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UNITS, ADMISSIBLE ORIENTED PARALELLOPIPEDS AND BASES

by

DANIEL AULICINO

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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ABSTRACT

UNITS, ADMISSIBLE ORIENTED PARALELLOPIPEDS AND BASES

by

DANIEL AULICINO

Adviser: Professor Harvey Cohn

1. The theory of unit calculation as developed by Minkowski and Voronoii is presented in a unified manner.
2. The geometry and neighboring processes of admissible oriented parallelopipeds is developed in origin symmetric discrete arrays, a more general setting than an irreducible lattice.
3. A new type of admissible oriented parallelopiped, the edge-face admissible oriented parallelopiped, is presented. Its geometry and neighboring process is developed.
4. The concept of rank matrix is introduced to aid in the proofs and understanding of the above theory. It is shown that in n-dimensions, there are p(n) types of n-dimensional admissible oriented paralleopipeds, where p(n) is the partition function.
5. Some new proofs are given of Voronoii's theorems.
6. The basis approach to calculating units is described and placed in a general setting.
7. Cohn's basis approach is described. A minor flaw is located: one of his theorems is invalid. Some new proofs of his theorems are given. Also, a clearer connection is established between MAOP and Cohn bases. Finally, a connection is shown between VAOP and Cohn bases. As a result of these connections a new proof that Cohn bases will generate units results. A possible new way of calculating fundamental units by Cohn bases results and also a possible way of calculating units in higher dimensions is suggested.

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INTRODUCTION

Part I: Classical Survey. In this part, I have attempted to unify the Minkowski and Voronoii general theory of the calculation of units in an irreducible multplicative lattice of dimension three with identity. The algebraic integers of the cubic field Q(0), whose monic irreducible equation has real roots is an example of such a lattice. To accomplish this unification, in Chapter I, I have introduced the more general setting of an origin symmetric discrete array satisfying two postulates. In this structure I have developed the geometry of the Voronoii and Minkowski admissible oriented parallelopipeds and of their neighboring processes. It is shown that there exists a third such admissible oriented parallelopiped: the edge-face for which I develop the geometry of it and its neighboring process. To develop these theories more clearly, in Chapter II, I have introduced the concept of rank matrix, which enables a clearer presentation of the theory of the Voronoii, Minkowski and Edge-face neighboring processes in Chapter III. Also rank matrices are used to show that there are p(n) types of n-dimensional oriented parallelopipeds in an n-dimensional origin symmetric discrete array, where p(n) is the partition function. In Chapter IV, an irreducible lattice is shown to be an origin symmetric discrete array. Finally, in this chapter it is shown that the algebraic integers of the cubic field Q(0), whose monic irreducible equation has real roots is an example of an irreducible multiplicative lattice of dimension three with identity. In Chapter V, the actual techniques that are used to calculate units, independent units and fundamental units are described and proven. Many of the proofs relating to Voronoii's work are my own. Finally, in Chapter VI, the actual algorithms are briefly described and used to calculate units.

Part II: Basis Approach. We explore a basis approach to calculating units. The concept of basis is linked to admissible oriented parallelopipeds and diagrams for three oriented parallelopipeds determined by three origin symmetric vectors are presented in Chapter VII. These enable a deeper understanding of the interrelationship of bases and admissible oriented parallelopipeds. In Chapter VIII, the basis approach is explained and Professor Harvey Cohn's work is presented. A flaw is pointed out in one of the proofs of a theorem. At the present, this theorem cannot be salvaged. In Chapter IX, I show a clearer connection than existed between Minkowski admissible oriented parallelopiped and the Cohn bases. Also, I establish a connection between Cohn bases and Voronoii admissible oriented parallelopipeds. These connections allow for a different way of establishing that Cohn bases can be used to calculate units. Also, an approach to calculating fundamental units by Cohn bases is discussed. Finally, a possible way of calculating units in higher dimensions is presented.

PART I.   A CLASSICAL SURVEY

CHAPTER 1

DISCRETE ARRAYS, ORIENTED PARALLELOPIPEDS AND NEIGHBORING PROCESSES IN THREE SPACE

In an origin symmetric discrete array, a more general setting than an irreducible lattice, we present the fundamental structures and neighboring processes of the Voronoii and Minkowski geometric approach to calculating units.

1.1. Discrete Arrays

Definition 1.1.1. A centered rectangle is a rectangle in coordinated three space such that: 1) A coordinate axis is perpendicular to the rectangle at its center, 2) The projection of the rectangle onto the plane of the remaining axes will have each of its parallel edges parallel to one of the remaining axes.

Definition 1.1.2. A uniformly discrete array or discrete array is a set of uniformly discrete points in three space such that coordinate axes can be centered at a point (origin) of the set so that: 1) Any centered rectangle whose center is the origin and which is translated away from the origin will encounter another point of the set. (Existence Property); 2) Any plane coincident or parallel to the coordinate plane will contain at most one point of the set. (Uniqueness Property).

Lemma 1.1.3. The coordinate axes contain only the origin in a discrete array. □

1.2. Oriented Parallelopides in Discrete Arrays

We shall now present those pertinent fundamental structures of discrete arrays. We begin by presenting altered definitions of the parts of a parallelopiped.

Definition 1.2.1. The edge of a parallelopiped is the open edge, that is, the edge without its boundary: vertices.

Definition 1.2.2. The face of a parallelopiped is the open face, that is, the face without its boundary: edges and vertices.

Definition 1.2.3. The side of a parallelopiped is the closed face of the parallelopiped.

Definition 1.2.4. An oriented parallelopiped or OP is a parallelopiped whose sides are centered rectangles and whose sides each contain a point of the discrete array.

Proposition 1.2.5. An OP can be described by a pairwise permutation of the following three ways only:

(1) A point at one of its vertices.

(2) A point on one face of each pair of opposite faces.

(3) A point on an edge and a point on a face perpendicular to that edge. □

Proposition 1.2.6. In any discrete array, an OP of Proposition 1.2.5. exists, though perhaps not all types exist. □

1.3. Raising and Lowering of Centered Rectangles and Admissible Oriented Parallelopides in Discrete Arrays

Definition 1.3.1. To "raise a centered rectangle along the positive x axis" means to translate a centered rectangle (whose center is on the positive x axis) along the positive x axis away from the origin. Similar definitions exist for raising centered rectangles along the negative x, positive y, negative y, positive z and negative z axes.

Definition 1.3.2. To "lower a centered rectangle along the positive x axis" means to translate a centered rectangle (whose center is on the positive x axis) along the positive x axis towards the origin. Similar definitions exist for lowering centered rectangles along the negative x, positive y, negative y, positive z and negative z axes.

Proposition 1.3.3. If in the process of raising a centered rectangle R along the positive x axis, where R is originally centered at the origin, the centered rectangle R encounters a point A of the discrete array on its vertex or edge, then by further raising R along the positive x axis, R will encounter an infinite number of points of the discrete array. Of these infinite points, there is a first point B encountered by R.

Proof: Consider a centered rectangle R' whose length is strictly less than the length of R and whose width is strictly less than the width of R. By the existence property of a discrete array, R' will encounter a point C when it is raised along the positive x axis away from the origin. Furthermore, it is clear that $A \neq C$. We can create an infinite set of rectangles $R^{(n)}$ and an infinite sequence of points $C^{(n)}$ that will be encountered by R when it is raised along the positive x axis. Hence, the first part of the proposition is proved. For the second part of the proposition, we note that the x coordinate of C will be greater than the x coordinate of A. Consider the parallelopiped AHIJKLMN defined by R at C and R at A. (Refer to Diagram 1.3.6. Vertex I is not indicated, but clearly understood.) This parallelopiped is a closed and bounded set, and hence, can contain only a finite set F of points of the uniformly discrete array. Let B be the first point of F encountered by R when it is raised from A along the positive x axis. Similar propositions exist for the negative x, positive, y, negative y, positive z and negative z directions. $\square$

Definition 1.3.4. An admissible oriented parallelopiped or AOP is an OP such that its interior contains no points of the discrete array except the origin.

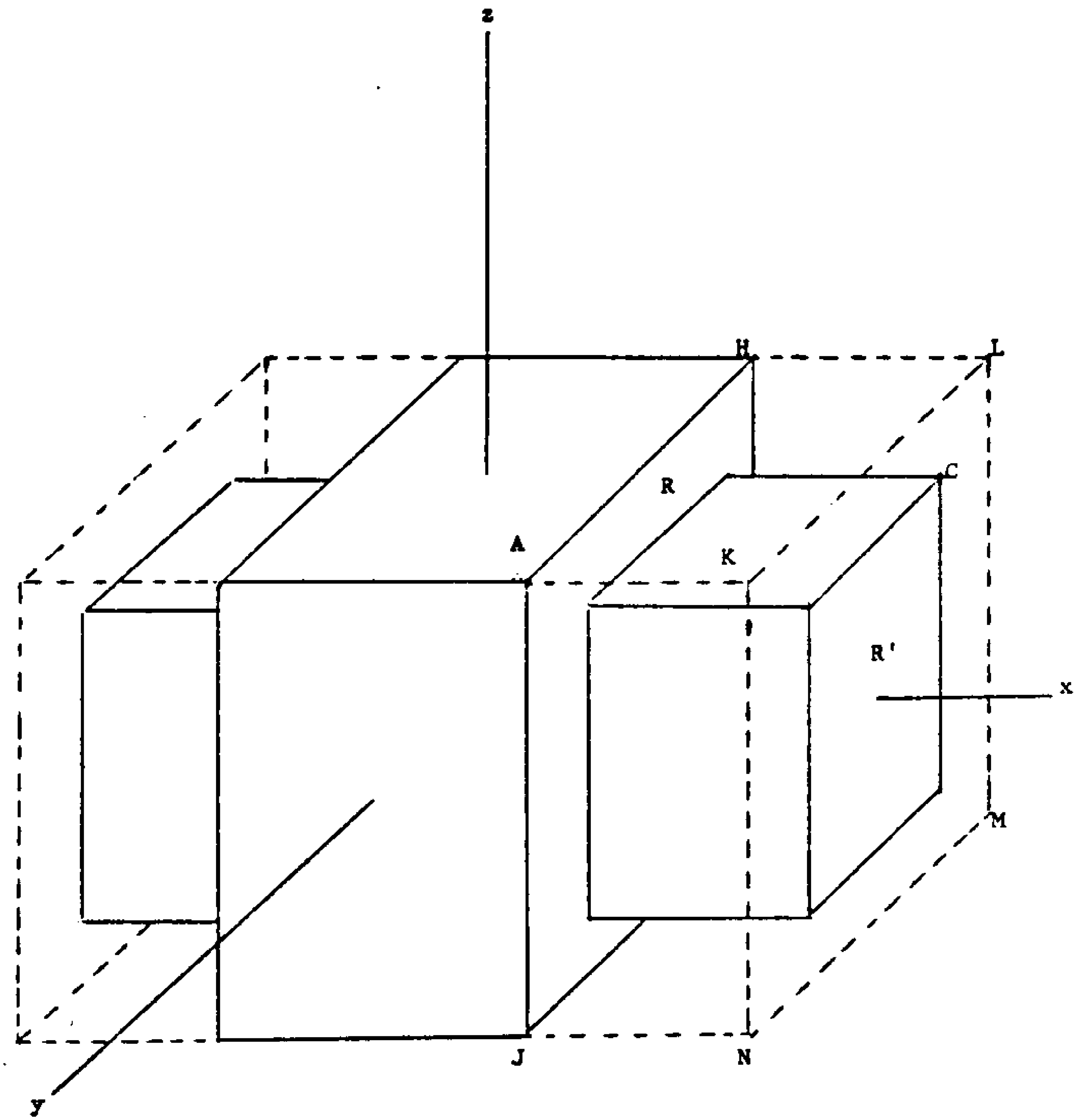
Proposition 1.3.5. In any discrete array, an AOP exists. $\square$

1.4. Symmetric Oriented Parallelopipeds in Origin Symmetric Discrete Arrays and Their Three Subdivisions

Definition 1.4.1. An origin symmetric discrete array or OSDA is a discrete array whose points are symmetric with respect to the origin.

Definition 1.4.2. A symmetric oriented parallelopiped or SOP is an OP whose opposite faces are symmetric with respect to

Diagram 1.3.6. Raising of a Centered Rectangle Along the x Axis



the coordinate plane they parallel.

Definition 1.4.3. A Voronoii oriented parallelopiped or VOP or a vertex oriented parallelopiped is an OP that is completely determined by two vertex points.

Definition 1.4.4. A Minkowski oriented parallelopiped or MOP or a face-face-face oriented parallelopiped is an OP that is completely determined by two sets of three points, each set consisting of points on three non-opposite faces.

Definition 1.4.5. A edge-face oriented parallelopiped or EFOP is an OP that is completely determined by two sets of two points, each set consisting of a point on its face and a point on an edge perpendicular to that face.

Proposition 1.4.6. In any OSDA, any SOP is either a VOP, MOP or EFOP. □

Proposition 1.4.7. In an OSDA, an admissible oriented parallelopiped is a symmetric oriented parallelopiped.

Because of the previous two propositions, we make the following definitions. □

Definition 1.4.8. A Voronoii admissible oriented parallelopiped or VAOP or a vertex admissible oriented parallelopiped is a VOP that is an admissible oriented parallelopiped. Similar definitions exist for a MAOP and EFAOP.

Proposition 1.4.9. An any OSDA, a VAOP, MAOP and EFAOP exist. □

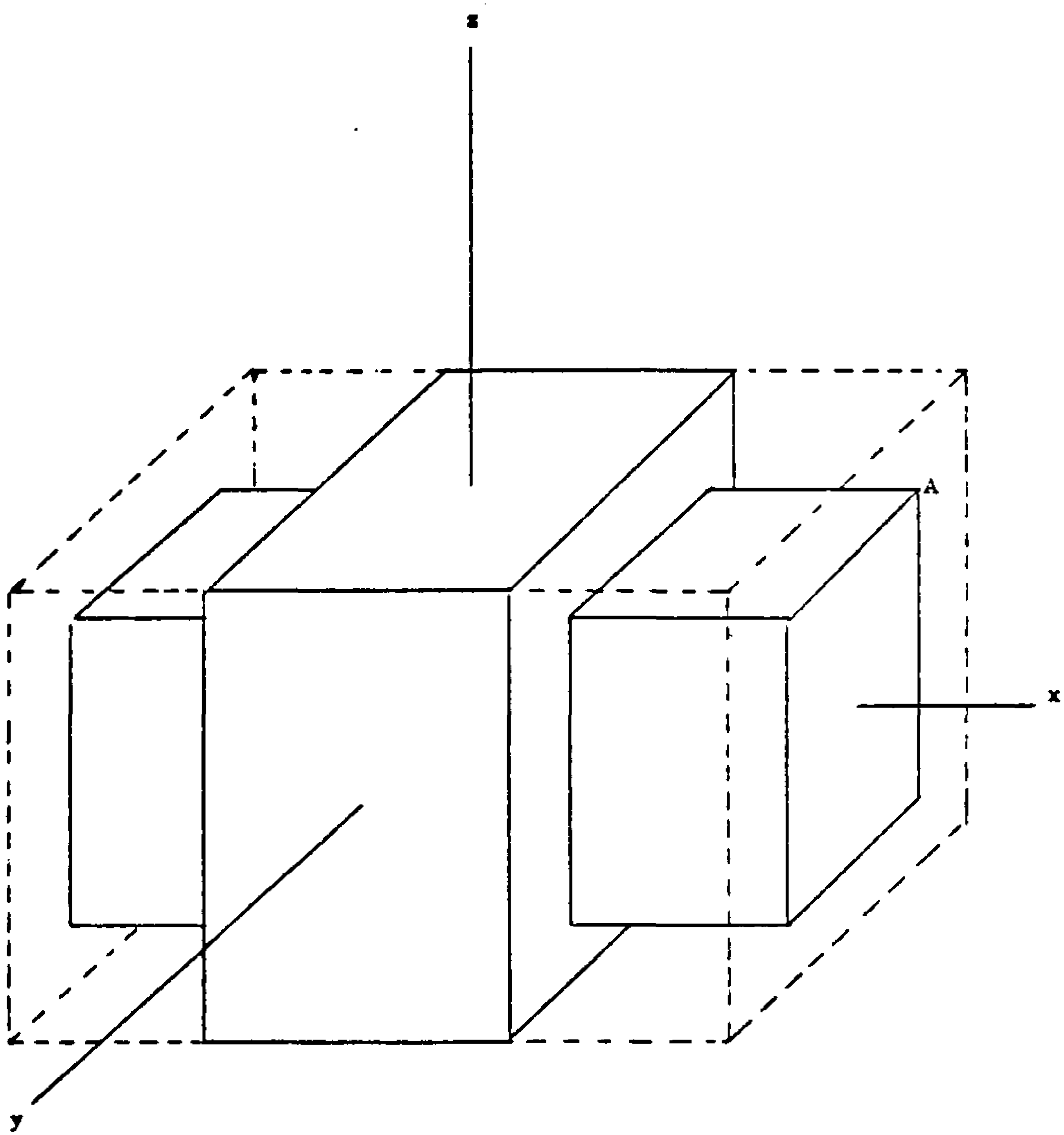
1.5. The Three Types of Admissible Oriented Parallelopipeds and Their Respective Neighboring Processes in Origin Symmetric Discrete Arrays

A VAOP, an MAOP, or an EFAOP can be extended by a "neighboring process" to another AOP of the same type in an OSDA. We describe each of the three processes below. Note that we will be making extensive use of Proposition 1.3.3. Also note that because we are concerned with symmetric oriented parallelopipeds in an OSDA, the Definitions 1.3.1. and 1.3.2. will be used without the words positive and negative. In short, it is only necessary to look at the half-space in which the non-negative x axes lie, when we raise a centered rectangle along the positive x axis. Similarly, this is true for the y and z coordinates. Throughout the remainder of the discussion we shall refer to the centered rectangle of the SOP perpendicular to the x axis as the x-side. Analagous meanings will hold for the y-side and z-side.

A. Voronoii Neighboring Process for a VAOP

Raise the x-side along the x axis until a point A is encountered on the new x-face. Point A exists because of the existence property of a discrete array. Point A is on the x-face, that is,it cannot be on an edge or vertex of the new x-side because it would contradict the uniqueness property of a discrete array. Point A cannot be on the x axis because of Lemma 1.1.3. Now lower the new z-side along the z axis until point A is encountered. Similarly, lower the new y-side along the y axis until point A is encountered. We have now created a new VAOP. Refer to Diagram 1.5.1. (We also could have begun the neigh-

Diagram 1.5.1. The Voronoii Neighboring Process



boring process by raising the y-side of the MAOP along the y axis or by raising the z-side of the MAOP along the z axis.)

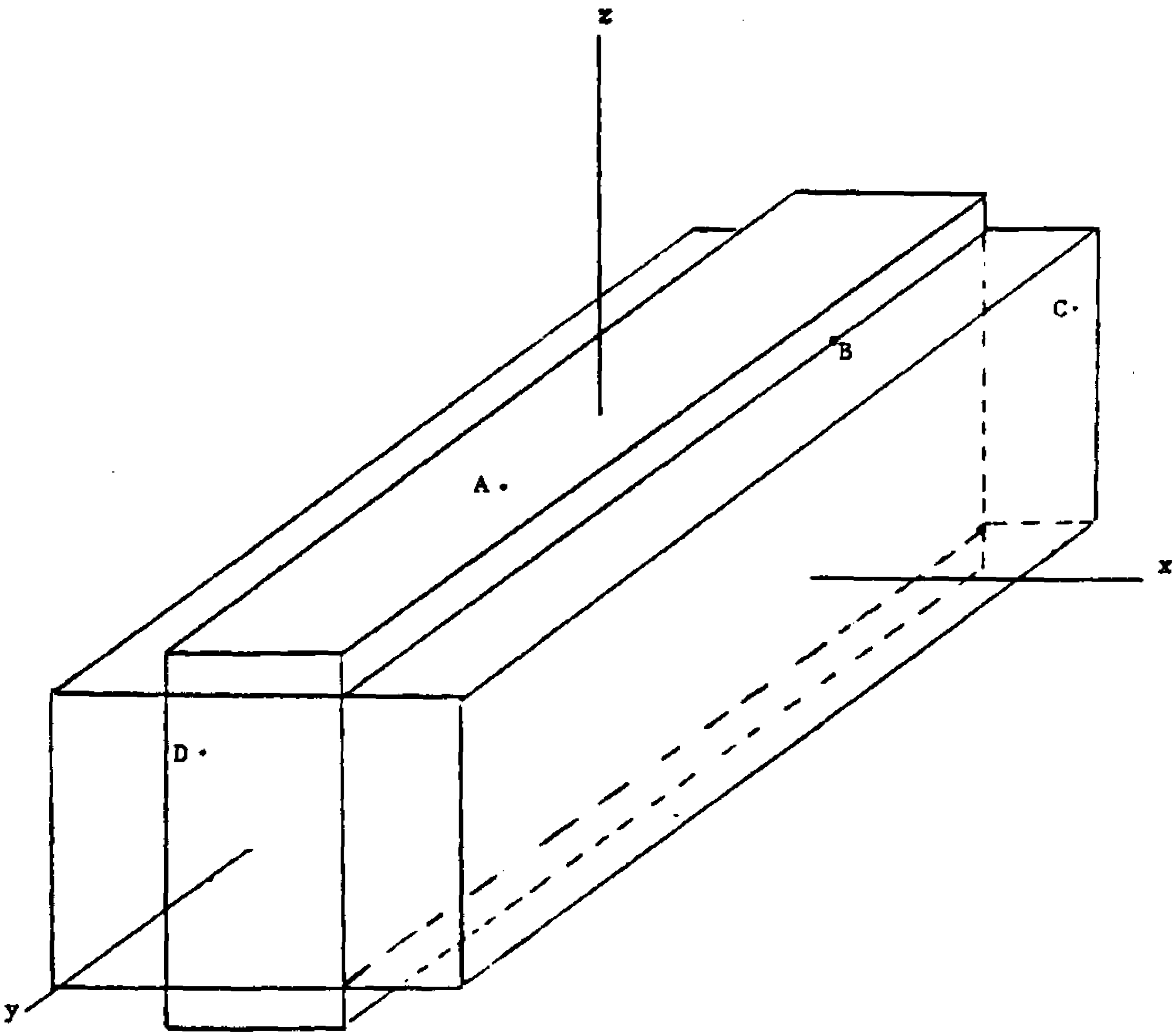
Note: We see that for each coordinate direction (say x), we have only one possible path: Raise the x-side along the x axis and lower the new z-side along the z axis and also lower the new y-side along the y axis.

B. Minkowski Neighboring Process for a MAOP

Lower the x-side of the MAOP along the x axis until a point of the y- or z-face is encountered. Let us say B of the z-face is encountered. B cannot be on the z-y plane because of Lemma 1.1.3. Hence, raise the new z-side along the z axis until a point A is encountered. Point A exists because of the existence property of a discrete array. Point A must be on the z-face, that is, it cannot be on an edge or vertex of the new z-side because it would contradict the uniqueness property of a discrete array. We have now created a new MAOP. Refer to Diagram 1.5.2. (We also could have begun the neighboring process by lowering the y-side of the MAOP along the y axis or by lowering the z-side of the MAOP along the z axis.)

Note: We see that for each coordinate direction (say x), we have only one of two possible paths: Either lower the x-side along the x axis and raise the new z-side along the z axis or lower the x-side along the x axis and raise the new y-side along the y axis.

Diagram 1.5.2 The Minkowski Neighboring Process



C. Edge-Face Neighboring Process for an EFAOP

This neighboring process may begin in either of two ways as described below. We assume that the x-face contains a point A and some edge perpendicular to the x-face contains point B in the following sections. Needless to say, point C of the following section exists because of the existence and uniqueness property of a discrete array.

1) (a) Lower the x-side along the x axis until a point B is encountered. Now raise the new z-side along the z axis until a point C is encountered on the z-face. We now have created a new EFAOP. Refer to diagram 1.5.3.

OR

1) (b) Lower the x-side along the x-axis until a point B is encounterd. Now raise the new y-side along the y-axis until a point C is encountered on the y-face. We now have created a new EFAOP.

2) (a) Raise the y-side along the y axis until a point C is encountered on the y-face. Now lower the z-side along the z axis until either A or C is encountered and a new EFAOP is formed. Refer to diagram 1.5.4.

OR

2) (b) Raise the y-side along the y-axis until a point C is encountered on the y-face. Now lower the x-side along the x-axis until either B or C is encountered and a new EFAOP if formed.

3) (a) Raise the z-side along the z axis until a point C is encountered on the z-face. Now lower the y-side

<u>Diagram 1.5.3.</u> The Edge-Face Neighboring Process

Type I

The following diagram below depicts only the case where the z-side is raised to point C after the lowered x side has encountered the point B. The case where the y-side is raised after point B is encountered can similarly be depicted.

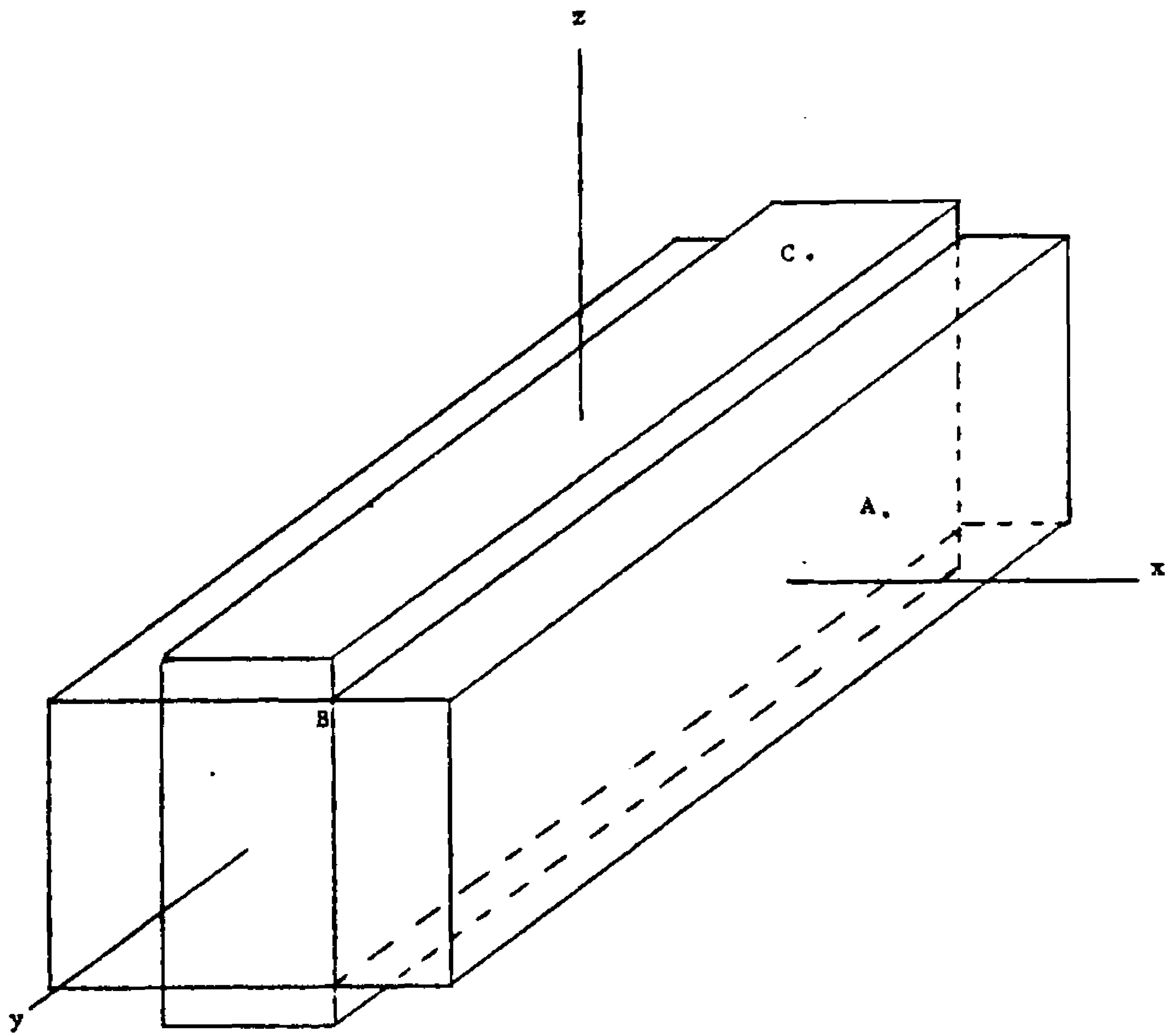
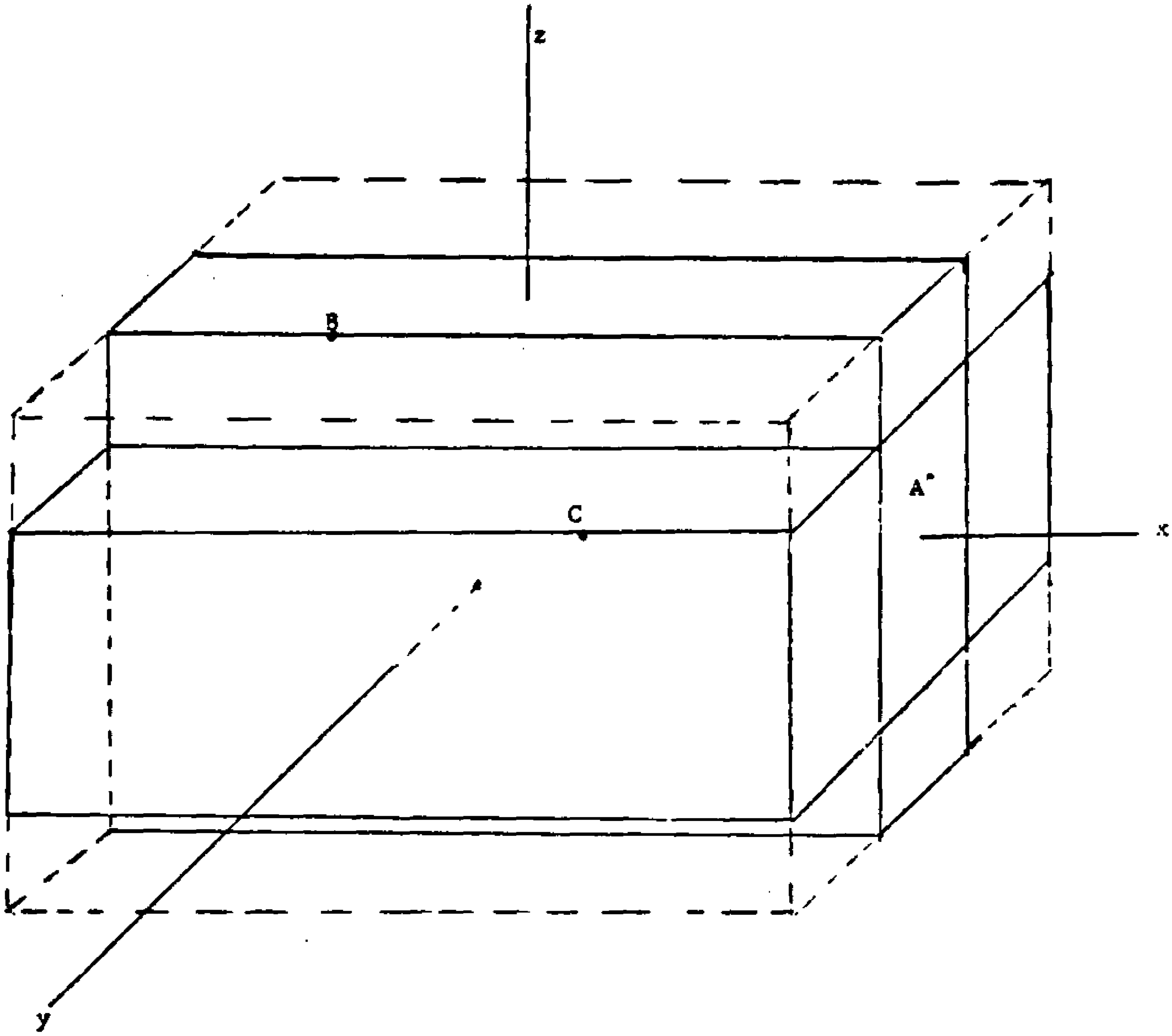


Diagram 1.5.4 The Edge-Face Neighboring Process
Type II

The following diagram below depicts only the case where the z-side is lowered to point C after y-side was raised to point C. The case where the x-side is lowered after point C is encountered can similarly be depicted.



along the y axis until either A or C is encountered and a new EFAOP is formed. Refer to diagram 1.5.5.

OR

3) (b) Raise the z-side along the z-axis until a point C is encountered on the z-face. Now lower the x-side along the x-axis until either B or C is encountered and a new EFAOP is formed.

4) (a) Lower the y-side along the y axis until point A is encountered. Now raise the z-side along the z axis until point C is encountered on the z-face and a new EFAOP is created. Refer to diagram 1.5.6.

OR

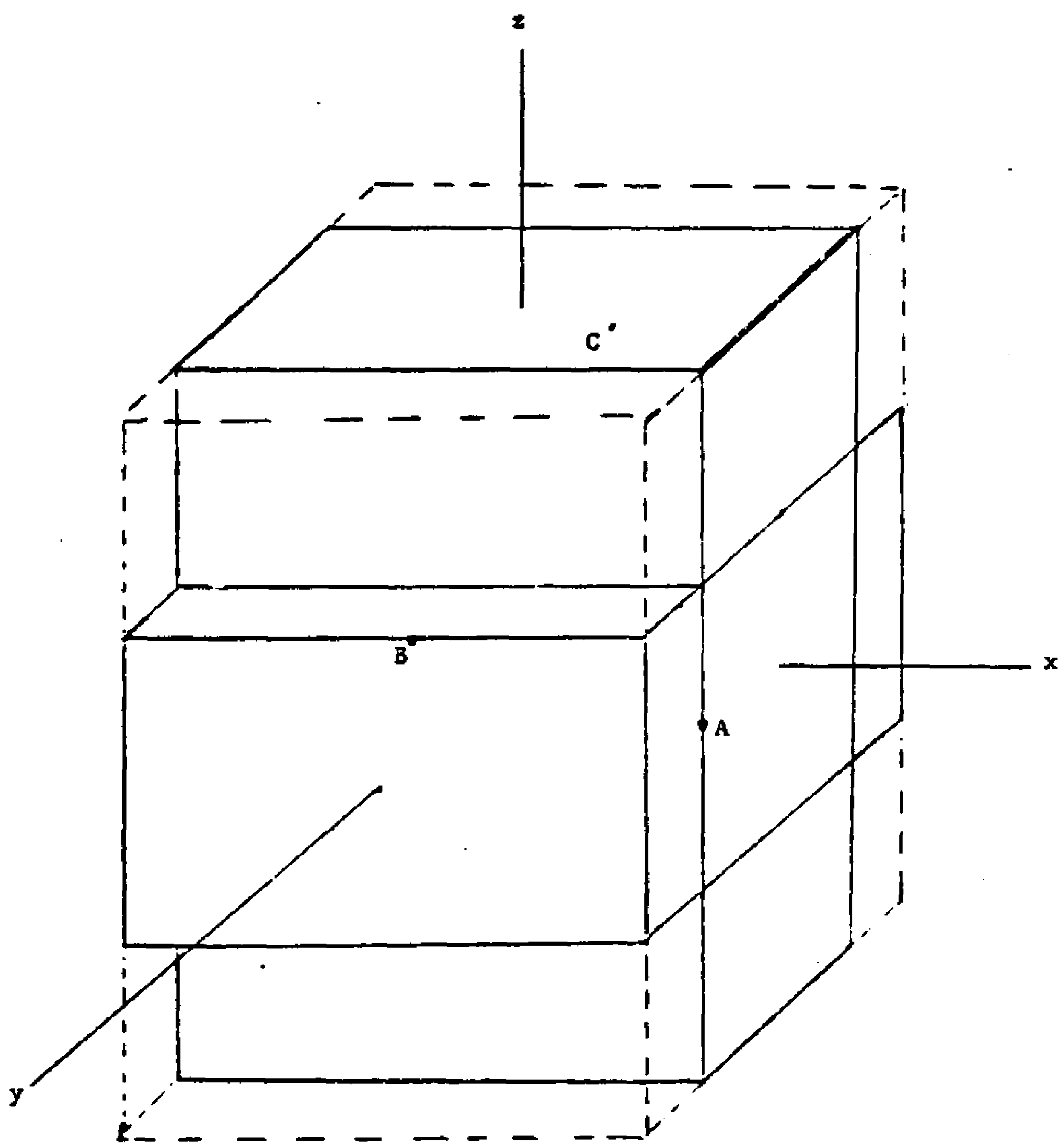
4) (b) Lower the y-side along the y axis unitl point A is encountered. Now raise the x-side along the x axis until a point C is encountered on the x-face. Finally, lower the z-side along the z axis until point B or C is encountered and a new EFAOP is formed.

5) (a) Lower the z-side along the z axis until point A is encountered. Now raise the y-side along the y axis until point C is encountered on the y-face and a new EFAOP is created.

OR

5) (b) Lower the z-side along the z axis until point A is encountered. Now raise the x-side along the x axis until a point C is encountered on the x-face. Finally, lower the y-side along the y axis until a point A or C is encountered and a new EFAOP is formed. Refer to diagram 1.5.7.

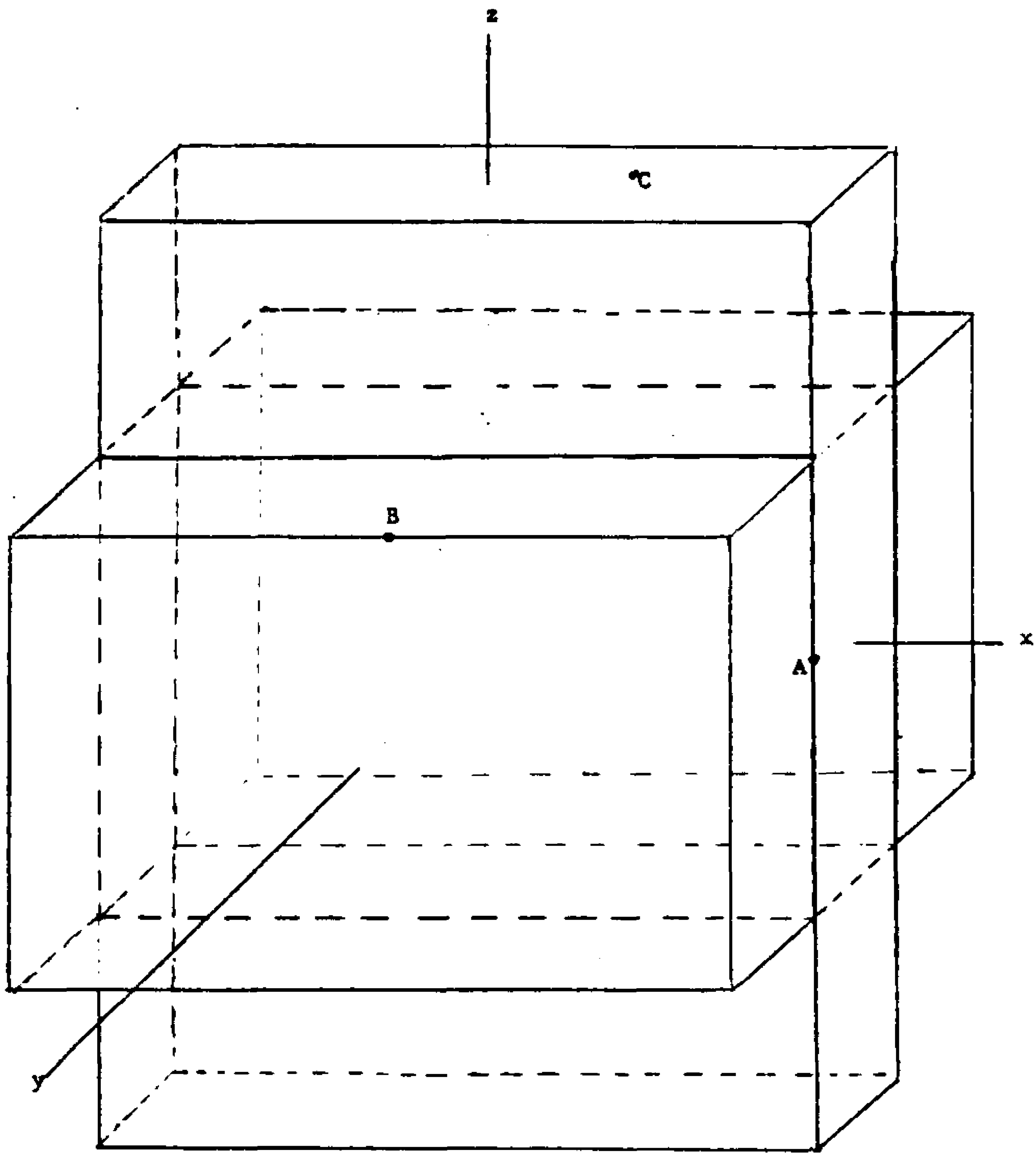
Diagram 1.5.5. The Edge-Face Neighboring Process
Type III



In this diagram above, we depict only the case where the y-side has been lowered to point A after the z-side has encountered the point C. The case where the x-side is lowered after point C is encounterd can be similarly depicted.

Diagram 1.5.6. The Edge-Face Neighboring Process

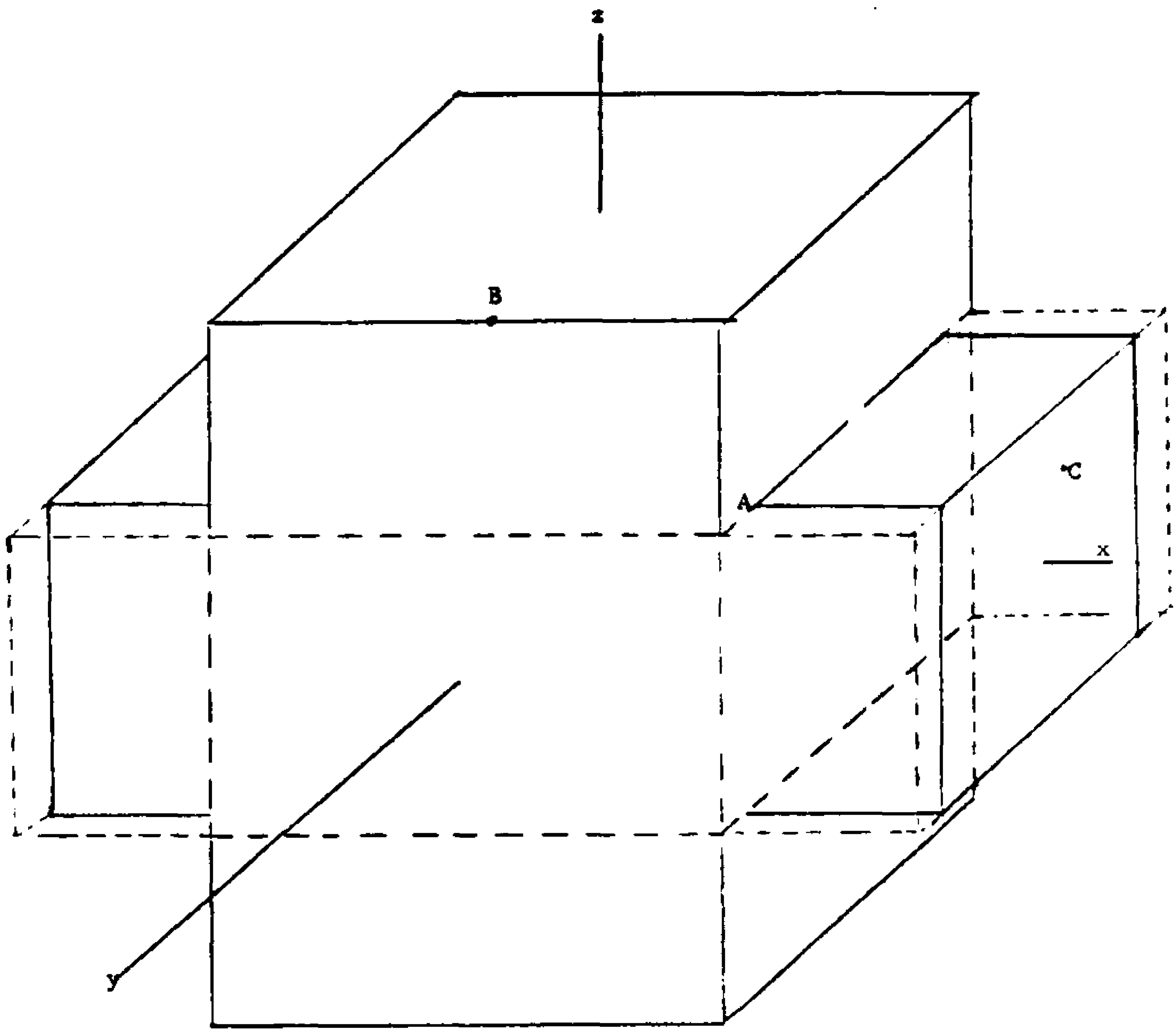
Type IV



In the diagram above, we depict only the case where the z-side is raised to point C after the lowered y-side has encountered the point A. The case 5) (a) of raising the y-side to point C after the lowered z-side has encountered A can be similarly depicted.

Diagram 1.5.7. The Edge-Face Neighboring Process

Type V



In the diagram above, we depict only the case where the x-side has been raised to point C after the z-side has been lowered to point A. Finally the y-side is lowered to point A. The case 4) (b) of raising the x-side to point C after the y-side has been lowered to A and finally, the z-side is lowered to point A can be similarly depicted.

Note: We see that we have a most ten possible paths to follow to create a new EFAOP: 1) Lower the x-side along the x axis and raise the z-side along the z axis. 2) Lower the x-side along the x axis and raise the y-side along the y axis. 3) Raise the y-side along the y axis and lower the z-side along the z axis. 4) Raise the y-side along the y axis and lower the x-side along the x axis. 5) Raise the z-side along the z axis and lower the y-side along the y axis. 6) Raise the z-side along the z axis and lower the x-side along the x axis. 7) Lower the y-side along the y axis and raise the z-side along the z axis. 8) Lower the y-side along the y axis, raise the x-side along the x axis and finally lower the z-side along the z axis. 9) Lower the z-side along the z axis and raise the y-side in the y direction. 10) Lower the z-side along the z axis, raise the x-side along the x axis and lower the y-side along the y axis.

We summarize some important features of the three processes. Minkowski neighbors share two points, edge-face neighbors share one point and Voronoii neighbors share no points. The creation of a Minkowski neighbor for an MAOP S requires that two sides of S be raised or lowered. The creation of a Voronoii neighbor of the VAOP T requires that three sides of S be raised or lowered. The creation of an edge-face neighbor for an EFAOP U requires that two or three sides of U be raised or lowered.

CHAPTER 2

PARAMETRIC NOTATION, GEOMETRIC EQUIVALENCE AND RANK MATRICES, DIAGRAMS OF TWO ORIENTED PARALLELOPIPEDS IN THE SAME ORIGIN SYMMETRIC DISCRETE ARRAY AND n-DIMENSIONAL ORIENTED PARALLELOPIPEDS IN AN n-DIMENSIONAL ORIGIN SYMMETRIC DISCRETE ARRAY

Throughout the remainder of this work, we shall consider only origin symmetric discrete arrays and symmetric oriented parallelopipeds and/or symmetric admissible oriented parallelopipeds. We introduce the concept of rank matrix to aid us in our understanding of the geometric interrelationship of VAOP, MAOP and EFAOP.

2.1. Parametric Notation

We will often have need to identify an OP by its parameters.

Definition 2.1.1. An oriented parallelopiped shall be designated by $(a,g,h)_p$, where the positive real numbers a,g and h represent the respective x, y and z parameters of the oriented parallelopiped.

Note: It is clear that the two vertices of a VOP will appear among the eight possible sign combinations of $(\pm a,\pm g,\pm h)$. In an EFOP some sign combination of two of the parameters will be two of the coordinates of the edge point and some sign combination of the third will be a face point coordinate. In an MOP some sign combination of each parameter is a member of each face point coordinates.

2.2. Geometric Equivalance and Rank Matrices

Definition 2.2.1. Let $S = \{(a_1,g_1,h_1)_p, (a_2,g_2,h_2)_p, \ldots, (a_n, g_n, h_n)_p\}$ and $T = \{(a_1,g_1,h_1)_p, (a_2,g_2,h_2)_p, \ldots, (a_n, g_n, h_n)_p\}$ S and T are geometrically equivalent if there exists a one to one mapping $f:S \to T$ and an axis permutation $p:T \to T'$ so that $pf:S \to T'$ is order preserving in each coordinate.

Proposition 2.2.2. Geometric equivalance is an equivalance relation.

Definition 2.2.3. The unarranged rank matrix of n oriented parallelopipeds $(a_1,g_1,h_1)_p, (a_2,g_2,h_2)_p, \ldots, (a_n, g_n, h_n)_p$ is an nx3 matrix $[a_{(i,j)}]$ whose entries are obtained in the following manner: Arbitrarily label the n parallelopipeds 1 through n, which shall be the domain of the i index of $a_{(i,j)}$. Now label the x, y and z axis as 1, 2 and 3 respectively which shall be the domain of the j index of $a_{(i,j)}$. The order 1,2,..., at most n (in increasing size) of the parameter of the j^{th} coordinate of the i^{th} parallelopiped among all n parallelopipeds's j^{th} coordinates is $a_{(i,j)}$. (These concepts can be extended to m-dimensional oriented parallelopipeds in m dimensions.)

Proposition 2.2.4. If the unarranged rank matrix of n oriented parallelopipeds S is changed by interchanging rows and/or columns into the unarranged rank matrix of n oriented parallelopipeds T, then S is geometrically equivalent to T.

Definition 2.2.5. The rank matrix of n oriented parallelopipeds is that unarranged rank matrix of n oriented parallelopipeds in which $a_{(1,1)} = 1$ and whose first column is non-decreasing and whose first row is non-decreasing.

2.3. Diagrams of Two Oriented Parallelopipeds in the same Origin Symmetric Discrete Array

We now apply the above concepts to two oriented parallelopipeds in the same OSDA. Because two VOP can share no OSDA points, we consider a rank matrix for two VOP in which each column is a permutation of 1 and 2. We have that there are at most two non-geometrically equivalent sets of two VOP which have the following rank matrices:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} .$$

See diagrams 2.3.1 and 2.3.2.

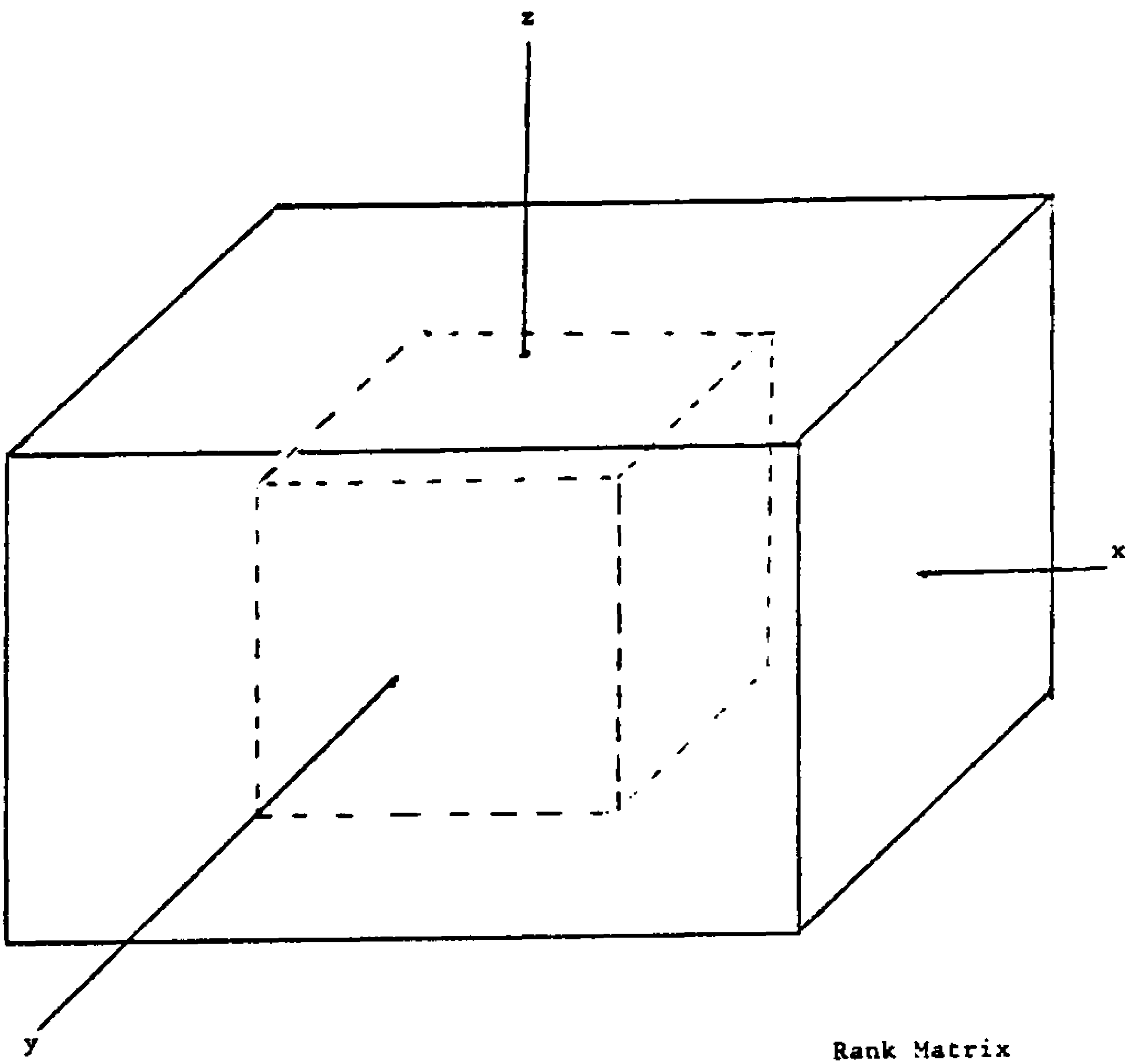
Because two EFOP can share at most one OSDA point, we consider additional rank matrices for two EFOP in which the first column entries are each one and the remaining columns are each permutations of 1 and 2. We now have that there are two more non-geometrically equivalent sets of two EFOP which have the following rank matrices:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} .$$

See diagrams 2.3.3 and 2.3.4

Finally, because two MOP can share at most two OSDA points, we consider additional rank matrices for two MOP in which

Diagram 2.3.1. Two Oriented Parallelopipeds with Distinct Parameters - Type I



A Configuration For
An AOP and Its Successor

V	M	EF
No	No	No

Number of Determining
Points If Both Are:

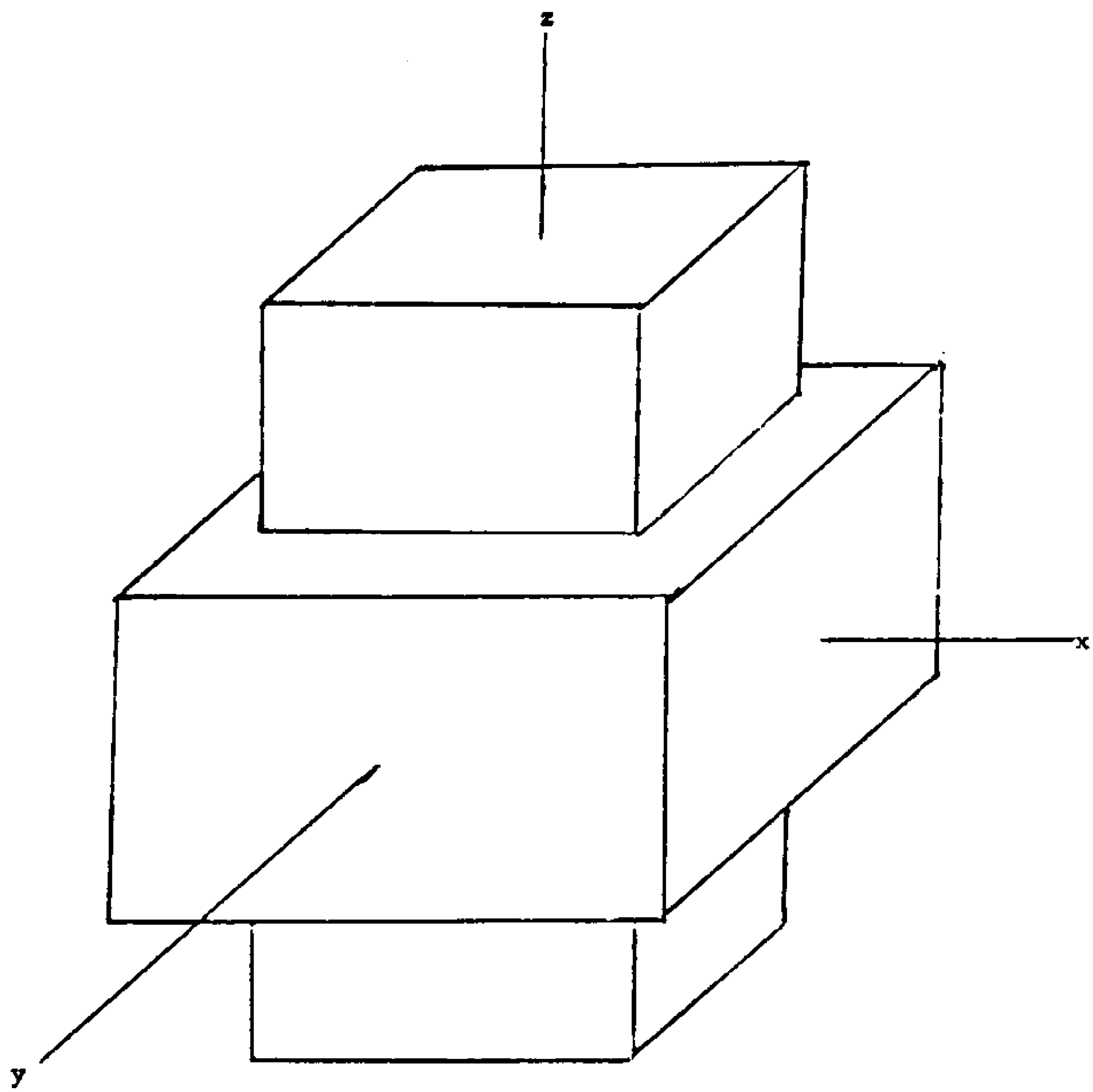
V	M	EF
2	6	4

Rank Matrix

$$\begin{array}{ccc} x & y & z \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Inadmissible
Configuration

Diagram 2.3.2 Two Oriented Parallelopipeds with Distinct Parameters - Type II



A Configuration For An AOP and Its Successor

V	M	EF
Yes	No	Yes

Number of Determining Points If Both Are:

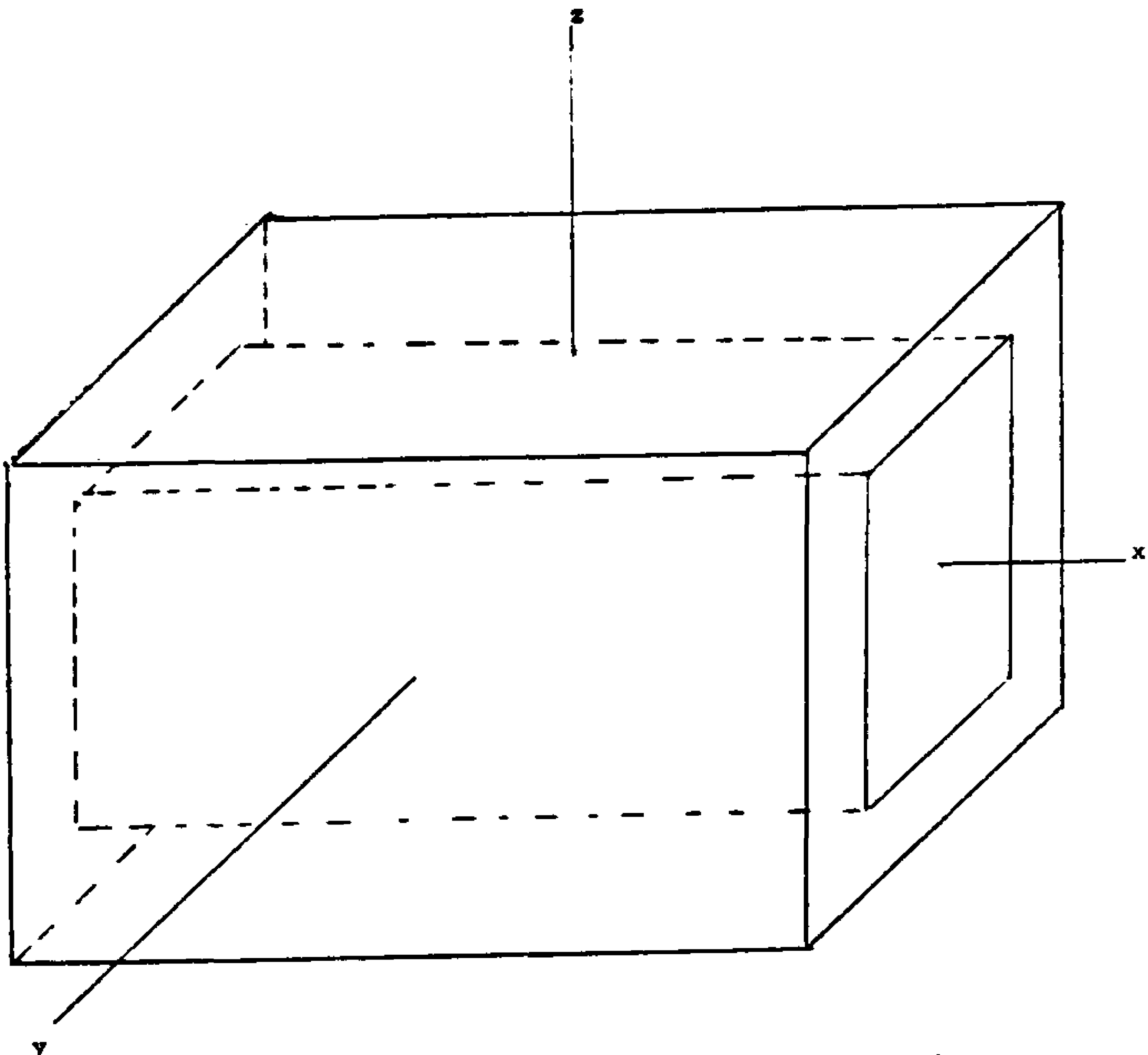
V	M	EF
2	6 or 5	4 or 3

Rank Matrix

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Possible Admissible Configuration

Diagram 2.3.3 Two Oriented Parallelopipeds with Distinct Parameters - Type III



Rank Matrix

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Inadmissible Configuration

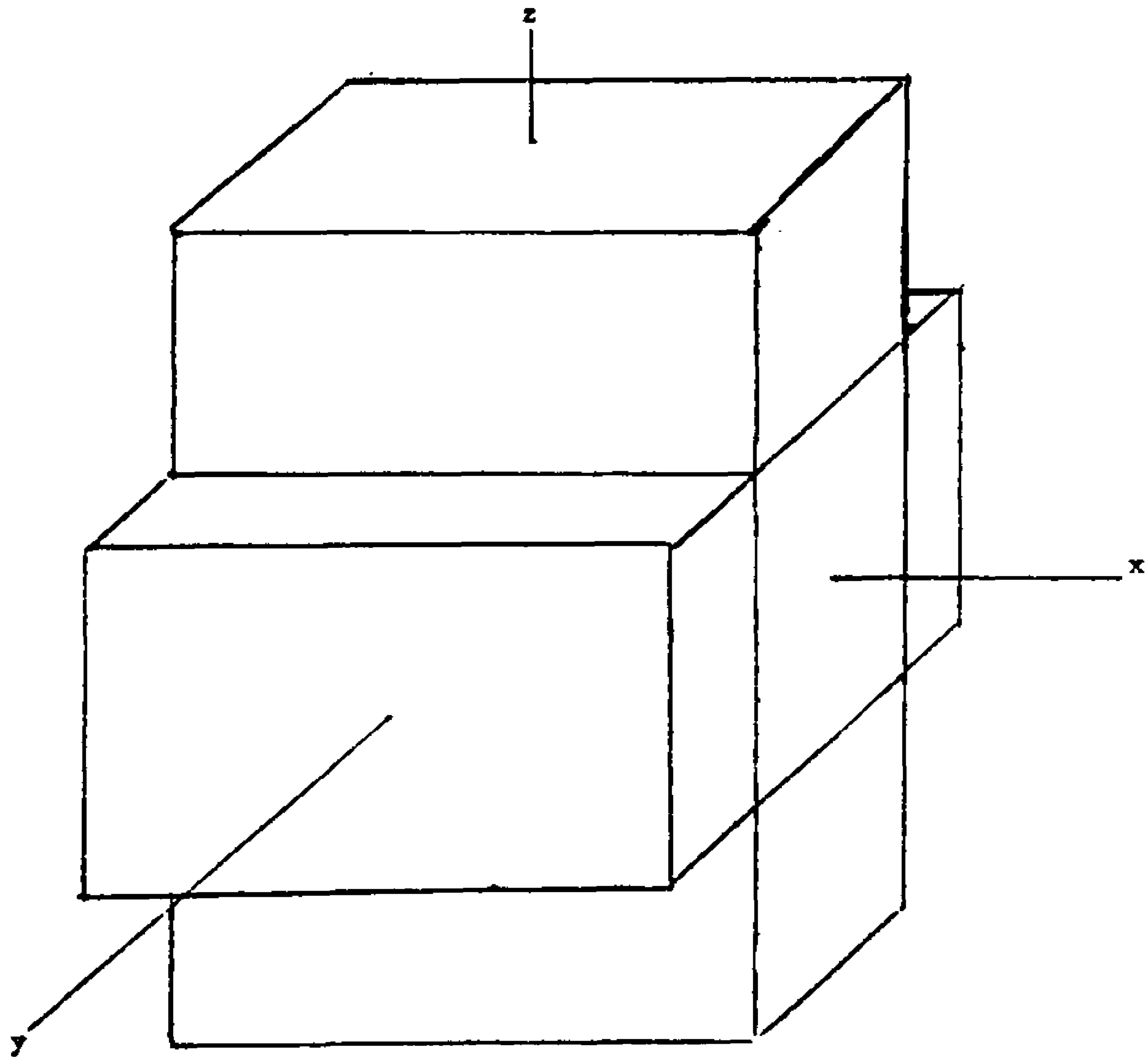
A Configuration For An AOP and its Successor

V	M	EF
No	No	No

Number of Determining Points If Both Are:

V	M	EF
X	5	3

Diagram 2.3.4. Two Oriented Parallelopipeds with Distinct Parameters - Type IV



Rank Matrix

x	y	z
1	1	2
1	2	1

Possible Admissible Configuration

A Configuration For An AOP and Its Successor

V	M	EF
No	Yes	Yes

Number of Determining Points If Both Are:

V	M	EF
X	4 or 5	3

the first two column entries are each one and the remaining column is a permutation of 1 and 2. We now have that there is one more non-geometrically equivalent set of two MOP which has the following rank matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

See diagram 2.3.5

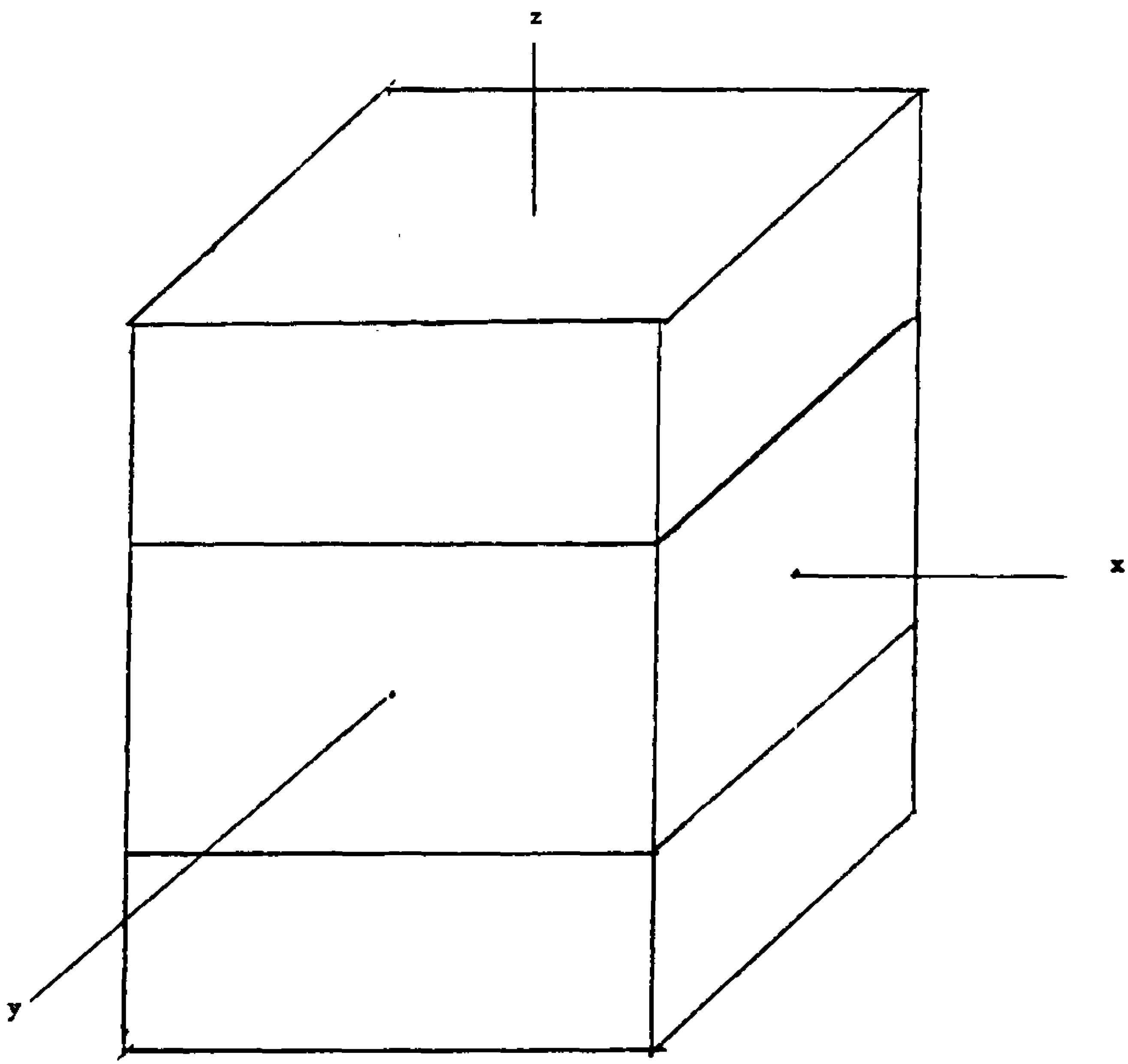
2.4 The Number Of Types Of n-Dimensional Parallelopipeds In An n-Dimensional Origin Symmetric Discrete Array

We now use a reinterpretation of rank matrices to calculate the number of types of n-dimensional oriented parallelopipeds in an n-dimensional OSDA. In a three dimensional OSDA, there were three: a VOP, an MOP and an EFOP.

Proposition 2.4.1. The number of ways an n-dimensional oriented paralleopiped in an n-dimensional OSDA may be determined by points on its surface is p(n), where p(n) is the partition function.

Proof: Assume that k points determine the parameters of the n-dimensional OP, then the parameters of the OP must appear among these points coordinates. We form an nxk matrix of the k points's n-coordinates, and label the determining parameter coordinates k. As in a rank matrix, we consider two such matrices equivalent up to row and column interchanges. The theorem is now clear. □

Diagram 2.3.5. Two Oriented Parallelopipeds with Distinct Parameters - Type V



A Configuration For
An AOP and Its Successor

V	M	EF
No	No	No

Rank Matrix

x	y	z
1	1	1
1	1	2

Inadmissible
Configuration

Number of Determining
Points If Both Are:

V	M	EF
X	4	3

CHAPTER 3

GEOMETRIC THEOREMS ABOUT NEIGHBORING PROCESSES

For each neighboring process, we shall answer each of the three following questions which are essential to our understanding of the processes and their ability to calculate units.

1) Given any AOP, how many "immediate predecessors" and "immediate successors" of its type are created by the appropriate neighboring process?

2) Given any AOP, if its neighboring process were unrestricted, would it be possible to return to the same AOP? If so, how can this be avoided?

3) Given any two AOP of the same type, is it possible to connect the two by means of that type's neighboring process?

3.1. Preliminary Definitions

Definition 3.1.1. A Voronoii predecessor A of an VAOP B is any VAOP such that if the Voronoii neighboring process is applied to B, A will result. Analoguous definitions exist for a Voronoii successor and for Minkowski predecessors and successors and for EFAOP predecessors and successors.

Definition 3.1.3. Given any VAOP P, then P, P_1 , P_2 ... is a Voronoii chain for P if P_i is a Voronoii successor of P_{i-1}. Analoguous definitions exist for Minkowski chains and EFAOP chains.

3.2. The Voronoii Neighboring Process

Proposition 3.2.1. Given any VAOP A, depending on the OSDA there can exist any number (zero, finite or infinite) of predecessors of A in any coordinate direction. However, there will exist three successors of A, one in each coordinate direction. □

Comment 3.2.2 Given any VAOP, the random use of the Voronoii neighboring process can return us to A.

We may avoid the "looping" of Comment 3.2.2 by demanding that throughout the Voronoii neighboring process we always begin by raising a side along the same axis. The resulting structure will be a needle-like sequence of VAOP which would have an increasing coordinate parameter while the other two coordinate parameters are decreasing.

It is not the case that any VAOP can be connected by means of a finite or infinite Voronoii chain to any other VAOP. The following theorem gives Voronoii's best answer to this question. First we give the definition of a Voronoii coordinate chain.

Definition 3.2.3. A Voronoii coordinate chain along the x direction is a Voronoii chain for a VAOP in which each successor Q' of an element Q of the chain is obtained by applying the Voronoii neighboring process to Q along the x axis. We denote these chains by $(Q)_x = Q, Q_1, Q_2 \ldots$ Analogous definitions exist for chains along the z and y axes.

Theorem 3.2.4. Given any two VAOP T and S, then either T is in a Voronoii coordinate chain of S or S is in a Voronoil coordinate chain of T, and if not, some Voronoii coordinate chain of S will intersect some Voronoii coordinate chain of T. □

Before we prove this theorem, it is necessary to present various definitions and propositions.

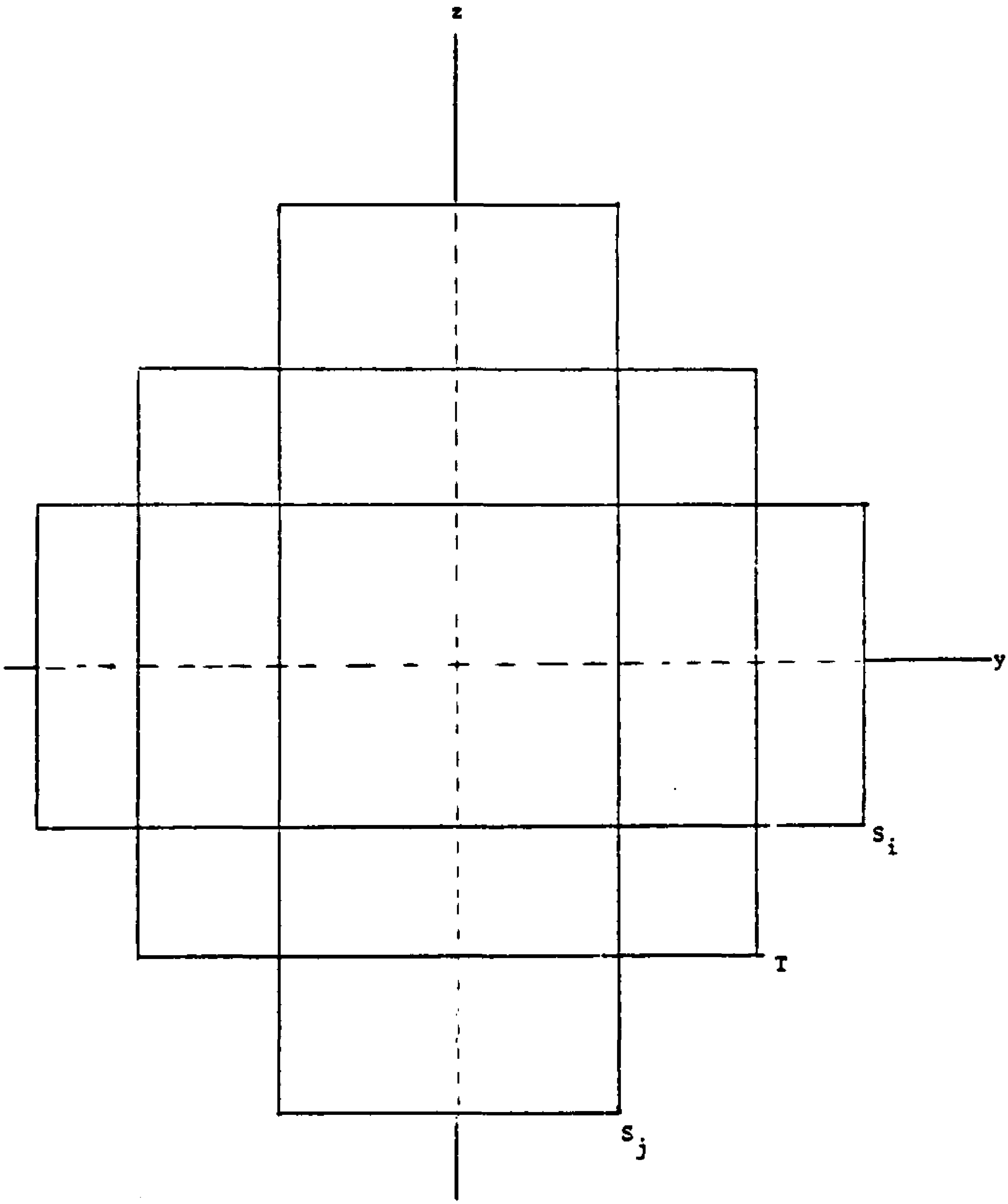
Definition 3.2.5. For VAOP T and VAOP S, *T pierces S along the x axis* means that the x parameter of T is greater than the x parameter of S, the y parameter of T is less than the y parameter of S, and the z parameter of T is less than the z parameter of S. Analogous definitions exist for T piercing S along the y axis and T piercing S along the z axis.

Definition 3.2.6. For VAOP T and $(S)_x$, *T is z lesser, y greater than* $(S)_x$ if there is an S_i whose z parameter is less than the z parameter if T and whose y parameter is greater than the y parameter of T. Analogous definitions exist for all permutations of x, y and z.

Proposition 3.2.7. For VAOP T not in $(S)_x$ and $T_x > S_x$, either T is z lesser, y greater than $(S)_x$ or T is z greater, y lesser than $(S)_x$. See diagram 3.2.9. Analogous propositions hold for T and $(S)_y$ and T and $(S)_z$. □

Proposition 3.2.8. If VAOP T has its x parameter greater than the x parameter of each of the elements of $(S)_z$ and T is z lesser, y greater than $(S)_x$, then $(S)_x$ and

Diagram 3.2.9. A VAOP Cannot Be Both z-Greater, y-Lesser and z-Lesser, y-Greater Than Another VAOP



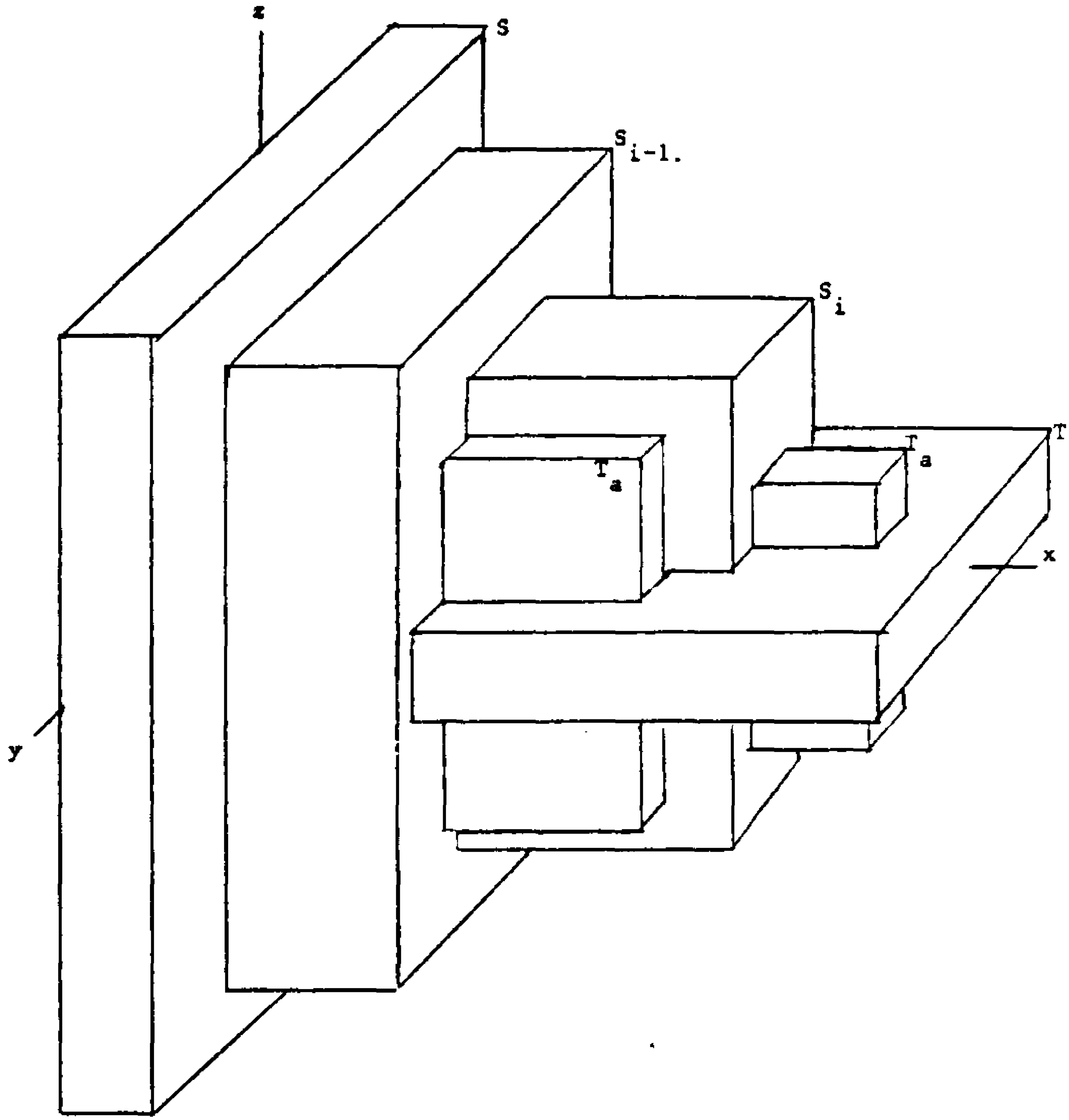
$(T)_z$ have a common element.

Proof: Two VAOP S and T may be situated with respect to each other in six different ways: T pierces S along the x axis, T pierces S along the y axis, T pierces S along the z axis, S pierces T along the x axis, S pierces T along the y axis, and S pierces T along the z axis. Only two of these six possibilities satisfy the hypothesis of the proposition: T pierces S along the x axis and S pierces T along the z axis. We shall only consider the first possibility for the second is similar.

Let T pierce S along the x axis. See diagram 3.2.10. Because T is z lesser, y greater than $(S)_x$, $(S)_x$ contains an S' which is z greater, y lesser than T and whose x parameter is greater than the x parameter of T. For if not, all the elements of $(S)_x$ whose x parameter is less than the x parameter of T had z parameter greater than z parameter of T and y parameter greater than y parameter of T would imply that T would belong to $(S)_x$ which contradicts the hypothesis. Let S_i be the first element of $(S)_x$ such that S_{i-1} is pierced by T along the x axis. Now S_i pierces T along the z axis. We now show that S_i either belongs to $(T)_z$ or is pierced by an element of $(T)_z$ along the x axis.

Consider the elements of $(T)_z$ whose z parameter is less than the z parameter of S_i. If S_i pierces each of them along the z axis, then S_i belongs to $(T)_z$

<u>Diagram 3.2.10.</u> Different Directed Chains of VAOP



The leftmost T_a is not possible as the proof indicates.

and the proof is complete. If the $(T)_z$ contains an element T_a that is not pierced by S_i along the z axis, then it is either y greater, x lesser than S_i or x greater, y lesser than S_i. But the first case is impossible because T_a would be the successor of S_{i-1} not S_i. The second case means that T_a pierces S_i along the x axis-as was claimed.

Now T and S_i again satisfy the hypothesis of the theorem, but this time S_i pierces T along the z axis. The same argument is repeated. The first element T_a of the chain $(T)_z$ which is not pierced by S_i along the z axis is shown to belong to the chain $(S_i)_x$ or is z lesser, y greater than $(S)_x$. In this latter case, we repeat the same argument, and so on. All VAOP introduced by these arguements are bounded by the x parameter of T and the y and z parameters of S. Because we are in a uniformly discrete array, there are only a finite number of VAOP within these bounds and hence $(S)_x$ must share a common element with $(T)_z$. ☐

Similar propositions may be proved for any pair permutation of x, y and z.

<u>Proposition 3.2.9</u> For any VAOP S and T, one of the following $(S)_x$, $(S)_y$ or $(S)_z$ will have an element in common with one of the $(T)_x$, $(T)_y$ or $(T)_z$.

<u>Proof:</u> We consider only one of the six cases listed in the beginning of the proof of proposition 3.2.8: T pierces S along the x axis. In this case, T has x parameter

greater than the x parameter of each of the elements of $(S)_z$ and T is z lesser, x greater than $(S)_y$. Consider the element S_i of $(S)_x$ which has the largest x parameter that is less than the x parameter of T. If S_i has y parameter greater than the y parameter of T and z parameter greater than the z parameter of T, then T belongs to $(S)_x$ and the proof is complete. If S_i has z parameter greater than z parameter of T and y parameter less than the y parameter of T, then T is y greater, z lesser than $(S)_x$ and by proposition 3.2.8, $(S)_x$ and $(T)_z$ have a common element. If S_i has z parameter greater than the z parameter of T and S_i has y para- greater than y parameter of T, then T is z greater, y lesser than $(S)_x$, and by one of the propositions analogous to proposition 3.2.8, $(S)_y$ and $(T)_x$ have a common element and the proof of the case T pierces S along the x axis is complete. All other five cases are analogous. ☐

As a direct consequence of proposition 3.2.9, theorem 3.2.4 is proved.

3.3. The Minkowski Neighboring Process

Proposition 3.3.1. Given any MAOP A, there exists three predecessors to A and three successors of A. ☐

Comment 3.3.2. Given any MAOP A, the random use of the Minkowski neighboring process can return us to A.

We note that we may avoid the "looping" of Comment 3.3.2

by demanding that throughout the Minkowski neighboring process we always begin by lowering along the same axis. The resulting structure would be MAOP's which would be flattened along one dimension and expanded along the other(s).

Lemma 3.3.3. Given any AOP P which we agree may not contain points on its closed faces, then the following is true. Let X be the x coordinate of the point encountered by raising the x-side of P along the x axis. Let Y and Z be similarly defined. Any MAOP $(a,g,h)_p$ which contains P is such that $a \leq X$, $g \leq Y$, $h < Z$. □

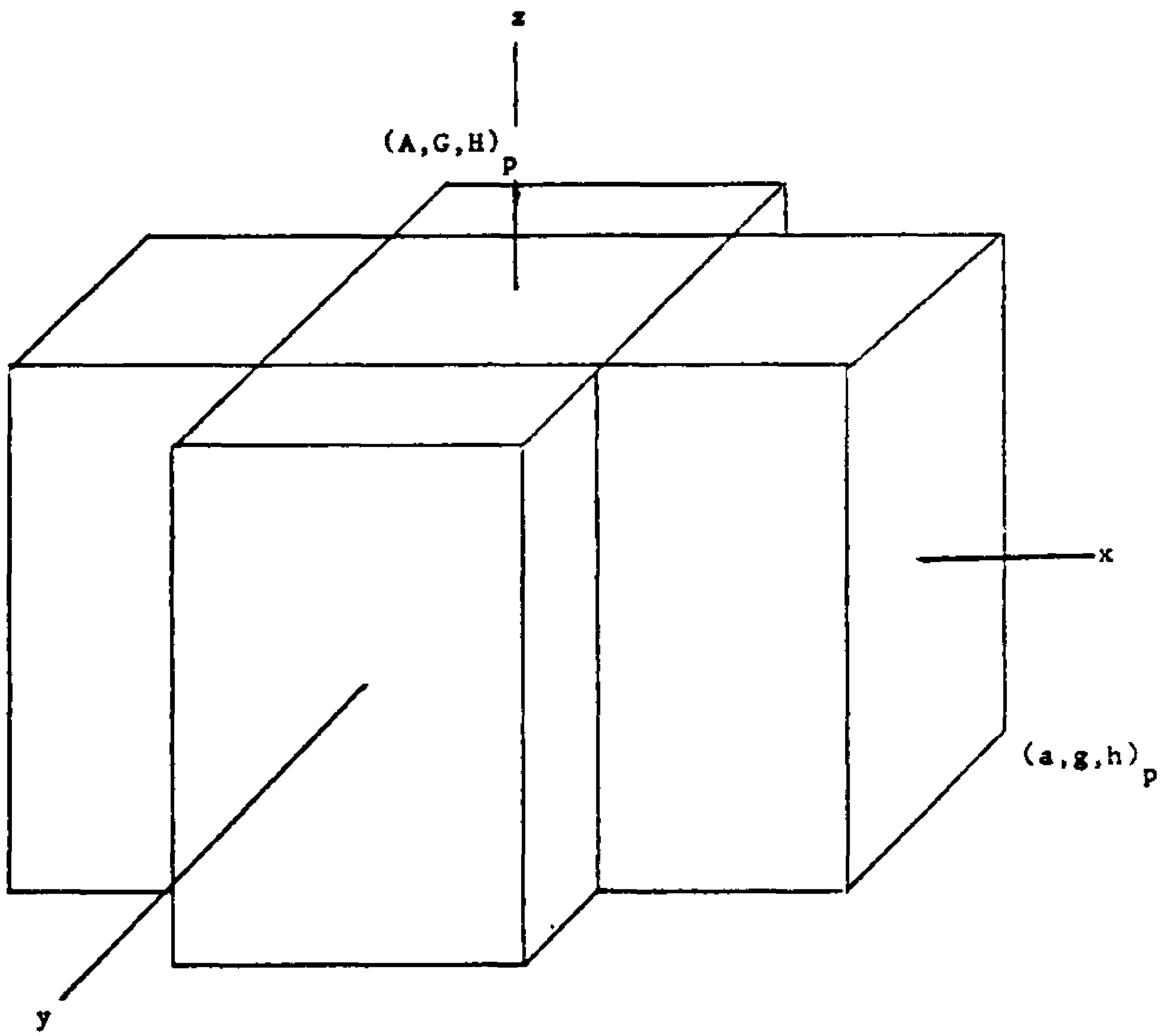
Theorem 3.3.4. Given any MAOP, one can construct a finite Minkowski chain that will lead to any other MAOP.

Proof: Given any two MAOP $(A,G,H)_p$ and $(a,g,h,)_p$, we see that up to geometric equivalence the two can only be geometrically interrelated as pictured in diagrams 2.3.2 and 2.3.4. Their corresponding parameters in a geometrically equivalent situation may have the following order relationships: $a > A$, $g < G$, $h \leq H$.

Diagram 3.3.5 which corresponds to $a > A$, $g < G$, $h = H$, can be determined by four or five points. Because of the uniform discreteness of an OSDA, two MAOP so related as in diagram 3.3.5., can be connected by the Minkowski neighboring process in a finite number of steps if we begin by lowering the x-side of $(a,g,h)_p$ along the x axis and raising the y-side along the y axis until a point is encountered. If necessary, we repeat the process

Diagram 3.3.5. MAOP Chains Between Two Different MAOPs - Type I

a > A g < G h = H



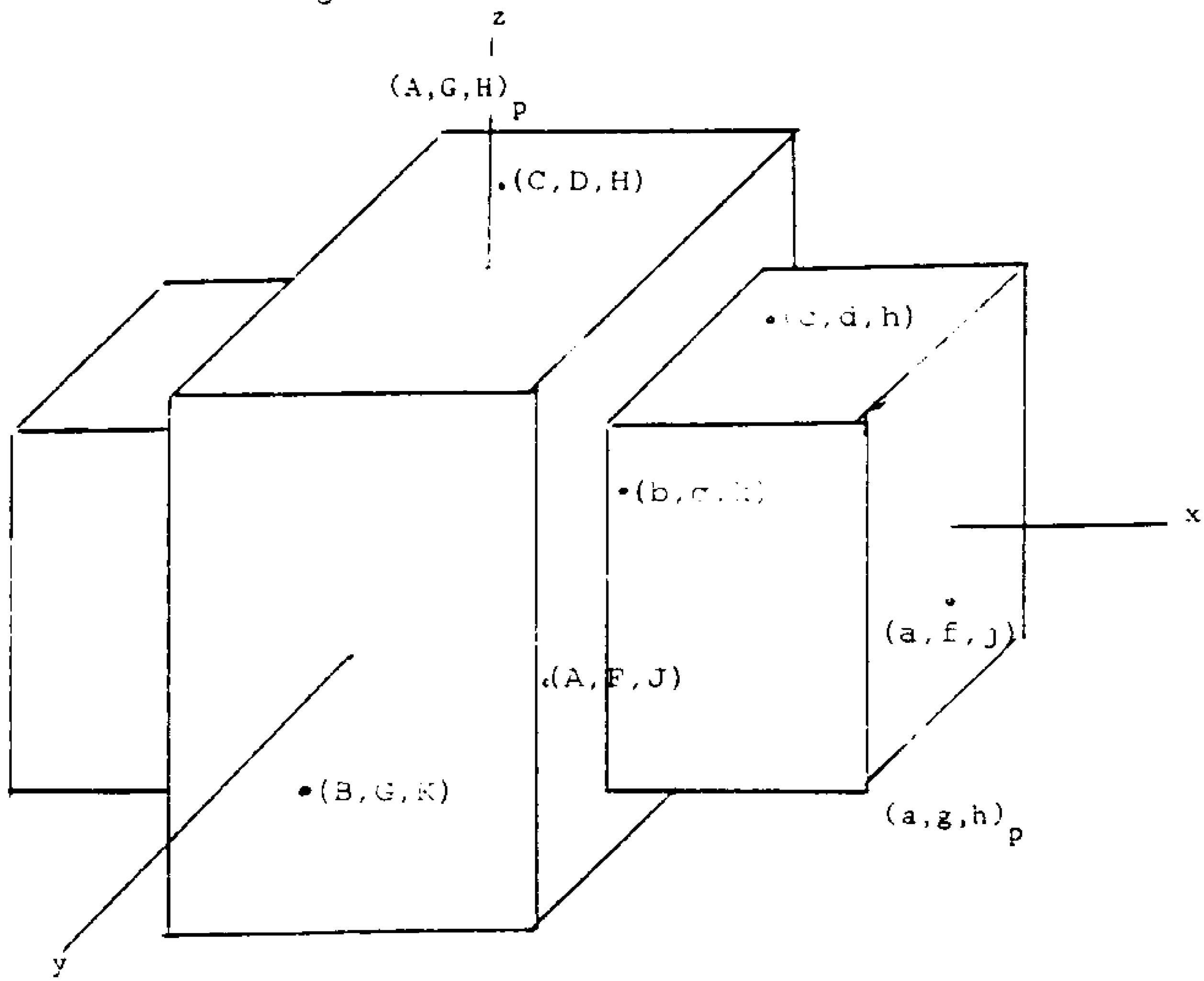
on the new $(a'_1, g'_1, h)_p$ until the process terminates with $(A,G,h)_p$.

Diagram 3.3.6. which corresponds up to geometric equivalence to diagram 2.3.2 has a A, g G, h H, and can be determined by five or six points. We consider the six point case only and label these points as in diagram 3.3.6: (a,f,j) , (b,g,k) , (c,d,h) , (A,F,J) , (B,G,K) and (C,D,H).

These points have been selected so that a, g, h, A, G, and H are positive. These two MAOP have a parallelopiped intersection Q with x, y, and z parameters A, g, and h respectively. Now lower the x-side of $(a,g,h)_p$ along the x axis until a point is encountered and raise the y-side along the y axis or the z-side along the z axis in accordance with $|b|>|c|$ or $|b|<|c|$. It is clear that diagram 3.3.5 or 3.3.6 can occur only. If diagram 3.3.5 occur we have finished by the preceeding paragraph. If diagram 3.3.6 occurs, we have a new $(b, g',h)_p$, such that $|g'|>|g|$ if $|b|>|c|$ or we have a new $(c,g,h')_p$ such that $|h'|>|h|$ if $|b|<|c|$. In each case, the intersection of either new MAOP with $(A,G,H)_p$ will be a parallelopiped intersection Q_1 which contains Q. Morever, the volume of Q_1 is greater than the volume of Q. If we continue the process, we cannot continually create diagram 3.3.6 configurations. If so, the sequence Q, Q_1.... insure us that distinct new MAOP are being created by the process. Because we are in a uniformly discrete space,

<u>Diagram 3.3.6.</u> MAOP Chains Between Two Different MAOPs - Type II

a > A g < G h < H



the number of such MAOP is finite since they are bounded by Lemma 3.3.3 if we set Q equal to P. Hence, we must eventually create the configuration of diagram 3.3.5 and the proof is complete. □

3.4. The Edge-Face Neighboring Process

Proposition 3.4.1 Given any EFAOP, there exists at most ten edge-face successors and at most ten edge-face predecessors. □ It is interesting to note that some of the predecessors may be arrived at by using different forms of the edge-face neighboring process depending on the origin symmetric discrete array and the EFAOP. AN Analogous statement is true for the successors of an EFAOP.

Comment 3.4.2 Given any EFAOP S, the random use of the edge-face neighboring process can return us to S. To avoid the "looping" of Comment 3.4.2, we can demand that the edge-face neighboring process always begin with the lowering of the side along the same axis. The resulting structure would be a sequence of EFAOP which would be flattened along one dimension and expanded along the other(s).

Proposition 3.4.3 Given any MAOP, there exists three possible EFAOP in the MAOP, one EFAOP for each of the three possible pair permutations of the three points of the MAOP. Each of these EFAOP is a edge-face successor of the other. □

Theorem 3.4.4 Given any EFAOP, one can construct a finite edge-face chain that will lead to any other EFAOP.

<u>Proof</u>: Theorem 3.4.4 is a direct consequence of theorem 3.3.4 and proposition 3.4.3. □

CHAPTER 4

ADMISSIBLE ORIENTED PARALLELOPIPEDS AND UNITS IN IRREDUCIBLE THREE DIMENSIONAL MULTIPLICATIVE LATTICES WITH IDENTITY

4.1. Origin Symmetric Discrete Arrays and Irreducible Three-Dimensional Lattices

A. Lattices, Basic Vectors, Basic Parallelopipeds, Discriminants and Norms

Definition 4.1.1. An *n-dimensional lattice* L is any uniformly discrete set L of points lying in an n-dimensional plane of R_n and not in any (n-1) -dimensional plane which is additive and subtractive.

Proposition 4.1.2. Any set L of points in R_n, $(n \geq 1)$, with coefficients in Z, of n R-linearly independent elements v_1, v_2, ..., v_n in R_n is logically equivalent to an n-dimensional lattice. $\square$

Definition 4.1.3. The vectors $v_1, v_2, \ldots, v_n$ of proposition 4.1.2. are called *basic vectors of L.*

Definition 4.1.4. In L, a *parallelopiped determined by vectors* v_1, $v_2, \ldots,$ v_n will be the parallelopiped

$$\left\{ \sum_{i=0}^{n} x_i v_i \;\middle|\; x_i \text{ in } R,\ 0 \leq x_i \leq 1 \text{ for } i=1, \ldots, n \right\}.$$

Definition 4.1.5. Let L be an n dimensional lattice and $v_1, \ldots, v_n$ be any n vectors of L. The determinant of $v_1, \ldots, v_n$ is:

$$\begin{bmatrix} v_1^{(1)} & v_2^{(1)} & \ldots & v_n^{(1)} \\ v_1^{(2)} & v_2^{(2)} & \ldots & v_n^{(2)} \\ \vdots & \vdots & & \vdots \\ v_1^{(n)} & v_2^{(n)} & \ldots & v_n^{(n)} \end{bmatrix} .$$

Propositon 4.1.6. Let L be an n dimensional lattice and let $v_1, \ldots, v_n$ be n vectors, then $|d|$, where d is the determinant of $v_1 \ldots, v_n$, is the volume of the parallelopiped determined by $v_1, \ldots, v_n$. $\square$

Definition 4.1.7 In an n dimensional lattice L a parallelopiped determined by basic vectors $v_1, \ldots, v_n$ is called a basic parallelopiped of L.

Proposition 4.1.8. In an n dimensional lattice L all basic parallelopipeds have the same volume. $\square$

Definition 4.1.9. The discriminant D of an n dimensional lattice L in R_n is d^2, where d is the determinant of any set of basic vectors $v_1, \ldots, v_n$.

Definition 4.1.10. The norm of a point v of lattice L denoted Nv shall be the product of the coordinates of v.

B. Minkowski's Convex Body Theorem in Lattices

Definition 4.1.11. A subset S of R_n is said to be convex if, for x and y in S, the line segment joining x and y is contained in S.

Definition 4.1.12. A subset S of R_n is said to be bounded if there exists an n-dimensional sphere with center the origin and radius r which contains S.

Definition 4.1 13. A subset S of R_n is said to be closed if every accumulation point of any sequence y_1, y_2, ... of elements of S is in S.

Definition 4.1.14. A subset S of R_n is said to be symmetric if for every y in S, $-y$ is in S.

Theorem 4.1.15 (The Minkowski Convex Body Theorem) Let $|d|$ be the volume of the basic parallelopiped of a lattice L in R_n, $(n \geq 1)$. If S is a non-empty, symmetric, convex, bounded and closed subset of R_n such that $vol(S) > 2^n |d|$, then there exists a point of L, district from the origin, which belongs to the interior of S. However, if $vol(S) = 2^n |d|$, then there is a point of L, distinct from the origin, which is in S, but not necessarily in the interior of S. $\square$

C. Irreducible lattices and Origin Symmetric Discrete Arrays

Definition 4.1.16 A lattice L in R_n is irreducible if none of the coordinates of its points is zero except the origin which has all coordinates equal to zero.

We come to an important consequence of theorem 4.1.15.

Theorem 4.1.17. Every irreducible, three-dimensional lattice L in R_3 is an origin symmetric discrete array.

Proof: A lattice is origin symmetric because if w is in L, then $-1w$ is in L. A lattice is uniformly discrete by proposition 4.1.2. An irreducible lattice is a discrete array because 1) If any centered rectangle is translated away from the origin, it will encounter a lattice point by theorem 4.1.15. 2) Furthermore, any plane coincident with a coordinate plane will contain only the origin by the irreducibility of the lattice. If a plane is parallel to the coordinate plane and contains more than one point, say v and w, then $v - w$ will have a coordinate equal to zero, contradicting the irreducibility of the lattice. Hence, existence - 1) and uniqueness - 2) properties of OSDA is established for irreducible three-dimensional lattices. □

D. Volume Bounds for MAOP, VAOP and EFAOP

The following proposition is another important consequence of the Minkowski Convex Body Theorem.

Proposition 4.1.18 For any MAOP, EFAOP or VAOP $(a, g, h)_p$, $agh \leq |d|$. □

Corrollory 4.1.19 For a VAOP associated with vector v, $Nv \leq |d|$. □

Minkowski proved that the inequality of proposition 4.1.18 is strict by establishing in [5,p.85] the following proposition.

Proposition 4.1.20 If a convex body has volume $8\,|d|$ in a three dimensional lattice, then it contains at least 14 points on its surface. □

Proposition 4.1.21. In an irreducible three dimensional lattice $agh < |d|$, for any MAOP, EFAOP or VAOP $(a,g, h)_p$. □

E. Sublattices of Lattices and Indices

Definition 4.1.22. The index i of a sublattice L' of a lattice L is the volume of the basic parallelopiped of L' divided by the volume of the basic parallelopiped of L.

Needless to say, i is an integer.

Proposition 4.1.23 The number of different n-dimensional sublattices of a given n-dimensional lattice of index i is finite. □

4.2. Units and Irreducible Multiplicative Lattices with Identity

A. Multiplicative Lattices with Identity

Definition 4.2.1. A multiplicative n-dimensional lattice L is an n-dimensional lattice in which the coordinatewise product of any two points is a point of the lattice.

Proposition 4.2.2 L is a ring. □

Definition 4.2.3 If L is a ring with identity, then v is a unit if an only if v^{-1} is in L.

Theorem 4.2.4 The coordinates of a point w in a multiplicative n-dimensional lattice L in R_n are all roots of some nth degree monic polynomial.

Proof: Consider:

$w_1 w = a_{(i,1)} w_1 + a_{(i,2)} w_2 + \ldots + a_{(i,n)} w_n,$ for $i=1, \ldots, n.$

We derive:

$$0 = a_{(i,1)} w_1 + a_{(i,2)} w_2 + \ldots + (a_{(i,i)} - w) w_i + \ldots + a_{(i,n)} w_n$$

for $i=1, \ldots, n$. The determinant of the last set of equations is the nth degree monic polymonial sought. □

Proposition 4.2.5 The norm of a point w, Nw, in a multiplicative lattice is an integer. □

Proposition 4.2.6 The discriminant D of a multiplicative lattice is an integer. $\square$

Proposition 4.2.7. v is unit in L with identity if and only if Nv=1.

Proof: The if direction. If v is a unit then v^{-1} is in L. $N\,v\,N\,v^{-1} = 1$. Because Nv is an integer we have that Nv=1. The only if direction. If Nv =1, we have:

$$v^n + a_{n-1}v^{n-1} + \ldots + 1 = 0.$$

Now $v\,(v^{n-1} + a_{n-1}v^{n-2} + \ldots a_1) = -1$ and v is a unit. $\square$

B. Dirichlet's Theorem on Fundamental Units in Irreducible Multiplicative Lattices with Identity

It is clear that the set of units A in L is a multiplicative group. Dirichlet proved that it is a finitely generated multiplicative group. He proved the following general theorem [4, pp.28-33].

Theorem 4.2.8. (Dirichlet Theorem on Fundamental Units). The units of an irreducible multiplicative n- dimensional lattice L with identity, wherein the coordinates of L consists of r_1 real conjugates and $2r_2$ complex conjugate coordinates (in r_2 pairs) and so n = $r_1 + 2r_2$, where n-1≥0 can be expressed as:

$$e = E_j e_1^{m_1} e_2^{m_2} \ldots e_{r_1+r_2-1}^{m_{r_1+r_2-1}}, \text{ where } m_i \text{ are in } Z. \quad \square$$

Definition 4.2.9 The e_i of Theorem 4.2.8 are the fundamental units of A. E_j is a root of unity in A.

It is these fundamental units of Dirichlet Theorem that we wish to calculate for a real irreducible three dimensional lattice. Hence, for these lattices we need find two fundamental units e_1 and e_2.

Inherent in the concept of fundamentality is independence.

Definition 4.2.10 Two units e_1 and e_2 are said to be independent if there exist no integers m and n such that $e_1^m = e_2^n$.

It will be seen that independent units will be easier to calculate than fundamental units.

C. Sublattices of Multiplicative Lattices

Proposition 4.2.11. Given any multiplicative lattice L and a lattice point v of L, then for the lattice point v, vL is a sublattice of L. (Note: vL is not necessarily multiplicative.) □

4.3. An Irreducible Multiplicative Three-Dimensional Lattice with Identity Associated with the Cubic Field Q(θ)

Definition 4.3.1 Let Q(θ) be the cubic field for a monic irreducible polynomial $p(x)=x^3+a_1x^2+a_2x+a_3=0$, where the a_i , i=1,2,3, are integers and where the roots θ, θ' , θ'' are real.

Definition 4.3.2. Let A be a subset of Q(θ) which consists of those numbers of Q(θ) that satisfy $q(x) = x^3+b_1x^2+b_2x+b_3 = 0$, where the b_i are in Z. We call A the algebraic integers of Q(θ) and Z the rational integers.

Proposition 4.3.3 A is an integral domain with identity.

Definition 4.3.4. Let L ⊂ A x A' x A'' , where A' and A'' are conjugate integral domains of A, be such that $\hat{L} = \{(a, a', a'') \text{ in } A\times A'\times A'' \mid a \text{ in } A\}$.

Definition 4.3.5. We define D_θ for the field $Q(\theta)$ (as well as for $Q(\theta')$ and for $Q(\theta''))$

$$D_\theta = \begin{bmatrix} 1 & \theta & \theta^2 \\ 1 & \theta' & \theta'^2 \\ 1 & \theta'' & \theta''^2 \end{bmatrix}^2 = (\theta'-\theta'')^2(\theta-\theta'')^2(\theta-\theta')^2 .$$

Proposition 4.3.6. For $Q(\theta)$ (and also $Q(\theta')$ and $Q(\theta'')$), the cubic field for a monic irreducible polynomial with rational integer coefficients and real roots, D_θ is a positive rational integer.

Proposition 4.3.7. Every integer of $\hat{L}$ is of the form $(aw^2 + bw + c)/D_\theta$, where $w = (\theta, \theta', \theta'')$ and a,b,c and D_θ are rational integers.

Proposition 4.3.8. (1,1,1), w and w^2 form a basis of $\hat{L}$, where w is as in proposition 4.3.7.

Proposition 4.3.9. D_θ is the discriminant of $\hat{L}$.

Proposition 4.3.10. $\hat{L}$ of definition 4.3.4 is an irreducible three-dimensional multiplicative lattice with identity.

Proof: First, note that $v_1 = (1,1,1)$, $v_2 = w$ and $v_3 = w^2$, where w is as in proposition 4.3.7, form a basis for L. Because of the basis v_1, v_2 and v_3 and the fact that $Q(\theta)$, $Q(\theta')$, $Q(\theta'')$ are field isomorphic, we have that v_1, v_2, and v_3 are a basis for a three-dimensional multiplicative lattice L. Furthermore, (d,d',d'') in L cannot have zero components unless (d,d',d'') = (0,0,0) because each coordinate is irrational of degree 3, we have that L is irreducible.

The fact that $v_1 = (1,1,1)$ establishes that L contains the identity. $\square$

We shall apply the methods that we will develop for calculating units for irreducible multiplicative real three dimensional lattices to $\hat{L}$.

CHAPTER 5

CALCULATION OF UNITS BY MEANS OF VAOP, MAOP AND EFAOP IN IRREDUCIBLE THREE DIMENSIONAL MULTIPLICATIVE LATTICES WITH IDENTITY

5.1. Unit Related - MAOP, VAOP and EFAOP

Definition 5.1.1. Two VAOPs (or EFAOPs or MAOPs) $(a,g,h)_p$ and $(a_1, g_1, h_1)_p$ in a multiplicative 3-dimensional lattice L with identity such that $w\,a = a_1$, $w'\,g = g_1$, $w''h = h_1$ and $w\,w'w'' = 1$, where (w, w', w'') is in L are called unit related VAOP (or EVAOP or MAOP).

A deeper insight into unit related AOPs is provided by the following propositions.

Proposition 5.1.2. If in an irreducible multiplicative lattice L with identity, two MAOPs, each determined by three points v_1, v_2, v_3, and v'_1, v'_2, v'_3, are unit related by u in L, then $v_1 = uv'_1$, $v_2 = uv'_2$ and $v_3 = uv'_3$. An analogous proposition holds for two unit related EFAOPs. For two unit related VAOPs, the definition is equivalent to the analogous proposition. □

Also, we present the converse.

Proposition 5.1.3. Give any unit $u = (w,w',w'')$, where $ww'w''=1$, of an irreducible multiplicative lattice L with identity and MAOP $(a, g, h)_p$, determined by v_1, v_2, v_3, then the OP B determined by uv_1, uv_2, uv_3, is an MAOP and $B = (wa,w'g,w''h)_p$ in L. Analogous propositions hold for EFAOP and VAOP. □

It is clear that for each unit there exist at least a pair of MAOP (or EFAP or VAOP) that is unit related by proposition 5.1.2.

If it is possible to locate unit related MAOP (or EFAOP or VAOP), then it will be possible to calculate units. The general procedure used that succeeds at doing this is the following:

1) Find a method to locate an initial MAOP (or VAOP of EFAOP).
2) Find a method to move from one MAOP to another MAOP (or VAOP or EFAOP) and within a finite number of steps to two unit related MAOPs (or VAOPs or EFAOPs).

Implicit in 2) is the need for a test to determine if two unit related MAOPs (or VAOPs or EFAOPs) have been created. As we shall see admissibility of the OP is essential to doing 1) and 2).

Both Minkowski and Voronoii presented entirely different methods for 1) and 2). We shall describe these methods in chapter VI. Here we assume that 1) and 2) exist and show the actual details of how to calculate units by Minkowski's method and Voronoii's method. The Minkowski test for unit related MAOP and the Voronoii test for unit related VAOP will be described here completely. Furthermore we show how independent units can be calculated by both Voronoii's and Minkowski's method. Also we shall show how fundamental units can be elegantly calculated by Voronoii's method and how they can be calculated by Minkowski's method.

At the present no simple device such as Minkowski's unit related MAOP test or Voronoii's unit related VAOP test exists for unit related EFAOP. Also, no separate algorithm exist for the edge face neighboring process.

Before beginning the presentation of Voronoii's and Minkowski's work on units it is important to note the following.

Geometrically it is clear that any MAOP creates EFAOP and VAOP, and that if unit related MAOP are found, then unit related EFAOP and unit related VAOP are found. So that any procedure that did 1) to 2) for MAOP, would do the same for EFAOP and VAOP. Similarly, geometrically it is clear that EFAOP creates VAOP, and that if unit related EFAOP are found, then unit related VAOP are found. Hence any procedure that did 1) to 2) for EFAOP would do the same for a VAOP.

It appears that Voronoii's work is a consequence of Minkowski's work. However, we shall see that this is not entirely true and that Voronoii's theory is a very rich and separate theory.

5.2. Voronoii Admissible Oriented Parallelopipeds, Units, Independent Units and Fundamental Units

A. Preliminary Concepts

1. Similarly Maps and VAOP

We begin by pursuing the direction of proposition 4.2.10.

Definition 5.2.1. For an n-dimensional lattice L in R_n, a mapping f:L f (L) in R_n that assigns to each point $(x_1,...,x_n)$

of L a point $(a_1x_1,\ldots,a_nx_n)$ of $f(L)$ where $(a_1,\ldots,a_n)$ is in R_n and $a_i \neq 0$ for all i is called a <u>similarity map.</u>

<u>Proposition 5.2.2.</u> Given a similarity map $f:L \to f(L)$, where L in R_n is n-dimensional lattice, then f is a lattice isomorphism and hence $f(L)$ in R_n is an n dimensional lattice. □

<u>Definition 5.2.3.</u> A mapping $g:L \to L'$ where L and L' are three dimensional lattices, is said to <u>preserve VAOP</u> if whenever S is a VAOP in L, $g(S)$ is a VAOP in L'.

<u>Definiton 5.2.4.</u> A mapping $g:L \to L'$ where L and L' are three-dimensional lattices is said to preserve Voronoii <u>neighbors</u> if whenever S is a successor of S' in L, $g(S)$ is a successor of $g(S')$ in L'.

<u>Proposition 5.2.5.</u> A similarity map $f:L \to L'$, where L and L' are three-dimensional lattices, preserves VAOP and Voronoii neighbors. □

<u>Definition 5.2.6.</u> An <u>automorphic similarity map</u> f for a lattice L is a similarity map such that $f(L)=L$.

Note: Similarity maps are very special maps. Their VAOP and Voronoii neighbors preserving properties are crucial to the remainder of the discussion. Even lattice automorphisms do not necessarily have these preserving properties; $g: L \to L$ and g is linear such that $v_1 \mapsto v_2$, $v_2 \mapsto v_1$, and $v_3 \mapsto v_3$, where the VOP of v_1 is a VAOP and the VOP of v_2 is not a VAOP.

<u>Definition 5.2.7.</u> Given any VOP or lattice point in a three-dimensional lattice L we shall mean by the phrases "VOP similarity map," "VAOP similarity map" or "lattice point similarity map" the similarity map defined by the vertex

point (a_1, a_2, a_3) of the VOP, VAOP or lattice point.

2. Units in Multiplicative Lattices with Identity and VAOP Automorphic Similarity Maps

Proposition 5.2.8. v is a unit of a multiplicative lattice L with identity if and only if the similarity map associated with v is an automorphic similarity map.

Proof: Only if direction: By proposition 4.2.7., v is a unit implies that Nv = 1, which in turn implies that the similarity map associated with v maps the basic parallelopiped of L to a basic paralleopiped of L. Hence, the index of the sub-lattice vL in L is one. We have that vL = L and our similarity map associated with v is an automorphic similarity map.
If direction: If the similarity map associated with v is an automorphic similarity map, then vL = L. Hence, a basic parallelopiped of vL is the same as a basic parallelopiped of L and hence, Nv=1. By proposition 4.2.7., we have that v is a unit. □

This proposition 5.2.8 is essential to our approach in finding units in L. We now seek automorphic similarity maps of L.

3. Bounds on Indices for Sublattices Associated with VAOP

Proposition 5.2.9 Let L be a multiplicative lattice with identity. There exists for any such L a real number M such that for any similarity map f associated with a vertex point of a VAOP, the index i of f(L) in L is such that i M.

Proof: i = (Basic Parallelopiped of f(L))/(Basic Parallelopiped of L) = Nf d by corollary 4.1.19. □

We can now locate unit related VAOP's and, hence, units.

B. Finding Units by Voronoii's Method - The Voronoii Test to Determine Unit Related VAOP Within a Finite Number of Steps

Let L be a irreducible multiplicative lattice with identity. Let S, S_1, S_2, ... be a chain of VAOP along the x axis and for the sake of notation, we let them represent the similarity maps associated with them so that $S_i(L)$ will be written as S_iL. Now we consider the sequence SL, S_1L, S_2L, ... of sublattices of L. Because of proposition 5.2.9, this sequence of sublattices consists of only a finite number of sublattices. Let $S_mL=S_nL$ for some non-negative integers m and n. We have that S_m/S_n , where this division is coordinatewise, is a automorphic similarity map of L. Moreover, it is not the identity automorphism nor the negative identity automorphism of L because the x coordinates of the S_i are different. Finally, because (1,1,1,) is in our lattice L, S_m/S_n is in L. S_m and S_n will be unit related VAOP and S_m/S_n is a unit. Moreover, the unit related VAOP have been found in a finite number of steps.

C. Finding Independent Units by Voronoii's Method

If we calculate a unit (w, w', w") by a chain of VAOP along the x axis, we would have $|w|<1$, $|w'|>1$ and $|w''|>1$. Analogously, if we calculate a unit by a chain of VAOP along the y axis, we would have $|w|>1$, $|w'|<1$ and $|w''|>1$. Finally, if we calculate a unit by a chain of VAOP along the z axis we would have $|w| > 1, |w'| > 1$ and $|w''| < 1$. It is clear that two of these three units are independent.

D. Finding Fundamental Units by Voronoii's Method

1. A Partition and Linear Ordering of the Set of VAOP Chains in a Coordinate Direction

The material of this subsection can be developed for OSDA which we shall now do. This material can be developed along any coordinate axis, however, here we choose the development along the x axis.

Proposition 5.2.10 If VAOP S does not pierce VAOP T along the x axis, the T is in $(S)_x$ or T is z lesser, y greater than $(S)_x$ or T is z greater, y lesser than $(S)_x$. □

Proposition 5.2.11. If VAOP T is z greater, y lesser than $(S)_x$, then the Voronoii successor of T along the x axis will be □reater, y lesser than $(S)_x$ or it will be in $(S)_x$. □

An analogous result is true for T z lesser, y greater than $(S)_x$.

As a result of the previous propositions, we may make the following definition even if S Pierces T in the x direction.

Definition 5.2.12. $(T)_x$ is a z greater, y lesser than $(S)_x$ if there exists an element in $(T)_x$ that is z greater, y lesser than $(S)_x$, denoted as $(T)_x > (S)_x$.

$(T)_x$ is z lesser, y greater than $(S)_x$ if there exists an element in $(T)_x$ that is z lesser, y greater than $(S)_x$, denoted $(T)_x < (S)_x$. If $(T)_x$ and $(S)_x$ are not related as described in the first two parts of this definition, then we say that $(T)_x$ and $(S)_x$ belong to each other, denoted as $(T)_x \sim (S)_x$.

Proposition 5.2.13. $\sim$ is an equivalance relation in the set of VAOP chains along the x axis.

Proposition 5.2.14. $\gtrsim$ is a linear ordering in the set of VAOP chains along the x axis.

Definition 5.2.15. We shall refer to the linear ordering of proposition 4.2.27 as the Voronoii chain ordering.

Definition 5.2.16. A mapping $f:L \to L'$ that preserves, preserves the Voronoii chain ordering if for any two Voronoii chains $(S)_x$ and $(T)_x$: $(f(S))_x > (f(T))_x$, when $(S)_x > (T)_x$; $(f(S))_x < (f(T))_x$, when $(S)_x < (T)_x$; and $(f(S))_x \sim (f(T))_x$, when $(S)_x \sim (T)_x$.

Proposition 5.2.17. Similarity maps preserves the Vornoii chain ordering. □

2. Prediocity and VAOP in Irreducible Multiplicative Lattices

We shall explore some of the directions pursued in chapter III for VAOP in an OSDA, in an irreducible multiplicative lattice. Though a lattice is a much more "uniformly symmetric" structure than an OSDA, many of the same results of this section will carry over to an OSDA.

Proposition 5.2.18. In a lattice, multiplicative or not, a VAOP need not necessarily have a unique VAOP Voronoii predecessor. In fact, a VAOP may have no Voronoii-predecessor or more than one.

Proof: For an elementary proof of this proposition see [4, pp. 248-250]. □

Definition 5.2.19. A Voronoii chain $(S)_x$ in an irreducible lattice L of dimension 3 becomes periodic at m if there exists a point e in L, where e has no coordinate zero, and an n such that $S_{i+n} = eS_i$ for $i \geq m$.

Proposition 5.2.20 Every Voronoii chain $(S)_x$ in a multiplicative lattice L with identity becomes periodic.

Proof: By the reasoning of section 5.2.2 there exists an m and a $k = m + n$ such that $S_m L = S_k L$. Now $S_m/S_k = e$ is an automorphic similarity map of L Furthermore, $S_{i+n} = eS_i$ for $i \geq m$, because e preserves Voronoii neighbors. □

Definition 5.2.21 A Voronoii chain $(S)_x$ in a lattice L of dimension 3 is purely periodic if it becomes periodic at S, that is, at m=0.

Definition 5.2.22 If for a Voronoii chain $(S)_x$ in a three-dimensional lattice L there exists an infinite set of VAOP S_{-1}, S_{-2}, S_{-3}, ..., such that $(S_{-k})_x$ is a Voronoii chain for all -k then we call $(S)_x$ a two-sided Voronoii chain.

The following theorem is one of the most revealing theorems on the structure of Voronoii chains in three-dimensional irreducible multiplicative lattices with identity.

<u>Theorem 5.2.23.</u> Every Voronoii chain in an irreducible multiplicative lattice L with identity is purely periodic if and only if it is a two-sided Voronoii chain.

<u>Proof</u>: Only if: Let $(S)_x$ be a purely periodic chain. Hence, $S_{i+n} = eS_i$ for all $i \geq 0$ and n is fixed. Now $e^{-1}S_{n-1}$ is in L because e^{-1} is an automorphic similarity map of L and is in L. Furthermore, because S_{n-1} is a VAOP, $e^{-1}S_{n-1}$ is a VAOP by the fact that similarity maps preserve VAOP. Finally, because S_{n-1} is a Voronoii predecessor of S_n, $e^{-1}S_{n-1}$ is a Voronoii predecessor of $e^{-1}S_n = S_0 = S$, by the fact that similarity maps preserve Voronoii neighbors. We create S_{-2}, S_{-3}, ... *ad infinitum* by means of this method, and we have finished the only if part of the proof.

If: By the reasoning of Section 5.2.2, there exists an m and a k=m+j such that $S_mL=S_{m+j}L$. Now $S_{m+j}/S_m = e^*$ is an automorphic simmilarity map of L such that $e^*S_h = S_{h+j}$ for h m . Now let e be the automorphic similarity map for the least positive integer n such that $S_{i+n} = eS_i$ where $i \geq m$. At the moment, however, we cannot assume that $e^{-1}S_{k-1}=S_{m-1}$ because of proposition 5.2.18: a VAOP may have two predecessors. However, in fact, it will be the case that $e^{-1}S_{k-1}=S_{m-1}$. Consider: there exist greatest integers r and s such that $S_rL = S_sL$, $s=r+t$, $t > 0$, $r < m$ and $s < m$ and r and s may be non-positive integers. Let $e' = S_r/S_s$. It is clear that $n|t$, for if not $t=un+v$, where $0 \leq v < n$. Now, $e'' = e'/e^u$ is an automorphic similarity map such that $e''S_i = S_{i+v}$ for $i \geq m$, contradicting our choice of n.

Hence, $t=an$. We now wish to show in fact that $a=1$, that is, $t=n$. Now there exists a p r so that $e'S_p=S_m$ and $q=p+n$ is such that $e'S_q = S_{m+n}$. Now we have $e=S_p/S_q$ and consequently $S_pL=S_qL$, contradicting our choice of r and s. We conclude that $t=n$. Because $e'S_j=S_{j+t}$ for $j>r$, we have in particular that $e'S_i=S_{i+n}$, and hence, $e'=e$. Finally, we have that $e'^{-1}S_{k-1}=S_{m-1}=e^{-1}S_{k-1}$. This discussion finally implies that $eS_i=S_{i+n}$ for all integers i. □

Proposition 5.2.24. In each class A of VAOP chains, there exists only one VAOP chain that is a two sided Voronoii Chain, and it is this chain that all Voronoii chains in class A will intersect. □

An interesting implication of proposition 5.2.24 is that if a VAOP possesses two VAOP Voronoii predecessors, then for at least one of these predecessors, any Voronoii chain that poscesses it cannot be extended infinitely backward and hence certain VAOP must not have prececessors as in proposition 5.2.18.

Notation: Given a Voronoii chain $(A)_x$, we indicate its two-sided Voronoii chain by $(A_i)'_x$, where A_i is that VAOP at which $(A)_x$ becomes purely periodic. If $(A)_x$ is purely periodic, then $(A)'_x$ indicates the two-sided Voronoii chain of $(A)_x$.

Proposition 5.2.25 If $(A)_x \sim (B)_x$, there exists a VAOP C such that $(C)_x=(A)_x \cap (B)_x$. However, if $(A)'_x \sim (B)'_x$, then there exists an i such that for any element C of $(A)'_x$, $B_i=C$. □

Because of this proposition, we now may write $(A)'_x = (B)'_x$ when $(A)'_x \quad (B)'_x$.

The following useful proposition, expressed in a new notation, will be valuable in the next sections.

<u>Proposition 5.2.26.</u> If $(A)'_x > (B)'_x$ and $(B)'_x > (C)'_x$, then $(A)'_x \cap (C)_z$ is in $((B)'_x \cap (C)_z)_z$. □

We now come to the following important theorem.

<u>Theorem 5.2.27</u> Two two-sided Voronoii chains along different axes will intersect.

<u>Proof</u>: The hypothesis of proposition 3.3.6 can be shown to be satisfied by these two two-sided Voronoii chains. □

3. <u>A Geometric Proof of the Existence of Fundamental Units e_1 and e_2 in an Irreducible Multiplicative Lattice of Dimension Three with $((1,1,1))_x$ Purely Periodic</u>

We begin by stating two elementary propositions.

<u>Proposition 5.2.28.</u> (1,1,1), the identity, is a VAOP in a multiplicative lattice with identity. □

<u>Propostion 5.2.29.</u> If the similarity map associated with a lattice point v in a multiplicative lattice L with identity is automorphic, then the VOP associated with v is a VAOP. □

Now, let L be an irreducible multiplicative lattice with identity. The identity (1,1,1) is a VAOP by proposition 5.2.28, however, $((1,1,1))_x$ is not necessarily purely periodic. For this section 5.2.D.3., we will assume that $((1,1,1))_x$ is purely periodic and give a geometric proof of the existence of the fundamental units e_1 and e_2. (In the following section 5.2.D.4., we will show that the proof can be given even if $((1,1,1))_x$ is not purely periodic.) These methods will imply techniques for calculating fundamental units which we shall present in 5.2.D.5. □

Proposition 5.2.30 $((1,1,1))_x$ is purely periodic if and only if $(e)_x$ is purely periodic where e is any automorphic similarity map of L. □

Proposition 5.2.31. Let e_1 be the automorphic similarity map for the least positive integer n such that $e_1S_i=S_{i+n}$ for i 0 where $(1,1,1)=S_0=S$, that is, $(S)_x=((1,1,1))_x$, then every automorphic similarity map e such that $eS_i=S_{i+k}$ for some k must be such that $e=e_1{}^n$. □

Proposition 5.2.32. For any automorphic similarity map e, the automorphic similarity map e^{-1} maps $(e)_x$ to $(S)_x=((1,1,1))_x$. Furthermore, e^{-1} maps the two-sided Voronoii chain $(e)'_x$ to the two-sided Voronoii chain $(S)'_x$. That is, in other words each element of the two-sided Voronoii chain $(S)'_x$ is unit related by e to the corresponding element of the two sided Voronoii chain $(e)'_x$.

Proof: It is clear that e^{-1}: $e_{-1}\to S_{-1}$. Use induction on k in e^{-1} $e_{-k}\to S_{-k}$ and the fact that two-sided chains of each class is unique. □

Proposition 5.2.33. For any automorphic similarity map e, the two-sided Voronoii chain $(e)'_x$ contains only automorphic similarity maps of the form $ee_1{}^n$, n an integer.

Proof: Consider the interrelationship of $((1,1,1,))'_x$ and $(e)'_x$, which the following diagram depicts:

$$\ldots, e_1^{-2}, \ldots, e_1^{-1}, \ldots, (1,1,1), \ldots, e_1, \ldots, e_1^{2}, \ldots\ldots\ldots, e_1^{n}, \ldots$$
$$\downarrow e$$
$$\ldots, ee_1^{-2}, \ldots, ee_1^{-1}, \ldots, e, \ldots, ee_1, \ldots, ee_1^{2}, \ldots\ldots\ldots, ee_1^{n}, \ldots$$

Any other automorphic similarity map e'' in $(e)_x$ would be such that $ee_1^{n} \ldots e'' \ldots ee_1^{n+1}$, for some n. Mapping $(e)_x$ to $((1,1,1))_x$ by $(ee_1^{n})^{-1}$ would create an automorphic similarity map $(ee_1^{n})^{-1}e''$ that would contradict the definition of e_1. □

Propoposition 5.2.33 essentially says that each element of the two sided chain $((1,1,1))'_x$ is unit related by ee_1^{k} for some k to a corresponding element of the two sided chain $(e)'_x$, where e is any unit. We now consider the chain $((1,1,1))_z$ and the two-sided chain $(e)'_x$ for each automorphic simlarity map such that $(e)'_x > ((1,1,1))'_x$. By proposition 3.2.8, $(e)'_x$ and $((1,1,1))_z$ share a common VAOP. This observation together with proposition 5.2.25 makes possible the following definition.

<u>Definition 5.2.34.</u> Let $((1,1,1)_z = (T_0)_z = (T)_z$ and let $N = \{n \mid e$, an automorphic similarity map of L, and $(e)'_x > ((1,1,1))_x$ and $(e)'_x \cap ((1,1,1))_z = T_n\}$.

Define e_{2_i} $(e_{2_1} = e_2)$ as any automorphic similarity map in that automorphic similarity map e two-side chain $(e)'_x$ where $(e)'_x > ((1,1,1))_x$ and $(e)'_x \cap (T)_z = T_{n_i}$ where n_i is the i^{th} least positive integer of N. To aid in understanding this definition, please see diagram 5.2.36.

<u>Proposition 5.2.35.</u> For any automorphic similarity map

Diagram 5.2.36 Unit VAOP Chains Along the x Axis Which Are z-Greater, y-Lesser Than the Periodic $(1,1,1)_x$ Chain

This diagram does not depict the general relationship of automorphic similarity maps and their chains in their Voronoi linear ordering relationship to each other and $((1,1,1))'_x$ and $((1,1,1))_z$ because the e_n may appear anywhere in the chain $(U^{[k]})'_x$. In the following, we set $((1,1,1))'_x=(S)'_x$ and $((1,1,1))_z=(T)_z$. → connects a VAOP to its successor.

. $U_{-1}^{[k]} \to e_{n_k} = U_0^{[k]} \to \cdots \to U_r^{[k]} = \overset{\uparrow}{T}_{n_k} \quad U_{r+1}^{[k]} \to U_{r+2}^{[k]} \to U_{r+3}^{[k]}$ ..

.... $U_{-q-1}^{[2]} \to \overset{\uparrow}{T}_{n_2} = U_{-q}^{[2]} \to \cdots U_{-1}^{[2]} \to U_0^{[2]} = e_{2_2} \to U_1^{[2]}$ ..

..$U_{-1}^{[1]} \to e_{2_1} = e_2 = U_0^{[1]} \to \cdots \to U_p^{[1]} = \overset{\uparrow}{T}_{n_1} \to U_{p+1}^{[1]}$ ....

.... $S_{-3} \to S_{-2} \to S_{-1} \to (1,1,1) = \overset{\uparrow}{T}_0 = S_0 \to S_1 \to S_2 \to S_3 \to S_4 \to S_5$ ......

x axis →

z axis ↑

such that $(e)_x > ((1,1,1))_x$, there exists a $k>0$ such that $(e_{2_k})'_x > (e)'_x$ or $(e_{2_k})'_x = (e)'_x$.

<u>Propositon 5.2.37</u>. The two-sided Voronoii chain $(e_{2_k})'_x$ contains e_2^k, that is, $(e_{2_k})'_x = (e_2^k)'_x$.

<u>Proof</u>: Consider $e_2^{-1} : L \to L$. Because e_2^{-1} is an automorphic similarity map, we have that $(e_{2_k})'_x$ is mapped to $(e_{2_{k-1}})'_x$ by e_2^{-1} for all $k \geq 1$, where $e_{2_0} = (1,1,1)$. Applying e_2^{-1} k times to L results in $(e_{2_k})'_x$ being mapped to $((1,1,1))'_x$ and the proof is complete.

<u>Theorem 5.2.38.</u> Every similarity automorphic map e (unit) of a multiplicative irreducible lattice of dimension three with $((1,1,1,))_x$ purely periodic possesses two automorphic similarity maps e_1 and e_2 (fundamental units) such that $e = e_1^m e_2^n$, where m and n are integers.

<u>Proof</u>: e is contained in some two-sided chain $(A)'_x$ whereby the Voronoii chain ordering: $(A)'_x = ((1,1,1))'_x$, $(A)'_x > ((1,1,1))'_x$ or $(A)'_x < ((1,1,1))'_x$.

If $(A)'_x \leq ((1,1,1))'_x$, that is, $(e)'_x < (1,1,1))'_x$ or $(e)'_x = ((1,1,1))'_x$. Let $(e)_x = (S)_x$. Now for some k, $S_k = (1,1,1)$ and we have $eS_i = S_{k+i}$. By proposition 5.2.31, we obtain: $e = e_1^n = e_1^n e_2^0$.

If $(A)'_x > ((1,1,1))'_x$, that is, $(e)'_x > ((1,1,1))'_x$. Now by propositons 5.2.35 and 5.2.37, $(e)'_x = (e_2^k)'_x$, for some k. By propostion 5.2.33, all automorphic similarity maps of $(e_2^k)'_x$ are of the form $e_2^k e_1^n$ for some integer n. Now, e is an automorphic similarity map of the two-sided chain $(e_2^k)'_x$ and so $e = e_2^k e_1^n$.

If $(A)'_x < ((1,1,1))'_x$, that is, $(e)'_x < ((1,1,1))'_x$. Now e^{-1} maps $(e)'_x$ to $((1,1,1))'_x$ and e^{-1} maps $((1,1,1))'_x$ to $(e^{-1})'_x$ and because similarity maps preserve the Voronoii chain ordering, we have $(e^{-1})'_x < ((1,1,1))'_x$. By the previous paragraph we have that $e^{-1} = e_2^q e_1^m$ or $e = e_2^{-q} e_1^{-m}$.

4. A Geometric Proof of the Existence of Fundamental Units e_1 and e_2 in an Irreducible Multiplicative Lattice of Dimension Three with $((1,1,1))_x$ Not Purely Periodic

We now assume that $(1,1,1,)_x$ is not purely periodic in L. This section is analogous to section 5.2.4.C, however, it is a bit more difficult. We begin by noting that by proposition 5.2.30 no automorphic similarity map e of L is such that $(e)_x$ is purely periodic.

Notation: For any chain $(T)_x$, let i be the positive integer at which $(T)_x$ becomes purely periodic, then in order to distinguish between the predecessors of T_i in the two sided chain of $(T_i)'_x$ and the partial chain $T_0, T_1, \ldots, T_{i-1}$, we shall denote the elements T_k of the two sided chain by T'_k.

Proposition 5.2.39 Let $(1,1,1)_x = (S)_x$ and let S_i be that VAOP at which $(1,1,1,)_x$ first becomes purely periodic. Let e_1 be the automorphic similarity map for the least positive integer n such that $e_1 S'_j = S'_{j+n}$ $j \geq i$, then every automorphic similarity map e^* such that $e^* S'_j = S'_{j+k}$ for some k must be such that $e^* = e_1^m$ for some m. $\square$

Proposition 5.2.40. Let $(1,1,1,)_x = (S)_x$ and let S_i be that

VAOP at which $(1,1,1)_x$ first becomes purely periodic. For any automorphic similarity map e, let $(e)_x = (T)_x$ and let T_j be that VAOP at which $(e)_x$ becomes purely periodic, then j=i. □

Proposition 5.2.41. Let $(S)_x$, $(T)_x$, i and e be as in proposition 5.2.40, then e: $S_k \to T_k$ for $0 \le k \le i$ and e: $S'_k \to T'_k$ for all k. That is, each element of the two-sided chain $((1,1,1))'_x$ is unit related by e to each element of the two-sided chain $(T)'_x$. □

The following proposition adds insight into the structure of the non-periodic chain.

Proposition 5.2.42. For the chain $(T)_x$, where T_i is the element at which $(T)_x$ becomes purely periodic, there exists no T'_k for any k in the two sided chain $(T_i)'_x$ such that $T'_k = e$, where e is an automorphic similarity map and there exixsts no T_k, where $0 \le k \le i$, such that T_k=e. □

Proposition 5.2.43 For any automorphic similarity map e, let $(e)_x = (T)_x$ and let T_i be the VAOP at which $(T)_x$ becomes purely periodic. Let $(1,1,1)_x = (S)_x$ and let S_i be the VAOP at which $(S)_x$ becomes purely periodic. Then any unit e', an automorphic similarity map, that unit relates an element of $(S)_x$ to an element of $(T)_x$, that is, $e'S_s = T_r$ for some integers r and s, is such that $e' = ee_1{}^j$ for some integer j, where e_1 is the automorphic similarity map of proposition 5.2.39.

Proof: Consider the interrelationship of $(T)_x$ and $((1,1,1))_x$, which the following diagram depicts:

$$(1,1,1) = S_0 \to S_1 \searrow \ldots \quad e_1 = e_1 S_0 = S_n \to e_1 S_1 \searrow \ldots$$

$$\ldots e_1^{-2} S'_{i-2n} \to \ldots \to e_1^{-1} S'_{i-n} \ldots S_i = S'_i \to \ldots \; e_1 S'_i = S'_{i+n} \to \ldots \; e_1^{2} S'_i = S'_{i+2n} \to \ldots$$

$$e = eS_0 = T_0 \to eS_1 = T_1 \searrow \ldots \qquad ee_1 = ee_1 S_0 = eS_n \searrow \ldots$$

$$ee_1^{-2} S'_{i-2n} = T'_{i-2n} \to \ldots \; ee_1^{-1} S'_{i-n} = T'_{i-n} \to \ldots \; eS'_i = T'_i = T_i \to \ldots \; ee_1 S'_i = eS'_{i+n} = T'_{i+n}$$

Case I: Suppose e' is such that for some m: $e_1^{m} eS'_i \ldots e'S'_k \ldots e_1^{m+1} S'_i$ in the periodic part of $(T)'_x = (eS)'_x$, then the map $(ee_1^{m})^{-1} e'$ would contradict the definition of e_1. Hence, $e' = ee_1^{q}$ for some q.

Case II: Suppose that e' is such tha for some r: $e \ldots e'S_r \ldots eS_i$ in the nonperiodic part of $(T)_x = (eS)_x$, then $e^{-1} e' S_r = S_r$, which would make $e' = e$, which is of the desired form: $e' = ee_1^{0}$. □

We now consider the chain $((1,1,1))_z$ and the two sided chains $(U_i)'_x$ where U_i is the VAOP at which $(U)_x$ becomes purely periodic and $(U)_x = (e)_x$ for any automorphic similarity map e where $(e)_x \; ((1,1,1,))_x$. By proposition 3.2.6., $(U_i)'_x$ and $((1,1,1))_z$ share a common VAOP. This observation together with proposition 5.2.25 makes possible the following definition.

<u>Definition 5.2.44</u> Let $((1,1,1,))_z = (T_0)_z = (T)_z$ and for an automorphic similarity map e, let $(e)_x = (U)_x$, and $(U_i)'_x$ is the two sided chain of $(e)_x$. Finally, let:

$$N = \left\{ n \,\middle|\, e, \text{ in automorphic similarity map of L and } (e)_x > ((1,1,1,))_x \text{ and } (U_i)'_x \cap ((1,1,1,))_z = T_n \right\}$$

Define $e_{2_i} (e_2 = e_2)$ as the automorphic similarity map e contained in the automorphic similarity map partial chain $e = U_0, \; U_1, \ldots U_i$, where $(U)_x = (e)_x > ((1,1,1))_x$ and

$(U_i)'_x \cap ((1,1,1))_z = T_n$ and n_i is the i^{th} least positive integer of N. To aid in the understanding of this definition, please see diagram 5.2.48.

Note: All units, automorphic similarity maps, will all be among the e_2 because L contains the identity.

Proposition 5.2.45. For any automorphic similarity map such that $(e)_x > ((1,1,1))_x$ there exists a $k > 0$ such that $(e_{2_k})_x$ $(e)_x$ or $(e_{2_k})'_x = (e_i)'_x$, where e_i is where $(e)_x$ becomes purely periodic. □

Proposition 5.2.46 In the Voronoii chain $(e_{2_k})_x$, we have that $e_{2_k} = e_2^k$.

Proof: Consider e_2^{-1}: L L. e_2^{-1} maps the partial chain $e_2 = U_0^{[k]}, U_1^{[k]}, \ldots, U_i^{[k]}$ to the partial chain e_2 = $U_0^{[k-1]}, U_1^{[k-1]}, \ldots, U_i^{[k-1]}$ for all k. Applying e_2^{-1} k times we have that the partial chain $e_2 = U_0^{[k]}, U_1^{[k]}, \ldots,$ $U_i^{[k]}$ is mapped onto the partial chain of (1,1,1), $(1,1,1)_1, \ldots,$ $(1,1,1)_i$ and the proof is complete. □

Theorem 5.2.47 Every automorphic similarity map e (unit) of a multiplicative irreducible lattice of dimension three with $((1,1,1,))_x$ not purely periodic possesses two automorphic similarity e_1 and e_2 (fundamental units) such that $e = e_1^m e_2^n$, where m and n are integers.

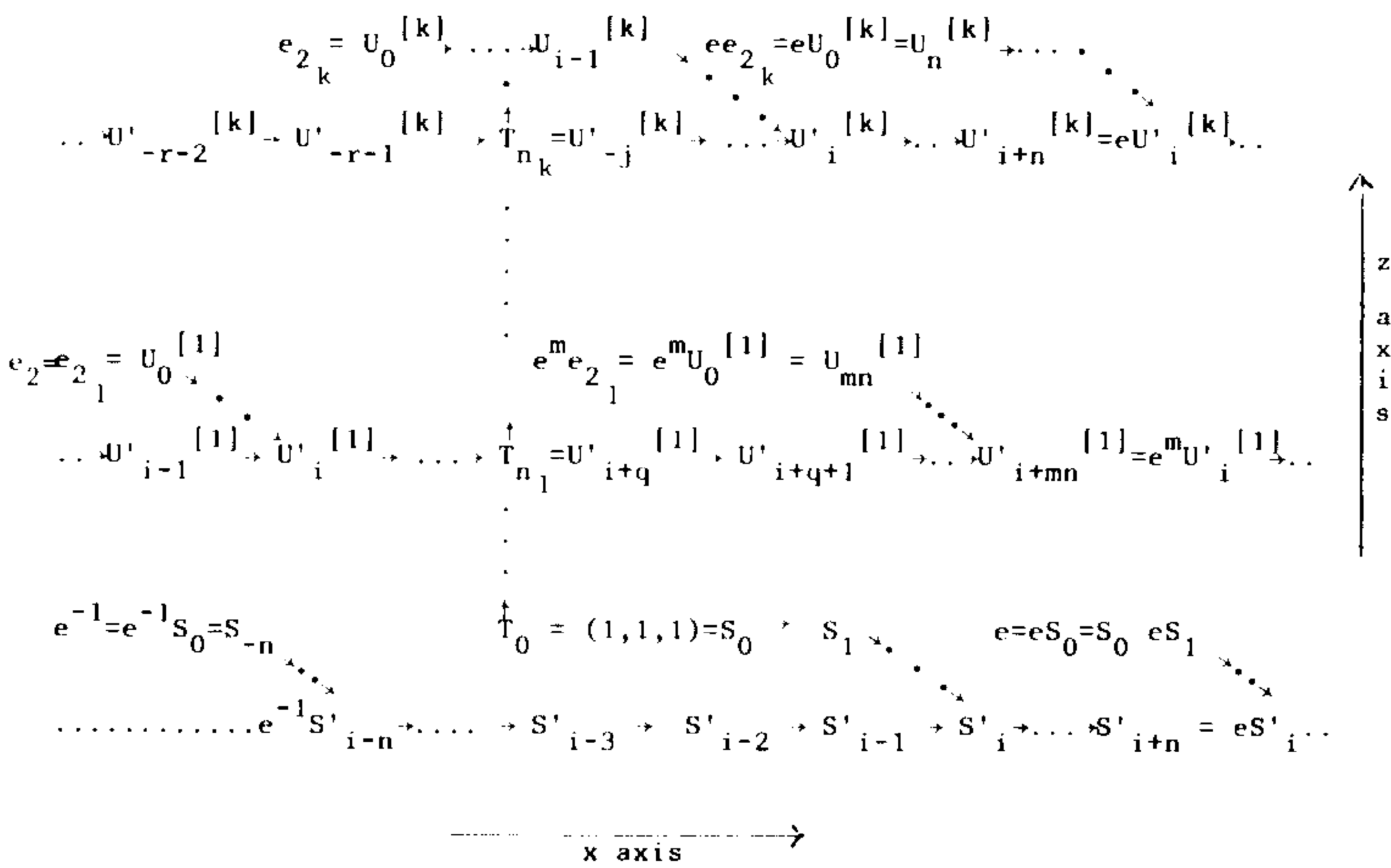
Proof: Analogous to theorem 5.2.38. □

5. Calculation of Fundamental Units e_1 and e_2 for Irreducible Multiplicative Lattice with Identity

1) We assume that $((1,1,1,)_x$ is purely periodic, then $((1,1,1,))_x$ will

Diagram 5.2.48 Unit VAOP Chains Along the x Axis Which Are z-Greater, y-Lesser Than the Non-Periodic $(1,1,1)_x$ Chain

This diagram does not depict the general relationship of automorphic similarity maps and their chains in their Voronoii linear ordering relationship to each other and $((1,1,1))_x$ and $((1,1,1))_z$ because the e_{n_k} may appear anywhere in the chain $(U^{[k]})_x$. In the following, we set $((1,1,1))_x=(S)_x$ and $((1,1,1))_z = (T)_z$. → connects a VAOP to its successor.



Note: $(e)_x \simeq ((1,1,1))_x \simeq (e^{-1})_x$; $(e_{2_1})_x \simeq (e_2)_x \simeq (e^m e_2)_x$; and $(e_{2_k})_x \simeq (ee_{2_k})_x$.

contain the automorphic similarity map e_1. Let $S = S_0 =$ (1,1,1,). Calculate S_1 by the Voronoii algorithm (Described in Chapter VI) and see if $S_1L = L$. If not, calculate S_2 by the algorithm and see if $S_2L = L$. Continue until S_k is found such that $S_kL = L$. Let $e_1 = S_k$. Now e_2 is such that $((e_2)_x \cap ((1,1,1)))_z)_z$ contains $(e)_x \cap ((1,1,1))_z$ for all e where $(e)_x > ((1,1,1))_z$. In other words if $(T)_z = ((1,1,1))_z$ and if $T_p = (e_2)_x \cap ((1,1,1))_z$ and if $(e)_x \cap ((1,1,1))_z = T_r$, then $r > p$. Now e_2^{-1}: $(e_2)_x \to ((1,1,1))_x$. Hence, $e_2^{-1}\ T_p = S_m$ for some m. We have that $e_2 = T_p/S_m$ or that $T_pL = S_mL$. We have calculated S_jL for all $j \geq 0$, because $(S)_x$ is purely periodic. Hence, to find T_p, we find T_1 by the algorithm and see if $T_1L = S_jL$ for all j. Repeat for T_2, T_3, until T_p is found so that $T_pL = S_mL$ for some m. T_p/S_m is e_2.

2) We assume that $((1,1,1,))_x = (S)_x$ is not purely periodic, then $((1,1,1,))_x$ will contain the automorphic similarity map e_1 as a quotient of an S_q and S_i. Calculate S_1, by the algorithm and record S_1L. Calculate S_2 by the algorithm and record S_2L. Check if $S_2\ L = S_1L$, continue until an S_q is found such that $S_qL = S_iL$ where $0 \leq i < q$, then $e_1 = S_q/S_i$. Now let (1,1,1,) $= T_0 = T$. Now e_2 is such that $((e_2)_x \cap (T)_z)_z$ contains $(e)_x \cap (T)_z$ for all e where $(e)_x > ((1,1,1))_x$. In other words if $T_n = (e_2)_x \cap (T)_z$ and if $(e)_x \cap (T)_z = T_k$, then $k > n$. Now e_2^{-1} $(e_2)_x \to ((1,1,1))_x$. Hence, $e_2^{-1}\ T_n = S_h$, where S_h is in $(S)_x$. We have that $e_2 = T_n/S_h$ or $T_nL = S_hL$. We have calculated all S_rL because we have calculated S_rL for $r < q$ and $(S_i)'_x$ is purely periodic. Hence, to find T_n, find T_1

by the algorithm and see if $T_1L = S_tL$ for all t q. Repeat for T_2, T_3 until T_n is found such that $T_nL=S_hL$ for some n and t. $T_n/S_h = e_2$.

5.3. Minkowski Admissible Oriented Parallelopipeds Units, Independent Units and Fundamental Units

A. Preliminary Concepts

1. Minkowski Matrices for MAOP and a Partition of MAOP into Six Classes

Proposition 5.3.1 In an irreducible lattice, no two lattice points on the surface of an MAOP lie in the same octant.

Proof: Assume v and s, two different lattice points on the surface of a MAOP $(a,g,m)_p$, are in the same octant.

$v-s = (\pm (v_x - s_x), \pm (v_y - s_y), \pm (v_z - s_z))$ for some sign combination. Now $0 \leq |v_x - s_x| < a$, $0 \leq |v_y - s_y| < g$ and $0 \leq |v_z - s_z| < m$ because $v \neq s$ and the lattice is irreducible. Therefore v-s is in $(a,g,m)_p$, which is impossible and we have our result.

Definition 5.3.2. Given any Minkowski admissible oriented parallelopiped $(a,g,m)_p$ and three of its surface points p_1, p_2, p_3, one from each of the three origin symmetric point pairs of the MAOP. We define a Minkowski Matrix of the MAOP as the 3x3 matrix.

$$\begin{bmatrix} +a & \pm b & \pm c \\ \pm f & +g & \pm h \\ \pm j & \pm k & +m \end{bmatrix}$$

Where a,f,j,b,g,k,c,h,m are positive and $p_1 = (\pm a, \pm f, \pm j)$,

$p_2 = (\pm b, \pm g, \pm k)$ and $p_3 = (\pm c, \pm h, \pm m)$ for some sign combination of the coordinates.

<u>Proposition 5.3.3</u> In an irreducible lattice, let MAOP $(a,g,m)_p$ have six surface points p_1, p_2, p_3, $-p_1$, $-p_2$, $-p_3$, then there is unique way of choosing three points q_1, q_2, q_3 one from each of the three symmetric point pairs so that the Minkowski matrix of q_1, q_2, q_3 has one of the following 24 sign systems:

1	2	3	4	5	6
+ + + + + - + - +	+ + - + + + + - +	+ - + + + - + + +	+ - - + + - + + +	+ - - + + + + - +	+ - - + + - + - +

7	8	9	10	11	12
+ + + + + - - + -	+ + - + + + - + +	+ - + + + + - + +	+ - + + + - - + +	+ - + + + + - - +	+ - - + + + - - +

13	14	15	16	17	18
+ + + - + - - + +	+ + - - + - - + +	+ + + - + + - - +	+ + + - + - - - +	+ - + - + + - - +	+ - - - + - - - +

19	20	21	22	23	24
+ + - - + + + + +	+ + - - + - + + +	+ + + - + + + - +	+ + - - + + + - +	+ - + - + + + + +	+ - - - + - + + +

. □

<u>Definition 5.3.4.</u> Given any MAOP $(a,g,m)_p$ and its surface points p_1, p_2, p_3, $-p_1$, $-p_2$, $-p_3$, then the unique Minkowski Matrix for appropriate q_1, q_2, q_3 whose main diagonal elements are positive shall be called the positive Minkowski Matrix for an MAOP.

Propostion 5.3.5. Let $|s_1| = |s_2| = |s_3| = 1$ and let $a_{(i,j)}$ be the positive Minkowski Matrix for appropriate q_1, q_2, q_3 of an MAOP $(a,g,m)_p$ the s_1, s_2, s_3 may be selected so that $s_1 s_2 s_3 = 1$ and $[s_i s_j a_{(i,j)}]$ will have one of the following six sign systems:

I	II	III	IV	V	VI
+ + +	+ - -	+ - -	+ + -	+ - +	+ - -
- + -	+ + +	- + -	- + +	+ + -	- + -
- - +	- - +	+ + +	+ - +	- + +	- - +

Proof: To prove this proposition consider the table below. The Arabic numbers are the same as in proposition 5.3.3. The s_i following these numbers are to be set equal to -1. The Roman numeral is the resulting case listed above.

1) s_2, s_3, VI;	2) s_1, s_2, III;	3) s_1, s_2, II;	4) s_1, s_3, IV;
5) s_1, s_2, V;	6) s_2, s_3, I;	7) s_2, s_3, III;	8) s_1, s_3, VI;
9) s_1, s_3, I;	10) None, V;	11) s_2, s_3, IV;	12) None, II;
13) s_1, s_2, IV;	14) s_1, s_3, II;	15) s_1, s_3, V;	16) None, I;
17) s_1, s_2, III;	18) None, VI;	19) s_1, s2, I;	20) s_2, s_3, V;
21) s_2, s_3, II;	22) None, IV;	23) s_1, s_2, VI;	24) None, III. □

Note: It must be understood that the 6 sign systems are quite different from the previous 24 sign systems. In the previous case the resulting matrices had columns which were points of the lattice. The $[s_i s_j a_{(i,j)}]$ columns do not necessarily yield points for the lattice. However, determinants and parameters of MAOP are preserved under this classification. Use of the six sign system makes for a simpler presentation of

Minkowski's Algorithm. Hence, we create the following definitions.

Definition 5.3.6. The six sign systems of proposition 5.3.4 shall be called the Minkowski algorithmic system or the algorithmic system.

Defintion 5.3.7. The partition of MAOP into the algorithmic system via positive Minkowski Matrices of MAOP shall be called the Minkowski partition of MAOP.

2. Parametric Intrarelationships of an MAOP for each of the Six Categories of the Algorithmic System

Proposition 5.3.8. According as an MAOP in an irreducible lattice is Minkowski partitioned into the algorithmic system I through VI, then a, b, c, f, g, h, j, k, m, the absolute value of the entries of the Minkowski matrix of the MAOP satisfy the conditions under the corresponding numerals.

I	II	III
$b + c > a$	$f + h > g$	$j + k > m$
$f > h$ and/or $j > k$	$k > j$ and/or $b > c$	$c > b$ and/or $h > f$

IV	V
$b > c$ and/or $h > f$ and/or $j > k$	$c > b$ and/or $f > h$ and/or $k > j$

VI

$$b + c = a \ , \ f + h = g \quad j + k = m$$

Proof: We prove VI only (all others are similar). Because the rows of the algorithmic system VI matrices are not points of the lattice. We look at each of the four Minkowski matrices and its unique mapping of Proposition 5.5.4. that map the four positive Minkowski matrices into the algorithmic system. Hence, we have:

$$\begin{bmatrix} s_1s_1a & s_1s_2b & s_1s_3c \\ s_2s_1f & s_2s_2g & s_2s_3h \\ s_3s_1j & s_3s_2k & s_3s_3m \end{bmatrix} = \begin{bmatrix} a & -b & -c \\ -f & g & -h \\ -j & -k & m \end{bmatrix}$$

for appropriate s_1, s_2, s_3, and $s_1s_2s_3 = 1$.

Let $q_1 = (s_1a, -s_2f, -s_3j)$, $q_2 = (-s_1b, s_2g, -s_3k)$ and $q_3 = (-s_1c, -s_2h, s_3m)$, which are each a row or negative of a row of the positive Minkowski matrix, that is transformed by proposition 5.3.4. into algorithmic system VI. Hence, q_i are points of the lattice. Consider $q=q_1+q_2+q_3$. We have $q_x = -s_1(-a+b+c)$, $q_y = -s_2(f-g-h)$ and $q_z = -s_3(j+k-m)$. Because $a > b > 0$, $a > c > 0$; $g > f > 0$, $g > h > 0$; and $m > j > 0$, $m > k > 0$; we have $|q_x| < a$, $|q_y| < g$ and $|q_z| < m$. We have that q is in the MAOP at this matrix, and hence $q=(0,0,0)$. This fact yields the result that $a = b + c$, $g = f + h$ and $m = j + k$. $\square$

3. <u>Determinants of Each of the Six Categories of the Algorithmic System</u>

a. Lattice Octahedra and Determinants

<u>Definition 5.3.9.</u> A lattice octahedron H in a three dimensional lattice is an octahedron that is symmetric about the origin with lattice points at its vertices (all of which are

non-planar with the origin) and no lattice points on its surface or interior (except the origin).

The following proposition was proved by Minkowski in [8, p.100].

Proposition 5.3.10 If H is a lattice octahera where p_1, p_2, p_3 are three veritices of H such that no two are origin symmetric, then either:

1) $\text{Det}\begin{bmatrix} p_1^{(1)} & p_2^{(1)} & p_3^{(1)} \\ p_1^{(2)} & p_2^{(2)} & p_3^{(2)} \\ p_1^{(3)} & p_2^{(3)} & p_3^{(3)} \end{bmatrix} = \pm d$, the volume of the basic parallelopiped of L,

or

2) $\text{Det}\begin{bmatrix} p_1^{(1)} & p_2^{(1)} & p_3^{(1)} \\ p_1^{(2)} & p_2^{(2)} & p_3^{(2)} \\ p_1^{(3)} & p_2^{(3)} & p_3^{(3)} \end{bmatrix} = \pm 2d$, in which case $p=(j+n/2)p_1+(k+n/2)p_2+(m+n/2)p_3$ is in L for j, k, m and n integers. □

b. Algorithmic Systems and their Determinant

We begin by stating the converse of Proposition 4.1.6.

Proposition 5.3.11 If v_1, v_2, v_3, have determinant $|d|$, then v_1, v_2, v_3 is a basis of L. □

Theorem 5.3.12. If an MAOP of an irreducible lattice is Minkowski partitioned into the algorithmic system I through V, then the determinant of its Minkowski matrix is $|d|$, and if it is partitioned into the algorithmic system VI, then the determinant of its Minkowski matrix is zero.

Proof: Case VI is obvious. Since all cases are essentially the same, we consider only case I.

Let $N = \text{Det}\begin{bmatrix} a & b & c \\ -f & g & -h \\ -j & -k & m \end{bmatrix} = agm - ahk + fbm + fck + jbh + jgc,$

Where $a > b > 0$, $a > c > 0$; $g > f > 0$, $g > h > 0$ and $m > j > 0$, $m > k > 0$. Then $agm > ahk > 0$, $agm > fbm > 0$, $agm > fck > 0$, $agm > jbh > 0$, $agm > gjc > 0$, so that we have that $N > 0$. Consider the points p_1, p_2 and p_3. Because we have $N > 0$, p_1, p_2, p_3, are non-coplanar with the origin. Together with the fact that these points are on an MAOP, we have that p_1, p_2, p_3 form a lattice octahedron. Upon inspection of case 2 of proposition 5.3.10, $p=\frac{1}{2}p_1-\frac{1}{2}p_2-\frac{1}{2}p_3$ is in our lattice. That is, by proposition 5.3.8:

$$p_x = \tfrac{1}{2}|a-b-c| < a,$$
$$p_y = \tfrac{1}{2}|-f-g+h| < g \text{ and}$$
$$p_z = \tfrac{1}{2}|-j+k-m| < m.$$

This implies that p is in the MAOP which is a contradiction so that $N = |d|$

4. The <u>Minkowski Neighboring Process and the Six Algorithmic Classes of MAOP</u>

Because we may think of MAOP of algorithmic Systems I through V as bases we have the following propositions.

<u>Proposition 5.3.13</u> In an irreducible lattice, let A be an MAOP of algorithmic system I through V, and let B be a successor of A such that B is an MAOP of algorithmic system I through V, then there exists a matrix T with integers entries such that AT = B and determinant of T = 1. □

We shall calculate this T for such A in section 6.2.

<u>Proposition 5.3.14</u> In an irreducible lattice, let a be an MAOP of algorithmic system I through V and let B be a successor of A such that B is an MAOP of algorithmic system VI and there exists a matrix T with integer entries such that AT = B and determinant of T = 0. □

We shall also calculate this T for such A in section 6.2.

We now come to the following very important theorem.

<u>Theorem 5.3.15</u> In an irreducible lattice, let A be an algorithmic system VI and if the MAOP B is a successor of A, then B is of algorithmic system I through V.

<u>Proof</u>: Because axis may be interchanged without affecting the neighboring process, we may assume without loss of generality that we find the successor along the x axis and that b c (See propopsition 6.2.7). We have also that $a > b$, $a > c$, $g > f$, $g > h$, $m > j$, $m > k$; $b+c=a$, $f+h=g$, $j+k=m$. If p_1, p_2, p_3 are the three lattice points of b in order of their occurence in the matrix of their MAOP, then the successor B in the x direction has on its surface the four lattice points p_2, $-p_2$, p_3, $-p_3$ and beside two other lattice points p'_2 and $-p'_2$. Because b c, the coordinants of p'_2 and $-p'_2$ will occupy the first column of the matrix for the MAOP B. We have the following:

$$\text{Matrix of B} = \begin{bmatrix} b & -b' & -c \\ -g & g' & -h \\ -k & -k' & m \end{bmatrix}, \quad \text{where } b = b'+c,\ g'=g+h \text{ and } m=k'+k.$$

But from the above, $m = k+j$. Therefore, $k'=j$. Since there is only one lattice point, either p_1 or $-p_1$ for whose y coordinate = j, one has $p'_2 = \pm\ p_1$; this yields a=b'=b-c, that is, a < b because b > c > 0, which contradicts a > b. □

Proposition 5.3.16 In an irreducible lattice, let MAOP A be of algorithmic system I through V and its successor MAOP B be of algorithmic system VI, then any MAOP C, a successor of MAOP B, can be found by creating a matrix T with integer entries such that AT = C and determinant of T=1. □

We shall calculate T for such A and B in section 6.2.

Note: It is important to realize that three vectors which form a basis of L need not form an MAOP, and conversely that three vectors which form an MAOP need not form a basis of L. Hence, that any T with integer entries such that determinant of T=1 need not preserve an MAOP. In section 6.2., we seek out the special T that do preserve MAOP and do create successors to an MAOP.

B. Finding Units by Minkowski's Method - The Minkowski Test to Locate Unit Related MAOP Within a Finite Number of Steps

Definitions 5.3.17 Let v_1, v_2, v_3 be three vectors of a lattice of dimension three, tnen the v_1, v_2, v_3 form is the polymonial:

$$(v_1^{(1)}x+v_1^{(2)}y+v_1^{(3)}z)(v_2^{(1)}x+v_2^{(2)}y+v_2^{(3)}z)(v_3^{(1)}x+v_3^{(2)}y+v_3^{(3)}z).$$

Proposition 5.3.18 Let v_1, v_2, v_3, be three vectors of a multiplicative lattice of dimension three, then the v_1, v_2, v_3 form has integer coefficients.

<u>Definition 5.3.19</u> An <u>MAOP form</u> is the v_1, v_2, v_3 form, where v_1, v_2, v_3 are the vectors of MAOP.

<u>Proposition 5.3.20.</u> An MAOP form has integers coefficients, which are each less than $9|d|$. □

<u>Proposition 5.3.21.</u> There exists at least two MAOP of determinant $\pm d$ with the same MAOP form. □

<u>Proposition 5.3.22.</u> Two MAOP of determinant $\pm d$ with the same MAOP form in a multiplicative lattice with identity are unit related by a unit (w, w', w"). Furthermore, (w, w', w") is in L. □

C. Finding Independent Units by Minkowski's Method

If we calculate a unit (w, w', w") by a chain of MAOP along the x axis, we would have $|w|<1$, $|w'|>1$ and $|w''|>1$. Analogously, if we calculate a unit by a chain of MAOP along the y axis, we would have $|w|>1$, $|w'|<1$ and $|w''|>1$. Finally, if we calculate a unit by a chain of MAOP along the z axis we would have $|w|>1$, $|w'|>1$ and $|w''|<1$. It is clear that two of these three units are independent.

D. Finding Fundamental Units by Minkowski's Method

There is no elementary method to calculate the fundamental units by Minkowski's method. We know that it is possible because his neighboring algorithm allows passage between any two MAOP by Theorem 3.3.4. Hence, the fundamental unit related MAOP are among the unit related MAOP and, therefore, calculable. It is conceivable that a development for MAOP along one axis analogous to that which was done in 5.3.D. and 5.3.E. for VAOP along one axis, can be created. The fact that two neighboring MAOP share a common point and the following theorem would be essential to such a theory:

Theorem 5.3.23 If $(a,g,m)_p$ is an MAOP in an irreducible multiplicative lattice L with identity, and if T is a 3 x 3 matrix with integral entries such that T:a wa, where w is a unit of L, then $T\,(a,g,m)_p = (wa,w'g,w''m)_p$, an MAOP. □

However, the creation of the initial MAOP for such a theory is not a simple matter as the initial VAOP for the Voronoii theory, where we are given that (1,1,1) is a VAOP.

CHAPTER 6

LOCATING INITIAL AOP OF EACH TYPE AND SUCCESSOR ALGORITHMS

6.1. Voronoii

A. An Initial VAOP

We see that by proposition 5.2.28 that given any irreducible multiplicative lattice with identity, the VOP associated with (1,1,1) is a VAOP. In such lattices, we may take this VAOP as our initial VAOP to which we apply the Voronoii Algorithm that we shall shortly describe. However, Voronoii's Algorithm will test whether the VOP associated with a basis vector v in any irreducible lattice (not necessarily multiplicative and not necessarily containing the identity) is admissible. If the VOP is not admissible, it will locate a point v' in the VOP whose associated VOP can be tested for admissibility. Eventually a VAOP will be found, and then the chain of this VAOP will be created.

B. The Voronoii Algorithm for Finding If a VOP is a VAOP and If Not, an Interior Point; And If Yes, Its Successor

We shall not prove the algorithm in detail, but rather give a brief sketch of it. We shall find the successor of the VAOP along the x axis.

1. Lattice Transformations

In this section, we indicate how Voronoii "collapses" the three dimensional irreducible lattice into a two dimensional irreducible lattice. The known theory of such two dimensional irreducible lattices is then related to the problem of finding the interior point of a VOP or if the VOP is a VAOP, the point associated with its succesor. Essentially, a finite set of points are located in the two dimensional lattice to which the desired point (the successor or interior point) "belongs," as we shall shortly explain. Let V represent the VOP associated with the basis vector v of an irreducible lattice L. We wish to test V for admissibility. If V is admissible, we wish to find its Voronoii successor along the x axis. If not we wish to find an interior point of V.

a. Similarity Transformation I

For v and some vectors v_1 and v_2, let v, v_1 and v_2 be a basis of L, whose points will be designated (x,y,z). Now (1,1,1), $v'_1=v_1/v$ and $v'_2=v_2/v$ defines a lattice L', whose points will be denoted by (x', y', z'), that is similar to L by proposition 5.2.5. Therefore, we may examine the numerically simpler VOP associated with (1,1,1) for inadmissibility, and hence, an interior point, or admissibility, and hence, a successor point.

b. Rotational Transformation

We now rotate clockwise the lattice L' about the x' axis 45° and obtain a new lattice L'' with points (x'', y'', z'').

Accordingly, all VOPs will be rotated and hence the rotation transformation will preserve admissibility and the Voronoii neighboring process. The transformation is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\ 2 & 1/\sqrt{2} \\ 0 & -1/\ 2 & 1/\sqrt{2} \end{bmatrix}$$

The new basis is $(1,0,\sqrt{2}\ )$, v'''_1 and v'''_2 for the new lattice L'''.

c. Similarity Transformation II

So that (1,1,1) of L' will go into (1, 0, 1) of a similar lattice L" after transformations a. and b., we divide the points of L''' by $(1, \sqrt{2}, \sqrt{2})$ to obtain the final basis v", v''_1 and v''_2 and a lattice L". The points of L" will be represented by (x", y",z").

2. <u>A Collapsing of L" Into A Two Dimensional Lattice</u>

a. Partition

<u>Definition 6.1.1</u> For any two points a, b of L", we define a b to mean a and b lie on the line G defined by x" = z" and y" = 0 or on a line parallel to G.

<u>Proposition 6.1.2</u> $\sim$ is an equivalance relation.

<u>Definition 6.1.3</u> If a and b of L" and $a \sim b$, then a is said to be <u>adjacent</u> to b with regard to $\sim$, if there exists no point c of L" such that c a and c is between a and b.

<u>Proposition 6.1.4</u> Two points a and b adjacent with regard to $\sim$ will have a projection of unit length on the x"-y" or x"-z" plane.

b. Two Dimensional Lattices

Proposition 6.1.5 Let $G^* = \{G \mid G$ is a line and G is parallel to the line defined by $x'' = z''$ and $y'' = 0$ and there is an a in L'' such that a is on $G\}$, $T^* = \{G \mid x''\text{-}y''$ plane of $L'' \mid G$ in $G^*\}$ is a two dimensional lattice. We shall denote points of T^* by (x^*,y^*,z^*) or $(x^*,y^*,0)$ or (x^*,y^*). □

Definition 6.1.6 T^* is called the lattice of punctures associated with L'' or the lattice of punctures.

Proposition 6.1.7 A basis v^*_1 and v^*_2 of T^* can be obtained from the basis (1,1,1), v'_1 and v'_2 of L' or actually from v'_1 and v'_2 by means of the transformation $((2x'-y'-z')/2,(y'-z')/2, 0)$.

Proof: The puncture $(x^*, y^*, z^*) = (x^*,y^*,0)$ or (x^*,y^*) of T^* associated with (x', y', z') in L' is $((2x'-y'-z')/2,(y'-z')/2,0)$. □

3. Two Dimensional Lattice Theory

Proposition 6.1.8 Given any two dimensional VAOP (rectangle), then the successor of the VAOP (can be found) by the algorithm of continued fractions. □

Proposition 6.1.9 The lattice points of a VAOP and its successor in the x direction in a two dimensional lattice lie on opposite sides of the x axis. □

Propositon 6.1.10 The vector w_1 associated with a VAOP in a two dimensional lattice and the vector w_2 with its successor form a basis of the lattice. □

4. The Voronoii Algorithm Theorem

a. Preliminary Concepts

Definition 6.1.11 a=(x',y',z') in L' belongs to a puncture b* in T* if ((2x'-y'-z')/2,(y'-z')/2,0)=b*.

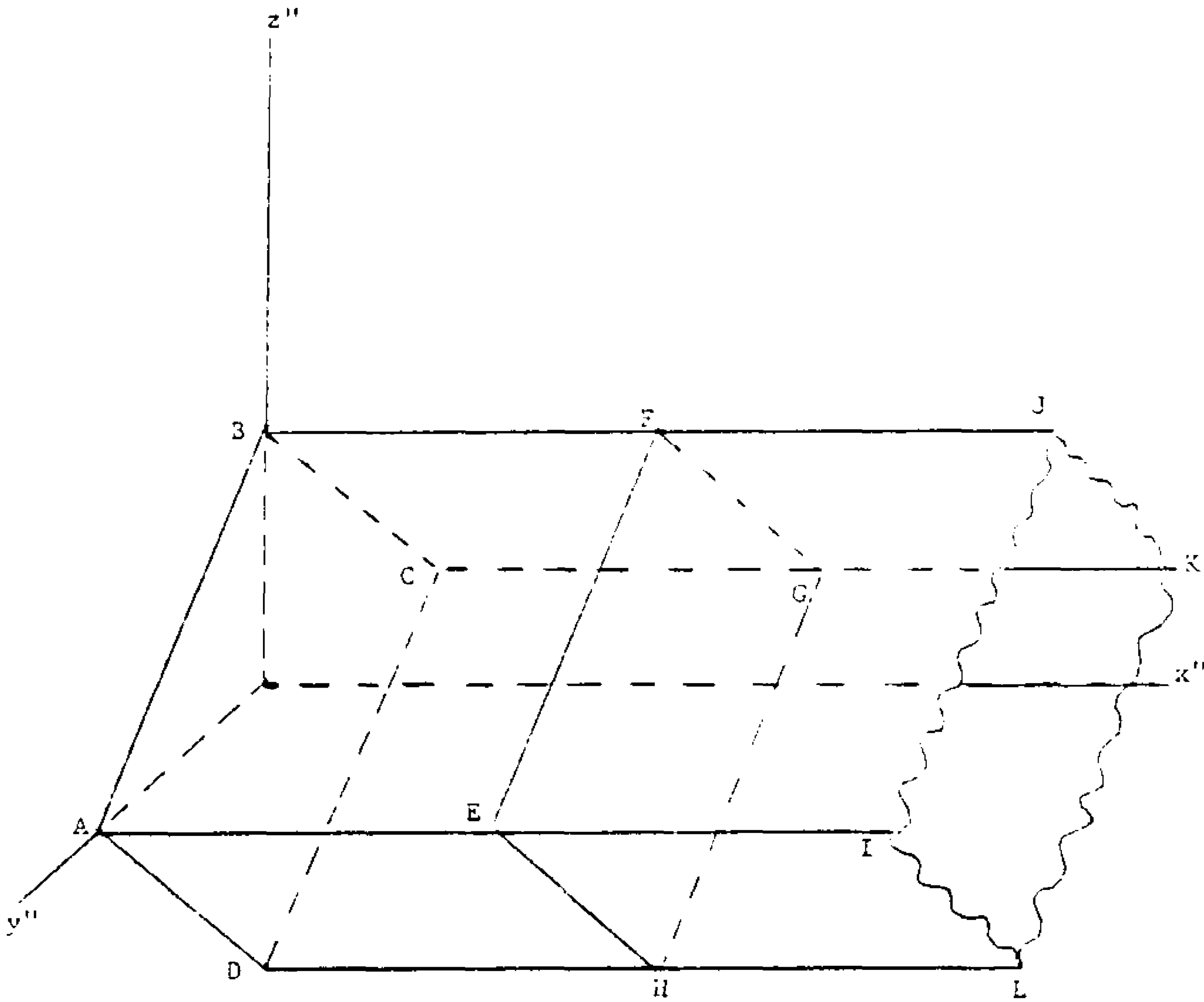
Definition 6.1.12 If a in L' belongs to puncture b* in T*, where $a_x > b*_x$, and there exists no other point c in L' belonging to b* such that c is between a and b*, then a is the upper point of puncture b*. A corresponding definition holds for a lower point of puncture b*. We represent the unit cube in diagram 6.1.14. ABCDEFGH will be positive x half of the transformed unit cube. EFGHIJKL will be adjacent space to the transformed unit cube in which the vertex point of its VAOP must lie. The space ABCDIJKL will be called the unit prism. We restate our problem. We seek a point within the unit prism whose distance from y''-z'' plane is less than one, which will be an interior point of VOP. If no such point exists, we seek a point closest to ABCD in the unit prism: the point associated with the successor VAOP. Obviously, these points must have their punctures (x*, y*,0) or (x*,y*) such that $y^* < 1$, and by proposition 6.1.3 their upper or lower points are the only points that can be in the unit prism. We now categorize punctures in two sets.

Definition 6.1.13 Category I is the set $\{a^* \mid a^*$ is a puncture such that $a^*_{y^*} < 1/2\}$. Category II is the set $\{a^* \mid a^*$ is a puncture such that $1/2 < a^*_{y^*} < 1\}$.

By proposition 6.1.3 upper and/or lower points of category I punctures must be in the unit prism. Also upper or lower or no point of a puncture of Category II may be in the prism.

b. Main Theorem of Voronoii's Algorithm

Diagram 6.1.14. The Unit Cube and the Unit Prism



Theorem 6.1.15 Let A^* be the puncture of category I which is closest to the y^* axis in T^* so that $A^*_{y^*} < \frac{1}{2}$. Let B^* be the puncture which is closest to the x^* axis so that $0 < B^*_{x^*} < A^*_{x^*}$. (Refer to diagram 6.1.16) The desired point, that is, an interior point of VOP (1,1,1) or the VAOP successor of (1,1,1) (if (1,1,1) is a VAOP) along the x axis belongs to one of the following punctures A^*, B^*, A^*+B^*, A^*-B^* or $2A^*+B^*$; moreover, it may belong to the last puncture only when both points belonging to the puncture A^*+B^* lie outside the prism and if the puncture B^*, and hence also A^*-B^* lie beyond the limits of the band $|y^*| < 1$.

Proof: It is known that A^* and B^* will form a basis for T^* and using the concepts above, please see [4,pp.265-269] for the lengthy derivation of this theorem. □

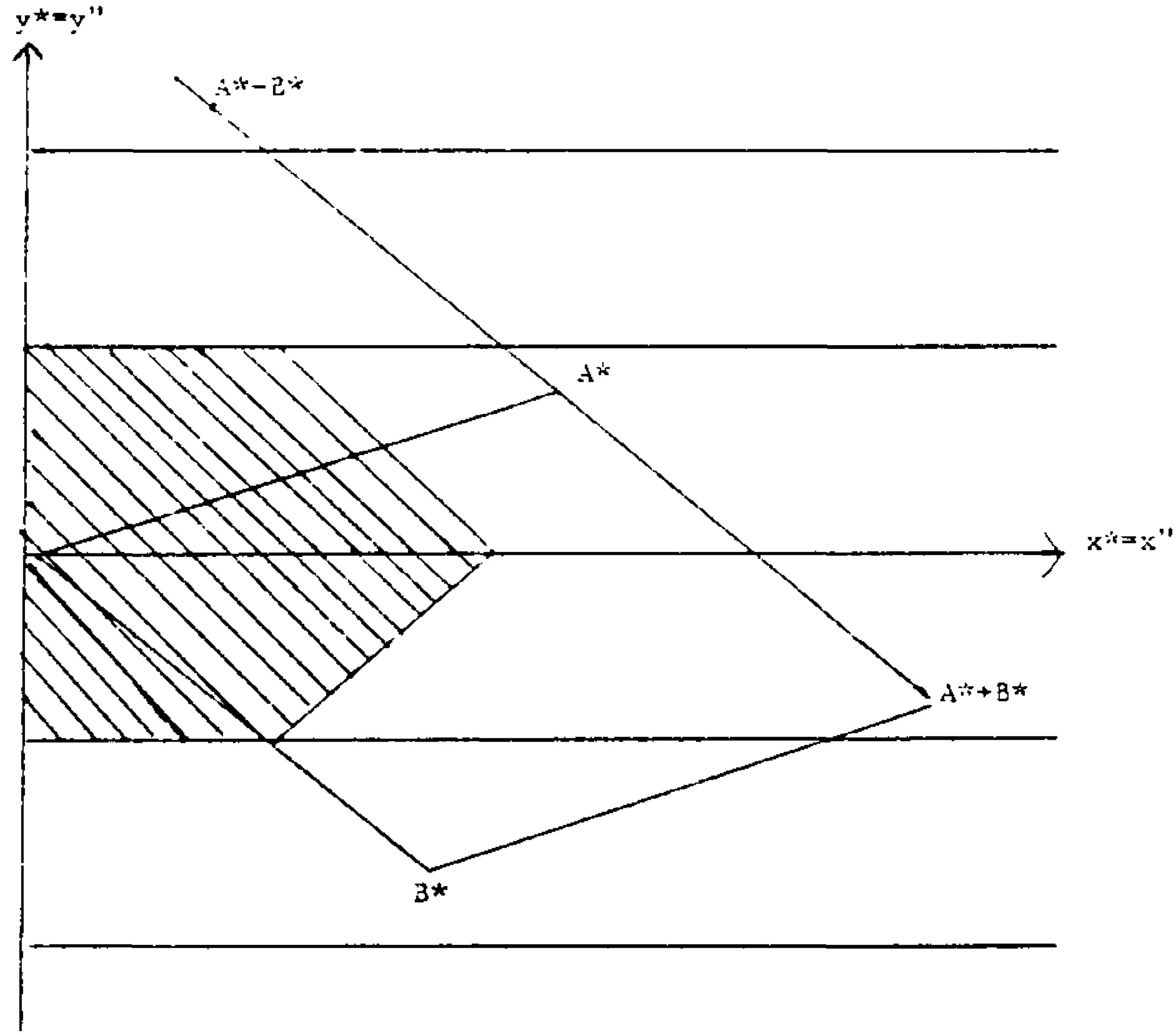
C. Programmatic Description of the Voronoii Algorithm

In the proof of theorem 6.1.15, it is revealed that the desired point can be obtained by the following procedure. This description will use the same notation and diagrams as above. We begin by testing whether the VAOP of a vector v, which we shall make part of a basis v, v_1 and v_2 of the lattice L.

1) Begin by dividing the basis v, v_1 and v_2 of L, by v so that a new lattice L', which is similar to the old lattice L, and a new basis (1,1,1), $v_1/v = v'_1$, $v_2/v = v'_2$ results. We express a lattice point coordinates of L by (x,y,z) and those of L' by (x',y',z').

2) Determine the coordinates (x^*,y^*) of the punctures for the

Diagram 6.1.16. The Lattice of Punctures, A*, B* and the Projection of the Unit Cube on the Lattice of Punctures



points v'_1 and v'_2 by the formulas:

$$x^*=(2x'-y'-z')/2 \text{ and } y^*=(y'-z')/2.$$

This results in computing v^*_1 and v^*_2.

3) By means by algorithm of continued fractions, find points A^* and B^* that are VAOP successors (rectangles) in the lattice of punctures so that: $A^*_{x^*} > 0$; $B^*_{x^*} > 0$; $|A^*_{y^*}| < \frac{1}{2}$; $|B^*_{y^*}| > \frac{1}{2}$. Express these points in terms of the points v^*_1 and v^*_2 of the basis of the lattice punctures: $A^* = m_1v^*_1+n_1v^*_2$; and $B^* = m_2v^*_1+n_2v^*_2$.

4) Calculate the coordinates of the points $A = m_1v'_1+n_1v'_2$; and $B = m_2v'_1+n_2v'_2$. Please note that A and B are in L', not T*.

5) Choose an integer t_1 so that the coordinates y' and z' of the point $A_0 = A+t(1,1,1)$ are less than unity in absolute value. It is certainly possible to do this in at least one way, and if it is possible in two ways, then take that value for t_1 which gives the least value for the x coordinate of the point $A+t(1,1,1)$.

6) Choose in the same way numbers t_2 and t_3 for the points B and A-B, respectively. If this can be done at all, it is possible in only one way.

7). If there is a suitable number t_2 for the point B, compare the abscissae of the points A_0, B_0 and $A-B+t_3(1,1,1)$. That point whose abscissa is least will be in the unit cube or will be the point associated with the VAOP successor of (1,1,1) along the x' axis.

8) If there does not exist a suitable number t_2 for B, then

try to find in the same way a number t_4 for the point A+B, and if such a t_4 exists, compare the abscissae of the points A_0, $A-B+t_3(1,1,1)$ and $A+B+t_4(1,1,1)$.

9) If there does not exist a suitable number t_4 for the point A+B, and if moreover, $B^*_{y^*} > 1$ and $(A^*+B^*)_{x^*} < 1$, then chose a t_5 in the same way for the point 2A+B and compare the abscissae of the points A_0 and $2A+B+t_5(1,1,1)$.

D. Sample Calculation of Fundamental Units by Voronoii's Method

We shall find the fundamental units for the algebraic integers of the real field Q(0), where 0 is given by the equation $x^3-6x+2=0$, whose roots are all real.

1. The Initial VAOP

We know that because Q(0) is an irreducible multiplicative lattice with identity, (1,1,1) is a VAOP in Q(0) by proposition 5.2.28. We, therefore, shall begin the algorithm with this VAOP.

2. Fundamental Units Calculated by Means of the Voronoii Algorithm for Successor VAOP

This lattice will have as a basis (1,1,1), (0,0',0"), $(0^2, 0'^2, 0''^2)$. Because (1,1,1)=v, (x,y,z)=(x',y',z'), $v_1=v'_1$ and $v_2=v'_2$. For the construction of the chains of the VAOP, it is necessary to know the approximate value of the co-ordinates of the points of the basis. The approximate value of the roots 0, 0' and 0" of $x^2-6x+2=0$ and its squares yields the following basis (1,1,1), v_1 = (0=2.6017, 0'=-2.2618, 0"=.3399) and v_2 = (0^2 = 6.7688, $0'^2$ = 5.1157, $0''^2$ = .1155). Let us agree to place the coordinate 0 on the x-axis, 0' on the y-axis and 0" on the z-axis. We will now construct an x-chain starting from (1,1,1). Using the basis of the lattice and the transformation: $x^*=(2x-y-z)/2$ and $y^*=(y-z)/2$, we compute the basis v^*_1 and v^*_2 of the lattice of punctures. v^*_1 = (3.90,-.96) and v^*_2=(4.15,2.50) which correspond to v_1 and v_2 respectively. We now make a reduction of the basis for the lattice of punctures by means of the algorithm of continued fractions, which it is necessary to apply to the coordinates of the punctures v^*_1 and v^*_2.

$$2v^*_1+v^*_2 = (11.96,.58);$$
$$3v^*_1+v^*_2 = (15.86,-.38).$$

We must take the puncture $3v^*_1+v^*_2$ for the A* and the puncture $2v^*_1+v^*_2$ for the point B*. The VAOP successor of (1,1,1) can only be a point belonging to the punctures A*-B*,A* or B*, since the puncture A*+B* is at a distance greater than 1 from A* in the x* direction. Because v_1 belongs to the puncture A*-B* and v_1 has lattice coordinates (2.60,-2.26,-0.34), we see that no point belongs to puncture A*-B* within the prism $|y| \le 1$, $|z| \le 1$. $B=v_2+2v_1$ belongs to the puncture B* and has lattice coordinates (11.97,0.59,-0.56). This will be a point belonging to B*, and in fact, the only point belonging to puncture B*, lying within the prism $|y| \le 1$, $|z| \le 1$ .Points corresponding to the puncture A* need not be investigated, since their abscissae will certainly be greater than the abscissa of B. Hence the point B is the lattice point associated with the VAOP successor of (1,1,1) in the x direction. In order to find the next VAOP successor of B, we divide the original lattice by B. We now obtain a new lattice whose basis is: (1,1,1), $v_1' = (0/2, 0'/2, 0''/2) = v_1/2$ and $v'_2 = (0^2/2, 0'^2/2, 0''^2/2) = v_2/2$. Now we repeat the same process with this lattice, the basis of the punctures will be: $v^*_1=(1.95,-.48)$, $v^*_2=(2.08,1.25)$. The reduced punctures will be: $v^*_1=A^*$, $-v^*_1+v^*_2=B^*$. The punctures B^*, B^*-A^* and B^*+A^* are found to be beyond the limit of the band x^* 1. Thus the lattice point associated with the VAOP successor to (1,1,1) belongs to A^*. Because v'_1 belongs to A^* and v'_1 has lattice coordinates (1.30,-1.13,-017)), then $v'_1+(1,1,1)$ will be the only point within the unit prism $|y'|< 1$, $|z'|< 1$. This point will be the point associated with the VAOP successor of (1,1,1) in the x' direction. Division by $v'_1+(1,1,1)$ takes the lattice with basis (1,1,1), v'_1 and v'_2 into lattice L with basis (1,1,1), v_1 and v_2. Consequently, the x chain as it continues will be periodic. The unit $e_1 = (v_2+2v_1)(v_1+(1,1,1))=2v_2+5v_1+(1,1,1)$. For the determination of e_2, we must construct the z chain by the same algorithm. Again, $v_1 = v'_1$, $v_2 = v'_2$, $v_3 = v'_3$ and $(x,y,z) = (x',y',z')$. The punctures of the basis in the corresponding lattice will be $v^*_1=(-.51, 2.43)$, $v^*_2=(-5.83,.83)$. The reduced basis of the lattice of punctures will be $v^*_1-3v^*_2=A^*$, $v^*_1-2v^*_2=B^*$. We must investigate points belonging to the punctures A^*-B^*, A^*

and B*. $-v_2$ belongs to the puncture A*-B* and its lattice coordinates are (-6.77,-5.12,-.12). Consequently, we see that there exists a point belonging to A*-B* within the unit prism $|x| \leq 1$, $|y| \leq 1$, namely, $C = v_2+(6,6,6)$. It is useless to investigate the punctures A* and B* now, since they are at a distance greater than 1 from the puncture A*-B*. Hence, the lattice point associated with the VAOP successor of (1,1,1) along the z axis is C. Division by C takes the original lattice into the lattice with basis (1,1,1), $v_1/2$ and $v_2/2$. Thus, C is associated with the element of v_2+2v_1 of the x chain. The unit e_2 is given by $(-v_2+(6,6,6))/(v_2+2v_1)=2v_2-v_1+(1,1,1)$. The problem is solved.

6.2. Minkowski

A. An Initital MAOP

To locate an initial MAOP is not a simple matter. This should not be a surprise for it is clear that any method that would generate distinct MAOP would enable one to calculate units. By means of trial and error methods and applying theorem 6.2.5 below to a bases, we can find an initial MAOP. To prove this theorem we use proposition 6.2.4, which is a consequence of an important theorem of Minkowski, see [5,pp.383-384].

1. Preliminary Concepts

Definition 6.2.1 f(x,y,z) is a radial distance function if it satisfies the following four conditions:

1) $f(x,y,z) > 0$, when $(x,y,z) \neq (0,0,0)$;

2) $f(tx,ty,tz) = t\, f(x,y,z)$, when $t > 0$;

3) $f(x+x',y+y',z+z') \leq f(x,y,z) + f(x',y',z')$ and

4) $f(-x,-y,-z) = f(x,y,z)$.

Proposition 6.2.2. For a radial distance function, $f(x,y,z) \leq 1$ defines a convex body with center the origin. Minkowski proved a converse of this proposition in [4, pp. 94-97]. □

Propositon 6.2.3 For any symmetric convex body with the center of symmetry at the origin, there exists a radial distance function. □

Proposition 6.2.4 If a lattice octahedron has all its vertices on the surface of a convex body K, which we shall assume equals $\{ (x,y,z) \mid f(x,y,z) \leq 1 \}$, where f is a radial distance function, and if these vertices are non-coplanar with the origin and are denoted by $(a_{(1,1)},a_{(1,2)},a_{(1,3)})$, $(a_{(2,1)},a_{(2,2)},a_{(2,3)})$ and $(a_{(3,1)},a_{(3,2)},a_{(3,3)})$, where $a_{(i,j)} = \pm d$ or $\pm 2d$, then

1) if the determinant of $a_{(i,j)}=\pm d$, then the convex body K contains in the interior no lattice points (x,y,z) other than the origin, if and only if, $f(x,y,z) \geq 1$, where $x=ra_{(1,1)}+ sa_{(2,1)}+ta_{(3,1)}$, $y=ra_{(1,2)}+sa_{(2,2)}+ta_{(3,2)}$ and $z=ra_{(1,3)}+ sa_{(2,3)}+ta_{(3,3)}$ and (r,s,t) assumes each of the following 22 sets of values: $(0,1,\pm1)$, $(1,1,\pm1)$, $(1,\pm1,0)$, $(1,\pm1,\pm1)$, $(1,\pm1,\pm2)$, $(1,\pm2,\pm1)$ and $(2,\pm1,\pm1)$, where all possible sign combinations are considered.

2) If the determinant $a_{(i,j)}=\pm 2d$, then the convex body K contains in its interior no lattice point other than the origin, if and only if, $f(x,y,z) \geq 1$ for each of the following four sets of values of (r,s,t): $(\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$, where

all possible sign combinations are considered. □

2. Main Theorem on Initial MAOP

Theorem 6.2.5 If one can find three lattice points p_1, p_2, p_3, whose matrix has determinant d, and if these points can be arranged (as in proposition 5.3.5) into a positive Minkowski matrix of algorithmic system I through V, then p_1, p_2, p_3 determine an MAOP.

Proof: Define a primitive basis of a lattice in the following manner. Consider a sphere of radius zero, centered at the origin. Let v_1 be the first lattice point encountered by the sphere when the radius is increased. If more than two points are encountered by this sphere simultaneously, choose v_2 and v_3 as any points non-coplanar with v_1 and the origin. If not, and only two points are encountered, choose v_2 as the point non-colinear with v_1 and the origin. Now expand the sphere until the first v is found which is non-coplanar with v_1 and v_2 and the origin. If only v_1 were encountered, continue expanding this sphere until v_2 is encountered which is not colinear with v_1 and the origin. If more than one point was encountered with v_2, choose it as v_3 if it is not coplanar with v_1 and v_2 and the origin. If no points are encountered with v_2, expand this sphere until a v_3 is found which is not coplanar with v_1 and v_2 and the origin. Certainly, this primitive basis determines a lattice octahedron. Let A be the matrix of the primitive basis v_1, v_2 and v_3. The determinant of A is d . Let B be the matrix of p_1, p_2 and p_3, whose determinant is also d . There exists a matrix T with integer entries such that AT = B and determinant of T=1. Now the entries of T^{-1} are also integers because the entries of T are integral and the determinant of T is 1. Because T maps the lattice points of L into the lattice points of L, and in fact, L onto itself, we conclude that T maps the lattice octahedron determined by the primitive basis to a lattice octahedron determined by p_1, p_2 and p_3. Consider $(a,g,m)_p$, where a, g and m are the main diagonal of the positive Minkowski matrix. Because this is a matrix of the algorithmic system I through V, one has that a b, a c, g f, g h, m j and m k. This establishes that p_1, p_2 and p_3 are in the faces of $(a,g,m)_p$, not just in the planes containing these faces. Now, let (x,y,z) be the coordinates of a lattice point in L, and define:

$$f(x,y,z)=\max(|x / a|, |y / g|, |z / m|)$$

and

$$C = \{ (x,y,z) | f(x,y,z) \leq 1 \} .$$

It is clear that f is a radial distance function and that $C=(a,g,m)_p$. Applying case 1 of proposition 6.2.4, we see that if we substitute the 22 different values for (x,y,z) in f that $f \geq 1$, and hence $(a,g,m)_p$ is an MAOP. □

To make certain that theorem 6.2.5 is not operating in a vacuum, we state the following theorem:

<u>Theorem 6.2.6.</u> Every irreducible lattice has a basis of three vectors that is an MAOP.

<u>Proof:</u> We know that an MAOP exists in each lattice by

proposition 1.4.9. Now by theorem 5.3.15, the desired MAOP and basis exists. □

B. The Minkowski Algorithm For Finding The Successor of an MAOP

1. Lattice Transformations

Proposition 6.2.7. Under a suitable renaming of the axes the successor A' of an MAOP A along the x axis, y axis or z axis can be viewed as a sucessor along the x axis in such a way that the sucessor A' will have as its matrix one of the types of the algorithmic system I through VI, in which the inequality $b' < c'$ is satisfied. The following lists the various conditions, the necessary transformations to change A of type of algorithmic system I through VI to the desired type A' of algorithmic system I through VI.

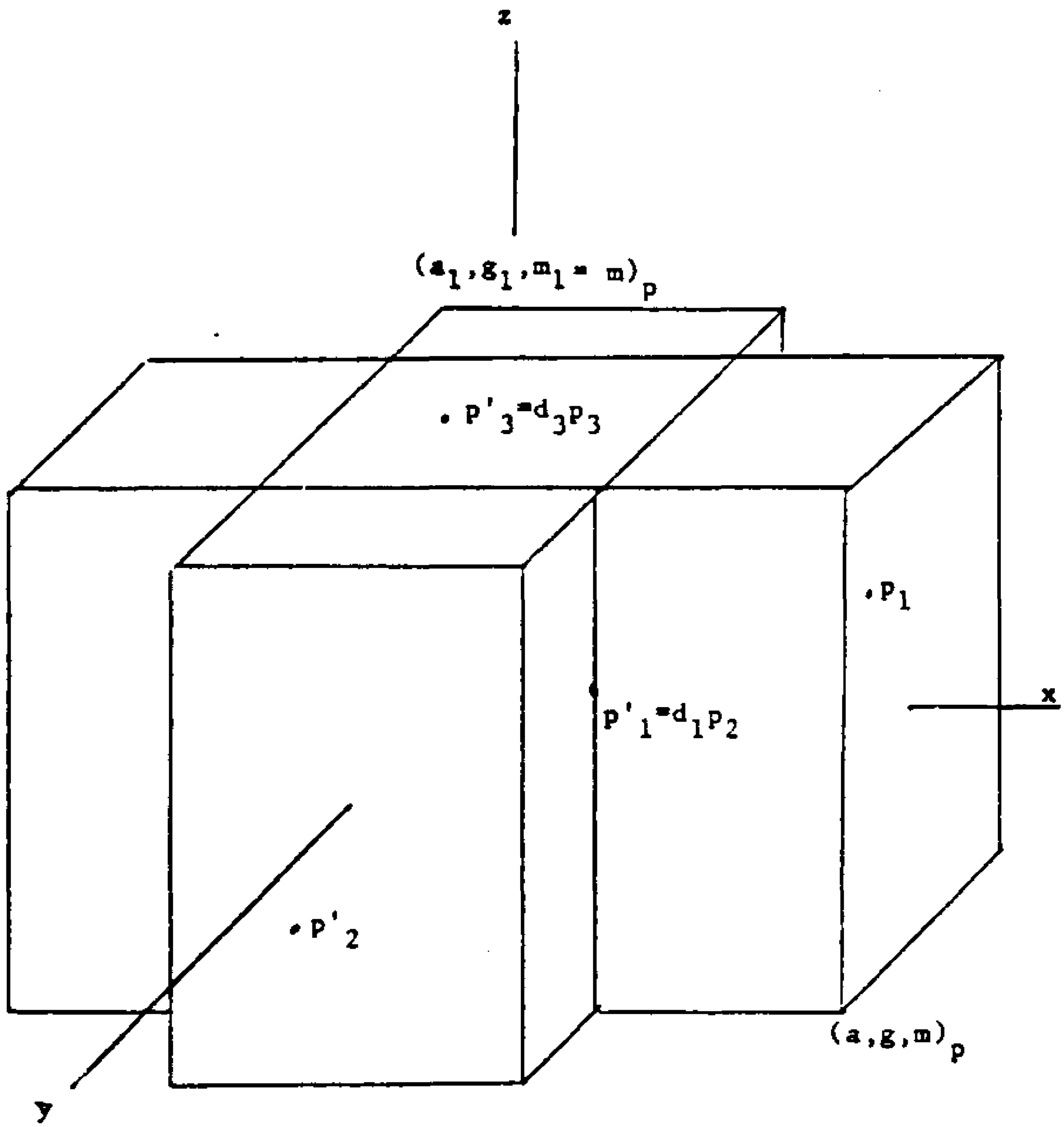
		The type of A' is in Column Headed by Type of A					
Condition	Transformation	I	II	III	IV	V	VI
x-neighbor $c > b$	$x=x', y=z', z=y'$	I	III	II	V	IV	VI
y-neighbor $f > h$	$x=y', y=x', z=z'$	II	I	III	V	IV	VI
y-neighbor $h > f$	$x=z', y=x', z=y'$	III	I	II	IV	V	VI
z-neighbor $j > k$	$x=y', y=z', z=x'$	II	III	I	IV	V	VI
z-neighbor $k > j$	$x=z', y=y', z=x'$	III	II	I	V	IV	VI

. □

2. Finding the MAOP Successor

As was indicated in proposition 5.3.13 through 5.3.16, we seek matrix T with integer entries of determinant 1 for MAOP A, whose points will be denoted p_1, p_2 and p_3, of category I through V such that AT = A', where A', whose points will be denoted p'_1, p'_2 and p'_3, is the MAOP successor of A. If A' is of category VI, then we seek a T of determinant 1 with integer entries such that AT = C, where C is an MAOP successor of A'. We begin by noting that to secure the x-neighbor when $b > c$, one lowers the x-face to the lattice points in the face $y=\pm g$ (one has $x=\pm b$ for these points) and subsequently raises the y face to the first lattice point in neither the x-face nor z-face. The first parameter is decreased, the second is increased and the third remains the same. (See diagram 6.2.8) For the x neighbor $(a_1, g_1, m_1)_p$ of $(a,g,m)_p$, one has then $a_1=b < a$, $g_1 > g$, and $m_1 = m$. Since there is at most one lattice points in each of the planes, $x=k_1$, a constant, $y=k_2$, a constant, it follows that two of the lattice points whose coordinates are in the same matrix for the neighbor A' are $p'_1=d_1p_2$, $p'_3 = d_3p_3$, where $d_1=\pm 1, d_3=\pm 1$. If $A = (a,g,m)_p$ is a parallelopiped of algorithmic system I through V, then A has determinant d and can be interpreted as a basis of a lattice. We seek out a matrix T with integer entries such that AT = A' and determinant of T=1. If one denotes by $q_i^{(j)}$ the co-factor of $p_i^{(j)}$ in A, one has $[q_j^{(i)}] \quad [p'^{(j)}_i] = [t_i^{(j)}]$. From the relation $p'^{(1)}_i=d_1p_i^{(2)}$, $p'^{(3)}_i = d_3p_i^{(3)}$ $(i=1,2,3)$, it follows that:

<u>Diagram 6.2.8.</u> The Minkowski Neighboring Process: b < c



$d_1 = \pm 1$ $d_3 = \pm 1$

$$T = \begin{bmatrix} 0 & \sum_{k=1}^{3} q_k^{(1)} p'_k{}^{(2)} & 0 \\ d_1 & \sum_{k=1}^{3} q_k^{(2)} p'_k{}^{(2)} & 0 \\ 0 & \sum_{k=1}^{3} q_k^{(3)} p'_k{}^{(2)} & d_3 \end{bmatrix}.$$

Then $\sum_{k=1}^{3} q_k^{(1)} p'_k{}^{(2)} = -d_1d_3 = \pm 1$ or 0 according as the neighbor of $(a_1, g_1, m)_p$ is of algorithmic system I through V or of algorithmic system VI. Set $\sum_{k=1}^{3} q_k^{(1)} p'_k{}^{(2)} = Kd_2$, $\sum_{k=1}^{3} q_k^{(2)} p'_k{}^{(2)} = Md_2$, $\sum_{k=1}^{3} q_k^{(3)} p'_k{}^{(2)} = Nd_2$, where $d_2 = -d_1d_3$ and where $K = 1$ or 0 according as $(a_1, g_1, m)_p$ is of algorithmic system I through V or of algorithmic system VI. Then:

$$A' = AT = \begin{bmatrix} d_1p_1^{(2)} & d_2(Kp_1^{(1)}+Mp_1^{(2)}+Np_1^{(3)}) & d_3p_2^{(3)} \\ d_1p_2^{(2)} & d_2(Kp_2^{(1)}+Mp_2^{(2)}+Np_2^{(3)}) & d_3p_2^{(3)} \\ d_1p_3^{(2)} & d_2(Kp_3^{(1)}+Mp_3^{(2)}+Np_2^{(3)}) & d_3p_3^{(3)} \end{bmatrix}.$$

The integers K, M, and N are not all zero, for in this case p'_2 would be the origin, which is impossible. Then $p'_2 = d_2(Kp_1+Mp_2+Np_3)$ and $p'_2{}^{(1)} = s_1d_2(Ka \pm Mb \pm Nc)$, $p'_2{}^{(2)} = s_2d_2(\pm Kf + Mg \pm Nh)$, $p'_2{}^{(3)} = s_3d_2(\pm Kj \pm Mk + Nm)$, where the system of signs of A' is that of algorithmic system I through V.

Due to proposition 4.1.18, the relation $a_1 = b, g_1 < g, m_1 = m$ yield to condition $g_1 < |d/(bm)|$. The lattice point p'_2 is to be found among those lattice points of the form $p = Kp_1 + Mp_2 + Np_3$ ($K = \pm 1$ or 0; M,N integers; $|K| + |M| + |N| \geq 1$), which satisfy $x < |b|$, $y < |d/(bm)|$, $|z| < m$, and in particular $\pm p'_2$ will be the two such points which make N a minimum. There are only two lattice points for which y will assume this minimum value, otherwise y=0 would be satisfied by lattice points other than the origin. It is clear then, that if the simultaneous inequalities just stated can be solved for the finite number of solutions which they possess, then p'_2 can be determed except for sign. The sign can be later determined to make A' a matrix of algorithmic system I through V. The lengthy solutions of these equations for the algorithmic system I through V are given in [5,pp.405-426].The results are as follows.

C. The Various T for MAOP A of Different Categories

1. The Matrix T For A of Algorithmic System I Through V

The matrix T for A of algorithmic system I through V is given in the following list. If A' = AT becomes an MAOP of category VI, see section 2. below for the calculation of T for its neighbors.

Algorithm for the x-neighbor ($b < c$)

I: 1] $j > k$,

$$T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad +, +, +; \text{ V.}$$

2] $j < k$,

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, +, -, -; \text{ III.}$$

The Roman numeral denotes which category of the algorithmic system to which A belongs; the three signs are the signs of the products $s'_i s_i$, $(i = 1,2,3)$, the s_i, being those for A in propositon 5.3.5 and the s'_i, those for A'.

II and V:

The upper sign is for Case II; the lower sign for Case V.

$M \equiv [G/F]$, $N \equiv [\pm H/F]$, $u \equiv a-Mb-Nc$, $v \equiv \pm j+Mk-Nm$.

Here F,G, and H are the signed minors of f, g, and ±h in A. The square brackets denote the largest integer function.

		m	n	s
1) $u < c$,	$v > k$	M-1	N+1	+1
2) $u < b-c$,	$v < 0$,	M	N-1	-1
3) $u < b$,	$v > 0$, but not 1)	M	N	-1
4) $u < b$,	$v < 0$, but not 2)	M	N	+1
5) $u > b$,	$v > 0$	M	N+1	+1
6) $u > b$,	$v < 0$	M+1	N	-1

$$T = \begin{matrix} 0 & \mp s & 0 \\ \pm s & \mp sm & 0 \\ 0 & -sn & 1 \end{matrix} \quad , \quad \mp s, \quad \mp s, \quad +1;$$

$s=+1$, I; $s=-1$, IV.

III: 1] $a+c < 2b$

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \quad , \quad -, \quad +, \quad -; \quad \text{II}.$$

2] $a+c > 2b$

$$T = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad , \quad +, \quad +, \quad +; \quad \text{VI}.$$

IV: 1] $a < 2b$, $f < h$, $j+k < m$

$$T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \ +, \ +, \ +; \ \ \text{II.}$$

2] $a > 2b$ or $f > h$ or $j+k > m$

$$T = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad , \ -, \ +, \ -; \ \ \text{VI.}$$

2. AT=C Where C is a Neighbor of A' Where A' is of Algorithmic SystemVI and a Neighbor of A

a. Algorithm for the x Neighbor of the x Neighbor In III 2], IV 2].

The upper sign is for Case III 2], the lower for IV 2].

1] $b-c > c$:

$$M \equiv [\pm G/F], \qquad N \equiv [(\pm G+H)/F];$$

$$u^{o} = b-c, \quad u'' = c; \quad v^{o} = m, \ v'' = m-k;$$

$$u \quad a-Mu^{o}-Nu'', \quad v \equiv -j+Mv''-Nv^{o}$$

		m	n	s
1) $u < u''$,	$v > v''$	M-1	N+1	+1
2) $u < u''$,	$v'' > v > 0$	M	N+1	-1
3) $u > u''$,	$v > 0$	M	N+1	+1
4) $u < u^{o}$,	$v < 0$	M	N	+1
5) $u > u^{o}$,	$v < 0$	M+1	N+1	-1

$$T = \begin{bmatrix} 0 & \mp s & 0 \\ -1 & -sm & 0 \\ \pm 1 & \pm s(m-n) & \mp s \end{bmatrix}, \quad \pm 1, \quad -s, \quad \mp s;$$

s= +1, V; s= -1, III.

2] $b-c < c$:

$M \equiv [(\pm K+L)/J]$, $N \equiv [\pm K/J]$;

$u^o = c$, $u'' = b-c$; $v^o = g+h$, $v'' = h$;

$u \equiv a-Mu^o-Nu''$, $v \equiv -f+Mv''-Nv^o$.

$$T = \begin{bmatrix} 0 & 0 & \mp s \\ 0 & -s & \mp sn \\ \mp 1 & \pm s & \mp s(m-n) \end{bmatrix}, \quad \pm 1, \quad -s, \quad \mp s;$$

s = +1, IV; s= -1, II.

b. Algorithm for the y Neighbor of the x Neighbor in III 2], IV 2]

When $b > c$, the y-neighbor of the x-neighbor is $(a, g, m)_p$ itself.

c. Algorithm for the z Neighbor of the x Neighbor In III 2], IV 2]

1] $k < m-k$:

$M \equiv [-H/F]$, $N \equiv [\mp G-H)/F]$;

$u^o = m-k$, $u'' = k$; $v^o = b$, $v'' = b-c$;

$u \equiv j-Mu^o-Nu''$, $v \equiv -a+Mv''-Nv^o$.

$$T = \begin{bmatrix} 0 & \mp s & 0 \\ s & \mp s(m-n) & -1 \\ 0 & \pm sm & \pm 1 \end{bmatrix}, \quad \mp s, \quad -s, \quad \pm 1;$$

s = +1, IV; s = -1, I.

2) $k > m-k$;

In case III:

$$T = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad -, \quad -, \quad +; \quad V.$$

In case IV:

1) $j > k$:

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad -, \quad +, \quad -; \quad II.$$

2) $j < k$:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad +, \quad -, \quad -; \quad V.$$

D. Programmatic Description of the Minkowski Algorithm

1) The lattice will have the basis $v_1 = (1,1,1)$, $v_2 = (\theta,\theta',\theta'')$ and $(\theta^2,\theta'^2,\theta''^2)$, where θ is located along the x axis, θ' is located along the y-axis and θ'' is located along the z-axis. To calculate this basis we need calculate the roots θ, $\theta'$and θ'' of the irreducible monic cubic polynomial with real roots whose field is $Q(\theta)$.

2) Select a T by trial and error so that for matrix B of the s_1v_1, s_2v_2 and s_3v_3, where $s_i=\pm1$, BT is a matrix of algorithmic system I through V, and hence BT is an MAOP. Compute its MAOP form.

3) Classify BT according to the algorithmic system to which it belongs, and use the above algorithm to calculate its neighbor

B'. Calculate its form and see if it is the same form as that of B. If it is, the ratio of their corresponding parameters, if they are algebraic integers, will be a unit. If not, repeat step 3 until identical forms are found.

E. Sample Calculation of Units by Minkowki's Method

We shall find the units for the algebraic integers of the real field $Q(\theta)$, where θ is given by the equation $x^3+x^2-2x-1=1$, whose roots are all real.

1. The Initial MAOP

We calculate the roots of $x^3+x^2-2x-1=0$, and they have approximate values $\theta=1.25$, $\theta'=-.45$ and $\theta''=-1.80$, which we locate along the x, y and z axis, respectively. These yield the basis $v_1 = (1,1,1)$, $v_2 = (1.25,-.45,-1.80)$ and $v_3 = (1.80,.20,3.64)$.

If we set $s_3=-1$ and $T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, we will have created a matrix U of category IV, which by proposition 6.2.5 is an MAOP.

This matrix $U = \begin{bmatrix} 1.25 & 1 & -.55 \\ -.45 & 1 & .80 \\ 1.80 & -1 & 2.25 \end{bmatrix}$ and its MAOP form is

$-x^3-y^3-z^3+2y^2z+2z^2x+xy^2+yz^2+zx^2+xyz$.

2. Units Calculated by Means of the Minkowski Algorithm for Successor MAOP

We shall calculate the x-neighbor, y-neighbor and z-neighbor of U. We must use the rule of case IV, 2. The resulting neighbors are all of category VI of the algorithmic system.

$$\text{x-neighbor of U} = P = \begin{bmatrix} 1 & -.45 & -.55 \\ -1 & 1.80 & -.80 \\ -1 & -1.25 & 2.25 \end{bmatrix},$$

whose MAOP form is $x^3+y^3+z^3-x^2y-y^2z-z^2x+2xy^2$ and the determinant of p=0.

$$\text{y-neighbor of U} = Q = \begin{bmatrix} 1.25 & -.55 & -.70 \\ 0.45 & .80 & -.35 \\ -1.80 & -2.25 & 4.05 \end{bmatrix},$$

whose MAOP form is $x^3+y^3+z^3-x^2y-y^2z-z^2x+2xyz$ and the determinant of Q=0.

$$\text{z neighbor of U} = R = \begin{bmatrix} 2.25 & -1 & -1.25 \\ -.55 & 1 & -.45 \\ -.80 & -1 & 1.80 \end{bmatrix},$$

whose MAOP form is $x^3+y^3+z^3-x^2y-y^2z-z^2x+zxyz$ and the determinant of R=0.

The x parameter of P, Q and R are 1, θ, $1+\theta$ and their quotients, algebraic integers, show that P, Q and R are unit related by the units θ, $-1/(1+\theta)$ and $(-1-\theta)/\theta$

To see that these units are independent, see section 5.3.A.

It is conceivable that a procedure may be developed for finding fundamental units by Minkowski's method, but it is doubtful that it would be as consistently simple as the Voronoii method.

PART II. A BASIS APPROACH

CHAPTER 7

ADMISSIBLE ORIENTED PARALLELOPIPEDS AND BASES

7.1. Bases

Definition 7.1.1 A Voronoii triplet is a set of three non-origin symmetric vectors v_1, v_2, v_3 such that each vector defines a VAOP.

Definition 7.1.2 If a Voronoii triplet is a non-planar with the origin, it is called a Voronoii basis, otherwise, it is called a degenerate Voronoii basis.

Definition 7.1.3 A Minkowski triplet is a set of three non-origin symmetric vectors v_1, v_2, v_3 which define an MAOP.

Definition 7.1.4 If a Minkowski triplet is non-planar with the origin, then it is called a Minkowski basis, otherwise, it is called a degerate Minkowski basis.

Definition 7.1.5. An Edge-Face triplet is a set of three non-origin symmetric vectors v_1, v_2, v_3 such that any pair of the three define an EFAOP.

Definition 7.1.6 If an Edge Face triplet is non-planar with the origin, then it is an Edge Face basis, otherwise, it is called a degenerate Edge Face basis.

Proposition 7.1.7 A Minkowski triplet implies an Edge Face triplet implies an Voronoii triplet. □

Proposition 7.1.8 A Minkowski basis, a Voronoii basis and Edge-Face basis exist in any lattice.

Proof: Use theorem 6.2.6 and proposition 7.1.7. □

We will attempt to obtain a better geometric understanding of the above concepts by calculating up to geometric equivalence the ways oriented parallelopipeds associated with three non-origin symmetric vectors can be represented in the same origin symmetric discrete array. We will provide diagrams of these equivalences.

Note: The more general question, the analagous question of section 2.3, that is, the number N of three oriented parallelopipeds up to geometric equivalence in the same origin symmetric discrete array, is a much more difficult number to calculate and at the moment irrelevant to our discussion. Similarly, the general question of the number K of n m-dimensional oriented parallelopiped in the same m-dimensional irreducible lattice is an interesting, but very difficult combinatoric problem. At the present, I have no solution.

7. 2. The Number of Oriented Parallelopipeds Determined by Three Non-Origin Symmetric Vactors In The Same Origin Symmetric Discrete Array

Proposition 7.2.1 The number of MOP in the same OSDA determined by three non-origin symmetric points v_1, v_2, v_3 is one. □

Propositon 7.2.2 There exists only one way up to geometric equivalent by which an MAOP determined by three non-origin symmetric points may exist in the same OSDA. □

Needless to say, no diagram is necessary for this special situation.

Proposition 7.2.3 The number of EFOP in an irreducible lattice determine by three non-origin symmetric points is two. □

Proposition 7.2.4 There exists four different ways up to geometric equivalence by which two EFAOP determined by three non-origin symmetric points may exist in the same OSDA.

Proof: See section 2.3. □

These diagrams are illustrated in Chapter 2.3 as diagrams 2.3.2, 2.3.3, 2.3.4, and 2.3.5.

Proposition 7.2.5. The number of VOP in the same OSDA determined by three non-origin symmetric points v_1, v_2, v_3 is three. □

Proposition 7.2.6 There exists ten different geometrically equivalent ways by which three VAOP may exist in the same OSDA determined by three non-origin symmetric points.

Proof: The following is a list of all twenty-four possible rank matrices.

1 $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ 2 $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}$ 3 $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 3 \end{bmatrix}$ 4 $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}$

5 $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ 6 $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 7 $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}$ 8 $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

9 $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 3 & 2 \end{bmatrix}$ 10 $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 3 & 1 \end{bmatrix}$ 11 $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ 12 $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 2 \end{bmatrix}$

13 $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ 14 $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ 15 $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ 16 $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

17 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 1 \end{bmatrix}$ 18 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ 19 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ 20 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}$

21 $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$ 22 $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ 23 $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ 24 $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$. $\square$

Under allowable column and row operations, the following equivalences hold: 1; $2 \simeq 3 \simeq 4$; $5 \simeq 14$; $6 \simeq 8 \simeq 20$; 7; 9; $10 \simeq 22$; $11 \simeq 12 \simeq 13 \simeq 19$; $15 \simeq 16 \simeq 17 \simeq 21 \simeq 23 \simeq 24$ and 18.

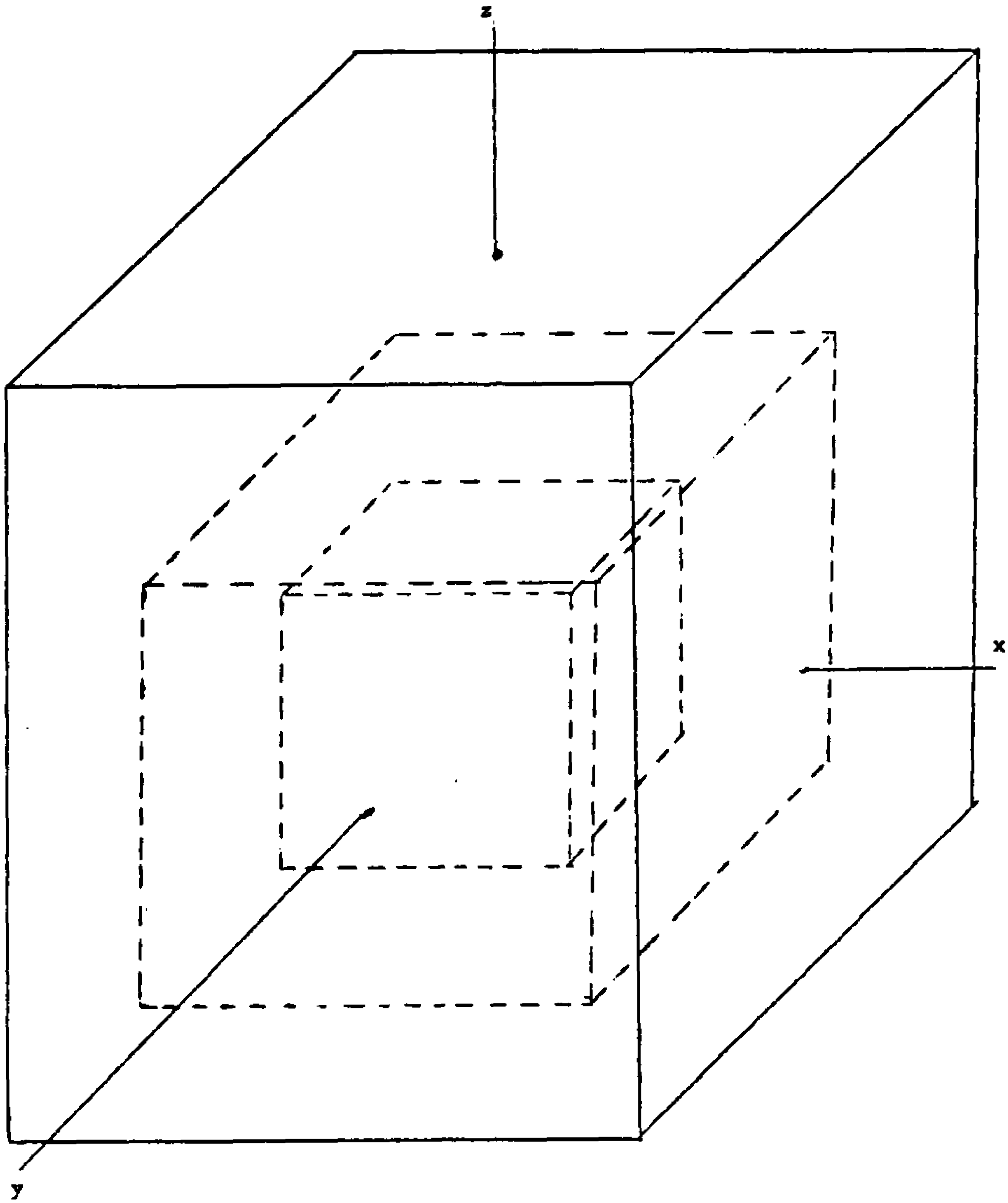
Taking a representative from each group yields the result.

These diagrams are illustrated below:

7.3. Diagrams of the Ways Three Oriented Three Dimensional Parallelopipeds Determined by Three Non-Origin Symmetric Points can be Interrelated

Three Space

Diagram 7.3.1. Three Oriented Parallelopipeds with Distinct Parameters - Type I

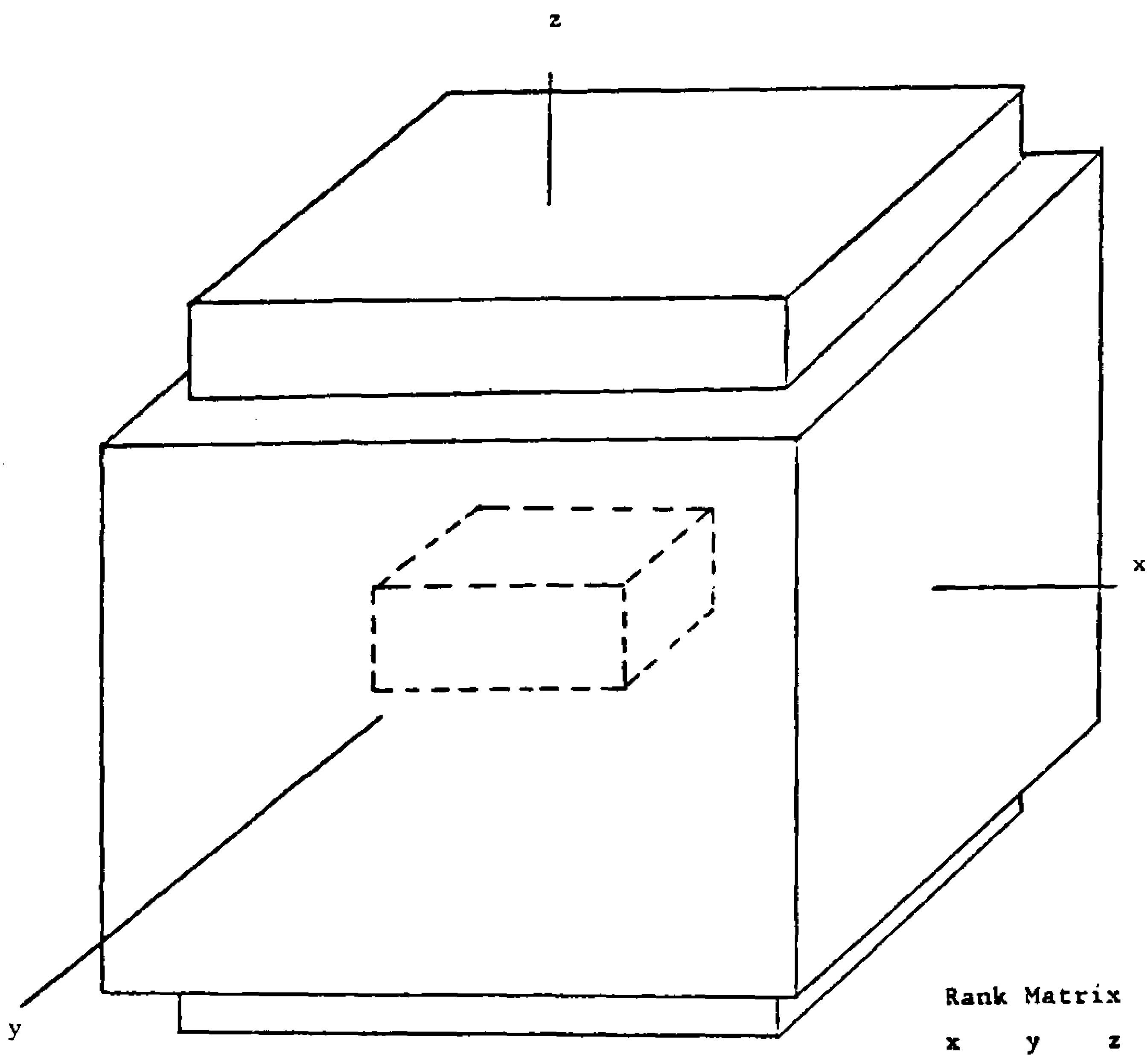


Inadmissible Configuration

Rank Matrix

x	y	z
1	1	1
2	2	2
3	3	3

Diagram 7.3.2. Three Oriented Parallelopipeds with Distinct Parameters - Type II

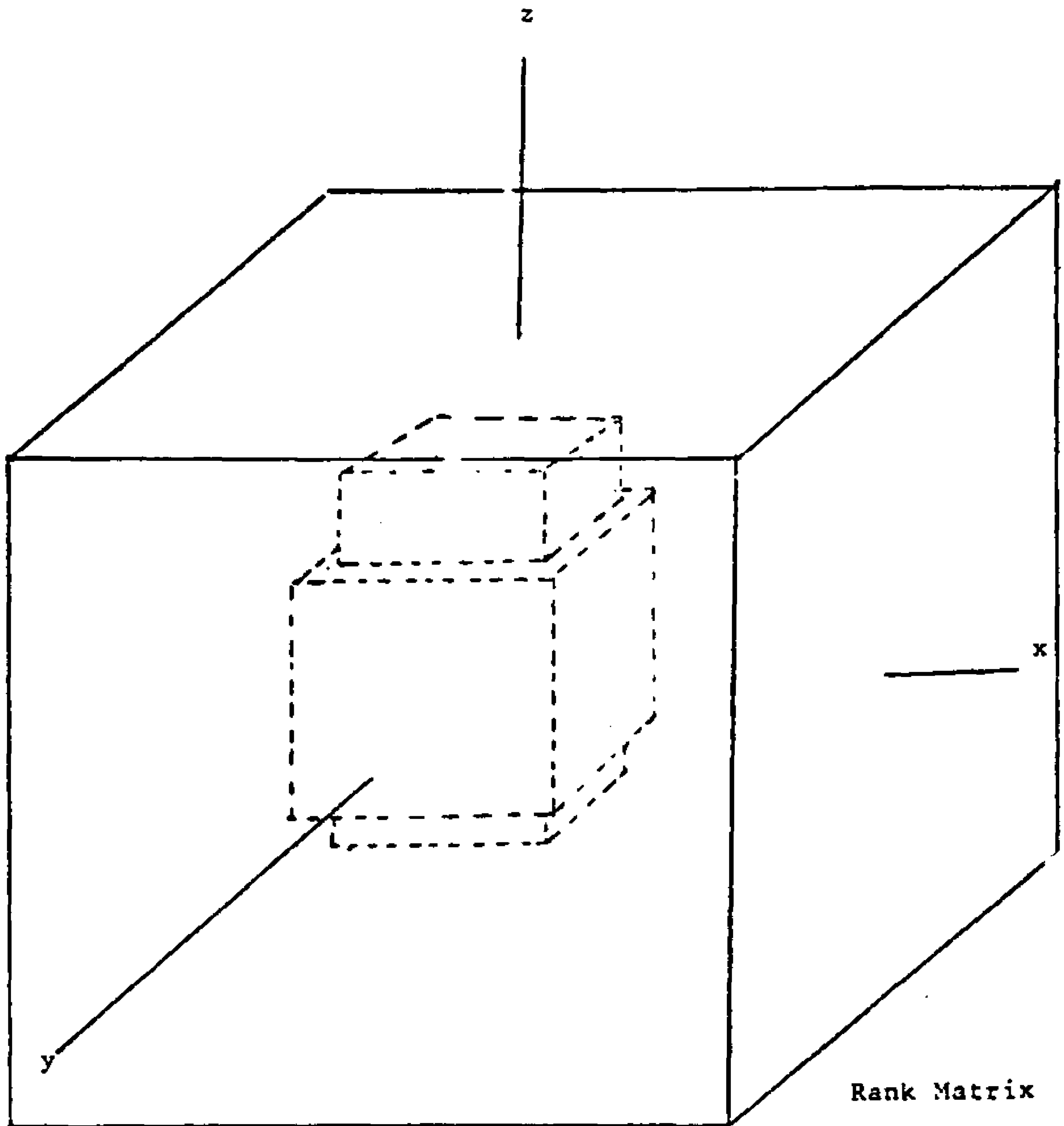


Rank Matrix

x	y	z
1	1	1
2	2	3
3	3	2

Inadmissible
Configuration

Diagram 7.3.3. Three Oriented Parallelopipeds with Distinct Parameters - Type III

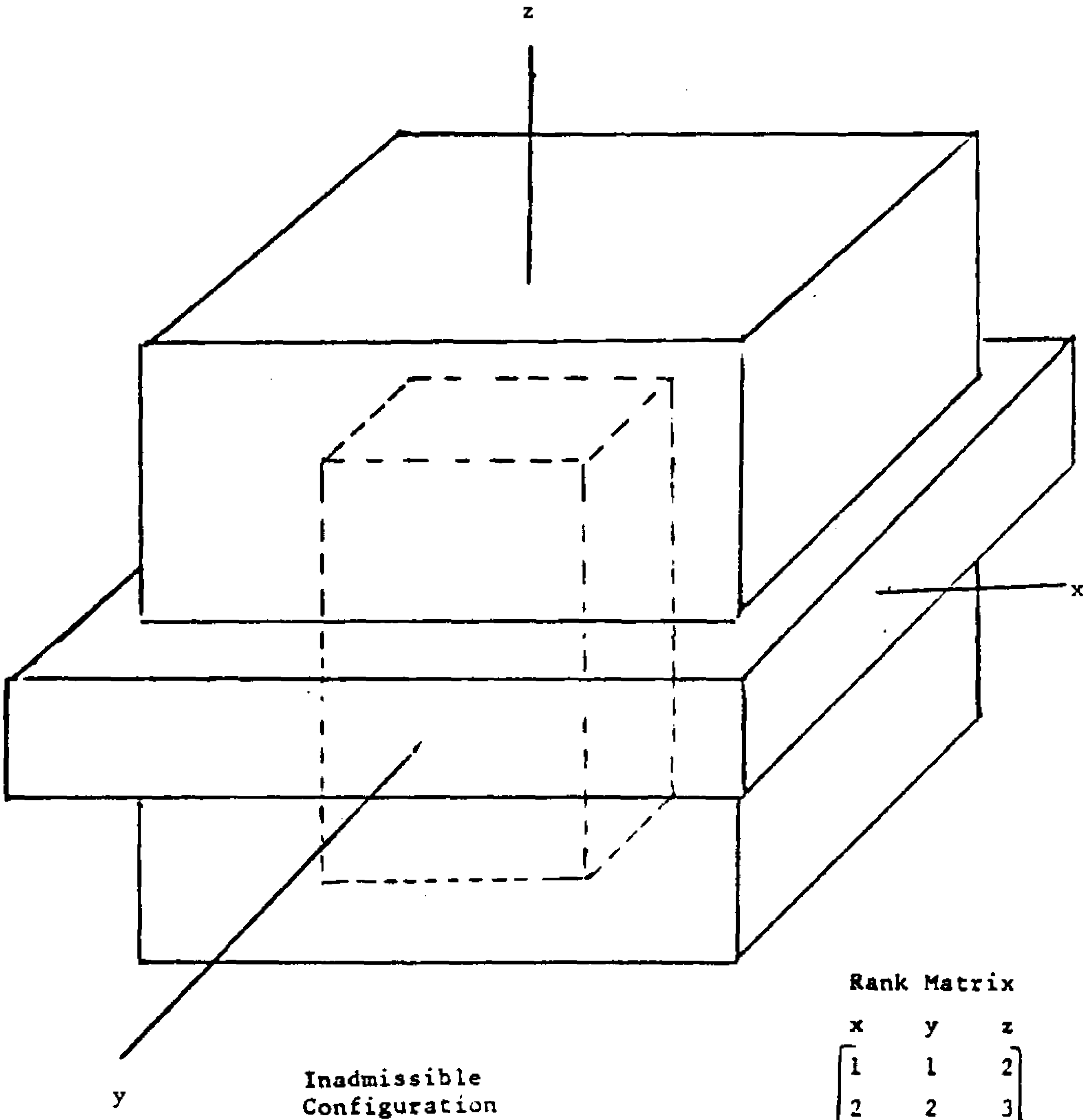


Inadmissible
Configuration

Rank Matrix

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$$

Diagram 7.3.4. Three Oriented Parallelopipeds with Distinct Parameters - Type IV



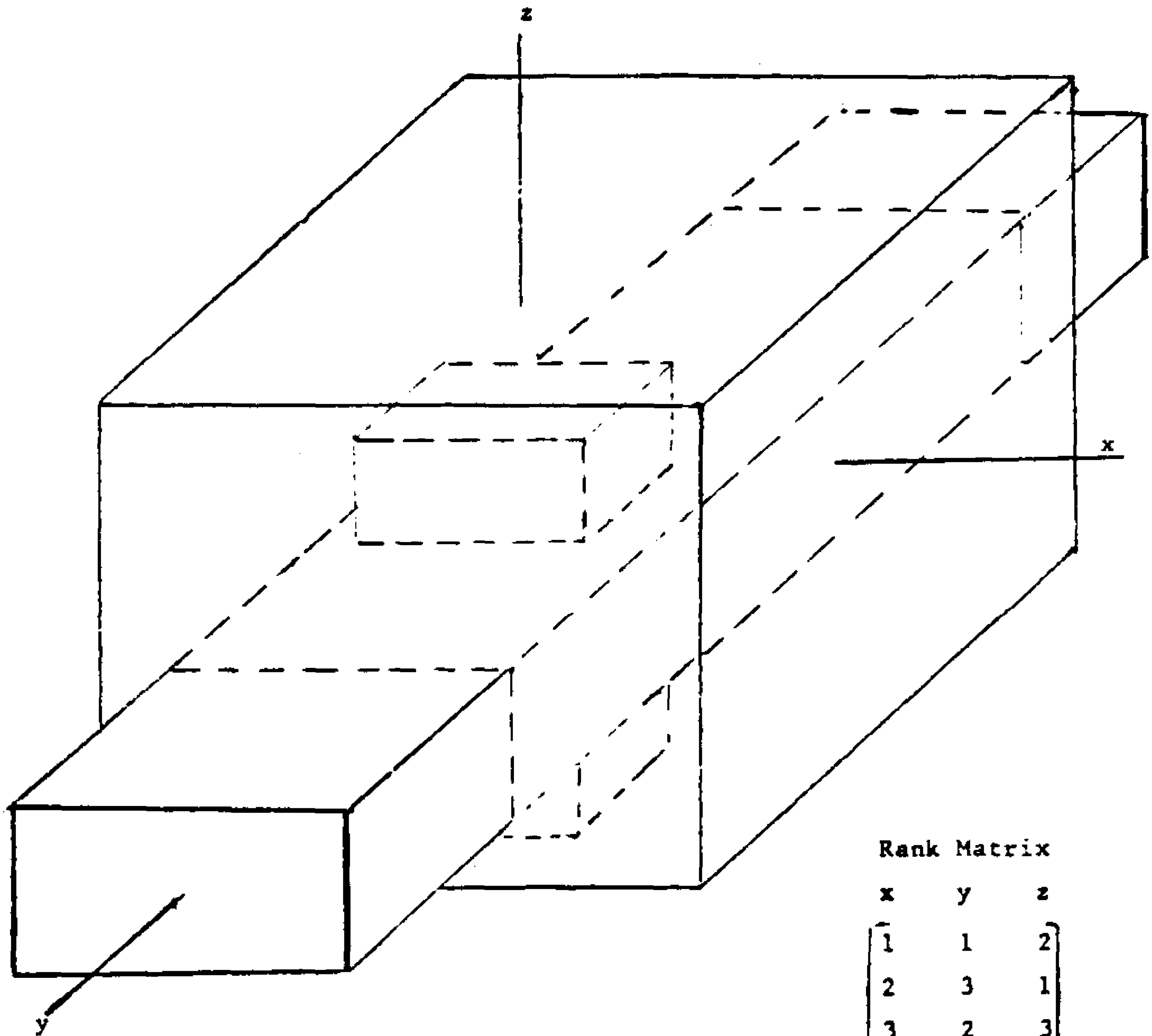
Inadmissible Configuration

Rank Matrix

$$
\begin{array}{ccc}
x & y & z \\
\end{array}
$$

$$
\begin{bmatrix}
1 & 1 & 2 \\
2 & 2 & 3 \\
3 & 3 & 1
\end{bmatrix}
$$

Diagram 7.3.5. Three Oriented Parallelopipeds with Distinct Parameters - Type V

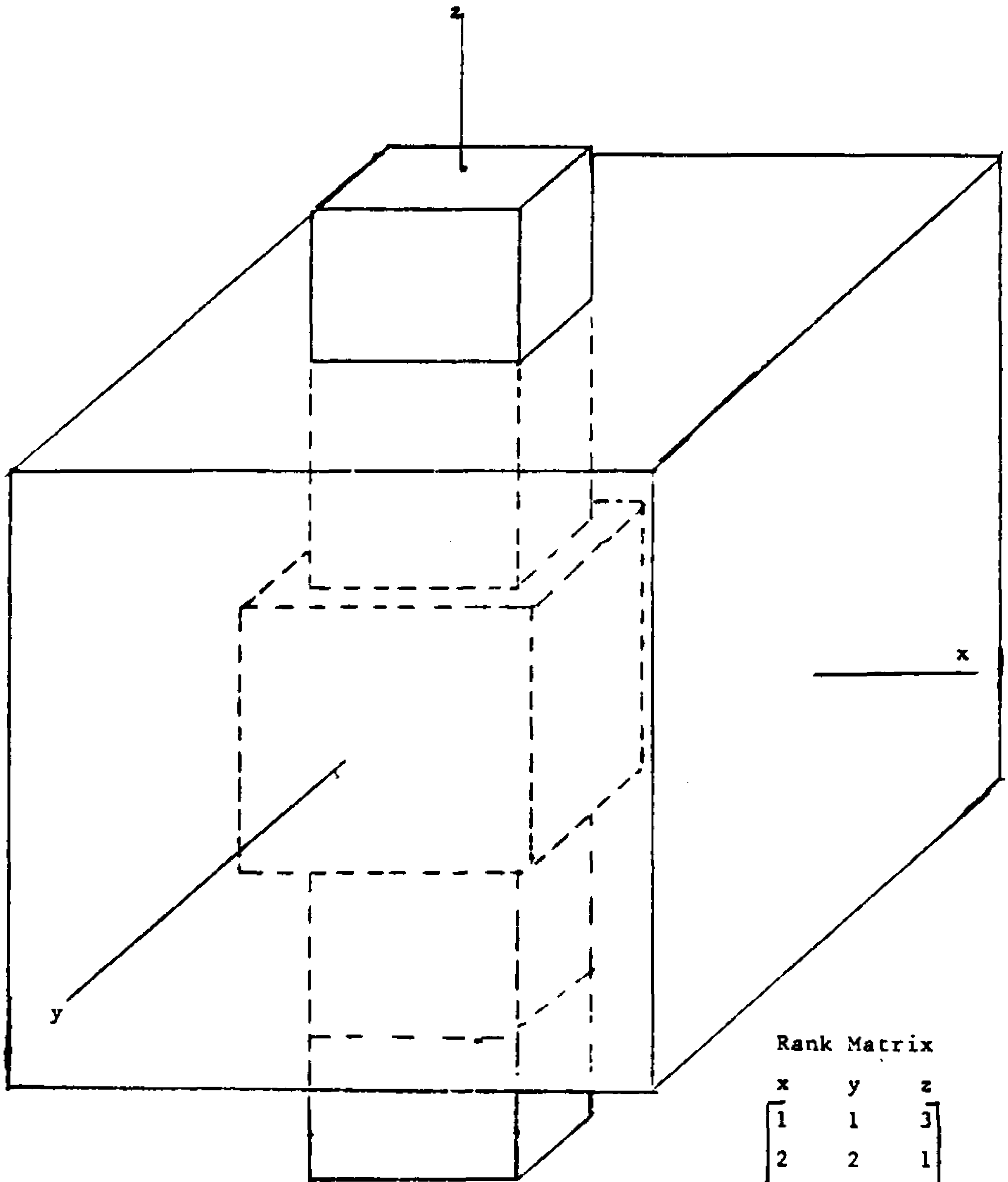


Rank Matrix

x	y	z
1	1	2
2	3	1
3	2	3

Inadmissible
Configuration

Diagram 7.3.6. Three Oriented Parallelopipeds with Distinct Parameters - Type VI

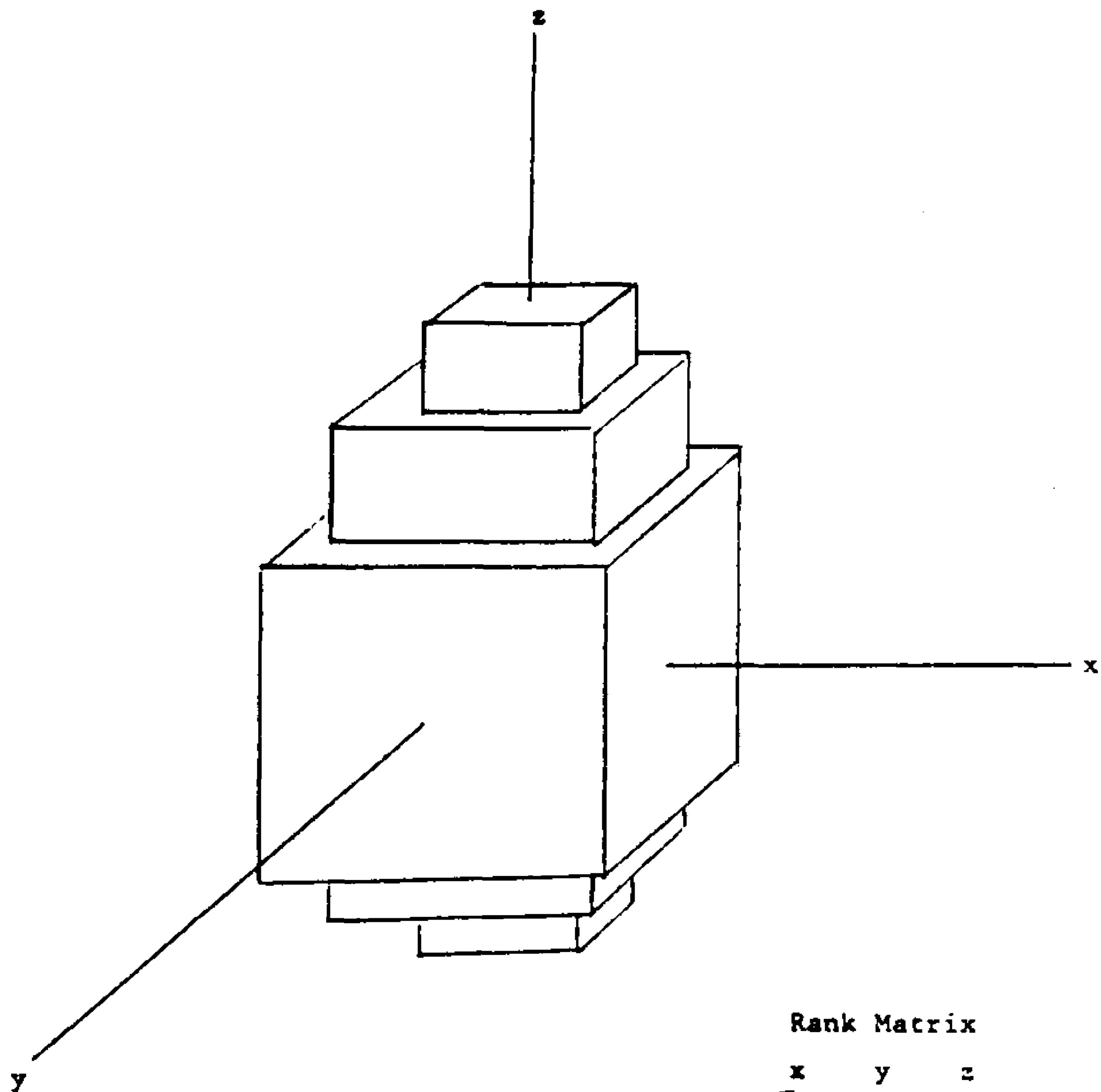


Rank Matrix

x	y	z
1	1	3
2	2	1
3	3	2

Inadmissible
Configuration

<u>Diagram 7.3.7.</u> Three Oriented Parallelopipeds with Distinct Parameters - Type VII

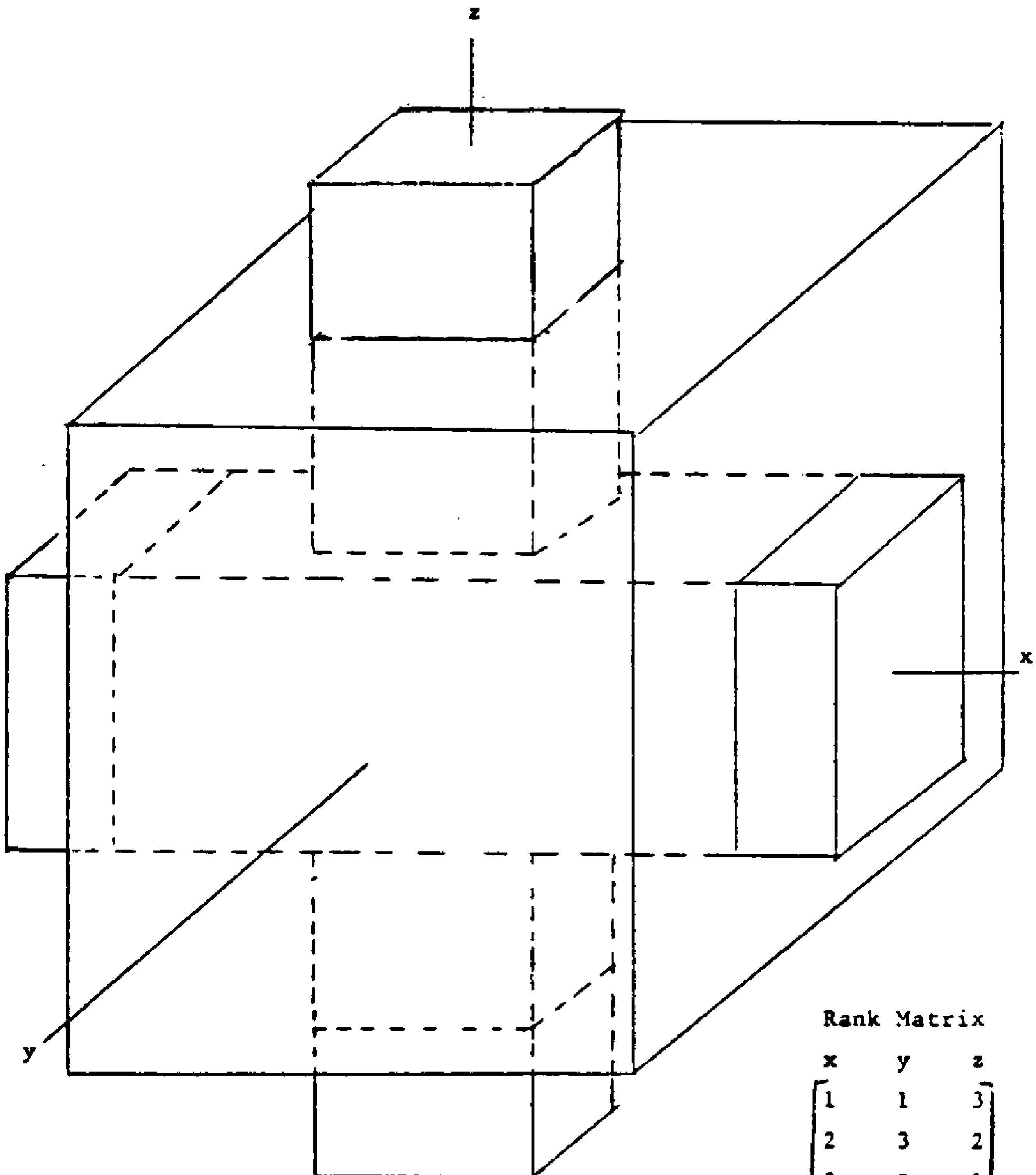


Rank Matrix

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Possibly
Admissible
Configuration

Diagram 7.3.8. Three Oriented Parallelopipeds with Distinct Parameters - Type VIJ†

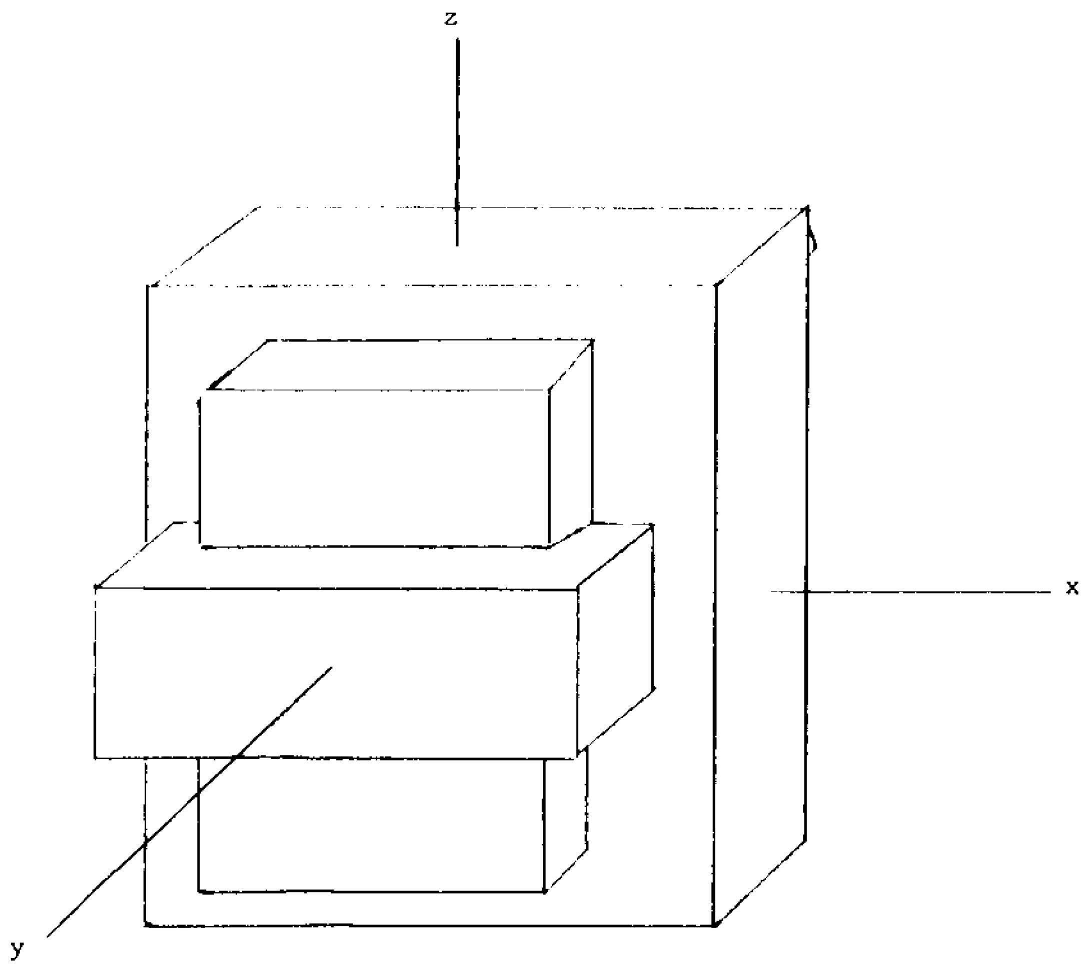


Rank Matrix

x	y	z
1	1	3
2	3	2
3	2	1

Possibly
Admissible
Configuration

Diagram 7.3.9. Three Oriented Parallelopipeds with Distinct Parameters - Type IX

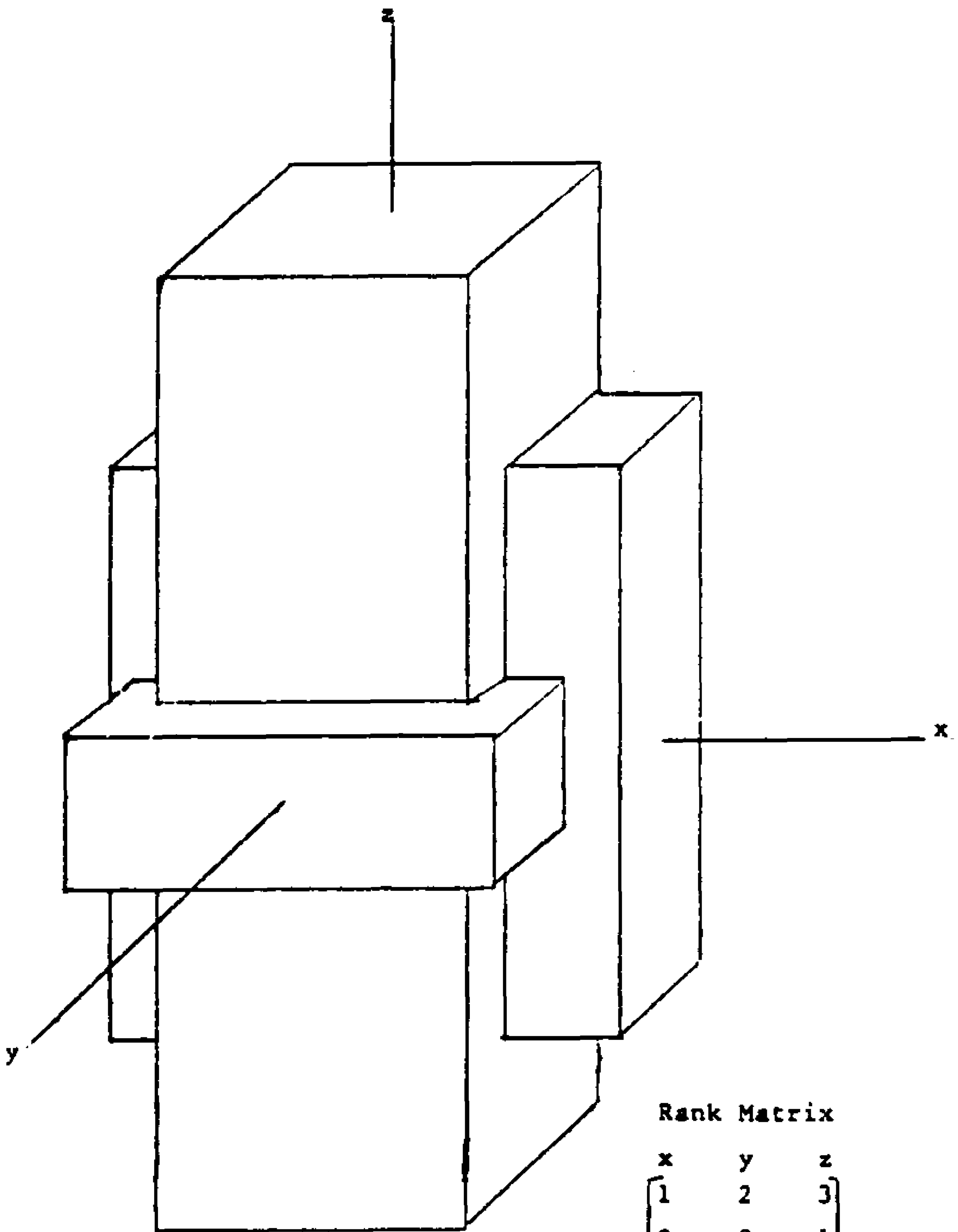


Rank Matrix

x	y	z
1	2	2
2	3	1
3	1	3

Possibly
Admissible
Configuration

Diagram 7.3.10. Three Oriented Parallelopipeds with Distinct Parameters - Type X



Rank Matrix

x	y	z
1	2	3
2	3	1
3	1	2

Possibly
Admissible
Configuration

CHAPTER 8

BASES AND UNITS

8.1 Unit Related Bases

We now create an approach that is analogous to section 5.1. calculating units. However, this approach will use bases instead of AOP. We begin with the analogous definition to Definition 5.1.2.

Definition 8.1.1. Two bases v_1, v_2, v_3 and v'_1 v'_2, v'_3 are said to be unit related in an multiplicative lattice with identity if there, exists a unit v such then $v_i = v_i$ for $i=1,2,3$. It is clear that for each unit v there exists at least a pair of unit related bases. Hence, if it is possible to locate unit related bases, it will be possible to calculate units. The following procedure (analogous to the procedure of Chapter V section 2) that does this is:

1) Find a method to locate an initial bases of a "type".
2) Find a method to move from basis of that "type" to another basis of that "type" and within a finite number of steps to two unit related bases of that "type".

The concept of "type" will be explained later. Implicit in 2) is the need for a test to determine if two unit related bases of the "type" have been created. The "type" should be essential to doing 1) and 2).

The motivation for this approach is quite simply the desire

to avoid the lengthy and difficult calculations required by both Minkowski's and Voronoii's method for computing the successor MAOP or VAOP. Often the multiplicative aspects of these approaches, blur the accuracy of the units. In a basis approach, the calculation of a successor basis for a basis, can be accomplished by simply "adding and subtracting vectors."
We shall now briefly describe Cohn's successful creation of a method using the basis approach.

8.2 Cohn

A. Cohn Bases and Cohn Neighboring Process

1. Cohn Bases

Definition 8.2.1. Given any three basis vectors v_1, v_2, v_3. Then we define a Cohn basis if v_1, v_2, v_3, $v_4 = -v_1 - v_2 - v_3$ have a 3 x 4 matrix:

$$\begin{bmatrix} v_1^{(1)} & v_2^{(1)} & v_3^{(1)} & v_4^{(1)} \\ v_1^{(2)} & v_2^{(2)} & v_3^{(2)} & v_4^{(2)} \\ v_1^{(3)} & v_1^{(3)} & v_3^{(3)} & v_4^{(3)} \end{bmatrix},$$

whose signs are:

$$\pm \begin{matrix} + & - & - & + \\ - & + & - & + \\ - & - & + & + \end{matrix}.$$

It is noted that if we multiply any row by a minus, or any two rows by a minus, or all three rows by a minus and then shift appropriate columns, we would still have the same sign config..ration above. Each of these eight possible configurations are to be identified with the above original configurations.

Notation: We shall indicate a Cohn basis for v_1, v_2, v_3 and v_4 by $[v_1, v_2, v_3, v_4]$.

These type of vectors are an example of the "type" of basis that is discussed in 1) and 2). We shall shortly see that 1) and 2) and the many analogous bases' questions to the geometric questions of section 3.1 will be answered by a Cohn basis.

Proposition 8.2.2. Any three vectors of a Cohn basis is a basis. □

Proposition 8.2.3 If two Cohn bases are unit related, then any three vectors of one basis will be unit related to the corresponding vectors of the other basis. □

We now need only find unit related Cohn bases and our problem of finding unit related bases, and hence, units is solved.

2. Cohn Neighboring Process

Among all the possible neighboring processes that may be considered we choose the most elementary.

Definition 8.2.4. Given a Cohn basis v_1, v_2, v_3, v_4, then a Cohn neighbor will be $v_i - v_j$, $v_k + v_j$, v_j, v_m where i, j, k, m, is a permutation of 1, 2, 3, 4 and which is itself a Cohn basis.

Notation: We shall indicate the neighboring process for $[v_1,v_2,v_3,v_4]$ by (v_i, v_j, v_k), which will create

$[v_i+v_j, v_k-v_j, v_j, v_m]$, where i, j, k, m is a permutation of 1,2,3,4.

We see there are 24 possible candidates for the neighbors of Cohn basis. The actual number that are Cohn neighbors and the other analogous questions of section 3.1 will be provided in the following section.

B. Basis Theorems about the Neighboring Process

1. Number of Predecessors and Successors of a Cohn Bases

Given any Cohn basis how many immediate predecessors and immediate successors does it have (where predecessor and successor have the analogous definitions of those for AOP)?

Proposition 8.2.5 For each Cohn basis the neighboring process create 3, 4, 5 or 6 Cohn neighbors.

We shall present a slightly stronger form of this proposition, and indicate its proof.

Definition 8.2.6 A type 1, type 2 or type 3 neighboring process for a Cohn basis [P,Q,R,S] is indicated by the following table:

Type 1	Type 2	Type 3
(QPS), (PQR)	(PQS), (QPR)	(PRS), (RPQ)
(RPS), (SQR)	(RQS), (SPR)	(QRS), (SPQ)
(QSP), (PRQ)	(PSQ), (QRP)	(PSR), (RQP)
(RSP), (SRQ)	(RSQ), (SRP)	(QSR), (SQP) .

Proposition 8.2.7. Every Cohn basis [P,Q,R,S] has either one or two Cohn neighbors of each type.

Proof: We only sketch the proof. Consider a subdivision into cases according to relative magnitudes of each row: p_i , q_i , r_i , s_i . There are twenty four different cases. We omit the details. □

2. Looping In a Cohn Basis

Given any Cohn basis, if its neighboring process were unrestricted would it be possible to return to the same Cohn basis? If so, how can this be avoided?

Comment 8.2.8. Given any Cohn basis S, the random use of the Cohn neighboring process can return us to S.

To avoid the "looping" of Comment 8.2.8., is not the simple matter that it is in the case of AOP. However, it can be done as will be explained in the following proposition.

Proposition 8.2.9 Let $h_i = u_i^{(i)}$, for i=1,2,3, where $u_i^{(j)}$ are the components of $U^{(j)}$ and $U^{(1)} = P+S$, $U^{(2)} = Q+S$, $U^{(3)} = R+S$. A neighbor operation of type i decreases h_i, and h_i is positive, while it increases or leaves the same the other two h_j, where $j \neq i$.

Proof: Because of the symmetries, we consider only the operation (QPS) of type i, which changes $h_1=p_1+s_1$ into $s_1 < h_1$, while it changes $h_3=r_3+s_3$ into $r_3+s_3-p_3 > h_3$ and leaves $h_2=q_2+s_2$ unchanged. □

3. Connecting Cohn Bases

Given any two Cohn bases, is it possible to connect the two by means of the Cohn neighboring process.

Cohn has claimed that the answer to this question is yes: There is a flaw in the proof as I will indicate. At the moment, the theorem is consequently false. I will provide a possible way of circumventing the flaw. However, the suggested method has not been established.

Theorem 8.2.10.(UNESTABLISHED) Any two Cohn bases can be connected by a chain of Cohn neighbors. (A Cohn chain has the analogous definition of the AOP chains of each type.)
In order to establish this theorem, Cohn related his bases to MAOP and then used the fact that two MAOP may be connected by a chain of MAOP.

Definition 8.2.11. Two sets of triple vectors or two sets of quadruple vectors are said to overlap if they share two or more vectors.

Proposition 8.2.12. Two overlapping Cohn bases can be connected by a chain of Cohn neighbors.

Proof: We only sketch the proof. Let the two Cohn bases be $A = [P,Q,R,S]$ and $B = [P,Q, R+aP+bQ, S-aP-bQ]$ for integers a and b. Now the Cohn basis sign conditions on B leads to three sign conditions on the third and fourth vector ($= -R-(a+1)P-(b+1)Q$) of B, yields:

$$\begin{array}{ll} f_1(a+1,b+1) < 0 & f_1(a,b) < 0 \\ f_2(a+1,b+1) < 0 & f_2(a,b) < 0 \\ f_3(a+1,b+1) < 0 & f_3(a,b) > 0 \quad , \end{array}$$

where $f_i(a,b) = r_i+ap_i+bq_i$, which are always under the sign restrictions of a Cohn basis. It is tedious, but elementary to show by representing the six inequalities graphically that if these relations hold for (a,b), then a and be are not both of the same sign, while the inequalities hold with (a,b) replaced by (a,b+sgn a) or (a+sgnb, b), where sgn 0 = 0. Proposition 8.2.12 follows by descent. □

Theorem 8.2.13. If v_1, v_2, and v_3 are a Minkowski basis, then $\pm v_1$, $\pm v_2$, $\pm v_3$, with an appropriate choice of signs together with the sum v_4 of these re-signed vectors, form a Cohn basis.

Proof: Please see section 9.1. where this important theorem is restated and proved. □

We now give the partial converse of theorem 8.2.10.

Theorem 8.2.14. In each chain of a Cohn basis of length n, there will exist an element of the chain which overlaps a Minkowski basis.

Proof: The lengthy proof of this theorem can be found in [2,pp.908-909]. □

Proposition 8.2.15. Any two neighboring Minkowski bases (and hence, overlapping) each overlap overlapping Cohn bases. □

To circumvent MAOP whose vectors form a degenerate basis, and hence make the connection between all MAOP whose vectors form non-degenerate bases Cohn stated the following theorem in [2, p.907].

Proposition 8.2.16(FALSE) Any two MAOP of algorithmic system I through V can be connected by a chain of such MAOP.

This proposition is false. A counterexample is provided by the initial MAOP used in the sample calculation of units by Minkowski's method in section 6.2.F. It is an MAOP, whose vectors form a non-degenerate basis, and each neighbor is an MAOP, whose vectors form a degnerate basis. A way to circumvent this flaw is to show that two given MAOP A and B, whose vectors form a non-degenerate basis, and who share as a common neighbor an MAOP C, whose vectors form a degenerate basis, have Cohn bases of A and B that can be connected by a chain of Cohn bases. However, this proposition has not been fully established. The consequence of the inability to establish this theorem is the inability to say for certain that all units and, hence, fundamental units can be calculated by Cohn bases.

C. Cohn Bases, Units, Independent Units and Fundamental Units

We now assume that 1) and 2) exist for Cohn bases. We shall describe these in section 8.2.D. below.

1. Units

Proposition 8.2.17. In an irreducible multiplicative lattice with identity, the number of v_1, v_2, v_3, forms, where v_1, v_2, v_3, is any three vectors from a Cohn basis is finite.

Proof: With an analogous approach that Minkowski used to establish the analogous proposition 5.3.20. for MAOP, Cohn proves this proposition in [1,pp.827-829].

Because of proposition 8.2.9, we can create an infinite number of Cohn bases, which by proposition 8.2.17 will create a finite number of bases forms. Using proposition 5.3.22, we have that the Cohn method calculates units. An alternate proof to that indicated by proposition 8.2.17 which will establish that Cohn's method does create units will be presented in section 9.3.

2. Independent Units

Letting the $U^{(1)}$, $U^{(2)}$ and $U^{(3)}$ of proposition 8.2.9. "play the role" of the x, y and z axes in the discussion of section 5.3.4., we can calculate independent units as indicated by that same discussion in section 5.3.4.

3. Fundamental Units

Because theorem 8.2.8 cannot be established at present, we cannot say for certain that Cohn basis has the ability to calculate fundamental units, though computational evidence implies that it can [3,pp.162-168].

D. Locating an Initial Cohn Basis and Cohn Basis Successor Algorithm

1. Initial Cohn Basis

One way to find an initial Cohn basis would be to find an initial MAOP of algorithmic system I through V by theorem 8.2.13. By theorem 6.2.5., we can find an initial MAOP. If this MAOP is of algorithmic system VI, its neighbors will be of algorithmic system I through V by theorem 5.3.15., and we have located our initial Cohn basis.

2. The Cohn Algorithm for Finding the Successor of a Cohn Basis

As was stated, the motivation of this basis approach is the non-multiplicative nature of calculating neighbors. We construct the 24 possible candidates for the Cohn basis neighbors and test which are Cohn bases. This algorithm requires no multiplicative calculations.

3. Programmatic Description of Cohn Algorithm

The programmatic description of the Cohn algorithm can be found in [3,pp.160-165]. Due to the severe limitations of the computer, only one type of the neighbor types were explored and not all the Cohn neighbors of this type were explored. On the modern computer and using the entire body of Cohn's theory, even more spectacular results should be obtained.

4. Sample Calculation of Units by Cohn's Method

See [3, pp.162-168].

CHAPTER 9

COHN BASIS AND INTERRELATIONSHIP WITH ADMISSIBLE ORIENTED PARALLELOPIPEDS

9.1 Cohn Bases And MAOP

Let us recall the very important theorem 8.2.11 for the sake of completeness of this section. We will offer a simpler proof than the one given by Cohn [1,pp.825-826].

Theorem 8.2.11 If v_1, v_2 and v_3 are a Minkowski basis, then $\pm v_1$, $\pm v_2$, $\pm v_3$, with an appropriate choice of signs together with the sum v_4 of these resigned vectors, form a Cohn basis.

Proof: By Proposition 5.3.5 an MAOP, whose vectors form a non-degenerate basis are of algorithmic system I through V where $a > b$, $a > c$, $g > f$, $g > h$, $m > k$, $m > j$. The following shows how to change each category I through V to a Cohn basis.

I. Multiply column 2 and 3 by 1 and switch. Use the fact that $b+c > a$ by proposition 5.3.8 for row 1 $g > h$ for row 2 $m > k$ for row 3 to show the fourth vector is of sign form + + +.

II. Multiply column 1 and 3 by 1 and switch. Use the fact that $f + h > g$ by proposition 5.3.8 for row 2, $a > c$ for row 1 $m > j$ for row 3 to show that the fourth vector has sign form + + +.

III. Multiply column 1 and 2 by 1 and switch. Use the fact that $j + k > m$ by proposition 5.3.8. for row 3, $a > b$ for row 1, and $g > f$ for row 2 to show that the fourth vector has sign form + + +.

IV. Multiply all columns by -1. Move column I to column 2, column 2 to column 3 and column 3 to colunm 1. Use the fact that $a > c$ for row 1, $g > f$ for row 2, $m > k$ for row 3 to show the fourth vector is of sign form + + +.

V. Multiply all columns by -1. Move column 1 to column 3, column 3 to column 2, column 2 to column 1. Use the fact that $a > b$ for row 1, $g > h$ for row 2, $m > j$ for row 3 to establish that the fourth vector is of sign form + + +. □

The converse of theorem 8.2.11 is not true, the following is a partial converse.

Proposition 9.1.1 If in a Cohn basis $\{v_1, v_2, v_3, v_4\}$ we let the X represent the largest in absolute value of the three x components of v_1, v_2, v_3 with similar definitions for Y and Z, then the first three vectors of a Cohn basis will be an MAOP of category I through V in accordance with the location of X, Y, Z in the first three columns of the Cohn basis matrix as the following table indicates:

I	II	III	IV	V
X - -	- - -X	- -X -	- -X -	- - -X
± - - -Y	± - -Y -	± -Y - -	± - - -Y	± -Y - -
- -Z -	-Z - -	- - -Z	-Z - -	- -Z - .

Proof: Use Theorem 6.2.5. □

There are many more Cohn bases than MAOP. The following matrix [3, p.160] is a Cohn basis, but no three of its vectors form an MAOP.

$$\begin{bmatrix} 2.507 & -2.285 & -1.221 & 1.000 \\ -1.221 & 2.507 & -2.285 & 1.000 \\ -2.285 & -1.221 & 2.507 & 1.000 \end{bmatrix}$$

However, we note that (1,1,1) is a VAOP, which motivates the next section 9.2.

9.2 Cohn Bases And VAOP

Theorem 9.2.1 For any Cohn basis v_1, v_2, v_3, and v_4, the VOP associated with at least one of the v_i is a VAOP.

In order to prove this theorem, some preliminary definitions and propositions will be presented.

Definition 9.2.2 For any triple of different numbers a_1, a_2 and a_3, $\text{mid}_i\ a_i$ = the second largest value of a_1, a_2 and a_3. $\text{Min}_i\ a_i$ and $\max_i\ a_i$ will have their usual meaninings.

Proposition 9.2.3 Given a Cohn basis v_1, v_2, v_3, and v_4, then the oriented parallelopiped P defined by $x_j \le \max_i v_j^{(i)}$, $x_k \le \min_i v_k^{(i)}$, $x_m \le \min_i v_m(i)$, where j, k, m is a permutation of 1, 2, 3, is an admissible oriented parallelopiped, if v_4 is not in P.

Proof: By the symmetry of the problem, we only consider j=1, k=2 and m=3. Let $X = \max_i v_1^{(i)}$, $y = \min_i v_2^{(i)}$ and $z = \min_i v_3^{(i)}$. Consider an arbitrary lattice point $p = av_1 + bv_2 + cv_3$, where a, b, c are integers. We examine P to see if p is in its interior. That is,

	1	$-X < av_1^{(1)}+bv_2^{(1)}+cv_3^{(1)} < X$
Inequalities I	2	$-y < av_2^{(2)}+bv_2^{(2)}+cv_2(2) < y$
	3	$-z < av_3^{(3)}+bv_3^{(3)}+cv_3^{(3)} < z$

Note: We do not allow equality in any of the inequalities because the lattice is irreducible.

The following sign-zero pattern account for all sign-zero pattern for a,b,c. The table below lists why p for the sign-zero combination fails to be in P.

Sign-0 Pattern			Impossible Because p Does Not Satisfy The Inequality i(=1,2,3) Of Inequality I's	Left(L) or Right(R) Side
a	b	c	i	L or R
0	0	0	Origin Exists In A VAOP	
0	0	-	3	L
0	0	+	3	R
0	-	0	3	R
0	-	-	1 See [i] below.	R
0	-	+	3	R
0	+	0	0	L
0	+	-	3	L
0	+	+	1 See [i] below.	L
-	0	0	3	R
-	0	-	2	R
-	0	+	3	R
-	-	0	3	R
-	-	-	By hypothesis and discussion [ii] below.	
-	-	+	3	R
-	+	0	2	R
-	+	-	2	R
-	+	+	1	L

+	0	0	3	L
+	0	-	3	L
+	0	+	2	L
+	-	0	2	L
+	-	-	1	R
+	-	+	2	L
+	+	0	3	L
+	+	-	3	L
+	+	+	By hypothesis and discussion [ii] below.	

[1] By symmetry we only consider 0 - - . We know $v_1^{(1)}+v_2^{(1)}+v_3^{(1)} < 0$ because we have a Cohn basis. We have for $b < 0$, $c < 0$: $bv_2^{(1)}+cv_3^{(1)} \geq -v_2^{(1)}-v_3^{(1)} \geq v_1^{(1)}$; $bv_2^{(1)}+cv_3^{(1)} \geq -v_2^{(1)}$; and $bv_2^{(1)}+cv_3^{(1)} \geq -v_3^{(1)}$, which yields the result.

[ii] By symmetry , we consider + + + only. Though by hypothesis, one of the following inequalities fails: $v_1^{(1)}+v_2^{(1)}+v_3^{(1)} < -X$ or $v_1^{(2)}+v_2^{(2)}+v_3^{(2)} < -y$ or $v_1^{(3)}+v_2^{(3)}+v_3^{(3)} < -z$. However, the question remains wether other vectors q = $dv_1+ev_2+fv_3$ for integers d, e, f of the + + + sign constraint are in the interior of P. Consider the following inequalities:

Inequalities II	1	$-X < dv_1^{(1)}+ev_2^{(1)}+fv_3^{(1)} < X$
	2	$-y < dv_1^{(2)}+ev_2^{(2)}+fv_3^{(2)} < y$
	3	$-z < dv_1^{(3)}+ev_2^{(3)}+fv_3^{(3)} < z$

Before beginning we introduce a trick notation: d > e = f will mean d[1] e[2] f[2]. We list the possible order relations among d,e,f.

Order Relations			Impossible Because q Does Not Satisfy The Inequality i(=1,2,3) of Inequalities II's Left(L) or Right(R) Side	
d[j]	e[k]	f[m]	i	Left or Right
j	k	m		
1	2	3	3	L
1	2	2	3	L
1	2	1	2	L
2	3	1	2	L
1	3	2	2	L
1	1	2	3	L
1	1	1	By hypothesis.	
2	2	1	2	L
2	1	3	3	L
2	1	2	3	L
3	1	2	1 See [j] below.	L
2	1	1	1 See [j] below.	L
3	2	1	1 See [j] below.	L

We consider 3 1 2 for the rest are similar. Let f=d+h and e=d+h+c. $dv_1^{(1)}+ev_2^{(1)}+fv_3^{(1)}=d(v_1^{(1)}+v_2^{(1)}+v_3^{(1)})+h(v_2^{(1)}+v_3^{(1)})+v_2^{(1)}$. Now, $v_1^{(1)}+v_2^{(1)}+v_3^{(1)} < 0$, by the Cohn basis hypothesis and $v_2^{(1)} < 0$ and $v_3^{(1)}+v_2^{(1)} < 0$. Now, $v_1^{(1)}+v_2^{(1)}+v_3^{(1)} < 0$, implies that $v_2^{(1)}+v_3^{(1)} < -v_1^{(1)}$. We have that $dv_1^{(1)}+ev_2^{(1)}+fv_3^{(1)} < -v_1^{(1)}$ and $h(v_2^{(1)}+v_3^{(1)}) < v_2^{(1)}$ or $h(v_2^{(1)}+v_3^{(1)}) < v_3^{(1)}$. We have that $dv_1^{(1)}+ev_2^{(1)}+fv_3^{(1)} <$

$v_2^{(1)}$ or $dv_1^{(1)}+ev_2^{(1)}+fv_3^{(1)} < v_3^{(1)}$. This completes the proof of this proposition 9.2.3. □

Proposition 9.2.4. If the vector v_4 is in the interior of P, where P is the same as in proposition 9.2.3, then the VOP K associated with v_4 is a VAOP.

Proof: Because of symmetry we consider only + + +.

Now for any vector $w = dv_1+ev_2+fv_3$, where d, e, f are positive integers, by the previous proof, we need only show that w is not in K, which is equivalent to showing that one of the inequalities i (=1,2,3) of inequalities III below fails.

Inequalities III

$$1 \quad -X < -v_4^{(1)} < dv_1^{(1)}+ev_2^{(1)}+fv_3^{(1)} < v_4^{(1)} < X$$
$$2 \quad -y < -v_4^{(2)} < dv_1^{(2)}+ev_2^{(2)}+fv_3^{(2)} < v_4^{(2)} < y$$
$$3 \quad -z < -v_4^{(3)} < dv_1^{(3)}+ev_2^{(3)}+fv_3^{(3)} < v_4^{(3)} < z \; .$$

In an analogous proof that used inequalities II above, we recall the order relations among the d, e,,f.

Order Relations			Impossible Because w Does Not Satisfy The Inequality i (=1,2,3) of Inequalities II's Left(L) or Right(R) Side and which particular value (X,y,...) it noes not satisfy.		
d[j]	e[k]	f[m]	i	Left or Right	Value
j	k	m			
1	2	3	3	L	-z
1	2	2	3	L	-z
1	2	1	2	L	-y
2	3	1	2	L	-y
1	3	2	2	L	-y

1	1	2	3	L	$-z$
1	1	1	1	L	$-v_4^{(1)}$
2	2	1	1	L	$-v_4^{(1)}$
2	1	3	3	L	$-z$
2	1	2	3	L	$-z$
3	1	2	1	L	$-v_4^{(1)}$
2	1	1	1	L	$-v_4^{(1)}$
3	2	1	1	L	$-v_4^{(1)}$

Combininig the facts derived from the inequalities I and II of the proof of proposition 9.2.3. and the facts of inequalities III, we have the proof of this proposition 9.2.4. □

Proposition 9.2.5 In a Cohn basis $[v_1, v_2, v_3, v_4]$, the three vectors v_1, v_2, and v_3 have rank matrices of diagrams 7.3.1 to 7.3.8, then there exists a vector from among the four vectors whose associated VOP is a VAOP.

Proof: We see that in each of these cases the rank matrices imply the existence of a vector from among v_1, v_2, v_3 which satisfy the constraints of proposition 9.2.3. □

Before proceding any further, let us reexamine the proof of proposition 9.2.3 and 9.2.4, but this time with $y' = \mathrm{mid}_i\ v_i^{(2)}$ substituted for y and see how our proof may fail. In other words, when does P', where P' is the VOP $\{(x'',y'',z'')\,|\, |x''| \leq X,\ |y''| \leq y',\ |z''| \leq z\}$, contain a lattice point in its interior. Only in inequality II 2 2 1 does the P' appear that it may have a point in its interior. Now let us reexamine proof of propositions 9.2.3 and 9.2.4, but this time with

$z' = \text{mid}_i\ v_i^{(3)}$ substitituted for z and again see how our proof may fail. In other words, when does P", where P" is the VOP $\{(x'',y'',z'') \mid |x''| \le X, |y''| \le y, |z''| \le z'\}$, appear that it may have a point in its interior. Only in inequality II 2 1 2 does it appear that P" may have a point in its interior. We now ask the question, can three vectors exist so that one of the vectors v_3 has a VOP of the type P' and another vector v_2 have a VOP of the type P" simultaneously. We shall construct all the possible rank matrices for such a set of three vectors.

It is clear that by studying the inequalities II, the components of v_2 must have in a rank matrix its value

	≤ 2		1		1		1		2		2	2
order	≤ 3	, that is,	1	or	2	or	3	or	1	or 2	or	3 .
	≤ 1		1		1		1		1		1	1

It is clear that by studying the inequalities II, the components of v_3 must have in a rank matrix its value

	≤ 2		1		1		1		2		2	2
order	≤ 1	, that is,	1	or	1	or	1	or	1	or 1	or	1
	≤ 3		1		2		3		1		2	3

Combining v_2 and v_3 component value orderings, we have that only the following 8 component order relationships make sense for v_2 and v_3.

	v_2	v_3		v_2	v_3		v_2	v_3		v_2	v_3
	1	2		1	2		1	2		1	2
1)	2	1	2)	2	1	3)	3	1	4)	3	1
	1	2		1	3		1	2		1	3

	v_2	v_3		v_2	v_3		v_2	v_3		v_2	v_3
	2	1		2	1		2	1		2	1
5)	2	1	6)	2	1	7)	3	1	8)	3	1
	1	2		1	3		1	2		1	3

Now adding to each of these 8 pairs the component orderings of v_1 (which will be automatically determined), the following matrices , followed by their equivalent rank matrices, results:

1) $\begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ Rank Matrix $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ 2) $\begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ Rank Matrix $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

3) $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ Rank Matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 3 & 2 \end{bmatrix}$ 4) $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ Rank Matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 2 \end{bmatrix}$

5) $\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ Rank Matrix $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ 6) $\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ Rank Matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 3 & 2 \end{bmatrix}$

					Rank Matrix					Rank Matrix
7)	$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$				$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}$	8)	$\begin{bmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$			$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 3 & 2 \end{bmatrix}$

We observe that none of these correspond to diagrams 7.3.9 or 7.3.10, which yields the following proposition.

Proposition 9.2.6. In a Cohn basis $\{v_1, v_2, v_3, v_4\}$, the three vectors v_1, v_2, and v_3 have rank matrices of diagrams 7.3.9 to 7.3.10, then there exists a vector from among the four vectors whose associated VOP is a VAOP. □

Combining proposition 9.2.5 and 9.2.6, we have theorem 9.2.1. □

9.3. Implications of the Interrelationship of AOP and Cohn Bases

It is easy to see that Cohn bases must generate units by theorem 9.2.1. Use the fact that we can creates an infinite number of VAOP by proposition 8.2.9. Voronoii's method assures that two are unit related. In a similar manner, we could have used the infinite number of MAOP generated by Cohn bases (proposition 8.2.9 and theorem 8.2.14) to show that Cohn bases can calculate units. Another implication of theorem 9.2.2 is that we may possibly use Cohn bases to calculate fundamental units via the generated VAOP. One final implication is that since theorem 9.2.2. is true in lower dimensions it may very well be true in higher dimensions. We may, therefore, be able to use this analogous theorem to calculate units in higher dimensions.

CONCLUSION

This dissertation has left many directions to be further pursued. In Part I, it would be valuable to create not only a method to locate an initial EFAOP, but also a test to find two unit related EFAOPs within a finite number of steps. It would also be appreciated, if a separate alogrithm for the edge-face neighboring process could be developed. Finally, it would be of great importance if any algorithm could be extended to higher dimensions.

In Part II, It would be important to show that any two Cohn bases can be connected by Cohn bases, which would complete the theory of Cohn's approach. It would be valuable to show if fundamental units can be calculated by means of the VAOPs generated by the Cohn basis. Also, to explore the basis approach for other such algorithms as Cohn's would be interesting. Also, it would be extremely important if a basis approach could be extended to higher dimensions. Finally, the solution of the combinatoric problem of the number K of n m-dimensional oriental parallelopipeds in the same m-dimensional irreducible lattice may shed light on the geometry of AOP and neighboring processes in higher dimensions.

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AUTOBIOGRAPHICAL STATEMENT

Daniel Aulicino was born on July 29, 1945 in Yonkers, New York. He attended the public schools of Yonkers. During the summer of his high school years, he attended Yeshiva University to study mathematics under a National Science Foundation grant. In 1963, he entered Columbia College in the City of New York on several scholarships. Upon graduation in 1967, he continued his graduate studies at C.C.N.Y. under a Meuhlstein Fellowship and National Science Foundation Assistantship. During this time, he published a paper on primitive roots in the American Mathematical Monthly and helped to edit a book on linear programming. After receiving his M.A. in 1969, he continued his studies at the C.U.N.Y. Graduate Center. During this time, he left his employment at C.C.N.Y. as a lecturer to join the faculty at LaGuardia Community College, where he received tenure as an Assistant Professor.

Presently, he resides in Yonkers with his wife, Anne. His non-mathematical interests include financial investment, music, collecting rare silver and travel.