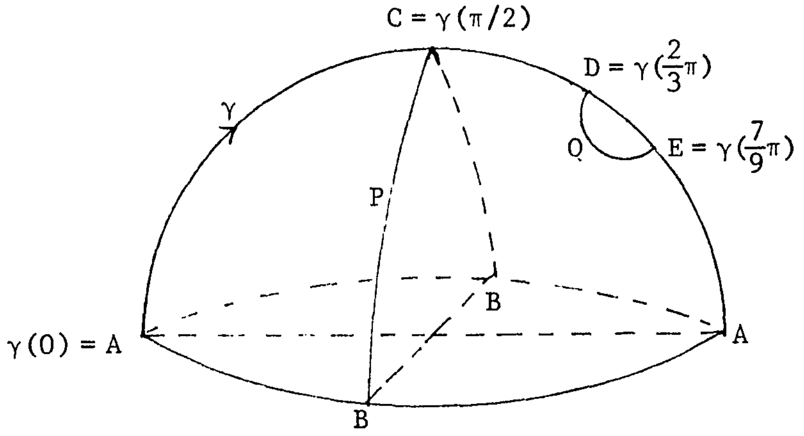
We place S^d with its south pole at the origin in R^{d+1} and S^{d-1} with its center along the x_{d+1} axis and the second S^{d-1} with its center not along the x_{d+1} axis. Let σ be stereographic projection which sends the south pole of S^d to the origin in R^d and the north pole to the point at infinity. This tells us there is one great circle on S^d which is orthogonal to P and Q. It is the one which σ sends into that coordinate axis which joins $\sigma(P)$ and $\sigma(Q)$ and is orthogonal to both. Along this geodesic we have (d-1) orthonormal parallel vector fields $E_1, \ldots, E_{d-1}$ and (d-1) Jacobi fields given by $J_i(t) = (\sin t)E_i(t)$; $i = 1, \ldots, d-1$, which are linearly independent. We will obtain the same computations as in example 1 for the index of A for each of the Jacobi fields, but this time for every complete revolution of a geodesic we will increase the number of P-focal points by 2(d-1). Therefore, we will have

$$\begin{aligned}
i(\gamma_1 + n(2\pi)) &= 2n(d-1)\\
i(\gamma_2 + n(2\pi)) &= (2n+1)(d-1)\\
i(\gamma_3 + n(2\pi)) &= (2n+1)(d-1)\\
i(\gamma_4 + n(2\pi)) &= (2n+2)(d-1)\\
i(\gamma_5 + n(2\pi)) &= (2n+1)(d-1)\\
i(\gamma_6 + n(2\pi)) &= (2n+2)(d-1)\\
i(\gamma_7 + n(2\pi)) &= (2n+2)(d-1)\\
i(\gamma_8 + n(2\pi)) &= (2n+3)(d-1).
\end{aligned}$$

In the special case for $d = 3$ we get $M = S^3$, $P = S^2$, and $Q = S^2$. The homotopy type of this path space will be

$$\begin{aligned}
&e^0 \cup e^2 \cup e^2 \cup e^4 \cup e^2 \cup e^4 \cup e^4 \cup e^6\\
&\cup e^4 \cup e^6 \cup e^6 \cup e^8 \cup e^6 \cup e^8 \cup e^8 \cup e^{10}\\
&\cup e^8 \cup \ldots \quad \text{etc.}
\end{aligned}$$

3. The homotopy type of the space of paths joining real projective space, P^1, to a non-intersecting circle, S^1, in real projective space P^2.



$M = P^2$; $P = P^1$;
$Q = S^1$

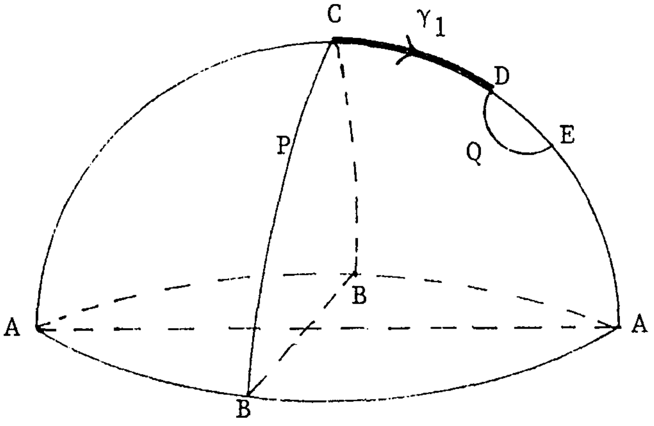
Let γ be the geodesic whose points lie along the great arc which passes through A and C such that $\gamma(0) = A$; $\gamma(\frac{\pi}{2}) = C$, and $\gamma(\frac{2}{3}\pi) = D$. Let E be a unit parallel vector field along γ and orthogonal to γ such that $E(t) \in P^2_{\gamma(t)}$. Let $J(t) = \sin t\, E(t)$. $J(t)$ vanishes for $t = 0, \pi, 2\pi, \ldots$ At $J(\frac{\pi}{2})$ we have

$$\begin{aligned} \langle J' - S_p J, J\rangle\Big|_{\pi/2} &= (\cos\tfrac{\pi}{2} + K_p \sin\tfrac{\pi}{2})(\sin\tfrac{\pi}{2}) \\ &= 0 \end{aligned}$$

since $K_p = \cot\frac{\pi}{2} = 0$.

The first four geodesics, orthogonal to P and Q, are listed in order of increasing length and their indices are computed.

$\underline{\gamma_1}$; CD



P-Jacobi field:

$$J(t);\ \frac{\pi}{2} \leqslant t \leqslant \frac{2}{3}\pi.$$

$$\begin{aligned} A(J(\tfrac{2}{3}\pi)) &\cong (\cos\tfrac{2}{3}\pi + (5.7)\sin\tfrac{2}{3}\pi)(\sin\tfrac{2}{3}\pi) \\ &\cong (-.5 + (5.7)(.9))(.9) \\ &> 0, \end{aligned}$$

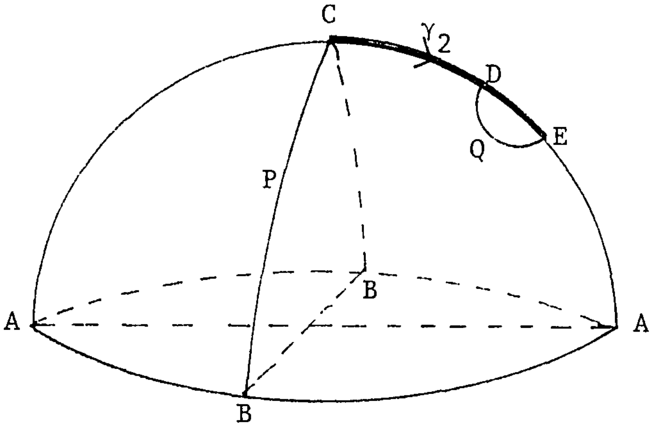
so $i(A) = 0$.

$$\begin{aligned} i(\gamma_1) &= (\text{number of P-focal points with multiplicity}) + i(A) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

and $i(\gamma_1 + n\pi) = n$, since each time γ_1 is extended a complete revolution, only one P-focal point is added to the index, while

$$A(J(t+\pi) = A(J(t)).$$

$\underline{\gamma_2}$; CE



P-Jacobi field:

$$J(t);\ \frac{\pi}{2} \leqslant t \leqslant \frac{7}{9}\pi.$$

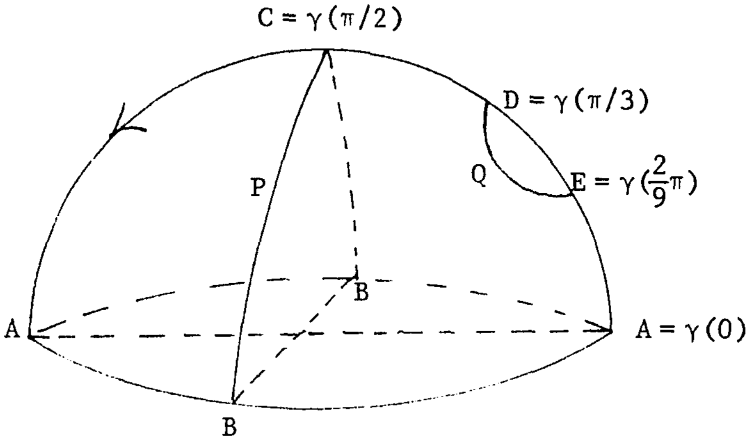
$$\begin{aligned} A(J(\tfrac{7}{9}\pi) &\cong (\cos\tfrac{7}{9}\pi + (-5.7)\sin\tfrac{7}{9}\pi)(\sin\tfrac{7}{9}\pi) \\ &\cong (-.8 - 4.7(.6))(.6) \\ &< 0, \end{aligned}$$

so $i(A) = 1$.

$$\begin{aligned} i(\gamma_2) &= 0 + 1 \\ &= 1 \end{aligned}$$

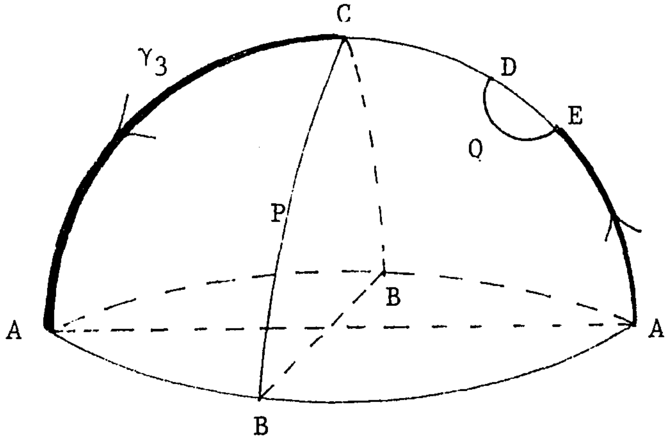
and

$$i(\gamma_2 + n\pi) = n + 1.$$



Define a different $\gamma(t)$; $E(t)$; $J(t)$. Let $\gamma(t)$ be the geodesic whose points lie on the great arc joining A and D such that $\gamma(0) = A$ and $\gamma(\frac{2}{9}\pi) = E$. Let $E(t)$ be a unit parallel vector field along $\gamma(t)$ and orthogonal to $\gamma(t)$ such that $E(t) \in P^2_{\gamma(t)}$, and let $J(t) = (\sin t)E(t)$.

<u>γ_3</u>; $-CE$



P-Jacobi field:

$$J(t);\ \frac{\pi}{2} \leqslant t \leqslant \frac{11}{9}\pi.$$

$$A(J(\tfrac{11}{9}\pi)) \cong (\cos\tfrac{11}{9}\pi + (5.7)(\sin\tfrac{11}{9}\pi))(\sin\tfrac{11}{9}\pi)$$

$$\cong (-.8 + (5.7)(-.6))(-.6)$$

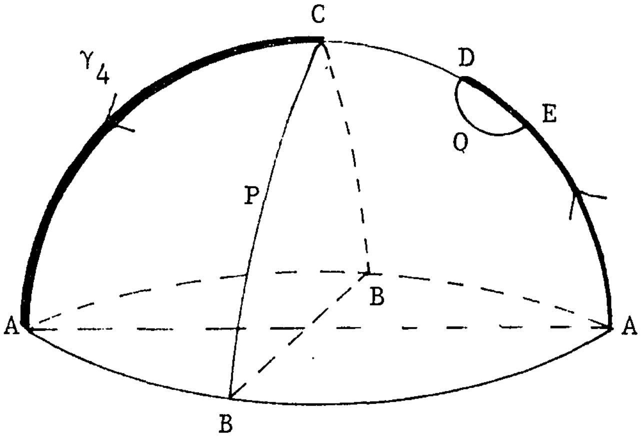
$$> 0,$$

so $i(A) = 0$, and there is one P-focal point at A.

$$i(\gamma_3) = 1 + 0$$

$$i(\gamma_3 + n\pi) = n + 1.$$

γ_4; -CD



P-Jacobi field:

$$J(t);\ \tfrac{\pi}{2} \leqslant t \leqslant \tfrac{4}{3}\pi.$$

$$A(J(\tfrac{4}{3}\pi)) \cong (\cos\tfrac{4}{3}\pi + (-5.7)\sin\tfrac{4}{3}\pi)(\sin\tfrac{4}{3}\pi)$$

$$\cong (-.5 + (-5.7)(-.8)(-.8)$$

$$< 0,$$

so $i(A) = 1$.

$$i(\gamma_4) = 1 + 1 = 2$$

and

$$i(\gamma_4 + n\pi) = n + 2.$$

Putting together the computations we have:

	n=0	n=1	n=2	... etc.
$i(\gamma_1 + n\pi) = n$	0	1	2	
$i(\gamma_2 + n\pi) = n+1$	1	2	3	
$i(\gamma_3 + n\pi) = n+1$	1	2	3	
$i(\gamma_4 + n\pi) = n+2$	2	3	4	

Therefore, the homotopy type of the space of paths joining P^1 to a non-intersecting S^1 in P^2 is given by:

$e^0 \cup e^1 \cup e^1 \cup e^2$
$\cup e^1 \cup e^2 \cup e^2 \cup e^3$
$\cup e^2 \cup e^3 \cup e^3 \cup e^4$
$\cup e^3 \cup e^4 \cup$... etc.

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