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ALGEBRAIC PROPERTIES OF THE  
REPRESENTATION RING OF COMPACT CONNECTED LIE GROUPS

BY

*ilton*  
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## TABLE OF CONTENTS

	<u>PAGE NUMBER</u>
ACKNOWLEDGMENTS	iii
CHAPTER I (Preliminaries)	1
Section 1: The Representation Ring	1
Section 2: The Weyl Group	4
Section 3: Alternating Elements	6
Section 4: Weyl Character Formula	7
Section 5: A Generalization	8
Section 6: Main Theorem	9
Section 7: Another Approach	11
Section 8: An Application	12
CHAPTER II (Algebraic Background)	13
Section 1: Galois Theoretic Properties of Ring Extensions	13
Section 2: An Observation	17
CHAPTER III ( $R(T)$ as an $R(G)$ -Module )	18
Section 1: $SU(n)$ and $U(n)$	18
Section 2: $Sp(n)$ and $SO(2n+1)$	21
Section 3: $SO(2n)$	24
Section 4: $Spin(n)$	25
Section 5: $G_2$	35
CHAPTER IV (The Bilinear Form)	36
Section 1: Definitions	36
Section 2: The Inductive Process	44
CHAPTER V (Computation of the Bilinear Form)	48
Section 1: $SU(n)$ and $U(n)$	48
Section 2: $Sp(n)$	54
Section 3: $Spin(n)$ and $Spin^c(2K)$	56
Section 4: $Spin^c(2K+1)$	72
CHAPTER VI (An Observation)	75
BIBLIOGRAPHY	77
AUTOBIOGRAPHY	78

## CHAPTER I

## (Preliminaries)

Section 1.

Let  $G$  be a compact connected Lie group, and  $T$  a maximal torus of  $G$ . If  $H$  is any connected subgroup of  $G$  which contains a maximal torus, then  $H$  is a subgroup of maximal rank. For a given  $G$ ,  $H$ , and  $T \leq H$ , as above, we have the following diagram:  $T \xrightarrow{\alpha} H \xrightarrow{\beta} G$ , where  $\alpha$  and  $\beta$  are the inclusion maps. Let  $i : T \rightarrow G$  be the map  $\beta \circ \alpha$ .

$R(G)$  will denote the Complex Representation Ring of  $G$ , which is defined as follows: Let  $S$  be the set of all equivalence classes of finite dimensional irreducible representations,  $\rho : G \rightarrow GL(n, \mathbb{C})$ , where by  $\rho \sim \rho'$  we mean that there exists a linear automorphism,  $\varphi$ , of  $\mathbb{C}^n \ni$  the following diagram commutes for all  $g \in G$ :

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\rho(g)} & \mathbb{C}^n \\ \varphi \downarrow & & \downarrow \varphi \\ \mathbb{C}^n & \xrightarrow{\rho'(g)} & \mathbb{C}^n \end{array}$$

(Note: We will always pick an irreducible representation to represent an equivalence class, and all future constructions will be invariant of this choice.)

$R(G)$  is the free abelian group on the set  $S$ . Ring multiplication is defined by taking tensor products of representations as follows: Let

$\rho_1 : G \longrightarrow GL(n, \mathcal{C})$  and  $\rho_2 : G \longrightarrow GL(m, \mathcal{C})$ ,  $\rho_1$  and  $\rho_2$  irreducible. Then form  $\mathcal{C}^n \otimes_{\mathcal{C}} \mathcal{C}^m$ , on which  $G$  acts by  $\rho_1 \otimes \rho_2(g) (v_1 \otimes v_2) = \rho_1(g) v_1 \otimes \rho_2(g) v_2$ , for  $v_1 \in \mathcal{C}^n$ ,  $v_2 \in \mathcal{C}^m$ . If  $\rho_1 \otimes \rho_2$  is irreducible then by the tensor product of  $\rho_1$  and  $\rho_2$  we mean the equivalence class of  $\rho_1 \otimes \rho_2$ . If  $\rho_1 \otimes \rho_2$  is reducible, then, since  $G$  is compact  $\rho_1 \otimes \rho_2 = \bar{\rho}_1 + \dots + \bar{\rho}_k$ , where the  $\bar{\rho}_i$  are irreducible representations. (See [1; 3.20, 3.24]). In this case the tensor product means the sum of the equivalence classes of the  $\bar{\rho}_i$ . By linearity we can extend this definition to any two elements of  $R(G)$ .

Let  $h : G \longrightarrow G'$  be a Lie group homomorphism, then  $h$  induces a homomorphism  $h^* : R(G') \longrightarrow R(G)$ . Let  $\rho$  be an irreducible representation of  $G'$ ,  $\rho : G' \longrightarrow GL(n, \mathcal{C})$ . Then  $\rho \circ h : G \longrightarrow GL(n, \mathcal{C})$  is a representation of  $G$ . As in the previous case, if  $\rho \circ h$  is reducible then write it as  $\bar{\rho}_1 + \dots + \bar{\rho}_s$ , where the  $\bar{\rho}_i$  are irreducible. Then  $h^*(\rho) = \bar{\rho}_1 + \dots + \bar{\rho}_s$ , an element of  $R(G)$ . By linearity we can extend  $h^*$  to act on all of  $R(G')$ . In particular if we consider the inclusion homomorphisms  $\alpha : T \longrightarrow H$  and  $\beta : H \longrightarrow G$ , then  $\alpha^*$  and  $\beta^*$  are monomorphisms. (See [9]). We therefore think of  $R(G)$  as a subring of  $R(H)$ . (i.e.,  $R(G) \leq R(H) \leq R(T)$ ).

Let  $W(G)$ , the Weyl Group of  $G$ , be defined as follows:  $W(G)$  is the group of automorphisms of  $T$ , a maximal torus of  $G$ , induced by those inner automorphisms of  $G$  which map  $T$  into  $T$ . Since all maximal tori of  $G$  are conjugates,  $W(G)$  is well defined [1; 4.23]. It is also quite easy to show that  $W(G) \cong \frac{N(T, G)}{T}$ , where  $N(T, G)$  is the normalizer of  $T$  in  $G$  [1; 4.29].  $T$  is also a maximal torus of  $H$ , and therefore we can define  $W(H) \cong \frac{N(T, H)}{T}$ . Furthermore,  $N(T, H) \leq N(T, G)$ , which implies that  $W(H) \leq W(G)$ .

Every element,  $\theta$ , of  $W(G)$  is an automorphism of  $T$ , and therefore induces  $\theta^* : R(T) \rightarrow R(T)$ . So, we can consider  $W(G)$  as a group of automorphisms of the Representation Ring of  $T$ . Any inner automorphism of  $G$  induces the identity map on  $R(G)$  (see [9]). Therefore  $W(G)$  acts trivially on  $R(G)$ , thought of as a subring of  $R(T)$ . In fact  $R(G) = R(T)^{W(G)}$ , where  $R(T)^{W(G)}$  is the fixed subring of  $R(T)$  under the action of  $W(G)$ . ([4]). Similarly,  $R(H) = R(T)^{W(H)}$ .

Section 2.

We now wish to describe the Weyl Group in another manner. In order to do so, we must introduce the concept of the roots of  $G$ .

Let  $\text{Ad} : T \rightarrow \text{Aut}(G_e)$  be the homomorphism induced from  $a : T \rightarrow \text{Aut}(G)$ , where  $a(t)(g) = tgt^{-1}$ ,  $t \in T$  and  $g \in G$ . By  $G_e$ , we mean the tangent space of  $G$  at the identity.  $G_e$  is identified, as usual, with  $L(G)$ , the Lie algebra of  $G$ .

Definition 1.1. The integer lattice of  $L(T)$  is  $\exp^{-1}(\text{Id}_T)$ , where  $\exp : L(T) \rightarrow T$  is the exponential map.

Since  $T$  is connected and abelian, the exponential map is a surjective homomorphism of groups. [1,2.19]. Therefore, we can identify  $T$  with  $L(T)/\text{integer lattice}$ .

Let us note the following important theorem:

Theorem 1.2 [1,4.12]

$L(G) = G_e$  splits in the form  $V_0 \oplus_{i=1}^m V_i$ , where  $\text{Ad}(T)$  acts trivially on  $V_0$ ,  $\dim(V_i) = 2$  for  $i > 0$ , and  $\text{Ad}(T)|_{V_i}$  acts as 
$$\begin{pmatrix} \cos 2\pi \theta_i(t) & -\sin 2\pi \theta_i(t) \\ \sin 2\pi \theta_i(t) & \cos 2\pi \theta_i(t) \end{pmatrix}$$
, where  $\theta_i : T \rightarrow \mathbb{R}/\mathbb{Z}$  is given by the linear form  $\theta_i : L(T) \rightarrow \mathbb{R}$ , taking integer values on the integer lattice, and no  $\theta_i$  is zero.

Definition 1.3. The set  $\{\pm\theta_i\}_{i=1}^m$  is called the set of roots of  $G$ .

If we view the  $\theta_i$  as linear forms on  $L(T) = \mathbb{R}^n$ , we can look at the  $(n-1)$ -dimensional hyperplane  $L(U_i)$ , defined as follows:  $L(U_i) = \{v \in \mathbb{R}^n \mid \theta_i(v) = 0\}$ . Let us note that this is just the Lie algebra of the closed subgroup of  $T$ ,  $U_i$ , defined by  $U_i = \{t \in T \mid \theta_i(t) \equiv 0 \pmod{1}\}$ .

These hyperplanes divide  $L(T) = \mathbb{R}^n$  into chambers, called the Weyl Chambers of  $G$ . We arbitrarily choose one of those chambers and call it the Fundamental Weyl Chamber. We say that a root,  $\theta$ , is positive if  $\theta(v) \geq 0$  for  $v$  in the F.W.C., and negative if  $\theta(v) \leq 0$  for all  $v \in \text{F.W.C.}$  This process divides the roots into 2 disjoint sets.

Theorem 1.4 [1,5.13]

- (1)  $W(G)$  permutes the Weyl Chambers simply transitively.
- (2) For each  $r$ ,  $1 \leq r \leq m$ ,  $W(G)$  includes the reflection in the plane  $L(U_r)$
- (3) Such reflections generate  $W(G)$ .

Note: We choose an inner product defined on  $L(G)$  which is invariant under  $\text{Ad}(g)$ , for all  $g \in G$ . By a reflection we mean relative to the norm induced by this inner product [1,5.12].

Section 3.

Since  $W(G)$  is generated by reflections in hyperplanes of  $L(T)$ ,  $W(G)$  also acts as a group of linear automorphisms of  $L(T) = \mathbb{R}^n$ . We can therefore make the following definitions:

Definition 1.5: Let  $\varphi \in W(G)$ , then  $\text{sgn}(\varphi) \equiv \text{sign of the determinant of the linear automorphism, } \varphi$ .

We call  $x \in \mathbb{R}(T)$  symmetric if  $\varphi(x) = x$  for all  $\varphi$  in  $W(G)$ , and  $x$  is anti-symmetric if  $\varphi(x) = (\text{sgn}\varphi)x$  for all  $\varphi$  in  $W(G)$ .

Any linear form,  $w : L(T) \rightarrow \mathbb{R}^1$ , which takes integer values on the integer lattice can be viewed as an element of  $\mathbb{R}(T)$ . It is just the 1-dimensional representation mapping  $t$  to multiplication by  $e^{2\pi i w(t)}$ . Such  $w$ 's are called weights of  $T$ . (e.g., the roots of  $G$  are weights.)

Given any  $x$  in  $\mathbb{R}(T)$ , let  $A(x) = \sum_{\varphi \in W(G)} (\text{sgn}\varphi) \varphi(x)$ . We call  $A(x)$  the elementary alternating sum of  $x$ . It is clear that for all  $x$  in  $\mathbb{R}(T)$ ,  $A(x)$  is anti-symmetric.

Proposition 1.6 [1,6.12]

Let  $w$  be a weight, then either  $A(e^{2\pi i w})$  contains  $|W(G)|$  distinct terms, or  $A(e^{2\pi i w}) = 0$ .

Similarly, we define  $S(x) = \sum_i x_i$ , where the  $x_i$  range over the orbit of  $x$  under  $W(G)$ .  $S(x)$  is called the elementary symmetric sum of  $x$ .

Notation: If  $w$  is a weight, then by  $A(w)$  we mean  $A(e^{2\pi i w})$ .

Section 4.

Previously, we introduced the homomorphism  $\text{Ad} : T \longrightarrow \text{Aut}(G_e)$  .  
 If we look at the more general map,  $\text{Ad} : G \longrightarrow \text{Aut}(G_e)$  , then the image  
 is contained in  $\text{SO}(N)$  , where  $\dim_{\mathbb{R}} G_e = N$  . [1,5.56]. Since  
 $\pi_1(\text{SO}(N)) \cong \mathbb{Z}_2$  ,  $N \geq 3$  ,  $\text{SO}(N)$  has a simply connected double cover.  
 This simply connected group is called  $\text{Spin}(N)$  , the Spinor Group of dimen-  
 sion  $\frac{N^2 - N}{2}$  . For a more detailed description see [9] . If  
 $\text{Ad}_*(\pi_1(G)) = 0$  , then by elementary covering space theory,  $\text{Ad}$  lifts  
 to  $\text{Spin}(N)$  . (i.e., there exists a homomorphism  $\tilde{\text{Ad}} : G \longrightarrow \text{Spin}(N)$   
 such that  $\text{Ad} = \pi \circ \tilde{\text{Ad}}$  , where  $\pi : \text{Spin}(N) \longrightarrow \text{SO}(N)$  is the covering  
 map.)

Theorem 1.7 [1,5.57]

If  $\text{Ad}$  lifts to  $\text{Spin}$  , then  $\beta_G = \frac{1}{2} \sum_{i=1}^m \theta_i$  is a weight, where  
 $\theta_i$  ranges over the positive roots.

We are interested in  $\delta \equiv A(\beta_G)$  , for if  $\text{Ad}$  lifts to  $\text{Spin}$  , we  
 get the following strong and important theorem:

Theorem 1.8 [1,6.6, 6.16, 6.18]

- (1) If  $x$  is symmetric then  $x \longrightarrow x \cdot \delta$  gives an isomorphism  
 onto from the additive group of symmetric elements to the  
 additive group of anti-symmetric elements.
- (2) Let  $x$  be an irreducible complex representation of  $G$  , and  
 think of  $R(G) \leq R(T)$  , then  $x \cdot \delta = A(y)$  for some weight  $y$  .
- (3) Conversely, every  $A(y)$ ,  $y$  is a weight, is equal to  $\frac{1}{2} x \cdot \delta$   
 for some irreducible  $x$  in  $R(G)$  .

Section 5.

A more general approach can be found in [4]. The claim there is that if  $\pi_1(G)$  has no 2-torsion, then there exists an alternating element  $\Omega(G) \ni$ , Theorem 1.8 is true if we substitute  $\Omega(G)$  for  $\delta$ . Some hints are given of a possible proof, but no proof has ever appeared in print. We wish to make use of this more general approach in this paper, and will therefore offer a proof of this fact later on.

Notation: The notation in [4] is slightly different from the one used here. Let us therefore make note of the following: The term irreducible G-Module is used instead of irreducible representation of G.  $A(G)$  stands for the alternating elements and  $I(G)$  is used for the subring of symmetric elements. Instead of considering  $R(G) \leq R(T)$ , the notation  $i^*(M) \in R(T)$  is used, where  $M$  is a G-Module and  $i^* : R(G) \xrightarrow{1-1} R(T)$ . With this in mind, we will freely translate all future statements appearing in [4] into our notation.

Section 6.

We have  $R(G) \leq R(H) \leq R(T)$  , and we can therefore consider each overring as a module over a subring. (e.g.,  $R(T)$  is an  $R(G)$ -Module.) If both  $G$  and  $H$  are such that  $\pi_1(G)$  and  $\pi_1(H)$  have no 2-torsion, then by the generalization of Theorem 1.8 ,  $\Omega(G)$  and  $\Omega(H)$  can be defined. It also follows from the Theorem that for any  $x$  in  $R(T)$  ,  $\frac{A(x)}{\Omega(G)}$  is an element of  $R(G)$  , where  $A$  denotes "alternating over  $W(G)$ ."

The analogous statement about  $\Omega(H)$  and  $R(H)$  is also true.

Let  $z$  be an element of  $R(H)$  and let  $\Omega(H)$  be as above. Let us look at the term  $A(\Omega(H) \cdot z) = \sum_{\varphi \in W(G)} \text{sgn}(\varphi) \varphi(\Omega(H) \cdot z)$  . Let  $\text{Id} = \alpha_0, \alpha_1, \dots, \alpha_k$  denote the elements of  $W(H)$  . If we pick a representative,  $\varphi \in W(G)$  , for each left coset of  $W(G)/W(H)$  , then we can write the elements of  $W(G)$  as  $\{\varphi_i \alpha_j\}$  .  $A(\Omega(H) z) = \sum_i (\text{sgn } \varphi_i) \varphi_i [\sum_j \text{sgn}(\alpha_j) \alpha_j (\Omega(H) \cdot z)] = |W(H)| \sum_i \text{sgn}(\varphi_i) \varphi_i (\Omega(H) w)$  , since  $z$  is fixed under  $W(H)$  and  $\text{sgn}(\alpha_j) \alpha_j (\Omega(H)) = \Omega(H) \forall_j$  . Therefore, picking any set of representatives  $\{\varphi_i\}$ , for the left cosets of  $W(G)/W(H)$  , we can define  $\bar{A}(\Omega(H) \cdot w)$  to be  $\sum_i \text{sgn}(\varphi_i) \varphi_i (\Omega(H) w)$  ,  $w \in R(H)$  . This is well defined by the previous argument. It is also true that for any  $w \in R(H)$  ,  $\frac{\bar{A}(\Omega(H) \cdot w)}{\Omega(G)}$  is in  $R(G)$  . (See [4]). Let

$F : R(H) \times R(H) \longrightarrow R(G)$  , an  $R(G)$ -bilinear form, be defined by

$$F(z, w) = \frac{\bar{A}(\Omega(H) \cdot z \cdot w)}{\Omega(G)} . \quad (\text{For } H = T , \Omega(T) = \text{id} , \text{ and } \bar{A} \text{ is } A ,$$

the usual alternating sum.)

Discussion: Let  $M$  be a module over a commutative integral domain,  $R$  , with identity. Suppose we have defined a bilinear form  $F : M \times M \longrightarrow R$  . This form induces an  $R$ -Module homomorphism ,  $\varphi$  , from  $M$  to  $\text{Hom}(M, R)$

defined as follows:  $\varphi(m)(m') = F(m, m') \in R$ . If  $\varphi$  is an isomorphism, then we call  $F$  strongly non-singular. In particular, if  $M$  is a finitely generated free module over  $R$ , and if  $\exists$  two bases

$\{a_i\}_{i=1}^N$  and  $\{b_j\}_{j=1}^N \ni$  determinant of  $((F(a_i, b_j)))$  is a unit in

$R$ , then  $F$  is strongly non-singular. For,  $\text{Hom}(M, R)$  is also a free  $R$ -Module generated by  $\{f_i\}_{i=1}^N$  where  $f_i$  is defined by  $f_i(b_j) = \delta_{ij}$ .

If we denote  $F(a_i, b_j)$  by  $C_{ij}$ , then in terms of the basis  $\{f_j\}_{j=1}^N$ ,

$\varphi(a_i) = \sum_j C_{ij} f_j$ . Since the determinant of  $((C_{ij}))$  is a unit, this

implies that the row vectors are independent. In particular, this means

that  $\{\sum_j C_{ij} f_j\}_{i=1}^N$  is also a basis for  $\text{Hom}(M, R)$  over  $R$ . Therefore,

$\varphi$  takes a basis of  $M$  to a basis of  $\text{Hom}(M, R)$  and is an isomorphism.

The object of this paper is to prove the following Theorem for a large number of cases.

Theorem 1.9 ( $\pi_1(G)$  and  $\pi_1(H)$  have no 2-torsion.)

(1)  $R(H)$  is a free  $R(G)$ -Module of rank  $N = \frac{|W(G)|}{|W(H)|}$ . (For

$H = T$ ,  $W(H) = 1$ )

(2) There exists two sets of bases  $\{a_i\}_{i=1}^N$  and  $\{b_j\}_{j=1}^N$

$\ni F(a_i, b_j) = \begin{cases} 1 & i=j \\ 0 & i < j \end{cases}$  (i.e.,  $F$  is strongly non-singular.)

We will prove this Theorem for all the classical groups. The proof

for  $\text{Spin}(N)$ , will also induce a proof for  $\text{Spin}^c(N)$ .

Section 7.

There is another way of viewing the form  $F$ , and although we will not be using this approach, it is still a useful point to mention.

$R(G)$  was defined as the free abelian group generated by the set of equivalence classes of irreducible complex representations. We can define an inner product on  $R(G)$  (having values in the integers) as follows: If  $w_\alpha, w_\beta$  are two irreducible representations, then  $\langle w_\alpha, w_\beta \rangle_G = \delta_{\alpha\beta}$  (the Kronecker delta). Extending by linearity, we get an inner product. Let  $\beta : H \rightarrow G$ , as before, then define  $\beta_* : R(H) \rightarrow \hat{R}(G)$ , a group homomorphism, where  $\hat{R}(G)$  denotes the abelian group whose elements are possibly infinite sums of irreducible representations, as follows:  $\beta_*(z) = \sum_{\alpha} \langle \beta^*(w_\alpha), z \rangle_H w_\alpha$ , where  $\{[w_\alpha]\} \alpha \in \Lambda$  is the set of irreducible representations of  $G$ . If  $\pi_1(G)$  and  $\pi_1(H)$  have no 2-torsion, then  $\beta_*(z \cdot \Omega^*(G/H))$  is an element of  $R(G)$ . Here,  $\Omega(G/H) = \Omega(G)/\Omega(H)$ , an element of  $R(H)$ , where  $R(H)$  is thought of as a subring of  $R(T)$ , and if  $\Omega = \varphi_1 + \dots + \varphi_n$ ,  $\varphi_i$  irreducible, then  $\Omega^* = \varphi_1^* + \dots + \varphi_n^*$  with  $\varphi_i^*$  equal to the dual representation of  $\varphi_i$  [4]. For  $H = T$ ,  $\Omega(G/T) \equiv \Omega(G)$ . In this notation  $F(z, w) = \beta_*(\Omega^* z \cdot w) \in R(G)$ . (This map  $\beta_*$  is used very strongly in the proof of the index Theorem for Homogeneous Elliptic Operators (see [4]) ).

Section 8.

There is a strong relation between the representation ring of a compact connected Lie group,  $G$ , and the ring  $K_G$ , of equivalence classes of  $G$  vector bundles. For  $T \leq H \leq G$  we have  $K_G(G/H) \cong R(H)$  and  $K_G(\text{pt}) \cong R(G)$ . Our Theorem will therefore give us information about  $K_G(G/H)$  and about  $K(G/H)$ . Furthermore, the strong non-singularity of the bi-linear form can be interpreted as a Poincare Duality result for  $K_G$ -theory. If the conjecture is true in general (i.e., if the bilinear form is strongly non-singular whenever  $\pi_1(G)$  and  $\pi_1(H)$  have no 2-torsion), then it will give very strong information about  $K_G(G/H)$  for a large class of homogeneous spaces.

## CHAPTER II

## (Algebraic Background)

Section 1.

As usual let  $G$  be a compact connected Lie group and  $T$  a maximal torus. We have  $R(G) \leq R(T)$ , where by  $R(G)$  we mean  $i^*(R(G))$ ,  $i$  being the inclusion map of  $T$  in  $G$ . We are interested in the structure of  $R(T)$  as an  $R(G)$ -Module. Let us start by noting that much can be said even for the most general case.

If  $T$  is a  $n$ -torus and if we pick a Euclidean Coordinate System for  $\mathbb{R}^n$ , we can represent  $T$  as the set of  $\{(y_1, \dots, y_n) \mid y_i \in \mathbb{R}/\mathbb{Z}\}$ . Every irreducible representation of an abelian group is one-dimensional, and therefore every irreducible  $T$ -Representation is of the form  $\chi : T \rightarrow S^1$ , where  $\chi$  is a continuous homomorphism. [1,3.73].

(Note: The compactness of  $T$  implies that the image lies in  $S^1$ .)

If  $T = S^1 \times \dots \times S^1$  then let  $p_i$  denote projection onto the  $i^{\text{th}}$  factor. By  $\chi_i$ , an element of  $R(T)$ , we will mean the irreducible representation  $t \rightarrow e^{2\pi i p_i(t)}$ .

Theorem 2.1 (1,3.77)

Let  $T_n$  be an  $n$ -torus, then  $R(T) \cong \mathbb{Z}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ .

Since in the most general case,  $R(T)$  has this explicit algebraic description, we are able to get a good hold on the first part of our problem through the following Algebraic Theorems.

Proposition 2.2 (11,V,#4)

Let  $A$  be an integrally closed domain,  $K$  its quotient field,  $F$  a finite separable algebraic extension of  $K$  and  $A'$  the integral closure of  $A$  in  $F$ . Then  $\exists$  a basis  $x_1, \dots, x_n$  of  $F$  over  $K \ni A'$  is contained in the  $A$ -Module  $\sum_{i=1}^n A x_i$ ,  $[F:K] = n$ .

Proposition 2.3

Given the above hypothesis. If  $A'$  is generated by  $y_1, \dots, y_i$  over  $A$ , for  $i \leq n$ , then  $i = n$  and  $A'$  is freely generated by  $\{y_j\}_{j=1}^n$ .

Proof.

In the proof of the first proposition it is shown that  $F = \{a'/a \mid a' \in A', a \in A\}$ . Therefore, for any  $f \in F$ ,  $f = a'/a$ , which equals  $\frac{a_1 y_1 + \dots + a_i y_i}{a} = \frac{a_1}{a} y_1 + \dots + \frac{a_i}{a} y_i$ ,  $\frac{a_i}{a} \in K$ . Therefore,  $i = n$ , and the set  $\{y_1, \dots, y_n\}$  is linearly independent over  $K$ . In particular if  $a_1 y_1 + \dots + a_n y_n = 0$ ,  $a_i \in A$ , then  $a_i = 0$  for all  $i$ . This implies that  $\{y_1, \dots, y_n\}$  freely generates  $A'$  over  $A$ .

Notation: If  $A$  is an integral domain, then  $Q(A)$  will denote its quotient field.

We have  $R(G) \leq R(T)$  and further that  $R(G) = R(T)^{W(G)}$ , where  $W(G)$  is the Weyl Group of  $G$ . Let us also note that  $R(T_n) \cong Z[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$  and that therefore  $R(T)$  is integrally closed. [11,V,#3].

Proposition 2.4

Let  $A'$  be an integrally closed domain and let  $G$  be a finite group acting on  $A'$ , as a group of automorphisms. Then if  $A = (A')^G$ ,  $A$  is also integrally closed and  $A'$  is the integral closure of  $A$  in  $Q(A')$ .

Proof.

1. Let  $G$  act on  $Q(A')$  by  $g\left(\frac{a'_1}{a'_2}\right) = \frac{g(a'_1)}{g(a'_2)}$ . Then  $Q(A) \leq Q(A')^G$ . Suppose  $q \in Q(A)$  and  $q$  is integral over  $A$ . Then  $q$  is in  $A'$ , since  $A'$  is integrally closed. But  $q$  is fixed by  $G$ , therefore,  $q \in (A')^G = A$ .

2.  $A'$  is integral over  $A$ . For let  $a' \in A'$ , then  $\sum_{i=0}^n \emptyset_i (x - \emptyset(a'))^i$  is an integral relation.  $A'$  is integrally closed, therefore, it is the integral closure of  $A$  in  $Q(A')$ .

Theorem 2.5

$$\begin{array}{ccc} Q(R(G)) & \cong & Q(R(T)) \\ \uparrow i & & \uparrow j \\ R(G) & \leq & R(T) \end{array}$$

This diagram satisfies the hypothesis of Proposition 2.2. Hence, if  $R(T)$  is finitely generated over  $R(G)$  by  $N = |W(G)|$  elements, then  $R(T)$  is free over  $R(G)$  of rank  $N$ .

Proof.

By Proposition 2.4,  $R(G)$  is integrally closed and  $R(T)$  is the integral closure of  $R(G)$  in  $Q(R(T))$ . By the proof of Proposition 2.2,  $Q(R(T)) = \left\{ \frac{s}{r} \mid s \in R(T), r \in R(G) \right\}$ , implying that  $Q(R(G)) = Q(R(T))^{W(G)}$ . By elementary Galois Theory (see [11], Chapter II),  $Q(R(T))$  is a normal finite algebraic extension of  $Q(R(G))$ . The final assertion follows from Proposition 2.3.

Proposition 2.6 [10,I,10.5]

Let  $R$  be a domain and let  $a \in R$ . Then if  $a^{-1} \in Q(R)$  is integral over  $R$ , then  $a^{-1} \in R$ .

Section 2.

In view of these results, we make the following observations:

Since  $R(G) = R(T)^{W(G)}$ , we need only prove that  $R(T)$  can be generated by  $|W(G)|$  elements over  $R(G)$ . This will imply that it is freely generated by these elements.

From Proposition 2.6 it follows, that, if we can generate a subring of  $R(T)$  which contains all the  $x_i$ , then it must contain the  $x_i^{-1}$ , and hence we will have generated all of  $R(T)$ .

Let  $W' \leq W(G)$  and let  $R' = R(T)^{W'}$ . Then we have  $R(G) \leq R' \leq R(T)$ , with  $R'$  integrally closed and  $R(T)$  the integral closure of  $R'$  in  $Q(R(T))$ . This provides us with an inductive method for proving  $R(T)$  is free over  $R(G)$ .

As a final note, let us suppose that we have two integral domains  $R \leq R'$  with  $R$  integrally closed and  $x \in R'$ , integral over  $R$ . (e.g.,  $R' = R(T)$  and  $R = R(G)$ ). Then the minimal polynomial of  $x$  over  $Q(R)$  is of the form  $X^n + a_1 X^{n-1} + \dots + a_n$ , where the  $a_i$  are in  $R$ . Therefore, if we let  $R[x]$  denote the smallest ring containing  $R$  and  $x$ , then  $R[x]$  is freely generated by  $\{1, x, \dots, x^{n-1}\}$  over  $R$ .

## CHAPTER III

Section 1. SU(n) and U(n)

$U(n)$  is the subgroup of  $GL(n, \mathbb{C})$  whose elements have the following property:  $A^{-1} = (\bar{A})^t$  (i.e., the inverse of any element is equal to the conjugate transpose.) This implies that  $\det(A) \in \mathbb{C}^1$  has norm 1 for all  $A \in U(n)$ .  $SU(n)$  is the subgroup of  $U(n)$  whose elements have determinant +1. A maximal torus for  $U(n)$  is the subgroup of diagonal matrices  $((e^{2\pi i y_i}))$   $1 \leq i \leq n$ . If we restrict ourselves to those diagonal matrices where  $y_n = -y_1 \dots -y_{n-1} \pmod{1}$ , we have an  $(n-1)$ -torus, which is a maximal torus for  $SU(n)$ . In particular  $SU(n)$  is not a subgroup of maximal rank.

We can also include  $U(n) \xrightarrow{i} SU(n+1)$  as follows: Let  $A \in U(n)$ , then  $i(A) = \begin{pmatrix} 0 \\ A & \vdots \\ 0 & \dots & 0 \\ 0 \dots 0 & * \end{pmatrix}$ , where  $* = 1/\det A$ . As can easily be seen  $i(A)$  is in  $SU(n+1)$ . In this case the maximal  $n$ -torus of  $U(n)$  which was previously described, is mapped isomorphically onto a maximal  $n$ -torus of  $SU(n+1)$ , and we have  $U(n)$  as a subgroup of maximal rank of  $SU(n+1)$ . As usual, this induces  $R(SU(n+1)) \leq R(U(n)) \leq R(T_n)$ .

The Weyl group of  $U(n)$ ,  $W(U(n))$ , acts on  $R(T_n)$  as the group of permutations of the set  $\{x_1, \dots, x_n\}$ , where  $R(T_n) \cong \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ . The order of  $W(U(n))$  is therefore  $n!$ .  $R(U(n))$  is the fixed subring of  $W(U(n))$ , and it is quite easy to show that  $R(U(n)) \cong \mathbb{Z}[\tau_1, \dots, \tau_n, \tau_n^{-1}]$ , where the  $\tau_i$  are the elementary symmetric functions of  $\{x_1, \dots, x_n\}$ .

[1,7.3]. As representations of  $U(n)$ ,  $\tau_1$  is  $U(n)$  acting on  $\mathcal{C}^n$  and  $\tau_i = \Lambda^i \tau_1$ , the  $i^{\text{th}}$  exterior power of  $\tau_1$  (see [9]).

$W(SU(n+1))$  acts on  $R(T_n)$  as the full group of permutations of the set  $\{x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}\}$ . As an abstract group, it is isomorphic to  $S_{n+1}$ , the symmetric group on the set of  $(n+1)$ -elements. Therefore,  $|W(SU(n+1))| = (n+1)!$ .  $R(SU(n+1)) \cong Z[\rho_1, \dots, \rho_n]$ , where  $\rho_1|_{T_n} = x_1 + \dots + x_n + x_1^{-1} + \dots + x_n^{-1}$  (i.e.,  $\rho_1$  is  $SU(n+1)$  acting on  $\mathcal{C}^{n+1}$ ) and  $\rho_i = \Lambda^i \rho_1$ . [1,7.4].)

Let  $W'$  be the subgroup of  $W(SU(n+1))$  which fixes  $x_1^{-1} \dots x_n^{-1}$ , then  $W' \leq W(SU(n+1))$  and  $R(U(n))$  is the fixed subring of  $R(T_n)$  under  $W'$ . (i.e.,  $W' \cong W(U(n))$ .)

### Theorem 3.1

$R(T_n)$  is a free Module over  $R(U(n))$  of rank  $n!$ . A set of generators for  $R(T_n)$  over  $R(U(n))$  is the set  $\{x_1^{a_1} \dots x_{n-1}^{a_{n-1}}\}$  where  $0 \leq a_i \leq n-i$ .

### Theorem 3.2

$R(T_n)$  is a free Module over  $R(SU(n+1))$  of rank  $(n+1)!$ . A set of generators is  $\{x_1^{a_1+j} \dots x_{n-1}^{a_{n-1}+j} x_n^j\}$  where  $0 \leq a_i \leq n-i$  and  $-n \leq j \leq 0$ .

### Lemma 3.3

$R(U(n))$  is a free  $R(SU(n+1))$  Module of rank  $n+1$ , with  $\{(x_1 \dots x_n)^j\}_{j=0}^{-n}$  a set of generators.

Note: Lemma 3.3 and Theorem 3.1 imply Theorem 3.2.

### Proof (Lemma 3.3):

$(x_1 \dots x_n)^{-1}$  is an element of  $R(T_n)$  and is therefore integral

over  $R(SU(n+1))$  . [Ch. II, 2].  $\tau_n^{-1} = (x_1 \dots x_n)^{-1}$  has  $n+1$  images under  $W(SU(n+1))$  and therefore its minimal polynomial, which is the same as its integral polynomial is  $p(y) = \prod_{\phi \in W} (y - \phi(\tau_n^{-1}))$  is of degree  $(n+1)$  . This implies that  $R(SU(n+1))[\tau_n^{-1}]$  is free over  $R(SU(n+1))$  and is generated by  $\{(\tau_n^{-1})^j\}_{j=0}^{n-1}$  . [See Ch. II, 2] .

We will now show that  $R(SU(n+1))[\tau_n^{-1}] = R(U(n))$  , completing the proof.  $\tau_1 = \rho_1 - (x_1 \dots x_n)^{-1}$  is in  $R(SU(n+1))[\tau_n^{-1}]$  and for  $i > 1$  ,  $\tau_i = \rho_i - \tau_{i-1} \cdot (x_1 \dots x_n)^{-1}$  , and therefore  $\tau_i \in R(SU(n+1))[\tau_n^{-1}]$  by finite induction. This implies  $R(SU(n+1))[\tau_n^{-1}] = Z[\tau_1, \dots, \tau_n, \tau_n^{-1}]$  which in turn equals  $R(U(n))$  .

Note: Let us look at  $S = Z[x_1, \dots, x_n]$  the polynomial ring over  $Z$  in  $n$  indeterminants. Let  $S_n$  , the symmetric group, act on  $S$  , then  $S^{S_n}$  is the ring  $R = Z[\tau_1, \dots, \tau_n]$  where the  $\tau_i$  , as before, are the elementary symmetric functions. Claim: If  $\{a_i\}_{i=1}^N$  is a set of generators for  $S$  over  $R$  , then  $\{a_i\}_{i=1}^N$  considered as elements of  $R(T_n) \cong Z[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  generate  $R(T_n)$  over  $R(U(n))$  . This follows trivially from the fact that  $R \leq R(U(n))$  and from Proposition 2.6 (i.e., if we can generate the  $x_i$  we also get all the  $x_i^{-1}$  .). The proof of Theorem 3.1 will, therefore, be complete provided we can prove the following lemma:

#### Lemma 3.4

Let  $S$  and  $R$  be as above. Then  $S$  is a free  $R$ -Module generated by  $\{x_1^{a_1} \dots x_{n-1}^{a_{n-1}}\}$  where  $0 \leq a_i \leq n-i$  .

Proof (induction on  $n$ ).

(a)  $n = 2$   $R = Z[x_1 + x_2, x_1 \cdot x_2] \leq Z[x_1, x_2] = S$ . Clearly  $S$  is freely generated by  $\{1, x_1\}$ .

(b) Assume true for  $n \leq k-1$ . Let  $S_{k-1}$  be the subgroup of  $S_k$  fixing  $x_1$ . Clearly  $R' = S_{k-1}^{S_k}$  is the ring  $Z[x_1][\tau'_1, \dots, \tau'_{n-1}]$ , where the  $\tau'_i$  are the elementary symmetric functions in  $\{x_2, \dots, x_n\}$ . By the inductive hypothesis  $Z[x_2, \dots, x_n]$  is freely generated over  $Z[\tau'_1, \dots, \tau'_{n-1}]$  by  $\{x_2^{a_2} \cdots x_n^{a_n}\}$   $0 \leq a_i \leq n-i$ . This implies that the same set generates  $Z[x_1][x_2, \dots, x_n]$  over  $Z[x_1][\tau'_1, \dots, \tau'_{n-1}]$  and does so freely. We have  $Z[\tau_1, \dots, \tau_n] \leq Z[x_1][\tau'_1, \dots, \tau'_{n-1}] \leq Z[x_1, \dots, x_n]$  and to complete the proof we will show that  $Z[x_1][\tau'_1, \dots, \tau'_{n-1}]$  is freely generated over  $Z[\tau_1, \dots, \tau_n]$  by  $\{1, x_1, \dots, x_1^{n-1}\}$ . By the reasoning in Chapter II, 2, the proof will be complete if we can show that  $Z[\tau_1, \dots, \tau_n][x_1] = Z[x_1][\tau'_1, \dots, \tau'_{n-1}]$ . But  $\tau'_1 = \tau_1 - x_1$  and  $\tau'_i = \tau_i - x_1 \cdot \tau'_{i-1}$ , therefore the result follows.

## Section 2. $Sp(n)$ and $SO(2n+1)$

We are going to consider  $Sp(n)$  and  $SO(2n+1)$  together, because their representation rings, as subrings of  $Z[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , are the same.

$Sp(n)$  is the subgroup of  $Gl(n, k)$ , whose elements satisfy  $A^{-1} = \bar{A}^T$ , where  $k$  is non-commutative field of Quaternions and conjugation is the canonical quaternionic conjugation (i.e.,

$\overline{a + ib + jc + kd} = a - ib - jc - kd$ ). A maximal torus for  $Sp(n)$  is the subgroup of diagonal matrices. (See [1, 4.18].)

$O(n)$  is the subgroup of  $Gl(n, R)$  whose elements satisfy  $A^{-1} = A^T$ . Therefore,  $\det(a) = \pm 1$  for  $A \in O(n)$ . This implies  $O(n)$  is not connected.  $\left\{ \begin{pmatrix} 1 & \\ & \cdot \\ & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & \\ & \cdot \\ & & -1 \end{pmatrix} \text{ are in different components} \right\}$ : Let



we have that  $R(\text{Sp}(n)) \cong \mathbb{Z}[\rho_1, \dots, \rho_n]$ . ([1, 7.6]).

If we think of  $\text{SO}(2n+1)$  as a group of real linear transformations of Euclidean  $2n+1$  - space,  $\mathbb{R}^{2n+1}$ , we get a real representation of  $\text{SO}(2n+1)$ . The complexification of this representation will be called  $\bar{\rho}_1$ .

$\bar{\rho}_1 \mid T$  is  $x_1 + \dots + x_n + x_1^{-1} + \dots + x_n^{-1} + 1$ . If we let  $\bar{\rho}_i = \Lambda^i \bar{\rho}_1$ , then  $R(\text{SO}(2n+1)) = \mathbb{Z}[\bar{\rho}_1, \dots, \bar{\rho}_n]$  (see [1, 7.7]). Let us note that if  $\{\tau_i\}_{i=1}^n$

are the elementary symmetric functions on the set  $\{(x_j + x_j^{-1})\}_{j=1}^n$ ,

previously discussed, then  $\bar{\rho}_1 = \tau_1 + 1$ , and in fact each  $\bar{\rho}_i$  can be

written as  $\tau_i + n_1 \tau_{i-1} + \dots + n_i$ , where  $n_i \in \mathbb{Z}$ .

We will prove the following theorem for  $\text{Sp}(n)$  and the result will follow for  $\text{SO}(2n+1)$  because of the isomorphism.

### Theorem 3.5

$R(T_n)$  is a free  $R(\text{Sp}(n))$ -Module of rank  $2^n \cdot n!$ . A set of generators is  $\{x_1^{a_1} \dots x_n^{a_n}\}$  where  $0 \leq a_i \leq 2(n-i) + 1$ .

Proof (Induction on  $n$ ):

(a)  $n = 1$   $R(\text{Sp}(1)) = \mathbb{Z}[x_1 + x_1^{-1}] \leq \mathbb{Z}[x_1, x_1^{-1}] = R(T_1)$ . Clearly  $\{1, x_1\}$  is a free basis.

(b) Assume, now, by the induction hypothesis that  $R(T_{k-1}) = \mathbb{Z}[x_2^{\pm 1}, \dots, x_k^{\pm 1}]$  is freely generated over  $R(\text{Sp}(k-1)) = \mathbb{Z}[\tau'_1, \dots, \tau'_{k-1}]$  by  $\{x_2^{a_2} \dots x_k^{a_k}\}$   $0 \leq a_i \leq 2(k-i) + 1$ . ( $\tau'_i$  is the  $i^{\text{th}}$  elementary symmetric function in the set  $\{(x_j + x_j^{-1})\}_{j=2}^k$ , (as previously noted.)

If we let  $W' \leq W(\text{Sp}(k))$  be the subgroup fixing  $\{x_1\}$ , and if we let  $R' = R(T_k)^{W'}$ . Then, as in the case of  $R(U(n))$ , we get

$R' = Z[x_1, x_1^{-1}][\tau_1', \dots, \tau_{k-1}']$  and furthermore we have  $R(\text{Sp}(k)) \leq R'$   
 $\leq Z[x_1, x_1^{-1}][x_2^{\pm 1}, \dots, x_k^{\pm 1}]$  and  $R(T_k)$  is freely generated over  $R'$  by  
 $\{x_2^{a_2}, \dots, x_k^{a_k}\}$  where  $0 \leq a_i \leq 2(k-i) + 1$ .

We also note, that since  $x_1$  has  $2 \cdot k$  images under the action  
 $W(\text{Sp}(k))$ ,  $R(\text{Sp}(k))[x_1]$  (which equals  $R(\text{Sp}(k))[x_1, x_1^{-1}]$ ) is free  
 over  $R(\text{Sp}(k))$  of rank  $2 \cdot k$ , generated by  $\{1, x_1, \dots, x_1^{2k-1}\}$ . The  
 proof will be complete if we can show that  $R(\text{Sp}(k))[x_1, x_1^{-1}] = R'$ .

Exactly as in the case  $U(n)$ , we have the following relations:  
 $\tau_1' = \tau_1 - (x_1 + x_1^{-1})$  and  $\tau_i' = \tau_i - (x_1 + x_1^{-1})\tau_{i-1}'$ . This, then, implies  
 that  $R(\text{Sp}(k))[x_1, x_1^{-1}] = R'$  and completes the proof.

### Section 3. $SO(2n)$ .

We can include  $SO(2n)$  in  $SO(2n+1)$  as a subgroup of maximal  
 rank as follows: If  $A \in SO(2n)$  map  $A \longrightarrow \begin{pmatrix} & 0 \\ A & \vdots \\ 0 \dots & 1 \end{pmatrix} \in SO(2n+1)$ .

A maximal torus for  $SO(2n)$  is the set of matrices of the form

$\begin{pmatrix} C(y_1) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & C(y_n) \end{pmatrix}$  where the  $C(y_i)$  are the elements of  $SO(2)$  previously

discussed. (See previous section.) Under the inclusion map this torus  
 is mapped isomorphically onto the maximal torus of  $SO(2n+1)$ .

Let  $\rho_1$  be the complexification of the real representation of  
 $SO(2n)$  acting as a group of real linear transformations on  $R^{2n}$ .

$\rho_1 | T = x_1 + x_1^{-1} + \dots + x_n + x_n^{-1}$ . Let  $\rho_i = \Lambda^i \rho_1$ , then  $R(SO(2n))$   
 $= Z[\rho_1, \dots, \rho_{n-1}, \rho_n^+, \rho_n^-]$  where  $\rho_n^+$  and  $\rho_n^-$  are defined as follows.

(See [9]). Analogous to the case,  $SO(2n+1)$ ,  $\rho_1 = \tau_1$  and  $\rho_i$ ,  $i > 1$ ,

can be written as  $\tau_i + n_1 \tau_{i-1} + \dots + n_i$ , where the  $\tau_i$  are the elementary symmetric function discussed in the previous section. In particular,  $\rho_n = \tau_n + k_1 \tau_{n-1} + \dots + k_n$ , where  $k_i \in \mathbb{Z}$ .  $\tau_n$  can be

written as  $\sum_{\epsilon_i = +1} x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ . Let us call  $\tau_n^+$  the sum,

$\sum_{\pi \epsilon_i = +1} x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$  and  $\tau_n^-$  the sum,  $\sum_{\pi \epsilon_i = -1} x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ . (Note:

$\tau_n = \tau_n^+ + \tau_n^-$ ). Then  $\rho_n^+ = \tau_n^+ + \sigma$ ,  $\rho_n' = \tau_n^- + \sigma$ ,  $\sigma \in R(\text{SO}(2n))$ , and

$\rho_n^+ + \rho_n^- = \rho_n$ , [1]. Let us note that  $\rho_n$  is also an element of the

ring  $R(\text{SO}(2n+1))$  and that in fact  $R(\text{SO}(2n+1)) \leq R(\text{SO}(2n)) \leq R(\mathbb{T}_n)$ .

(See [1]).

An arbitrary element of  $W(\text{SO}(2n+1))$  can be thought of as acting on  $\{x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}\}$  as follows. (As usual,  $R(\mathbb{T}_n) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ ). It first permutes the subscripts, as an element of  $S_n$ , and then makes an arbitrary number of sign changes of the form  $x_i \longleftrightarrow x_i^{-1}$ . Thinking of  $W(\text{SO}(2n+1))$  as acting on  $R(\mathbb{T}_n)$  in this way,  $W(\text{SO}(2n))$  is the subgroup of  $W(\text{SO}(2n+1))$  where only an even number of sign changes are allowed.

We will not prove a theorem about the freeness of  $R(\mathbb{T}_n)$  over  $R(\text{SO}(2n))$  because  $\pi_1(\text{SO}(2n)) = \mathbb{Z}_2$  and our main theorem will not apply. The only reason we introduced this case is because of its usefulness in the discussion of  $\text{Spin}(2n)$ .

#### Section 4. Spin(n)

As mentioned in the introduction,  $\pi_1(\text{SO}(n)) \cong \mathbb{Z}_2$ , for  $n > 2$ , and therefore  $\text{SO}(n)$  has a simply connected double cover. This double cover is called  $\text{Spin}(n)$ , the Spinor Group.

We will take as a maximal torus of  $\text{Spin}(n)$  the cover of the standard maximal torus that we used for  $\text{SO}(n)$ .  $L(\tilde{T})$  and  $L(T)$  are isomorphic as vector spaces, where  $\tilde{T}$  and  $T$  are the maximal tori of  $\text{Spin}(n)$  and  $\text{SO}(n)$ , respectively, but the integer lattices of each differ. (See [1, 5.56]). If the integer lattice of  $T$  is represented by  $\{(y_1, \dots, y_n) \mid y_i \in \mathbb{Z}\}$ , then the integer lattice of  $\tilde{T}$  is the subset, where  $y_1 + \dots + y_n = 2k$ , for some integer  $k$ .

Let  $\pi : \text{Spin}(n) \rightarrow \text{SO}(n)$  be the double cover and let  $\epsilon$  be the non-identity element in the kernel of  $\pi$ . Then with the previous interpretation,  $\epsilon$  can be represented by any  $(y_1, \dots, y_n) \in L(\tilde{T})$  with  $y_1 + \dots + y_n = 2k + 1$ .

The walls of the Weyl Chambers of  $L(T)$ , considered as hyperplanes in  $L(\tilde{T})$ , yield the Weyl Chambers of  $\text{Spin}(n)$ . In particular,  $W(\text{Spin}(n)) \cong W(\text{SO}(n))$ , because both groups are generated by reflections in the same hyperplanes.

Let  $x_1^{\frac{1}{2}} \dots x_k^{\frac{1}{2}}$  be the element of  $R(\tilde{T}_k)$  defined as follows:

Send  $(y_1, \dots, y_k) \rightarrow e^{\pi i(y_1 + \dots + y_k)}$ . Because  $y_1 + \dots + y_k = 2k$  whenever  $(y_1, \dots, y_k)$  is in the integer lattice,  $x_1^{\frac{1}{2}} \dots x_k^{\frac{1}{2}}$  is well defined. Furthermore,  $R(\tilde{T}_k) \cong \mathbb{Z}[x_1^{\pm 1}, \dots, x_k^{\pm 1}, x_1^{\frac{1}{2}} \dots x_k^{\frac{1}{2}}]$ . (See [9]).

Let  $\rho : \text{SO}(n) \rightarrow \text{Aut}(V)$ , be a representation of  $\text{SO}(n)$ , then  $\rho \circ \pi : \text{Spin}(n) \rightarrow \text{Aut}(V)$  is a Spinor representation. In particular, the  $\rho_i$ , (recall:  $R(\text{SO}(2k+1)) = \mathbb{Z}[\rho_1, \dots, \rho_k]$ ), are elements of  $R(\text{Spin}(n))$ . Because of the projection map, in any representation induced from  $\text{SO}(n)$ ,  $\epsilon$  acts as the identity. There is a special Spinor representation,  $\Delta$ , under which  $\epsilon$  does not act as the identity.  $\Delta|_{\tilde{T}_k} = \prod_{i=1}^k (x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}}) = \sum_{\epsilon_i = \pm 1} x_1^{\frac{\epsilon_1}{2}} \dots x_k^{\frac{\epsilon_k}{2}}$ . (See [9]).

As in the case of  $SO(n)$  it makes a difference whether  $n$  is of the form  $2k$  or  $2k+1$ .  $R(\text{Spin}(2k+1)) \cong Z[\rho_1, \dots, \rho_{k-1}, \Delta]$ , where the generators are algebraically independent. For the case  $\text{Spin}(2k)$ ,  $\Delta$  is not irreducible, but it splits into two irreducible representations

$$\Delta^+ \text{ and } \Delta^- . \quad \Delta^+ |_{\tilde{T}_k} = \prod_{\epsilon_i=+1} x_1^{\frac{\epsilon_1}{2}} \dots x_k^{\frac{\epsilon_k}{2}} \quad \text{and} \quad \Delta^- |_{\tilde{T}_k} = \prod_{\epsilon_i=-1} x_1^{\frac{\epsilon_1}{2}} \dots x_k^{\frac{\epsilon_k}{2}} .$$

$R(\text{Spin}(2k))$  is the polynomial ring  $Z[\rho_1, \dots, \rho_{k-2}, \Delta^+, \Delta^-]$ .

In either case, we have the following important relations:

- (a)  $\Delta^+ \cdot \Delta^+ = \rho_k^+ + \rho_{k-2} + \dots$
- (b)  $\Delta^+ \cdot \Delta^- = \rho_{k-1} + \rho_{k-3} + \dots$
- (c)  $\Delta^- \cdot \Delta^- = \rho_k^- + \rho_{k-2} + \dots$
- (d)  $\Delta \cdot \Delta = \rho_k + \rho_{k-1} + \dots + \rho_1 + 1$

Note: For (a), (b), and (c), let us understand  $\rho_0$  as equal to  $+1$ .

Since  $SO(2k)$  is contained in  $SO(2k+1)$  as a subgroup of maximal rank,  $\text{Spin}(2k)$  is of maximal rank in  $\text{Spin}(2k+1)$ . This yields:

$R(\text{Spin}(2k+1)) \leq R(\text{Spin}(2k)) \leq R(\tilde{T}_k)$ . Our approach will be to first prove that  $R(\tilde{T}_k)$  is free over  $R(\text{Spin}(2k))$  and then prove that  $R(\text{Spin}(2k))$  is free over  $R(\text{Spin}(2k+1))$ . (See [9] for entire previous assertion.)

Discussion: In the ring  $R(\tilde{T}_k) \cong Z[x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_1^{\frac{1}{2}}, \dots, x_n^{\frac{1}{2}}]$  there are two types of monomials. One type is a monomial consisting of a product of  $x_i$  with integral powers (e.g.,  $x_1 x_2$ ); the second type is one in which all the  $x_i$  have  $\frac{1}{2}$  integral powers (e.g.,  $x_1^{\frac{1}{2}} \dots x_k^{\frac{1}{2}}$ ). We will call these the "integral" and "non-integral" monomials, respectively. A poly-

nomial in  $R(\tilde{T}_k)$  is said to be homogeneous if it is a sum of monomials, all of which are either integral or non-integral. It is clear that as an abelian group  $R(\tilde{T}_k) = I \oplus N$ , where  $I$  and  $N$  represent the homogeneous integral and homogeneous non-integral polynomials, respectively. This also gives us a  $Z_2$  graded ring related to the ring  $R(\tilde{T}_k)$ .

Since, if  $p_i$  is in  $I$  and  $q_j$  is in  $N$  then  $p_i q_j \in N$ ,  $p_{i_1} \cdot p_{i_2} \in I$  and  $q_{j_1} \cdot q_{j_2} \in I$ .

Theorem 3.6.

$R(\tilde{T}_k)$  is a free module over  $R(\text{Spin}(2k))$ , freely generated by  $|W| = 2^{k-1} \cdot k!$  homogeneous polynomials.

Proof (by induction on  $k$ )

Lemma 3.7

Suppose  $R(\tilde{T}_{k-1}) = Z[x_1^{\pm 1}, \dots, x_{k-1}^{\pm 1}, x_{\frac{1}{2}}^{\pm 1}, \dots, x_{\frac{k-1}{2}}^{\pm 1}]$  is generated over  $R(\text{Spin}(2k-2)) = Z[\rho_1', \dots, \rho_{k-3}', \Delta_{k-1}^+, \Delta_{k-1}^-]$  by the set  $\{a_1, \dots, a_s\}$  where the  $a_i$  are homogeneous polynomials in  $R(\tilde{T}_{k-1})$ . Let  $R' \leq R(\tilde{T}_k)$  be the following subring.  $R' = Z[x_k^{\pm 1}, \rho_1', \dots, \rho_{k-3}', x_k^{\pm \frac{1}{2}} \Delta_{k-1}^+, x_k^{\pm \frac{1}{2}} \Delta_{k-1}^-]$ . If  $a_i' \in R(\tilde{T}_k)$  is defined by:  $a_i' = a_i$  if  $a_i$  is integral and  $a_i' = x_k^{\frac{1}{2}} a_i$  if  $a_i$  is non-integral. Then  $R(\tilde{T}_k)$  is generated by  $\{a_i'\}_{i=1}^s$  over  $R'$ .

Explanation: The notation  $\rho_i'$  is used to differentiate these elements from the  $\rho_i$ , where  $R(\text{Spin}(2k)) = Z[\rho_1, \dots, \rho_{k-2}, \Delta_k^+, \Delta_k^-]$ . Secondly, any homogeneous integral element in  $R(\tilde{T}_{k-1})$  can be considered as an element of  $R(\tilde{T}_k)$ . This explains why the  $\rho_i'$  are elements of  $R(\tilde{T}_k)$  and the definition of  $a_i'$ .

Proof of Lemma 3.7

We will show that any monomial,  $m$ , of  $R(\tilde{T}_k)$  can be generated by the  $a'_i$ , and this will imply the Lemma.

Case 1: ( $m$  is integral and has no factor of  $x_k$ ).

In this case  $m$  is in  $R(\tilde{T}_{k-1})$  and  $m = \sum r_i a_i$ , where  $r_i \in R(\text{Spin}(2k-2))$ . By homogeneity we may assume that each  $r_i a_i$  must be integral. This implies that both  $r_i$  and  $a_i$  are in  $I$  or they both are in  $N$ . Let  $r'_i \in R(\text{Spin}(2k))$  be equal to  $r_i$ , if  $r_i$  is integral and it be equal to  $r_i \cdot x_k^{-\frac{1}{2}}$  if  $r_i$  is non-integral. It is then clear that  $m = \sum_{i=1}^s r'_i a'_i$ .

Case 2: ( $m$  is integral and contains a factor of  $x_k$ ).

$m = \bar{m} x_k^p$  where  $p \in \mathbb{Z}$ ,  $p \neq 0$ . By Case 1,  $\bar{m} = \sum r'_i a'_i$ . But  $x_k^p$  is in  $R'$ , therefore  $m = \sum_{i=1}^s (r'_i x_k^p) \cdot a'_i$ .

Case 3: ( $m$  is non-integral).

By the reasoning of Case 2, we need only consider  $m$ , where  $x_k$  appears to the  $\frac{1}{2}$  power.

$m = \bar{m} \cdot x_k^{\frac{1}{2}}$ , where  $\bar{m} = \sum r_i a_i$  is a non-integral monomial in  $R(\tilde{T}_{k-1})$ . By homogeneity we may assume that each  $r_i a_i$  is in  $N$ .

This implies that for each  $i$ ,  $r_i$  and  $a_i$  are in different parts of the  $Z_2$  grading. Let  $r''_i = r_i$  if  $r_i$  is in  $I$ , and let  $r''_i = x_k^{\frac{1}{2}} r_i$  if  $r_i \in N$ . Then  $m = \sum_{i=1}^s r''_i \cdot a'_i$ .

Lemma 3.8

Let  $W'$  be the subgroup of  $W(\text{Spin}(2k))$  which fixes  $x_k$ . Then  $R' = R(\tilde{T}_k)^{W'}$  is the ring  $R'$  described in the previous Lemma.

Proof

Let us first note that  $W' \cong W(\text{Spin}(2k-2))$ , and both have the same action on the set  $\{x_1, \dots, x_{k-1}, x_1^{-1}, \dots, x_{k-1}^{-1}\}$ . It is therefore clear that  $R' \leq R(\tilde{T}'_k)^{W'}$ .

Let  $p \in R(\tilde{T}'_k)$  and suppose that  $\emptyset(p) = p \forall \emptyset \in W'$ .  $p = \sum x_k^j \overline{p_j}$ , where the  $\overline{p_j}$  are elements of  $R(\tilde{T}'_{k-1})$  and  $-\frac{N}{2} \leq j \leq \frac{M}{2}$ ,  $N, M \in \mathbb{Z}$ . Since  $W'$  fixes  $x_k$ , this implies that  $\overline{p_j} \in R(\tilde{T}'_{k-1})^{W'} \forall j$ . Therefore  $p \in R'$ .

Proof of Theorem 3.6 (induction on  $k$ ).

1.  $k = 2$ .  $R(\text{Spin } 4) = \mathbb{Z}[\Delta_2^+, \Delta_2^-] \leq \mathbb{Z}[x_1^{\frac{1}{2}}, x_2^{\frac{1}{2}}, x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}] = R(\tilde{T}'_2)$ .

a.  $x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$  has 2 images under  $W(\text{Spin } 4)$ , therefore,  $R'' = R(\text{Spin } 4)[x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}}]$  is free over  $R(\text{Spin } 4)$ , generated by  $1, x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ .

b.  $\sigma_1$  and  $\sigma_2$  are elements of  $R''$ , where  $\sigma_1, \sigma_2$  are the elementary symmetric functions in  $\{x_1, x_2\}$ . This follows because  $\sigma_2 = (x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}})^2$  and  $x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}}(\Delta_2^+ + \Delta_2^-) = \sigma_1 + \sigma_2 + 1$ .

c. By Lemma 3.4 we can generate  $\{x_1, x_2\}$  over  $\mathbb{Z}[\sigma_1, \sigma_2]$  by  $\{1, x_1\}$ . This implies that  $\{1, x_1\}$  generate  $R(\tilde{T}'_2)$  over  $R''$ .

d. Putting these steps together, we get that  $1, x_1, x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}, x_1^{\frac{1}{2}}(x_1^{\frac{1}{2}} \cdot x_2^{\frac{1}{2}})$  generate  $R(\tilde{T}'_2)$  over  $R(\text{Spin}(4))$ . Since  $|W(\text{Spin}(4))| = 4$ , we have that these elements freely generate  $R(\tilde{T}'_2)$  over  $R(\text{Spin}(4))$ .

(See Ch. II, 2.)

2. Assume the theorem is true for  $k < s$ . Let  $a_1, \dots, a_r$  be the homogeneous elements which generate  $R(\tilde{T}'_{s-1})$  over  $R(\text{Spin}(2s-2))$ , where  $r = |W(\text{Spin}(2s-2))| = 2^{s-2}(s-1)!$ . If  $W'$  is the subgroup of

$W(\text{Spin}(2s))$  fixing  $\{x_s\}$  and  $R'$  is  $R(\tilde{T}_s)^{W'}$ . Then by Lemma 3.7 and 3.8, we have  $R(\text{Spin}(2s)) \leq R' \leq R(\tilde{T}_s)$  and  $R(\tilde{T}_s)$  is generated over  $R'$  by  $\{a_i\}_{i=1}^r$ . The proof will be complete provided  $R'$  can be generated over  $R(\text{Spin}(2s))$  by  $2 \cdot s$  homogeneous elements.

Lemma 3.9

$R'$  is generated over  $R(\text{Spin}(2s))$  by the set  $\{1, x_s, \dots, x_s^{s-1}, x_s^{-1}, \dots, x_s^{-(s-2)}, x_s^{\frac{1}{2}} \Delta_{s-1}^+, x_s^{\frac{1}{2}} \Delta_{s-1}^-\} \equiv S$ . (Note: these are exactly  $2 \cdot s$  homogeneous elements).

Proof

Let us first note that  $R(\text{Spin}(2s))[S]$ , the smallest ring containing  $R(\text{Spin}(2s))$  and the set  $S$ , is equal to  $R'$ . Let us recall that  $R' = \mathbb{Z}[x_s^{\pm 1}, \rho_1', \dots, \rho_{s-3}', x_s^{\frac{1}{2}} \Delta_{s-1}^+, x_s^{\frac{1}{2}} \Delta_{s-1}^-]$ , and  $R(\text{Spin}(2s)) = \mathbb{Z}[\rho_1, \dots, \rho_{s-2}, \Delta_S^+, \Delta_S^-]$ . In the proof of  $SO(2s+1)$  we showed that any  $\rho_i'$  could be generated over  $\mathbb{Z}[\rho_1, \dots, \rho_i]$  by the set  $\left\{ (x_s + x_s^{-1})^j \right\}_{j=1}^{i-1}$ . (Let us note, that in the case  $SO(2k)$  we let  $W'$  be the subgroup fixing  $x_1$ , the first coordinate, and here we are fixing  $x_s$ , the last coordinate. Obviously, nothing changes if we rename coordinates.) Since  $R(\text{Spin}(2s))[S]$  contains  $\left\{ (x_s + x_s^{-1})^j \right\}_{j=1}^{s-2}$ , we get that  $\rho_1', \dots, \rho_{s-3}'$  is in  $R(\text{Spin}(2s))[S]$ . The other generators of  $R'$  over  $\mathbb{Z}$  are already in  $S$ , therefore  $R(\text{Spin}(2s))[S] = R'$ . To complete the proof of the Lemma, we will show that the Module generated by  $S$  over  $R(\text{Spin}(2s))$  is closed under ring multiplication.

We introduce the following notation to make the combinatorics less cumbersome:

$$(i) \quad F_j = \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_j \leq s \\ \epsilon_i = \pm 1}} \epsilon_{i_1}^1 \dots \epsilon_{i_j}^j \quad (1 \leq j \leq s). \quad \text{Note that } F_j$$

is fixed under  $W(\text{Spin}(2s))$  (hence the letter  $F$ ), and therefore  $F_j \in R(\text{Spin}(2s))$ . [e.g.,  $F_1 = x_1 + \dots + x_s + x_1^{-1} + \dots + x_s^{-1}$ ].

$$(ii) \quad F_j^f = \sum_{\substack{1 \leq i_1 < i_2 \dots < i_j \leq s-1 \\ \epsilon_i = +1}} x_{i_1}^{\epsilon_1} \dots x_{i_j}^{\epsilon_j} \quad (1 \leq j \leq s-1).$$

$F_j^f$  is fixed under  $W^f$  and therefore  $F_j^f \in R^f$ . [e.g.,  $F_1^f = x_1 + \dots + x_{s-1}^{-1} + x_s^{-1}$ ]. Let  $M$  denote the Module generated by  $S$  over  $R(\text{Spin}(2s))$ .

Step 1:  $x_s^S$  is an element of  $M$ .

$$(i) \quad x_s^S = x_s^{S-1}(F_1) - x_s^{S-2} - x_s^{S-1}(F_1^f)$$

The first two terms on the right side are in  $M$ . Therefore, to complete Step 1, we must show that  $x_s^{S-1}(F_1^f)$  is in  $M$ .

(ii) Multiply the equation in (i) by  $x_s^{-1}$ , and we see that  $x_s^{S-2}[F_1^f]$  is in  $M$ .

(iii) Multiplying by the appropriate power of  $(x_s^{-1})$ , we see that  $x_s^j[F_1^f] \in M$  for  $-(s-3) \leq j \leq (s-2)$ .

(iv) We have the following important inductive formula:

$$F_i = x_s(F_{i-1}^f) + x_s^{-1}(F_{i-1}^f) + F_i^f. \quad \text{This formula implies that if}$$

$x_s^j F_{i-1}^f$  is in  $M$  for  $-(s-2) \leq -n \leq j \leq m \leq s-1$ , then  $x_s^k \cdot F_i^f$  is in

$M$  for  $-(n-1) \leq j \leq (m-1)$ , and conversely. (To see this, multiply

the formula by  $x_s^{j-1}$ , and we get  $x_s^{j-1} \cdot F_i = x_s^j \cdot F_{i-1}^f + x_s^{j-2} F_{i-1}^f + x_s^{j-1} \cdot F_i^f$ )

(v) By finite induction on  $i$ , we see that  $x_s^{S-1}(F_1^f)$  is in  $M \Leftrightarrow x_s \cdot F_{s-1}^f$  is in  $M$ . This reduces the problem to showing that

$x_s \cdot F_{s-1}^f$  is in  $M$ . Also, since  $x_s^{s-2} \cdot F_1^f$  is in  $M$  (see (ii)), this implies that  $x_s F_i^f$  is in  $M$  for  $1 \leq i \leq s-2$ .

(vi) We have the following formulas:

$$\Delta_s^+ = x_s^{\frac{1}{2}} \Delta_{s-1}^+ + x_s^{-\frac{1}{2}} \Delta_{s-1}^-$$

$$\Delta_s^- = x_s^{\frac{1}{2}} \Delta_{s-1}^- + x_s^{-\frac{1}{2}} \Delta_{s-1}^+$$

Therefore: (a)  $x_s^{\frac{1}{2}} \Delta_{s-1}^+ (\Delta_s^+) = x_s (\Delta_{s-1}^+ \cdot \Delta_{s-1}^+) + \Delta_{s-1}^+ \cdot \Delta_{s-1}^-$

$$= x_s (\rho_{s-1}^{'+} + \rho_{s-3}^{'+} + \dots) + \rho_{s-2}^{'+} + \rho_{s-4}^{'+} + \dots$$

(b)  $x_s^{\frac{1}{2}} \Delta_{s-1}^- (\Delta_s^-) = x_s (\Delta_{s-1}^- \cdot \Delta_{s-1}^-) + \Delta_{s-1}^- \cdot \Delta_{s-1}^+$

$$= x_s (\rho_{s-1}^{'-} + \rho_{s-3}^{'-} + \dots) + \rho_{s-2}^{'-} + \rho_{s-4}^{'-} + \dots$$

(vii)  $p = x_s^{\frac{1}{2}} \Delta_{s-1}^+ (\Delta_s^+) + x_s^{\frac{1}{2}} \Delta_{s-1}^- (\Delta_s^-)$  is in  $M$ . Therefore,

$$x_s (\rho_{s-1}^{'+} + \rho_{s-1}^{'-}) + x_s (2\rho_{s-3}^{'+} + \dots) + 2\rho_{s-2}^{'+} + \dots \text{ is in } M.$$

Note, that any  $\rho_i^f$  is a sum of the  $F_j^f$  for  $1 \leq j \leq i$ . Also,

$$\rho_{s-1}^{'+} + \rho_{s-1}^{'-} = F_{s-1}^f + \text{a sum of terms from } \{F_i^f\} \text{ with } 1 \leq i \leq s-2.$$

This implies that  $p = F_{s-1}^f + p^f$  where  $p^f$  is in  $M$  by (v). Therefore

$x_s (F_{s-1}^f)$  is in  $M$ , and by (v) the same is true for  $x_s^{s-1} (F_1^f)$ , which

completes Step 1.

Step 2: We will show that  $x_s^j \cdot (x_s^{\frac{1}{2}} \cdot \Delta_{s-1}^+)$  is an element of  $M$  for  $0 \leq j \leq s-1$ .

(i) for  $j = 0$  the result is trivial

(ii) assume true for  $j < r \leq s-1$ . We have:

$$x_s^r (x_s^{\frac{1}{2}} \Delta_{s-1}^+) = x_s^r (\Delta_s^+) - x_s^{r-1} (x_s^{\frac{1}{2}} \cdot \Delta_{s-1}^-)$$

and  $x_s^r (x_s^{\frac{1}{2}} \Delta_{s-1}^-) = x_s^r (\Delta_s^-) - x_s^{r-1} (x_s^{\frac{1}{2}} \cdot \Delta_{s-1}^+)$

proving Step 2.

Step 3:  $(x_s^{\frac{1}{2}} \cdot \Delta_{s-1}^+)^2 = x_s [\rho_{s-1}'^+ + \rho_{s-3}'^+ + \dots]$  which is in  $M$  by part v. in Step 1.

Similarly,  $(x_s^{\frac{1}{2}} \Delta_{s-1}^-)^2 = x_s [\rho_{s-1}'^- + \rho_{s-3}'^- + \dots]$  and  $(x_s^{\frac{1}{2}} \Delta_{s-1}^+) (x_s^{\frac{1}{2}} \Delta_{s-1}^-) = x_s [\rho_{s-2}' + \rho_{s-4}' + \dots]$  are in  $M$ .

Step 4: The previous three steps imply that  $M$  contains

$R(\text{Spin}(2s))[x_s, x_s^{\frac{1}{2}} \Delta_{s-1}^+, x_s^{\frac{1}{2}} \Delta_{s-1}^-]$ . As usual this implies that  $x_s^{-1}$  is in this ring and hence  $M = R(\text{Spin}(2s))[S]$ .

This then completes the proof that  $R(\tilde{T}_n)$  is free over  $R(\text{Spin}(2n))$  of rank  $|W(\text{Spin}(2n))| = 2^{n-1} n!$ .

Proposition 3.10

$R(\text{Spin}(2n))$  is freely generated over  $R(\text{Spin}(2n+1))$  by the set  $\{1, \Delta^+\}$ .

Proof

$R(\text{Spin}(2n+1)) = \mathbb{Z}[\rho_1, \dots, \rho_{n-1}, \Delta] \leq \mathbb{Z}[\rho_1, \dots, \rho_{n-2}, \Delta^+, \Delta^-] = R(\text{Spin}(2n))$ . It is therefore clear that  $R(\text{Spin}(2n+1))[\Delta^+] = R(\text{Spin}(2n))$ .  $\Delta^+$  has two images under  $W(\text{Spin}(2n+1))$ , therefore  $\{1, \Delta^+\}$  are a set of generators.

Theorem 3.11

$R(\tilde{T}_n)$  is free over  $R(\text{Spin}(2n+1))$  of rank  $2^n \cdot n! = |W(\text{Spin}(2n+1))|$ .

Proof

This follows trivially from Theorem 3.6 and Proposition 3.10.

Section 5.  $G_2$

In the classification of simple complex Lie Algebras, it has been shown that besides the four general classes, there are seven "exceptional" algebras. The exceptional algebra whose diagram is made up of two simple roots and three connecting lines is known standardly as  $G_2$ . (See [8].) In an abuse of notation we will call  $G_2$  the simply connected compact Lie Group representing the local structure of the algebra,  $G_2$ .  $G_2$  contains  $SU(3)$  as a subgroup of maximal rank. (See [2].) As usual we have  $R(G_2) \leq R(SU(3)) \leq R(T_2) = Z[x_1^{\pm 1}, x_2^{\pm 1}]$ .

It also follows that  $W(G_2) \geq W(SU(3))$  where  $W(SU(3))$  acts on  $R(T_2)$  as the group of permutation of the set  $\{x_1, x_2, x_1^{-1}, x_2^{-1}\}$ . As was noted in Section 1 of this Chapter,  $R(SU(3)) = Z[\rho_1, \rho_2]$  where  $\rho_1 = x_1 + x_2 + x_1^{-1} \cdot x_2^{-1}$ .  $\rho_2 = x_1^{-1} + x_2^{-1} + x_1 \cdot x_2$ .  $W(G_2)$  contains an element,  $\varphi$ , sending  $x_i \rightarrow x_i^{-1}$  (for  $i = 1, 2$ ) and  $|W(G_2)| = 12$ . (See [8] Chapter VII). This implies that  $W(G_2)$  is the group generated by  $\{\varphi, SU(3)\}$ . (The order of  $W(SU(3)) = 6$ .) Since  $\varphi : \rho_1 \leftrightarrow \rho_2$ , and since  $\rho_1, \rho_2$  are fixed by  $SU(3)$  we have that  $R(G_2) = Z[\Lambda_1, \Lambda_2]$  where  $\Lambda_1 = \rho_1 + \rho_2$  and  $\Lambda_2 = \rho_1 \cdot \rho_2$ .

Theorem 3.12:  $R(SU(3))$  is freely generated over  $R(G_2)$  by  $\{1, \rho_1\}$ .

Theorem 3.13:  $R(T_2)$  is a free  $R(G_2)$  Module of rank  $12 = |W(G_2)|$ .

Proof

Both Theorems follow trivially from the preceding discussion.

## CHAPTER IV

## (The Bi-linear Form)

We are now ready to work with the bi-linear form introduced in Chapter I. Let us recall that if  $\beta : H \rightarrow G$  is the inclusion of a subgroup of maximal rank, then we have  $R(G) \xrightarrow{\beta^*} R(H)$ .  $\beta^*$  is a monomorphism, and we consider  $R(H)$  as an  $R(G)$ -Module.

Under special conditions for  $\pi_1(G)$  and  $\pi_1(H)$  there exists elements  $\Omega(G), \Omega(H) \in R(T)$ , and  $\Omega(G/H) = \frac{\Omega(G)}{\Omega(H)} \in R(H)$ , where  $T$  is a maximal torus for  $H$  and  $G$ . These elements have the following property. For  $x \in R(T)$ , if we let  $A(x) = \sum_{\phi \in W(G)} (\text{sgn } \phi) \cdot \phi(x)$ , the elementary alternating sum of  $x$ , then  $\frac{A(x)}{\Omega(G)}$  is in  $R(G)$ . Similarly, if we let  $A'(x) = \sum_{\phi \in W(H)} (\text{sgn } \phi) \phi(x)$ , for  $x \in R(T)$ , then  $\frac{A'(x)}{\Omega(H)}$  is in  $R(H)$ . Finally, for any  $z \in R(H)$ , we have  $\frac{A''(\Omega(H) \cdot z)}{\Omega(G)}$  is in  $R(G)$ . (See [4].)

The existence of these properties are proved in [4] under the condition that  $\pi_1(G)$  and  $\pi_1(H)$  have no 2-torsion. The proof depends heavily on the following theorem.

**Theorem 4.1:** Let  $\pi_1(G)$  have no 2-torsion. Then:

(1) there exists an element  $\Omega(G)$  in  $R(T)$  such that the mapping  $x \rightarrow x \cdot \Omega(G)$  gives an isomorphism onto, from the additive group of symmetric elements to the additive group of anti-symmetric elements.

(2) Let  $x$  be an irreducible complex representation of  $G$ ,  $(R(G))$

$\leq R(T)$  , then  $x \cdot \Omega(G) = A(y)$  for some representation,  $y$  , in  $R(T)$  .

(3) Conversely, every  $A(y)$  is equal to  $\pm x \Omega(G)$  for some irreducible representation,  $x$  , in  $R(G)$  .

Note: (a) For  $\pi_1(G) = 0$  ,  $\Omega(G)$  is the  $\delta$  introduced in Chapter I, and this Theorem is Theorem 1.8.

(b) Recall, that the notation in [4] is different from the one used above, and that in Chapter I we "translated" the notation of [4].

(c) In [4] Bott gives no proof of this Theorem, and he only gives a few hints as to a possible proof.

(d) In [4] it is also asserted that the condition on  $\pi_1$  is "hereditary." (i.e., if  $\pi_1(G)$  has no 2-torsion, then  $\pi_1(H)$  also has no 2-torsion for  $H \leq G$  of maximal rank.) I am indebted to Father L. Conlon for the following counter example. Let  $G$  be  $\text{Spin}(p+q)$  where  $p$  and  $q$  are even integers,  $(p, q \geq 4)$ . Let  $H$  be the  $Z_2$  quotients of  $\text{Spin}(p) \times \text{Spin}(q)$  by identifying  $(\epsilon_p, \epsilon_q)$  to  $(\text{Id.}, \text{Id.})$ . Here,  $\epsilon_p$  and  $\epsilon_q$  are the respective non-identity elements of the kernel of  $\pi : \text{Spin}(n) \rightarrow \text{SO}(n)$  . Then  $H$  is a subgroup of maximal rank contained in  $G$  . In this case  $\pi_1(G) = 0$  and  $\pi_1(H) \cong Z_2$  providing a counter example.

We will now prove Theorem 4.1, following closely the proof given in [1] for the case  $\pi_1(G) = 0$  and the hints provided in [4].

#### Lemma 4.2

Suppose  $\beta$  is a weight, where  $\beta$  is one-half the sum of the positive roots of  $G$  , or, if  $\beta$  is not a weight, suppose that there exists a one-dimensional complex representation,  $\rho$  , of  $G$  ,  $\rho|T = e^{2\pi i w}$  , with  $\beta + \frac{1}{2}w$  a weight. Then if we set  $\Omega(G) = A(\beta)$  , in the first case, and set  $\Omega(G) = A(\beta + \frac{1}{2}w)$  , in the second case, then  $\Omega(G)$

will satisfy the conditions of Theorem 4.1.

Proof

If  $\beta$  is a weight (e.g., if  $\pi_1(G) = 0$ ), then  $A(\beta) = \delta$  (in the notation of Theorem 1.8) and the Lemma is just Theorem 1.8.

For the case  $\beta$  is not a weight, let us call  $\beta + \frac{1}{2}w$ ,  $\beta'$ , and let  $A(\beta') = \delta'$ . Then, following the same line of argument used in [1] to prove Theorem 1.8, we will show that if we substitute  $\beta'$  for  $\beta$ , and  $\delta'$  for  $\delta$ , all the conclusions remain valid.

Sub-Lemma 4.3 ([1,6.5])

If  $\bar{x}$  in  $R(T)$  is anti-symmetric, then  $\bar{x}$  is equal to  $\prod_{j=1}^m (e^{2\pi i \theta_j} - 1) \cdot \psi$ , where  $\psi$  is in  $R(T)$  and  $\{\theta_1, \dots, \theta_m\}$  is the set of positive roots.

We will now prove Lemma 4.2:

A. (1)  $\delta'$  is anti-symmetric, therefore  $\psi \cdot \delta'$  is anti-symmetric for  $\psi \in R(G) = R(T)^{W(G)}$ .

$$(2) \quad \delta' = \delta \cdot e^{\pi i w}, \quad \delta \neq 0 \quad [1,6.6], \quad \text{and} \quad e^{\pi i w(t)} \neq 0 \quad \forall t \in T.$$

Therefore  $\psi \longrightarrow \psi \cdot \delta'$  is a monomorphism.

(3) (onto). Suppose  $\bar{x}$  is an anti-symmetric element of  $R(T)$ .

Then by the Sub-Lemma  $\bar{x} = \prod_{j=1}^m (e^{2\pi i \theta_j} - 1) \psi$ . Since  $\frac{1}{2}(\theta_1 + \dots + \theta_m) + \frac{1}{2}w$  is a weight and  $\frac{1}{2}(\theta_1 + \dots + \theta_m)$  are weights, it follows that  $-\frac{1}{2}(\theta_1 + \dots + \theta_m) + \frac{1}{2}w$  and  $\frac{1}{2}(\theta_1 + \dots + \theta_m) - \frac{1}{2}w$  are also weights. Multiply  $\prod_{j=1}^m (e^{2\pi i \theta_j} - 1)$  by  $e^{-\pi i (\theta_1 + \dots + \theta_m - w)}$  and multiply  $\psi$  by  $e^{\pi i (\theta_1 + \dots + \theta_m - w)}$  and we get that  $\bar{x} = \prod_{j=1}^m (e^{\pi i \theta_j} - e^{-\pi i \theta_j}) \cdot e^{\pi i w} \cdot \psi'$ , where  $\psi' = \psi \cdot e^{\pi i (\theta_1 + \dots + \theta_m - w)}$

is an element of  $R(T)$ . Therefore we have  $\bar{x} = \delta \cdot e^{\pi i w} \cdot \psi' = \delta' \cdot \psi'$ .

We will now show that  $\psi'$  is symmetric. (Therefore, the map  $x \rightarrow x \delta$  is an epimorphism.)  $\phi(\bar{x}) = \text{sgn}(\phi) \bar{x} = \text{sgn}(\phi) \cdot \delta' \cdot \psi'$ , for  $\phi \in W(G)$ .  $\phi(\bar{x}) = \phi(\delta' \cdot \psi') = \text{sgn}(\phi) \delta' \cdot \phi(\psi')$ . Since  $\delta'$  is not equal to zero and since  $R(T)$  is an integral domain, this implies that  $\psi' = \phi(\psi') \forall \phi \in W(G)$ . Part A therefore implies the first of the conclusions of Theorem 4.1.

B. If we note that  $\delta' \cdot \bar{\delta}' = \bar{\delta} \cdot \delta$ . (This follows because  $e^{\overline{\pi i w}} = e^{-\pi i w}$ ) . Then the other two conclusions follow by the same proof as in [1,6.16,6.18] if we change every  $\delta$  to  $\delta'$  in the statements and proofs.

Part A and Part B therefore imply Lemma 4.2. We will now show that if  $\pi_1(G)$  has no 2-torsion then a one-dimensional element of  $R(G)$  exists having the properties described in Lemma 4.2. This will then conclude the proof of Theorem 4.1.

Discussion: Let  $\text{Spin}^c(N)$ ,  $N \geq 4$ , be the following  $Z_2$  quotient of  $\text{Spin}(N) \times S^1$ . Identify  $(\epsilon, -1)$  to  $(\text{Id.}, \text{Id.})$ , where  $\epsilon$  is as usual and  $-1 = e^{-\pi i} \in S^1$ . Let  $q : \text{Spin}(N) \times S^1 \rightarrow \text{SO}(N) \times S^1$  be the projection of  $\text{Spin}(N)$  onto  $\text{SO}(N)$  on the left factor and the usual double cover of  $S^1$  on the right factor.  $q$  factors through  $\text{Spin}^c(N)$  and we have  $\bar{q} : \text{Spin}^c(N) \rightarrow \text{SO}(N) \times S^1$ . Let  $\pi_1 : \text{SO}(N) \times S^1 \rightarrow \text{SO}(N)$  be projection onto the first factor. Then define  $\pi : \text{Spin}^c(N) \rightarrow \text{SO}(N)$  to be  $\pi_1 \circ \bar{q}$ .

If  $G$  is a compact connected Lie group, we have previously noted that  $\text{Ad} : G \rightarrow \text{Aut}(G_e)$  is actually a homomorphism from  $G$  to  $\text{SO}(N)$  for some  $N$ . If we can find a homomorphism  $\tilde{\text{Ad}} : G \rightarrow \text{Spin}^c(N)$  making the following diagram commute:

$$\begin{array}{ccc}
 & \tilde{\text{Ad}} & \\
 & \nearrow & \\
 G & \xrightarrow{\text{Ad}} & \text{SO}(N) \\
 & \searrow & \\
 & \downarrow \pi & \\
 & \text{Spin}^c(N) & 
 \end{array}
 , \text{ then we say that}$$

"Ad lifts to  $\text{Spin}^c(N)$ ."

If  $\lambda$  is any homomorphism,  $\lambda : G \rightarrow S^1$ , then  $\text{Ad} \times \lambda : G \rightarrow \text{SO}(N) \times S^1$  is a "partial lift" of  $\text{Ad}$ , in the sense that

$$\begin{array}{ccc}
 & \text{Spin}^c(N) & \\
 & \downarrow \bar{q} & \\
 & \text{SO}(N) \times S^1 & \\
 \nearrow \text{Ad} \times \lambda & & \downarrow \pi_1 \\
 G & \xrightarrow{\text{Ad}} & \text{SO}(N)
 \end{array}$$

this diagram commutes.

Since  $\bar{q} : \text{Spin}^c(N) \rightarrow \text{SO}(N) \times S^1$  is just a covering space map, the problem of lifting  $\text{Ad} \times \lambda$  resolves itself into a homotopy problem. (i.e.,  $\text{Ad} \times \lambda$  lifts to  $\tilde{\text{Ad}} \Leftrightarrow \bar{q}_*(\pi_1(\text{Spin}^c(N))) \subseteq (\text{Ad} \times \lambda)_*(\pi_1(G))$ . (See for example [5].)

The questions that must first be answered are: (1) What is  $\pi_1(\text{Spin}^c(N))$  and, (2) What is  $\bar{q}_*(\pi_1(\text{Spin}^c(N)))$ ?

(1) Let  $\bar{\Phi} : \frac{\text{Spin}(N) \times S^1}{Z_2} \rightarrow S^1/Z_2$  be the following fiber bundle. The space  $S^1/Z_2$  is the quotient space of  $S^1$  gotten by identifying  $e^{-\pi i}$  with  $1$ .  $\frac{\text{Spin}(N) \times S^1}{Z_2}$  is just  $\text{Spin}^c(N)$ . The fiber is clearly  $\text{Spin}(N)$ . From the homotopy sequence for a fibration we get  $0 = \pi_1(\text{Spin}(N)) \rightarrow \pi_1(\text{Spin}^c(N)) \xrightarrow{\cong} \pi_1\left(\frac{S^1}{Z_2}\right) \rightarrow \pi_0(\text{Spin}(N)) = 0$ .  $S^1/Z_2 \cong S^1$  and therefore  $\pi_1\left(\frac{S^1}{Z_2}\right) \cong Z$ . A generator for  $\pi_1(S^1/Z)$  is the curve  $f : I \rightarrow S^1/Z_2$  defined by  $f(t) = [e^{\pi i t}]$ , where the brackets indicate the equivalence class with respect to the  $Z_2$  action. This implies that  $\pi_1(\text{Spin}^c(N))$  is  $Z$  and that a generator is the curve  $\bar{f} : I \rightarrow \frac{\text{Spin}(N) \times S^1}{Z_2}$  defined by  $\bar{f}(t) = [(\sigma(t), f(t))]$  where

$\sigma : I \rightarrow \text{Spin}(n)$  is any curve with  $\sigma(0) = \text{Id}$ .  $\sigma(1) = e$ .

It also follows directly that  $\bar{q}_*(\bar{f})$  is  $(\text{gen.}, \text{gen.}) \in \pi_1(\text{SO}(N) \times S^1) \cong Z_2 \oplus Z$ . This implies that  $\text{Ad} \times \lambda$  will lift to  $\tilde{\text{Ad}} \ni \lambda_* : \pi_1(G) \rightarrow \pi_1(S^1) \cong Z$  is equal to  $\text{Ad}_* : \pi_1(G) \rightarrow \pi_1(\text{SO}(N)) = Z_2, \text{Mod } 2$ . (Note: In [4] it was hinted that this was an essential criteria for lifting.) We therefore have the following lemma:

Lemma 4.4

Let  $h : G \rightarrow \text{SO}(N)$  be any homomorphism. Then  $h$  lifts to  $\tilde{h} : G \rightarrow \text{Spin}^c(N)$ ,  $\tilde{h}$  a homomorphism,  $\ni \mathbb{H}$  a homomorphism  $\lambda : G \rightarrow S^1 \ni \lambda_* : \pi_1(G) \rightarrow Z$  is equal to  $h_* : \pi_1(G) \rightarrow Z_2, \text{Mod } 2$ .

Proof (See preceeding discussion).

Lemma 4.5

If  $\pi_1(G)$  has no 2-torsion, then  $\mathbb{H}$  a map  $\lambda : G \rightarrow S^1 \ni \lambda_*(\text{Mod } 2) = h_*$  where  $h : G \rightarrow \text{SO}(N)$  is any homomorphism.

Proof

$h_* : \pi_1(G) \rightarrow Z_2$ . Since  $\pi_1(G)$  has no 2-torsion, this implies that we can lift  $h_*$  to a homomorphism  $k : \pi_1(G) \rightarrow Z$ . We must therefore show that  $k$  can be induced from a homomorphism  $\lambda : G \rightarrow S^1$ . In fact, we will show that any homomorphism from  $\pi_1(G)$  to  $Z$  can be induced from a homomorphism of  $G$  into  $S^1$ .

For  $G = T_n$ , an  $n$ -torus, the result is trivial.  $\pi_1(T_n) = Z \oplus \dots \oplus Z$  ( $n$ -times), and  $T_n = S^1 \times \dots \times S^1$ . For the general case  $G$ , compact connected, we have that  $G \cong \frac{\bar{G} \times T}{D}$ , where  $\bar{G}$  is semi-simple,  $T$  is torus, and  $D$  is a discrete normal subgroup which intersects  $\bar{G}$  and  $T$  only at the identity. (See [6].)

Let  $\frac{\bar{G} \times T}{D} \xrightarrow{\pi} T/\bar{D}$  be the obvious fiber space, where  $\bar{D}$  is the image of  $D$  under projection onto the second factor. The fiber is then  $\bar{G}$ , and we get the following homotopy sequence:

$$\pi_1(\bar{G}) \longrightarrow \pi_1(G) \longrightarrow \pi_1(T/D) \xrightarrow{\pi_*} \pi_0(\bar{G}) = 0 .$$

Since  $\bar{G}$  is semi-simple,  $\pi_1(\bar{G})$  is a finite group and hence completely torsion.  $\pi_1(T/D)$  is free, and therefore the sequence splits. Since  $\pi_1(\bar{G})$  is completely torsion, any homomorphism from  $\pi_1(G)$  to  $Z$  factors through  $\pi_*$ . This, together with the proof for  $G = T$ , implies the result. (e.g., Let  $K : \pi_1(G) \rightarrow Z$  be a homomorphism. Let  $H \circ \pi_*$  be the factoring. By the case  $G = T$ ,  $H = h_*$  where  $h : T/D \rightarrow S^1$ . Therefore  $(h \circ \pi)_* = K$ .

We are now assured that  $Ad$  lifts to  $Spin^c(N)$  provided  $\pi_1(G)$  has no 2-torsion. The proof of Theorem 4.1 will be complete with the following lemma.

Lemma 4.6

If  $Ad$  lifts to  $Spin^c(N)$ , then  $\Xi$ , a one-dimensional complex representation of  $G$ ,  $\lambda : G \rightarrow S^1$ ,  $\exists \lambda|_T = e^{2\pi i w}$ ,  $w$  is a weight and  $\exists \beta + \frac{1}{2}w$ , is a weight.

Proof

$$\begin{array}{ccccc}
 & & Spin^c(N) & & \\
 & & \downarrow & & \\
 & & SO(N) \times S^1 & \xrightarrow{\frac{1}{2}\Sigma\alpha_1} & R^1 \\
 & \nearrow Ad \times \lambda & \downarrow & & \\
 G & \xrightarrow{Ad} & SO(N) & & 
 \end{array}$$

Let  $x_i$  represent the coordinate functions of the canonical torus of  $SO(N)$ . Then if we take  $(\frac{1}{2}\sum_i x_i) \circ (\text{Ad})|_T$  we get  $\frac{1}{2}(\sum_i \pm \theta_i)$  where the

$\{\theta_i\}$  are the positive roots of  $G$ . (See [1,5.57].) Therefore

$\frac{1}{2}\sum_i x_i \circ \text{Ad} \times \lambda|_T$  is equal to  $\frac{1}{2}\sum_i \pm \theta_i + \frac{1}{2}w$  where  $\lambda|_T = e^{2\pi i w}$ . It is

clear that if the above element is a weight, then so is  $\frac{1}{2}\sum_i \theta_i + \frac{1}{2}w = \beta + \frac{1}{2}w$ .

(See [1,5.57].) We will show that if  $T'$  is the canonical maximal torus of  $SO(N) \times S^1$ , then we can find  $T''$  a maximal torus for  $\text{Spin}^c(N)$ ,  $\pi(T'') = T'$ , such that the following is true. If the integer lattice for  $L(T')$  is the usual (i.e.,  $\{(y_1, \dots, y_n) \mid y_i \in \mathbb{Z}\}$ ), then the integer lattice of  $L(T'')$  has the property that if  $(y_1, \dots, y_n)$  is in the integer lattice, it follows that  $y_1 + \dots + y_n$  is an even integer. As in [1,5.57] this will imply that  $\frac{1}{2}\sum_i \theta_i + \frac{1}{2}w$  is a weight.

From the diagram  $\text{Spin}(N) \times S^1 \longrightarrow \text{Spin}^c(N) \longrightarrow SO(N) \times S^1$ , we get  $\tilde{T}' \longrightarrow T'' \longrightarrow T'$ , where  $\tilde{T}'$  is a maximal torus of  $\text{Spin}(N) \times S^1$ .  $\tilde{T}'$  is equal to  $\tilde{T} \times S^1$  where  $\tilde{T}$  is the torus of  $\text{Spin}(N)$  previously discussed. (See [1,5.56]), and  $S^1$  is represented by  $\mathbb{R}^1/2\mathbb{Z}$ . Therefore  $\tilde{T}'$  has the desired property. When we project, we get  $T'' \leq \text{Spin}^c(N)$ .

By the projection map we get the following "new" integer lattice pts:

$\{(\lambda_1, \lambda_2) \mid \lambda_1, \lambda_2 \text{ are integer lattice points of } L(\tilde{T}), L(\mathbb{R}^1/2\mathbb{Z}), \text{ respectively, with } \lambda_1 = y_1 + \dots + y_n \text{ an odd integer, and } \lambda_2 \text{ an odd integer}\}$ .

Therefore the added lattice points also have the property, that  $\lambda_1 + \lambda_2$  is an even integer. This completes the proof of Theorem 4.1.

Section 2.

The final section of this Chapter will deal with the bi-linear form introduced in the first Chapter. Let us recall how this form was defined and see what general information we have before we actually begin the calculations in Chapter V.

For  $H \leq G$ ,  $H$  of maximal rank, we wish to define  $F : R(H) \times R(H) \rightarrow R(G)$ , a bi-linear form. Let  $x, y$  be elements of  $R(H)$ , then  $F(x, y)$  is defined to be  $\frac{A(\Omega(H)x \cdot y)}{\Omega(G)}$ , where  $A$  is the alternating sum over  $\frac{W(G)}{W(H)}$  (see Chapter I), and  $\Omega(H)$ ,  $\Omega(G)$  are the elements discussed in the previous section. For the case  $H = T$ ,  $\Omega(T) = 1$  and  $A$  is the usual elementary alternating sum over  $W(G)$ . In [4] it was proven that  $F(x, y)$  actually takes values in  $R(G)$ . The bilinearity follows trivially from the fact that  $R(G)$  is fixed by  $W(G)$ .

In the previous section, we showed that  $\Omega(H) = A_H(\beta')$ , where  $\beta'$  is in  $R(T)$  and  $A_H$  represents "alternation with respect to  $W(H)$ ." Similarly, let  $A_G$  and  $A_{G/H}$  denote alternation with respect to  $W(G)$  and  $\frac{W(G)}{W(H)}$ , respectively. Then  $A_{G/H}(\Omega(H) \cdot x) = A_{G/H}(A_H(\beta' \cdot x)) = A_G(\beta' \cdot x)$ , whenever  $x$  is an element of  $R(H)$ . (Since  $x$  is then fixed by  $W(H)$ .)

Our form  $F : R(H) \times R(H) \rightarrow R(G)$  can now be rewritten as  $F(x, y) = \frac{A(\beta' \cdot x \cdot y)}{\Omega(G)}$ , where  $\Omega(H) = A_H(\beta')$ .

In Chapter I we stated Theorem 1.9, the main Theorem of this paper. Assuming that  $R(H)$  is a free  $R(G)$ -Module, we wish to find two sets of bases  $\{a_i\}$  and  $\{b_j\} \ni F(a_i, b_j) = \begin{cases} \pm 1 & i=j \\ 0 & i < j \end{cases}$ . We have previously noted that if a bi-linear form has this property, it implies

that the form is strongly non-singular (see Chapter I).

Let us also note that if such bases exist, it implies the existence of  $\{a'_i\}$  and  $\{b'_j\}$ , two other sets of bases  $\ni F(a'_i, b'_j) = \delta_{ij}$ . Using this fact, we will be able to prove the following strong inductive tool:

Suppose we have  $T \leq H \leq G$ . Then we can define

$$F_1 : R(T) \times R(T) \longrightarrow R(H) , \quad F_2 : R(H) \times R(H) \longrightarrow R(G) \quad \text{and}$$

$F : R(T) \times R(T) \longrightarrow R(G)$ . Although the domain of definition of these forms will not allow us to claim that  $F = F_2 \circ F_1$ , in some sense this is the case. For, let us define  $f : R(T) \longrightarrow R(G)$  as follows:

$$f(x) = \frac{A(x)}{A(\beta)} , \quad x \in R(T) . \quad \text{Then it is clear that } F(x, y) = f(x \cdot y) \quad \text{for}$$

$x, y \in R(T)$ . Similarly, we can define  $f_1 : R(T) \longrightarrow R(H)$  and

$f_2 : R(H) \longrightarrow R(G)$ . For these functions it follows trivially from the calculations at the beginning of this Section that  $f = f_2 \circ f_1$ .

Lemma 4.7 (inductive Lemma)

If  $F_1 : R(T) \times R(T) \longrightarrow R(H)$  and  $F_2 : R(H) \times R(H) \longrightarrow R(G)$  are both strongly non-singular, then so is  $F : R(T) \times R(T) \longrightarrow R(G)$ .

Proof

By the previous observations we may assume that  $\Xi \{x_1, \dots, x_n\}$  and  $\{y_1, \dots, y_n\}$ , two sets of bases for  $R(T)$  as an  $R(H)$ -Module,

$\ni f_1(x_i \cdot y_j) = \delta_{ij}$ . Similarly,  $\Xi \{z_1, \dots, z_m\}$  and  $\{w_1, \dots, w_m\}$ , two sets of bases for  $R(H)$  as an  $R(G)$ -Module,  $\ni f_2(z_\alpha \cdot w_\beta) = \delta_{\alpha\beta}$ .

Clearly,  $\{x_i \cdot z_\alpha\}_{i=1}^N \{z_\alpha \cdot w_\beta\}_{\alpha=1}^M$  and  $\{y_j \cdot w_\beta\}_{j=1}^N \{w_\beta\}_{\beta=1}^M$  are bases for  $R(T)$

as an  $R(G)$ -Module. Furthermore,  $F(x_i \cdot z_\alpha, y_j \cdot w_\beta) = f(x_i \cdot y_j \cdot z_\alpha \cdot w_\beta) =$

$$f_2 \circ f_1(x_i y_j \cdot z_\alpha \cdot w_\beta) = f_2(z_\alpha \cdot w_\beta (f_1(x_i \cdot y_j))) = f_2(z_\alpha \cdot w_\beta) \cdot \delta_{ij} =$$

$\delta_{\alpha\beta} \cdot \delta_{ij}$ . Therefore the result follows:

Remark: This inductive Lemma is not quite strong enough. Given  $T \leq G$ , there are not enough subgroups of  $G$ , of maximal rank, to put in between  $T$  and  $G$ .

Suppose  $W'$  is any subgroup of  $W(G)$ . Let  $R'$  be equal to  $R(T)^{W'}$ . Then we have  $R(G) \leq R' \leq R(T)$ . If  $x$  is in  $R(T)$ , we let  $A'(x)$  denote the alternating sum under the action of  $W' \leq W(G) \cong W$ . If  $z$  is alternating with respect to  $W'$ , (i.e.,  $\emptyset(z) = \text{sgn}(\emptyset) \cdot z$ ), then let  $A''(z)$  be the alternating sum with respect to  $W/W'$ .

We will now make an assumption on  $R'$ , which will allow us to improve the inductive Lemma, by replacing  $R(H)$  with such an  $R'$ .

Suppose there exists an element  $\beta'$  in  $R(T)$  such that  $\frac{A'(x)}{A'(\beta')}$  is

an element of  $R'$  for all  $x$  in  $R(T)$ . Then we can define

$F_1 : R(T) \times R(T) \rightarrow R'$  by  $F_1(x,y) = \frac{A'(x \cdot y)}{A'(\beta')}$   $x,y \in R(T)$ , and define

$F_2 : R' \times R' \rightarrow R(G)$  by  $F_2(z,w) = \frac{A(\beta' \cdot z \cdot w)}{\Omega(G)}$  for  $z,w \in R'$ , and the

proof of the inductive Lemma will apply to this case as well. We therefore have the following "strong" inductive Lemma.

#### Lemma 4.8

Suppose  $R'$  is as above. Then, if  $F_1 : R(T) \times R(T) \rightarrow R'$  and  $F_2 : R' \times R' \rightarrow R(G)$  are both strongly non-singular, we also have that  $F : R(T) \times R(T) \rightarrow R(G)$  is strongly non-singular.

#### Proof

Just note that  $A(\beta' \cdot z \cdot w) = A''(A'(\beta')zw)$  for  $z,w$  in  $R'$ .

Substituting  $A'(\beta')$  for  $\Omega(H)$ , the proof is the same as in Lemma 4.8.

Note: (1) This seems to be a very strong condition on  $R'$ . Yet, for

the cases computed in this paper, there will be enough  $R'$ 's to make the induction much easier.

(2) Even in the case  $R(G) \leq R(H) \leq R(T)$ , we will use the notation  $W'$  for  $W(H)$  and  $W$  for  $W(G)$ . Furthermore,  $A_G$  will be called  $A$ ,  $A_H$  will be notated  $A'$ , and we will use  $A''$  for  $A_{G/H}$ .

(3) As a final note, if  $z$  is an element of  $R(T)$  and if there exists an element  $\emptyset$  in  $W(G)$ ,  $\text{sgn}(\emptyset) = -1$ , such that  $\emptyset(z) = z$ , this implies that  $A(z) = 0$ , where  $A$  denotes alternation with respect to  $W(G)$ . This follows because  $A(\emptyset(z)) = -A(z)$ .

## CHAPTER V.

Section 1. SU(n) and U(n)

Let us recall that  $U(n)$  can be imbedded in  $SU(n+1)$ , and that if  $T_n$  is the canonical torus of  $U(n)$ , we have  $R(SU(n+1)) \leq R(U(n)) \leq R(T_n)$ . If  $R(T_n)$  is described in its usual manner as  $Z[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , then  $R(T_n)$  is freely generated over  $R(U(n))$  by  $\{x_1^{a_1}, \dots, x_{n-1}^{a_{n-1}}\}$  ( $0 \leq a_i \leq n-i$ ). (See Chapter III.)

If we let  $\tau_i$  be the  $i^{\text{th}}$  elementary symmetric function on the set  $\{x_1, \dots, x_n\}$ , we recall that  $R(U(n)) = Z[\tau_{-1}, \dots, \tau_n, \tau_n^{-1}]$  and  $R(U(n))$  is freely generated over  $R(SU(n+1))$  by  $\{\tau_n^j\}$   $-n \leq j \leq 0$ .

In this Section we will show that the bilinear forms

$$F : R(T_n) \times R(T_n) \longrightarrow R(U(n)) \quad \text{and} \quad F : R(U(n)) \times R(U(n)) \longrightarrow R(SU(n+1))$$

are s.n.s. . As we have seen (Lemma 4.7), this will imply that

$$F : R(T_n) \times R(T_n) \longrightarrow R(SU(n+1)) \quad \text{is also s.n.s. .}$$

Before we can calculate these forms, we must first know the elements  $\Omega(SU(n+1))$  and  $\Omega(U(n))$ .

$$\pi_1(SU(n+1)) = 0 \quad [1, 5.49], \quad \text{therefore } \beta \text{ is a weight and}$$

$\Omega(SU(n+1)) = A(\beta)$ . In [1] it is shown that the roots of  $SU(n+1)$  are

the  $(x_i - x_j)$   $i \neq j$   $1 \leq i, j \leq n+1$ , where by  $x_{n+1}$  we mean  $-(x_1 + \dots + x_n)$ .

If we choose the Fundamental Weyl Chamber to be  $x_1 > x_2 > \dots > x_n > x_{n+1}$ ,

then the positive roots are the  $(x_i - x_j)$  with  $i < j$ .  $\beta$  is then just

$nx_1 + (n-1)x_2 + \dots + x_n$ . We will abuse notation and let  $\beta$  also

stand for  $e^{2\pi i \beta}$ , the representation of  $T$  "associated" to the weight  $\beta$ .  $\beta = x_1^n \dots x_n$ , where, as previously noted  $x_i$  stands for the representation  $(x_1, \dots, x_n) \rightarrow e^{2\pi i x_i} \in S^1$ .

$\pi_1(U(n))$  is  $Z$ , (see [1]); therefore we do not expect  $\beta$  to be a weight. The roots of  $U(n)$  are the  $(x_i - x_j)$   $i \neq j$   $1 \leq i, j \leq n$ . Let the Fundamental Weyl Chamber be described by  $x_1 > x_2 > \dots > x_n$ , then the positive roots are the  $(x_i - x_j)$  with  $i < j$ . It is easy to see that  $\beta$  is equal to  $\frac{1}{2}[(n-1)x_1 + (n-3)x_2 + \dots - (n-1)x_n]$ . Therefore  $\beta$  will be a weight if and only if  $n$  is odd.

Case i ( $n = 2K+1$ )

$\beta = Kx_1 + \dots - Kx_n$ , (note that  $x_{K+1}$  has coefficient zero).  $\beta$  is a weight, and the associated representation is  $\beta = x_1^K x_2^{K-1} \dots x_n^{-K}$ .

Case ii ( $n = 2K$ )

$\beta = (K - \frac{1}{2})x_1 + \dots - (K - \frac{1}{2})x_n$ , and  $\beta$  is not a weight. Recalling Lemma 4.2, we want to find a one-dimension representation of  $U(n)$ ,  $\rho$ , such that  $\rho|_T = e^{2\pi i w}$  and such that  $\beta + \frac{1}{2}w = \beta'$  is a weight. It will then follow that  $A(\beta') = \Omega(U(n))$ .

$\tau_n^{-1}$  is a one dimensional representation of  $U(n)$  and  $\tau_n^{-1}|_T = e^{2\pi i(-x_1 - \dots - x_n)}$ . Let  $w = -x_1 - \dots - x_n$ , then  $\beta' = \beta + \frac{1}{2}w$  is equal to  $(K-1)x_1 + (K-2)x_2 + \dots - Kx_n$ , a weight. The associated representation is  $\beta' = x_1^{K-1} \dots x_n^{-K}$ , and by Lemma 4.2  $A(\beta') = \Omega(U(n))$ .

Theorem 5.1

$F : R(T_n) \times R(T_n) \rightarrow R(U(n))$  is strongly non-singular.

Proof (induction on  $n$ )

Suppose the Theorem is true for  $n < r$ .  $R(U(r-1)) =$

$Z[\bar{\tau}_1, \dots, \bar{\tau}_{r-1}, \bar{\tau}_{r-1}^{-1}] \leq Z[x_2^{\pm 1}, \dots, x_r^{\pm 1}] = R(T_{r-1})$ , where the  $\bar{\tau}_i$  are the elementary symmetric function on the set  $\{x_2, \dots, x_r\}$ . By the induction hypothesis, we have  $\{a_i\}$  and  $\{b_j\}$ , two bases for  $R(T_{r-1})$  over  $R(U(r-1))$ , such that  $F(a_i, b_j) = \begin{cases} \pm 1 & i=j \\ 0 & i < j \end{cases}$ .

$W(U(r-1))$  acts as the group of permutations of  $\{x_2, \dots, x_r\}$ . Let  $W'$  be that subgroup of  $W(U(r))$  which fixes  $x_1$ . [Note that  $W'$  acts on  $\{x_2, \dots, x_r\}$  in the exact manner as  $W(U(r-1))$ .]

If we think of  $R(T_r) = Z[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$  as the polynomial ring in the indeterminants  $x_2^{\pm 1}, \dots, x_r^{\pm 1}$  with coefficients in  $Z[x_1^{\pm 1}]$ . (i.e.,  $Z[x_1^{\pm 1}][x_2^{\pm 1}, \dots, x_r^{\pm 1}]$ ), then it is clear that  $R' = R(T_r)^{W'}$  is just  $Z[x_1^{\pm 1}][\bar{\tau}_1, \dots, \bar{\tau}_{r-1}, \bar{\tau}_{r-1}^{-1}]$ . It also follows trivially that the sets  $\{a_i\}$  and  $\{b_j\}$  generate  $R(T_r)$  over  $R' = Z[x_1^{\pm 1}, \bar{\tau}_1, \dots, \bar{\tau}_{r-1}, \bar{\tau}_{r-1}^{-1}]$ .

Let  $\beta'$  be the element of  $R(T_{r-1})$  such that  $A(\beta') = \Omega(U(r-1))$ .

In particular, if  $x$  is any element of  $R(T_{r-1})$ , then  $\frac{A(x)}{A(\beta')}$  is an element of  $R(T_{r-1})^{W(U(r-1))}$ . Now let  $y$  be any element of  $R(T_r)$ .  $\beta'$  can also be considered as an element of  $R(T_r)$  in a natural way.

If we assume  $y$  is a monomial, then  $\frac{A'(y)}{A'(\beta')}$  is an element of  $R'$ .

Since, if  $y$  contains  $x_1^p$ , we have  $y = x_1^p \cdot \bar{y}$ , with  $\bar{y}$  in  $R(T_{r-1})$ .

Then  $\frac{A'(y)}{A'(\beta')} = \frac{x_1^p A'(\bar{y})}{A'(\beta')} = x_1^p \cdot w$ , where  $w$  is in  $R(U(r-1))$ . (This

follows from the association of  $W'$  and  $W(U(r-1))$ .  $x_1^p \cdot w$  is an

element of  $R'$  completing the proof of the assertion. If  $y$  is a

polynomial, just use the fact that  $A'$  is linear. The upshot is that

$R'$  satisfies the hypothesis of Lemma 4.8.

We can therefore define  $F' : R(T_r) \times R(T_r) \rightarrow R'$  (see Lemma 4.8),

and we will now show that the induction hypothesis implies that  $F$  is s.n.s. . As pointed out, the bases  $\{a_i\}$  and  $\{b_j\}$  generate  $R(T_r)$  over  $R'$  .  $F'(a_i, b_j) = \frac{A'(a_i \cdot b_j)}{A'(\beta')}$  is equal to  $\frac{A(a_i b_j)}{A(\beta')}$  , where  $A$  represents alternation with respect to  $W(U(r-1))$  . (This follows because  $a_i, b_j, \beta'$  are elements of  $R(T_{r-1})$ .) This in turn equals  $F(a_i, b_j)$  , where  $F : R(T_{r-1}) \times R(T_{r-1}) \longrightarrow R(U(r-1))$  . But this has the desired property by the induction hypothesis. It now follows by the "strong" inductive Lemma, that if we can show that  $F : R' \times R' \longrightarrow R(U(r))$  is s.n.s. , then the result will follow for  $F : R(T_r) \times R(T_r) \longrightarrow R(U(r))$  .

$$R(U(r)) = Z[\tau_1, \dots, \tau_r, \tau_r^{-1}] \leq Z[x_1^{\pm 1}, \bar{\tau}_1, \dots, \bar{\tau}_{r-1}, \bar{\tau}_{r-1}^{-1}] = R' . \text{ As}$$

in Chapter III, it is clear that  $R(U(r))[x_1^{\pm 1}]$  equals  $R'$  . Since  $x_1$  has  $r$  images under  $W(U(r)) = W$  , we know that the set  $\{1, x_1, \dots, x_1^{+(r-1)}\}$  generates  $R'$  over  $R(U(r))$  . Let this basis, in the order written, be denoted by  $a_i$  . We could have equally well generated  $R'$  over  $R(U(r))$  by  $\{1, x_1^{-1}, \dots, x_1^{-(r-1)}\}$  . Let us denote this set  $\{b_j\}$  . Let us recall that  $F : R' \times R' \longrightarrow R(U(r))$  is defined by  $F(z, w) = \frac{A(\beta' \cdot z \cdot w)}{A(\beta)}$  , where  $\beta'$  is the element induced from the case  $U(r-1)$  .

Case 1: (  $r = 2K + 1$  )

We have  $\beta = x_1^K \dots x_r^{-K}$  and  $\beta' = x_2^{K-1} \dots x_r^{-K}$  . Let  $a_i$  be as described and let  $b'_j$  be  $b_j \cdot x_1^K$  .  $x_1^K$  is a unit in  $R'$  , therefore

$\{b_j \cdot x_1^K\} = \{b'_j\}$  is also a basis for  $R'$  over  $R(U(r))$  .  $F(a_i, b'_j) =$

$$\frac{A(\beta' \cdot a_i b'_j)}{A(\beta)} = \frac{A(\beta \cdot a_i b_j)}{A(\beta)} .$$

Case 2 (  $r = 2K + 2$  )

Then,  $\beta = x_1^K, \dots, x_r^{-(K+1)}$  and  $\beta' = x_2^K, \dots, x_r^{-K}$ . Let  $a_i$  be as described and let  $b'_j$  be  $b_j \cdot x_1^K (x_2^{-1} \dots x_r^{-1})$ .  $x_1^K (x_2^{-1} \dots x_r^{-1}) = x_1^K \tau_{r-1}^{-1}$  is a unit in  $R'$ , therefore  $\{b'_j\}$  is a basis.  $F(a_i, b'_j) =$

$$\frac{A(\beta' \cdot a_i \cdot b'_j)}{A(\beta)} = \frac{A(\beta \cdot a_i \cdot b_j)}{A(\beta)} : \text{ It therefore follows that in either case,}$$

$r$  even or odd, that we must calculate  $\frac{A(\beta \cdot a_i \cdot b_j)}{A(\beta)}$ .

If  $i=j$ , then  $a_i = b_j^{-1}$  and we have  $\frac{A(\beta)}{A(\beta)} = 1$ .

If  $i < j$ , then we have  $\frac{A(\beta \cdot x_1^\ell)}{A(\beta)}$ , where  $-(r-1) \leq \ell < 0$ . In

either case,  $r$  even or odd, we can see that there is some  $x_s$ ,  $s \neq 1$ , where in  $\beta \cdot x_1^\ell$ ,  $x_1$  and  $x_s$  are raised to the same power. Let  $\varphi$  be the element of  $W(U(r))$  permuting  $x_1$  and  $x_s$ , then  $\text{sgn}(\varphi) = -1$  and  $\varphi(\beta \cdot x_1^\ell) = \beta \cdot x_1^\ell$ . By the note at the end of Chapter IV this implies  $A(\beta \cdot x_1^\ell) = 0$ . We must now start the induction.

$n = 2$  ( $U(1)$  is just a torus).

$R(U(2)) = Z[x_1+x_2, x_1 \cdot x_2, (x_1 \cdot x_2)^{-1}] \cong Z[x_1^{\pm 1}, x_2^{\pm 1}] = R(T_2)$ . Two bases for  $R(T_2)$  over  $R(U(2))$  are  $\{1, x_1^{-1}\}$  and  $\{1, x_1\}$ . In this case  $\beta$  is  $x_2^{-1}$ . Let us change the first basis, above, by multiplying by  $x_2^{-1}$ , a unit. We then get  $\{x_2^{-1}, x_1^{-1} \cdot x_2^{-1}\}$ . Let  $\{1, x_1\}$  be denoted by  $\{a_i\}$  and  $\{x_2, x_1^{-1} \cdot x_2^{-1}\}$  by  $\{b_j\}$ , then  $F(a_i, b_j) = \begin{cases} 1 & i=j \\ 0 & i < j \end{cases}$ .  $F(a_i, b_i) =$

$\frac{A(\beta)}{A(\beta)} = 1$ , proving the diagonal part of the statement. If  $i < j$ , we have  $F(x_2^{-1}, x_1^{-1}) = \frac{A(x_1^{-1} \cdot x_2^{-1})}{A(\beta)}$ .  $(x_1 \cdot x_2)^{-1}$  is fixed under  $\varphi$ , the

element of  $W(U(2))$  permuting  $x_1$  and  $x_2$ , therefore  $A(x_1 \cdot x_2) = 0$ .

This concludes the proof of Theorem 5.1.

Let us now note the following. Consider  $U(n-1) \times S^1 \xrightarrow{\alpha} U(n)$ , with  $\alpha(A, e^{2\pi i \theta}) = \begin{pmatrix} A & 0 \\ 0 & e^{2\pi i \theta} \end{pmatrix}$ ,  $A \in U(n-1)$ . Then  $U(n-1) \times S^1$  is a subgroup of maximal rank in  $U(n)$ . Furthermore, (see [1]),  $R(U(n-1) \times S^1) \cong R(U(n-1)) \otimes_{\mathbb{Z}} R(S^1) = \mathbb{Z}[\bar{\tau}_1, \dots, \bar{\tau}_{n-1}, \bar{\tau}_{n-1}^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[x_n, x_n^{-1}] \cong \mathbb{Z}[\bar{\tau}_1, \dots, \bar{\tau}_{n-1}, \bar{\tau}_{n-1}^{-1}, x_n^{\pm 1}] = R'$ , using the notation of Theorem 5.1. Therefore our abstract ring  $R' = R(T_n)^{W'}$  is actually the Representation Ring of a subgroup of maximal rank.

Theorem 5.2:  $F : R(U(n)) \times R(U(n)) \longrightarrow R(SU(n+1))$  is strongly non-singular.

Proof

In Chapter III we proved that  $R(U(n))$  is freely generated over  $R(SU(n+1))$  by the set  $\{1, \tau_n^{-1}, \dots, \tau_n^{-n}\}$ . From the proof in Chapter III it can be easily seen that  $\{1, \tau_n, \dots, \tau_n^n\}$  would also be a basis. Let the first basis be denoted by  $\{b_j\}$  and the second by  $\{a_i\}$ .

The element  $\beta$ , for  $SU(n+1)$ , is  $x_1^n \dots x_n$ . The corresponding  $\beta'$  for  $U(n)$  depends on whether  $n$  is even or odd. In either case, we look for  $u$ , a unit in  $R(U(n))$ , such that  $\beta' \cdot u = \beta$ . If we can find such a  $u$ , let  $b'_j = b_j \cdot u$ . Then  $F(a_i, b'_j)$  is just

$$\frac{A(\beta' \cdot a_i \cdot b'_j)}{A(\beta')} = \frac{A(\beta \cdot a_i \cdot b_j)}{A(\beta)}. \quad \text{For } n = 2K \quad \beta' = x_1^{K-1} \dots x_n^{-K}, \text{ so we let}$$

$$u \text{ be } \tau_n^{K+1}. \quad \text{For } n = 2K + 1, \quad \beta' = x_1^K \dots x_n^{-K} \text{ and let } u \text{ be } \tau_n^K.$$

In either case, to compute  $F(a_i, b'_j)$  we must really compute

$$\frac{A(\beta \cdot a_i \cdot b_j)}{A(\beta)}.$$

$$A(\beta)$$

If  $i = j$ ,  $a_i = b_j^{-1}$  and we get  $\frac{A(\beta)}{A(\beta)}$  which equals  $+1$ .

Suppose  $i < j$ , then we have  $\frac{A(\beta \cdot \tau_n^{-r})}{A(\beta)}$  where  $1 \leq r \leq n$ .

If  $r < n$ ,  $\beta \cdot \tau_n^{-r} = (x_1^n \dots x_n^n) (x_1 \dots x_n)^{-r}$  is a monomial where for some pair  $(i, i+1)$ ,  $x_i$  appears to the first power and  $x_{i+1}$  to the zero power. Let  $\varphi$  in  $W(SU(n+1))$  be that element permuting  $x_i$  and  $x_{i+1}$ . Then, in the monomial  $\varphi(\beta \cdot \tau_n^{-r})$ ,  $x_i$  and  $x_{i+1}$  both appear raised to the  $(-1)$ -power. Therefore  $A(\beta \cdot \tau_n^{-r}) = -A(\varphi(\beta \cdot \tau_n^{-r})) = 0$ , since  $\varphi(\beta \cdot \tau_n^{-r})$  is fixed by  $\alpha$  in  $W(SU(n+1))$ , where  $\alpha$  permutes  $x_i$  and  $x_{i+1}$  (see [1, 6.12]).

This concludes the proof of Theorem 5.2, and implies the following

### Theorem 5.3

$F : R(T) \times R(T) \rightarrow R(SU(n+1))$  is strongly non-singular.

### Proof

The Theorem follows from Theorems 5.2, 5.3, and the inductive lemma.

### Section 2. Sp(n)

The roots of  $Sp(n)$  are  $\pm 2x_i$ ,  $(x_i - x_j)$  and  $\pm(x_i + x_j)$ ,  $i \neq j$ ,  $1 \leq i, j \leq n$ . (See [1, 4.18].) Choose the Fundamental Weyl Chamber by setting  $x_1 > x_2 \dots > x_n > 0$ . The positive roots are then  $\{2x_i\} \cup \{x_i - x_j \mid i < j\} \cup \{x_i + x_j \mid i \neq j\}$ .  $\beta = nx_1 + (n-1)x_2 + \dots + x_n$ , a weight. (This is to be expected since  $\pi_1(Sp(n)) = 0$  (see [1])). As an element of  $R(T_n)$ ,  $\beta = x_1^n \dots x_n$ .

In Chapter III we showed that  $R(T_n)$  is generated over  $R(Sp(n))$  by  $\{x_1^{a_1} \dots x_n^{a_n}\}$  where  $0 \leq a_i \leq 2(n-i) + 1$ . Let us recall that we also

showed that  $R(\text{Sp}(n))[x_1] = R'$ , where  $R'$  is the subring of  $R(T_n)$  fixed by  $W'$ , and  $W'$  is the subgroup of  $W(\text{Sp}(n))$  fixing  $\{x_1\}$ . We noted there that  $W'$  was isomorphic to  $W(\text{Sp}(n-1))$ , and that in fact if we consider  $R(\text{Sp}(n-1))$  as a subring of  $Z[x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ , then  $R'$  is just  $R(\text{Sp}(n-1))[x_1, x_1^{-1}] \leq Z[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ . Exactly as in the case of  $U(n)$ , the upshot of this is the following. If we let  $\beta'$  be the element of  $R(T_{n-1})$  such that  $A(\beta') = \Omega(\text{Sp}(n-1))$ , then considering  $\beta'$  as in  $R(T_n)$ ,  $\beta'$  is the element which makes  $R'$  satisfy Lemma 4.8. Furthermore, suppose that  $F : R(T_{n-1}) \times R(T_{n-1}) \rightarrow R(\text{Sp}(n-1))$  is strongly non-singular, and that  $\{a_i\}$  and  $\{b_j\}$  are the bases making the form s.n.s., then, clearly, these elements are bases for  $R(T_n)$  over  $R'$  and make  $F : R(T_n) \times R(T_n) \rightarrow R'$  s.n.s..

#### Theorem 5.4

$F : R(T_K) \times R(T_K) \rightarrow R(\text{Sp}(K))$  is strongly non-singular.

#### Proof (induction on $K$ )

Assume true for  $K < n$ . Since the theorem is true for  $K = n-1$ ,  $F : R(T_n) \times R(T_n) \rightarrow R'$  is strongly non-singular, by the previous discussion. By the strong inductive Lemma, we need only prove that  $F : R' \times R' \rightarrow R(\text{Sp}(n))$  is strongly non-singular to complete the general inductive step.

In Chapter III we also showed that  $\{1, x_1, \dots, x_1^{2n-1}\}$  generates  $R'$  over  $R(\text{Sp}(n))$ . Let us call this set  $\{a_i\}$ . It is also clear that the set  $\{1, x_1^{-1}, \dots, x_1^{-(2n-1)}\}$  will yield a basis. Let us denote this set by  $\{b_j\}$ .  $\beta = x_1^n \dots x_n$  and  $\beta' = x_2^{n-1} \dots x_n$ , so we will again look for a unit,  $u$ , in  $R'$  such that  $\beta' \cdot u = \beta$ . Such a unit is the element  $x_1^K$ . As usual, let  $b'_j = b_j \cdot x_1^K$ . Then

$$F(a_i, b'_j) = \frac{A(\beta \cdot a_i \cdot b_j)}{A(\beta)} .$$

If  $i = j$ , then  $a_i = b_i^{-1}$ , therefore we get  $\frac{A(\beta)}{A(\beta)} = +1$ .

Suppose  $i < j$ . Then  $F(a_i, b'_j) = \frac{A(\beta \cdot x_1^r)}{A(\beta)}$  where  $-(2n-1) \leq r \leq -1$ .

In the monomial  $\beta \cdot x_1^r$ , there appears some  $x_\ell$ , such that the power that  $x_1$  is raised to and the power that  $x_\ell$  is raised to are the same in absolute value. Since there is a  $\varphi$  in  $W(\text{Sp}(n))$  which permutes  $x_\ell$  and  $x_1^{-1}$ , it follows that  $\beta \cdot x_1^r$  is fixed by some element of  $W(\text{Sp}(n))$ . This implies that  $A(\beta \cdot x_1^r) = 0$ , completing this part of the proof (see [1, 6.12]).

$G = \text{Sp}(1)$

$R(\text{Sp}(1)) \leq Z[x_1, x_1^{-1}] = R(T_1)$ .  $R(T_1)$  is generated by  $\{1, x_1\}$  over  $R(T_1)$ . The set  $\{1, x_1^{-1}\}$  is also a basis. Call the first set  $\{a_i\}_{i=1}^2$  and the second set  $\{b_j\}_{j=1}^2$ .  $\beta = x_1$ , so, as usualy, multiply  $b_j$  by  $x_1$ , a unit, to get  $\{b'_j\}$ . Then  $F(a_i, b'_j) = \begin{cases} +1 & i = j \\ 0 & i < j \end{cases}$ .

[Note:  $A(1) = 0$  implies  $F(x_1, x_1^{-1}) = 0$ ].

$$\left( \left( F(a_i, b'_j) \right) \right) = \begin{pmatrix} F(1, x_1) & F(1, 1) \\ F(x_1, x_1) & F(x_1, 1) \end{pmatrix} .$$

Section 3.  $\text{Spin}(n)$  and  $\text{Spin}^c(2K)$

Let us recall that we have  $R(\text{Spin}(2K+1)) \leq R(\text{Spin}(2K)) \leq R(\tilde{T}_K)$ . We noted in Chapter III that  $R(\text{Spin}(2K))$  is freely generated over  $R(\text{Spin}(2K+1))$  by the set  $\{1, \Delta^+\}$ . To compute the matrix of  $F : R(\text{Spin}(2K)) \times R(\text{Spin}(2K)) \rightarrow R(\text{Spin}(2K+1))$  will therefore only involve a two-by-two matrix, and will be quite easy. Our main task will therefore be to prove the strong non-singularity of  $F : R(\tilde{T}_K) \times R(\tilde{T}_K) \rightarrow R(\text{Spin}(2K))$ .

Let us recall that  $\text{Spin}(2K)$  is the simply connected double cover of  $\text{SO}(2K)$ . We pointed out in Chapter III that this implied that  $W(\text{Spin}(2K)) \cong W(\text{SO}(2K))$ . In fact, if we let  $T$  be the maximal torus of  $\text{SO}(2K)$  previously described, and let  $\tilde{T}$  be the subgroup of  $\text{Spin}(2K)$  covering  $T$ , then  $\tilde{T}$  is a maximal torus of  $\text{Spin}(2K)$  (Chapter III). If  $R(T) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_K^{\pm 1}]$ , then we can regard the  $x_i$  as representations of  $\tilde{T}$  by first projecting onto  $T$  and then following by  $x_i$ . In Chapter III we noted that  $R(\tilde{T}) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_K^{\pm 1}, x_1^{\frac{1}{2}} \dots x_K^{\frac{1}{2}}]$  and that  $W(\text{Spin}(2K))$  acts on the set  $\{x_1^{\pm 1}, \dots, x_K^{\pm 1}\}$  in the same way that  $W(\text{SO}(2K))$  acts on this set. Furthermore, we noted that  $L(T) \cong L(\tilde{T})$ , as Euclidean spaces, and that the walls of the Weyl Chambers were described by the "same" linear forms. In particular, this implies that the roots of  $\text{Spin}(2K)$ , regarded as linear forms on  $L(\tilde{T})$ , are just  $\{x_i - x_j\} \cup \{-(x_i + x_j)\}$ ,  $i \neq j$ ,  $1 \leq i, j \leq K$ . [1]. (Notice, if the  $x_i$  are regarded as linear forms on  $L(T)$ , then these are just the roots of  $\text{SO}(2K)$ .) Let the fundamental Weyl Chamber be chosen such that  $x_k > x_{k-1} > \dots > x_1 > x_2$ . Then the positive roots are  $\{x_i - x_j\}$   $i > j$  and  $\{(x_i + x_j)\}$   $i \neq j$ .  $\beta$  is equal to  $(K-1)x_k + \dots + x_2$  and is a weight. This is to be expected, since  $\pi_1(\text{Spin}(2K)) = 0$ . Viewing  $\beta$  as a representation of  $\tilde{T}_K$ ,  $\beta = x_2 \dots x_K^{K-1}$ .

In Chapter III we introduced  $W'$ , the subgroup of  $W(\text{Spin}(2K))$  which fixes  $\{x_K\}$ , and  $R'$ , the fixed subring of  $W'$ . We will now show that  $R'$  is actually the representation ring of  $\text{Spin}^c(2K-2)$ , a subgroup of  $\text{Spin}(2K)$  of maximal rank.

We can view  $\text{SO}(2K-2) \times \text{SO}(2)$  as a subgroup of  $\text{SO}(2K)$  as follows.

If  $(A,B)$  is an element of  $SO(2K-2) \times SO(2)$ ,  $A \in SO(2K-2)$ ,  $B \in SO(2)$ , then sending  $(A,B)$  to  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in SO(2K)$  yields an injection. Since the image of  $SO(2K-2) \times SO(2)$  in  $SO(2K)$  contains the maximal torus of  $SO(2K)$  previously described (see Chapter III), it is of maximal rank. Then,  $\text{Spin}^c(2K-2)$  is the subgroup of  $\text{Spin}(2K)$ , double covering  $SO(2K-2) \times SO(2)$ . (See [2].) It then follows, as in the case of  $\text{Spin}(2K)$ , via  $SO(2K)$ , that the Weyl group of  $\text{Spin}^c(2K)$  is isomorphic to  $W(SO(2K-2) \times SO(2))$ . The roots of  $SO(2K-2) \times SO(2)$  are the same as the roots of  $SO(2K-2)$ , because  $SO(2) \cong S^1$  is a torus. Therefore, the roots of  $\text{Spin}^c(2K-2)$  are just the roots of  $SO(2K-2)$ . In particular, one-half the sum of the positive roots,  $\beta$ , is equal to  $(K-2)x_{K-1} + \dots + x_2$ , a weight. Therefore, the previous theory applies and  $F : R(\tilde{T}) \times R(\tilde{T}) \rightarrow R(\text{Spin}^c(2K-2))$  can be defined, where  $\tilde{T}$  is the maximal torus for  $\text{Spin}(2K)$  previously described. (Note:  $\text{Spin}^c(2K-2)$  is of maximal rank and contains  $\tilde{T}$ .) We have previously noted that  $W'$  acts on  $\{x_1^{\pm 1}, \dots, x_{K-1}^{\pm 1}\}$  in the same manner as  $W(\text{Spin}(2K-2))$  acts on this set. Therefore,  $\text{Spin}^c(2K-2)$ , having  $W(\text{Spin}(2K-2))$  as its Weyl group, implies that  $R' = R(\tilde{T}_K)^{W'}$  is equal to  $R(\text{Spin}^c(2K-2))$ .

In Chapter III we proved that if  $a_1, \dots, a_N$  was a basis for  $R(\tilde{T}_{K-1})$  over  $R(\text{Spin}(2K-2))$ ,  $N = |W(\text{Spin}(2K-2))|$ , then the set  $\{a'_i\}$  generates  $R(\tilde{T}_K)$  over  $R'$ . (Recall that the  $a'_i$  are the images of  $a_i$  under the map fixing an integral monomial and sending a non-integral monomial  $m$ , to  $m \cdot x_K^{\frac{1}{2}}$ .) Since  $N = |W(\text{Spin}^c(2K-2))|$ , this implies that the  $a'_i$  freely generate  $R(\text{Spin}(2K-2)) \cong R'$ . (See Chapter II.)

We will now show that information about the form for the case  $G = \text{Spin}(2K-2)$ , will yield information for the case  $\text{Spin}^c(2K-2)$ .

Let us suppose that  $F : F(\tilde{T}_{K-1}) \times R(\tilde{T}_{K-1}) \longrightarrow R(\text{Spin}(2K-2))$  is strongly non-singular. Then  $\mathbb{H} \{a_i\}$  and  $\{b_j\} \ni F(a_i, b_j) = \begin{cases} \pm 1 & i = j \\ 0 & i < j \end{cases}$ .

Let us furthermore assume that these  $a_i$  and  $b_j$  are homogeneous elements. By Lemma 3.7 the sets  $\{a'_i\}$  and  $\{b''_j\}$  are a basis for  $R(\tilde{T}_K)$  over  $R(\text{Spin}^c(2K-2))$ . By inspecting the proof of Lemma 3.7, we can see that if we multiply the non-integral  $b_j$  by  $x_K^{-\frac{1}{2}}$ , the proof works just as well. Let us call this new basis  $\{b''_j\}$ .

Let us now inspect  $\bar{F} : R(\tilde{T}_K) \times R(\tilde{T}_K) \longrightarrow R(\text{Spin}^c(2K-2))$ .

$$\bar{F}(a'_i, b''_j) = \frac{A'(a'_i \cdot b''_j)}{A'(\beta')} , \quad \beta' = x_2 \dots x_{K-1} .$$

Notice that  $a'_i \cdot b''_i = a_i \cdot b_i$ ,

since we multiplied one by  $x_K^{\frac{1}{2}}$  and the other by  $x_K^{-\frac{1}{2}}$ . Since  $a'_i \cdot b''_i = a_i \cdot b_i$  is an element of  $R(\tilde{T}_K)$ , and since no factor of  $x_K$  appears, this implies that  $a'_i \cdot b''_i = a_i \cdot b_i$  is a homogeneous integral element of  $R(\tilde{T}_K)$ . Let us note that there is an isomorphism onto from

$$S = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{K-1}^{\pm 1}] \leq R(\tilde{T}_{K-1}) \quad \text{to} \quad \bar{S} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{K-1}^{\pm 1}] \leq R(\tilde{T}_K)$$

described

by sending  $x_i \longrightarrow x_i$ ,  $i \leq K-1$ , and that this isomorphism commutes with the action of  $W(\text{Spin}(2K-2))$  on  $S$  and the action of  $W'$  on  $\bar{S}$ .

By the assumption on  $F$ ,  $A(a_i \cdot b_i) = \pm A(\beta')$ , where  $A$  is alternation under  $W(\text{Spin}(2K-2))$  and  $a_i b_i$  and  $\beta'$  are in  $S$ . Under the above isomorphism, we then have that  $A'(a'_i \cdot b''_i) = A'(a_i \cdot b_i) = \pm A'(\beta')$ , where these elements lie in  $\bar{S}$ . This implies that  $\bar{F}(a'_i, b''_i) = \pm 1$ .

Now let us consider  $\bar{F}(a'_i, b''_j)$  for  $i < j$ , where  $\bar{F}(a'_i, b''_j) =$

$$\frac{A'(a'_i \cdot b''_j)}{A'(\beta')} .$$

We pass to  $\bar{R}$ , the ring extension of  $R(\tilde{T}_K)$  with  $x_K^{\frac{1}{2}}$  adjoined, and let  $W'$  act on  $\bar{R}$  by its usual action on  $\{x_1^{\pm 1}, \dots, x_K^{\pm 1}\}$ . Then  $A'(a_i' \cdot b_j'')$  is an element of  $\bar{R}$  and in  $\bar{R}$  we can factor out the  $x_K$  term.  $A'(a_i' \cdot b_j'') = x_K^{\epsilon/2} \cdot A'(a_i' \cdot b_j)$ ,  $\epsilon = \pm 1$ , since  $W'$  fixes  $x_K^{\epsilon/2}$ . Since  $\bar{R}$  contains  $x_K^{\frac{\pm 1}{2}}$ , this implies that  $R(\tilde{T}_{K-1})$  can be isomorphically imbedded in  $\bar{R}$  and furthermore this isomorphism commutes with the actions of  $W(\text{Spin}(2K-2))$  and  $W'$ , respectively. By the assumption on  $F$ ,  $A(a_i' \cdot b_j) = 0$ ,  $i < j$ . By the above identification of  $R(\tilde{T}_{K-1})$  with a subring of  $\bar{R}$ , we have that  $A'(a_i' \cdot b_j) = 0$  in  $\bar{R}$ . Therefore  $A'(a_i' \cdot b_j'') = x_K^{\epsilon/2} A'(a_i' \cdot b_j) = 0$  in  $\bar{R}$ . But  $A'(a_i' \cdot b_j'')$  is an element of  $R(\tilde{T}_K) \leq \bar{R}$  and therefore  $A'(a_i' \cdot b_j'') = 0$  in  $R(\tilde{T}_K)$ . This finally implies that  $\bar{F}(a_i', b_j'') = 0$  and yields the following Lemma.

Lemma 5.5

Suppose  $F : R(\tilde{T}_{K-1}) \times R(\tilde{T}_{K-1}) \longrightarrow R(\text{Spin}(2K-2))$  is strongly non-singular. Then the same is true for  $\bar{F} : R(\tilde{T}_K) \times R(\tilde{T}_K) \longrightarrow R(\text{Spin}^c(2K-2))$ , where, as usual, we assume that there exists homogeneous bases,  $\{a_i\}$  and  $\{b_j\}$ ,  $\ni F(a_i, b_j) = \begin{cases} \pm 1 & i = j \\ 0 & i < j \end{cases}$ .

Theorem 5.6

$F : R(\tilde{T}_n) \times R(\tilde{T}_n) \longrightarrow R(\text{Spin}(2n))$  is strongly non-singular for all  $n \in \mathbb{Z}$   $n \geq 2$ . Furthermore, the bases  $\{a_i\}$  and  $\{b_j\}$  which make  $F$  s.n.s. are homogeneous elements of  $R(\tilde{T}_n)$ .

Proof (induction on  $n$ )

Inductive Step: Suppose the theorem is true for  $n < K$ . By Lemma 5.5 this implies that  $\bar{F} : R(\tilde{T}_K) \times R(\tilde{T}_K) \longrightarrow R(\text{Spin}^c(2K-2))$  is

strongly non-singular. To complete the inductive step we must show that  $F : R(\text{Spin}^c(2K-2)) \times R(\text{Spin}^c(2K-2)) \rightarrow R(\text{Spin}(2K))$  is strongly non-singular.

Let us recall that a basis for  $R(\text{Spin}(2K))$  over  $R(\text{Spin}^c(2K-2))$  is the set  $\{x_K^{-(K-2)}, \dots, x_K^{-1}, 1, x_K^{\frac{1}{2}} \Delta_{K-1}^+, x_K^{\frac{1}{2}} \Delta_{K-1}^-, x_K, \dots, x_K^{(K-1)}\}$ . Let us refer to this basis as  $\{a_i\}$  where the ordering is as written. These rings are completely symmetric in the  $x_i, x_i^{-1}$ . Therefore if we take the image of this basis under  $\varphi : x_i \rightarrow x_i^{-1} \forall i$ , we will get a second basis  $\{b_j\}$ . To be explicit  $\{b_j\} = \{x_K^{(K-2)}, \dots, x_K, 1, x_K^{-\frac{1}{2}} \bar{\Delta}_{K-1}^+, x_K^{-\frac{1}{2}} \bar{\Delta}_{K-1}^-, x_K^{-1}, \dots, x_K^{-(K-1)}\}$  in the order written.  $\bar{\Delta}^\pm$  represents the image of  $\Delta^\pm$  under the action of  $\varphi$ . Unless otherwise stated  $\Delta^\pm$  will stand for  $\Delta_{K-1}^\pm$ .

The  $\beta$  associated to  $\text{Spin}(2K)$  is  $\beta = x_2 \dots x_K^{K-1}$  and the  $\beta'$  associated to  $\text{Spin}^c(2K-2)$  is  $\beta' = x_2 \dots x_{K-1}^{K-2}$ . As in the previous cases, there exists a unit in  $R(\text{Spin}^c(2K-2))$ ,  $x_K^{K-1}$ , such that

$\beta' \cdot x_K^{K-1} = \beta$ . Therefore, let  $\{b'_j\}$  be a new basis, where

$b'_j = b_j \cdot x_K^{K-1}$ . We will show that  $F(a_i, b'_j) = \begin{cases} \pm 1 & i = j \\ 0 & i < j \end{cases}$ .

$$F(a_i, b'_j) = \frac{A(\beta' \cdot a_i \cdot b'_j)}{A(\beta)} = \frac{A(\beta' \cdot a_i \cdot b_j \cdot x_K^{K-1})}{A(\beta)} = \frac{A(\beta \cdot a_i \cdot b_j)}{A(\beta)}, \text{ because}$$

$\beta' \cdot x_K^{K-1} = \beta$ . So we have to compute  $\frac{A(\beta \cdot a_i \cdot b_j)}{A(\beta)}$ , where  $A$  is

alternation over  $W(\text{Spin}(2K))$ . By the choice of the two bases, we can see that for almost all  $i$ ,  $a_i \cdot b_i = 1$ . This will imply that for

all those  $i$ 's  $\frac{A(\beta \cdot a_i \cdot b_i)}{A(\beta)} = +1$ . The only two diagonal terms which

give any trouble are the  $\Delta$  terms.

Let us recall that  $\Delta^+ = \sum_{\substack{\epsilon_i = +1 \\ \prod \epsilon_i = 1}} x_1^{\frac{\epsilon_1}{2}} \dots x_{K-1}^{\frac{\epsilon_{K-1}}{2}}$ , and

$\Delta^- = \sum_{\substack{\epsilon_i = +1 \\ \prod \epsilon_i = -1}} x_1^{\frac{\epsilon_1}{2}} \dots x_{K-1}^{\frac{\epsilon_{K-1}}{2}}$ . We must now differentiate between two cases.

i)  $K-1$  is an even integer: Then

$$\bar{\Delta}^+ = \Delta^+ \quad \text{and} \quad \bar{\Delta}^- = \Delta^-$$

ii)  $K-1$  is an odd integer: Then

$$\bar{\Delta}^+ = \Delta^- \quad \text{and} \quad \bar{\Delta}^- = \Delta^+.$$

In Chapter III we introduced the following formulas:

$$\Delta^+ \cdot \Delta^+ = \rho_{K-1}^+ + \rho_{K-3}^+ + \dots$$

$$\Delta^- \cdot \Delta^- = \rho_{K-1}^- + \rho_{K-3}^- + \dots$$

$$\Delta^+ \cdot \Delta^- = \rho_{K-2} + \dots$$

In these relations, determining which sums end with the term  $+1$  depends on whether  $(K-1)$  is an even or odd integer. In either case, the terms  $\bar{\Delta}^+ \cdot \bar{\Delta}^+$  and  $\bar{\Delta}^- \cdot \bar{\Delta}^-$  will be of the form  $1 + \rho_2 + \rho_4 + \dots$ .

To conclude the proof that the diagonal is  $\pm 1$ , we will show that

$A(\beta \cdot \rho_i) = A(\beta \cdot \rho_{K-1}^{\pm})$  is zero. This will imply that the diagonal terms,

$F(x_K^{\frac{1}{2}} \Delta^+, x_K^{-\frac{1}{2}} \bar{\Delta}^+)$  and  $F(x_K^{\frac{1}{2}} \Delta^-, x_K^{-\frac{1}{2}} \bar{\Delta}^-)$ , are equal to 1. Since,

$$A(\beta \cdot x_K^{\frac{1}{2}} \Delta^+ \cdot x_K^{-\frac{1}{2}} \bar{\Delta}^+) = A(\beta \cdot \Delta^+ \cdot \bar{\Delta}^+) = A[\beta(1 + \rho_2 + \dots)] = A(\beta) + \sum_i A(\beta \rho_i) = A(\beta).$$

A similar computation shows that  $A(\beta \cdot x_k^{\frac{1}{2}} \Delta^- \cdot x_K^{-\frac{1}{2}} \Delta^-) = A(\beta)$ . (Note: As we have done previously, the factoring actually takes place in an extension ring which contains  $x_K^{\frac{1}{2}}$ , and in which we can embed  $R(\overline{T}_{K-1})$ ).

Lemma 5.7

$A(\beta \cdot \rho_{K-1}^{\pm})$  and  $A(\beta \rho_i)$  for  $1 \leq i \leq K-2$  both equal zero.

Proof

Suppose  $m$  is a monomial in  $Z[x_1^{\pm 1}, \dots, x_K^{\pm 1}, x_1^{\frac{1}{2}} \dots x_K^{\frac{1}{2}}]$ ,  $m = x_1^{a_1} \dots x_K^{a_K}$ . Suppose  $a_i = \pm a_j$  for some  $i, j$ . Then  $m$  is fixed under some  $\varphi$  in  $W(\text{Spin}(2K))$ . For suppose  $a_i = a_j$ , then let  $\varphi$  be the element permuting  $x_i, x_j$  and fixing the other variables. If  $a_i = -a_j$ , let  $\varphi$  be the element permuting  $x_i$  and  $x_j^{-1}$ . This has an even number of sign changes and is therefore in  $W(\text{Spin}(2K))$ . Call such a monomial symmetric in  $(i, j)$ . (i.e., either if  $a_i = a_j$  or  $a_i = -a_j$ .) As we have noted before,  $A(m) = 0$  if  $m$  is "symmetric in  $(i, j)$ ."

Through an abuse of language, we will call the following  $m$  a "monomial."  $m$  is of the form  $x_{j_1}^{\epsilon_1} \dots x_{j_i}^{\epsilon_i}$   $\epsilon_i = \pm 1$  and  $1 \leq j_1 \leq j_2 \dots \leq j_i \leq K-1$  with the following understanding. If  $j_s = j_{s+1}$ , then  $\epsilon_s = +1, \epsilon_{s+1} = -1$ , and no other  $j_r$  is equal to  $j_s$ . (i.e., we allow terms like  $x_i \cdot x_i^{-1}$  to appear and we do not cancel.)

$\beta \cdot m = (x_2 \dots x_K^{K-1}) (m)$  is therefore a monomial of the form  $x_1^{a_1} \dots x_K^{a_K}$  where the  $0 \leq |a_j| \leq K-1$ . (Just note that the  $\epsilon_i$  can only raise or lower the exponent by 1.) By the Pigeon Hole principle we have one of two possibilities.

(1) each  $a_i$  is different, and the  $\{|a_i|\}$  fill the range

between zero and  $K-1$  .

(2) At least two of the  $a_i$  agree, in absolute value, in which case  $\beta \cdot m$  is symmetric in some  $(i, j)$  .

Let us write  $\rho_i$  as  $\rho_i' + \rho_i''$  where  $\rho_i'$  is made up of monomials of the first type and  $\rho_i''$  is made up of monomials of the second type.  $A(\beta \cdot \rho_i'') = 0$  because each  $m$  in  $\rho_i''$  is of type 2 . Therefore  $A(\beta \cdot \rho_i) = A(\beta \cdot \rho_i')$  .

Since for each  $m$  in  $\rho_i'$  ,  $\beta \cdot m$  is equal to  $x_1^{a_1} \dots x_K^{a_K}$  where the  $\{ |a_j| \}$  fill out the range zero to  $K-1$  , it follows that  $\beta \cdot m = \varphi(\beta)$  for some  $\varphi$  in  $W(\text{Spin}(2K))$  . (Just permute the indices and change signs, until you get  $x_2 \dots x_K^{K-1}$  . If, in the process, you make an odd number of sign changes, just follow the process by  $x_1 \leftrightarrow x_1^{-1}$  and we have added an extra sign change which does not alter the term  $\beta$  .) It follows that for any  $m$  in  $\rho_i'$  ,  $A(m \cdot \beta) = A(\varphi(\beta)) = \pm A(\beta)$  , the plus or minus depending on whether sign  $\varphi$  is positive or negative. We will show that  $\rho_i'$  has an even number of terms,  $\rho_i' = m_1 + \dots + m_s + \bar{m}_1 + \dots + \bar{m}_s$  , such that  $\beta \cdot m_i = \varphi(\beta)$  with  $\text{sgn}(\varphi) = +1$  and  $\beta \cdot \bar{m}_j = \varphi(\beta)$  with  $\text{sgn}(\varphi) = -1$  . This will imply that  $A(\beta \cdot \rho_i') = 0$  , completing the proof.

Let us suppose that  $m$  is a monomial in  $\rho_i'$  (i.e.,  $A(\beta m) \neq 0$ ) .

$m = x_{j_1}^{\epsilon_1} \dots x_{j_i}^{\epsilon_i}$  with the previous conditions on the  $x_{j_r}$  . Suppose

$j_1 \neq j_2$  and furthermore suppose that either  $j_2 \neq j_1 + 1$  or that

$j_2 = j_1 + 1$  but  $\epsilon_1 = \epsilon_2$  . (We will not consider the case  $j_1 = 1$

and  $\epsilon_1 = \epsilon_2 = -1$  .) Then  $(\beta \cdot m)$  is symmetric in  $(j_1 - 1, j_1)$  if

$\epsilon_1 = -1$  , and is symmetric in  $(j_1, j_1 + 1)$  if  $\epsilon_1 = +1$  . This would

imply  $A(\beta \cdot m) = 0$  contradicting our assumption. Therefore we must have  $j_1 = j_2$  (implying  $\epsilon_1 = -\epsilon_2$ ), or  $j_2 = j_1 + 1$  and  $\epsilon_1 = 1$ ,  $\epsilon_2 = -1$ . (The only exception is if  $j_1 = 1$  we can have  $\epsilon_1 = -1$ ,  $\epsilon_2 = -1$ .) Now, if  $j_2 = j_1 + 1$ , we must have  $j_3 \neq j_2$ . For, if  $j_3 = j_2$  (implying  $\epsilon_3 = -\epsilon_2$ ), we again have symmetricity in  $(j_1, j_1+1)$ . (Except in the case  $j_1 = 1$  and  $\epsilon_1 = \epsilon_2 = -1$ , in which case we would get symmetricity in  $(1,2)$ .) The same line of reasoning applied to  $(j_1, j_2)$  applies to  $(j_3, j_4)$  and we must have either  $j_3 = j_4$  or  $j_4 = j_3 + 1$  and  $\epsilon_3 = 1$ ,  $\epsilon_4 = -1$ .

Following this line of reasoning we see that  $m$  must be a monomial of the following type.  $m$  has an even number of terms (i.e.,  $i = 2r$ ), and they come in "pairs." (i.e., for any  $1 \leq j \leq r$ ,  $x_{2j-1}^{\epsilon_{2j-1}} \cdot x_{2j}^{\epsilon_{2j}}$  is of the form  $x_s x_s^{-1}$  or of the form  $x_s \cdot x_{s+1}^{-1}$ , with the usual exception regarding  $x_1^{-1} \cdot x_2^{-1}$ .) The first conclusion of this assertion is that for  $\rho_i$ ,  $i = 2r+1$ ,  $\rho'_i$  is empty and therefore  $A(\beta \cdot \rho_i) = 0$ .

For the case  $i = 2r$ , we will set up a one-to-one correspondence among the  $m_j$ 's in  $\rho_i$  as previously described. In each case, note that multiplying  $\beta$  by  $m_j$  and  $\bar{m}_j$ , respectively, give images of  $\beta$  under elements of  $W(\text{Spin}(2K))$  of opposite sign.

Correspondence 1: Let  $m$  be a monomial of the form  $x_j \cdot x_j^{-1} \dots$  where  $j$  is  $\geq 2$ . Let  $\bar{m}$  be equal to  $x_{j-1} \cdot x_j^{-1} \dots$  where the rest of the monomial is exactly the same as the rest of  $m$ . (Note: For any possible end for  $m$ , we have the same possibility for  $\bar{m}$ .) If we

take  $m \cdot \beta$  and add the extra reflection  $x_{j-1} \longleftrightarrow x_j$  we get  $\bar{m}\beta$ .

Therefore  $m\beta = \varphi(\bar{m}\beta)$  where  $\text{sgn}(\varphi) = -1$ . This sets up our first correspondence. This case covers all possibilities except monomials beginning with  $x_1 \cdot x_1^{-1}$  or with  $x_1^{-1} \cdot x_2^{-1}$ .

Correspondence 2: (Monomials beginning with  $x_1 \cdot x_1^{-1}$ .)

Suppose  $m$  is of the form  $x_1 \cdot x_1^{-1} x_j x_{j+1}^{-1} \dots$  ( $j \geq 2$ ). Then let  $\bar{m}$  be  $x_1 \cdot x_1^{-1} x_{j+1} x_{j+1}^{-1} \dots$ , where, as in Case One, the dots imply, that otherwise, the corresponding monomials have the same terms. (Note that  $\beta \cdot m$  and  $\beta \bar{m}$  yield images of  $\beta$  with "opposite signs.") This will cover all cases except monomials beginning with  $x_1 \cdot x_1^{-1} \cdot x_2 \cdot x_2^{-1}$ . For this set use the following correspondence.

$$x_1 \cdot x_1^{-1} \cdot x_2 \cdot x_2^{-1} x_j \cdot x_{j+1}^{-1} \dots (j \geq 3) \longleftrightarrow x_1 \cdot x_1^{-1} \cdot x_2 \cdot x_2^{-1} x_{j+1} x_{j+1}^{-1} \dots$$

This will cover all cases except monomials beginning with  $x_1 \cdot x_1^{-1} \cdot x_2 \cdot x_2^{-1} x_3 \cdot x_3^{-1}$ .

Following this line of reasoning we can "match up" all monomials beginning with  $x_1 \cdot x_1^{-1}$ , except  $x_1 \cdot x_1^{-1} x_2 x_2^{-1} \dots x_r \cdot x_r^{-1}$ . We will deal with this "exception" in Case 4.

Correspondence 3: (Monomials beginning with  $x_1^{-1} \cdot x_2^{-1}$ )

We use analagous correspondences to those used in Case 3, but we substitute  $x_1^{-1} \cdot x_2^{-1}$  in place of  $x_1 \cdot x_1^{-1}$ . (e.g.,

$$\text{a) } x_1^{-1} \cdot x_2^{-1} x_j \cdot x_{j+1}^{-1} \dots (j \geq 3) \longleftrightarrow x_1^{-1} \cdot x_2^{-1} x_{j+1} \cdot x_{j+1}^{-1} \dots$$

$$\text{b) } x_1^{-1} \cdot x_2^{-1} x_3 \cdot x_3^{-1} x_j \cdot x_{j+1}^{-1} \dots (j \geq 4) \longleftrightarrow x_1^{-1} \cdot x_2^{-1} \cdot x_3 \cdot x_3^{-1} x_{j+1} \cdot x_{j+1}^{-1} \dots )$$

etc.

This will cover all cases except  $x_1^{-1} \cdot x_2^{-1} \cdot x_3^{-1} \cdot x_3^{-1} \cdot \dots \cdot x_{r+1}^{-1} \cdot x_{r+1}^{-1}$ , which will be covered in Case 4.

Correspondence 4:

$$x_1 \cdot x_1^{-1} \cdot x_2 \cdot x_2^{-1} \cdot \dots \cdot x_r \cdot x_r^{-1} \longleftrightarrow x_1^{-1} \cdot x_2^{-1} \cdot x_3 \cdot x_3^{-1} \cdot \dots \cdot x_{r+1} \cdot x_{r+1}^{-1} .$$

(Note: In all cases, the corresponding monomials yield images of  $\beta$  of "different signs.") For the case  $A(\beta \cdot \rho_{K-1}^{\pm}) = 0$ , let us first note that by our previous arguments, if  $K-1$  is odd, the result is trivial.

To check the proof for  $\rho_{K-1}^{\pm}$ , where  $K-1 = 2r$ , just note the following.  $A(\beta \cdot \rho_{K-1}^{\pm}) = 0$  by the preceding argument.  $\rho_{K-1}^{\pm} =$

$$\sum_{\pi \epsilon_i = +1} x_1^{\epsilon_1} \cdot \dots \cdot x_{K-1}^{\epsilon_{K-1}} + \sigma, \text{ where } \sigma \text{ is a sum of monomials, in the usual sense, with } K-3 \text{ or fewer } x_i^{\pm} \text{'s. Similarly, } \rho_{K-1}^{\pm} = \sum_{\pi \epsilon_i = -1} x_1^{\epsilon_1} \cdot \dots \cdot x_{K-1}^{\epsilon_{K-1}} + \sigma,$$

where  $\sigma$  is the same polynomial. (See [1, 7.7].) Furthermore,

$$\rho_{K-1} = \rho_{K-1}^+ + \rho_{K-1}^- . \text{ Let us also note that in } \rho_{K-1}^{\pm} \text{ there are only two}$$

monomials which appear in  $\rho_{K-1}^{\pm}$ . They are  $m_1 = x_1 \cdot x_2^{-1} \cdot x_3 \cdot x_4^{-1} \cdot \dots \cdot x_{K-2} \cdot x_{K-1}^{-1}$ ,

and  $m_2 = x_1^{-1} \cdot x_2^{-1} \cdot x_3 \cdot x_4^{-1} \cdot \dots \cdot x_{K-2} \cdot x_{K-1}^{-1}$ , one appearing in  $\rho_{K-1}^+$  and the

other in  $\rho_{K-1}^-$ . (Depending on whether  $\frac{K-1}{2}$  is even or odd.)

$$\begin{aligned} \text{We now have that } 0 &= A(\beta \cdot \rho_K) = A(\beta[\rho_K^+ + \rho_K^-]) = A(\beta \rho_K^+) + A(\beta \rho_K^-) \\ &= A(\beta \cdot m_1) + A(\beta \cdot m_2) + 2A(\beta \cdot \sigma) . \text{ Note that if } \beta \cdot m_1 = \varphi_1(\beta) \text{ and} \\ \beta \cdot m_2 &= \varphi_2(\beta) , \text{ then } \text{sgn}(\varphi_1) = \text{sgn}(\varphi_2) . \text{ Therefore } 0 = A(\beta \cdot \rho_K) \\ &= 2 \text{sgn}(\varphi_1) A(\beta) + 2A(\beta \cdot \sigma) = 2[\text{sgn}(\varphi_1)A(\beta) + A(\beta \cdot \sigma)] = 0 . \end{aligned}$$

Therefore  $A(\beta \cdot \rho_{K-1}^+) = A(\beta \cdot \rho_{K-1}^-) = \text{sgn}(\varphi_1)A(\beta) + A(\beta \cdot \sigma) = 0$ . This

completes the proof that  $A(\beta \cdot \rho_{K-1}^{\pm}) = 0$ .

This completes the proof of the Lemma 5.7, and as we noted previously, this implies that  $F(a_i, b'_i) = 1$ . Therefore we now know that the diagonal terms are 1, and we will proceed to show that the terms above the diagonal are zero. The following chart will make it easier to follow the cases.

	$x_K^{K-2}, \dots, 1,$	$x_K^{-\frac{1}{2}} \Delta^+$	$x_K^{-\frac{1}{2}} \Delta^- ,$	$x_K^{-1}, \dots, x_K^{-(K-1)}$
$x_K^{-(K-2)}$ ⋮ 1	CASE 1	CASE 2		CASE 1
$x_K^{\frac{1}{2}} \Delta^+$			CASE 4	CASE 3
$x_K^{\frac{1}{2}} \Delta^-$				
$x_K$ ⋮ $x_K^{(K-1)}$				CASE 1

Case 1:  $(a_i = x_K^S, b_j = x_K^{-r} \quad s < r)$ . Therefore,  $\beta \cdot a_i \cdot b_j$   
 $= \beta x_K^{S-r}$  where  $-2K+3 \leq S-r \leq -1$ . For some  $\ell$ ,  $\beta \cdot x_K^{S-r}$  is  
symmetric in  $(\ell, K)$ . Therefore  $A(\beta \cdot a_i b_j) = 0$ .

Case 2:  $(a_i = x_K^S, -(K-2) \leq S \leq 0, b_j = x_K^{-\frac{1}{2}} \bar{\Delta}^+)$ . Therefore  
 $\beta \cdot a_i \cdot b_j$  is equal to  $\beta \cdot x_K^{S-\frac{1}{2}} \cdot \bar{\Delta}^+$ . The terms of  $\bar{\Delta}^+$  are of  
the form  $x_1^{\frac{\epsilon_1}{2}} \dots x_{K-1}^{\frac{\epsilon_{K-1}}{2}}$ . We then have that  $\beta \cdot a_i \cdot b_j$  is a sum  
of terms of the form  $x_1^{\frac{\epsilon_1}{2}} \cdot x_2^{1+\frac{\epsilon_2}{2}} \dots x_K^{\frac{(K-1)+2S-1}{2}}$ . The range of  
the absolute value of the exponents are  $\frac{1}{2}$  integers between  $\frac{1}{2}$  and  
 $\frac{2K-3}{2}$ . By the Pigeon Hole Principle we have that each term is symmetric  
in some  $(i, j)$ . This implies  $A(\beta \cdot a_i b_j) = 0$  for Case 2.

Case 3:  $(a_i = x_K^{\frac{1}{2}} \cdot \Delta^+, b_j = x_K^S \quad -1 \leq S \leq -(K-1))$ .

Exactly as in the previous case, we use the Pigeon Hole Principle on

terms of the form  $x_1^{\frac{\epsilon_1}{2}} \dots x_K^{\frac{K+2S+1}{2}}$ , to get  $A(\beta a_i b_j) = 0$ .

Case 4:  $(a_i = x_K^{\frac{1}{2}} \Delta^+ \quad b_j = x_K^{-\frac{1}{2}} \bar{\Delta}^-)$

$A(\beta \cdot a_i b_j) = A(\beta \cdot \Delta^+ \cdot \bar{\Delta}^-)$ . As in the case of the diagonal element,  
whether  $K-1$  is even or odd,  $\Delta^+ \cdot \bar{\Delta}^- = \rho_1 + \rho_3 + \dots$ . Therefore  
by the previous Lemma  $A(\beta \Delta^+ \cdot \bar{\Delta}^-) = 0$ .

This completes the inductive step of the Theorem. We must now  
start our induction with the case  $n = 2$ .

Case:  $G = \text{Spin}(4)$ .

$\beta = x_2 \in R(\tilde{T}_2)$  and two bases for  $R(\tilde{T}_2)$  over  $R(\text{Spin}(4))$  are

$a_i = \{1, x_2^{\frac{1}{2}}(x_1^{-\frac{1}{2}}), x_2^{\frac{1}{2}}(x_1^{\frac{1}{2}}), x_2\}$ ,  $b_j = \{1, x_2^{-\frac{1}{2}}(x_1^{\frac{1}{2}}), x_2^{-\frac{1}{2}}(x_1^{-\frac{1}{2}}), x_2^{-1}\}$ . The terms in the bracket take on the role of  $\Delta_1^{\pm}$ . Let  $b'_j = b_j \cdot x_2$ , then

$$F(a_i, b'_j) = \frac{A(\beta \cdot a_i b'_j)}{A(\beta)}$$

Suppose  $i = j$ , then  $a_i = b_i^{-1}$  and  $F(a_i, b_i) = \frac{A(\beta)}{A(\beta)} = +1$ .

Suppose  $i < j$ , then each of the six terms above the diagonal is symmetric in some  $(r, s)$  and are therefore zero.

	1	$x_2^{-\frac{1}{2}}(x_1^{\frac{1}{2}})$	$x_2^{-\frac{1}{2}}(x_1^{-\frac{1}{2}})$	$x_2^{-1}$
1	1	$\frac{A(x_2^{\frac{1}{2}}x_1^{\frac{1}{2}})}{A(\beta)}$	$\frac{A(x_1^{-\frac{1}{2}}x_2^{\frac{1}{2}})}{A(\beta)}$	$\frac{A(1)}{A(\beta)}$
$x_2^{\frac{1}{2}}(x_1^{-\frac{1}{2}})$		1	$\frac{A(x_2 \cdot x_1^{-1})}{A(\beta)}$	$\frac{A(x_2^{\frac{1}{2}}x_1^{-\frac{1}{2}})}{A(\beta)}$
$x_2^{\frac{1}{2}}(x_1^{\frac{1}{2}})$			1	$\frac{A(x_2^{\frac{1}{2}}x_1^{\frac{1}{2}})}{A(\beta)}$
$x_2$				1

This completes the proof of Theorem 5.6. Furthermore, as a consequence of the proof of Theorem 5.6, we get the following additional Theorem.

Theorem 5.8

$F : R(\tilde{T}_k) \times R(\tilde{T}_k) \rightarrow R(\text{Spin}^c(2K-2))$  is strongly non-singular for  $2K-2 \geq 4$ .

We remarked at the beginning of this Section that  $R(\text{Spin}(2K+1)) \leq R(\text{Spin}(2K))$  and that  $R(\text{Spin}(2K))$  is generated by  $1, \Delta^+$  over

$R(\text{Spin}(2K+1))$  .

Theorem 5.9

$F : R(\text{Spin}(2K)) \times R(\text{Spin}(2K)) \longrightarrow R(\text{Spin}(2K+1))$  is strongly non-singular.

Proof

The roots of  $\text{Spin}(2K+1)$  are the same as the roots of  $\text{SO}(2K+1)$  . They are  $\pm x_r$  ,  $x_i - x_j$  and  $\pm(x_i + x_j)$   $i \neq j$   $1 \leq i, j, r \leq K$  . Let the Fundamental Weyl Chamber be described by  $0 < x_1 < \dots < x_K$  , then the positive roots are  $x_r$  ,  $x_i - x_j$  for  $i > j$  and  $x_i + x_j$  for  $i \neq j$  . Therefore  $\beta$  is  $\frac{1}{2}(x_1 + 3x_2 + \dots)$  and as a representation

$$\beta = x_1^{\frac{1}{2}} \dots x_K^{\frac{2K-1}{2}} , \text{ an element of } R(\hat{\Gamma}_K) .$$

Let the set  $\{1, \Delta^+\}$  be called  $\{a_i\}_{i=1}^2$  , and let  $\{b_j\}_{j=1}^2$  be the set  $\{\Delta^+, 1\}$

	$\Delta^+$	1
1	1	0
$\Delta^+$	*	1

$$\text{We will show that } F(a_i, b_j) = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases} \cdot F(a_i, b_j) = \frac{A(\beta' \cdot a_i b_j)}{A(\beta)} .$$

The first thing to note is that  $A(\beta') = 0$  .  $\beta'$  is equal to  $x_2 \dots x_{K-1}$  , and if we let  $\varphi$  be the element of  $\text{Spin}(2K+1)$  permuting  $x_1$  and  $x_1^{-1}$  , then  $\varphi(\beta') = \beta'$  , implying that  $A(\beta') = 0$  .

To complete the proof we will show that  $A(\beta' \cdot \Delta^+) = A(\beta)$  .

$$\Delta^+ = \prod_{i=1}^K x_i^{\frac{\epsilon_i}{2}} . \text{ One summand of this sum is the term } x_1^{\frac{1}{2}} \dots x_K^{\frac{1}{2}} . \text{ We also have that } \beta' \cdot x_1^{\frac{1}{2}} \dots x_K^{\frac{1}{2}} = \beta . \text{ We will now}$$

prove that for any other summand,  $\gamma$ ,  $A(\beta'\gamma) = 0$ .

$\gamma$  is equal to  $x_1^{\frac{\epsilon_1}{2}} \dots x_K^{\frac{\epsilon_K}{2}}$ . Suppose  $\epsilon_1 = -1$ , then  $\epsilon_2$  must be equal to  $+1$  if we wish  $A(\beta'\gamma)$  to be non zero. (Otherwise we would have  $x_1^{-\frac{1}{2}} \cdot x_2^{\frac{1}{2}} \dots$ , which is symmetric in  $(1,2)$ ). Furthermore, all  $\epsilon_3$  must be  $+1$  or  $\beta' \cdot \gamma$  would be symmetric in  $(2,3)$ . Following this line of reasoning, the only term  $\gamma$ , which would yield  $\beta'\gamma$ , non-symmetric, is  $\gamma = x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} \dots x_K^{\frac{1}{2}}$  which is not an element of  $\Delta^+$ . Suppose now that  $\epsilon_1 = +1$ , then by the same reasoning, the only element,  $\gamma$ , which gives  $A(\beta'\gamma) \neq 0$  is  $\gamma = x_1^{\frac{1}{2}} \dots x_K^{\frac{1}{2}}$  which we have already considered. Therefore,  $A(\beta' \Delta^+) = A(\beta)$  completing the proof.

By the inductive Lemma we also have the following:

Theorem 5.10

$F : R(\tilde{T}_K) \times R(\tilde{T}_K) \longrightarrow R(\text{Spin}(2K+1))$  is strongly non-singular.

Section 4.  $\text{Spin}^c(2K+1)$

We have previously discussed the properties of  $\pi : \text{Spin}^c(n) \longrightarrow \text{SO}(n)$ . (See Chapter IV.) We can include  $\text{SO}(2K) \times S^1$  in  $\text{SO}(2K+1) \times S^1$  as a subgroup of maximal rank. This in turn induces an inclusion of  $\text{Spin}^c(2K)$  in  $\text{Spin}^c(2K+1)$ , of maximal rank. In particular, we have  $R(\text{Spin}^c(2K+1)) \leq R(\text{Spin}^c(2K)) \leq R(\tilde{T}_{K+1}) \cong \mathbb{Z}[x_1^{\pm 1}, \dots, x_{K+1}^{\pm 1}, x_1^{\frac{1}{2}}, \dots, x_{K+1}^{\frac{1}{2}}]$ .

We have already noted that  $R(\text{Spin}^c(2K))$  is the ring

$\mathbb{Z}[x_{K+1}^{\pm 1}, \rho_1, \dots, \rho_{K-2}, x_{K+1}^{\frac{1}{2}} \Delta_K^+, x_{K+1}^{\frac{1}{2}} \Delta_K^-]$ , where the  $\rho_i$  are the canonical generators for  $\text{SO}(2K)$ .

In Chapter IV we also noted that the roots and Weyl group of  $\text{Spin}^c(n)$  were the same as the roots and Weyl group of  $\text{SO}(n)$ . [By the roots

being the same, recall that we have  $L(\tilde{T}) \cong L(T)$  and we view the roots as linear forms on this Euclidean Space.]

Recall that  $W(SO(2k+1))$  is the group of permutations of the set  $\{x_1, x_1^{-1}, \dots, x_K, x_K^{-1}\}$ . This group acts on  $R(\tilde{T}_{K+1})$ , by fixing  $x_{K+1}$ , and permuting the other  $x_i, x_j^{-1}$ . In particular, then,  $R(\text{Spin}^C(2K+1)) = R(\tilde{T}_{K+1})^W$ , where  $W$  is  $W(\text{Spin}^C(2K+1))$ .

Lemma 5.11

$$R(\text{Spin}^C(2K+1)) \cong \mathbb{Z}[x_{K+1}^{\pm 1}, \rho_1, \dots, \rho_{K-1}, x_{K+1}^{\frac{1}{2}}, \Delta_K] \cong \bar{R}.$$

Proof

Let us look at  $R(\text{Spin}(2K+1)) \leq \mathbb{Z}[x_1^{\pm 1}, \dots, x_K^{\pm 1}, x_1^{\frac{1}{2}}, \dots, x_K^{\frac{1}{2}}]$ . Then  $R(\text{Spin}(2K+1)) \cong \mathbb{Z}[\rho_1, \dots, \rho_{K-1}, \Delta]$  and furthermore,  $R(\text{Spin}(2K+1))$  is equal to  $R(\tilde{T}_K)^W$ , where  $W$  is as above. Since  $W$  fixes  $x_{K+1}^{\pm 1}$ , this implies that  $\bar{R} \leq R(\tilde{T}_{K+1})^W = R(\text{Spin}^C(2K+1))$ .

Suppose  $p$  is an element of  $R(\tilde{T}_{K+1})$ , which is fixed under  $W$ . As in the case of  $\text{Spin}^C(2K)$ , let us write  $p$  as a polynomial in  $x_{K+1}$ . (i.e.,  $p = \sum x_{K+1}^i p_i$ , where the  $i$  are integers or half-integers.) Since  $W$  fixes  $x_{K+1}$ , this implies that each  $p_i$  is fixed under  $W$ . The  $p_i$  are elements of  $R(\tilde{T}_K)$ , thus the  $p_i$  are in  $\mathbb{Z}[\rho_1, \dots, \rho_{K-1}, \Delta]$ . This implies that  $p$  is in  $\bar{R}$ . Therefore  $R(\text{Spin}^C(2K+1)) = \bar{R}$  completing the proof.

Theorem 5.12

$$F : R(\text{Spin}^C(2K)) \times R(\text{Spin}^C(2K)) \longrightarrow R(\text{Spin}^C(2K+1)) \text{ is s.n.s.}$$

Proof

The roots of  $SO(2K+1)$  are  $\pm x_r, x_i - x_j, \pm(x_i + x_j)$ ,  $i \neq j$ ,

$1 \leq i, j, r \leq K$ . Define the same positive Weyl Chamber as in the Case  $\text{Spin}(2K+1)$ . We then have, as in  $\text{Spin}(2K+1)$ ,  $\beta = \frac{1}{2}(x_1 + 3x_2 + \dots + (2K-1)x_K)$ .  $\pi_1(\text{Spin}^c(2K+1)) \cong \mathbb{Z} \neq 0$ , therefore as we might expect,  $\beta$  is not a weight.  $x_{K+1}$  is a weight, and it is the restriction to  $\tilde{T}_{K+1}$  of a 1-dimensional  $\text{Spin}^c(2K+1)$  representation. Furthermore,  $\beta + \frac{1}{2}x_{K+1}$  is a weight. Therefore the element  $\bar{\beta}$ , we need, is  $x_1^{\frac{1}{2}} \dots x_K^{\frac{2K-1}{2}} \cdot x_{K+1}^{\frac{1}{2}}$ , and  $A(\bar{\beta}) = \Omega(\text{Spin}^c(2K+1))$ . (See Chapter IV.)

$\beta'$ , as previously described, for the Case  $H = \text{Spin}^c(2K)$  is equal to  $x_2 \dots x_K^{K-1}$ . A basis for  $R(\text{Spin}^c(2K))$  over  $R(\text{Spin}^c(2K+1))$  is clearly the set  $\{1, x_{K+1}^{+\frac{1}{2}} \Delta^+\}$ . Let a second basis be the set  $\{x_{K+1}^{+\frac{1}{2}} \Delta^+, 1\}$ . Call the first set  $\{a_i\}$  and the second set  $\{b_j\}$ . Then  $F(a_i, b_j) = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases}$ .

	$x_K^{+\frac{1}{2}} \Delta^+$	1
1	1	0
$x_K^{+\frac{1}{2}} \Delta^+$	*	1

As in the case  $R(\text{Spin}(2K+1)) \leq R(\text{Spin}(2K))$  we will get  $A(\beta') = 0$ .

## CHAPTER VI.

## An Observation

Theorem 1.9 was stated under the hypothesis that  $\pi_1(G)$  have no two-torsion. This hypothesis implied that one-half the sum of the positive roots was a weight and allowed us to define the bilinear form. One might then conjecture that for any compact connected Lie group,  $G$ , if one-half the sum of the positive roots is a weight, then the bilinear form defined in Chapter I is strongly non-singular. A counter-example is the Case  $G = SO(6)$ .

For  $G = SO(6)$ , one-half the sum of the positive roots,  $\beta$ , is equal to  $x_2 + 2x_3$ , a weight, where  $R(T_3) = Z[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ . (See Chapter V, Section 3.)

Let us note that  $R(SO(6))$  can be isomorphically imbedded in  $R(\text{Spin}(6))$  and  $R(T_3)$  can be imbedded in  $R(\tilde{T}_3)$ , where  $\tilde{T}_3$  is the torus of  $\text{Spin}(6)$  previously described. (Both maps are induced from the covering space projection.) This isomorphism (into) commutes with the actions of the respective Weyl groups, and in fact, the bilinear form for the Case  $G = SO(6)$  can be interpreted as the bilinear form for  $\text{Spin}(6)$ , restricted to a subring. (See Chapter V.)

If the bilinear form for  $SO(6)$  was strongly non-singular, there would exist two bases  $\{a_i\}$  and  $\{b_j\}$ , making the matrix of the form lower triangular with units on the diagonal. Under the isomorphism connecting  $R(\text{Spin}(6))$  and  $R(SO(6))$  this would imply that the sets  $\{a_i\}$  and  $\{b_j\}$  were a bases for  $R(\tilde{T}_3)$  over  $R(\text{Spin}(6))$ . (Since  $|W(\text{Spin}(6))| = |W(SO(6))|$ ) . Since these bases are made up of homogeneous integral elements, this would, in particular, imply that

$R(\text{Spin}(6))[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$  equals  $Z[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} x_3^{\frac{1}{2}}] = R(\tilde{T}_3)$  .

To complete this counter-example, we will show that this last statement is not true.

Recall that  $R(\text{Spin}(6)) = Z[\rho_1, \Delta^+, \Delta^-]$  (See Chapter V) . Let  $A$  be the following subring of the reals:  $A = \left\{ \frac{n}{3^r} + \frac{m\sqrt{3}}{3^s} \mid n, m, r, s \in Z \right\}$  .

Let  $B$  be the subring of  $A$  defined by:

$B = \left\{ \frac{n}{3^r} + \frac{4 \cdot m \cdot \sqrt{3}}{3^s} \mid n, m, r, s \in Z \right\}$  . We can define a ring homomorphism,

$\varphi$  , from  $R(\tilde{T}_3)$  to  $A$  by setting  $\varphi(x_i) = 3$   $i = 1, 2, 3$ . We will show

that  $\varphi$  maps  $R(\text{Spin}(6))[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$  into  $B$  , while  $\varphi(x_1^{-\frac{1}{2}} \cdot x_2^{\frac{1}{2}} \cdot x_3^{\frac{1}{2}})$

$= \sqrt{3} \notin B$  . This will complete the proof that the bilinear form of  $SO(6)$  is not strongly non-singular.

Note that  $\varphi(\rho_1) = 10$  ,  $\varphi(\Delta^+) = 4\sqrt{3}$  ,  $\varphi(\Delta^-) = \frac{28\sqrt{3}}{3^2}$  and  $\varphi(x_i) = 3$  . Since  $\varphi$  is a ring homomorphism, this proves the claim and provides our counter-example.

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## AUTOBIOGRAPHY

Jack Shapiro was born in New York City on October 2, 1942. After elementary and high school education in that city, he attended Yeshiva University from which he received both his Bachelor of Arts degree in 1964 and his Master of Science degree in 1966.

Jack entered the Ph.D. Program in Mathematics at the Graduate Center of The City University of New York in September, 1966. In December of that year, he married Miss Phyllis Schlissel, and in March, 1969 a son, Michael Leonard, was born to them.