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RENORMALIZATION, RIGIDITY, AND UNIVERSALITY IN BIFURCATION THEORY

by

Jun Hu

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, the City University of New York.

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Abstract

RENORMALIZATION, RIGIDITY, AND UNIVERSALITY IN BIFURCATION THEORY

by

Jun Hu

Adviser: Professor Dennis P. Sullivan

We aim to give a possible explanation why smooth one-parameter families of *threadlike mappings*, which pass from *simple* dynamical systems to *chaotic* dynamical systems, generically exhibit the asymptotic geometric rigidity in period-doubling bifurcation: there are infinite sequences $\{\mu_n\}$ of parameter values such that at $\{\mu_n\}$ there is a loss of a stable periodic trajectory of period 2^n and a rise of a stable periodic trajectory of period 2^{n+1}, and the ratios of adjacent parameter changes tend to a constant $4.6692\cdots$. We show that this statement is true in the space of smooth one-dimensional maps with finitely many critical points.

We also show that in the space of real-coefficient one-variable polynomials of degree $d \geq 1$, the set of the maps at the accumulations of period-doubling bifurcations forms the boundary of *chaos*, where the chaos is the set of the maps with *positive topological entropy*.

To my parents and my brother, to my wife Xia.

To the memory of my grandmother.

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Introduction

In the study of how dynamical systems depending on a parameter pass from a stable type of motion (called laminar) to an unstable type that appears turbulent, one finds an appearance of an infinite sequence $\{\mu_n\}$ of parameter values such that at μ_n there is a loss of a stable periodic trajectory of period 2^n and a rise of a stable periodic trajectory of period 2^{n+1}, and a more remarkable phenomenon where the bifurcation parameters tend to a critical value with an asymptotic constant ratio of adjacent parameter changes and after the critical value the turbulence appears. We refer to this phenomenon as the asymptotic geometric rigidity in period-doubling bifurcation. This discovery ([F], [CT]) lies on the boundary of mathematics and physics in the sense that the statement is rather mathematical but the approach to its solution is motivated by the renormalization-group method in theoretical physics. In this thesis we offer a possible explanation of this phenomenon by using the ideas of *threadlike mappings* and *renormalization* (see Conjecture and Main Theorem 1). Part of the techniques provide a method to investigate the microscopic structures of the dynamical systems at the accumulations of the period-doubling bifurcations (see theorem 0.6), and also give a way to describe the boundary of *chaos* in the space of *smooth* one-dimensional maps with finitely many critical points (see Main Theorem 2).

A *threadlike mappings* is a smooth endomorphism of R^n ($n \geq 1$) which coherently curls, in an arbitrary way, and folds, finitely many times, lines in one direction, sufficiently compresses the other $n-1$ transversal directions, and puts the images near one line in the preferred direction (for the precise definition, see section 1.2). For instance, one-dimensional smooth maps with finitely many critical points are threadlike mappings where the dimension $n = 1$.

A dynamical system f is *simple* if all points except countably many lower dimensional manifolds converge to an attracting periodic orbit of period 2^n for some n. A dynamical system f is *chaotic* if some iterate of f has a full horseshoe.

Conjecture *The asymptotic geometric rigidity in period-doubling bifurcation with constant ratio* $4.6692\cdots$ *generically exists in the families of uniformly sufficiently compressed threadlike mappings which pass from simple dynamical systems to chaotic dynamical systems.*

We presume that an approach to the conjecture requires an apriori verification and two main steps. The appriori verification is the generic existence of the period-doubling bifurcations in such families. The first main step is to reduce the proof of the conjecture to a special version (namely Main Theorem 1) which is to study the families of smooth one-dimensional maps with finitely many critical points. The second main step is to prove the Main Theorem 1.

We first give ideas in chapter 1 to explain why it is possible to carry out the first main step. Then we begin to prove the Main Theorem 1.

Theorem 0.1 (Main Theorem 1) *The asymptotic geometric rigidity in period-doubling bifurcation with constant ratio* $4.6692\cdots$ *generically exists in the families of smooth multimodal maps with finitely many critical points which pass from simple dynamical systems to chaotic dynamical systems.*

Ideas and techniques, developed by Sullivan [S1], are adopted to prove this theorem. And it is a long way to get to the end of the proof. It contains the work from chapter 2 to chapter 5.

In chapter 2 we show so-called real bound's properties for the renormalizations of smooth multimodal maps.

Let I be an interval. A map $f : I \rightarrow I$ is called *renormalizable* if there exists a proper subinterval J of I and an integer k such that (1) $f^i(J), i = 0, 1, \cdots, k-1$, have no intersection pairwisely except at endpoints, and (2) $f^k(J) \subset J$. Then $f^k|_J : J \rightarrow J$ is a renormalization of f. The permutation induced by f on the set $\{f^i(J) : i = 0, 1, \cdots, k-1\}$ is called the *combinatorial* type of the renormalization. Usually we take J to be the maximal subinterval satisfying (1) and (2) and denote $R(f) = f^k|_J$. A multimodal map f is a *smooth polynomial-like* map bounded by B if

$$f = P \circ h : I \rightarrow I,$$

where P is a real-coefficient polynomial and h is bounded by B, i.e., h is a diffeomorphism with $\phi = \log h'$ satisfying

$$|\frac{\phi(x) + \phi(y)}{2} - \phi(\frac{x+y}{2})| \leq B|x-y|, |\phi(x) - \phi(y)|^2 \leq B|x-y|$$

for any x, y in I.

Theorem 0.2 (Real bound's properties)

Let f be a smooth polynomial-like map bounded by B and infinitely renormalizable. Suppose $\{I_n : n \in N\}$ is the sequence of renormalization intervals. Then

(1) $\sum_{i=0}^{k(1)k(2)\cdots k(n)-1} |f^i(I_n)|$ *decreases geometrically fast.*

(2) The Lebesgue measure of the Cantor set

$$\cap_{n=1}^{\infty} \cup_{i=0}^{k(1)k(2)\cdots k(n)-1} f^i(I_n)$$

is 0*, and its Hausdorff dimension is greater than* 0 *and less than* 1 *if* $k(n)$ *is bounded.*

(3) ("beau" property) All renormalizations $R^n(f), n = 1, 2, \cdots$, *are smooth polynomial-like maps with a bound only depending on* B*. After a number of renormalizations (depending on* B*) further renormalizations are bounded universally.*

(4) Any C^0 *limit of* $\{f_n = R^n(f) : n \in N\}$*is a map of the form* $h \circ Q$ *where* Q *is a polynomial and* h^{-1} *has a complex analytic injective extension to the complex plane minus the complement of a universal neighborhood of* I *in the real line (such a map is said to be of* Epstein class*).*

In chapter 3, we show so-called complex bound's property.

Theorem 0.3 (Complex bound's property)

Let $f = h \circ P : I \to I$ *be an infinitely renormalizable map of Epstein class and* $k(n) \leq T$*. Then there exists* $n(T) \in N$ *and* $m(T) > 0$ *such that for any* $n \geq n(T)$*, the* n^{th} *renormalization* $g = R^n(f)$ *has a complex extension* G *to some neighborhood* U *in the complex plane so that* $G : U \to G(U)$ *is a polynomial-like mapping and the modulus of* $G(U) \setminus U$ *is greater than* $m(T)$*.*

The theory of polynomial-like mappings is also applied in this part. *Internal* classes and *external* classes are used to label the maps in the unstable manifold and the stable manifold of the renormalizaton operator at the fixed point. Therefore we know the codimensions of the stable manifolds, which are related to the number of the critical points (counted with multiplicity) involved in the renormalizations. For infinitely renormalizable unimodal maps one gets a codimension-1 stable manifold associated with it. For infinitely renormalizable bimodal maps one gets codimension-2 stable manifold, and so on.

Chapter 4 contains an introduction of the Teichmüller contraction principle [S1] of the renormalization operator acting on the stable manifolds.

Theorem 0.4 (Teichmüller Contraction Principle) [*S*1]

Assume F and G are two infinitely renormalizable maps of Epstein class, have the same bounded combinatorial types, and the closures of cirtical orbits of F and G are respectively same. Then

$$D(R^n(F), R^n(G)) \to 0$$

as $n \longrightarrow \infty$.

In chapter 5, we show the Main Theorem 1. We also study the set of smooth multimodal maps f whose set of periods

$$P(f) = \{2^n : n \in Z_+\} = \{1, 2, 3, \cdots, 2^n, \cdots\}.$$

We first show

Lemma 0.1 [*HT*] *All C^2 smooth maps f of an interval, with finitely many critical points and with set of periods $P(f) = \{2^n : n \in Z_+\}$, are infinitely renormalizable.*

In the space $C^r(I, I), r \geq 1$, we define *nonchaos* as the set of maps with only finite many periods (equivalent to simple maps), and *chaos* as the set of maps which has a periodic point of period not a power of 2 (equivalent to chaotic maps), and the *boundary* of the chaos is defined as the complement of these two sets. The second main result is

Theorem 0.5 (Main Theorem 2) *In the space of real-coefficient polynomials of degree d, the boundary of chaos has no interior point, i.e., nonchaos and chaos share the same boundary.*

We also prove the following theorem.

Theorem 0.6 (Coullet-Tresser rigidity) *All infinitely renormalizable maps, of Epstein class and same bounded combinatorial types, have corresponding critical orbital Cantor sets of same asymptotic scaling functions.*

Chapter 1

Threadlike Mappings

This chapter aims to study a type of high-dimensional maps (called *threadlike mappings*) whose dynamics strongly contract under iterations to one-dimensional maps. More precisely, threadlike mappings have lines in one important direction coherently curled and folded, (n-1) transversal directions sufficiently compressed, and the image lines lied near one of the preferred direction. We presume that smooth one-parameter families of such mappings generically exhibit the asymptotic constant ratios of adjacent parameter changes among period-doubling bifurcations and asymptotic rigid Cantor-set attractors for the dynamical systems at the accumulations of the period-doubling bifurcations (see Theorem 0.6 in the introduction).

In the context *renormalization* means replacing one dynamical system defined by f by the new dynamical system defined by the first return map of f restricted on a proper subdomain, and followed by a rescaling conjugacy. In this chapter we are going to see under the iterates of renormalization operator, a one-parameter family of threadlike mappings gets "closer" and "closer" to a family of singular endomorphisms of R^n which map a line in one direction to itself and map all parallel lines into the same line. There is evidence [YA] (also see [Ro] and [Fr]) for that period-boubling bifurcations generically exist in the families of threadlike mappings passing from *simple* to *chaotic* dynamical systems. The work of [CEK], which generalizes the structure of the renormalization operator near the fixed point [La] from the space of one-dimensional maps to the space of high dimensional maps, gives us the hint that the proof of the conjecture possibly reduces to show a special version (see Main Theorem 1 in the introduction).

1.1 Renormalization from the Viewpoint of Functional Analysis

Let us have a quick short trip in the theory of renormalization of maps from the viewpoint of functional analysis.

I. One-dimensional theory.

Let M be the space of real-analytic function $g : [-1,1] \rightarrow [-1,1]$, equipped with the supremum norm, satisfying

(1) g has a unique critical point at 0, and increases on $[-1,0)$ and decreases on $(0,1]$,

(2) $g(0) = 1$ and $g''(0) \neq 0$,

(3) $g(1) < 0$.

In M we define a functional operator $\mathcal{R}$ by

$$\mathcal{R}(g)(x) = g(1)^{-1} g \circ g(g(1)x).$$

The map $g \rightarrow \mathcal{R}(g)$ is called the renormalization operator for period doubling. The first main result of this theory is (for a nice review, see [CE]):

Theorem A [La] The functional operator $\mathcal{R}$ has a fixed point $\phi \in M$ satisfying:

1. ϕ is an analytic function defined in a neighborhood of $[-1, 1]$,
2. ϕ is a symmetric function, i.e., $\phi(x) = h(x^2)$, where h is an analytic homeomorphism,
3. The linearized operator $D\mathcal{R}_\phi$ is a compact operator with one eigenvalue δ of modulus > 1 and the remainder of the spectra being inside the open unit disk.

From the theory of invariant manifolds, there exists a local stable manifold $W^s_{loc}(\phi)$ of codimension 1 in a neighborhood of ϕ in M. Then all maps in $W^s_{loc}(\phi)$ belong to the domain of all iterates of $\mathcal{R}$. This implies that they must have the same topological dynamics, in other words, they have a periodic orbit of period 2^n for each n, a single invariant Cantor set on which the dynamics is well understood, and nothing else in the nonwandering set. Since maps in $W^s_{loc}(\phi)$ converge to the fixed point ϕ under the iteration of $\mathcal{R}$, it follows that the geometry of the invariant Cantor set is unique, i.e., for any map in $W^s_{loc}(\phi)$, we will observe asymptotically the same ratios when comparing successive scales of the invariant Cantor set. $W^s_{loc}(\phi)$ is the accumulation set of a sequence of local manifolds $W^{2^n}_{loc}(\phi)$ which corresponds to the bifurcation where a 2^n cycle gets unstable and a 2^{n+1} stable cycle is created.

Theorem A' [La] If a family $t \mapsto f_t$ is transverse to $W^s_{loc}(\phi)$, then

(1) The family $t \mapsto f_t$ has infinitely many bifurcation points which correspond to successive bifurcations from a stable period 2^n to a stable period 2^{n+1}.

(2) If $\{t_i\}_{i \in N}$ is the sequence of the bifurcation values in (1) then

$$\lim_{n \to \infty} \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = \delta$$

where $\delta = 4.669...$ does not depend on f_t.

So far, we have described the local dynamics of the period-doubling renormalization operator in a neighborhood of the fixed point ϕ. A global picture, which is deep, is given by Sullivan [S1]. Here we just mention a simple version of his result.

Theorem [S1] Any map in M, which possesses the topological dynamics of the accumulation of period doubling, converges to the map ϕ under iteration of the renormalization operator. In particular, the orbit of the map $x \longrightarrow 1 - a_{2^\infty} x^2$, which is at the accumulation of the period-doubling bifurcations of the quadratic family $f_a(x) = 1 - ax^2$, under the action of renormalization, converges to the fixed point ϕ.

Consequently, all maps in M have the same asymptotical ratios for the invariant Cantor set. In another word, topology implies geometry.

II. n-dimensional theory ($n \geq 2$).

For $\Delta > 0$, we define

$$D(\Delta) = \{(z_1, \vec{z_2}) \in C^n : \exists x_0 \in [-1,1] \; s.t. \; ||z_1 - x_0|| + ||\vec{z_2}|| < \Delta\}$$

and

$$R(\Delta) = D(\Delta) \cap R^n,$$

where C^n denotes the product of n complex planes and R^n denotes the product of n real planes.

Let $H(\Delta)$ be the set of analytic functions from $D(\Delta)$ to C^n mappimng the reals into the reals and let M^* be the subset of $H(\Delta)$ of maps of the form:

$$(x, \vec{y}) \rightarrow (g(x), \vec{0}),$$

where g is a map in M, we are going to consider the set $E_{\epsilon,\Delta}$ of maps of $H(\Delta)$ that are $\epsilon - close$, in the C^0 topology, to a map in M^* and whose restriction to $R(\Delta)$ is an embedding of $R(\Delta)$ into R^2.

The following property gives a nice bridge between the renormalizations of one-dimensional and n-dimensional maps.

Prop. 1.1 *If an unimodal map $f(x) = h(x^2)$ is fixed point of the renormalization operator*

$$\mathcal{R}(f) = \lambda^{-1} \circ f \circ f \circ \lambda,$$

where $\lambda(x) = \lambda x$, then $F_{\vec{\alpha}}(x, \vec{y}) = (h(x^2 - \vec{\alpha} \cdot \vec{y}), \vec{0})$ is a fixed point of the renormalization operator

$$\mathcal{R}(F_{\vec{\alpha}}) = \Lambda^{-1} \circ F_{\vec{\alpha}} \circ F_{\vec{\alpha}} \circ \Lambda$$

for any $\vec{\alpha}$, where $\Lambda(x, \vec{y}) = (\lambda x, \lambda^2 \vec{y})$.

Proof:

$$\mathcal{R}(F_{\vec{\alpha}})(x, \vec{y}) = \Lambda^{-1} \circ F_{\vec{\alpha}} \circ F_{\vec{\alpha}}(\lambda x, \lambda^2)\vec{y}$$

$$= \Lambda^{-1} \circ F_{\vec{\alpha}}(h(\lambda^2 x^2 - \lambda^2 \vec{\alpha} \cdot \vec{y}), 0)$$
$$= \Lambda^{-1}(h([h(\lambda^2 x^2 - \lambda^2 \vec{\alpha} \cdot \vec{y})]^2), 0)$$
$$= (\frac{1}{\lambda} h([h(\lambda^2(x^2 - \vec{\alpha} \cdot \vec{y}))]^2), 0)$$
$$= (h(x^2 - \vec{\alpha} \cdot \vec{y}), 0)$$

since

$$\mathcal{R}(f)(x) = \lambda^{-1} f \circ f(\lambda x) = \lambda^{-1} f(h(\lambda^2 x^2))$$
$$= \frac{1}{\lambda} h([h(\lambda^2 x^2)]^2)$$
$$= f(x) = h(x^2). \square$$

Remark: If the exponent of f at the turning point is $2n$, $n \in N$, then the rescaling factor for the second coordinate is λ^{2n}.

Thoerem B [CEK] For ϵ small, there is a map $\Phi \in E_{\epsilon,\Delta}$, and a local submanifold $W^s_{loc}(\Phi)$ (of codimension 1) such that

(1) If a continuously differentiable one-parameter family $t \mapsto F_t$ in $E_{\epsilon,\Delta}$ crosses transversally through W^s_{loc} then near Φ F_t has infinitely many bifurcations from a stable eriod 2^n to a stable period 2^{n+1}.

(2) If $\{t_i\}_{i \in N}$ is the sequence of the bifurcation values in (1) then

$$\lim_{n \to \infty} \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = \delta$$

where $\delta = 4.669...$ does not depend the family F_t.

We will see in section 1.3 that under the iterates of renormalization operator, a family of threadlike mapping (see section 1.2) gets "closer" and "closer" to a family of maps in $E_{\epsilon,\Delta}$.

1.2 Threadlike mappings

We first give the definition for a threadlike mapping on a n-dimensional space, $n \geq 1$.

Definition 1.1 *A differentiable endomorphism $F : R^n \to R^n$ is a (b, a, δ)-threadlike mapping, where $n \geq 1$, if it satisfies:*

(1) There exists a straight line $R^{(\Delta)}$ in R^n such that

$$f(F) = P_{R^{(\Delta)}} \circ F|_{R^{(\Delta)}} : R^{(\Delta)} \to R^{(\Delta)}$$

is a smooth map with finitely many critical points, where $P_{R^{(\Delta)}}$ is the projection to the line $R^{(\Delta)}$;

(2) There exists a box $B = [-1, 1]^n$ around the line $R^{(\Delta)}$ such that

$$F(B) \subset B, \ \ where \ \ [-1, 1] \times \{\vec{0}\} \subset R^{(\Delta)};$$

(3) Denote $R^n = R^{(\Delta)} \bigoplus R^{(\perp)}$, then

$$F : R^{(\Delta)} \bigoplus R^{(\perp)} \to R^{(\Delta)} \bigoplus R^{(\perp)} : (x, \vec{y}) \mapsto (u(x, \vec{y}), v(x, \vec{y}))$$

satisfies for any $x_1, x_2 \in [-1, 1]$,

$$|v(x_1, y) - v(x_2, y)| < b|x_1 - x_2|,$$

where $0 \leq b < 1$;

(4) For each critical point of $f(F)$ there exists $0 \leq \delta < 1$ such that for any cylinder $R_l = [-l, l] \times D_{n-1}$, where the radius of the disk D_{n-1} is l^k, where k is the exponent of $f(F)$ at the critical point, $F(R_l)$ is contained in

a cylinder with length $\leq |(P_{R(\Delta)})([-l,l] \times \{0\})|(1+\delta)$ *and the radius of the disk* $\leq b(2l)$*;*

(5) There exist infinite cylinders $S_{l_i} = [-l_i, l_i] \times R^{n-1}$, $i = 1, 2, \cdots, m$ *and* m *is the number of the critical points of* $f(F)$*, such that* F *compresses in the disk direction any horizontal cylinder outside* $\cup_{i=1}^{m} S_{l_i}$ *by a factor* $0 \leq a < 1$.

Remark 1: If a map F only satisfies the conditions (1)-(4), We still call F a (b, a, δ) threadlike mapping in the sense that $\cup_{i=1}^{m} S_{l_i} \supset B = [-1,1]^n$ hence $\cup_{i=1}^{m} S_{l_i} \setminus B = \emptyset$. In another word, the condition (5) automatically holds for any F. We put it in the definitions just for the brevity of language in later discussion.

Remark 2: In the definition, b, a and δ are invariant under conjugation by translations and linear maps with same scaling factor in every direction. In general, they are not invariant for affine conjugation. It seems to be reasonable for a nonlinear transformation defined on a n-dimensional space with $n > 1$.

Remark 3: The condition (4) in the definition is very technical, but it is invariant under renormalization, hence it is convenient in later discussion. The rectangle R_l is defined so because of the quadratic turning of the map $P_{(\Delta)} \circ F|_{R(\Delta)}$ at 0, the condition (4) also helps to control the horizontal size of renormalization box.

Examples of threadlike mappings:

0. All one-dimensional smooth maps with finitely many critical points are threadlike mappings.

1. $F : R \times R^{n-1} \to R \times R^{n-1} : (x, \vec{y}) \mapsto (f(x) + \vec{\alpha} \cdot \vec{y}, \vec{0})$

where $\vec{\alpha}$ is a vector in R^{n-1} with norm $||\vec{\alpha}|| < 1$ and f is a smooth map with finitely many critical points.

2. $F : R \times R^{n-1} \to R \times R^{n-1} : (x, \vec{y}) \mapsto (f(x + \vec{\alpha} \cdot \vec{y}), \vec{0})$

where $\vec{\alpha}$ and f are the same as above.

3. $F : R \times R^{n-1} \to R \times R^{n-1} : (x, \vec{y}) \mapsto (f(x), \vec{b}x)$

where $\vec{b}$ is a vector in R^{n-1} with norm $||\vec{b}|| < 1$ and f is an unimodal map with quadratic turning.

4. $F : R \times R^{n-1} \to R \times R^{n-1} : (x, \vec{y}) \mapsto (f(x) + \vec{\alpha} \cdot \vec{y}, \vec{b}x)$

where $\vec{\alpha}$ and $\vec{b}$ are two vectors in R^{n-1} with norms < 1 and f is an unimodal map wtih quadratic turning.

5. Modified Henón map

$$H : R \times R \to R \times R : (x, y) \mapsto (1 - \mu x^2 + ay, bx)$$

where $|\mu| < 2$, $|a| < 1$ and $|b| < 1$.

1.3 Renormalization of threadlike mappings

Intuitively, threadlike mappings are thickenings of one-dimensional maps embedded into high dimensional space. Their dynamics are naturally related to the dynamics of one-dimensional systems, especially in finite time. Threadlike mappings become renormalizable if it is close to a renormalizabl one-dimensional system. The interesting thing is that renormalizations of renormalizable threadlike mappings are more threadlike, in another word, they get closer to one-dimensional systems. We are going to treat the threadlike mappings F with one critical point for $f(F)$. The results are also valid

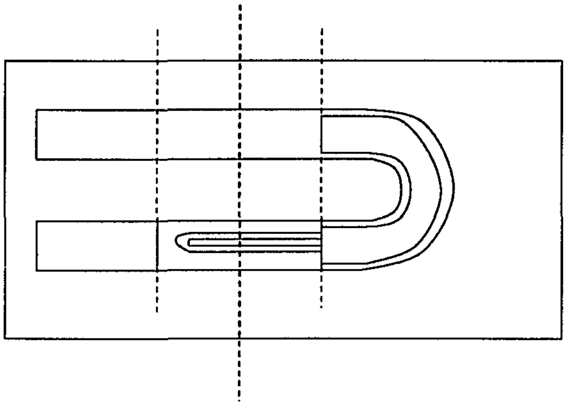


Figure 1.1

for the threadlike mappings F with finitely many critical points for $f(F)$. Their proofs are finitely many compositions of the following steps.

In section 1.1, for ϵ sufficiently small, there exists in $E_{\epsilon,\Delta}$ a ball B_{ϕ^*} surrounding the map $\phi^* : (x,y) \to (\phi(x), 0)$ in which the real dynamics is as follows:

Any map G in B_{ϕ^*} possesses a unique fixed point P_G which is hyperbolic and of saddle type. Let W_1^s be the connected component of the stable manifold of G at P_G in $R(\Delta)$. There exists another component of the stable manifold W_1^s on the left of W_1^s such that W_2^s is mapped into W_1^s.

Consider now the rectangle R_1, containing the fixed point P_G on its boundary, whose vertical sides are included in W_2^s and W_1^s and whose horizontal sides are determined by the image of the boundary of $R(\Delta)$. The ball B_{ϕ^*} is chosen so that $G \circ G(R_1) \subset R_1$, and consequently we can define a renormalization operator

$$\mathcal{N} : B_{\phi^*} \to H(\Delta) : G \mapsto (G) = \Lambda^{-1} \circ G \circ G \circ \Lambda,$$

where Λ is an affine map sending the boundary of $R(\Delta)$ to the boundary of

R_1.

Definition 1.2 *A threadlike mapping F is renormalizable if there exists a box R and an integer n such that*

(1) $F^j(R)$, $j = 0, 1, 2, \cdots, n-1$, are disjoint except boundaries,

(2) $F^n(R) \subset R$.

Definition 1.3 *Let f be a renormalizable smooth quadratic-like one-dimensional map and I is the domain of the renormalization of f. A (b, a, δ)-threadlike mapping matches with f if*

(1) $P_{R^{(\Delta)}} \circ F|_{R^{(\Delta)}} = f$, and

(2) in the condition (5) of the definition of a threadlike mapping, the interval $[-l_0, l_0]$, which is the projection of S_{l_0} to $R^{(\Delta)}$, at most contains I but contains no other iterates of I under f before the first return.

Lemma 1.1 *Let $T > 0$ and $B > 0$. There exists $\bar{b}$, $\bar{a}$ and $\bar{\delta} > 0$ (only depending on T and B) such that for any $0 \leq b < \bar{b}$, $0 \leq a < \bar{a}$ and $0 \leq \delta < \bar{\delta}$, a (b, a, δ)-threadlike mapping F is renormalizable if it matches with a smooth quadratic-like map f bounded by B (defined in Chapter 2) with period of renormalization bounded by T.*

Proof: Clearly f has the Real Bound's properties shown in Chapter 2. Let λ_0 be the upper bound for the ratio of the scales of the renormalization domain I and the domain $[-1, 1]$ of f.

The renormalization cylinder for F is $I \times D$, where the radius of D is the square of the radius of I.

Let us modify a little bit the renormalization domain I of f (compared to what we do in Chapter 2). We used to have one of the end points of I fixed by $\mathcal{R}(f)$, which is also a hyperbolic fixed point of $\mathcal{R}(f)$. Now we shrink I symmetrically by a definite factor from both sides. Still denote it by I. Then $\mathcal{R}(f)$ maps I into I and leaves at least δ_0-proportional gaps on both sides. Then F maps $I \times D$ into a cylinder with size of $|P_{R(\Delta)} \circ F)(I \times \{\vec{0}\})|(1+\delta)$ cross a disk D_1 of radius $\leq b\frac{|I|}{2} \leq b\lambda_0$.

Let n be the period of the renormalization of f.

Since the composition of f's, after the iterate till the first return, has bounded nonlinearity (only depending on B and T), there exists a constant M (only depending on B and T) such that F^{n-1} maps $F(I \times D)$ into a cylinder with size $|f^n(I)|(1 + M\delta) \times D_n$, where D_n has a very small radius because of the strong compression in the disk direction, in fact the radius of D_n is less than $a^{n-1}b\lambda$.

Let $a = a_0\lambda_0$ and $b = b_0\lambda_0$, where $0 \leq a_0 < \lambda_0$ and $0 \leq b_0 < \lambda_0$. Then

$$a^{n-1}b\lambda_0 = (a_0\lambda_0)^{n-1}b_0\lambda_0\lambda_0 = a_0^{n-1}b_0\lambda_0^{n+1} \leq a_0b_0\lambda_0^3$$

since $n \geq 2$.

Let δ be small enough such that $M\delta < \delta_0$, then

$$F^n(I \times D) \subset I \times D.\square$$

Corollary 1.1 *When parameters have very small norms, all examples of threadlike mappings given in the section 1.2 are renormalizable if related one-dimensional systems are renormalizable.*

Theorem 1.1 *The renormalization $\mathcal{R}(F)$ of a threadlike mapping in Lemma 1.1 becomes more threadlike in the sense that the contraction factors b and a for the renormalization are smaller. But the factor δ for the renormalization usually gets larger. After rescaling by $\Lambda(x,\vec{y}) = (\lambda x, \lambda^2\vec{y})$, where $\lambda = \frac{|I|}{2}$, the factors b and a for $\Lambda^{-1} \circ \mathcal{R}(F) \circ \Lambda$ are still smaller than the factors b and a for f, and the factor δ for $\Lambda^{-1} \circ \mathcal{R}(F) \circ \Lambda$ remains the same as for $\mathcal{R}(F)$. Furthermore the smaller are a and b, the more threadlike is the renormalization $\mathcal{R}(F)$ (or δ for $\Lambda^{-1} \circ \mathcal{R}(F) \circ \Lambda$).*

Proof: From the proof of Lemma 1.1,

$$b(\mathcal{R}(F)) \approx a^{n-1}b\lambda, \quad \frac{b(\mathcal{R}(F))}{b} = a^{n-1}\lambda < 1.$$

Denote $a = a_0\lambda_0$ and $b = b_0\lambda_0$. Then

$$b(\mathcal{R}(F)) \approx a_0^{n-1}b_0\lambda_0^{n+1}.$$

Hence

$$b(\Lambda^{-1} \circ \mathcal{R}(F) \circ \Lambda) \approx \frac{b(\mathcal{R}(F))}{\lambda_0} \approx a_0^{n-1}b_0\lambda_0^{n}.$$

Then

$$\frac{b(\Lambda^{-1} \circ \mathcal{R}(F) \circ \Lambda)}{b} \approx \frac{a_0^{n-1}b_0\lambda_0^{n}}{b_0\lambda_0} = a_0^{n-1}\lambda_0^{n-1} < 1$$

since $n \geq 2$.

If the factor a exists for $\mathcal{R}(F)$, then

$$a(\mathcal{R}(F)) = a^n, \quad \frac{\mathcal{R}(F)}{a} = a^{n-1} < 1$$

since $n \geq 2$ and $0 \leq a < 1$. After rescaling,

$$\frac{a(\Lambda^{-1} \circ \mathcal{R}(F) \circ \Lambda)}{a} \approx \frac{a^2/\lambda_0^2}{a}$$

$$= \frac{a}{\lambda_0^2} = \frac{a_0\lambda_0}{\lambda_0^2} = \frac{a_0}{\lambda_0} < 1$$

since $a_0 < \lambda_0$. Clearly

$$\delta(\Lambda^{-1} \circ \mathcal{R}(F) \circ \Lambda) = \delta(\mathcal{R}(F)) \leq M\delta. \ \Box$$

Remark: For the renormalization of a threadlike mapping F, the condition (5) in the definition can be weakened. Because of this reason, further renormalization can be carried on for the renormalization $\mathcal{R}(F)$ of F even though $\mathcal{R}(F)$ doesn't satisfy all conditions in Lemma 1.1. For instance, example 1, 2 and 3 in the section 1.2 are infinitely renormalizable if $\vec{a}$ and $\vec{b}$ have small norms and f is infinitely renormalizable.

Lemma 1.2 *Under iterates of renormalization operator (if it can be done over and over), the factor δ, induced for the renormalizations of a threadlike mappings F, converges to a constant which is less than* 1.

Proof: Let f be the one-dimensional map matched with F. from one of the Real Bound's Properties of the renormalization of one-dimensional maps, under iterates of renormalization operator, the nonlinearity (i.e., logarithm of the ratio of derivatives at two points) of the composition, after the first iterate till the first return, under the map f becomes smaller and smaller, hence the proportions of the gaps at both sides of the modified dynamical intervals remains almost same in deep renormalizations. $\Box$

Remark: Theorem 1.1 and Lemma 1.2 have shown partially why the renormalizations of a map in $W^s_{loc}(\Phi)$ converges to the singular map Φ of the form described in Lemma 1.1.

From Theorem 1.1 and Lemma 1.2, we get

Theorem 1.2 *There exists $\bar{b}$, $\bar{a}$ and $\bar{\delta}$ such that if a (b, a, δ)-threadlike mapping F, where $0 \leq b < \bar{b}$, $0 \leq a < \bar{a}$ and $0 \leq \delta < \bar{\delta}$, is infinitely renormalizable and its matched one-dimensional maps only have period-doubling periodic orbits, then the sequence of renormalizations $\mathcal{R}^n(F)$ converges to a map F_α of the form*

$$F_\alpha(x, \vec{y}) = (h(x^2 - \vec{\alpha} \cdot \vec{y}), \vec{0}),$$

where $f(x) = h(x^2)$ is the fixed point of the renormalization operator on the space of unimodal maps and $\vec{\alpha}$ is a $(n-1)$ vector.

1.4 Existence of Period-Doubling Bifurcation

When a horseshoe is created in *a natural manner* (see [YA]) as a parameter is varied, the process involves the appearance of attracting periodic orbits of period 2^n for all $n \in N$, hence there exists a parameter value which is at the accumulation of period-doubling bifurcations. This statement is proved to be true for any continuous family of smooth one-dimensional maps no matter how the horseshoe is created [BH]. One expects this property for some family of threadlike mappings. So we make the following conjecture.

Conjecture 1 *Continuous one-parameter families of uniformly controlled threadlike mappings, which possess two parameter values where the dynamics are respectively simple (all points converge to an attracting fixed point) and chaotic (it has a homoclinic point), generically exhibit a cascade of period-doubling bifurcations as the parameter is varied from simple to chaotic dynamics.*

Theorem 1.3 *Under the iterates of renormalization operator, a continuous one-parameter families of uniformly controlled threadlike mappings, which possesses two parameter values where the dynamics are respectively simple and chaotic, gives rises of new families of more and more threadlike mappings, which also exhibit simple and chaotic dynamics.*

If one can show that Theorem A (or A') ([La]) is the generic structure picture for the renormalization operator on the space M of smooth one-dimensional maps with finitely many critical points. Then it is natural to see that Theorem B ([CEK]) is the generic structure picture for the renormalization operator on the space $E_{\epsilon,\Delta}$ of high dimensional maps which are close to the one-dimensional maps in M. This idea and Theorem 1.3 reduce the proof of the main conjecture in the introduction to study the renormalization on the space M. It is set up in the following four chapters.

A consequence of Conjecture 1 and Theorem 1.1 and 1.2 is as follows:

There exist $\bar{b}$, $\bar{a}$ and $\bar{\delta} > 0$ such that any smooth one-parameter family of (b, a, δ)-threadlike mappings, where $0 \leq b < \bar{b}$, $0 \leq a < \bar{a}$ and $0 \leq \delta < \bar{\delta}$, which possesses simple and chaotic dynamics, has a cascade of period-doubling bifurcations with asymptotic constant ratio of two successive changes of parameters among the bifurcations, which is the same as what happens to the quadratic family $f_a = 1 - ax^2$, $0 \leq a \leq 2$.

Henón family is a simple model of a family of two-dimensional diffeomorphisms with very rich and complicated dynamics. Although one cannot apply the results of this chapter to the Henón family directly, we believe, as a matter of fact, that the techniques are useful in the study of this family.

In the following, we are going to make another conjecture with a slightly different family, which is relatively simpler than the Henón family from the viewpoint of renormalization, given by the formular

$$H_{\mu,a,b}(x,y) = (1 - \mu x^2 + ay, bx).$$

From Conjecture 1, there exists $\bar{a} > 0$ and $\bar{b} > 0$ such that for any $0 \leq a < a_0$ and $0 \leq b < b_0$ there exists μ_∞ such that $H_{\mu_\infty,a,b}(x,y)$ has and only has all period-doubling periodic orbits, and $H_{\mu_\infty,a,b}$ is an infinitely renormalizable threadlike mapping. Hence another consequence of Conjecture 1 is the existence of Kupka-Smale phenomenon, i.e., there exists a C^∞-diffeomorphism from sphere S^2 into itself with neither sinks nor sources and all periodic orbits are hyperbolic points are transversal.

We also know from [La] and [S1] that when $a = b = 0$, there is one and only one μ_∞ such that $H_{\mu_\infty,a,b}$ has and only has all period-doubling periodic orbits.

Conjecture 2 *There exists $\bar{a} > 0$ and $\bar{b} > 0$ such that for any $0 \leq a < a_0$ and $0 \leq b < b_0$ there exists one and only one μ_∞ such that $H_{\mu_\infty,a,b}$ has and only has all period-doubling periodic orbits.*

The study of threadlike mappings may give a possible way to attack this conjecture.

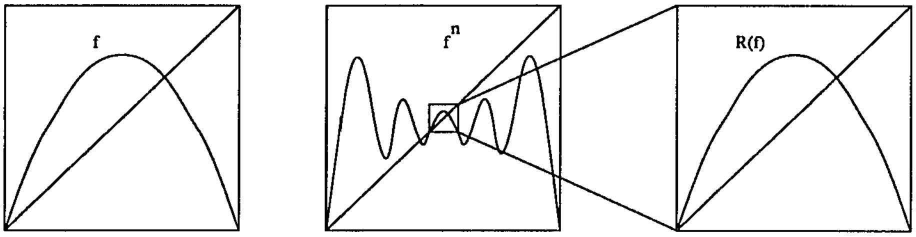
Chapter 2

Real Bound's Properties

The original idea of renormalization was introduced into one-dimensional dynamics independently by Feigenbaum [F] and Coullet-Tresser [CT] to explain some quantitative and universal phenomena appearing in the bifurcations of one-parameter families of unimodal maps. Recently new techniques developed by Dennis Sullivan [S1] are very helpful in studying the subject. In this chapter we will adopt the techniques of [S1] to find real bound's properties of renormalizations of smooth multimodal maps.

2.1 Introduction

Beginning with a folding mapping f of an interval I (figure 1a), if there exists a smaller interval $I_1 \subset I$ about the turning point of f such that $f^n(I_1) \subset I_1$ for some natural number n, then the restriction of f^n on I_1 is again a folding mapping of the interval I_1. After rescaling, we get a new folding mapping of the interval I, denoted by $f_1 = Rf$ (figure 1b). This procedure defines a renormalization operator from the space of renormaliz-



1a 1b

Figure 2.1

able folding mappings into the space of the folding mappings. A map f is infinitely renormalizable if the renormalization operator can be used on f and its image forever.

Let the length of interval I be the scale of the renormalization, the first fact about the infinitely renormalizable unimodel map is that the scales of the renormalizations decrease geometrically fast. We will first show this property is also true for smooth multimodal maps, denoted by Γ, which will be precisely defined as follows.

Definition 2.1 *Let I be a closed interval of R^1. A continuous map $h : I \longrightarrow R^1$ is of bounded Zygmund variation if there exists $B > 0$ such that*

$$\sup_{\{x_0,x_1,\cdots,x_n\}} \sum_{i=0}^{n-1} |h(x_i) + h(x_{i+1}) - 2h(\frac{x_i + x_{i+1}}{2})| \leq B,$$

where $\{x_0, x_1, \cdots, x_n\}$ is a partition of the interval I.

Definition 2.2 *Let I be a closed interval of R^1. A continuous map $h : I \longrightarrow R^1$ is of bounded quadratic variation if there exists $B > 0$ such that*

$$\sup_{\{x_0,x_1,\cdots,x_n\}} \sum_{i=0}^{n-1} (h(x_{i+1}) - h(x_i))^2 \leq B,$$

where $\{x_0, x_1, \cdots, x_n\}$ is a partition of the interval I.

Definition 2.3 *We say a map* $f \in C^{1+b.Z.v+b.q.v}$ *if* f *is* C^1 *smooth and the logarithm of its derivative,* $\log f'$ *has bounded Zygmund variation and bounded quadratic variation.*

Definition 2.4 *A map* $f : I \to I$ *belongs to the class* Γ *if*

a) f *is* $C^{1+b.Z.v+b.q.v}$ *away from turning points;*

b) Let K_f *be the set of the turnig points of* f*. For every* $x_0 \in K_f$*, there exists* $\alpha > 1$*, a neighborhood* $U(x_0)$ *of* x_0 *and a* $C^{1+b.Z.v+b.q.v}$*-diffeomorphism* $\phi : U(x_0) \to (-1, 1)$ *such that* $\phi(x_0) = 0$ *and*

$$f(x) = f(x_0) \pm |\phi(x)|^\alpha, \quad \forall x \in U(x_0).$$

Clearly, f has only finite many turning points.

The renormalization of a multimodel map is given by the following.

Definition 2.5 *Let* I *be an interval. A map* $f : I \to I$ *is renormalizable if there exists a proper subinterval* J *of* I *and an integer* p *such that*

(1) $f^i(J), i = 0, 1, \cdots, p-1$*, have no pairwise interior intersection,*

(2) $f^p(J) \subset J$*.*

Then $f^p|_J : J \to J$ *is called a renormalization of* f *and* p *is called the period of the renormalization. The permutation induced by* f *on the set* $\{f^i(J) : i = 0, 1, \cdots, p-1\}$ *is called the combinatorial type of the renormalization.*

Usually we take J to be the maximal proper subinterval satisfying (1) and (2), and denote $R(f) = f^p|_J$.

Definition 2.6 *Let* I *be an interval. A map* $f : I \to I$ *is infinitely renormalizable if there exists an infinite sequence* $\{I_n : n \in N\}$ *of nested renormalization intervals and an infinite sequence* $\{p(n) : n \in N\}$ *of periods of the renormalizations such that*

$$\cap_{n=1}^{\infty} \cup_{i=0}^{p(1)p(2)\cdots p(n)-1} f^i(I_n)$$

is a Cantor set.

Remark: The existence of C^2-smooth infinitely renormalizable multimodal maps is proved in [HT] (also see section 5.3).

Theorem 2.1 *Let* $f \in \Gamma$ *be infinitely renormalizable. Then*

$$\sum_{i=0}^{p(1)p(2)\cdots p(n)-1} |f^i(I_n)|$$

decreases geometrically fast as $n \to \infty$*.*

Theorem 2.2 *Let* $f \in \Gamma$ *be infinitely renormalizable. Then the Lebesgue measure of the Cantor set*

$$\cap_{n=1}^{\infty} \cup_{i=0}^{p(1)p(2)\cdots p(n)-1} f^i(I_n)$$

is 0*, and its Hausdorff dimension is greater than* 0 *and less than* 1 *if* $p(n)$ *is bounded.*

Definition 2.7 *A continuous function* $\phi : R^1 \to R^1$ *satisfies Zygmund (resp. zygmund) condition if*

$$|\frac{\phi(x) + \phi(y)}{2} - \phi(\frac{x+y}{2})|$$

is a big O (resp. a little o) of $|y-x|$*, where* x *and* y *are two points in the real line. The best constant corresponding to the big O will be called the Zygmund norm of* ϕ*.*

Definition 2.8 *A multimodal map* f *is called a Zygmund-smooth (resp. zygmund-smooth) polynomial-like map if*

$$f = P \circ h : I \to I,$$

where P *is a real-coefficient polynomial and* h *is a* C^1 *diffeomorphism with* $\log h'$ *satisfying Zygmund (resp. zygmund) condition.*

The following theorem is proved by Dennis P Sullivan for unimodal maps, the general cases are proved in this paper.

Theorem 2.3 ("beau" property) *Let* f *be a Zygmund-smooth polynomial-like map and infinitely renormalizable. Then all renormalizations* $R^n(f)$*,* $n = 1, 2, \cdots$*, are Zygmund-smooth polynomial-like maps with Zygmund norms which have a bound only depending on the Zygmund norm* B *of* $\log h'$*. After a number of renormalizations (depending on* B*) further renormalizations have a universal Zygmund bound.*

Theorem 2.4 *Let* f *be a Zygmund-smooth polynomial-like map and infinitely renormalizable with* $\{p(n)\}_{n=1}^{\infty}$ *bounded. Then any* C^0 *limit of* $\{f_n = R^n(f) : n \in N\}$ *is a map* $\tilde{f}$ *of the form*

$$H_m \circ P_m \circ \cdots \circ H_2 \circ P_2 \circ H_1 \circ P_1 \circ \Lambda$$

satisfying

1) $m \leq 2^d - 1$, *where* d *is the modality of* f,

2) $P_i = a_i(x - b_i)^{2d_i} + c_i$ *for an integer* d_i *and three reals* a_i, b_i *and* c_i, *where* $i = 1, 2, \cdots, m$,

3) H_i^{-1} *has a complex analytic injective extension to the complex plane minus the complement of a definitely scaled neighborhood of an interval, which contains a critical value of the map, in the real line, where* $i = 1, 2, \cdots, m$.

Definition 2.9 *Any map* $\tilde{f}$ *of the form*

$$H_m \circ P_m \circ \cdots \circ H_2 \circ P_2 \circ H_1 \circ P_1 \circ \Lambda$$

satisfying the conditions in the Theorem 2.4 will be said to belong to Epstein Class.

All theorems will be proved after the following "Gap" Lemma is set up.

Lemma 2.1 ("Gap" Lemma) *There exists* $\epsilon > 0$ *such that for any* $f \in \Gamma$, $f(\partial I) \subset \partial I$, *there exists* $\delta > 0$, *such that* $\forall n \in N$, *if* $I_0, i = 0, 1, 2, \cdots, n-1$, *satisfy*

(1) $I_i = f(I_{i-1}), i = 1, 2, \cdots, n-1$, *and* $f^n(I_0) \subset I_0$,

(2) $\max_{i=0}^{n-1}\{|I_i|\} < \delta$, *and*

(3) at least one critical point is in the interior of $\cup_{i=0}^{n-1} I_i$,

then

$$\frac{\sum_{i=0}^{n-1} |I_i|}{|I|} \leq \frac{1}{1+\epsilon},$$

where ϵ *is independent of* f *and* n.

Remark: Actually, replacing each critical interval by the smallest symmetric interval around the critical point containing the critical interval, the above inequality is still true.

The main tool used to obtain the "Gap" Lemma is the Macroscopic Koebe Principle which has been studied by J. Guckenheimer [G3], W. de Melo and S. J. van Strien [MvS], M. Martens [Ma], A. M. Blokh and M. Yu. Lyubich [BloLy] and etc..

Remark: The condition $f(\partial I) \subset \partial I$ can be omitted in the theorem since we can apply the technique used in the proof of the "Gap" Lemma directly into the proof of the theorem.

2.2 Macroscopic Koebe Distortion Principle

First we introduce cross-ratio distortion and some results related to bounded cross-ratio distortions.

Definition 2.10 *Let T be a bounded closed interval, M be a subinterval of T and $T \setminus M$ consists of nontrivial intervals L and R, the cross ratio of intervals L, M, R and T is defined as*

$$D(M,T) = \frac{|M||T|}{|L||R|},$$

where $|I|$ denotes the length of an interval I.

If $f : T \to R$ is continuous and monotone, the cross-ratio distortion of f on the intervals L, M, R and T is defined as

$$B(f,M,T) = \frac{D(f(M),f(T))}{D(M,T)}.$$

If $f : I \to I$ is continuous, $M \subset T$ as above and $f^n|_T$ is monotone, then

$$B(f^n,M,T) = \prod_{i=0}^{n-1} B(f,f^i(M),f^i(T)).$$

Prop. 2.1 *Any Möbius transformation preserves the cross ratio.*

Definition 2.11 *The Schwarzian derivative of a C^3-map $f : I \longrightarrow R^1$ is defined as*

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

where

$$f'(x) \neq 0 .$$

Prop. 2.2 *The schwarzian derivative of $f : I \longrightarrow R^1$ is identically equal to zero if and only if f is the restriction of a Möbius transformation on I.*

Prop. 2.3 *If f and g are composable C^3-diffeomorphisms, then*

$$S(g \circ f) = S(g) \circ f \; (f')^2 + S(f) .$$

Prop. 2.4 *If $f : I \longrightarrow R^1$ has negative (resp. positive) Schwarzian derivative, ϕ and ψ are Möbius transformations, then $f \circ \phi$ and $\psi \circ f$ also have negative (resp. positive) Schwarzian derivatives.*

Prop. 2.5 (Singer's lemma) *If $f : I \longrightarrow R^1$ is C^3-monotone and $S(f) < 0$, then $|f'|$ can not have a positive minimum in the interior of I.*

Proof: If $f' > 0$, and $f'' = 0$, then $S(f) < 0$ implies $f''' < 0$. This contradicts with the minimum property.

Corollary 2.1 *If $f : I \longrightarrow R^1$ is C^3-monotone and $S(f) \leq 0$ but not identically equals to zero on I, then $|f'|$ has minimum at the end point of interval I.*

Prop. 2.6 *If $f : I \to R^1$ is a C^3-diffeo with negative Schwarzian deirvative, then*

$$B(f, M, T) \geq 1$$

for all intervals $M \subset T \subset I$.

Proof: Let $T = [x_0, x_1], M = [y_0, y_1]$. Let ϕ be a Möbius transformation such that $\phi \circ f$ fixes the endpoints of T and y_0. We claim that $(\phi \circ f)(y_1) \geq y_1$. In fact, suppose that $(\phi \circ f)(y_1) < y_1$, by mean value theorem, there exists $z_0 \in [x_0, y_0], z_1 \in [y_0, y_1]$ and $z_2 \in [y_1, x_1]$ such that

$$(\phi \circ f)'(z_0) = 1, \quad (\phi \circ f)'(z_1) < 1 \quad \textit{and} \quad (\phi \circ f)'(z_2) > 1.$$

They contradict with the Singer's lemma. Hence $(\phi \circ f)(y_1) \geq y_1$, thus

$$B(\phi \circ f, M, T) \geq 1.$$

So

$$B(f, M, T) = B(\phi \circ f, M, T) \geq 1.$$

By Corollary 2.1 and Prop. 2.6, the following is also true.

Corollary 2.2 *If $f : I \to R^1$ is C^3-diffeo with non-positive Schwarzian derivative, then*

$$B(f, M, T) \geq 1$$

for any $M \subset T \subset R^1$.

Notice that if f has nonpositive Schwarzian derivative, then $|f'|$ has the minimum at the end points of the interval and its cross-ratio distortion is bounded from below. What kind of property does $|f'|$ have if f is only C^1-diffeomorphism and the cross-ratio distortion is bounded from below? It is the Minimum Principle. Furthermore if a C^1-diffeomorphism of an interval has its cross-ratio distortion bounded from below (resp. from above) by a positive constant, then its inverse (resp. it) behaves regularly in some sense, i.e., the ratio of the derivatives at two points in a proper subinterval will be bounded from below and from up. The complete statement is the Koebe Principle. In the following content the notation Df means the derivative of f, i.e., f'.

Theorem 2.5 (Minimum Principle) *Let* $f : I = [a,b] \to R^1$ *be a* C^1-*diffeomophism, if*

$$B(f,M,T) \geq c > 0$$

for any $M \subset T \subset I$*, then*

$$|Df(x)| \geq c^3 min\{|Df(a)|, |Df(b)|\}$$

for any $x \in I = [a,b]$.

Proof: See page 275 of [MvS].

Remark: If $B(f,M,T)$ has a bound from above, then $|Df^{-1}(x)|$ has the similar low bound, hence $|Df(x)|$ has a upper bound.

For maps with negative Schwarzian derivative, one version of the next principle was first used and proved by S. J. van Strien [St] and later rediscovered by S. Johnson and J. Guckenheimer [JG]. This principle states that the maps having bounds on the cross-ratio distortions have bounded non-linearity in a proper subdomain.

Theorem 2.6 (Real Koebe Principle I) *For each* $c > 0, \tau > 0$, *there exists* $K(c,\tau) < \infty$ *with the following property: Let* $f : I \to R^1$ *be* C^1-*diffeomorphism satisfying*

$$B(f, M^*, T^*) \geq c$$

for any intervals $M^* \subset T^* \subset I$, *then for any intervals* $M \subset T \subset I$, *let* L *and* R *be the components of* $T \setminus M$, *if* $\frac{|f(L)|}{|f(M)|} \geq \tau$ *and* $\frac{|f(R)|}{|f(M)|} \geq \tau$ *then*

$$\frac{1}{K(c,\tau)} \leq \frac{|Df(x)|}{|Df(y)|} \leq K(c,\tau)$$

for any $x, y \in M$. *Moreover* $K(c,\tau)$ *tends to 1 as* $c \to 1$ *and* $\tau \to \infty$.

Proof: See page 277 of [MvS].

Till now we have seen that some good nonlinearity estimates appear if bounds on the cross-ratio distortion are known. Roughly speaking to control a dynamical system is equivalent to control the nonlinearity under the iterate of the system. Hence it is very important to control cross ratio distortion of high iterates of a smooth map.

Next in this paper we will control cross ratio distortion for standard 4-tuples in terms of Zygmund variation and quadratic variation. The description may be very analytic, but involved ideas are extremely geometric. We will also give a simple proof of a version of the Real Koebe Principle.

Let $a, b, c, d \in R^1$ and $a < b < c < d$.

Another cross ratio $[a,b,c,d] = \frac{(d-b)(c-a)}{(c-b)(d-a)}$ can be computed by

$$log[a,b,c,d] = \int\int_S \frac{dxdy}{(x-y)^2},$$

where S is $\{(x,y) : a \leq x \leq b, c \leq y \leq d\}$.

The cross ratio (a,b,c,d), which we consider in the above, is $\frac{(c-b)(d-a)}{(b-a)(d-c)}$ and, obviously,

$$[a,b,c,d] = 1 + \frac{1}{(a,b,c,d)}.$$

Given a homeomorphism h, the distortion of the first cross ratio under h is

$$\frac{(ha,hb,hc,hd)}{(a,b,c,d)}.$$

In this paper, by the cross ratio distortion we mean the distortion of the first cross ratio.

We call a 4-tuple $a < b < c < d$ standard if $b-a = c-b = d-c$. The cross ratio distortion under h of a standard 4-tuple is bounded away from zero and from above if and only if (ha,hb,hc,hd) is also. If h is C^1 diffeomorphism, then

$$log[1 + \frac{1}{(ha,hb,hc,hd)}] = log[ha,hb,hc,hd] = \int\int_S (h \times h)^* \mu,$$

where μ is the measure $\frac{dxdy}{(x-y)^2}$.

Clearly the cross ratio distortion under f of a standard 4-tuple is bounded away from zero and from above if and only if $log[ha,hb,hc,hd]$ is also. Calculating the integrand, we get

$$\frac{h'xh'y}{(hx-hy)^2} = \frac{1}{(x-y)^2}\frac{h'xh'y}{[h']^2_{xy}},$$

where $[h']_{xy}$ is the average of h' over the interval $[x,y]$.

Since $b-a = c-b = d-c$, $\int\int_S \frac{dxdy}{(x-y)^2} = log([a,b,c,d]) = log\frac{4}{3}$. Thus a bound on $\frac{h'xh'y}{[h']^2_{xy}}$ yields a bound on the cross ratio distortions for standard 4-tuples.

We say h satisfies the *bounded Koebe condition* if one of the following equivalent conditions hold:

$$1)\ \frac{1}{M} \leq \frac{h'xh'y}{[h']^2_{xy}} \leq M \ for\ some\ M > 0,$$

$$2)\ |log\frac{h'xh'y}{[h']^2_{xy}}| \leq M' \ for\ some\ M' > 0.$$

The following proposition is trivial.

Prop. 2.7 *If h satisfies the bounded Koebe condition then the cross ratio distortion under h of a standard 4-tuple is bounded away from zero and from above.*

In order to estimate the log in 2), i.e.,

$$logh'x + logh'y - 2log[h']_{xy},$$

let us consider the following two terms:

$$a)\ logh'x + logh'y - 2[logh']_{xy}$$

and

$$b)\ log[h']_{xy} - [logh']_{xy}\ .$$

Remark: If both a) and b) are bounded, then 1) and 2) hold.

Expression a) can be controlled by the Zygmund variation of $logh'$ on the interval $[x, y]$ because of the following proposition.

Prop. 2.8 *Let ϕ be a continuous function from R^1 to R^1. Then*

$$|\phi(x) + \phi(y) - 2[\phi]_{xy}|$$

is no more than the Zygmund variation of ϕ over $[x, y]$, $ZV(\phi|_{[x,y]})$.

Remark: Conversely the Zygmund variation is no more than twice of the average Zygmund variation. Hence the above two conditions are actually equivalent.

Proof: Without loss of generality, assume $[x, y] = [0, 1]$. Then

$$[\phi]_{01} = \int_0^1 \phi dx.$$

The integral can be expressed as a limit of Riemann sums in the following way,

$$\int_0^1 \phi(x)dx = \lim_{n \to \infty} \frac{1}{2^{n+1}}[\phi(0) + 2\sum_{i=1}^{2^n-1} \phi(\frac{i}{2^n}) + \phi(1)] \ .$$

Since

$$\phi(0) + \phi(1) - 2\int_0^1 \phi(x)dx$$

$$= \lim_{n \to \infty} \frac{1}{2^n}(2^{n-1}[\phi(0) + \phi(1) - 2\phi(\frac{1}{2})]$$

$$+2^{n-2}\{[\phi(0) + \phi(\frac{1}{2}) - 2\phi(\frac{1}{4})] + [\phi(\frac{1}{2}) + \phi(1) - 2\phi(\frac{3}{4})]\}$$

$$+\cdots\cdots+$$

$$+2\{[\phi(0) + \phi(\frac{1}{2^{n-2}}) - 2\phi(\frac{1}{2^{n-1}})] + \cdots + [\phi(\frac{2^{n-2}-1}{2^{n-2}}) + \phi(1) - 2\phi(\frac{2^{n-1}-1}{2^{n-1}})]\}$$

$$+\{[\phi(0) + \phi(\frac{1}{2^{n-1}}) - 2\phi(\frac{1}{2^n})] + [\phi(\frac{1}{2^{n-1}}) + \phi(\frac{2}{2^{n-1}}) - 2\phi(\frac{3}{2^n})]$$

$$+ \cdots + [\phi(\frac{2^{n-1}-1}{2^{n-1}}) + \phi(1) - 2\phi(\frac{2^n-1}{2^n})]\}) \ ,$$

$$|\phi(0) + \phi(1) - 2[\phi]_{01}| \leq \frac{1}{2^n}[2^{n-1} + 2^{n-2} + \cdots + 2 + 1]ZV(\phi|_{[0,1]})$$

$$\leq ZV(\phi|_{[0,1]}).$$

Next we estimate the expression b) in terms of the quadratic variation.

Lemma 2.2 *If $\epsilon \geq \delta > -1$, assume*

$$log(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2}\Delta(\epsilon),$$

then there exists $B(\delta) > 0$ depending on δ such that $\Delta(\epsilon) \leq B(\delta)$.

Proof: Since

$$\Delta(\epsilon) = \frac{log(1+\epsilon) - \epsilon}{\epsilon^2/2},$$

the proof is an elementary calculation.

Prop. 2.9 *Suppose the derivative h' satisfies $1/M \leq h' \leq M$ for some $M > 0$. Then the expression b) is equal to the big O of the quadratic variation of $logh'$ over the interval $[x, y]$.*

Proof: Let $h'(x) = a$. The expression b) is unchanged if we multiply $h'x$ by $1/a$. Write $(1/a)h'$ on $J = [x, y]$ as $1+\epsilon$ where ϵ is a function of $(t-x), t \in J$. Expand the two terms of b)

$$\begin{aligned} & log\frac{1}{|J|}\int_J(1+\epsilon) - \frac{1}{|J|}\int_J log(1+\epsilon) \\ &= log(1 + \frac{1}{|J|}\int_J \epsilon) - \frac{1}{|J|}\int_J [\epsilon - \frac{\epsilon^2}{2}\Delta(\epsilon)] \\ &= [\frac{1}{|J|}\int_J \epsilon - \frac{1}{2}(\frac{1}{|J|}\int_J \epsilon)^2\Delta(\frac{1}{|J|}\int_J \epsilon)] - [\frac{1}{|J|}\int_J \epsilon - \frac{1}{|J|}\int_J \frac{\epsilon^2}{2}\Delta(\epsilon)] \\ &= -\frac{1}{2}(\frac{1}{|J|}\int_J \epsilon)^2\Delta(\frac{1}{|J|}\int_J \epsilon) + \frac{1}{|J|}\int_J \frac{\epsilon^2}{2}\Delta(\epsilon). \end{aligned}$$

By Cauchy's inequality, $(\frac{1}{|J|}\int_J \epsilon)^2 \leq \frac{1}{|J|}\int_J \epsilon^2$. Since $1/C \leq h' \leq C$ for some $C > 0$, there exists $\delta(C) > -1$ such that $\epsilon = \frac{h' t}{h' x} - 1 \geq \delta$ for any $t \in J$, $J = [x, y]$. Hence $\frac{1}{|J|}\int_J \epsilon \geq \delta$. By Lemma 1, there exists $B(\delta) > 0$ such that

$|\Delta(\epsilon)| \leq B(\delta)$. Hence $|\Delta(\frac{1}{|J|}\int_J \epsilon)| \leq B(\delta)$. Furthermore, we can get that $\epsilon = \frac{h't}{h'x} - 1$ is a big O of $log\frac{h't}{h'x} = logh't - logh'x$. So the expression b) is a big O of the quadratic variation of $logh'$ over J.

The following proposition estimates the cross ratio distortion of a standard 4-tuple under high iterates of a diffeomorphism f.

Prop. 2.10 *Suppose* $h : I \rightarrow R^1$ *is a* C^1 *diffeomorphism with* $h' > 0$*, and* $logh'$ *has bounded Zygmund variation and bounded quadratic variation over* I*. Assume* $J_0 \subset I$ *and* J_0 $J_1 = h(J_0), \cdots, J_n = h^n(J_0)$ *are pairwise disjoint. Then the cross ratio distortion under* h^n *of a standard 4-tuple in the interval* J_0 *is the big* O *of the summation of the bounds of Zygmund variation and quadratic variation of* h *on* $\cup_{i=0}^n J_i$.

Proof: From the expression 2) above the Prop. 2.7, we want to estimate

$$log\frac{(h^n)'(x)(h^n)'(y)}{[(h^n)']_{xy}^2}.$$

By the chain rule of calculating the derivative of h^n,

$$log\frac{(h^n)'(x)(h^n)'(y)}{[(h^n)']_{xy}^2} = \sum_{i=0}^{n-1} log\frac{h'(h^i(x))h'(h^i(y))}{[h']_{h^i(x)h^i(y)}^2}.$$

Each summand can be decomposed into the expression a) and expression b), by Prop. 2.8 and Prop. 2.9, each summand is a big O of the sum of the Zygmund variation and the quadratic variation of $logh'$ over the interval $[h^i(x), h^i(y)]$, where $i = 0, 1, 2, \cdots, n-1$. So the cross ratio distortion under h^n of a standard 4-tuple in J_0 is a big O of the sum of the Zygmund variation and the quadratic variation of $logh'$ on $\cup_{i=0}^{n-1} J_i$.

Theorem 2.7 (Real Koebe Principle II) *If $h : I \rightarrow R^1$ does not increase the cross ratio distortions for standard 4-tuples too much then the quasisymmetric distortions for standard interior triples are controlled. More precisely, if x, $y \in I$ satisfy $|x-y|$ is as small as the distance to the boundary ∂I of I and $z = (x+y)/2$, then*

$$\frac{1}{C} \leq |h(x) - h(z)|/|h(z) - h(y)| \leq C,$$

where C only depends on the bound of the cross ratio distortions for standard 4-tuples.

Proof: See §2 of [S1]. The idea to prove this lemma is to use the four interval argument. Let J, L, M, R be four contiguous equal lenth intervals. Suppose the lenth of $h(L)$ is much smaller than $h(M)$. Since the cross ratio distortion $\frac{|h(M)||h(T)|}{h(L)||h(R)|}/3$ on L, M, R is greater than the ratio distortion $\frac{|h(M)|}{|h(L)|}$, no bound of ratio distortions implies no bound of cross ratio distortions.

It is easy to use the Real Koebe Principle II to get the following Macroscopic Koebe Distortion Principle.

Definition 2.12 *Let M and T be two intervals with $M \subset T$, and L and R be components of $T \setminus M$. If $\epsilon > 0$ we say T is an ϵ-scaled neighborhood of M if*

$$\frac{|L|}{|M|} \geq \epsilon \textit{ and } \frac{|R|}{|M|} \geq \epsilon.$$

Theorem 2.8 (Macroscopic Koebe Distortion Principle) *Given any $B > 0$, $\epsilon > 0$, there exists $\delta > 0$ only depending on B and ϵ such that, for any homeomorphism f, any subintervals $M \subset T$ and any $n \geq 0$, if the cross* *ratio distortion under f^n of any standard 4-tuple in T is bounded by B and $f^n(T)$ contains an ϵ-scaled neighborhood of $f^n(M)$ then T contains a δ-scaled neighborhood of M.*

Proof: Let $T \setminus M = L \cup R$. Without loss of generality, we only need to prove $\frac{|M|}{|L|}$ cannot be very large. Suppose $\frac{|M|}{|L|}$ is large, we cut M into pieces L_i from left to right with lengths $2^{i-1}|L|, i = 1, 2, 3, \cdots$. We also denote $L_0 = L$. From the Real Koebe Principle, there exists a constant C only depending on B such that

$$\frac{|f^n(L_i)|}{|\bigcup_{j=0}^{i-1} f^n(L_j)|} \geq \frac{1}{C},$$

where $i = 1, 2, 3, \cdots$. Hence

$$\frac{|\bigcup_{j=0}^{i} f^n(L_i)|}{|\bigcup_{j=0}^{i-1} f^n(L_j)|} \geq 1 + \frac{1}{C},$$

where $i = 1, 2, 3, \cdots$. So

$$\frac{|\bigcup_{j=0}^{i} f^n(L_i)|}{|f^n(L_0)|} \geq (1 + \frac{1}{C})^i,$$

where $i = 1, 2, 3, \cdots$. This means

$$\frac{|\bigcup_{j=1}^{i} f^n(L_i)|}{|f^n(L_0)|} \geq (1 + \frac{1}{C})^i - 1,$$

where $i = 1, 2, 3, \cdots$.

Clearly i cannot be very large, otherwise $f^n(T)$ cannot be an ϵ-scaled neighborhood of $f^n(M)$. Hence we can find a bound of i only depending on B and ϵ, which means there exists $\delta > 0$ only depending on B and ϵ such that T contains a δ-scaled neighborhood of M.

Definition 2.13 *The intersection multiplicity of a collection of sets* $X_{\alpha\in\Lambda}$ *is the maximal cardinality of a subcollection with non-empty intersection.*

Corollary 2.3 *Let* $f \in \Gamma$, *m be a positive interger, there exist* $\delta_0, \epsilon_0, B_0$ *such that for any intervals* $M \subset T$, *any* $n \geq 0$ *and any* $0 < \epsilon < \epsilon_0$, *if the following conditions hold*

a) $\max_{i=0}^{n}\{|f^i(T)|\} < \delta_0$;

b) $f^n|_T$ *is monotone ;*

c) $f^n(T)$ *contains a* ϵ*-scaled neighborhood of* $f^n(M)$; *and*

d) the intersection multiplicity of $\{T, f(T), \cdots, f^n(T)\}$ *is at most m,*

then T is a $B_0 \cdot \epsilon$*-scaled neighborhood of M.*

Remark: Compared to the work in [MvS], the smooth condition used here is weaker (See [HS] for details).

Definition 2.14 *A continuously injective map* $f : R^1 \to R^1$ *is quasisymmetric if there exists* $M > 0$ *such that*

$$\frac{1}{M} \leq \sup_{x,h\in R^1} |\frac{f(x+h)-f(x)}{f(x)-f(x-h)}| \leq M.$$

Clearly, if a C^1-diffeomorphism has cross-ratio distortions bounded from below, then it is quasisymmetric when restricted on a proper subdomain.

Let

$$f(x) = sign(x) \cdot |x|^\alpha, \alpha > 0,$$

then f is a quasisymmetric map.

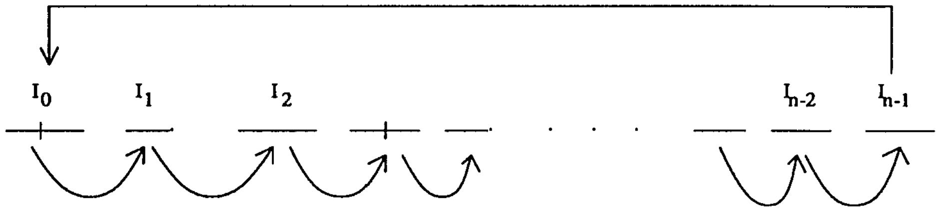


Figure 2.2

2.3 Proof of the “Gap” Lemma

The picture is drawn according to the dynamical order. “|” represents the critical point and “$\cdot$” denotes the critical value.

Proof: Let $\delta = \delta_0/6$, where δ_0 is what we have selected in the Macroscopic Koebe Principle, and $\delta_0 \leq \inf\{d(x,y) : x,y \in C(f) \cup V(f) \cup \partial I\}$, where $C(f)$ and $V(f)$ denote the set of critical points and critical values respectively, ∂I denotes the boundary points of the interval I. Now each I_i contains at most one element of $C(f) \cup V(f) \cup \partial I$. Furthermore, if V_i is a 2-scaled neighborhood of I_i, then at most one of adjacent components of $V_i \setminus I_i$ contains one critical value.

Assume I_0 contains one critical point and it is symmetric about the critical point. Pick up I_{i_0} with the smallest length among $\{I_i : 0 \leq i \leq n-1\}$. The proof can be divided into two cases.

Case 1 (Figure 2.3): Suppose I_{i_0} is between the other two elts. of $\{I_i : i \neq i_0\}$, denote them by $I_{i_0}^{\wedge}$ and $I_{i_0}^{\sim}$.

Let V be 1-scaled neighborhood of I_{i_0}, then each component of $V \setminus I_{i_0}$ intersects at most one of $\{I_i : i \neq i_0\}$. Let U be the component of $f^{-1}(V)$ containing I_{i_0-1}.

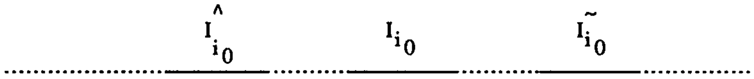


Figure 2.3

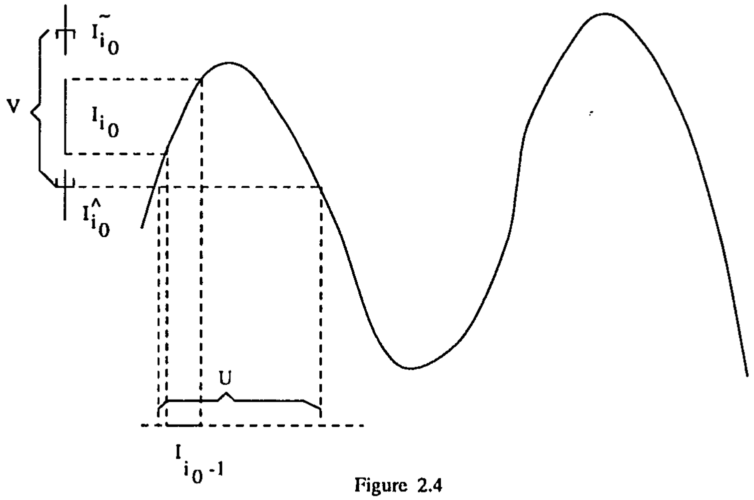


Figure 2.4

Subcase 1 (Figure 2.4): If U contains a critical point and the critical value at this critical point is in the right component of $V \setminus I_{i_0}$, then U is $\epsilon_1 -$ *scaled* neighborhood of I_{i_0-1}. Precisely by the quasisymmetricity and the symmetricity of f around the critical point, at least one of $U \setminus I_{i_0-1}$ will be in the gap. In another word, at least one side of I_{i_0-1} contains a gap whose length is a definite proportion of the length of the interval I_{i_0-1}.

Subcase 2 (Figure 2.5): Suppose U contains a critical point and the critical value at this critical point is in the left component of $V \setminus I_{i_0}$.

By the same reason of subcase 1, again U is $\epsilon_1 -$ *scaled* neighborhood of I_{i_0-1} and at least one side of I_{i_0-1} contains a gap whose length is a definite proportion of the length of the interval I_{i_0}.

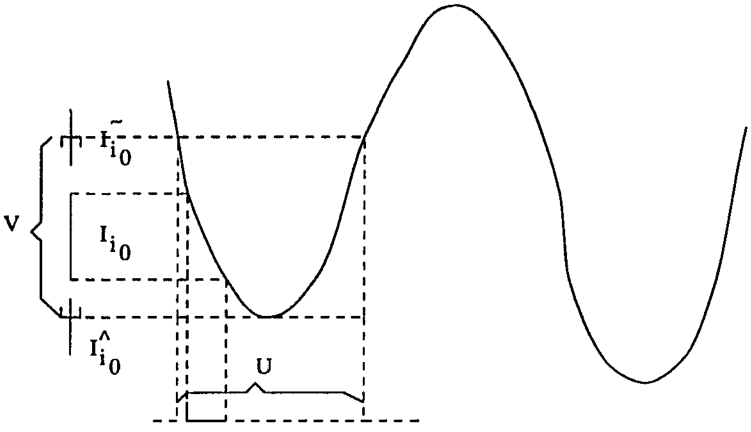


Figure 2.5

Subcase 3: U contains no critical point, then this case will be eventually reduced to one of the above subcases. Precisely, take the component of $f^{-1}(V)$ containing I_{i_0-1}, if U contains no critical point, do the pullback again untill U contains a critical point. It is guaranteed since at least one of $\{I_i : 0 \le i \le n-1\}$ contains a critical point. The multiplicity of maximal monotone pullbacks in the consideration is bounded. By the Macroscopic Koebe Principle, the quasisymmetricity and the symmetricity around the critical point in the last step, U will be a ϵ_1-scaled neighborhood of some element of $\{I_i : 0 \le i \le n-1\}$, let's still denote it by I_{i_0-1}, then at least one componenet of $U \setminus I_{i_0-1}$ is in the gap. That means at least one side of I_{i_0-1} contains a gap whose length is a definite proportion of the length of I_{i_0-1}.

Now we restrict our consideration on U, go through the procedure of subcase 1, subcase 2 and subcase 3. Repeat this kind of consideration. Since f has only finite many critical points and finite critical values, except finite many cases, all considerations belong to subcase 3. Applying the Macroscopic

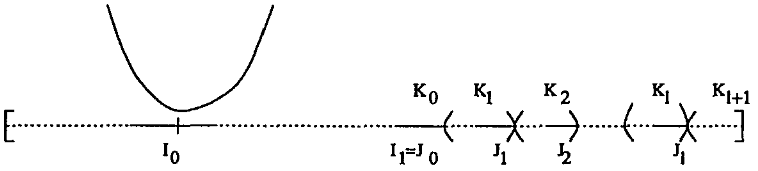


Figure 2.6

Koebe Principle, there exists $\epsilon_0 > 0$, which is independent of n, such that for each I_i there is a $\epsilon_0 -$ *scaled* neighborhood U_i of I_i such that at least one of the components of $U_I \setminus I_i$ is in the gap, the proportion is ϵ_0, compared to the length of I_i. This implies that the summation of gaps in $I \setminus \cup_{i=0}^{n-1} I_i \geq \frac{\epsilon_0}{2} \sum_{i=0}^{n-1} |I_i|$. So

$$\frac{\sum_{i=0}^{n-1} |I_i|}{|I|} \leq \frac{\sum_{i=0}^{n-1} |I_i|}{\sum_{i=0}^{n-1} |I_i| + \frac{\epsilon_0}{2} \sum_{i=0}^{n-1} |I_i|} \leq \frac{1}{1 + \frac{\epsilon_0}{2}}.$$

Let $\epsilon = \frac{\epsilon_0}{2}$, we finished the proof for this case.

Case 2: I_{i_0} is on the left (or right) hand side of anyone else in $\{I_i : 0 \leq i \leq n-1\}$. We will find a definite scaled neighborhood V of some I_i such that each component of $V \setminus I_i$ intersects at most one of $\{I_i : i \neq i_0\}$. Then the rest of proof goes back to the case 1. Without loss of generality, we assume the image $I_1 = f(I_0)$ of I_0 is on the right hand side of I_0. There are two more situations we need to consider.

Subcase 1 (Figure 2.6): The left end point of I_1 is the critical value.

By the assumption on δ, the length of I_1 is less than one half of the distance between this critical value and the right end point of I, if there is no element of $\{I_i : 0 \leq i \leq n-1\}$ on the right hand side of I_1, then take 1-scaled half neighborhood of I_1 on the right and one step of pullback will

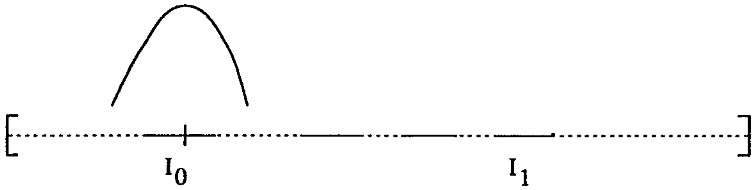


Figure 2.7

give us needed neighborhood V of I_0. Now suppose there are several elements of $\{I_i : 0 \le i \le n-1\}$ between the I_1 and the right end point of I. Denote them by J_1, J_2, $\cdots$, J_l from left to right and denote I_1 by J_0. Let K_0 be J_0, K_i be the interval from the right end point of J_{i-1} to the right point of J_i, $j = 1, 2, \cdots, l$ let K_{l+1} be the space left if there is. Claim: there is $j \in \{0, 1, \cdots, l, l+1\}$ such that $\frac{|K_{j+1}|}{|K_j|} \ge \frac{1}{2}$. Otherwise the summation of the lengths of all K_j will be strictly less than the distance between the left end point of I_1 and the right end point of I. This is a contradiction. Now let j be the smallest such j. If $j = 0$, then one step of pullpack gives the V. Suppose $j > 0$, let T be the interval between the midpoints of K_{j-1} and K_{j+1} and L, M be the left and right components of $T \setminus J_j$. Hence

$$\frac{|R|}{|J_j|} \ge \frac{|R|}{|K_j|} \ge \frac{1}{4} \;\; and \;\; \frac{|L|}{|J_j|} \ge \frac{1/2|K_{j-1}|}{|K_j|} \ge 1.$$

Let $V = T$, we finished the construction of V.

Subcase 2 (Figure 2.7): The right end point of I_1 is the critical value.

Clearly if there are no other dynamical intervals between I_0 and I_1, then take 1-scaled half neighborhood of I_1 on the left and one step of pullback will give the V. If there are several dynamical intervals between I_0 and I_1, then we can use the same argument used in the subcase 1 of this case, again we can find V.

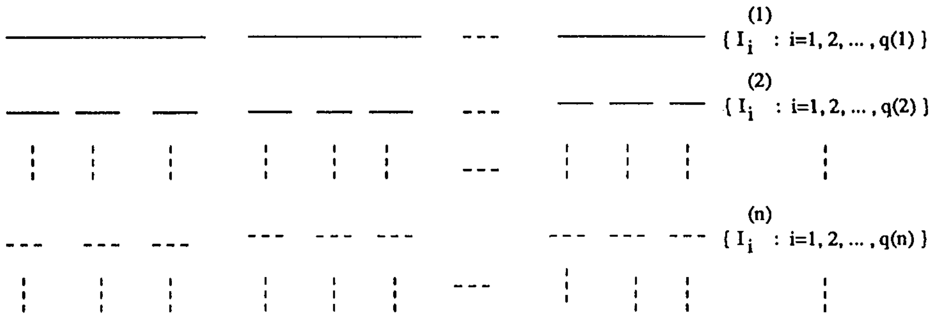


Figure 2.8

We finish the proof of the "Gap" Lemma.

Remark: (1) The proof of the lemma looks like a programming for a computer to check.

(2) Since f has finite critical points and finite critical values, in deep renormalization, the conditions in the "Gap" Lemma will be automatically satisfied.

2.4 Real Bound's Properties

Proof of Theorem 2.1: Let $q(n) = p(1)p(2)\cdots p(n)$, where $p(i)$ is the peried of the i^{th} renormalization $i = 1, 2, \cdots, n$. Let $I_0^{(n)}, I_1^{(n)}, \cdots, I_{q(n)-1}^{(n)}$ be the whole dynamical intervals in the first n's renormalizations of f, where $I_i^{(n)} = f^i(I_0^{(n)}), i = 1, 2, \cdots, q(n) - 1$. See figure 2.8.

Since f is infinitely renormalizable, at least one of $\{I_i^{(n)} : i = 0, 1, \cdots, q(n)-1\}$ contains a critical point. Otherwise, the map $f^{q(n)}$ will be a monotone map from $I_0^{(n)}$ into itself, it cannot be renormalizable any more. Let $I_0^{(n)}$ contain a critical point and it is symmetric around the critical point. Assume we have replaced each critical interval by the smallest symmetric interval around the critical point containing the critical interval. Clearly after replacing, the dynamical intervals are still disjoint. Let

$$F^{(n)} = I_0^{(n)} \cup I_1^{(n)} \cup \cdots \cup I_{q(n)-1}^{(n)},$$

where each $I_i^{(n)}$ is called an interval of generation n. Each component of $F^{(n-1)} \setminus F^n$ is called a gap of generation n.

Now let us go through the arguments again in the proof of the "Gap" Lemma, by the quasisymmetricity and symmetricity around each critical point, there exist a definite gap of generation n at one side of $I_0^{(n)}$ compared to the length of $I_0^{(n)}$. Once this gap appears, use the pullback argument again and again, by the finiteness of critical points and the Macroscopic Koebe Principle, there exists at least a definite gap of generation n at one side of each $I_i^{(n)}$ compared to the length of $I_i^{(n)}$. Let ϵ be the same in the "Gap" Lemma, then

$$\frac{|I_0^{(n)}|}{|I_0^{(n-1)}|} \leq \frac{1}{1+\epsilon}$$

and

$$\frac{\sum_{i=0}^{q(n)-1} |I_i^{(n)}|}{\sum_{i=0}^{q(n-1)-1} |I_i^{(n-1)}|} \leq \frac{1}{1+\epsilon}.$$

This implies that the scales of the renormalizations decrease geometrically fast and the summation of the lengths of all dynamical intervals in the n^{th} renormalization also decreases geometrically fast when $n \to \infty$. □

Proof of Theorem 2.2: Let $C = \cap_{n=1}^{\infty} F^n$, where F^n is defined in the proof of the above theorem. It is an immediate corollary that the Lebesgue measure of the Canter set C is zero. We will prove that if $p = \max_{i \in N} p(i) < \infty$ then the Hausdorff dimension of the Cantor set C is strictly less than 1. The proof for the Hausdorff dimension of the Cantor set C greater than 0 is similar.

By the proof of the theorem,

$$\sum_{I_i^{(n)} \subset I_j^{(n-1)}} |I_i^{(n)}| \leq \frac{1}{1+\epsilon} |I_j^{(n)}|,$$

where $i = 0, 1, \cdots, q(n) - 1$, and $j = 0, 1, \cdots, q(n-1) - 1$. Hence

$$\sum_{I_i^{(n)} \subset I_j^{(n-1)}} \frac{|I_i^{(n)}|}{|I_j^{(n-1)}|} \leq \frac{1}{1+\epsilon}.$$

We can show the following element fact by contradiction. There exists $\alpha \in (0, 1)$ such that

$$\sum_{i=1}^{p} x_i \leq \rho < 1 \quad \Longrightarrow \quad \sum_{i=1}^{p} x_i^{\alpha} \leq 1$$

where $x \in [0, 1], i = 1, 2, \cdots, p$ and $\rho \in (0, 1)$ is fixed. So

$$\sum_{I_i^{(n)} \subset I_j^{(n-1)}} |I_i^{(n)}|^{\alpha} \leq |I_j^{(n-1)}|^{\alpha}.$$

Consequently,

$$\sum_{i=0}^{q(n)-1} |I_i^{(n)}|^{\alpha} \leq \sum_{i=0}^{q(n-1)-1} |I_i^{(n-1)}|^{\alpha}.$$

So $\mu_{\alpha}(C) < \infty$, where μ_{α} denotes the α-dimensinal Hausdorff measure. Thus

$$dimC \leq \alpha < 1.$$

□

Remark: M. Marten's thesis proved the following result: For $f \in \Gamma$, there exists n_0 and $\rho > 0$ such that for every periodic point x of f with period $n \geq n_0$ one has

$$|Df^n(x)| \geq 1 + \rho.$$

Hence every point in the attracting orbital Cantor Set is an accumulation point of repelling periodic orbits.

Bounds on Zygmund variation and quadratic variation of the logarithm of the derivative of the map give a bound on the cross-ratio distortion of the monotone iterates restricted on an interval whenever the multiplicity of the orbit of the interval is bounded, and hence implies the bounded geometry of renormalization scales and attracting orbital Cantor sets. It is quite natural that stronger local regularities of maps will induce better asymptotical behaviours of the renormalizations and the attracting orbital Cantor sets. The Zygmund condition will be employed. A continuous map $\phi : I \rightarrow R^1$ satisfies Zygmund condition if there exists $B > 0$ such that

$$|\frac{\phi(x) + \phi(y)}{2} - \phi(\frac{x+y}{2})| \leq B$$

for any $x, y \in I$. Clearly ϕ has bounded Zygmund variation if it satisfies Zygmund condition. Furthermore, the Zygmund condition implies α-Hölder continuous for any $0 < \alpha < 1$. The 1/2-Hölder continuity implies bounded quadratic variation.

Lemma 2.3 *($See[HS]or[MvS]$) If $\phi : I \rightarrow R^1$ satisfies the Zygmund condition: there exists $B > 0$ such that*

$$\sup_{x,t} |\frac{\phi(x+t) + \phi(x-t) - 2\phi(x)}{t}| \leq B,$$

then ϕ is α-Hölder continuous for any $0 < \alpha < 1$, i.e., there exists a constant $M > 0$ such that

$$|\phi(x) - \phi(y)| \leq M|x-y|^{\alpha}|.$$

In fact the norm of α-Hölder continuity goes to zero as $|x-y| \rightarrow 0$.

Remark: The norm of α-Hölder continuity goes up as α tends to 1 and goes down as the Zygmund norm decreases for any fixed $0 < \alpha < 1$.

Corollary 2.4 *If f is a Zygmund-smooth polynomial-like mapping then f belongs to Γ.*

A very useful fact is that the logarithm of cross-ratio distortion is a big O of the scale of the interval if and only if the logarithm of the derivative satisfies the Zygmund condition. More precisely:

Lemma 2.4 (See §1 of S1 or page 294 of MvS) *Assume that $h : (0,1) \to R$ is a C^1-diffeomorphism. Then*

$$|B(h, J, T) - 1| \leq O(|T|)$$

for any pairs $J \subset T \subset (0,1)$ if and only if $\log h'$ satisfies the Zygmund condition.

Lemma 2.5 *Let $[u_0^{(n)}, v_0^{(n)}]$ be the domain of the n^{th} renormalization without rescaling. Asssume $f^{q(n)}$ is monotone on an subinterval $[x_0, y_0]$ of $[u_0^{(n)}, v_0^{(n)}]$. Denote $[u_i, v_i] = f^i([u_0^{(n)}, v_0^{(n)}])$ and $[x_i, y_i] = f^i([x_0, y_0])$ where $i = 0, 1, 2, \cdots, q(n) - 1$. Then there exists a contant $M > 0$ such that*

$$\sum_{i=0}^{q(n)-1} |x_i - y_i| \leq M|\lambda(n)x_0 - \lambda(n)y_0|$$

where $\lambda(n)$ is the scaling factor of the n^{th} renormalization compared to the original dynamical interval. In fact M tends to zero as $n \to \infty$.

Proof: It is a direct corollary of the Real Koebe Principle. We outline it as follows. Since

$$\frac{\sum_{i=0}^{q(n)-1}|x_i - y_i|}{\sum_{i=0}^{q(n)-1}|u_i - v_i|} \approx O(\frac{|x_0 - y_0|}{|u_0 - v_0|}),$$

$$\begin{aligned}\sum_{i=0}^{q(n)-1}|x_i - y_i| &\approx O(\frac{|x_0 - y_0|}{|u_0 - v_0|}\sum_{i=0}^{q(n)-1}|u_i - v_i|)\\ &\approx O(\frac{|x_0 - y_0|\lambda(n)}{|u_0 - v_0|\lambda(n)}\sum_{i=0}^{q(n)-1}|u_i - v_i|)\\ &\approx O(|\lambda(n)x_0 - \lambda(n)y_0|\frac{\sum_{i=0}^{q(n)-1}|u_i - v_i|}{|u_0 - v_0|\lambda(n)})\\ &\approx O(|\lambda(n)x_0 - \lambda(n)y_0|).\end{aligned}$$

The idea in the proof of Lemma 2.5 also proves the following corollary.

Corollary 2.5 *With the same assumption in Lemma 2.5, if $f \in \tau$ is infinitely renormalizable of bounded types then there exists $0 < \alpha(f) < 1$ (which is the Hausdorff dimension of the attracting orbital Cantor set) such that for any $\alpha(f) < \alpha \leq 1$*

$$\sum_{i=0}^{q(n)-1}|x_i - y_i|^\alpha \leq M\frac{|x_0 - y_0|^\alpha}{|u_0 - u_0|^\alpha}$$

for some constant $M > 0$ independent of n and α.

Lemma 2.6 *If $\phi(x) = \log x$, $0 < x < y$ and $y - x < Mx^2$ for some constant M, then*

$$|\frac{\phi(x) + \phi(y)}{2} - \phi(\frac{x+y}{2})| \leq \frac{M}{4}(y - x).$$

Proof:

$$\frac{\phi(x) + \phi(y)}{2} - \phi(\frac{x+y}{2})$$

$$
\begin{aligned}
&= \frac{1}{2}[(\phi(y) - \phi(\frac{y+x}{2})) - (\phi(\frac{y+x}{2}) - \phi(x))] \\
&= \frac{1}{2}[\phi'(\zeta)\frac{y-x}{2} - \phi'(\eta)\frac{y-x}{2}] \\
&= \frac{1}{4}(\frac{1}{\zeta} - \frac{1}{\eta}) = \frac{1}{4}(y-x)\frac{\eta - \zeta}{\zeta\eta} \\
&\leq \frac{1}{4}(y-x)\frac{Mx^2}{x^2} \leq \frac{M}{4}(y-x),
\end{aligned}
$$

where $x \leq \eta \leq \zeta \leq y$.

Proof of Theorem 2.3: Suppose $f = P \circ h$ where P is a polynomial and h is a diffoemorphism satisfying the conditions in the theorem. Decompose $R^n(f)$ as

$$R^n(f) = \Lambda^{-1} \circ f^{q(n)}|_{I_n} \circ \Lambda$$

$$= \Lambda^{-1} \circ H_m \circ P_m \circ \cdots \circ H_2 \circ P_2 \circ H_1 \circ P_1 \circ h \circ \Lambda,$$

where $P_i's$ are the restrictions of the original polynomial in the small neighborhoods around critical points which are involved in renormalizations, $H_i's$ are monotone compositions of $h's$ and the branches of P, and Λ is the rescaling map.

Let $\tilde{H}_i = H_i \circ \Lambda$ and $\tilde{P}_i = \Lambda^{-1} \circ P_i$ where $i = 1, 2, \cdots, m$. Then

$$R^n(f) = \Lambda^{-1} \circ f^{q(n)}|_{I_n} \circ \Lambda$$

$$= \Lambda^{-1} \circ \tilde{H}_m \circ \tilde{P}_m \circ \cdots \circ \tilde{H}_2 \circ \tilde{P}_2 \circ \tilde{H}_1 \circ \tilde{P}_1 \circ h \circ \Lambda.$$

The logarithm of the derivative of $\tilde{P}_i$ around each critical point is equal to $\log x$ up to a constant, and the logarithm of the derivative of $\tilde{H}_i$, $\log \tilde{H}_i'$, consists of two parts, one is the sum of $\log P'$ over corresponding dynamical intervals and the other is the sum of $\log h'$ over the corresponding dynamical intervals. By the remark after Prop. 2.8 and Lemmas 2.5 and 2.6, $\log \tilde{H}_i'$ satisfies Zygmund condition, $i = 1, 2, \cdots, m$.

Since the kneading data of polynomials with real coefficients are complete, there exists a polynomial $P^{(n)}$ and a homeomorphism $h^{(n)}$ such that

$$R^n(f) = \Lambda^{-1} \circ f^{q(n)}|_{I_n} \circ \Lambda$$

$$= \Lambda^{-1} \circ \tilde{H}_m \circ \tilde{P}_m \circ \cdots \circ \tilde{H}_2 \circ \tilde{P}_2 \circ \tilde{H}_1 \circ \tilde{P}_1 \circ h \circ \Lambda$$

$$= P^{(n)} \circ h^{(n)}.$$

In fact $h^{(n)} = (P^{(n)})^{-1} \circ R^n(f)$, where $(P^{(n)})^{-1}$ denotes one branch of the inverse map of $P^{(n)}$. It is easy to see that $h^{(n)}$ is smooth piecewise and the logarithm of its derivative satisfies Zygmund condition piecewise except in the small neighborhoods around the turning points of $R^n(f)$. We will see that $\log(h^{(n)})'$ satisfies Zygmund condition in the small neighborhoods around turning points by using the chain rule as follows. Suppose $h \circ Q = P \circ h_1$ where h is a diffeomorphism, Q, P are quadratic polynomials and h_1 is a homeomorphism. Let c be the critical point of Q. Then

$$h'(Q(c^-))Q'(c^-) = P'(h_1(c^-))h_1'(c^-)$$

and

$$h'(Q(c^+))Q'(c^+) = P'(h_1(c^+))h_1'(c^+)$$

where c^- and c^+ denote the points near c from left and right respectively. Hence

$$\lim_{c^- \to c} h_1'(c^-) = \lim_{c^+ \to c} h_1'(c^+).$$

Furthermore $\log h_1'$ satisfies Zygmund condition in a small neighborhood around the critical point c because the Zygmund norms of $\log Q'$ and $\log P'$ almost cancel in the small neighborhood since they have the same quadratic turnings around the critical points. From the above argument one can see $\log(h^{(n)})'$ satisfies the zygmund condition globally.

Check through the proof of the Lemma 2.5, one can see the constant M in this lemma tends to zero as n tends to the infinity. This shows after a number of renormalizations (in fact only depending on B) further renormalizations have a universal bound on the Zygmund norm of $\log h'$ for the $h's$ in the decompostions. □

Proof of the Theorem 2.4: Assume $f = P \circ h$ be a Zygmund-smooth polynomial-like map. As the same as we did in the proof of the Theorem 2.3, we write

$$R^n(f) = \Lambda^{-1} \circ f^{q(n)}|_{I_n} \circ \Lambda$$

$$= \Lambda^{-1} \circ H_m \circ P_m \circ \cdots \circ H_2 \circ P_2 \circ H_1 \circ P_1 \circ h \circ \Lambda,$$

where $P_i's$ are the restrictions of the original polynomial in the small neighborhoods around critical points which are involved in renormalizations, $H_i's$ are monotone compositions of $h's$ and the branches of P, and Λ is the rescaling map.

We know that $\log h'$ is α-Hölder continuous for any $0 < \alpha < 1$ and $\log P'$ is a Lipschitz function. For each $i \in \{1, 2, \cdots, m\}$, let $[u, v]$ be the maximal domain for the monotone composition H_i and let $x, y \in [u, v]$. Through the chain rule of calculating the derivative and by applying Corollary 2.5, one can get for any $\alpha(f) < \alpha < 1$,

$$|\log H_i'(x) - \log H_i'(y)| \approx O(|\frac{x-y}{u-v}|^\alpha)$$

where $\alpha(f)$ is the same as in the Corollary 2.5. This means

$$|\frac{H_i'(x)}{H_i'(y)} - 1| \approx O(|\frac{x-y}{u-v}|^\alpha).$$

By rescaling, one gets

$$|H_i'(x) - H_i'(y)| \approx O(|x-y|^\alpha).$$

Let $\tilde{H}_i = H_i \circ \Lambda$ and $\tilde{P}_i = \Lambda^{-1} \circ P_i$ where $i = 1, 2, \cdots, m$. Then

$$R^n(f) = \Lambda^{-1} \circ f^{q(n)}|_{I_n} \circ \Lambda$$

$$= \Lambda^{-1} \circ \tilde{H}_m \circ \tilde{P}_m \circ \cdots \circ \tilde{H}_2 \circ \tilde{P}_2 \circ \tilde{H}_1 \circ \tilde{P}_1 \circ h \circ \Lambda.$$

The above arguement shows $\tilde{H}_i$, $i = 1, 2, \cdots, m$, are $C^{1+\alpha}$ smooth for any $\alpha(f) < \alpha < 1$.

From the "Gap" Lemma, there exists $\tau > 0$ and τ-scaled neighborhood $T_i^{(n)}$ of $I_i^{(n)}$ such that $f(T_i^{(n)}) \subset T_{i+1}^{(n)}, i = 1, 2, \cdots, q(n)-1$, and $f(T_i^{(n)}) = T_{i+1}^{(n)}$ if $f|_{T_i^{(n)}}$ is monotone. Let

$$g_i = f|_{T_i^{(n)}} = P \circ h|_{T_i^{(n)}}.$$

If g_i is monotone, then denote by $\bar{h}_i$ the affine map form $T_i^{(n)}$ onto $h(T_i^{(n)})$ and let

$$\bar{g}_i = P \circ \bar{h}_i|_{T_i^{(n)}}.$$

Denote

$$H_k = g_{i+j} \circ \cdots \circ g_{i+1} \circ g_i$$

and

$$\bar{H}_k = \bar{g}_{i+j} \circ \cdots \circ \bar{g}_{i+1} \circ \bar{g}_i,$$

where $1 \leq k \leq m$ and $1 \leq i \leq i+j \leq q(n) - 1$.

For the same reasons, $\bar{H}_i$ is $C^{1+\alpha}$ smooth as H_i after rescaling, where $i = 1, 2, \cdots, m$ and $\alpha(f) < \alpha < 1$. In the following context $\bar{H}_i$ and H_i denote the corresponding rescaled maps, $i = 1, 2, \cdots, m$. The important thing is that the $C^{1+\alpha}$ distance between $\bar{H}_i$ and H_i goes to zero as the renormalization level n goes to ∞, i.e.,

$$|[H_i'(x) - H_i'(y)] - [\bar{H}_i'(x) - \bar{H}_i'(y)]| \approx O(|x-y|^{\alpha})$$

where $x, y \in I_k^n$ for some $1 \leq k \leq q(n) - 1$.

Since each $\bar{H}_i$ is a composition of the monotone branches of the polynomial P and affine maps, $\bar{H}_i$ is holomorphic and its inverse is defined a slit complex plane containing a definitely scaled interval neighborhood of the corresponding dynamical interval. By Montel's theorem, the limit of any convergent subsequence of $(\bar{H}_i^{(n)})_{n=1}^{\infty}$ will we an univalent function with its inverse defined on a slit complex plane containing a definitely scaled interval neighborhood of the corresponding dynamical interval. Since H_i is $C^{1+\alpha}$ close to $\bar{H}_i$, the limit of any subconvergent sequence of $(H_i^{(n)})_{n=1}^{\infty}$ will have the same properties. □

Chapter 3

Complex Bound's Property

In this chapter, we first introduce the sector lemma (due to Dennis Sullivan), which has been applied to get a complex bound for the renormalizations of symmetric quadratic-like mappings [S1]. We will also use this lemma to find a complex bound of the renormalizations of infinitely renormalizable multimodal maps of Epstein class.

3.1 Sector Lemma

Let $I = [a, b]$ be an interval on the real axis. Let $S = S(a, b)$ be the set of holomorphic injections (Schlicht mappings) defined on $C \setminus (-\infty, a] \cup [b, +\infty)$ which are homeomorphisms of $[a, b]$ to itself and preserve the two half planes.

Clearly $S(a, b)$ contains left and right square roots, branches of $\sqrt{z}$ pre and post composed by affine maps defined on a slit domain.

Theorem 3.1 *Suppose a composition of elements of* $S(a, b)$ *of the form*

$$A_n B_n \cdots A_2 B_n A_1 B_1$$

satisfying:

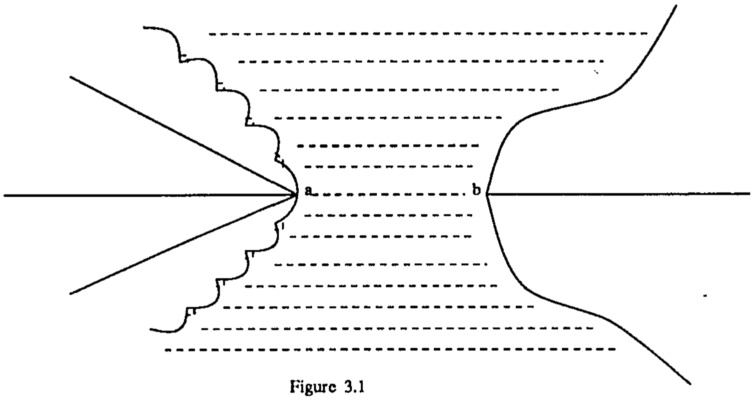


Figure 3.1

1) $A_1, A_2, \cdot, A_n$ are left square roots and $B_1, B_2, \cdots, B_n$ are general elements of $S(a,b)$.

2) A_1 has the singularity at a (i.e. the slit for A_1 is $(-\infty, a)$), and the singularity a_i of A_i moves to $-\infty$ exponentially fast, i.e., if $|a-b| = 1$, $k \leq \frac{|a_i - a|}{|a_{i-1} - a|} \leq K$ for $i = 2, 3, \cdots, n$, where $1 < k < K < \infty$.

3) Let I_i denote the maximal open interval on which B_i extends to a diffeomorphism into reals, then $B_i(I_i)$ contains (a_i, a). Moreover I_i contains $B_i^{-1}((a_i, a))$ and a λ-proportional space on both sides.

Then there exists θ only depending on (k, K, λ) so that the image of $C \setminus (-\infty, a] \cup [b, \infty)$ under the composition is contained in the sector

$$-(\pi - \theta) \leq \arg(z - a) \leq \pi - \theta$$

for any z in the image.

Proof: 1) Since any element or composition from S preserves the two half planes, it is enough to show that the image of the upper half plane by the composition $A_n B_n \cdots A_2 B_2 A_1 B_1$ is contained in the sector $0 \leq \arg(z - a) \leq \pi - \theta$ for any z in the image.

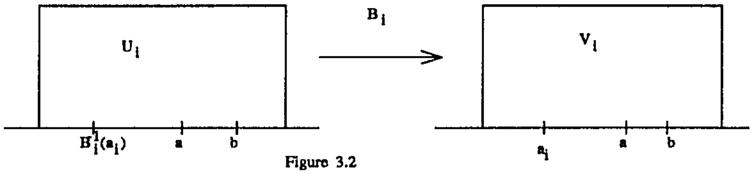


Figure 3.2

2) Let $J_i = B_i^{-1}((a_i, a))$, $J_i' = J_i$ plus λ-proportional space on both sides, then $J_i' \subset I_i$. Since B_i is a Schlicht map on $\mathcal{C} \setminus \{real\ x\ not\ in\ I_i\}$, by Koebe distortion lemma, B_i maps a bounded shape neighborhood U_i of J_i in the upper plane to a bounded shape neighborhood V_i of (a_i, a).

Since B_i fixes the two points a and b, it only distorts the Euclidean metric on the neighborhood by a bounded factor.

Now we begin to prove the sector lemma. The statement is obvious for $n = 1$, so assume $n \geq 2$. Start with any point P_1 in the upper half plane and define $P_{i+1} = A_i B_i(P_i)$, $i = 1, \cdots, n-1$. First note that P_2 lies to the right of the vertical line at a. There are two cases.

Case (1) P_i is away from (a, b) relative to the singularity a_i of A_i. Precisely P_i does not belong to U_i.

Then P_{i+1} is not in $A_i B_i(U_i) = A_i(V_i)$. Now $A_i(V_i)$ contains a rectangle resting on $(A_i(a_i), a)$ with height a definite fraction of $|A_i(a_i) - a|$. Since P_{i+1} lies to the right of the vertical line at $A_i(a_i)$ and outside of the rectangle, the angle of P_{i+1} as viewed from a is larger than a constant determined by the ratio of the two sides of the rectangle.

Before we get into the Case (2), we need to make the following observations.

Observation 1. The domains of the upper half plane bounded by circles passing through a and b are hyperbolic (Poincaré metric) neighborhood of

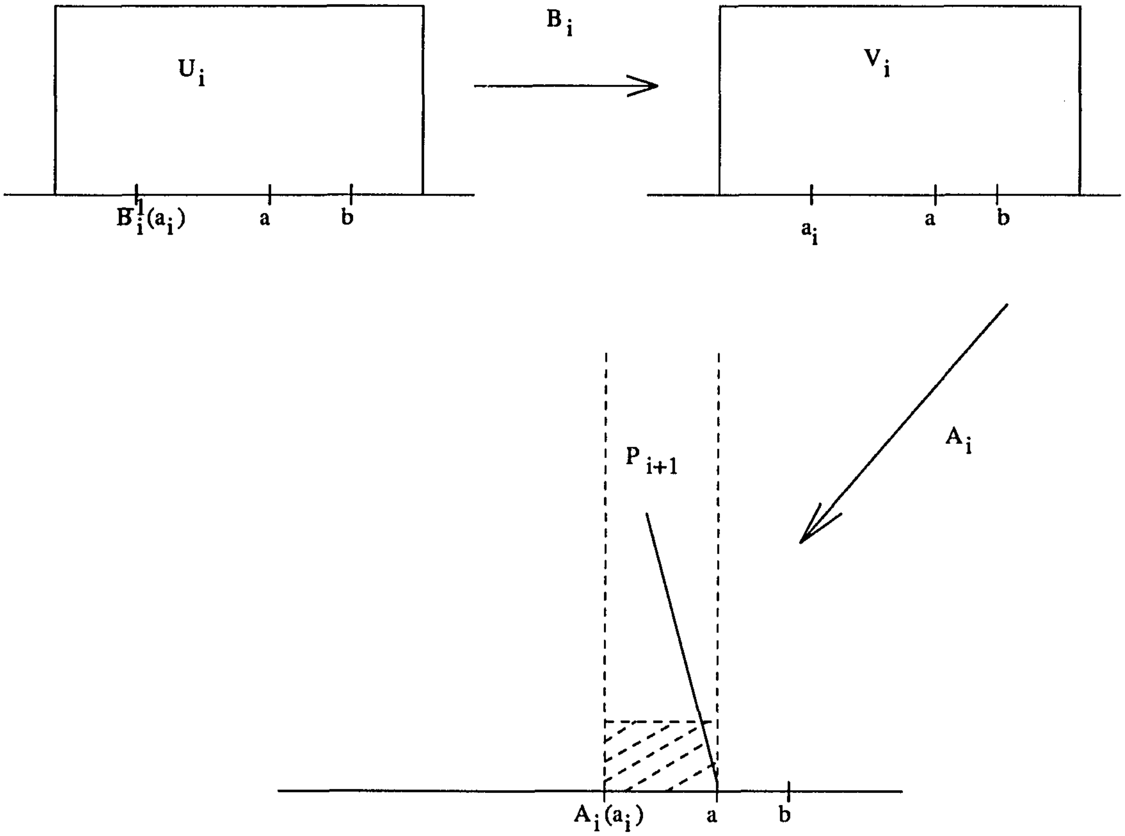


Figure 3.3

the geodesics (a, b) in $\mathcal{C} \setminus (-\infty, a] \cup [b, +\infty)$. By Schwarzian lemma they are mapped into themselves by any element or any composition from S.

Observation 2. Any Schlicht map defined on $\mathcal{C} \setminus (-\infty, a] \cup [b, +\infty)$ has bounded distortion on any bouned region R_0 as the following.

The distortion depends on the geometry of R_0. Also it has exponentially small nonlinearity on a region such as R_n which is exponentially small.

Case (2) Suppose there is a first i such that $P_i \in U_i$. We may assume that the angle of P_i as viewed from a is large because P_{i-1} does not belong to U_{i-1}

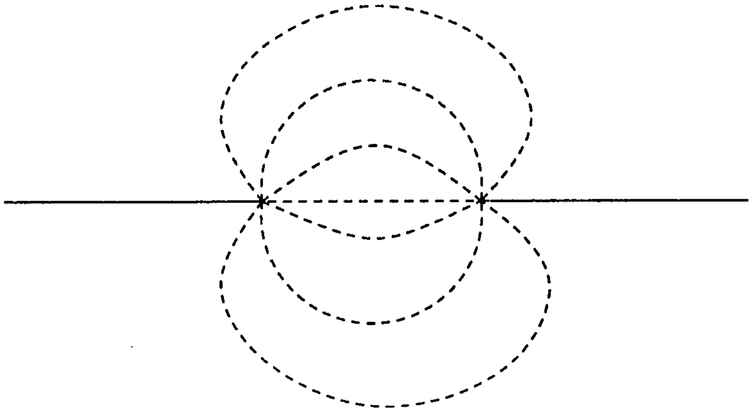


Figure 3.4

and the reason is used in the Case (1). Now we apply a fixed number l of the factors $A_iB_i, A_{i+1}B_{i+1}, \cdots, A_{i+l}B_{i+l}$ until a_{i+l} is much further away from a than P_{i+1}. This happens because of the assumption 2) and observation 1. By observation 2, the angle as viewed from a is only bounded distorted during these l itrates and after that the subsequent factors A_jB_j for $j > i+l$ only cause a sequence of distortions which are exponentially small and decay geometrically slow. Thus the angles of $P_2, P_3, \cdots$ as viewed from a remain large in all cases.

3.2 Sector Inequality

For the proof of the complex bound's property, we need the following sector inequality.

Suppose $a < a' < b' < b < c$ are reals and F is a Schlicht map of a hemidisk D of radius R and center b' into a sector with angles $\theta, \pi/2$ resting on a', b' in the upper half plane H. Let N be the Poincaré neighborhood of

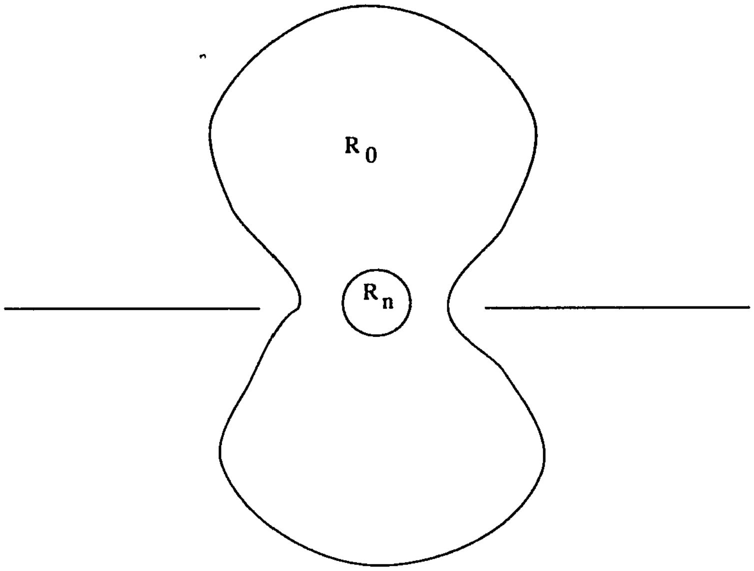


Figure 3.5

(a, c) in the upper half place H corresponding to the Euclidean disk of radius $\bar{R}$ whose boundary passes through a and c. Assume θ is fixed, all non-zero distances between $\{a, b, c, a', b', c'\}$ are of order 1, and F carries $\{a, b, c\}$ to $\{a', b', c'\}$ in order.

Theorem 3.2 (Sector Inequality) *For $\bar{R}$ sufficiently large and $R/\bar{R}$ sufficiently large compared to $\bar{R}$, N contains $F(N)$ plus all points of the upper half plane within a definite Euclidean distance to $F(N)$.*

Proof: 1) Let ψ be the Riemann mapping of the upper half plane to the sector carrying ∞, a, c to ∞, a', c' in order. Let $U(R)$ denote $\psi(D(R))$ where $D(R)$ is the hemidisk of radius R centered at b'.

2) Suppose U is a simply connected domain with an arc γ on its boundary and F is a complex analytic map of $U \rightarrow U$ which continuously extends to

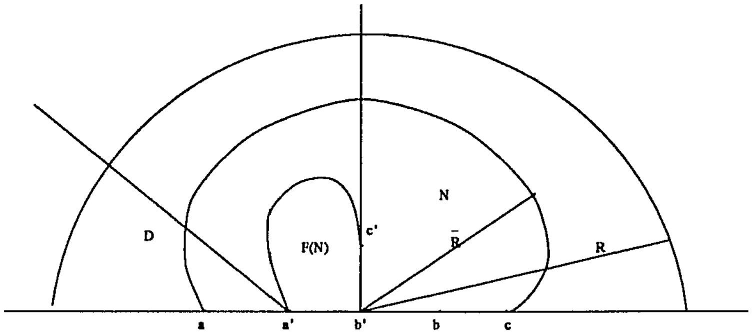


Figure 3.6

γ and preserves γ. Let $\tilde{U}$, $\tilde{F}$ denote the double of U and F along γ. Then γ is a geodesic in the Poincaré metric of $\tilde{U}$ preserved by $\tilde{F}$. By Schwartz's lemma $\tilde{F}$ preserves the neighborhoods of γ of Poincaré distance $\leq c$ for all $c > 0$. Clearly F preserves the restrictions of these neighborhoods to U. We go on calling them Poincaré geodesic neighborhoods of γ in U.

3) The Riemann map ψ, outside a fixed neighborhood of the corner (a, b, c), is a composition of the fractional power z^{α} of z and a mapping of uniformly bounded distortion of the Euclidean metric, where $\alpha = 1/2 - \theta/\pi$. When R is large enough $U(R)$ is almost the intersection of the sector with the hemidisk of radius R^{α} centered at b'.

4) Now outside $D(R)$ when R is large enough, F is the composition of the above power map and some Schlicht map G of $U(R)$ into the sector region S which fixes a' and c'.

5) Let γ be the arc of the boundary of the sector from a' to c'. By the arguement of 2) G takes the Poincaré geodesic neighgorhoods of γ in $U(R)$ into the poincare neighborhoods of γ in the sector region S preserving or decreasing the distance to γ.

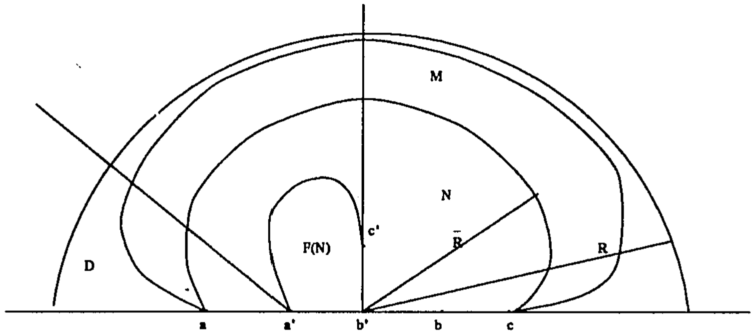


Figure 3.7

6) The boundary of S and the boundary of $U(R)$ only differ a very little at the Euclidean distance R^α from γ. Thus the geodesics much closer to γ than R^α are about the same for $U(R)$ and for S. So G doesn't move these neighborhoods too much.

Now 3), 4) and 6) give the proof of the theorem.

Replace the hemidisk D of raduis R centered at b' in the previous theorem by the largest "Poincaré neighborhood" M of (c, a) in the upper half plane contained in D (it is defined in 2) of the above proof. Suppose F is a Schlicht map of M into the sector which continuously extends to the boundary arc (a, c) and carries a, b, c to a', b', c' in order. Let N be the same as the above, which is the Poincaré neighborhood of (a, c) of Euclidean diameter $\tilde{R}$.

Theorem 3.3 (Modified Sector Inequality) *For $\tilde{R}$ large and $R/\tilde{R}$ sufficiently large compared to $\tilde{R}$, N contains $F(N)$ plus all points of the upper half plane within a definite Euclidean distance to $F(N)$.*

The following corollary will be used in the proof of the complex bound's property of renormalizations of bimodal maps.

Suppose $a < a' < o < b'$ and $c < b < O$ are real. Actually a and c

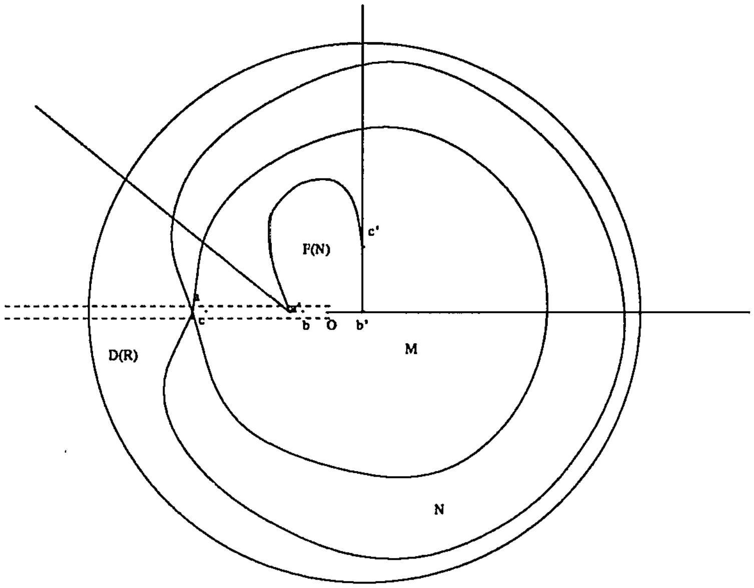


Figure 3.8

are at the same place on the real line. Let D be a slit disk D of radius R and center o. Let S be a sector with angles θ, $\pi/2$ resting at a', b' in the upper plane H. Let N be the Poincaré neighborhood $a\bar{o}c$ in the slit plane $\mathcal{C} \setminus (-\infty, o]$ corresponding the Euclidean disk of radius $\bar{R}$ whose boundary through a and c. Let M be the largest Poincaré neighborhood of $a\bar{o}c$ in the slit plane $\mathcal{C} \setminus (-\infty, o]$ contained in D. Assume θ is fixed and all non-zero distances between $\{a, a', o, c'\}$ are of order 1.

Corollary 3.1 *Suppose F is a Schlicht map from the slit plane $\mathcal{C} \setminus (-\infty, o]$ into the sector S and carries $\{a, o, b, c\}$ to $\{a', o, b', c'\}$ in order. For $\bar{R}$ large and $R/\bar{R}$ sufficiently large compared to $\bar{R}$, N (or M) contains $F(N)$ (or $F(M)$) plus all points of the slit plane $\mathcal{C} \setminus (-\infty, o]$ within a definite Euclidean distance to $F(N)$ (or $F(M)$).*

3.3 Complex Bound's Property

Let us start with a review of the complex bound's property of renormalizations of unimodal maps.

Definition 3.1 *An unimodal map f belongs to the Epstein class if $f = h \circ Q$, where Q is a quadratic polynomial, h is a homeomorphism and there exists a neighborhood J of I such that h^{-1} has a complex analytic injective extension to the region $\mathcal{C} \setminus \{x : x \text{ is real but not in } J\}$. We wirte $f \in E(J)$.*

In [S1] Sullivan proves the following theorem.

Theorem 3.4 *For any integer $T > 0$, there exist $N(T) > 0$ and $m(T) > 0$ such that for any infinitely renormalizable unimodal map $f = h \circ Q : I \to I$ of combinatorial types bounded by T of the Epstein class $E(J)$, $I \subset J$, and for any integer $n \geq N(T)$ the n^{th} renormalization $g = R^n f$ has a complex analytic extension G to some disk $D \subset \mathcal{C}$ so that $G : D \to G(D)$ is a degree 2 polynomial-like mapping and $G(D) \setminus D$ has conformal modulus $> m(T)$.*

The purpose of this section is to prove the complex bound's property for renormalizations of infinitely renormalizable multimodal maps of Epstein class. This section will be set in this way: Details will be given to get a complex bound of renormalizations of maps with two turning points and the theorem will will be stated in a general case (the proof is just a modification).

Definition 3.2 *An infinitely renormalizable bimodal map $f : I \to I$ is of the Epstein class if there exist neighborhoods U_i and V_i of critical points c_i, $i = 1, 2$, and $h_1 : U_1 \to V_2$, $h_2 : U_2 \to V_1$ such that $f = h_1 \circ Q_1$ for $x \in U_1$,*

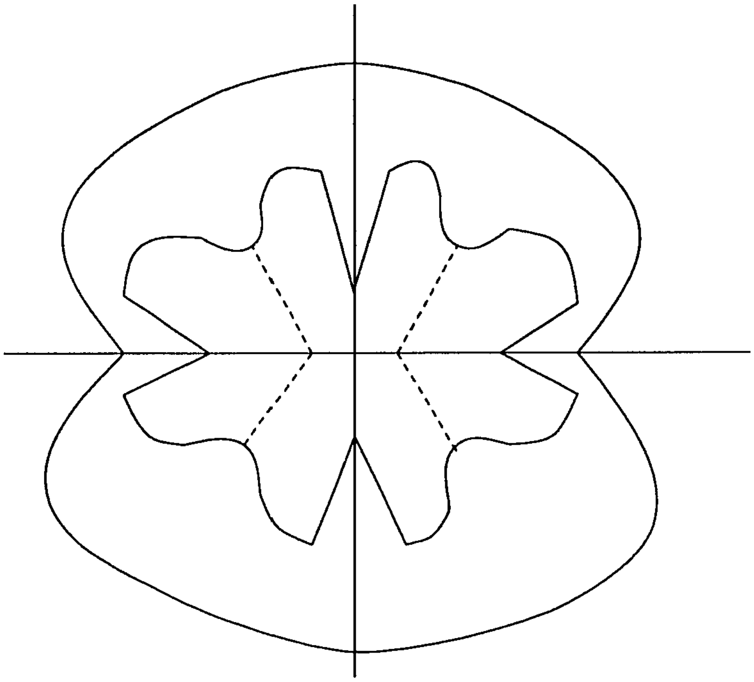
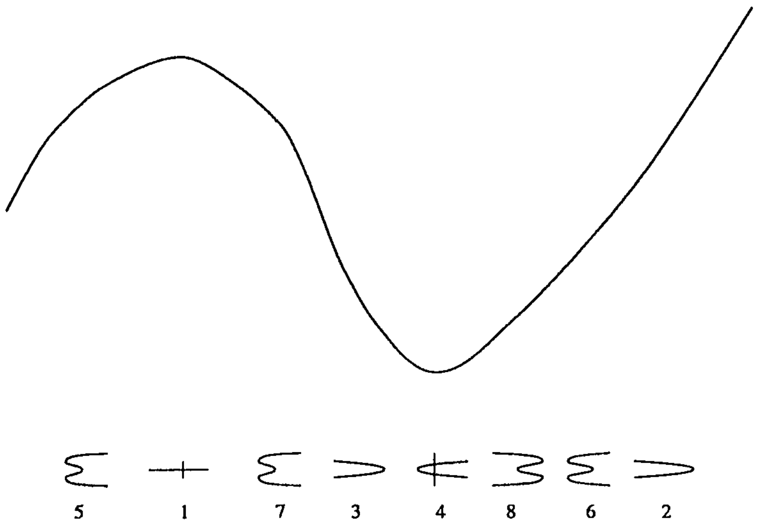


Figure 3.9

$f = h_2 \circ Q_2$ for $x \in U_2$ and h_i has a complex injective analytic extension in $\mathcal{C} \setminus \{x : x$ is real but not in $V_i\}$, $i = 1, 2$.

We will give all the details to prove the following thoerem.

Theorem 3.5 *For any integer $T > 0$ there exist an integer $N(T) > 0$ and $m(T) > 0$ such that for any infinitely renormalizable bimodal map $f : I \longrightarrow I$ of combinatorial types bounded by T of the Epstein class and for any $n \geq N(T)$ the n^{th} renormalization $g = R^n f$ has a complex analytic extension G to some disk $D \subset \mathcal{C}$ so that $G : D \longrightarrow G(D)$ is a symmetric complex quartic-like mappings and $G(D) \setminus D$ has conformal modulus $> m(T)$.*



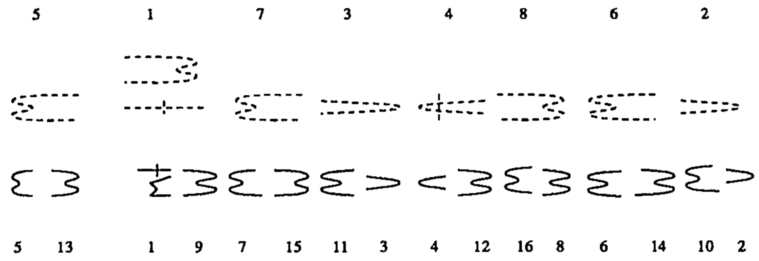
Here numbers denote the order of the dynamical intervals under iterates.

Figure 3.10

In order to prove the above complex bound's property, the first step is to factor the pullbacks into the composition satisfying the conditions of the Sector Lemma.

Consider an infinitely renormalizable bimodal map $f : I \to I$ of combinatorial types bounded by T with critical points c_1, c_2 and critical values v_1, v_2 respectively. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_n, \cdots$ be interval collections at each level of renormalizations. Let $I_n(c_1)$, $I_n(c_2)$ and $I_n(v_1)$, $I_n(v_2)$ denote the intervals of Γ_n containing the critical points c_1, c_2 and critical values v_1, v_2 respectively.

Assume f also belongs to the Epstein class. The following figure 3.10 and 3.12 will illustrate the epoch decompositions which will be defined immediately, and figure 3.11 and 3.13 (at the end of this chapter) will illustrate the proof of Prop. 3.3.



The numbers denote the order of the dynamical intervals under iterates.

Figure 3.12

Without loss of generality, we select the first interval in Γ_n around the critical point c_1. Consider the backwards composition $f(n)$ from $I_n(c_1)$ to $_n(v_1)$ passing through each n^{th} level interval in Γ_n. Define the scale of a factor f^{-1} of $f(n)$ to be the largest s so that its domain at leval n belongs to $I_s(v_1)$. Note that $f(n)$ is monotonic if it is restricted on the interval $R^n(f)(I_n(c_1))$.

Divide the composition $f(n)$ into epochs by: epoch $(n-1)$ is from the beginning of the composition up to and including the last factor of scale $n-1$, epoch $(n-2)$ is from there up to and including the last factor of scale $n-2$, etc.

Among the pullbacks of $R^n(f)(I_n(c_1))$ along the pullback composition $f(n)$, a pullback is called left (right) if it is dynamically related to $I_n(v_1)$ by an orientation reversing (preserving) map. The interval of Γ_n containing a left (right) pullback is called a left (right) interval.

Definition 3.3 *The backwards composition from $R^s(f)(I_s(c_1))$ to $I_s(v_1)$ at level s restricted to an n^{th} level interval in $I_n(c_1)$ is called a basic map at level s, where $s \leq n$.*

Prop. 3.1 *The last visit during epoch (j) to an n^{th} level interval in $I_j(v_1)$ lands on a left interval.*

Proof: Assume f is an increasing, decreasing then increasing bimodal map. Consider the case when the composition $f(j)$ from $R^j(f)(I_j(c_1))$ to $I_j(v_1)$ is an orientation preserving map. Now R^j is an increasing, decreasing, increasing then decreasing trimodal map. If we consider $R^j(f)$ as a map from $I_j(v_1)$ into itself then it maps $I_n(v_1)$ to the interval J_n furthest to the left in $I_j(v_1)$ by an orientation reversing map. Thus J_n is a left interval and the backwards composition of $f(n)$ in the epoch (j) is the composition of inverse branches of $R^j(f)$ running through all the intervals of Γ_n in $I_n(v_1)$ arriving last at J_n.

A left (right) root is the part of the factor f^{-1} starting at a left (right) interval corresponding to Q_i^{-1} in the factoring $f = h_i \circ Q_i, i = 1.2$. In epoch (j) mark all the left intervals of scale j. Let B_α denote the part of basic composition between two marked left roots.

The following proposition is trivial.

Prop. 3.2 *Suppose the marked left root just after some composition B_α has scale j. Then B_α is a finite composition of basic maps at level j, right roots and restrictions of $h_i^{-1}, i = 1, 2$. The number of each is bounded in terms of the bound T of the combinatorial types.*

Consider renormalizable increasing-decreasing-increasing bimodal maps. Let c_1, c_2 and v_1, v_2 be critical points and corresponding critical values. Assume $v_1 > c_1$ and $v_2 < c_1$. Without loss of generality, we assume in the second renormalization, $f(I_2(c_1))$ does not contain c_2. Now let A_1 be the marked left square root at scale $(n-1)$ furthest to $I_n(v_1)$ in $I_{n-1}(v_1)$, which is also the last visit in the backwards pullbacks during epoch $(n-1)$. Let $A_2, A_3, \cdots, A_{n-1}$ be the subsequent marked left square roots. Then we can decompose $f(n)$ as follow

$$f(n) = A_n B_n A_{n-1} B_{n-1} \cdots A_1 B_1,$$

where B_i, $i = 1, 2, \cdots, n$, are Schlicht maps, $A_i, i = 1, 2, \cdots, n-1$, are left square roots and A_n is a right square root related to the singular point v_2.

Prop. 3.3 *Assume f is an infinitely renormalizable increasing-decreasing-increasing bimodal map of combinatorial types bounded by T of the Epstein class. Then the composition*

$$A_{n-1} B_{n-1} \cdots A_1 B_1$$

in the above $f(n)$ satisfies the hypothesis 1), 2) and 3) of the Sector Lemma.

Remark: The dependence of the constants k, K, λ on f and n are "beau" as the same in the real bound's property.

Proof: 1) is true by the construction. 2) follows from the real bound's property. The proof of 3) needs some explanation.

We express two copies of complex planes in the polar coordinate system by

$$2\mathcal{C}_{\{0\}} = \{(\rho, \theta) : \rho \geq 0, -3\pi < \theta \leq \pi\}$$

where 0 is the origin in the polar coordinate system. We could select any point in the complex plane as this origin. We also consider it as the one-to-one image of a complex plane under a quadratic polynomial. The image of

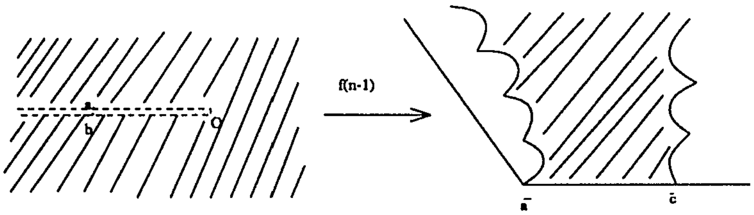


Figure 3.14

$2\mathcal{C}_{\{0\}}$ under the square root from $I_n(v_2)$ to $I_n(c_2)$ will be a complex plane. Here we take v_2 as the origin 0 in $2\mathcal{C}_{\{0\}}$. After certain pullbacks it will meet the second square root from $I_n(v_1)$ to $I_n(c_1)$. This maps a complex plane to $\mathcal{C}_{\{v_1\}}$ into a half plane. In the subsequent pullbacks the real bound's property makes 3) satisfied (Figure 2.11 and Figure 2.13 illustrate the proof).

Now we can begin to prove the main theorem of this section, i.e., the complex bound's property of renormalizations of infinitely renormalizable bimodal maps of combinatorial types bounded by T of Epstein class.

Proof: Let $f(n)$ denote the backwards pullbacks going from the critical interval $I_n(c_1) \in \Gamma_n$ to the critical value interval $I_n(v_1) \in \Gamma_n$. By the Prop. 2.3,

$$f(n) = A_n B_n A_{n-1} B_{n-1} \cdots A_1 B_1,$$

where $A_i, i = 1, 2, \cdots, n-1$ are left square roots related to the critical value v_1, $B_i, i = 1, 2, \cdots, n$, are Schlicht maps, and A_n is the right square root related to the critical value v_2.

We denote

$$f(n-1) = A_{n-1} B_{n-1} \cdots, A_1 B_1.$$

Let $a\bar{o}c$ denote the maximal interval where $f(n)$ is a "homeomorphism" into reals and let $(\bar{a}, \bar{c}) = f(n-1)(a\bar{o}c)$. By the Sector Lemma the im-

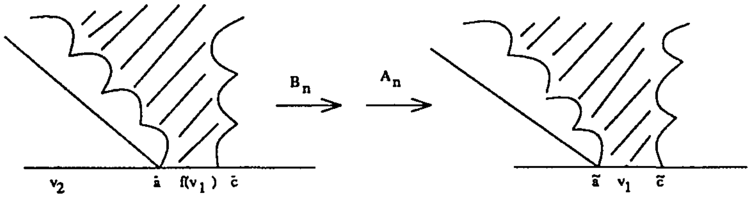


Figure 3.15

age under $f(n-1)$ of the slit plane $C \setminus (-\infty, o]$ is contained in the sector $S_1 : 0 \leq \arg(z-\bar{a}) \leq \pi - \theta(T)$. Now we apply $A_n B_n$ to the sector S_1. By the real bound's property $\bar{a}\bar{c}$ is an interval small enough containing the point $f(v_1)$ in the deep renormalizations. B_n is well-defined on a large Poincaré neighborhood U of $\bar{a}\bar{c}$ in the upper half plane H corresponding to an Euclidean disk of radius R. So B_n just reduces the angle of the sector by a definite amount in Poincaré neighborhood U' of $\bar{a}\bar{c}$ inthe upper half plane H corresponding to an Euclidean disk of radius $R/2$. Since the singularity v_2 of A_n is also far away from $\bar{a}\bar{c}$ compared to the length of $\bar{a}\bar{c}$ in the deep renormalizations, A_n just reduces the angle of the sector by a definite amount. Hence $A_n B_n$ maps the sector S_1 into a sector $S_2 : \theta'(T) \leq \arg(z-\tilde{a}) \leq \pi$.

Now apply two branches of f^{-1} around v_1 to the sector S_2, we get the following figure.

There are several points to make.

(1) It is simple that $g_+^{-1}(M)$ in the above figure lies in a sector $\pi/2 \leq (z-b') \leq \pi\theta_1$ because we have only applied a bounded distortion map h^{-1} to the sector S_2 and then a right square root related to v_1. Here $g_+^{-1} = f_+^{-1} f(n)$.

(2) The remark (1) applies to $g_-^{-1} = f_-^{-1} f(n)$ and $g_-^{-1}(M)$.

(3) The distance $\overline{ab'}$ is a definite factor greater than the distance $\overline{a'b'}$ and $\overline{a'o}$, where b', o and $-o$ are three critical points of $g = R^n(f)$.

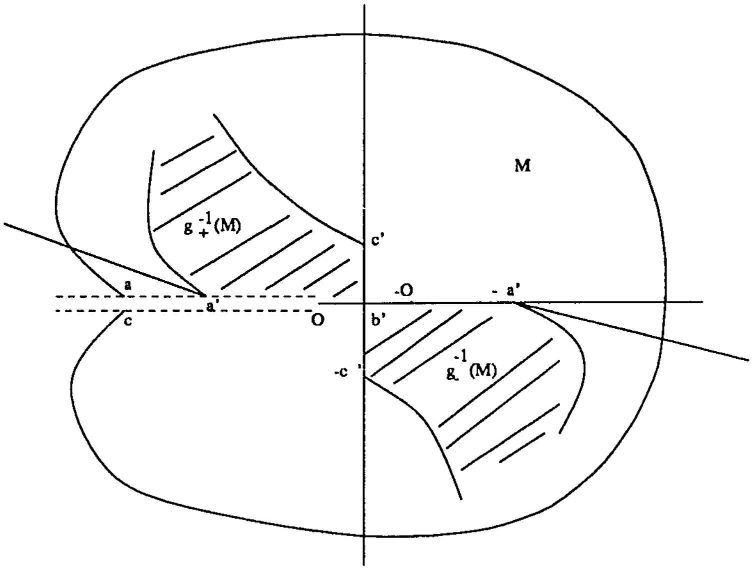


Figure 3.16

(4) If $n(T)$ is big enough then M can be taken sufficiently large compared to N in the Sector Inequality for some $N \subset M$ with N large relative to $a\bar{o}c$. This yields the following final figure.

The proof of the following theorem is just a modification of this section.

Definition 3.4 *A map $f : I \longrightarrow I$ is called to be of Epstein class if it is of the form $h \circ P$, where P is a real-coefficient polynomial and h^{-1} has a complex analytic injective extension to the complex plane minus the complement of a definite neighborhood of I in the real line.*

Theorem 3.6 (Complex Bound's Property) *Let $f = h \circ P : I \longrightarrow I$ be infinitely renormalizable of Epstein class and $k(n) \leq T$. Then there exists $n(T) \in N$ and $m(T) > 0$ such that for any $n \geq n(T)$, the n^{th} renormalization $g = R^n(f)$ has a complex extension G to some neighborhood U in the complex*

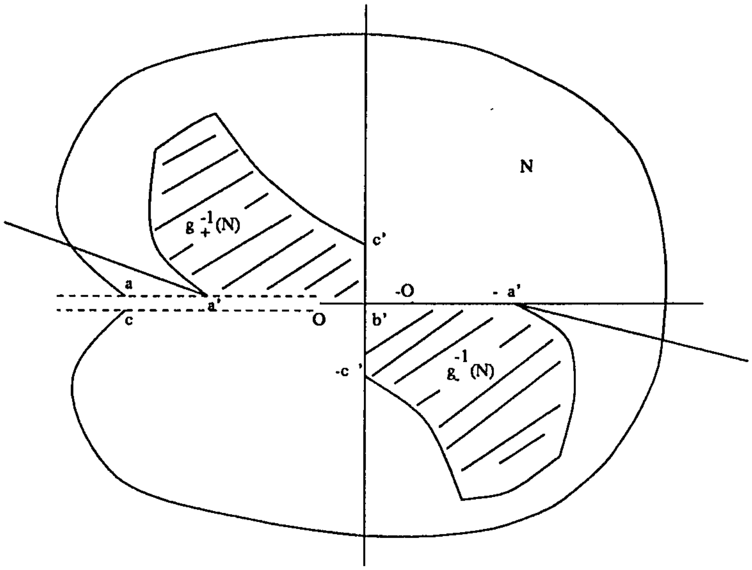


Figure 3.17

plane so that $G : U \longrightarrow G(U)$ *is a polynomial-like mapping and the modulus of* $G(U) \setminus U$ *is greater than* $m(T)$.

3.4 Polynomial-like Mappings

The bridge between the complex bound of renormalizations and a Teichmüller theory of the space of infinitely renormalizable maps with the same combinatorial types is Douady-Hubbard's theory of polynomial-like mappings. A polynomial-like mapping is defined as follows.

Definition 3.5 *A polynomial-like mapping of degree* d *is a triple* (U, U', f) *where* U *and* U' *are open subsets of the complex plane* C *holomorphically homeomorphic to disks with* U' *relatively compact in* U, *and* $f : U' \longrightarrow U$ *an analytic and proper mapping of degree* d.

Let $P : C \rightarrow C$ be a polynomial of degree d and let $U = D_r$ be the disk of radius r. If r is large enough, then $U' = P^{-1}(U)$ is relatively compact in U and homeomorphic to a disk, and $P : U' \rightarrow U$ is annlytic and proper of degree d. Hence (U, U', P) is a polynomial-like mapping.

Definition 3.6 *Let $f : U' \rightarrow U$ be a polynomial-like mapping of degree d. The set*

$$K_f = \cap_{n \geq 0} f^{-n}(U'),$$

consisting of all $z \in U'$ such that $f^n(z)$ is defined and belongs to U' for $n \geq 0$, is called the filled-in Julia set of f. The Julia set J_f of f is the boundary of K_f.

The filled-in Julia set is a compact subset of U'. As a dynamical system, f is mainly interesting near K_f. The following statements are standard for polynomials and they are also valid for polynomial-like mappings.

Prop. 3.4 *Every attractive cycle has at least one critical point in its immediate basin.*

Prop. 3.5 *The filled-in Julia set K_f is connected if and only if all critical orbits of f belong to K_f. If none of critical points belongs to K_f then K_f is a Cantor set.*

Let $f : U' \rightarrow U$ and $g : V' \rightarrow V$ be two polynomial-like mappings. We will say that f and g are topologically equivalent (denoted by $f \sim_{top.} g$) if there is a homeomorphism ϕ from a neighborhood of K_f onto a neighborhood of K_g such that $\phi \circ f = g \circ \phi$ near K_f. If ϕ is quasiconformal (resp. holomorphic) then we will say f and g are quasiconformal (resp. holomorphic) equivalent (denoted by $f \sim_{qc} g$ and $f \sim_{hol.} g$). We will say that f and g are hybrid equivalent (denoted by $f \sim_{hb} g$) if they are quasiconformally equivalent and ϕ almost has no complex dilatation, i.e., $\overline{\partial}\phi = 0$ almost everywhere on K_f. We see that

$$\begin{gathered} f \sim_{hol.} g \Longrightarrow f \sim_{hb} g \\ \Longrightarrow f \sim_{qc} g \Longrightarrow f \sim_{top.} g. \end{gathered}$$

If J_f is of measure 0 (as said in [DH], no example is known for which this does not hold), the condition $\overline{\partial}\phi = 0$ a.e. on K_f just mean that ϕ is analytic on interior of K_f. The following theorem give the relation between polynomial-like mappings and polynomials.

Theorem 3.7 (The Straightening Theorem) *[DH]*

1) Every polynomial-like mapping $f : U' \to U$ of degree d is hybrid equivalent to a polynomial P of degree d.

2) If K_f is connected, P is unique up to conjugation by an affine map.

Remark: By this theorem, Sullivan's nonwandering domain theorem and classification theorem of Fatou components are also valid for Fatou components inside K_f for any polynomial-like mapping f.

The hybrid-equivalent classes of polynomial-like mappings are called internal classes. The following is going to classify the polynomial-like mappings in the same internal class. We are only concerned with the cases in which the filled-in Julia sets are connected.

Definition 3.7 *Let $f : U' \to U$ and $g : V' \to V$ be two polynomial-like mappings, with K_f and K_g connected. f and g are externally equivalent* *(denoted by* $f \sim_{ext.} g$*) if there exist connected open sets* U_1, U_1', V_1 *and* V_1' *such that*

$$K_f \subset U_1' \subset U_1 \subset U,\ f^{-1}(U_1) = U_1',$$

$$K_g \subset V_1' \subset V_1 \subset V,\ g^{-1}(V_1) = V_1',$$

and a complex-analytic isomorphism

$$\phi : U_1 \setminus K_f \to V_1 \setminus K_g$$

such that

$$\phi \circ f = g \circ \phi.$$

Any degree d polynomial-like map $f : U' \to U$ with connected filled-in Julia set can be associated with a real analytic degree d expanding circle map $h_f : S^1 \to S^1$ which is unique up to conjugation by a rotation [DH]. h_f is called the external map of f. Since we are only interested in the case in which the filled-in Julia set is connected, the construction of h_f is very simple.

Let α be an isomorphism of $U \setminus K_f$ onto $W_+ = \{z : 1 < |z| < R\}$ ($\log R$ is the modulus of $U \setminus K_f$) such that $|\alpha(z)| \to 1$ when $d(z, K_f) \to 0$. Set $W_+' = \alpha(U' \setminus K_f)$ and $h_+ = \alpha \circ f \circ \alpha^{-1} : W_+' \to W_+$. Let $\tau : z \to \frac{1}{\bar{z}}$ be the reflection with respect to the unit circle, and set $W_- = \tau(W_+)$, $W_-' = \tau(W_+')$, $W = W_+ \cup S^1 \cup W_-$ and $W' = W_+' \cup S^1 \cup W_-'$. By the Schwartz reflection principle, h_+ extends to an analytic map $h : W' \to W$, the restriction of h to S^1 is h_f. The mapping is strongly expanding for degree $d \geq 2$.

Prop. 3.6 *Let* $f : U' \to U$ *and* $g : V' \to V$ *be two polynomial-like mappings with connected filled-in Julia set* K_f *and* K_g*,* h_f *and* h_g *be their external* *maps. Then f and g are externally equivalent if and only if h_f and h_g are conjugated by a real analytic circle map.*

Proof: $\Longrightarrow$) It is straight forward.

$\Longleftarrow$) h_f, h_g and ϕ are real analytic and $\phi \circ h_f = h_g \circ \phi$ on the unit circle. h_f, h_g and ϕ can extend to complex analytic maps in a small annulus neighborhooh of the unit circle and the above identity can also extend to the above neighborhood by the property of complex analytic maps. After conjugation by α constructed as the above, $\alpha^{-1} \circ \phi \circ \alpha$ conjugate f to g in a annulus neighborhood of K_f (outside). This proves they are externally equivalent.

Prop. 3.7 *Let $f : U' \longrightarrow U$ be a polynomial-like mapping of degree d with connected filled-in Julia set and $h : S^1 \longrightarrow S^1$ be an expanding analytic map of the same degree d. Then there exists a polynomial-like mapping $g : V' \longrightarrow V$ with connected filled-in Julia set which is hybrid equivalent to f and it is in the same external class of h.*

Proof: Suppose α is constructed as the above which maps $U \setminus K_f$ onto an annulus domain whose inner boundary is the unit circle. We still denote by h the extension of h to an annulus domain outside the unit disk. It suffices to construct a quasiconformal map ϕ defined on a neighborhood of K_f with no complex dilatation almost everywhere in K_f and such that

$$\phi \circ f = \alpha^{-1} \circ h \circ \alpha \circ \phi$$

holds on $W \setminus K_f$, where W is a neighborhood of K_f.

Let denote $\tilde{h} = \alpha^{-1} \circ h \circ \alpha$. We can construct ϕ as follows. Without loss of generality, we assume $U \subset W$. Pick up any elliptic vector field μ on $U \setminus U'$, where $||\mu||_\infty < l$. Pull back the elliptic vector by f, i.e., $\mu(z) = \mu(f(z))\frac{\overline{f'(z)}}{f'(z)}$, we get an elliptic vector field on $U \setminus K_f$. Let $\mu = 0$ on K_f, by the measurable Riemann mapping theorem, we will get ϕ with $\overline{\partial}\phi / \partial\phi = \mu$, which is unique up to pre or post composion with a Möbius transformation.

Since μ is invariant under f, $\phi \circ f \circ \phi^{-1}$ is analytic on $\phi(U)$. Indeed it is a polynomial-like mapping of degree d which is hybrid equivalent to f and it is in the same external class of h.

Definition 3.8 *The Mandelbrot set of the quadratic family $f_c(z) = z^2 + c$, $c \in \mathcal{C}$, is defined as*

$$M_c = \{c \in \mathcal{C} : \textit{the critical orbit of } f_c \textit{ is bounded}\}$$

$$= \{c \in \mathcal{C} : \textit{the Julia set } J(f_c) \textit{ is connected}\}.$$

From the Straightening Theorem and the Props. 2.6 and 2.7, it is easy to get the following theorem.

Theorem 3.8 (DH) *1) Every hybrid equivalent class cuts the Mandelbrot set once and only once.*

2) Each hybrid equivalent class is bijectively equivalent to the set of real analytic degree 2 expanding maps of S^1 up to real analytic conjugation.

Related to the renormalizations of infinitely renormalizable bimodal maps, we need to consider a special family of quartic maps.

Definition 3.9 *A symmetric complex quartic-like mapping is a degree* 4 *polynomial-like mapping* f *which is symmetric with respect to one of its critical points, i.e., there exists a critical point* c *of* f *such that* $f(z-c) = f(c-z)$ *for any* z *in the domain.*

Prop. 3.8 *Any symmetric complex quartic-like mapping* f *has one critical point as a middle point of the other two critical points where* f *has the same critical values.*

Prop. 3.9 *For a complex quartic polynomial* f, *if one of critical point is the middle point of the other two critical points then* f *is a complex symmetric polynomial. Furthermore any complex symmetric quartic polynomial* f *is conjugate to a polynomial of the form of*

$$z^4 + pz^2 + q$$

by an affine map, where $p, q \in C$.

Proof: Up to conjugation by a translation, we assume $-c$, 0 and c are three critical points of f. Then $f'(z) = a(z+c)z(z-c)$. So

$$f(z) = \frac{a}{4}z^4 - \frac{ac^2}{2}z^2 + r.$$

Clearly we can conjugate f to a polynomial of the form

$$z^4 + pz^2 + q$$

by a linear map, where $p, q \in C$.

We define the Mandelbrot set of the quartic family $f_{p,q}(z) = z^4 + pz^2 + q$ as the following.

$$M_{p,q} = \{(p,q) \in \mathcal{C} \times \mathcal{C} : J(f_{p,q}) \textit{ is connected}\}$$

$$= \{(p,q) \in \mathcal{C} \times \mathcal{C} : \textit{critical orbits of } f_{p,q} \textit{ are bounded}\}.$$

And let the intersection of $M_{p,q}$ with the real plane be

$$\dot{M}_{p,q} = \{(p,q) \in \mathcal{R} \times \mathcal{R} : J(f_{p,q}) \textit{ is connected}\}$$

$$= \{(p,q) \in \mathcal{R} \times \mathcal{R} : \textit{critical orbits of } f_{p,q} \textit{ are bounded}\}.$$

Related to the renormalizations of infinitely renormalizable bimodal maps, we are interested in the symmetric quartic polynomials $f_{p,q} = z^4 + pz^2 + q$ with $p \leq 0$ and $q \in \mathcal{R}$. Before we give the theorem to see how the internal classes of complex symmetric quartic-like mappings cut $M_{p,q}$ (or $\dot{M}_{p,q}$), we need the following modified Douady-Hubbard straightening theorem.

Theorem 3.9 *1) Every complex symmetric quartic-like mapping f is hybrid equivalent to a complex symmetric quartic polynomial P.*

2)If $K(f)$ is connected then P is unique up to conjugation by an affine map.

We also need the following modification of the Prop. 3.7.

Theorem 3.10 *Let f be a complex symmetric quartic-like mapping and $h : S^1 \to S^1$ be an expanding analytic map of degree 4. Then there exists a complex symmetric quartic-like mapping g which is hybrid equivalent to f and is in the same external class of h.*

Now we have the following result.

Theorem 3.11 *1) Every hybrid equivalent class of complex symmetric quartic-like mappings cuts* $\dot{M}_{p,q}$ *once and only once, but it cuts* $M_{p,q}$ *three times.*

2) Each hybrid equivalent class of complex symmetric quartic-like mappings is bijectively equivalent to the set of real analytic degree 4 *expanding maps of* S^1 *up to real analytic equivalence.*

We will refer to the various "submanifolds" of symmetric complex quartic-like mappings with constant internal classes as the prestable manifolds of the renormalization operator on infinitely renormalizable bimodal maps. Any $\mathcal{C}$-analytic family fixing the internal class defines a $\mathcal{C}$-analytic submanifold [DH]. Therefore in the context of symmetric complex quartic-like mappings, the stable manifolds of the renormalization operator have at least a real analytic structure.

3.5 Quasiconformal Conjugacy

In this section we prove all infinitely renormalizable Epstein-class maps with the same bounded combinatorics are actually hybrid equivalent.

First we introduce Thurston equivalences and the pullback conjugacy.

Given two polynomial-like mappings $F : U \to F(U)$ and $G : V \to G(V)$ with connected filled-in Julia sets. Assume X_F and X_G are contractible regions containing the post critical orbits C_F and C_G respectively for F and G.

Consider isotopic classes of pairs

$$(X_F, C_F) \xrightarrow{H_0} (X_G, C_G)$$

which send the critical orbits C_F to the critical orbits C_G in order. Start with such an isotopic class H_0 and lift it to H_1 through the branched covers so that the critical points of F go to the critical points of G in order.

$$(U, C_F) \xrightarrow{H_1} (V, C_G)$$

$$(F(U), C_F) \xrightarrow{H_0} (G(V), C_G).$$

Now we have defined a map $H_0 \to H_1$ on isotopic classes which is called the Thurston map. A Thurston equivalence is an isotopic class which is fixed by the Thurston map.

For the pullback conjugacy theorem we start with a representative of a Thurston equivalence $H_0 : F(U) \to G(V)$ which a) is also a conjugacy between the maps $F : \partial U \to \partial F(U)$ and $G : \partial V \to \partial G(V)$ and b) is a quasiconformal homeomorphism.

Then $H_0 : F(U) \to G(V)$ and the pullback $H_1 : U \to V$ restrict to maps $\partial H_i : \partial U \to \partial V$ satisfying $G \circ \partial H_i = \partial H_0 \circ F, i = 0, 1$. So either $\partial H_0 = \partial H_1$ or $\partial H_0 = \partial H_1 \circ \tau$ where τ is the involution of the degree d-cover $F : \partial U \to \partial G(U)$. Since H_0 and H_1 have the same isotopic class relative to C_F and C_G, $\partial H_0 = \partial H_1 : \partial U \to \partial V$. Thus we may add to H_1 the restriction of H_0 to the outer annulus $F(U) \setminus U$ to get an extension of H_1 to a homeomorphism between $F(U)$ and $G(V)$. This homeomorphism called $H_1 : F(U) \to G(V)$ will be quasiconformal if we make the assumption that ∂U in $F(U)$ is a quasicircle. Its maximal complex dilatation will be the same as that of H_0 because it is the union of H_0 on $F(U) \setminus U$ and a "complex analytic conjugacy" of H_0. We iterate the process to construct an infinite sequence of K-quasiconformal homeomorphisms with C_G carried to C_G in order and not changing on the outer rings $F(U) \setminus U$, $U \setminus F^{-1}(U)$ and etc. By a theorem of quasiconformal homeomorphisms, we can get convergent subsequences on whole $F(U)$. The limit actually exists on the union of these rings. So if the union is dense, all limits over subsequences are equal and are equal to the continuous extension of what is on the rings.

The union of the outer rings is dense in $F(U)$ if the filled-in Julia set has no interior point. In this case, the conformal distortion of the limiting conjugacy depends on the conformal distortion of H_0 on the fundamental domain $F(U) \setminus U$ and the conformal distortion of H_0 on the Julia set. The latter will be zero if there is no "measurable invariant Beltrami line bundle on the Julia set". Now we summrize the above discussion into the following theorem.

Theorem 3.12 *Suppose F and G are polynomial-like mappings which are Thurston equivalent (along their post critical orbits) via a quasiconformal homeomorphism. If $\partial U \subset F(U)$ and $\partial V \subset G(V)$ are quasicircles then there exists a quasiconformal conjugacy $H : F(U) \to G(V)$ between F and G. The conformal distortion of H only depends on the pairs $(F(U) \setminus U, F|_{\partial U})$ and $(G(V) \setminus V, G|_{\partial V})$ if the filled-in Julia set has no interior and the Julia set has no "measurable invariant Beltrami line bundle".*

Proof: We can deform and extend a quasiconformal homeomorphism, representing a Thurston equivalence, by an isotopy to make it satisfy a) and b). the above discussion amounts the proof.

Remark: In the case of infinitely renormalizable symmetric quadratic-like mappings, Curt McMullen proves that the Julia set cannot carry a "measurable invariant Beltrami line bundle" in [Mc]. I believe the strategy there can also used to prove the Julia set of infinitly renormalizable symmetric quartic-like mappings (or bimodal maps) cannot carry a "measurable invariant Beltrami line bundle". The classification theorem of Fatou components [S3] tells us the filled-in Julia set cannot have interior if the polynomial-like mapping is infinitely renormalizable. So in the context of infinitely renormalizable quadratic-like or quartic-like mappings, the assumptions in the above theorem are automatically satisfied.

In the next part the conformal distortions of the quasiconformal conjugacies between two external classes will be studied.

Let $m(x)$ denote the supremum of the conformal modulus of $F(N) \setminus N$ for some representative F of x, where $F : N \rightarrow F(N)$ is a degree m holomorphic mapping defined in a neighborhood N of $\{z : |z| = 1\}$ in $\{z : |z| \geq 1\}$ and x denotes the complex analytic equivalence class of the germ F. For example, if F is given by $z \rightarrow z^m$ then $m(x)$ is infinity. Let $distance(x, y)$ denote the infimum of the maximal conformal dilatations of the quasiconformal conjugacies between representatives of x and y on some neighborhoods of $\{z : |z| = 1\}$ in $\{z : |z| \geq 1\}$.

Theorem 3.13 *For any $\epsilon > 0$, there exists $0 < d(\epsilon) < \infty$ such that if x and y satisfies $m(x) > \epsilon$ and $m(y) > \epsilon$ then $distance(x, y) \leq d(\epsilon)$.*

Proof: Let $F : F^{-1}(N) \rightarrow N$ and $G : G^{-1}(N) \rightarrow N$ be representatives for x and y. Since the conformal modulus of $F^{-1}(N) \setminus F^{-2}(N)$ is half of the modulus of $N \setminus F^{-1}(N)$, there exists $0 < d(\epsilon) < \infty$ such that we can take suitable N_1 and N_2 for F and G so that the ratio of the moduli of

$N_1 \setminus F^{-1}(N_1)$ and $N_2 \setminus F^{-2}(N_2)$ is between $1/d(\epsilon)$ and $d(\epsilon)$. After coordinate changes we can assume $N_1 = N_2 = N$. Now start with the idnetity map on the outer boundary of N, pull it back through F and G, we get a map h which carries the outer boundary of $F^{-1}(N)$ to the out boundary of $G^{-1}(N)$. There exists a quasieconformal map H from $N \setminus F^{-1}(N)$ onto $N \setminus G^{-1}(N)$ such that the restriction of H to the boundary equals to the idntity map on the outer boundary and h on the inner boundary. The maximal conformal dilatation of H is controlled by the ratio of the moduli of $N \setminus F^{-1}(N)$ and $N \setminus G^{-1}(M)$. The pull back technique gives us a quasiconformal mapping $\tilde{H}$ between F and G, and the maximal conformal dilatations of $\tilde{H}$ and H are equal. We finish the proof.

The following theorem says that in the cases of bounded types the combinatorial types of infinitely renormalizable symmetric quadratic-like mappings, quartic-like mappings or bimodal maps of the Epstein class uniquely determine their internal classes. We state the theorem in the case of bimodal maps.

Theorem 3.14 *If two infinitely renormalizable multimodal maps F and G of the Epstein class have the same bounded combinatorial type $(\sigma_0, \sigma_1, \sigma_2, \cdots)$ then they are in the same internal class.*

Proof: All the renormalizations are bounded smooth quartic-like maps by the real bound's property. The critical orbital Cantor sets can be expressed as the intersection of the interval collections at all levels corresponding to F and G respectively. Futhermore, we can present the critical orbital Cantor sets as an intersection of the symmetric pictures with respect to the real axis

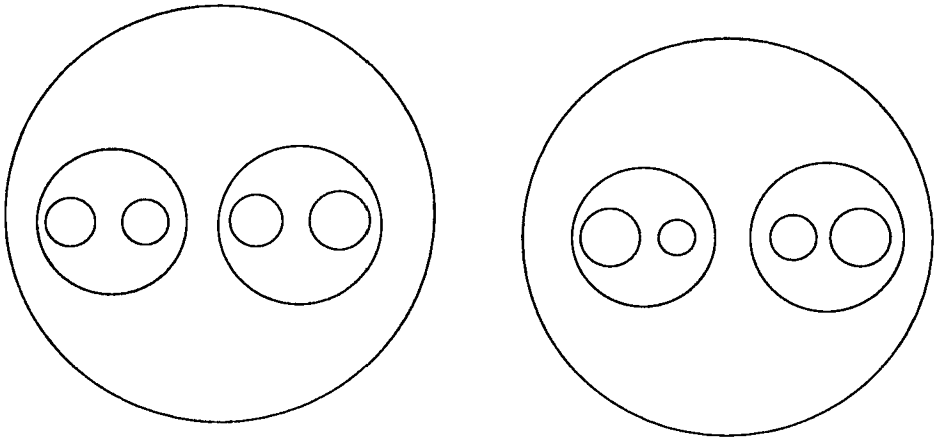


Figure 3.18

in plane of disks within disks as the following pictures.

The geometry of intermediate "pairs of pants" are bounded and eventually uniformly bounded because of the real and complex's properties. This leads us to construct a quasiconformal homeomorphism between two disks containing the critical orbital Cantor sets by choosing the standard symmetric rigid maps between corresponding circles and extending them to a diffeomorphism between intermediate "pairs of pants" regions with bounded maximal conformal dilatation. Use the pullback conjugacy technique to get a symmetric quasiconformal conjugacy between F and G.

Suppose that the above conjugacy between F and G has conformal dilatation on the Julia set of F. Let the complex polynomial $P(z) = z^3 + pz + q$ (p and q are two reals) be hybrid equivalent to F. Then P admits a symmetric invariant conformal structure on its Julia set. This conclusion is valid for all the polynomials $z \longrightarrow z^3 + pz + q$ with this kneading invariants. There is a closed line segment of such polynomials in the parameter space R^2 if theorem is false (restrict the consideration on a curve of symmetric Beltrami coefficients if necessary). Apply this statement about conformal structures to an end point of this line segment. Construct a quasiconformal deformation of the endpoint, which represents a cubic polynomial, to see it is not the endpoint. Contradiction.

Remark: In the context of quadratic polynomials, since Yoccoz proves a quadratic polynomial which carries an invariant line bundle on its Julia set is infinitely renormalizable, the above result implies the Julia set of a real quadratic polynomial carries no invariant line bundle. From the λ-lemma of [MSS], one obtains that every component of the interior of the Mandelbrot set meeting the real axis is hyperbolic. These conclusions are stated in the work of Curt McMullen in [Mc].

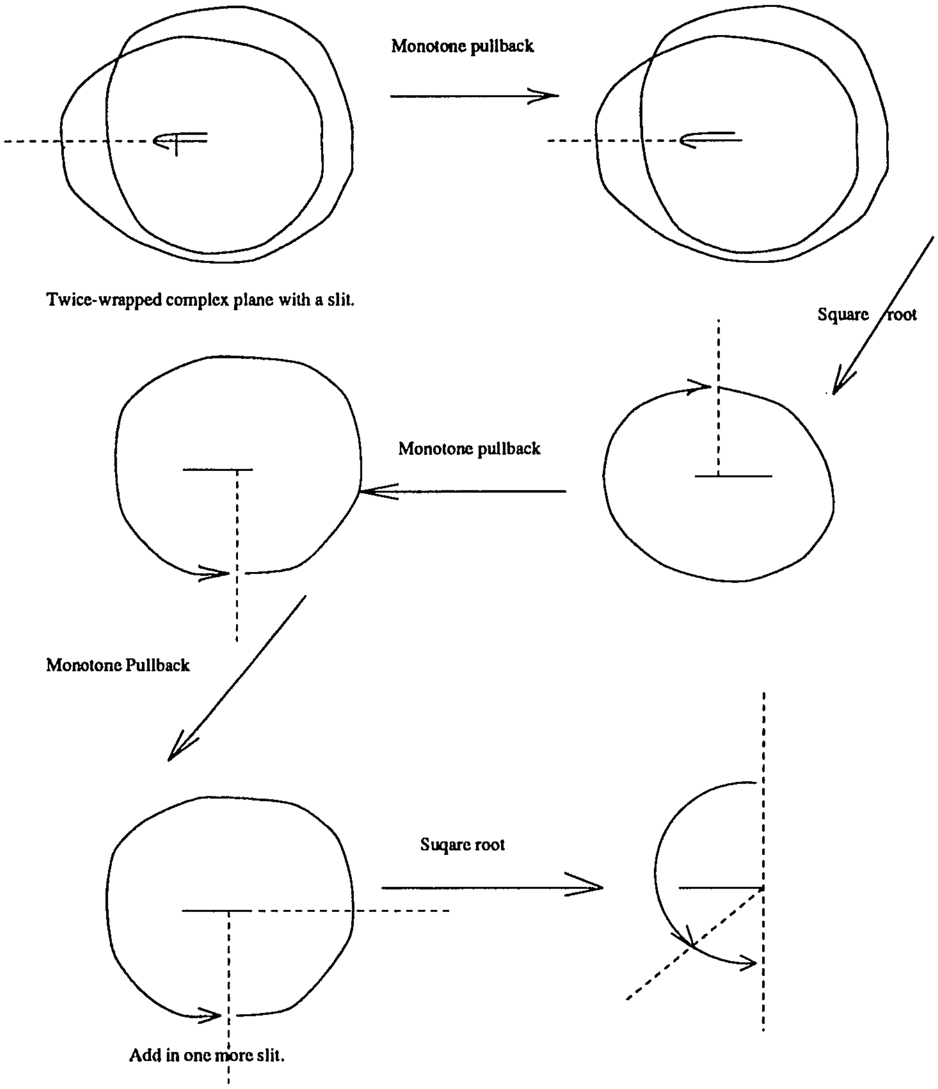

Monotone pullback
Twice-wrapped complex plane with a slit.
Square root
Monotone pullback
Monotone Pullback
Suqare root
Add in one more slit.

Figure 3.11

This figure continues the Figure 2.11.

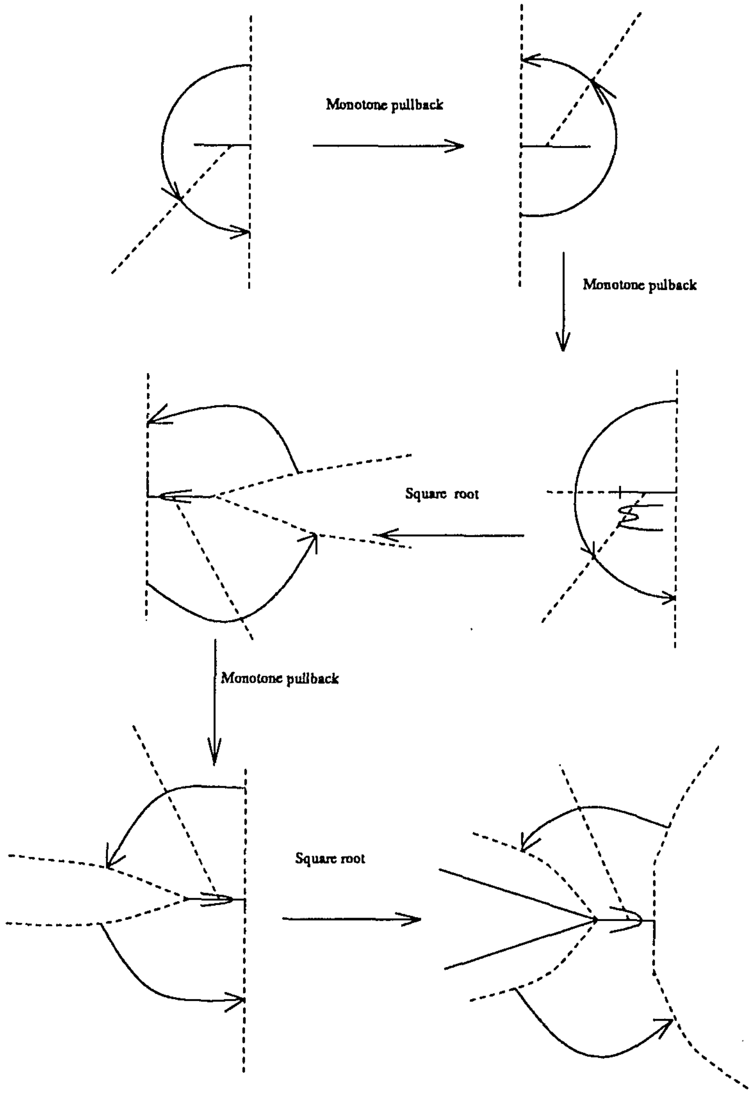


Figure 3.13

Chapter 4

Contraction of Renormalization in Teichmüller Metric

Laminations are somehow like foliations. The inverse limit space of a self map of a set is the space of backwards orbits. If the self map is a covering map of a manifold, then the inverse limit space can be viewed as a folliation with "sheets" are manifolds and fibres are Cantor sets. What we will be concerned with are the inverse limit spaces of expanding circle maps and analytic covering maps between two plane annuli. The space of the orbits of the natural extension map on an inverse limit space is called a dynamical lamination. A Teichmüller theory can be introduced to the laminations with sheets (will be called leaves) are hyperbolic Riemann surfaces. The space of conformal structures on dynamical lamination can be used to describe the space of dynamical systems. In fact, the Teichmüller theory on dynamical laminations and the complex bound's property of renormalizations lead to the contraction property of the renormalizations of quadratic-like mappings, due to Dennis P. Sullivan. Without any difficulty, we will see the same Teichmüller theory on the set of the external classes in a same internal class with connected Julia sets of any degree. The contraction property is also valid for renormalizations of multimodal maps of Epstein class since they also have a complex bound's property.

4.1 Inverse Limit Spaces as Laminations

Definition 4.1 *For any set X, any self map f of X, the inverse limit space $\tilde{X}_f$ is defined as follows:*

$$\tilde{X}_f = \{\tilde{x} = (..., x_n, ..., x_1, x_0) : f(x_n) = x_{n-1}, n \in N\}.$$

Furthermore, let $\pi_f : \tilde{X}_f \longrightarrow X$ be defined by $\pi_f((..., x_n, ..., x_1, x_0)) = x_0$, and $\tilde{f} : \tilde{X}_f \longrightarrow \tilde{X}_f$ be defined by $\tilde{f}((..., x_n, ..., x_1, x_0)) = (..., x_{n-1}, ..., x_1, x_0, f(x_0))$.

Prop. 4.1 *If X is a topological space, f is a continuous self surjection of X, then $\tilde{f}$ is a homeomorphism on $\tilde{X}_f$ with the weak product topology.*

Remark: If X is a topological space, f is a homeomorphism of X, then π_f conjugates $\tilde{f} : \tilde{X}_f \longrightarrow \tilde{X}_f$ with $f : X \longrightarrow X$.

If X is good enough, $f : X \longrightarrow X$ is a regular covering map, then we can define a leaf equivalent relation on the points of $\tilde{X}_f$. Every leaf equivalent class will be called a leaf.

Definition 4.2 *Let X be a connected manifold, $f : X \longrightarrow X$ be a regular covering map, two points $(..., x_n, ..., x_1, x_0)$ and $(..., y_n, ..., y_1, y_0)$ are leaf equivalent if there exists a curve γ_0 in X between x_0 and y_0 such that the* *lifting of* γ_{i-1} *with the initial point at* x_i *will end at the point* y_i, $i \in N$. *Every leaf equivalent class is called a leaf.*

Prop. 4.2 *If* X *and* $f : X \to X$ *are the same as the above, then every leaf is dense in* $\tilde{X}_f$.

Every leaf is actually a manifold. We will give the definition of a manifold lamination and put the manifold lamination structure on the above inverse limit space.

Definition 4.3 *Let* $\mathcal{L}$ *be a Hausdorff topological space. A* n*-dimensional manifold lamination atlas on* $\mathcal{L}$ *is a collection of open homeomorphisms:* $\phi_i : U_i \to R^n \times \Lambda$*(each of which is called a local coordinate chart or a flow box), where* Λ *is a topological space and* $\{U_i\}$ *is an open cover of* $\mathcal{L}$*, satisfying that the overlap maps* $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ *are of the form*

$$(x, \lambda) \mapsto (\phi_{ij}(x, \lambda), \Lambda_{ij}(\lambda)).$$

A n*-dimensional manifold lamination structure on* $\mathcal{L}$ *is an equivalent class of* n*-dimensional manifold lamination atlases on* $\mathcal{L}$*, where two atlases are equivalent if their union is again an atlas. We call* $\phi_i^{-1}(R^n \times \{\lambda\})$ *a local sheet. Furthermore we say* x *and* y *of* $\mathcal{L}$ *are equivalent if there exists a chain* $x_0 = x, x_1, ..., x_n = y$ *such that* x_i *and* x_{i+1} *belong to the same local sheet. An equivalent class* L *of this equivalent relation in* $\mathcal{L}$ *is called a leaf.*

Clearly, the leaf defined in a n-dimensional manifold lamination is a manifold. If an inverse limit space $\tilde{X}_f$ has a n-dimensional manifold structure, then the above two leaf equivalent classes are same in $\tilde{X}_f$.

Suppose X is a connected n-dimensional manifold, $f : X \to X$ is a proper covering map with degree m. For any point x_0 of X, x_0 has m preimages. The set of backward orbits of x_0 is a m-nary Cantor set. We say x_0 allow a thick Cantor set of backward orbits if there exists a neighborhood U of x_0 such that $f^{-n}(U)$ has m^n disjoint components for any $n \in N$. Clearly f maps any component of $f^{-n}(U)$ homeomorphically onto one of the component of $f^{-n+1}(U)$. Hence

$$\pi_f^{-1}(U) \cong U \times Z_m^\infty \cong U \times \pi_f^{-1}(x_0),$$

where $\pi_f : X_f \to X : (..., x_n, ..., x_1, x_0) \mapsto x_0$.

We will take $\phi_i : \pi_f^{-1}(U) \to U \times Z_m^\infty$ as local coordinate charts, where Λ is a m-nary Cantor set. If $\pi_f^{-1}(U) \cap \pi_f^{-1}(V) \neq \emptyset$, then $U \cap V \neq \emptyset$. The overlap map restricted to a leaf is the overlap map between U and V. The collection of all thick Cantor sets gives a manifold lamination structure atlas. An equivalent class of the atlases gives a manifold lamination structure on the inverse limit set $\tilde{X}_f$.

Theorem 4.1 *Let X be a connected n-dimensional manifold and f be a self covering map of X with degree m. Suppose every point $x \in X$ allows a thick Cantor set of backward orbits. Then the inverse limit space $\tilde{X}_f$ is a n-dimensional lamination. And every leaf of this lamination is a covering space of X.*

Proof: The first statement comes from the above discussion. One can easily see the second statement is true from the first defination of a leaf in a lamination.

Corollary 4.1 *Let* Y *be an open subset of* X *and* $f : Y \to X$ *be a surjective covering map. If there is a thick Cantor set at each point of* X*, the inverse limit space is also a manifold lamination.*

Remark: 1) If $f : X \to X$ is an expanding covering map or a Markov covering map, then every point of X allows a thick Cantor set of backward orbits.

2) In the most cases we treat, every leaf of $\tilde{X}_f$ is a universal covering space of X. Hence the lamination $\tilde{X}_f$ is sometimes called the quantized universal covering space of $f : X \to X$.

For $x_0 \in X$, $\pi_f^{-1}(x_0)$ is called the fibre of $\pi_f : \tilde{X}_f \to X$ over x_0. Let $\tilde{x} = (....,x_n,...,x_1,x_0) \in \pi_f^{-1}(x_0)$, for any element g of the fundermental group $\pi_1(X,x_0)$ of X with the base at the point X_0, we have

$$\pi_f^{-1}(x_0) \cap \textit{the leaf passing through } \tilde{x}$$

$$= \{\tilde{g}(\tilde{x}) : g \in \pi_1(X,x_0)\},$$

where $\tilde{g}(\tilde{x}) = (...,y_n,...,y_1,y_0)$, y_n are end points of the lifts of g_n with initial points at x_n, $n \in N$, $g_0 = g$.

Theorem 4.2 *Let* X *and* f *be the same as in the above Theorem 3.1 and* $\tilde{f}$ *be the natural extension map of* f *to the space* $\tilde{X}_f$*. The orbital space* $\tilde{X}_f/\tilde{f}$ *is a maifold lamination.*

In the following context, what we will be concerned with are 1-dimensional manifold laminations (called solenoids) and 2-dimensional manifold laminations with each leaf is a Riemann Surface and overlap maps are holomorphisms in leaf directions (called Riemann surface laminations or solenoidal Riemann surfaces).

Example 1: Let $f : S^1 \to S^1$ be a $C^{1+\alpha}$ expanding map of degree m. The inverse limit space

$$\tilde{S}_f = \{(..., x_n, ..., x_1, x_0) : f(x_n) = x_{n-1}, n \in N\}$$

with the weak product topology has a one-dimensional manifold lamination structure. It is a solenoid.

The natural extension

$$\tilde{f} : \tilde{S}_f \to \tilde{S}_f : (..., x_n, ..., x_1, x_0) \mapsto (..., x_{n-1}, ..., x_0, f(x_0))$$

is a homeomorphism. Its inverse is a shift map, and $\tilde{f}$ covers f, i.e., $\pi_f \circ \tilde{f} = f \circ \pi_f$, where $\pi_f : \tilde{S}_f \to S^1 : (..., x_n, ..., x_1, x_0) \mapsto x_0$. Indeed, π_f is a locally trivial fibration whose fibres are m-nary Cantor sets. Each orbit of the adding-one machine in the m-nary Cantor set Z_m^∞ is dense in Z_m^∞. All points in a same orbit of the adding-one machine are in a same leaf. Each leaf is dense in $\tilde{S}_f$. Each leaf of $\tilde{S}_f$ is homeomorphic to R^1. Through the local homeomorphism π_f it naturally inherits a C^1 smooth structure. In fact, because $f \in C^{1+\alpha}$, $\tilde{S}_f$ carries a natural affine structure. More precisely, one can make overlaps between local coordinates to be $C^{1+\alpha}$ in the leaf direction and such that $\tilde{f}$ becomes affine under those local coordinate charts.

A periodic leaf L of period k under $\tilde{f}$ contains a unique periodic point $\tilde{x}$ of $\tilde{f}$ which is mapped by π_f to a periodic point x of f with the same period k. The multiplication factor of $\tilde{f}^k$ at $\tilde{x}$ is equal to the eigeinvalue of f^k at the periodec point x.

A Riemann surface lamination can be defined as follows.

Let $\mathcal{L}_f = \{(x,v) : x \in \tilde{S}_f, v \in T_xL_x\}$, where L_x denotes the leaf of $\tilde{S}_f$ through x and T_xL_x its tangent line through x. One can give a Riemann surface lamination structure on $\mathcal{L}_f$ by covering it by two flow boxes. Let p be the fixed point of f and $f(p') = f(p)$, $p' \neq p$. Let $x \in \tilde{S}_f \backslash \pi_f^{-1}(p)$. Let x_- and x_+ be the only two points on the leaf L_x such that the interval (x_-, x_+) of L_x containing x is mapped homeomorphically onto $S^1\backslash\{p\}$ by π_f. Select the affine structure on $\tilde{S}_f$. Let $h_x : L_x \to R$ be the $C^{1+\alpha}$ diffeomorphism which maps (x_-, x_+) to $(0,1)$. We define $\Phi : \mathcal{L}_f \setminus \{(y,v) : y \in \pi_f^{-1}(p), v \in T_yL_y\} \longrightarrow Z_m^\infty \times (0,1) \times R$ to be

$$\Phi(x,v) = (\rho(x), h_x(x), \pm|Dh_x(x) \cdot v|),$$

where ρ is the labeling function of sheets in the local coordinate chart, we take the "+" sign if $D\pi_f(x) \cdot v$ is a positively orientated vector in S^1 and the "-" if otherwise. We get the other flow box by the same construction, just using p' instead of p.

The lamination $\tilde{\mathcal{L}}_f$ is defined as

$$\tilde{\mathcal{L}}_f = \{(x,v) \in \mathcal{L}_f : D\pi_f(x) \cdot v > 0\}$$

with local coordinate charts are the restrictions of the above local coordinate charts on it.

Let $\tilde{F} : \tilde{\mathcal{L}}_f \to \tilde{\mathcal{L}}_f$ be the map

$$\tilde{F}(x,v) = (\tilde{f}(x), D\tilde{f}(x) \cdot v),$$

where $D\tilde{f}(x)$ is the derivative of $\tilde{f}$ to the leaf through x. In terms of the local coordinate chart Φ , the expansion of $\tilde{F}$ is an affine map on each leaf.

Clearly $\tilde{F}$ is an expansion of $\tilde{f}$ and hence is holomorphic in the leaf direction. So $\tilde{L}_f$ is a Riemann surface lamination.

The orbital space $\tilde{\mathcal{L}}_f/\tilde{F}$ is a compact Riemann surface lamination. Each leaf of $\tilde{\mathcal{L}}_f/\tilde{F}$ is the quotient of a leaf of $\tilde{\mathcal{L}}_f$ under $\tilde{F}$. A leaf of $\tilde{\mathcal{L}}_f/\tilde{F}$ is a cylinder if its preimage in $\tilde{\mathcal{L}}_f$ is periodic under $\tilde{F}$. Furthermore the conformal modulus of each cylindrical leaf corresponds to the eigeinvalue of f at the corresponding periodic point. For the brevity of notations, we will still denote $\tilde{\mathcal{L}}_f/\tilde{F}$ by $\mathcal{L}_f$.

Example 2: Let U be a simply connected domain in the complex plane and $F : U \longrightarrow F(U) \supset \bar{U}$ be a polynomial-like mapping of degree m whose filled Julia set $J(F)$ is connected. As before we consider the inverse space

$$\tilde{\mathcal{L}}_F = \{\tilde{z} = (..., z_n, ..., z_1, z_0) : z_0 \in F(U) \backslash J(F) : F(z_n) = z_{n-1}, n \in N\},$$

the fibration $\pi_F : \tilde{\mathcal{L}}_F \longrightarrow F(U) \backslash J(F)$ is defined by $\pi_F(\tilde{z}) = z_0$ and the natural extension $\tilde{F}$ is defined from $\pi_F^{-1}(U \backslash J(F))$ to $\tilde{\mathcal{L}}_F$.

The quotient of the backward orbital space $\mathcal{L}_F = \tilde{\mathcal{L}}_F/\tilde{F}$ is a Riemann surface lamination, actually it is holomorphically equivalent to $\mathcal{L}_f$ in the example 1.

Indeed the lamination $\mathcal{L}_F$ only depends on the germ of F near the Julia set. If $W \subset U$is a smaller neighborhood of $J(F)$ with $F(W) \supset \bar{W}$, then $\mathcal{L}_{F|_W}$ is the backward orbital space of $\tilde{F}$ restricted on $F(W) \setminus J(F)$ and it is clearly equivalent to the backward orbital space of $\tilde{F}$ on $\tilde{\mathcal{L}}_F$.

Definition 4.4 *: Two polynomial-like mappings* $F_i : U_i \longrightarrow F_i(U_i) \supset \bar{U}_i, i = 1, 2$, *with connected filled-in Julia sets are external (or exterior) equivalent if* *there exist $V_i \subset U_i, i = 1, 2$ and a conformal map $h : V_1 \to V_2$ such that h conjugates $F_1 : V_1 \to F_1(V_1)$ to $F_2 : V_2 \to F_2(V_2)$.*

By Doudy and Hubbard, the external equivalent classes are represented by analytic expanding circle maps up to analytic conjugations.

Fix an expanding analytic circle map f_0, suppose another such map f is quasiconformally conjugate to f_0 (on a neighborhood of the circle), then there exists a quasiconformal homeomorphism $h_{f_0,f}$ which sends $\mathcal{L}_{f_0}$ to $\mathcal{L}_f$. We say that two Riemann surface laminations $\mathcal{L}_{f_1}$ and $\mathcal{L}_{f_2}$ are conformally equivalent if $h_{f_0,f_1} \circ h^{1}_{f_0,f_1}$ is isotopic to a homeomorphism between $\mathcal{L}_{f_1}$ and $\mathcal{L}_{f_2}$ which sends leaves to leaves and which is conformal on each leaf.

If two polynomial-like mappings F_1 and F_2 are external equivalent, then the Riemann surface laminations $\mathcal{L}_{F_1}$ and $\mathcal{L}_{F_2}$ are conformally equivalent. The following theorem says the converse is also true.

Theorem 4.3 *Two analytic expanding circle maps f_1 and f_2 are analytically conjugate if and only if the corresponding Riemann surface laminations $\mathcal{L}_{f_1}$ and $\mathcal{L}_{f_2}$ are conformally equivalent.*

Proof: Without loss of generality, we assume the degrees of f_1 and f_2 are bigger that 1. The analytic conjugation between f_1 and f_2 can be lifted to a homeomorphism between $\mathcal{L}_{f_1}$ and $\mathcal{L}_{f_2}$ which is conformal on each leaf. Conversely if two Riemann surface laminations $\mathcal{L}_{f_1}$ and $\mathcal{L}_{f_2}$ are conformally equivalent then the equivalence sends a leaf associated to a periodic orbit of f_1 to a leaf associated to the corresponding periodic orbit of f_2. Since these leaves are cylinders and they are conformal equivalent, the moduli of these cylinders are equal. It follows that the eigenvalues of the corresponding periodic orbits of f_1 and f_2 are equal. The paper [SS] implies that f_1 and f_2 are analytically conjugate.

4.2 A Teichmüller Theory on Laminations

The renormalization on complex quadratic-like mappings with the same combinatorial types can be interpreted as an operator on the spaces of external classes, namely the spaces of expanding analytic maps of the unit circle up to analytic conjugacy. A metric will be introduced to these spaces via a metric on the spaces of quasiconformal Riemann surface laminations $\mathcal{L}_f$ up to conformal equivalence. Till now it is quite natural to think of a Teichmüller theory on the space of the quasiconformal Riemann surface laminations.

One can describe the classical Teichmüller theory of Riemann surfaces in terms of Beltrami coefficients and holomorphic quadratic differentials [Gardiner, Lehto]. A similar Teichmüller theory on Riemann surface laminations related to dynamical systems in the renormalization theory has been studied by Sullivan [S1, S2], de Melo and van Strien [MvS] and etc.. The completion of the Teichmüller space of external classes of expanding analytic circle maps was explicitly studied by Gardiner and Sullivan in [GS].

In the following, we give the definitions of concepts and the statements of theorems, the details of their proofs can be found in the above references.

The following special vector fields V on Riemann surface laminations will be used.

A continuous vector field V on a Riemann surface is expressed by continuous functions $V_i : z_i(U_i) \rightarrow \mathcal{C}$ on local coordinate charts $z_i : U_i \rightarrow \mathcal{C}$ satisfying the transformation rule

$$V_j(z_j) = \frac{\partial z_j}{\partial z_i} V_i(z_i)$$

where $\frac{\partial z_j}{\partial z_i}$ is the derivative of the overlapping map $z_j \circ z_i^{-1}$. The transformation rule is indicated by $V\frac{\partial}{\partial z}$.

A vector field on a Riemann surface lamination $\mathcal{L}$ associates to each flow box $z_i : U_i \rightarrow \mathcal{D} \times \Lambda$ a function $V_i : \mathcal{D} \times \Lambda \rightarrow \mathcal{C}$ such that the transformation in the overlapping boxes z_i and z_j satisfies

$$V_j(z_{i,j}(z, \lambda)) = \frac{\partial z_j}{\partial z_i} V_i((z, \lambda))$$

where $z_{i,j}(z, \lambda) = z_j \circ z_i^{-1}(z, \lambda)$ for $(z, \lambda) \in z_i(U_i \cap U_j)$.

In particular, restricted to each leaf it is a vector field on the leaf. V is continuous if it is continuous in each flow box.

A Beltrami vector μ on a Riemann surface S is expressed by measurable functions $\mu_i : z_i(U_i) \rightarrow \mathcal{C}$ on local coordinate charts with the transformation rule

$$\mu_j(z_j) = \mu_i(z_i) \frac{\overline{\partial z_i / \partial z_j}}{\partial z_i / \partial z_j}$$

where $\frac{\partial z_i}{\partial z_j}$ is the derivative of $z_i \circ z_j^{-1}$.

The transformation rule is denoted by $\mu\frac{\overline{\partial z}}{\partial z}$. We define

$$||\mu||_\infty = ess\ sup_{p \in S} |\mu(p)|$$

$$= \inf\{k : the\ Lebesgue\ measure\ of\ \{p : |\mu(p)| > k\} = 0\}.$$

A Beltrami vectro is called a Beltrami coefficient if $||\mu||_\infty < 1$. The space of all Beltrami coefficients is denoted by $L_1^\infty(S)$.

Let $h(w_1, w_2)$ denote the Poincaré distance between two points w_1 and w_2 in the unit disk. Given two Beltrami coefficients μ_1 and μ_2, the hyperbolic distance $h(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as the essential supremum of $h(\mu_1(p), \mu_2(p))$, where $p \in S$.

The Beltrami coefficient of a quasiconformal map f in the plane is given by

$$\mu_f(z) = \frac{\bar{\partial} f(z)}{\partial f(z)}.$$

The Beltrami coefficient of the composition of two quasiconformal maps f and g in the plane is expressed by the following formular:

$$\mu_{fog}(z) = \frac{\mu_g(z) + \mu_f(g(z)) \cdot \overline{\partial g(z)}/\partial g(z)}{1 + \mu_f(g(z))\mu_g(z)\overline{\bar{\partial} g(z)}/\bar{\partial} g(z)}.$$

The maximal dilatation K_f of a quasiconformal map f is equal to

$$ess\ sup_z K_f(z) = ess\ sup_z \frac{1 + \mu_f(z)}{1 - \mu_f(z)}.$$

It is easy to check

$$K_{fog^{-1}} = exp(h(\mu_f, \mu_g)).$$

Definition 4.5 *A Beltrami vector μ on a Riemann surface lamination $\mathcal{L}$ is expressed by measurable functions $\mu_i : z_i(U_i) \to C$ on flow boxes $z_i : U_i \to D \times \Lambda$ with the transformation rule*

$$\mu_j(z, \lambda) = \frac{\overline{\partial z_i/\partial z_j}}{\partial z_i/\partial z_j} \mu_i((z_i \circ z_j^{-1})(z, \lambda))$$

for all $(z, \lambda) \in z_j(U_i \cap U_j)$, and μ is weakly transversally continuous, i.e., in each flow box μ is weakly continuous in the leaf direction, in other wards, $\lambda_n \to \lambda$ implies

$$\lim_{n\to\infty} \int \mu_i(z, \lambda_n)\phi dz d\bar{z} = \int \mu_i(z, \lambda)\phi dz d\bar{z}$$

for each $\phi \in L^1(D) = \{\phi : \int |\phi| dz d\bar{z} < \infty\}$.

Let $L^\infty(\mathcal{L})$ denote the space of Beltrami vectors on $\mathcal{L}$ with the essential supremum norm. If V is a continuous vector field, $\bar{\partial} V$ is a well defined L^∞-function in the sense of distribution in each leaf, then $\bar{\partial} V \in L^\infty(\mathcal{L})$. A Beltrami vector which can be written in the form $\bar{\partial} V$ is called a infinitesimally trivial Beltrami vector.

A quadratic differential is holomorphic (meromorphic) if all its local expressions are holomorphic (meromorphic) functions. If a quadratic differential ϕ is holomorphic and non-zero in a neighborhood of a point, then there exists a chart in the neighborhood such that $\phi = dz^2$. In such a local coordinate chart the metric $\sqrt{|\phi|}$ is the euclidean metric and its geodesics are straight lines. A parametrized curve γ is called a horizontal (vertical) trajactory of ϕ if $\phi(\gamma(t))\gamma(t)^2 > 0$ (respectively < 0) in the local coordinate charts. The horizontal and vertical trajactories define two transversal foliations in the complement of the zeros of ϕ. At a zero of ϕ of order m, there are $m+2$ horizontal and $m+2$ vertical trajactories emanating from this point.

Let

$$L^1(S) = \{\phi : \phi \; is \; a \; quadratic \; differential \; with \; ||\phi|| = \int |\phi| < \infty\}.$$

The quadratic differentials in $L^1(S)$ are called integrable.

For a holomorphic covering map from a Riemann surface S' to another Riemann surface S, the pullbacks of Beltrami coefficients and quadratic differentials on S are Beltrami coefficients and quadratic differentials on S' which are invariant under deck transformations.

The pairing

$$L^{\infty}(S) \times L^{1}(S) \to \mathcal{C} : (\mu, \phi) \mapsto \int_S \mu\phi$$

gives a duality between L^{∞} and L^{1}. This duality plays a fundermental role in characterizing the infinitesimally trivial Beltrami vectors.

Definition 4.6 *A Riemann surface lamination $\mathcal{L}$ has a quadratic differential if there exist a transversally invariant measure class $[m_i]$ and measurable functions $\phi_i : z_i(U_i) \to \mathcal{C}$ defined on flow boxes $z_i : U_i \to D \times \Lambda$ and depending on the choice of measures in $[m_i]$ such that*

1) the function

$$\Lambda_{ij} : (\Lambda, [m_i]) \to (\Lambda, [m_j])$$

induced by

$$z_j \circ z_i^{-1}(z, \lambda) = (z_{ij}(z, \lambda), \Lambda(\lambda))$$

is absolutely continuous for any two overlapping flow boxes z_i and z_j,

2) given a measure m_i in the measure class $[m_i]$, ϕ_i satisfies the following transformation rule on any two overlapping flow boxes z_i and z_j

$$\phi_i(z, \lambda) = \phi_j(z_{ij}(z, \lambda), \Lambda_{ij}(\lambda))(\frac{\partial z_{ij}(z, \lambda)}{\partial z})^2 \cdot Jac(\Lambda_{ij})(\lambda)$$

where $Jac(\Lambda_{ij})$ is the Jacobian of Λ_{ij} with respect to the measure m_i and m_j.

Let

$$L^1(\mathcal{L}) = \{\phi : quadratic\, differentials\, satisfying \int |\phi| dz d\bar{z} dm < \infty\}.$$

Any element of $L^1(\mathcal{L})$ is called a integrable quadratic differential on $\mathcal{L}$. ϕ is called holomorphic if ϕ is holomporphic on almost all leaves (with respect to the measure class). The quadratic differentials (even holomorphic) does not have the continuity in the transversal direction. The existence of the holomorphic quadratic differentials on $\mathcal{L}$ can be found in [MvS].

The pairing

$$L^{\infty}(\mathcal{L}) \times L^{1}(\mathcal{L}) \rightarrow \mathcal{C} : (\mu, \phi) \mapsto \int \mu\phi dz d\bar{z} dm$$

defines a bilinear map. It is not known whether $L^{\infty}(\mathcal{L})$ is the dual space of $L^{1}(\mathcal{L})$. But the space does seperate the points in $L^{\infty}(\mathcal{L})$ through the above pairing. This is enough to characterize infinitesimally trivial Beltrami vectors.

The classical Teichmüller theory of Riemann surfaces studies conformal structures on Riemann surfaces via quasiconformal mappings, Beltrami coefficients and quadratic differentials.

By the measurable Riemann mapping theorem, each Beltrami coefficient μ of a Riemann surface S defines a conformal structure S_{μ} on S. Topologically S_{μ} is equal to S. The identity map $g : S \rightarrow S_{\mu}$ is quasiconformal with respect to the initial and the new conformal structures on S. Clearly μ is the pullback of the Beltrami coefficient 0 on S_{μ}, i.,e., $\mu = g^{*}(0)$. Since the isotopic class of a homeomorphism between two Riemann surfaces contains a quasiconformal mapping, we see that a conformal structure on S is a new Riemann surface S_1 with a quasiconformal map $F_1 : S \rightarrow S_1$. Two conformal structures $F_i : S \rightarrow S_i$, $i = 1, 2$, are equivalent in the sense of Teichmüller if there exists a conformal map $F : S_1 \rightarrow S_2$ such that $F \circ F_1$ is homotopic to F_2. The Teichmüller space $T(S)$ of S is the set of equivalent classes of conformal structures on S. Two Beltrami coefficients μ_1 and μ_2 on S are equivalent if there exists a quasiconformal homeomorphism $g : S \to S$ homotopic to the identity such that $g^*(\mu_2) = \mu_1$, where

$$g^*(\mu) = \frac{\mu_g + \mu \circ g \cdot \overline{\partial g(z)}/\partial g(z)}{1 + \mu \circ g \cdot \overline{\mu_g} \cdot \overline{\bar{\partial} g(z)}/\bar{\partial} g(z)}.$$

The Teichmüller space $T(S)$ of S is isomophic to the space of quasiconformal self-mappings of S quotient by the space of the maps which are homotopic to the identity, and it is also isomorphic to the quotient space of $L^\infty_1(S)$ under the above equivalent relation. The quotient map projects a Beltrami path of $L^\infty_1(S)$ to a curve in the Teichmüller space $T(S)$, we still call it a Beltrami path. A Beltrami coefficient $\mu$is called trivial if it is equivalent to 0, i.e., there exists g such that g is homotopic to the identity and $\mu(g) = \mu$. A Beltrami path does not deform a conformal structure in the sense of Teichmüller if all its Beltrami coefficients are trivial.

Let $g : S \to S$ be a quasiconformal homeomorphism. The conformal distortion (the maximal dilatation) of g with respect to the conformal structures μ_1 and μ_2 is defined to be

$$K_g = ess\ sup_z K_g(z)$$

where $K_g(z) = exp(h(\mu_1(z), g^*(\mu_2)(z)))$ and h is the poincae metric of the unit disk.

The Teichmüller distance between two points $[\mu_1]$ and $[\mu_2]$ is defined as the infimum of $\log K_g$ over all quasiconformal homeomorphisms $g : S \to S$ which is homotopic to the identity. This defines a complete metric on $T(S)$. It is true that if a Beltrami path is a geodesics betwweem two points of $T(S)$ then it remains a geodesics between any other two points on this path. A quasiconformal homeomorphism g between two Riemann surfaces is extremal if its maximal dilatation is the smallest in its homotopy class. Furthermore an extremal mapping f of a Riemann surface S is either conformal or its complex dilatation is of the form $k\frac{\overline{\phi}}{|\phi|}$, where $0 < k < 1$ and ϕ is a holomorphic quadratic differential of S.

Let $f : S \longrightarrow S'$ be an extremal mapping determined by a pair (ϕ, k), i.e., ϕ is a holomorphic quadratic differential on S, $0 < k < 1$ and $\mu_f = k\frac{\overline{\phi}}{|\phi|}$. Then there exists a unique holomorphic quadratic differential ψ on S' with the following properties: 1) if ϕ has a zero of order $m \geq 0$ at p, then ψ has a zero of the same order at $f(p)$; 2) if z is the natural local coordinate of ϕ at a regular point p, then the mapping f has the representation

$$w \circ f = \frac{z + k\bar{z}}{1 - k}$$

in a neighborhood of p in terms of a natural coordinate w of ψ at $f(p)$. In another word, in a neighborhood of a regular point of ϕ, the extremal mapping is a conformal transformation followed by a stretching in the direction of the horizontal trajectories and another conformal transformation.

We can define a stretching function for any quasiconformal mapping g between S and S' (S' is determined by (ϕ, k)). Let z and w be the natural local coordinates of ϕ and ψ (as the above) at the regular points p of ϕ and its image $f(p)$. Then in a neighborhood of p, g is expressed as

$$w = g(z) = g(x + iy).$$

The stretching map $D(p)$ is defined as $|w_x|$, generally it is expressed by

$$D(p) = |\frac{\partial w(z)}{\phi(z)^{1/2}} + \frac{\overline{\partial} w(z)}{\overline{\phi}(z)^{1/2}}||\psi(w(z))|^{1/2}.$$

Let $K(p)$ denote the conformal dilatation of g. Let $J(g)$ be the Jacobian of g between two metrics. Then

$$|\int D(p)d\phi| \leq \int \sqrt{K(p)}\sqrt{|J(p)|}d\phi \leq \sqrt{\int Kd\phi}\sqrt{\int Jd\phi}$$

where $D(p) \leq \sqrt{K(p)}\sqrt{|J(p)|}$. Since $|\int D(p)d\phi| \geq |S'|$,

$$\int |K(p)|d\phi \geq |S'|$$

where the area $|S'|$ of S' is equal to $\int |\psi|$.

We need to generalize some of the above properties to Riemann surface laminations. Given a quadratic differentials Φ on a Riemann surface lamination $\mathcal{L}$, $\frac{\overline{\Phi(P)}}{|\Phi(p)|}$ doesn't have to be a Beltrami vector on $\mathcal{L}$, indeed it may be neither defined on each leaf nor continuous transversally in the weak topology. But it does transform like a Beltrami vector when we change the local coordinates and hence it defines a Beltrami vector on almost every leaves of $\mathcal{L}$ with respect to the transversal measure class. So Φ still defines a Beltrami path $t \rightarrow \mu_t$ starting at μ with Beltrami vector $\frac{\overline{\Phi(p)}}{|\Phi(p)|}$ on almost all leaves (with respect to transversal measure class corresponding to Φ). Let g a quasiconformal mapping between the Riemann surface lamination $\mathcal{L}[\mu]$ and the stretched Riemann surface lamination $\mathcal{L}'$ determined by $(k, \frac{\overline{\Phi(p)}}{|\Phi(p)|})$. The stretching function $D(p)$ is defined as the above for almost all points of $\mathcal{L}'$ (with respect to the measure $|\Phi|$). Let $K(p)$ be the conformal dilatation of g and $J(p)$ be the Jacobian of g. Then we get the same inequality as the above on $D(p)$, $K(p)$ and $J(p)$. We summarize them into the following proposition.

Prop. 4.3 *Let g be a quasiconformal homeomorphism of a Riemann surface or a Riemann surface lamination $\mathcal{S}$ which is isotopoic to the identity and let ϕ be a holomorphic quadratic differential on $\mathcal{S}$. Let $|\phi|$ be the measure on almost all leaves induced by ϕ. The stretching function*

$$D(p) = \frac{|Dg(p)v|_\phi}{|v|_\phi}$$

is defined almost everywhere, where v is tangent to the horizontal trajectories of ϕ. Then

$$\int_S D(p) d\phi dm \geq ||\mathcal{S}||$$

where $||\mathcal{S}|| = \int |\phi| dz d\bar{z} dm$.

Now we are ready to give the statements of some other parallel results on Riemann surfaces and on Riemann surface laminations. A Riemann surface lamination is called hyperbolic if every leaf is a hyperbolic Riemann surface.

Theorem 4.4 *Let $\mathcal{S}$ be a compact Riemann surface or a compact Riemann surface lamination. Then*

1) The set of infinitesimally trivial Beltrami vectors is closed in $L^\infty(\mathcal{S})$ with the weak topology (this means $\mu_n \longrightarrow \mu$ when $\int \mu_n \phi \longrightarrow \int \mu \phi$ for all quadratic differentials in $L^1(\mathcal{S})$).

2) Two Beltrami vectors μ and ν differ by a trivial deformation, i.e., $\mu - \nu = \overline{\partial} V$ for some continuous vector field V on $\mathcal{S}$, if and only if $\int \mu\phi = \int \nu\phi$ for all holomorphic quadratic differentials $\phi \in L^1(\mathcal{S})$.

2) is true because of the following equality

$$\inf |\mu + \overline{\partial} V| = \sup_{|\phi|=1} \int \mu\phi$$

where the infimum runs over all continuous vector fields V on $\mathcal{S}$ and the supremum runs over all holomorphic quadratic differentials ϕ on $\mathcal{S}$with L^1-norm 1.

But the above supremum may not be realizable. This gives the rise to ϵ-efficient Beltrami vector.

Definition 4.7 *The infinitesimal Teichmüller norm of a Beltrami vector* ν *is defined by*

$$|\nu|_{IT} = \sup_{|\phi|=1} |\int \nu\phi|$$

where the supremum runs over all holomorphic quadratic differentials ϕ *with* L^1*-norm* 1.

A Beltrami vector ν *is called* ϵ*-efficient if*

$$|\nu|_{IT} \geq (1-\epsilon)||\nu||_\infty.$$

We say a Beltrami vector ν is efficient if it is 0-efficient, i.e.,

$$|\nu|_{IT} = ||\nu||_\infty.$$

Theorem 4.5 *1) Suppose* S *is a Riemann surface. Then given a Beltrami vector* μ*, there exists a Beltrami vector* ν *such that* $\mu - \nu$ *is infinitesimally trivial and such that*

$$||\nu||_\infty = \sup_{|\phi|} |\int \mu\phi|$$

where the supremum runs over all holomorphic quadratic differentials with respect to a given conformal structure.

2) Let $\mathcal{L}$ *be a compact hyperbolic Riemann surface lamination. Given a Beltrami vector* μ *and any* $\epsilon > 0$*, there exists a Beltrami vector* ν *such that*

$\mu - \nu$ is infinitesimally trivial and

$$(1-\epsilon)\cdot ||\nu||_\infty \leq \sup_{|\phi|=1} |\int_{\mathcal{L}} \mu\phi|$$

where the supremum runs over all holomorphic quadratic differentials with respect to a given complex structure.

Now we can state the first version of Almost Geodesic Principle.

Theorem 4.6 *Given $\epsilon, l > 0$, there exists $\delta > 0$ such that if ν is δ-efficient Beltrami vector at a Beltrami coefficient μ_0 on a Riemann surface or on a compact Riemann surface lamination $\mathcal{S}$ and $g_t, 0 \leq t \leq 1$, is a quasiconformal isotopy between $[\mu_0]$ and $[\mu_l]$, where μ_l is the Beltrami coefficient which is obtained from μ_0 by stretching a Poincaré distance in the direction of ν in the unit disk, then the maximal dilatation K of g_1 satisfies*

$$K \geq \frac{l}{1+\epsilon}.$$

Before we give the Almost Geodesic Characterization which includes the Almost Geodesic Principle and its converse, Let us see some difference between the space of $L_1^\infty(\mathcal{L})$ of the Beltrami coefficients on a Riemann surface lamination $\mathcal{L}$ and the space of conformal structures on $\mathcal{L}$. By a quasiconformal map between two Riemann surface laminations $\mathcal{L}$ and $\mathcal{L}'$ we mean there exists a homeomorphism g between $\mathcal{L}$ and $\mathcal{L}'$ which is quasiconformal on every leaf and locally uniformly convergent in transversal directions. Hence the Beltrami coefficient μ of g is locally continuous in L^∞ topology in transversal directions. The space of conformal structures on $\mathcal{L}$ can be represented by the space of such kinds of μ's. As we difined, a Beltrami coefficient on $\mathcal{L}$ is weakly transversally continuous, but it is not necessory to be locally continuous in L^∞ topology in transversal directions. So the space $L_1^\infty(\mathcal{L})$ is bigger than the space of conformal structures on $\mathcal{L}$.

Fix a Riemann surface lamination $\mathcal{L}$, a conformal structure on $\mathcal{L}$ is a new Riemann surface lamination $\mathcal{L}_1$ and a quasiconformal map $g_1 : \mathcal{L} \rightarrow \mathcal{L}_1$. Two conformal structures $g_i : \mathcal{L} \rightarrow \mathcal{L}_i, i = 1, 2$, are equivalent in the sence of Teichmüller if there exists a conformal map $g : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ such that $g \circ g_1$ is homotopic to g_2. The Teichmüller space $T(\mathcal{L})$ of $\mathcal{L}$ is the equivalent classes of conformal structures on $\mathcal{L}$. The Teichmüller metric is defined exactly as the same as before.

Definition 4.8 *A Beltrami coefficient μ on a Riemann surface lamination $\mathcal{L}$ is ϵ-extremal if it represents a conformal structure on $\mathcal{L}$ and the Teichmüller distance from $\mathcal{L}$ to $[\mu]$ (expressed by $\log \frac{1+|\mu|_T}{1-|\mu|_T}$) satisfies*

$$|\mu|_T \geq (1-\epsilon)||\mu||_\infty.$$

We say a Beltrami coefficient μ is extremal if it is 0-extremal, i.e., $|\mu|_T = ||\mu||_\infty$.

Theorem 4.7 *(Almost Geodesic Characterization)*

Let ϵ, $\delta > 0$, $0 < l < 1$. Then there are universal functions $\epsilon' = \epsilon'(\delta, l)$ and $\delta' = \delta(\epsilon, l)$ such that

1) If ν represents a conformal structure on $\mathcal{L}$ and is δ'-efficient then the Beltrami coefficients $\mu_t = t\nu$ for all $0 \leq t \leq \frac{l}{||\nu||_\infty}$ are ϵ-extremal.

2) If ν represents a conformal structure on $\mathcal{L}$ and the Beltrami coefficient $\mu_t = t\nu$ is ϵ'-extremal for some $0 < t \leq \frac{l}{||\nu||_\infty}$ then ν is δ-efficient.

In fact for a fixed l $\epsilon' \to 0$ *as* $\delta \to 0$ *and* $\delta' \to 0$ *as* $\epsilon \to 0$.

Let d_P denote the Poincaré metric on the unit disk and d_T denote the Teichmüller metric on $T(\mathcal{L})$. We can immediately get the following Teichmüller Contraction Principle.

Theorem 4.8 *Assume* $0 < l_0 < 1$, $0 < \lambda_0 < 1$. *Then there exists a* $0 < \lambda < 1$ *only depending on* l_0 *and* λ_0 *such that*

$$d_T(0, l\mu) \leq \lambda d_P(0, l)$$

for all $0 \leq l \leq l_0$ *whenever this inequality holds for some* $0 \leq l \leq l_0$ *and some* μ *with* $||\mu||_\infty = 1$.

Roughly speaking, the Teichmüller Contraction Principle says that if the map $t \mapsto [t\mu]$, as a map from the unit disk with the Poincare metric into the Teichmüller space with the Teichmüller metric, is strictly contracting at two given points then it is strictly contracting on all pairs of points in the same Beltrami disk and within a specified distance from the two given points. Moreover the contracting factor is independent of μ.

4.3 Contraction of Renormalization in Teichmüller Metric

As the last part of this chapter, we will construct some Beltrami vectors according to the dynamics which define the Riemann surface laminations of Example 2. The corresponding Beltrami paths define not only the paths of laminations but also the paths of dynamical systems.

Let $F : U \to V$ be a polynomial-like mapping with connected Julia set. Let $\mathcal{L}$ be the Riemann surface lamination of the germ $[F]$, i.e., $\mathcal{L}$ is the orbital space of the inverse natural extension $\tilde{F}^{-1} : \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}$ of F^{-1} to the inverse limit space of $\tilde{\mathcal{L}}$ of

$$F : U \setminus J(F) \to V \setminus J(F).$$

A Beltrami vector μ on $\mathcal{L}$ can be pulled back to a Beltrami vector $\tilde{\mu}$ on $\tilde{\mathcal{L}}$, which is invariant under $\tilde{F}$, i.e., $\tilde{F}^* \tilde{\mu} = \tilde{\mu}$. Conversely any $\tilde{F}$-invariant Beltrami vector $\tilde{\mu}$ gives rise to a Beltrami vector μ on $\mathcal{L}$. Similarily the Beltrami paths in $\mathcal{L}$ correspond to $\tilde{F}$-invariant Beltrami paths in $\tilde{\mathcal{L}}$.

The dynamical vector on $\mathcal{L}$ is defined as follow: Taking a representative F of the germ $[F]$ which defines $\mathcal{L}$, starting with any measurable and essentially bounded function μ' from a fundermental domain $V \backslash U = F(U) \backslash U$ to $\mathcal{C}$, then pulling it back to $V \backslash J(F)$ by F, we get a L^∞-function $\hat{\mu} : V \backslash J(F) \to \mathcal{C}$ which is invariant under F, i.e., $\hat{\mu}(z) = \hat{\mu}(F(z)) \frac{\overline{F'(z)}}{F'(z)}$. Now we take the pullback of $\hat{\mu}$ to $\tilde{\mathcal{L}}$ via the natural projection, it follows an $\tilde{F}$-invariant Beltrami vector $\tilde{\mu}$ on $\tilde{\mathcal{L}}$ and hence a Beltrami vector μ on $\mathcal{L}$.

It is easy to get the following properties about the dynamical Beltrami coefficients on $\mathcal{L}$.

1. Notice that $\tilde{\mathcal{L}}$ can be covered by flow boxes such that the expression of $\tilde{\mu}$ in each of these flow boxes is independent of the transversal coordinate. The dynamical Beltrami coefficients give rise to transversally locally constant conformal structures on $\tilde{\mathcal{L}}$ and hence on $\mathcal{L}$.

2. In contrast to general Beltrami vectors on a Riemann surface lamination, a Beltrami path generated by a dynamical Beltrami coefficient can

be "integrated" and give rise to a one-parameter family of Riemann surface laminations $\mathcal{L}_t$, $0 \leq t < 1$, which are quasiconformally equivalent to $\mathcal{L}$. And $\mathcal{L}_t$, $0 \leq t < 1$, define a path of dynamical systems.

3. Given two dynamical Beltrami coefficients μ_0 and μ_1, there exists a dynamical Beltrami path μ_t, $0 \leq t \leq 1$, between them.

4. The integral of the product of a dynamical Beltrami vector and a quadratic differential can be intepreted in the dynamical plane. An integrable quadratic differential ϕ on $\mathcal{L}$ can be pulled back to a quadratic differential $\tilde{\phi}$ on $\tilde{\mathcal{L}}$, which is invariant under $\tilde{F}$. But $\tilde{\phi}$ is not integrable over all $\tilde{\mathcal{L}}$. We define a push-forward map

$$\pi_* : L^1(\tilde{\mathcal{L}}) \longrightarrow L^1(V \setminus U)$$

by taking $(\pi_*\phi)(z)$ to be the average of $\tilde{\phi}$ on the fibre $\pi^{-1}(z)$ with respect to the transversal measure of the quadratic differential. More precisely, let W be a small disk around $z \in V \setminus U$ such that $F^{-n}(W) \cap W = \emptyset$ for all $n \in N$. Then there exists a flow box $z_i : \pi_F^{-1}(W) \to W \times \{0,1\}^N$ that maps each fiber $\pi_F^{-1}(z)$ onto $\{z\} \times \{0,1\}^N$. Let the quadratic differential $\tilde{\phi}$ in this flow box is represented by $\tilde{\phi}(z, \lambda)$ and let m be a measure on $\Lambda = \{0,1\}^N$ in the measure class of the quadratic differential. Then

$$(\pi_*\tilde{\phi})(z) = \int_\Lambda \tilde{\phi}(z, \lambda) dm(\lambda).$$

Let μ be a dynamical Beltrami vector constructed from μ' as before, where μ' is a L^∞-function defined on $V \setminus U$. Then

$$\int_{\mathcal{L}} \mu\phi = \int_{V \setminus U} \mu'(\pi_*\phi).$$

Remark: The set of dynamical Beltrami vectors is dense in $L^\infty(\mathcal{L})$ with respect to weak topology (see page 44, [MoS]).

We finish this chapter with the proof of contration property of renormalization operator in Teichmüller metric. With this theorem, one can label the points in the infinitely image under the renormalization operator by using internal classes (or kneading data) and external classes. Furthermore we can describe the stable manifolds and unstable sets of the renormalization operator, each stable manifold consists of all external classes in a same internal class and the unstable set consists of all internal classes (or kneading data).

Theorem 4.9 *(Contraction of Renormalization)*

If $[F]$ *and* $[\tilde{F}]$ *are germs of two infinitely renormalizable maps of Epstein class and with the same bounded combinatorial types, then there exists* $0 < \lambda < 1$ *and* $n(T)$ *depending on* $[F]$, $[\tilde{F}]$ *and the bound of the combinatorics such that*

$$d_T([R^{2n}(F)], [R^{2n}(\tilde{F})]) \leq d_T([R^n(F)], [R^n(F)])$$

for any $n \geq n(T)$.

Proof: By the complex bound, there exists $n(T)$ only depending on the the bound of the combinatorics such that, for any $n \geq n(T)$, $G = R^n(F) : D \to G(D)$ and $\tilde{G} = R^n(\tilde{G}) : D \to \tilde{G}(D)$ are two polynomial-like maps with the conformal moduli of $G(D) \setminus D$ and $\tilde{G}(D) \setminus D$ greater than $m(t) > 0$. Through the pullback arguement, one can construct a quasiconformal conjugacy g between G and $\tilde{G}$ with the maximal dilatation bounded by a number $K(T)$ depending on $[F]$, $[\tilde{F}]$ and the bound of the combinatorics.

So $d_T([G],[\tilde{G}]) \leq \log K(T)$. The "$g$" gives rise to a dynamical Beltrami coefficient with L^{∞}-norm $\leq \frac{K(T)-1}{K(T)+1}$. Assume $d_T([G],[\tilde{G}]) \neq 0$. We take a quasiconformal conjugacy which almost realizes the Teichmüller distance between G and $\tilde{G}$. The corresponding Beltrami coefficient μ gives an almost geodesic path $t\mu, 0 \leq t \leq 1$, between $[G]$ and $[\tilde{G}]$. Let $l = \frac{K(T)^{10}-1}{K(T)^{10}+1}$. Then $t\mu, 0 \leq t \leq \frac{l}{||\mu||_\infty}$, is a path between $[0]$ and $[\frac{l}{||\mu||_\infty}\mu]$. By the complex bound of the renormalization again, R^n contracts two points of the Beltrami path $t\mu, 0 \leq t \leq \frac{l}{||\mu||_\infty}$, such that

$$d_T(R^n([0]), R^n([\frac{l}{||\mu||_\infty}\mu])) \leq \log K(T) = \frac{1}{10} d_P(0, \frac{l}{||\mu||_\infty}\mu).$$

By the Teichmüller Contraction Principle, there exists $0 < \lambda < 1$ only depending on l and $K(T)$ such that

$$d_T(R^n([t_1\mu]), R^n([t_2\mu])) \leq= \lambda d_P(t_1\mu, t_2\mu)$$

for any $0 \leq t_1 \leq t_2 \leq \frac{l}{||\mu||_\infty}$.

In particular,

$$d_T(R^n([0]), R^n([\mu])) \leq \lambda d_P(0, \mu).$$

Take $d_P(0,\mu)$ close to $d_T([G],[\tilde{G}])$ enough, we finish the proof.

Corollary 4.2 *If $[F]$ and $[\tilde{F}]$ are germs of two infinitely renormalizable maps of Epstein class and with the same bounded combinatorial types then*

$$d_T(R^n([F]), R^n([\tilde{F}])) \to 0$$

as $n \to \infty$.

Chapter 5

Smooth Transition from nonchaos to chaos

In this chapter, it first includes an brief introduction of topological entropy and mathematical understanding of chaos. Then a criterion for infinite renormalizability of smooth multimodal maps is developed. We show that all smooth multimodal maps are infinitely renormalizable if they are on the boundary of chaos (the set of maps with positive topological entropy) or on the boundary of nonchaos (the interior of the set of maps with zero topological entropy). In the last part all the results in the above three chapters are employed to prove two main results in this thesis. In the space of real-coefficient polynomials of degree d, the Feigenbaum-Coullet-Tresser period-doubling bifurcations generically exist in the smooth transition from nonchaos into chaos, and the chaos and nonchaos share the exact same boundary. The second main result solves a folklore conjecture, in the space of real coefficient polynomials, the combinatorial description of the boundary of chaos (or nonchaos) is equivalent to its topological description, i.e., any real polynomial f with set of periods

$$P(f) = \{2^i : i \in Z_+\} = \{2^i : i = 1, 2, 4, \cdots, 2^n, \cdots\}$$

can approximated by polynomials with positive entropy and by polynomials with only finitely many periods.

5.1 Periodic Orbit, Topological Entropy and Chaos

In the section, we summarize some combinatorial properties of dynamics of continuous maps from an interval into itself, such as Šarkovskii's theorem, characteristic theorem of periods and periodic orbits of a topological entropy-zero endomorphism of an interval, and a brief history about mathematical definitions of chaos.

In 1964, Šarkovskii showed for endomorphisms of an interval, the existence of some periodic points implies the existence of many others.

Definition 5.1 *The Šarkovskii's order on the set of natural numbers is defined as follows:*

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2n+1 \rhd \cdots \rhd 6 \rhd 10 \rhd 14 \rhd \cdots \rhd 2(2n+1) \rhd$$

$$\cdots \rhd 2^m \times 3 \rhd 2^m \times 5 \rhd \cdots \rhd 2^m \times (2n+1) \rhd$$

$$\cdots \rhd \cdots \rhd 2^n \rhd \cdots \rhd 2^2 \rhd 1.$$

A point x is a *periodic point* of period n of a map f if $f^i(x) \neq f^j(x), 0 \leq i < j < n$, and $f^n(x) = x$. If $n = 1$, x is called a *fixed point.*

Theorem 5.1 [*Šar*] *Let $f : [0,1] \to [0,1]$ be a continuous map. If f has a periodic point of period n, then f has a periodic point of period m for any $n \rhd m$ in the Šarkovskii's order.*

Remark: Replace $[0,1]$ by the real line, theorem 5.1 still holds.

Topological entropy is an quantity describing the complexity of dynamics of a continuous self-map on a compact space or a metric space. It first appeared in [AKM] in 1965. There are three definitions for the topological entropy. One is used for continuous self-maps of compact spaces, the other two are used for continuous self-maps of metric spaces. They coincide with each other for the continuous self-maps of compact metric spaces [H1]. For a continuous piecewise monotone map of an interval, Misurewicz and Szlenk (1980) and Young (1981) showed that the topological entropy is equal to the growth of the "lap number" of the intervals of the map, i.e., the number of maximal intervals of the monotonicity of the iterate.

Let M be a compact metric sapce and $f : M \to M$ be a continuous self-map of M.

Definition 5.2 *Let α, β be open covers of M, denote*

$$\alpha \vee \beta = \{A \cap B : A \in \alpha\},$$

$$f^{-1}\alpha = \{f^{-1}(A) : A \in \alpha\},$$

$$N(\alpha) = \min\{Cardinality\ of\ \gamma : \gamma \subset \alpha, M \subset \cup_{A \in \gamma} A\}$$

and

$$h(f,\alpha) = \lim_{n \to \infty} \frac{1}{n} \log N(\vee_{i=1}^{n} f^{-i+1}\alpha).$$

The topological entropy $h(f)$ *is defined as*

$$h(f) = \sup_{\alpha} h(f, \alpha),$$

where the supremum is taken with respect to all open covers of M.

Definition 5.3 *A subset* E *of* M *is called* $(n, \epsilon) -$ *separate for* f *if for any* $x, y \in E$ *with* $x \neq y$,

$$\max_{0 \leq j \leq n} d(f^j(x), f^j(y)) > \epsilon.$$

Denote

$$s(n, \epsilon) = \max\{Cardinality\ of\ E :\ E\ is\ (n, \epsilon) - separate\ for\ f\}.$$

The topological entropy $h(f)$ *is defined as*

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon).$$

Definition 5.4 *A subset* F *of* M *is said to be* $(n, \epsilon) -$ *span for* f *if for any* $x \in M$, *there exists* $y \in F$ *such that*

$$\max_{0 \leq i \leq n} d(f^i(x), f^i(y)) < \epsilon.$$

Denote

$$r(n, \epsilon) = \min\{Cardinality\ of\ F :\ F\ is\ (n, \epsilon) - span\ for\ f\}.$$

The topological entropy $h(f)$ *is defined as*

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \log r(n, \epsilon).$$

Prop. 5.1 [*H*1] *The above three definitions of topological entropy coincide with each other for any continuous self-map of a compact metric space.*

Prop. 5.2 [*BF*] *The topological entropy of a map f is invariant under topological conjugacies.*

Definition 5.5 *Let $f : I \to I$ be a continuous piecewise monotone map. The lap number of f, denoted by $l(f)$, is the number of maximal intervals on which f is monotone. In other words, $l(f) - 1$ is the number of turning points of f.*

Theorem 5.2 [*MS*] *Let $f : I \to I$ be a continuous piecewise monotone map. Then*

$$h(f) = \lim_{n \to \infty} \frac{1}{n} \log l(f^n).$$

In particular, if f is $l-$modal then $h(f) \leq l$.

Prop. 5.3 [*MS*] *Let $f : I \to I$ be a continuous monotone map. Denote the variation of f by $Var(f)$. Then*

$$h(f) = \lim_{n \to \infty} \frac{1}{n} \log Var(f^n).$$

The lower bounds for the topological entropy were studied by A. N. Šarkovskii, L. Jonker, D. A. Rand, J. Milnor, P. Štefan, L. BLock, J. Guckenheimer, L. S. Young, W. Szlenk and etc.. The main object is to study the relation between periodic orbits and lower bounds of the topological entropy. One of the famous results is the characteristic preperty of a map with topological entropy zero. It is the following.

Theorem 5.3 ($[BF], [M1]$) *Suppose f is a continuous self-map of an interval I. Then $h(f) = 0$ if and only if f has no periodic orbit of a period which is not a power of 2.*

Furthermore, L. Bloch found that the permmutations induced by a topological entropy-zero map on its periodic orbits have to be simple. And he also proved that entropy-zero maps have no homoclinic points.

Definition 5.6 *Let P be a periodic orbit of $f \in C^0(I, I)$ of period m, where m is a power of 2 and $m \geq 2$. We say P is simple if the following is satisfied: for any subset $\{q_1, \cdots, q_n\}$ of P where n divides m and $n \geq 2$, and any positive interger l which divides m, if $\{q_1, \cdots, q_n\}$ is a periodic orbit of f^l with $q_1 < q_2 < \cdots < q_n$, then*

$$f^l(\{q_1, \cdots, q_{n/2}\}) = \{q_{n/2+1}, \cdots, q_n\}.$$

Definition 5.7 *Let I be an interval, $f \in C^0(I, I)$ and p be a fixed point of f. A point x of I belongs to the unstable manifold $W^u(p, f)$ of p if for every neighborhood of V of p, $x \in f^n(V)$ for some positive interger n. A point $x \in I$ is a homoclinic point of f if there is a periodic point p of f of period n such that $x \neq p$, $x \in W^u(p, f^n)$ and $f^{mn}(x) = p$ for some $m \in N$.*

Theorem 5.4 ($[Bl1], [Bl2]$) *A map $f \in C^0(I, I)$ has positive topological entropy if and only if it has a homoclinic point, and if and only if it has a periodic point which is not simple.*

In 1975, the first mathematical definition of chaos was given by T. Y. Li and J. A. Yorke. They proved the following theorem.

Theorem 5.5 [*LY*] *Assume that $f \in C^0(I, I)$ and f has a periodic orbit of period 3. Then f has the following properties:*

(1) The set of periods of f is N;

(2) There exists an uncountable subset S in the complement of the set of periodic orbits $PP(f)$ of f such that

(a) $\forall x, y \in S$, $x \neq y$, $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$,

(b) $\forall x, y \in S$, $x \neq y$, $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$,

(c) $\forall x \in S, \forall y \in PP(f)$, $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$.

Usually, we say f is chaotic in the sense of Li-Yorke if f satisfies the second property of the above theorem.

Nathanson [Na] proved that f is chaotic in the sense of Li-Yorke if f has a period 5 or 7. Oono [O] proved that f is chaotic in the sense of Li-Yorke if f has a period not a power of 2. But a continuous self-map of an interval is chaotic in the sense of Li-Yorke is not equivalent to its topological entropy is equal to zero since Chu and Xiong [CX] constructed an example which is of topological entropy zero but it is chaotic in the sense of Li-Yorke. Because of this reason, Zhou [Zh] modified the definition of chaos.

Definition 5.8 [*Zh*] *Let $f \in C^0(I, I)$. Let $\Omega(f)$ be the nonwandering set of f and $PP(f)$ be the set of periodic orbits of f. An uncountable subset of $\Omega(f) \setminus PP(f)$ is called chaotic under f if*

(a) $\forall x, y \in S$, $x \neq y$, $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$,

(b) $\forall x, y \in S$, $x \neq y$, $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$,

(c) $\forall x \in S, \forall y \in PP(f)$, $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$.

We say f is chaotic in the sense of Zhou if there exists a chaotic subset S under f.

Theorem 5.6 [*Zh*] *Let $f \in C^0(I,I)$. f is chaotic in the sense of Zhou if and only if $h(f) > 0$.*

In this paper, we select the definition of chaos given by Coppel.

Definition 5.9 *Let $f \in C^0(I,I)$. f is chaotic if f has a period not a power of 2, i.e., $h(f) > 0$.*

5.2 Period-doubling, Boundary of Chaos and Sawtooth Family

In this section we treat the maps f whose set of periods

$$P(f) = \{1,2,4,\cdots,2^n,\cdots\} = \{2^i : i \in Z_+\}.$$

Theorem 5.7 ([*BlH*] *or* [*M*3]) *In the space $C^k(I,I)$, $k \geq 1$, of C^k endomorphisms of an interval I, if a map f is on the boundary of positive entropy then the set $P(f)$ of its periods is $\{2^n : n \in Z_+\}$. The same is true for f on the boundary of the interior of the set of zero entropy.*

The converse is the following folklore conjecture (see also [OT]):

Conjecture A *All maps $f \in C^k(I,I)$, $k \geq 1$, with set of periods (of its periodic orbits)*

$$P(f) = \{2^n : n \in Z_+\}$$

are on the boundary of the set of positive entropy and on the boundary of the interior of the set of zero entropy.

Another conjecture related to the Conjecture A is the following.

Conjecture B *A polynomial map f with set of periods (of its periodic orbits)*

$$P(f) = \{1, 2, 4, \cdots, 2^n, \cdots\} = \{2^i : i \in Z_+\}$$

can be C^k approximated ($k \geq 1$) by polynomials with positive entropy and by polynomials with finitely many periods.

Conjecture B has been established for quadratic polynomials (as a consequence of [S1] or [La]) and will be set up for high degree polynomials in the next two sections.

We first show in this section that any *stunted sawtooth map*, whose set of periods is the set of all nonnegative integer powers of 2, can be approximated by stunted sawtooth maps with positive entropy and stunted sawtooth maps with only finitely many periodic orbits. In another word, in C^0 topology, the boundary of positive entropy consists of all members with set of periods $\{2^i : i \in Z_+\}$, and the same is true for the boundary of the interior of the set of zero entropy.

By the *sawtooth map* of shape $s_1 s_2 \cdots s_{d+1}, s_i \in \{+, -\}, s_{i+1} = -s_i$, we mean the unique map $S_d : I \to I$ which is piecewise linear with slope $s_1 d, s_2 d, \cdots s_{d+1} d$ ($d+1$ alternate values). This is a d-modal map for which topological entropy takes the largest possible value for d-modal maps, which is $\log(d+1)$. Given any critical value vector $w = (w_1, w_2, \cdots, w_d)$ satisfying the usual inequalities $(w_j - w_{j+1}) sign(j+1) > 0, w_j \in I, j = 1, 2, \cdots, d$, we obtain the *stunted sawtooth map* $S_{d,w}$ from S_d by cutting off the tops and bottoms of the graph at heights $w_j, j = 1, 2, \cdots, d$. When we do not insist on a value of d, we denote $S_{d,w}$ by S_w.

Theorem 5.8 *Suppose S_w is a stunted sawtooth map with the set of periods*

$$P(S_w) = \{2^i : i \in Z_+\}.$$

Then for any $\epsilon > 0$, there exist w' and w'' such that

$$|w' - w| < \epsilon, \quad |w'' - w| < \epsilon,$$

$h(S_{w'}) > 0$ and $S_{w''}$ has only finitely many periods.

We will prove this theorem after a few lemmas. Let us first state a corollary.

Corollary 5.1 *In the parameter space, the set $\{w : P(S_w) = \{2^n : n \in N\}\}$ has no interior point, i.e, the combinatorial description of boundary of chaos coincides with its topological description. This set forms the boundary of chaos and the boundary of the set of zero entropy.*

Remark: Theorem 5.8 (or Corollary 5.1) solves the symbolic version of of the conjectures in the sense that stunted sawtooth maps carry all kneading invariants of multimodal maps (see section 5.4).

Lemma 5.1 *If $f : I \to I$ is a continuous map with isolated turning places (points or plateaus) and $P(f) = \{2^i : i \in Z_+\}$, let $\Omega(f)$ be the set of accumulating points of period-doubling periodic points of f. Then each point in $\Omega(f)$ is not periodic, hence $\Omega(f)$ is not finite.*

Proof: Let $p \in \Omega(f)$ be a periodic point of period 2^n. Denote $g = f^{2^n}$. Then p is a fixed of g. Look at the map g around p. Because f hence g has isolated turning places (points or plateaus), there are only three types of local behaviors for g. 1. g is monotone in a small neighborhood of p (if g is monotone reversing then g^2 is monotone preserving in a small neighborhood of p). 2. p is in the interior or at the end of one of the plateaus of g. 3. p is a turning point of g. Look at three iterates of g or g^2 at one point very near p, one can easily see that neither g nor g^2 has simple periodic points with higher periods in a small neighborhood of p. This contradicts with that p is an accumulating point of period-doubling periodic points of f. □

Remark: Lemma 5.1 is false if the turning places of f are not isolated.

$f \in C^0(I, I)$ and the set of periods $P(f) = \{2^n : n \in N^+\}$. Suppose that K is a closed invariant set under f, $f|_K$ is topologically transitive and K is not finite. It is shown in [M2] that $f|_K$ is semiconjugate to the adding-one machine on dyadics. More precisely, let $\sum = \prod_1^\infty \{0, 1\}$ be the product space of countable many copies of the space $\{0, 1\}$ with weak product topology. The adding-one machine σ is defined as

$$\sigma((x_1, x_2, \cdots, x_n, \cdots)) = (x_1 + 1, x_2, x_3, \cdots, x_n, \cdots),$$

which is exactly the same as one does addition in dyadic number system. For instance $\sigma((1, 0, 1, \cdots)) = (0, 1, 1, \cdots)$.

Theorem 5.9 [*M*2] *There exists a continuous map* $h : K \to \sum$ *such that*

$$h \circ f = \sigma \circ h.$$

Furthermore $h^{-1}(s)$ *contains at most two points for any point* $s \in \sum$.

In fact in the proof of Theorem 5.9, K can be expressed as a disjoint union $K = K_0 \cup K_1$, where the supporting intervals of K_0 and K_1 have no interior intersection and $f(K_i) = K_{1-i}$, where $i = 0, 1$. Hence K_0 and K_1 are invariant under f^2. K_0 and K_1 have the same bisections for f^2 and so on.

We usually call K a period-doubling Cantor set of f. The types of symbolic renormalizations indicate how the map $f : K \rightarrow K$ goes to $f^2 : K_0 \rightarrow K_0$ ([GLOT], [OT]).

One can express K as disjoint unions

$$K = \cup_{i=1}^{2^n} K_i^{(n)}$$

for $n \in N$. Therefore at least one point of each *fiber* $h^{-1}(s)$, $s \in \sum$, is recurrent (not periodic) to itself.

Lemma 5.2 *Suppose that S_w is a stunted sawtooth map with set of periods*

$$P(S_w) = \{2^i : i \in Z_+\}.$$

Let $\Omega(S_w)$ be the set of accumulating points of period-doubling periodic points of S_w and $\Sigma(S_w)$ be the set of the endpoints of the interval I and the closures of turning places of S_w. Then the intersection of $\Omega(S_w)$ and $\Sigma(S_w)$ is not empty.

Proof: Suppose that the intersection of $\Omega(S_w)$ and $\Sigma(S_w)$ is empty. Then there exists a positive distance ϵ between these two closed sets. Let K be a closed invariant subset of $\Omega(S_w)$ which is not finite and $S_w|_K$ is topologically transitive K. Clearly the distance between K and $\Sigma(S_w)$ is greater or equal to ϵ. Suppose that $x \in K$ and its orbit is dense in K. There exists $l = 2^k, k \geq 1$, (from Theorem 5.9) such that $|f^l(x) - x| < \frac{1}{8}\epsilon$. Again by Theorem 5.9, one can assume that $f^l(x) > x$. Let V be the largest neighborhood of x on which

f^l is monotone. The slope of f^l on V is d^l and clearly $V \supset (x-\epsilon/d^l, x+\epsilon/d^l)$. We seperate our considerations in two cases. Case 1. Assume that f^l is monotone preserving on U. Since $f^l(x) > x$ and $f^l(x-\frac{7}{8}\epsilon/d^l) = f^l(x)-\frac{7}{8}\epsilon < x+\frac{1}{8}\epsilon-\frac{7}{8}\epsilon = x-\frac{3}{4}\epsilon < x-\epsilon/d^l$, there exists a point $p \in (x-\frac{7}{8}\epsilon/d^l, x)$ such that $f^l(p) = p$. The unstable manifold $U(f^l, p)$ of f^l at p contains $(p, f^l(x)+\epsilon)$. We know that $f^l(f^l(x)) < f^l(x)$ from Theorem 5.9. Let W be the largest neighborhood of $f^l(x)$ on which f^l is monotone. Then $W \supset (f^l(x)-\epsilon/d^l, f^l(x)+\epsilon/d^l)$. No matter whether f^l is preserving or reversing, $f^l(f^l(x)-\epsilon/d^l)$ (or $f^l(f^l(x)+\epsilon/d^l)$) is equal to $f^l(f^l(x))-\epsilon$, which is less than $f^l(x)-\epsilon < x+\frac{1}{8}\epsilon-\epsilon = x-\frac{7}{8}\epsilon < p$. This implies that there exists $y \in (x, f^l(x)) \subset U(f^l, p)$ such that $y \neq p$ and $f^l(y) = p$, which is a homoclinic point, a contradiction. Case 2. Assume that f^l is monotone reversing on V. Use the same argument in case 1, one can see that there exists a point $p \in (x, f^l(x))$ fixed by f^l. By the assumption and Theorem 5.9, $f^l(x)$ is recurent (non-periodic) to itself under iterates by f^l. There exists $n = 2^m, m > 0$, such that $|(f^l)^n(x) - f^l(x)| < \frac{1}{8}\epsilon$. Do similar estimates as we do in case 1, one can find a point y in the unstable manifold of f^l at the fixed point p such that $y \neq p$ and $f^{nl}(y) = p$, which is a homoclinical point, a contradiction again. $\square$

For any stunted sawtooth map, since the endpoints of the interval I are eventually fixed by S_w, from Lemma 5.1, they cannot be accumulating points of simple perioidc points. Because S_w is constant on each plateau, any interior point of every plateau cannot be an accumulating point of periodic points. Hence one has the following corollary.

Corollary 5.2 *Suppose S_w, $\Omega(S_w)$ and $\sum(S_w)$ are the same as in Lemma 5.2. Let $E(S_w)$ be the set of the endpoints of turning places of S_w. Then $\Omega(S_w)$ has no intersection with the interior of $\sum(S_w)$ and the intersection of $\Omega(S_w)$ and $E(S_w)$ is not empty.*

Proof of Theorem 5.8: All turning places, of S_w, which reduce to turnings points, are eventually fixed by S_w, from Lemma 5.1, they are not in $\Omega(S_w)$. Let $\Lambda = E(S_w) \bigcap \Omega(S_w)$. For any $\epsilon > 0$, we push the concave horizontal plateaus up a little bit and the convex horizontal plateaus down a little bit, if one of their end points belongs to Λ, to get another stunted sawtooth map $S_{w'}$ with $|w - w'| < \epsilon$. If a periodic orbit of S_w has no point in the interior of any plateau of S_w then it is also a periodic orbit of $S_{w'}$. Thus the accumulating set of periodic points of S_w is contained in the accumulating set of periodic points of $S_{w'}$ of periods $2^n, n \in Z_+$. The arguments in the proof of Lemma 5.2 show that $S_{w'}$ has a homoclinic point, which means $S_{w'}$ has positive topological entropy.

We next push all concave horizontal plateaus down a little bit and all convex horizontal up a little bit to get another stunted sawtooth map $S_{w''}$ with $|w - w''| < \epsilon$. Suppose that $P(S_{w''}) = \{2^n : n \in Z_+\}$. Then by the proof of Lemma 5.2, S_w has positive topological entropy, a contradiction. □

5.3 Renormalizability of Multimodal Maps

This a preparation for the next section.

Definition 5.10 *Let I be an interval. A map $f : I \to I$ is called renormalizable if there exists a proper subinterval J of I and an integer p such* *that*

(1) $f^i(J), i = 0, 1, \cdots, p-1$, *have no pairwise interior intersection,*

(2) $f^p(J) \subset J$.

Then $f^p|_J : J \to J$ *is called a renormalization of* f *and* p *is called the* period *of the renormalization. The permutation induced by* f *on the set* $\{f^i(J) : i = 0, 1, \cdots, p-1\}$ *is called the combinatorial type of the renormalization.*

Definition 5.11 *A map* $f : I \to I$ *is infinitely renormalizable if there exist an infinitely sequence* $\{I_n\}_{n=1}^{\infty}$ *of nested intervals and an infinitely sequence* $\{u(n)\}_{n=1}^{\infty}$ *of integers such that* $f^{u(n)}|_{I_n} : I_n \to I_n$ *are renormalizations of* f *and the length of* I_n *tends to zero as* $n \to \infty$.

In this section we are going to prove that maps f with $P(f) = \{2^i : i \in Z_+\}$, which satisfy some smooth conditions, are infinitely renormalizable.

Definition 5.12 *Let* $I \subset R$ *be an interval and* $\{x_0, x_1, \cdots, x_n\}$ *be a partition of* I. *A continuous map* $f : I \to R$ *is*

– *of bounded Zygmund variation if there exists* $B > 0$ *such that*

$$\sup_{\{x_0, x_1, \cdots, x_n\}} \sum_{i=0}^{n-1} |f(x_i) + f(x_{i+1}) - 2f(\frac{x_i + x_{i+1}}{2})| \leq B,$$

– *of bounded quadratic variation if there exists* $B > 0$ *such that*

$$\sup_{\{x_0, x_1, \cdots, x_n\}} \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i))^2 \leq B.$$

Definition 5.13 *A homeomorphism* $f : I \to R^1$ *is said to be* $C^{1+b.Z.v+b.q.v}$ *if* f *is* C^1 *smooth and the logarithm of the derivative* f' *has bounded Zygmund variation and bounded quadratic variation.*

Definition 5.14 *Let K_f be the set of the turning points of a map $f : I \to I$. We say that f belongs to $\Gamma(\lambda)$ for $\lambda \in \{2, 1 + b.Z.v. + b.q.v.\}$ if*

a) f is C^λ away from the turning points;

b) For every $x_0 \in K_f$, there exists $\alpha > 1$, a neighborhood $U(x_0)$ of x_0 and a C^λ-diffeomorphism $\phi : U(x_0) \to (-1, 1)$ such that $\phi(x_0) = 0$ and

$$f(x) = f(x_0) \pm |\phi(x)|^\alpha, \quad \forall x \in U(x_0).$$

Notice that $\Gamma(2) \subset \Gamma(1 + b.Z.v + b.q.v)$ (see [HS]). Using nonwandering interval theorem ([MMS], [HS]) and finiteness of attractors [Ma], one has

Theorem 5.10 *Let f be a d-modal map with $P(f) = \{2^n : n \in Z_+\}$. Then if $d = 1$ and $f \in \Gamma(1 + b.Z.v. + b.q.v)$ or $d > 1$ and $f \in \Gamma(2)$, f is infinitely renormalizable.*

In fact it will be shown that for any unimodal map in $\Gamma(1+b.Z.v+b.q.v)$ or any multimodal map in $\Gamma(2)$ the supporting intervals of $K_i^{(n)}, i = 1, 2, \cdots, 2^n$ (see section 5.2) can be taken as dynamical intervals in renormalization when n is large enough. In fact it is sufficient to prove that the semiconjugacy in the Theorem 5.9 can be improved to be a conjugacy if f satisfies the smooth conditions.

$f \in C^0(I, I)$, a periodic orbit $\{x_i : i = 0, 1, \cdots, n-1\}$ is called an *attracting cycle* (resp. an *one-side attracting cycle*) if there exists a neighborhood (resp. an one-side neighborhood) U of x_0 such that for any $x \in U$, $f^{nl}(x) \to x_0$ as $l \to \infty$. An interval $J \subset I$ is called a *homoterval* if $f^n|_J$ is monotone for any $n \in N$. An open interval $J \subset I$ is called a *wandering interval* of f if 1) $f^n(J) \cap f^m(J) = \emptyset$ for any $n \neq m, n, m \in N$ and 2) $f^n(J)$ does not converge to a periodic orbit.

Theorem 5.11 ([*Ma*], [*MMS*], [*MvS*], [*HS*])

(1) If $f \in \Gamma(1 + b.Z.v + b.q.v)$ *then* f *has no wandering intervals.*

(2) (Finiteness of Attractors) For any $f \in \Gamma(2)$ *there exist* n_0 *and* $\rho > 0$ *such that for any periodic point* x *of* f *of period* $n \geq n_0$ *one has*

$$|(f^n)'(x)| > 1 + \rho.$$

Lemma 5.3 *Suppose* $f \in C^0(I, I)$ *has no attracting cycles and the set of its period* $P(f) = \{2^n : n \in N^+\}$. *Let* K *be the same as in Theorem 4.9, and* $K = K_0 \cup K_1$ *with* $f(K_i) = K_{1-i}, i = 0, 1$. *Assume that* $\sup\{x : x \in K_0\} < \inf\{x : x \in K_1\}$. *Let* $[K_i]$ *denote the smallest closed interval containing* $K_i, i = 0, 1$. *Then for* $i = 0, 1$ *there exists a periodic point* q *of* f *with period* $n = 1$ *or* 2 *in the gap between* $[K_0]$ *and* $[K_1]$ *such that the unstable manifold* $U(f^n, q)$ *of* f^2 *at* q *contains* $[K_i]$.

Proof: By Theorem 5.9, there exists a fixed point p of f in the gap between $[K_0]$ and $[K_1]$. Clearly the unstable manifold $U(f, p)$ is connected and is invariant under f. For $i = 1$, let $s = \inf\{x : x \in K_1\}$, consider the map f^2 on the interval $[p, s]$. p is fixed by f^2. Let q be the largest fixed point of f^2 in $[p, s]$. Because of Lemma 5.1, $q \neq s$ and then $f^2(y) > y$ for any $y \in (q, s)$. Clearly $f^2(K_1) \subset K_1$ and $f^2(s) > s$. Therefore s, hence $[K_1]$, is in the unstable manifold $U(f^2, q)$ of f^2 at q. Similar argument applies to the case when $i = 0$. □.

From Theorem 5.9 and Lemma 5.3, one has

Corollary 5.3 *Suppose* $f \in C^0(I, I)$ *has no attracting cycles and the set of its period* $P(f) = \{2^n : n \in N^+\}$. *Let* K *be the same as in Theorem 5.9, and*

$K = \bigcup_{i=1}^{2^n} K_i^{(n)}$, $n \in N$. Then for any $n \in N$ there exists a periodic point p of period m, where $m = 2^n$ or 2^{n+1}, satisfying that the orbit $\{f^i(p) : i = 0, 1, 2, \cdots, m-1\}$ is contained in the set $\bigcup_{i=1}^{2^n}[K_i^{(n)}] \setminus \bigcup_{i=1}^{2^{n+1}}[K_i^{(n+1)}]$ and the unstable manifold $U(f^m, p)$ contains some $K_i^{(n+1)}$, where $1 \leq i \leq 2^{n+1}$.

Theorem 5.12 *A continuous map $f : I \to I$ with set of periods $P(f) = \{2^n : n \in Z_+\}$ has no homotervals, no plateaus, only finitely many turning points and finitely many periodic attractors. K is the same as in Theorem 5.9. Then the semiconjugacy h in Theorem 5.9 is actually a conjugacy between $f : K \to K$ and $\sigma : \sum \to \sum$.*

Proof: Suppose that semiconjugacy is not a conjugacy. Then there exists a point $s \in \sum$ such that $h^{-1}(s) = \{x, y\}$, $x \neq y$. We claim that $h^{-1}(\sigma^n(s))$ eventually contains a single point when n is large enough. Let I_n denote the supporting interval of $h^{-1}(\sigma^n(s))$, $n \geq 0$. Since there are only finitely many turning points and I_n, $n \geq 0$, are pairwise disjoint, there exists $m > 0$ such that I_n contains no turning points for any $n \geq m$. If the claim is not true, then I_m is a homoterval, a contradiction. Therefore one can assume that $f(x) = f(y)$, where $x, y \in K$ and $x \neq y$. We separate our considerations into two cases.

Case 1. Suppose that f is monotone preserving in some neighborhoods of x and y (the proof is same when f is monotone reversing around x and y). From Corollary 5.3, there exists a periodic point p of f of period n, $n = 2^k$, $k > 0$, such that the unstable manifold $U(f^n, p)$ of f^n at p contains the interval $[x, y]$, p is not in $[x, y]$ and p is very near x or y. From the continuity of f, when p is near x (or q) enough, $f(p)$ is very near $f(x) = f(y)$. By the intermediate value theorem, there are at least two points $a, b \in (x, y)$ such that $f(a) = f(b) = f(p)$. Then $f^n(a) = f^n(p) = p$. Hence a is a homoclinic point, a contradiction.

Case 2. Suppose that f is monotone preserving in a neighborhood of x and monotone reserving in a neighborhood of y (the proof is same when the situation is reversed). Denote $x_t = f^t(x) = f^t(y)$, $t \in N$. Let $\{c_i\}_{i \in \alpha}$ denote the set of turning points of f, where α is a finite set. Clearly there exists $t_0 > 0$ such that $(x_t, c_i) \bigcap K \neq \emptyset$ for any $t \geq t_0$, denoted by $K_{(t,i)}$. In the interval (x, y) we select v and w near x and y respectively with $f(v) = f(w)$. Because of the nonexistence of homotervals, the itineraries of $f(p) = f(q)$ and $f(v) = f(w)$ under f are eventually different. Let $v_t = f^t(v)$, $t \in N$. Then there exists $t > t_0$ such that there is at least one $c_i \in (x_t, v_t)$. Clearly $K_{(t,i)} \subset (x_t, v_t)$ and $f^t((x, v)) = f^t((w, y)) \supset (x_t, v_t)$. From Theorem 5.9 and Corollary 5.3, there exists a periodic point p of period $n = 2^k$, $k > 0$, where p is not in $[x, y]$, such that p is very near x (or y), $f^j(p)$ is in the supporting interval $[K_{(t,i)}]$ and the unstable manifold $U(f^n, p) \supset [x, y]$. Clearly there exists a point $u \in (x, v)$ (or $u \in (w, y)$) such that $f^t(u) = f^j(p)$. Hence $u \in U(f^n, p)$, $u \neq p$ and $f^{nt}(u) = f^n(f^j(p)) = p$. So u is a homoclinic point, a contradiction.

Case 3. There are other cases besides Case 1 and Case 2, in which at least one of x and y is a turning point. The proofs for these cases are just slight modifications of Case 1 or Case 2. □

Theorem 5.13 *A continuous map $f : I \to I$ with set of periods $P(f) = \{2^n : n \in Z_+\}$ has no homotervals, no plateaus, only finitely many turning* *points and finitely many periodic attractors. Then f is infinitely renormalizable.*

Proof: Let K be the same as in Theorem 5.9. From Theorem 5.12, $f : K \to K$ is conjugate to $\sigma : \sum \to \sum$ by h. Separate the turning points of f into two parts. Let S_1 (resp. S_2) denote the turning points of f contained (resp. not conatined) in K. Denote $K = \cup_{i=1}^{2^n} K_i^{(n)}, n \in N$. There exists $n_0 \in N$ such that for any $n > n_0$, $S_2 \cap \cup_i^{2^n}[K_i^{(n)}] = \emptyset$. It is sufficient to prove for any $x \in S_2$ there exists n_0 such that for any $n > n_0$, x is not in $\cup_i^{2^n}[K_i^{(n)}]$. Suppose not, there exists an infinite sequence of nested intervals $[K_{i(n)}^{(n)}]$ containing x. Hence $x \in \cap_{n \in N}[K_{i(n)}^{(n)}]$. Since the end points of $\cap_{n \in N}[K_{i(n)}^{(n)}]$ are in the same fiber of h, which has to be a point, hence x is in K, a contradiction. Now let $n > n_0$. Then f maps each $[K_i^{(n)}]$ onto another $[K_j^{(n)}]$ in the same level. Therefore f is infinitely renormalizable. □

Proof of Theorem 5.10: Suppose f is an unimodal map in $\Gamma(1 + b.Z.v. + b.q.v)$. From the proof of Theorem 5.12, it is enough to show the conjugacy h (as the same as in Theorem 5.9) is a homeomorphism. Suppose not, then there exists a point $s \in \sum$ such that $h^{-1}(s) = \{p, q\}$, $p \neq q$. $h^{-1}(\sigma^n(s))$ cannot contain two points for all $n \geq 0$. Otherwise let I_n be the supporting intervals of $h^{-1}(\sigma^n(s))$, $n \geq 0$. Since I_n, $n \geq 0$ are pairwise disjoint, only one member of $\{I_n : n \geq 0\}$ can contain the turning point. Hence there exists $m > 0$ such that none of $\{I_{m+i} : i \geq 0\}$ contains the turning point. So I_m is a homoterval which give rise to a wandering interval, it contradicts with (1) of Theorem 5.11. Therefore we can assume that $f(p) = f(q)$ and $p < q$. Clearly $(p, q) \cap K = \emptyset$. Let c be the turning point of f. Then $c \in (p, q)$. Compare the itenaries of $f(p)$ and $f(c)$, if they are same then produce a homoterval, hence a wandering interval, if they are not same then either f has positive topological entropy or f has only finite many periodic orbits because of the monotonicity of kneading data of unimodal maps (see [MiT], [MaT]). We have shown that h is a homeomorphism.

Suppose f is a multimodal map in $\Gamma(2)$. (1) and (2) of Theorem 5.11 tell us that f satisfies the conditions in Theorem 5.12. From Theorem 5.12 we finish the proof. □

The techniques of [S1], [BloLy] and [H2] (or see chapter 2) show the following geometric property of period-doubling Cantor sets.

Theorem 5.14 *Suppose that f is an unimodal map in $\Gamma(1+b.Z.v+b.q.v)$ or a multimodal map in $\Gamma(2)$ with set of its periods $P(f)=\{2^n : n\in N^+\}$, and a closed invariant set K is not finite and $f|_K : K \to K$ is topologically transitive. Then the Lebesgue measure of K is zero and the Hausdorff dimension of K is greater than 0 and less than 1.*

5.4 Rigidity Results

An interesting question, from the point of view of mathematics and physics, is how a dynamical system, depending on parameters, make a transition from simple to complicated behaviour, according to whether the system is in the interior of zero entropy or it has positive entropy.

For any one-parameter family of smooth unimodal maps, for example, the quadratic family

$$f_a = ax(1-x),\ \in [0,4],$$

there exists a special map f_∞ which is on the boundary of chaos and is an accummulation of period-doubling bifurcations. A lot of other properties have been stated in the introduction (or see chapter 2). f_∞ is infinitely renormalizable.

Kneading theory [MiT] provides a way to classify piecewise monotone maps from the viewpoint of the combinatorics of the iterates of turning points. It can also be used to characterize the combinatorical types of the maps on the boundary of chaos in the space of maps with finitely many turning points.

Let $I = [C_0, C_{d+1}]$ be an interval and $f : I \longrightarrow I$ be a d-modal map, i.e., a piecewise monotone continuous map with $d \geq 1$ turning points $C_1 < C_2 < \cdots < C_d$ seperating I into $d+1$ intervals $J_0, J_1, \cdots, J_d$ in order, on which f alternates between increasing and decreasing. A d-modal map is called *multimodal* if $d > 1$ and *unimodal* if $d = 1$. To each point $x \in I$ one can associate the *itinerary* $I(x)$, defined to be the sequence of symbols $I_n \in \{J_0, C_1, J_1, \cdots, C_d, J_d\}, n \in Z_+$ such that $f^n(x) \in I_n$. The *kneading sequence* k_j are defined to be the itineraries of $f(C_j), j \in \{0, 1, \cdots, d+1\}$, and the *kneading invariant* of a map f to be the $(d+2)$-tuple $K(f) = (k_0, k_1, \cdots, k_{d+1})$. The kneading invariant is preserved under orientation preserving topological conjugacy. For most purposes, the itineraries of the end points are not important, so only $k_1(f), \cdots, k_d(f)$ are required.

Kneading theory provides a partition of the space of piecewise monotone maps with finite number of turning points into monotone equivalent classes [G3]. As is known, maps on the boundary of chaos with only one turning point ("unimodal maps") all belong to a single monotone eqivalent class [CE].

The kneading invariant $k(f_\infty)$ is

$$X^{(\infty)} = k(f) = RLRRRLRLRLRRRLRR\cdots.$$

$X^{(\infty)}$ can be written down inductively. Let

$$X^{(\infty)} = x_1 x_2 \cdots x_n \cdots,$$

then $x_1 = R$, $x_2 = L$, the symbols from x_{2^k+1} to $x_{2^{k+1}-1}$ is the copy of the symbols from x_1 to x_{2^k-1} and $x_{2^{k+1}}$ is the opposite of x_{2^k}, where $k = 1, 2, 3, \cdots$.

There is a partial order on the space of the kneading sequences of maps with the same number of turning point. The paper [MiT] (see also [MaT]) gives a complete description of the boundary of topological entropy-zero maps in the space of C^p unimodal maps, $p \geq 1$, in terms of kneading datas.

Theorem 5.15 *Let f be an increasing-decreasing unimodal map of an interval to itself. Then*

(1) f has positive topological entropy $\Longleftrightarrow k(f) > X^{(\infty)}$,

(2) f is on the boundary of chaos in C^p, $p \geq 1 \Longrightarrow k(f) = X^{(\infty)}$,

(3) $k(f) < X^{(\infty)} \Longrightarrow f$ is in the interior of the set of topological-entropy-zero maps in $C^p, p \geq 1$.

Different from the case of unimodal maps, for multimodal maps, for instance bimodal maps, the boundary of chaos intersects uncountably many monotone equivalent classes: [MaT] gives a combinatorial description of the set of monotone equivalent classes of bimodal maps f whose set of periods $P(f) = \{2^n : n \in Z_+\}$. An analogue of the above theorem is also given in [MaT], but it is very complicated, we would rather not state here.

Let us finish the introduction of kneading theory with the following theorem.

Theorem 5.16 [*DGMT*1, 2] *For any integer d, the stunted sawtooth family* $S_{d,w}$ *is complete in the sense that for any d-modal map f, there is a canonical stunted sawtooth map* $S_{d,w}$ *which has exactly the same kneading invariant as f.*

Remark: By this theorem, we say that in some sense Theorem 5.8 (or Corollary 5.1) solves the symbolic version of the Conjecture A and B (see section 5.2).

If a system goes along a smooth curve which connects nonchaos with chaos in $C^k(I, I)$, $k \geq 1$, how does the dynamics of the system changes? Roughly thinking, any continuous curve from nonchaos into chaos has hit a point at the accumulation of an infinite sequence of period-doubling bifurcations. The paper [BlH] studies how the set of periods of a map changes as the map goes through a continuous arc in $C^1(I, I)$ in terms of the Šarkovskii's ordering.

Let $F(n)$ denote the set of maps in $C^1(I, I)$ which have a periodic point of period n. Let $\lhd$ denote the Šarkovskii's ordering on nature numbers. Let $G(n)$ denote those $f \in F(n)$ such that f is not contained in $f(m)$ if $m \rhd n$. We may include the symbol ∞ by defining $k \lhd \infty$ if $k = 2^i$ for some integer $i \geq 0$, and $\infty \lhd k$ otherwise. Those maps in $F(\infty)$, but not in $F(m)$ for any m with $m \rhd \infty$, will be denoted by $G(\infty)$. Block and Hart proved the following intermediate value result for continuous arcs in $C^1(I, I)$, in terms of the Šarkovskii's ordering.

Theorem 5.17 [*BlH*] *Let* f_t *be a continuous arc in* $C^1(I, I)$, $0 \leq t \leq 1$, *with $f_0 \in G(n)$ and $f_1 \in G(m)$. Then for any k such that $n \lhd k \lhd m$, there exists t_0 such that $f_{t_0} \in G(k)$.*

It follows that any continuous arc f_t in $C^1(I,I), 0 \leq t \leq 1$, with $h(f_0) = 0$ and $h(f_1)$, must pass through a point in $G(\infty)$. In other words, any continuous arc in $C^p(I,I), p \geq 1$, from nonchaos to chaos must contains a map whose $P(f) = \{2^i : i \geq 0\}$.

Theorem 5.17 tells us that in the space $C^k(I,I)$, $k \geq 1$, of C^k endomorphisms of an interval I, if a map f is on the boundary of positive entropy then the set $P(f)$ of its periods is $\{2^n : n \in Z_+\}$, and the same is true for f on the boundary of the interior of the set of zero entropy. The converse of this theorem is stated as Conjecture A (see section 5.2).

It is studied in [Be], [GLOT] and [OT] how period-doubling periodic orbits evolve as a system walks along a curve from nonchaos into chaos. what we mainly concern are the asymptotic ratios of successive changes among parameters corresponding to the period-doubling bifurcations, and the structure of the set of maps $P(f)$ with set of period $\{2^i : i \in Z_+\}$. So far we cannot give definite answers in general to these questions, but we are able to solve them in the space of real polynomials. The techniques of geometric analysis will be applied.

The renormalization idea naturally appeared in the study of the dynamics of maps in $G(\infty)$. A self-map of an interval is called renormalizable if there exists some subinterval which will be mapped into itself under certain iterates. The infinite renormalizability of smooth maps with finitely critical points is given in section 5.2.

For a real polynomial $f \in G(\infty)$, by Theorem 5.10, we know that it is infinitely renormalizable. Its renormalizations are of bounded types. The real bound's properties (see chapter 2) and the complex bound's property (see chapter 3) follow. Before we prove several rigidity results stated in the introduction, let us review a little bit in polynomial-like mapping theory.

In the space of polynomial-like mappings of degree d with filled-in Julia sets, an internal class consists of those that are quasiconformally conjugate in the neighborhoods of filled-in Julia sets and the conjugacies have no complex dilatations in the filled-in Julia sets. Two polynomial-like mappings are externally equivalent if they are conjugated by an analytic map in an annulus neighborhood along the filled-in Julia set.

The following theorem is a corollary of the Straightening Theorem [DH] (see section 3.4).

Theorem 5.18 *1) The real-coefficient polynomials with connected filled-in Julia sets cut each internal class at most once.*

2) The external classes of an internal class are bijectively equivalent to the set of real analytic expanding maps of the unit circle S^1 up to real analytic conjugacy.

A Beltrami coefficient μ (see chapter 4) is *an invariant Beltrami line bundle* for a complex analytic map F is

$$\mu(z) = \mu(F(z))\frac{\overline{F'(z)}}{F'(z)}$$

for almost all z in the domain.

The following theorem is a general version of the main theorem in [Mc].

Lemma 5.4 $[Mc]$ *Assume P is an infinitely renormalizable real polynomial and the closures of critical orbits are the same. Then there is no invariant Beltrami line bundle lying on the filled-in Julia set of P.*

Theorem 5.19 *If two real polynomials P, $Q \in G(\infty)$ have the same kneading invariants, and the closures of critical orbits of P (respectively Q) are the same, then P and Q are conjugate by an affine map.*

Proof: It is a consequence of Theorem 5.10, 3.10 and the Straightening Theorem. □

Let us repeat the main theorem of chapter 4.

Theorem 5.20 (Teichmüller Contraction Principle) $[S1]$

Let F and G be two infinitely renormalizable maps of Epstein class and with the same bounded combinatorial types. Then

$$D(R^n(F), R^n(G)) \to 0$$

as $n \longrightarrow \infty$.

An immediate consequence of this theorem is

Theorem 5.21 (Coullet-Tresser Rigidity) *All infinitely renormalizable maps, of Epstein class and the same bounded combinatorial types, have corresponding critical orbital Cantor sets of the same asymptotically scaling ratios.*

Theorem 3.10, the Straightening Theorem and Teichmüller Contraction Principle of the renormalization operator give a rise to the following theorem.

Theorem 5.22 *The codimension of stable manifold, associated with each point F in the image of all iterates of renormalization operator, is the number of critical points of F (counted with multiplicity). If F is a unimodal map one gets a codimension 1 stable manifold associated with it. If F is a bimodal map one gets a codimension 2 stable manifold.*

In another word, we can use internal classes to label the points in the unstable set of the renormalization operator and external classes to label the points in the stable manifolds.

Now the following two rigidity results are naturally of the consequences of Thoerem 5.19 and 5.22.

Theorem 5.23 *The Feigenbaum-Coullet-Tresser period-doubling bifurcations generically exist in the smooth transition from nonchaos into chaos in the space of real-coefficient polynomials of degree d.*

Theorem 5.24 *In the space of real-coefficient polynomials of degree d, nonchaos and chaos share the same boundary.*

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